Scale Symmetry and the Non-Equilibrium Quantum Dynamics of Ultra-Cold Atomic Gases

by

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Scale Symmetry and the Non-Equilibrium Quantum Dynamics of Ultra-Cold Atomic Gases

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Abstract

The study of the quantum dynamics of ultra-cold atomic gases has become a forefront of atomic research. Experiments studying dynamics have become routine in laboratories, and a plethora of phenomena have been studied. Theoretically, however, the situation is often intractable unless one resorts to numerical or semiclassical calculations. In this thesis we apply the symmetry associated with scale invariance to study the dynamics of atomic gases, and discuss the implications of this symmetry on the full quantum dynamics. In particular we study the time evolution of an expanding two-dimensional Bose gas with attractive contact interactions, and the three-dimensional Fermi gas at unitarity. To do this we employ a quantum variational approach and exact symmetry arguments. It is shown that the time evolution due to a scale invariant Hamiltonian produces an emergent conformal symmetry. This emergent conformal symmetry has implications on the time evolution of an expanding quantum gas. In addition, we examine the effects of broken scale symmetry on the expansion dynamics. To do this, we develop a non-perturbative formalism that classifies the possible dynamics that can occur. This formalism is then applied to two systems, an ensemble of two-body systems, and for the compressional and elliptic flow of a unitary Fermi gas, both in three spatial dimensions.
Lay Summary

A theoretical understanding of the motion of ultra-cold atomic gases is quite complex. In order to understand this, a thorough knowledge of strongly interacting systems is required. Without the aid of special tools, like symmetries, such a problem would simply be impossible. In this work we investigate how the symmetry associated with fractals, i.e. scale symmetry, allows one to make concrete predictions about the dynamics of ultra-cold gases, and how their density profiles evolve with time, even for strongly interacting systems.
Preface

All of the research was done by the author, J. Maki.

The results in Chapter 3 are based on two published works. The first of which was done with the assistance of Mohammadreza Mohammadi: J. Maki, M. Mohammadi, and F. Zhou, Phys. Rev. A, 90, 063609 (2014). In this work I acted as the lead investigator, responsible for the primary analysis. M. Mohammadi, worked on the early stages of the project. F. Zhou was the supervisory authority on the project, and aided on both the analysis, and the manuscript composition.

Chapter 3 is also based off the work: J. Maki, S.J. Jiang, and F. Zhou, Euro. Phys. Letts. 118, 5 (2016). In this work, I was the lead investigator, responsible for the primary analysis of these works. S.J. Jiang helped develop the analysis, and contributed to the manuscript edits. Again, F. Zhou was the supervisory authority on the project, and aided on both the analysis, and the manuscript composition.

The results contained in Chapters 4, 5, and 6 were published in the work: J. Maki, L.M. Zhao, and F. Zhou, Phys. Rev. A 98, 013602 (2018). In this work, I acted as the lead investigator, responsible for the development of the theory, the analysis, and the manuscript. L.M. Zhao aided with the analysis and the numerics. F. Zhou was the supervisory authority on the project, and aided on both the analysis, and the manuscript composition.

The results of Chapter 7 are unpublished results which will appear in an upcoming article written by myself, and my supervisor, Fei Zhou. In this work, I acted as the lead investigator, under the aid of my supervisor F. Zhou. In this work we both worked on the analysis and the manuscript.
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I would like to thank my supervisor Fei Zhou for his continued support over the course of my study. In addition, my heartfelt thanks go to Shao-Jian Jiang and Li-Ming Zhao for the numerous discussions and help.
Dedication

I would like to thank my parents, my family, Jarett, Josh, Marcel, Tali, and all my friends back home for their constant love, good cheer, and support. Keep being the lovely people you are.
Cold atom systems have been praised for the extreme amount of control available to experiments. In experiments, one is able to tune almost every key feature of a system using lasers or magnetic fields; including the dimensionality of the gas [5], the interaction [6], and many other facets. It is not surprising that people have considered atomic gas systems as a candidate for new quantum mechanical technologies [7–9], and as quantum simulators [10–13], i.e. a quantum system that one can fine tune to replicate a more realistic, or complicated, quantum system. Already atomic systems have been used to simulate the physics inside an atom’s nucleus [12], and high temperature superconductors [11, 13].

In order to truly create new technologies from atomic gases, a comprehensive understanding of atomic gases is needed. Not only does one need to have knowledge of the energetic and thermodynamic properties of these gases, but even their dynamical properties. In recent years, it has become routine for experiments to create non-equilibrium atomic systems and to study how they evolve in time. Almost any conceivable non-equilibrium dynamical experiment can be performed in a laboratory. This has led to studies on expansion dynamics [14–17], breathing modes [18–21], quench dynamics [6, 15, 22–24], time dependent trap dynamics [4, 25], thermalization and localization [23, 26–29], periodic driving [29–34], and more.

Although dynamical experiments are routine, theoretical and numerical studies of dynamics can be quite difficult. In principal, to determine the dynamics of a quantum system with \(N\) atoms, it is necessary to solve the many body Schrodinger equation:

\[
i\partial_t \psi(\{\vec{r}_i\}, t) = H\psi(\{\vec{r}_i\}, t),
\]

where \(\psi(\{\vec{r}_i\}, t)\) is the many body wave function describing a system of \(i = 1, 2, ...N\) particles with positions, \(\{\vec{r}_i\}\), and Hamiltonian, \(H\). Solving Eq. (1.1) is an extremely complex problem. In order to study the dynamics, it is necessary to obtain the full spectrum of the \(N\)-particle many body system. This is usually impossible, as even understanding the ground state of an arbitrary many body system can be daunting. For this reason, one
numerical simulations, or semiclassical approximations.

Numerically, the study of dynamics is often limited to one dimensional models, where the Bethe ansatz can be employed \cite{35,38}. An extension of this ansatz to higher dimensions simply does not exist. For higher dimensions, different schemes, such as time dependent density functional theory \cite{39,41}, need to be applied to dynamics. These numerical schemes often utilize the scaling properties of the equation of state, and rely on phenomenological parameters imported from experiments or other numerical simulations. For this reason, a complete analytic understanding of dynamical phenomena, from microscopic first principles, is not always possible. In terms of theoretical studies, the study of dynamics is often limited to semiclassical approximations \cite{42,45}. These approximations render the problem tractable, but do not describe all the quantum effects present in dynamics. Beyond these treatments, the understanding of non-equilibrium quantum dynamics remains a difficult problem.

One important tool for simplifying the complexities of many body physics is symmetry. If a symmetry is present in a system, certain aspects of the physics will be governed by the symmetry alone, regardless of the microscopic details of the system. That is, many conceivably different systems can exhibit the same physics if they obey the same symmetry. The notion of symmetry has been paramount in other areas of modern physics, and has been used to understand phenomena such as phase transitions \cite{46,47}, the standard model of particle physics \cite{48}, and crystallography \cite{49}.

For the study of dynamics, it is possible to employ the same procedure; to look for symmetries to simplify the problem. In this case, it is necessary to discuss symmetries that simultaneously transform both the spatial and temporal coordinates. A specific dynamical symmetry of interest is scale symmetry, i.e. the invariance associated with scale transformations. Scale symmetry is the invariance of the physics under dilations of spatial and temporal coordinates. In the context of quantum gases, a system is scale invariant if the governing equation of motion, the many body Schrödinger equation, Eq. [1.1] is invariant under the scale transformation:

\[
\vec{r}'_i = e^{-b}\vec{r}_i \quad i = 1, 2, ..., N \quad t' = e^{-2b}t,
\]

where the set of coordinate \(\{\vec{r}_i\}\) are the position of the \(N\) particles in the system, and \(b\) is a scaling factor.

Generally speaking, scale symmetries have had resounding success in studying a wide array of time independent problems, such as the physics at
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Phase transitions [46, 47, 50, 51], as well as complex biological systems [52]. For dynamical systems, the dynamical scale symmetry defined in Eq. 1.2 can lead to drastic simplifications. In particular, dynamical scale symmetry will equate the dynamics at two points in space and time, up to an overall scaling factor, if those two points can be equated by a scale transformation. For this reason, as well as the resounding success in thermodynamics, it is fitting to apply scale symmetry to the problem of dynamics.

Thankfully, there are a number of experimentally accessible cold atom systems which are, or nearly are, scale invariant. Beyond the trivial non-interacting quantum gas, systems such as the two dimensional quantum gas [53–55], the high dimensional degenerate Bose gas at unitarity, [24, 56, 57], the three dimensional Fermi gas at unitarity [3, 16, 58], the Tonks-Girardeau gas in one dimension [59–61], the two dimensional Fermi gas at a p-wave resonance [62], also exhibit, or nearly exhibit, scale symmetry. For these systems, scale symmetry has been applied to quantities such as the equation of state [24, 56, 57, 62–65], frequencies and damping rates of collective modes [20, 21, 66], and expansion dynamics [16, 42, 45].

Apart from numerical methods, there have been two main theoretical approaches for studying the effect of scale invariance on dynamics. The first method is to apply a scaling solution to the dynamics [42–45]. The scaling solution is a variational approach that is often limited to the semi-classical approximation. However, this approximation does not necessarily require scale invariance, as it also provides an approximate description of the dynamics for non-scale invariant systems. Although the scaling solution is generally applied to semiclassical systems, in Refs. [2, 67] we utilized a scaling solution in order to capture the effects of quantum fluctuations on the N-body bound states of low dimensional Bose gases, and the dynamics of two dimensional Bose gases.

The second major approach for studying scale invariant dynamics is to use the Heisenberg equation of motion. This approach is useful for addressing the role of scale invariance on the full quantum dynamics of global observables [4, 25, 66, 68]. When the scale symmetry is broken, the Heisenberg equation of motion becomes much more difficult, and potentially intractable without any additional knowledge about the microscopic properties of the system. As well, the Heisenberg equation of motion is very specialized to a given observable; it lacks any microscopic, local, information that is carried in the many body wave function.

A microscopic approach which utilizes the scaling property of the Schrödinger equation, Eq. 1.1, was put forward in Refs. [69, 70]. In this approach, one uses a transformation that studies the dynamics of the N-body wave func-
tion in an expanding non-inertial reference frame. Although this microscopic wave function approach is valid for a variety of systems [71], it is quite beneficial for scale invariant systems. This approach can still be difficult as one still needs to determine the full spectrum of an effective $N$-body Hamiltonian, but it can give insights into the dynamics of scale invariant systems. A simple application of this approach is for the dynamics of systems initially prepared in the ground state of a harmonic potential. In this case, it was shown that the expansion dynamics of the many body wave function is equivalent to the scaling solution [71]. As a result, the dynamics of all local observables are equivalent to a time dependent rescaling. This method was used to study the dynamics of three dimensional Fermi gases at unitarity [69, 70, 72], Bose gases [71], the Tonks gas in one dimension [73], and for harmonically trapped scale invariant systems [68, 74].

In general, a full microscopic categorization of scale invariant dynamics, for arbitrary initial conditions, is lacking. This is especially true for strongly interacting scale invariant systems, where the many body wave function is intractable. Even the energetics of these systems are not completely understood, although tremendous progress has been made [57, 62, 64, 75–77]. For scale invariant dynamics, a number of outstanding questions remain; what are the signatures of scale invariance in dynamics, are these signatures independent of the interactions present in the system in question, how do the dynamics differ for different scale invariant systems, and is there any relationship between the dynamics and the energetics when one breaks the scale invariance?

In this thesis we address these issues and categorize the effects of scale invariance on the quantum dynamics of cold atom systems, in the presence, or absence of, resonant interactions. To do this, we utilize the fact that scale symmetry is intimately connected to another dynamical symmetry, conformal symmetry [78]. Conformal symmetry corresponds to the invariance of the equations of motion under the following transformation:

$$r_i' = \frac{r_i}{1 - bt}, \quad i = 1, 2, ..., N, \quad t' = \frac{t}{1 - bt}. \quad (1.3)$$

The first application of this symmetry was in the context of non-relativistic field theory [78, 79]. For these field theories, it was shown that the spectra can be arranged into a series of towers, where the level spacing in a given tower is a constant [79]. Following the success of conformal symmetry in field theory, later this symmetry was applied to understand the dynamics of cold atomic gases. The first application of conformal symmetry in cold gases was presented in Ref. [66], where the authors showed that the breathing modes
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of scale invariant gases in harmonic traps form a tower of excitations, and the frequency of the oscillations occur at twice the trap frequency.

Inspired by the previous symmetry arguments, [66, 70, 79], we show that the quantum dynamics due to scale invariant Hamiltonians are constrained by an emergent conformal symmetry. The presence of this new symmetry allows one to make analytical statements about the quantum dynamics, independent of the microscopic details of the system, and the initial conditions. In addition, we develop a non-perturbative formalism that categorizes the breaking of scale invariance, using both exact symmetry arguments, and the microscopic wave function method [69, 70]. This formalism allows one to relate the effect of the broken scale invariance on the dynamics to how a perturbation behaves under a scale transformation. This result allows one to delineate the dynamics near different scale invariant points, and to construct a scaling argument for the dynamics of nearly scale invariant systems.

The remainder of this thesis is organized as follows. In chapter 2, we define scale and conformal transformations. The relationship between these two symmetries and their action on quantum operators is discussed. In chapter 3 we study the dynamics of a two dimensional Bose gas using an approach we developed, the quantum variational approach. In this quantum variational approach, it is possible to elucidate the effects of scale invariance on the expansion dynamics of an initially inhomogeneous Bose gas. In chapter 4 we put forward the exact implications of scale symmetry. This can be done by exploiting the connection between scale transformations, conformal transformations, and an overarching symmetry, the SO(2,1) symmetry. This overarching symmetry allows one to make concrete predictions about the dynamics of any scale invariant system. In chapter 5 this analysis is extended to the case of nearly scale invariant systems, where scale symmetry is explicitly, but slightly, broken. Here we develop a non-perturbative approach that categorizes the possible deviations from scale invariant dynamics. In chapter 6 we apply our non-perturbative formalism to the two-body system interacting with s-wave interactions in three spatial dimensions. In Chapter 7 we apply this formalism to the expansion dynamics of a resonant Fermi gas of N particles in three spatial dimensions. In particular, we examine the moment of inertia for the gas, and how scale and conformal symmetry affect the dynamics of compressional and elliptic flows. We then conclude and summarize our results in Chapter 8.
Chapter 2

Definition of Scale and Conformal Invariance

Before discussing the effects of scale symmetry, it is necessary to review the notion of symmetry transformations in quantum mechanics. In particular, how to perform a generic transformation in quantum mechanics and how to show that a transformation is a symmetry of the system.

In general, a given transformation can be associated with a unitary operator, $U$, that enacts the transformation. We will be focused on transformations that can be done continuously. These transformations have unitary operators of the form:

$$U(b) = e^{-ibG}, \quad (2.1)$$

where $b$ can be thought of as a generalized angle that parametrizes the magnitude of the transformation, while $G$ is a Hermitian operator that is called the generator of the transformation.

In this thesis we will be primarily focused on operators and how they transform under scale and conformal transformations. For this reason we will work in the Heisenberg representation where operators have time dependence, but the quantum state does not:

$$O(\vec{r}, t) = e^{iHt}O(\vec{r})e^{-iHt}. \quad (2.2)$$

where $O(\vec{r})$ is an arbitrary operator.

In this representation all the quantum dynamics are contained in the operator, and the Schrodinger equation is replaced with the equivalent Heisenberg equation of motion:

$$i\partial_t O(\vec{r}, t) = [H, O(\vec{r}, t)], \quad (2.3)$$

where $H$ is the Hamiltonian for the system.

For a transformation to be a symmetry of the system, the Heisenberg equation of motion for the symmetry transformed operator:
2.1 Scale Symmetry

\[ O_b(\vec{r}, t) = U^\dagger(b)O(\vec{r}, t)U(b), \quad (2.4) \]

must be equivalent. To be more explicit, the Heisenberg equation of motion after the symmetry change:

\[ U^\dagger(b)i\partial_iO(\vec{r}, t)U(b) = U^\dagger(b)[H, O(\vec{r}, t)]U(b), \]
\[ i\partial_tO_b(\vec{r}, t) = [U^\dagger(b)HU(b), O_b(\vec{r}, t)], \quad (2.5) \]

must be equivalent to the original equation of motion, Eq. 2.3.

2.1 Scale Symmetry

We are now in a position to define scale symmetry. The generator of scale transformations is the operator:

\[ D = -i \int d^d r \psi_\dagger(\vec{r}) \left( \frac{d}{2} + \vec{r} \cdot \nabla_r \right) \psi(\vec{r}), \quad (2.6) \]

where \( \psi^{(i)}(\vec{r}) \) is the annihilation (creation) operator which annihilates (creates) a particle at position \( \vec{r} \). For concreteness we consider Bosonic particles, such that the creation and annihilation operators satisfy:

\[ [\psi(\vec{r}, t), \psi_\dagger(\vec{r}', t)] = \delta(\vec{r} - \vec{r}'). \quad (2.7) \]

The following discussions are equally valid for Fermions.

Let us consider the Heisenberg equation of motion for the field operator, \( \psi(\vec{r}, t) \). As shown in Appendix A, the action of the scale transformation on the field operator is:

\[ \psi_b(\vec{r}, t) = e^{iD_b}\psi(\vec{r}, t)e^{-iD_b} \]
\[ = e^{-db/2} \psi \left( \vec{r} e^{-b}, t e^{-2b} \right). \quad (2.8) \]

As one can see, the generator of scale transformations does indeed perform the transformations defined in Eq. 1.2. The overall rescaling of the field operator is present to conserve normalization. The equation of motion for the field operator then transforms to:

\[ e^{-2b}i\partial_i\psi(\vec{r}, t') = \left[ e^{iD_b}He^{-iD_b}, \psi(\vec{r}', t') \right], \quad (2.9) \]
2.1. Scale Symmetry

where the coordinates \( \vec{r}' \) and \( t' \) are defined in Eq. 1.2.

To see if the equation of motion is left invariant, we need to consider the action of a scale transformation on the Hamiltonian. Consider a generic Hamiltonian with two body interactions \( V(\vec{r}) \):

\[
H = \int d^d\vec{r} \psi^\dagger(\vec{r}) \left( -\frac{\nabla^2}{2} \right) \psi(\vec{r}) + \frac{1}{2} \int d^d\vec{r} d^d\vec{r}' V(\vec{r} - \vec{r}') \psi^\dagger(\vec{r}') \psi^\dagger(\vec{r}) \psi(\vec{r}) \psi(\vec{r}').
\]

The commutator between this Hamiltonian and the generator of scale invariance is found to be:

\[
[D, H] = 2iH - i \int d^d\vec{r} d^d\vec{r}' \left[ \left( 1 + \frac{\vec{r} \cdot \nabla \vec{r} + \vec{r}' \cdot \nabla \vec{r}'}{2} \right) V(\vec{r} - \vec{r}') \right] \psi^\dagger(\vec{r}') \psi^\dagger(\vec{r}) \psi(\vec{r}) \psi(\vec{r}').
\]

If a system is said to be scale invariant, we require the commutator between the generator of scale transformations and the Hamiltonian to be proportional to the Hamiltonian. Otherwise, new terms in the Hamiltonian will be generated under a scale transformation. Therefore, we require that the second term on the right hand side of Eq. 2.11 vanishes. If this is true, all terms in the Hamiltonian physically scale like the kinetic energy operator. That is, a scale invariant Hamiltonian, \( H_s \), is defined to be:

\[
[D, H_s] = 2iH_s.
\]

In general, we note that the scaling dimension, \( \Delta_O \), of a given operator, \( O \), as:

\[
[D, O] = \Delta_O iO.
\]

Given Eq. 2.12 one can show that a scale invariant Hamiltonian transforms as:

\[
e^{iDb} H_s e^{-iDb} = e^{-2b} H_s.
\]

For now let us consider the Hamiltonian in Eq. 2.9 to be scale invariant. Upon substituting this result into Eq. 2.9 one can see that the Heisenberg equation of motion in the new coordinates is equivalent to the original equation of motion. In this way we define scale invariance.
2.2 Conformal Symmetry

In this section we define a closely related symmetry, conformal symmetry. This transformations is more abstract than scale symmetry, and is generated by the operator:

$$C = \int d^d r \, r^2 \psi^\dagger(\vec{r}) \psi(\vec{r})$$ \hspace{1cm} (2.15)

The generator of conformal symmetry, alongside the scale invariant Hamiltonian, $H_s$, and the generator of scale transformations, $D$, form a representation of the group $SO(2,1)$ [66]:

$$[H_s, C] = -iD,$$
$$[D, H_s] = 2iH_s,$$
$$[D, C] = -2iC.$$ \hspace{1cm} (2.16)

A more detailed discussion of this symmetry will be presented in Chapter 4.

For now, consider the action of a conformal transformation on a field operator $\psi(\vec{r}, t)$ [78]. A calculation equivalent to the case of a scale transformation gives:

$$\psi_b(\vec{r}, t) = e^{iCb} \psi(\vec{r}, t) e^{-iCb}$$
$$= (1 - bt)^{-d/2} e^{-i\frac{d}{2} \frac{b}{1-bt}} \psi \left( \frac{\vec{r}}{1 - bt}, \frac{t}{1 - bt} \right).$$ \hspace{1cm} (2.17)

One can again compute the effect of a conformal transformation on the equation of motion for a field operator. If the Hamiltonian governing the system is scale invariant, one can show that the equation of motion is left invariant under the transformation:

$$\vec{r}' = \frac{\vec{r}}{1 - bt} \quad t' = \frac{t}{1 - bt}.$$ \hspace{1cm} (2.18)

For more details see Appendix A.

In general the presence of these two symmetries will restrict the dynamics of any operator $O$ with scaling dimension $\Delta_O$. I.e. the operator will transform as:

$$O_b(\vec{r}, t) = e^{-\Delta_O b} O \left( \vec{r} e^{-b}, t e^{-2b} \right),$$ \hspace{1cm} (2.19)
under scale transformations, and as:

\[ O_b(\vec{r}, t) = (1 - bt)^{-\Delta \phi} O \left( \frac{\vec{r}}{1 - bt}, \frac{t}{1 - bt} \right), \quad (2.20) \]

under conformal transformations.

As one can see, scale and conformal transformations affect both the spatial and temporal coordinates. These two symmetries relate the dynamics at two different points of space and time. That is, with some previous knowledge of the system at one point in space and time, it is possible to predict the dynamics at a different point in space and time\(^1\). It is for this reason we expect scale invariant (and conformal invariant) dynamics to be analytically tractable.

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\(^1\)Here we ignore the effect of initial conditions, which can add another length scale that breaks these symmetries. However, as we will see below, the initial conditions will be irrelevant in the long time dynamics.
Chapter 3

Dynamics of Two-Dimensional Bose Gases

We begin our study of scale invariant systems by considering a so-called quintessential example of a scale invariant quantum gas, the two-dimensional Bose gas with short ranged, attractive, interactions. This system can be readily created in experiments [53, 54], and is a prime candidate for the investigation of scale invariant dynamics. The Hamiltonian for the interacting two dimensional Bose gas is given by:

\[ H = \int d^2r \psi^\dagger(\vec{r}) \left( -\frac{\nabla^2}{2} \right) \psi(\vec{r}) + \frac{g}{2} \int d^2r \psi^\dagger(\vec{r}) \psi^\dagger(\vec{r}) \psi(\vec{r}) \psi(\vec{r}), \]  

(3.1)

Naively one expects this Hamiltonian to be scale invariant as the coupling constant, \( g \), is dimensionless. However, this is not true when one examines the system.

For attractive interactions, \( g < 0 \), the Hamiltonian is not well defined at short length scales, i.e. in the ultraviolet limit. The process of regularizing the potential results in the presence of a new length scale associated with the two-body bound state. The energy of this bound state is given by:

\[ E_b = -\Lambda^2 e^{-4\pi/g}, \]

(3.2)

where \( \Lambda \) is the ultraviolet cut-off for the theory [67]. This new bound state explicitly breaks the scale invariance, as this bound state energy will change under a scale transformation. Equivalently, one can say that the interaction strength, \( g \), is no longer a constant, but is a function of this new energy scale:

\[ g = -\frac{4\pi}{\log(\Lambda^2/|E_b|)}. \]

(3.3)

This is known as a quantum anomaly: the act of quantization breaks the classical symmetry of the system.
Chapter 3. Dynamics of Two-Dimensional Bose Gases

The effects of the quantum anomaly on the thermodynamics of two-dimensional Bose gases have been studied theoretically [80–83]. A number of experiments have probed the magnitude of scale invariance breaking in the energetics and dynamics. In terms of energetics, experiments determining the equation of state have not been able to detect a significant effect from the quantum anomaly at weak interactions. As discussed in Refs. [53, 54], an energetic signature of scale invariance is that the equation of state can be expressed as a homogeneous function of the chemical potential and temperature. As a result, the equation of state will have a scaling form similar to Eq. 2.19, but in terms of the chemical potential and temperature. This has been reproduced quite well in experiments and is evidence for the anomaly being a weak correction to the scale invariance.

In terms of dynamics, it was first shown in Ref. [66] that a harmonically trapped two-dimensional Bose gas would exhibit breathing modes, at exactly twice the trap frequency. This is due to the presence of a hidden symmetry in the system, namely the SO(2,1) symmetry. The presence of the quantum anomaly will naturally break the scale invariance, and the SO(2,1) symmetry, which will result in a correction to the breathing mode frequency. Although a number of theoretical works have examined how the anomaly affects this frequency [65, 84–87], the experimental evidence has also shown that the dynamics are only weakly perturbed by the quantum anomaly [55]. Since both the experiments and dynamics provide strong evidence for the scale invariance of a two-dimensional Bose gas, we will neglect the effect of the quantum anomaly on the dynamics and examine the signatures of scale invariance in the expansion dynamics of an inhomogeneous two-dimensional Bose gas.

In the following discussions, we will show using an approach we developed, the quantum variational approach, that the continuous scale invariance is broken. This breaking is not due to the quantum anomaly, but rather due to the fact that it is experimentally impossible to create a true two-dimensional system; there is always a finite confinement radius. Although the continuous scale invariance is broken, it is replaced by a discrete scale invariance. This discrete scale invariance will affect the expansion dynamics of an initially inhomogeneous Bose gas. In particular, we show that the scale invariance manifests in a logarithmic rise in the density profile near the center of the condensate, and a set of discrete scale invariant beat frequencies.
3.1 A Formal Solution to Dynamics

To begin, we will write a formal solution to the problem. In order to determine the density, \( n(\vec{r}, t) \), it is necessary to evaluate:

\[
n(\vec{r}, t) = \langle \psi_0 | e^{iHt} \hat{n}(\vec{r}) e^{-iHt} | \psi_0 \rangle. \tag{3.4}
\]

where the density operator: \( \hat{n}(\vec{r}) = \hat{\psi}^\dagger(\vec{r}) \hat{\psi}(\vec{r}) \) \( H \) is the Hamiltonian defined in Eq. [3.1], and \( |\psi_0 \rangle \) is the initial many body state. Although this expression is completely general, it is beneficial to insert a complete set of coherent states \( \{|\psi(\vec{x})\rangle\} \):

\[
n(\vec{r}, t) = \int D\psi D\psi' \langle \psi_0 | e^{iHt} \{\psi(\vec{x})\} \langle \{\psi(\vec{x})\} | e^{-iHt} | \psi_0 \rangle \nonumber
\]

\[
\left[ \psi^*(\vec{r}) \psi'(\vec{r}) \right] \langle \{\psi(\vec{x})\} | \{\psi'(\vec{x})\} \rangle. \tag{3.5}
\]

The coherent states are eigenstates of the annihilation operator:

\[
\hat{\psi}(\vec{r}) |\{\psi(\vec{x})\}\rangle = \psi(\vec{r}) |\{\psi(\vec{x})\}\rangle, \tag{3.6}
\]

while the eigenvalues satisfy the normalization condition:

\[
\int d^2 r |\psi(\vec{r})|^2 = N. \tag{3.7}
\]

In addition, it is also possible to write the transition amplitude, \( \langle \{\psi(\vec{x})\} | e^{-iHt} | \psi_0 \rangle \), in terms of functional integrals [88]:

\[
\langle \{\psi(\vec{x})\} | e^{-iHt} | \psi_0 \rangle = \int' D\psi e^{iS}, \tag{3.8}
\]

where \( S \) is the action for a non-relativistic Bose gas:

\[
S = \int d^2 x \int_0^t dt' \psi^*(\vec{x}, t') \left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2} \right) \psi(\vec{x}, t) - \frac{g}{2} \psi(\vec{x}, t)^4, \tag{3.9}
\]

and \( \int' D\psi \) denotes the sum over all field configurations \( \psi(\vec{x}, t) \) which satisfy the following boundary conditions:

\[
\psi(\vec{x}, T) = \psi(\vec{x}) \quad \psi(\vec{x}, 0) = \psi_0(\vec{x}). \tag{3.10}
\]

\(^2\)In this section, the annihilation operators are labelled by hats. They are not to be confused with their corresponding eigenvalue. This notation is only present in this chapter as we will exclusively deal with the eigenvalues of the annihilation operator, and not the operator itself.
3.2 Semiclassical Solution

Although everything stated so far is general, it is impossible to evaluate the density profile by means of brute force. For this reason the problem is often treated semiclassically. To do this, we minimize the action in Eq. [3.9], and examine only the fields that minimize the action. This results in the well known Gross-Pitaevskii equation [89]:

\[
i\partial_t \psi_{sc}(\vec{r}, t) = \left( -\frac{\nabla^2 r}{2} - g|\psi_{sc}(\vec{r}, t)|^2 \right) \psi_{sc}(\vec{r}, t).
\]

Eq. [3.11] is scale invariant, and possesses a scaling solution of the form:

\[
\psi_{sc}(\vec{r}, t) = \sqrt{N} \lambda^2(t) f\left(\frac{r}{\lambda(t)}\right) e^{i \frac{\lambda(t)}{2} \frac{r^2}{\lambda(t)}},
\]

where \( \lambda(t) \) is a yet to be determined function. The function, \( f(x) \), is chosen to match the initial density profile of the Bose gas, while the phase is chosen to satisfy conservation of probability. For more details we refer the reader to Appendix B.

