Annular displacement flows in turbulent and mixed flow regimes

by

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Abstract

This thesis presents a comprehensive, yet practical, two-dimensional model for the displacement of viscoplastic fluids in eccentric annuli in laminar, turbulent and mixed flow regimes. The motivations originally stem from primary cementing of oil and gas wells, as well as other types of wells such as those in Carbon Capture and Storage applications. During primary cementing, cement slurries are placed in an annular region between a steel casing and a wellbore to provide mechanical stability and hydraulic isolation. Several complications may arise due to the eccentricity of the annular region, as well as the viscoplastic nature of the fluids involved.

The existing 2D and 3D models of primary cementing assume the flow is laminar, while in practice, turbulent and more importantly, mixed flow regimes are common. In this thesis, we fill this gap in knowledge. More specifically, we expand the laminar model of Bittleston et al. [24] and develop a new formulation that includes turbulent and mixed flow regimes. This new formulation considers scaling based on the disparity of length-scales, which allows a narrow-gap averaging approach to be effective. With respect to the momentum equations, the leading-order equations correspond to a turbulent shear flow in the direction of the modified pressure gradient. With respect to the mass transport equations that model the miscible displacement, to leading-order turbulence effectively mixes the fluids. Changes in concentrations within the annular gap arise due to the combined effects of advection with the mean flow, anisotropic Taylor dispersion (along the streamlines) and turbulent diffusivity.

This new extension allows us to understand the process of cementing more deeply, and resolve several questions that have been left unanswered for many
years. In particular, we show that many simple statements/rules that are often employed in industry do not stand up to serious analysis. Instead, modelling approaches such as the one developed here can incorporate specific features of wells in the simulations, and therefore, yield more accurate predictions.
Lay Summary

Primary cementing is a process by which cement slurries (dry cement + water) are placed outside of a metal pipe through which oil and gas are extracted. When cured, the cement will provide mechanical stability and a hydraulic seal. Failure of primary cementing operations can have severe economical and environmental consequences, which can range from leakage of gas (in varying degrees), through to rarer but more extreme well control incidents. In this study, we derive a mathematical model that accounts for several complexities of this process. For example, cement slurries behave like solid materials when the force they experience is below some threshold. Above that threshold, they behave like liquids. Further complications may arise from the varying and complex annular geometry between the wellbore and the metal pipe. Having derived our cementing model and solved it numerically, we then study a wide range of practically interesting cementing scenarios in order to understand limitations in current practice and make recommendations. The results of this study have immediate implications for oil and gas producers, as well as government regulators.
Preface

The research presented in the current thesis is conducted by the author, Amir Maleki, under the supervision of Professor Ian Frigaard. The following papers are published or in preparation for publication:

  Both authors had equal contributions in this paper, both in terms of generating the contents as well as writing the draft.

  A. Maleki primarily generated the content and prepared the initial draft. I. Frigaard supervised model derivation, and assisted with writing the paper.

  N. Hanachi was the principal contributor for this paper. A. Maleki assisted in deriving the model and developing the computational codes. I. Frigaard supervised model derivation, and assisted with writing the paper.

  A. Renteria was the principal contributor for this paper. A. Maleki assisted in running the annular displacement code. I. Frigaard supervised the project,
and assisted with writing the paper. B. Lund, A. Taghipour and J. Ytrehus performed the experiments and aided with data analysis.

  A. Maleki was the principal contributor for this study. I. Frigaard supervised model derivation, and assisted with writing the paper.

• A. Maleki, I. Frigaard, “Turbulent displacement flows in primary cementing of oil and gas wells”, Phys. Fluids (accepted).
  A. Maleki was the principal contributor for this study. I. Frigaard supervised model derivation, and assisted with writing the paper.

  A. Maleki was the principal contributor for this study. I. Frigaard supervised model derivation, and assisted with writing the paper.

Several parts of this study have been disseminated in forms of oral or poster presentation at the following conferences:

• 37th International Conference on Ocean, Offshore & Arctic Engineering, Madrid, Spain.

• 24th International Conference on Theoretical and Applied Mathematics (IC-TAM), Montreal, 2016.

• The XVIIth International Congress on Rheology (ICR2016), Kyoto, 2016.

A. Maleki has also benefited from a summer undergraduate student, N. Heim, who worked on turning the existing code into a standalone software.
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Nomenclature

\( A_{DM}, B_{DM} \): constants of log-law (turbulent velocity profile)

\( Bu \): Buoyancy number

\( c_k \): concentration of fluid \( k \)

\( D \): Pipe diameter

\( \hat{D}_e \): eddy diffusivity

\( \hat{D}_t \): turbulent diffusivity

\( \bar{D} \): averaged dimensionless turbulent diffusivity

\( \hat{D}_T \): Taylor dispersivity

\( \hat{D}_T^* \): secondary dispersivity (originated by geometry variations)

\( \hat{d} \): dimensional mean gap width

\( E \): Axillary variable defined by \( \hat{\kappa}_p \hat{N}_L/\hat{\tau}_w \)

\( e_s \): local direction of streamlines

\( e \): eccentricity

\( Fr \): Froude number

\( f_f \): Fanning friction factor

\( H \): dimensionless mean gap width

\( He \): Hedström number

\( H_w \): Hedström number based on wall shear stress

\( H_{w,1} \): \( H_w \) when flow is not laminar anymore

\( H_{w,2} \): \( H_w \) when flow is fully turbulent

\( n \): power-law index

\( n' \): flow dependent power-law index defined by (2.3)

\( Pe \): Péclet number

\( \hat{Q}_0 \): Representative flow rate
Re$_{MR}$: Metzner-Reed Reynolds number
Re$_p$: power-law Reynolds number
Re$_1$: Metzner-Reed Reynolds number when flow is not laminar anymore
Re$_2$: Metzner-Reed Reynolds number when flow is fully turbulent
Ri: Richardson number
\( \hat{r} \): dimensional radial coordinate
\( \hat{r}_i \): inner radius of annulus
\( \hat{r}_o \): outer radius of annulus
\( \hat{r}_a \): mean radius of annulus (local)
\( \hat{r}_{a,0} \): axially averaged mean radius of annulus (global)
\( r_a \): dimensionless mean radius of annulus (local)
r$_y$: ratio of yield stress to wall shear stress
r$_{ALG}$, \( \rho_{ALG} \): augmented Lagrangian parameters
\( \hat{t} \): time
\( \hat{t}_{ACT} \): contact time
\( \hat{W}_0 \): mean velocity (global scale)
\( \hat{W}_L \): mean velocity in a laminar flow
\( \hat{W}_f \): friction velocity
\( y \): local annular gap coordinate
\( y^+ \) and \( x^+ \): dimensionless wall coordinate defined based on friction velocity
\( \hat{z} \): axial coordinate
\( \beta \): inclination angle from vertical
\( \hat{\rho}_{eff} \): effective viscosity
\( \hat{\eta}_L \): laminar strain rate at the wall
\( \hat{\eta}_N \): Newtonian strain rate at the wall
\( \hat{\gamma}^* \): nominal strain rate
\( \hat{\rho} \): density
\( \hat{\kappa} \): consistency
\( \hat{\kappa}_p \): power-law consistency
\( \hat{\kappa}' \): flow-dependent consistency defined by (2.3)
\( \psi \): wall layer scaling parameter
\( \Psi \): stream function
\( \theta \): azimuthal coordinate
φ: scaled azimuthal coordinate (θ/π)
\hat{\tau}: dimensionless time (only in chapter 8)
\hat{\tau}_Y: yield stress
\hat{\tau}_w: wall shear stress
δ: mean aspect ratio of annulus (local)
\delta_0: axially averaged mean aspect ratio of annulus (global)
η: displacement efficiency
\eta_N: narrow side displacement efficiency
ξ: dimensionless coordinate axis along the well (measuring upward)
\hat{\xi}: coordinate axis along the well (measuring downward)
\hat{\xi}_{bh}: bottom hole axial coordinate
Glossary

The following technical terms are used throughout the thesis:

• Bottom plug: A mechanical device to isolate the spacer from the cement slurry, while moving downwards inside the casing. When the cement slurry has reached to the bottom of well, the plug is ruptured by pressure.

• Casing: A metallic pipe inserted in, and cemented to, the borehole. Wellbore equipments, such as production packers and blowout preventers are installed on the casing. The casing provides mechanical stability and hydraulic isolation and controls pressure.

• Cement slurry: A mixture of cement powder, water and several additives. The additives modify the physical properties of the cement such as its viscosity, yield stress, thickening time, etc.

• Centralizer: A device that keeps the casing or liner at the center of the wellbore. Centralizers are fitted to the casing and typically have bowsprings to keep the casing at the center.

• Conditioning (Mud Conditioning): For the sake of a better displacement, mud properties are changed by adding additives and circulating the mud around the flowpath before cementing. This process is called mud conditioning. Often water or dispersants are added to the mud to change its density and rheology. It is recommended to circulate the mud before and after removal of drill pipe.

• Dogleg: Places in the wellbore where the trajectory of the wellbore changes rapidly, faster than anticipated or desired.
- Drilling mud: Drilling mud is muddy liquid that is left after drilling. It is primarily made up of drilling fluid and formation cuttings. Drilling fluid is used during drilling to facilitate drilling. Drilling fluid suspends and carries the drill cuttings and keeps the drill bit cool. In addition, it stops the formation fluids from invading the wellbore.

- Drillpipe: Tubular steel pipes are fitted together to form a drillpipe. The drillpipe is connected to the rig surface from one end and to the drill bit at the other end. Drilling fluid is pumped through the drillpipe during drilling.

- Eccentricity: In the region between two parallel cylinders, eccentricity is a measure of offset of the two cylinders’ axes. When the two cylinders are co-axial, the eccentricity is zero (stand off is 100%). The other extreme is when the inner cylinder touches the outer cylinder. In this case eccentricity is 1 (stand off is 0%). See §3.1 for the precise definition.

- Frictional pressure: The pressure losses due to the wall shear stresses are called frictional pressure losses.

- Gel strength: A property of drilling mud, working analogously to a yield stress, which increases when the mud is static until it reaches a plateau after 10-60 minutes. The gel strength has to be broken for the mud to flow. Gel strengths occur in drilling fluids due to the presence of electrically charged molecules and clay particles (Drilling Fluid Reference Manual, Baker Hughes, 2006).

- Liner: A liner is a casing that extends downwards from just above the previous casing.

- Preflush: Primary cementing typically starts with conditioning the mud and then proceeds to pumping a sequence of the so called preflushes inside the casing. Preflush can be a spacer or a wash. Preflushes are intended to break the static gelation in the mud, clean the wellbore from the cuttings and provide a buffer layer between the mud and cement slurry.

- Spacer: Spacers are heavier preflushes with viscosifying constituents. Because of their larger density and viscosity, spacers are more often pumped
in laminar regime. Spacers are placed to separate cement slurry from the
drilling mud. Often cement slurries and the drilling mud are chemically in-
compatible, meaning that their contact can damage highly tuned properties
of cement slurry (such as thickening time) [167]. Furthermore, spacers rhe-
ological parameters can be designed carefully to aid the displacement of the
drilling mud [232].

• Wash: Washes may be water-based or oil-based. Rheologically, they are
generally Newtonian fluid solutions (e.g. water). They are designed to wash
the walls of the annulus free from residual fluids (and any remaining solids),
to leave the annulus water-wet for the cement slurry. In addition, they should
break any static gelation of the mud, mobilizing the mud in general. The low
viscosity of these fluids allows them to be pumped in turbulent flow regimes.
Most common wash is water with a range of chemical additives.

• Wellbore: A hole that is drilled to extract natural resources such as oil and
gas.

• Yield stress fluid: Yield stress fluids, also call viscoplastic fluids, are a cat-
egory of non-Newtonian fluid which exhibit yield stress behavior; i.e. they
do not flow, unless they are sufficiently stressed.
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Dedication

I dedicate this thesis to myself for not giving up, for believing in my own abilities, and for my perseverance. I am proud of myself.
Chapter 1

Introduction

In this work we study the displacement of one fluid by another in a narrow, eccentric annulus in laminar, turbulent and mixed flow regimes. The study is motivated by the processes involved in the primary cementing of oil and gas wells. The main focus of this work is to understand the primary cementing from a fluid mechanic perspective, identify the current industrial misunderstandings, and propose means to improve the efficiency of this critical operation.

In the following sections, we briefly review the primary cementing and different procedures that it entails and explain why this operation plays a critical role, not only economically (in terms of oil and gas production), but also environmentally. We continue the chapter by outlining the types of fluids that are common in the primary cementing operations, and explain how their physical properties may influence the outcome of a cement job. Subsequently, we proceed to reviewing the existing literature on the displacement of yield stress fluids. In doing so, we briefly touch on displacement flows in pipe geometry, in Hele-Shaw cells and in porous media, but the main focus remains on those in annular or plane channel geometries. Moreover, we review the industrial design rules that have been developed for successful annular displacements between 1970-1990. These rules are only derived for laminar displacement in vertical wells. Finally, we turn our attention to the more recent developments, including 2D and 3D models of primary cementing. We will then close the chapter by outlining the objectives of the thesis.
1.1 What is primary cementing?

Primary cementing is the process by which oil and gas wells are sealed during construction. The seal is achieved by placing cement into a narrow gap formed between the drilled borehole and the outside of a steel casing\(^1\) (or liner), that is placed in the well. The cement not only seals the well hydraulically, preventing fluids from migrating axially along the wellbore between fluid-bearing zones, but also provides mechanical support, resisting geo-mechanical stresses. Primary cementing in the conventional form of pumping and displacement was first used in 1910 in shallow wells in California [167]. The process proceeds as follows; see Figure 1.1. A new section of the well is drilled. The drillpipe is removed from the wellbore, leaving drilling mud inside the wellbore. A steel tube (casing or liner) is inserted into the wellbore, typically leaving an average annular gap of \(\approx 2\text{-}3\) cm. The tubing is inserted in sections of length roughly 10 m each, threaded together so that cemented sections can extend 100 to 1000 meters. So-called centralizers are fitted to the outside of the tube, to prevent the heavy steel tubing from slumping.

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\(^1\)Throughout this chapter, several technical terms are highlighted with a bold font, which are fully defined in the glossary provided on page xxv.
to the lower side of the wellbore. However, even in (nominally) vertical wells it is common that the annulus is eccentric and this is especially true in inclined and horizontal wells. With the steel casing in place and drilling mud on the inside and outside, the operation begins. First, the drilling mud is conditioned by circulating around the flow path. Next a sequence of fluids are circulated down inside of the casing and returning up the outside of the annulus. Preflushes (washes or spacer fluids) are followed by one or more cement slurries. The fluid volumes are designed so that the cement slurries fill the annular space to be cemented. Drilling mud follows the final cement slurry to be pumped and the operations ends with the cement slurry held in the annulus (with a valve system). The cement is then set over a period of many hours. With reference to Figure 1.1, it can be seen that the completed well often has a telescopic arrangement of casings and liners. Thus, the operation is repeated more than once on most wells. Typically, annulus inner diameters can start at anything up to 50 cm and can end as small as 10 cm in the producing zone.

1.2 Why is primary cementing important?

The whole process of primary cementing takes place over the span of 1 or 2 days, but its economical and environmental impacts will remain for tens or maybe hundreds of years. In Canada, there is an estimate that 5-20% of wells leak to various degrees [63], due to the failure of the primary cementing. Similar estimates are reported for other countries too [51]. This has both economical and environmental consequences.

From an economical point of view, the recovery rates for conventional reservoirs are typically 20-40%, which means losing even a few percent is expensive. In addition, a well with a poor cementing will normally have further complications, such as necessary remedial treatments, which imposes a significant economical burden [244]. As a result, both industry and regulators have advocated for “Doing it right the first time” [213, 63].

More critically, the environmental impacts of oil and gas wells may span several hundreds of years. That means a leaking well will be a continuous source of environmental damage and safety hazards. Such environmental concerns include
groundwater contamination due to wellbore leakage [38, 12] as well as greenhouse gas emission [126]. An example of a massive consequence of a poor cementing job is the notorious Deep Water Horizon blowout in the Gulf of Mexico.

1.3 Why do primary cementing operations fail?

To understand the sources of failure of primary cementing, and the subsequent leakage of wells, we begin to consider primary cementing as a fluid mechanics problem. The fluids involving in primary cementing are drilling muds, washes, spacers and cement slurries. With the exception of washes being Newtonian, the other fluids mentioned above typically exhibit yield stress behavior; i.e. they do not flow, unless they are sufficiently stressed. These fluids are called yield stress, or interchangeably viscoplastic, fluids.

Commonly, the fluids in the oil and gas well cementing are described by the “Herschel-Bulkley” (HB) model. The HB model is defined by the following constitutive relation between the stress and strain rate:

$$\hat{\tau}_{ij} = \left( \hat{\kappa} \hat{\gamma}_{ij}^{n-1} + \frac{\hat{\tau}_Y}{\hat{\gamma}} \right) \hat{\gamma}_{ij} \iff \hat{\tau} > \hat{\tau}_Y$$

$$\hat{\gamma} = 0 \iff \hat{\tau} \leq \hat{\tau}_Y$$

(1.1)

Here $\hat{\tau}_{ij}$ is the deviatoric stress tensor and $\hat{\gamma}_{ij}$ is the strain rate tensor:

$$\hat{\gamma}_{ij} = \frac{\partial \hat{u}_i}{\partial \hat{x}_j} + \frac{\partial \hat{u}_j}{\partial \hat{x}_i},$$

(1.2)

the $\hat{u}_j$’s are the components of the velocity field, and $\hat{\tau}$ and $\hat{\gamma}$ are the second invariant of their respective tensor; i.e.

$$\hat{\gamma} = \sqrt{\frac{1}{2} \sum_{i,j=1}^{3} \left| \hat{\gamma}_{ij}^2 \right|}, \quad \hat{\tau} = \sqrt{\frac{1}{2} \sum_{i,j=1}^{3} \left| \hat{\tau}_{ij}^2 \right|}.$$  

(1.3)

The three rheological parameters of HB model are the consistency ($\hat{\kappa}$), the power-
law index \((n)\) and the yield stress \(\hat{\tau}_Y\). If we set \(\hat{\tau}_Y = 0\), the HB model reduces to the so called power-law model. The power-law model takes into account that viscosity of some fluids varies with the shear rate. Instead, if we set \(n = 1\), we recover the Bingham model. Bingham model is the simplest model for yield stress fluids, which basically assumes the fluid behaves Newtonian (in a sense that the plastic viscosity remains constant), once it is yielded. In addition, there are a handful of other rheological models, such as Casson model or Carreau model, which are less commonly used in the oil and gas industry.

Cement slurries also show time-dependent behavior, such as aging or thixotropic behavior, which is primarily due to the micro-structure of cement particles within the suspending fluid [167]. These structures build up or break down which then results in change in macro-properties of the fluid. The time-dependent nature of physical properties of the fluids is neglected in this thesis.

An unsuccessful cement job allows the hydrocarbon to leak. Leakage during the primary cementing operation can lead to gas pockets and channels, that compromise the well. A number of other defects may arise either during the cementing of a well, or afterwards during cement hydration, that allow the well to leak later. The most common fluid-related defects include:

- Residual mud channeling: This is where the yield stress of the mud holds it in place, typically on the narrow side of the annulus, as preflushes and cement slurry by-pass. This is a bulk flow feature predicted well by simple mechanical arguments [156], as well as more sophisticated models [24].

- Wet micro-annulus: This is a local mechanical effect, where the displacing fluid does not generate sufficient shear stress to mobilize the mud at the wall. To some extent this is predictable in model flows [9, 85, 247, 254]. For example, in a channel of half width \(\hat{H}\), where a Newtonian fluid displaces a Bingham fluid, Zare et al. [254] suggested that if

\[
B_N = \frac{\hat{\tau}_{Y, \text{displaced}} \hat{H}}{\hat{\mu}_{\text{displacing}} \hat{W}_0} < 3, \tag{1.4}
\]

the thickness of micro-annulus will gradually diminish, as the displacement progresses. Here \(\hat{W}_0\) is displacement mean velocity. At higher values of \(B_N\),
some mud layer may be left behind, depending on the viscosity ratio, density difference between the fluids as well as the flow rate. Of course, features such as mud dehydration and uneven wellbores create some complications with these results.

- Mixing/contamination of the slurry: Mixing (and consequent contamination) occur in different scenarios such as downwards displacement within the casing [8, 223], fluid instabilities in laminar annular flows [181, 162, 163] or in turbulent annular displacement flows. The latter scenario will be extensively studied in Chapter 2. In combination with mud channels, micro-annuli or even mud pockets left behind in irregular wellbores (e.g. washouts), residual drilling fluid can be partially eroded/dispersed in a passing slurry and continue to contaminate cement over long lengths.

Examples of these features can be found in Watson [244].

While the above mechanisms are flow-related, not all wellbore leakage has a fluid-mechanical cause. The most common cause is formation of dry micro-annulus which is due to cement-to-casing or cement-to-formation bonding defects. De-bonding can happen as a result of cement volumetric shrinkage, downhole thermal/hydraulic/mechanical stresses or lack of casing and formation roughness [167]. The interested reader is also referred to Watson and Bachu [245, 246], which analyze different leakage factors for significant data sets.

From a fluid mechanics perspective, one of the main operational questions is whether it is preferable to cement a well in turbulent or laminar flow. To explain this, displacement flow regime depends on local geometry and fluid properties as well as the overall imposed flow rate. It is relatively common within the annulus that one fluid can be fully turbulent (e.g. a chemical wash or low-viscous spacer) while others are laminar. Indeed, as it will be shown later, this also can occur on a single section of the annulus, e.g. turbulent on the wide side, laminar or even static on the narrow side. Furthermore, although some fluids can be strongly turbulent, the more viscous fluids (muds, viscous spacers and slurries) are often only weakly turbulent, transitional or laminar. These flow regimes have become more prevalent in recent decades as extended reach and horizontal wells, require reduced flow rates to control friction pressures.
The early literature on cementing generally stated that the turbulent displacement is more successful than those in laminar in removing the mud during primary cementing [200, 166, 130]. While some recent studies have been less definitive and warned that certain conditions must be satisfied for turbulent displacement to succeed [167, 132], there still appears to be a widely accepted perception in the cementing community that turbulent displacement is necessarily superior to laminar displacement. See for example Kelessidis et al. [127] and the very recent Lavrov and Torsæter [140] and Enayatpour and van Oort [70]. However, the scientific evidence to support this appears to be scant. Two papers by Howard and Clark [119] and Smith and Ravi [216] are often cited as references, but although these have observed that displacement experiments with higher flow rate led to better displacement efficiency, they did not compare turbulent and laminar regimes. Similarly, two separate studies by Smith [215] and Haut and Crook [115] suggest that “as the annular velocity is increased there is no sharp increase in the displacement efficiency at the transition from laminar to turbulent flow” and “high flow rates, whether or not the cement is in turbulent, provide better displacement than plug flow rates”. The other work that is often cited is Brice and Holmes [32], in which 26 wells are experimented where “turbulent flow techniques were applied in primary cementing”. The quality of the cement jobs was then evaluated with different metrics, including cement bond-log and pressure test, all of which give a bulk assessment of the cement, and are rather insensitive to small features such as micro-annuli. In addition, the study does not use laminar displacement in any of the wells, so it is not entirely clear if the success of cement jobs can be attributed to the turbulent flow regime. Moreover, this study was performed in an era before our understanding of laminar displacements evolved. Ideally, we would like to confirm if the above perception is correct or not. Comparing the laminar vs turbulent displacement flows is the objective of Chapter 6. Before we state our other objectives more explicitly, we review the available literature on primary cementing, and more generally annular displacement flows.
1.4 Literature review

Primary cementing involves the displacement of one yield stress fluid with another yield stress fluid. The displacement happens in the pipe (casing) and then in the annulus. The key parameters here are: i) physical properties of the displacing fluid; ii) physical properties of the displaced fluid; iii) local geometry and iv) flow parameters. These dozen parameters form a multi-dimensional parameter space which is practically impossible to fully study. Instead, to understand the underlying physics, we may start by looking at simpler problems, e.g. single flows of a yield stress fluid in an annular geometry, and then build complexity.

1.4.1 Flows of a single yield stress fluid in annular geometry

Despite its apparent simplicity, analytical solution for flows of a yield stress fluid in eccentric annulus has proved to be controversial, primarily due to the existence of the yield stress. It is known that a naive application of classical fluid mechanics approaches, such as the lubrication approximation or thin film analysis, can lead to solutions that exhibit velocity gradients within unyielded plug regions. This is inconsistent with the yield stress closure model. The inconsistency is called the lubrication paradox, which has led to many confusing and incorrect statements [141]. The lubrication paradox was later resolved in several cases; see for example Balmforth and Craster [14], Frigaard and Ryan [83], Maleki et al. [152]. In particular, Walton and Bittleston [240] analytically solved the laminar flow of a Bingham fluid in an eccentric annulus. They adopted a systematic perturbation technique and cleverly defined a plug-like region where stress exceeds yield stress, but only by a very small amount and called it pseudo-plug. They also showed that true plugs can exist on both narrow and wide sides of the annulus. The prediction of Walton and Bittleston [240] was later confirmed numerically by Szabo and Hassager [218]. Another approach in analyzing the flow in an eccentric annulus is to use the so-called slot approximation, where the curvature of the annulus is neglected, and the annulus is modeled as a slot of variable height; see for example Iyoho and Azar [121], Fordham et al. [76], Bittleston and Hassager [23]. When the annulus is concentric, the flow is axisymmetric and reduces to 1D. Several studies have considered such a configuration under different conditions. Recent examples consider
these flows with wall slip [125, 172] or with linear [142] and rotational [18] motion of the inner cylinder.

1.4.2 Experimental works on the displacement flows of yield stress fluids

We now move to displacement flows, where one fluid displaces another fluid. In addition to primary cementing, displacement flow problems appear in other industrial applications such as drilling, mining and water treatment [34, 46, 151] as well as biomedical [120] and geophysical [210] applications.

In the context of primary cementing, there exists two distinctly different categories of displacement flows: i) displacement in the casing (pipe) where mostly the flow is downward and a lighter fluid displaces a heavier fluid. ii) displacement in the annulus where the flow is upward and generally a heavier fluid displaces a lighter fluid. In both cases, the main objective of any analysis is to predict displacement efficiency, and to provide an estimate for the mixing.

Downward displacement flows in pipes have been extensively studied by Frigaard and coworkers [9, 221, 222, 223, 219, 6, 7, 8] as well as by Gabard and Hulin [86]. In particular, Taghavi et al. [223] classified the near horizontal displacement flow of Newtonian iso-viscous fluids as viscous and inertial. The former is characterized by lack of interfacial instabilities whereas the latter is characterized by the existence of interfacial instabilities and mixing. A third class of fully-diffusive displacement was then proposed in Alba et al. [8] to include displacement flows in inclined pipes. This new class denotes displacement flows where the fluids completely mix across the pipe. More recently, Etrati and Frigaard [73] extend this analysis to displacement flow with different viscosities for both near horizontal and near vertical orientations. They report that viscosifying the less dense fluid tends to significantly destabilize the flow. More interestingly, Hasnain et al. [114] experimentally investigated pipe displacement for immiscible fluids, to mimic displacement of oil-based mud with water-based slurries. They found that immiscibility can significantly enhance the efficiency of the displacement.

A slightly different class of displacement flows is exchange flows where there is no imposed flow. Exchange flows in pipe have been extensively studied by Hulin
and coworkers [204, 205, 206, 207, 208] as well as by Kerswell [129], Beckett et al. [17], Varges et al. [238]. These groups found that the relative magnitude of viscous and inertial velocity scales informs of the leading order behavior of the flow.

There are relatively few experimental studies in the annulus. Many of experimental studies rely on measuring the electrical conductivity of the fluids around the annulus at multiple axial locations. The conductivity values can then be translated into the concentration of each fluid. Typically, salt is added to one of the fluids to increase the conductivity jump across the interface. Other studies take pictures/movies during the process of displacement. Early experimental studies such as those in McLean et al. [156], Lockyear and Hibbert [143] considered easy displacement scenarios; i.e. no mud with extreme rheological parameters. More interestingly, Tehrani et al. [231] conducted several annular displacement experiments with non-Newtonian fluids. They found eccentricity as the most critical parameter in determining the displacement efficiency. Furthermore, they also observe that a thin layer of fluid may be left behind due to development of interfacial instabilities. More recently, Malekmohammadi et al. [153] conducted a similar study and elucidated the roles of eccentricity, viscosity, density as well as flow rate in annular displacement of Newtonian fluids. In another work, Deawwanich [56] analyzed the role of inner cylinder rotation and reported significant displacement improvement in the presence of casing rotation, specially when the annulus is highly eccentric.

Another relevant family of displacement flows is the displacement of a yield stress fluid with air, which is common in injection molding or oil recovery applications. de Souza Mendes et al. [54] conducted a series of displacement experiments where air pushes Carbopol in a capillary tube. They identified a critical flow rate below which Carbopol is perfectly displaced. Above the critical flow rate however, liquid film start to deposit on the wall tube. Similar behavior was observed by Poslinski et al. [185] who reported that the deposition layer thickness can approach as high as 35% of the pipe radius at high flow rates of gas.

Finally, flows in porous media and flows in Hele-Shaw cells resemble similarities with those in annular displacement. In particular, we will show that our 2D governing equations are similar to those in porous media as well as Hele-Shaw cells. Fluid displacement in porous media has been studied for several
years. The early literature is summarized in Barenblatt et al. [15]. More recent works concern the stability of miscible displacement and development of viscosity-driven and density-driven instabilities [224, 118, 154, 39]. Displacement flows in Hele-Shaw cells is rather more recent. A representative list of examples is: [175, 176, 199, 47, 139, 74, 13, 239].

1.4.3 Rule-based systems of primary cementing

More broadly in the context of primary cementing, there exists an extensive industrial technical literature. Unfortunately, this literature is less relevant/useful for several reasons: i) Much of the literature do not concern displacement flows at all. ii) Even those that deal with the displacement flows rarely focus on the fluid mechanics aspects of the problem. iii) Finally, much of the discussions are based on sparse studies on a dozen or so cement jobs with anecdotal evidence, which can hardly be generalized. Nonetheless, some of these studies bear more attention which we will review here. In particular, we look at a number of rule-based systems that evolved, from 1970s to early 1990s, to guide laminar mud displacement in vertical wells. These studies often lead to conservative and occasionally contradictory predictions.

- McLean et al. [156] suggested the cement slurry should be thicker than the mud. Otherwise, the displacement shall be “aided by the motion of the casing or buoyant forces”. To account for the eccentricity, a rule of thumb is to have

$$\hat{\tau}_{Y,\text{mud}}(1 + e) < \hat{\tau}_{Y,\text{cement slurry}}(1 - e)$$

where here $e$ is the eccentricity of the well (see §3.1). The study discouraged thinning a cement slurry for the purpose of achieving turbulent displacement, as it would “reduce the efficiency of the displacement” and increase the possibility of viscous fingering.

- Smith [214] defined “cementable wellbore” as the one “as nearly gauge as possible (without washouts), as straight as possible (without severe dog-legs) and stabilized and properly conditioned” with a gap width of 1.5 in (3.8 cm). The absolute minimum gap width is 0.75 in (1.9 cm). The author
also suggested “the maximum slurry density to prevent losing circulation to be 2096 kg/m$^3$ for the production liner, 1797 kg/m$^3$ for the drilling liner, and 1378 kg/m$^3$ to 1498 kg/m$^3$ filler slurry followed by 1893 kg/m$^3$ tail-in slurry for the intermediate string.” In regard to mud condition, the paper recommended that “cement pumping should not begin until at least 95% of the hole volume is being circulated”.

- The work of Lockyear and Hibbert [143] and Lockyear et al. [144] concluded that three main rules should be satisfied:
  - The wall shear stress has to be large enough to break the gel strength of the mud. More explicitly,
    
    $$\hat{\tau}_w > \hat{\tau}_g$$

    has to be satisfied everywhere around the annulus (and particularly on the narrow side). Here $\hat{\tau}_g$ is the gel strength of the mud and $\hat{\tau}_w$ is the wall shear stress.
  - The wall shear stress on the narrow side must exceed the yield stress of each fluid (mud, spacer, and cement).
  - The interface velocity shall be uniform around the annulus (we will later call this the steady flow condition). To quantify this condition more precisely, the ratio of
    
    $$\frac{\hat{w}_{narrow}}{\hat{w}_{wide}}$$

    must be calculated at the interface. Ideally the ratio ought to be 1. However, the problem is that the interface velocities on the wide and narrow sides cannot be determined using any 1D analysis.

The three criteria discussed above were then implemented in simulation software by Ryan et al. [197].

- A number of rules were developed by Couturler et al. [48] and then listed more explicitly in Brady et al. [31] and Theron et al. [232]. These rules are
known as the *Effective Laminar Flow* (ELF) rule system, which has some overlap with those of Lockyear and Hibbert [143] and Lockyear et al. [144]. ELF states:

- The displacing fluid must be at least 10% heavier than the displaced fluid:
  \[ \hat{\rho}_{\text{displacing}} > 1.1 \hat{\rho}_{\text{displaced}} \]
  where here \( \hat{\rho} \) is the density of each fluid.

- The frictional pressure gradient exerted by the displacing fluid should be at least 20% larger than that of displaced fluid.
  \[ \frac{\partial \hat{p}_f}{\partial \hat{z}}_{\text{displacing}} > 1.2 \frac{\partial \hat{p}_f}{\partial \hat{z}}_{\text{displaced}} \]
  here \( \partial \hat{p}_f/\partial \hat{z} \) is the *frictional* pressure gradient along the well axis.

- The shear stress on the narrow side of the annulus should exceed the yield stress of the displaced fluid. To formulate this, notice that the shear stress depends on the total pressure gradient. There are two factors contributing to wall shear stress: frictional pressure gradient and buoyancy. Thus,
  \[ \frac{\partial \hat{p}_f}{\partial \hat{z}}_{\text{displacing}} + (\hat{\rho}_{\text{displacing}} - \hat{\rho}_{\text{displaced}}) \hat{g} \cos \beta > 2 \frac{\hat{\tau}_{Y,\text{displaced}}}{\hat{d}_{\text{min}}} \]
  Here \( \hat{g} \) is the gravitational acceleration, \( \beta \) is the inclination angle measured from vertical, and \( \hat{d}_{\text{min}} \) is the gap width on the narrow side.

- The displacing fluid speed on the wide side must be equal or smaller than the displaced fluid speed on the narrow side (steady flow condition).
  \[ \hat{w}_{\text{narrow}} \geq \hat{w}_{\text{wide}}. \]

Again the last rule which is often called *differential velocity criteria* may not be so useful, because 1D models cannot estimate wide and narrow side velocities at the interface.
1.4.4 2D and 3D models of primary cementing

Since the 1990s, the industry has been able to access model-based simulators. Model based simulators are now actively used in cementing case studies, where they compare favourably with post-placement logging of the wells [173, 26, 96, 101]. Model-based simulations have been shown to improve the rule-based systems. In particular, Pelipenko and Frigaard [181] reviewed these rules and compared them with predictions of their model-based simulations. They concluded that there is a general agreement between the model-based simulations and rule-based systems, although the rule-based systems are often too conservative. Here we review the most recent model-based simulators.

A 2D model for annular displacement was first introduced by Bittleston et al. [24]. The key assumption of the model is that the annular gap is narrow compared to the mean circumference. This assumption has several implications: i) It allows to average the radial profile of velocity and fluid concentration, and therefore, reduces the problem to 2D. ii) It justifies neglect of the local curvature of the annulus which then allows to unwrap the annulus into a channel of varying width. iii) It allows us to assume the flow is locally a shear-flow. Since the flow is locally a shear-flow, 1D closures are derived and employed in the model. This model was analyzed numerically in Pelipenko and Frigaard [179, 180]. In particular, [180] provided a computational algorithm that is guaranteed to converge and does not require any viscosity regularization of the yield stress fluid. Using this computational framework, the model has been extensively studied for analyzing nearly vertical annuli [181], nearly horizontal annuli [37], as well as exchange flows [81].

A new generation of 2D models for primary cementing is introduced by Tardy and Bittleston [226]. Although the model derivation is somewhat different, it is essentially based on the same principles of Bittleston et al. [24]. In particular, it assumes a narrow gap and employs a lubrication-type approximation. However, Tardy and Bittleston [226] work with the pressure formulation and require the flow law. More recently, Tardy [225] modified the earlier formulation and reconstructed the radial velocity profiles. To this end, instead of integrating across the entire gap width, the gap width is divided into several smaller sections and integration is performed across each small section. The new formulation provides an economical
extension of the previous models to 3D, where the velocity field is still updated by solving a 2D Poisson equation, and the radial variations can be computed \textit{a posteriori}.

Apart from the above two families of models, other 2D cementing models exist in the literature. However, these models fail to properly model the yield stress of the mud or cement. For example, Carvalho and coworkers’ models \cite{90, 91, 52} are generally derived for Newtonian fluids and then extended to non-Newtonian fluids via an effective viscosity. The Lattice-Boltzman based model of Zhao et al. \cite{256} also assumes the fluids are power-law. In addition to the above openly accessible models, many service companies, such as Schlumberger and Halliburton, have in-house simulators that are mostly confidential and unfortunately, make few details publicly accessible.

The simulators above are all based on lubrication-type 2D models of primary cementing. In fact, 3D simulation of primary cementing is generally not economical, because of the large aspect ratio of the problem. Recall the typical gap width of the annulus is 2 cm, but the length of a cemented section is up to 1000 m. Resolving such a skewed geometry numerically requires a large number of computational cells (based on resolving the gap scale) and might need to run for hour-long pumping schedules to complete some operations. Given also the additional iterations required for most non-Newtonian fluid models, this scale of computation is still practically infeasible even with multi-processor machines and parallel codes. In addition to lower computational cost, 2D models are effective at reproducing large scale process features. However, there are some limitations in using 2D models:

i) These models are based on shear-flows, which relies on the gap being narrow with respect to the annular circumference.

ii) These models neglect inertial effects under the assumption that the aspect ratio times the Reynolds number is small, which is not always the case. Thus, inertial effects are not accurately represented.

iii) Averaging across the gap means that these models predict only the mean concentrations of fluids and not the transverse distribution.

iv) These models are based on the flows being fully developed locally, i.e. rela-
Putting these into perspective, 3D simulations can be useful to understand particular features of displacement flows, for example those in a washout where the flow is inherently 3D.

Unfortunately, several such 3D analyses in the literature lack a deep physical understanding. For example, many of these studies do not consider or properly model rheological effects (see Savery et al. [201], Zulqarnain [258], Chen et al. [40], Gomes et al. [92]). Some other 3D analysis investigates features that can be more readily studied using a 2D model. For example, Shadravan et al. [209] concluded that rheological hierarchy is critical in horizontal wells in preventing intermixing and viscous fingering. Even worse, some other studies have led to erroneous conclusions. For instance, Dutra et al. [64] concluded that “The influence of eccentricity on the interface shape is rather small”, while it is now widely accepted (and will be demonstrated later) that eccentricity greatly influences the shape of interface. Another example of a false conclusion can be found in the work by Savery et al. [201], where the authors argued that for water-based muds, “the [molecular] diffusion term likely will have a great impact” compared to the convective term. This statement is easily debunked, because a simple dimensional analysis shows that in laminar regimes $Pe \gg 1$, suggesting that molecular diffusion does not play any significant role in the mixing. Perhaps more interestingly, there are a number of controversial conclusions in Zulqarnain [258]. In particular, this study argued that

- “Fresh water will be the most effective means of displacing mud and detaching the adhered mud layer to walls.” On the same note, Aranha et al. [11] concluded: “In the case of vertical wells with good centralization, the absence of hierarchy for rheology and specific weight between the drilling fluid and washer did not lead [to] the formation of interface mixing because the displacement happened in turbulent regime”. Our results in Chapter 7 as well as the work of Guillot et al. [99] will disprove these statements.

- In horizontal displacement “There are no significant changes observed when the spacer viscosity was changed.” This contradicts several previous studies (e.g. see Shadravan et al. [209]).
• “For vertical wells the final cement fraction slightly decreases with increasing displacement rate for spacer having density less than cement, while for the spacer density equal to cement the opposite is true”. We will show this is not necessarily true.

Nonetheless, several other studies can be found that elucidate the 3D dynamics of cementing. Gomes et al. [92] looked at the 3D flow patterns near a liner hanger and suggested design modifications that improve flow recirculation. More recently, Kragset et al. [135] have investigated the flows in a nearly horizontal well with a washout. In particular, they explore the competition between buoyancy and eccentricity in driving the displacement on the wide or narrow side of the annulus. The results of this study confirms that the flow is 3D inside a washout, emphasizing the weakness of 2D models in modeling such scenarios. Furthermore, they show displacement inside the irregular section is facilitated by lowering the flow rates and increasing the density difference. In the absence of washout, the general observations agree well with those of the 2D model in Carrasco-Teja et al. [37] and the results of Appendix C.

1.5 Thesis objectives

The literature review presented above concerned entirely laminar displacement flows, with the exception of the work of de Araujo et al. [52] in which the authors model turbulent displacement of two Newtonian fluids in annular geometry. While this study has shed light to our understanding of turbulent displacement flows, fully turbulent flows are relatively uncommon in primary cementing. It is more likely to have one fluid fully turbulent (e.g. a chemical wash or low-viscosity spacer), while others are laminar. Indeed, as it will be shown later, this also can occur on a single section of the annulus, e.g. turbulent on the wide side, laminar or even static on the narrow side. To the best of our knowledge, there is no study that systematically investigated turbulent displacement flows in the annulus and accounted for regime transition. The aim of the present study is to fill this gap in knowledge. More specifically, we aim to derive a 2D model for the turbulent displacement of yield stress fluids in annular geometry. Although the main target is turbulent flows, the
model has to be consistent with the previous laminar models, e.g. Bittleston et al. [24], so that mixed flow regimes can be modeled. In doing so, we may modify the existing model to account for (i) Turbulent stresses and (ii) Diffusive and dispersive mixing.

Evidently, a model for turbulent stresses is needed. This seems particularly challenging, because, as shown by a number of studies [171, 80, 43, 169], all components of turbulent stress have similar magnitude (the fluctuating velocity is inherently three-dimensional). This is in contrast to laminar stresses in narrow geometries in which the normal components of stress can be neglected. In addition, although diffusion exists in laminar displacement, it is molecular, and therefore it can be legitimately neglected. Turbulent diffusion however is significantly larger and therefore, has to be taken into account. Even more importantly, as it will be shown later, weakly turbulent displacement flows fall into the Taylor dispersion regime, where the mixing by dispersion is at least one order of magnitude larger than that by turbulent diffusion. Therefore, a key aspect of the thesis is to model both turbulent diffusion and Taylor dispersion in turbulent and transitional regimes. Finally, the model has to be implemented using robust and fast algorithms. Of particular concern is to properly model the yield stress fluid to capture the unyielded immobile regions.

Our goal is to simulate displacement scenarios that are found in real cementing operations. We focus particularly on the types of primary cementing job performed in British Columbia, Canada. Thanks to the data collected from British Columbia Oil and Gas Commission, we look to provide insight about the following cementing practices:

1. It is common to start primary cementing with a light wash (usually water). This is believed to provide cleaning and enhance displacement quality. Scientific intuition however, does not support this argument. More explicitly, is a low viscous lightweight wash able to remove a heavy viscous mud at all? If not, a large volume of water is being used ineffectively.

2. There is a perception that turbulent displacement is always better than laminar displacement. Which displacement regime should we generally prefer: laminar or turbulent? This question is particularly suited for numerical stud-
ies such as the present study, as we can rarely find wells with comparable geometries and geophysical specifications that have been cemented with different displacement flow regimes. In other words, no systematic experimental data exists that favour or disfavour laminar displacements over turbulent displacements.

3. Where adequate pump capacity is present and where there are no risks of either an influx nor fracturing the well, how does laminar displacement compare with a weak turbulent displacement? What about strong turbulence?

4. Is it possible at all to derive a simple and computationally cheap model to quickly check any displacement design?

5. Currently, the quality of a cement job is evaluated using Cement Bond-Log (CBL) measurements. These measurements are costly, insensitive and often inaccurate. Can we use simulations to identify new means of post job evaluation?

1.6 Thesis outline

The content of the thesis, broadly speaking, can be divided into two parts.

1. In the first part, which includes Chapters 2-4 we extend the existing laminar model of Bittleston et al. [24] to transitional and turbulent regimes and derive the closure laws needed for the model. We then analyze the model from a computational point of view. More specifically,

   • In Chapter 2 we lay down a consistent framework to perform one-dimensional (1D) hydraulic calculations for yield stress shear-thinning materials. More specifically, we find the relationship between the mean velocity and the wall shear stress, as a function of fluid rheology, channel (or pipe) geometry and flow parameters. In addition, we extend the classical turbulent Taylor dispersion analysis to Herschel-Bulkly fluids. In doing so, we derive an accurate profile of velocity in turbulent regimes and give estimates for turbulent diffusivity using Reynolds analogy.
• In Chapter 3, we derive a comprehensive two-dimensional (2D) model for displacement flows of yield stress fluids in annular geometry. The novelty of this new formulation is in inclusion of turbulent and mixed flow regimes, which is absent in all similar developments. The final model consists of a nonlinear elliptic equation for the momentum equations as well as a transport equation which governs the evolution of the concentration of different fluids. The model is based on narrow-gap approximation, which allows the momentum equations to be simplified to locally 1D shear-flow hydraulic calculations (as laid down in Chapter 2).

• In Chapter 4, we analyze our annular model from a computational point of view. For the momentum equation, we establish a variational form and show that the system has a unique solution. This variational form is then employed to construct a robust augmented Lagrangian algorithm, particularly suitable for the modeling of yield stress fluids. We will also analyze and compare different algorithms that exist in the literature for solving our 2D transport equation.

2. In the second part, we systematically investigate many of the questions mentioned above. More specifically,

• In Chapter 5, we focus on fully turbulent displacement flows and show how different parameters such as rheology or buoyancy may or may not influence the outcome of the cement job.

• In Chapter 6, we explore displacement flows in mixed flow regimes; i.e. either when one fluid is in laminar and the other is turbulent or when one fluid undergoes a regime change around the well. In particular, we are interested to understand if one flow regime is more favorable compared to the other one.

• In Chapter 7, we turn our attention to density unstable displacement flows. In particular, through a set of simulations, we consider the effectiveness of using washes for primary cementing.
• In Chapter 8, we present a feasibility analysis on using particles to track the displacement front. The main idea is to exploit the density difference between successive fluids pumped in order to design a tracer particle to sit at the interface. Although apparently trivial, such particles must overcome viscous drag and strong secondary flows in order to reach and remain at the interface. Here we present simplistic models that show such technique can be indeed employed to track the interface in displacement flows.

The main body of the thesis is closed with a list of conclusions in Chapter 9.

The thesis is accompanied by four appendices. The details of our software prototype are presented in appendix A. Appendix B presents our group’s most recent development of 2D models to include (weakly compressible) foamed cement slurries. This work is the core subject of a master thesis in which A. Maleki has been directly involved. Appendices C and D present some additional studies of our 2D model. The analysis is primarily conducted by A. Rentaria, who is another PhD student in our group.
Chapter 2

Cementing Hydraulics

In this chapter, we lay down a consistent framework within which to perform hydraulic calculations for shear-thinning yield stress fluids. All fluids involved in conventional cementing operations (drilling muds, washes, spacers, cement slurries) are rheologically included in this description, which is widely employed in industry. To characterize turbulent flows, we adopt the widely popular Dodge-Metzner-Reed approach. Although the focus is on turbulent and transitional flows, the framework is consistent with the laminar regime too. This computational framework is later utilized in the development of our cementing model in Chapter 3.

The velocity profile in laminar regimes is directly integrable from the constitutive law. For turbulent flows however, hydraulic-style calculations have been studied since the 1950’s. In particular, Ryan and Johnson [198] and Hank and coworkers [105, 106, 107, 110, 111] have focused on shear thinning yield stress fluids. Although not universally accepted, the phenomenological method of Dodge-Metzner-Reed [158, 58] is popular in many process industries. In this method a generalized Reynolds number is defined based on the local power-law parameters. Then, a closure relationship is established for the frictional pressure drop as a function of the generalized Reynolds number, calibrated with available data. The Dodge-Metzner-Reed approach was intended to apply to all generalized Newtonian fluids. The extension to yield stress fluids can be found in Reed and Pilehvari [188], Pilehvari et al. [182], Founargiotakis et al. [77], as well as internally within
technical literature of many petroleum companies. Tests against experimental data are described by Guillot and Denis [98]. More recently, comparisons with direct numerical simulation data were made by Rudman et al. [196].

As it will be explained later, the original version of Dodge-Metzner-Reed formulation loses its simplicity when applied to complex generalized Newtonian fluids, such as Herschel-Bulkley fluids. To retain this simplicity, we overhaul and reconstruct this formulation here. Hydraulic calculations were historically developed for pipe flows and then extended to other flows such as those in channel. Although channel flow hydraulics is of main interest in this work, it is more natural to derive the equations for pipe flows first. We then briefly outline the formulation for channel flows. Finally, we end the chapter by estimating the streamwise dispersion and diffusion effects in fully turbulent flows. A version of this chapter is published in Maleki and Frigaard [146].

2.1 Pipe flow hydraulics

Consider fully developed steady flow of a Herschel-Bulkley fluid along a pipe. The axial momentum balance relates the axial gradient of frictional pressure \( \hat{p}_f \) to the wall shear stress \( \hat{\tau}_w \), which is then described in terms of the inertial stress scale \( \hat{\rho} \hat{\bar{W}}_0^2 / 2 \) and (Fanning) friction factor \( f_f \):

\[
- \frac{\hat{D}}{4} \frac{\partial \hat{p}_f}{\partial \hat{z}} = \hat{\tau}_w = \frac{\hat{\rho} \hat{W}_0^2}{2} f_f,
\]

(2.1)

where \( \hat{W}_0 \) is the mean velocity and \( \hat{\rho} \) is the fluid density. Herschel-Bulkley fluids are defined rheologically by 3 parameters: the yield stress \( \hat{\tau}_Y \), the consistency \( \hat{\kappa} \), and the power law index \( n \). In the hydraulic calculations that are generally performed, the fluid properties: \( \hat{\rho}, \hat{\tau}_Y, \hat{\kappa}, n \), and the pipe diameter \( \hat{D} \) are known. The aim is to define the closure relationship between the wall shear-stress \( \hat{\tau}_w \) and the mean velocity \( \hat{W}_0 \) for the different flow regimes.

A widely used approach is that of Dodge and Metzner [58] in defining \( f_f \) as a function of the generalized (Metzner-Reed) Reynolds number and power law index, with an additional dimensionless parameter needed to quantify yield stress ef-
fects. Although we are concerned with turbulent flows, the Metzner-Reed approach requires the laminar flow relations. The Metzner-Reed generalized Reynolds number is defined:

\[ Re_{MR} = \frac{8\hat{\rho}\hat{W}_0^2}{k'(\hat{\gamma}_N)^n} \]  

(2.2)

where the primed variables are:

\[ k' = \frac{\hat{\tau}_w}{(\hat{\gamma}_L)^n}, \quad n' = \frac{d\ln \hat{\tau}_w}{d\ln \hat{\gamma}_L}. \]  

(2.3)

The Newtonian strain rate at the wall is \( \hat{\gamma}_N \) and \( \hat{\gamma}_L \) is the laminar strain rate:

\[ \hat{\gamma}_N = \frac{8\hat{W}_0}{D}, \quad \hat{\gamma}_L = \frac{8\hat{W}_L}{D}. \]  

(2.4)

The velocity \( \hat{W}_L \), used to define \( \hat{\gamma}_L \), is the mean velocity that the fluid would have in a laminar flow, driven by the wall shear-stress \( \hat{\tau}_w \). Note that \( \hat{W}_L \) and \( \hat{\gamma}_L \) are defined by the wall shear stress \( \hat{\tau}_w \) across all flow regimes, but will only equal \( \hat{W}_0 \) and \( \hat{\gamma}_N \) in the case that the flow is laminar.

For laminar flows, the Buckingham-Reiner equation can be derived, which is an algebraic equation relating the flow rate to the wall shear stress. The Rabinowitsch-Mooney procedure results in the same expression. For Herschel-Bulkley fluids the result is:

\[ \hat{\gamma}_L = \frac{4n}{3n+1}(1-r_Y)^{1/n+1} \left[ \frac{\hat{\tau}_w}{k'} \right]^{1/n} \times \left[ (1-r_Y)^2 + \frac{2(3n+1)(1-r_Y)r_Y}{2n+1} + \frac{(3n+1)r_Y^2}{n+1} \right]. \]  

(2.5)

Here \( r_Y = \hat{\tau}_Y / \hat{\tau}_w \), which also represents the dimensionless radial position of the yield surface. Combining (2.3) with (B.9) we find:

\[ n' = n(1-r_Y) \frac{(n+1)(2n+1) + 2n(n+1)r_Y + 2n^2r_Y^2}{(n+1)(2n+1) + 3n(n+1)r_Y + 6n^2r_Y^2 + 6n^3r_Y^3}, \]  

(2.6)

and hence can define \( Re_{MR} \) etc. Note that the expression for \( n' \) in Zamora and Bleier [253] is incorrect. Figure 2.1a illustrates the variation of \( n' \) with \( (n, r_Y) \): increasing
the yield stress (and hence \( r_Y \)) reduces \( n' \) and the laminar velocity profiles become increasingly plug-like.

The complicated derivation of \( Re_{MR} \) has the virtue of ensuring that \( f_f = 16/Re_{MR} \) in the laminar regime for all generalized Newtonian fluids. The original derivation was for power law fluids, where \( n' = n \) and

\[
\hat{n} = \frac{4n}{3n + 1} \left[ \frac{\hat{\tau}_w}{\hat{\kappa}} \right]^{1/n} \implies \hat{k}' = \hat{k} \left[ \frac{3n + 1}{4n} \right]^n. \tag{2.7}
\]

Thus, for power law fluids, in all flow regimes, \( Re_{MR} \) is explicitly defined in terms of the mean velocity, making it straightforward to work with \( f_f, Re_{MR} \) and \( n \) in defining the mapping between \( \hat{\tau}_w \) and \( \hat{W}_0 \). The simplicity of the Metzner-Reed formulation however is lost once we move more complex generalized Newtonian fluids and study different flow regimes.

### 2.1.1 Choice of dimensionless groups

From dimensional considerations, we expect the relation between \( \hat{\tau}_w \) and \( \hat{W}_0 \) to be expressible in terms of \( n \) and 3 other dimensionless groups. Although different expressions have been used to define \( f_f \) in terms of \( n' \) & \( Re_{MR} \), when these are expressed in terms of \( \hat{W}_0 \) the definition is typically implicit, which makes these variables less appealing for characterising \( \hat{\tau}_w \mapsto \hat{W}_0 \). Instead, we feel it is more
convenient to work with a Reynolds number that can be defined explicitly in terms of \( \hat{W}_0 \) and that is independent of \( \hat{\tau}_w \). Motivated by (2.7) we use a rescaled consistency \( \hat{\kappa}_p \), referred to as the power-law consistency, and use \( n \), to define the power law Reynolds number \( Re_p \), as follows:

\[
Re_p = \frac{8\hat{\rho}\hat{W}_0^2}{\hat{\kappa}_p(\hat{\gamma}_N)^n}, \quad \hat{\kappa}_p = \hat{\kappa}\left[\frac{3n+1}{4n}\right]^n.
\] (2.8)

For a power-law fluid, \( Re_{MR} = Re_p \), and \( Re_p \) is always an explicit function of \( \hat{W}_0 \). The Buckingham-Reiner equation (B.9) may now be simplified to:

\[
\frac{\hat{\kappa}_p \hat{\gamma}_N^n}{\hat{\tau}_w} = E(n, r_Y):
\]

\[
E(n, r_Y) = (1 - r_Y)^{1+n} \left( (1 - r_Y)^2 + \frac{2(3n+1)(1-r_Y)r_Y}{2n+1} + \frac{(3n+1)r_Y^2}{n+1}\right)^n,
\]

see Figure 2.1b.

It is common to represent yield effects with either the Bingham number, with \( r_Y \) or with the Hedström number. The Bingham number involves \( \hat{W}_0 \), and \( r_Y \) involves \( \hat{\tau}_w \). Thus, we select the Hedström number, the definition of which varies in the literature for \( n \neq 1 \). We choose to normalize so that the Hedström number has a linear variation in yield stress and use \( \hat{\kappa}_p \) for later convenience:

\[
He = \hat{\tau}_Y \left( \frac{\hat{\rho}\hat{D}^2n}{\hat{\kappa}_p^2} \right)^{1/(2-n)}.
\] (2.10)

This definition agrees with other common definitions at \( n = 1 \).

Finally, for a dimensionless group that depends on \( \hat{\tau}_w \), but is independent of \( \hat{W}_0 \) we mimic the definition of \( He \), replacing yield stress with wall shear stress:

\[
H_w = \hat{\tau}_w \left( \frac{\hat{\rho}\hat{D}^2n}{\hat{\kappa}_p^2} \right)^{1/(2-n)},
\] (2.11)

noting that \( r_Y = He/H_w \).

Our aim has been to isolate effects of \( \hat{\tau}_w \) and \( \hat{W}_0 \) from other targeted physi-
cal effects (e.g. \( \hat{\tau}_Y \)) in our dimensionless description, achieved with \((Re_p, H_w, He)\), which independently represent the effects of increasing \( \hat{W}_0, \hat{\tau}_w \) and \( \hat{\tau}_Y \). Other variables (such as \( f_f, Re_{MR} \) and \( r_Y \)) may be economical for expressing specific analytical or empirical relationships, but their utility has partially eroded with the advent of modern computing power and there is no gain in simplicity once we consider yield stress fluids and different flow regimes. It is also worth mentioning here that \( He \) is a system dependent parameter (depends only on pipe diameter and the fluid properties), whereas \( Re_p \) and \( H_w \) are flow dependent and thus the relationship between them uniquely specifies the problem.

### 2.1.2 Flow regimes

As the flow rate (and wall shear stress) increases the flow changes from laminar through a transitional regime to fully turbulent flow. In each regime the mapping between \( \hat{\tau}_w \) and \( \hat{\bar{W}}_0 \) is to be defined, represented dimensionlessly by the mapping between \( H_w \) and \( Re_p \). For laminar flows \( \hat{\gamma}_N = \hat{\gamma}_L \), and from (2.8): 
\[
\hat{\gamma}_N = [8 \hat{k}_p Re_{p,Lam}/(\hat{\rho} \hat{D}^2)]^{1/(2-n)},
\]
and on using (2.9) we have:
\[
\frac{(8Re_{p,Lam})^{n/(2-n)}}{H_w} = E(n, r_Y) = E\left(n, \frac{He}{H_w}\right).
\]
This defines \( Re_{p,Lam} \) explicitly in terms of \( H_w \) and vice-versa if \( Re_{p,Lam} \) is specified, solving (2.12) iteratively, e.g. by finding \( r_Y \in [0, 1] \) to any required precision. Thus, we may readily compute the mapping \( Re_{p,Lam} \leftrightarrow H_w \), after which we may define:
\[
f_f = \frac{2 \hat{\tau}_w}{\rho \hat{W}_0^2} = \frac{16 \hat{\tau}_w}{8 \rho \hat{W}_0^2} = \frac{16}{Re_{p,Lam} E\left(n, \frac{He}{H_w}\right)} = \frac{16}{Re_{MR,Lam}}.
\]

For fully turbulent flows, following Dodge and Metzner [58]:
\[
\frac{1}{\sqrt{f_f}} = \frac{4.0}{(n')^{0.75}} \log(Re_{MR} f_f)^{1-n'/2} - 0.4 \left(\frac{n'}{n'}\right)^{1.2}.
\]
We will use (2.14) (and its counterpart for channel flow (2.27)) in the following
analysis wherever needed. Noting that:
\[
\hat{\gamma}_N = \left[ Re_p \frac{8 \hat{k}_p}{\rho D^2} \right]^{1/(2-n)} \quad \text{and} \quad \hat{\gamma}_L = \left[ E \left( n, \frac{He}{H_w} \right) \frac{\hat{\tau}_w}{\hat{k}_p} \right]^{1/n},
\]

after a little algebra we find:
\[
Re_{MR} = \frac{8^{n-n'} \Re_p^{\frac{2-n'}{n}} E \left( n, \frac{He}{H_w} \right)^{n'/n}}{H_w^{1-n'/n}}, \quad f_f = \frac{2H_w 8^{\frac{2-n}{2-n'}}}{Re_p^{\frac{1}{n'}}}. \tag{2.15}
\]

Substituting into (2.14) and simplifying leads to:
\[
\Re_p = H_w\left[ 2^{4-\frac{n'}{2}} \left( \frac{4.0}{(n')^0.75} \right)^{\log \left( \left( \frac{4^{1-n'/2} E \left( n, \frac{He}{H_w} \frac{\hat{\tau}_w^2}{\hat{k}_p^2} \right)}{0.4 (n')^1.2} \right) \right)^{2-n} \right. \tag{2.16}
\]

Again, in the case that \( H_w \) is specified (i.e. \( \hat{\tau}_w \)), then (2.16) defines \( \Re_p \) explicitly. If instead \( \Re_p \) is specified (i.e. \( \hat{\tau}_w \)), then \( H_w \) is found iteratively from (2.16).

Note that the 2 nonlinear equations that must be solved in the case that \( \Re_p \) is specified, (2.12) & (2.16), can be straightforwardly written as monotone functions of \( H_w \) within specified bounds. Such equations can be solved with simple but robust root-finders such as the bisection method, Ridder’s method, Brent’s method etc.

Transitional flows are found for \( \Re_1(n') < \Re_{MR} < \Re_2(n') \). The choice of \( \Re_1(n') \), \( \Re_2(n') \) and transition to turbulence are discussed at length in §2.3. Recall that \( n' = n'(n, He/H_w) \), and since \( \Re_{MR} \) depends on \( (n, He) \) and either of \( H_w \) or \( \Re_p \), the critical values \( \Re_1 \) and \( \Re_2 \) can be used to define critical (transitional) values of either \( H_w \) or \( f_f \), e.g. we solve the equation:
\[
\Re_1(n, He/H_w) = \Re_{MR}(n, He, H_w), \tag{2.17}
\]
by iterating with respect to \( H_w \) and using the laminar flow closure expression, thus defining \( f_{f,1} \) and \( H_{w,1} \). Similarly, on solving:
\[
\Re_2(n, He/H_w) = \Re_{MR}(n, He, H_w), \tag{2.18}
\]
by iterating with respect to $H_w$ and using the turbulent flow closure expression, we define $f_{f,2}$ and $H_{w,2}$.

For representing hydraulic quantities in transitional flows it is common to use some form of interpolation. Here we choose to interpolate $\log f_f$ linearly with respect to $\log Re_{MR}$. More explicitly:

$$\log f_f = \frac{\log f_{f,1} \log Re_2 - \log Re_{MR}}{\log Re_2 - \log Re_1} + \log f_{f,2} \left[ \frac{\log Re_{MR} - \log Re_1}{\log Re_2 - \log Re_1} \right].$$

Figure 2.2a illustrates the 3 flow regimes in $(Re_{MR}, f_f)$-space at $He = 500$ for $n = 0.2, 0.4, \ldots 0.8, 1$. We observe the usual collapse of data in the laminar regime. For $n$ closer to 1 we see a sharp change in $f_f$ at transition, but not for smaller $n$. The transitional curves are linear in the log-log plot, as shown. Figure 2.2b & c plots $Re_p$ and $Re_{MR}$ against $(H_w - H_{w,1})/(H_{w,2} - H_{w,1})$ for $n = 0.2, 0.4, \ldots 0.8, 1$ at $He = 500$, i.e. this is the same data as Figure 2.2a. We see large relative difference between $Re_p$ and $Re_{MR}$ at smaller values of $n$ and $H_w$ (laminar and transitional), which corresponds to those parameters where $n'$ is smallest. Qualitatively similar plots are found at other $He$.

### 2.2 Plane channel flows

A broadly similar analysis to that for the pipe can be performed for a plane channel flow. This flow is often used to locally approximate flow along a narrow annulus. We outline here only the main results, highlighting any differences with the pipe flow. Here we briefly repeat the above computational procedure for channel flows.

We consider axial flow in a 2D channel of width $2\hat{H}$. In order to define Metzner-Reed and power law Reynolds numbers as well as Hedström numbers we need to replace $\hat{D}$ with $2\hat{H}$ and the prefactor 8 with 6; i.e.

$$Re_{MR} = 6\hat{\rho}\hat{W}_0^2 \frac{2n}{\hat{\kappa}'(\hat{\gamma}_N)^{n'}}$$

$$Re_p = 6\hat{\rho}\hat{W}_0^2 \frac{2n}{\hat{\kappa}_p(\hat{\gamma}_N)^n}$$

$$He = \hat{\tau}_Y \left( \frac{\hat{\rho}^{\alpha}(2\hat{H})^{2n}}{\hat{\kappa}_p} \right)^{1/(2-n)}$$

$$H_w = \hat{\tau}_w \left( \frac{\hat{\rho}^{\alpha}(2\hat{H})^{2n}}{\hat{\kappa}_p} \right)^{1/(2-n)}$$

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where $\kappa'$ and $n'$ are still defined as (2.3) and

$$\hat{\eta}_N = \frac{6\hat{W}_0}{2\hat{H}}, \quad \hat{\eta}_L = \frac{6\hat{W}_L}{2\hat{H}}.$$  \hfill (2.22)

The power law consistency is defined similarly to (2.8) as:

$$\hat{\kappa}_p = \hat{\kappa} \left(\frac{2n + 1}{3n}\right)^n$$  \hfill (2.23)

Figure 2.2: Examples for $He = 500$ and $n = 0.2, 0.4, \ldots, 0.8, 1$: a) $f_f$ vs $Re_{MR}$; b) $Re_p$ against $(H_w - H_{w,1})/(H_{w,2} - H_{w,1})$; c) $Re_{MR}$ against $(H_w - H_{w,1})/(H_{w,2} - H_{w,1})$. Regimes are denoted: laminar (green), transitional (red), turbulent (black).
The Rabinowitsch-Mooney procedure applied to the laminar flow results in the following expressions for \( n'(n, y_Y) \) and \( E(n, y_Y) \):

\[
n' = n(1 - y_Y) \frac{n y_Y + n + 1}{2n^2 y_Y^2 + 2n y_Y + n + 1}
\]

\[
E = \frac{\hat{k}_p \hat{\gamma}^p}{\hat{\tau}_w} = (1 - y_Y)^{(n+1)} \left( \frac{n}{n+1} y_Y + 1 \right)^n
\]

where \( y_Y = \hat{\tau}_Y / \hat{\tau}_w \) represents the dimensionless (laminar) plug width.

In the laminar regime, the mapping from \( H_w \leftrightarrow Re_p \) (i.e. \( \hat{\tau}_w \leftrightarrow \hat{W}_0 \)) is:

\[
\frac{(6Re_{p,Lam})^{n/(2-n)}}{H_w} = E(n, y_Y),
\]

from which we then define

\[
f_f = \frac{2 \hat{\tau}_w}{\hat{\rho} \hat{W}_0^2} = \frac{12 \hat{\tau}_w}{6 \hat{\rho} \hat{W}_0^2} = \frac{12}{Re_{p,Lam} E \left( n, \frac{He}{H_w} \right)} = \frac{12}{Re_{MR,Lam}}.
\]

In fully turbulent regime, the Dodge-Metzner relation is

\[
\frac{1}{\sqrt{f_f}} = \frac{4.0}{n^{0.75}} \log \left( Re_{MR} f_f^{1-\frac{\nu}{\tau}} \right) - 0.395 \frac{0.395^{2-n}}{n^{1.2}}
\]

which leads to the following equation defining \( H_w \leftrightarrow Re_p \):

\[
Re_p = H_w^{1-\frac{\nu}{\tau}} 6^{1-n} 2^{1-\frac{\nu}{\tau}} \left[ \frac{4.0}{n^{0.75}} \log \left( 6^{1-n'} 2^{1-\frac{\nu'}{\tau}} E \frac{\nu'}{H_w} - \frac{\nu}{\tau} \right) \right] - \frac{0.395^{2-n}}{n^{1.2}}.
\]

As with the laminar flows, this must be solved iteratively if \( Re_p \) is specified, but is explicit if \( H_w \) is specified. In the following section, we will discuss transition to turbulence and the choices of \( Re_1 \) and \( Re_2 \).

### 2.3 Turbulent transition in generalized Newtonian fluids

The question of transition from laminar into fully turbulent flows for yield stress fluids was first considered by Hedström [116] who advocated a criterion based on
Figure 2.3: Example of the hydraulic quantities for channel flow $He = 200$ and $n = 0.2, 0.4, ..., 1$. a) $f_f$ against $Re_MR$. Regimes are denoted: laminar (green), turbulent (black). Broken line is extrapolation into transitional range; b) $f_f$ against $Re_M$. Regimes are denoted: laminar (green), transitional (red) and turbulent (black); c) $Re_p$ against $(H_w - H_{w,1})/(H_{w,2} - H_{w,1})$; d) $Re_{MR}$ against $(H_w - H_{w,1})/(H_{w,2} - H_{w,1})$; e) $f_f$ against $(H_w - H_{w,1})/(H_{w,2} - H_{w,1})$
the point of intersection of the laminar and turbulent friction factor curves (intersection method). A later approach by Metzner-Dodge-Reed was to use $f_f \approx 0.0076$, as the transition parameter (expressed equivalently with generalized Reynolds number); Metzner and Reed [158], Dodge and Metzner [58]. This concept was extended to the Bingham model by Govier and Aziz [94] and can be applied to any purely viscous non-Newtonian fluids, assuming transition takes place at $f_f \approx 0.0076$. A related approach followed more recently is due to Desouky and Al-Awad [57], which combines the Metzner-Reed and intersection methods.

Furthermore, a number of approaches have evolved that balance stabilizing and destabilizing effects on the flow, setting a criterion based on when this balance exceeds some critical value. Two identical predictions of transitional Reynolds numbers have been made by Ryan and Johnson [198] and by Hanks & co-workers [105, 106, 107, 108, 109, 110, 111], although arrived at using different rationale. A slightly different balance approach is advanced by Mishra and Tripathi [159], balancing the mean kinetic energy and the wall shear stress. Güzel et al. [103] have developed another local balance approach that shares similarities. Wilson & Thomas [248, 233, 249] have evolved analyses based on estimates of the viscous sub-layer, postulating that transition depends only on $He$. Other approaches have evolved that are industry-specific, e.g. those of Slatter [211], Slatter and Wasp [212] are predominantly developed for mining applications. Pilehvari and co-workers have reviewed available data and many of the existing phenomenological criteria [188, 182] and advocate a type of intersection method using the Metzner-Reed approach.

It is noteworthy that many of the above non-Newtonian approaches have developed from roots 40-60 years old. Over this same period our understanding of Newtonian fluid transition has evolved considerably. Although transitional $Re$ for Newtonian fluids are typically quoted at $Re \approx 2100$, theoretically pipe flow is subcritical and believed to be linearly stable at all $Re$. Indeed, with careful control it has been possible to achieve stable laminar pipe flows at $Re$ in the $20,000 - 60,000$ range; see for example Draad et al. [61], Hof [117]. The transitional $Re$ essentially gives a measure of the quality of the experimental flow loop. Thus, the common engineering perspective that “transition” (meaning the end of the laminar regime) will occur at a given $Re$ is flawed, even for a Newtonian pipe flow.
More detailed experimental studies of transition in non-Newtonian fluids have appeared e.g. Pinho and Whitelaw [183], Draad et al. [61], Escudier et al. [71], Peixinho et al. [177, 178], Esmael and Nouar [72], Güzel et al. [102]. Coupled to these are a range of theoretical and computational studies, e.g. Frigaard et al. [84], Nouar and Frigaard [170], Rudman et al. [195], Frigaard and Nouar [82], Rudman et al. [196], Esmael and Nouar [72]. This list covers only those studies focused at inelastic fluids. Although by comparison to Newtonian fluids, our understanding of transition in shear-thinning yield stress fluids remains limited, it has significantly evolved in the past 20 years. Some aspects of this understanding can now be applied pragmatically to improve the common descriptions of transition.

Firstly, despite the existence of stability at elevated $Re$ for Newtonian flows, application requires criteria that are approximately correct for typical hydraulic settings, i.e. industrial pumps and pipes. In essence, there is a critical flow parameter at which stability of the laminar flow is lost. Secondly, it is observed that transition occurs over an extended range of $Re$ for shear-thinning yield stress fluids. Thirdly, all this is modulated by realization of experimental factors not all known 40-60 years ago, including the following. (i) The initial loss of stability is often hard to detect at the pipe centre, but is visible at the walls. (ii) Sharp changes in $f_f$ are also not always evident, especially in more strongly shear-thinning fluids. (iii) A wide range of different phenomena are found in transitional flows (e.g. puffs & slugs, coherent structures, flow asymmetry...) and these specific phenomena have rheological dependencies that are not fully explored. However, eventually all flows transition into full turbulence.

In the above context, we advocate an approach that uses 2 critical $Re$: the smaller one reflecting loss of stability and the larger reflecting onset of full turbulence. Although we dismiss the intersection method (as it predicts only a single transition), we must recognise that such approaches are in some sense robust as the friction factor relationships extrapolated are based on data valid over ranges of laminar and turbulent flow rates, instead of at a particular transition point which may be hard to detect at smaller $n$.

There are in fact a number of approaches in usage that adhere to the above picture, and this is reasonably common in oilfield application; e.g. Nelson and Guillot [167]. Here we assume 2 critical Reynolds numbers: $Re_1(n') < Re_2(n')$, 

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depending only on the local power law index \( n' = n'(n, \text{He}/H_w) \), and use these critical values to delineate laminar, transitional and turbulent flow regimes. The first critical value is given by:

\[
Re_1(n') = 3250 - 1150n',
\]  

as advocated by Nelson and Guillot [167], i.e. laminar flow for \( Re_{MR} \leq Re_1(n') \). For the second critical Reynolds number, one option is that of Guillot and Denis [98], Founargiotakis et al. [77], which is algebraically similar to (2.29). Unfortunately, although well behaved for power law fluids (\( \text{He} = 0 \)), as the yield stress \( \text{He} \) is increased the Dodge-Metzner expression (2.14) loses monotonicity for smaller \( n \) and eventually ceases to be single valued, as illustrated in Figure 2.4a. Thus, expressions that extend (2.29) algebraically such as that of Guillot and Denis [98], Founargiotakis et al. [77] tend to fail to produce physically realistic transition criteria at small \( n \) once we have a significant yield stress.

Mathematically, at fixed \((n, \text{He})\) the variable \( n' \) varies with \( Re_{MR} \). The expression (2.14) gives \( f_f \) monotone with respect to \( Re_{MR} \) only for fixed \( n' \). This behaviour does not agree with experimental observation. Consequently if (2.14) represents the frictional behaviour for fully turbulent flows, it is necessary to restrict the transitional range approximately to those for which (2.14) is well-behaved. An expression which does this effectively is the following:

\[
Re_2(n') = \begin{cases} 
1.328529 \times 10^{(6.00-7.84n')} & n' < 0.31, \\
3000 + \left[ \frac{1}{a(n')} \right]^{\frac{1}{n'(n'=1)}} - \left[ \frac{1}{a(n'=1)} \right]^{\frac{1}{n'(n'=1)}} & n' \geq 0.31,
\end{cases}
\]  

\[
a(n') = 0.078504 + 0.0098085 \log n',
\]

\[
b(n') = -0.24984 + 0.059646 \log n'.
\]

The dependency of \( Re_1 \) and \( Re_2 \) on \( n \) and \( r_Y = \text{He}/H_w \) is shown in Figs. 2.4b & c. Note that the effects of the yield stress at fixed \( n \) are felt wholly through \( n' \). In particular full turbulence \((Re_{MR} \geq Re_2)\) is significantly delayed by a strong yield stress, as is observed experimentally. An example of the flow regimes, plotted as \( f_f \) vs \( Re_{MR} \) and computed using \( Re_1 \) and \( Re_2 \) defined above, has been given in
Figure 2.4: a) $f_f$ vs $Re_{MR}$ for $He = 2000$ and $n = 0.2, 0.4, 0.6, 0.8, 1$. Regimes are denoted: laminar (green), turbulent (black); broken line is an extrapolation into transitional range using the turbulent closure. b) $Re_1(n'(n, r_Y))$ for $n = 0.1, 0.2, \ldots, 0.9, 1$. c) $Re_2(n'(n, r_Y))$ for $n = 0.1, 0.2, \ldots, 0.9, 1$. 

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Figure 2.2a. This figure illustrates the effectiveness of (2.30)-(2.32) in truncating the fully turbulent regime, ensuring a single-valued $f_f$.

For channel flows (§2.2), the same issues arise with extrapolating the turbulent $f_f(Re_{MR})$ at small $n$ and large $He$. The criterion (2.30)-(2.32) is replaced by:

\[
Re_2(n') = \begin{cases} 
1.106969 \times 10^{(6.00-8.19n')} \\
3000 + \left[ \frac{24}{a(n')} \right]^{1/b(n')} - \left[ \frac{24}{a(n'=1)} \right]^{1/b(n'=1)} 
\end{cases}, \quad n' < 0.28, \tag{2.33}
\]

\[
a(n') = 0.5^{b(n')} \times \left[ 0.096045 + 0.0082711 \log n' \right], \quad (2.34)
\]

\[
b(n') = -0.27103 + 0.063985 \log n'. \quad (2.35)
\]

The same expression (2.29) is used for $Re_1$.

A final comment regarding transitional flows is more pragmatic. On the one hand, applications involve real fluids. Frequently, models such as the Herschel-Bulkley model only describe a limited range of shear rates and are fitted to rheological data from viscometric flows. Although shear-thinning and yield stress aspects may be the dominant rheological behaviours observed over these shear rate ranges, invariably the fluids used experimentally have other rheological behaviours depending on the flow history and fluid micro-structure. As fully turbulent shear flows are characterized by broad ranges of time and length-scales it is unknown if and how smaller-order rheological features may influence turbulence phenomena. Analytical and computational study of inelastic generalized Newtonian fluid models is one approach that explicitly removes other rheological influences. On the other hand, for industrial application even the use of models such as the Herschel-Bulkley fluid presents problems. Rheological measurements in application are frequently dictated by industry protocol and standardization. Thus, the fitting of rheological parameters to application data is an imperfect science, and these errors propagate into whatever predictions we make.

### 2.4 Dispersion and diffusion of passive scalars

As mentioned earlier in Chapter 1, the fluids used in primary cementing are drilling fluids, washes, spacer fluids and cement slurries, all of which are characterized
within the industry as shear-thinning yield stress fluids. If water-based, these fluids are miscible. In turbulent flows they rapidly mix transversely and then disperse longitudinally, presumably driven by the Taylor dispersion mechanism, [229, 230]. Although Zhang and Frigaard [255] have considered dispersion of such fluids in laminar regimes, for laminar flows primary cementing does not typically fall into the Taylor-regime.

Axial dispersion in turbulent flows of Newtonian fluids was initially studied by Taylor [230]. Upon applying the Reynolds analogy to model the turbulent dispersivity, he then integrated the relative velocity profile across the pipe to calculate the axial bulk dispersivity. Taylor used tabulated data from the universal distribution of velocity which is known to be valid only at high Reynolds number and therefore his results significantly deviate from experimental data [234, 67, 113]. Taylor’s analysis was later revisited by Tichacek et al. [234] and Flint and Eisenklam [75] who utilized experimental velocity profiles to give better estimates. Nonetheless, both of these studies deviate from experimental results at low Reynolds number ($Re < 10^4$) mainly because the experimental velocity profile was unable to capture the wall layer. In another study Ekambara and Joshi [67] estimated the axial dispersion with a velocity profile obtained computationally using the $k – \varepsilon$ model. A comparison of these approaches with the experimental data can be found in Hart et al. [113].

For inelastic non-Newtonian fluids, axial dispersion in laminar [28, 4, 3, 255] and turbulent [241, 137, 220] flows has been studied. In the case of turbulent regimes, Wasan and Dayan [241], Krantz and Wasan [137] studied dispersion of power-law fluids using the turbulent velocity profile of Bogue and Metzner [27]. Wasan and Dayan [241] predicted the axial dispersion to increase with Reynolds number, contradicting Taylor’s model for dispersion. Krantz and Wasan [137] modified the earlier results by adding a wall layer to the velocity profile. However, the validity of their results is questionable since the velocity scale used appears to be different from that of Bogue and Metzner [27].

As noted by Tichacek et al. [234], Krantz and Wasan [137], Ekambara and Joshi [67], Hart et al. [113], good estimation of the Taylor dispersion demands an accurate velocity profile. Laminar velocity profiles are integrable from the constitutive law. As seen above, the Metzner-Reed generalized Reynolds number provides
an economical description of the hydraulic closure relationship. In particular, the Dodge-Metzner-Reed approach is attractive in that the hydraulic calculations (and closure) are linked to a universal log-law velocity profile, proposed by Dodge and Metzner [58]. Such profiles may be used directly to calculate Taylor dispersion coefficients. However, two common deficiencies occur: (i) the log-law is not valid at the centreline of the pipe/duct; (ii) the log-law must be matched/patched to a different velocity approximation close to the wall. Various centreline corrections have been suggested, including the correction of Reichardt [189] and exponential correction of [27]. Near the wall, Krantz and Wasan [136] argued that Reynolds stresses decay as the cube of the distance, and therefore suggested that the wall layer effect could be significant. Krantz and Wasan [137] developed the analysis framework to evaluate the wall layer for power-law fluids.

In laminar flows, increasing the yield stress tends to flatten the velocity profile and hence reduce Taylor dispersion. In turbulent flows it is generally thought that the yield stress has little influence on the velocity profile in the turbulent core, but is known to retard turbulent transition. Equally, since the yield stress contributes to the effective viscosity we might expect that wall-layer effects are significant as the yield stress increases. Hence the interest in weak turbulence where wall-layers are thicker and occupy a larger proportion of the duct area, also where the velocity changes are greatest. This study explores the subtlety of this relationship.

Here we aim to estimate streamwise dispersion and diffusion effects, focusing on fully turbulent flows. Pragmatically, we are unable to easily model diffusion and dispersion in transitional flow regimes. Furthermore, in the laminar flows of industrial interest we are typically far from the laminar Taylor dispersion regime. In fully developed turbulent flows the dominant transport mechanism is invariably Taylor dispersion, which is modelled straightforwardly once the turbulent diffusivity and velocity profile are known.

### 2.4.1 Velocity profiles in turbulent pipe flows

Dodge and Metzner [58] derive the following velocity profile in the turbulent core:

\[
\frac{\hat{W}(\hat{r})}{\hat{W}_*} = A_{DM} \log \hat{y}^+ + B_{DM},
\]  

(2.36)
where the friction velocity is defined by: 
\[
\hat{W} = \sqrt{\hat{\tau}_w/\hat{\rho}} = \sqrt{f_f/2\hat{\bar{W}}_0},
\]
and
\[
\hat{y}^+ = (1 - r)^{n'} \hat{\bar{R}} n \hat{W}_w^{2-n} = (1 - r)^{n'} f_f^{1-\frac{n}{2}} \hat{\bar{R}} n \left[ \frac{3n + 1}{4n} \right]^{n} 8^{n-1} \frac{2^{1+\frac{3}{2}}}{2^{1+\frac{3}{2}}},
\]  
(2.37)

for \( r = \hat{r}/\hat{R} \). This velocity profile, when averaged across the pipe should give an expression equivalent to (2.14), thus defining \( A_{DM} \) & \( B_{DM} \). More precisely, since the dimensionless velocity \( W(r) \) has mean value 1, we have:

\[
1 = 2 \int_0^1 r W(r) \, dr
\]
\[
= \sqrt{\frac{f_f}{2}} \int_0^1 r [A_{DM} \log \left( (1 - r)^{n'} f_f^{1-\frac{n}{2}} \hat{\bar{R}} n \left[ \frac{3n + 1}{4n} \right]^{n} 8^{n-1} \frac{2^{1+\frac{3}{2}}}{2^{1+\frac{3}{2}}} \right) + B_{DM}] \, dr
\]
(2.38)

In order that (2.38) is equivalent to (2.14), we find:

\[
A_{DM} = 4.0\sqrt{2} \frac{n'}{(n')^{0.75}},
\]  
(2.39)

\[
B_{DM} = -0.4\sqrt{2} \frac{n'}{(n')^{1.2}} - A_{DM} \left( \log \left( f_f^{\frac{n'}{2}} \frac{Re_p}{Re_{MR}} 2^{\frac{n}{2}} \left[ 3 + \frac{1}{n} \right]^{n} \sqrt{\frac{3n'}{2\ln(10)}} \right) - \frac{3n'}{2\ln(10)} \right),
\]  
(2.40)

which can be verified with those in Dodge and Metzner [58] for a power law fluid \((n' = n)\). With a little algebra, the Dodge-Metzner velocity profile is given in terms of \( r \) by:

\[
W(r) = \sqrt{\frac{f_f}{2}} \left[ A_{DM} \left( \log [(1 - r)^{n'} f_f^{1-\frac{n'}{2}} Re_{MR}] + \frac{3n'}{2\ln(10)} \right) - \frac{0.4\sqrt{2}}{(n')^{1.2}} \right]
\]  
(2.41)

Two common deficiencies of such log-law profiles are the centreline behaviour and a correction for the wall layer, as we now describe.

\(^1\)Note that there is an errata to the formula in equation (48) of Dodge and Metzner [58]; the corrected coefficients in the velocity profile may be found in Dodge and Metzner [59].
Centreline correction

Firstly, it is common to adjust the profile near the pipe centre so that the mean turbulent velocity \( W(r) \) has zero gradient at centreline. This correction is purely empirical and there are many suggested forms in the literature. As these different corrections work on the derivative of a smooth velocity profile at the pipe centre, it is hard to differentiate between these expressions by comparing with experimental data, even for Newtonian fluids. The condition that any correction function \( c(r, f_f, n') \) should satisfy is:

\[
\frac{d}{dr}c(0, f_f, n') = \sqrt{\frac{f_f n' A_{DM}}{2 \ln 10}}, \tag{2.42}
\]

which ensures that the corrected velocity has zero derivative at the pipe centre. We also expect that the maximum velocity is at the pipe centre (an inequality constraint on the 2nd derivative of \( c \)), and that the correction remains relatively small for \( r \in [0, 1] \).

We consider the following 2 candidates for the centreline correction function and proceed our analysis:

\[
c_1(r, f_f, n') = \sqrt{\frac{f_f n' A_{DM}}{2 \ln 10}} \left( 0.375 e^{\frac{0.44 - (r - 0.2)^2}{0.15}} \right), \tag{2.43}
\]

as suggested by Bogue and Metzner [27], and

\[
c_2(r, f_f, n') = \sqrt{\frac{f_f n' A_{DM}}{2 \ln 10}} r(1 - r)^2. \tag{2.44}
\]

which is a modified version of the one suggested by Reichardt [189]. Although the correction is assumed small relative to the dominant term in (2.41), it still contributes to the flow rate. This contribution must be subtracted from the constant \( B_{DM} \), to balance the flow rate of the corrected profile. The corrected dimensionless turbulent core velocity becomes \( W(r) = W_0(r) \):

\[
W_0(r) = \sqrt{\frac{f_f}{2}} [A_0 \ln(1 - r) + B_0 + B_{0,c}(r)] \tag{2.45}
\]
\[ A_0 = \frac{A_{DMn'}}{\ln 10} \quad (2.46) \]

\[ B_0 = -\frac{0.4\sqrt{2}}{(n')^{1/2}} + A_0 \left( \frac{1}{n'} \ln(f_f^{1/2} R e_{MR}) + \frac{3}{2} \right) \quad (2.47) \]

where \( B_{0,c}(r) \) is a zero-mean correction:

\[
B_{0,c}(r) = A_0 \left( 0.375 e^{0.04 \left( r - 0.2 \right)^2} - 0.1581529 \right), \text{ associated with (2.43),}
\]

\[
B_{0,c}(r) = A_0 \left( r(1-r)^2 - \frac{1}{15} \right), \text{ associated with (2.44).}
\]

We shall see in §2.4.2 that these small corrections may have a significant effect on the turbulent diffusivity. Below, unless otherwise stated, all the figures are based on the correction function of (2.44).

**Wall-layer correction**

The second correction concerns the pipe wall, where viscous effects come into play. The velocity (2.45) clearly does not satisfy the boundary conditions at \( r = 1 \).

Equally (2.14) is based on a velocity profile such as (2.45), that ignores the wall layer but conserves the flow rate. These approximations are reasonable in highly turbulent flows where we expect the wall layers to be very thin. However, in weakly turbulent flows we expect thicker wall layers to emerge, that may affect both the flow rate and the Taylor dispersion coefficient. To analyse these effects we follow the approach of Krantz and Wasan [136, 137].

We first introduce wall coordinates, \( \hat{y} = \hat{R} - \hat{r} \). Using the wall shear stress we define a wall shear rate scale \( \hat{\gamma}_w \) to satisfy the constitutive law, i.e.

\[
\hat{\gamma}_w = \left[ \frac{\hat{\tau}_w - \hat{\tau}_y}{\hat{k}} \right]^{1/n} = \left[ \frac{\hat{\tau}_w}{\hat{k}} \right]^{1/n} [1 - r_y]^{1/n}. \quad (2.48)
\]

The viscous wall layer length-scale \( \hat{\gamma}_w \) is then defined using \( \hat{\gamma}_w \) and the friction
velocity \( \hat{W}_s \), i.e. \( \hat{y}_s = \hat{W}_s / \hat{y} \). The wall layer length and velocity variables are:

\[
y^+ = \frac{\hat{y}}{\hat{y}_s}, \quad W^+(y^+) = \frac{\hat{W}(\hat{r})}{\hat{W}_s} = W(r) \frac{\hat{W}_0}{\hat{W}_s} = \sqrt{\frac{2}{f_f}} W(r). \tag{2.49}
\]

The pressure gradient is independent of \( \hat{r} \) and consequently we may integrate the axial momentum equation with respect to \( \hat{r} \) to give:

\[
\frac{\hat{R} - \hat{y} \frac{\partial \rho f}{\partial z}}{2} = -\rho \overline{u^+ w^+} + \overline{\tau}_{zr} - \left(1 - \frac{\hat{y}_s}{\hat{R}} y^+\right) = -\overline{u^+ w^+} (y^+) - (1 - r_Y) \left[ \left( \frac{\partial W^+}{\partial y^+} \right)^n + \frac{r_Y}{1 - r_Y} \right]. \tag{2.50}
\]

Note here that the Reynolds stress term, \( \overline{u^+ w^+} \), has been scaled with \( \hat{W}_s^2 \). Equation (2.50) is valid across the wall layer and into the turbulent core. Only in the wall layer are we justified in evaluating \( \overline{\tau}_{zr} \) in terms of the mean turbulent velocity using the leading order constitutive laws, i.e. because the wall layer is dominated by shear.

Within the wall layer we may deduce that \( \overline{u^+ w^+} \to 0 \) as \((y^+)^3\). We expand velocity profile and Reynolds stress in wall layer as polynomial series in \( y^+ \):

\[
W(y^+) = W_0 + W_1 y^+ + W_2 (y^+)^2 + W_3 (y^+)^3 + W_4 (y^+)^4 + W_5 (y^+)^5 \tag{2.52}
\]

\[
\overline{u^+ w^+} = \overline{u^+ w^+}_3 (y^+)^3 + \overline{u^+ w^+}_4 (y^+)^4 \tag{2.53}
\]

Upon substituting (2.52) and (2.53) in (2.51) we get:

\[
0 = 1 - \psi y^+ - \left( \overline{u^+ w^+}_3 (y^+)^3 + \overline{u^+ w^+}_4 (y^+)^4 + \ldots \right) + r_Y - (1 - r_Y) \times (W_1^+ + 2W_2^+ y^+ + 3W_3^+ (y^+)^2 + 4W_4^+ (y^+)^3 + 5W_5^+ (y^+)^4 + \ldots)^n \tag{2.54}
\]

where \( \psi = \hat{y}_s / \hat{R} \) gives the wall layer scaling, and the various coefficients are constants with subscript denoting the power of \( y^+ \) in the expansions. Equating at successive powers of \( y^+ \) we find:

\[
W_0^+ = 0, \quad W_1^+ = 1, \quad W_2^+ = -\frac{\psi}{2n(1 - r_Y)}, \quad W_3^+ = (1 - n) \frac{\psi^2}{6n^2(1 - r_Y)^2}. \tag{2.55}
\]

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Figure 2.5: The wall-layer scaling parameter $\psi$ for $n = 0.2, 0.4, ... 0.8, 1$: a) $He = 5$; b) $He = 100$; c) $He = 2000$. The red part of the curves shows the transitional regime.

These expressions match those in Krantz and Wasan [136] for $r_Y = 0$. The scaling parameter $\psi$ is defined\(^2\) by:

$$\psi = \frac{\gamma_s}{R} = \frac{2^{4/n-1/2}}{(1-r_Y)^{1/n}(3+1/n)} \left[ Re p f_f^{1-\frac{4}{n}} \right]^{-1/n}. \quad (2.56)$$

Figure 2.5 plots representative $\psi$ for wall shear stresses just above and below full turbulence. We see that $\psi < 10^{-2}$, and that $\psi$ decreases rapidly with wall

\(^2\)Note a factor of $n$ different in our $\psi$, compared to Krantz and Wasan [136].
shear stress, particularly for smaller \( n \). Note that \( H_{w,2} \gg 1 \), and therefore \( He = 5 \) (Figure 2.5a) is close to power law fluid behaviour: \( \psi \) is only sensitive to \( H_w \) for smaller \( n < 0.3 \). As the yield stress becomes significant (Figure 2.5b \& c), we see that for \( n \leq 0.5 \), \( \psi \) becomes extremely small. Again this is largely the effect of the yield stress on \( n' \) that we are seeing. Certainly, the very thin wall layers predicted at small \( n' \) are physically unrealistic. Values within the transitional regime that are plotted in Figure 2.5 indicate that choices of other \( Re_2 \) in place of (2.30) are still likely to result in very small \( \psi \) at modest \( n \) for any significant yield stress.

The wall layer ends at \( r = r_c = 1 - y_c = 1 - y_c^+ \hat{y}_s / \hat{R} = 1 - \psi y_c^+ \). This is to be found by matching with the core velocity. First however, on integrating the core velocity \( W_0(r) \) across the pipe we deduce that the wall layer perturbs the flow rate by a term of order \( \sqrt{f f_2 y_c^+ \psi y_c^+} \). This suggests that the core velocity (2.45) must itself be corrected to take account of the flux in the wall layer. More explicitly:

\[
W(r) = W_0(r) + \sqrt{f f_2} B_{w,c}.
\]

where we expect \( B_{w,c} \) to scale with the critical layer thickness, \( y_c = [\psi y_c^+] \). The term \( W_0(r) \) also satisfies:

\[
2 \int_0^1 rW_0(r) \, dr = 1.
\]

Subtracting \( W_0(r) \) from \( W(r) \) and integrating across the pipe, we find:

\[
r_c^2 B_{w,c} = 2 \int_{r_c}^1 r[A_0 \ln(1 - r) + B_0 + B_{0,c}(r)] \, dr
- 2 \psi \sum_{j=1}^5 \left( \frac{[\psi y_c^+]^{j+1}}{j+1} - \psi \frac{[\psi y_c^+]^{j+2}}{j+2} \right) W_j^+
= r_c^2 [B_{w,core} - B_{w,wall}]
\]

(2.57)

\[
B_{w,core} = \frac{A_0 [\psi y_c^+] \left( (2 - [\psi y_c^+]) \ln[\psi y_c^+] - 2 + \frac{[\psi y_c^+]}{2} \right)}{(1 - [\psi y_c^+])^2}
+ \frac{B_0 [\psi y_c^+] (2 - [\psi y_c^+]) + 2 B_{0,c} [\psi y_c^+]}{(1 - [\psi y_c^+])^2},
\]

(2.58)
\[
B_{w,\text{wall}} = \frac{2\psi}{(1 - \psi y_c^+)^2} \sum_{j=1}^{5} \left( \frac{|y_c^+|^{j+1}}{j+1} - \frac{|y_c^+|^{j+2}}{j+2} \right) W_j^+, \quad (2.59)
\]

\[
\bar{B}_{0,c} = \frac{1}{\psi y_c^+} \int_{r_c}^{1} r B_{0,c}(r) \, dr \quad (2.60)
\]

The correction term \(\bar{B}_{0,c}\) is generally small. The leading order terms come from \(A_0[\psi y_c^+] \ln[\psi y_c^+]\) and the term \([\psi y_c^+] \log(f_j^{1-\frac{d}{2}} Re_{MR})\), contained within \(B_0\). The corrected core velocity is:

\[
W(r) = \sqrt{\int f \left[ A_0 \ln(1-r) + B_0 + B_{0,c}(r) + B_{w,\text{core}} - B_{w,\text{wall}} \right]}, \quad (2.61)
\]

which we note is defined in terms of \(y_c^+\) and the coefficients of the wall velocity, \(W_j^+\), which are as yet unknown for \(j > 3\).

To find \(y_c^+\) we follow the procedure outlined by Krantz and Wasan [136, 137], using the above core velocity. We match \(W^+(y^+\pm)\) and the first 2 derivatives at the edge of the viscous sublayer: \(y^+ = y_c^+\):

\[
A_0 \ln[\psi y_c^+] + B_{0,c} |_{r=1-\psi y_c^+} + B_0 + B_{w,\text{core}} - B_{w,\text{wall}} = \sum_{j=1}^{5} |y_c^+|^{j} W_j^+ \quad (2.62)
\]

\[
\frac{A_0}{y_c^+} - \psi \frac{dB_{0,c}}{dr} |_{r=1-\psi y_c^+} = \sum_{j=1}^{5} j |y_c^+|^{j-1} W_j^+ \quad (2.63)
\]

\[
- \frac{A_0}{(y_c^+)^2} + \psi^2 \frac{d^2 B_{0,c}}{dr^2} |_{r=1-\psi y_c^+} = \sum_{j=1}^{5} j(j-1) |y_c^+|^{j-2} W_j^+ \quad (2.64)
\]

The last 2 of these equations are used to express the unknown \(W_4^+\) and \(W_5^+\) in terms of \(y_c^+\):

\[
W_4^+ = 1.25 A_0 (y_c^+)^{-4} - \psi(y_c^+)^{-3} \frac{dB_{0,c}}{dr} |_{r=1-\psi y_c^+} - 0.25 \psi^2(y_c^+)^{-2} \frac{d^2 B_{0,c}}{dr^2} |_{r=1-\psi y_c^+}
\]

\[
- (y_c^+)^{-3} \sum_{j=1}^{3} j |y_c^+|^{j-1} W_j^+ + 0.25 (y_c^+)^{-2} \sum_{j=1}^{3} j(j-1) |y_c^+|^{j-2} W_j^+, \quad (2.65)
\]
Figure 2.6: The critical layer thickness $y_c = \psi y_c^+$, for $n = 0.2, 0.4, \ldots, 0.8, 1$: a) $He = 5$; b) $He = 100$; c) $He = 2000$.

\[
W_5^+ = -0.8A_0(y_c^+)^{-5} + 0.6\psi(y_c^+)^{-4}\frac{dB_{0,c}}{dr} \bigg|_{r=1-\psi y_c^+} + 0.2\psi^2(y_c^+)^{-3}\frac{d^2B_{0,c}}{dr^2} \bigg|_{r=1-\psi y_c^+} \\
+ 0.6(y_c^+)^{-4} \sum_{j=1}^{3} j[y_c^+]{j-1} W_j^+ - 0.2(y_c^+)^{-3} \sum_{j=1}^{3} j(j-1)[y_c^+]{j-2} W_j^+.
\] 

(2.66)

These expressions are substituted into the first equation to give a single nonlinear equation for $y_c^+$, which may be solved iteratively. Figure 2.6 shows the results of this calculation, in terms of $y_c = \psi y_c^+$, for the same $(He, n, H_w)$ as in Figure 2.5.

Although for smaller $n < 0.3$ the critical layer is insignificant, in full turbulence
we see that the critical layer thickness can be 5-15% of the pipe radius. This radial thickness (at the wall) corresponds to a larger area fraction of the pipe and may significantly affect Taylor dispersion, being close to the wall where the velocity variation is maximal. Note that in the transitional regime, it is to be expected that the log-law profile loses validity progressively with decreasing flow rate; for Figure 2.6 we have simply extended the calculations into the transitional regime. Again it is observed that an increasing yield stress \( (He) \) reduces the effective power law index and hence reduces the critical layer thickness, so that for significant yield stresses, we see wall layers at 5-15% of the pipe radius only for \( n \geq 0.5 \). Note however, that \( \gamma^+ \) increases with \( He \), but this is masked by the decrease in \( \psi \).

Having found \( \gamma^+ \) we can evaluate \( W_4^+ \) & \( W_5^+ \) and hence the contributions to the Reynolds stresses in the wall layer, \( \overline{u'^w w'^w} \) & \( \overline{u'^w w'^w} \):

\[
\overline{u'^w w'^w}_{3} = -4 \left( n(1 - r_Y) W_4^+ + \psi^3 \frac{2n^2 - 3n + 1}{24n^2(1 - r_Y)^2} \right),
\]

\[
\overline{u'^w w'^w}_{4} = -5 \left( n(1 - r_Y) W_5^+ - 0.8(n - 1) \psi W_4^+ + \psi^4 \frac{2n^3 - 9n^2 + 10n - 3}{120n^3(1 - r_Y)^3} \right).
\]

These expressions are now used to define the turbulent diffusivity within the wall layer.

In Figure 2.7 we plot some example velocity profiles, lying just within the fully turbulent regime: \( H_w = 1.05H_{w,2} \) (i.e. with wall shear stress 5% larger than that required for full turbulence), for \( He = 5, \ 100, \ 2000, \) and \( n = 0.2, \ 0.4, \ ... \ 0.8, \ 1 \). The main differences within the wall layer profiles as \( He \) is increased are found for smaller \( n \), which is of course also where the layer thickness is insignificant. The turbulent core profiles appear to vary only modestly with \( He \), being mostly dependent on \( n \). This coincides with both computational and experimental observations, Rudman et al. [196], Güzel et al. [102].

There is little data regarding the velocity distribution in the wall-layer for shear-thinning fluids. However, we have compared the velocity profile with Newtonian fluid data from the DNS computations of [250]. Figure 2.8 shows this comparison. We can see that velocity profiles are matched very well, both close to the wall and in the core. They deviate at the edge of the wall layer, which is partly to be
Figure 2.7: Example velocity profiles in wall coordinate \((W^+(y^+/(\nu^+)))\) for \(H_w = 1.05H_{w,2}\) and \(n = 0.2, 0.4, \ldots, 0.8, 1\). Insets show velocity profiles in global coordinate. The black dots show the case of \(n = 0.2\). Velocity profiles within the wall layer are marked red. a) \(He = 5\); b) \(He = 100\); c) \(He = 2000\).
expected, as we have simply “patched” the wall layer to the core region here.

2.4.2 Diffusivity and dispersivity in turbulent pipe flows

We now follow a classical path towards estimating streamwise spreading of a passive tracer by the turbulent flow via diffusive and dispersive mechanisms, e.g. Taylor [230]. The net diffusivity is denoted \( \hat{D}_D = \hat{D}_m + \hat{D}_t \), representing molecular and turbulent terms respectively. The turbulent diffusivity \( \hat{D}_t \) is usually modelled using the Reynolds analogy for the turbulent transport of mass and momentum, and the axial momentum balance to evaluate the shear stress, i.e.

\[
\hat{D}_t = \frac{1}{S_{Ct}} \hat{D}_e = \frac{1}{S_{Ct} \hat{\rho}} \frac{\hat{W}}{\hat{R}} \left( \frac{\hat{r}}{\hat{R}} \hat{\tau}_w - \left| \hat{\tau}_r \right| \right). \tag{2.69}
\]

Here \( \hat{D}_e \) and \( S_{Ct} \) are the eddy diffusivity and the turbulent Schmidt number respectively, and on the right-hand side we have the total shear stress minus the mean viscous shear stress.

We work primarily with dimensionless diffusivities, scaled by \( \hat{W}_0 \hat{D} \). In the wall
layer we can evaluate (2.69) directly from our approximate solution:

\[
D_D(y^+) = D_m + \frac{\psi}{2S \epsilon_f} \left( \frac{f_f}{2} \right)^{1/2} \left\{ 1 - \frac{\psi y^+}{r} - \left( 1 - \frac{r_1}{r} \right) \frac{dW}{dy^+} \right\}^{n} \tag{2.70}
\]

In the turbulent core the velocity is given by (2.61). The velocity gradient and \(D_t\) are continuous at \(r = r_c\). However for \(r < r_c\), the averaged viscous stress \(\bar{\tau}_{cr}\) in (2.69), is not simply defined by inserting the strain rates of the averaged velocity into the constitutive law, (it is the average of the shear stress, not the shear stress of the average). In the core we expect that velocity fluctuations will be of size \(\hat{W}_* \sim \sqrt{\hat{f}_f \hat{W}_0}\), which would be the same size as the strain rate evaluated from the mean flow. Since the strain rate tensor is assumed locally isotropic, at most we get an order of magnitude for \(\bar{\tau}_{cr}\). It is unclear how to approximate this term.

Krantz and Wasan [136] argue that there is no theoretically justified form for the molecular diffusion of vorticity in the turbulent core for the power law fluids they consider, so they simply neglect \(\bar{\tau}_{cr}\). On the other hand this seems at odds with the significant effects of \(n\) on the mean velocity profile and of both \((n, He)\) in affecting transition. In Güzel et al. [102] it is shown that full turbulence waits for the average Reynolds stresses to exceed the yield stress, i.e. breaking the laminar plug. Thus, at least close to transition and for weak turbulence, there are suggestions that viscous stresses are still relevant within the core. For simplicity, we assume that \(\bar{\tau}_{cr}\) vanishes at the centreline (from symmetry) and approximate (2.69) by assuming that \((\hat{\tau}_u/\hat{R} - |\bar{\tau}_{cr}|)\) varies linearly with \(r\) across the core. We then use values as \(r \to r_c^-\) to match with the wall layer:

\[
D_D(r) = D_m + \frac{1}{2S \epsilon_f} \left( \frac{f_f}{2} \right)^{1/2} \frac{r}{r_c} \frac{G(r)}{G(r)} \times \\
\left[ r_c - r - \frac{8}{Re_p} \left[ \frac{n}{3n+1} \right]^n \right] \\
\left[ G(r_c) \right]^n \left( \frac{f_f}{2} \right)^{n/2 - 1}, \tag{2.71}
\]

\[
G(r) = \left| -\frac{A_0}{1-r} + \frac{d}{dr} B_{O,e}(r) \right| = \left| \frac{dW}{dr} \right| \sqrt{\frac{2}{f_f}}, \tag{2.72}
\]

For numerical robustness, in the case of very small \(\psi\), for \(r \to r_c^-\) we evaluate at
\( r = 0.99r_c \). Note also that \( G(r) \) vanishes at \( r = 0 \), due to the centreline correction. Thus, a Taylor series and l’Hôpital’s rule are used to resolve \( D_t(r) \) as \( r \to 0 \). In (2.71) \( D_m \) is the dimensionless molecular diffusivity, equal to the inverse of the Péclet number, \( Pe = \frac{W_0 D}{D_m} \gg 1 \) (for our flows of interest).

Examples of \( D_t(r) \) are illustrated in Figure 2.9 for the same parameters as the velocity profiles in Figure 2.7. We observe that \( D_t \) is reduced both by decreasing \( n \) and by increasing \( He \). The wall layer variation is characteristically cubic with \( y^+ \). The variation of \( D_t(r) \) is curious. Via the Reynolds analogy (2.69) and our assumed linear variation of stresses with \( r \), this variation is clearly related to dividing through by the velocity gradient. Since the velocity gradient vanishes as \( r \to 0 \), we see that the variation in \( D_t(r) \) close to the pipe centreline is directly related to the choice of centreline correction function. The first derivative of the correction function is fixed and the size of correction is small. Thus, it is essentially the second derivative of the correction function that is important!

We remark that the existence of a local maximum in the radial profile of turbulent diffusivity somewhere away from the centreline is found in the literature; see e.g. Seagrave [203], Koo [133], Travis et al. [235]. It seems the correction function (2.44) captures this feature qualitatively. The more exotic variations in \( D_t(r) \) that correspond to the correction function (2.43) of Bogue and Metzner [27] are not supported by any computational or experimental data that we have found. Of course, this is not conclusive, but favours (2.44).

It may be of concern that the correction function can influence \( D_t(r) \) to this extent. A more rudimentary analyses would simply approximate \( D_t(r) \) as constant, perhaps evaluated from the wall layer, (hence vanishing for small \( n \) and large \( He \) as \( \psi \to 0 \)). Alternatively, if we ignore the centreline and wall layer corrections, just using the Reynolds analogy and the logarithmic velocity profile leads to a variation: \( D_t(r) \propto r(1 - r) \). This clearly does not represent either the expected diffusivity behaviour at the centreline, nor can it resolve any effects of weak turbulence on wall layers as the cubic variation is gone. Practically speaking, although the correction function can influence \( D_t(r) \), the effects are primarily in the central part of the pipe, which does not contribute greatly to either the averaged turbulent diffusivity nor to the Taylor dispersivity.