The function \( \lambda(t) \) can be determined by substituting Eq. [3.12] into Eq. [3.11] and integrating out the position dependence. The result is a differential equation for \( \lambda(t) \):

\[
m\ddot{\lambda}(t) = \frac{V}{\lambda(t)^3}
\]

where \( m = C_1 N \) and \( V = NC_2 + C_3 g N^2 \), and \( C_1, C_2, C_3 \), are constants that depend on the specific shape of the Bose gas [67].

Eq. [3.13] is a classical, scale invariant, equation of motion for a particle in an inverse square potential: \( V(\lambda) \propto \lambda^{-2} \). The semiclassical variational approach reduces a quantum many body problem to a single particle classical mechanics problem.

3.3 Quantum Variational Approach

The results presented in the previous section neglect quantum fluctuations. However, we show in Refs. [1, 67] that it is possible to treat the quantum fluctuations in low dimensions by means of coarse graining. This can be done by noting two things; the first is that the inner products between two different field configurations is approximately given by:
3.3. Quantum Variational Approach

\[ \langle \{ \psi(\vec{r}) \} | \{ \psi'(\vec{r}) \} \rangle \approx \delta (\{ \psi(\vec{r}) \} - \{ \psi'(\vec{r}) \}) \cdot (3.14) \]

This is valid for dense condensates, \( n(\vec{r}, t) \gg 1 \), as small deviations in many-body coherent states lead to nearly orthogonal states, i.e. the orthogonality catastrophe.

The second simplification is to express the fields \( \psi(\vec{r}, t) \) in terms of slow and fast degrees of freedom:

\[ \psi(\vec{r}, t) = \psi_\lambda(\vec{r}, t) + \delta \psi(\vec{r}, t). \cdot (3.15) \]

The fields \( \delta \psi(\vec{r}, t) \) represent short wavelength, fast, many body fluctuations, such as phonons, and the fields \( \psi_\lambda(\vec{r}, t) \) represent the isotropic, long wavelength, slow, degrees of freedom that parametrize the motion of the gas as a whole. We parametrize the slow degree of freedom using a single parameter ansatz, identical to the semiclassical solution:

\[ \psi_\lambda(\vec{r}, t) = \sqrt{\frac{N}{\lambda^2(t)}} f\left(\frac{r}{\lambda(t)}\right) e^{i e^2 \lambda(t) \frac{\lambda^2(t)}{4m(t)}} \cdot (3.16) \]

The separation of the slow-isotropic and fast-anisotropic degrees of freedom leads to a controllable expansion of the many body fluctuations. This is valid in the limit of dense condensates \[1\]. A full calculation is shown in Appendix \[B\] here we quote the result; an effective action for the parameter \( \lambda(t) \):

\[ n(\vec{r}, t) = \int_0^\infty d\lambda \ \frac{N}{\lambda^2} f\left(\frac{r}{\lambda}\right) | \langle \psi_\lambda | e^{-iHt} | \psi_0 \rangle |^2 \]

\[ \langle \psi_\lambda | e^{-iHt} | \psi_0 \rangle = \int_{\lambda(0)=\lambda_0}^{\lambda(t)=\lambda} D\lambda(t) e^{i \int_0^t dt' \frac{1}{2m} \lambda^2(t') + \frac{V}{2m(t')^2}}. \cdot (3.17) \]

where \( |\psi_\lambda\rangle \), is a state of a condensate with width \( \lambda \), and again \( m = C_1 N \) and \( V = C_2 N + C_3 g N^2 \). In Eq. \[3.17\], we assume the initial wave function is well described as a condensate with size \( \lambda_0 \). It is important to note that if one takes the semiclassical approximation to Eq. \[3.17\], one can reproduce Eq. \[3.13\].

The key thing to note is that Eq. \[3.17\] is equivalent to a single particle quantum mechanical problem:

\[ H_\lambda = \frac{\hat{P}_\lambda^2}{2m} + \frac{V}{2\lambda^2}. \cdot (3.18) \]
3.3. Quantum Variational Approach

In this effective description, $|\lambda\rangle$ is an eigenstate of the operator $\hat{\lambda}$: $\hat{\lambda}|\lambda\rangle = \lambda|\lambda\rangle$, representing a condensate with size $\lambda$, while $\hat{P}_\lambda$ is the momentum conjugate to $\hat{\lambda}$.

The calculation of the density profile reduces to:

$$n(\vec{r}, t) = \int_0^\infty d\lambda \frac{N}{\lambda^2} f \left( \frac{\vec{r}}{\lambda} \right) |\psi(\lambda, t)|^2$$

(3.19)

where $\psi(\lambda, t) = \langle \lambda | e^{-iH_\lambda t} | \psi_0 \rangle$, is the wave function associated with the size of the condensate. In this quantum variational approach, we replace the many body dynamics with a single particle quantum mechanical problem, where the single particle represents the macroscopic size of the condensate.

Although Eq. 3.17 is equivalent to Eq. 3.19, a natural question to ask is whether quantum fluctuations in the size of the condensate are important. To examine this, it is necessary to consider the magnitude of the fluctuations, $\delta \lambda$, around the semiclassical solution of size $\lambda_0$: $\langle \delta \lambda(t) \delta \lambda(t) / \lambda_0^2 \rangle$. If the fluctuations are small compared to the semiclassical path, then Eq. 3.13 is an accurate description of the dynamics. As shown in Appendix B for attractive interactions with $V < 0$, the amplitude of fluctuations is not controlled by the size of the semiclassical path, but by the scale independent parameter $1 / \sqrt{m|V|} \propto 1 / (N \sqrt{gN})$. This states that for mesoscopic condensates, quantum fluctuations are important and it is best to consider the full quantum mechanical problem associated with Eq. 3.18. For the remainder of this chapter we will focus on this limit.

We end this section with a brief discussion on the spectrum of Eq. 3.18 for attractive interactions. The spectrum consists of a continuous set of scattering states, $\psi_s^{(1)}$ and $\psi_s^{(2)}$, with energies $E = k^2 / 2m$, and a discrete set of bound states, $\psi_b$, with energies $E_n = -k_n^2 / 2m$, where (up to normalization factors):

$$\begin{align*}
\psi_s^{(1)} &= Re \sqrt{k\lambda} J_a(k\lambda) \\
\psi_s^{(2)} &= Re \sqrt{k\lambda} Y_a(k\lambda) \\
\psi_b &= \sqrt{k\lambda K_a(k_n\lambda)} \\
k_n &= k_0 \exp \left( -\frac{n\pi}{\sqrt{m|V|}} \right).
\end{align*}$$

(3.20)

The functions $J_a(x)$, $Y_a(x)$, and $K_a(x)$ are the Bessel J, Bessel Y and modified Bessel K functions of order $a = i / \sqrt{m|V| - 1/4}$, respectively, and $n = 1, 2, 3...$ Here we focus on condensates with $m|V| > 1/4$ and $|g| \ll 1$, or equivalently, $C_2/C_3N < |g| \ll 1$. 

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It is important to notice that Eq. [3.18] is singular near $\lambda = 0$, and is strictly speaking ill defined. As a result, when the motion of $\lambda$ is quantized, it is necessary to introduce an ultraviolet scale, $k_0$, which regularizes the singular potential. This new length scale is not associated with the quantum anomaly, but rather with the confinement radius of a quasi-two-dimensional geometry [82]. Practically, this length scale breaks the continuous scale invariance. However, a discrete scale invariance can be induced as a result of the original, continuous, scale invariance. This piece of physics is reminiscent of Efimov physics, where the three body bound states also satisfy a discrete scaling relation [90]. The discrete scale invariance is explicit in the bound state spectrum which is equally spaced on a logarithmic scale.

3.4 Dynamics of an Inhomogeneous Bose Gas

At this stage one can consider the dynamics of a condensate which is initially prepared with size $\lambda_0$. The initial amplitude can be represented as:

$$\psi(\lambda, t = 0) = \langle \lambda | \psi_0 \rangle = \frac{1}{(\pi)^{1/4}} \frac{e^{-(\lambda - \lambda_0)^2 / 2\sigma^2}}{\sqrt{\sigma}},$$

where the spreading, $\sigma$, is fixed by requiring that the energy of the effective model is identical to the microscopic model: $\sigma = \lambda_0 / (\sqrt{C_1} N)$. Below we present our numerical solutions of Eqs. [3.19] and [3.18] with the initial state given by Eq. [3.21].

In order to evaluate the transition amplitude at time $t$, it is necessary to examine how the initial state in Eq. [3.21] is projected onto the complete set of eigenstates. The amount of probability projected onto the bound states depends on the ratio of the potential energy to the kinetic energy: $\sqrt{m|V|/N}$. The more kinetic (potential) energy the system possesses, the more probability will be concentrated in the scattering (bound) states. Once the projection of the initial state is known, the unitary evolution can be carried out to obtain the probability density, $|\psi(\lambda, t)|^2$. We focus on the limit when the time $t \gg \sqrt{m|V|\lambda_0^2}$, when the initial transient dynamics have disappeared. We present the probability density in Figs. 3.1a-b).

3.4.1 Dynamics of the Density Profile

It is now possible to evaluate the dynamics of the density profile. We find two robust features due to the scale invariance of Eq. [3.18]. The first feature depends on the shape of the density profile. Near the origin, the density
3.4. Dynamics of an Inhomogeneous Bose Gas

Figure 3.1: The numerical solution of the probability density, $|\psi(\lambda,t)|^2$, and the resulting density profile. Here the semiclassical solution is $\lambda_{sc}(t) = (m|V|)^{1/4}\sqrt{t}$. a) For $\lambda \ll \lambda_{sc}(t)$ (only the scattering state contribution is shown, see main text) when $\lambda_0/\sigma = 50$, $m|V| = 50$ and $t/(m\sigma^2) = 1000$. b) For $\lambda \ll \lambda_0$ when $\lambda_0/\sigma = 10$, $m|V| = 27.2$ and $t/(m\sigma^2) = 1000$. The linear depletion in the probability density is specifically shown by the red dashed lines. c) The density profile as $r \to 0$, Eq. [3.24] (blue solid) and the semiclassical solution (red dashed). This figure first appeared in Ref. [1].
will approximately be \(n(0,t) \approx \langle \hat{\lambda}^{-2} \rangle(t)\). First one might consider approx-
ing \(\langle \hat{\lambda}^{-2} \rangle(t)\) with the most probable value of \(\lambda\). Often this value cor-
responds to the semiclassical value, which would be a solution to Eq. 3.13. This methodology is equivalent to the semiclassical and hydrodynamical methods studied previously [42–44].

However, there are anomalous contributions to \(\langle \hat{\lambda}^{-2} \rangle\) due to small \(\lambda\). To see this, note that all the eigenfunctions for this Hamiltonian have a linear depletion in the probability density:

\[
|\psi_{s,b}(k\lambda \ll \sqrt{mV})|^2 \propto \lambda.
\] (3.22)

The reason for this can be understood semiclassically. A particle falling into a potential has a probability density proportional to the inverse of the momentum at that position. For small \(\lambda\), the momentum is entirely governed by the potential energy:

\[
|\psi(\lambda)|^2 \propto \frac{1}{p(\lambda)} \approx \sqrt{\frac{2\lambda^2}{|V|}}.
\] (3.23)

The fact that the effective model has an inverse square potential is a signature of the original scale invariance of the problem. It means that the probability density must deplete linearly with \(\lambda\). This is true for all eigen-
states, both scattering and bound states.

The consequence of the linear depletion in the probability density is that the dominant contributions to \(\langle \hat{\lambda}^{-2} \rangle\) come from the least probable condensate sizes. This results to a logarithmic singularity in the density profile:

\[
\lim_{r/\sqrt{t} \to 0} \rho(\vec{r},t) = \frac{1}{\pi} \frac{m}{|V|} \frac{\lambda_0^2}{t^2} \log \left( \frac{\sqrt{t}}{r} \right).
\] (3.24)

This logarithmic rise in the density is shown in Fig. 3.1 c). This feature is robust as it only depends on the linear depletion of the probability density for all eigenstates.

The second major consequence concerns condensates with larger inter-
action energies, i.e. when the initial state has a larger projection onto the bound states. The presence of bound states in Eq. 3.18 will undoubtedly lead to oscillations in the density profile, see Fig. 3.2 a). The frequencies of these oscillations correspond to the difference of two bound state energies:

\[
\omega_{n,\nu} = E_{n+\nu} - E_n,
\]

with \(E_n\) given by Eq. 3.20, and \(n,\nu = 1,2,3...\). The exact location of these beat frequencies and their spectral weight will specifically depend on the ultraviolet parameter, \(k_0\), and the initial conditions, \(\lambda_0\).
However, the effect of the induced discrete scale invariance of the system is manifest in the organization of these frequencies. From Eq. [3.20] the beat frequencies can be written as:

\[
\log \left( \frac{k^2_0 \sigma^2}{2} \right) - \frac{2\pi}{\sqrt{m|V|}} n - \log \left( 1 - e^{2\pi \nu \sqrt{m|V|}} \right).
\] (3.25)

Upon observation, one can see that the beat frequencies are arranged into a series of families. The separation between frequencies in a given family is a constant universal value: \(2\pi/\sqrt{m|V|}\). However, there are a number of families present, and each family is shifted with respect to one another by the final term in Eq. [3.25]. This final term, which depends on the bound state spacing, \(\nu\), denotes the family of beat frequencies. Our numerical simulations of the oscillations in the density profile, shown in Figs. 3.2 b-c), are completely consistent with this general analysis.

The universal scaling found in Eq. [3.25] is to be contrasted with the semiclassical solution for a Bose gas with an initial size \(\lambda_0\). In Appendix B, the semiclassical solution for a particle oscillating in an inverse square potential has been worked out; here we quote the result:

\[
n(\vec{r},t) = \frac{N}{\lambda_{sc}^2(t)} f \left( \frac{r}{\lambda_{sc}(t)} \right),
\]

\[
\lambda_{sc}(t) = \lambda_0 \sqrt{1 - \left( 2(t - nT)/T \right)^2},
\] (3.26)

for \(t \in [nT - T/2, nT + T/2]\) with \(n = 0, 1, 2, ...\) and period, \(T\):

\[
T = 2\lambda_0^2 \sqrt{\frac{m}{|V|}}.
\] (3.27)

We note that the period depends only on the initial conditions of the problem, which is a consequence of the scale invariance of the system. Since this solution oscillates with a period \(T\), the frequency spectrum only contains frequencies \(\omega_n = 2\pi n/T\) and \(n = 1, 2, 3, ...\) These points are shown alongside the quantum frequency spectrum in Figs. 3.2 b-c). For smaller values of \(\sqrt{m|V|}\), there is a major discrepancy between the quantum and semiclassical oscillations. As noted previously, for larger values of \(\sqrt{m|V|}\), the role of quantum fluctuations diminish. In this limit, the oscillation spectrum will collapse onto the semiclassical solution.
3.4. Dynamics of an Inhomogeneous Bose Gas

Figure 3.2: a) The temporal evolution of the density profile at a fixed position $r \ll \lambda_0$. For this calculation $r/\sigma = 0.1$, $m|V| = 27.2$, $\lambda_0/\sigma = 10$. b) The frequency spectrum (see Eq. 3.25, blue solid line) is shown alongside the semiclassical frequencies (red dashed line). Only two families are shown explicitly with labels 1) and 2) corresponding to families with $\nu = 1$ and $\nu = 2$, respectively. c) The spectra for $r/\sigma = 0.1$, $m|V| = 32$ and $\lambda_0/\sigma = 10$. This figure first appeared in Ref. [1].
3.4. Dynamics of an Inhomogeneous Bose Gas

3.4.2 Dynamics of the Moment of Inertia

In this section we consider another important observable, the moment of inertia:

$$\langle r^2 \rangle(t) = \int d^2r \ r^2 n(r,t).$$  (3.28)

The moment of inertia effectively describes the size of the expanding Bose gas as a function of time. Using the effective model, one can write:

$$\langle r^2 \rangle(t) \approx \langle r^2 \rangle(0) \lambda^2 \int d\lambda \lambda^2 |\psi(\lambda,t)|^2.$$  (3.29)

Unlike the average over density, this average will not contain anomalous contributions from small condensate sizes. Therefore the dominant contribution to the moment of inertia will be from the semiclassical solution.

Although the dominant contribution may be due to the semiclassical solution, the dynamics will not be equivalent to the semiclassical solution. The projection of the wave function onto the bound states will lead to the discrete scale invariant oscillations at frequencies given by Eq. 3.25. In this section we show an alternative approach, using the Heisenberg equation of motion, to examine the difference between the semiclassical and quantum solutions.

For a scale invariant system, the equation of motion for the moment of inertia is found to be:

$$\frac{\partial}{\partial t} \langle r^2 \rangle(t) = 2 \langle D \rangle(t)$$

$$\frac{\partial^2}{\partial t^2} \langle r^2 \rangle(t) = 4 \langle H_\lambda \rangle.$$  (3.30)

The solution to the Heisenberg equations of motion are equivalent to the semiclassical solution presented in Eq. 3.26. In this case, the condensate oscillates with a period given by Eq. 3.27. The question then arises, where do the discrete scale invariant beats come from?

The answer is that Eq. 3.30 is incomplete. In particular, one can show that $H_\lambda$ is not self adjoint, i.e. $H_\lambda$ and $H_\lambda^\dagger$ act on different vector spaces. If one were to take into account this discrepancy, one would find a modified Heisenberg equation of motion [91–93]:

$$i \partial_t O(t) = i [H, O(t)] + i \left( H^\dagger - H \right) O(t),$$  (3.31)
3.4. Dynamics of an Inhomogeneous Bose Gas

Figure 3.3: Comparison between the Heisenberg and Schrodinger calculations for the moment of inertia, $\langle \lambda^2(t) \rangle$. Here we use the initial conditions set in Fig. 3.2 a). Since energy is conserved: $d^3\langle \lambda^2(t) \rangle / dt^3 = d\langle A_D(t) \rangle / dt$. The quantum beats are due to the anomalous term, $\langle A_D(t) \rangle$, defined in Eq. 3.32. When this term is included in the Heisenberg equation of motion, Eq. 3.33, one can show the presence of the discrete invariant beats. This anomaly highlights the non-trivial nature of the Hamiltonian.

where $O(t)$ is some generic operator. The last term we identify as the anomaly associated with the operator $O$:

$$A_O(t) = i \left( H_\dagger - H \right) O(t). \quad (3.32)$$

In calculating the moment of inertia one can show that:

$$\frac{\partial^2}{\partial t^2} \langle r^2 \rangle(t) = 4 \langle H_{\lambda} \rangle + 2A_D(t), \quad (3.33)$$

where $A_D$ is the anomaly associated with the generator of scale transformations. The anomaly term depends on the microscopic details of the system, and will be the source of the discrete scale invariant quantum beats. In Fig. 3.3 we have calculated the moment of inertia using the Schrodinger representation as well as by means of Eq. 3.33 for an initially Gaussian wave function. As one can see, the two approaches are identical.

For a more detailed discussion of the anomalous term to the equation of motion, and its application to the quantum anomaly in two dimensions, we refer the reader to Appendix C.
3.4.3 Dynamics of a Repulsively Interacting Bose Gas

In the previous discussion, we focused primarily on attractive condensates as quantum fluctuations are important. For repulsively interacting condensates, the situation is different. By examining the fluctuations around the semiclassical solution, \( \langle \delta \lambda(t) \delta \lambda(t) \rangle / \lambda_0^2 \), one can show that the strength of the fluctuations is inversely proportional to the semiclassical path size, \( \lambda_0 \).

In the long time limit, the semiclassical path size grows as \( \lambda_0 \propto t \). Therefore, the dynamics can be treated semiclassically. The density profile will not exhibit a logarithmic singularity at short distances, since the probability of small condensate sizes are exponentially suppressed, and there are no bound states to produce discrete scale invariant beats. In Fig. 3.4 we show a schematic for the time evolved wave function and for the resulting density profile using the quantum variational approach. This solution is consistent with the semiclassical description of the dynamics.

3.5 Summary

In this chapter we have studied the expansion dynamics of an initially inhomogeneous Bose gas in two spatial dimensions. Although the system is classically scale invariant for repulsive interactions, for attractive interactions there is a quantum anomaly. In this work we neglected the effects of
the quantum anomaly, and looked for the signatures of scale invariance on the expansion dynamics of an initially inhomogeneous Bose gas.

For repulsive interactions, it was shown that quantum fluctuations in the dynamics can be neglected. In this case, the dynamics of the Bose gas are well described by a time dependent scaling ansatz that satisfies the semiclassical and hydrodynamic equations of motion. On the other hand, for attractive interactions, and weak renormalization effects, quantum fluctuations in the size of the condensate can not be neglected. We employ a quantum variational approach that replaces the many body quantum dynamics with an effective single particle quantum mechanical problem of a particle in an inverse square potential. This model is naively scale invariant, but requires regularization which is due to the confinement radius of the system [82]. The act of regularization breaks the continuous scale invariance, but replaces it with a discrete scale invariance. The discrete scale invariance is manifest in a logarithmic singularity of the density at short distances, and in quantum beats that obey a discrete scaling relation.

This approach is not only limited to the dynamics of Bose gases, but can be used to study their energetics as well. In Ref. [67] we applied this formalism to study $N$-body solitonic states of low dimensional Bose gases. The results of this formalism are consistent with previous studies on the spectrum of one-dimensional [94] and two-dimensional [95, 96] Bose gases.

Although the quantum variational approach works quite well for the energetics and dynamics of low-dimensional Bose systems, it has two limitations. The first is that the extension to Fermionic systems is not straightforward since Fermionic fields are expressed in terms of Grassmann fields. Secondly, this approach is a variational approach. It’s application depends on the specific system of interest, and for this reason it does not tell one about the general features of scale invariant dynamics. The validity of such a method depends on the variational ansatz, and should be compared to experiment. For this reason, a microscopic approach to the dynamics would be ideal. In the next chapter we will develop a microscopic formalism that can elucidate the exact signatures of scale invariance.
Chapter 4

Non-Relativistic Dynamics and Scale Symmetry

In this chapter we answer the question what are the effects of scale invariance on the dynamics of non-relativistic quantum systems? To do this, we exploit a useful symmetry, the SO(2,1) symmetry. This symmetry is important as it incorporates both scale and conformal transformations.

The first application of the SO(2,1) symmetry in cold atoms was presented in Ref. [66], where the authors showed that the breathing modes occur at twice the trap frequency. Although the two dimensional Bose gas is not scale invariant due to the quantum anomaly, their analysis applies to any scale invariant system placed in an isotropic harmonic trap. As we will review, the presence of this symmetry means that the spectrum of a scale invariant gas in an isotropic harmonic oscillator potential can always be decomposed into a series of evenly spaced states called conformal towers [70, 79].

To supplement the algebraic approach, we use an exact microscopic formalism based on a many body wave function that involves a time dependent rescaling of the position coordinates with a gauge transformation. This method was put forward in Refs. [69, 70], where they examined the dynamics and excitation spectrum of the unitary fermi gas in time dependent harmonic traps. This wave function approach is useful for understanding the expansion dynamics of scale invariant gases prepared in the ground state of an isotropic harmonic trap. In this case, the dynamics at all times are equivalent to a time dependent rescaling. This trivial rescaling dynamics has been exploited for studying the dynamics of a number of scale invariant atomic systems prepared in the ground state of an harmonic oscillator [4, 25, 71-74].

In this chapter, we extend the previous analyses to arbitrary initial conditions. We begin by showing that the conformal tower states are an ideal basis to study dynamics, as their dynamics are equivalent to a trivial time dependent rescaling at all times. We then show that this algebraic approach is equivalent to understanding the dynamics of the many body wave function in an expanding, non-inertial reference frame. Using these two techniques,
we show that scale invariance implies that the long time dynamics of a local observable are equivalent to a time dependent rescaling. Although this is not true for short times, the trivial rescaling dynamics in the long time limit is true for arbitrary initial conditions, and arbitrary observable; it depends only on the scale invariance of the Hamiltonian.

4.1 The so(2,1) Algebra

In general, the Schrödinger equation has a number of symmetries consistent with Galilean invariance [97]. The full list of symmetries is given in Appendix D, but here we focus on three specific symmetries: time translations, scale transformations, Eq. 2.6, and conformal transformations, Eq. 2.15. As noted in chapter 2, the generator for these three transformations form a representation of the SO(2,1) symmetry; i.e., they form a Lie algebra, Eq. 2.16, the non-relativistic conformal algebra, or so(2,1) algebra [66, 70, 79]. In this section we will study quantum dynamics by exploiting this algebra.

We begin by discussing the implications of the so(2,1) algebra, Eq. 2.16, on the Hamiltonian, \( H_s + \omega^2 C \), a scale invariant gas placed in an isotropic harmonic trap with frequency \( \omega \). Here we explicitly define \( H_s \) as:

\[
H_s = \int d^d r \psi^\dagger (\vec{r}) \left( -\frac{\nabla^2}{2} \right) \psi (\vec{r}) + \frac{1}{2} \int d^d r d^d r' \psi^\dagger (\vec{r}) \psi^\dagger (\vec{r}') V_s (\vec{r} - \vec{r}') \psi (\vec{r}') \psi (\vec{r}),
\]

while \( C \) is defined in Eq. 2.15, and we note \( V_s (\vec{r}) \) is a scale invariant potential satisfying: \( V_s (\vec{r} e^{-b}) = e^{2b} V_s (\vec{r}) \). Although \( H_s + \omega^2 C \) is not scale invariant, it does possess SO(2,1) symmetry, which was first pointed out in Ref. [66]. Since the conformal algebra includes both scale and conformal transformations, it restricts both the spatial and temporal coordinates. For this reason, it will be beneficial to explore the explicit consequences of the conformal algebra.

In terms of energetics, the spectrum of \( H_s + \omega^2 C \) is composed of a series of conformal towers [66, 70, 79]. Each state in a given conformal tower will have energy:

\[
E_{n,\nu} = (2n + E_\nu) \omega.
\]

The intra-tower level spacing is fixed to be two harmonic units. The number of towers and their ground state energies will depend on the specific system being investigated. The quintessential example of this is the one dimensional
4.1. The so(2,1) Algebra

harmonic oscillator. For a single particle, there are two towers: one of even parity with energy $E_{n,e} = (2n + 1/2)\omega$, and one of odd parity, $E_{n,o} = (2n + 3/2)\omega$. For higher dimensions, one can show that each conformal tower can be labelled by the angular momentum quantum numbers. The proof of this is shown in Appendix D.

These conformal towers turn out to be an exceptional basis to study dynamics. Consider the time evolution of a conformal tower state, $|n\rangle$, with collective quantum number $n$, under the scale invariant Hamiltonian, $H_s$:

$$|n(t)\rangle = e^{-iH_st}|n\rangle$$
$$E_n|n(t)\rangle = e^{-iH_st}(H_s + \omega^2C)e^{iH_st}|n(t)\rangle. \quad (4.3)$$

From the commutation relations, Eq. 2.16, one can show:

$$e^{-iH_st}(H_s + \omega^2C)e^{iH_st} = (1 + \omega^2t^2)H_s - \omega^2tD + \omega^2C. \quad (4.4)$$

Although this operator may seem random, it can be rewritten in a more convenient form:

$$\left(1 + \omega^2t^2\right)\int d^dr\psi^\dagger(\vec{r})\left[\frac{1}{2}\left(-i\nabla_r - \vec{r}\frac{\omega^2t}{1 + \omega^2t^2}\right)^2 + \frac{\omega^2}{2}\left(\frac{r^2}{1 + \omega^2t^2}\right)^2\right]\psi(\vec{r})$$

$$+ \frac{1}{2}\int d^drr'\psi^\dagger(\vec{r})\psi^\dagger(\vec{r}')V_s\left(\frac{\vec{r} - \vec{r}'}{\sqrt{1 + \omega^2t^2}}\right)\psi(\vec{r}')\psi(\vec{r}). \quad (4.5)$$

Eq. 4.5 is nothing more than the operator, $H_s + \omega^2C$, but defined in new coordinates:

$$\vec{r} = \frac{r}{\sqrt{1 + \omega^2t^2}} \quad \vec{p} = \sqrt{1 + \omega^2t^2}\left(-i\nabla_r - \vec{r}\frac{\omega^2t}{1 + \omega^2t^2}\right). \quad (4.6)$$

Eqs. 4.5 and 4.6 specifies that the state $|n(t)\rangle$ is an eigenstate of the Hamiltonian, $H_s + \omega^2C$, if one uses the coordinates defined in Eq. 4.6.