In computing mean dispersive and diffusive transport along the pipe, 3 compo-
Figure 2.9: Example profiles of $D_t(r)$ for $H_w = 1.05H_{w,2}$ and $n = 0.2, 0.4, ..., 0.8, 1$, with $Sc_t = 1$: a) $He = 5$; b) $He = 100$; c) $He = 2000$. Solid and broken lines are associated with centreline corrections (2.44) and (2.43), respectively. Profiles within the critical wall layer are marked red.
ments contribute. The first component is the molecular diffusivity \((D_m)\), which is typically much smaller than the second component, the radially averaged turbulent diffusivity \(\bar{D}_t\):

\[
\bar{D}_t = 2 \int_0^1 r D_t(r) \, dr = 2 \int_0^{r_c} r D_t(r) \, dr + 2 \psi \int_0^{y^+} (1 - \psi y^+) D_t(y^+) \, dy^+.
\]

\[
= \frac{1}{Sc_T} \left( \frac{f_f}{2} \right)^{1/2} \left[ \frac{r_c - r_Y - 8}{r_c} \left[ \frac{n}{\sigma_{w+1}} \frac{\hat{D}_T(r)}{G(r)} \right]^{n/2} \right] \int_0^{r_c} \frac{r^2}{G(r)} \, dr
\]

\[
+ \frac{\psi^2}{Sc_T} \left( \frac{f_f}{2} \right)^{1/2} \int_0^{y^+} \left( 1 - \psi y^+ \right) \left[ \frac{1 - \psi y^+ - r_Y - (1 - r_Y) \left| \frac{dW}{dy^+} \right|^{n} \right] \, dy^+
\]

Both integrals above must be evaluated numerically. Although potentially time consuming, the integrands have been normalised and are well-behaved. Thus, a relatively coarse mesh can be used for the integration with a high order approximation, e.g. Simpson’s rule.

The calculation of \(\bar{D}_t\) is sensitive to the approximation of \(\bar{\epsilon}\) in (2.69) and also to the velocity gradient, hence correction function. Examples of the variations in \(\bar{D}_t\) are shown in Figure 2.10. We see that \(\bar{D}_t\) is not particularly sensitive to either wall shear stress nor \(He\) (yield stress) over these ranges. The main variation is with \(n\).

The third (and usually dominant) component is the Taylor dispersion coefficient, which is defined as:

\[
D_T = \frac{\hat{D}_D}{\hat{W}_0 \hat{D}} = \frac{1}{2} \int_0^1 \frac{\left( \int_0^r \left[ W(\tilde{r}) - 1 \right] \tilde{r} \, d\tilde{r} \right)^2}{r D_D(r)} \, dr = \bar{I}_c(r_c) + \psi^3 I^+(y^+) \tag{2.74}
\]

where

\[
\bar{I}_c(r_c) = \frac{1}{2} \int_0^{r_c} \frac{\left( \int_0^r \left[ W(\tilde{r}) - 1 \right] \tilde{r} \, d\tilde{r} \right)^2}{r D_D(r)} \, dr
\]
and

$$\Gamma^+(y^+_c) = \frac{1}{2} \int_0^{y^+_c} \left( \int_0^{y^+_c} \frac{\sqrt{0.5 f_j W^+(s) - 1} (1 - \psi s) ds}{1 - \psi y^+} D_D(y^+) \right)^2 dy^+$$

Both these terms require numerical integration. However, the integral terms in the numerator can be evaluated explicitly, which accelerates computation, i.e. only one numerical integration is needed.

Figure 2.10: Examples of $\bar{D}_t$ for $n = 0.2, 0.4, ... 0.8, 1$, with $Sc_t = 1$: a) $He = 5$; b) $He = 100$; c) $He = 2000$. Broken and solid lines are associated with the centreline correction functions (2.43) and (2.44), respectively.
For the core integral \( I_c(r_c) \), the velocity is given by (2.61), so that we see:

\[
\int_{r_0}^{r_0} \left[ W(r) - 1 \right] d\tilde{r} = \sqrt{\frac{f_f}{2}} \int_{r_0}^{r_0} \left[ A_0 \ln(1 - \tilde{r}) + B_0 + B_{w,core} - B_{w,wall} \right] d\tilde{r}
\]

\[
= \left( \sqrt{\frac{f_f}{2}} \left[ A_0 + B_{w,core} - B_{w,wall} \right] - 1 \right) \frac{r^2}{2} + \frac{\sqrt{f_f}}{2} \int_{r_0}^{r_0} \left[ A_0 \ln(1 - \tilde{r}) + B_{0,c} \right] d\tilde{r}
\]

(2.75)

Integrating the first term on the right hand side, we have:

\[
\int_{0}^{r} A_0 \tilde{r} \ln(1 - \tilde{r}) \, d\tilde{r} = A_0 \frac{1}{4} \left( 2r^2 \ln(1 - r) - (r + 2)r - 2 \ln(1 - r) \right)
\]

(2.76)

For the third term on the right hand side, we have two expressions:

\[
\int_{0}^{r} \tilde{r} B_{0,c}(\tilde{r}) \, d\tilde{r} = \int_{0}^{r} A_0 \tilde{r} \left( 0.375e^{0.04 - (\tilde{r} - 0.2)^2} - 0.1581529 \right) \, d\tilde{r}
\]

\[
= \frac{3A_0}{1600} \left( 2\sqrt{15} \pi e^{4/15} \left[ \text{erf} \left( \frac{10r - 2}{\sqrt{15}} \right) - \text{erf} \left( \frac{2}{\sqrt{15}} \right) \right] \right)
\]

\[
- \frac{45A_0}{1600} \left( e^{(8\tilde{r} - 20r^2)/3} - 1 \right) + 0.079076A_0 r^2
\]

which is associated to the centreline correction function (2.44) and

\[
\int_{0}^{r} \tilde{r} B_{0,c}(\tilde{r}) \, d\tilde{r} = \int_{0}^{r} A_0 \left[ \tilde{r}^2 (1 - \tilde{r})^2 - \frac{1}{15} \tilde{r} \right] \, d\tilde{r} = A_0 \left( \frac{r^5}{5} - \frac{r^4}{2} + \frac{r^3}{3} - \frac{r^2}{30} \right)
\]

which is associate to the centreline correction function (2.44).

In the wall layer integral \( I^+(y_c^+) \):

\[
\int_{0}^{y_+} \left[ \sqrt{0.5f_f W^+(s)} - 1 \right] (1 - \psi s) \, dx = \quad (2.77)
\]

\[
\sqrt{\frac{f_f}{2}} \sum_{j=1}^{5} W_j^{+} (y^+)^{j+1} - y^+ + \psi \left( \frac{(y^+)^2}{2} - \sqrt{\frac{f_f}{2}} \sum_{j=1}^{5} W_j^{+} (y^+)^{j+2} \right)
\]

Note that the molecular diffusivity contributes very close to the wall in removing a logarithmic singularity from the calculation of \( I^+(y_c^+) \), i.e. the eddy viscosity terms in the denominator vanish cubically and the numerator vanishes quadratically as
y^+ \to 0.

Examples of the variations in $D_T$ are shown in Figure 2.11a-c. The main observations that we see are: (i) $D_T$ decreases significantly with $n$; (ii) $D_T$ decreases with the yield stress $He$; (iii) for larger $n$ we see a very significant rise in $D_T$ as the wall shear stress approaches its transitional value. The overall trend of increasing $D_T$ with $n$ and the size of $D_T$ are similar to those of Krantz and Wasan [136] for power law fluids. In computing $D_T$ we divide by the diffusivity $[D_i(r) + D_m]$ in the integrands. Although we have seen significant differences in the turbulent diffusivities $D_i(r)$ within the core, according to the choice of centreline correction function, the numerator in the core involves integrals of $[W(r) - 1]$, which is of order $\sqrt{f}$, and these terms scale with $r^4$. Thus, the choice between corrections functions such as (2.43) & (2.44) is not critical insofar as calculating $D_T$ is concerned.

In the original work on dispersion, Taylor [230] used a coarse approximation of the (universal) velocity distribution taken from available measurements and performed a numerical integration. This gave $\hat{D}_T = 10.06 \hat{W}_s \hat{D}/2$ and $\hat{D}_r = 0.052 \hat{W}_s \hat{D}/2$, giving $D_T/D_i \approx 193$. A comparison of Taylor’s coefficients with ours for $n = 1$ and $He = 0$ (Newtonian fluid) is shown in Figure 2.11d for increasing Reynolds number in the weak turbulent range. Our computed $D_T$ is significantly larger than that of Taylor in the weak turbulent range, but converges as the wall layers thin. The main reason for the difference is (of course) including our analysis of the wall layers, where we expect to have a significant contribution to $D_T$ for weakly turbulent flows.

It is interesting to understand where the main contributions to $\overline{D}_i$ and $D_T$ come from. This is explored in Figure 2.12 for $He = 10$ (although analogous effects are found at other $He$). Firstly, Figure 2.12a shows that the contribution of the core region to $\overline{D}_i$ is always dominant; typically at least 90%. This explains the large differences in $\overline{D}_i$ according to the corrections functions. On the other hand, Figure 2.12b shows the wall-layer contribution to computing $D_T$. The wall layers correspond to regions where $[W(r) - 1]$ is of order 1 and where the diffusivity is small. In the core, $[W(r) - 1]$ is of order $\sqrt{f}$ and the diffusivity is of size $\overline{D}_i$. Thus we see an interesting transition in Figure 2.12b. Where the wall layer is relatively thick, it gives the dominant contribution to $D_T$. As $n$ decreases sufficiently, or simply as we move further into the fully turbulent regime, the wall layer scaling
Figure 2.11: Examples of $D_T$ for $n = 0.2, 0.4, ..., 0.8, 1$, with $Sc_T = 1 \& D_m = 10^{-6}$; a) $He = 5$; b) $He = 100$; c) $He = 2000$. Broken and solid lines are associated with centreline correction functions (2.43) and (2.44), respectively. d) Newtonian fluid ($He = 0, \; n = 1$). Broken and solid lines: our results for centreline correction functions (2.44) and (2.43); solid thick line: Taylor’s prediction [230]; black point-line; numerical results of Ekambara and Joshi [67]; diamonds: experimental results of Flint and Eisenklam [75]; filled squares: experimental results of Hart et al. [113]; hollow squares: experimental results of Keyes [131]; circles: experimental results of Fowler and Brown [79]. All data are taken from Hart et al. [113].
parameter $\psi$ becomes extremely small and the wall layer contribution reduces significantly due to the small thickness of the wall layer. This effect occurs at more moderate $n$ for larger yield stresses, $He$. We see a corresponding effect on $D_T$, which decreases significantly as $n$ decreases at any fixed $He$. As $H_w$ increases the wall layer effects diminish, but relatively slowly for $n \approx 1$.

Figure 2.11 and the comparison with Taylor [230] and other data in Figure 2.11d indicate clearly the importance of modelling the wall layers in estimating streamwise dispersion in weakly turbulent flows. Although the effects are significant we must regard our analysis as approximate. Quantitative values rely on the mean velocity profile. Earlier authors, e.g. Taylor [230], Tichacek et al. [234], have used empirical values of the mean turbulent velocity profile. These profiles are available for Newtonian fluid flows, but are lacking for non-Newtonian fluids (for which one must consider at least some range of dimensionless $(n, He/H_w)$). Partly this is because experimental studies use real fluids for which rheological models like the Herschel-Bulkley model have limitations at high shear rates. Also experimental studies with such fluids in order to accurately measure pointwise velocity values are time intensive and often involve a degree of rheological degradation (and/or other rheological effects that deviate from simple model descriptions). Thus, we
are pushed towards expressions such as the log-law, which do arise naturally from a dimensional analysis, but nevertheless need correcting. The matching procedure used to define the wall layer thickness (and hence the velocity coefficients) is dependent on the core velocity profile.

2.4.3 Velocity profile in turbulent channel flows

Following the same procedure explained in §2.4.1, we correct the velocity profile near the centreline and the wall. To do so, we introduce the wall layer coordinate $\hat{x} = \hat{H} - \hat{y}$, $x^+ = \hat{x}/\hat{x}_*$ and $W^+(x^+) = \hat{W}(\hat{y})/\hat{W}_* = \sqrt{2/\bar{f}} W(y)$, where $\hat{W}_*$ is the friction velocity and

$$\hat{x}_* = \frac{\hat{W}_*}{\bar{y}}$$

$$\hat{y}_* = \left[ \frac{\bar{y}}{\bar{K}} \right]^{1/2} (1 - y)^{1/2}$$

We eventually find the core velocity profile:

$$W(y) = \sqrt{\frac{\bar{f}}{2}} [A_0 \ln(1 - y) + B_0 + B_{0,c}(y) + B_{w,core} - B_{w,wall}], \quad (2.78)$$

where

$$A_0 = \frac{A_{DM}}{\ln 10}, \quad A_{DM} = \frac{4.0\sqrt{2}}{(n')^{0.75}} \quad (2.79a)$$

$$B_0 = A_0 \left( \frac{1}{n'} \ln \left( Re_{MR} f^{1-\frac{1}{2}} \right) + 1 \right) - 0.395 \sqrt{2} \quad (2.79b)$$

$$B_{0,c}(y') = A_0 \left( (1 - y'y^2) - \frac{1}{12} \right) \quad (2.79c)$$

$$B_{0,c} = \frac{1}{1 - y_c} \int_{y_c}^1 B_{0,c} \, dy \quad (2.79d)$$

$$B_{w,core} = \frac{\psi x_c^+}{1 - \psi x_c^+} \left[ A_0 \left( \ln(\psi x_c^+) - 1 \right) + B_0 + \bar{B}_{0,c} \right] \quad (2.79e)$$

$$B_{w,wall} = \frac{\psi}{1 - [\psi x_c^+]} \sum_{j=1}^5 \frac{[x_c^+]^{j+1}}{j+1} W_j^+ \quad (2.79f)$$

60
Figure 2.13: Critical wall layer thickness $x_c = \psi x_c^+$ for $n = 0.2, 0.4, \ldots, 0.8, 1$: a) $He = 10$; b) $He = 400$. Inset figures show the wall-layer scaling parameter $\psi$. The black dot is associated with $n = 0.2$.

Note that the centreline correction function we used here is of form (2.44). The wall scaling parameter in the channel flow is:

$$\psi = \frac{6^{\frac{1}{n}}.2^{\frac{3}{2}-\frac{1}{n}}}{2 + \frac{1}{n}} \left( (1 - yr)Re_p f^{1-\frac{4}{n}} \right)^{-\frac{1}{n}}$$ (2.80)

We are left to determine the position of the wall layer thickness ($x_c^+$) using an analogous matching procedure to that of (2.62-2.64). Examples of the wall layer thickness and scaling parameter $\psi$ are shown below in Figure 2.13 at two values of $He$. We again observe significant wall layers for weakly turbulent flows provided $n'$ is not too small. The scaling parameter decreases rapidly at small $n$ or as $He$ is increased significantly.

Example velocity profiles are shown in Figure 2.14 for wall shear stresses just above full turbulence ($H_w = 1.05H_{w,2}$), for the same $He$ and $n$ as in Figure 2.13. The trends observed are quite similar to those in the pipe flow.
2.4.4 Diffusivity and dispersivity in turbulent channel flows

A similar approach as in §2.4 is taken here. Using the Reynolds analogy, the turbulent diffusivity $\hat{D}_t$ can be written as:

$$\hat{D}_t = \frac{1}{Sc_t} \hat{D}_e = \frac{1}{Sc_t \hat{\rho}} \left[ \frac{\hat{\gamma}}{\hat{H}} - \frac{\hat{\gamma}_y}{|\hat{\gamma}_y|} \right].$$

(2.81)

We scale $\hat{D}_t$ with $\hat{T}_0 \hat{H}$, to give dimensionless expressions in the core and wall layers, as follows:

$$D_t(x^+) = \frac{\psi}{2Sc_t} \left( \frac{f_f}{2} \right)^{1/2} \left\{ \frac{1 - \psi x^+ - yy - (1 - yy) \left| \frac{dw^+}{dx^+} \right|^n}{\frac{dw^+}{dx^+}} \right\},$$

(2.82)

and

$$D_t(y) = \frac{1}{2Sc_t} \left( \frac{f_f}{2} \right)^{1/2} \frac{y}{y_c} \frac{1}{G(y)} \left[ y_c - yy - \frac{6}{Re_p} \left[ \frac{n}{2n+1} \right]^n \left[ G(y_c) \right]^n \left( \frac{f_f}{2} \right)^{n/2-1} \right],$$

(2.83)
where

\[ G(y) = \left| \frac{A_0}{1-y} + \frac{d}{dy} B_{0,c}(y) \right| = \left| \frac{dW}{dy} \right| \sqrt{\frac{2}{f_f}}. \]

Integrating \( D_t(y) \) across the channel gives the average turbulent diffusivity (\( \bar{D}_t \)), exploiting symmetry:

\[
\bar{D}_t = \int_0^1 D_t(y) \, dy = \int_0^{y_c} D_t(y) \, dy + \psi \int_0^{x_c} D_t(x^+) \, dx^+.
\]

\[
= \frac{1}{Sc_t} \left( \frac{f_f}{2} \right)^{1/2} \int_0^{y_c} \left[ \frac{y_c - y Y - 6}{\kappa'_{\epsilon}} \left[ \left( \frac{G(y_c)}{n} \right)^n \left( \frac{f_f}{f_f^2} \right)^{n/2-1} \right] \right] \, dy
\]

\[
+ \frac{\psi^2}{Sc_t} \left( \frac{f_f}{2} \right)^{1/2} \int_0^{x_c} \left[ 1 - \frac{\psi x^+ - y Y - (1 - y Y) \left| \frac{dW}{dx^+} \right|}{\psi x^+} \right] \, dx^+
\]

(2.84)

The Taylor dispersion coefficient is also evaluated straightforwardly from the velocity profile and turbulent diffusivity, as below.

\[
D_T = \frac{\hat{D}_T}{2W_0 H} = \frac{1}{2} \int_0^1 \left( \int_0^{y_c} \frac{[W(\tilde{y}) - 1]}{D_D(y)} \, d\tilde{y} \right)^2 \, dy = \bar{I}_c(y_c) + \psi^3 I^+(x_c^+) \quad (2.85)
\]

where

\[
\bar{I}_c(y_c) = \frac{1}{2} \int_0^{y_c} \left( \int_0^{y_c} \frac{[W(\tilde{y}) - 1]}{D_t(y)} \, d\tilde{y} \right)^2 \, dy
\]

and

\[
I^+(x_c^+) = \frac{1}{2} \int_0^{x_c} \left( \int_0^{x_c} \left[ \sqrt{0.5 f_f W^+(s) - 1} \right] ds \right)^2 \, dx^+.
\]

Again the calculations involved in \( \bar{D}_t \) and \( D_T \) involve a single numerical integration in core and wall layer.

The Taylor dispersion term is again dominant in diffusing/dispersing mass axially. Computed examples are shown in Figure 2.15. Again we see the main sensitivity is to \( n \) although \( He \) does decrease the dispersivity slightly (acting through
Figure 2.15: Examples of $D_T$ for channel flow, with $n = 0.2, 0.4, ... 0.8, 1$, $Sc_t = 1$ & $D_m = 10^{-6}$: a) $He = 10$; b) $He = 400$

$n'$. In the weakly turbulent regime we again see a significant increase in $D_T$, associated with the wall layers. This is an $O(1)$ increase in $D_T$, but is a smaller effect than in the pipe flows. The reasons for this difference are largely geometric. First, computed $x_c$ are slightly smaller than $y_c$ (pipe). Secondly, thick wall layers in the pipe represent a larger area fraction than in the channel flow. We can also compare the expressions for the core contributions to the $D_T$ integrals, close to the centrelines: in pipe flow we effectively integrate $r^3 f_f / D_D$ and in channel flow we integrate $y^2 f_f / D_D$.

### 2.5 Discussion and Summary

The aim of this chapter was two-fold: i) To analyze turbulent flows of shear-thinning yield stress fluids in both pipe and channel geometries and lay down a consistent procedure for hydraulic calculation of Herschel-Bulkley fluids. This means finding the relationship between the mean velocity and the wall shear stress. ii) To extend the classical turbulent Taylor dispersion analysis [230] to shear-thinning yield stress fluids and find estimates of turbulent diffusivity and dispersivity.

The chapter can be summarized into the following concluding remarks:

- We have explored the effects of the yield stress on turbulent transport of
mass along pipes and plane channels. The yield stress produces competing effects in the wall layers. The critical layer thickness in wall coordinates is increased, but the scaling parameter $\psi$ decreases rapidly with $n'$. Thus, we find that for very large yield stresses ($He$) and small $n$ the wall layer thickness is vanishingly small, indeed unrealistically so. This is however dictated by the friction factor closure and delayed transition.

- For larger $n \leq 1$ and a wide range of practical $He$ the wall layer thickness can be over 10% of the pipe radius. Following the procedure of Wasan & Krantz we have developed approximations to the velocity and turbulent diffusivity in the wall region, and for these parameter ranges we show a significant increase in the (dominant) Taylor dispersivity in weakly turbulent flows. In pipe flows this effect can be an $\mathcal{O}(10)$ increase, compared to values of highly turbulent regimes where the wall layers thin. In plane channel flows it is a more modest $\mathcal{O}(1)$ increase.

- This demonstrates that in weakly turbulent regimes (as found in the primary cementing applications of interest), it is necessary to include the effects of the wall layers. Our predictions, when applied to Newtonian fluids in weakly turbulent regimes, bound above the available data and show similar variation with $Re$ (see Figure 2.11d). The classical expression of Taylor under-predicts the same data.

- It is shown that for weakly turbulent flows it is necessary to include an analysis of wall layers in studying dispersion. In particular, in pipe flows, we observe an $\mathcal{O}(10)$ increase in Taylor dispersion coefficients, compared to highly turbulent values. This arises from a combination of large velocity and small turbulent dispersivity, acting over a wall layer that can represent $\gtrsim 20\%$ of the pipe area. In channel flows the wall layer effect is more modest, but still represents an $\mathcal{O}(1)$ increase in Taylor dispersion coefficient. The preceding effects are negated for small power law index, due to rapid reduction of the wall layer, and are observed to reduce modestly as the the yield stress increases.
Chapter 3

Primary cementing modelling

This chapter presents a detailed derivation of a practical two-dimensional model for displacement flows of viscoplastic fluids in an annular geometry. The two-dimensional formulation presented here models laminar, turbulent and mixed regime annular flows. Such mixed regimes, including those in which different regimes exist in the same annular cross-section are relatively common in primary cementing. A version of this chapter is published in Maleki and Frigaard [147].

3.1 Dimensional governing equations

A cylindrical coordinate system $(\hat{r}, \theta, \hat{\xi})$ is used to describe the well geometry: $\hat{\xi}$ measures distance along the central axis of the casing $\hat{r} = 0$ which is assumed to be inclined to the vertical with angle $\beta(\hat{\xi})$. The local cross-section of the well, outside the casing, is assumed to be that of an eccentric annulus, with inner radius $\hat{r}_i(\hat{\xi})$, equal to the outer radius of the casing and outer radius $\hat{r}_o(\hat{\xi})$ equal to the inner radius of the hole (or previous casing). At each depth $\hat{\xi}$, the mean radius $\hat{r}_a(\hat{\xi})$ and the mean half-gap width $\hat{d}(\hat{\xi})$ are defined by:

$$\hat{r}_a(\hat{\xi}) \equiv \frac{1}{2}[\hat{r}_o(\hat{\xi}) + \hat{r}_i(\hat{\xi})], \quad \hat{d}(\hat{\xi}) \equiv \frac{1}{2} [\hat{r}_o(\hat{\xi}) - \hat{r}_i(\hat{\xi})]. \quad (3.1)$$

As well as inner and outer radii, the distance between the centres of the two cylinders is given by $\hat{e}(\hat{\xi})$, see Figure 3.1. It is assumed that $\hat{e}(\hat{\xi}) < 2\hat{d}(\hat{\xi})$ (the cylinders...
do not touch), and the variations in the geometry along $\xi$ are slow. For simplicity, it is also assumed that the narrow side of the annulus will be found on the lower side of the well and that the casing remains stationary. Both these assumptions can be relaxed with care, e.g. see Carrasco-Teja and Frigaard [36], Tardy and Bittleston [226].

The annular displacement is modeled as a concentration dependent multi-fluid flow that is fully turbulent and incompressible. The fluid constituents contribute to the mixture density $\hat{\rho}$ and to the rheological properties of the mixture. The latter are used in closure expressions that define the deviatoric stress tensor $\hat{\tau}_{ij}$. In primary cementing a sequence of $K$ fluids is pumped around the flow path, from the bottom of the annulus to the top. A typical sequence would be: mud, wash, spacer, lead slurry, tail slurry, mud. When circulating, drilling muds, spacer fluids and cement slurries are predominantly shear thinning, nonlinearly viscous and inelastic, often also with a significant yield stress. Washes are Newtonian fluids. In general each constituent fluid can be effectively modeled as a Herschel-Bulkley fluid. We shall address evolution of the mixture concentrations at length in §3.4, but for now we assume that all fluid properties are approximated effectively by closures that depend on the local mean fluid concentrations: both ensemble averaged and averaged across the annular gap.
We adopt the usual Reynolds decomposition for turbulent flows, into mean and fluctuating parts. Because the flows will be time-varying as the displacement proceeds, the mean part (denoted with an overbar below) is interpreted as an ensemble average. The velocity and pressure are \( \hat{u} = (\hat{u}, \hat{v}, \hat{w}) \) and \( \hat{p} \), respectively. In the local coordinate system the Reynolds-decomposed Navier-Stokes system is:

\[
\begin{align*}
\frac{\partial}{\partial t} [\hat{\rho} \hat{\bar{u}}] + \nabla \cdot [\hat{\rho} \hat{\bar{u}}] &= \frac{1}{r} \frac{\partial}{\partial r} [r \hat{\bar{\tau}}_{rr}] + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\bar{z}}_{\theta r} + \frac{\partial}{\partial \xi} \hat{\bar{z}}_r - \frac{\hat{\bar{t}}_{\theta r}}{r} - \frac{\partial \hat{\bar{p}}}{\partial r} + \hat{\rho} \hat{\bar{g}}_r + \\
&+ \frac{1}{r} \frac{\partial}{\partial r} [r \hat{\bar{\tau}}_{\theta \theta}] + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\bar{z}}_{\theta \theta} + \frac{\partial}{\partial \xi} \hat{\bar{z}}_{\theta} - \frac{\hat{\bar{t}}_{\theta \theta}}{r} , \quad (3.2) \\
\frac{\partial}{\partial t} [\hat{\rho} \hat{\bar{v}}] + \nabla \cdot [\hat{\rho} \hat{\bar{v}}] &= \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 \hat{\bar{z}}_{\theta r}] + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\bar{z}}_{\theta \theta} + \frac{\partial}{\partial \xi} \hat{\bar{z}}_{\theta} - \frac{1}{r} \frac{\partial \hat{\bar{p}}}{\partial \theta} + \hat{\rho} \hat{\bar{g}}_{\theta} + \\
&+ \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 \hat{\bar{z}}_{\theta \theta}] + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\bar{z}}_{\theta \theta} + \frac{\partial}{\partial \xi} \hat{\bar{z}}_{\theta} \quad (3.3) \\
\frac{\partial}{\partial t} [\hat{\rho} \hat{\bar{w}}] + \nabla \cdot [\hat{\rho} \hat{\bar{w}}] &= \frac{1}{r} \frac{\partial}{\partial r} [r \hat{\bar{\tau}}_{\xi r}] + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\bar{z}}_{\xi \theta} + \frac{\partial}{\partial \xi} \hat{\bar{z}}_{\xi} - \frac{\hat{\bar{t}}_{\xi r}}{r} - \frac{\partial \hat{\bar{p}}}{\partial \xi} + \hat{\rho} \hat{\bar{g}}_{\xi} + \\
&+ \frac{1}{r} \frac{\partial}{\partial r} [r \hat{\bar{\tau}}_{\xi \xi}] + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\bar{z}}_{\xi \xi} + \frac{\partial}{\partial \xi} \hat{\bar{z}}_{\xi} \quad (3.4) \\
0 &= \frac{1}{r} \frac{\partial}{\partial r} [r \hat{\bar{\tau}}_{\xi \xi}] + \frac{1}{r} \frac{\partial \hat{\bar{v}}}{\partial \theta} + \frac{\partial \hat{\bar{w}}}{\partial \xi} . \quad (3.5)
\end{align*}
\]

The components \( \hat{\bar{\tau}}_{ij} \) are those that come from ensemble averages of the viscous stress tensor (which itself is nonlinear). The components \( \hat{\bar{\tau}}_{ij} \) are the turbulent (Reynolds) stresses, resulting from the fluctuating components of the velocity field \( \hat{u}' \), defined as:

\[
\hat{\bar{\tau}}_{ij} = -\hat{\rho} \hat{u}' \hat{u}'_j. \quad (3.6)
\]

The gravitational acceleration, \( \hat{g} = (\hat{g}_r, \hat{g}_\theta, \hat{g}_\xi) \), is given by

\[
\hat{g}_r = -\hat{g} \sin \beta (\hat{\xi}) \cos \theta, \quad \hat{g}_\theta = \hat{g} \sin \beta (\hat{\xi}) \sin \theta, \quad \hat{g}_\xi = -\hat{g} \cos \beta (\hat{\xi}), \quad (3.7)
\]

where \( \hat{g} = 9.81 \text{ m/s}^2 \).
3.2 Scaling and simplification

We wish to reduce our model to something more tractable than (3.2)-(3.5), by exploiting the aspect ratio of the annulus. The annulus geometry is typically long and narrow, with the typical gap half-width (\(\sim 1\)cm) being much smaller than a typical azimuthal distance (\(\sim 30\)cm), which in turn is much smaller than a typical length of the annulus (\(\sim 500\)m).

Following Bittleston et al. [24], let the mean radius (\(\hat{r}_{a,0}\)), the local and global aspect ratios (\(\delta(\hat{\xi})\) and \(\delta_0\) respectively) be defined by:

\[
\hat{r}_{a,0} = \frac{1}{Z} \int_{\hat{\xi}_{bh}}^{\hat{\xi}_{tz}} \hat{r}_{a}(\hat{\xi}) \, d\hat{\xi}, \quad \delta(\hat{\xi}) = \frac{\hat{d}(\hat{\xi})}{\hat{r}_{a}(\hat{\xi})}, \quad \delta_0 = \frac{1}{Z} \int_{\hat{\xi}_{bh}}^{\hat{\xi}_{tz}} \delta(\hat{\xi}) \, d\hat{\xi},
\]

where \(Z\) is the length of the zone of the well to be cemented, extending upwards from bottom hole, \(\hat{\xi}_{bh}\), to the top of the zone, \(\hat{\xi}_{tz}\). Scaled axial and azimuthal coordinates \(\xi\) and \(\phi\) are then:

\[
\xi = \frac{\hat{\xi} - \hat{\xi}_{bh}}{\pi \hat{r}_{a,0}}, \quad \phi = \frac{\theta}{\pi}.
\]

In each cross-section, we define the local average radius, \(r = r_a(\xi)\), and local annulus eccentricity, \(e(\xi)\), by:

\[
r_a(\xi) = \frac{\hat{r}_a(\hat{\xi})}{\hat{r}_{a,0}}, \quad e(\xi) = \frac{\hat{e}(\hat{\xi})}{2\hat{d}(\hat{\xi})}.
\]

The centreline of the annular gap is at \(\hat{r} = \hat{r}_{a,0} r_a(\xi) r_c(\phi, \xi)\). The radial coordinate is scaled relative to the distance from the centreline of the annulus, as follows:

\[
y = \frac{\hat{r} - \hat{r}_{a,0} r_a(\xi) r_c(\phi, \xi)}{\hat{r}_{a,0} \delta_0},
\]

i.e. \(y\) is a local annular gap coordinate.

We assume a narrow annulus approximation: \(\delta(\xi) \sim \delta_0 \ll \pi\); noting that \(\delta_0/\pi\), denotes the ratio of radial (gap) length-scales to azimuthal length-scale. To leading order in \(\delta_0/\pi\), we have \(r_c(\phi, \xi) \sim 1\) and find that the inner and outer walls are at
$y = \mp H(\phi, \xi)$, where:

$$H(\phi, \xi) = \frac{\delta(\xi) r_a(\xi) [1 + e(\xi) \cos \pi \phi]}{\delta_0}.$$  \hspace{1cm} (3.12)

More complex geometries are readily accommodated by specifying any $H(\phi, \xi)$ of $O(1)$ that varies slowly with $\xi$, e.g. $H(\phi - \phi_0(\xi), \xi)$ with $H$ as above, retains the eccentric annular shape but shifts the wide-side of the annulus to $\phi = \phi_0(\xi)$. Helically varying well eccentricity, elliptic cross sections and irregular washouts are each fairly common deviations away from (3.12).

Times are scaled with an advective timescale: $\pi^\hat{r}_a(\hat{t})/\hat{W}$ where $\hat{W}$ is representative of a mean axial velocity. For simplicity, we assume that the fluids are pumped following a schedule of pump rates: $\hat{Q}_{\text{pump}}(\hat{t})$, typically a step function. The pump schedule is used to define a representative flow rate $\hat{Q}_0$, e.g. the maximum flow rate. The mean axial velocity is: $\hat{W} = \hat{Q}_0/\hat{A}_0$, where $\hat{A}_0 = 4\pi \delta_0 |\hat{r}_{a,0}|^2$ is a typical cross-sectional area of the annulus.

We now consider the relative sizes of the different terms in (3.2)-(3.5), with a view to simplification. Given that $\delta_0/\pi \ll 1$, we may assume that the dominant components of mean velocity will be in the $(\phi, \xi)$-directions, scaling approximately with $\hat{W}$. The incompressibility condition (3.5) suggests the radial component of mean velocity scales with $\delta_0 \hat{W}/\pi$. Therefore, we can see that the acceleration and inertial terms on the left-hand side of (3.2), (3.3) & (3.4) have respective sizes:

$$\frac{\delta_0 \rho \hat{W}^2}{\pi \pi \hat{r}_{a,0}}, \quad \frac{\delta_0 \rho \hat{W}^2}{\pi \hat{r}_{a,0}}, \quad \frac{\delta_0 \rho \hat{W}^2}{\pi \hat{r}_{a,0}}.$$

Simplifying the stress terms on the right-hand side of (3.2), (3.3) & (3.4) is less straightforward. A number of authors have computed turbulent flows of Newtonian fluids in uniform eccentric annuli and calculated the Reynolds stresses, e.g. Nouri and Whitelaw [171], Freund et al. [80], Chung et al. [43], Nikitin et al. [169]. Insofar as we are concerned here, the main point is that each component of $\hat{\tau}_{ij}$ has similar magnitude, because the fluctuating velocity is inherently three-dimensional. In considering the viscous stresses, we need to consider two regions separately: the turbulent core of the annular flow and the wall region. In the turbulent core the mean viscous stresses $\bar{\tau}_{ij}$, are the average of the viscous stress and not the viscous
stress of the averaged strain rate. Although the strain rate associated with the mean velocity can be estimated using scaling arguments, the fluctuating velocity also contributes to each component of the strain rate and indeed this contribution may be dominant in the core. Nonlinearity of the constitutive relations together with the probabilistic nature of the strain rate tensor prevents any easy simplification. A nice discussion may be found in chapters 4 & 5 of Sawko [202]. All that we assume here is that the viscous stress components in the core all have similar magnitude, say $\tilde{\tau}_{ij} \sim \tilde{\tau}_{v,0}$ for some viscous stress scale $\tilde{\tau}_{v,0}$.

Conventionally, the Reynolds stresses in the turbulent core have similar size to the wall shear stresses (which will vary azimuthally in an eccentric annulus). In the wall region the Reynolds stresses vanish and the viscous stresses increase over a thin layer to match the wall shear stress. In this wall region it is possible to estimate relative sizes of the different shear rates and approximate the flow. Thus, both turbulent and viscous stresses have roles to play in describing these flows. This is particularly true in cementing flows which are generally not highly turbulent, with the exception of low viscosity washes. For now we adopt a stress scale $\tilde{\tau}_0$ that we apply to mean turbulent and viscous stresses, and discuss the order of magnitudes of dimensionless terms in different flow regimes later.

Next we consider the gravitational terms. It is common to exploit density differences in creating buoyancy forces to aid in displacing the in-situ drilling mud. To capture this aspect we scale all densities with the density $\rho_1$ of the first fluid in the pumping sequence, i.e. the in-situ drilling mud, and subtract the hydrostatic pressure from $\bar{p}$. It is assumed that the pressure remaining balances the dominant stress gradients: $\bar{p} = \bar{p}_{bh}(\hat{r}) + \rho_1 \mathbf{g} \cdot \mathbf{x} + (\pi \tilde{\tau}_0 / \delta_0) \bar{p}$, where $\bar{p}_{bh}(\hat{r})$ denotes the bottom-hole pressure.

We now substitute the above variables into (3.2), (3.3) & (3.4) and divide through by the largest dimensional scales, to give the following.

$$
O \left( \frac{\delta_0^3 \rho \bar{W}^2}{\pi^3 \tilde{\tau}_0} \right) = -\frac{\partial \bar{p}}{\partial y} + O \left( \frac{\delta_0 \rho - 1}{\pi F r^2} \right) + O \left( \frac{\delta_0}{\pi} \right) + O \left( \frac{\delta_0}{\pi} \right), \quad (3.13)
$$
$$O\left(\frac{\delta_0 \hat{\rho} \hat{\tau}^2}{\pi \hat{\tau}_0}\right)_{\text{IT}} = -\frac{1}{r_a} \frac{\partial \hat{p}}{\partial \phi} + \frac{(\rho - 1) \sin \beta \sin \pi \phi}{Fr^2} + \frac{\partial}{\partial y} \left[ \tau_{\phi y} + \hat{\tau}_{\phi y} \right] + O\left(\frac{\delta_0}{\pi}\right)_{\text{ST}} + O\left(\frac{\delta_0}{\pi}\right)_{\text{CT}}$$

(3.14)

$$O\left(\frac{\delta_0 \hat{\rho} \hat{\tau}^2}{\pi \hat{\tau}_0}\right)_{\text{IT}} = -\frac{\partial \hat{p}}{\partial \xi} - \frac{(\rho - 1) \cos \beta}{Fr^2} + \frac{\partial}{\partial y} \left[ \tau_{\xi y} + \hat{\tau}_{\xi y} \right] + O\left(\frac{\delta_0}{\pi}\right)_{\text{ST}} + O\left(\frac{\delta_0}{\pi}\right)_{\text{CT}}$$

(3.15)

$$0 = \frac{\partial \hat{u}}{\partial y} + \frac{1}{r_a} \frac{\partial \hat{v}}{\partial \phi} + \frac{\partial \hat{w}}{\partial \xi} + O\left(\frac{\delta_0}{\pi}\right)_{\text{CT}}$$

(3.16)

where $\rho$ is the scaled density and $Fr$ is a Froude number:

$$Fr = \sqrt{\frac{\tau_0}{\hat{\rho} \hat{g} \delta_0 r_{a,0}}}.$$  

Our leading order model is the narrow gap limit, $\delta_0/\pi \to 0$, with other parameters fixed. The different terms that will vanish in this limit are identified by the under-braces as follows: $\text{IT}$ denotes the leading order inertial terms; $\text{BT}$ denotes the next order buoyancy terms; $\text{ST}$ denote the next order stress terms (turbulent and viscous); $\text{CT}$ denote terms associated with curvature.

If the flow is fully turbulent, the stress scale $\hat{\tau}_0$ is the wall shear stress, so the ratio $\hat{\rho} \hat{\tau}^2/\hat{\tau}_0$ that appears in the $\text{IT}$ under-braces is effectively $\sim 1/f_f$, for a representative turbulent friction factor $f_f$. Although multiplied by $\delta_0/\pi$ and formally vanishing in the narrow gap limit, these are likely to be the next largest terms. In the case that the flow is laminar, the Reynolds stress terms vanish and the expression $\rho \hat{\tau}^2/\tau_0$ becomes effectively the Reynolds number, since in this case $\tau_0$ is correctly interpreted as a viscous stress scale. Taking $\delta_0/\pi \to 0$ leads to the laminar displacement model of Bittleston et al. [24].

The curvature terms arise both due to replacing the $r$-derivatives with $y$-derivatives (changing $1/r$ to $1/r_a + O(\delta_0/\pi)$), and due to slow variations in $\xi$ as the well trajectory changes (assumed of $O(\delta_0/\pi)$). The next order stress terms are also only $O(\delta_0/\pi)$ smaller in the turbulent flow: from scaling using the geometric aspect ratio, i.e. the partial derivatives are smaller in the $(\phi, \xi)$-plane than with respect to
If the flow were laminar, then the Reynolds stresses would vanish and scaling arguments can be used to estimate the size of the viscous stresses; the next largest stresses appear at $O((\delta_0/\pi)^2)$.

To summarise, if we consider the next order terms we see that there is a proliferation of terms at order $\delta_0/\pi$: buoyancy terms, inertial, stress and curvature. The main point here is that to include the next order of terms in $\delta_0/\pi$ is prohibitively complex. On the other hand, considering the formal narrow gap limit, $\delta_0/\pi \to 0$, although the scaling arguments are different the leading order equations are similar to those of Bittleston et al. [24].

### 3.3 Narrow gap approximation

Proceeding with the narrow gap approximation, we take $\delta_0/\pi \to 0$ in (3.13) – (3.16). To eliminate $\bar{u}$ we integrate (3.16) across the gap width, using conditions of no-slip at the annulus walls:

$$\frac{\partial}{\partial \phi} [2H\bar{v}] + \frac{\partial}{\partial \xi} [2raH\bar{w}] = 0,$$

where

$$\bar{v}(\phi, \xi, t) = \frac{1}{2H} \int_{-H}^{H} \bar{v} \, dy,$$

$$\bar{w}(\phi, \xi, t) = \frac{1}{2H} \int_{-H}^{H} \bar{w} \, dy.$$  

Equation (3.17) is satisfied using a stream function:

$$2raH\bar{w} = \frac{\partial \Psi}{\partial \phi}, \quad 2H\bar{v} = -\frac{\partial \Psi}{\partial \xi}.$$  

For later convenience we introduce 2D annular divergence and gradient operators as follows:

$$\nabla_a \cdot \mathbf{q} = \frac{1}{ra} \frac{\partial q_\phi}{\partial \phi} + \frac{\partial q_\xi}{\partial \xi}.$$  

$$\nabla_a q = \left( \frac{1}{ra} \frac{\partial q}{\partial \phi}, \frac{\partial q}{\partial \xi} \right).$$

Turning now to the momentum balance, from (3.13) we see the pressure is independent of $y$, as is $\rho$ (see below in §3.4). Equations (3.14) & (3.15) may be integrated across the annular gap, assuming symmetry at $y = 0$ for this leading
order approximation:

\[
\left( \tau_{\phi y} + \bar{\tau}_{\phi y}, \tau_{\xi y} + \bar{\tau}_{\xi y} \right) = y \left( \frac{1}{r_a} \frac{\partial \bar{p}}{\partial \phi} - \frac{(\rho - 1) \sin \beta \sin \pi \phi}{Fr^2}, \frac{\partial \bar{p}}{\partial \xi} + \frac{(\rho - 1) \cos \beta}{Fr^2} \right),
\]

The leading order stresses are only non-zero in the direction of the modified pressure gradient. Viewed in the \((\phi, \xi)\)-plane this is a one-dimensional (1D) turbulent shear flow through a channel of width \(2H(\phi, \xi)\), driven in the direction of:

\[- \left( \frac{1}{r_a} \frac{\partial \bar{p}}{\partial \phi} - \frac{(\rho - 1) \sin \beta \sin \pi \phi}{Fr^2}, \frac{\partial \bar{p}}{\partial \xi} + \frac{(\rho - 1) \cos \beta}{Fr^2} \right), \]

which is therefore also the direction of the streamlines, say \(e_s\):

\[
e_s = \left( \frac{\bar{v}, \bar{w}}{\sqrt{\bar{v}^2 + \bar{w}^2}} \right) = \frac{1}{|\nabla_a \Psi|} \left( \frac{- \partial \Psi}{\partial \xi}, \frac{1}{r_a} \frac{\partial \Psi}{\partial \phi} \right).
\]

The integrated momentum balance (3.20) can now be resolved along the streamlines, in the \(e_s\) direction:

\[
\tau_{sy} + \bar{\tau}_{sy} = - \frac{y}{H} \tau_w
\]

where the dimensionless wall shear stress \(\tau_w\) is:

\[
\tau_w = H \left| \left( \frac{1}{r_a} \frac{\partial \bar{p}}{\partial \phi} - \frac{(\rho - 1) \sin \beta \sin \pi \phi}{Fr^2}, \frac{\partial \bar{p}}{\partial \xi} + \frac{(\rho - 1) \cos \beta}{Fr^2} \right) \right|.
\]

Combining the above we have:

\[
\frac{1}{|\nabla_a \Psi|} \left( \frac{- \partial \Psi}{\partial \xi}, \frac{1}{r_a} \frac{\partial \Psi}{\partial \phi} \right) = - H \frac{\tau_w}{\tau_w} \left( \frac{1}{r_a} \frac{\partial \bar{p}}{\partial \phi} - \frac{(\rho - 1) \sin \beta \sin \pi \phi}{Fr^2}, \frac{\partial \bar{p}}{\partial \xi} + \frac{(\rho - 1) \cos \beta}{Fr^2} \right)
\]

which can be re-arranged and cross-differentiated to eliminate the pressure:

\[
\nabla_a \cdot \left( \mathbf{S} + \mathbf{b} \right) = 0
\]

(3.24)
in which
\[ S = \frac{r_a \tau_w(\|\nabla_a \Psi\|)}{H \|\nabla_a \Psi\|} \nabla_a \Psi \quad \text{and} \quad b = \frac{r_a (\rho - 1)}{Fr^2} (\cos \beta, \sin \pi \phi \sin \beta). \] (3.25)

The term \( \nabla_a \cdot S \) in (3.24) is a quasilinear elliptic operator on \( \Psi \) and the term \( \nabla_a \cdot b \) provides a source term that is driven by the spatial gradients of the buoyancy vector \( b \). Note that (3.24) contains no time derivatives: time enters only via (i) boundary data, e.g. if the flow rate changes; (ii) through the fluid concentrations, which affect both fluid rheology and buoyancy.

The following remarks are to be noted:

i) We have still not fixed the shear stress scale \( \hat{\tau}_0 \), and this is not particularly important as all terms in (3.24) are scaled with \( \hat{\tau}_0 \). Although the main target here is to model turbulent flows, laminar flows are more prevalent. The scale \( \hat{\rho}_1 \hat{\bar{W}} \) also over-estimates the turbulent stresses (by using the mean velocity and not a friction velocity), and these in turn are larger than the typical laminar viscous stresses. In addition, the objective has been to develop a model capable of dealing with mixed flow regimes. Consequently, we define a \( \hat{\tau}_0 \) that is relevant to the laminar viscous stresses:

\[
\hat{\tau}_0 = \max_k \left\{ \hat{\tau}_{k,Y} + \hat{\kappa}_k \hat{\gamma}_0^n \right\}, \quad \hat{\gamma}_0 = \frac{3\hat{\bar{W}}}{\delta_0 \hat{\tau}_{a,0}},
\] (3.26)

with \( \hat{\tau}_{k,Y} \), \( \hat{\kappa}_k \), and \( n_k \), respectively the yield stress, consistency and power-law index of fluid \( k \) in the pumped sequence. Using this scale we define the functions \( \tau_w(\|\nabla_a \Psi\|) \) and \( S \), by using the closure expressions in Chapter 2, as outlined below in §3.6.1.

ii) The term \( S/|\nabla_a \Psi| \) is singular as \( |\nabla_a \Psi| \to 0 \), which is the limit where the yield stress of the fluid is not exceeded at the walls of the channel and \( \tau_w < \tau_Y \) is indeterminate. In the laminar displacement model of Bittleston et al. [24], Pelipenko and Frigaard [180] the vector \( S \) is defined explicitly to reflect
this yielding phenomenon, i.e.

\[
S = \left[ \frac{r_a \chi(|\nabla_a \Psi|)}{|\nabla_a \Psi|} + \frac{r_a \tau_Y}{H \nabla_a |\Psi|} \right] \nabla_a \Psi \iff |S| > \frac{r_a \tau_Y}{H} \quad (3.27)
\]

|\nabla_a \Psi| = 0 \iff |S| \leq \frac{r_a \tau_Y}{H} \quad (3.28)

To connect the model derivation here with that of Bittleston et al. [24], Pelipenko and Frigaard [180], note that the function \(\chi(|\nabla_a \Psi|)\) is simply:

\[
\chi(|\nabla_a \Psi|) = \tau_w (|\nabla_a \Psi| - \tau_Y), \quad (3.29)
\]

where \(\tau_w\) is defined implicitly (2.25). Furthermore, provided that |\nabla_a \Psi| > 0, we can write:

|S|(|\nabla_a \Psi|) = r_a \chi(|\nabla_a \Psi|) + \frac{r_a \tau_Y}{H}.

The function \(\chi(|\nabla_a \Psi|)\) represents the contribution to the modified pressure gradient that is surplus to that needed to yield the fluid locally. It is continuous and strictly monotone. As the flow transitions through regimes, from laminar through to turbulent, the gradient of \(\chi\) is continuous within any flow regime (but discontinuous when the flow transitions between regimes). Recall that \(\chi(|\nabla_a \Psi|)\) also has a local dependency on \((\phi, \xi, t)\) through the local geometry and fluid concentrations present. However, in general we may represent |S| graphically at any \((\phi, \xi, t)\) as in Figure 3.2a.

iii) The function \(\chi(|\nabla_a \Psi|)\) increases strictly monotonically (as does \(\tau_w(|\nabla_a \Psi|)\)). We can examine the limits of (2.25), both as \(H_w \to \infty\) and as \(H_w \to He\) (yield limit). For the latter we find:

\[
Re_p^{n/(2-n)} \sim [H_w - He]^{n+1} \Rightarrow \chi \sim |\nabla_a \Psi|^{n/(n+1)},
\]

see Figure 3.3b. As \(H_w \to \infty\) we find \(Re_p^{n/(2-n)} \sim H_w\), i.e. \(\chi \sim |\nabla_a \Psi|^n\), reflecting the shear-thinning behaviour. These limiting behaviours agree with those in Pelipenko and Frigaard [180], where the laminar model is analysed in more depth. The difference here though is that the limit \(H_w \to \infty\) is not physically attained in the laminar regime: we transition to turbulent flow.
iv) The function \( \tau_w(\|\nabla_a \Psi\|) \) is found to increase monotonically in the turbulent regime. Considering \( He \) fixed (the rheology) and taking \( H_w \rightarrow \infty \), we find \( y \rightarrow 0, n' \rightarrow n \) and \( E \rightarrow 1 \). Thus, we find that

\[
Re_p \sim H_w^{1 - \frac{n}{2}} \log H_w \Rightarrow |\nabla_a \Psi| \sim \sqrt{\tau_w} \left[ \log \tau_w \right]^{1/(2-n)},
\]

as \( \tau_w \rightarrow \infty \). Thus, \( \tau_w \) grows slightly less fast than \( |\nabla_a \Psi|^2 \), which would be the expectation in a fully rough turbulent regime, and the rheological dependency on \( n \) is minimal (in the exponent of the log term only), as would also be expected. Thus, we see essentially parallel curves in Figure 3.3a at large \( H_w \), independent of \( n \).

v) In deriving (3.24), we cross differentiate to eliminate pressure in favour of stream function. Alternatively, we may derive the pressure equation directly by reorganizing (3.21) to eliminate \( \Psi \):

\[
0 = \nabla_a \cdot \left[ r_a^2 \frac{|\nabla_a \Psi|(|S|)}{|S|} (\nabla_a p + b_p) \right]
\]

(3.30)

\[
b_p = \frac{\rho - 1}{Fr^2} \left( -\sin \beta \sin \pi \phi, \cos \beta \right)
\]

(3.31)

\[
|S| = \left| \left( -r_a \frac{\partial \tilde{p}}{\partial \xi} - \frac{r_a(\rho - 1) \cos \beta}{Fr^2}, \frac{\partial \tilde{p}}{\partial \phi} - \frac{r_a(\rho - 1) \sin \beta \sin \pi \phi}{Fr^2} \right) \right|.
\]

(3.32)

The function \( |\nabla_a \Psi|(|S|) \) is qualitatively illustrated in Figure 3.2b.

### 3.4 Mass transport

We now turn to transport of the different fluids along the annulus, assuming that at each time the elliptic problem (3.24) can be solved to give the gap-averaged velocity field. This suggests a similar model reduction will be appropriate for the different fluids and now proceed to derive this.

The concentrations of each individual fluid component \( c_k \) are modeled by an
advection-diffusion equation:

\[
\frac{\partial c_k}{\partial t} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} [\hat{r} \hat{u} c_k] + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{\theta}} [\hat{v} c_k] + \frac{\partial}{\partial \hat{\xi}} [\hat{w} c_k] = \nabla \cdot [(\hat{D}_k, m) \hat{\nabla} c_k],
\]  

(3.33)

where \( \sum_{k=1}^{K} c_k = 1 \) and \( \hat{D}_{k,m} \) represents the molecular diffusivity of species \( k \) within the mixture. For the turbulent flow, we apply the usual Reynolds decomposition and introduce the closure:

\[
-\bar{\hat{u}}' c_k' = \hat{D}_t \hat{\nabla} \bar{c}_k.
\]  

(3.34)

where \( \hat{D}_t \) is the turbulent diffusivity of species \( k \) (assumed the same for each species). Equation (3.33) becomes:

\[
\frac{\partial \bar{c}_k'}{\partial t} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} [\hat{r} \hat{u} \bar{c}_k] + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{\theta}} [\hat{v} \bar{c}_k] + \frac{\partial}{\partial \hat{\xi}} [\hat{w} \bar{c}_k] = \hat{\nabla} \cdot [(\hat{D}_t + \hat{D}_{k,m}) \hat{\nabla} \bar{c}_k].
\]  

(3.35)

We now apply the scaling introduced earlier. We anticipate that the main diffusive term is \( \hat{D}_t \) which will scale with the local gap width and friction velocity, but here introduce a global scale: \( \delta_0 \hat{r}_a, \delta \hat{\gamma} \) for the purposes of simplification. The
The scaled system is:

\[
\frac{\delta_0}{\pi} \left( \frac{\partial \tilde{c}_k}{\partial t} + \frac{\partial}{\partial y} [\tilde{u}_k \tilde{c}_k] + \frac{1}{r_a} \frac{\partial}{\partial \phi} [\tilde{v}_k \tilde{c}_k] + \frac{\partial}{\partial \xi} [\tilde{w}_k \tilde{c}_k] \right) = \frac{\partial}{\partial y} \left[ D_k \frac{\partial \tilde{c}_k}{\partial y} \right] + \left( \frac{\delta_0}{\pi} \right)^2 \nabla \cdot [D_k \nabla \tilde{c}_k] \tag{3.36}
\]

with \( D_k = D_t + 1/Pe_k \) and \( Pe_k = \delta_0 \hat{r}_{a,0} \hat{W}/\hat{D}_{k,m} \), \( D_t = \hat{D}_t/\delta_0 \hat{r}_{a,0} \hat{W} \). On the left-hand side of (3.36) we have neglected terms that come from approximating geometry/curvature effects, which are \( O(\delta/\pi) \) smaller than those considered. On the right-hand side we have also neglected terms of \( O(\delta/\pi) \) that come from approximating the radial diffusion term.

Eliminating these curvature/geometry terms only, while retaining the other terms may appear questionable as a perturbation procedure. However, note that the intention is to include the leading order effects of all physically relevant transport processes. It is evident that (3.36) represents a singular perturbation, in which the leading order concentration will be constant across the annular gap (see below). We thus retain the first order advective component on the left-hand side as this is responsible both for advection of the mean concentration and dispersive effects, within the plane of the narrow annulus. We also wish to evaluate the balance between turbulent diffusion and dispersion within the \((\phi, \xi)\)-plane and consequently retain the diffusive terms in \((\phi, \xi)\) directions. Lastly, although curvature may effect cross-gap diffusion the leading order effect is included in the order 1 terms.

### 3.4.1 Dispersion effects

We look for a perturbation approximation to (3.36) in terms of the parameter \( \delta_0/\pi \ll 1 \). The velocity is assumed to have the following form

\[
\hat{u} = \tilde{u}_0 + \hat{u}_0 + \left( \frac{\delta_0}{\pi} \right) \hat{u}_1 + \left( \frac{\delta_0}{\pi} \right)^2 \hat{u}_2 + \ldots \tag{3.37}
\]

\[
0 = \int_{-H}^{H} \hat{u}_j \, dy, \quad j = 0, 1, 2, \ldots \tag{3.38}
\]
i.e. the velocity (which we recall is anyway ensemble-averaged) is decomposed into a gap-averaged component $\bar{u}_0$ and successive components at each order that describe the $y$-variation. We assume that $\bar{u}_0 = (0, \bar{v}, \bar{w})$, as defined in (3.18), and that $\tilde{u}_0 = 0$, i.e. the radial component of velocity only arises at the first order. Similarly we write:

$$\bar{c}_k = \bar{c}_{k,0} + \left(\frac{\delta_0}{\pi}\right) \bar{c}_{k,1} + \left(\frac{\delta_0}{\pi}\right)^2 \bar{c}_{k,2} + ....$$  \hspace{1cm} (3.39)

$$0 = \int_{-H}^{H} \bar{c}_{k,j} \, dy, \quad j = 1, 2...$$  \hspace{1cm} (3.40)

Here following Zhang and Frigaard [255], Taghavi and Frigaard [220], we use method of multiple timescales suggested by Fowler [78] in which we assume that the variables respond on both the advective time $t$ and on a slower timescale $T = (\delta_0/\pi)t$, where we expect diffusive effects to come into play. Note that no-slip boundary conditions are satisfied by $\bar{u}$ at the walls, where also the diffusive fluxes of $\bar{c}_k$ are zero (Neumann condition).

Substituting these expressions into (3.36) we find that at leading order $\bar{c}_{k,0}$ is independent of $y$, as we have already assumed previously in analysing the momentum balance. From (3.40) we interpret $\bar{c}_{k,0}$ as the gap-averaged mean concentration. The first order equations are as follows:

$$\left(\frac{\partial \bar{c}_{k,0}}{\partial t} + \frac{\partial}{\partial y} [u_0 \bar{c}_{k,0}] + \nabla_a \cdot [\bar{c}_{k,0} (\bar{v} + \bar{v}_0, \bar{w} + \bar{w}_0)]\right) = \frac{\partial}{\partial y} \left[ D_{k,0} \frac{\partial \bar{c}_{k,1}}{\partial y} \right]$$  \hspace{1cm} (3.41)

Integrating across the channel shows that:

$$\frac{\partial \bar{c}_{k,0}}{\partial t} + (\bar{v}, \bar{w}) \cdot \nabla_a \bar{c}_{k,0} = 0.$$  \hspace{1cm} (3.42)

and on substituting back into (3.41) and using the continuity equation:

$$\frac{\partial}{\partial y} \left[ D_{k,0} \frac{\partial \bar{c}_{k,1}}{\partial y} \right] = (\bar{v}_0, \bar{w}_0) \cdot \nabla_a \bar{c}_{k,0}, \quad \Rightarrow \quad \frac{\partial \bar{c}_{k,1}}{\partial y} = \frac{\nabla_a \bar{c}_{k,0} \cdot \left( \int_{-H}^{H} (\bar{v}_0, \bar{w}_0) \, dy' \right)}{D_{k,0}}.$$  \hspace{1cm} (3.43)

The expression (3.42) says that on the advective timescale $t$ the leading order con-
centration is simply advected along the streamlines.

To understand evolution on the slow timescale $T$ we move to a frame of reference moving along the gap-averaged streamlines. The coordinates $(s, n)$ align locally with the directions $e_s$ and $e_n$: tangential and normal to the streamlines, respectively. The gap-averaged speed in the direction of the streamline is denoted

$$\bar{s}_0 = \sqrt{\bar{v}_0^2 + \bar{w}_0^2} = |\nabla_a \Psi|/(2H).$$

To further integrate (3.43), recall that in analyzing the momentum balance in the previous section we have shown that the leading order turbulent velocity is in the direction of the pressure gradient. Therefore, the two vectors $(\bar{v}_0, \bar{w}_0)$ and $(\tilde{v}_0, \tilde{w}_0)$ are parallel and in the direction of $e_s$, i.e.

$$(\bar{v}, \bar{w}) \cdot \nabla_a \bar{c}_{k,0} = \bar{s}_0 \frac{\partial \bar{c}_{k,0}}{\partial s} \quad \text{and} \quad (\tilde{v}_0, \tilde{w}_0) \cdot \nabla_a \tilde{c}_{k,0} = \tilde{s}_0 \frac{\partial \tilde{c}_{k,0}}{\partial s},$$

where $\tilde{s}_0(y)$ gives the variation in the mean speed across the narrow gap. Substituting into (3.41) and integrating the first order terms, we get:

$$\tilde{c}_{k,1} = \tilde{c}_{k,1}(-H) + \frac{\partial \tilde{c}_0}{\partial s} \int_{-H}^y \frac{1}{D_{k,0}} \int_{-H}^{y'} \tilde{s}_0 dy'' dy' = \tilde{c}_{k,1}(-H) + \frac{\partial \tilde{c}_0}{\partial s} k(y) \quad (3.44)$$

in which

$$k(y) = \int_{-H}^y \frac{1}{D_{k,0}} \int_{-H}^{y'} \tilde{s}_0 dy'' dy'.$$

By construction, $\tilde{c}_{k,1}(-H) = -k \frac{\partial \tilde{c}_0}{\partial s}$ where $\bar{k}$ is the average of $k(y)$ across the gap. Thus, $\tilde{c}_{k,1}$ is expressed in terms of $\frac{\partial \tilde{c}_0}{\partial s}$ and the distribution of velocity across the gap. Before proceeding, note that the leading order velocity is based on the narrow channel approximation, which leads to an even function: $\tilde{s}_0(y)$ is symmetric about $y = 0$. Since also

$$0 = \int_{-H}^H \tilde{s}_0(y) dy = 2 \int_0^H \tilde{s}_0(y) dy = 2 \int_{-H}^0 \tilde{s}_0(y) dy,$$

we may write:

$$k(y) = \int_{-H}^y \frac{1}{D_{k,0}} \int_{0}^{y'} \tilde{s}_0 dy'' dy'.$$
and note that the integral of $\tilde{s}_0(y)$ will be an odd function. The function $D_{k,0}(y)$ is also defined by the leading order velocity and can be assumed to be an even function. The integrand above is therefore also an odd function. From this we may conclude that $k(y)$ is an even function and that $k(-H) = k(H) = 0$. Similarly, $\tilde{c}_{k,1}$ is an even function of $y$.

At the next order of asymptotic expansion, in the moving frame of reference we collect terms of $O((\delta_0/\pi)^2)$:

$$
\frac{\partial \tilde{c}_{k,0}}{\partial T} = -\nabla_a \cdot [\tilde{c}_{k,1}(\bar{v}_0,\bar{w}_0)] + \nabla_a(D_{k,0} \nabla_a \tilde{c}_{k,0})
- \frac{\partial \tilde{c}_{k,1}}{\partial t} - \tilde{s}_0 \frac{\partial \tilde{c}_{k,1}}{\partial s} - \frac{\partial}{\partial y}(\bar{u}_1 \tilde{c}_{k,0}) - \nabla_a \cdot [\tilde{c}_{k,0}(\bar{v}_1,\bar{w}_1)]
+ \frac{\partial}{\partial y}(D_{k,0} \frac{\partial \tilde{c}_{k,2}}{\partial y}) + \frac{\partial}{\partial y}(D_{k,1} \frac{\partial \tilde{c}_{k,1}}{\partial y}) + \frac{\partial}{\partial y}(D_{k,2} \frac{\partial \tilde{c}_{k,0}}{\partial y}).
$$

(3.45)

We integrate (3.45) across the gap width. The terms in the first line of (3.45) do not vanish. In the second line, the first two terms are linear in quantities that integrate to zero. For the last two terms we use the incompressibility of $\bar{u}_1$:

$$
\frac{\partial}{\partial y}(\bar{u}_1 \tilde{c}_{k,0}) + \nabla_a \cdot [\tilde{c}_{k,0}(\bar{v}_1,\bar{w}_1)] = \bar{u}_1 \cdot \nabla \tilde{c}_{k,0} = \nabla_a \tilde{c}_{k,0} \cdot (\bar{v}_1,\bar{w}_1).
$$

These terms now integrate to zero across the gap. In the 3rd line the terms vanish as there is no flux through the walls.

On substituting from (3.44) we see that the slow time evolution of $\tilde{c}_{k,0}$ in the frame of reference moving along the streamline is governed by:

$$
2H \frac{\partial \tilde{c}_{k,0}}{\partial T} = -\int_{-H}^{H} \frac{\partial}{\partial s} \left[ \tilde{s}_0(k(y) - \tilde{k}) \frac{\partial \tilde{c}_{k,0}}{\partial s} \right] dy + \int_{-H}^{H} \nabla_a(D_{k,0} \nabla_a \tilde{c}_{k,0}) dy.
$$

(3.46)

The first term on the right-hand side of (3.46) is the Taylor dispersion term, which we write as follows:
\[
\int_{-H}^{H} \frac{\partial}{\partial s} \left[ \tilde{s}_0(k - \bar{k}) \frac{\partial \tilde{c}_{k,0}}{\partial s} \right] dy = \frac{\partial}{\partial s} \left[ \left( \int_{-H}^{H} \tilde{s}_0 k dy \right) \frac{\partial \tilde{c}_{k,0}}{\partial s} \right] + 2 \frac{\partial H}{\partial s} \frac{\partial \tilde{c}_{k,0}}{\partial s} [\tilde{s}_0(H)\bar{k}]
\]
\[
= -\frac{\partial}{\partial s} \left[ 2H D_T \frac{\partial \tilde{c}_{k,0}}{\partial s} \right] + 2 \frac{\partial H}{\partial s} \frac{\partial \tilde{c}_{k,0}}{\partial s} [\tilde{s}_0(H)\bar{k}]
\]
(3.47)

where
\[
D_T = \frac{1}{2H} \int_{-H}^{H} \frac{1}{D_{k,0}(y)} \left( \int_{-H}^{y} \tilde{s}_0(y')dy' \right) dy.
\]
(3.48)

The second term on the right-hand side of (3.46) reflects the average effect of the diffusivity \(D_{k,0}\). At \(y = \pm H\) the turbulent term vanishes, leaving only a negligible molecular contribution \(1/Pe \ll 1\). Therefore, we can write:

\[
\int_{-H}^{H} \nabla_a \cdot (D_{k,0}(y)\nabla_a \tilde{c}_{k,0}) dy = \nabla_a \cdot \int_{-H}^{H} (D_{k,0}(y)\nabla_a \tilde{c}_{k,0}) dy - \nabla_a H \cdot \nabla_a \tilde{c}_{k,0}[D_{k,0}(H) + D_{k,0}(-H)]
\]
\[
= \nabla_a \cdot [2H \tilde{D} \nabla_a \tilde{c}_{k,0}] + O(1/Pe)
\]

where
\[
\tilde{D} = \frac{1}{2H} \int_{-H}^{H} D_{k,0} dy.
\]

Combining the above expressions with (3.42), reverting back to the single timescale \(t\) and transforming back into the fixed frame of reference, we arrive at the following equation for the evolution of the leading order concentration:

\[
\frac{\partial \tilde{c}_{k,0}}{\partial t} = -(\bar{v}_0, \bar{w}_0) \cdot \nabla_a \tilde{c}_{k,0} + \tilde{\delta}_0 \left( \frac{1}{2H} \mathbf{e}_s \cdot \nabla_a [2H D_T \mathbf{e}_s \cdot \nabla_a \tilde{c}_{k,0}] \right)
\]
\[
- \frac{k\tilde{s}_0(H)}{H} (\mathbf{e}_s \cdot \nabla_a H)(\mathbf{e}_s \cdot \nabla_a \tilde{c}_{k,0}) + \frac{1}{2H} \nabla_a \cdot [2H \tilde{D} \nabla_a \tilde{c}_{k,0}]
\]
(3.49)

Equation (3.49) describes how the leading order concentrations of fluid \(k\) change. The right-hand side has four terms. Firstly, we have advection with the mean flow. Secondly we have a pure Taylor-dispersion term, which we can see takes the form of an anisotropic diffusivity, i.e. only along the streamlines (in direction \(\mathbf{e}_s\)). The
third term results from variations in width of the annulus. The fourth term on the right-hand side of gives the averaged effect of the diffusivity. In §2.4.4 we have modelled the velocity profiles for the flow along a uniform plane channel, i.e. $\bar{s}_0 + \bar{s}_0(y)$ and have used this, together with estimates of the turbulent diffusivity, to compute the Taylor dispersivity $D_T$. In general it is found that $D_T \ll \bar{D}$. In highly turbulent flows, $D_T$ decreases, but still remains two orders of magnitude larger than $\bar{D}$. Thus, it is the first two dispersive terms that are of most interest.

We may extract the variation of $H$ from the first term and rewrite (3.49) as:

$$\frac{\partial \bar{c}_{k,0}}{\partial t} = -(\bar{v}_0, \bar{w}_0) \cdot \nabla_a \bar{c}_{k,0} + \frac{\bar{s}_0}{\pi} \left( e_s \cdot \nabla_a [D_T e_s \cdot \nabla_a \bar{c}_{k,0}] + \frac{1}{2H} \nabla_a \cdot [2H \bar{D} \nabla_a \bar{c}_{k,0}] \right)$$

$$+ \frac{\bar{s}_0}{\pi} (e_s \cdot \nabla_a H)(e_s \cdot \nabla_a \bar{c}_{k,0}) \frac{D_T - D_{T^*}}{H},$$

(3.50)

where

$$D_{T^*} = \bar{k} \bar{s}_0(H) = -\frac{\bar{s}_0(H)}{2H} \int_{-H}^{H} \frac{y}{D_{k,0}(y)} \int_{-H}^{H} \bar{s}_0(y')dy' dy.$$  

(3.51)

We see that nominally $D_{T^*}$ has the same size as $D_T$. Note that the last term in (3.50) will vanish when the annulus is concentric. Also we should note that in most displacement flows through long annuli the streamlines are pseudo-parallel to the $\xi$-axis for most of the annulus and $H$ generally varies slowly with $\xi$, so this last term is mostly insignificant. However, in interfacial regions between two displacing fluids we often see azimuthal velocities of a similar size to the axial velocity. These regions are of course also where the main Taylor dispersion term is active. Local closure expressions for $D_T$, $D_{T^*}$ and $\bar{D}$, in terms of the expressions derived in §2.4.4 need to be rescaled as shown in §3.6.2.

### 3.5 Boundary conditions

The elliptic second order equation (3.24) determines the stream function, and hence gap-averaged velocity, at each time. It requires suitable boundary conditions in order to be solved. The annular domain has been reduced via the scaling to rectangular domain $\Omega$, representing the unwrapped gap-averaged annulus. At each timestep it is necessary to specify suitable boundary conditions on $\partial \Omega$ in order to
solve (3.24). We suppose that $\partial \Omega$ can be split into $\partial \Omega_{\Psi}$ and $\partial \Omega_{S}$:

$$
\begin{align*}
\Psi &= \Psi_{b}, \quad (\phi, \xi) \in \partial \Omega_{\Psi}, \quad (3.52) \\
S \cdot \mathbf{n} &= f, \quad (\phi, \xi) \in \partial \Omega_{S}, \quad (3.53)
\end{align*}
$$

where $\Psi_{b}$ and $f$ are specified boundary data. The conditions are explained below.

First, in the azimuthal direction if the geometry is fixed so that the narrow side of the annulus is the lowest side, then one may simplify the model by assuming that the flow is symmetric in $\phi$ along both wide and narrow sides:

$$
\begin{align*}
\Psi(0, \xi, t) &= 0, \quad (3.54) \\
\Psi(1, \xi, t) &= 2Q(t), \quad (3.55)
\end{align*}
$$

where $Q(t)$ is the dimensionless flow rate (and $\Omega = [0, 1] \times [0, Z]$, $Z$ being the dimensionless well depth). Note that by rearranging (3.23), $S$ can also be expressed in terms of the pressure gradients as:

$$
S = \left( -r_{a} \frac{\partial \bar{p}}{\partial \xi} - \frac{r_{a}(\rho - 1) \cos \beta}{Fr^2} \frac{\partial \bar{p}}{\partial \phi} - \frac{r_{a}(\rho - 1) \sin \beta \sin \pi \phi}{Fr^2}, \right).
$$

Thus, the symmetry condition $\bar{v} = 0$, which gives $S_{\xi} = 0$, also implies that

$$
\frac{\partial \bar{p}}{\partial \phi} = 0.
$$

On the other hand, suppose we consider a full annulus, with no symmetry imposed at wide and narrow sides ($\Omega = [0, 2] \times [0, Z]$). Then an alternate to (3.54) & (3.55) would be:

$$
\Psi(\phi + 2, \xi, t) = \Psi(\phi, \xi, t) + 4Q(t), \quad (3.56)
$$

fixing only the total flow rate. In using (3.56), if one wanted to work with the pressure, the pressure would be 2-periodic in $\phi$.

Secondly for the end conditions, following Pelipenko and Frigaard [179] we might expect to impose Dirichlet conditions at the inflow, $\xi = 0$:

$$
\begin{align*}
\Psi(\phi, 0, t) &= \Psi_{0}(\phi, t), \quad (3.57)
\end{align*}
$$
e.g. a uniform inflow velocity can be specified, reflecting the fact of some kind of entry/development region, following local mixing as the fluids enter the annulus. Similarly, at the outflow ($\xi = Z$), it might be appropriate to assume a fully developed flow profile:

$$\Psi(\phi, Z, t) = \Psi_Z(\phi, t).$$

(3.58)

The fully developed profile above of course needs to be specified. The natural way to do this is by neglecting $\xi$-derivatives in (3.24), due to the length of the annulus, which leads to

$$\partial_{\phi} \left[ S_\phi + b_\phi \right] = 0,$$

which in turn implies that $\partial \bar{p} / \partial \xi$ is independent of $\phi$. The fully developed profile would be found computationally by decreasing $\partial \bar{p} / \partial \xi$, which increases the axial velocity at each $\phi$ (hence the flowrate), until the net imposed flowrate through the exit section is attained. Although this appears a convoluted procedure, it is straightforward numerically.

Alternatively to (3.57) & (3.58), one might impose Neumann conditions

$$\frac{\partial \Psi}{\partial \xi} = 0$$

at each end of the annulus, e.g. as in Pelipenko and Frigaard [181]. Note that the Neumann condition, implies that $S_\xi = 0$, which specifies $S \cdot n$ on the boundary with outward normal $n$, i.e. a condition of type (3.53). Depending on the density gradients at inflow and outflow we might choose to specify $S \cdot n$ in terms of the buoyancy $b \cdot n$.

Boundary conditions for (3.50) are generally that $\bar{c}_{k,0}$ is specified at the inflow to the annulus, either from a pump schedule or from coupling with a predictive model of the downwards displacement flow in the casing. Along the sides of the annulus, either a symmetry condition is imposed, i.e.

$$\frac{\partial \bar{c}_{k,0}}{\partial \phi} = 0,$$

or potentially a periodicity condition in the case that the full annulus is resolved
and no symmetry is assumed. At the outflow we generally assume that

$$\frac{\partial \tilde{c}_{k,0}}{\partial \tilde{\xi}} = 0,$$

although in cases of large density differences we may have counter-current flows with denser fluids entering the annulus and need to specify accordingly.

### 3.6 One-dimensional closures

To close the model derived above, we need to introduce closure relations to evaluate wall shear stress as well as diffusion and dispersion coefficients.

#### 3.6.1 Hydraulic closure

In Chapter 2 we solved (3.21) to give an expression for the leading order turbulent mean velocity for a Herschel-Bulkley fluid, based on the phenomenological approach of Dodge-Metzner-Reed [58, 158]. This analysis provides a closure expression for $\tau_w = \tau_w(|\nabla \psi|; \phi, \xi, t)$. The local dependency $\phi, \xi)$ explicitly reflects the local geometric variables and the fluid rheology and density are represented implicitly with dependency $\phi, \xi, t)$ as the fluids are displaced.

Following the notation introduced in Chapter 2, the dimensional wall shear stress is

$$\hat{\tau}_w = \hat{\tau}_0 \tau_w$$

and the dimensional mean speed (averaged across the local gap) and the dimensional local annular gap are given as:

$$\hat{W}_0 = \hat{W} \left( \frac{1}{2H} |\nabla \psi|, \right), \quad 2\hat{H} = 2H \delta_0 \hat{\rho}_{a,0}$$

recall that $\hat{W}$ is the velocity scale for the entire annulus. We also assume that, according to the concentrations of the fluids at $(\phi, \xi, t)$ we may construct the dimensional local mixture density $\hat{\rho}$ and the rheological parameters: $\hat{\tau}_Y, \hat{k}, n$. We can the proceed to find $Re_p, He$ and $H_w$ as defined by (2.20) and (2.21).

The procedure in Chapter 2 gives a detailed description of the mapping from
Figure 3.3: The closure $H_w(Re_p)$, showing asymptotic behaviour: a) $H_w \to \infty$; b) $H_w \to He$. The closure is plotted for $He = 100$ and $n = 0.2, 0.4, 0.6, 0.8, 1$: green - laminar; red - transitional; black - turbulent.

$Re_p$ to $H_w$ and vice versa, which is parameterized by $(n, He)$. Observe that $H_w \propto \tau_w$ and $Re_p \propto |\nabla_a \Psi|^{2-n}$, so the mapping $H_w \leftrightarrow Re_p$ defines our closure relation. Figure 3.3 shows an example of this mapping for different $n$ at $He = 100$. The sensitivity to $He$ is not extreme.

### 3.6.2 Dispersion and diffusion closures

Similar to §3.6.1, we can find the averaged turbulent diffusivity and Taylor dispersion coefficients ($\bar{D}, D_T, D_T^*$) using the method introduced in Chapter 2. We first construct the dimensional parameters; e.g.

$$\hat{D}_T = D_T \delta_0 \hat{\tau}_a \hat{\bar{W}}_0$$

and then rescale it using the scaling defined in Chapter 2. We eventually find:

$$D_T = \frac{1}{2} |\nabla_a \Psi| D_{T,1D} \quad (3.59)$$

where $D_{T,1D}$ is the dispersion coefficient obtained assuming a locally one-dimensional channel flow (see §2.4.4). Similar relations can be derived for $\bar{D}$ and $D_T^*$, i.e. mul-
Multiplying the one-dimensional results from §2.4.4 by $|\nabla_a \Psi|/2$.

It is worthwhile to compare the values of $\bar{D}_{1D}$, $D_{T,1D}$ and $D_{T,1D}^*$ (or equivalently $\bar{D}$, $D_T$ and $D_T^*$). Figure 3.4 plots turbulent diffusivity and dispersion coefficients as a function of wall shear stress for fully turbulent flows. $H_{w,1}$ and $H_{w,2}$ are defined in §2.2. As Figure 3.4 shows, $\bar{D}$ is 2 to 3 orders of magnitude smaller than $D_T$. This is a typical feature of turbulent Taylor dispersion [230]. In addition, $D_T^*$ is almost always larger than $D_T$. This is interesting, although the results shown later have not revealed where these terms become important.

### 3.7 Model summary and conclusions

We derived a practical leading-order model for simulating 2D turbulent and mixed regime displacement flows, as encountered in the process of primary cementing.
Leading order annular displacement flows are governed by the coupled system:

\[
\nabla_a \cdot [S + b] = 0 \tag{3.60}
\]

\[
\frac{1}{2H} \nabla_a \Psi = (\bar{w}_0, -\bar{v}_0) \tag{3.61}
\]

\[
\frac{\partial \bar{c}_{k,0}}{\partial t} + (\bar{v}_0, \bar{w}_0) \cdot \nabla_a \bar{c}_{k,0} = \frac{\delta_0}{\pi} \left( e_s \cdot \nabla_a [D_T e_s \cdot \nabla_a \bar{c}_{k,0}] + \frac{1}{2H} \nabla_a \cdot [2H \bar{D} \nabla_a \bar{c}_{k,0}] \right) + \frac{\delta_0}{\pi} (e_s \cdot \nabla_a H) (e_s \cdot \nabla_a \bar{c}_{k,0}) \frac{D_T - D^*}{H} \tag{3.62}
\]

with associated boundary conditions (§3.5) and the one-dimensional closures (§3.6).

Compared to the laminar model of Bittleston et al. [24], the turbulent model has two main differences: I) The treatment of the momentum equations differs. Unlike the laminar flow, the turbulent Reynolds stress components all have similar size. The leading-order flow is a turbulent shear flow, but only due to differential scaling of the lengths. At the next order of approximation many more unknowns enter the model. Having derived the turbulent shear flow the analysis is similar to Bittleston et al. [24] in developing the field equations for the stream function (or pressure), but with the closure expressions coming from turbulent-flow hydraulics. II) The second principal difference comes in the treatment of the fluid concentrations. First we note that the assumed decoupling of averaged concentration from velocity, in the advective part of (3.62), is more valid here than for the laminar model in Bittleston et al. [24]. Second, following our analysis, we find that turbulent flows are governed by complex diffusive and dispersive transport processes (absent in the laminar flows). The largest effect is Taylor dispersion, which diffuses only along the streamlines. The gap-averaged turbulent diffusivity acts isotropically but appears to be relevant only in sufficiently long wells. We have also derived terms that describe the influence of annulus gap variations on dispersion, again in the streamwise direction. For the laminar displacement flow, only molecular diffusion should be present, which results in smaller effects than those here.
Chapter 4

Computational Analysis

In the previous chapter, we showed that the leading order annular displacement flows (across mixed flow regimes) is governed by this coupled system of partial differential equations:

\[
\nabla_a \cdot (S + b) = 0 \tag{4.1a}
\]

\[
\frac{1}{2H} \nabla_a \Psi = (\vec{w}_0, -\vec{v}_0) \tag{4.1b}
\]

\[
\frac{\partial \tilde{c}_{k,0}}{\partial t} + (\vec{v}_0, \vec{w}_0) \cdot \nabla_a \tilde{c}_{k,0} = \frac{\delta_0}{\pi} \left( e_s \cdot \nabla_a [D_T e_s \cdot \nabla_a \tilde{c}_{k,0}] + \frac{1}{2H} \nabla_a \cdot [2HD_a \tilde{c}_{k,0}] \right) + \frac{\delta_0}{\pi} (e_s \cdot \nabla_a H)(e_s \cdot \nabla_a \tilde{c}_{k,0}) \frac{D_T D^c_T}{H} \tag{4.1c}
\]

with associated boundary conditions (§3.5) and the one-dimensional closures (§3.6).

This system of equations is to be solved numerically. In this chapter, we analyze this system of equations from a computational point of view, discuss the common numerical challenges and provide robust algorithms to accurately solve them.

4.1 Discretization and variable storage

First of all, it is important to discuss the choice of discretization. It is well-known that the choice of discretization scheme plays a key role in the accuracy and robustness of any numerical algorithm. In particular, features such a checkerboard...
pressure field are possible consequences of poor discretization choices. Our choice of discretization is primarily motivated by the fact that the velocity field is computed via differentiating stream functions.

We discretize the system of (4.1) on a staggered mesh. The domain \((\phi, \xi) \in [0, \phi_{\text{end}}] \times [0, \xi_{\text{end}}]\) is divided into \(n_\phi \times n_\xi\) mesh cells. Here \(\phi_{\text{end}}\) can take value of 1 or 2, depending on whether the half or full annulus is modeled. \(\xi_{\text{end}}\) is the dimensionless length of cemented interval. Vertices of the mesh cells are at:

\[
(\phi_i, \xi_j) = (i\Delta \phi, j\Delta \xi),
\]

where the mesh cell sizes \((\Delta \phi, \Delta \xi)\) are given by

\[
\Delta \phi = \frac{1}{n_\phi} \text{ and } \Delta \xi = \frac{\xi_{\text{end}}}{n_\xi};
\]
i.e. we have uniform structured mesh cells. Notice here, in the case of a full annulus, the number of mesh cells is \(2n_\phi n_\xi\). The stream function is approximated at the vertices:

\[
\Psi_{i,j} \approx \Psi(\phi_i, \xi_j, t)
\]

Naturally, this leads us to store the velocity fields at the edge centres:

\[
w_{i+1/2,j} \approx w(\phi_{i+1/2}, \xi_j, t), \text{ and } v_{i,j+1/2} \approx v(\phi_i, \xi_{j+1/2}, t).
\]

This choice enables central differencing to give a second order approximation to the mean velocity components. Besides, the velocity components are readily available for finite-volume-type discretization of \(c_k\) at cell centre:

\[
c_{k,i+1/2,j+1/2} \approx c(k, \phi_{i+1/2}, \xi_{j+1/2}, t).
\]

In reference to Figure 4.1, \(\Psi, v, w, c_k\) are stored at circles, diamonds, squares and triangles, respectively. Note, had we chosen to work with the pressure formulation, instead of the stream function, we would also represent \(p\) at the cell centres.
4.2 Solving for $\Psi$

4.2.1 Variational inequality and weak solution

Equation (4.1a) is an elliptic second order partial differential equation, in which time evolution enters indirectly only via the fluid concentrations (see §3.4) or via flow rate changes. Our ultimate goal is to use some form of augmented Lagrangian algorithms, which are suitable for computing the flow of viscoplastic fluids and are guaranteed to converge. Therefore, we begin our analysis by establishing a variational form of (4.1a) and demonstrating certain features of the problem.

Consider a rectangle $\Omega$ with boundary $\partial\Omega \cup \partial S$, on which boundary conditions (3.52) & (3.53), respectively are satisfied. We regard any suitably smooth $\bar{\Psi}$ as an admissible stream function provided that (3.52) is satisfied. Similarly, $\bar{S}$ will be regarded as admissible provided that:

\[
\nabla_a : [\bar{S} + b] = 0,
\]

and that (3.53) is satisfied. The following statements are easily proven using Green’s theorem in the plane.

- For any admissible $\bar{\Psi}$ & $\bar{S}$:

\[
0 = \int_{\Omega} \bar{\Psi} \nabla_a \cdot b - \nabla_a \bar{\Psi} \cdot \bar{S} \, d\Omega + \int_{\partial\Omega_p} \bar{\Psi} \bar{S} \cdot n \, ds + \int_{\partial\Omega_s} \bar{\Psi} f \, ds. \tag{4.2}
\]
• For $\Psi$ & $S$ that solve (3.24) with boundary conditions (3.52) & (3.53):

$$0 = \int_{\Omega} \Psi \nabla_a \cdot b - \nabla_a \Psi \cdot S \ d\Omega + \int_{\partial\Omega_{\Psi}} \Psi b \cdot n \ ds + \int_{\partial\Omega_{S}} \Psi f \ ds. \quad (4.3)$$

• For the solution $\Psi$ & $S$, and any other admissible $\tilde{\Psi}$:

$$0 = \int_{\Omega} [\tilde{\Psi} - \Psi] \nabla_a \cdot b - [\nabla_a \tilde{\Psi} - \nabla_a \Psi] \cdot S \ d\Omega + \int_{\partial\Omega_{S}} [\tilde{\Psi} - \Psi] f \ ds. \quad (4.4)$$

• For the solution $\Psi$ & $S$, and any other admissible $\tilde{S}$:

$$\int_{\Omega} \nabla_a \Psi \cdot [\tilde{S} - S] \ d\Omega = \int_{\partial\Omega_{\Psi}} S [\tilde{S} - S] \cdot n \ ds. \quad (4.5)$$

Now we consider the closure relationship defining $S$. Provided that $|\nabla_a \Psi| > 0$ or equivalently $|S| > r_a \tau_Y/H$, we can write this as:

$$|S||\nabla_a \Psi| = \frac{r_a}{H} \tau_w (|\nabla_a \Psi|) = r_a \chi (|\nabla_a \Psi|) + \frac{r_a \tau_Y}{H}. \quad (4.6)$$

The function $\chi (|\nabla_a \Psi|)$ represents the contribution to the modified pressure gradient that is surplus to that needed to yield the fluid locally. It is continuous and strictly monotone. As the flow transitions through regimes, from laminar through to turbulent, the gradient of $\chi$ is continuous within any flow regime (but discontinuous when the flow transitions between regimes). Recall that $\chi (|\nabla_a \Psi|)$ also has a local dependency on $(\phi, \xi, t)$ through the local geometry and fluid concentrations present. However, in general we may represent $|S|$ graphically (at any $(\phi, \xi, t)$) as in Figure 4.2a.

### 4.2.2 Stream function and pressure potential functionals

The stream function potential functional $J(\Psi)$ is defined as:

$$J(\Psi) = \int_{\Omega} \int_{0}^{\nabla_a \Psi} |S|(x) \ dx - \Psi \nabla_a \cdot b \ d\Omega - \int_{\partial\Omega_{S}} \Psi f \ ds, \quad (4.7)$$

which has the following property.
Lemma 4.2.1. The solution $\Psi$ minimizes $J(\tilde{\Psi})$ over all admissible $\tilde{\Psi}$.

Proof. We look at:

$$J(\tilde{\Psi}) - J(\Psi) = \int_{\Omega} \left( \int_{|\nabla_a \Psi|} |S|(x) \, dx \right) - (\tilde{\Psi} - \Psi) \nabla_a \cdot b \, d\Omega - \int_{\partial \Omega} (\tilde{\Psi} - \Psi) f \, ds$$

$$= \int_{\Omega} \left( \int_{|\nabla_a \Psi|} |S|(x) \, dx \right) - \int_{\Omega} [\nabla_a \tilde{\Psi} - \nabla_a \Psi] : S(|\nabla_a \Psi|) \, d\Omega$$

$$\geq \int_{\Omega} \left( \int_{|\nabla_a \Psi|} |S|(x) \, dx \right) - \int_{\Omega} (|\nabla_a \tilde{\Psi}| - |\nabla_a \Psi|) |S|(|\nabla_a \Psi|) \, d\Omega$$

$$= \int_{\Omega} \left( \int_{|\nabla_a \Psi|} (|S|(x) - |S|(|\nabla_a \Psi|)) \, dx \right) \geq 0.$$ 

We have used (4.4) and then the Cauchy-Schwarz inequality above. In this last expression note that $|S|(x) > |S|(|\nabla_a \Psi|)$ whenever $x > |\nabla_a \Psi|$ due to monotonicity. Thus the sign of the integrand changes according to the limits and the integral is always positive. □

The minimization of $J(\tilde{\Psi})$ can also be expressed as a variational inequality, which is the basis of our augmented Lagrangian method; see §4.2.4. Considering now Figure 4.2b, we can define the function $|\nabla_a \Psi|(|S|)$ by effectively inverting
as illustrated, i.e.

\[
\begin{align*}
|\nabla_a \Psi|(|S|) &= \begin{cases} 
|S|^{-1}|\nabla_a \Psi|, & |S| > \frac{\alpha_0 \gamma}{H}, \\
0, & |S| \leq \frac{\alpha_0 \gamma}{H}.
\end{cases}
\end{align*}
\]

We now define the pressure potential function \( K(\tilde{S}) \) for any admissible \( \tilde{S} \) as follows.

\[
K(\tilde{S}) = -\int_\Omega \int_{\nabla_a \Psi} |\tilde{S}| \nabla \Psi(y) \, dy \, d\Omega + \int_{\partial \Omega} \Psi_b \tilde{S} \cdot n \, ds.
\] (4.8)

Analogous to Lemma 4.2.1 we have the following.

**Lemma 4.2.2.** The solution \( S \) maximizes \( K(\tilde{S}) \) over all admissible \( \tilde{S} \).

**Proof.** We look at:

\[
K(S) - K(\tilde{S}) = \int_\Omega \int_{\nabla_a \Psi} |\nabla_a \Psi| (y) \, dy \, d\Omega + \int_{\partial \Omega} \Psi_b [S - \tilde{S}] \cdot n \, ds
\]
\[
= \int_\Omega \int_{\nabla_a \Psi} |\nabla_a \Psi| (y) \, dy \, d\Omega - \int_\Omega |\nabla_a \Psi| (\tilde{S} - S) \, d\Omega
\]
\[
\geq \int_\Omega \int_{\nabla_a \Psi} (|\nabla_a \Psi| (y) - |\nabla_a \Psi|) \, dy \, d\Omega \geq 0.
\]

Here we have used (4.5) and then the Cauchy-Schwarz inequality. In the last expression note that if \( |\tilde{S}| > |S| \) then \( |\nabla_a \Psi| (y) > |\nabla_a \Psi| \) due to monotonicity; similarly when \( |\tilde{S}| < |S| \). Thus the sign of the integrand changes according to the limits and the integral is always positive. \( \square \)

Finally, since the shaded areas in Figures. 4.2a & b, sum to give \( |S| |\nabla_a \Psi| \), we have:

\[
\int_\Omega |S| |\nabla_a \Psi| \, d\Omega = \int_\Omega \int_{\nabla_a \Psi} |\nabla_a \Psi| (x) \, dx \, d\Omega + \int_{\partial \Omega} \int_{\nabla_a \Psi} |\nabla_a \Psi| (y) \, dy \, d\Omega,
\]

which can be combined with (4.3). In combination with the minimization and maximization principles above we have the following minimax principle:
Lemma 4.2.3. The solution pair \((S, \Psi)\) satisfy:

\[
K(\tilde{S}) \leq K(S) = J(\Psi) \leq J(\tilde{\Psi}),
\]

for all admissible \(\tilde{S}\) and \(\tilde{\Psi}\).

In the porous media context, similar variational principles are used to describe nonlinear filtration, see e.g. Barenblatt et al. [15], Spena and Vacca [217]. The first (integral) terms in both \(J(\cdot)\) and \(K(\cdot)\) are referred to as dissipation potentials. In a porous media flow, one is often more concerned with determining the pressure field and a stream function formulation is restrictive in only applying to 2D flows. Thus, typically \(K(\cdot)\) is referred to as the primal potential and \(J(\cdot)\) as dual potential. Here however, we treat Lemma 4.2.1 as the primal principle as it leads to a unique stream function (see below). Note that the terminology dissipation results from (4.3) which is essentially a mechanical energy balance, equating the dissipation within the system to the work done by buoyancy forces and by the boundary terms.

4.2.3 Existence and uniqueness

Lemma 4.2.1 is the basis of an existence and uniqueness result. Firstly, note that \(J(\tilde{\Psi})\) can be split as follows:

\[
J(\tilde{\Psi}) = \int_{\Omega} r_a \int_0^{\nabla \cdot \tilde{\Psi}} \chi(x) \, dx \, d\Omega + \int_{\Omega} \frac{r_a Y}{H} |\nabla \cdot \tilde{\Psi}| \, d\Omega - \int_{\Omega} \tilde{\Psi} \nabla \cdot b \, d\Omega - \int_{\partial \Omega} \tilde{\Psi} f \, ds = J_0(\tilde{\Psi}) + J_1(\tilde{\Psi}) - L(\tilde{\Psi}).
\]

The functional \(J_0\) is strictly convex as the integrand has second derivative equal to the derivative of \(\chi\), which is a strictly monotone function. The functional \(J_1\) containing the yield stress is convex, but not strictly. Finally, \(L\) denotes the linear parts that are in \(\tilde{\Psi}\).

This problem structure is in a format where standard results may be applied, (e.g. Theorem 2.1 in chapter 5 of Glowinski [89]), to guarantees existence of a unique weak solution. The relevant function space is determined by the behaviour
of $J_0$ as $||\Psi|| \to \infty$. In the fully turbulent regime $|\nabla_a \Psi| \sim \sqrt{\tau_w} [\log \tau_w]^{1/(2-n)}$ as $|\nabla_a \Psi| \to \infty$, and therefore also

$$\chi [\log \chi]^{2/(2-n)} \sim |\nabla_a \Psi|^2.$$ 

This suggests $\chi \geq O(|\nabla_a \Psi|^{2-\varepsilon})$ for any small $\varepsilon > 0$ as $|\nabla_a \Psi| \to \infty$, i.e. the log term is less significant than any power.

Proceeding now as in Pelipenko and Frigaard [181] we can infer that $\Psi \in W^{1,3-\varepsilon}(\Omega)$, with further details specific to the boundary conditions to be considered. It is interesting to compare with the results for the purely laminar case considered in Pelipenko and Frigaard [181], where the growth of $\chi$ using only the laminar closure resulted in $\Psi \in W^{1,1+n_{\text{min}}}(\Omega)$. It appears that the turbulent closure results in a smoother weak solution and a function space largely independent of the rheology.

### 4.2.4 Computational algorithm

Equation (4.1a) can be solved using Newton’s-like iterative algorithms; see Tardy and Bittleston [226], Tardy [225]. However these algorithms are not guaranteed to converged; i.e a poor initial guess may result in divergence. Therefore, instead we opt to derive and implement an augmented Lagrangian (AL) algorithm. The advantages of AL algorithms is two-fold: i) By the convexity of (4.1a), as shown above, AL algorithm is guaranteed to converge. ii) It is common that inside an eccentric annulus, flow regime varies from turbulent in the wider side to laminar in the narrow side. More crucially, we may even have mud stuck on the narrow side (= unyielded viscoplastic fluid). AL algorithms can fully resolve the unyielded regions. This is superior to the e.g. a typical viscosity regularisation techniques common in other models, e.g. in Tardy and Bittleston [226], Tardy [225].

An alternative simpler approach for solving (4.1a) is the so called “slice model”. In the slice model, we assume the gradients in the axial direction are much smaller than those in azimuthal direction which then simplifies (4.1a) to a 1D equation. This approximate equation can be straightforwardly solved iteratively. We explain both algorithms below:
Augmented Lagrangian algorithm

We expand $S(\nabla \Psi)$ to

$$S = \left[ \frac{r_a \chi |\nabla_a \Psi|}{|\nabla \Psi|} + \frac{r_a \tau_Y}{H|\nabla_a \Psi|} \right] \nabla_a \Psi$$

(4.10)

where $\chi(\nabla \Psi)$ is given by (4.6):

$$\chi = \frac{\tau_w (|\nabla_a \Psi|) - \tau_Y}{H}.$$  

(4.11)

The advantage of using $\chi$ instead of $\chi_w$ in the formulation is that the effect of yield stress can be explicitly seen. We develop the algorithm for both notations.

In §3.5, we discussed in length different choices of boundary conditions. For simplicity, we adopt the following boundary conditions: For half annulus simulations, the boundary conditions are:

$$\Psi(0, \xi, t) = 0, \quad \Psi(1, \xi, t) = 2Q(t), \quad \Psi(\phi, 0, t) = \Psi_0(\phi, t), \quad \text{and} \quad \Psi(\phi, 1, t) = \Psi_Z(\phi, t).$$

(4.12)

For full annulus simulations, the boundary conditions are

$$\Psi(\phi + 2, \xi, t) = \Psi(\phi, \xi, t) + 4Q(t), \quad \Psi(\phi, 0, t) = \Psi_0(\phi, t), \quad \text{and} \quad \Psi(\phi, 1, t) = \Psi_Z(\phi, t)$$

(4.13)

We intend to homogenize the boundary conditions, therefore we decompose the stream function $\Psi$ into homogeneous ($\tilde{\Psi}$) and particular ($\Psi^*$) parts.

$$\Psi = \Psi^* + \tilde{\Psi},$$

(4.14)

where the particular parts ($\Psi^*$) satisfies the boundary condition (4.12) or (4.13) and the homogeneous part ($\tilde{\Psi}$) satisfies the homogeneous Dirichlet boundary conditions:

$$\Psi(0, \xi, t) = \Psi(1, \xi, t) = \Psi(\phi, 0, t) = \Psi(\phi, 1, t) = 0 \quad \text{half annulus}$$

(4.15)

$$\Psi(\phi, 0, t) = \Psi(\phi, 1, t) = 0 \quad \text{full annulus}$$

(4.16)
The particular part can be constructed through different approaches. In [180], the slice model (§4.2.4) is employed at the bottom and the top of annulus ($\xi = 0, \xi_{bh}$) and then the particular solution was constructed via a linear interpolation. Here we choose to use the full slice model to compute a particular solution.

For $\tilde{\Psi}$, we use the minimisation problem introduced in Lemma 4.2.1. Lemma 4.2.1 showed that the solution of (4.1a) can be found by minimising the functional introduced in 4.7. Substituting (4.14), we find that $\tilde{\Psi}$ minimises the following functional:

$$J(\tilde{\psi}) = \int_{\Omega} \int_0^{\xi_bh} \left| \nabla \tilde{\Psi}^* + \nabla_{a\tilde{\psi}} \right| r_a \frac{\tau_w(s)}{H} \, ds - \tilde{\psi} \nabla_{b\tilde{\psi}} \, d\Omega$$

over all admissible homogeneous solutions $\tilde{\psi}$. Notice that $\partial \Omega_s = \emptyset$ (an empty space) with either choice of boundary conditions. To find the minimiser, we relax $|\nabla \tilde{\psi}| \rightarrow q$ and augment the functional $J(\tilde{\psi})$ with a Lagrange multiplier term

$$\int_{\Omega} T. (\nabla_{a\tilde{\psi}} - q) \, d\Omega$$

as well as a stabilizer term

$$\frac{r_{ALG}}{2} \int_{\Omega} |\nabla_{a\tilde{\psi}} - q|^2 \, d\Omega.$$

This effectively replaces the minimisation problem with the following saddle point problem:

$$\min_{\bar{\Psi}} \max_{q, T} J_r(\tilde{\psi}, q, T).$$

where the augmented function $J_r(\tilde{\psi}, q, T)$ is defined by:

$$J_r(\tilde{\psi}, q, T) = \int_{\Omega} \int_0^{\xi_bh} \left| \nabla_{a\tilde{\psi}} + q \right| r_a \frac{\tau_w(s)}{H} \, ds + \tilde{\psi} \nabla_{b\tilde{\psi}} \, d\Omega +$$

$$\frac{r_{ALG}}{2} \int_{\Omega} |\nabla_{a\tilde{\psi}} - q|^2 \, d\Omega + \int_{\Omega} T. (\nabla_{a\tilde{\psi}} - q) \, d\Omega.$$

The following Uzawa-type algorithm iteratively solves this saddle point problem:
1. Updating $\tilde{\psi}$: solve

$$r_{ALG}\nabla^2_a\tilde{\psi} = \nabla_a \cdot (r_{ALG}q - T - b)$$

with the boundary conditions given by (4.15).

2. Updating $q$:

$$q = \theta \frac{M}{|M|} - \nabla_a \Psi^*$$

where

$$M = T + r_{ALG}\nabla_a\tilde{\psi} + r_{ALG}q, \quad m = |M| \quad (4.21)$$

and

$$\frac{r_a}{H} \tau_w(\theta) + \theta r_{ALG} = |M|$$

This equation clearly has a solution between $0 < \theta < |M|/r_{ALG}$ which can be found iteratively.

3. Updating $T$:

$$T = T + \rho_{ALG} (\nabla_a\tilde{\psi} - q)$$

The parameters $r_{ALG}$ and $\rho_{ALG}$ are two free parameters in this algorithm. The convergence speed depends on the choice of these parameter. However, there is no systematic way to choose the optimum values. A sufficient requirement for convergence is given in Glowinski [89]:

$$\rho_{ALG} < \frac{1 + \sqrt{5}}{2} r_{ALG}.$$ 

Recently, more advanced algorithms have been developed where the value of these free parameters are automatically chosen; see Treskatis et al. [236, 237].

Remark 1: Each iteration of the Uzawa algorithm above requires solving a Poisson equation for $\tilde{\psi}$. This may sound computationally expensive. However the discretization matrix associated to this equation only depends on mesh sizes and mean radius which do not change during the simulation. This effectively means the discretization matrix needs to be computed and factorized only once.
Remark 2: The second step of Uzawa algorithm requires finding \( \tau_w \) as a function of \( \theta = \| \nabla_a \Psi^* + \nabla_a \tilde{\psi} \| \). This operation is computationally heavy, because \( \tau_w \) (or equivalently \( H_w \)) is only known implicitly as a function of \( \theta \) (or equivalently \( Re_p \)).

To circumvent this problem, instead of having \( \theta \) the independent variable, we can think of \( \tau_w \) as the independent variable. i.e. write:

\[
\frac{r_a}{H} \tau_w + \theta (\tau_w) r_{\text{ALG}} = |M|
\]

This formulation allows us to iterate on the value of \( \tau_w \) to compute \( \theta \), which is a much lighter operation because the hydraulic relation is known explicitly.

Remark 3: With the above formulation, the effect of yield stress is hidden inside the second step of the Uzawa algorithm where the hydraulic module is called. It is not obvious under what condition the flow will be partially unyielded. However, notice that inside the hydraulic module, the first condition investigated is whether \( He < H_w \). If this conditions is not satisfied, then the fluid is unyielded. To see the role of yield stress more clearly, it can be straightforwardly shown that \( \tilde{\psi} \) also minimises the following functional:

\[
J_Y(\tilde{\psi}) = \int \Omega \int_0^{\| \nabla_a \Psi^* + \nabla_a \tilde{\psi} \|} \chi(s) ds d\Omega + \int \Omega \frac{r_a \tau_Y}{H} |\nabla_a \tilde{\psi}| d\Omega - \int \Omega \tilde{\psi} \cdot \nabla b d\Omega. \quad (4.22)
\]

A similar procedure can be repeated to turn this minimisation problem into a saddle point problem. The Uzawa algorithm above will solve this problem if we replace the second step with:

\[
q = \begin{cases} 
\frac{\theta M}{|M|} - \nabla_a \Psi^* & \text{if } |M| > \frac{r_a \tilde{\psi}}{H} \\
(0,0) & \text{if } |M| \leq \frac{r_a \tilde{\psi}}{H}
\end{cases}
\]

The condition \( |M| > \frac{r_a \tilde{\psi}}{H} \) is the yielding criteria.
Slice model

Let’s assume the gradients in axial direction are much smaller than those in azimuthal direction; i.e.
\[
\frac{\partial}{\partial \xi} \ll \frac{\partial}{\partial \phi}
\]  
(4.23)
then (4.1a) simplifies to:
\[
\frac{\partial}{\partial \phi} \left( \frac{r_a}{H} \tau_w \text{sgn} \left( \frac{\partial \psi}{\partial \phi} \right) + b \phi \right) = 0
\]
(4.24)
where \( \text{sgn}(\chi) \) is the sign function. Integrating with respect to \( \phi \) and rearranging the equation, we obtain:
\[
\tau_w = \frac{H}{r_a} |G - b \phi|
\]
(4.25)
where \( G \) is the modified pressure gradient which is constant in \( \phi \); i.e. \( G = G(\xi) \). The wall shear stress can be uniquely calculated with \( G \), given the local rheological and geometrical parameters. The procedure is as follows: On the slice \( \xi = \xi_i \),

1. Guess some \( G \) and find \( \tau_w \).

2. Compute \( Re_p = Re_p(n, He, Hw) \). To compute \( H_w \), we need to dimensionlize \( \tau_w \) with our global scaling and then non-dimensionlize it with the local scaling of our hydraulic module. More specifically,
\[
He = \frac{\tau_y}{c_\tau}
\]
\[
H_w = \frac{\tau_w}{c_\tau} = \frac{H}{r_a c_\tau} |G - b \phi|
\]
\[
Re_p = Re_p(n, He, Hw)
\]
where the rescaling factor \( c_\tau \) is
\[
c_\tau = \frac{\hat{\tau}^\pi_{\text{c}}}{\hat{\tau}^\pi \left( \hat{\rho} \hat{H}^{2n} \right)^{\frac{1}{2n}}}
\]
In the above expression \( \hat{\tau}^\pi \) is the global scale for stress, and other parameters \( (\hat{\rho}, \hat{H}, n, \hat{\kappa}_p) \) are evaluated locally. Notice that \( Re_p \) is explicitly given as a
function of its parameters. Therefore, this step is fast and straightforward.

3. Construct mean velocity and then integrate to find stream function. More specifically,

\[
Re_p = \frac{6^{1-n} \hat{\rho} \hat{W}^n}{\hat{k}_p} (2\hat{H})^n = \frac{6^{1-n} \hat{\rho} \hat{W}^n}{\hat{k}_p \hat{H}^{2-n} r_0^{2-n}} \frac{1}{2-n} = B \frac{\partial \Psi}{\partial \phi} |^{2-n} \Rightarrow
\]

\[
\left| \frac{\partial \Psi}{\partial \phi} \right| = \left( \frac{Re_p}{B} \right)^{\frac{1}{2-n}} \Rightarrow
\]

\[
\frac{\partial \Psi}{\partial \phi} = \left( \frac{Re_p}{B} \right)^{\frac{1}{2-n}} \text{sgn}(G - b_\phi)
\]  

(4.26)

4. Enforce the flow rate by requiring the following residual function to vanish:

\[
R = \Psi(1, \xi_i) - \Psi(0, \xi_i) - 2Q \quad \text{half annulus}
\]

\[
R = \Psi(2, \xi_i) - \Psi(0, \xi_i) - 4Q \quad \text{full annulus}
\]

5. If $|R| > \varepsilon$, change $G$ accordingly. Otherwise, the algorithm is converged. $\varepsilon$ is the tolerance of iterations.