The dynamics of these conformal tower states are equivalent to the dynamics of a Gaussian wave packet in free space. The probability density of the wave function will maintain its shape, and all the dynamics are contained in a time dependent rescaling factor: $\lambda(t) = \sqrt{1 + \omega^2t^2}$. As a result the conformal tower basis is an excellent basis to consider scale invariant dynamics, as the trivial rescaling dynamics are encapsulated in the rescaling factor, $\lambda(t)$.
4.2 The Comoving Reference Frame

The previous discussions were based on the conformal algebra, and connects scale invariant dynamics in free space, to the states in a time dependent isotropic harmonic oscillator potential. Here we relate this connection to a change of reference frame by examining the $N$-body Schrodinger equation for particles interacting through a scale invariant two-body potential, in $d$ spatial dimensions:

$$i \partial_t \psi \left( \{ \vec{r}_i, \sigma_i \}, t \right) = H_s \psi \left( \{ \vec{r}_i, \sigma_i \}, t \right),$$

where

$$H_s = \sum_{i=1}^{N} -\frac{1}{2} \nabla_i^2 + \frac{1}{2} \sum_{i,j=1}^{N} V_s(\vec{r}_i - \vec{r}_j),$$

(4.7)

Consider the following many body wave function:

$$\psi(\{ \vec{r}_i, \sigma_i \}, t) = \frac{1}{\lambda(t)^{dN/2}} \exp \left[ \frac{i}{2} \sum_{i=1}^{N} \nabla_i^2 \frac{\dot{\lambda}(t)}{\lambda(t)} \right] \phi \left( \{ \vec{x}_i, \sigma_i \}, \tau(t) \right),$$

(4.8)

where $\lambda(t)$ and $\tau(t)$ are yet to be determined functions of time, $\vec{r}_i$ and $\sigma_i$ are the position and spin of the $i = 1, 2, \ldots, N$ particles. This ansatz was first put forward in Ref. [69, 70], and it combines a time dependent rescaling with a gauge transformation. The purpose of this wave function is to separate the trivial rescaling dynamics from any non-trivial dynamics. Substitution of this many body wave function into Eq. 4.7 yields:

$$i \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} \phi \left( \{ \vec{x}_i, \sigma_i \}, \tau \right) = \left( \sum_i \left[ -\frac{1}{2} \lambda^2(t) \nabla_i^2 + \frac{\dot{\lambda}(t)}{2} \lambda(t) \lambda(t) \right] \right)$$

$$+ \frac{1}{2} \sum_{i,j} V_s(\lambda(t) (\vec{x}_i - \vec{x}_j)) \phi \left( \{ \vec{x}_i, \sigma_i \}, \tau \right),$$

(4.9)

where we have defined new coordinates:

$$\vec{x} = \frac{\vec{r}_i}{\lambda(t)} \tau(t)$$

(4.10)
4.2. The Comoving Reference Frame

At this stage, $\lambda(t)$ and $\tau(t)$ are undefined functions of time. Although the choice is arbitrary, the most logical choice for expansion dynamics is to define $\lambda(t)$ and $\tau(t)$ according to:

$$
\ddot{\lambda}(t) \lambda^3(t) = \omega^2 \quad \lambda(t) = \sqrt{1 + \omega^2 t^2},
$$
(4.11)

$$
\lambda(0) = 1 \quad \dot{\lambda}(0) = 0.
$$
(4.12)

$$
\frac{\partial \tau}{\partial t} = \frac{1}{\lambda^2(t)} \quad \tau(t) = \frac{1}{\omega} \arctan(\omega t).
$$
(4.13)

For this choice, the Schrodinger equation reduces to:

$$
i \frac{\partial}{\partial \tau} \phi(\{\vec{x}_i, \sigma_i\}, \tau) = \tilde{H} \phi(\{\vec{x}_i, \sigma_i\}, \tau),
$$

$$
\tilde{H} = \sum_{i=1}^{N} \left[ -\frac{1}{2} \nabla_i^2 + \frac{1}{2} \omega^2 \vec{x}_i^2 \right] + \frac{1}{2} \sum_{i,j=1}^{N} V_s(\vec{x}_i - \vec{x}_j),
$$
(4.14)

where we have used the fact that the interaction is scale invariant: $V_s(e^{-b \vec{r}}) = e^{2b} V_s(\vec{r})$. Eq. 4.14 is nothing more than the Schrodinger equation for a scale invariant gas in the presence of a Harmonic trap.

Eq. 4.14 can be understood as studying the dynamics of a quantum system in an expanding, non-inertial reference frame, or comoving frame, with coordinates defined in Eq. 4.10. Since the comoving frame is a non-inertial reference frame, a fictitious force must appear. This fictitious force is nothing more than a harmonic restoring force with frequency $\sqrt{\dot{\lambda}(t) \lambda^3(t)} = \omega$.

The resulting Hamiltonian in this comoving reference frame is the physical interpretation of the conformal algebra and Eq. 4.5. The eigenstates of Eq. 4.14 are just the conformal tower states which were discussed in the previous section. If one prepares a quantum system in a given conformal tower state, the dynamics of the wave function in the laboratory frame will be a simple time dependent rescaling:

$$
\psi_n(\{\vec{r}_i, \sigma_i\}, t) = \frac{1}{\lambda(t)^{dN/2}} \exp \left[ \frac{i}{2} \sum_{i=1}^{N} \vec{r}_i^2 \frac{\dot{\lambda}(t)}{\lambda(t)} \right] \phi_n \left( \{ \frac{\vec{r}_i}{\lambda(t)}, \sigma_i \} \right) e^{-iE_n \tau(t)},
$$
(4.15)

where $\phi_n(\{\vec{r}_i, \sigma_i\}) = \langle \{\vec{r}_i\} | n \rangle$ is a conformal tower state with energy $E_n$.

The dynamics of scale invariant systems in a given conformal tower state have been used to study the dynamics of resonant three dimensional gases.
4.2. The Comoving Reference Frame

Figure 4.1: The conformal tower states in both a) the laboratory frame, and b) in the co-moving frame. A single conformal tower is depicted with ground state energy $E_0$. In the laboratory frame, the conformal tower states are evenly spaced but contract like $t^{-2}$ in the long time limit, $\omega t \gg 1$, see Eq. 4.16. In the co-moving frame the spectrum does not evolve with time. This plot first appeared in Ref. [2].

The phase factor $-iE_n \tau(t)$, is the dynamical phase. Although these conformal tower states are not eigenstates of the Hamiltonian in the laboratory frame, $H_s$, they can still be described by a global dynamical phase, as if they were eigenstates. Physically, the dynamical phase is equivalent to the adiabatic phase for an eigenstate with quasi-energy:

$$E_n(t) = \frac{E_n}{\lambda^2(t)}, \quad (4.16)$$

The conformal tower spectrum is shown in Fig. 4.1, in both the comoving and laboratory frames. The main difference is that in the laboratory frame it is necessary to deal with the quasi-energies of Eq. 4.16, while in the comoving frame, the spectrum is time independent.

It is important to note that this phase factor freezes at large times:

$$\tau(\omega t \gg 1) \approx \frac{1}{\omega} \left[ \frac{\pi}{2} - \frac{1}{\omega t} + O\left(\frac{1}{\omega^2 t^2}\right) \right]. \quad (4.17)$$

As a result, the dynamics of an arbitrary superposition of conformal tower states will eventually be governed only by $\lambda(t)$. To see this consider an
arbitrary superposition of conformal tower states with expansion coefficients, $C_n$:

$$
\psi(\{\vec{r}_i, \sigma_i\}, t) = \sum_n C_n \frac{1}{\lambda^d N/2(t)} \exp \left[ \frac{i}{2} \sum_{i=1}^n r_i^2 \frac{\dot{\lambda}(t)}{\lambda(t)} \right] \phi_n \left( \{\frac{\vec{r}_i}{\lambda(t)}\}, \sigma_i \right) e^{-iE_n \tau(t)}. \tag{4.18}
$$

In the comoving frame the wave function will freeze in some configuration when the dynamical phase saturates after a few initial time scales. At this point, the probability density will maintain its shape, and simply rescale according to $\lambda(t)$, exactly like an individual eigenstate would. The only difference is that for an arbitrary superposition, the asymptotic wave function in the comoving frame is not necessarily equivalent to the initial wave function.

We can use the saturation of the dynamical phase to state a consequence of scale symmetry on the dynamics of an arbitrary non-relativistic quantum system: the dynamics of some local observable, $\langle O(\vec{r}) \rangle(t)$, will be equivalent to a simple time dependent rescaling at long times:

$$
\lim_{\omega t, \omega t' \to \infty} \langle O(\vec{r}) \rangle(t) \approx \left( \frac{\lambda(t')}{\lambda(t)} \right)^{\Delta_O} \left\langle O \left( \frac{\lambda(t')}{\lambda(t)} \vec{r} \right) \right\rangle(t'), \tag{4.19}
$$

where $\Delta_O$ is the scaling dimension of the operator, $O(\vec{r})$. In the long time limit, $\lambda(t) \approx \omega t$, and thus $\langle O(\vec{r}) \rangle(t) \propto t^{-\Delta_O}$. As we will prove in Chapter 7, this approximate long time rescaling is due to an emergent conformal symmetry.

To end this section, we comment that the use of the comoving frame is not only limited to connecting the dynamics in free space to that in a harmonic trap. The transformation shown in Eq. 4.8 can be applied to a number of situations including the dynamics of time dependent traps. For this situation, we can use the comoving frame to map the system onto a time independent harmonic trap. For this situation, the equation for $\lambda(t)$ is given by:

$$
\ddot{\lambda}(t) = \frac{\omega_0^2}{\lambda^3(t)} - \omega^2(t)\lambda(t), \tag{4.20}
$$

where $\omega_0$ is a reference trap frequency, and $\omega(t)$ is the time dependent trap frequency. This has been applied to the dynamics of scale invariant gases placed in the ground state [4, 25, 68, 74]. In Appendix E, we review this approach, and determine the time dependent scaling factor, $\lambda(t)$, for a sudden quench in the trap, [74], and for a broadening time dependent trap [4].

32
4.3 Summary

In summary, the dynamics of scale symmetric systems are intimately tied to conformal symmetry. Here we used the fact that for scale invariant systems, there exists a set of dynamical states which remain eigenstates of a time-dependent harmonic oscillator potential, at all times. The existence of these states rely on the non-relativistic conformal, or SO(2,1), symmetry, which is valid for Galilean invariant systems with scale invariant Hamiltonians [66,70,79,97]. Physically the so(2,1) algebra is equivalent to studying the dynamics in an expanding, non-inertial, comoving, reference frame. The Schrodinger equation in the comoving frame is simply the scale invariant Hamiltonian placed in an isotropic harmonic potential. This system possesses the SO(2,1) symmetry, and the spectrum is given by a series conformal towers.

We examined the dynamics using this conformal tower basis, and showed that the quantum state necessarily freezes in the comoving frame. Consequently, all the dynamics at long times are governed by the time dependent rescaling factor, $\lambda(t)$. This has immediate implications for the dynamics of local observables. The freezing of the dynamical phase implies that the long time dynamics of any local observable is simply a time dependent rescaling, independent of the initial conditions. These predictions are robust and do not depend on the microscopic details of the system, just the scale invariance of the Hamiltonian, and the conformal algebra. In the next chapter we will study the robustness of these results to the explicit breaking of scale invariance.
Chapter 5

The Breaking of Scale Invariance

For a generic quantum system, scale invariance requires some form of fine tuning. In the context of atomic gases, it is possible to tune the s-wave interactions by means of a Feshbach resonance [6]. The Feshbach resonance allows one to tune the interactions from zero to positive or negative infinity. For quantum gases with s-wave interactions, we can relate the interaction strength to a new length scale, the d-dimensional scattering length, $a$. For scale invariant systems, this length scale will either be zero or diverge, and as a result, will not be present in the physics. In this situation, the dynamics of the quantum gas are well described by the results in Chapter 4.

However, an important question remains. How do the dynamics for a system with explicitly broken scale invariance differ from the scale invariant case. If the scale invariance is only broken slightly, we can imagine two situations. The first is that the breaking of scale invariance is irrelevant, and the long time dynamics is equivalent to a scale invariant system; and the second, the interactions destroy the long time scaling behaviour seen in Eq. [4.19].

In this chapter, we will answer this question by examining a d-dimensional quantum gas with either nearly resonant, or weak s-wave interactions. Although we consider a specific system, our methods will be quite general, and can apply to a number of cold atom systems. Here we show that the relevancy of the interactions in the long time limit depends on the thermodynamic relevancy of the deviation. We explicitly put down a condition that delineates a relevant from an irrelevant deviation from scale invariance.

For relevant interactions, we show that it is possible to find a non-perturbative solution for the long time dynamics. This approach primarily depends on the universal scaling properties of the operator that breaks the scale invariance, and only marginally on the microscopic details of the system. In this situation, the time dependent rescaling dynamics is modified by a non-trivial time dependence. The form of this time dependence is fixed by the scaling of the deviation from scale invariance.
To address the effects of the broken scale invariance for nearly resonant, and weakly interacting scale invariant quantum gases, we consider the Hamiltonian:

\[
H = \int d^d r \psi^\dagger(\vec{r}) \left( -\frac{\nabla^2}{2} \right) \psi(\vec{r}) + \frac{1}{2} \int d^d r d^d r' \psi^\dagger(\vec{r}) \psi^\dagger(\vec{r}') V(\vec{r} - \vec{r}') \psi(\vec{r}') \psi(\vec{r}), \tag{5.1}
\]

where \( V(\vec{r}) \) is a nearly scale invariant two-body potential. In order to study the breaking of scale invariance, it is beneficial to split Eq. [5.1] into a scale invariant piece, \( H_s \), and a deviation, \( \delta H \):

\[
H_s = \int d^d r \psi^\dagger(\vec{r}) \left( -\frac{\nabla^2}{2} \right) \psi(\vec{r}) + \frac{1}{2} \int d^d r d^d r' \psi^\dagger(\vec{r}) \psi^\dagger(\vec{r}') V_s(\vec{r} - \vec{r}') \psi(\vec{r}') \psi(\vec{r}),
\]

\[
\delta H = \frac{1}{2} \int d^d r d^d r' \psi^\dagger(\vec{r}) \psi^\dagger(\vec{r}') (V(\vec{r} - \vec{r}') - V_s(\vec{r} - \vec{r}')) \psi(\vec{r}') \psi(\vec{r}). \tag{5.2}
\]

For this Hamiltonian, the time evolved wave function is given by:

\[
|\psi(t)\rangle = e^{-iHt} |\psi_0\rangle, \tag{5.3}
\]

where \(|\psi_0\rangle\) is the initial state. Here we note that the state \(|\psi(t)\rangle\) is related to the many body wave function through:

\[
\langle \{\vec{r}_i\} | \psi(t) \rangle = \psi(\{\vec{r}_i, \sigma_i\}, t) = \frac{1}{\lambda^{dN/2}(t)} \exp \left[ \frac{i}{2} \sum_{i=1}^{N} \vec{r}_i^2 - \lambda(t) \right] \phi \left( \{ \vec{r}_i, \sigma_i \}, \tau(t) \right). \tag{5.4}
\]

For nearly scale invariant systems, it is ideal to expand the total Hamiltonian around the scale invariant Hamiltonian. This can be done by using the interaction picture with respect to the scale invariant Hamiltonian, \( H_s \). In the interaction picture, the time dependent wave function is given by:

\[
|\psi(t)\rangle = \sum_n C_n(t) e^{-iE_n \tau(t)} |n(t)\rangle
\]

\[
C_n(t) = \langle n | T e^{-i \int_0^t dt' \delta H_I(t')} | \psi_0 \rangle
\]

\[
\delta H_I(t) = e^{iH_{st}} \delta H e^{-iH_{st}}, \tag{5.5}
\]

where \( T \) is the time ordering operator, and we have used the notation:
5.1. Explicit Form of the Deviation From Scale Invariance

\[ \langle \{ \vec{r}_i \} | n(t) \rangle = \frac{1}{\lambda^{dN/2}(t)} e^{\frac{i}{2} \sum_{i=1}^{N} r_i^2 \frac{\lambda(t)}{\lambda(t)} \phi_n \left( \{ \vec{r}_i \}, \sigma_i \right)} . \] (5.6)

The presence of the deviation will induce couplings between the different conformal tower states. As a result, the occupation of a given conformal state is no longer time independent, as in the scale invariant case. The remainder of this chapter will investigate this additional time dependence, and determine the modification to the scale invariant dynamics.

5.1 Explicit Form of the Deviation From Scale Invariance

In order to solve Eq. [5.5] it is necessary to understand the explicit form of the deviation operator, Eq. [5.2]. The breaking of scale invariance means the physics depends on an additional length scale. As mentioned previously, for atomic gases, this length scale is the scattering length. The question is then how to relate the deviation to this new length scale. To study this we consider a model contact interaction of strength \( g(\Lambda) \):

\[ V = \frac{1}{2} \int_{r > \Lambda^{-1}} d^d r g(\Lambda) \psi^\dagger(\vec{r}) \psi(\vec{r}) \psi^\dagger(\vec{r}) \psi(\vec{r}), \] (5.7)

where \( \Lambda \) is the ultraviolet momentum cut off for the theory. Although we have chosen a specific model to study, it is important to note that \( g(\Lambda) \) is chosen such that the low energy physics is independent of the UV cut off, and this effective model reproduces the low energy scattering of the true interatomic potential; i.e. both the true potential and the contact interaction can be parametrized by the same scattering length, \( a \). The scale invariant value of the coupling constant is denoted by, \( g^*(\Lambda) \). When the coupling constant is fine tuned to this value, the scattering length will either disappear or be infinite.

For this interaction, the deviation operator can then be written as:

\[ \delta H = \int_{r > \Lambda^{-1}} d^d r (g(\Lambda) - g^*(\Lambda)) \psi^\dagger(\vec{r}) \psi^\dagger(\vec{r}) \psi(\vec{r}) \psi(\vec{r}) . \] (5.8)

In order to proceed further, it is important to note that \( g(\Lambda) - g^*(\Lambda) \) can be related to the beta-function [46, 47]. The beta-function describes the effects of perturbations and their relevancy to thermodynamic properties, as one coarse grains and examines the system on larger and larger length scales. For this reason, the beta-function is intimately connected to how the
5.1. Explicit Form of the Deviation From Scale Invariance

system responds to a scale transformation. In particular, if the system is scale invariant, the system should not change under a scale transformation; in terms of the beta-function, a scale invariant system corresponds to a zero of the beta-function. The definition of the beta-function is:

\[ \beta(\tilde{g}(\Lambda)) = \frac{\partial}{\partial \log(\Lambda)} \tilde{g}(\Lambda) \]

\[ \tilde{g}(\Lambda) = C_d \Lambda^{d-2} g(\Lambda), \tag{5.9} \]

where \( C_d \) is a dimensional dependent constant. For a \( d \)-dimensional quantum gas with short range interactions, the beta-function is given by [67]:

\[ \beta(\tilde{g}(\Lambda)) = (d - 2)\tilde{g}(\Lambda) + \tilde{g}^2(\Lambda). \tag{5.10} \]

For three spatial dimensions, Eq. 5.10 has two zeros corresponding to the non-interacting system, \( \tilde{g}^*(\Lambda) = 0 \), and the resonantly interacting system, \( \tilde{g}^*(\Lambda) = -1 \).

In order to relate the deviation to the beta-function, it is convenient to introduce a physical length scale associated with the interaction, \( a \). In terms of atomic gases, this length scale is simply the s-wave scattering length. The scattering length is related to the coupling constant via:

\[ a = \Lambda^{-1} f(\tilde{g}(\Lambda)). \tag{5.11} \]

Since this is a physical length scale, it must be independent of the ultraviolet cutoff of the theory:

\[ \frac{\partial a}{\partial \log(\Lambda)} = 0 = -a + \frac{\partial a}{\partial \tilde{g}(\Lambda)} \beta(\tilde{g}(\Lambda)). \tag{5.12} \]

Near a scale invariant fixed point, it is possible to expand the beta-function to linear order:

\[ \beta(\tilde{g}(\Lambda)) \approx \beta'(\tilde{g}^*(\Lambda)) (\tilde{g}(\Lambda) - \tilde{g}^*(\Lambda)), \tag{5.13} \]

where \( \beta'(\tilde{g}^*(\Lambda)) \) is the derivative of the beta-function evaluated at the scale invariant point. Substituting Eq. 5.13 into Eq. 5.12 relates the physical length scale to the deviation of the coupling constant from its scale invariant value:

\[ g(\Lambda) - g^*(\Lambda) \propto \frac{1}{\Lambda^{d-2}} \left( \frac{1}{(\Lambda a)^{-3\beta'(\tilde{g}^*(\Lambda))}} \right). \tag{5.14} \]
5.2. Classification of Deviations

At this point we can write down the deviation operator as:

$$
\delta H = \frac{1}{a^{-\beta'(\tilde{g}^*(\Lambda))}} \frac{1}{\Lambda^{d-2-\beta'(\tilde{g}^*(\Lambda))}} \int_{r>\Lambda^{-1}} d^d r \psi^\dagger (\vec{r}) \psi^\dagger (\vec{r}) \psi (\vec{r}) \psi (\vec{r}).
$$

(5.15)

It is instructive to consider how $\delta H$ transforms under a scale transformation, $\psi(\vec{r}) \rightarrow e^{-bd/2} \psi(\vec{r}e^{-b})$. In this case, by changing the variables $\vec{r} \rightarrow \vec{r}e^{-b}$ and $\Lambda \rightarrow \Lambda e^b$, one can show:

$$
\delta H \rightarrow e^{-(2+\beta'(\tilde{g}^*(\Lambda)))b} \delta H.
$$

(5.16)

Eq. 5.16 states that the scaling dimension of $\delta H$ is no longer 2, but $2 + \beta'(\tilde{g}^*(\Lambda))$. The derivative of the beta-function will be an important quantity for the remainder of these discussions. For this reason we define:

$$
\alpha = -\beta'(\tilde{g}^*(\Lambda)).
$$

(5.17)

For the three dimensional gas with s-wave contact interactions, one can show that $\alpha = 1$ near resonance, and $\alpha = -1$ for weak interactions.

Now that we know the scaling dimension of the deviation, the only thing left is to show that $\delta H$ is independent of the ultraviolet physics. This has been done previously by examining the short distance physics of Fermi systems [75,77,98]. In short we note that $\delta H$ is regularized with the scaling dimension $2 - \alpha$.

5.2 Classification of Deviations

Now that the explicit form of the deviation, $\delta H$, determined, let us consider how the deviation couples different conformal tower states, and how the deviation alters the dynamics of a quantum system. To do this, it is necessary to determine the time dependence of the expansion coefficients defined in Eq. 5.5:

$$
C_n(t) = \sum_{m=0}^{\infty} C_n^{(m)}(t)
$$

$$
C_n^{(m)}(t) = (-i)^m \int_0^t dt_1...\int_0^{t_{m-1}} dt_m \langle n|e^{iHs_1}\delta He^{-iHs_1}...e^{iHs_m}\delta He^{-iHs_m}|\psi_0\rangle.
$$

(5.18)
5.2. Classification of Deviations

A fundamental component of Eq. [5.18] is the matrix element of the deviation operator between two conformal tower states, in the interaction picture. Utilizing the dynamical properties of the conformal tower states, see Eq. [4.5], one obtains:

$$\langle n | e^{iH_s t} \delta H e^{-iH_s t} | l \rangle = \frac{1}{\lambda(t)^{2-\alpha}} a^a e^{i(E_n - E_l)\tau(t)} \langle n | \delta h | l \rangle,$$ (5.19)

where \( |n \rangle \), and \( |l \rangle \) are two conformal tower states, \( \lambda(t) = \sqrt{1 + \omega^2 t^2} \), and \( \delta H = a^{-\alpha} \delta h \).

As an estimate of the relevancy of the deviation, consider the number of conformal tower states mixed by the interaction at time \( t \). The number of coupled conformal tower states can be estimated by calculating the ratio of the interaction strength to the quasi-energy spacing of the conformal tower states, see Eq. [4.16]:

$$N_{\text{coupled}} \approx \left( \frac{1}{\sqrt{1 + \omega^2 t^2}} \right)^{2-\alpha} \frac{1 + \omega^2 t^2}{2} \approx (1 + \omega^2 t^2)^{\alpha/2}. \quad (5.20)$$

In the long time limit, \( N_{\text{coupled}} \propto t^\alpha \). Therefore the relevancy of the deviation to the dynamics depends on the derivative of the beta-function near a given fixed point.

We note that this same analysis can be performed in the comoving frame. In this frame, the conformal tower states have no time dependence, but the deviation will scale as: \( \delta H \propto a^{-\alpha}(1 + \omega^2 t^2)^{\alpha/2} \). This leads to the same condition as Eq. [5.20]. The strength of the deviation as a function of time, compared to the conformal tower spectrum, is shown in Fig. 5.1 for two deviations with scaling \( \alpha = 1 \), and \( \alpha = -1 \).

In order to obtain an exact condition on whether a deviation is relevant or irrelevant, let us consider first order perturbation theory, in the long time limit:

$$C_n(t) \approx C_n(0) - i \int_0^t dt' \frac{1}{(\omega t')^{2-\alpha}} a^a e^{i(E_n - E_l)\tau(t')} \langle n | \delta h | \psi_0 \rangle. \quad (5.21)$$

For \( 2 - \alpha > 1 \), or equivalently \( \alpha < 1 \), the dominant contribution to the expansion coefficient is from short times. In the large time limit, the effect
5.2. Classification of Deviations

Figure 5.1: Here we show the conformal tower spectrum and perturbation in both a) the laboratory frame, and b) the comoving frame. The blue (dotted) lines correspond to the conformal tower states. The red (solid) and black (dash-dotted) lines correspond to deviations with scaling, $\alpha = 1$ and $\alpha = -1$, respectively. For scaling $\alpha = 1$, the interaction eventually couples more and more states, see Eq. 5.20, with $N_{\text{coupled}} \propto t$. For $\alpha = -1$, the interaction vanishes with time, i.e. for long times, fewer and fewer states are coupled together with, $N_{\text{coupled}} \propto 1/t$. We therefore expect a breakdown of time-dependent perturbation theory for $\alpha \geq 1$. This figure first appeared in Ref. [2].
of the interaction vanishes like $t^{-1+\alpha}$. Therefore, the effects of the deviation vanish at long times, and the dynamics are equivalent to the scale invariant dynamics discussed in Chapter 4. The only difference between the scale invariant dynamics and the perturbed dynamics is that the deviation will alter the expansion coefficients at short times. However, this time dependence will quickly saturate, and the resulting dynamics will be governed by the time dependent rescaling. If the deviation is sufficiently weak, perturbation theory and the conformal dynamics will provide an accurate description of the dynamics.

On the other hand, for $\alpha \geq 1$, the contribution at long times diverges as:

$$C_n(t) \approx 1 - i\frac{(\omega t)^{\alpha-1}}{\alpha - 1} \frac{1}{(\sqrt{\omega a})^\alpha} \langle n|\tilde{V}|\psi_0 \rangle \quad \alpha > 1$$

$$\approx 1 - i\log(\omega t) \frac{1}{\sqrt{\omega a}} \langle n|\tilde{V}|\psi_0 \rangle \quad \alpha = 1,$$

where we have defined the matrix, $\tilde{V}$, by its matrix elements between two conformal tower states:

$$\langle n|\tilde{V}|l \rangle = \frac{1}{\omega^{1-\alpha/2}} e^{i(E_n-E_l)} \frac{\omega^\alpha}{\sqrt{\omega a}} \langle n|\delta h|l \rangle.$$

Therefore, the effect of the interaction becomes increasingly important in the long time limit. As a result, perturbation theory is inadequate at describing the dynamics. At this point, one can posit that the rescaling dynamics at long times will be broken by a non-trivial time dependence. In order to capture this non-trivial time dependence, a non-perturbative approach is needed.