4.3 Solving for $c_k$

Equation (4.1c) is an advection diffusion equation. Let us first rewrite (4.1c) in a conservative form. For the sake of simplicity, we drop the subscripts and bars and write:

\[
\frac{\partial c}{\partial t} + (v, w) \cdot \nabla a c = \frac{\delta_0}{\pi} \left( e_s \cdot \nabla [D_T e_s \cdot \nabla a c] + \frac{1}{2H} \nabla a \cdot [2H \hat{D} \nabla a c] \right) + \frac{\delta_i}{\pi} (e_s \cdot \nabla a H)(e_s \cdot \nabla a c) \frac{D_T - D_T^*}{H}
\]

(4.27)
We multiply the left hand side of (4.27) by an $H_r a$ and write:

$$H_r a \left( \frac{\partial c}{\partial t} + (v, w) \cdot \nabla_a c \right) = \frac{\partial (r_a H c)}{\partial t} + \nabla_a \cdot [r_a H (v, w)] - c \nabla_a \cdot [r_a H (v, w)]$$

$$= \frac{\partial (r_a H c)}{\partial t} + \nabla_a \cdot [r_a H (v, w)]$$

(4.28)

Note that we invoked the continuity equation to simplify the above relation. Similarly on the right hand side, we can write:

$$H_r a (e_s \cdot \nabla_a [D_T e_s \cdot \nabla_a c]) = \frac{1}{s} \nabla_a \cdot (D_T \nabla_a \cdot [r_a H (v, w)] e_s)$$

(4.29)

where

$$s = \sqrt{v^2 + w^2}$$

Unfortunately, the last term on the right hand side cannot be written in a conservative form, so we have:

$$\frac{\partial (r_a H c)}{\partial t} + \nabla_a \cdot [r_a H (v, w)] = \frac{\delta_0}{\pi s} \nabla_a \cdot (D_T \nabla_a \cdot [r_a H (v, w)] e_s)$$

$$+ \frac{\delta_0}{\pi} r_a \nabla_a \cdot [\bar{D} H \nabla_a c]$$

$$+ \frac{\delta_0}{\pi} r_a (e_s \cdot \nabla_a H)(e_s \cdot \nabla_a c)(D_T - D_T^*)$$

(4.30)

There are a number of challenges in solving this equation:

- The velocity field in the left hand side of (4.30) implicitly depends on $c$. This means that simplistically speaking, the left hand side of (4.30) is a quasi-linear conservative law, which is known that can develop shocks. Therefore, any method employed need a shock-capturing feature.

- Numerical diffusion is inevitably present here. This is particularly problematic, because numerical diffusive will contaminate the natural diffusion present here due to turbulent diffusion and Taylor dispersion.
• With the choice of discretization explained earlier, implicit time discretization of (4.1c) leads to non-sparse matrices. On the other hand, explicit implementation requires inhibively small time steps for stability.

We analyze these challenges separately:

4.3.1 Advection

In order to study the advection separately, for now we ignore the diffusive and dispersive terms on the right hand side of (4.30) and write:

$$\frac{\partial U}{\partial t} + \nabla_a (F, G) = 0$$  \hspace{1cm} (4.31)

where

$$U = Hr_a c, \quad F = Hr_a c_v \quad \text{and} \quad G = Hr_a c_w.$$  

We test three common numerical algorithms for simulating conservation laws. These three algorithms are:

1. Flux Corrected Transport (FCT) algorithm of Zalesak [251]: FCT scheme relies on calculating the advective fluxes using two methods: I) method one which is highly accurate, but may produce unphysical values in the case of a non-smooth concentration. II) method two which is a lower order method, but is guaranteed to produce physical values at all times. When the solution is not differentiable, the algorithm chooses method (II), otherwise method (I) is employed. This switch is conducted via calculating anti-diffusive fluxes. The algorithm proceeds as follows:

(a) Compute low order fluxes: $F_{i+1/2,j}^{low}$ and $G_{i,j+1/2}^{low}$

(b) Compute high order fluxes: $F_{i+1/2,j}^{high}$ and $G_{i,j+1/2}^{high}$

(c) Compute anti-diffusive fluxes:

$$A_{i+1/2,j} = F_{i+1/2,j}^{high} - F_{i+1/2,j}^{low}, \quad A_{i,j+1/2} = G_{i,j+1/2}^{high} - G_{i,j+1/2}^{low}$$

(d) Limit anti-diffusive fluxes such that the solution does not get unphysical values.

$$A_{i+1/2,j}^c = C_{i+1/2,j} A_{i+1/2,j}, \quad A_{i,j+1/2}^c = C_{i,j+1/2} B_{i,j+1/2}$$
(e) Compute the time advanced low order solution

\[ q_{i,j}^{lo} = q_{i,j}^n - \frac{1}{\Delta \phi \Delta \xi} \left( F_{i+1/2,j} - F_{i-1/2,j} + G_{i,j+1/2} - G_{i,j-1/2} \right) \]

(f) Apply the limited antidiffusive fluxes:

\[ q_{i,j}^{n+1} = q_{i,j}^{lo} - \frac{1}{\Delta \phi \Delta \xi} \left( A_{i+1/2,j}^C - A_{i-1/2,j}^C + A_{i,j+1/2}^C - A_{i,j-1/2}^C \right) \]  

The subtlety of the algorithm lies in finding suitable choice of limiting functions \( 0 \leq C_{i+1/2,j}, C_{i,j+1/2} \leq 1 \). As \( C \) varies from 0 to 1, the algorithm moves from a purely low order scheme to a purely high order scheme. Several choices of limiting functions are recommended in the literature; see Zalesak [251, 252].

In this work, we use a 1st order upwind difference and a 2nd order central difference for the low and high order schemes, respectively. As for the limiting functions, we first compute the upper and lower bounds:

\[ P_i^+ = \max \left( A_{i-1/2,j}, 0 \right) - \min \left( A_{i+1/2,j}, 0 \right) \]  

\[ Q_i^+ = \left( q_i^{\text{max}} - q_i^{lo} \right) \Delta \phi \]  

\[ R_i^+ = \min \left( 1, Q_i^+ / P_i^+ \right) \text{ if } P_i^+ > 0, \text{ otherwise } R_i^+ = 0 \]  

\[ P_i^- = \max \left( A_{i+1/2,j}, 0 \right) - \min \left( A_{i-1/2,j}, 0 \right) \]  

\[ Q_i^- = \left( q_i^{lo} - q_i^{\text{min}} \right) \Delta \phi \]  

\[ R_i^- = \min \left( 1, Q_i^- / P_i^- \right) \text{ if } P_i^- > 0, \text{ otherwise } R_i^- = 0 \]

and then

\[ C_{i+1/2,j} = \begin{cases} 
\min(R_{i+1}^+, R_i^-) & \text{when } A_{i+1/2,j} > 0 \\
\min(R_i^+, R_{i+1}^-) & \text{when } A_{i+1/2,j} \leq 0 
\end{cases} \]
$C_{i,j+1/2}$ can be calculated similarly. In the above procedure, two physical values are employed: $q_i^{\text{max}}$ and $q_i^{\text{min}}$. These are “physically-motivated upper and lower bounds on the solution in the next timestep” [252].

2. Algorithm of Nessyahu and Tadmor [168] (NT): This is an improvement of the well-known Lax-Friedrichs (LxF) method. LxF method discretizes (4.31) as following: (for simplicity, we drop $r_a$ here)

$$U_{i+1/2,j+1/2}^{n+1} = \frac{1}{4} \left( U_{i,j}^n + U_{i+1,j}^n + U_{i,j+1}^n + U_{i+1,j+1}^n \right) - \frac{\Delta t}{2\Delta \xi} \left[ F(U_{i+1,j}^n) - F(U_{i,j}^n) + F(U_{i+1,j+1}^n) + F(U_{i,j+1}^n) \right]$$ (4.35)
$$- \frac{\Delta t}{2\Delta \phi} \left[ G(U_{i,j+1}^n) - G(U_{i,j}^n) + G(U_{i+1,j+1}^n) + G(U_{i+1,j}^n) \right]$$

LxF is simple, but only first order accurate. Improvement of the accuracy, while keeping the algorithm still simple, has been made by Nessyahu and Tadmor [168]. The details of multi-dimensional implementation are fully explained in Jiang et al. [123]. It basically modifies (4.35) as follows:

$$U_{i+1/2,j+1/2}^{n+1} = \frac{1}{4} \left( U_{i,j}^n + U_{i+1,j}^n + U_{i,j+1}^n + U_{i+1,j+1}^n \right)$$
$$- \frac{\Delta t}{2\Delta \xi} \left[ F(U_{i+1,j+1}^{n+1}) - F(U_{i,j}^{n+1}) + F(U_{i+1,j}^{n+1}) + F(U_{i,j+1}^{n+1}) \right]$$
$$- \frac{\Delta t}{2\Delta \phi} \left[ G(U_{i,j+1}^{n+1}) - G(U_{i,j}^{n+1}) + G(U_{i+1,j+1}^{n+1}) + G(U_{i+1,j}^{n+1}) \right]$$
$$+ \frac{1}{16} \left( U_{i,j}^n - U_{i+1,j}^n + U_{i,j+1}^n - U_{i+1,j+1}^n \right)$$
$$+ \frac{1}{16} \left( U_{i,j}^n - U_{i+1,j}^n + U_{i,j+1}^n - U_{i+1,j+1}^n \right)$$

(4.36)

where

$$U_{i,j}^{n+1/2} = U_{i,j} - \frac{\Delta t}{2\Delta \xi} f_{i,j}^n - \frac{\Delta t}{2\Delta \phi} g_{i,j}^n.$$  

In this notation $\triangledown$ and $\triangledown$ denote discrete slopes in $\xi$ and $\phi$ directions. In order to avoid oscillatory behavior and to ensure the scheme is total variation diminishing (TVD), the slopes are found with a slope limiter. Here we use
min-mod limiter defined below

\[ v'_j = MM\left(\theta(v_{j+1} - v_j), \frac{1}{2}(v_{j+1} - v_{j-1}), \theta(v_j - v_{j-1})\right) \]  

(4.37)

where

\[
MM(x_1, x_2, ...) := \begin{cases} 
  \min_j x_j & \text{if } x_j > 0 \text{ for all } j \\
  \max_j x_j & \text{if } x_j < 0 \text{ for all } j \\
  0 & \text{otherwise}
\end{cases}
\]

(4.38)

\( \theta \) is a parameter that is introduced to enhance stability of the scheme. Normally, \( 1 \leq \theta \leq 2 \). Note that this is a staggered scheme. In order to turn this into a nonstaggered scheme a reconstruction of solution may be necessary.

3. Semi-discrete method of Kurganov and Tadmor [138]: This is the next family of central difference scheme for simulation of hyperbolic equations. In this method, the numerical fluxes are calculated to form nonlinear system of ODE’s for the discrete unknowns \( U_{i,j}(t) \). Then, higher-order stable time discretizations, such as 2-step or 3-step Runge-Kutta algorithms are employed to integrate these ODE’s in time. More specifically,

\[
\frac{d}{dt} U_{i,j}(t) = -\frac{H_{i+1/2,j}(t) - H_{i-1/2,j}(t)}{\Delta \phi} - \frac{K_{i,j+1/2}(t) - K_{i,j-1/2}(t)}{\Delta \xi} 
\]

(4.39)

where the numerical fluxes are given by

\[
H_{i+1/2,j}(t) = \frac{F\left(U^+_{i+1/2,j}(t)\right) + F\left(U^-_{i+1/2,j}(t)\right)}{2} - \frac{a_{i+1/2,j}(t)}{2} \left(U^+_{i+1/2,j}(t) - U^-_{i+1/2,j}(t)\right)
\]

\[
K_{i,j+1/2}(t) = \frac{G\left(U^+_{i,j+1/2}(t)\right) + G\left(U^-_{i,j+1/2}(t)\right)}{2} - \frac{a_{i,j+1/2}(t)}{2} \left(U^+_{i,j+1/2}(t) - U^-_{i,j+1/2}(t)\right)
\]

(4.40)

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in which the intermediate terms are

\[ U_{i+1/2,j}(t) = U_{i+1,j}(t) - \frac{\Delta \phi}{2} U'_{i+1,j} \]  
(4.41a)

\[ U_{i-1/2,j}(t) = U_{i,j}(t) + \frac{\Delta \phi}{2} U'_{i,j} \]  
(4.41b)

and

\[ U_{i,j+1/2}(t) = U_{i,j+1}(t) - \frac{\Delta \xi}{2} U'_{i,j+1} \]  
(4.42a)

\[ U_{i,j-1/2}(t) = U_{i,j}(t) + \frac{\Delta \xi}{2} U'_{i,j}. \]  
(4.42b)

Here, \( a_{i+1/2,j}(t) \) and \( a_{i,j+1/2}(t) \) denote the local maximum speed, i.e. \( \max(U') \) or \( \max(U') \), respectively and the derivative terms are computed using the minmod limiter (4.37) to avoid oscillatory behavior. This system of ODE’s is then integrated in time using a 2-step or 3-step Runge-Kutta algorithm.

**Stability Condition**

Stability of the formulations given above requires that one satisfies the so called \( CFL \)-condition:

\[ CFL < CFL_{\text{critical}}. \]  
(4.43)

For the FCT algorithm \( CFL_{\text{critical}} = 1 \), and for SD, \( CFL_{\text{critical}} = 1/8 \). We define our CFL number, quite conservatively, by:

\[ CFL = \left( \frac{\max(v)}{\Delta \phi} + \frac{\max(w)}{\Delta \xi} \right) \Delta t. \]  
(4.44)

In the following, we test these three methods using two benchmark problems (see the Problem 1 and 2 defined below and Tables 4.1 and 4.2). Generally we can say all these method are better than a first order scheme, but they are not second order. Quite surprisingly, FCT was found to be noticeably faster and equally accurate. Contours of \( c \) for the *Problem 2* are plotted in Figure 4.3 for different choice of mesh size and scheme. Interestingly, the FCT appears to be slightly better at low mesh sizes, compared to the other two schemes. We therefore chose this scheme for the rest of our computational examples.
**Problem 1:** We take \((\xi, \phi) = [0, 1] \times [0, 1], (v, w) = (0.1, 0.1)\) and \(r_a = H = 1\) with periodic boundary. The initial condition is

\[
U = c = \begin{cases} 
1 & \phi, \xi \leq 0.25 \\
0 & \text{otherwise}
\end{cases}
\]  

(4.45)

i.e. a square blob that moves diagonally.

<table>
<thead>
<tr>
<th>Mesh Size \ Method</th>
<th>FCT</th>
<th>NT</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 \times 20</td>
<td>0.0056</td>
<td>0.008</td>
<td>0.0075</td>
</tr>
<tr>
<td>40 \times 40</td>
<td>0.0023</td>
<td>0.0027</td>
<td>0.0025</td>
</tr>
<tr>
<td>80 \times 80</td>
<td>7.6 \times 10^{-4}</td>
<td>9.4 \times 10^{-4}</td>
<td>9.3 \times 10^{-4}</td>
</tr>
<tr>
<td>160 \times 160</td>
<td>2.7 \times 10^{-4}</td>
<td>3.3 \times 10^{-4}</td>
<td>3.4 \times 10^{-4}</td>
</tr>
<tr>
<td>320 \times 320</td>
<td>1.0 \times 10^{-4}</td>
<td>1.2 \times 10^{-4}</td>
<td>1.2 \times 10^{-4}</td>
</tr>
</tbody>
</table>

**Problem 2:** We take \((\phi, \xi) = [0, 1] \times [0, 1], (v, w) = (0.0, 0.1)\) and \(r_a = H = 1\) with inflow boundary conditions at the bottom boundary:

\[
c = 1 \text{ at } \xi = 0, \quad \frac{\partial c}{\partial \xi} = 0 \text{ at } \xi = 1, \quad \frac{\partial c}{\partial \phi} = 0 \text{ at } \phi = 0, 1
\]  

(4.46)

The initial condition is \(c = 0\) everywhere.

### 4.3.2 Diffusion

Now we consider the diffusion and dispersion terms on the right hand side of (4.1c). We define intermediate variables

\[
J = D_T \nabla_a \cdot [r_a H c(v, w)] e_t
\]  

(4.47a)

\[
K = \tilde{D} H \nabla_a c
\]  

(4.47b)
Table 4.2: L2-norm of error for the three methods in problem 2 (computed solution - analytical solution) calculated at $t = 1$.

<table>
<thead>
<tr>
<th>Mesh size \ Method</th>
<th>FCT</th>
<th>NT</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20 \times 20$</td>
<td>0.0037</td>
<td>0.0068</td>
<td>0.0066</td>
</tr>
<tr>
<td>$40 \times 40$</td>
<td>0.0013</td>
<td>0.0028</td>
<td>0.0026</td>
</tr>
<tr>
<td>$80 \times 80$</td>
<td>$4.77 \times 10^{-4}$</td>
<td>0.001</td>
<td>0.0010</td>
</tr>
<tr>
<td>$160 \times 160$</td>
<td>$1.69 \times 10^{-4}$</td>
<td>$3.79 \times 10^{-4}$</td>
<td>$4.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>$320 \times 320$</td>
<td>$5.0 \times 10^{-5}$</td>
<td>$1.63 \times 10^{-4}$</td>
<td>$1.58 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

which we discretize as:

$$J_{i,j} = D_{T,i,j}(v_{i,j}, w_{i,j}) \left[ \frac{1}{\Delta \phi} \left( H_{i+1/2,j} c_{i+1/2,j} v_{i+1/2,j} - H_{i-1/2,j} c_{i-1/2,j} v_{i-1/2,j} \right) + \frac{1}{\Delta \xi} \left( H_{i,j+1/2} c_{i,j+1/2} v_{i,j+1/2} - H_{i,j-1/2} c_{i,j-1/2} v_{i,j-1/2} \right) \right]$$

(4.48)

$$K_{i,j} = D_{i,j} H_{i,j} \left[ \frac{1}{r_{a,j}} \frac{1}{\Delta \phi} \left( c_{i+1/2,j} - c_{i-1/2,j} \right) + \frac{1}{\Delta \xi} \left( c_{i,j+1/2} - c_{i,j-1/2} \right) \right]$$

(4.49)

Notice that as our discretization is staggered (see Figure 4.1), several variables above shall be reconstructed with interpolation. For example, azimuthal velocity vector is computed at positions $(i, j + 1/2)$ (i.e. diamonds in Figure 4.1). To calculate $v$ at positions $(i, j)$, as needed above, we may simply write:

$$v_{i,j} = \frac{1}{2} \left( v_{i,j-1/2} + v_{i,j+1/2} \right).$$

Similarly, we can find $w_{i,j}$, and so on.
Figure 4.3: Contour of $c$ in benchmark Problem 2 at $t = 9$. Columns from left to right are FCT, NT and SD schemes. Rows from top to bottom are mesh sizes: $20 \times 20$, $40 \times 40$, $80 \times 80$, $160 \times 160$ and $320 \times 320$. 
Now the right hand side of (4.30) discretizes like:

\[
RHS_{i+1/2,j+1/2} =
\frac{\delta_0}{\pi} \frac{1}{s_{i+1/2,j+1/2}} \left[ \frac{1}{r_{a,j+1/2}} \left( \frac{1}{\Delta \phi_i} \left( J_{i+1,j+1/2} - J_{i,j+1/2} \right) + \frac{1}{\Delta \xi_i} \left( J_{i+1/2,j+1} - J_{i+1/2,j} \right) \right) + \frac{\delta_0}{\pi} \left( K_{i+1,j+1/2} - K_{i,j+1/2} \right) + r_{a,j+1/2} \right]
\]

\[
\left( D_{r,i+1/2,j+1/2} - D^*_{r,i+1/2,j+1/2} \right) r_{a,j+1/2} \times
\left[ \frac{1}{r_{a,j+1/2}} \frac{1}{\Delta \phi_i} \left( h_{i+1,j+1/2} - h_{i,j+1/2} \right) + w_{i+1/2,j+1/2} \frac{1}{\Delta \xi_i} \left( h_{i+1/2,j+1} - h_{i+1/2,j} \right) \right] \times
\left[ \frac{1}{r_{a,j+1/2}} \frac{1}{\Delta \phi_i} \left( c_{i+1,j+1/2} - c_{i,j+1/2} \right) + w_{i+1/2,j+1/2} \frac{1}{\Delta \xi_i} \left( c_{i+1/2,j+1} - c_{i+1/2,j} \right) \right]
\]

(4.50)

To include the diffusion and dispersion terms in the FCT scheme, we need to modify (4.51) to

\[
q_{i,j}^{n+1} = q_{i,j}^{n} - \frac{1}{\Delta \phi_i \Delta \xi_i} \left( A_{i+1/2,j}^{C} - A_{i-1/2,j}^{C} + A_{i,j+1/2}^{C} - A_{i,j-1/2}^{C} \right) + RHS_{i+1/2,j+1/2}^{nn} \Delta t.
\]

(4.51)

where

\[
nn = \begin{cases} 
n & \text{for explicit scheme} \\
n + 1 & \text{for implicit scheme.}
\end{cases}
\]

(4.52)

**Approach 1:** If we take \(nn = n\), the algorithm is explicit in time, meaning that (4.51) can be solved at each time step to update the solution. However, in addition to the CFL condition (4.43), stability of diffusive terms requires:

\[
\alpha = \frac{\max(D_r, D^*_r, D^*_\delta) \Delta t}{\left[ \min(\Delta \phi, \Delta \xi) \right]^2} < \alpha_{\text{critical}}.
\]

(4.53)
The stability condition (4.53) is rather conservative, because the dominant diffusion term is Taylor dispersion, which is active in the direction of streamlines, which is in turn mostly axial (in vertical wells). Therefore, the relevant length scale is $\Delta \xi$ (and not $\min(\Delta \phi, \Delta \xi)$). Nonetheless, we proceed with the above definition.

Theoretically, $\alpha_{\text{critical}} = 0.5$, but we take values between 0.2 to 0.4 for robustness. Recall that the variables in (4.53) are all dimensionless. In a dimensionless setting, typically we get $\tilde{D} \ll D_T \sim D_T^* \sim 0.001 - 0.01$, $\Delta \phi \simeq 0.01$ and $\Delta \xi \simeq 1$. This means that $\Delta t \lesssim 0.001$. Roughly, in a dimensional setting, the stability condition requires $\Delta \tilde{t} \lesssim 0.01$ s. This stringent stability condition makes the simulations prohibitively slow.

**Approach 2:** Alternatively, if we take $nn = n + 1$, the algorithm is implicit in time, and a nonlinear system of equations is needed to be solved. This approach has the benefit that $\Delta t$ is governed only by the CFL condition (4.43). However, the system of equations is astronomically large. In addition, as we need to interpolate $(v, w, c)$ at spatial position where they are not defined, the mass matrix is no longer tri-diagonal.

**Approach 3:** To alleviate this problem, we chose a third approach:

- We separately compute the largest $\Delta t$ that satisfy (4.43) and (4.53). We call these two values $\Delta t_{\text{CFL}}$ and $\Delta t_\alpha$, respectively.
- If $\Delta t_{\text{CFL}} < \Delta t_\alpha$, we set $\Delta t = \Delta t_{\text{CFL}}$ and use the explicit formulation. This normally happens when the diffusive terms are very small, and $\Delta t_{\text{CFL}}$ is suitably large.
- If $\Delta t_\alpha < \Delta t_{\text{CFL}}$, we take $\Delta t = \Delta t_\alpha$, but we will only update the velocity field every $m$ time steps, where $m = \text{div}(\Delta t_{\text{CFL}}, \Delta t_\alpha)$, where the function $\text{div}(x, y)$ computes the quotient of $x$ divided by $y$. In other words, the concentration equation is updated by $\Delta t_\alpha$, but the streamfunction equation is updated by $\Delta t_{\text{CFL}}$. Notice that this only introduces a second order splitting error in time. One might decide to choose a cap for $m$. This approach allows us to circumvent the stringent condition (4.53) without the hassle of working with large matrices.
4.3.3 Displacement example

To explore the accuracy of our computational approach in solving the conservation equation (4.1c), we showcase a displacement example, and consider two diagnostic tests, as discussed below.

We take annular dimensions that are more typical of a laboratory flow loop setting, with \( \hat{r}_i = 6.5 \text{ cm} \), \( \hat{r}_o = 9 \text{ cm} \), \( \hat{Z} = 20 \text{ m} \). The annulus is assumed to be vertical \( \beta = 0 \) and uniformly eccentric with \( e = 0.5 \). We restrict the simulation to only one half of the annulus: assuming symmetry at the wide and narrow sides. The displaced and displacing fluids both have identical properties:

\[ \hat{\rho}_1 = 1100 \text{ kg/m}^3, \quad \hat{\kappa}_1 = 0.002 \text{ Pa.s}^{0.9}, \quad n_1 = 0.9, \quad \hat{\tau}_Y,1 = 0 \text{ Pa}. \]

The flow rate is constant and equal to \( Q = 0.01 \text{ m}^3/\text{s} \), equivalent to a mean velocity of \( \hat{\bar{W}} = 0.82 \text{ m/s} \). The flow rate is large enough that the displacement is fully turbulent.

At each timestep the stream function is found by solving (4.1a) using the Uzawa algorithm presented §4.2.4. The concentration enters the stream function equation through the buoyancy field \( b \) and through the local fluid properties. Having found the stream function we construct the velocity field from (4.1b) and advance the fluid concentrations in time by solving (4.1c). We use \( 30 \times 200 \) meshcells in the azimuthal and axial directions, respectively. The concentration equation, in particular, is solved using FCT scheme under the following conditions:

i) We ignore the diffusion and dispersion terms on the right hand side of (4.1a). The timesteps are chosen dynamically, to satisfy the CFL condition (4.43).

ii) We include the diffusion and dispersion terms on the right hand side of (4.1a) and use Approach 1 of §4.3.2 to compute their flux. The timesteps are chosen dynamically, to satisfy both stability conditions (4.43) and (4.53). We take \( CFL_{\text{critical}} = \alpha_{\text{critical}} = 0.4 \).

iii) Same as ii), except we use Approach 3 of §4.3.2.

Our goal is to test the accuracy of our advection and diffusion schemes, as well as to demonstrate the effect of diffusion and dispersion on the displacement...
Figure 4.4: Turbulent displacement in an eccentric annulus: $e = 0.5$. Color maps of the concentration are shown in dimensionless coordinate system $(\phi, \xi)$ at different dimensionless times as the front progresses along the annulus. White lines are streamlines with spacing $\Delta \Psi = 0.25$. a) No dispersion or diffusivity ($\bar{D} = D_T = D_T^* = 0$); b) Diffusion and dispersion are present, and computed using the Approach 1 of §4.3.2; c) Diffusion and dispersion are present, and computed using the Approach 3 of §4.3.2.

Figure 4.4 presents a colourmap of the fluid concentrations as they advance along the annulus: fluid 1 (red) is displaced by fluid 2 (blue). The wide and narrow sides are denoted with W and N. As will be explained later in Chapter 5, the eccentricity of the annulus has led to faster flows on the wider side (W) than the narrow side (N). The interface constantly elongates as a consequence. In Figure 4.4a we have set to zero all terms on the right hand side of (4.1c); i.e. no diffusion or dispersion. In Figures 4.4b and 4.4c, the diffusive and dispersive fluxes are computed using Approach 1 and Approach 3 of §4.3.2, respectively.

Figure 4.4a essentially shows the performance of the FCT scheme in advecting the front. Numerical smearing (diffusion/dispersion) of the front is present but is kept to a few cells in width. For this flow in particular (1D) there are no secondary flows that can advectively mix intermediate concentrations. The front is sharpened by mesh refinement of course, at the expense of longer computational times.
Figures 4.4b-c show the significant effect of $D_T$, in comparison to Figure 4.4a. Notice that as commented and illustrated earlier in Chapter 2, turbulent diffusivity is one or two orders of magnitude smaller than Taylor dispersion, and in primarily responsible for the smearing of interface. Indeed, if we switch the turbulent diffusion on or off, no discernible difference can be identified. Furthermore, although the term $D^*_T$ is in fact larger than $D_T$ (see Figure 3.4), since the streamlines are parallel and $\mathbf{e}_x \cdot \nabla_a c_{k,0}$ is only in the $\xi$ direction, the last term of (4.1c) does not have any contribution in this example.

In order to measure the accuracy of our FCT advection scheme, we compute the average concentration of displacing fluid:

$$\bar{c}_2(t) = \frac{\int_0^1 r_a(\xi) H(\phi, \xi)c_2(\phi, \xi, t) \, d\phi}{\int_0^1 r_a(\xi) H(\phi, \xi) \, d\phi}. \tag{4.54}$$

Notice that the denominator is equal to unity, by construction. (4.54) is in fact the efficiency of the displacement (which is used later in Chapter 6) and is defined based on the conserved quantity: $r_a H_c^2$. Alternatively, we can also calculate the averaged concentration, by computing the inlet and outlet flux:

$$\bar{c}_{2,q}(t) = \int_0^t q_{in}(\tilde{t}) - q_{out}(\tilde{t}) \, d\tilde{t}, \tag{4.55}$$

where here $q_{in}$ and $q_{out}$ are the azimuthally-averaged volumetric flux of displacing fluid entering and leaving the annulus from the bottom and top, respectively. More specifically,

$$q_{in} = \int_0^1 G^{\text{low}}(\phi, 0) + A^C(\phi, 0) \, d\phi,$$

$$q_{out} = \int_0^1 G^{\text{low}}(\phi, \xi_{bh}) + A^C(\phi, \xi_{bh}) \, d\phi;$$

see §4.3.1.

Figure 4.5 plots $\bar{c}_2$ and $\bar{c}_{2,q}$ as a function of time. The green line marks the theoretical breakthrough time with piston-like displacement, hence the curved line in Figure 4.5a is indicating some dispersion ahead of the breakthrough. More importantly, we observe that the two averaged concentrations are almost identical, which proves the FCT scheme is properly implemented (i.e. we are not losing mass).
Figure 4.5: a) Profile of $\tilde{c}_2$ (black circles) and $\tilde{c}_{2,q}$ (red dashed line) vs time for the displacement example shown in Figure 4.4a. The green line marks the arrival time. b) Red line is the $L_2$ norm of difference in the two solutions shown in Figures 4.4a and 4.4b. Blue line is the $L_2$ norm of difference in the two solutions shown in Figures 4.4b and 4.4c.

The second test we conduct is to see if the approximation of Approach 3 is accurate, compared to Approach 1. In Figure 4.5b, the blue line shows the $L_2$ norm of the difference between the two solutions of $c_2$ shown in Figure 4.4b (Approach 1) and Figure 4.4c (Approach 3). As this figure demonstrates, the error is $\lesssim O(10^{-7})$. To gain some insight about this value of error, we have also computed a similar error between Figures 4.4a and 4.4b, which is shown by the red line in Figure 4.5b. Comparison confirms that error associated with the approximation of Approach 3 is suitably negligible.

In order to evaluate the accuracy of our scheme in a more complicated displacement regime, we consider an example where the displacement regime is mixed. Here the properties of displaced fluid is given by

$$\hat{\rho}_1 = 1100 \text{ kg/m}^3, \quad \hat{k}_1 = 0.005 \text{ Pa.s}, \quad n_1 = 1, \quad \hat{\tau}_{Y,1} = 5 \text{ Pa},$$

and the properties of displacing fluid is given by

$$\hat{\rho}_2 = 1200 \text{ kg/m}^3, \quad \hat{k}_2 = 0.005 \text{ Pa.s}, \quad n_2 = 1, \quad \hat{\tau}_{Y,2} = 0 \text{ Pa}.$$
Figure 4.6: Same as Figure 4.5a, except the displaced and displacing fluid properties are changed to (4.56) and (4.57), respectively.

Notice that the displaced fluid now has a yield stress and displacing fluid is now heavier. The density difference here assists the displacing fluid to yield the displaced fluid around the annulus. In addition, both fluids are slightly more viscous, so that the displacement is in mixed regime.

Figure 4.6 plots $\overline{c_2}$ and $\overline{\dot{c}_2}$ as a function of time. Again the green line marks the theoretical breakthrough. We observe that the difference between the two averages is slightly more amplified, which is due to the complexity of displacement example (e.g. the velocity field).

4.4 Conclusions

In this chapter, we analyzed the 2D displacement model that we derived earlier in Chapter 3 from a computational point of view. The following remarks can be made here:

- We derive a weak solution for the stream function equation (4.1a). In addition, we proved that the weak problem has a unique solution.

- We construct an algorithm based on the variational form of the stream function equation (4.1a). The algorithm is guaranteed to converge, and is partic-
ularly suitable for yield stress fluids.

- A less computationally expensive algorithm for solving the stream function equation (4.1a) is also derived which rely on neglecting the axial variations in the annulus.

- As for the concentration equation (4.1c), we have reviewed some of the common approaches in solving conservation equation, and show that they are comparable in terms of accuracy and speed.
Chapter 5

Fully turbulent displacement flows

In the previous chapters we derived our 2D displacement model, and studied the model from a computational point of view. In particular, we designed robust algorithms to solve the model accurately. Starting from this chapter, we begin to explore various primary cementing scenarios, with the focus on annular displacement in turbulent and mixed regimes. The cementing parameters chosen here are all in ranges commonly found in industrial applications, which we review more explicitly in Chapter 6.

This chapter specifically discusses displacement examples that are fully turbulent. Although fully turbulent displacement flows are relatively uncommon in practice, the examples studied here elucidate the key differences between laminar and turbulent displacements and paves the way to understand more complicated mixed regime displacement that will be studied later in Chapters 6 and 7.

A version of this chapter is published in Maleki and Frigaard [150].

5.1 Simulation parameters

All simulations presented in this chapter are restricted to only one half of the annulus; i.e. assuming symmetry at the wide and narrow sides. This choice is not justified if we have unstable displacement, as it will favor one azimuthal location.
against others in terms of the onset and growth of instabilities. We will study unstable displacement flows later in Chapter 7. We are particularly interested in cementing scenarios pertinent to surface casing where the flow is more likely to be turbulent. As explained earlier, surface casings normally have larger radii, and there is often a wider pore-frac envelope, so more scope for high flow rates. Therefore, we take an annulus with inner and outer radii of \( \hat{r}_i = 16.5 \) cm (\( \hat{D}_i = 13'' \)) and \( \hat{r}_o = 19.0 \) cm (\( \hat{D}_o = 15'' \)). A cemented section is typically a few hundred meters, but the flow develops in a much shorter distance along the well, which suggests as far as the displacement flows and efficiency are concerned, a much shorter simulated cementing section suffices. Therefore, we take the cemented section length to be \( \hat{\xi}_{bh} = 150 \) m. We have used \( n_\phi = 30 \) mesh cells in the azimuthal direction and \( n_\xi = 300 \) cells along the annulus. This will give an azimuthal and axial mesh sizes of \( \Delta \theta = 6^\circ \) and \( \Delta \hat{\xi} = 0.5 \text{ m} \). Similar spatial resolution will be used in the following chapter too, where we study mixed regime displacement flows. Such resolution is commonly found in the literature; e.g. see Tardy [225], Kragset et al. [135].

In the examples presented below, we show the contours of concentration of fluids present in the displacement. These can be thought as snapshots of the displacement. In all figures, the annulus is unwrapped into a channel with varying width and the narrow and wide sides are marked with N and W on the horizontal axis. Only half of the annulus is shown. Unless specified otherwise, time is reported in dimensionless units; see the derivations in Chapter 3. The displaced and displacing fluids are indexed with 1 and 2, and colored red and blue, respectively. Streamlines are depicted with white lines. Several figures are accompanied by a map of displacement regime. These maps highlight laminar, transitional and turbulent regions in dark gray, light gray and white, respectively. The regions with immobilized (unyielded) mud are highlighted in black.

### 5.2 Rheology is not very relevant

Our first example establishes a general rule that we will use throughout the thesis. The rule says as long as the displacement is fully turbulent, the rheology of the fluids does not play any significant role in the displacement dynamics; i.e. the displacement front is not affected much by varying the rheology. To show this, we
consider the displacement example shown in Figure 5.1. This example describes a displacement job in a vertical well with eccentricity of $e = 0.5$. The rheological parameters of the displaced fluid are kept constant, while those of displacing fluid are varied. Notice that as we change the power-law index of the displacing fluid, the consistency is also modified such that the nominal effective viscosity remains constant. The nominal effective viscosity is computed using the mean velocity and mean gap width as the nominal shear rate:

$$\hat{\gamma}^* = \frac{\hat{W}}{\hat{r}_o - \hat{r}_i}, \quad \hat{\mu}_{eff} = \frac{\hat{k}\hat{\gamma}^n + \hat{\tau}_Y}{\hat{\gamma}^*}. \quad (5.1)$$

In our first example, the properties of displaced fluids are:

$$\hat{\rho}_1 = 1200 \text{ kg/m}^3, n_1 = 1, \hat{k}_1 = 0.001 \text{ Pa.s} \quad \text{and} \quad \hat{\tau}_{Y,1} = 0 \text{ Pa}.. \quad (5.2)$$

while, those of displacing fluids are

Fluid A: $\hat{\rho}_2 = 1250 \text{ kg/m}^3, n_2 = 1.0, \hat{k}_2 = 0.0010 \text{ Pa.s} \quad \text{and} \quad \hat{\tau}_{Y,2} = 0 \text{ Pa}.$

Fluid B: $\hat{\rho}_2 = 1250 \text{ kg/m}^3, n_2 = 0.7, \hat{k}_2 = 0.0024 \text{ Pa.s}^{0.7} \quad \text{and} \quad \hat{\tau}_{Y,2} = 0 \text{ Pa}.$ \quad (5.3)

Fluid C: $\hat{\rho}_2 = 1250 \text{ kg/m}^3, n_2 = 0.4, \hat{k}_2 = 0.0059 \text{ Pa.s}^{0.4} \quad \text{and} \quad \hat{\tau}_{Y,2} = 0 \text{ Pa}.$

Note that there is a positive density difference between the displacing and displaced fluids ($\hat{\rho}_2 > \hat{\rho}_1$), but we kept it small enough so that it does not mask any effect caused by the variation of rheological parameters. The flow rate in all three cases is $\hat{Q} = 0.0142 \text{ m}^3/\text{s} \quad (= 5.46 \text{ bbl/min})$ or equivalently the mean velocity is $\hat{W} = 0.5 \text{ m/s}$. The flow rate is large enough that both displaced and displacing fluids are fully turbulent around the annulus. The displacement snapshots are shown in Figure 5.1. We observe despite of the variations in $n$ and $\hat{k}$, no noticeable changes happen in how the interface of displacement advances.

If we reduce the flow rate such that the flow regime falls into laminar, then the rheology of either fluid starts to become more relevant. For example, in Figure 5.2, we repeat the previous simulation in Figure 5.1a, except we have reduced the flow rates to $\hat{Q} = 0.00071 \text{ m}^3/\text{s} \quad (= 0.273 \text{ bbl/min})$, or equivalently $\hat{W} = 0.025 \text{ m/s}$. Here both fluids are fully laminar. We observe the displacement front is slightly more
Figure 5.1: Effect of rheological parameter in fully turbulent displacement flows. Panels in each subfigure show displacement snapshots at three different times, with white lines denoting the streamlines $\Delta \Psi = 0.25$. The displacements are fully turbulent. $e = 0.5$ and $\hat{Q} = 0.0142 \text{ m}^3/\text{s} (= 5.46 \text{ bbl/min})$, $\hat{W} = 0.5 \text{ m/s}$. Displaced and displacing fluids properties are given by (5.2) and (5.3), respectively. a) Fluid A; b) Fluid B; c) Fluid C.

flat, and is much more dispersive. This presumably means stronger secondary flows develop near the interface in the laminar case (see the large deviation of streamlines near the interface) compared to the turbulent case, which are responsible for the large dispersive mixing at the interface.

To analyze this more carefully, in Figure 5.3 we have plotted contours of axial velocity and wall shear stress at $t = 100$ for all three fluids in Figure 5.1. In Figure 5.3a, it appears that the axial velocities are identical for all three fluids. In fact, further inspection confirms that $L^2$ norm of the differences of axial velocities among the three simulations are of the order of iteration tolerance ($\approx \mathcal{O}(10^{-5})$). Surprising here is that behind the displacement front, where the fluids have different shear-thinning indices $n_2$, the azimuthal distribution of turbulent velocities is very insensitive to $n_2$, although clearly varying with $\phi$. On the other hand, the
Figure 5.2: Same as Figure 5.1 a, except the flow rate is reduced to $\dot{Q} = 0.00071 \text{ m}^3/\text{s} = 0.273 \text{ bbl/min}$, $\dot{W} = 0.025 \text{ m/s}$

wall shear stress contours (Figure 5.3b) show that behind the interface the values of wall shear stress for the three fluids are quite different; i.e. larger values for the Fluid A and smaller values for the Fluid C. This is not surprising, because different rheological parameters results in different values of Reynolds number ($Re_p$), which in turn leads to having variations of wall shear stress among the three simulations. However, what is surprising is that the interface front in the three simulation has identical position, despite the difference in the wall shear stress. To answer this, in Figure 5.4 we plot the profiles of wall shear stress behind the interface (solid lines) and at the interface (dashed lines). Quite remarkably, we see the difference in the wall shear stresses almost disappears at the interface, which could be the reason the displacement progresses similarly in all three examples.

We remark here that the scaling we used to non-dimensionalize the wall shear stress fields in Figure 5.3b is a laminar stress scale. More specifically, we have:

$$\tau_w = \frac{\hat{\tau}_w}{\hat{\tau}_{lam}}, \quad \hat{\tau}_{lam} = \hat{k}_1 \hat{\gamma}^{	ext{m}1} + \hat{\tau}_{Y,1}. \quad (5.4)$$
**Figure 5.3**: a) Contour of axial velocity \(w\), corresponding to the displacement example shown in Figure 5.1 at \(t = 100\). From left to right, panels are Fluid A, Fluid B and Fluid C. b) Same as a), except the contours show wall shear stress \(\tau_w\). Wall shear stress is non-dimensionalized with a laminar velocity scale given by (5.4).

Recall that the nominal shear rate \(\dot{\gamma}^*\) is defined based on the averaged gap width and mean velocity (5.1). The laminar stress scale is based on the displaced fluid, and does not change. That is why the stress field remains unchanged for the displaced fluid in Figure 5.3. We may choose instead to scale the wall shear stress with a turbulent stress scale:

\[
\hat{\tau}_{turb}^* = \frac{1}{2} \hat{\rho}_2 \hat{W}^2 f_f,
\]

(5.5)

where here

\[
f_f = f_f \left( n_2, \hat{\kappa}_2, \hat{\tau}_{Y,2}, \hat{\dot{\gamma}} \right)
\]

is the turbulent friction factor calculated for the displacing fluid based on a repre-

\[127\]
Figure 5.4: Profile of wall shear stress behind the interface (solid lines) and at the interface (dashed lines), corresponding to the displacement example of Figure 5.1. Black, red and blue colors denote Fluid A, Fluid B and Fluid C, respectively.

Figure 5.5: Same as Figure 5.3, except the wall shear stress is scaled with a turbulent stress scale.
sentative mean velocity. Here the stress scale is based on the displacing fluid, and varies for each set of displacement examples shown. The contours of wall shear stress, scaled with the turbulent stress scale above are plotted in Figure 5.5. The turbulent stress scale defined in (5.5) encapsulates any variation of stress, and a result the dimensionless wall stress field are identical in the displacing fluid. On the hand, we see that the dimensionless stress fields in the displaced fluid vary quite significantly.

In our second example, we consider a slightly more complicated displacement example. Here we viscosify the displaced fluid and add some moderate yield stress:

\[
\hat{\rho}_1 = 1200 \text{ kg/m}^3, n_1 = 1, \hat{\kappa}_1 = 0.005 \text{ Pa.s and } \hat{\tau}_{Y,1} = 8 \text{ Pa}, \quad (5.6)
\]

while the properties of the displacing fluid are:

Fluid D: \( \hat{\rho}_2 = 1300 \text{ kg/m}^3, n_2 = 1.0, \hat{\kappa}_2 = 0.0010 \text{ Pa.s and } \hat{\tau}_{Y,2} = 0 \text{ Pa}. \)

Fluid E: \( \hat{\rho}_2 = 1300 \text{ kg/m}^3, n_2 = 0.7, \hat{\kappa}_2 = 0.0034 \text{ Pa.s}^{0.7} \text{ and } \hat{\tau}_{Y,2} = 0 \text{ Pa}. \quad (5.7)

Fluid F: \( \hat{\rho}_2 = 1300 \text{ kg/m}^3, n_2 = 0.4, \hat{\kappa}_2 = 0.0115 \text{ Pa.s}^{0.4} \text{ and } \hat{\tau}_{Y,2} = 0 \text{ Pa}. \)

Notice that larger density difference is employed here, because the mud is more viscous and has a moderate yield stress; i.e. more challenging. In addition, we run the simulations at a higher flow rate: \( \hat{Q} = 0.0426 \text{ m}^3/\text{s} (= 16.38 \text{ bbl/min}) \), which is equivalent to the mean velocity of \( \hat{\bar{W}} = 1.5 \text{ m/s} \). The consistency of displacing fluid is chosen according to its value of power-law index, such that the effective viscosity based on mean shear rate (5.1) is equal among the three simulations.

The snapshots of displacements together with the contours of flow regime are shown in Figure 5.6. Notice here that since the displaced fluid is more viscous, and has some yield stress, it undergoes a regime change around the annulus, as a result of variation of gap thickness (eccentricity of the annulus). On the wide side with larger velocity, the displacement is in the transitional regime, while on the narrow side, the displacement is laminar (or even static). One interesting feature in Figure 5.6a-b is that although the mud ahead of interface remains unyielded, at the interface it yields, and therefore, the interface progresses.
Figure 5.6: Effect of rheological parameter in fully turbulent displacement flows. Left panels in each subfigure show displacement snapshots at three different times, with white lines denoting the streamlines $\Delta \Psi = 0.25$. The right panels at each subfigure show the contour of regime: white, dark gray and light gray represent regions turbulent, transitional and laminar flow regimes. The black regions denote unyielded fluid. $e = 0.5$ and $\dot{Q} = 0.0426 \text{ m}^3/\text{s} \left(= 16.38 \text{ bbl/min}\right)$, or equivalently $\dot{W} = 1.5 \text{ m/s}$. Displaced and displacing fluids properties are given by (5.6) and (5.7), respectively. a) Fluid D; b) Fluid E; c) Fluid F.
Quite interestingly, although the displaced fluid regime changes around the annulus, the displacement front remains identical in the case of Fluids D and E. Further changing the rheology of displacing fluid to those of Fluid F results in having the displacing fluid then being unable to mobilize the mud on the narrow side. This is an interesting observation, because the displacing fluid is still turbulent, but is not able to move the mud on the narrow side. This could also be inferred from Figure 5.4. Here Fluid C (which has properties similar to Fluid F) generates the smallest wall shear stress. The generated wall shear stress is not sufficient to yield the mud on the narrow side.

The conclusion is that as long as the two displacing and displaced fluids are fully turbulent, varying their rheological parameters has little effect on the displacement front. However, if either fluids’ flow regime becomes laminar, then the rheological parameters of either of fluid may become relevant. In the example of Figure 5.6, it is the displaced fluid that is flowing partially laminar, but then changing the rheology of the displaced fluid influences how the displacement front moves. This suggests that the flows of the two (or more) fluids are strongly coupled.

Of course, the notion that rheology is not important in fully turbulent flows is to some extent inherent in the scaling of the stresses. The notion that rheology becomes relatively irrelevant in the fully turbulent flows is reported for the single flows of yield stress fluids [102]. In the above we have simply verified that this is also the case for displacement flows. The interesting aspect for cementing is that frequently we do deal with mixed regime flows (see Chapter 6), simply because the fluids are fairly viscous and often we have restrictions on the frictional pressures in the wellbore.

5.3 Turbulence vs buoyancy

In an eccentric well, the flow tends to be faster on the wide side, because in a narrow annulus, the wall shear stress scales linearly with the gap width and a larger wall shear stress induces a larger flow rate. For the very same reason, the displacement is slower on the narrow side. It is perhaps astonishing that when this same feature is present both far downstream and far upstream of an “interface”, the interface
may itself propagate at uniform (mean) speed along the annulus. However, these steady displacements have been shown to arise for laminar flows, both theoretically [179, 181] and numerically [180, 37]. In the laminar flow these arise via positive gradients of density and frictional pressure. In turbulent flows, we have just seen that rheology is not important.

A positive density difference ($\hat{\rho}_2 - \hat{\rho}_1 > 0$) competes against the tendency for the interface to elongate along the wide side, promoting a steady displacement. Here we provide a simple explanation that elucidates the role of buoyancy stresses, and the type of motions they induce near the interface. Recall that the buoyancy vector was defined as

$$b = \frac{r_a(\rho - 1)}{Fr^2} (\cos \beta, \sin \pi \phi \sin \beta).$$

In a vertical well with uniform geometry, $r_a = 1$ and $\beta = 0$, therefore

$$b = (b_\phi, b_\xi) = \left(\frac{\rho - 1}{Fr^2}, 0\right),$$

from which it follows that

$$\nabla \cdot b = \frac{1}{r_a} \frac{\partial b_\phi}{\partial \phi} + \frac{\partial b_\xi}{\partial \xi} = \frac{1}{Fr^2} \frac{\partial \rho}{\partial \phi}.$$ 

If the displacing fluid is heavier than the displaced fluid, elongation of the interface on the wide side creates a negative azimuthal gradient of density, thus $\nabla \cdot b < 0$.

Let’s now consider the following simple toy model

$$\nabla^2 \Psi + f = 0 \quad (5.9a)$$
$$0 \leq \phi \leq 1, \quad 0 \leq \xi \leq L \quad (5.9b)$$
$$\Psi(0, \xi) = 0, \quad \Psi(1, \xi) = 2 \quad (5.9c)$$
$$\frac{\partial \Psi}{\partial \xi} = 0 \quad \text{at} \quad \xi = 0, L \quad (5.9d)$$

Problem (5.9a) is a simpler version of (3.60), where $f = f(\phi, \xi)$ plays the role of $\nabla \cdot b$ and all other nonlinearities are thrown out. However, (5.9a) is elliptic just as is (3.60) and we expect qualitatively similar behavior, i.e. we expect that the solution
of (5.9a) will respond in the same mechanistic way according to the sign of the source term $f$. Using $f$ to mimic the buoyancy gradient should illustrate how the density stable configuration promotes stabilizing fluid motions. For the sake of simplicity, we take:

$$f(\phi, \xi) = \begin{cases} -1 & \text{if } \frac{L-\varepsilon}{2} \leq \xi \leq \frac{L+\varepsilon}{2}, \\ 0 & \text{otherwise} \end{cases}, \quad (5.10)$$

i.e. the interface lies on a band with thickness $\varepsilon$ and its center at height $L/2$ (see Figure 5.7a). The problem (5.9) with the source term given by (5.10) can be readily solved numerically. We used a simple second order finite difference scheme to solve this PDE. The solution $\Psi$ is plotted in Figure 5.7b. Notice here that the effect of the source term here is obscured by the mean axial flow. To see the buoyancy driven motions more clearly, in Figure 5.7c, we subtract the mean velocity and plot arrows that represent magnitude and direction of the secondary flow near the interface. Of course, as (5.9a) is linear we can also explicitly deconstruct and solve for the secondary flow, but this is only illustrative. We see that counter-clockwise flows develop near the interface that move the displacing fluid to the narrow side and displaced fluid to the wide side. This mechanism clearly competes against the bias that eccentricity creates in the flow and is the source of stabilization.

Notice that the sign of the source term $f$ was crucial here. If we flip the sign (i.e. the lighter fluid displaces the heavier fluid), then the secondary flow moves in the opposite direction, which means these buoyancy-induced flows intensify the biasing effect of eccentricity. This is the primary reason that Effective Laminar Flow method recommends to have the displacing fluid at least 10% heavier than the displaced fluid [48].

We will now explore this in a more realistic displacement case. In laminar displacement, the wall shear stress is determined by not only the gap thickness (i.e. eccentricity), but also by the rheology of the two fluids, leaving the displacement dynamics more complicated. In turbulent regime however, the competition is rather simpler, because as shown earlier, the rheology is only weakly relevant.
and the turbulent wall shear stress at fixed flow rate is predominantly influenced by the eccentricity. Figure 5.8 investigates the competition between the eccentricity (through turbulent wall shear stress) and buoyancy stresses. We use the displacement scenario of Figure 5.1a, except here we vary the density of displacing fluid to illustrate different possible scenarios. Starting from no density difference (\( \hat{\rho}_1 = \hat{\rho}_2 = 1200 \text{ kg/m}^3 \)) in Figure 5.8a, we gradually increase the density of displacing fluid to \( \hat{\rho}_2 = 1215 \) and finally 1230 kg/m³.

In Figure 5.8a the two fluids have identical densities. As a result, we observe the interface elongates. As we increase the density difference in Figure 5.8b-c, the interface velocity become more and more azimuthally uniform (the difference in wide and narrow side velocities diminishes at the interface). Elongation of the interface is suppressed and the flow is more steady.\(^1\) To see this more clearly,

\(^1\)Since Turbulent displacement flows fall into Taylor dispersion regime, the interface continuously
Figure 5.8: Competition between turbulent stresses and Buoyancy stresses. Panels in each subfigure show displacement snapshots at three different times, with white lines denoting the streamlines $\Delta \Psi = 0.25$. The displacements are fully turbulent: $e = 0.5$ and $\hat{Q} = 0.0142 \text{ m}^3/\text{s}$ ($= 5.46 \text{ bbl/min}$), $\hat{W} = 0.5 \text{ m/s}$. Displaced and displacing fluids properties are given by (5.2) and (5.3), except the displacing fluid density is a) $\rho_2 = 1200 \text{ kg/m}^3$; b) $\rho_2 = 1215 \text{ kg/m}^3$ and c) $\rho_2 = 1230 \text{ kg/m}^3$.

we look at the interface velocity on the wide and narrow sides. Figure 5.9 plots the difference in wide and narrow side axial velocities $(w_W - w_N)$ along the well for the three simulations shown in Figure 5.8. The colors black, red and blue denote $\hat{\rho}_2 = 1200, 1215$ and 1230 kg/m$^3$, respectively. The solid lines are plotted at $t = 50$ and dashed lines are at $t = 100$. For the black line, the two fluids have no density difference. Thus, the wide side velocity is always larger than the narrow side velocity. However, for the two other cases, we can see the differential velocity shrinks, as we cross the interface. More interestingly, although not clear from Figure 5.8b, the displacement example with $\hat{\rho}_2 = 1215$ is still not fully steady, because as Figure 5.9 shows, the narrow side velocity is slightly smaller than the diffuses. Therefore, strictly speaking, the interface will always be unsteady. Nonetheless, turbulent flows can still be steady in the sense that the interface moves uniformly around the annulus. Here, we have adopted the latter notion of a steady displacement.
Figure 5.9: Profile of differential velocity ($w_W - w_N$) along the annulus, corresponding to the simulations in Figure 5.8 at $t = 50$ (solid lines) and $t = 100$ (dashed lines). The colors black, red and blue denote $\hat{\rho}_2 = 1200, 1215$ and 1230 kg/m$^3$, respectively.

wide side velocity at the interface. Perhaps this would have been more apparent, if the simulation was conducted in a longer annulus, and the displacement had more time to elongate. Further increasing the density $\hat{\rho}_2 = 1230$ kg/m$^3$ completely eliminates any differential velocity at the interface, indicating that a fully steady displacement is achieved.