### 5.3 Non-Perturbative Solution for Relevant Deviations

In this section, we develop a non-perturbative approach for dealing with nearly scale invariant systems with deviations that have scalings: $\alpha \geq 1$. We begin by examining the $m$th order of $C_n(t)$, Eq. 5.18. Inserting $m$ conformal tower states, and utilizing Eq. 5.19 gives:

$$\langle n|C_n^{(m)}(t)|\psi_0 \rangle = \left(-\frac{i}{a^\alpha}\right)^m \sum_{l_1...l_m} \int_0^t dt_1... \int_0^{t_{m-1}} dt_m$$
5.3. Non-Perturbative Solution for Relevant Deviations

\[
\left( \frac{1}{\sqrt{1 + \omega^2 t^2}} \right)^{2-\alpha} \cdots \left( \frac{1}{\sqrt{1 + \omega^2 t_m^2}} \right)^{2-\alpha} e^{-i(E_n - E_1)t} \cdots e^{-i(E_{m-1} - E_m)t_m} \langle n|\delta h|l_1 \rangle \cdots \langle l_{m-1}|\delta h|l_m \rangle \langle l_m|\psi_0 \rangle,
\]

(5.24)

Eq. 5.24 can be simplified by noting that in the long time limit, \( \omega t \gg 1 \), the dominant contribution to these integrals comes from long times. In this limit the dynamical phase saturates, see Eq. 4.17, and the time dependence of the deviation comes from the simple rescaling factor, \( \lambda(t) \). In this case, the summation over the intermediate conformal states can be collapsed:

\[
\sum_{l_1, \ldots, l_m} e^{-i(E_n - E_m)\tau(t_m)} \langle n|\delta h|l_1 \rangle \cdots \langle l_{m-1}|\delta h|l_m \rangle \langle l_m|\psi_0 \rangle = \omega^m (1 - \alpha/2) \langle n|\tilde{V}^m|\psi_0 \rangle,
\]

(5.25)

where \( \tilde{V} \) is defined in Eq. 5.23. The remaining integrations can be carried out to give:

\[
\int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m \left( \frac{1}{\sqrt{1 + \omega^2 t_1^2}} \right)^{2-\alpha} \cdots \left( \frac{1}{\sqrt{1 + \omega^2 t_m^2}} \right)^{2-\alpha} \approx \frac{1}{m!} \frac{(\omega t)^{\alpha-1}}{\alpha - 1} \frac{1}{\omega^m},
\]

(5.26)

Eqs. 5.25 and 5.26 allow one to non-perturbatively solve for the expansion coefficients, in the long time limit:

\[
C_n(t \gg \omega^{-1}) \approx \exp \left[ -i \frac{1}{(\sqrt{\omega a})^\alpha} \frac{(\omega t)^{\alpha-1}}{\alpha - 1} \tilde{V} \right] \quad \alpha > 1
\]

\[
\approx \exp \left[ -i \frac{1}{\sqrt{\omega a}} \log(\omega t) \tilde{V} \right] \quad \alpha = 1.
\]

(5.27)

Eq. 5.27 is nothing more than the unitary evolution operator under a Hamiltonian, \( \tilde{V} \), with time coordinate \( t^{\alpha-1}/a^\alpha \). As stated in the previous section, the deviation operator, \( \tilde{V} \), is a regularized, dimensionless, and universal operator. It only depends on the statistics and the number of particles.

The scaling of the effective temporal coordinate, \( t^{\alpha-1}/a^\alpha \), is dictated by the derivative of the beta-function evaluated at the nearby scale invariant fixed point. It is thus universal, and only depends on basic symmetry properties, and the dimension of the system.
5.4 Summary

The presence of this non-perturbative time dependence will lead to non-trivial dynamics for a local observable, $O$. In particular, due to the scaling of the deviation, one can posit that the dynamics of a local observable will have the form:

$$\langle O \rangle(t >> \omega^{-1}) \approx (\omega t)^{-\Delta \alpha} F \left( \frac{1}{\alpha} \frac{(\omega t)^{\alpha-1}}{\alpha - 1} \right)$$  

Eq. 5.28 states that the scale invariant dynamics are modified by some function of $t^{\alpha-1}/\alpha$. Although the function, $F(t)$, depends on the specific observable, the number of particles, and more importantly, the matrix $\tilde{V}$, the scaling of Eq. 5.28 is robust.

5.4 Summary

In this chapter we examined the dynamical consequences of explicitly breaking the scale invariance. When the scale invariance is explicitly broken, a new length scale appears. For quantum gases with short ranged interactions, this length scale is the $s$-wave scattering length. This length scale can be used to parametrize the strength of the deviation from scale invariance: $\delta H \propto a^{-\alpha}$, where $\alpha$, defined in Eq 5.17, is given by the negative of the derivative of the beta-function evaluated at the nearby scale invariant point.

The scaling of the deviation, $\alpha$, dictates whether a deviation is relevant, $\alpha \geq 1$, or irrelevant, $\alpha < 1$, to the study of dynamics. For irrelevant interactions, the number of conformal states mixed by the deviation vanishes for long times. Therefore, the long time dynamics are governed by the time dependent rescaling. For relevant interactions, the system becomes strongly interacting in the long time limit, and a non-perturbative approach is required. The results of such an approach is contained in Eqs. 5.27. These results are non-perturbative, and are the main theoretical results of this thesis. This relevant breaking of scale invariance will modify the dynamics of local observables, as seen in Eq. 5.28.

In the remaining sections, we explore the physical implications of these results. In particular, we employ this formalism to the expansion of an ensemble of two-body systems in Chapter 6 and to the compressional and elliptic flow of resonantly interacting Fermi gases in three dimensions in Chapter 7.
Chapter 6

Application to Two-Body and One-Body Systems

In this chapter we employ the formalism of Chapter 5 on a toy model; the relative dynamics of a two-body system, or equivalently, a single particle in the presence of a short ranged potential, in three spatial dimensions. The two particles can be either two bosons, or fermions in the spin singlet channel.

For both these systems, the two scale invariant points correspond to the non-interacting, $a = 0$, and resonantly interacting, $a = \infty$, limits. The dynamics at these points can be understood using the formalism in Chapter 4. However, in advance it is not known how the dynamics differ for finite scattering lengths. As stated in Chapter 5, the relevancy of the interaction can be determined by means of the beta-function. For a three dimensional quantum gas the relevancy of the deviation from resonance was marginally relevant, $\alpha = 1$, while the deviation from scale invariance was irrelevant for weak interactions, $\alpha = -1$. Since the beta-function only depends on basic properties of the system, and not the number of particles, we expect the dynamics near resonance to exhibit a non-perturbative logarithmic time dependence, while the asymptotic dynamics for weak interactions to be equivalent to a scale invariant system. This can be shown analytically for our two toy models.

6.1 The Two-Body Problem: Schrodinger Equation in the Comoving Frame

To start, we consider the two-body problem, and write down the radial Schrodinger equation for the relative coordinates, in the comoving frame:

$$i\partial_{\tau}\chi_l(x,\tau) = \left[-\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 + \frac{l(l+1)}{2x^2} + \lambda^2(\tau)V(x\lambda(\tau))\right]\chi_l(x,\tau)$$

$$\lambda(t) = \sqrt{1 + \omega^2 t^2} = \sec(\omega \tau)$$
6.2 Near Resonance

\[ \tau(t) = \frac{1}{\omega} \arctan(\omega t) \] (6.1)

where we have set the (reduced) mass to unity, and \( Y_{l,m}(\vec{x}) \) is the spherical harmonic with angular quantum number, \( l \), and projection quantum number, \( m \). The radial wave function, \( \chi_l(x, \tau) \), is related to the full wave function via: \( \phi_{l,m}(\vec{x}, \tau) = Y_{l,m}(\hat{\vec{x}}) \chi_l(x, \tau)/x \), and is properly normalized:

\[ \int_0^\infty dx |\chi_l(x, \tau)|^2 = 1. \] (6.2)

In what follows we will only focus on the zero angular momentum, or s-wave, scattering of this potential, as higher angular momentum scattering is suppressed by a factor of \((\sqrt{E}r_0)^{2l}\), where \( r_0 \) is the range of the potential, and \( E \) is the relative energy.

For specificity, we will consider the potential to be a square well of depth: \( V_0 \lambda^2(\tau) \), and range: \( r_0/\lambda(\tau) \). This potential is consistent with the time dependence of the interaction in the comoving frame: \( \lambda^2(\tau)V(x\lambda(\tau)) \), and captures all the essential physics at low energies. The scattering length for this potential depends on the depth and range via:

\[ a(\tau) = a(\lambda(\tau)) = \frac{r_0}{\lambda(\tau)} \left( 1 - \frac{\tan(\sqrt{V_0r_0})}{\sqrt{V_0r_0}} \right). \] (6.3)

As can be seen from Eq. 6.3, at either of the scale invariant points, \( a(\tau) \), will remain at the scale invariant point. However, if the scattering length is finite, it will naturally flow to smaller values as time increases. The question now is, how do the dynamics for nearly resonant and weakly interacting two-body systems differ from their scale invariant counterparts.

6.2 Near Resonance

For large scattering lengths, we expand Eq. 6.1 around the resonant, or \( a = \infty \), solution. The spectrum of the resonant Hamiltonian will be composed of conformal towers, each characterized by their angular momentum quantum number. Since we only consider s-wave scattering, we only need to focus on the matrix element of the deviation operator between s-wave conformal tower states.

Naturally, we expect the deviation from scale invariance to be proportional to \( a^{-1} \), or equivalently, has scaling \( \alpha = 1 \). In appendix F, we show that this is indeed the case, and derive an analytical expression for the deviation operator, see Eq. 5.23.
6.2. Near Resonance

\[ \langle n|\hat{V}|m \rangle = f_n f_m, \quad (6.4) \]

where:

\[ f_n = \frac{\sqrt{2}}{\pi^{1/4}} \frac{(2n-1)!!}{\sqrt{(2n)!}}. \quad (6.5) \]

The deviation operator, \( \hat{V} \), has only a single non-zero eigenvalue:

\[ v = \sum_{n=0}^{n_{\text{max}}} f_n^2 \langle n|v \rangle = \frac{f_n}{\sqrt{v}}. \quad (6.6) \]

The eigenvalue, \( v \), diverges with the harmonic quantum number as: \( \sqrt{n_{\text{max}}} \).

We note that this sum is controlled by the energy scale set by the range of the potential: \( n_{\text{max}} = 1/(\omega r_0^2) \), where \( r_0 \) is the range of the potential with \( r_0 \ll a \).

The formal solution for the wave function in the laboratory frame is given by:

\[
\psi(\vec{r},t) = \frac{1}{\lambda^{3/2}(t)} \exp \left[ i\frac{\lambda(t)}{2} \sum_n C_n(t) e^{-iE_n\tau(t)} \phi_n \left( \frac{\vec{r}}{\lambda(t)} \right) \right] 
\]

\[
C_n(t) = \langle n| \exp \left[ -i \frac{1}{\sqrt{\omega a}} \log (\omega t) \hat{V} \right] |\psi_0 \rangle 
= \langle n|\psi_0 \rangle + \langle n|v \rangle \langle v|\psi_0 \rangle \left( e^{-\sqrt{\omega a} \log(\omega t)v} - 1 \right). \quad (6.7) 
\]

where \( \phi_n(\vec{r}) \) are the resonant conformal tower states, see Appendix [F]. Here we have assumed real initial expansion coefficients to specifically highlight the effect of the deviation.

The first term in Eq. (6.7) is the initial expansion coefficients, and it produces the scale invariant dynamics. The presence of the second term will produce a beat in the probability density of the form: \( \sin^2(v/(\sqrt{\omega a}) \log(\omega t)/2) \), in the laboratory frame. The amplitude of this beat is proportional to \( v^{-1} \). In the limit of zero range interactions, \( r_0 \to 0, v \) diverges and the beat amplitude vanishes. As a result, the scale invariant dynamics of zero range models is robust to deviations. Only models with a finite range will exhibit non-perturbative corrections to the dynamics. The beat is illustrated in Fig. [6.1(a)] where we show the numerical solution for the probability to be in the resonant ground state, \( |C_0|^2 \), for a wave function initially in the resonant
6.2. Near Resonance

Figure 6.1: The probability for the particle to remain in the resonant ground state in the comoving frame, for large finite scattering lengths, \( a \gg \lambda_0 \), as a function of \(-\lambda_0/(4\pi a) \ln(\pi/2 - \tau)\). Here we note, \( \lambda_0 = 1/\sqrt{\omega} \), and \( \pi/2 - \tau = 1/\omega t \). This result has been obtained by numerically solving Eq. 6.1 with \( \lambda_0/(4\pi a) = 0.015 \) and \( r_0/\lambda_0 = 10^{-3/2} \). The system is initially prepared in the ground state of the resonant model. Very quickly the probability satisfies Eq. 6.7, and develops oscillations at the frequency \( \nu/2 = 20.14 \). This figure first appeared in Ref. [2].

The solution is obtained by numerically solving the time-dependent Schrödinger equation, Eq. 6.1. The oscillations at frequency, \( \nu/2 \), are easily observed and are well described by Eq. 6.7.

This logarithmic beat will be present in all the dynamics of the system. As an example of the breaking of scale invariance, consider the time evolution of the moment of inertia for the two-body system. The moment of inertia is defined as:

\[
\lim_{t \to \infty} \langle r^2 \rangle(t) = \langle r_1^2 + r_2^2 \rangle(t) \approx \lambda^2(t) \left( \langle X^2 \rangle(\pi/2) + \langle x^2 \rangle(\tau(t)) \right). 
\]

In Eq. 6.8, we have assumed that the center of mass motion is not entangled with the relative motion, allowing for the center of mass and relative moments of inertia to decouple. The contribution to the center of mass,
6.2. Near Resonance

$\langle X^2 \rangle (\tau)$, will saturate as it is trivially scale invariant. The relative motion is complicated due to the breaking of scale invariance. In Fig. [6.2] we show the relative moment of inertia. In the comoving frame, the log-periodic oscillations are clearly visible. In the laboratory frame, the dynamics of the relative moment of inertia are given by:

$$\lim_{t \to \infty} \omega \langle x^2 \rangle (t) \approx A + B \sin \left( v \frac{1}{\sqrt{\omega a}} \log(\omega t) \right) \frac{1}{\omega t} + D \sin^2 \left( v \frac{1}{2 \sqrt{\omega a}} \log(\omega t) \right).$$

(6.9)

In Eq. (6.9) the coefficients $A$, $B$, and $D$ depend on the initial conditions, and the range of the potential, $r_0$, through the state $|v\rangle$. Explicit expressions for these coefficients are given in Appendix [F]. This result is consistent with Eq. (5.28) and the time dependent perturbation theory, Eq. (5.22) if one expands the sine functions to first order in $v/(\sqrt{\omega a}) \log(\omega t)$. The presence of the beat in the expansion coefficients can lead to significant deviations from the scale invariant dynamics, which can be seen in Fig. [6.2].

In addition to these results, we note that the two-body interaction does not break translational invariance in the laboratory frame. Thus, it is meaningful to study the dynamics of the momentum distribution, $n(k,t)$. In particular, we examine the dynamics of the contact. The contact is defined by the asymptotic behaviour of the momentum distribution: $C(t) = \lim_{k \to \infty} k^4 n(k,t)$ [76, 77]. The dynamics of the contact at resonance, and for weak interactions have been discussed previously in Ref. [72]. Here we extend their analysis to study the dynamics of the contact away from resonance.

In order to obtain the momentum distribution near resonance, we will again assume that the initial wave function is a product state between the center of mass, and relative motion. One can then integrate out the center of mass coordinate to unity, and examine the momentum distribution for the relative coordinate. The relative momentum distribution for this case is related to the Fourier transform of the solution for the relative wave function, Eq. (6.7):

$$\psi(k, t) = \sum_n e^{-iE_n \tau} \lambda^{3/2}(t) \langle n|e^{-i \frac{1}{\sqrt{\omega a}} \log(\omega t)} \mathcal{V} |\psi_0\rangle \cdot \int d^3r e^{i \frac{\lambda(t)}{2} \frac{\lambda(t)}{\lambda(t)} - i k \cdot \mathbf{r}} \phi_n \left( \frac{\mathbf{r}}{\lambda(t)} \right).$$

(6.10)

For large momenta, the integrand of Eq. (6.10) will be dominated by the contribution at short distances, $r \ll k^{-1}$. It is possible to obtain an analytical expression for the contact:
6.2. Near Resonance

Figure 6.2: The time evolution of $\langle x^2 \rangle(\tau)$ as a function of $-\lambda_0/(4\pi a) \ln(\pi/2-\tau)$, where $\lambda_0 = 1/\sqrt{\omega}$, and $\pi/2 - \tau = 1/(\omega t)$. This has been obtained by numerically solving the near resonant wave function, Eq. 6.1. In this calculation, the system was prepared in the ground state with $\lambda_0/(4\pi a) = 0.015$ and $r_0/\lambda_0 = 10^{-3/2}$. The dynamics can be fit to Eq. 6.9 with oscillations at frequency $v/2 = 20.14$. In the inset, the dynamics over the entire range is shown. This figure first appeared in Ref. [2].
6.3 Weakly Interacting

\[ C(t) = \lim_{k \to \infty} k^4 n(k, t) = \lim_{k \to \infty} k^4 |\psi(k, t)|^2 = \frac{1}{\lambda(t)} \tilde{C}(t) \]

\[ \tilde{C}(t) = \left| \sum_n \langle n | e^{-\frac{i}{\sqrt{2\omega}} \ln(\omega t)} \tilde{V} | \psi_0 \rangle e^{-iE_n \tau(t)} \frac{\sqrt{\pi}}{2} f_n \right|^2, \quad (6.11) \]

where the matrix elements of \( \tilde{V} \) and \( f_n \) are given in Eqs. 6.4 and 6.5.

At resonance, the contact in the comoving frame tends to a constant which depends on the initial conditions. The resulting dynamics are given by the scaling parameter, \( \lambda(t) \); this result was obtained in Ref. [72]. Near resonance, however, the expansion coefficients are time dependent due to the log-periodic beat, Eq. 6.7. The beat in the expansion coefficients will translate to a beat in the contact:

\[ \lim_{t \to \infty} C(t) \approx \frac{E}{\lambda(t)} + \frac{F}{\lambda(t)} \sin^2 \left( \frac{\nu}{2} \frac{1}{\sqrt{\omega a}} \log (\omega t) \right), \quad (6.12) \]

where \( E \) and \( F \) are two coefficients, which are given explicitly in Appendix F and we have neglected terms that vanish as \( 1/(\omega t)^2 \). The first term is the resonant scale invariant result, while the second term is the deviation that arises from breaking the scale invariance. Again, the presence of the log-periodic beat only depends on the deviation from resonance. The amplitude of the beat is controlled by the constant \( F \) which depends on the finite range, \( r_0 \), through the state \( |v\rangle \).

6.3 Weakly Interacting

Let us compare the dynamics near resonance to the weakly interacting case. In this case, the scattering length is small, and we expect the deviation to be proportional to \( a \). As shown in Appendix F, this is indeed the case, and the scaling is: \( \alpha = -1 \). Therefore, the deviation is irrelevant to the dynamics.

The system will be well described by first order perturbation theory:

\[ C_n(t) = 1 - i \int_0^t dt' \frac{a}{\lambda^3(t')} \langle n | \tilde{V} | \psi_0 \rangle. \quad (6.13) \]

As shown in Appendix F, the matrix, \( \tilde{V} \), in the weakly interacting limit is given by:
6.4 Experimental Application

\[ \langle m | \hat{V} | n \rangle = g_m g_n \]
\[ g_n = (-1)^n \frac{1}{\pi^{1/4}} \frac{1}{2^{n-1}} \frac{\sqrt{(2n+1)!}}{n!}, \quad (6.14) \]

where \( \Gamma(n) \) is the gamma function.

6.4 Experimental Application

In this section, we propose a future experiment composed of an ensemble of two-body states, tightly confined in micro-traps that are periodically arranged in an optical lattice, see Fig. 6.3. The trap frequency, \( \omega \), of each micro-trap depends on the laser intensities:

\[ \omega^2 = \frac{1}{3m} \nabla^2 V(\vec{r})|_{\vec{r}=\vec{r}_0} \] (assuming cubic lattice symmetry), where the derivatives are evaluated at the lattice sites of the optical lattice, which are formed by the minima of the confining potential energy, \( V(\vec{r}) \), see Eq. 6.15. To enhance the effect of broken scale invariance and for the convenience of experimental observation, we further propose to use three pairs of coplanar lasers angled at a small \( \theta \) to create a lattice with a controllable and relatively large lattice constant, \( a_l(\theta) \).

The laser set up is shown in Fig. 6.3 a). For each dimension, we use two coplanar beams with wave vectors \( \vec{k}_1 \), and \( \vec{k}_2 \). These two beams will then interfere and create a standing wave with an effective wave vector, \( \delta \vec{k}_\alpha = (\vec{k}_1 - \vec{k}_2)_\alpha = k \sin(\theta/2) \hat{e}_\alpha \), where \( \hat{e}_\alpha \) is the unit vector for direction \( \alpha = x, y, z \). In this experiment we assume that each pair of coplanar beams are constructed to produce a periodic potential of the form:

\[ V(\vec{r}) = V_0 \cos^2 \left( k \sin \frac{\theta}{2} x \right) \cos^2 \left( k \sin \frac{\theta}{2} y \right) \cos^2 \left( k \sin \frac{\theta}{2} z \right) \quad (6.15) \]

where \( V_0 \) is proportional to the laser intensity. The lattice constant for this potential is given by:

\[ a_l = \frac{\pi}{k \sin \frac{\theta}{2}}. \quad (6.16) \]

In practice, the lattice constant can be tuned up to the order of millimetres by decreasing \( \theta \).

Now it is possible to create an ensemble of two-body systems via an optical lattice [99, 100] by downloading pre-cooled atoms either a) in the
presence of Feshbach resonance, or b) in the absence of scattering. The
two situations, a) and b), correspond to the resonant and free particle fixed
points, respectively. As the tunnelling is negligible in the limit of tight
confinement, each approximately harmonic micro-trap will host conformal
tower states. If the atoms are at rest in the ground state, the initial state
will be at the bottom of the tower. At \( t = 0 \), the lattice is then turned off
but the magnetic field is simultaneously varied so that the systems are no
longer at resonance (a), or zero scattering (b).

The expansion of the two-body ensemble, shown in Fig. 6.3 b), can then
be observed to verify the main conclusions about the relevance or irrelevance
of deviations from a scale invariant fixed point. Near resonance the dynamics
should be modified by a non-perturbative log-periodic time dependence, see
Eqs. 6.7 and 6.9 as well as Figs. 6.1 and 6.2, while for weak interactions the
long time dynamics are equivalent to a rescaling. As far as \( \lambda(t) \ll a_t/2 \), each
two-body system will expand independently in free space. For systems with
initially tight confinement, \( 1 \ll \sqrt{\omega a} \), the effect of broken scale invariance
on the dynamics will then be visible for times: \( \omega^{-1} \ll t \ll a_t/\sqrt{\omega} \).

6.5 Application to a Mobile Impurity

As mentioned previously, the relative motion of the two-body problem is
equivalent to a particle in an external potential. However, a more realistic
system is a massive impurity placed in a quantum gas. The effect of the
impurity is to create an external potential for the quantum gas.

Consider a quantum gas interacting with an impurity of mass, \( M \), that
is subjected to a harmonic trap of frequency \( \omega_I \). In the laboratory frame,
the Hamiltonian for the system is:

\[
H = \sum_i -\frac{1}{2} \nabla_i^2 - \frac{1}{2M} \nabla_i^2 + \frac{1}{2} M \omega_I^2 R_i^2 + \sum_i U(\vec{r}_i - \vec{R}_I),
\]

where \( U(\vec{r}) \) is the short range atom-impurity interaction. In this Hamil-
tonian, the motion of the trapped impurity atom is included alongside the
quantum gas. The operators, \( \vec{R}_I \) and \(-i \nabla_I\) are the coordinate and momenta
operators for the impurity atom, while \( \vec{r}_i \) and \(-i \nabla_i\) are still the coordinates
and momenta for the quantum gas.

One can still examine the dynamics in the expanding co-moving frame.
In order to examine the dynamics of the quantum gas, it is ideal to choose:
\( \lambda(t) = \sqrt{1 + \omega^2 t^2} \). This choice of \( \lambda(t) \) then gives the modified effective
Hamiltonian:
6.5. Application to a Mobile Impurity

Figure 6.3: Proposed experimental set up for examining broken scale invariance on two-body dynamics. a) To create a three dimensional lattice, with a lattice constant, $a_l$, much larger than the optical wavelength, three pairs of coplanar beams are needed. Each pair of beams will have the same wave number, $k$, but an angle, $\theta$ between them. The resulting optical lattice will be due to the difference between the two beams: $\delta k_\alpha = \sin(\theta_\alpha/2)k_\alpha$, for $\alpha = x, y, z$. For large lattice constants, the optical lattice will look like an ensemble of harmonic traps with harmonic lengths: $\lambda_0 = 1/\sqrt{\omega}$. If each of the different pairs of beams lie in different orthogonal planes, the result will be a square lattice. b) A schematic of the experiment. Here we show a single dimension of the resulting optical lattice. At $t = 0$ the lattice is removed so the two-body systems can expand in free space. For times, $\omega^{-1} \ll t \ll a_l/\sqrt{\omega}$, the dynamics of the whole system will be equivalent to an ensemble of independent two-body systems. This figure first appeared in Ref. [2].
6.5. Application to a Mobile Impurity

\[ \hat{H} = \sum_i \left( -\frac{1}{2} \hat{\nabla}_i^2 + \frac{1}{2} x_i^2 + \lambda^2(\tau) U(\lambda(\tau)(\vec{x}_i - \vec{X}_I)) \right) \]
\[ - \frac{1}{2M} \hat{\nabla}_I^2 + \frac{1}{2} M X_I^2 \left( \omega_I^2 \lambda^4(\tau) + 1 \right) \quad (6.18) \]

where \( \vec{X}_I = \vec{R}_I/\lambda(t) \).

Eq. (6.18) states that the impurity is subject to a harmonic trap of frequency: \( \sqrt{\omega_I^2 \lambda^4(\tau) + 1} \). As \( \tau \) approaches \( \pi/2 \), the frequency of the trap will diverge. In this case, we expect that the motion of the impurity to be adiabatic, and that the impurity will become more and more localized near the origin.

To test this hypothesis, we show in Fig. 6.4 a) the probability for the impurity to be in the instantaneous ground state of the trap, when it was initially prepared in the ground state. It is easy to see that the adiabatic approximation works extremely well in the long time limit. The fluctuations in the position of the trapped impurity, \( \sqrt{\langle X_I^2 \rangle} \), can then be related to the instantaneous trap size:

\[ \sqrt{\langle X_I^2 \rangle} = \sqrt{\frac{1}{M\omega_I}} \cos(\tau), \quad (6.19) \]

which vanishes in the long time limit. The dynamics of this expectation value is shown in Fig. 6.4 b).

The adiabatic dynamics in the comoving frame is intuitive when one considers the motion in the laboratory frame. For heavy impurities, or large trapping potentials, the interaction between the impurity and the gas will not excite the impurity. As a result, when the gas expands further away from the impurity, the impurity will remain near the origin, and it’s exact position in the trap will become more and more irrelevant to the expanding gas. The adiabaticity of the impurity results in a form of coarse grained dynamics for the quantum gas. In the long time limit, the dynamics of the quantum gas will be insensitive to the initial preparation of the impurity. All the results obtained for the two-body problem, see Eqs. 6.7 and 6.9 as well as Figs. 6.1 and 6.2 will be equally valid.

Although these results were obtained for a single particle, the extension to \( N \) particles is straightforward. For multiple cold atoms, the dynamics can be understood by examining how \( N \) particles occupy the eigenstates of \( \hat{V} \). For fermions, the Pauli-exclusion principle prevents multiple fermions with the same spin to occupy the state, \( |v\rangle \). This results in the appearance
6.6 Summary

of a single frequency $v$ in the dynamics. On the other hand, if the system is composed of bosons which have condensed, the dynamics will be identical to a single particle, and will still have oscillations at the frequency, $v$.

In general, for a $N$-body non-interacting quantum gas in the presence of an external potential, the wave function in the comoving frame can be written in terms of:

$$\phi(x_1, ..., x_N, \tau) = \sum_{\{n_i\}} \psi(\{n_i\}, \tau) e^{-i \sum_{i=1}^{N} 2n_i \tau} \times$$

$$\frac{1}{\sqrt{N!}} \sum_{P} (\pm 1)^P \phi_{n_{P_1}}(x_1) \phi_{n_{P_2}}(x_2) ... \phi_{n_{P_N}}(x_N)$$

(6.20)

where the first summation runs over all distinct combinations of single particle conformal states, $n_1, ..., n_N$, where each individual $n_i$, runs from $n_i = 0, 1, ..., n_{\text{max}}$, and $i = 1, 2, ..., N$. The second summation is over all the permutations of the set $n_1, ..., n_N$. The factor of $(\pm 1)^P$ ensures the correct particle exchange symmetry for either bosons, or fermions, with $P$ being the number of exchanges to reach the given permutation. The expansion coefficients, $\psi(\{n_i\}, \tau)$, are normalized to unity.

For this general wave function, one can show that the moment of inertia for a non-interacting quantum gas has the same form as Eq. 6.9. In Appendix F, explicit expressions for the the coefficients, $A$, $B$, and $D$ are given for $N$ particles.

6.6 Summary

In this chapter we discussed the dynamics of two systems near resonance and for weak interactions; the two-body system, and a non-interacting quantum gas in the presence of a massive impurity. Both of these problems are analytically tractable. For $s$-wave interactions, the deviation from scale invariance is marginally relevant, $\alpha = 1$, near resonance, and irrelevant, $\alpha = -1$, for weakly interacting systems. Near resonance, the dynamics are modified by a logarithmic beat, with a frequency given by Eq. 6.6. This can be clearly seen in the moment of inertia, see Fig. 6.2 and Eq. 6.9.

Although these results were obtained for $s$-wave interactions in three spatial dimensions, it is possible to use the scaling of the deviation to classify the dynamics of a variety of cold atom systems. In particular, one can
Figure 6.4: The time evolution of the trapped impurity according to Eq. 6.18 in units of $\omega$, for $M\omega I_\lambda^2 = 3$. a) The probability of being in the instantaneous ground state of Eq. 6.18. b) The fluctuations of the trapped impurity position, Eq. 6.19, in both the laboratory (dashed line) and co-moving frames (solid line). This figure first appeared in Ref. [2].
show that for s-wave interactions, the dynamics are marginally relevant near resonance in three spatial dimensions, and for weak interactions in one spatial dimension. Similarly, one-dimensional systems with p-wave interactions behave identically to a three dimensional quantum gas with s-wave interactions; near resonance the deviation from scale invariance is marginally relevant, and irrelevant for weak interactions. We summarize these results in Fig. [6.5] where we show the dynamic and thermodynamic relevancy of a d-dimensional s-wave gas near resonance.

The only caveat to this discussion is for systems that possess a quantum anomaly, for example, the two-dimensional quantum gas. For these systems, the quantum anomaly will break the SO(2,1) symmetry, and the structure of the conformal towers. As a result, our formalism can only be applied to the non-interacting fixed point.

It is interesting to note there is a difference between the dynamical and thermodynamic relevancy of a given perturbation. Therefore, it is an important question to ask why is there a difference? The answer to this is based on the difference between scale transformations and conformal transformations on the action. In thermodynamics, the relevancy of a deviation is determined by examining how the action changes under a scale transformation. Let us consider how the deviation operator affects the scaling of the action. The deviation term of the action can be written as:

$$\delta S = -\int_{r \geq \Lambda^{-1}} d^d r \int dt \frac{1}{a(\Lambda)} \frac{1}{\Lambda^{d-2}} \psi^\dagger(\vec{r}, t) \psi^\dagger(\vec{r}, t) \psi(\vec{r}, t) \psi(\vec{r}, t).$$

(6.21)

where we have used the explicit form of the deviation operator, Eq. [5.16].

Under a scale transformation, one can show that the deviation of the action is given by:

$$\delta S' = e^{-b\beta'(\bar{g}(\Lambda))} \delta S.$$

(6.22)

as one can see, the effect of the deviation is entirely captured by the beta function, and how the action changes under scale transformations.

In terms of dynamics, it is conformal symmetry that one must consider. As discussed in Chapter [4], the long time dynamics are governed by the trivial time dynamics of the conformal tower states. We will show in the following chapter that this is due to the emergence of conformal symmetry. Therefore, in order to understand the effect of the deviation on the dynamics, it is necessary to consider how the deviation operator responds to
conformal transformations. Using Eq. (6.21), one can show under a conformal transformation:

$$\delta S' = (1 - bt)^{-1 - \beta'(\tilde{g}^*(\Lambda))} \delta S.$$  \hspace{1cm} (6.23)

Upon comparison, one can see that scale transformations and conformal transformations have different effects on the deviation term of the action. This difference is the reason why the condition for relevancy in dynamics differs from the thermodynamic relevancy. In the long time limit, one needs to consider conformal symmetry, not scale symmetry.
6.6. Summary

Figure 6.5: Thermodynamic and dynamic relevancy for a d-dimensional quantum gas with s-wave interactions near resonance. The thermodynamic relevancy differs from the dynamic relevancy by one spatial dimension.
Chapter 7

Application to Many-Body Systems

Previously we have highlighted the effects of scale invariance on the dynamics of arbitrary quantum systems in Chapter 4 and how the dynamics are modified in the presence of an explicit deviation from scale symmetry in Chapter 5. These previous discussions are true for arbitrary Galilean invariant, non-relativistic, quantum systems. Here we apply the formalisms developed in those two chapters to the dynamics of a three dimensional Fermi gas near resonance, with $N$ particles. This system is strongly interacting, and has been the focus of a number of both experimental and theoretical studies [4, 16, 17, 20, 25, 39, 40, 45, 63, 64, 101, 102].

One particular phenomenon of interest to experimentalists is the expansion dynamics of the unitary Fermi gas. This can be examined by considering the motion of the moment of inertia in a given direction:

$$
\langle r_i^2 \rangle(t) = \int d^3r \ r_i^2 \langle \psi_0 | \psi^\dagger(\vec{r}, t) \psi(\vec{r}, t) | \psi_0 \rangle,
$$

(7.1)

where $i = x, y, z$. The moment of inertia is important as it tells one about the size of the expanding gas in a given direction, as a function of time.

In Ref. [3], the expansion dynamics of a resonant Fermi gas, from an initially anisotropic harmonic trap, were studied experimentally. The resulting flow is denoted as elliptic flow, as it is ultimately anisotropic. This is to be contrasted with compressional flow, where the dynamics are isotropic. This experiment was analysed using a variational ansatz and the hydrodynamic equations of motion, a semiclassical technique. Although this approach was able to reproduce the experimental observations, this approach does not identify the effect of scale invariance. In this chapter we will examine the exact consequences of scale and conformal invariance on the dynamics.

In order to investigate the expansion dynamics, it is ideal to consider the more general quantity:

$$
\langle r_i r_j \rangle(t) = \frac{1}{3} \langle r^2 \delta_{i,j} + Q_{i,j} \rangle(t),
$$

(7.2)
7.1 Definitions for the One-Body Density Matrix

where $r^2$ is the moment of inertia, or monopole moment, and $Q_{i,j}$ is a traceless matrix, known as the quadrupole moment:

$$\langle Q_{i,j}(t) \rangle = \int d^3 r \left( 3r_i r_j - r^2 \delta_{i,j} \right) \langle \psi_0 | \psi^\dagger(\vec{r}, t) \psi(\vec{r}, t) | \psi_0 \rangle.$$  \hspace{1cm} (7.3)

This decomposition is ideal as it allows one to calculate the moment of inertia in any direction, and it splits the dynamics into two angular momentum channels: $l = 0$ and $l = 2$.

Although in previous chapters we studied the dynamics using a wave function approach in an expanding reference frame, it can be difficult to obtain quantitative predictions for many body systems. This is because one still needs to solve for the eigenstates of a resonantly interacting Fermi gas in an harmonic potential in order to use the comoving frame.

Instead, it is better to introduce a new quantity, the one-body density matrix:

$$\rho(\vec{r}, \vec{r}', t) = \langle \psi_0 | e^{iHt} \psi^\dagger(\vec{r}') \psi(\vec{r}) e^{-iHt} | \psi_0 \rangle.$$  \hspace{1cm} (7.4)

As we will see in the following section, it is possible to explicitly evaluate the role of scale invariance on the dynamics of the density matrix. In particular, we show that the density matrix satisfies an emergent conformal symmetry in the long time limit. This emergent symmetry is equivalent to the long time dynamics in the comoving frame, when the dynamical phase freezes, and is the reason for the time dependent rescaling of local observables. In the remainder of this chapter, we will apply the density matrix to the problem of compressional and elliptic flow and make concrete predictions about the possible dynamics that can occur. We will show that this conformal symmetry fixes the leading long time behaviour of the moment of inertia. The corrections to this leading behaviour will depend on the initial conditions, and has not been examined previously. We will conclude this chapter by comparing our methods to the semiclassical approach employed in Ref. [3].

7.1 Definitions for the One-Body Density Matrix

We begin by explicitly determining the role of scale and conformal invariance on the dynamics of the one-body density matrix. We can once again split the motion into a term governed by the scale invariant Hamiltonian, $H_s$, and a term governed by the deviation, $\delta H$. To do this, we use the interaction representation discussed in Chapter [5] and insert two complete sets of conformal tower states:
\[ \rho(\vec{r}, \vec{r}', t) = \sum_{m,n} \tilde{\rho}_{s} n,m(\vec{r}, \vec{r}', t) \tilde{\Gamma}_{m,n}(t) \]

\[ \tilde{\rho}_{s} n,m(\vec{r}, \vec{r}', t) = e^{-i(E_n-E_m)\tau(t)} \langle n | e^{iH_s t} \psi(\vec{r}') \psi(\vec{r}) e^{-iH_s t} | m \rangle \]

\[ \tilde{\Gamma}_{m,n}(t) = e^{i(E_n-E_m)\tau(t)} \langle m | U(t) | \psi_0 \rangle \langle \psi_0 | U^\dagger(t) | n \rangle, \] (7.5)

where:

\[ U(t) = T e^{-i \int_0^t dt' e^{iH_s t'} \delta H e^{-iH_s t'}} \]

\[ U(t \gg \omega^{-1}) \approx \exp \left[ -i \frac{1}{(\sqrt{\omega})^\alpha} \frac{(\omega t)^{\alpha-1}}{\alpha - 1} \tilde{V} \right], \alpha > 1 \]

\[ \approx \exp \left[ -i \frac{1}{\sqrt{\omega}} \log(\omega t) \tilde{V} \right], \alpha = 1. \] (7.6)

In Eqs. 7.5 and 7.6 \( \tau(t) = \omega^{-1} \tan(\omega t) \) and \( T \) is the time ordering operator. Finally we note that:

\[ \langle n | \tilde{V} | m \rangle = \frac{1}{\omega^{1-\alpha/2}} e^{i(E_n-E_m)\pi/2\omega} \langle n | \delta \delta h | m \rangle. \] (7.7)

A full derivation of the effects of scale and conformal symmetry on the one-body density matrix is shown in Appendix G. Here we note that one can use the fact that the conformal tower states are eigenstates of \( H_s + \omega^2 C \), to obtain a differential equation for the one-body density matrix near resonance:

\[ \left( 1 + \omega^2 t^2 \right) \partial_t + \omega^2 t (\vec{r} \cdot \nabla_r + \vec{r}' \cdot \nabla_r + d) + i\omega \frac{r'^2 - r^2}{2} \right) \rho(\vec{r}, \vec{r}', t) \]

\[ = \frac{1 + \omega^2 t^2}{t \log(\omega t)} \frac{1}{a \partial a^{-1}} \rho(\vec{r}, \vec{r}', t) + \sum_{m,n} i (E_n - E_m) \tilde{\rho}_{s} n,m(\vec{r}, \vec{r}', t) \tilde{\Gamma}_{m,n}(t). \] (7.8)

The first line of Eq. 7.8 is nothing more than the generator of conformal transformations [78]. It states that the for scale invariant systems, the density matrix is an eigenfunction of the generator of conformal transformations, with zero eigenvalue. As a result, the long time dynamics are constrained by conformal symmetry, not scale symmetry.
7.2. Isotropic Trap and Compressional Flow

Using Eq. [7.8] it is possible to write a differential equation describing the dynamics of $\langle r_i r_j \rangle (t)$:

$$
\left[ (1 + \omega^2 t^2) \partial_t - 2 \omega^2 t - \frac{1 + \omega^2 t^2}{t \log(\omega t)} \frac{\partial}{\partial a} \right] \langle r_i r_j \rangle (t) = \omega^2 t^2 \sum_{m,n} i (E_n - E_m) \int d^3 r \ r_i r_j \ \tilde{\rho}_{s \ n,m}(\vec{r}, \vec{r}, 0) \tilde{\Gamma}_{m,n}(t). \tag{7.9}
$$

In order to obtain Eq. [7.8] we have used the conformal property of the density matrix:

$$
\tilde{\rho}_{s \ n,m}(\vec{r}, \vec{r}, t) = \left( \frac{1}{\omega t} \right)^3 \tilde{\rho}_{s \ n,m} \left( \frac{\vec{r}}{\omega t}, \frac{\vec{r}}{\omega t}, \omega^{-1} \right) \approx \left( \frac{1}{\omega t} \right)^3 \tilde{\rho}_{s \ n,m} \left( \frac{\vec{r}}{\omega t}, \frac{\vec{r}}{\omega t}, 0 \right). \tag{7.10}
$$

In the remainder of this chapter, we will investigate the solutions of Eq. [7.9] for compressional and elliptic flows.

7.2 Isotropic Trap and Compressional Flow

First consider the case of a resonantly interacting Fermi gas initially placed in an isotropic harmonic trap. In this case, angular momentum is a good quantum number, and will be conserved throughout the expansion. In particular, we will be focused on isotropic initial conditions.

7.2.1 Scale Invariance and Compressional Flow

Since the initial gas is isotropic, the quadrupole moment vanishes, and will remain so for all times. All that remains is the isotropic moment of inertia. The differential equation for the moment of inertia is given by:

$$
\left[ (1 + \omega^2 t^2) \partial_t - 2 \omega^2 t \right] \langle r^2 \rangle (t) = \omega^2 t^2 \sum_{m,n} i (E_n - E_m) \int d^3 r \ r^2 \ \tilde{\rho}_{s \ n,m}(\vec{r}, \vec{r}, 0) \tilde{\Gamma}_{m,n}(t). \tag{7.11}
$$
7.2. Isotropic Trap and Compressional Flow

The leading long time solution can be obtained by focusing on the emergent conformal symmetry, or equivalently by neglecting the term on the right hand side. The solution reads:

\[
\langle r^2 \rangle(t) = \left( \frac{t}{t'} \right)^2 \langle r^2 \rangle(t'),
\]

(7.12)

where \( t, t' \gg \omega^{-1} \). This is consistent with the analysis in the comoving reference frame, Eq. [4.19] since the moment of inertia has a scaling dimension of \( \Delta_{r^2} = -2 \). Moreover, we can now see that the approximate rescaling is due to the emergence of conformal symmetry in the long time limit, and is independent of the microscopic details of the system, and the initial conditions.

The corrections to these long time asymptotics will depend on the matrix, \( \tilde{\Gamma}(t) \). For scale invariant systems, the time dependence of \( \tilde{\Gamma}(t) \) is trivial as the dynamical phase will saturate. As a result, the occupation of a given conformal tower state is conserved during the expansion. For the purpose of our discussions, we will be focused on initial conditions that can be described by a real symmetric matrix:

\[
\tilde{\Gamma}_{m,n}(0) = \tilde{\Gamma}_{n,m}(0),
\]

(7.13)

which is true for initial states that are invariant under time reversal symmetry.

With this in mind, we can now consider different initial conditions and how the long time dynamics differ. First we examine a diagonal ensemble of conformal tower states: \( \tilde{\Gamma}_{m,n}(0) = \delta_{m,n}P(n) \). This can be easily achieved for Fermi gases initially in thermal equilibrium at resonance. In this case, the right hand side of Eq. [7.11] vanishes for all times. The exact solution is then:

\[
\langle r^2 \rangle(t) = (1 + \omega^2 t^2) \langle r^2 \rangle(0).
\]

(7.14)

This is just the trivial rescaling dynamics, and is equivalent to the hydrodynamic solution, see Appendix [H].

In general, the initial density matrix will involve states from multiple conformal towers. Here we will assume the case that the system is in an arbitrary superposition of s-wave states. This assumption is used to ensure that the Fermi gas is initially isotropic. Such a system can be obtained by quenching a non-interacting Fermi gas in the ground state to resonance. For this situation, Eq. [7.11] simplifies to:
where we have used the fact that \( \tau(t) \approx \pi/(2\omega) \) and \( E_n - E_m = 2(n - m)\omega \) for states within a conformal tower. A generic solution to Eq. (7.15) is of the form:

\[
\langle r^2 \rangle(t) \approx \left[ v^2 t^2 + \left( \frac{v^2}{\omega^2} - \frac{F_0}{2\omega} \right) \right] \langle r^2 \rangle(0).
\] (7.16)

where we have neglected higher powers of \((\omega t)^{-1}\), and defined:

\[
F_0 = \frac{1}{\langle r^2 \rangle(0)} \sum_{n,m} (-1)^{n-m} 4(n - m)^2 \int d^3r \ r^2 \tilde{\rho}_{s\ n,m}(\vec{r},\vec{r},0) \tilde{\Gamma}_{m,n}(0).
\] (7.17)

The primary difference between Eqs. (7.14) and (7.16) is the relative velocity:

\[
v_{rel} = \lim_{t \to \infty} \sqrt{\frac{\langle r^2 \rangle(t)}{\langle r^2 \rangle(0)t^2}}.
\] (7.18)

For generic initial conditions, the relative velocity is no longer pinned to the trap frequency, \( \omega \). In fact, in Appendix [H] we show using the Heisenberg equation of motion that the relative velocity is given by:

\[
v_{rel} = \sqrt{\frac{2\langle H_s \rangle}{\langle r^2 \rangle(0)}}.
\] (7.19)

### 7.2.2 Broken Scale Invariance and Compressional Flow

When scale invariance is broken, it is necessary to use Eq. (7.8). Again, we only focus on low energy, \( s \)-wave interactions, which will only mix conformal tower states with the same angular momentum. For diagonal ensembles, the solution is given by:

\[
\langle r^2 \rangle(t) \approx \left[ 1 + \omega^2 t^2 \right] \left[ 1 + G \left( \frac{1}{\alpha} \log(\omega t) \right) \right] \langle r^2 \rangle(0).
\] (7.20)
for some function $G(x)$. For generic initial conditions, the solution Eq. 7.8 is given by:

$$ \langle r^2 \rangle(t) \approx \left[ v^2 t^2 + \left( \frac{v^2}{\omega^2} - \frac{F_0}{2 \omega} \right) \right] \left[ 1 + G \left( \frac{1}{a} \log(\omega t) \right) \right] \langle r^2 \rangle(0). \tag{7.21} $$

Both these solutions are modified by some function of $\log(\omega t)/a$, which is consistent with the analysis presented in Chapter 5.

## 7.3 Elliptic Flow

If the system is initially prepared in an anisotropic trap, the dynamics in each direction will differ. However, due to the emergent conformal symmetry, the long time dynamics for each direction must still be given by Eq. 7.12. The question then is how do the long time dynamics differ in comparison to compressional flow?

### 7.3.1 Scale Invariance and Elliptic Flow

To begin, we note that for an anisotropic trap, there will be at most three differing trap frequencies, labeled as $\omega_i$ with $i = x, y, z$. For this situation, angular momentum is no longer a good quantum number initially. As a result, the initial state must be a superposition of different conformal towers. The density matrix can no longer be diagonal in this case. Although this could be true for compressional flow, we note that the integral:

$$ \int d^3 r \, r_i^2 \rho_s n,m(\vec{r}, \vec{r}, t), \tag{7.22} $$

can couple states together with the same angular momentum, or with angular momentum that differ by two. The zero angular momentum channel will be identical to the isotropic trap as the conformal towers are decoupled from one another. However, the quadrupole moment will now be non-zero due to the anisotropy of the initial trap. For the remainder of this section, we will focus on the dynamics of the quadrupole moment. For scale invariant systems, the differential equation for the quadrupole moment is:

$$ [(1 + \omega^2 t^2) \partial_t - 2 \omega^2 t] \langle Q_{i,j} \rangle(t) = -\omega^2 t^2 \sum_{m,n} (E_n - E_m) \sin \left( (E_n - E_m) \frac{\pi}{2 \omega} \right) \int d^3 r \, Q_{i,j} \tilde{\rho}_s n,m(\vec{r}, \vec{r}, 0) \tilde{\Gamma}_{m,n}(0), $$
7.3. Elliptic Flow

\[ + \omega t \sum_{m,n} (E_n - E_m)^2 \cos \left( (E_n - E_m) \frac{\pi}{2\omega} \right) \int d^3r \, Q_{i,j} \, \bar{\rho}_s \, n_{m}(\vec{r}, \vec{r}, 0) \bar{\Gamma}_{m,n}(0). \]  

(7.23)

A generic solution to Eq. 7.23 is given as:

\[ \langle Q_{i,j} \rangle (t) \approx \left[ v_{i,j}^2 t^2 + F_{i,j}^2 t + \frac{v_{i,j}^2}{\omega^2} - \frac{F_{i,j}^2}{2w} \right], \]  

(7.24)

where we note that:

\[ F_{i,j}^2 = \sum_{m,n} (E_n - E_m) \sin \left( (E_n - E_m) \frac{\pi}{2\omega} \right) \cdot \int d^3r \, Q_{i,j} \, \bar{\rho}_s \, n_{m}(\vec{r}, \vec{r}, 0) \bar{\Gamma}_{m,n}(0) \]

\[ F_{0}^{i,j} = \sum_{m,n} (E_n - E_m)^2 \cos \left( (E_n - E_m) \frac{\pi}{2\omega} \right) \cdot \int d^3r \, Q_{i,j} \, \bar{\rho}_s \, n_{m}(\vec{r}, \vec{r}, 0) \bar{\Gamma}_{m,n}(0), \]  

(7.25)

and \( v_{i,j}^2 \) are traceless symmetric tensors.

The leading correction to the asymptotic dynamics is no longer of \( O(1) \), but now linear in \( t \). The linear term arises from the interference between different conformal towers, something that is not present in an isotropic system. This is the main difference between the two cases, and allows one to delineate the different initial conditions.

In the case of azimuthal symmetry, we can simplify Eq. 7.24 appreciably. One can show that the quadrupole tensor will simplify to the form:

\[ \langle Q_{i,j} \rangle (t) = 3Q(t) \langle e_z e_z \rangle \]  

\[ \approx v_Q^2 t^2 + A_Q t + B_Q, \]  

(7.26)

where \( v_Q, A_Q, \) and \( B_Q \) are constants, and \( e_z \) is the unit vector in the azimuthally symmetric direction.

7.3.2 Broken Scale Invariance and Elliptic Flow

If the Fermi gas is slightly away from resonance, the results of the previous section will be modified by a log-periodic beat.
7.4. Comparison to the Scaling Solution Ansatz

\[ \langle Q_{i,j} \rangle (t) = \left[ v_{i,j}^2 t^2 + F_{i,j}^2 t + \left( \frac{v_{i,j}^2}{\omega^2} - \frac{F_{i,j}^2}{2\omega} \right) \right] \left[ 1 + G \left( \frac{1}{a} \log(\omega t) \right) \right]. \quad (7.27) \]

for some function \( G(t) \).

7.4 Comparison to the Scaling Solution Ansatz

The results presented previously provide the scaling of the long time dynamics of (nearly) scale invariant systems initially confined in harmonic traps. In this section we will contrast the density matrix calculation with the scaling solution of the hydrodynamic equations of motion. In order to do this, we will consider an azimuthally symmetric system. We define the aspect ratio as the ratio of the moments of inertia in the two independent directions:

\[ \sigma_{z,x} = \sqrt{\frac{\langle r_z^2 \rangle (t)}{\langle r_x^2 \rangle (t)}}. \quad (7.28) \]

For a non-interacting Fermi gas released from an anisotropic trap, the aspect ratio saturates to unity. This is because the kinetic energy in a given direction is proportional to the trap frequency in that direction, \( \omega_i \) while the initial size scales as \( \omega_i^{-2} \) in the Thomas-Fermi limit. However, at resonance, the situation is different, the interaction distributes the energy anisotropically, and the emergent conformal symmetry states that the aspect ratio saturates to a finite number:

\[ \sigma_{z,x} \approx \frac{\langle r_z^2 \rangle (t) - Q(t)}{\langle r_z^2 \rangle (t) + 2Q(t)} \]

\[ \approx \frac{v^2 \langle r_z^2 \rangle (0) - v_Q^2}{v^2 \langle r_z^2 \rangle (0) + 2v_Q^2} \]

\[ - \left( \frac{A_Q}{v^2 \langle r_z^2 \rangle (0) - v_Q^2} + \frac{2A_Q}{v^2 \langle r_z^2 \rangle (0) + 2v_Q^2} \right) \frac{1}{t}. \quad (7.29) \]

In Ref. [3], a unitary Fermi gas was prepared in an anisotropic trap with frequencies: \( \omega_y = 2.7\omega_x \), and \( \omega_z = 33\omega_x \). In their experiment they considered the aspect ratio between the \( x \) and \( y \) directions. In their experiment, the aspect ratio for a non-interacting gas was observed to saturate at unity. For the resonant case, the aspect ratio was not observed to saturate at a
finite value, but its growth was indeed slowing down at large times. The analysis provided here is consistent with the experimental findings. Moreover, we show that the aspect ratio will indeed saturate, due to the emergent conformal symmetry.

In order to give more credence to the density matrix approach, we now compare Eq. [7.29] to a time dependent scaling ansatz that satisfies the hydrodynamic equations of motion. This approach was used in Ref. [3] to analyse their experimental findings. Here we reproduce their calculation.

The time-dependent scaling ansatz for the moment of inertia in a given direction is given by:

$$\langle r_i^2(t) \rangle = b_i^2(t) \langle r_i^2 \rangle (0),$$

where $b_i(t)$ is an unspecified scaling factor. In Appendix [H] we show that the scaling ansatz is a solution to the hydrodynamic equations of motion if the $b_i(t)$ factors satisfy:

$$\ddot{b}_i(t) = \frac{\omega_i^2}{(b_x(t)b_y(t)b_z(t))^{2/3} b_i(t)} \left( \frac{\langle \alpha_s \rangle}{\langle r_i^2 \rangle (0)} - \frac{1}{3} \left( \frac{\dot{b}_x(t)}{b_x(t)} + \frac{\dot{b}_y(t)}{b_y(t)} + \frac{\dot{b}_z(t)}{b_z(t)} \right) \right),$$

where $\langle \alpha_s \rangle$ is the trap averaged shear viscosity coefficient. This equation was studied in Refs. [3] for finite shear viscosity and in Ref. [45], in the absence of shear viscosity. The solution for the moment of inertia in a given direction is qualitatively similar in both cases.

The solution to the aspect ratio for an anisotropy in the x-y plane is shown in Fig. [7.1]. The saturation at a non-unity value is clearly visible, and the leading correction is consistent with Eq. [7.29]. In the inset, we show the solution for $\langle y^2 \rangle (t)$. A numerical fit for the moment of inertia along the y-direction shows that it is also consistent with Eq. [7.24].

### 7.5 Summary

In this chapter we examined the expansion dynamics of a Fermi gas either at or near unitarity from an initial harmonic trap. For a scale invariant Hamiltonian, the long time dynamics are ultimately controlled by the emergence of conformal symmetry. This emergent conformal symmetry is the explanation for the long time dynamics being equivalent to a time dependent rescaling. This conclusion is independent of the microscopic details of the system, and the initial conditions. However, the correction to this limit will
7.5. Summary

Figure 7.1: The solution to the aspect ratio for various shear viscosities for a unitary Fermi gas. These results were obtained by numerically solving Eq. 7.31. We have used the experimental parameters in Ref. [3], \( \omega_x = 2\pi \times 230 \, \text{Hz}, \omega_y = 2.7\omega_x, \omega_z = 33\omega_x \). For any value of the shear viscosity, the aspect ratio saturates. In the inset, the dynamics of the moment of inertia in the \( y \) direction is shown. This solution fits well with Eq. 7.24.

sensitively depend on the initial conditions, and whether the harmonic trap is isotropic or not. For the case of isotropic traps, the interference between states within a single conformal tower give a \( O(t^{-2}) \) correction, while for anisotropic traps, the interference between differing conformal towers gives a \( O(t^{-1}) \) correction. The leading correction to the asymptotic dynamics are summarized in Table 7.1.

We compared these results to the experimental study presented in Ref. [3]. In this study, the experimental observations for the aspect ratio were compared to a scaling ansatz to the hydrodynamic equations of motion. We showed that the semiclassical approach is indeed consistent with the density matrix approach. However, the hydrodynamical approach does not clearly state the role of conformal invariance; namely the long time behaviour of the moment of inertia, and the connection between the leading long time correction and the interference of conformal towers.

Although Ref. [3] also studied the effect of broken scale invariance on elliptic flow was studied experimentally, the theoretical description was left at the hydrodynamic level. It would be interesting to see if the deviation from scale invariance follows the scaling shown in Eqs. 7.27.
7.5. Summary

\[
\langle r_i^2 \rangle(t)/\langle r_i^2 \rangle(0) = \begin{cases} 
1 + \omega^2 t^2 & \text{Isotropic Trap} \\
\text{NA} & \text{Anisotropic Trap}
\end{cases}
\]

<table>
<thead>
<tr>
<th></th>
<th>Diagonal $\Gamma(0)$</th>
<th>Generic $\Gamma(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1 + \omega^2 t^2$</td>
<td>$v^2 t^2 + B$</td>
</tr>
<tr>
<td></td>
<td>$v^2 t^2 + B$</td>
<td>$v^2 t^2 + At + B$</td>
</tr>
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</table>

Table 7.1: The leading and next leading order time dependence for the moment of inertia: $\langle r_i^2 \rangle(t)$, for a unitary Fermi gas. The time dependence depends on whether the initial harmonic confinement is isotropic, with frequency $\omega$, or anisotropic, and whether the initial conditions are a diagonal, or generic non-diagonal ensemble of conformal tower states. The exact coefficients, $v$, $A$, and $B$, can be found in Eqs. [7.14], [7.16] and [7.24].
Chapter 8

Conclusions

The focus of this thesis was to understand the role of scale invariance on non-relativistic quantum dynamics, and the effects of explicitly breaking scale invariance. We addressed this using a quantum variational approach for two dimensional bosons, and by using exact symmetry arguments applied to Fermi gases in three spatial dimensions.

For two dimensional Bose gases, we developed a quantum variational approach that assigned a wave function to the size of the condensate. This approach allows one to treat the quantum fluctuations that are present for attractive interactions. For this system, we showed that the continuous scale invariance was broken, but replaced by a discrete scale invariance. This discrete scale invariance is manifest in the dynamics by a logarithmic rise in the density at short distances and log periodic beats.

We improved the preceding analysis by identifying the exact implications of scale invariance on the non-relativistic dynamics of quantum systems in Chapter 4. We used the SO(2,1) symmetry, which connects the dynamics to scale and conformal transformations, to show that there is an ideal basis to expand the dynamics in, the conformal tower basis. The time evolution of these states is related to a trivial time dependent rescaling. Using this basis, we showed that the long time dynamics of local observables is equivalent to a time dependent rescaling.