Figure 5.10 plots the contours of wall shear stress, scaled with the turbulent stress scale introduced in (5.5), and the $\phi$-component of the buoyancy vector ($b_\phi$), as well as the divergence of buoyancy vector $\partial b_\phi / \partial \phi$, all corresponding to the simulations shown in Figure 5.8. In Figure 5.10a, once there is a density difference between the two fluids, we can see the wall shear stress contours change rapidly at the interface, apparently becoming close to uniform along the interface. As the fluids are miscible the interface is a line of intermediate fluid properties. The dominant turbulent stresses scale with the square of the velocity and the friction factor.
If the interface moves steadily the only variation in wall shear stress along the interface comes from the fully turbulent friction factor, which is slowly changing with $H$ for fixed rheology and velocity. This explains the apparent uniformity.

The jump in $b_\phi$ is also apparent across the interface. The mechanism that makes the interface steady is, simplistically speaking, governed by $\partial b_\phi / \partial \phi$. The contours of $\partial b_\phi / \partial \phi$ are reminiscent of the $f$ function (5.10) which is plotted in Figure 5.7a. We see an increasingly negative buoyancy gradient focused around the narrow side interface, which we have seen has the effect of inducing a stabilizing secondary flow.

Our second example probes into the competition between buoyancy and turbulent stresses in a more complicated displacement setting. Here the displaced fluid has the following properties:

$$\hat{\rho}_1 = 1200 \text{ kg/m}^3, n_1 = 1, \hat{k}_1 = 0.005 \text{ Pa.s and } \hat{\tau}_{y,1} = 3 \text{ Pa}. \quad (5.11)$$
We consider three displacing fluids with three different values of density:

\[ \dot{\rho}_2 = 1200, 1250 \text{ and } 1300 \text{ kg/m}^3, n_2 = 1, \dot{\kappa}_2 = 0.001 \text{ Pa.s and } \dot{\tau}_{Y,2} = 0 \text{ Pa}. \]  

(5.12)

Snapshots of the displacement are shown in Figure 5.11. Here, since the displaced fluid is more viscous, and has a yield stress, it experiences a change in regime around the interface. In Figure 5.11a, there is no density difference, therefore the displacing fluid is unsteady, but nevertheless we see that the turbulent stresses in fluid 2 are sufficient to remove the laminar fluid 1 on the narrow side. The displaced fluid on the narrow side moves far slower than on the wide side to the extent that a secondary displacement front develops on the narrow side. Adding some density difference in Figure 5.11b, the interface moves more uniformly. Eventually, in Figure 5.11c, the density difference is sufficient to make the displacement steady. This dynamic is more clearly depicted in Figure 5.12. The differential velocity is plotted at \( t = 50 \) (solid lines) and \( t = 100 \) (dashed lines) and the colors black, red and blue denote \( \dot{\rho}_2 = 1200, 1250 \) and \( 1300 \text{ kg/m}^3 \), respectively. It is apparent that two displacement fronts are developing in the case of \( \dot{\rho}_2 = 1200 \text{ kg/m}^3 \). The differential velocity shrinks when \( \dot{\rho}_2 = 1250 \text{ kg/m}^3 \) and then finally vanishes when \( \dot{\rho}_2 = 1300 \text{ kg/m}^3 \).

### 5.4 Is turbulence necessarily good?

In the cementing community it is widely believed that the more turbulent the flow, the more efficient the displacement (e.g. see Nelson and Guillot [167], Lavrov and Torsæter [140]). However intuitively, as the flows becomes more turbulent (e.g. by increasing the flow rate), the turbulent wall shear stress increases and eventually dominates the flow, rendering both rheology and density differences unimportant in the displacement. As a result, no mechanism remains present in the displacement to make the displacement front steady. Therefore, one would expect the displacement to become unsteady. We will investigate this hypothesis here. We consider the displacement example in Figure 5.1a and change the flow rate. In Figure 5.13a the flow rate is reduced to \( \dot{Q} = 0.00071 \text{ m}^3/\text{s} = 0.273 \text{ bbl/min} \), or equivalently \( \dot{W} = 0.025 \text{ m/s} \). As a result, the effect of buoyancy is stronger and the displacement
Figure 5.11: Competition of turbulent stresses vs Buoyancy stresses. Right panels in each subfigure show displacement snapshots at three different times, with white lines denoting the streamlines $\Delta \Psi = 0.25$. Left panels in each window show the contours of displacement regime: white, light gray and dark gray represent turbulent, transitional and laminar flow regimes. $e = 0.5$ and $\dot{Q} = 0.0426 \text{ m}^3/\text{s} (= 16.38 \text{ bbl/min})$, or equivalently $\dot{W} = 1.5 \text{ m/s}$. Displaced and displacing fluids properties are given by (5.11) and (5.12), respectively. a) $\rho_2 = 1200 \text{ kg/m}^3$; b) $\rho_2 = 1250 \text{ kg/m}^3$ and c) $\rho_2 = 1300 \text{ kg/m}^3$. 

\[ \rho_1 \]
Figure 5.12: Profile of differential velocity \((w_W - w_N)\) along the annulus, corresponding to the simulations in Figure 5.11 at \(t = 50\) (solid lines) and \(t = 100\) (dashed lines). The colors black, red and blue denote \(\hat{\rho}_2 = 1200, 1250\) and \(1300\) kg/m\(^3\), respectively.

is perfectly steady. Notice that the displacement regime is also fully laminar here. In 5.13b, we increase the flow rate by a factor of 20. The displacement regime becomes fully turbulent, and the displacement is still perfectly steady. Finally, in Figure 5.13c, we increase the flow rate to \(\hat{Q} = 0.142\) m\(^3\)/s (= 54.6 bbl/min), or equivalently \(\hat{W} = 5.0\) m/s. Interestingly, we observe the displacement is now qualitatively similar to the case where there was no density difference, illustrated in Figure 5.8a, suggesting that the density difference is no longer effective.

To explain this more comprehensively, in Figure 5.14 we have plotted the contours of \(\tau_w, b_\phi\) and \(\partial b_\phi/\partial \phi\) for the displacement examples shown in Figure 5.13b-c. The sharp change in the wall shear stress (Figure 5.14a) has disappeared in the case of the displacement with largest flow rate, which has left the wall shear stress contours very similar to those in Figure 5.10a, for the case of no density difference. Additionally, we can see the jump in \(b_\phi\) has also disappeared, which leads
to $\partial b_\phi / \partial \phi$ being insignificant. Recall that $\partial b_\phi / \partial \phi$ is responsible for making the interface steady, and in its absence, the interface becomes unsteady again.

It is worth mentioning here that a mean velocity as large as $\hat{W} = 5 \text{ m/s}$, as in Figure 5.13c, is uncommon in industrial practice. However, the flow can locally achieve velocities as high as this. For example, this can easily happen, when a heavy mud is to be displaced with a low viscous lightweight wash, as will be shown later in Chapter 7.

The question is now when is the displacement too turbulent? In other words, when is velocity too large that the buoyancy vector and its azimuthal gradient vanish. The buoyancy vector in the vertical wells has only a $\phi$-component. If we use a turbulent stress scale like (5.5) to calculate the Froude number, we can further simplify $b_\phi$:

$$ b_\phi = \frac{\rho - 1}{Fr^2} = (\rho - 1) \frac{\hat{g}(\hat{r}_o - \hat{r}_i)}{2\hat{W}^2 f_f} = Ri c_2, \quad (5.13) $$

**Figure 5.13:** Same as Figure 5.1a, except we have changed the flow rates: a) $\hat{Q} = 0.00071 \text{ m}^3/\text{s} (= 0.273 \text{ bbl/min})$, or equivalently $\hat{W} = 0.025 \text{ m/s}$; b) $\hat{Q} = 0.0142 \text{ m}^3/\text{s} (= 5.46 \text{ bbl/min})$, or equivalently $\hat{W} = 0.5 \text{ m/s}$; c) $\hat{Q} = 0.142 \text{ m}^3/\text{s} (= 54.6 \text{ bbl/min})$, or equivalently $\hat{W} = 5.0 \text{ m/s}$.
Figure 5.14: a) Contour of wall shear stress ($\tau_w$), scaled using a turbulent stress scale (5.5), corresponding to the displacement example shown in Figure 5.13 at $t = 100$. The left and right panels in each subfigure belong to the displacement examples shown in Figure 5.13b ($\hat{W} = 0.5$ m/s) and Figure 5.13c ($\hat{W} = 5$ m/s), respectively. b) Same as (a), except the contours show $b_\phi$; c) Same as (a), except the contours show $\partial b_\phi / \partial \phi$.

where $Ri$ is the Richardson number defined by:

$$Ri = \frac{(\hat{\rho}_2 - \hat{\rho}_1) g (\hat{r}_o - \hat{r}_i)}{\frac{1}{2} \hat{\rho}_1 \hat{W}^2 f_f}.$$  \hspace{1cm} (5.14)

$Ri$ is the ratio of buoyancy stresses over turbulent stresses. Evidently, we expect displacement flows become too turbulent, when $Ri$ is sufficiently small. To see the role of $Ri$ more clearly, we conduct a parametric study in which we vary the eccentricity and flow rate of the displacement example in Figure 5.13. We are interested to identify different displacement flows, e.g. steady or unsteady laminar, steady or unsteady mixed regime and steady or unsteady turbulent (too turbulent).

Figure 5.15 shows the result of our parametric study. Here the displacement parameters are exactly those in Figure 5.13, except we play with eccentricity and...
flow rate. The density difference is kept constant, and as we increase the flow rate (or equivalently mean velocity), Richardson number decreases. A number of remarks can be inferred here:

- It appears that the criteria to avoid too turbulent flows is simply given by:

  \[ Ri \geq 1. \]  \hspace{1cm} (5.15)

  In other words, if the turbulent stress scales are larger than representative buoyancy scale, then the flow become too turbulent, and therefore unsteady.

- To see why (5.15) is approximately correct, we can see from 5.13 than any stable interface will produce a buoyancy gradient of size \( Ri \), since then \( c_2 \) changes slowly from wide to narrow. The wall shear stresses scale with the gap width, meaning the gradient is proportional to \( e \). Hence the balance of these terms leads to (5.15).

- The condition (5.15) is only a guideline and a necessary condition, and not sufficient, for steady flows. When the eccentricity is too large, even at large values of \( Ri \), the flow becomes unstable.

- Although (5.15) is a useful prediction, we need to be a little cautious about its generality and that of Figure 5.13. We constructed this example by increasing \( Ri \) through lowering the flow rates - leading eventually to laminar flows. An alternate would be to vary the density difference at fixed flow rate, and probably there are other possibilities.

Finally, before we close this section, we aim to analytically derive the axial velocity profile for the case of highly turbulent flows. Notice that the streamlines in Figure 5.13c are almost parallel even at the interface, as opposed to those in Figure 5.13 b, which have a tweak near the interface. This suggests that the flow is so turbulent that the density difference is not appreciated. For such highly turbulent flows, the displacement dynamics is solely determined by the gap size (eccentricity). In addition, since the streamlines are nearly parallel everywhere, the azimuthal
Figure 5.15: Steady (green symbols) vs unsteady (red symbols) displacement for different values of eccentricity $e$ and Richardson number $Ri$. Circles denote displacement flows where both displaced and displacing fluids are turbulent. Squares denote displacement flows where the displacing fluid is turbulent, and displaced fluid is laminar. Diamonds denote displacement flows where either displaced (with no star) or displacing (with a star) fluid has a mixed regime.

velocity vanishes and the flow is unidirectional everywhere. Following the closure model presented in §2.2, we see as $\tau_w \to \infty$,

$$Re_p \sim H_w^{1-n/2} \log H_w \propto \tau_w^{1-n/2} \log \tau_w, \quad (5.16)$$

where $H_w$ is another dimensionless form of the given by (2.21). Neglecting the log term,

$$Re_p \propto \left( \frac{\nabla \Psi}{H} \right)^{2-n} \sim \tau_w^{1-n/2} \Rightarrow \nabla \Psi \propto H \sqrt{\tau_w} \quad (5.17)$$

Now assuming no density difference and fully developed flow, the wall shear stress
linearly scales with the gap size, therefore

$$\nabla \Psi \propto H^{3/2}. $$

Upon satisfying the imposed flow rate, we find

$$W = \frac{\nabla \Psi}{2H} = \frac{\sqrt{H}}{\int_0^1 H^{3/2} d\phi}. \quad (5.18)$$

Notice that the velocity profile above can only be obtained if we neglect the log term in (5.16). The alternative approach for deriving the velocity profile semi-analytically is to use the slice model approach described in §4.2.4. In fact, the difference between the above derivation and the slice model is that the slice model does include the log term in (5.16). To test the accuracy of this analysis, we compute and plot the velocity field using our 2D simulations for a highly turbulent displacement. Then, we compare this with the velocity profile of the slice model as well as the velocity profile given by (5.18). For the two displacement examples in Figure 5.13b-c, we compute the velocity profile far from the interface and near the interface using both full 2D simulation and slice model. The results are plotted in Figure 5.16. We see that the velocity profile given by (5.18) is somewhat correct, at least for the case where the flow is too turbulent (i.e. Figure 5.13c). The discrepancy between the actual and the analytical velocity profile can be attributed to neglect of log term in the estimation of $Re_p$. In fact, inclusion of these terms is equivalent to the slice model, which happens to be very accurate, compared with the solution obtained by the 2D simulations.

5.5 Conclusions

In this chapter, we primarily focused on fully turbulent displacement flows. Although fully turbulent flows are relatively uncommon in primary cementing, where laminar or mixed regimes flows are more prevalent, our analysis in this chapter will help us to understand mixed regimes flows, which we address in Chapter 6, following.

Our numerical experiments in this chapter evolved around three main themes:
1. **Role of rheology**: We show that if the flow is fully turbulent, varying the rheology of either fluids does not significantly influence the displacement outcome. This statement holds as long as both fluids remain fully turbulent. More interestingly, we show that if either fluid falls into laminar or mixed regimes, then the rheological parameters begin to become relevant. This means that controlling turbulent displacement flows and ensuring steady displacement is more difficult than laminar displacement flows, because in laminar flows one can play with both the viscous and buoyancy forces to ensure a steady displacement. In turbulent flows however, the only tool left is buoyancy forces. An implication of this is that in highly inclined and horizontal wells, turbulent flows are bound to be unsuccessful at removing the mud from eccentric annuli.

2. **Competing effects of buoyancy and eccentricity**: We explained how eccentricity of the annulus creates an azimuthal bias in the flow. In addition, we constructed a simple problem analogous to our 2D model, to illustrate how density differences can aid or compete against the biasing effect of eccentricity. More specifically, given a positive density difference (heavier displacing fluid), a counterclockwise velocity field developed about the in-
terface that move displacing fluid from the wide side to the narrow side, and displaced fluid from narrow side to the wide side. This mechanism allows the interface to stabilize and progress steadily. More complicated examples are also studied, where we gradually increase the density difference and observe how this changes the dynamic of displacement. Simplistically speaking, the role of buoyancy driven secondary flows is more or less similar in laminar and turbulent displacement flows. The results of our study debunk the earlier presumption that “gravitational forces are not important when displacing in turbulent flow” [166].

3. **Too much turbulence**: We demonstrate that if the flow becomes too turbulent, the density difference loses its ability to compete against the biasing effect of eccentricity and as a result, the interface becomes unstable. In addition, we identified a dimensionless parameter that characterizes the notion of *too turbulent*. Finally, in the case of a *too turbulent* flow, we derive a velocity profile analytically, and compared it with those computed using our 2D model.

A key conclusion therefore of this chapter is that the long-standing industry practice of preferring a turbulent displacement flow over laminar is oversimplistic and questionable. If the flow rate is increased too much when turbulent, steady displacements do not occur.
Chapter 6

Mixed regimes displacement flows

In the previous chapters we focused on fully turbulent displacement flows. More likely however, is that primary cementing displacement flows fall onto laminar or mixed regimes. Displacement flows with mixed regime can happen because the two fluids have different rheological parameters, resulting in one being laminar and the other being turbulent. In this case, the flow regime changes across the displacement front. More interestingly, the change in flow regime can also happen in the azimuthal direction, governed by the annulus eccentricity. As pointed out earlier in §5.3, displacement flows tend to be faster on the wide side of annulus and slower on the narrow side. Given enough variation in gap thickness (i.e. large enough eccentricity), the flow can be turbulent on the wide side, transitional or laminar on the narrow side [167]. Examples of such flow configuration were presented in the previous chapter. More dramatically, in some cases with large eccentricity values (e.g. \( e \gtrsim 0.6 \)), the flow can be turbulent on the wide side, and stuck on the narrow side. Such azimuthal variations are a key aspect of 2D simulators compared to the rule based system, which are generally based on 1D hydraulic calculations.

It is widely agreed upon that one of the most critical parameters in displacement flows during primary cementing is the eccentricity (standoff) of the annulus [167]. In Chapter 5, Figure 5.15 presented examples where a successful displacement with a steady front is turned into an unsteady unsuccessful displacement,
when the eccentricity is increased. Moreover, even after primary cement is finished and the cement is set, stress distribution inside the cement sheath depends on the eccentricity of the annulus [100]. This means a more eccentric annulus is more vulnerable to thermal or hydraulic stresses, and as a result more likely to develop cracks that compromise the integrity of the cement.

Eccentricity is controlled via the use of centralizers, which are fitted to the outer wall of the casing, designed to exert normal forces when in contact with the borehole wall to align the casing with the borehole. A range of centralizers exist and there is no standard geometry/mechanical design. Centralizers spacing varies quite substantially, typically from as close as 9 m apart to as far as 40 m apart, depending on the operator’s design choices. In addition, operational realities often override design choices, e.g. in long cemented sections the risk of the casing getting stuck as it is lowered into the well is significant and centralizers represent mechanical obstructions. The effectiveness of centralization can be inferred from logging measurements taken after the cement job. Positioning of centralizers is designed using a range of models; see Juvkam-Wold and Wu [124], Blanco et al. [25], Guillot et al. [100]. Figure 6.1 shows a typical profile of eccentricity measured by ultrasonic logs [100]. The centralizers position are indicated by the blue diamonds. It may seem surprising that even in a vertical section of wellbore the annulus is not fully concentric. In fact, eccentricity is relatively large and varies considerably along the well. The maximum API standard recommendation for eccentricity is 33% [29].

Our model derived in Chapter 3 is capable of simulating wells with varying eccentricity (see Appendix D, as an example). However, for the sake of simplicity of analysis, we choose to fix the value of eccentricity. It is important to realize that eccentricity can dominate any other effects. Eccentricity values are rarely reported in primary cementing jobs in the literature, nor is there any routinely applied post-placement test that measures eccentricity. Also, depending on jurisdiction the mechanical design of the centralization program might not be documented and stored external to the operator or service company, (e.g. with a regulator). This makes it hard to realistically assess the actual eccentricities in wells. To account for these, we will use two ranges of eccentricity in our simulations of this chapter: $e = 0.3 - 0.4$ (mildly eccentric annulus, standoff = 70-60%) and $e = 0.6$ (highly
eccentric annulus, standoff = 40%). Of course, in a horizontal well this could be much larger.

In addition to the eccentricity, well geometry (inner and outer radii) varies along the well. Annulus inner diameters can start at anything up to 20” (51 cm) and can end as small as 4” (10 cm) in the producing zone. In this chapter we will consider two sizes of casing; described below.

Another important family of parameters relevant in the primary cementing is fluids density and rheology, as well as the pumping rates. We have surveyed the literature to collect a range of realistic parameters that are reported by several operators in the technical literature, as case studies, and also from private communications. The results are collected in Table 6.1. A number of remarks are worth mentioning here:

- The table has unfortunately many blank cells, indicating the lack of data in...
the literature. As an example, many operators register and report the total volume of the pumped fluids, whereas from a fluid mechanics point of view, the total volume is largely irrelevant (it is the flow rate that matters).

- As pointed out above, eccentricity measurements (if there are any) are almost never reported. Ironically, eccentricity is perhaps the most critical parameter needed for the simulations, and also affecting the actual cement job.

- The rheological measurements reported here are typically collected using FANN viscometers and industry standard techniques. In such techniques, the consistency and yield stress are fitted using relatively few shear rate points, there are in-house variations in fitting methods, systematic extrapolation errors, etc. Therefore, some generosity of interpretation is needed, on top of natural variations in the fitted parameters due to variations in the fluids.

Our focus in this chapter is to identify how flow regime can influence efficiency of displacement. It is widely believed in the industry that turbulent flows are more effective in terms of mud removal and cement placement. However, the scientific evidence supporting this appears to be scant. In Chapter 1 we reviewed the literature on this and pointed out the weakness in the scientific evidence. In the following we primarily limit our analysis to displacement of mud with a spacer. This is because cement slurries are typically relatively dense and viscous, and are mostly pumped in laminar regime. Thus, fluid design possibilities are more focused at having the “right” spacer relative to the mud. At the end of the chapter, we will briefly explore spacer-cement displacement too.

In this chapter, we fix the properties of the mud to

\[
\hat{\rho}_1 = 1200 \text{ kg/m}^3, n_1 = 1, \hat{\kappa}_1 = 0.01 \text{ Pa.s and } \hat{\tau}_{Y,1} = 10 \text{ Pa}
\]  

(6.1)

and play with the properties of the spacer and the flow rate. Notice that the mud has somewhat typical properties (i.e. it is not rheologically too extreme or too heavy). The key question is that, given a mud with a moderate viscosity and yield stress, which spacer displaces the mud more efficiently? A heavy highly viscous spacer that flows in laminar regime or a low viscous lightweight spacer that flows in turbulent regime or something in between?
Table 6.1: Range of density and rheological parameters as well as pumping rates for the mud, preflush and cement slurry. Red readings are in SI.

<table>
<thead>
<tr>
<th>Ref</th>
<th>Geometry</th>
<th>Mud</th>
<th>Preflush</th>
<th>Cement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$h_1$ (m)</td>
<td>$h_2$ (m)</td>
<td>$\beta$</td>
<td>$e$ (%)</td>
</tr>
<tr>
<td>[10]</td>
<td>9.625</td>
<td>12.25</td>
<td>0</td>
<td>9.2, 1100</td>
</tr>
<tr>
<td>[10]</td>
<td>10.75</td>
<td>12.25</td>
<td>0</td>
<td>9.2, 1100</td>
</tr>
<tr>
<td>[15]</td>
<td>5.5</td>
<td>8.75</td>
<td>0</td>
<td>10-10.5, 1198-1258</td>
</tr>
<tr>
<td>[15]</td>
<td>7</td>
<td>8.75</td>
<td>0</td>
<td>10-10.5, 1198-1258</td>
</tr>
<tr>
<td>[66]</td>
<td>10.75</td>
<td>13.625</td>
<td>0</td>
<td>14.9, 1785</td>
</tr>
<tr>
<td>[66]</td>
<td>10.75</td>
<td>13.625</td>
<td>0</td>
<td>14.9, 1785</td>
</tr>
<tr>
<td>[29]</td>
<td>9.625</td>
<td>10.75</td>
<td>90</td>
<td>1490</td>
</tr>
<tr>
<td>[29]</td>
<td>9.625</td>
<td>10.75</td>
<td>90</td>
<td>1490</td>
</tr>
<tr>
<td>[29]</td>
<td>9.625</td>
<td>10.75</td>
<td>90</td>
<td>1490</td>
</tr>
<tr>
<td>[69]</td>
<td>0.875</td>
<td>15</td>
<td>1800</td>
<td></td>
</tr>
<tr>
<td>[18]</td>
<td>10.179</td>
<td>14.75</td>
<td>0</td>
<td>10, 1198</td>
</tr>
<tr>
<td>[18]</td>
<td>11.87</td>
<td>16</td>
<td>12, 1437</td>
<td></td>
</tr>
<tr>
<td>[95]</td>
<td>13.79</td>
<td>16</td>
<td>12, 1437</td>
<td></td>
</tr>
<tr>
<td>[95]</td>
<td>10</td>
<td>14.75</td>
<td>0</td>
<td>10, 1198</td>
</tr>
<tr>
<td>[95]</td>
<td>10</td>
<td>16</td>
<td>12, 1437</td>
<td></td>
</tr>
<tr>
<td>[33]</td>
<td>11.26</td>
<td>12.3, 5.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[33]</td>
<td>8.75</td>
<td>300</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[180]</td>
<td>8.75</td>
<td>1490</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[180]</td>
<td>8.75</td>
<td>300</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[243]</td>
<td>8.3</td>
<td>3, 5, 1000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[1]</td>
<td>11.7-19.1, 400-2500</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>152</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
As we change the physical properties and flow rate to test different flow regimes, it is important to notice that spacer design is typically constrained by the formation fracture pressure and the pore pressure (the pore-frac envelope). This means that if the frictional pressure drop is too large (i.e. pumping a highly viscous spacer at a large flow rate), the spacer can fracture the formation or alternately allow an influx. Which of these is more likely is very well dependent, but in either case there is a frictional pressure constraint. In order to perform a more realistic analysis in comparing fluid designs, we keep the pumping capacity constant. More specifically, we impose that the total frictional pressure drop generated by the displacing fluid, over the length of well, down the pipe and up in the annulus, should be less than 150 psi (= 1034 kPa). The value 150 psi is representative of a typical safety margin, but is nominal in that different well plans would have different total frictional pressure losses.

To proceed, we will consider two casing sizes, representing a surface casing and a production casing.

### 6.1 Surface casing

We consider an annulus with the following geometrical parameters:

\[ \hat{D}_i = 13'' (\hat{r}_i = 16.5 \text{ cm}), \quad \hat{D}_o = 15'' (\hat{r}_o = 19 \text{ cm}), \quad \hat{z}_{bh} = 500 \text{ m} \quad (6.2) \]

Notice that displacement flows typically develop within a few diameters from the entrance, which is much shorter than the cementing section length. Therefore, for the sake of a better spatial resolution, we will only simulate the bottom 150 meters of the well.

Seven fluids with different properties are listed in Table 6.2 as the displacing fluid. These candidates represent a wide range of parameters, covering from laminar low Reynolds displacement to highly turbulent high Reynolds displacements. For each candidate, the flow rate is maximum flow rate possible without violating the pressure constraint. The flow rate is computed using the 1D hydraulic procedure, as laid down in Chapter 2.

Fluids A₁, A₂ and A₃ are all significantly heavier than the mud and they all have...
Table 6.2: Candidate preflush fluids for displacement in the surface casing.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>(\rho_2) (ppg)</th>
<th>(n_2)</th>
<th>(\kappa_2) (Pa·s)</th>
<th>(\dot{\Omega}) (bbl/min)</th>
<th>(\mu_e) (Pa·s)</th>
<th>Features</th>
<th>Turbulent when (e = 0.3)?</th>
<th>Turbulent when (e = 0.4)?</th>
<th>Turbulent when (e = 0.6)?</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>11.3, 1350</td>
<td>1</td>
<td>0.04</td>
<td>0, 0</td>
<td>0.039, 1.38</td>
<td>0.04</td>
<td>highly viscous, no yield stress</td>
<td>no</td>
<td>transitional</td>
</tr>
<tr>
<td>A2</td>
<td>11.3, 1350</td>
<td>1</td>
<td>0.01</td>
<td>4.2, 2</td>
<td>0.043, 1.50</td>
<td>0.043</td>
<td>moderately viscous, small yield stress</td>
<td>partially turbulent</td>
<td>partially turbulent</td>
</tr>
<tr>
<td>A3</td>
<td>11.3, 1350</td>
<td>0.5</td>
<td>0.30</td>
<td>0, 0</td>
<td>0.049, 1.72</td>
<td>0.036</td>
<td>shear thinning, no yield stress</td>
<td>partially turbulent</td>
<td>partially turbulent</td>
</tr>
<tr>
<td>A4p</td>
<td>10.0, 1200</td>
<td>1</td>
<td>0.04</td>
<td>0, 0</td>
<td>0.039, 1.38</td>
<td>0.04</td>
<td>no density difference, highly viscous, no yield stress</td>
<td>transitional</td>
<td>transitional</td>
</tr>
<tr>
<td>B1</td>
<td>11.3, 1350</td>
<td>1</td>
<td>0.001</td>
<td>0, 0</td>
<td>0.056, 1.99</td>
<td>0.001</td>
<td>low viscous, no yield stress</td>
<td>fully turbulent</td>
<td>fully turbulent</td>
</tr>
<tr>
<td>B2p</td>
<td>10.0, 1200</td>
<td>1</td>
<td>0.001</td>
<td>0, 0</td>
<td>0.060, 2.13</td>
<td>0.001</td>
<td>no density difference, low viscous, no yield stress</td>
<td>partially turbulent</td>
<td>highly turbulent</td>
</tr>
<tr>
<td>C</td>
<td>11.3, 1350</td>
<td>1</td>
<td>0.04</td>
<td>10.6, 5</td>
<td>0.016, 0.55</td>
<td>0.27</td>
<td>highly viscous, high yield stress</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

approximately similar effective viscosity based on their flow rates. The effective viscosity is computed using the mean velocity and mean gap width as the nominal shear rate, as illustrated by (5.1).

Figure 6.2 shows the snapshots of displacement together with the contours of flow regime at three different times, when the annulus is highly eccentric \((e = 0.6)\). The details on how to interpret the simulation figures, together with the numerical parameter used in the simulation were explained in §5.1. We observe that the flow regime is in transition to turbulence for Fluid A1 and partially turbulent for Fluids A2 and A3. In all cases however, the mud remains either in laminar or transitional regime, due to its larger yield stress. The change in the flow regime, both axially along the well and azimuthally around well is clearly depicted here. Despite the change in the flow regime from laminar and transitional in the case of Fluid A1 to turbulent in the case of Fluids A2 and A3, the displacement outcome does not appear to have improved significantly.

The displacement scenarios discussed above are all unsteady, meaning that the interface is faster on the wide side and slower on the narrow side. This leads to continuous elongation of the interface and accumulation of mud that is left behind on the narrow side. Ideally, we would like to avoid this. Two different directions may be pursued to improve the displacement efficiency: i) reduce the viscosity.
of the spacer and enhance turbulence (Fluid B) and ii) increase the yield stress of the spacer and rely on viscoplastic stresses (Fluid C). The displacement snapshots for these two choices are shown in Figure 6.3. In case of Fluid B (Figure 6.3a) the turbulent regime expands and is found all around the annulus within Fluid B. The interface is still progressing unsteadily, however the wide and narrow side velocity difference has shrunk slightly (differential velocity criteria has improved), as can be seen by the large volume of mud that is displaced on the narrow side. The displacement is of course improved, which appears to be due to the turbulent regime. On the other hand, for Fluid C, the displacement has deteriorated, as the mud on the narrow side barely moves.

Before analyzing the displacement efficiency more closely, we also consider two preflushes that are not any heavier than the mud (fluids A_p and B_p). The displacement snapshots for these two fluids are plotted in Figure 6.4. Compared to their counterpart examples with density difference, we observe that these two fluids displace the mud very poorly, leaving a large layer of mud unyielded on the narrow side. This may seem intuitive, but bear in mind that one strategy to enhance displacement quality that is often cited in literature [258] is to use a lightweight preflush that can be pumped in turbulent regime. Figure 6.4 disproves this idea entirely. In Chapter 7, we will further investigate the use of lightweight preflushes. Also notice that the pressure limit is less of a concern for these two fluids, meaning that we might be able to pump them at higher flow rates, as we have decreased the static pressure component. However, this will not improve the displacement, because the displacement here enters the too turbulent regime (see the study in §5.4), where increasing the flow rate has no effect on the displacement.

To compare the preflush candidates in Table 6.2 more precisely, it is customary in the literature to quantify the displacement using a volumetric efficiency \( \eta(t) \), which is the percentage of mud that is displaced. Here we compute the efficiency in the bottom 100 meters of the well. Mathematically, this is equivalent to:

\[
\eta(t) = \frac{\int_{0}^{\xi} \int_{\phi}^{\xi} r_{a} H c_{2}(\phi, \xi, t) d\phi d\xi}{\int_{0}^{\xi} \int_{\phi}^{\xi} r_{a} H d\phi d\xi}, \quad \xi_{\eta} = \frac{100}{(\pi r_{a})}
\] (6.3)
Figure 6.2: Effect of flow regime in a largely eccentric surface casing. For each subfigure, left panels show displacement snapshots at three different times, with white lines denoting the streamlines $\Delta \Psi = 0.25$ and right panels shows the corresponding flow regime map. In the regime maps, dark gray, light gray and white regions are laminar, transitional and turbulent, respectively and black regions are unyielded fluid. Well geometry is given by (6.2) with $e = 0.6$, displaced fluid properties are given by (6.1) and displacing fluid properties are given in Table 6.2: a) case $A_1$; b) case $A_2$ and c) case $A_3$. 

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Recall that $\hat{\xi} = \xi \times (\pi \hat{r}_a)$. For a uniform well, (6.3) is simplified to

$$\eta(t) = \frac{1}{\xi \eta} \int_{\tilde{\xi}_\eta}^{\xi_\eta} \int_0^1 Hc_2(\phi, \xi, t) d\phi d\xi. \quad (6.4)$$

Notice that the above definition of efficiency might be somewhat deceptive. This is because the volume of annulus on the narrow side is smaller than the wide side, therefore when the mud on the wide side is displaced successfully, the value of
volumetric efficiency grows rapidly. This might happen in spite of having the mud left behind on the narrow side, but that will not be noticed, because the volume of narrow side is smaller, and does not influence volumetric efficiency as much. Nevertheless, from the perspective of well leakage, a residual mud channel is a severe problem. As an example, for an annulus with $e = 0.6$, the widest quartile of annulus has a volume 3.25 times larger than that of the narrowest quartile. This number grows to 6.15, if the eccentricity is $e = 0.8$. This is particular problematic, because in annuli with high eccentricity, the value of volumetric efficiencies can reach as high as 80-90%, even if the displacement is poor on the narrow side. In fact, this is the case for the displacement example shown above. Figure 6.5a plots the volumetric efficiency $\eta$ as a function of time ($t$) for all the seven preflush candidates in Table 6.2. Although none of displacement examples can be called successful, as clearly illustrated in Figures 6.2-6.4, efficiency values are as high as 90%.

To account for this factor, we define a more stringent measure of efficiency, which is solely based on the displacement on the narrow side. More specifically, we only look at the displacement in the narrowest quartile of the annulus:

$$\eta_N(t) = \frac{\int_0^{\xi_0} \int_{\frac{1}{2}}^{\frac{1}{4}} H c_2(\phi, \xi, t) d\phi d\xi}{\int_0^{\xi_0} \int_{\frac{1}{2}}^{\frac{1}{4}} H c_2(\phi, \xi, t) d\phi d\xi} = \frac{4\pi}{\pi - 2\sqrt{2}\pi} \int_0^{\xi_0} \int_{\frac{1}{2}}^{\frac{1}{4}} H c_2(\phi, \xi, t) d\phi d\xi$$

(6.5)

Figure 6.5b plots the narrow side displacement efficiency $\eta_N$ vs time for all the seven preflush candidates in Table 6.2. As expected, the narrow side efficiency reflects a better picture of the displacement quality. We observe roughly two-third of the mud in the narrowest quartile of the annulus is left behind. In fact, the best score is for Fluid B, and then Fluids C and A_2, all at around 30-35%. This is interesting, because the laminar displacement (Fluid C) performed almost equally good as the partially turbulent displacements (Fluids A_1 and A_2), and fully turbulent displacement (Fluid B). More critically, the Fluids A_p and B_p which are both flowing in fully turbulent regime did not move the mud on the narrow side at all, and their efficiency score remains zero. These observations suggest that the notion that “turbulent flow cementing yields improved results and reduces the amount of remedial work required”[32] needs some adjustment.
Figure 6.5: Displacement efficiency as a function of time. Well geometry is given by (6.2) with \( e = 0.6 \), mud properties are given by (6.1) and preflush properties are given in Table 6.2. The green line indicates the (dimensionless) arrival time, based on the mean velocity. a) volumetric efficiency \( \eta \); b) narrow side efficiency \( \eta_N \).

Upon closer inspection, it appears that the single parameter that has made the displacement examples above unsuccessful is the eccentricity of the annulus. To elucidate the critical role of eccentricity, we have repeated the above simulations for a slightly less eccentric annulus (\( e = 0.4 \)). Figure 6.6 shows the snapshots of displacement examples for five fluid candidates in Table 6.2, when \( e = 0.4 \). The Fluids A\(_p\) and B\(_p\) are not shown here, as their displacement performance remains poor. Because the annulus is less eccentric, the velocity profiles are slightly more uniform, but still similar flow regimes are found for the different fluid candidates. For example, the flow regime varies from fully turbulent for Fluid B to partially turbulent for Fluid A\(_2\) to transitional for Fluid A\(_1\), and finally to fully laminar for the Fluid C. Although the displacement regimes remain relatively unchanged, the displacement efficiency is improved significantly, as shown in Figure 6.7 a. Here four candidate fluids have reached a narrow side efficiency of 90% or higher. Namely, Fluids A\(_2\) and B efficiency reaches 99%, which could be attributed to the turbulent regime. Fluid C also did almost as well. More critically, Fluid A\(_3\) and A\(_p\) and B\(_p\) did not perform well, despite the fact that they were in turbulent or
Figure 6.6: Same as Figure 6.2, except $e = 0.4$ and a) case A1; b) case A2; c) case A3; d) case B and e) case C.
transitional regimes. In Figure 6.7 b, we further decrease the eccentricity to $e = 0.3$ and Fluids $A_1$, $A_2$, $B$ and $C$ reach to 0.95% narrow side efficiency or higher.

### 6.2 Production casing

For the second part of our analysis, we take a smaller annulus with the following geometrical parameters:

$$\hat{D}_i = 4''(\hat{r}_i = 5.1\text{cm}), \quad \hat{D}_o = 6''(\hat{r}_o = 7.6\text{cm}), \quad \hat{\xi}_{bh} = 1500\text{m} \quad (6.6)$$

This represents a production casing. Notice that the annulus length is now much longer, which will result in having smaller flow rates, as the total pressure drop is still limited to 150 psi. Similar to the previous section, we only simulate the bottom 150 m of the well to study displacement (although the entire flowpath is considered in calculating the maximal flow rates).

Five fluids with different properties are listed in Table 6.3 as the displacing fluid. Similar to the previous section, the maximum allowed flow rates are computed using the 1D hydraulics model of Chapter 2. Although the physical pa-
rameters vary quite significantly, the flow rates remain relatively small due to the pressure drop limit, and as a result, in most cases only laminar flows are present in the annulus.

Table 6.3: Candidate preflushes for displacement in the production casing.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>$\rho_2$ (ppg)</th>
<th>$\kappa_2$ (Pa s)</th>
<th>$\nu_2$ (htbl$^3$/Pa)</th>
<th>$Q$ (bbl/min)</th>
<th>$\mu_{eff}$ (Pa s)</th>
<th>features</th>
<th>turbulent when $e = 0.3$?</th>
<th>turbulent when $e = 0.6$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>1350</td>
<td>1</td>
<td>0</td>
<td>0.0039, 0.38</td>
<td>0.04</td>
<td>highly viscous, no yield stress</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>A2</td>
<td>1350</td>
<td>1</td>
<td>0.01</td>
<td>0.0063, 0.48</td>
<td>0.051</td>
<td>moderately viscous, small yield stress</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>A3</td>
<td>1350</td>
<td>0.5</td>
<td>0.25</td>
<td>0.0057, 0.56</td>
<td>0.05</td>
<td>shear thinning, no yield stress</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>B</td>
<td>1350</td>
<td>1</td>
<td>0.001</td>
<td>0.0085, 0.84</td>
<td>0.001</td>
<td>low viscous, no yield stress</td>
<td>fully turbulent</td>
<td>partially turbulent</td>
</tr>
<tr>
<td>C</td>
<td>1350</td>
<td>1</td>
<td>2.5</td>
<td>0.0005, 0.05</td>
<td>1.3</td>
<td>highly viscous, high yield stress</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

Figure 6.8 shows the narrow side displacement efficiency $\eta_N$ as a function of time for two values of eccentricity $e = 0.6$ and $e = 0.3$. For the more eccentric annulus, the displacement is relatively poor on the narrow side for all preflush candidates. Notice that here only Fluid B is in turbulent regime, and the rest are in laminar regime. Nonetheless, even Fluid B does not reach any satisfactory efficiency. More interestingly, the fluid that outperforms the other candidates is Fluid C, which has the largest rheological parameters and smallest Reynolds number. Fluid C even performs better than Fluid B, which is in turbulent. When the eccentricity is reduced to $e = 0.3$, the displacement outcome is substantially improved. Here, Fluid B reaches perfect displacement ($\eta_N = 1$), and then Fluids A1, A2 and C with narrow side efficiency of 95%.

Similar to the previous section, it appears that the displacement regime only marginally influences the displacement outcome. Importantly, it is not the turbulent regime that outperforms other displacement regime. Indeed, in the more eccentric annulus, the highly viscous low Reynolds displacement displaced the mud better than other candidates.
6.3 Removing the preflush

In the two previous sections, we were investigating how to design an ideal preflush based on its ability to remove mud from the annulus and achieve a high displacement efficiency. Another aspect of the design is to see how these preflush candidates are removed by the cement slurry. In particular, is there one that is removed easier or harder compared to the other ones? We choose a cement slurry with the following properties:

\[ \hat{\rho}_2 = 1550 \text{ kg/m}^3, n_2 = 1, \hat{\kappa}_2 = 0.05 \text{ Pa.s and } \hat{\tau}_{Y,2} = 5 \text{ Pa}. \]  (6.7)

The cement is highly viscous and has a moderate yield stress. Therefore, we would expect that it flows only in laminar regime. However, the preflush candidates can still flow in laminar or turbulent regimes, depending on the displacement flow rate. For simplicity, we opt to work with same geometry as in §6.1. The displaced fluids (the preflush that is to be removed by the cement) are those listed in Table 6.2. We keep the flow rate as indicated in Table 6.2. We aim to see whether the flow regime of the displaced fluid influences the displacement or not.
Figure 6.9: Same as Figure 6.2, except the displacing fluid properties are given by (6.7) and the displaced fluid properties are given in Table 6.2: a) case $A_1$; b) case $A_2$; c) case $A_3$; d) case $B$ and e) case $C$. 
**Figure 6.10:** Displacement efficiency as a function of time. Well geometry is given by (6.2) with $e = 0.6$, displacing fluid properties are given by 6.7 and displaced fluid properties are given in Table 6.2. The green line indicates the (dimensionless) arrival time, based on the mean velocity. a) volumetric efficiency $\eta$; b) narrow side efficiency $\eta_N$.

Figure 6.9 plots the displacement snapshots together with contours of flow regime for all seven preflush candidates in Table 6.2. Although the annulus is relatively largely eccentric, the displacement outcomes are satisfactory. More precisely, Figure 6.10 shows the volumetric efficiency ($\eta$) as well as the narrow side efficiency ($\eta_N$). Although $\eta > 0.95$ is achieved in all displacement cases, the displacement on the narrow side is poorer, as indicated in Figure 6.10 b. The best preflush candidates for removal are Fluids A$_3$ and C. Notice that Fluid A$_3$ performed very poorly in terms of mud removal. More interestingly, the fully turbulent candidates, Fluids B and B$_p$ are harder to remove than the more viscous laminar candidate, Fluid C.

Compared to the two previous sets of example in annuli with $e = 0.6$, the displacements show in Figure 6.9 have higher scores, both in terms of the overall efficiency and the narrow side efficiency. This is primarily due to the large density difference between the cement slurry and the preflush. The other contributing factor is the larger rheological parameters of the cement compared to those of the preflushes. These two factors compete against the effect of eccentricity.
6.4 Conclusion

This chapter presented a number of interesting simulations, mostly in mixed flow regimes. We explored different displacement scenarios to identify if the displacement regime has any effect on the displacement. In particular, we tested the notion that turbulent displacement is always preferred than laminar. Our analysis shows that

- Far more important than the flow regime is the effect of annulus eccentricity. Our simulations consistently confirmed that in a largely eccentric annulus (e.g. $e \gtrsim 0.6$), displacement of a mud with moderate yield stress and 10% density difference is generally unsuccessful, regardless of flow displacement regimes. On the other hand, in a mildly eccentric annulus (e.g. $e \lesssim 0.3$) displacement is typically successful.

- The other key parameter in achieving successful displacement is having sufficient density difference between the displacing and displaced fluids. In particular, reducing displacing fluid density to achieve turbulent displacement was shown to be extremely unsuccessful, no matter how turbulent the flow is.

- There is no clear indication that turbulent displacement always outperforms laminar displacement. On the contrary, we showcased examples that the highly viscous low Reynolds displacement flow achieved a better displacement efficiency compared to any other choice, including partially and fully displacement flows.

We feel the value of this chapter is not specifically in the example considered, as arguably slightly different parameters might favour particular fluid design strategies. We see the contribution as threefold: i) First, we hope that this leads to other researchers correctly describing the fluid design problem in terms of operational constraints and then experimenting within those constraints to see which design performs better. ii) Second, in selecting measures of success, volumetric bias in the displacement efficiency needs to be countered. Our narrow side efficiency is
offered as one sensible measure that targets the typical problem area. iii) Lastly, although industrial practice likes simple statements/rules, selecting a turbulent flow regime as being “better” does not stand up to serious analysis and this is an area where engineers need to work hard on specific wells with simulation tools such as these, before making a design decision.
Chapter 7

Using washes for primary cementing

In the two previous chapters, we primarily focused on density stable displacement flows, meaning that a heavier fluid displaces a lighter fluid in the annulus. In this chapter, we consider the opposite scenario where a lighter fluid displaces a heavier fluid (i.e. density unstable). This is particularly relevant in understanding the effect of low viscous lightweight preflushes in primary cementing. Intuitively, it is difficult to assume how such a fluid, which is in most cases water, can displace a heavy mud in the annulus. A version of this chapter has appeared in Maleki and Frigaard [148].

Washes are a category of preflushes. Preflushes are often used as part of the sequence of fluids pumped in primary cementing. Usually two functions are served by preflushes: I) to wash the drilling fluid ahead, by a combination of turbulence and chemical reaction; II) to provide a chemically compatible spacer between the lead slurry and the drilling mud. Preflushes can be generally divided into two categories: I) Light weight, low viscosity preflushes known as washes and II) weighted and viscosified preflushes known as spacers. In some cases a wash precedes a spacer, but often only a single preflush is used. In using a wash, protocols suggest that a minimum contact time is met. The contact time here refers to the time taken for the wash to pass a position in the annulus. Typically, this minimum contact time is 10 minutes, although shorter contact times (e.g. 5 minutes) may be recom-
mended when the flow is fully turbulent [167].

Washes may be water-based or oil-based. Rheologically, they are generally Newtonian fluid solutions (e.g. water). They are designed to wash the walls of the annulus free from residual fluids (and any remaining solids), to leave the annulus water-wet for the cement slurry. In addition, they should break any static gelation of the mud, mobilizing the mud in general. The low viscosity and density of these fluids allows them to be pumped in turbulent flow regimes. Turbulent displacement is believed to be more effective in terms of cleaning the annulus walls [167, 140], although in operational practice, turbulent flow may not be achieved by all the fluid in the sequence, e.g. due to pump limitation or pore/frac restrictions.

Spacers are heavier preflushes with viscosifying constituents. Because of their larger density and viscosity, spacers are more often pumped in laminar regime. The two roles of a spacer are: (i) to separate cement slurry from the drilling mud; (ii) to aid displacement of the drilling mud by careful rheological design. Mud-slurry separation is often necessary because the drilling mud and cement slurry can be chemically incompatible.

In this chapter, we primarily focus on washes. Our intention here is to evaluate the efficiency of displacements involving a wash. In particular, we challenge the notion that light weight low viscous preflushes provide effective chemical cleaning of the annulus and address the following questions: I) Does the preflush mobilize the mud around the annulus? II) In the context of using washes, is turbulent displacement more successful in displacing the drilling mud? III) Does the preflush provide the pre-designed contact time needed for chemical cleaning?

### 7.1 Displacement parameters

To focus our study we adopt a range of physical and geometrical parameters typical of wells drilled in North Eastern British Columbia, when setting surface casing. We consider a vertical annulus of length 450 m, with inner and outer radii of \( r_i = 24.5 \) cm and \( r_o = 31.0 \) cm, respectively. For simplicity we assume the annulus is uniformly eccentric, and 3 values of eccentricity are studied; \( e = 0.1, 0.3 \) and 0.5.

As we emphasize the role of the preflush here, we keep the rheology and density of mud and cement constant throughout this chapter. It is assumed that all ce-
menting fluids can be characterized rheologically as Herschel-Bulkley fluids. The following physical parameters are assumed:

**Mud:** $\hat{\rho}_1 = 1200 \text{ kg/m}^3, n_1 = 1, \hat{\kappa}_1 = 0.02 \text{ Pa.s}, \hat{\tau}_{\gamma,1} = 5 \text{ Pa},$

**Cement:** $\hat{\rho}_3 = 1700 \text{ kg/m}^3, n_3 = 0.6, \hat{\kappa}_3 = 0.4 \text{ Pa.s}^{0.6}, \hat{\tau}_{\gamma,3} = 7 \text{ Pa}.$

The subscripts “1” and “3” denote mud and cement, respectively. We have saved the superscript “2” for the preflush.

The cement slurry properties are within the typical range for oilfield cements and the mud is not particularly extreme, rheologically speaking. Indeed, in the absence of the preflush the cement slurry is able to effectively displace the mud, even at $\epsilon = 0.5$. We now vary the density and viscosity of the wash to explore different flow scenarios.

**Preflush:** $\hat{\rho}_2 = 1050, 1100 \text{ and } 1200 \text{ kg/m}^3, n_2 = 1, \hat{\kappa}_2 = 0.001, 0.02 \text{ Pa.s and } \hat{\tau}_{\gamma,2} = 0 \text{ Pa}.$

For all simulations we assume the annulus is initially filled with the drilling mud. The cementing commences with pumping a preflush volume calculated to give 10 minutes of contact time. To study the displacement in both laminar and turbulent regimes, we consider two flow rates: $\hat{Q}_{pf} = 0.015$ and $0.075 \text{ m}^3/\text{s}$. The associated mean velocities are 0.13 and 0.66 m/s. The cement slurry follows the preflush at fixed flow rate $\hat{Q}_{cs} = 0.015 \text{ m}^3/\text{s}$. Note that this reduction in flow rate (for the turbulent case) is to ensure that only the dynamics and behaviour of the preflush displacement are varied. We make comparisons between different cases at the time when the cement slurry has progressed half way along the annulus. We start below by using a relatively viscous preflush ($\hat{\kappa}_2 = 0.02 \text{ Pa.s}$). The final examples use a more realistic value of viscosity ($\hat{\kappa}_2 = 0.001 \text{ Pa.s, i.e. water}$) and show the dynamic of displacement is unchanged.

### 7.2 Laminar wash

In this section we consider several displacement scenarios in which the preflush is in laminar flow regime. Our first three examples start with a rather heavy and viscous preflush (green fluid in all figures). The density and viscosity of the preflush are $\hat{\rho}_2 = 1200 \text{ kg/m}^3$ and $\hat{\kappa}_2 = 0.02 \text{ Pa.s}$ and the preflush flow rate is $\hat{Q}_{pf} = 0.015 \text{ m}^3/\text{s}$. Figure 7.1 shows snapshots of the displacement (top panels) and flow regime.
Figure 7.1: Laminar pre-flush displacement in a nearly concentric annulus; \( e = 0.1 \). The wide and narrow sides of annulus are marked with a W and N, and only half of the annulus is shown. Preflush density and viscosity are \( \hat{\rho}_2 = 1200 \text{ kg/m}^3 \) and \( \hat{\kappa}_2 = 0.02 \text{ Pa.s} \) and the pre-flush flow rate is \( \hat{Q}_{pf} = 0.015 \text{ m}^3/\text{s} \). Top panels: Red, green and blue fluids are mud, preflush and cement slurry respectively. Bottom panels: static regions are shaded black and moving laminar regions are highlighted in gray. In future figures, we show turbulent regions in white.

(bottom panels) when the well is nearly concentric (\( e = 0.1 \)). In the top panels, the red, green and blue fluids represent mud, preflush and cement slurry, respectively. In the bottom panels, the black regions are stationary (where yield stress is not surpassed and the fluid is, therefore, stuck). The gray regions are moving laminar regions (and in following figures, we show turbulent regions in white).

The preflush high shear viscosity and density is identical to those of mud. However, the mud has a yield stress (of 5 Pa.) and the annulus is slightly eccentric.
These combine to favour flow of the preflush along the wide side. We observe that where the preflush flows on the wide side, the wall shear stress generated by the preflush is insufficient to mobilize the mud on the narrow side: a static mud channel is formed. Interestingly, the mud flowing alone (before/after) the wash is in fact mobile on the narrow side. When the cement slurry enters the annulus, both fluids are fully displaced. This interesting effect is due to buoyancy, i.e. the pressure gradient is approximately equal around the annulus. The lighter fluid requires pressure gradient to move upwards and hence the pressure gradient is reduced in the mud also.

What this example illustrates is that even if the well is only slightly eccentric, the preflush displacement is likely to be restricted to the wide side of the annulus. As might be expected, if the annulus is more eccentric, the preflush will be contained within a tighter region of the annulus. Figures 7.2 and 7.3 confirm this. In
Figure 7.2 we observe as the annulus becomes more eccentric (here $e = 0.3$), the preflush moves more rapidly in a narrower channel on the wide side of annulus. Effectively, this leaves a larger mud layer stuck on the narrow side as the mud passes. However now at these larger eccentricities, the mud is also static on the narrow side after the wash passes. Interestingly, the cement slurry itself, is stationary on the narrow side. When we further increase the eccentricity, the preflush moves faster in a narrower channel and becomes partially turbulent as shown in Figure 7.3 (notice the turbulent regions are highlighted in white in the lower panels).

In the previous three examples the preflush is effectively contained within the wide side of annulus and appears to have no impact on the mud on the narrow side. In the next few example, we will see that once we reduce the density of preflush (more representative of preflushes used in the industry), the preflush displacement is even less successful. Figure 7.4 and 7.5 show displacement where the preflush
density is reduced to 1100 and 1050 kg/m$^3$ respectively. The value of eccentricity is now fixed at $e = 0.3$ and all other parameters are kept constant. We observe in both cases, the preflush displacement becomes unstable which leads to fingering type of instabilities. The instability originates with the lower density of the preflush. The fingering instabilities spread the preflush randomly around the annulus. Although a larger volume of mud is mobilized as a result of the spreading compared to previous examples in Figure 7.2, the notion that the preflush might provide a buffer layer, separating the mud and cement slurry, has proved to be flawed. Notice here that although our laminar model does not have any mixing mechanism (because $Pe \gg 1$; see Bittleston et al. [24]), there is still significant dispersion due to secondary flows driven by the fingering instabilities. This dispersive mechanism is clearly stronger in Figure 7.5 where the density of the preflush is lowest.