All of these results were for scale invariant systems. An important question to ask is what happens to the scale invariant dynamics when the symmetry is explicitly broken. Will the asymptotic time dependent rescaling persist or will it be broken by the broken scale invariance? In Chapter 5 we showed that there are two types of deviations from a scale invariant point: irrelevant deviations which do not break the scaling dynamics in the long time limit, and relevant dynamics which do. The relevancy of a deviation can be related to the beta-function; a thermodynamic property. In particular we showed that for deviations with scaling dimensions, \( \alpha \geq 1 \), where \( \alpha \) is related to the derivative of the beta-function, see Eq. the effect of the interactions can not be ignored in the long time limit. We performed a non-perturbative calculation that gives the leading long time behaviour; the scale
invariant dynamics are modified by a function of $t^{\alpha-1}/a^\alpha$. The quantitative aspects of the dynamics are contained in the so-called V-matrix, Eq. [5.23], which is a dimensionless, universal matrix that depends only on the number of particles. This is the main result of this thesis, as it categorizes the types of deviations, and allows for a scaling analysis of non-equilibrium dynamics.

The formalism developed here was applied to two systems, the expansion of an ensemble of two-body systems, Chapter 6, and to the compressional and elliptic flow of a Fermi gas, Chapter 7, both in three spatial dimensions. Both these systems are experimentally viable, and we examined the signatures of scale invariance, and its breaking on the mean size of the systems. In particular, we showed that for the unitary Fermi gas, this approach is equivalent to the hydrodynamic approach studied in Ref. [3]. However, our approach highlights the exact consequences of scale and conformal symmetry on the asymptotic dynamics and the leading long time correction, which is hidden in the scaling solution.

Our approach for studying dynamics is quite general and can be applied to a variety of atomic systems, such as quantum gases in other dimensions, and with p-wave interactions. For these systems, the non-equilibrium dynamics can be categorized by the relevancy of the deviation from scale invariance. Again, all the quantitative information on the dynamics will be contained in the V-matrix.

For future studies, there are a number of open questions pertaining to this V-matrix. First of all, it would be interesting to investigate what kind of structure is present in the V-matrix, and whether some form of approximations, like random matrix theory, can be applied to the dynamics. Secondly, it would be interesting to study different dynamical phenomena. In this thesis we focused primarily on the effect scale invariance has on the expansion of quantum gases. However, the field of dynamics is rich with a countless number of phenomena. The implications of conformal symmetry on transport dynamics, Floquet physics and others has still not been fully addressed. It will be interesting to press further and to have a clear picture how conformal symmetry restricts other dynamical phenomena.
Bibliography


Bibliography


Bibliography


Appendix A

The Action of Scale and Conformal Transformations

In this appendix we present explicit calculations showing the action of scale and conformal transformations on the field operator, \( \psi(\vec{r}, t) \), and its equation of motion:

\[
i \partial_t \psi(\vec{r}, t) = [H_s, \psi(\vec{r}, t)]. \tag{A.1}
\]

Here we consider a scale invariant Hamiltonian, as well as the generator of scale transformations:

\[
D = -i \int d^d r \psi^\dagger(\vec{r}) \left( \frac{d}{2} + \vec{\nabla}_{\vec{r}} \right) \psi(\vec{r}), \tag{A.2}
\]

and the generator of conformal transformations:

\[
C = \int d^d r \, r^2 \psi^\dagger(\vec{r}) \psi(\vec{r}). \tag{A.3}
\]

These three operators have the following commutation relations:

\[
[H_s, C] = -iD \quad [D, H_s] = 2iH_s \quad [D, C] = -2iC. \tag{A.4}
\]

A.1 Scale Transformation

First, consider a scale transformation:

\[
\psi_b(\vec{r}, t) = e^{iD_b} \psi(\vec{r}, t)e^{-iD_b}. \tag{A.5}
\]

Taking the derivative with respect to \( b \) will give:

\[
\frac{d}{db} \psi_b(\vec{r}, t) = ie^{iD_b} e^{iH_s t} \left[ e^{-iH_s t} De^{iH_s t}, \psi(\vec{r}) \right] e^{-iH_s t} e^{-iD_b}. \tag{A.6}
\]

Using the commutation relations in Eq. [A.4] and the identity:
\[ e^{ibA}Be^{-ibA} = B + ib[A, B] + \frac{(ib)^2}{2!} [A, [A, B]] + ..., \] (A.7)

it is possible to write:

\[ e^{-iH_s t} De^{iH_s t} = D - 2tH_s. \] (A.8)

Eq. (A.8) simplifies the calculation of the equation of motion to commutators between \( \psi(\vec{r}) \) and time independent operators, \( D \) and \( H_s \). These commutators can be easily worked out to:

\[
\begin{align*}
[D, \psi^{(i)}(\vec{r})] &= -i \left( \frac{d}{2} + \vec{r} \cdot \nabla \right) \psi^{(i)}(\vec{r}), \\
[H_s, \psi^{(i)}(\vec{r})] &= -ie^{-iH_s t} \partial_t \psi^{(i)}(\vec{r}, t) e^{iH_s t}.
\end{align*}
\] (A.9)

Substituting Eq. (A.9) into Eq. (A.6) yields the following differential equation:

\[
\left( \frac{\partial}{\partial b} + \frac{\partial \vec{r}(b)}{\partial b} \cdot \nabla_{\vec{r}} + \frac{\partial t(b)}{\partial b} \partial_t \right) \psi_b(\vec{r}, t) = - \left( \frac{d}{2} + \vec{r} \cdot \nabla_{\vec{r}} + 2t \partial_t \right) \psi_b(\vec{r}, t).
\] (A.10)

It is possible to equate like terms on each side of Eq. (A.10) and define new coordinates via:

\[
\frac{\partial \vec{r}(b)}{\partial b} = -\vec{r}, \quad \frac{\partial t(b)}{\partial b} = -2t.
\] (A.11)

The above equations are solved by:

\[
\vec{r}(b) = \vec{r} e^{-b} \quad t(b) = t e^{-2b}.
\] (A.12)

Finally, combining this all together, one obtains the desired result:

\[
\psi_b(\vec{r}, t) = e^{-db/2} \psi \left( \vec{r} e^{-b}, t e^{-2b} \right). \] (A.13)

We now consider applying the scale transformation to the equation of motion. It is straightforward to show that the transformed equation of motion is:

\[
i \partial_t \psi_b(\vec{r}, t) = \left[ e^{iDb} H_s e^{-iDb}, \psi_b(\vec{r}, t) \right]. \] (A.14)

Substituting Eq. (A.13) into the transformed equation of motion gives:
A.2. Conformal Transformation

\[ e^{-2b} \partial_t \psi(\vec{r}', t') = \left[ e^{i Db} H_s e^{-i Db}, \psi(\vec{r}', t') \right] , \]  
\( \text{(A.15)} \)

where \( \vec{r}' = e^{-b} \vec{r} \), and \( t' = te^{-2b} \).

At this point, it is necessary to see how the Hamiltonian changes under a scale transformation. This can be readily worked out through Eq. [A.7]

\[ e^{i Db} H_s e^{-i Db} = e^{-2b} H_s , \]  
\( \text{(A.16)} \)

Eq. (A.16) states that the Hamiltonian has the same scaling as the time derivative. The result is that the equation of motion will remain unchanged during a scale transformation. This is the definition of scale invariance.

A.2 Conformal Transformation

We apply the same technique to the conformal transformation. Defining the transformed field as:

\[ \psi_b(\vec{r}, t) = e^{iCb} \psi(\vec{r}, t) e^{-iCb} , \]  
\( \text{(A.17)} \)

one can obtain the relation:

\[ \frac{d}{db} \psi_b(\vec{r}, t) = ie^{iCb} e^{iH_st} \left[ e^{-iH_st} C e^{iH_st}, \psi(\vec{r}) \right] e^{-iH_st} e^{-iCb} . \]  
\( \text{(A.18)} \)

Following the identities in Eq. [A.4] and Eq. [A.7] one obtains:

\[ \frac{d}{db} \psi_b(\vec{r}, t) = ie^{iCb} e^{iH_st} \left[ C - Dt + t^2 H_s, \psi(\vec{r}) \right] e^{-iH_st} e^{-iCb} , \]  
\( \text{(A.19)} \)

The commutators between the field operator and \( H_s \) and \( D \) were evaluated in Eq. [A.9]. In addition one can show:

\[ [C, \psi(\vec{r})] = - \frac{r^2}{2} \psi(\vec{r}) \]

\[ [C, \psi^j(\vec{r})] = \frac{r^2}{2} \psi^j(\vec{r}) . \]  
\( \text{(A.20)} \)

Substituting in these commutators gives a differential equation for the transformed field operator:
A.2. Conformal Transformation

\[
\left( \frac{\partial}{\partial b} + \frac{\partial t(b)}{\partial b} \delta_t + \frac{\partial \vec{r}(b)}{\partial b} \right) \psi_b(\vec{r}, t) = \left( t^2 \partial_t + t \left( \frac{d}{2} + \vec{r} \cdot \nabla_r \right) - i \frac{r^2}{2} \right) \psi_b(\vec{r}, t). \tag{A.21}
\]

As in the case for scale transformations, we can define new coordinates that satisfy:

\[
\frac{\partial \vec{r}(b)}{\partial b} = rt \quad \frac{\partial t(b)}{\partial b} = t^2. \tag{A.22}
\]

Solving the coordinates gives:

\[
\vec{r}(b) = \frac{\vec{r}}{1 - bt} \quad t(b) = \frac{t}{1 - bt}. \tag{A.23}
\]

The last piece is to determine how the field rescales. This is also straightforward to do [78], and in summary the conformal transformation acting on the field gives:

\[
\psi_b(\vec{r}, t) = (1 - bt)^{-d/2} e^{-i \frac{r^2}{2(1 - bt)}} \psi \left( \frac{\vec{r}}{1 - bt}, \frac{t}{1 - bt} \right). \tag{A.24}
\]

Now let us consider the equation of motion:

\[
i \partial_t \psi_b(\vec{r}, t) = \left[ e^{iCb} H_s e^{-iCb}, \psi_b(\vec{r}, t) \right]. \tag{A.25}
\]

The time derivative acting on \( \psi_b(\vec{r}, t) \) will generate spurious terms:

\[
i \partial_t \psi_b(\vec{r}, t) = (1 - bt)^{-d/2} e^{-i \frac{r'^2}{2(1 - bt)}} \left( \frac{b}{1 - bt} \left( \frac{d}{2} + \vec{r}' \cdot \nabla_{r'} \right) - i \frac{r'^2 b^2}{2} + \frac{1}{(1 - bt)^2} \partial_{r'} \right) \psi(\vec{r}', t'), \tag{A.26}
\]

where \( \vec{r}' = r/(1 - bt) \) and \( t' = t/(1 - bt) \). These spurious terms will be cancelled by the transformed Hamiltonian.

In order to study the transformed Hamiltonian, it turns out to be advantageous to write the Hamiltonian in terms of the transformed field operators \( \psi_b(\vec{r}, t) \). The result is:
A.2. Conformal Transformation

\[ e^{iCb} H_s e^{-iCb} = e^{iCb} e^{iH_s t} H_s e^{-iH_s t} e^{-iCb} \]
\[ = \frac{1}{(1 - bt)^2} H_s + \frac{b}{1 - bt} D(t) + b^2 C(t), \quad (A.27) \]

where we note \( H, C(t), \) and \( D(t), \) are now written in terms of the operators: \( \psi^{(t)}(\vec{r}', t'). \) Performing the commutators gives:

\[
\left[ e^{iCb} H_s e^{-iCb}, \psi_b(\vec{r}, t) \right] = (1 - bt)^{-d/2} e^{-\frac{b r'^2}{2}} \frac{i}{(1 - bt)^2} \left[ H_s, \psi(\vec{r}', t') \right] \\
+ \left( \frac{b}{1 - bt} \left( \frac{d}{2} + \vec{r}' \cdot \vec{\nabla}_{\vec{r}'} - i \frac{r'^2 b^2}{2} \right) \right) \psi(\vec{r}', t') \right] \\
\quad (A.28)
\]

Comparison between Eqs. [A.26] and [A.28] show that all the spurious terms vanish and the equation of motion is left invariant:

\[ i \partial_{t'} \psi(\vec{r}', t') = [H_s, \psi(\vec{r}', t')] \]  
\quad (A.29)
Appendix B

Derivation of the Quantum Variational Approach

B.1 The Effective Action

In the following sections of this appendix, we provide a detailed derivation of the effective theory:

\[
\frac{d\lambda}{X} = \frac{\int d\lambda \frac{N}{2} f(\frac{\lambda}{X}) |\psi(\lambda, t)|^2}{\int d\lambda |\psi(\lambda, t)|^2}.
\]  

(B.1)

starting from:

\[
\begin{align*}
n(\vec{r}, t) &= \int D\psi(\vec{x}) D\psi'(\vec{x}) \langle \psi_0 | e^{iHt} | \{\psi(\vec{x})\} \{\psi'(\vec{x})\} \rangle \langle \{\psi'(\vec{x})\} | e^{-iHt} | \psi_0 \rangle \\
&\quad \left[ \langle \psi^*(\vec{r}) \psi'(\vec{r}) | \{\psi(\vec{x})\} \{\psi'(\vec{x})\} \rangle \right].
\end{align*}
\]  

(B.2)

The states \(|\{\psi(\vec{x})\}\rangle\) are the eigenstates of the annihilation operator \(\hat{\psi}(\vec{r})\):

\(\hat{\psi}(\vec{r}) |\{\psi(\vec{x})\}\rangle = \psi(\vec{r}) |\{\psi(\vec{x})\}\rangle\), defined on a discretized lattice with sites: \(\vec{x}\).

For the current system, we are only interested in the coherent states normalized to the number of particles \(N\). That is, in the continuum limit:\n
\(\int d^2r |\psi(\vec{r})|^2 = N\).

The matrix element \(\langle \{\psi(\vec{x})\} | e^{-iHt} | \psi_0 \rangle\) can be written in terms of a functional integral [88]:

\[
\langle \{\psi(\vec{x})\} | e^{-iHt} | \psi_0 \rangle = \int' D\phi e^{iS},
\]  

(B.3)

where \(S\) is the action for a non-relativistic Bose gas:

\[
S = \int d^2r \int_0^t dt' \psi^*(\vec{r}, t) \left(i\partial_t + \frac{\nabla^2}{2}\right) \psi(\vec{r}, t) - \frac{g}{2} |\psi(\vec{r}, t)|^4
\]  

(B.4)
B.1. The Effective Action

and $\int' D\psi$ denotes the sum over all field configurations $\psi(\vec{r}, t)$ which satisfy the following boundary conditions:

$$\begin{align*}
\psi(\vec{r}, T) &= \psi(\vec{r}), \\
\psi(\vec{r}, 0) &= \psi_0(\vec{r}).
\end{align*}$$

(B.5)

When the number of particles at each point in our discretized space is large $|\psi(\vec{r})|^2 \gg 1$, it is possible to simplify Eq. B.2 by noting that the overlap between two coherent states approaches a functional delta function:

$$\langle \{\psi(\vec{x})\} | \{\psi'(\vec{x})\} \rangle = \exp \left[ \int d^2 x \left( \psi^*(\vec{x}) \psi'(\vec{x}) - \frac{1}{2} |\psi(\vec{x})|^2 - \frac{1}{2} |\psi'(\vec{x})|^2 \right) \right] \approx \prod_{\vec{x}} \delta (\psi(\vec{x}) - \psi'(\vec{x}))$$

(B.6)

Eq. B.6 states that the value of the two fields $\psi(\vec{r})$ and $\psi'(\vec{r})$ are equivalent at each point in space. In the continuum limit, this is equivalent to a delta function enforcing the two field configurations to be identical. This result simplifies Eq. B.2 to:

$$n(\vec{r}, t) = \int D\psi(\vec{x}) |\langle \{\psi(\vec{x})\} | e^{-iHt} |\psi_0\rangle|^2 |\psi(\vec{r})|^2.$$

(B.7)

In this work it will be advantageous to rewrite Eqs. B.3, B.4, and B.7 in terms of two new fields; the density field, $\rho(\vec{r})$, and phase field $\theta(\vec{r})$. These two fields are related to $\psi(\vec{r})$ by:

$$\psi(\vec{r}) = \sqrt{\rho(\vec{r})} e^{i\theta(\vec{r})}$$

(B.8)

In terms of the density and phase field, Eq. B.7 can be written as:

$$n(\vec{r}, t) = \frac{\int D\rho(\vec{x}) \int D\theta(\vec{x}) |\langle \{\rho(\vec{x})\}, \{\theta(\vec{x})\} | e^{-iHt} |\psi_0\rangle|^2 \rho(\vec{r})}{\int D\rho(\vec{x}) \int D\theta(\vec{x}) |\langle \{\rho(\vec{x})\}, \{\theta(\vec{x})\} | e^{-iHt} |\psi_0\rangle|^2}.$$

(B.9)

where $|\{\phi(\vec{x})\}\rangle = |\{\rho(\vec{x})\}, \{\theta(\vec{x})\}\rangle$. The matrix element $\langle \{\rho(\vec{x})\}, \{\theta(\vec{x})\} | e^{-iHt} |\psi_0\rangle$ can also be expressed in terms of the density and phase fields:
\[ \langle \{ \rho(\vec{x}) \}, \{ \theta(\vec{x}) \} | e^{-iHt} | \psi_0 \rangle = \int D\rho(\vec{x}, t) D\theta(\vec{x}, t) e^{iS}, \]  
where the action is given by:

\[
S = -\int_0^t dt' \int d^2r \left[ \rho(\vec{r}, t) \partial_t \theta(\vec{r}, t) + \frac{1}{2} \nabla_r \sqrt{\rho(\vec{r}, t)} \cdot \nabla_r \sqrt{\rho(\vec{r}, t)} + \frac{\rho(\vec{r}, t)}{2} \nabla_r \theta(\vec{r}, t) \cdot \nabla_r \theta(\vec{r}, t) + \frac{g}{2} \rho^2(\vec{r}, t) \right] \]  
(B.11)

These manipulations are exact and do not require any approximations.

## B.2 Scale Invariance at the Semiclassical Level

In this section we examine the semiclassical solution to the dynamics and the role of scale invariance. The semiclassical solution is obtained by minimizing this action and only considering the semiclassical contribution to the dynamics. This approach gives the standard hydrodynamic description of a Bose gas:

\[
0 = \partial_t \theta(\vec{r}, t) - \frac{1}{2} \left( \frac{\nabla_r^2 \sqrt{\rho(\vec{r}, t)}}{\sqrt{\rho(\vec{r}, t)}} \right) + \frac{1}{2} (\nabla_r \theta(\vec{r}, t))^2 + g \rho(\vec{r}, t), \\
0 = \partial_t \rho(\vec{r}, t) + \nabla_r \cdot (\rho(\vec{r}, t) \nabla_r \theta(\vec{r}, t)).
\]  
(B.12)

Both of the hydrodynamic equations are invariant under the transformation:

\[
\vec{r}' = e^{-b} \vec{r}, \quad t' = e^{-2b} t,
\]  
(B.13)

and are scale invariant. This implies that a generic solution to this equation of motion also satisfies:

\[
\rho(\vec{r}', t', \{ \lambda' \}) = e^{-2b} \rho(\vec{r}, t, \{ \lambda \}). \\
\theta(\vec{r}', t', \{ \lambda' \}) = \theta(\vec{r}, t, \{ \lambda \}).
\]  
(B.14)
The set of parameters \( \{ \lambda \} \) represent any additional scales introduced by the initial conditions. These length scales explicitly break the scale invariance, and need to be rescaled alongside the spatial and temporal coordinates in the problem: \( \{ \lambda' \} = e^{-b} \{ \lambda \} \). That is, the scale invariance relates the dynamics from different initial conditions to one another.

For the remainder of this discussion we consider the case when the initial conditions introduces a single length scale into the problem, \( \lambda_0 \). This case was previously studied for repulsive interactions in Ref. [42]. At \( t = 0 \), Eq. \[14] implies that the density field can be written as:

\[
\rho(\vec{r},0,\lambda_0) = \frac{N}{\lambda_0^2} f \left( \frac{r}{\lambda_0} \right).
\]  

The scale invariance does not predict the function \( f(x) \) or \( \lambda_0 \), but they can be determined by the initial conditions of the system.

We then choose a time dependent scaling solution for the density and phase fields:

\[
\rho(\vec{r},t) = \frac{N}{\lambda(t)} f \left( \frac{r}{\lambda(t)} \right),
\]

\[
\theta(\vec{r},t) = \frac{r^2 \lambda(t)}{2 \lambda(t)}.
\]  

This solution can be shown to be consistent with Eq. \[12\] if \( \lambda(t) \) satisfies:

\[
m \ddot{\lambda}(t) = \frac{V}{\lambda^3(t)}.
\]  

This is the scale invariant equation of motion for a classical particle in a \( r^{-2} \) potential. The general solution for attractive interactions is:

\[
\lambda(t) = \left[ \frac{|V|}{2E} \left( 2E \sqrt{\frac{1}{m|V|} t + \sqrt{1 + \frac{2E\lambda_0^2}{|V|}}} - 1 \right) \right]^{1/2}.
\]  

### B.3 Coarse Grained Dynamics

To understand the quantum dynamics contained in Eq. \[7\], we perform a coarse graining procedure. This is accomplished by splitting the fields in Eqs. \[4\] and \[7\] into long wavelength isotropic degrees of freedom, \( n(\vec{r},t) \approx \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ll
\textbf{B.3. Coarse Grained Dynamics}

\( \rho_{\lambda}(\vec{r}, t) \) and \( \theta_{\lambda}(\vec{r}, t) \), and short wavelength fluctuations, \( \delta \rho(\vec{r}, t) \) and \( \delta \theta(\vec{r}, t) \). This expansion is controllable in the limit of dense condensates, \( \rho_{\lambda}(\vec{r}, t) \gg 1 \), and leads to an effective theory describing the isotropic long wavelength dynamics of the system.

Motivated by the discussion in the previous appendix, we choose to work with a single parameter ansatz for the isotropic long wavelength degrees of freedom:

\begin{align*}
\rho_{\lambda}(\vec{r}, t) &= \frac{N}{\lambda^2(t)} f \left( \frac{r}{\lambda(t)} \right), \\
\theta_{\lambda}(\vec{r}, t) &= \frac{r^2 \dot{\lambda}(t)}{2 \lambda(t)} + \eta(t),
\end{align*}

(B.19)

where \( f(x) \) is a normalizable isotropic function that is regular at the origin, and \( \eta(t) \) is a time dependent phase that is irrelevant to the following discussion. The parameter \( \lambda(t) \) is the time dependent size of the condensate. In this approach, all the dynamical information is encoded in \( \lambda(t) \), and we wish to derive an effective theory for this single parameter.

This specific form of the phase field, \( \theta_{\lambda}(\vec{r}, t) \), is chosen in order to satisfy the conservation law, the second line in Eq. (B.12). The dynamics of the phase field are treated semiclassically, and is of little importance for the remainder of this discussion. However, no restrictions are placed on the density field.

Applying Eq. (B.19) to Eq. (B.11) results in the following zeroth order action:

\[ S_{\lambda} = \int_0^t dt' \frac{1}{2} m \dot{\lambda}^2 - \frac{V}{2 \lambda^2} \]  

(B.20)

where \( m = C_1 N \) and \( V = C_3 g N^2 + C_2 N \). The coefficients \( C_1 \), \( C_2 \), and \( C_3 \) can be calculated once the function \( f(x) \) has been specified. Some examples of these coefficients can be found in Ref. \[67\].

The main effect of the short wavelength fluctuations to the dynamics is to modify the matrix element, Eq. (B.10), or equivalently the action, Eq. (B.20). To generate the correction to Eq. (B.20) we expand Eq. (B.11) to second order in \( \delta \rho(\vec{r}, t) \) and \( \delta \theta(\vec{r}, t) \). Since the phase field is chosen to satisfy the semiclassical equation of motion, Eq. (B.12), the fluctuations \( \delta \theta(\vec{r}, t) \) will appear at \( O(\delta \theta^2(\vec{r}, t)) \), while the fluctuations in the density, \( \delta \rho(\vec{r}, t) \) will appear at \( O(\delta \rho^2(\vec{r}, t)) \).

In principle there are fluctuations of linear order in \( \delta \rho(\vec{x}, t) \) since we do not minimize the action with respect to density. However, these short
B.3. Coarse Grained Dynamics

wavelength fluctuations represent anisotropic modes that are orthogonal to the motion described by $\lambda$. Therefore any term linear in the fluctuations must vanish.

The remaining quadratic fluctuations can then be integrated out in order to derive an action in terms of the slow degrees of freedom, $\lambda(t)$. In principle there is no limitation to integrating out these fluctuations, however in practice this can be quite a challenge. In order to obtain an estimate of these fluctuations, we assume that the isotropic modes $\rho(\vec{x},t)$ and $\theta(\vec{x},t)$ are approximately constant over the length and time scales associated with the fluctuations. By neglecting the spatial and temporal dependence of the slow degrees of freedom, the action for the fluctuations becomes diagonal when expanded in the definite angular momentum basis:

$$
\delta \rho(\vec{x}, t) = \sum_{k, \ell} N_{k, \ell} J_\ell(kx) \frac{e^{i\ell\phi}}{\sqrt{2\pi}} \delta \rho_{k, \ell}(t) \\
\delta \theta(\vec{x}, t) = \sum_{k, \ell} N_{k, \ell} J_\ell(kx) \frac{e^{i\ell\phi}}{\sqrt{2\pi}} \delta \theta_{k, \ell}(t)
$$ (B.21)

where $k$ and $\ell$ specify the radial mode and angular momentum, respectively, $N_{k, \ell}$ is the normalization factor associated with each radial mode, and $J_\ell(kx)$ is the Bessel function of order $\ell$. An explicit calculation of the fluctuations is given in Ref. [67].

The effect of the fluctuations is to act as a background field upon which the long wave dynamics occur. These fluctuations introduce a correction to the action, $\delta S$. $\delta S$ contains both real and imaginary terms. The real part of $\delta S$ renormalizes the coefficients $C_1$, $C_2$, $C_3$, and the coupling constant $g$, while the imaginary part implies that the system under consideration has a finite lifetime. These corrections to the action are suppressed in the limit $|g| \ll 1$ which is the focus of this work. These corrections are thoroughly discussed in Ref. [67].

After integrating out the fluctuations, the expression for the time evolved density is given by:

$$
n(\vec{r}, t) = \int d\lambda \frac{N}{\lambda^2} f\left(\frac{r}{\lambda}\right) \left| \langle \psi_\lambda | e^{-iHt} | \psi_0 \rangle \right|^2 \\
\langle \psi_\lambda | e^{-iHt} | \psi_0 \rangle = \int_{\lambda(0)=\lambda_0}^{\lambda(t)=\lambda} D\lambda(t) e^{i \int_0^t dt' \frac{1}{2} m \lambda^2(t) - \frac{V_2}{2\lambda(t)^2} + i \delta S}
$$ (B.22)

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B.4. Need for Quantization

where $\lambda_0$ represents the size of the condensate which is in the state $|\psi_0\rangle$.

Eq. [B.22] is equivalent to the quasi-unitary time evolution of a wave function $\psi(\lambda, t) \equiv \langle \lambda | e^{-iH_{\lambda} t} | \lambda_0 \rangle$ under the Hamiltonian $H_{\lambda}$:

$$H_{\lambda} = \frac{P_{\lambda}^2}{2m} + \frac{V}{2\lambda^2} + i \text{Im} \delta H_{\lambda}$$  \hspace{1cm} (B.23)

where $\lambda$ is now an operator with eigenstates $|\lambda\rangle$ and $P_{\lambda}$ the conjugate momentum: $[\lambda, P_{\lambda}] = i \cdot m$ and $V$ are now defined via the renormalized constants $C_1$, $C_2$, and $C_3$. The correction, $i \text{Im} \delta H_{\lambda}$, is the imaginary contribution due to the anisotropic fluctuations:

$$\text{Im} \delta H_{\lambda} = \frac{C_4 g^2 N^2}{2\lambda^2}.$$  \hspace{1cm} (B.24)

Finally, this equivalence between the full dynamics and the effective quantum mechanical model, Eq. [B.23] allows one to recast Eq. [B.9] into the desired result:

$$\rho(\vec{r}, t) = \frac{\int d\lambda \rho(\vec{r}, \lambda)|\psi(\lambda, t)|^2}{\int d\lambda |\psi(\lambda, t)|^2}. \hspace{1cm} (B.25)$$

B.4 Need for Quantization

In this section, we highlight the need for quantization. This is most readily done by examining the fluctuations around a semiclassical path. Here we consider the semiclassical path of a particle with no initial kinetic energy. The solution to Eq. [B.18] will be periodic. For half a period, the solution is given by:

$$\lambda_{sc}(t) = \lambda_0 \sqrt{1 - (2t/T)^2} \hspace{1cm} (B.26)$$

where $T = 2 \frac{\lambda_0^2}{|V|}$ is the semiclassical period.