**Figure 7.4:** Same as Figure 7.1, except $e = 0.3$ and $\hat{\rho}_2 = 1100$ kg/m$^3$
7.3 Turbulent wash

In this section we study the displacement of mud using a preflush in turbulent flow regime. In particular, we are interested to understand if the turbulent displacement can make the preflush displacement successful, i.e. mobilize the mud around the annulus and separate the mud and the cement slurry. Similar to previous section, we start with a preflush with density and rheology identical to those of mud. To ensure the displacement is turbulent, the flow rate is increased to 0.075 m$^3$/s during pre-flushing. All other physical parameters are kept constant for now.

The first example starts with the heaviest preflush ($\hat{\rho}_2 = 1200$ kg/m$^3$) in a nearly concentric annulus ($e = 0.1$). Figure 7.6 shows the snapshots of the displacement as well as the flow regime. In contrast to the laminar case, here we observe that the preflush mobilizes the mud on both wide and narrow sides of annulus. The
narrow side, however, moves at a lower speed which leads to the elongation of the interface. Comparing Figures 7.1 and 7.6 clearly shows that the turbulent displacement is more effective in mobilizing the mud and providing cleaning on the walls. Once the eccentricity is increased however, the static mud channel again forms on the narrow side. Figure 7.7 shows the turbulent displacement of preflush when $e = 0.3$. The preflush on the wide side rapidly travels the annulus in turbulent regime, and leaves the mud on the narrow side stationary. Note that in these two examples, as the cement slurry enters the annulus the flow rate is reduced to $\hat{Q}_{cs} = 0.015 \text{ m}^3/\text{s}$, which is why the preflush becomes laminar. Without this reduction the preflush advances more rapidly along the wide side by-passing the mud, as in the first few panels of Figures 7.6 and 7.7.
Figure 7.7: Same as Figure 7.1, except $e = 0.3$ and $\dot{\varphi}_{pf} = 0.075$ m³/s.

The previous example considered a dense preflush, which resulted in having a relatively sharp interface between the fluids. The next examples investigates turbulent displacement when the preflush is lighter than the mud ($\bar{\rho}_2 = 1100$ kg/m³). Figure 7.8 shows snapshots of the displacement and flow regime. As expected the interface between the preflush and mud becomes unstable and fingering instabilities mix the preflush around the annulus leaving a mixture of preflush and mud. We may expect that this mixture has a smaller yield stress stress than the mud alone and therefore is more easily mobilized.

7.4 Actual contact time

As discussed earlier, industrial protocols suggest a minimum contact time that the wash must be pumped in the annulus such that cleaning objectives are met. A typ-
ical designed minimum contact time is 10 minutes [167]. To analyse the notion of contact time, we measured the actual contact time ($\hat{t}_{ACT}$) everywhere inside the annulus, during each displacement. As the fluids mix, contact with the preflush needs defining. At any position within the annulus and at any time during the displacement, if the concentration of preflush is larger than 20% of the total concentration, we regard that the mud is in contact with the preflush.

Figure 7.9 shows the contours of contact time associated with laminar preflushing, as studied. From left to right, the density of preflush is reduced, which resulted in fingering instabilities (illustrated previously in Figures 7.4 and 7.5). We observe that the localised nature of the preflush confines contact time to a narrow region on the wide side (left panel). The buoyancy-induced instabilities do mix the preflush more effectively around the annulus and as a result, the 10-minute minimum contact time criterion is met in a larger area, although coverage is still marginal (centre and right panel).

While actual contact time gives a better picture of the displacement provided by the preflush, it is critically important to realize that the contact time itself may not guarantee cleaning objectives are met. The idea of the wash is based on having turbulent flow at the same time as contact. Therefore a second and more telling measure of contact time is the actual turbulent contact time ($\hat{t}_{ATCT}$), which considers the contact time at which preflush is flowing in turbulent regime.

Figure 7.10 compares the actual contact time and the actual turbulent contact
Figure 7.9: Actual contact time (in seconds) for laminar preflush. $e = 0.3$ and $\hat{\kappa}_2 = 0.02$ Pa.s. From left to right, $\hat{\rho}_2 = 1200, 1100$ and $1050$ kg/m$^3$. Top row: $\hat{Q}_{pf} = 0.015$ m$^3$/s (laminar). Bottom row $\hat{Q}_{pf} = 0.075$ m$^3$/s (turbulent)

...time for a turbulent displacement with two different preflush densities. Looking at the top row, we see once the preflush density is reduced, the minimum (10-minute) contact time is satisfied in a wider area. As explained earlier, this is due the bulk dispersion driven by the viscous fingering and secondary flows. However, the bottom row shows that when we consider actual turbulent contact time, apart from a thin layer on the wide side, nowhere else in the annulus satisfies the minimum 10-minute contact time. Reducing the density of preflush does not improve the actual turbulent contact time, perhaps even exacerbates it. A similar story can be found in Figure 7.11 where the actual contact time and actual turbulent contact time are compared in a turbulent displacement in which vary the annulus eccentricity. Here again we observe the actual turbulent contact time is barely met only in the wide side of annulus, even if the well is only slightly eccentric.

7.5 Low viscous wash

The examples illustrated above are all for a preflush with a relatively high viscosity ($\hat{\kappa}_2 = 0.02$ Pa.s), while in many situations washes have viscosities similar to the
Figure 7.10: Actual contact time (top row) vs actual turbulent contact time (bottom row) in seconds. $\kappa_2 = 0.02$ Pa.s. $e = 0.3$. Left column: $\bar{\rho}_2 = 1200$ kg/m$^3$; Right column: $\bar{\rho}_2 = 1100$ kg/m$^3$

viscosity of water ($\sim 0.001$ Pa.s). The final few examples we study investigate a wash with such low viscosity and low density. Figure 7.12 and 7.13 show snapshots of the displacements two different flow rates.

Figure 7.12 is a turbulent displacement with a low viscous lightweight preflush, the closest example to common current industrial practice. Here the interface between the preflush and mud is doubly unstable, because of both the lower density and lower viscosity of the preflush compared to those of the mud (at high shear). Consequently, the fingering instabilities and bulk dispersion effects are
Figure 7.11: Contours of the actual contact time (top row) vs actual turbulent contact time (bottom row) in seconds. \( \hat{k} = 0.02 \) Pa.s. \( \hat{Q}_{pf} = 0.075 \) m\(^3\)/s. From left to right, \( e = 0.1, 0.3 \) and 0.5.

much stronger, rapidly mixing the preflush around the annulus. The resulting mixture has a lower yield stress (see Figure 7.12, bottom row). Note that here the preflush is turbulent at the lower imposed flow rates, due to the lower viscosity.

Interestingly, when we increase the flow rate, we see the mixing mechanism largely disappears, suggesting that the interface is now not as unstable as the previous example. This is because when the flow rate is increased, the turbulent stresses dominate the buoyancy stress. Effectively, this means the density difference between the preflush and mud is not felt. Thus the interface remains fairly stable.
Figure 7.12: Same as Figure 7.1, except $\rho_2 = 1050 \text{ kg/m}^3$ and $\kappa_2 = 0.001 \text{ Pa.s}$. The bottom row indicates the yield stress of the mixture at the different times.

(Figure 7.13).

The above observation can also explain the actual contact time and the actual turbulent contact time shown in Figure 7.14. When the flow is more turbulent here, mixing is significantly reduced and therefore the contact time is only high in the wide side.

7.6 Conclusion

In this chapter, we have demonstrated that when using low viscosity and density washes, the wash progressively advances ahead of the lead slurry, channeling rapidly up the wide side of the annulus. Even when fully turbulent, it is ineffective
**Figure 7.13:** Same as Figure 7.1, except $\hat{\rho}_2 = 1050 \text{ kg/m}^3$, $\hat{\kappa}_2 = 0.001 \text{ Pa.s}$ and $\hat{Q}_{pf} = 0.075 \text{ m}^3/\text{s}$.

**Figure 7.14:** Actual contact time (top row) vs actual turbulent contact time (bottom row) in seconds. $\hat{\rho}_2 = 1050 \text{ kg/m}^3$ and $\hat{\kappa}_2 = 0.001 \text{ Pa.s}$. left column $\hat{Q}_{pf} = 0.015 \text{ m}^3/\text{s}$. right column $\hat{Q}_{pf} = 0.075 \text{ m}^3/\text{s}$.
at displacing mud from around the annulus. Furthermore, the advance along the wide side of the annulus drains the volume of fluid which separates the cement from the drilling mud. Thus, the idea that the preflush provides a barrier between slurry and and mud, shielding incompatibility is largely invalid.

We have also computed the actual contact time and actual turbulent contact time from our simulations. Our threshold of > 20% wash concentration is quite generous, but still we are unable to ensure a uniform distribution of contact times. One should note too that due to the localization observed, increasing wash volumes does not necessarily help with contact time. This largely invalidates the motivation of measuring chemical cleaning efficiency through a bulk contact time. If this concept is to be rescued it needs local computations, as here, and more attention need to be paid to the fluid mechanics of the actual displacement flow.

Also, still missing in such designs is a mechanical measure of local wall shear stresses. Studies such as Allouche et al. [9], Zare et al. [254] show that static residual mud layers can persist at the wall wherever the yield stress is not overcome by the wall shear stress of the displacing fluid. Thus, it is not only advisable to compute a contact time but also a measure of effectiveness of mechanical removal, based on wall shear stress and yield stress. As illustrated in Guo et al. [101], such measures often agree well with post-job evaluation techniques.

Unlike in previous similar studies such as Guillot et al. [99] where relatively complex scenarios were studied, here the setting is about as simple as can be: a vertical surface casing in a mildly eccentric well with a modest drilling fluid to be displaced. This reinforces the suggestions of Guillot et al. [99] that washes are often ineffective. The parameters chosen for this study were representative of local wells. In the context of typical wells currently drilled in British Columbia, a second issue with lightweight washes comes from cementing the production casing. Commonly the production casing runs to surface, but only a 200m overlap of cement with the surface casing is required. Therefore, if the production casing is not cemented to surface the interval above top of cement will contain a mix of wash and drilling mud, the former of which offers no protection against casing corrosion and is typically chemically loaded, i.e. ideally washes should be circulated from the well.
Chapter 8

Tracking displacement interface using suspending particles

The outcome of a primary cementing operation may have many defective features that are not detected during the cementing operation. For example, mud channels and wet micro-annuli may form, annular volumes are only approximately known, fluid losses may occur, eccentricity may result in a top of cement that is non-uniform (higher on the wide side) and the fluids may move significantly post-placement. Such imperfections call for a robust and sensitive assessment technology. Currently, the quality of a cement job quality is evaluated using a variety of logging methods (CBL, VDL, USIT, etc). The most common form is CBL (cement bond log), which measures acoustic amplitude, travel time and attenuation. A variety of tools are available providing different spatial resolutions, depths, logging speeds etc. Interpretation of the signals infers information about how well the cement is bonded to the inside of the casing. CBL readings generally indicate the top of cement fairly clearly and other large-scale features/defects, see e.g. Tardy et al. [227]. However, the CBL is only run when the cement has fully set, which may involve a costly time delay. Thus, CBL is not run on the majority of cement jobs, directly afterwards, although many wells will have CBL run later in their lifetime, before decommissioning. In addition, since logging is only performed after the cement is set, no corrective measures can be pursued if the cement job went wrong, except performing remedial cementing, which are expensive and can jeopardize
8.1 A simplified view of annular displacements and interface tracking

8.1.1 Annular cementing fluid mechanics

As mentioned earlier, most cemented annuli are eccentric, even when wells are vertical; see e.g. Guillot et al. [100]. Nevertheless, a moderately eccentric well can still be cemented successfully. Pelipenko and Frigaard [181] showed that successful laminar primary cementing is synonymous with a steady state traveling wave displacement front. In this chapter, we simply accept that, for some combination of density and frictional pressure hierarchies between displacing and displaced fluids, there is a steady state. The interface between the two fluids advances at steady...
speed \( \hat{w}_i \) upwards along the well. Since the fluids are incompressible, \( \hat{w}_i = \hat{w} \), the mean velocity pumped. As discussed in depth earlier, both ahead of the interface and behind the interface, the annular eccentricity results in velocities that vary around the annulus, i.e. a wider gap means larger wall shear stresses which then lead to larger shear rates and a larger flow rate (vice versa on the narrow side). Therefore, in the far-field the fluid velocity along the annulus \( \hat{w} \) varies from \( \hat{w}_W \) on the wide side to \( \hat{w}_N \) on the narrow side of the annulus, where:

\[
\hat{w}_W \geq \hat{w}_i = \hat{w} \geq \hat{w}_N.
\]

As we approach the steadily advancing interface, this implies a redistribution of fluid from the far-field flows to match with the steady interface motion. This is achieved via secondary flows near the interface, which flow from the wide side towards the narrow side in the displacing fluid and from narrow to wide side in the displaced fluid, downstream of the interface. Note that the precise \( \hat{w}_W \) and \( \hat{w}_N \) will be different in displaced and displacing fluids, due to rheology differences between the fluids.

### 8.1.2 Particle migration

For now we neglect the secondary flows near the interface and address the question of whether a particle, released into the fluid upstream of the interface can catch the interface? We therefore suppose that the flow domain consists of two fluids with densities \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) where fluid 2 is below fluid 1 and the system is density stable (i.e. \( \hat{\rho}_1 < \hat{\rho}_2 \)). The interface moves with steady speed \( \hat{w}_i \), having position \( \hat{z}_i = \hat{w}_i \hat{t} \).

We assume that the fluid advances at speed \( \hat{w} \), which may be different from the interface velocity; see Figure 8.1.

The particle diameter is \( \hat{d}_p \), its position is denoted \( \hat{z}_p(\hat{t}) \) and its vertical motion is determined by the following momentum balance:

\[
\hat{m}_p \frac{d^2 \hat{z}_p}{d\hat{t}^2} = \hat{F}_D + \hat{F}_B,
\]

where \( \hat{m}_p \) denotes the particle mass, \( \hat{F}_D \) and \( \hat{F}_B \) are drag and buoyancy forces, respectively. The particle’s initial position and initial velocity are \( \hat{z}_{p,0} \) and \( \hat{w}_{p,0} \).
respectively. The particle volume is \( \hat{V}_p = \frac{\pi \hat{d}_p^3}{6} \) and the particle density is \( \hat{\rho}_p \). For simplicity, we model \( \hat{F}_D \) using Stokes drag:

\[
\hat{F}_D = -3\pi \hat{d}_p \hat{\mu} \left( \frac{d\hat{z}_p}{dt} - \hat{w} \right),
\]

with \( \hat{\mu} \) the fluid viscosity. Equation (8.1) becomes:

\[
\hat{\rho}_p \hat{V}_p \frac{d^2\hat{z}_p}{dt^2} = -3\pi \hat{d}_p \hat{\mu} \left( \frac{d\hat{z}_p}{dt} - \hat{w} \right) + \Delta \hat{\rho} \hat{g} \hat{V}_p, \tag{8.2}
\]

where \( \Delta \hat{\rho} = \hat{\rho} - \hat{\rho}_p \), in which \( \hat{\rho} \) is the fluid density. Considering Figure 8.1, the density of the fluid can be written as:

\[
\hat{\rho} = \hat{\rho}_2 + \mathcal{H}(\hat{z}_p - \hat{w} \hat{t}) (\hat{\rho}_1 - \hat{\rho}_2), \tag{8.3}
\]

where \( \mathcal{H}(x) \) is the Heaviside function, (i.e. \( \mathcal{H}(x) = 1 \) for \( x > 0 \), and zero otherwise). Let us assume that the particle density is taken intermediate between that of the two fluids: \( \hat{\rho}_p = 0.5(\hat{\rho}_1 + \hat{\rho}_2) \). The density difference in (8.2) simplifies to

\[
\Delta \hat{\rho} = \frac{\hat{\rho}_2 - \hat{\rho}_1}{2} \left[ 1 - 2 \mathcal{H}(\hat{z}_p - \hat{w} \hat{t}) \right]. \tag{8.4}
\]

We see that there are two competing effects in (8.2). The drag term drives the particle to move at the speed of the fluid \( \hat{w} \), and in the absence of the buoyancy force...
force we observe that the particle speed converges exponentially to \( \hat{w} \). On the other hand the term \( \Delta \hat{\rho} \) above switches sign as the interface is crossed, which keeps the particle accelerating towards the interface position.

We define dimensionless variables as follows:

\[
z_p = \frac{\hat{z}_p}{d_p}, \quad t = \frac{\hat{t}}{d_p/\hat{w}_i}, \quad w = \frac{\hat{w}}{\hat{w}_i}
\]

The dimensionless form of (8.2) is:

\[
\frac{Re_p}{18} \frac{d^2 z_p}{dt^2} + \frac{dz_p}{dt} - 1 = -\Delta w + \frac{1}{18} Bu (1 - 2 \mathcal{H}(z_p - t))
\]

(8.5)

where

\[
Re_p = \frac{\hat{\rho}\hat{w}d_p}{\mu}, \quad Bu = \frac{(\hat{\rho}_2 - \hat{\rho}_1)\hat{g}d_p^2}{\hat{\omega}_i\hat{\mu}}, \quad \Delta w = 1 - \frac{\hat{w}}{\hat{w}_i} = 1 - w.
\]

The particle Reynolds number \( (Re_p) \) is a measure of the ratio of inertial to viscous forces over the scale of the particle. Since \( d_p \approx 1 \text{ mm} \), \( Re_p \approx 1 - 50 \). The second dimensionless parameter is the Buoyancy number \( (Bu) \) which is a ratio of buoyancy to viscous forces. The last dimensionless parameter is the velocity deficit \( (\Delta w) \) which measures the difference between the interface velocity and far field velocity: \( \Delta w < 0 \) (interface slower than far field velocity) happens in the wide side of the annulus and \( \Delta w > 0 \) happens in the narrow side.

We can further simplify to study convergence to the interface position, by defining:

\[
X = \frac{18(z_p(t) - t)}{Re_p} \quad \text{and} \quad \tau = \frac{18}{Re_p} t.
\]

The momentum balance becomes

\[
\frac{d^2 X}{d\tau^2} + \frac{dX}{d\tau} = -\Delta w + \frac{Bu}{18} (1 - 2 \mathcal{H}(X)),
\]

(8.6)

which we can write (8.6) in the form of two first order ODEs:

\[
X' = V,
\]

(8.7a)

\[
V' = -V - \Delta w + \frac{Bu}{18} (1 - 2 \mathcal{H}(X))
\]

(8.7b)
Evidently $X(\tau)$ measures distance from the interface and $V(\tau)$ is the relative velocity of particle and interface. When $(X(\tau), V(\tau)) \to (0, 0)$ the particles are moving at the interface position and velocity. We consider two cases below.

### 8.1.3 Concentric annulus ($\hat{w} = \hat{w}_i$)

First of all, in a concentric annulus we must expect that $\hat{w} = \hat{w}_i$, which means that $\Delta w = 0$. We find that the system (8.7) always has an equilibrium point $(X_{EP}, V_{EP}) = (0, 0)$, which is globally stable. To show this we can use as a Lyapunov function

$$\Gamma(X, V) = \frac{1}{2} V^2 + \frac{Bu}{18} |X|. \quad (8.8)$$

The function $\Gamma$ is positive in the neighbourhood of the equilibrium point $(0, 0)$ and consists of an increasing sequence of nested curves. We take the time derivative

$$\frac{\partial \Gamma}{\partial \tau} = \frac{\partial \Gamma}{\partial X} \frac{\partial X}{\partial \tau} + \frac{\partial \Gamma}{\partial V} \frac{\partial V}{\partial \tau}$$

$$= -\frac{Bu}{18} \left[ 1 - 2\mathcal{H}(X) \right] V + V \left[ -V + \frac{Bu}{18} (1 - 2\mathcal{H}(X)) \right]$$

$$= -V^2 \leq 0. \quad (8.9)$$

This proves that the equilibrium point $(0, 0)$ is Lyapunov stable, i.e. the phase paths of the system (8.7) cross the level sets of $\Gamma$ inwards only.

To illustrate this, Figure 8.2 shows the phase plane associated with the system (8.7) when $\Delta w = 0$. The (blue) arrows represent the direction field, always tangent to the solutions of (8.7). The red dots are different examples of initial position and velocity $(X_0, V_0)$. Particles starting at different initial condition follow different phase paths to reach to the equilibrium point, but all converge asymptotically.

Changing the value of $Bu$ will affect the particle phase paths followed, but not the equilibrium point, nor the asymptotic behaviour of the system. Increasing $Bu$ increases the amplitude of the velocity $V$ and consequent oscillations about the equilibrium point. The particle Reynolds number plays a role primarily in the
Figure 8.2: Phase plane for the system of (8.7) when $\Delta w = 0$, showing the direction field (blue arrows) and sample phase paths emanating from selected initial position and velocity (indicated by red dots): a) $Bu = 1$; b) $Bu = 10$; c) $Bu = 20$; d) $Bu = 100$.

length- and timescales. In dimensional terms:

$$\dot{z}_p - \dot{\hat{w}} \hat{t} = \frac{X}{18} \frac{\tilde{\hat{w}} d_p^2}{\tilde{\nu}}, \quad \hat{t} = \frac{\tau d_p^2}{18 \tilde{\nu}},$$

where $\tilde{\nu} = \hat{\mu} / \hat{\rho} \approx 10^{-6} - 10^{-4}$ m$^2$/s. For $d_p \sim 1$mm, this viscous-diffusive timescale varies: $d_p^2 / \tilde{\nu} \approx 10^{-2} - 1$s, and the length-scale is advective: the distance moved by the interface over this timescale.
8.1.4 Eccentric annulus ($\hat{w} \neq \hat{w}_i$)

We now consider $\hat{w} \neq \hat{w}_i$, for which $\Delta w$ is non-zero. As discussed, this corresponds to an eccentric annular flow with interface moving at steady state speed $\hat{w}_i$ and particle within fluid moving at a different speed. If we again consider the system (8.7), we see that $X' = 0$ when $V = 0$. Along $V = 0$ we see that $V'$ changes sign from positive to negative, provided that:

$$\frac{Bu}{18} > |\Delta w|.$$  \hfill (8.10)

Assuming (8.10), if we were to regularize the Heaviside function, we could compute a position $(X_{EP}, 0)$ of an equilibrium point and it is evident that $X_{EP} \to 0$ as the buoyancy term becomes discontinuous. Note too that a smoothed buoyancy term would anyway be physically reasonable for miscible fluids.

We would normally expect $|\Delta w| < 1$, although in cases of extreme eccentricity the wide side velocity could violate this. Therefore, an approximate stability criterion is:

$$Bu > 18.$$  \hfill (8.11)

Note that for an extremely eccentric annulus we likely would not have a steady displacement anyway.

We can explore this behaviour by plotting the phase plane associated with (8.7) for various $Bu$ and $\Delta w$; see Figure 8.3. In Figures 8.3a & b, we have $Bu = 1$ and $\Delta w = \pm 0.1$, so that (8.10) is not satisfied. Although some of the phase paths appear initially to approach $(0, 0)$, they move past and eventually asymptotically $X(\tau) \to \mp\infty$ with small constant $V(\tau)$. For $\Delta w > 0$ this represents the narrow side of the annulus and $\Delta w < 0$ represents the wide side.

For larger $Bu = 10$ with $\Delta w = 0.1$, (8.10) is satisfied and $(0, 0)$ is again a globally stable equilibrium point that particles with different initial conditions eventually approach (Figure 8.3c). The behaviour is similar to that for $\Delta w = 0$. For larger $\Delta w = 0.7$, e.g. the narrow side on a significantly eccentric well, Figure 8.3d shows that the phase paths again have $X(\tau) \to -\infty$ as the particle is left behind the interface. Finally, $Bu = 20$ and $\Delta w = 1$ restores the globally stable equilibrium point (Figure 8.3e).
Figure 8.3: Phase plane and sample solutions for the system (8.7). The blue arrows show the direction field and the red curves show solutions starting from different initial position and velocity (indicated by red dots): a) $Bu = 1, \Delta w = 0.1$; b) $Bu = 1, \Delta w = -0.1$; c) $Bu = 10, \Delta w = 0.1$; d) $Bu = 10, \Delta w = 0.7$; e) $Bu = 20, \Delta w = 1$. 
8.2 Two-dimensional model

Having understood the dynamics of the toy model of the previous section, we now couple this directly to our annular displacement model (3.60-3.62), derived in Chapter 3. We refer to Chapter 4 for the details of our computational approach. Throughout this chapter we impose symmetry on the wide and narrow sides of the annulus and only solve the problem in half of the annulus \( 0 \leq \phi \leq 1 \). We also restrict our examples to laminar flows for simplicity. Upon solving (3.60-3.62), the 2D gap averaged fluid velocity field \( (\hat{v}, \hat{w}) \) is obtained at all times during the displacement. This will be then fed into the particle motion equations.

To treat particle motion in 2D is similar to (8.2), except here we have include the azimuthal direction too. It is emphasized that our approach in this chapter is preliminary in adopting only the simplest model possible, to establish feasibility of the interface tracking concept. In particular, we assume:

- Particles do not interact with each other.
- The local effect of particle on the flow field can be neglected, i.e. the momentum coupling is one-way.
- Particle forces are simplified to only drag and buoyancy. The former is modeled using a modified Stokes law, in which the (Newtonian) viscosity is replaced with a representative effective viscosity \( \hat{\mu}_{\text{eff}} \).

With the above assumptions, the particle momentum equations are:

\[
\begin{align*}
\hat{\rho}_p \hat{V}_p \frac{d^2 \phi_p}{dt^2} &= -3\pi \hat{d}_p \hat{\mu}_{\text{eff}} \left( \frac{d\phi_p}{dt} - \frac{\hat{v}}{\pi \hat{r}_a} \right) \quad (8.12a) \\
\hat{\rho}_p \hat{V}_p \frac{d^2 \xi_p}{dt^2} &= -3\pi \hat{d}_p \hat{\mu}_{\text{eff}} \left( \frac{d\xi_p}{dt} - \hat{w} \right) + \Delta \hat{\rho} \hat{g} \hat{V}_p \quad (8.12b)
\end{align*}
\]

where \( (\phi_p, \xi_p) \) denote the azimuthal and axial coordinates of the particle position. For the effective viscosity we simply evaluate the Herschel-Bulkley viscosity using an effective (Newtonian) wall shear rate:

\[
\hat{\mu}_{\text{eff}} = \frac{\hat{\tau}_Y + \kappa \hat{\gamma}_L^p}{\hat{\gamma}_L}, \quad \hat{\gamma}_L = \frac{3\sqrt{\hat{v}^2 + \hat{w}^2}}{\hat{d}(1 + e \cos \pi \phi)}. \quad (8.13)
\]
8.3 Results

For the sake of demonstration we consider only a short vertical section of well ($\hat{Z} = 50$ m), with inner and outer radii $\hat{r}_i = 22.75$ cm and $\hat{r}_o = 25.25$ cm. Although the length of this annular section is much shorter than in an actual cement job, it is still long enough that particles can reach their equilibrium state, which we will see is either at the interface or alternatively at some intermediate velocity. As discussed in §8.1.3 for the toy model, the relevant timescale for the particle transient is viscous and for typical cementing fluids and parameters this scale is of the order of seconds. Having attained their equilibrium state, the results here can be extended to any length of annulus. Equally, the mean radius of the annulus has not been found to much affect the results.

We assume the annulus is initially filled with fluid 1 which is displaced by fluid 2. We consider two fluids with identical rheological parameters (for the sake of simplicity) and explore the role played by fluids density difference, rheological parameters and eccentricity. In all simulations we launch and track five identical particles with diameter $\hat{d}_p = 2$ mm and density

$$\hat{\rho}_p = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2).$$

In each simulation we compute the displacement flow by solving (3.60-3.62) throughout the displacement, and then integrate (8.12) numerically to track the motion of each tracer particle. In this way, the displacement problem is completely decoupled from the particle motion. The particles are released within the displacing fluid, from rest and at a height $\hat{z}_{p,0} = 75$ cm (from the bottom), at a time when the interface has reached a height $\approx 8$ m from bottom hole. The particles are placed only in half of the annulus, because the problem is symmetric. We do however show the full annulus, because in integrating the particle momentum equations particles can cross the symmetry line, i.e. due to timestepping.

Similar to the previous chapters, in all examples presented the horizontal axis is along the azimuth of the well. The wide and narrow sides of the well are marked with letters W and N in the colourmaps. Snapshots of the advancing interface are shown in each example with particles shown as green circles. We also show particle
trajectories (right panel) in the $(\hat{x}, \hat{\xi})$-plane where $\hat{x} = \pi \hat{r} \phi$ measures azimuthal distance.

8.3.1 Newtonian fluids

Our first displacement example is shown in Figure 8.4. The well is vertical and eccentric with $e = 0.2$. We take two Newtonian fluids as the displaced and displacing fluid. We keep the properties of the displaced fluid constant at

$$\hat{\rho}_1 = 1500 \text{ kg/m}^3, n_1 = 1, \hat{\kappa}_1 = 0.05 \text{ Pa.s and } \hat{\tau}_{Y,1} = 0 \text{ Pa}.$$  

We consider a heavy displacing fluid:

$$\hat{\rho}_2 = 1700 \text{ kg/m}^3, n_2 = 1 \text{ and } \hat{\tau}_{Y,2} = 0 \text{ Pa}.$$

and play with the value of viscosity. We try three values: $\hat{\kappa}_2 = 0.001, 0.0035$ and $0.01 \text{ Pa.s}$. Starting with a small flow rate of $\hat{Q} = 1 \text{ lit/s}$ or equivalently an average velocity of $\hat{W}_0 = 2.6 \text{ cm/s}$. This combination of parameters corresponds to $Bu = 300, 100, 30$. Figure 8.4 shows the snapshot of displacement together with the trajectory of particles motions.

Condition (8.11) is well satisfied in Figure 8.4, and we observe all five particles reach to and travel with the interface, regardless of the value of viscosity. This is in fact due to the small flow rate used here. Although the particles in all three cases successfully tag the interface, we observe as the viscosity of the carrier fluid becomes larger, it takes a longer distance for particle to reach to the interface. This is clearly reflected in the value of Buoyancy number $Bu$. The larger the value of $Bu$, the more buoyant the particle, and hence quicker the particle travels. The other interesting dynamic is that particles on the wide side are quicker to reach to the interface than those located near the narrow side. This difference is clearly noticeable in Figure 8.4c. This is because $|\Delta w|$ is not uniform around the annulus. Recall that

$$\Delta w = 1 - \frac{\hat{w}}{\hat{w}_i}.$$  

Here $\hat{w}$ is the far field velocity and $\hat{w}_i$ is the interface velocity, both of which are
Figure 8.4: Snapshot of the displacement with particles shown with green circles in a vertical eccentric well with $e = 0.2$. Flow rate is $\hat{Q} = 1$ lit/s. Displaced fluid: $\hat{\rho}_1 = 1500 \text{ kg/m}^3, n_1 = 1, k_1 = 0.05 \text{ Pa.s}$ and $\hat{\tau}_{Y,1} = 0 \text{ Pa}$. Displacing fluid: $\hat{\rho}_2 = 1700 \text{ kg/m}^3, n_2 = 1$ and $\hat{\tau}_{Y,2} = 0 \text{ Pa}$. a) $k_2 = 0.001 \text{ Pa.s}$; b) $k_2 = 0.0035 \text{ Pa.s}$; c) $k_2 = 0.01 \text{ Pa.s}$
functions of $\phi$. Since the far field velocity on the wide side is faster than on the narrow side, $\Delta w_W \leq \Delta w_N$. When released behind the interface the viscous drag and the buoyancy accelerate towards the interface, whereas on the narrow side the viscous drag retards the particles, competing against buoyancy. As we will see later, the narrow side particles are the first to not reach to the interface, as we increase the flow rate.

In Figure 8.5 we repeat the same example, except we increase the flow rate to $\dot{Q} = 4$ lit/s, which is equivalent to a mean velocity of $10.4$ cm/s. Since we are only interested in laminar flows, we will look at the displacement at two larger values of viscosity: $\kappa_3 = 0.0035$ and $0.01$ Pa.s. The corresponding values of buoyancy number are $Bu = 25, 7.5$. Figure 8.5 shows snapshots of displacement together with the particles trajectories. In Figure 8.5a, all 5 particles reach the interface, whereas in Figure 8.5b only the two particles on the wide side reach the interface. The other three particle reach to a secondary equilibrium point and travel at an intermediate speed between the far field and interface speed. The could be readily predicted by (8.11). In particular, in Figure 8.5b, where the condition (8.11) is not satisfied, the particles on the narrow side do not reach the interface. As pointed out earlier, note that although (8.11) is not strictly a sufficient condition, in many cases it is conservative. As an example, the particles on the wide side in Figure 8.5b do reach to the interface despite the fact condition (8.11) is not satisfied. This is because on the wide side $|\Delta w_W| < 1$, and therefore (8.10) can be satisfied without satisfying (8.11).

Moving to a displacement scenario with an even larger flow rate, we now test the previous example when $\dot{Q} = 6$ lit/s. To keep the flow regime laminar, we can only take the largest viscosity into account. Compared to the previous case, the particles buoyancy number is even smaller ($Bu = 5$), suggesting that fewer particles will reach to the interface, and more of them will be trapped in the intermediate equilibrium state. This is in fact the case as shown in Figure 8.6. Only one particle reaches to the interface, and the rest are left behind.

The example shown in Figures 8.5b and 8.6 are closer to real cementing scenarios involving laminar flows. It is rather unfortunate that the particles are not able to track the interface, although the earlier examples were. We now test a few ideas that might improve the applicability of the method. The first question is: does
Figure 8.5: Same as Figure 8.4, except $\hat{Q} = 4$ lit/s. a) $\hat{\kappa}_2 = 0.0035$ Pa.s; b) $\hat{\kappa}_2 = 0.01$ Pa.s

Figure 8.6: Same as Figure 8.4, except $\hat{Q} = 6$ lit/s. $\hat{\kappa}_2 = 0.01$ Pa.s.
it help if the particles are released nearer to the interface? In Figures 8.4-8.6 the particles were released $\approx 8 \text{ m}$ behind the interface. We now reduce this to just a few centimetres.

Figure 8.7a shows the earlier example shown in 8.6, except the particles are released just behind the interface. Remarkably, in this case all particles are locked and travel with the interface. More interestingly, in figure 8.7b the viscosity of the displacing fluid is increased by a factor 5 to $\hat{\kappa} = 0.05 \text{ Pa.s}$ and the particles still successfully reach the interface and travel with it. Note that the value of buoyancy number in this example is $Bu = 1$, much lower that the critical value condition (8.11) requires. To understand this, considering (8.10) note that if the displacement is steady then close to the interface the vertical component of velocity in the fluid approaches $w_i$ everywhere, meaning that if we release the particles close enough then we effectively have $|\Delta w| \sim 0$ and buoyancy can overcome drag.

We also speculate that eccentricity plays a critical role in tracking dynamics, as well as on the final distribution of particles. Figure 8.8 tests three different examples. In Figure 8.8a, we repeat the example in Figure 8.5b, except that the well is concentric, $e = 0$. Comparing with Figure 8.5b clearly shows the effect of eccentricity. In a concentric well, as shown earlier in §8.1.3 the only equilibrium state for the particles is when they reach and travel with the interface. In addition, there is no azimuthal gradient in the flow, so the particles move on parallel paths. In Figure 8.8a, at the end of the simulation they are closing the distance to the interface. In a slightly longer annulus they would reach to the interface. Notice that even a slight eccentricity breaks down this dynamic, as shown in Figure 8.5b.

In Figure 8.8b, we repeat the example in Figure 8.4b, except here we increase the eccentricity to $e = 0.5$. Although the well is more eccentric, and $\Delta w$ is possibly larger, the particles still reach to the interface. This is in fact because of the large buoyancy number ($Bu = 100$). We expect a displacement case with smaller value of $Bu$ is more sensitive to eccentricity. Indeed, this is shown in Figure 8.8c. Here the displacement is the same shown in Figure 8.5a, except the eccentricity is increased to $e = 0.5$. At a lower value of eccentricity (Figure 8.5a), the particles successfully catch and track the interface. However, as we increase the eccentricity (Figure 8.8c), the particles on the narrow side are no longer able to catch the interface. Instead, they reach an intermediate equilibrium state. Note that this happens,
Figure 8.7: a) Same as Figure 8.6, except particles are released just behind the interface; b) Same as Figure 8.6, except viscosity is increased to $\hat{\kappa}_2 = 0.05$ Pa.s and the particles are released at the interface.

despite the fact that condition (8.11) is still satisfied. This reveals the 2D nature of the problem under study, which is not quite captured by the 1D stability criteria (8.10) & (8.11).

We have also tested two further speculative questions. First, does it help to release the particle with an initial axial velocity. We have tested this in various scenarios and it turns out that the particle initial velocity plays little role in determining the final equilibrium state of the particles. Secondly, we have explored varying the particles density between those of the two fluids. Certainly particles with a lower density generate larger buoyancy force and are therefore more likely to reach to the interface (essentially we can double $Bu$ in this way). On the other hand, once they reach the interface, they may overshoot if they are not heavier than the dis-
Figure 8.8: a) Same as Figure 8.5b, except eccentricity is changed to $e = 0$; b) Same as Figure 8.4b, except eccentricity is changed to $e = 0.5$; c) Same as Figure 8.5a, except eccentricity is changed to $e = 0.5$

placed fluid. There is some trade-off here between reaching the interface (beating viscous drag) and finally converging to the interface (reducing oscillations about the interfacial position).
8.3.2 Non-Newtonian fluids

We start our analysis here by considering shear thinning fluids. Shear thinning can increase or decrease the effective viscosity, depending on the magnitude of shear rate relative to a reference case. There is also an inherent azimuthal bias. In an eccentric annular flow the wall shear stress (hence shear rate) is higher on the wide side than the narrow side; hence the effective viscosity on the wide side is smaller than on the narrow side. This effectively means the particles on the wide side experience a higher $Bu$ than those on the narrow side, which will favour their migration to the interface. Once the particles reach near to a steadily advancing interface however, the velocity distribution becomes uniform in $\phi$, and therefore the effect of shear thinning in favouring the wide side disappears.

Here we consider similar examples illustrated in the previous section, except that we make the displacing fluid shear thinning, (i.e. we retain $\dot{\rho}_1 = 1500 \text{ kg/m}^3$, $n_1 = 1$, $\kappa_1 = 0.05 \text{ Pa.s}$, $\tau_{Y,1} = 0 \text{ Pa}$; $\dot{\rho}_2 = 1700 \text{ kg/m}^3$, $\tau_{Y,2} = 0 \text{ Pa}$). We follow the examples in Figures 8.8b and 8.5b, with now $\kappa_2 = 0.0035 \text{ Pa.s}^{n_2}$ and a power law index of $n_2 = 0.5$. Thus, for effective strain rates larger than $1 \text{ s}^{-1}$ fluid 2 is less viscous than the previous Newtonian examples.

In Figure 8.9a, we take the example in Figure 8.8b, which we recall has high eccentricity ($e = 0.5$) and low flow rate ($\dot{Q} = 1 \text{ lit/s}$). Even at this low flow rate, introducing shear thinning effectively reduces the effective viscosity on both wide and narrow sides. More specifically, the wide and narrow side effective viscosity is reduced by a factor of 3.2 and 1.8, respectively, compared to the Newtonian counterpart. As a result, this case gives a larger local buoyancy number, which results in particles reaching to the interface much quicker (see Figure 8.9a and compare with Figure 8.8b).

In Figure 8.9b we compare with the example in Figure 8.5b, which has less eccentricity ($e = 0.2$) but larger flow rate ($\dot{Q} = 4 \text{ lit/s}$). Since the flow rate is larger, the shear thinning effect is considerably more significant. The effective viscosity on the wide and narrow sides is reduced by a factor of 6.6 and 5.2, respectively, equivalent to local buoyancy numbers of $Bu_W = 50$ and $Bu_N = 40$, on wide and narrow sides respectively. Recall that the buoyancy number in the equivalent Newtonian case (Figure 8.5b) was $Bu = 7.5$. Such large buoyancy forces relative to...
viscous force are enough for the particles, on the wide and narrow side, to reach and track the interface unlike the Newtonian example.

We now turn our attention to fluids with yield stress. We can envision the following 3 scenarios with yield stress fluids:

(i) There is a combination of eccentricity, density differences and rheology, such that: (a) we still have a steady displacement; (b) the fluids on the narrow side of the annulus are mobilised.

(ii) There is a combination of eccentricity, density differences and rheology, such that: (a) we no longer have a steady displacement and the interface elongates along the well; (b) the fluids on the narrow side of the annulus are mobilised.
There is a combination of eccentricity, density differences and rheology, such that: (a) we no longer have a steady displacement and the interface elongates along the well; (b) the displaced fluid on the narrow side of the annulus is static (a mud channel).

In a vertical well, the transition from (i)-(iii) can be a combination of increased eccentricity, reduced density difference or increased yield stress of the displaced fluid.

We investigate these 3 scenarios here. To keep the analysis simpler, we consider the displaced fluid to be a Bingham fluid and the displacing fluid to be Newtonian. We vary the displaced fluid yield stress. Except the yield stress of displaced fluid, all other parameters are those in Figure 8.5b, (i.e. \( n_1 = n_2 = 1 \), \( \hat{\kappa}_1 = 0.05 \text{ Pa} \), \( \hat{\kappa}_2 = 0.01 \text{ Pa} \), \( \hat{\tau}_{Y,1} = 5,10,20 \text{ Pa} \) and \( \hat{\tau}_{Y,2} = 0 \text{ Pa} \)). The results are shown in Figure 8.10.

In the first scenario (i), since the fluids are yielded everywhere and we still have a steady displacement, as far as the motion of particles is concerned, the effect of yield stress is just simply to change the local effective viscosity. The yield stress reduces the local \( Bu \) which lowers the possibility of particles reaching the interface. The expression of effective viscosity given by (8.13) suggests the increase in the viscosity due to the yield stress is smaller on the wide side than the narrow side. Therefore, similar to the shear-thinning effect, the yield stress would favour particles on the wide side to reach the interface.

Figures 8.10a & b represent scenario (i). Note however, that the displaced fluid has been enhanced by the yield stress and not the displacing fluid. The heavy displacing fluid steadily displaces the yield stress displaced fluid. The particles however, distribute slightly differently compared to Figure 8.5b. This may have two reasons. (a) The yield stress downstream appears to create larger secondary motions near the interface, which wash the particles from wide side to the narrow side. (b) The interface is now slightly mixed due the dispersive mechanism caused by the secondary flows near the interface.

In Figure 8.10c we explore scenario (ii). The displacement is unsteady displacement, meaning that the interface is continually elongating. In an unsteady displacement, the density difference is not large enough to displace the yield stress...
Figure 8.10: Same as Figure 8.5b, except the displaced fluid has a yield stress
a) $\hat{\tau}_{Y,1} = 5 \text{ Pa}$; b) $\hat{\tau}_{Y,1} = 10 \text{ Pa}$; c) $\hat{\tau}_{Y,1} = 20 \text{ Pa}$.
displaced fluid uniformly around the annulus. Interestingly, we observe that the particles more or less approach and track the interface at its different azimuthal positions. This may appear counter intuitive, in light of our previous analysis. The explanation for this is that in an unsteady displacement, as the interface elongates the axial velocities become closer to the far-field velocities. Thus paradoxically, the local interface speed is similar to the far field speed at the same azimuthal position, resulting in significant reduction in the differential velocity ($\Delta w \sim 0$), at least insofar as the axial component is concerned. The 1D (axial) dynamics are therefore similar to that studied in §8.1.3, in which the particles stably approach the interface.

To illustrate scenario (iii) we need to change the fluid parameters such that the displaced fluid is left immobilized on the narrow side. Therefore, we change the displacing fluid properties to $\hat{\rho}_2 = 1600 \text{ kg/m}^3$, $n_2 = 1$, $\hat{\kappa}_2 = 0.001 \text{ Pa}$ and $\hat{\tau}_{Y,2} = 0 \text{ Pa}$, while the displaced fluid is same as in Figure 8.10c. Note that the density difference is reduced here as well as the viscosity. The flow rate is $\hat{Q} = 1 \text{ lit/s}$. Figure 8.11 shows that the particles again approach and track the interface (as in Figure 8.10c), but now the displaced fluid is fully static on the narrow side of the annulus, so that the particles there are fully stationary.

Figure 8.11: Same as Figure 8.6a, except the displacing fluid is lighter $\hat{\rho}_2 = 1600 \text{ kg/m}^3$ and the displaced fluid has a yield stress $\hat{\tau}_{Y,1} = 20 \text{ Pa}$.
8.4 Summary and conclusions

In this chapter, we presented a feasibility study for a novel technique to track the interface between two fluids in an annular displacement flow, commonly found in primary cementing of oil and gas wells. Our technique is based on exploiting the density difference between successive fluids pumped in order to design a tracer particle to sit at the interface. Tracking the particles sitting at the interface allows us to track the displacement flow interface, which in turn provides a more accurate assessment of the cement job. Depending on the form of tracer particle and the detection method, which could potentially operate during pumping of the slurry, a number of exciting opportunities exist. These include: (i) real-time tracking of interfaces to detect displacement and geometric anomalies in the annulus; (ii) instantaneous determination of top of cement, together with azimuthal disparities; (iii) monitoring of post-placement fluid motions. Evidently, much work is needed on tracer particle detection and interpretation to realize these technological advances.

Relying on our annular displacement model of Chapter 3, we have computed the flow field as a function of the annulus geometry, fluid properties and flow rate. We then track the (passive) particles and show if they can catch the interface or not. For steady displacements we have shown how a simple criterion (8.11), obtained on scaling grounds, can approximately predict if the particles can reach and travel with the interface. This criterion, expressed in terms of a dimensionless buoyancy number $Bu$, represents the competition between buoyancy and viscous drag. The
criterion (8.11) appears effective, except in the case of extreme eccentricity when the secondary flows near the interface are strong. For unsteady displacement, buoyancy and drag do not compete and particles also appear to migrate to the interface, even when (8.11) is not satisfied.

Although we have established that the particle tracking concept is valid, for the full range of parameters relevant to laminar primary cementing the criterion (8.11) is only marginally satisfied. To extend the range of feasibility, in place of (8.11) the less conservative but more complex (8.10) can be applied locally. In dimensional terms, this amounts to:

\[
\frac{Bu}{18} > |\Delta w| \Rightarrow (\hat{\rho}_2 - \hat{\rho}_1)g\hat{d}_p > 18\hat{\mu}_{eff}|\hat{w}_i - \hat{w}|, \tag{8.14}
\]

where we have substituted the effective viscosity \( \hat{\mu}_{eff} \) to account for the non-Newtonian examples. Nearly all our results in the annulus can be understood by reference to (8.14) considered locally. The complexities come from not knowing a priori either \( |\hat{w}_i - \hat{w}| \) or \( \hat{\mu}_{eff} \) (in the non-Newtonian case, which would be the normal case).

The criterion (8.14) helps us understand how the process might be engineered in order to enable tracking. The two ideas that we have proposed and discussed are: (i) adjust the particle density which we have fixed to the mean density of the 2 fluids; (ii) start the particles closer to the interface. The former of these acts to increase \( (\hat{\rho}_2 - \hat{\rho}_1) \) by up to a factor of 2 in (8.14), with the risk of losing stability in the displaced fluid, should the particles be disturbed from the interface. The latter of these acts by reducing \( |\hat{w}_i - \hat{w}| \) at the point of release. Evidently, there will be some entry/development length for any steady interface to form as fluid exit the float shoe into the annulus, so release precisely at the interface is tricky. On the other hand, transporting tracer particles within the bottom plug (or any preceding plug) offers an easy way to deliver the particles downhole and into the flow when the plug bursts. Indeed this aspect appears quite promising.

The chief issue with (8.14) is that in laminar cementing it is common to design for both a density and viscosity hierarchy between fluids. Although the density difference is usually the more significant of the two, this means that spacer fluid viscosities are comparable to the drilling mud viscosities. Because the drilling mud
is generally quite viscous and *laminar* flow rates in cementing are not creeping flows, the right-hand side of (8.14) can be large due to viscous stresses (although helped by shear-thinning). On the other hand, density differences of 200 kg/m$^3$ or more are not uncommon.

One possibility might be to use a low viscosity but dense spacer fluid, i.e. using buoyancy to make it effective in displacing, but retaining low viscosity to help with (8.14). Releasing less dense particles within the spacer would migrate upwards to the mud-spacer interface and denser particles would fall back through the spacer to the top of cement interface. Combining this with additional plugs to release near the interfaces should greatly improve the range of feasible flows. Note that in this preliminary study we have largely looked at particle motion within a single fluid and have not really exploited the opportunities of different fluid properties.

Next, we must acknowledge that our model has been intentionally simplistic. We have used an established annular displacement modelling methodology, but the focus has primarily been on proof of concept in vertical wells with steady displacement flows, and we have only considered whether particles may track the interface. It is evident that inclined wells present challenges for this method. An alternate would be to release more particles and try to track (or seed) the bulk flow for later detection.

Our particle momentum balance is also minimalist. Certainly we can extend the drag force closure $\mathbf{F}_D$ to encompass regimes that are inertial on the particle scale. This will introduce some nonlinearity into the model, but the underlying buoyancy-drag competition remains. Of course there is a wealth of knowledge on particle migration in pipes and channels, including lift, added mass, Magnus, Basset and other force contributions. Interaction with the carrier fluid rheology is another serious complication, particularly in the case of a yield stress. Indeed our treatment of the non-Newtonian aspects is simplistic. Lastly, with the inherent interest in lower viscosity fluids, there is the possibility of at least the fluid flow attaining turbulent regimes, although the particle Reynolds numbers are likely to remain laminar.

Finally, we have focused at tracking the interfaces *during* cementing. Cement placement is designed with a safety factor of some hours before the cement truly thickens. During this post-placement period the interfaces may rearrange, i.e. settle.
under buoyancy. Equally, particles that have not attained the interface position may still migrate. In the post-placement phase $|\hat{\mathbf{w}}_i - \hat{\mathbf{w}}| = 0$ as both terms are zero, which naively means that (8.14) will be unconditionally satisfied. The only question is the time required to reach the interface through the static fluids. To demonstrate this more clearly, in Figure 8.12 we take an unsuccessful example presented earlier in Figure 8.6 and assume the pump is shut down at $t = 172$ s and the fluids instantaneously come to rest. We see that it takes few minutes for the particles to rise and tag the interface. Although our prototype well is rather short, it appears the typical thickening time of cement slurries is enough for the particles to settle down at the position of the interface. This observation makes the technique presented above more attractive, because the technique appears to be informative even in cases where the particles are unsuccessful in reaching to the interface.
Chapter 9

Summary and conclusions

In this thesis we have studied displacement flows of yield stress fluids in long narrow eccentric annuli, in laminar, turbulent and mixed flow regimes. Such flows are commonly found in the processes involved in the primary cementing of oil and gas wells. This operation is key to well integrity: affecting structural integrity, well productivity, well leakage (surface casing vent flow and other environmental issues), as well as impacting well abandonment.

The novelty of our work relies on developing a new formulation that includes turbulent and mixed flow regimes, something that has not been addressed before. This new extension allows us to understand the process of cementing more deeply, and resolve several questions that have been left unanswered for many years. Broadly speaking, the thesis can be divided into two parts. i) The chapter two to four contain a detailed derivation of our model, together with our computational approach for solving the model. ii) Subsequently, in chapters five to eight, we systematically investigated several practical questions in regard to primary cementing. The four appendices extend the applications, by outlining the software development and presenting work on cementing that has been conducted in collaboration with other members of the research group.

We begin this chapter by summarizing the results and contributions of individual chapters in §9.1. Several such results have immediate industrial implications, which we will discuss in length in §9.2. Thirdly, we try to look back at our approach critically in §9.3 and identify possible limitations of our methodology. This
leads naturally to recommendations for future improvements and research directions in §9.4.

9.1 Contributions of the individual chapters

Chapter 2

In Chapter 2, we accomplished two main objectives: Firstly, we laid down a consistent formulation for the hydraulic calculations involving yield stress shear thinning materials. In particular, we generalized the phenomenological approach of Dodge-Metzner-Reed [158, 58] to generalized Newtonian fluids and derived a computational procedure for calculating the necessary pressure gradient, given a desired flow rate (or vice versa). Secondly, we have extended the classical turbulent Taylor dispersion problem to Herschel-Bulkley fluids.

To account for the effect of dispersion, specially in the ranges of weakly turbulent flows such as those in primary cementing, we showed that it is necessary to obtain a correct velocity profile. This means that the well-known log-law of turbulent flows needs to be corrected both at the centerline as well as near the walls. In order to correct the velocity near the wall, two conditions must be satisfied: i) wall shear stress and ii) cubic decay of turbulent stresses. We employed an asymptotic approach in constructing a wall layer velocity that meets these two conditions.

Additionally, we found that although the yield stress produces competing effect in the wall layer, because of friction factor closure and delayed transition, the thickness of wall layer decreases as the yield stress drops. Similar behavior is also observed, not surprisingly, when the fluid become more shear thinning. Upon implementing the corrections indicated above, we used the Reynolds analogy to give estimates for the turbulent diffusivity, and then Taylor dispersivity. Interestingly, we found that in weakly turbulent flows the mixing due to the Taylor dispersion is one or two orders of magnitude larger than that of turbulent diffusion.

Chapter 3

Chapter 3 contains the detailed derivation of our displacement model. The model is a natural extension of Bittleston et al. [24], where only laminar displacement flows are considered. Here we extend this to turbulent and mixed flow
regimes. This derivation is a classic example of several mathematical techniques and modeling approaches such as lubrication approximation, multi-time scale and asymptotic analysis and Reynolds decomposition.

More specifically, we showed how turbulent flows can be dimensionally reduced using lubrication-type geometrical scaling. This is particularly challenging, because otherwise, the turbulent flows are inherently 3D and require solving large complex systems of equations. The other major difference compared to laminar displacement flows is our treatment of mixing. Laminar flows in the length and time scales of primary cementing fall in high Péclet number regime \( (Pe \gg 1) \), indicating that molecular diffusion is insignificant. However, in the turbulent displacement flows, through a rigorous asymptotic derivation, we show that mixing is considerable, and is governed by a complicated diffusive and dispersive transport processes which are absent in laminar regime.

The key point here is that the Taylor dispersive effects are dominant (from chapter 2), but are anisotropic in diffusing only along the streamlines. There is also an additional dispersive term active only close to the interface where streamlines have a strong azimuthal component. While it is evident in the technical literature that a number of in-house industrial codes do consider turbulent flows, details are lacking and we believe that only the (less significant) turbulent diffusivity terms are modeled.