To understand the effect of fluctuations around this semiclassical path, we calculate the autocorrelation of the fluctuations and compare that to the semiclassical path: $\langle \delta \lambda(t) \delta \lambda(t) \rangle / \lambda_0^2$. To calculate this it is necessary to expand Eq. [B.20] around this solution to quadratic order:

$$\delta S = \int_0^t dt' \frac{m}{2} \delta \dot{\lambda}^2(t) - \frac{3}{2} \frac{V}{\lambda_{sc}^4(t)} \delta \lambda^2(t). \hspace{1cm} (B.27)$$
B.4. Need for Quantization

In order to evaluate the correlator, it is ideal to expand the fluctuations in sine modes;

\[ \delta \lambda(t) = \frac{1}{\sqrt{T}} \sum_n \sin \left( \frac{n \pi t}{T} \right) \delta \lambda_n. \]  

(B.28)

From expanding the action in terms of sine modes one finds that the action is proportional to \( T^{-2} \). This implies that:

\[ \langle \delta \lambda(t) \delta \lambda(t) \rangle \propto \frac{1}{T} \frac{T^2}{\lambda_0^2} = \frac{1}{\sqrt{m|V|}} \]  

(B.29)

where \( \sqrt{m|V|} = \sqrt{C_1 N (C_2 N + C_3 g N^2)} \).

We note that the strength of the fluctuations is not controlled by the size of the classical path, but by a scale independent parameter. Therefore, depending on the value of \( 1/\sqrt{m|V|} \), full quantization might be necessary. We choose to work in this regime.
Appendix C

Quantum Anomaly and the Heisenberg Equation of Motion

In this section we derive the quantum anomaly using the equation of motion. To begin we define an operator in the Heisenberg representation:

\[ O(t) = e^{iH^\dagger t}Oe^{-iHt}. \]  

(C.1)

Here we note that although the Hamiltonian, \( H \), is Hermitian, it may not be self-adjoint, i.e. \( H \) and \( H^\dagger \) may not act on the same subspaces. Therefore, a more correct form of the Heisenberg equation of motion is [91–93]:

\[ i\partial_t O(t) = i[H, O(t)] + i\left(H^\dagger - H\right)O(t). \]

(C.2)

We define the final term as the anomalous piece:

\[ A = \left(H^\dagger - H\right)O(t). \]

(C.3)

Here we consider the Hamiltonian for a two-dimensional Bose (or Fermi) gas:

\[ H = \int d^2r \psi^\dagger(\vec{r}) \left(-\frac{\nabla_r^2}{2}\right) \psi(\vec{r}) + \frac{g}{2} \int d^2r \psi^\dagger(\vec{r})\psi^\dagger(\vec{r})\psi(\vec{r})\psi(\vec{r}). \]

(C.4)

The anomaly associated with scale transformations will then have the form:

\[ A = i \int d^2r \int d^2r' \frac{\nabla_r}{2i} \left(\psi^\dagger(\vec{r}) \nabla_r \psi(\vec{r}) - \nabla_r \psi^\dagger(\vec{r}) \psi(\vec{r})\right) \psi^\dagger(\vec{r}') \left(\frac{d}{2} + r' \cdot \nabla_{r'}\right) \psi(\vec{r}'). \]

(C.5)

As one can see the anomaly depends on the current operator:
Appendix C. Quantum Anomaly and the Heisenberg Equation of Motion

\[ J = \int d^2 r \frac{1}{2i} \left( \psi^\dagger(\vec{r}) \nabla_r \psi(\vec{r}) - \nabla_r \psi^\dagger(\vec{r}) \psi(\vec{r}) \right). \]  \hspace{1cm} (C.6)

For potentials with ill-defined ultraviolet physics, the current operator can be singular, leading to non-zero anomalies. For the remainder of this section, we will illustrate this physics by applying this formalism to two-body problem s-wave scattering in two spatial dimensions.

Consider two bosons interacting via a short range square well potential:

\[ V(r) = -V_0 \theta(r_0 - r). \]  \hspace{1cm} (C.7)

The s-wave wave function for the system can be written in terms of the center-of-mass, \( \vec{R} \), and relative, \( \vec{r} \), coordinates:

\[ \psi(\vec{R}, \vec{r}) = \frac{1}{\sqrt{\Omega}} e^{i\vec{P} \cdot \vec{R}} \chi(r), \]  \hspace{1cm} (C.8)

where \( \Omega \) is the volume of the system, \( P \) is the center of mass momentum, and \( \chi(r) \) is the relative wave function. The s-wave relative wave function satisfies the Schrodinger equation:

\[ \left( -\partial_r^2 + V(r) \right) \chi(r) = E \chi(r). \]  \hspace{1cm} (C.9)

Although the physical potential has a finite range, this physical potential is often replaced with a contact interaction. In order to do this, one must apply an boundary condition at \( r = r_0 \) that reproduces the low energy scattering of the true potential. For two-dimensional systems this condition is given by:

\[ \chi(r_0) \propto \log \left( \frac{r_0}{a_{2D}} \right), \]  \hspace{1cm} (C.10)

where \( a_{2D} \) is the effective scattering length for the two bosons. The effective scattering length is related to the bare parameters by:

\[ a_{2D} = \frac{r_0}{2} e^{\gamma_E} e^{\frac{2}{V_0 r_0^2}}, \]  \hspace{1cm} (C.11)

where \( \gamma_E \) is Euler’s gamma constant. Note that this length scale is related to the bound state energy by \( E_b = -\frac{a_{2D}^2}{2} \).

Since we replaced the physical non-scale invariant potential with a scale invariant contact interaction, the consequence will be a non-zero anomaly. It is only necessary to focus on the relative coordinate’s contribution to the
anomaly, as the center of mass motion is non-interacting. For this two-body system the anomaly is given by:

\[
A = \int d^2r \chi(r) (\nabla^2_r)(\vec{r} \cdot \nabla_r) \chi(r) - \nabla^2_r \chi(r)(\vec{r} \cdot \nabla_r) \chi(r)
\]

\[
= \int d\vec{S} \left[ \chi(r) \nabla_r (\vec{r} \cdot \nabla_r) \chi(r) - \nabla_r \chi(r) \cdot \vec{r} \cdot \nabla_r \chi(r) \right], \quad (C.12)
\]

where the integrals are over the surface of the space.

As mentioned previously, the anomaly is caused by neglecting the physics inside the potential. Therefore we focus on the contribution from short distances. Applying Eq. (C.10) to Eq. (C.12), one can show that [65, 84–87]:

\[
A = 2\pi. \quad (C.13)
\]

The fact that \( A \) is non-zero implies that the scale invariance is broken. However, if one were to consider the physics inside the potential, the scale invariance would be explicitly broken, but the anomaly would be zero. The anomaly comes from treating the system with a scale invariant effective model.
Appendix D

Existence of Conformal Towers

In this appendix, we show that conformal symmetry means that the eigenstates of a quantum system are organized in terms of towers, where the states within a tower are evenly spaced by two harmonic oscillator units.

To begin we note that for non-relativistic quantum systems. The conformal symmetry is a subgroup of the Galilean group, which is the largest symmetry class for the non-relativistic Schrodinger equation [97]. The algebra is formed by the following operators:

\[
N = \int d^3r \ n(\vec{r}) \quad P_i = \int d^3r \ \vec{r}_i(\vec{r}) \quad M_{i,j} = \int d^3r \ (r_{ij}(\vec{r}) - r_{ji}(\vec{r})) , \tag{D.1}
\]

\[
K_i = \int d^3r \ r_i n(\vec{r}) \quad C = \int d^3r \ \frac{r^2}{2} n(\vec{r}) \quad D = \int d^3r \ \vec{r} \vec{j}(\vec{r}) , \tag{D.2}
\]

where:

\[
n(\vec{r}) = \psi^\dagger(\vec{r})\psi(\vec{r})
\]

\[
\vec{j}(\vec{r}) = -\frac{i}{2} \left( \psi^\dagger(\vec{r}) \nabla \psi(\vec{r}) - \nabla \psi^\dagger(\vec{r}) \psi(\vec{r}) \right) \tag{D.3}
\]

are the particle density and probability current, and the field operators, \(\psi(\vec{r})\) satisfy either bosonic or fermionic statistics:

\[
[\psi(\vec{r}), \psi^\dagger(\vec{r}')]_\pm = \delta^3(\vec{r} - \vec{r}'). \tag{D.4}
\]

The remaining operators are associated with particle number, \(N\), momentum, \(P_i\), angular momentum \(M_{i,j}\), Galilean boosts, \(K_i\), conformal transformations, \(C\), and scale transformations, \(D\).

A subgroup of these operators is called the conformal group. The conformal group consists of three operators, a scale invariant Hamiltonian, \(H_s\), the dilation operator, \(D\), and the conformal operator, \(C\). This conformal group
group forms a subalgebra, the so(2,1) algebra. For the conformal group, one can show the following commutation relations:

\[ [H_s, C] = -iD, \quad [D, C] = -2iC, \quad [D, H_s] = 2iH_s. \]  (D.5)

Next consider an operator:

\[ O(\vec{r}) = e^{i\vec{P} \cdot \vec{r}}O(0)e^{-i\vec{P} \cdot \vec{r}}, \]  (D.6)

such that: \([D, O(0)] = i\Delta_O O(0),\) and \([C, O(0)] = [K_i, O(0)] = 0.\) Such an operator is called a primary operator. Here we consider the state:

\[ |\psi\rangle = e^{-H_s/\omega}O(0)|\text{vac}\rangle, \]  (D.7)

where \(|\text{vac}\rangle\) is the vacuum state. This state is an eigenstate of the Hamiltonian, \(H_s + \omega^2 C.\) For the following discussion we will set the trap frequency, \(\omega,\) to unity. To see that \(|\psi\rangle\) is an eigenstate of \(H_s + C,\) note:

\[ (H_s + C)|\psi\rangle = e^{-H_s}(H_s + e^{H_s}Ce^{-H_s})O(0)|\text{vac}\rangle \]  (D.8)

Using the identity:

\[ e^{H_s}Ce^{-H_s} = C + [H_s, C] + \frac{1}{2}[H_s, [H_s, C]] + \ldots \]  (D.9)

and using the commutation relations contained in Eq. (D.5) and the definition of \(O(0),\) one obtains:

\[ (H_s + C)|\psi\rangle = e^{-H_s}(C - iD)O(0)|\text{vac}\rangle = \Delta_O O(0)|\psi\rangle. \]  (D.10)

Therefore, \(|\psi\rangle\) is an eigenstate of \(H_s + C,\) with eigenvalue equal to \(\Delta_O.\) This is true for any primary operator \(O(0).\)

Given the states \(|\psi\rangle,\) it is possible to generate the remaining spectrum of a conformal tower. Consider the operators:

\[ L_{\pm} = H_s - C \pm iD. \]  (D.11)

These operators satisfy:

\[ [L_-, L_+] = 4(H_s + C), \quad [L_\pm, L_{\pm} + C] = 2L_{\pm}. \]  (D.12)

Eq. (D.12) is identical to the commutation relations for the ladder operators of a non-interacting harmonic oscillator. The only difference is that they raise and lower the energy by two harmonic units.
To show that $|\psi\rangle$ is the base of the tower, let us consider the action of $L_-$ on this state:

$$L_- |\psi\rangle = e^{-H_s}e^{H_s}(H_s - C - iD)e^{-H_s}O(0)|\text{vac}\rangle = -e^{-H_s}CO(0)|\text{vac}\rangle = 0.$$  \hspace{1cm} (D.13)

As can be seen by Eq. (D.13), all states $|\psi\rangle$ constructed from a primary operator, $O$, form the bases of the conformal towers. The remaining states in the spectrum can be constructed by the application of the raising operator. This proves that the spectrum of a conformally symmetric quantum system can be decomposed into a series of conformal towers. The states in each tower are evenly spaced by two harmonic units.

In order to label the towers, we note that:

$$[L_\pm, N] = 0,$$ \hspace{1cm} (D.14)

where $N$ is the total number of particles. As well, for rotationally invariant systems:

$$[H_s, M_{i,j}] = [C, M_{i,j}] = [D, M_{i,j}] = 0,$$ \hspace{1cm} (D.15)

or:

$$[L_\pm, M_{i,j}] = 0.$$ \hspace{1cm} (D.16)

Therefore a given conformal tower can be labelled by the total number of particles, and their angular momentum, or equivalently for one spatial dimension, by the parity. As a result there are infinitely many towers. In our analysis we will always consider a fixed number of particles. In this case, the conformal towers can be labelled by their angular momentum, or parity, alone.

### D.1 Application of Conformal Tower Spectrum to a 1D Harmonic Oscillator

The spectrum for a 1D harmonic oscillator is given by:

$$E_n = \left(n + \frac{1}{2}\right)\omega.$$ \hspace{1cm} (D.17)
As can be seen, for a fixed number of particles, the spectrum can be decomposed into two conformal towers: one for even and odd parity. For a single particle, the energies for the even and odd single particle conformal towers, respectively, are:

\[ E_{n,e} = \left(2n + \frac{1}{2}\right)\omega \]
\[ E_{n,o} = \left(2n + \frac{3}{2}\right)\omega. \]  

(D.18)

The creation operators for a particle in the lowest even and odd conformal tower states:

\[ \psi^\dagger_{0,e} = \int dx \left(\frac{\omega}{\pi}\right)^{1/4} e^{-\frac{x^2}{2}} \psi^\dagger(x) \]
\[ \psi^\dagger_{0,o} = \int dx \sqrt{2} \left(\frac{\omega}{\pi}\right)^{1/4} x\sqrt{\omega} e^{-\frac{x^2}{2}} \psi^\dagger(x). \]  

(D.19)

We note that the Fourier Transform of Eq. (D.19) is:

\[ \psi^\dagger_{0,e} = \int \frac{dk}{2\pi} \sqrt{2} \left(\frac{\pi}{\omega}\right)^{1/4} e^{-\frac{k^2}{2\omega}} \psi^\dagger(k) \]
\[ \psi^\dagger_{0,o} = \int \frac{dk}{2\pi} - 2i \left(\frac{\pi}{\omega}\right)^{1/4} \frac{k}{\sqrt{\omega}} e^{-\frac{k^2}{2\omega}} \psi^\dagger(k). \]  

(D.20)

Upon comparison to Eq. (D.7), one can read off:

\[ O_{e}(0) = \sqrt{2} \left(\frac{\pi}{\omega}\right)^{1/4} \psi^\dagger(0) \]
\[ O_{o}(0) = 2 \left(\frac{\pi}{\omega}\right)^{1/4} \frac{1}{\sqrt{\omega}} \lim_{x \to 0} \frac{\partial}{\partial x} \psi^\dagger(x). \]  

(D.21)

A straightforward calculation shows that:

\[ [D, O_{e}(0)] = i\frac{1}{2} O_{e}(0) \]
\[ [D, O_{o}(0)] = i\frac{3}{2} O_{o}(0), \]  

(D.22)

which are the correct ground state energies for the even and odd conformal towers.
Appendix E

Comoving Reference Frame and Time Dependent Traps

In this appendix we discuss the use of the comoving reference frame to scale invariant gases placed in time dependent traps. The Hamiltonian for this system is given by:

\[ H = \sum_{i=1}^{N} -\frac{1}{2} \nabla_{i}^{2} + \frac{1}{2} \sum_{i,j=1}^{N} V_{s}(\vec{r}_{i} - \vec{r}_{j}) + \sum_{i=1}^{N} \frac{\omega^{2}(t)}{2} r_{i}^{2} \]

\[ = H_{s} + \omega^{2}(t)C. \quad (E.1) \]

We employ the trial many body wave function:

\[ \psi(\{\vec{r}_{i},\sigma_{i}\},t) = \frac{1}{\lambda^{3N/2}(t)} e^{\frac{i}{2} \sum_{i=1}^{N} r_{i}^{2} \frac{\lambda(t)}{\lambda^{3}(t)} \phi \left( \{\vec{r}_{i},\sigma_{i}\}, \lambda(t) \right) }. \quad (E.2) \]

Upon substituting into the Schrödinger equation, one obtains:

\[ i \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} \phi(\{\vec{x}_{i},\sigma_{i}\},\tau(t)) = \left( \sum_{i} \left[ -\frac{1}{2} \frac{1}{\lambda^{2}(t)} \tilde{\nabla}^{2}_{i} + \frac{x_{i}^{2}}{2} \left( \tilde{\lambda}(t) \lambda(t) + \omega^{2}(t) \lambda^{2}(t) \right) \right] \right. \]

\[ + \left. \frac{1}{2} \sum_{i,j} V_{s}(\lambda(t) (\vec{x}_{i} - \vec{x}_{j})) \right) \phi(\{\vec{x}_{i},\sigma_{i}\},\tau(t)). \quad (E.3) \]

To make further simplifications, we define:

\[ \frac{\partial \tau(t)}{\partial t} = \frac{1}{\lambda^{2}(t)} \tilde{\lambda}(t) = \frac{\omega_{0}^{2}}{\lambda^{3}(t)} - \omega^{2}(t) \lambda(t), \quad (E.4) \]

noindent where the boundary conditions for \( \lambda(t) \) are chosen such that:

\[ \text{103} \]
\[ \lambda(0) = 1 \quad \dot{\lambda}(0) = 0 \quad (E.5) \]

Solving Eq. [E.4] and substituting into Eq. [E.3] gives a Schrödinger equation for the effective wave function:

\[ i \frac{\partial}{\partial \tau} \phi(\{ \vec{x}_i, \sigma_i \}, \tau) = \tilde{H} \phi(\{ \vec{x}_i, \sigma_i \}, \tau), \]

\[ \tilde{H} = \sum_{i=1}^{N} \left[ -\frac{1}{2} \tilde{\nabla}_i^2 + \frac{\sigma_i^2 x_i^2}{2} \right] + \frac{1}{2} \sum_{i,j=1}^{N} V_s(\vec{x}_i - \vec{x}_j). \quad (E.6) \]

The Hamiltonian in the comoving frame is simply a scale invariant gas in a time independent harmonic trap.

Eq. [E.6] is conformally invariant, and the spectrum of the system can be written in terms of the conformal tower spectra discussed in Appendix D. Therefore, in the laboratory frame, any system prepared in one of these conformal tower states will exhibit the trivial dynamics contained in Eq. [E.4].

### E.1 Quench of The Trapping potential

As a first example, consider a quench of the trapping potential:

\[ \omega(t) = \omega_i \quad t < 0 \]
\[ = \omega_f \quad t \geq 0, \quad (E.7) \]

where \( \omega_f < \omega_i \). The solution was derived in Ref. [74]. Here we show the result for \( \lambda(t) \) and \( \tau(t) \):

\[ \lambda(t) = \lambda_0 \sqrt{1 + \frac{\omega_i^2 - \omega_f^2}{\omega_f^2} \sin^2(\omega_f t)} \]
\[ \tau(t) = \frac{1}{\omega_i \lambda_0^2} \arctan \left( \frac{\omega_i}{\omega_f} \tan(\omega_f t) \right). \quad (E.8) \]

The solution for \( \lambda(t) \) is shown in Fig. [E.1] has two main features. First the solution oscillates at frequency \( \omega_f \), and secondly, for \( \omega_f t \ll 1 \), the solution is equivalent to free space expansion.
E.2 Efimovian Expansion

Figure E.1: Solution for $\lambda(t)$ for the dynamics of a quench trapping potential. The ratio of the trapping potentials is given by $\omega_i = 5\omega_f$. For convenience we plot $\lambda^2(t)$ as a function of time.

In this section, we consider a trap that has a trap frequency:

\[
\omega(t) = \begin{cases} 
\omega_i & t < t_0 \\
\frac{t_0}{t} \omega_i & t \geq t_0.
\end{cases}
\]  

(E.9)

The solution for $\lambda(t)$ and $\tau(t)$ are:

\[
\lambda(t) = \lambda_0 \left[ \frac{(1 + s_0^2)t}{t_0 s_0^2} \left[ 1 - \frac{1}{\sqrt{1 + s_0^2}} \cos \left( s_0 \log \left( \frac{t}{t_0} \right) - \arctan(s_0) \right) \right] \right]^{1/2}
\]

\[
\tau(t) = \frac{t_0}{\lambda_0^2} \frac{2}{\sqrt{1 + s_0^2}} \arctan \left[ \left( 2 + s_0^2 \right) \tan \left( \frac{s_0}{2} \log \left( \frac{t}{t_0} \right) - s_0 \right) \right],
\]  

(E.10)

where $s_0 = \sqrt{4(\omega_i t_0)^2 - 1}$.
E.2. Efimovian Expansion

Figure E.2: The dynamics of the moment of inertia for a scale invariant quantum gas inside an expanding trap. The time dependence of the trap is given by Eq. [E.9]. The dynamics of the moment of inertia are given by Eq. [E.11], and are in strong agreement with the observations in Ref. [4].

E.2.1 Experimental Detection of Efimovian Expansion

The dynamics of a scale invariant Fermi gas in a time dependent harmonic trap have been studied experimentally in Ref. [4], and theoretically in Ref. [68]. In this experiment the trap evolves according to Eq. [E.9], and the dynamics of the moment of inertia were examined. In this section we discuss the dynamics using the comoving reference frame.

Consider a scale invariant Fermi gas of \(N\) particles placed in an isotropic harmonic potential of frequency \(\omega_i\). At zero temperatures, the gas can be considered to be in the many body ground state, which necessarily is a conformal tower state. The moment of inertia is then defined to be:

\[
\langle \hat{r}^2 \rangle(t) = \int \frac{d^d r_1}{\lambda^d(t)} \int \frac{d^d r_2}{\lambda^d(t)} \cdots \int \frac{d^d r_N}{\lambda^d(t)} r_1^2 \frac{2}{\phi_0 \left( \left\{ \frac{r_i}{\lambda(t)} \right\} \right)}
\]

\[
= \lambda^2(t) \langle \hat{r}^2 \rangle(0),
\]

(E.11)

where \(\phi_0(\{\vec{x}_i\})\), is the many body wave function for the ground state.

The dynamics of the moment of inertia are shown in Fig. [E.2]. Eq. [E.11] is an accurate description of the dynamics, and faithfully reproduces the experiments in Ref. [4].
Appendix F

Two-Body Solution

In this appendix we derive the deviation, $\delta\tilde{H}$, from the transformed scale invariant Hamiltonian, $\tilde{H}_s$, in the expanding comoving frame. The approach we employ here is applicable for both the non-interacting quantum gas with an impurity, and the relative dynamics of the two-body problem. The only difference is that in the two-body problem, one uses the reduced mass for the two particles. In both cases, the physical interaction will be some short ranged, spherically symmetric potential, $V(r)$.

We begin with the radial Schrödinger equation in the co-moving frame:

$$i\partial_\tau \chi_l(x, \tau) = \tilde{H}\chi_l(x, \tau),$$

$$\tilde{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2 + \frac{l(l+1)}{2x^2} + \lambda^2(\tau)V(x\lambda(\tau)),$$

$$\lambda(\tau) = \lambda_0\sec(\tau),$$

where we have set the (reduced) mass to unity, and $Y_{l,m}(\vec{x})$ is the spherical harmonic with angular quantum number $l$ and projection quantum number, $m$. The radial wave function, $\chi_l(x, \tau)$, is related to the full wave function via: $\phi_{l,m}(\vec{x}, \tau) = Y_{l,m}(\vec{x})\chi_l(x, \tau)/x$, and is properly normalized:

$$\int_0^\infty dx |\chi_l(x, \tau)|^2 = 1. \quad (F.2)$$

In what follows we will only focus on the zero angular momentum, or s-wave, scattering of this potential, as higher angular momentum scattering is suppressed by a factor of $(\sqrt{E}r_0)^{2l}$, where $r_0$ is the range of the potential, and $E$ is the energy.

For specificity, we will consider the potential to be a square well of depth: $V_0\lambda^2(\tau)$, and range: $r_0/\lambda(\tau)$. This potential is consistent with the time dependence of the interaction in the coming frame: $\lambda^2(\tau)V(x\lambda(\tau))$, and captures all the essential physics at low energies. It is important to note that the range and depth of the potential are changing at a rate set by $\lambda(\tau)$, which is much slower than the energy scale set by the finite range of the
potential, \( r_0 \). This implies that we can use the adiabatic approximation. In this approximation, the effect of the finite scattering length is to impose the time-dependent boundary condition at the range of the potential [76, 77]:

\[
\frac{\chi'(r_0/\lambda(\tau))}{\chi(r_0/\lambda(\tau))} = -\frac{\lambda(\tau)}{a}.
\] (F.3)

As discussed in the main text, we split the effective Hamiltonian in the comoving frame, \( \tilde{H} \), into the effective Hamiltonian at the scale invariant fixed point, \( \tilde{H}_s \), and a deviation, \( \delta \tilde{H} \):

\[
\tilde{H} = \tilde{H}_s + \delta \tilde{H}
\]

\[
\tilde{H}_s = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} x^2 + \lambda^2(\tau) V_s(\lambda(\tau)x),
\] (F.4)

and \( V_s(x) \) is a scale invariant potential, and we have set the trap frequency to unity. In this analysis the quantities of interest are the matrix elements of the deviation:

\[
\delta \tilde{H}(\tau) = \lambda^2(\tau) V(x\lambda(\tau)) - \lambda^2(\tau) V_s(\lambda(\tau)x)
\]

\[
\delta \tilde{H}(\tau) = \lambda^2(\tau) V(x\lambda(\tau)) - V_s(x),
\] (F.5)

with respect to the eigenstates of \( \tilde{H}_s \). In Eq. (F.5) we have used the fact that the system possesses scale invariance at a fixed point, i.e. \( V_s(\lambda x) = \lambda^{-2} V_s(x) \).

We first evaluate the deviation from the resonant fixed point. The matrix elements of Eq. (F.5) near resonance can be determined by examining the zero angular momentum Schrödinger equation at, and near, resonance:

\[
E_{r,n} \chi_{r,n}(x) = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} x^2 + V_{res}(x) \right) \chi_{r,n}(x)
\]

\[
E_m(\tau) \chi_m(x, \tau) = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} x^2 + \lambda^2(\tau) V \left( \frac{x}{\lambda(\tau)} \right) \right) \chi_m(x, \tau). \] (F.6)

The top and bottom lines correspond to the resonant and off resonant Schrödinger equations, respectively. The states \( \chi_{r,n}(x) \) and \( \chi_m(x, \tau) \) are the eigenstates of the system with energy \( E_{r,n} \) and \( E_m(\tau) \), and quantum numbers \( n \) and \( m \), for the resonant, and off resonant Hamiltonians, respectively.
Appendix F. Two-Body Solution

At this stage one can multiply the resonant (off-resonant) Schrodinger equation by the state \( \chi_m(x,\tau) (\chi_{r,n}(x)) \), and integrate over the range of the potential, \( r_0/\lambda(\tau) \). The difference between the two Schrodinger equations is:

\[
\int_{r_0/\lambda(\tau)}^{r_0/\lambda(\tau)} dx \chi_{r,n}(x) (E_m(\tau) - E_{r,n}) \chi_m(x,\tau) = 
\int_{r_0/\lambda(\tau)}^{r_0/\lambda(\tau)} dx \chi_{r,n}(x) \left[ -\frac{1}{2} \partial_x^2 + \frac{1}{2} x^2 + \lambda^2(\tau) V(x,\lambda(\tau)) \right] \chi_m(x,\tau)
- \int_{r_0/\lambda(\tau)}^{r_0/\lambda(\tau)} dx \chi_{m}(x,\tau) \left[ -\frac{1}{2} \partial_x^2 + \frac{1}{2} x^2 + V_{res}(x) \right] \chi_{r,n}(x) \tag{F.7}
\]

To obtain the deviation operator, we expand the difference to first order in \( 1/a \). To this order the expansion of Eq. \( \text{F.7} \) gives:

\[
\int_{0}^{r_0/\lambda(\tau)} dx (\lambda^2(\tau) V(x,\lambda(\tau)) - V_{res}(x)) \chi_{r,m}(x) \chi_{r,n}(x) = 
\int_{0}^{r_0/\lambda(\tau)} dx \{(E_m(\tau) - E_{r,n}) \chi_m(x,\tau) \chi_{r,n}(x) \}
- \frac{\lambda(\tau)}{2a} \chi_{r,m}(r_0/\lambda(\tau)) \chi_{r,n}(r_0/\lambda(\tau)).
\]

In Eq. \( \text{F.8} \) we have used Eq. \( \text{F.3} \) to evaluate the difference in kinetic energies.

In the second line of Eq. \( \text{F.8} \) the quantities \( E_m(\tau) - E_{r,n} \) and \( \chi_{r,n}(x) \chi_{m}(x,\tau) \) are to be expanded to first order in \( \lambda(\tau)/a \). The integrals themselves will be proportional to \( r_0/\lambda(\tau) \), for \( r_0 \ll \lambda_0 \). Therefore the second line in Eq. \( \text{F.8} \) will be a correction of order \( r_0/a \) to the matrix elements, which is negligible in the large scattering length limit. After neglecting the terms proportional to \( O(r_0/a) \), the expression for the deviation becomes:

\[
\langle m | \delta H | m \rangle = \int_{0}^{r_0/\lambda(\tau)} dx \chi_{r,m}(x) \chi_{r,n}(x) \left( \lambda^2(\tau) V(x,\lambda(\tau)) - V_{res}(x) \right) 
- \frac{\lambda(\tau)}{2a} \chi_{r,m}(r_0/\lambda(\tau)) \chi_{r,n}(r_0/\lambda(\tau)).
\]

To simplify the deviation further, we note that the resonant eigenstates outside the potential are given by:
\[ \chi_{r,n}(x) = \langle x|n \rangle = \sqrt{2} \phi_{h.o,2n}(x) \quad E_{r,n} = 2n + 1/2, \]  
\( \text{(F.10)} \)

where \( \phi_{h.o,n}(x) \) is the normalized one-dimensional harmonic oscillator wave function with quantum number \( n = 0, 1, 2, \ldots \), and with the harmonic length scale set to unity. For \( x \ll 1 \), the resonant eigenstates are constant near the origin. The continuity of the wave function at the boundary allows one to simplify the deviation to:

\[
\langle m|\delta \tilde{H}|n \rangle = -\frac{\lambda(\tau)}{2a} \chi_{r,m}(0) \chi_{r,n}(0) = \frac{\lambda(\tau)}{a} f_m f_n \]  
\( \text{(F.11)} \)

where:

\[
f_n = \frac{\sqrt{2}}{\pi^{1/4}} \frac{(2n - 1)!!}{\sqrt{(2n)!}}. \]  
\( \text{(F.12)} \)

Therefore the scaling of the deviation is \( \alpha = 1 \).

For weak interactions, we compare the non- and weakly- interacting Schrödinger equations in the co-moving frame:

\[
E_{0,n} \chi_{0,n}(x) = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} x^2 \right) \chi_{0,n}(x)
\]

\[
E_m(\tau) \chi_m(x, \tau) = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} x^2 + \lambda^2(\tau) V(x \lambda(\tau)) \right) \chi_m(x, \tau). \]  
\( \text{(F.13)} \)

Here \( \chi_{0,n}(x) \) is a non-interacting eigenstate with energy \( E_{0,n} \) and quantum number \( n = 0, 1, 2, \ldots \):

\[
\chi_{0,n}(x) = \langle x|n \rangle = \sqrt{2} \phi_{h.o,2n+1}(x) \quad E_{0,n} = 2n + 3/2. \]  
\( \text{(F.14)} \)

A calculation identical to the resonant case yields:

\[
\langle m|\delta \tilde{H}|n \rangle = \frac{a}{2\lambda(\tau)} \chi'_{0,m}(0) \chi'_{0,n}(0) = \frac{a}{\lambda(\tau)} g_m g_n, \]  
\( \text{(F.15)} \)

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where:

\[ g_n = (-1)^n \frac{1}{\pi^{1/4}} \frac{1}{2^n - 1} \frac{\sqrt{(2n+1)!}}{n!}. \]  

\[ \text{(F.16)} \]

The approach used here is similar to Refs. [75–77]. The matrix elements of Eq. F.11 and F.15 have an important connection to the thermodynamic contact first examined in Ref. [76, 77]. If one were to consider just the ground state expectation value of the deviation, Eq. [F.5] one would simply obtain the contact. Eq. [F.5] is then a natural extension of the idea of contact to a matrix. The relationship between the breaking of scale invariance and the contact in equilibrium physics has been discussed in Ref. [86].

### F.1 Beat Amplitudes for Moment of Inertia and Contact

In this appendix we report the analytic expressions for the long time, \( \omega t \gg 1 \), or equivalently, \( \tau \approx \pi/2 \), dynamics for the moment of inertia of an non-interacting quantum gas in the presence of a scale invariant external potential, and for the contact in the two-body problem, both near resonance.

The moment of inertia for \( N \)-particles is defined as:

\[
\hat{r}^2 = \frac{1}{N} \sum_{i=1}^{N} r_i^2 = \frac{\lambda^2(t)}{N} \sum_{i=1}^{N} x_i^2
\]

\[
= \lambda^2(t) \langle x^2 \rangle (\tau(t)),
\]  

\[ \text{(F.17)} \]

where \( \langle x^2 \rangle \) is the moment of inertia calculated in the comoving frame. The long time dynamics of the moment of inertia, in the comoving frame, has the following form:

\[
\lim_{t \to \infty} \langle x^2 \rangle(t) \approx A + B \sin \left( v \frac{1}{\sqrt{\omega a}} \log(\omega t) \right) \left( \frac{1}{\omega t} + D \sin^2 \left( \frac{v}{2} \frac{1}{\sqrt{\omega a}} \log(\omega t) \right) \right).
\]

\[ \text{(F.18)} \]

The coefficients \( A, B, \) and \( D \) are found by explicitly evaluating the expectation value. Here we quote the result:

\[
A = \sum_{\{n_i\},\{m_i\}=0}^{n_{\text{max}}} \sum_{P,Q} \frac{(\pm 1)^{P+Q}}{N!} \psi(\{m_i\})\psi(\{n_i\})
\]

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\[
\frac{1}{2N} \sum_{j=1}^{N} (4n_{Q_j} + 1) \prod_{k=1,\neq j}^{N} \delta_{m_{P_k},n_{Q_k}}
\]

\[
\frac{1}{N} \sum_{i,j}^{N} \sqrt{(2n_{Q_j} + 1)(2n_{Q_j} + 1)} \prod_{k\neq j}^{N} \delta_{m_{P_k},n_{Q_k}} \delta_{m_{P_j},n_{Q_j} + 1}
\]

\[
B = \sum_{(n_i),\{m_i\}=0}^{N} \frac{(\pm 1)^{P+Q}}{N!} \frac{2}{N} \sum_{j=1}^{N} \left[ (4n_{Q_j} + 1) \prod_{k=1,\neq j}^{N} \delta_{m_{P_k},n_{Q_k}} \right.
\]

\[
(\langle n_{Q_1},...,n_{Q_N}|P_v|\psi_0\rangle \langle m_{P_1},...,m_{P_N}|P_v|\psi_0\rangle - \langle n_{Q_1},...,n_{Q_N}|P_v|\psi_0\rangle \psi(\{n_i\} - \langle n_{Q_1},...,n_{Q_N}|P_v|\psi_0\rangle \psi(\{m_i\}))
\]

\[
+ \sum_{(n_i),\{m_i\}=0}^{N} \frac{(\pm 1)^{P+Q}}{N!} \frac{2}{N} \sum_{j=1}^{N} \left[ \sqrt{(2n_{Q_j} + 1)(2n_{Q_j} + 1)} \right.
\]

\[
\prod_{k\neq j}^{N} \delta_{m_{P_k},n_{Q_k}} \delta_{m_{P_j},n_{Q_j} + 1} [\langle m_{P_1},...,m_{P_N}|P_v|\psi_0\rangle \psi(\{n_i\})
\]

\[
+ \langle n_{Q_1},...,n_{Q_N}|P_v|\psi_0\rangle \psi(\{m_i\}) - \langle n_{Q_1},...,n_{Q_N}|P_v|\psi_0\rangle \langle m_{P_1},...,m_{P_N}|P_v|\psi_0\rangle]
\]

\[
(F.19)
\]

\[
(F.20)
\]

In Eqs. (F.19) \(\psi_0\) and (F.21), \(\psi_0\), is the fully symmetrized initial state:

\[
|\psi_0\rangle = \sum_{\{n_i\}} \left[ \psi(\{n_i\}) \frac{1}{\sqrt{N!}} \sum_{P} (\pm 1)^{P} |n_{P_1},...,n_{P_N}\rangle \right]
\]

\[
(F.22)
\]

The many body states are expanded in the single particle basis: \(|n_1,...,n_N\rangle\), where \(n_i = 0,1,...,n_{\max}\), with \(n_{\max} = 1/(\omega r_0^2)\). The quantities \(\psi(\{n_i\})\) are the expansion coefficients for the many body states; they do not depend on the specific permutation of the indices, but rather the combination of indices. For this reason, the summations over the many body states are restricted to distinct combinations of single particle indices. The symmetry is fixed by summing over all the permutations of a given set of single particle indices, specified by the summation over \(P\).
F.1. Beat Amplitudes for Moment of Inertia and Contact

For the non-interacting quantum gas, the deviation from scale invariance, $\tilde{V}$ will only have a single non-zero eigenvalue of value $v$:

$$v = \sum_{n=0}^{n_{\text{max}}} f_n^2, \quad f_n = \frac{\sqrt{2}}{\pi^{1/4}} \frac{(2n-1)!!}{\sqrt{(2n)!}}.$$  \hspace{1cm} (F.23)

For $N = 1$, the state with eigenvalue $v$ is non-degenerate, however for multiple particle there are a number of degenerate states with eigenvalue, $v$. The operator, $P_v$, in Eqs. F.19, F.20, and F.21 is simply the projection operator onto all the eigenstates of the $N$-body deviation, $\tilde{V}_I$, with eigenvalue $v$.

Similarly, we quote the results for the contact of the relative motion in the interacting two-body problem. Near resonance, the asymptotic form of the contact is:

$$\lim_{t \to \infty} C(t) \approx \frac{E}{\lambda(t)} + \frac{F}{\lambda(t)} \sin^2 \left( \frac{v^2}{2} \frac{1}{\sqrt{\omega a}} \ln \left( \frac{t}{\lambda_0^2} \right) \right).$$  \hspace{1cm} (F.24)

The coefficients, $E$ and $F$, are given by:

$$E = \left| \sum_{n=0}^{n_{\text{max}}} f_n \frac{\sqrt{\pi}}{2} (-1)^n C_n(0) \right|^2,$$  \hspace{1cm} (F.25)

$$F = \left| \sum_{n=0}^{n_{\text{max}}} f_n \sqrt{\pi} (-1)^n \langle n|v\rangle \langle v|\psi_0 \rangle \right|^2$$

$$- \sum_{n,n'=0}^{n_{\text{max}}} \frac{\pi}{2} f_n f_n' (-1)^{n-n'}$$

$$\cdot \left( C_n'(0) \langle n|v\rangle \langle v|\psi_0 \rangle + C_{n'}(0) \langle n'|v\rangle \langle v|\psi_0 \rangle \right).$$

where again, $C_n(0)$ here are the expansion coefficients for the relative motion, $v$ and $f_n$ are defined in Eq. [F.23].
Appendix G

The Density Matrix and Conformal Symmetry

G.1 The Density Matrix and Scale Invariance

Here we consider the dynamics of the one-body density matrix, or simply density matrix:

\[ \rho(\vec{r}, \vec{r}', t) = \langle \psi_0 | e^{iHt} \psi^\dagger(\vec{r}') \psi(\vec{r}) e^{-iHt} | \psi_0 \rangle, \]  

(G.1)

where \( H = H_s + \delta H \) is the Hamiltonian of the nearly scale invariant system, \( H_s \) is the scale invariant Hamiltonian, and \( \delta H \) is the deviation. As we did previously, it is ideal to expand the full Hamiltonian around \( H_s \):

\[ \rho(\vec{r}, \vec{r}', t) = \langle \psi_0 | e^{iH_s t} e^{-iH_t} \psi^\dagger(\vec{r}', t) \psi(\vec{r}, t) e^{iH_s t} e^{-iH_t} | \psi_0 \rangle, \]  

(G.2)

where \( \psi(\vec{r}, t) = e^{iH_s t} \psi(\vec{r}) e^{-iH_s t} \) is the field operator time evolved by the scale invariant Hamiltonian, \( H_s \). The operator \( e^{iH_s t} e^{-iH_t} \) can be shown to satisfy:

\[ U(t) = e^{iH_s t} e^{-iH_t} = T e^{-i \int_0^t dt' e^{iH_s t'} \delta H e^{-iH_s t'}}. \]  

(G.3)

where \( T \) is the time ordering operator. Inserting two sets of conformal tower states allows one to separate the scale invariant dynamics from the dynamics governed by the deviation:

\[ \rho(\vec{r}, \vec{r}', t) = \sum_{m,n} \rho_{s,n,m}(\vec{r}, \vec{r}', t) \Gamma_{m,n}(t) \]

\[ \rho_{s,n,m}(\vec{r}, \vec{r}', t) = \langle n | e^{iH_s t} \psi^\dagger(\vec{r}') \psi(\vec{r}) e^{-iH_s t} | m \rangle \]

\[ \Gamma_{m,n}(t) = \langle m | U(t) | \psi_0 \rangle \langle \psi_0 | U(t) | n \rangle. \]  

(G.4)
G.1. The Density Matrix and Scale Invariance

G.1.1 Scale Invariant Dynamics

First consider the scale invariant piece of the density matrix, $\rho_{s,n,m}(\vec{r},\vec{r}',t)$. Here it is again possible to exploit the properties of the conformal tower states. Since the conformal tower states are eigenstates of the Hamiltonian, $H_s + \omega^2 C$, one can write the scale invariant piece as:

$$\rho_{s,n,m}(\vec{r},\vec{r}',t) = e^{-i(E_n - E_m)\eta} \langle n| e^{i(H_s + \omega^2 C)\eta} e^{iH_s t} \psi(\vec{r}) e^{-iH_s t} e^{-i(H_s + \omega^2 C)\eta}|m\rangle.$$

(G.5)

However, Eq. (G.5) is valid for arbitrary value of $\eta$. This leads to the following relation:

$$\frac{\partial}{\partial \eta} \rho_{s,n,m}(\vec{r},\vec{r}',t) = 0$$

$$= -i(E_n - E_m)\rho_{s,n,m}(\vec{r},\vec{r}',t)$$

$$+ i\langle n| e^{iH_s t} \left[ e^{-iH_s t}(H_s + \omega^2 C) e^{iH_s t}, \psi(\vec{r}) \right] e^{-iH_s t} e^{-i(H_s + \omega^2 C)\eta}|m\rangle.$$

(G.6)

One can use the relation:

$$e^{iH_s t} \left( H_s + \omega^2 C \right) e^{-iH_s t} = \left( 1 + \omega^2 t^2 \right) H_s - \omega^2 t D + \omega^2 C,$$

(G.7)

to obtain:

$$\frac{\partial}{\partial \eta} \rho_{s,n,m}(\vec{r},\vec{r}',t) = 0$$

$$= \left[ \left( 1 + \omega^2 t^2 \right) \partial_t + \omega^2 t (\vec{r} \cdot \nabla_r + \vec{r}' \cdot \nabla_{r'} + d) \right.$$

$$+ i \omega^2 \frac{r'^2 - r^2}{2} - i(E_n - E_m) \right] \rho_{s,n,m}(\vec{r},\vec{r}',t).$$

(G.8)

Note to derive Eq. (G.8) we have employed the commutators:

$$[C, \psi^{(1)}(\vec{r})] = \frac{\pm r^2}{2} \psi^{(1)}(\vec{r})$$

$$[D, \psi^{(1)}(\vec{r})] = \frac{i}{2} (2\vec{r} \cdot \nabla_r + d) \psi^{(1)}(\vec{r}),$$

(G.9)
G.1. The Density Matrix and Scale Invariance

where \( \pm \) refers to whether the operator is a creation operator or annihilation operator, respectively.

Eq. (G.8) is reminiscent of the operator for the generator of time dependent conformal transformations:

\[
G_C(\vec{r}, \vec{r}', t) = t^2 \partial_t + t (\vec{r} \cdot \nabla \vec{r} + \vec{r}' \cdot \nabla \vec{r}') + i \frac{r'^2 - r^2}{2}. \tag{G.10}
\]

In fact, if we perform a gauge transformation:

\[
\tilde{\rho}_{s n,m}(\vec{r}, \vec{r}', t) = e^{-i(E_n - E_m)\tau(t)} \rho_{s n,m}(\vec{r}, \vec{r}', t), \tag{G.11}
\]

where \( d\tau(t)/dt = 1/(1 + \omega^2 t^2) \), the equation of motion for \( \tilde{\rho}_{s n,m}(\vec{r}, \vec{r}', t) \) is simply the generator of conformal transformations. Stated differently, the density matrix is an eigenfunction of the generator for conformal transformations, with zero eigenvalue. Therefore the density matrix is a conformally invariant function. In conclusion, although the system obeys a scale invariant Hamiltonian, the dynamics produce an emergent conformal symmetry.

The price for this gauge transformation is that the matrix, \( \Gamma_{m,n}(t) \), acquires a trivial time dependence:

\[
\tilde{\Gamma}_{m,n}(t) = e^{i(E_n - E_m)\tau(t)} \Gamma_{m,n}(t). \tag{G.12}
\]

G.1.2 Broken Scale Invariance

Although Eq. (G.8) puts a restraint on the scale invariant part of the density matrix, it doesn’t restrict the full density matrix. In order to obtain a differential equation for the full density matrix, it is necessary to evaluate the time dependence of the matrix, \( \Gamma(t) \), or equivalently, the time dependence of the operator, \( U(t) \). Thankfully, the evaluation of \( U(t) \) is identical to the evaluation of the expansion coefficients in Chapter 5. The result is:

\[
U(t \gg \omega^{-1}) \approx \exp \left[ -i \frac{1}{(\sqrt{\omega}a)^{\alpha-1}} (\alpha - 1) \tilde{V} \right] \quad \alpha > 1
\]

\[
\approx \exp \left[ -i \frac{1}{\sqrt{\omega}a} \log(\omega t) \tilde{V} \right] \quad \alpha = 1. \tag{G.13}
\]

where again:

\[
\tilde{V} = \frac{1}{\omega^{1-\alpha/2}} e^{iH_s \frac{\pi}{2}\delta h e^{-iH_s \frac{\pi}{2}}} \tag{G.14}
\]

See Appendix A for the derivation of the generator of conformal transformations.
G.1. The Density Matrix and Scale Invariance

As was the case for the expansion coefficients, the matrix, $\Gamma(t)$, is a function of $t^{\alpha-1}/a^\alpha$, where $\alpha$ is the scaling of the deviation. This allows one to make the identification

$$
(1 + \omega^2 t^2) \partial_t \Gamma(t) = \frac{\alpha - 1}{t} \frac{1}{a^\alpha} (1 + \omega^2 t^2) \frac{\partial}{\partial a^{-\alpha}} \Gamma(t) \quad \alpha > 1
$$

$$
= \frac{1 + \omega^2 t^2}{t \log(\omega t) \alpha} \frac{\partial}{\partial a^{-1}} \Gamma(t) \quad \alpha = 1.
$$

(G.15)

The differential equation for the density matrix then satisfies:

$$
\left[ (1 + \omega^2 t^2) \partial_t + \omega^2 t (\vec{r} \cdot \nabla_r + \vec{r} \cdot \nabla_r + d) + i \omega \frac{r'^2 - r^2}{2} \right] \rho(\vec{r}, \vec{r}', t)
\]

$$
= \frac{\alpha - 1}{t} + \frac{\omega^2 t^2}{a^\alpha} \frac{\partial}{\partial a^{-\alpha}} \rho(\vec{r}, \vec{r}', t) + \sum_{m,n} i (E_n - E_m) \tilde{\rho}_{s \, n,m} (\vec{r}, \vec{r}', t) \tilde{\Gamma}_{m,n}(t).
$$

(G.16)

for $\alpha > 1$, or:

$$
\left[ (1 + \omega^2 t^2) \partial_t + \omega^2 t (\vec{r} \cdot \nabla_r + \vec{r} \cdot \nabla_r + d) + i \omega \frac{r'^2 - r^2}{2} \right] \rho(\vec{r}, \vec{r}', t)
\]

$$
= \frac{1 + \omega^2 t^2}{t \log(\omega t) \alpha} \frac{\partial}{\partial a^{-1}} \rho(\vec{r}, \vec{r}', t) + \sum_{m,n} i (E_n - E_m) \tilde{\rho}_{s \, n,m} (\vec{r}, \vec{r}', t) \tilde{\Gamma}_{m,n}(t).
$$

(G.17)

for $\alpha = 1$.

G.1.3 Dynamics of Local Observables

Consider an observable, $O$, with scaling dimensions, $\Delta_O$:

$$
O = \int d^3 r \, O(\vec{r}) \psi^\dagger(\vec{r}) \psi(\vec{r}).
$$

(G.18)

The dynamics for this operator can be written in terms of the density matrix:

$$
\langle O \rangle(t) = \int d^3 r \, O(\vec{r}) \, \rho(\vec{r}, \vec{r}, t).
$$

(G.19)
Using Eq. [G.17] or equivalently Eq. [G.16] it is possible to obtain a differential equation for \( \langle O \rangle(t) \), which is valid in the limit of long times. For example, if \( \alpha = 1 \), one would obtain:

\[
\left[ (1 + \omega^2 t^2) \partial_t - \Delta O \omega^2 t \right] \langle O \rangle(t) = \frac{1 + \omega^2 t^2}{t \log(\omega t)} \left( \frac{1}{a} \partial_a \right) \langle O \rangle(t) \\
+ i \sum_{n,m} (E_n - E_m) \int d^d r O(\vec{r}) \tilde{\rho}_{s,n}(\vec{r},t) \tilde{\Gamma}_{m,n}(t).
\]

(G.20)

For scale invariant systems, the long time solution to Eq. [G.20] is:

\[
\langle O \rangle(t) \approx \left( \frac{t'}{t} \right)^{\Delta O} \langle O \rangle(t'),
\]

(G.21)

for some \( t, t' \gg \omega^{-1} \). This result is equivalent to the results in the co-moving reference frame, Eq. [4.19] Similarly, for systems with broken scale invariance, the result is:

\[
\langle O \rangle(t) = t^{-\Delta O} F \left( \frac{\log(\omega t)}{a} \right),
\]

(G.22)

again, for \( \alpha = 1 \). Similar results for \( \alpha \geq 1 \) are straightforward to obtain.
Appendix H

Hydrodynamic and Heisenberg Equation of Motion for Compressional and Elliptic Flow

In this appendix we examine alternate approaches to understanding compressional and elliptic flow. The results here are consistent with the density matrix formalism discussed in Chapter 7, but do not highlight the facets of scale and conformal symmetry.

In both compressional and elliptic flow, the quantities of interest are the moments of inertia along a given direction:

$$\langle r_i^2 \rangle(t) = \frac{1}{N} \int d^3 r r_i^2 n(r,t), \tag{H.1}$$

where $i = x, y, z$, and $n(r,t)$ is the density of the fluid. For compressional flow, the motion is isotropic, while it is anisotropic for elliptic flow.

We will study the dynamics of this operator using the hydrodynamic and Heisenberg equations of motion.

H.1 Scaling Solution to the Hydrodynamic Flow

The hydrodynamic equations of motion are given by:

$$\partial_t n(\vec{r}, t) + \nabla \cdot (\vec{v}(\vec{r}, t)n(\vec{r}, t)) = 0 \tag{H.2}$$

$$n(\vec{r}, t) \left[ \partial_t + \vec{v}(\vec{r}, t) \cdot \nabla \right] v_i(\vec{r}, t) = -\partial_i P + \sum_j \partial_j \left( \eta \sigma_{ij} + \zeta_B \sigma'_i \delta_{ij} \right)$$

$$- n(\vec{r}, t) \partial_i U(\vec{r}, t) \tag{H.3}$$

where $\vec{v}(\vec{r}, t)$ is the velocity field, $P$ is the pressure, $\eta$ and $\zeta_B$ are the shear...
H.1. Scaling Solution to the Hydrodynamic Flow

and bulk viscosity coefficients, respectfully, $U(\vec{r}, t)$ is an external potential, and:

$$\sigma_{i,j} = \partial_j v_i(\vec{r}, t) + \partial_i v_j(\vec{r}, t) - \frac{2}{3} \delta_{i,j} \nabla \cdot \vec{v}(\vec{r}, t)$$

$$\sigma' = \nabla \cdot \vec{v}(\vec{r}, t).$$  \hspace{1cm} (H.4)

Using Eqs. H.2 and H.3 one can show that:

$$\frac{d \langle r^2_i(t) \rangle}{dt} = \langle r_i v_i(t) \rangle$$

$$\frac{1}{2} \frac{d^2 \langle r^2_i(t) \rangle}{dt^2} = \frac{1}{N} \int d^3r P(t) + \frac{1}{N} \int d^3r \left( \eta \sigma_{i,i} + \zeta B \sigma' \right)(t) - \langle r_i \partial_i U \rangle(t).$$  \hspace{1cm} (H.5)

We will use this equation to understand the dynamics of a scale invariant Fermi gas. We will assume that the density is initially of the Thomas-Fermi form, and that the dynamics can be captured by a time dependent rescaling:

$$n(\vec{r}, t) = \frac{1}{b_x(t)b_y(t)b_z(t)} \frac{1}{6\pi^2} \left( 2\mu - \omega_x^2 \frac{x^2}{b_x^2(t)} - \omega_y^2 \frac{y^2}{b_y^2(t)} - \omega_z^2 \frac{z^2}{b_z^2(t)} \right)^{3/2}$$

$$v_i(\vec{r}, t) = \frac{\dot{b}_i(t)}{b_i(t)} r_i$$

$$\mu = (6N\omega_x^2\omega_y^2\omega_z^2)^{1/3}.$$  \hspace{1cm} (H.6)

where $b_i(t)$ are time dependent scaling factors that satisfy:

$$b_i(0) = 1 \quad b'_i(0) = 0.$$  \hspace{1cm} (H.7)

The dynamics of the moment of inertia is solely encapsulated in the time dependent rescaling factors:

$$\langle r^2_i(t) \rangle = b_i^2(t) \langle r^2_i \rangle(0)$$

$$\langle r^2_i \rangle(0) = \frac{\mu}{4\omega_i^2}.$$  \hspace{1cm} (H.8)

For scale invariant systems initially in harmonic traps, it is possible to make a number of simplifications. The first is that the bulk viscosity, $\zeta_B$, vanishes for scale invariant systems [101]. Secondly, the Pressure can be related to the energy density, $E$ via:
H.1. Scaling Solution to the Hydrodynamic Flow

\[ P(t) = \frac{3}{2} E \propto n^{2/3}(r, t) = \frac{1}{(b_x(t)b_y(t)b_z(t))^{2/3}} P(0) \quad (H.9) \]

Initially, the gas is in equilibrium in the harmonic trap. Therefore, Eq. [H.5] can be used to relate the initial pressure to the trap energy:

\[ \frac{3}{N} \int d^3r \ P(0) = \omega_i^2 \langle r_i^2 \rangle (0). \quad (H.10) \]

Combining all this information gives a set of coupled differential equations for the scaling factors, \( b_i(t) \):

\[ \ddot{b}_i(t) = \frac{\omega_i^2}{(b_x(t)b_y(t)b_z(t))^{2/3} b_i(t)} - 2 \langle \alpha_s \rangle \langle \dot{r}_i^2 \rangle (0) \left( \frac{\dot{b}_i(t)}{b_i(t)} - \frac{1}{3} \left( \frac{\dot{b}_x(t)}{b_x(t)} + \frac{\dot{b}_y(t)}{b_y(t)} + \frac{\dot{b}_z(t)}{b_z(t)} \right) \right), \quad (H.11) \]

where \( \langle \alpha_s \rangle \) is the trap averaged shear viscosity coefficient:

\[ \langle \alpha_s \rangle = \frac{1}{N} \int d^3r \ \eta. \quad (H.12) \]

**H.1.1 Isotropic Expansion**

For isotropic expansion, Eq. [H.11] reduces to:

\[ \ddot{b}(t) = \frac{\omega^2}{b^3(t)}. \quad (H.13) \]

We have encountered this equation of motion before, and the solution for the moment of inertia is given by:

\[ \langle r^2 \rangle (t) = (1 + \omega^2 t^2) \langle r^2 \rangle (0). \quad (H.14) \]

This is just the dynamics for a system either prepared in an conformal tower state, or prepared in a thermal ensemble.

**H.1.2 Anisotropic Expansion**

For anisotropic expansion, it is necessary to evaluate Eq. [H.11] in its entirety. This has been done in Fig. [7.1] for a variety of shear viscosity coefficients. For all cases, the long dynamics of the moment of inertia can be fit to:
\[ \langle r_i^2 \rangle(t) \approx (v^2 t^2 + A t + B) \langle r_i^2 \rangle(0). \] (H.15)

This is consistent with the emergent conformal symmetry, and the density matrix analysis presented in Chapter 7.

H.2 Heisenberg Equation of motion

We now analyse the expansion dynamics of the moment of inertia under a scale invariant Hamiltonian, using the Heisenberg equations of motion. First consider the equation of motion for the moment of inertia, \( \langle r^2 \rangle(t) \). After the trap is released, the equation of motion for the moment of inertia can be found to be:

\[ \frac{d^2}{dt^2} \langle r^2 \rangle(t) = 4 \langle H_s \rangle. \] (H.16)

The solution for this equation is simply:

\[ \langle r^2 \rangle(t) = \left(1 + \frac{2 \langle H_s \rangle}{\langle r^2 \rangle(0)} t^2\right) \langle r^2 \rangle(0). \] (H.17)

If the system is in equilibrium, it possible to relate the energy of the gas to its initial size. Consider the initial Hamiltonian:

\[ H_s + \int \frac{1}{2} \left[ \sum_{i=x,y,z} \omega_i^2 r_i^2 \right] \psi^{\dagger}(\vec{r}) \psi(\vec{r}). \] (H.18)

The resulting equation of motion for the moment of inertia is then:

\[ \frac{d^2}{dt^2} \langle r^2 \rangle(t) = 4 \left( H_s - \frac{1}{2} \sum_{i=x,y,z} \omega_i^2 \langle r_i^2 \rangle(t) \right). \] (H.19)

However, in its initial state, the gas is stationary, which gives the following condition:

\[ \langle H_s \rangle = \frac{1}{2} \sum_{i=x,y,z} \omega_i^2 \langle r_i^2 \rangle(0), \] (H.20)

For isotropic traps this simply reduces to \( \langle H_s \rangle = \omega^2 \langle r^2 \rangle(0)/2 \), or equivalently:

\[ \langle r^2 \rangle(t) = (1 + \omega^2 t^2). \] (H.21)
H.2. Heisenberg Equation of motion

This result is a consequence of the Feynman-Hellmann theorem, and is again akin to the dynamics of a quantum system in a diagonal ensemble of conformal tower states. For generic conditions, we cannot relate the initial energy to the trapping potential. In this case, the relative velocity is no longer pinned to the trap frequency.