**Chapter 4**

In Chapter 4 we analyzed the model derived in Chapter 3 from a computational point of view. We start by deriving a variational formulation for the stream function equation (3.60). This formulations proceeds to a proof of existence and uniqueness of the solution. The proof is fairly comprehensive, in that only mild conditions are required, which are satisfied in almost any situation encountered in primary cementing. In terms of novelty, the methodology and framework of the proof is standard (see e.g. [179]). The interest is twofold: first, in including all flow regimes. Secondly, in the observation that by including the turbulent regime the solutions naturally lie in a more regular function space than for the laminar flows considered alone (i.e. in \( W^{1,3-\varepsilon} \) instead of \( W^{1,1+n} \)), which is nearly independent of the rheology.
The variational formulation is then employed to establish a convex optimization problem, which can be solved using Uzawa-type augmented Lagrangean algorithms. Although not directly motivated by the turbulent flows, as other faster algorithms can be implemented, the algorithm is particularly suitable for mixed flow regimes, as it resolves those parts of the flow where fluids are immobile due to the yield stress. In addition, since it is based on the convexity of dissipation functional, the algorithm is guaranteed to converge. The alternatives would be to regularize our wall shear stress closure in the laminar-static regime and use a Picard or Newton’s type iterative algorithm. This approach may be faster, but may also be less robust, and leaves open the question of interpretation of static regions. This could also be done within the pressure formulation of the problem.

The second part of the chapter describes our treatment for the concentration equation. Here we address the advection and diffusion, separately. We compare a number of choices for solving the advection, and then modify the one with the best performance to include the diffusive and dispersive fluxes.

Chapter 5

In Chapter 5 we began to study turbulent displacement flows. Through conducting a range of simulations, we show the followings.

i) When the displacement flow is fully turbulent, the rheology of either fluids does not play any significant role in the displacement outcome.

ii) In the absence of density difference, the displacement is faster on the wide side and slower on the narrow side. This biasing effect can be countered by a positive density difference, i.e. heavier fluid displacing light fluid.

iii) Sufficient density difference is required so that the displacement is steady. In particular, we find a regime where the flow is too turbulent, meaning that the density difference is not sufficient to give a steadily advancing interface. We showed that these competing effects can be quantified by Richardson number \((R_i)\) defined by (5.14). As a rough guide, the critical value of Richardson number is

\[
R_i^c \approx 1, \tag{9.1}
\]

below which flow is too turbulent.
Chapter 6

In Chapter 6, we started analyzing mixed flow regimes. Here we primarily focused on density stable flows, where the positive density difference assists us in having steady displacement flows. Our approach here was to investigate if any displacement regime is more preferred compared to others, in terms of displacement outcome. To answer this equation, we surveyed the literature and identified typical ranges of physical properties of drilling muds, spacers and cement slurries. Then, we chose a mud with typical physical properties and came up with several choices of spacers with different physical parameter within the range identified earlier. We now compare the effectiveness of the different spacers at displacing the mud. We do this under a constraint- total frictional pressure drop, i.e. mimicking the real constraints in job design by e.g. remaining below the fracture pressure of the formation. The goal was to find which admissible spacer design performs better in terms of mud displacement, i.e. does a low viscous turbulent spacer displace mud better/worse than a highly viscous one?

Our analysis showed that the effect of displacement regime is very minimal, and it is more likely dominated by the effect of eccentricity. That is, in a mildly eccentric annulus, most spacer designs that have sufficient density difference worked well, regardless of the displacement regime. Conversely, in a highly eccentric annulus, most of our spacer designs performed relatively poorly. In opposite to what is perceived in the cementing community, we did not find any consistent indication that turbulent displacement improves displacement outcome.

Chapter 7

In Chapter 7 we address the problem of density unstable displacement flows. Specifically, we probe into the effectiveness of displacement of mud with a lightweight low viscous washes, which is a fairly standard procedure in primary cementing. Our simulations consistently demonstrate that in such displacement flows, the wash rapidly channels up the wide side of the annulus. Even when fully turbulent, it is ineffective at displacing mud from around the annulus. We also illustrated that the usual notion of “contact time” is an inaccurate measure of cleaning. The contact time is very local, and a uniform distribution of contact time is not to be expected. More critically, even increasing the wash volume does not necessarily
help with contact time, as this type of flow naturally channels along the wide side. This largely invalidates the motivation of measuring chemical cleaning efficiency through a bulk contact time.

**Chapter 8**

In Chapter 8, we presented a feasibility study for a novel technique to track the interface between two fluids in annular displacement flow. Our technique is based on exploiting the density difference between successive fluids pumped in order to design a tracer particle to sit at the interface. Tracking the particles sitting at the interface allows us to track the displacement flow interface, which in turn provides a more accurate assessment of the cement job. Although apparently trivial, such particles must overcome viscous drag and strong secondary flows in order to reach and remain at the interface. We showed that for the particles to reach to the interface, the following heuristic criteria may be sufficient, provided that the well is not too eccentric:

\[ Bu = 18. \]

For a range of parameters relevant to laminar primary cementing, the criteria above is marginally satisfied, but we have suggested ways by which this method may be improved and used more widely in cementing.

### 9.2 Industrial implications

The contribution of the thesis from an industrial point of view are listed below:

- Shear thinning and yield stress fluids are ubiquitous in several industries, such as oil and gas, mining, drilling, paper as well as food and cosmetic industries. Designing such processes requires, at least, a gross understanding of hydrodynamic of the problem. Chapter 2 provides a straightforward framework for conducting hydraulic calculations for shear thinning yield stress fluids across different flow regimes.

- The problem of mixing is of great concern in several process industries. Here our treatment of diffusivity and dispersivity may give a practical estimate.
More critically, in the case of weakly turbulent flows, we showed that the Taylor dispersion massively influences the mixing paradigm, and is more important than turbulent diffusion.

- Although our model development is born out of complex mathematical analysis, our derivation, specially the treatment of the concentration equation, is still of interest to oil and gas service companies that have an in-house computer simulators and design tools based on similar 2D annular flow models. In this context, the anisotropy of the dispersivity modelling is the key improvement and one that is lacking in the literature.

- There is a widely accepted belief in the primary cementing community that the displacement should be conducted in turbulent, whenever possible. Our results however disagree. We did not find any consistent indication that turbulent flows perform much better than laminar displacement. As a matter of fact, we identified cases where a highly viscous displacing fluid that flows in laminar performs better than its turbulent low viscous counterpart.

- The main point of comparing fluid designs, that we hope to have reinforced is that such comparisons must be made within an operationally constrained scenario. Our analysis is just one such example where we have restricted the total frictional pressure. In real wells the constraints, geometries and pumping schedules are more complex. If our analysis leads industrial practitioners to question "is fluid A better than fluid B for this well and at these flow rates?" then this will be a step forward. Inevitably, answering such questions requires the use of model simulators of the complexity of that we have developed.

- Instead of flow regime, we found that the single most influential parameter in primary cementing is eccentricity. Wells with moderate eccentricity are able to be cemented, with a reasonable choice of spacer/cement slurry physical parameters. Whereas, it is a great challenge to successfully displacement muds in highly eccentric wells, even at large density differences. We strongly advocate for the use of centralizers in a systematic way and the subsequent evaluation of the actual eccentricity in wells that are cemented.
• In addition to the eccentricity, the sufficient density difference between the successive fluids are found to be critical. Sufficient density difference depends on various parameters, including the consistency and yield stress of mud, displacement flow rate, and most importantly, the well eccentricity. We find that the 10% density minimum difference criteria that is commonly found in industrial protocols such as in the “Effective Laminar Flow” rule may indeed not be sufficient.

• Having discussed the role of density difference, we introduced the notion of *too turbulent*. We showed that in vertical wells if the flow rate increases, the ability of buoyancy stresses to provide steady displacement reduces. This is clearly shown in the dimensionless Richardson number, defined by (5.14) and illustrated in Figure 5.15. The higher the Richardson number, the more likely the displacement is steady. More critically, the critical Richardson number criteria (9.1) denotes a necessary (and not necessarily sufficient) condition for the displacement to be steady.

• Perhaps our most consequential industrial contribution in this thesis is to clearly show the idea of using a lightweight low viscous preflush (known as *wash*) is debunked. In opposite to common belief that washes clean the walls of wellbore and assist the displacement, instead they rapidly channel up the wide side of the annulus, leaving a very poor displacement behind.

• Furthermore, we showed that the notion of “contact time” is a poor measure of displacement time. Our analysis confirms that contact time is not uniformly distributed. This means contact time may not even be correlated with the flow rate, because increasing flow rate just intensifies the non-uniformity of its distribution. Equally, increasing wash volume does nothing to help the regions that are by-passed by the wash.

• In Chapter 8, our analysis showed that the idea of tagging the interface with particles with intermediate densities for tracking may indeed be feasible. This opens up various interesting opportunities. For example, we can track the interface in real-time and identify displacement and/or geometric
anomalies in the annulus. Moreover, the top of cement, together with azimuthal disparities can potentially be detected. Finally, after the pumps are shut down, we can monitor post-placement fluid motions.

9.3 Limitations

As with every model-based study, there are limitations in terms of the assumptions made. Effective modeling of industrial processes is always a balance between including sufficient physical complexity that the model can predict meaningful behaviours/quantities and having a model that one is able to resolve computationally and/or analyze, in whatever manner practical constraints determine.

In Chapter 2, we presented a new analysis to predict/correct the turbulent velocity profile and its variations near the wall. We then employed the Reynolds analogy to derive expressions for the turbulent diffusion, subsequently Taylor dispersion. Our method appears to be effective. We do however acknowledge several limitations in predicting the turbulent diffusivity. Firstly, there is the influence of the centreline correction function discussed in Figure 2.9. Secondly, applying the Reynolds analogy in the wall layer is questionable, due to the different boundary conditions for mass and momentum transport. Thirdly, the coefficients of the Reynolds stress are sensitive to the matching procedure, which is admittedly crude. However, notice that the Taylor dispersion is shown to be the dominant effect, and in particular in weak turbulent regimes. This is not particularly sensitive to variations of turbulent diffusivity in the core, where the relative velocity is small. In the wall layers the relative velocity contributions to dispersivity coefficient are largest, but here turbulent diffusivity is constrained to decay cubicly to zero from matched values in the core. The precise shape of turbulent diffusivity has minor effect. Ideally, a direct prediction of turbulent diffusivity should be used to improve the methodology. However, we are not aware of data or a model that would adequately cover the range of rheological parameters. There are limited DNS studies. Experimental studies exist, but they use real fluids which have other rheological characteristics than the inelastic Herschel-Bulkley parameter fit, e.g. some elasticity. It is not known how such features may effect turbulent diffusivity.

In Chapter 3, we derived our 2D model for the displacement of yield stress
fluids in an annular geometry. While effective at reproducing large scale process features, 2D models also have restrictions due to their assumptions. Firstly, the model derivation is based on the assumption that

\[ \delta Re \ll 1, \quad Re = \frac{\hat{\rho} \hat{W}^2}{\varepsilon_0} \]  

(9.2)

see (3.14) and (3.15). This is required to ensure terms like

\[ \partial \frac{\partial}{\partial \hat{r}} [\hat{\rho} \hat{u}], \quad \partial \frac{\partial}{\partial \hat{r}} [\hat{\rho} \hat{v} \hat{u}] \quad \text{or} \quad \partial \frac{\partial}{\partial \hat{r}} [\hat{\rho} \hat{w} \hat{u}] \]

can be neglected, and assume the flow is dominantly shear-driven. Notice that these terms are the inertial stresses due to the mean velocity gradients. The turbulent stresses are accounted in the model. Simplistically speaking, the Reynolds number range varies from 100 to 2000 in laminar flows, and above in turbulent flows. This means that the condition (9.2) may not be fully satisfied, when the annulus is not narrow. Therefore, the use of the model in annuli that have larger aspect ratio may not be as accurate.

Additionally, the model is based on eliminating radial variations through averaging. Averaging across the gap means that these models predict only the mean concentrations of fluids and not the transverse distribution. Another assumption embedded in the model is that the flows are locally fully developed, i.e. relative to the gap with. In addition, any variation along the well has to be slow, which limits the use of the model in setting such as a washout, where these conditions are not met.

Finally, our model assumes the fluids are simple viscoplastic fluids, whereas real fluids have several other features such have elasticity, aging, thixotropy, etc. Having said this, the oil and gas industry uses 3-parameter models such as the Herschel-Bulkley model routinely and as we have seen, such models still allow for considerable complexity of behaviour. Until this behaviour is better understood and implemented into standard practices, it is unlikely that the industry will look for further complexity.

In Chapter 4, we derived an augmented Lagrangian algorithm to solve the non-linear elliptic equation found for the stream function (3.60). While the algorithm
is robust and guaranteed to converge, it is also notoriously slow, compared to alternative algorithms that are based on viscosity regularization.

In Chapters 5-7, we performed a number of simulations to analyze different displacement scenarios. A number of limitations can be mentioned here: i) We used a linear mixing closure to define the properties of a mixture from those of its constituents. There is no particular study that addresses this problem and provides more sophisticated closures, and it is likely that this would be quite specific to the fluids in question. ii) We always assume the displacement starts with an annulus initially filled with the drilling mud/spacer, and pure spacer/cement slurry enters the annulus. This may indeed be far from reality, as these fluid can be mixed in the casing. iii) Throughout the thesis we have focused at vertical wells. This is not a restriction in terms of our model, but we have had to restrict the focus for the applications.

In Chapter 8, we proposed a new methodology for tracking the interface. Our approach in analyzing this new tracking technique was rather simplistic. For example, we relied on a simple Stokesian drag law to compute the drag force on the particle. In the case of non-Newtonian fluids, we simply take the effective viscosity into account. More sophisticated closure in terms of drag law, as well as other effects, such shear induced particle migration, can be considered. Other limitations of the technique will evidently arise in strongly inclined and horizontal wells. Since the method relies on the buoyancy force, it fails to be effective, as the well becomes more inclined.

9.4 Future directions

There are many possible future directions, following from the previous section. We outline the main ones here.

First, to improve the turbulent diffusivity description for the Taylor dispersion calculations we favour computation because it can exactly represent the rheological models (without additional effects in real fluids). Using data generated from Direct Numerical Simulation (DNS), e.g. Rudman et al. [196], is certainly one approach worth exploring.

Perhaps the most immediate improvement of the annular displacement model
is to include some estimates of the inertial terms. Of primary concern is whether the system retains its nice features, if the inertial terms are included. For example, Navier-Stokes equations, in general, have non-unique solutions, as we increase the inertial effects. Does adding inertial terms make the solutions to the velocity non-unique, presumably because the problem is not elliptic anymore? If so, can we identify conditions that guarantee the uniqueness of the model?

As mentioned above, in the simulations presented, we neglect the fact that the displacing fluid may be already contaminated by the displaced fluid, as they are traveling down the casing. A nice improvement of the simulations is to include some form of “mixing box”, where the two fluids are allowed to be mixed, and then mixture is sent inside the annulus, or better still to model the fluids as they displace and mix along the casing.

In terms of employment of augmented Lagrangian algorithm, a possible research direction is to include more advanced algorithms, such as FISTA introduced in Treskatis et al. [236]. Out attempt to use FISTA did not improve the speed, but since then other studies such as Treskatis et al. [237] have been published that looked at numerical complexities of implementation of FISTA in viscoplastic-type problems. Insights from these studies may be helpful in successfully implementing faster optimization algorithms.

Our 2D annular displacement model was derived on the assumption that the inner cylinder does not rotate. However, many studies show that rotation of inner cylinder can significantly improve displacement outcome in laminar regime; e.g. see [56]. It is therefore of interest to modify the formulation to include the motion of inner cylinder too. Similar approaches to that of [35, 36] is a viable option.

Successful laminar primary cementing is synonymous with a steady state traveling wave displacement front. For vertical wells, Pelipenko and Frigaard [181] derived conditions that straightforwardly and quickly predicted whether the displacement front is steady or not, without fully resolving the 2D flow. To obtain these condition, Pelipenko and Frigaard [181] established a lubrication-type equation which is obtained from the kinematic equation for the interface. In the case of turbulent displacement, diffusion and dispersion are present, and therefore the evolution of interface is not governed by a simple advection equation, but rather has some complex diffusive terms which spread the interface. Is there any way...
that some analysis similar to those in Pelipenko and Frigaard [181] can be applied here?

Finally, there is a wide range of application-oriented questions that come from other cementing scenarios not considered here. In particular here, the role of turbulent flow in strongly inclined and horizontal cementing has not been studied.
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Appendix A

OGRE Package

A.1 General description

The main module is run_sim_fixed. In implementing this module we have assumed:

- Casing is stationary.

- Casing does not touch the wellbore. Though it can get as close as we want.

- Narrow side of annulus is at the bottom of well. This can be simply relaxed by changing $H$ and adding a shift phase.

- Density and rheology of the mixture is a linear function of constituents concentration.

Upon execution, the code reads data throughout read_sim_data.m. Details of data input are explain in §A.2. Following the scaling defined in §3.2, we non-dimensionalize the physical and geometrical parameters. This happens in scale_sim_data.m where all dimensional physical and geometrical parameters are passed in, get processed, and their dimensionless counterparts are returned.\(^1\) scale_sim_data.m also returns a structure called Scalar. This structure stores scaling parameters. More specifically, it is defined as below:

\(^1\)All dimensional parameters’ names start with a capital ‘D’ (e.g. Dyieldstress is the dimensional yield stress).
Scales.rho = Drho_star; % scaling factor for density
Scales.tau = Dtype_star; % scaling factor for stress
Scales.w = Dw_star; % scaling factor for velocity
Scales.t = Dt_star; % scaling factor for velocity

For more information about the output of scale_sim_data.m, see §A.3.

After reading and scaling data, the program calculates some computational parameters and then the main loop starts. The loop runs until the computational time is over. The loop will also end if 90% of the last displaced fluid exits the annulus.

Within the main loop, we first compute the rheological parameters as a function of concentration. This takes place in the following four functions: rhoC.m, kappaC.m, powerindexC.m and tauYC.m which account for density, power law consistency, power law index and yield stress, respectively. Currently, all these closure are linear with concentration; e.g.

\[ \rho = \rho_1 c_1 + \rho_2 c_2 + \ldots + \rho_k c_k. \]

The next step is to solve the elliptic equation for stream functions. The equation is solved using either the 2D augmented Lagrangian algorithm which is implemented in updatePsi_vector_corrected.m or the slice model algorithm which is implemented in updatePsi_slice_vector.m. The details can be found in §A.4. Once the stream function is updated, the new velocity field is obtained by differentiating stream function. This velocity field feeds the hyperbolic equation solver (stepadvance_c.m or stepadvance_c_diffusion_star.m). The output of this program is the concentration field at the new time step. This new concentration field will be used in the next time step to find density and rheology. Details of hyperbolic equation solver can be found in §A.6.

### A.2 Data input and output

Rheological and geometrical parameters, flow rate profile as well as numerical parameters are all given by the user using 5 text files. The text files are all in folder case_studies. The input text files are:
• **fluids.txt**: stores physical parameters of pure fluids. Each row belongs to one fluids and contains 4 numbers which are density (kg/m$^3$), consistency (Pa.s$^n$), power law index and yield stress (Pa), respectively.

• **geom1.txt**: stores profile of inner and outer radii as well as eccentricity. Each row contains the measurements at a certain depth with this order: vertical coordinate or depth (m), inner radius (m), outer radius (m) and eccentricity, respectively.

• **geom2.txt**: stores profile of inclination with respect to vertical axis. Each row contains measurement of inclination at a certain depth with this order: vertical coordinate (m) and inclination angle (degrees), respectively.

• **schedule.txt**: stores pumping schedule. Each row contains 2+nfluids numbers which are time (s), flow rate (m$^3$/s), concentration of fluid 1 in inlet, concentration of fluids 2 in inlet, etc.

• **intparams.txt**: stores numerical parameters. The rows are number of mesh in azimuthal direction $N_\phi$, number of mesh in axial direction $N_\xi$, maximum number of Uzawa iteration, tolerance of convergence, CFL number, ALPHA number, augmented Lagrangian $r_{ALG}$ parameter and augmented Lagrangian $\rho_{ALG}$ parameter.

• **options.txt**: stores the options that user chooses for the simulations. The options available are (orderly)

  – Simulation of $\Psi$: 1 = slice model, 2 = ALG model and 3 = consistently discretized ALG model. The recommended option is 3.

  – Hydraulic Closure: 1 = fully laminar, 2 = fully turbulent and 3 = multi-regime

  – Full or Half Annulus: 1 = half, 2 = full

  – Simulation of $c$: 1= only advection, 2 = advection + turbulent diffusion, 3 = Advection+ turbulent diffusion+ Taylor dispersion, 4= Advection+ turbulent diffusion+ Taylor dispersion + Taylor dispersion due to geometry.
– Initial guess for \( \Psi^* \), 1 = simple uniform flow, 2 = slice model at two ends and 3 = full slice model.

– FastDispersion: 0 = nofastdispersion, 1 = fastdispersion. see §4.3.2 for more details

– ALG Option, 1 = uses the Uzawa algorithm introduced in §4.2.4, 2 = uses a different version of the Uzawa algorithm where the gradient on \( q \) on the right hand side is translated onto \( b \).

– verbose option. 0 = only print the the current time of simulation, 1 = see the current time and the last step of ALG iteration results + figure of displacement, 2 = see the current time and every step of iterating results + figure of displacement.

The options.txt can have two additional optional parameters which be at line 9 and 10 in the text file. Line 9 takes the value of 1 for using bisection method and 2 for using Brent’s method. Brent’s method is typically slightly faster, but not by large margin. The default is bisection method. The code can run on multiple processors. The number of processors can be indicated in line 10. This number has to be a multiple of 2 (1, 2, 4, 8, ...). If the entry is blank or equal to 1, serial implementation is invoked.

All input text files are read by read_sim_data.m. After reading, the code checks if the input data are physically acceptable. The following requirements are considered:

• power law index must be between \((0, 1]\).

• time sequences must be strictly positive.

• flow rates must be strictly positive.

• measured depths must be increasing.

• inner radius must be smaller than outer radius.

• eccentricity must be \([0, 1)\).

• inflow concentrations must sum to 1 at all times.
Once the simulation is over, the output text file will be saved in the same folder. Writing data as a text file is done by `Write_To_Text.m`. The complementary code that can read output text file for future post processing is `Read_From_Text.m`. For example, the following command writes a vector variable $t$, matrix variable $v$ and cube variable $c$ and then reads it:

```matlab
Write_To_Text('output.txt',t,1,'time',v,2,'vel',c,3,'con'); % writing
[t,v,c]= Read_From_Text ('output.txt'); % reading
```

**Remark 1:** In the current implementation, `Read_From_Text.m` reads the data starting from top to bottom, so if you need something which is stored at the bottom of the file, you need to read everything above it too.

**Remark 2:** In order for the reading function to work properly, as you move towards the bottom of file, the order variables stored should not reduce. For example, the following command does not work:

```matlab
Write_To_Text('output.txt',v,2,'vel',c,3,'con',t,1,'time'); % wrong writing
[v,c,t]= Read_From_Text ('output.txt'); % reading won't work
```

### A.3 Data storage

The numerical algorithm is implemented on a staggered mesh, as shown in Figure 4.1. In this formulation, stream function ($\Psi$) is stored in the cell nodes. As velocity field is the derivative of stream function, azimuthal ($\nu$) and axial ($w$) components of velocity are naturally stored at cell interfaces. To ensure a high order of accuracy, we often needed to define different variables at various places in our discretization stencil. Assuming that a cell vertex is shown with index $(i,j)$ (circles in Figure 4.1), then interface mid-points are $(i+1/2,j)$ and $(i,j+1/2)$ (squares and diamonds in Figure 4.1). The cell center is also at $(i+1/2,j+1/2)$ (triangles in Figure 4.1).

To make the code more readable, we have used similar notation for naming the variables. For example,
• $H_{\text{zero} \_ \text{zero}}$: value of $H$ at cell nodes ($H(\phi_i, \xi_j)$).
• $H_{\text{half} \_ \text{half}}$: value of $H$ at cell center ($H(\phi_{i+1/2}, \xi_{j+1/2})$).
• $H_{\text{half} \_ \text{zero}}$: value of $H$ at cell interface ($H(\phi_{i+1/2}, \xi_j)$).
• $H_{\text{zero} \_ \text{half}}$: value of $H$ at cell interface ($H(\phi_i, \xi_{j+1/2})$).

As for $r_a$, since it is only function of $\xi$, we only have two cases:

• $r_{a \_ \text{zero}}$: value of $r_a$ at nodes ($r_a(\xi_j)$).
• $r_{a \_ \text{half}}$: value of $r_a$ at cell interface ($r_a(\xi_{j+1/2})$).

### A.4 Elliptic equation solver

We have implemented two algorithms in order to solve elliptic equation for stream function.

#### A.4.1 Slice model implementation

The 1D slice model is derived based on the assumption that the gradients in $\xi$ direction are much smaller than those in $\phi$ direction. This allows to simplify the 2D model into a 1D model that can be solved iteratively; for more details see §4.2.4. In the case of one fluid displacing another identical fluid, the slice model is exact (i.e. equivalent to the 2D model).

The slice model is implemented in `updatePsi_slice_vector.m`. To find $\Psi$ in each slice, the algorithm described in §4.2.4 is implemented using bisection method in `solve_sliceeq.m`. We start the algorithm by an initial guess

$$G_{\text{initial}} = \text{mean}(b_\phi).$$

Given the value of $G$, the slice stream function is constructed the subroutine within `find_dPsidphi_slice.m`. After finding $\Psi$, we evaluate

$$f = \Psi(\phi_{\text{end}}, \xi_j) - \Psi(1, \xi_j) - 2Q\phi_{\text{end}}.$$
As explained earlier, \( \phi_{\text{end}} \) takes the value of 1 or 2 for half or full annulus, respectively. \( f \) here is the residual function, and determines whether the initial guess is an upper bound or lower bound for the value of \( G \). We then accordingly search for the other bound by gradually decreasing or increasing \( G \). Once the solution is bracketed, we use bisection method to find the solution within the given tolerance. This means we successively halve the interval where the solution is located. We stop the iteration, once a prescribed threshold is satisfied. The convergence threshold is prescribed by user in text file \texttt{intparam.txt}; see §A.2.

### A.4.2 Augmented Lagrangian implementation

The second algorithm solves the full 2D model using augmented Lagrangian algorithm. The Uzawa algorithm, derived in §4.2.4 is implemented in the subroutine \texttt{updatePsi_vector_corrected}. In this algorithm, each iteration involves solving a Poisson equation. This may sound computationally expensive, however the good news is that the discretization matrix only depends on mesh sizes \((\Delta\phi, \Delta\xi)\) and the mean radius \((r_a)\) which do not change during the simulation. That means that the discretization matrix is only computed and factorized once. The entries of the matrix is computed outside the main loop using the function \texttt{make_sparse_A_11.m} for half annulus where the boundary conditions for \( \tilde{\Psi} \) is all homogeneous Dirichlet or \texttt{make_sparse_A_11_fullperiodic.m} for full annulus simulation, where we have periodic boundary conditions in \( \phi \). Finite Difference with 11-point stencil is implemented.

Before the iterations start, we need to find the particular part \((\Psi^*)\) of the solution \((4.14)\); see \((4.2.4)\). There are three options available for this:

- **Simple**: We simply assume a uniform flow enters the annulus, and therefore:

  \[
  \Psi^* = \int_0^{\phi_{\text{end}}} 2H \, d\phi
  \]

  This simple condition does not capture back flow at the entrance.

- **Slice at the two ends**: This option computes the stream function at the entrance and exit slice using \texttt{solve_sliceeq.m}. \( \Psi^* \) is constructed via a linear interpolation between the two sliced stream functions. Since this process
is time consuming, we only use this for the first time step or if the flow rate changes significantly. Otherwise, we use the previous time step. This operation is done in function `find_Psisistar.m`.

- **Full slice**: We solve full slice model as described in §A.4.1. Since this process is time consuming we only use it for the first time step or if the flow rate changes significantly. Otherwise, we use the previous time step.

The second step of ALG involves solving the following non-linear equation:

\[ \mathbf{q} = \theta \frac{\mathbf{M}}{|\mathbf{M}|} - \nabla_a \Psi^* \]

where

\[ \mathbf{M} = \mathbf{T} + r_{ALG} \nabla_a \Psi + r_{ALG} \mathbf{q}, \quad m = |\mathbf{M}| \quad (A.1) \]

and

\[ r_{a} \frac{\tau_w(\theta)}{H} + \theta r_{ALG} = |\mathbf{M}| \]

This equation clearly has a solution between \(0 < \theta < |\mathbf{M}|/r_{ALG}\) which can be found iteratively. Since the solution is already bracketed, a simple bisection algorithm is implemented to find the solution. What makes this operation expensive is that \(\tau_w\) (or equivalently \(H_w\)) is only implicitly defined as a function of \(\theta\) (or equivalently \(Re_p\)) via (2.25) or (2.28). In order to circumvent this problem, we consider two procedures which led to a factor of 100 improvement in computation speed.

- As explained earlier in 4.2.4, we take \(\tau_w\) as the independent variable and solve for \(\theta\). i.e. solve

  \[ r_{a} \frac{\tau_w}{H} + \theta (\tau_w)r_{ALG} = |\mathbf{M}|. \]

  With this formulation, the hydraulic calculations are explicit and require no iterative process.

- In spite of this massive improvement, the hydraulic calculations are still slow, because at each call, critical values of \(H_w, H_{w,1}\) and \(H_{w,2}\), must be calculated **iteratively** to identify if the flow is laminar, transitional or turbulent.
Both $H_{w,1}$ and $H_{w,2}$ are function of $(n, He)$, so conveniently, we generate a lookup table consisting of values of $H_{w,1}$ and $H_{w,2}$ for a wide range of $n$ and $He$. Later during the simulations, we only lookup the table to calculate $H_{w,1}$ and $H_{w,2}$.

Notice that in the OGRE package there also exists older implementations of the Uzawa algorithm. One older implementation is in $(updatePsi\_vector)$ which does not take into account the position where each variable is stored, so some of the numerical differentiations are only first order accurate. As a result, in cases with complicated rheology, the algorithm may not converge. The advantage of this implementation is that it is roughly twice faster than the one in $updatePsi\_vector\_corrected$.

A.5 Hydraulic module

Hydraulic calculation relates wall shear stress to Reynolds number (or equivalently mean velocity). Finding Reynolds number from wall shear stress (forward hydraulic calculation) is explicit, and therefore straightforward. On the other hand, in the general case of a yield stress fluid, wall shear stress is only implicitly known as a function of Reynolds number. This means backward hydraulic calculation may require numerical inversion.

In the current version of the code, the hydraulic calculation happens within the subroutine $find\_Rep\_from\_Hw.m$. The inputs of this function are $n$: power law index, $He$: Hedstrom number (or dimensionless yield stress), $Hw$: dimensionless wall shear stress and finally $Option$. $Option$ contains the options that user chooses to use for the simulation. As for hydraulic calculations, three options are available for the user: 1) fully laminar closure; 2) fully turbulent closure and 3) multi-regime closure. User can choose each of this three using $Option.HydClosure.type$; see §A.2.

Calculations are generally fast when fully laminar or fully turbulent options are selected. In the case multi-regime closure, however, we need to first find two indicators $H_{w,1}$ and $H_{w,2}$. These two indicators determine the flow regime and can only be found iteratively. This will be problematic if you notice that during
one complete simulation, the hydraulic module is called millions of times. Therefore, for the sake having faster calculation, a lookup table containing values of $H_{w,1}$ and $H_{w,2}$ as a function of $n$ and $He$ is generated. With this table, instead of iterating, the values of $H_{w,1}$ and $H_{w,2}$ are read from the table. A fast linear interpolation algorithm is implemented for this purpose. The hydraulic table is created by `hydraulic_table2.m`. Right now, we tabulated $H_{w,1}$ and $H_{w,2}$ for $n$ and $He$ discretized as below:

```matlab
n = linspace(0.3,1,71);
He1 = linspace(0,1000,1000);
He2 = logspace(3,6,1000);
He3 = logspace(6,8,1000);
He = [He1 He2 He3];
```

Reading and interpolating tabulated data are done by `interpolate2d.m`. Through this function, we find a pair of $n_{table}$ and a pair of $He_{table}$ in the table closest to the value of $n$ and $He$ at which we want to find $H_{w,1}$ and $H_{w,2}$. Then a simple linear interpolation between these four points gives us the answer.

In the following, I will quickly review the functions we have in our hydraulic module. All these function can be found in the directory `Hydraulics`.

**evalRe_withHws.m**: performs forward channel hydraulic calculation; i.e. given wall shear stress, it computes Reynolds number (= mean velocity). The two indicator $H_{w,1}$ and $H_{w,2}$ are also passed to the function. $H_{w,1}$ is the largest laminar wall shear stress and $H_{w,2}$ is smallest turbulent wall shear stress. If $H_w < H_{w,1}$, then the flow is laminar. If $H_w > H_{w,2}$ we have fully turbulent flow and finally, if $H_{w,1} < H_w < H_{w,2}$, the flow is in transitional regime. The relation between $H_w$ and $Re_p$ and with $Re_{MR}$ is clearly stated in Maleki and Frigaard [146]. Upon finding the flow regime, the respected relation is used to find $Re_p$. In the case of transitional flow, no relation is known. In this case, we use a logarithmic interpolation for $f_f$ at $H_{w,1}$ and $H_{w,2}$ and then use it to find $f_f$. Having known friction factor and wall shear stress, we can straightforwardly find Reynolds numbers.

**find_Hw.m**: performs backward channel hydraulic calculation; i.e. given Reynolds number (= mean velocity), it computes wall shear stress. This function is compu-
tationally expensive, so we avoid using it, at least not in the main loop.

Using the given rheological parameters, the code computes $Re_p,1$ and $Re_p,2$ which are the largest laminar power law Reynolds number and smallest turbulent power law Reynolds number, respectively. If $Re_p < Re_p,1$, then the flow is laminar. If $Re_p > Re_p,2$ we have fully turbulent flow and finally, if $Re_p,1 < Re_p < Re_p,2$, the flow is in transitional regime. The relation between $H_w$, $Re_p$ and $Re_{MR}$ is clearly stated in Maleki and Frigaard [146].

The relation between $Re_p$ and $H_w$ is implicit for $H_w$; i.e. in order to find $H_w$, had we known $Re_p$, we need to numerically invert a function. When laminar, $H_w$ is bracketed between $He$ and $H_{w,1}$. When transitional, $H_{w,1} < H_w < H_{w,2}$. However when turbulent, we only know a lower bound ($H_{w,1}$). In this case, we first need to find an upper bound. Once the solution is bracketed, then we solve for $H_w$ using bisection method.

**compute_dispersion_diffusion_directly.m:** Through this module, the turbulent diffusivity coefficient and Taylor dispersion coefficient is calculated from the wall shear stress and rheological data. Both of these coefficient are zero when the flow is laminar, so this is only active when we have turbulent or transitional flows. In the case of turbulent flows, we need to first the corrected turbulent velocity profile. This is done in VP_Coefficients_vec.m. Then we do the integrations introduced in Maleki and Frigaard [146](equations 71 and 72) to find the diffusion and dispersion coefficients. When the flow is in transitional, a linear interpolation based on $H_w$ is performed.

**other sub-routines:** A number of sub-routines are also implemented within this module:

- **nprime.m:** Calculates $n'$ as a function of $n$ and $y$. equation (59a) in Maleki and Frigaard [146].
- **EyYn.m:** Calculates $E$ as a function of $n$ and $y$. equation (59b) in Maleki and Frigaard [146].
- **Re2anpbnp.m:** Calculates $Re_2$ as a function of $n'$ as prescribed in equation (A.5-A.7) in Maleki and Frigaard [146].
• **find_Hw1.m**: This function compute $H_{w,1}$ (the largest laminar wall shear stress possible given the rheological parameters).

• **find_Hw2.m**: This function compute $H_{w,2}$ (the smallest turbulent wall shear stress possible given the rheological parameters).

• **evalReLam.m**: This function performs the forward *channel* hydraulic calculation when the flow is laminar; i.e. given wall shear stress, it finds Reynolds number assuming a laminar flow.

• **find_Hw_Lam.m**: This function performs the backward *channel* hydraulic calculation in laminar flow regime; i.e. given Reynolds number, it computes wall shear stress assuming the flow is laminar.

• **VP_Coefficients_vec.m**: This module finds the coefficients of velocity profile in the core and wall layer. In the core, the velocity is characterised by $A_0, B_0, B_0_{cr}$. In the wall layer, the coefficients are stored in $W_{plus}$. In addition, the width of wall layer is calculated by iteratively solving equation (46) in Maleki and Frigaard [146].

• **Fyplus_c_vec.m**: This function computes both sides of equation (46) in Maleki and Frigaard [146]. This is part of iterations to find width of wall layer ($x^+_c$).

• **CentreCorrection.m**: This function calculates the centreline correction function introduced in the velocity profile to fix the symmetry problem of logarithmic velocity profile. More specifically, it is equation (65c) in Maleki and Frigaard [146].

• **Dturb_profile_vec.m**: This function computes the integral given in equation (69) in Maleki and Frigaard [146].

• **DT_compute_vec.m**: This function computes the integral given in equation (72) in Maleki and Frigaard [146].
A.6 Hyperbolic equation solver

The hyperbolic equation solver is implemented in `stepadvance_c.m` for solely advective flows and `stepadvance_c_diffusion_star.m` for advective, diffusive, dispersive flows. Since the concentration equation to the leading order is governed by the advection, we may assure advection is implemented accurately.

One particular problem in dealing with this equation is due to the jump in concentration across the interface. This requires the solver to have shock capturing feature. In addition, in the absence of diffusion or dispersion, the code should keep the interface sharp which equivalently means numerical diffusion should be controlled. After testing different algorithms, the Flux Corrected Transport (FCT) algorithm was selected. FCT scheme relies on calculating the fluxes using two method: I) method one which is highly accurate, but may produce unphysical values in the case of a non-smooth concentration. II) method two which is a lower order method, but is guaranteed to produce physical values at all times. In both `stepadvance_c.m` and `stepadvance_c_diffusion_star.m` we use a second order central difference scheme and a first order upwind as the high and low order algorithms.

Diffusive and dispersive fluxes are calculated in the nested functions within `stepadvance_c_diffusion_star.m`. Multiple interpolations were necessary here.

User can switch on or off different components of this equation. The options are to have I) only advection, II) advection + turbulent diffusion, III) Advection+ turbulent diffusion+ Taylor dispersion and IV) Advection+ turbulent diffusion+ Taylor dispersion + Taylor dispersion due to geometry. These options can be set in `Option.c`; see §A.2.

The time advance scheme implemented here is explicit. Therefore, stability enforces small enough time steps. In particular, there are two time scales imposed by advection and diffusion:

\[
dt_{adv} = CFL \left[ \frac{d_x}{\text{max}(|w|)} + \frac{d_\phi}{\text{max}(|v|)} \right], \quad dt_{diff} = ALPHA \left[ \frac{d_\phi^2}{\text{max}(D,D_T,D_{\phi})} \right]
\]

stability requires \( CFL < 1 \) and \( ALPHA < 0.5 \). User can choose these values in
intparams.txt. Typically both are set to 0.4. For simple displacement flows, the typical value of these time steps are $dt_{diff} \sim 0.001 \ll dt_{adv} \sim 0.1$. For more complicated flows $dt_{adv}$ is much smaller. Even though, the bottle neck would be still $dt_{diff}$.

Implementing a implicit time advance scheme may sound attractive here. In fact, we tried this initially, but our choice of staggered mesh meant that a non-sparse matrix needs to be solved at every time step. Instead, we implemented a different algorithm in stepadvance_c_diffusion_star_fast.m where we modify the explicit scheme and update the concentration by $dt_{diff}$ and stream function by $dt_{adv}$. Effectively, this operation means we only update velocity field every few time steps; see §4.3.2.
Appendix B

A model for foamed cementing of oil & gas wells

I co-authored this paper with N. Hanachi and I. Frigaard. The paper is published in the Journal of Engineering Mathematics [104].

Abstract

We present a two-dimensional model of the primary cementing process for foamed cement slurries. Foamed cement slurries have a number of claimed advantages, but also have a pressure dependent density and rheology. The rheology is hard to quantify fully over all ranges of foam quality, which compromises the accuracy of models. The density variation is due to expansion/compression of the gas phase along the well, caused by variations in the static pressure. We show that in the absence of careful control, buoyancy-driven instabilities can result in the annulus, as the foamed slurry expands and the density drops below that of the displaced drilling mud. These instabilities appear to be of a classic porous media/Hele-Shaw cell fingering type, triggered by a threshold unstable density difference. We show that these instabilities are amplified by wellbore eccentricity, occurring lower in the well than in a concentric annulus. Our results question the safe usage of foamed cements in primary cementing.
B.1 Introduction

Primary cementing is a critical operation in the construction of oil/gas wells. The main purposes of primary cementing are to provide mechanical support and to provide a hydraulic seal in the annulus, between casing and rock formation. This serves to isolate the fluid-bearing zones and to prevent any uncontrolled flow behind the casing [167]. Wells are typically cemented using conventional cement slurries, which operate with densities in the range 1750 – 1950\( \text{kg/m}^3 \). In areas where the formation is weak and cannot support the hydrostatic pressure of conventional cement slurries, lightweight alternatives must be used: foamed cement is one of these.

Foamed cements consist of a base cement slurry that suspends a gas phase, typically nitrogen. Their usage was first proposed in the early 1980s [50, 19] and by the late 1980s a number of case study/field applications had been performed [112, 45, 30, 164]. This was also an era where nitrified drilling fluids were being developed and used in underbalanced drilling techniques. Both techniques have experienced a dormant phase but in the past 10-15 years there has been a resurgence of interest. In the drilling arena MPD/CPD (managed or controlled pressure drilling) methods are more common and foamed cementing has also benefitted from the increased comfort with pressure management technology.

Foamed cements have low density, are weakly compressible and when they dehydrate, they exhibit higher ductility and higher tensile strength, compared to their conventional counterparts. Some studies show improved displacement quality and therefore, better zonal isolation in foamed cement wells [134], particularly in wells with high temperature and pressure [20, 97]. This is supposedly because foamed cements are more resilient to cracking caused by temperature and/or pressure-induced stress cycles during production [20, 97]. Furthermore, the expansion of the gas phase during dehydration may compensate the volume loss that typically happens in conventional cements. This lowers the chance of gas migration in foamed cements [20].

Despite these apparent advantages, the disastrous Deepwater Horizon oil spill in the Gulf of Mexico has left major concerns in using foamed cements. Major themes of the investigation of the incident [2] concerned the stability of the foamed
slurry, its testing and suitability for this well. The hydrodynamic stability down-
hole was also questioned: “because of the great difference in density between the
two types of cement used in the pumping sequence, the heavy unfoamed cement
pumped in after the foamed cement probably fell into, and perhaps through, the
foamed slurry” [242]. Such incidents expose knowledge gaps in so far as foamed
cementing is concerned. Although hydrodynamically driven motions in the an-
nulus occur for conventional cements, two factors increase their importance in
foamed systems. (i) Buoyancy-driven instabilities result in foam expansion and
density reduction, i.e. the instabilities are self-reinforcing. (ii) Usage of foamed
cements relies on *foam stability*, (i.e. that the bubbles remain dispersed in the slurry
and do not coalesce significantly into larger gas pockets), which is less certain in
an accelerating unstable flow. Indeed, either of these factors may lead to a loss of
primary control of the well.

This paper is concerned with the hydrodynamics of well cementing using foamed
cements. To introduce the underlying physical systems we start with a one-dimensional
*toy* model. This model is similar to several other 1D models in the literature that
deal with foamed wellbore fluids, e.g. [22, 174]. Regarding foamed cement de-
signs explicitly, the compressible hydrostatic aspect has been dealt with in [155]
and the frictional pressure losses, which are the pressure losses due to the wall
shear stresses, depend on the assumed rheological closure model; an implementa-
tion example being [88]. We discuss this aspect further in §B.2.1. There are many
proprietary 1D hydraulics-style simulators used for job design by different cement-
ing companies. Our 1D model is less complete and meant only to be illustrative of
typical density variations downhole in foamed cementing, so as to highlight oper-
ational concerns.

The main novelty of the present study is the extension of our 2D gap-averaged
model [24, 147] for the annular displacement flow to compressible cement slur-
ries and its use to understand potential drawbacks of the use of foamed cements.
The derivation follows along the same lines as in [24]. The incompressible model
consists of a 2D elliptic equation to determine the gap-averaged stream function
coupled to a series of advective mass conservation equations for the different con-
stitutive fluids. The model has been extensively analysed for vertical and horizontal
well cementing [37, 179, 180, 181] and more recently extended to model displace-
ment flows in turbulent and mixed flow regimes [147]. Many cementing companies use proprietary model simulators for annular cementing design that are based on this same methodology.

Our development here is confined to laminar flow regimes. The main focus is to introduce a preliminary model that allows the study of hydrodynamic effects in the annulus resulting from use of the foamed cement. This model can be used to understand the complex influences on displacement flows of gas fraction, fluid density, rheology, wellbore geometry and imposed pressure controls. An outline of the paper is as follows. In §B.2 we introduce foamed cementing modeling through development of a 1D hydraulic model. In §B.3 the 2D annular displacement model is derived. We present several representative examples in §B.4, which expose serious concerns for foamed cementing. The paper closes with a brief discussion and conclusions in §B.5.

**B.2 Foamed cement hydraulics**

Displacement flow aspects of the primary cementing operation using foamed cements are conceptually similar to that using an incompressible cement slurry: a sequence of fluids is pumped down inside the casing to bottom-hole, through the float collar/shoe, and returns up the annulus (see Figure B.1). As with conventional cementing, the foamed slurry is preceded by one or more preflushes (e.g. wash, spacer) according to the job specifics. Additional operational complications are physico-chemical, mechanical and equipment related, summarised as follows.

(i) Fluid-mechanical complications arise from compressibility of the foamed slurry and from the complex dependence of the rheological properties of the slurry on the gas fraction.

(ii) To ensure foam stability it is critical to control the exact proportions of stabilizer, foaming agent, gas (nitrogen), and base slurry (mixture of dry cement and water). The preflushes may also need to be designed for chemical compatibility, avoiding degradation of the foam during setting, as a degree of mixing inevitably occurs.
Figure B.1: (a,b) Schematic of the primary cementing. (c,d) Unwrapping the annulus into a channel of varying width

(iii) A typical foamed cementing system contains the following: cement slurry pump, liquid nitrogen storage tank, nitrogen pumping device, foam generator, injection pumps for foaming and stabilizing agents. Injection rates should be controlled and automated, which needs an accurate process control system to measure, calculate and keep track of injected foam quality at given temperature and pressure.

In this paper we consider only (i) insofar as relevant to our models of the fluid flow, assuming that the foamed cement remains as a stable mixture, ignore chemical compatibility and equipment issues.
B.2.1 Description of foamed cements

Foamed cement is created when a gas (usually nitrogen) is injected into the pure cement slurry at high pressure. In order to have a light stable foam, adequate amounts of foam agent and stabilizer should be added to the mixture as well. The created foam contains small separate bubbles that should not coalesce and are not interconnected [49]. All foam properties strongly depend on the volume fraction of gas \( \alpha \). However, apart from the density, there is significant difficulty in characterizing these properties. At least two reasons underlie this difficulty.

First, much of our understanding of foam mechanics comes from study of classical systems such as soap film foams, which are characterized by very high gas fractions \( \alpha > 0.95 \) and bubbles separated by thin films of liquid. The bulk flow rheology of these systems is often characterised as a yield stress fluid, but the source of the foam structure is capillarity. In contrast, although foamed cement slurries may start at surface with high volume fractions as they are formed (e.g. 75-90%), these are then pumped at pressure downhole, where hydrostatic pressure compresses the gas phase significantly with depth. Downhole gas fractions in foamed slurries are more commonly in the range 5-40%, with a rapid increase in \( \alpha \) to surface values occurring in the first few hundred metres.

Second, unlike many other aqueous foams, here the liquid phase is itself non-Newtonian. Oilfield cements are fine colloidal suspensions with yield stress and shear-thinning behaviour, often modelled as Herschel-Bulkley fluids. Part of the structure of the foamed slurry, especially at lower \( \alpha \), is due to the cement rheology. Thus, to properly model foamed cement rheology we need to bridge between limiting models for conventional foams \( (\alpha \lesssim 1) \) and for bubbly liquids \( (\alpha \sim 0.5) \), while also taking into account the liquid phase rheology. This study is however not primarily about the rheology of foamed cement; see §B.2.2 below.

As a preliminary model we are concerned mainly with vertical wells. In such wells the dominant contribution to the pressure comes from static pressure and not the frictional pressure (typically 10% or less of the total pressure drop). Operationally, most of the technical literature uses the quality, \( q \), to describe the foamed slurry:

\[
q = \frac{\dot{Q}_g}{\dot{Q}_g + \dot{Q}_c},
\]  

(B.1)
where $\hat{Q}_g$ and $\hat{Q}_c$ denote volumetric flowrates of gas and liquid phases, respectively. Let’s suppose that a sequence of $K$ fluids are pumped, with the last fluid being the foamed cement. Assuming that the bubbles in the foam are well dispersed and that the two phases do not slip or coalesce, then $q = \alpha$. This is the common assumption made for stable foamed cements. With this mixture assumption, the density of the foamed slurry $\hat{\rho}_K$, is given by:

$$
\hat{\rho}_K = \alpha \hat{\rho}_g + (1 - \alpha) \hat{\rho}_c.
$$

(B.2)

The cement density $\hat{\rho}_c$ is constant, whereas the gas density $\hat{\rho}_g$ is given by its equation of state. For simplicity here we take:

$$
\hat{\rho}_g = \frac{\hat{\rho} \hat{M}}{\hat{R} \hat{T}},
$$

(B.3)

where $\hat{M}$ is the molar mass of the gas, $\hat{R}$ is the universal gas constant, and $\hat{T}$ is the temperature. Adoption of the ideal gas law for $\text{N}_2$ is reasonable for well depths up to $\approx 3$ km. To improve this description the $Z$-compressibility factor can be used (see e.g. [228]), as is done in most industry models.

As the foam compresses, $\alpha$ (or $q$) changes along the flow path: the gas flow rate $\hat{Q}_g$ at depth is not the same as at surface. It is instead the mass fraction $Y_g$ of gas within the foam which is transported with the foamed mixture. In terms of $Y_g$, the cement density is given by:

$$
\hat{\rho}_K = \frac{1}{Y_g/\hat{\rho}_g + (1 - Y_g)/\hat{\rho}_c},
$$

(B.4)

where

$$
Y_g = \frac{\alpha \hat{\rho}_g}{\alpha \hat{\rho}_g + (1 - \alpha) \hat{\rho}_c} = \frac{\hat{Q}_g \hat{\rho}_g}{\hat{Q}_g \hat{\rho}_g + \hat{Q}_c \hat{\rho}_c} = \frac{\hat{M}_g}{\hat{M}_g + \hat{M}_c}.
$$

(B.5)

The bracketed expressions are true for a homogeneous mixture. Here $\hat{M}_g$ and $\hat{M}_c$ are the mass flow rates of gas and liquid phases.
B.2.2 Foam rheology

As discussed above, foamed cements vary significantly in quality as they are pumped downhole, so that for much of the wellbore they resemble bubbly liquids rather than true foams. There is broad consensus that the foamed cements behave as shear-thinning yield stress fluids. The viscosity of conventional foam could be significantly larger than the viscosity of suspending fluid. Experimental results have indicated that the viscosity of the foam, which depends on foam quality and nitrogen injection rate, rises with foam quality at low to medium foam quality. The growth at small gas volume fraction is slight and at medium volume fractions can become more dramatic. However, for foam at high gas volume fraction, the viscosity falls sharply with increasing foam quality [65, 87].

A recent review of available models for foamed cement is given in [5]. The rheological parameters of these models depend on shear, volume fraction $\alpha$, and bubble size (dimensionlessly on the Capillary number). Depending on $\alpha$ and the borehole roughness, the foam may slip partially at the walls. These mechanistic features are also influenced by pressure and temperature. This complexity means that the literature often produces rheological results and trends that run counter to our intuition. There is not, to our knowledge, any comprehensive model that accounts fully for foamed cement rheology.

In this paper, for the purposes of model closure, we adopt the following constitutive equations, which are based on a micro-mechanical estimation of the behaviour of suspensions of bubbles in yield stress fluid [62]. The yield stress and consistency of foam, as a function of gas volume fraction, are

\[
\hat{\tau}_Y(\alpha) = \hat{\tau}_Y(0)\sqrt{\frac{(1-\alpha)(5+3\alpha)}{5-2\alpha}}, \quad \text{and} \quad (B.6)
\]

\[
\hat{\kappa}(\alpha) = \hat{\kappa}(0)\left(\frac{5+3\alpha}{5-2\alpha}\right)^{\frac{n+1}{2n}}(1-\alpha)^{\frac{1-2n}{2n}}, \quad (B.7)
\]

where $\hat{\tau}_Y(0)$ and $\hat{\kappa}(0)$ are the yield stress and consistency factor of the pure fluid (slurry). Equations (B.6) and (B.7) are valid for cases with $\alpha < 50\%$. These $\alpha$-dependent (effectively pressure-dependent) rheological parameters are used within a conventional Herschel-Bulkley constitutive model. We do not include any varia-
tion in the power-law index with \( \alpha \).

### B.2.3 Wall shear stress

We shall evaluate the wall shear stress (\( \hat{\tau}_w \)) in the pipe and annulus, as a function of the rheological parameters. We consider only laminar flows here. For a mean velocity \( \hat{W}_0 \) in a pipe of diameter \( \hat{D} \), by integrating the axial momentum balance across the pipe, the Buckingham-Reiner equation can be derived, which is an algebraic equation relating the flow rate to the wall shear stress. The Rabinowitsch-Mooney procedure results in the same expression. For Herschel-Bulkley fluids the result is:

\[
\frac{8 \hat{W}_0}{\hat{D}} \left[ \frac{\hat{\kappa}}{\hat{\tau}_w} \right]^{1/n} = \frac{4n}{3n+1} (1 - r_Y)^{1/n+1} \times \\
\left[ (1 - r_Y)^2 + \frac{2(3n+1)(1 - r_Y)r_Y}{2n+1} + \frac{(3n+1)r_Y^2}{n+1} \right].
\]  

(B.8)

Here \( r_Y = \hat{\tau}_Y / \hat{\tau}_w \), which also represents the dimensionless radial position of the yield surface. This nonlinear equation is solved numerically using the bisection method. Essentially we find that \( r_Y = r_Y(B,n) \), where \( B = \hat{\tau}_Y / [\hat{\kappa}(8\hat{W}_0/\hat{D})^n] \).

Similarly for the annulus, we will model that locally as a channel of width \( 2\hat{H} \) flowing with mean velocity \( \hat{W}_0 \). The Rabinowitsch-Mooney procedure applied to the laminar channel flow results in the following expression:

\[
\frac{6 \hat{W}_0}{2\hat{H}} \left[ \frac{\hat{\kappa}}{\hat{\tau}_w} \right]^{1/n} = \frac{3n}{2n+1} (1 - y_Y)^{1/n+1} \left( \frac{n}{n+1} y_Y + 1 \right),
\]  

(B.9)

where \( y_Y = \hat{\tau}_Y / \hat{\tau}_w \) represents the dimensionless plug thickness in the channel. Closures for other regimes are more complex; see e.g. [146].

### B.2.4 One-dimensional model

We now develop a one-dimensional hydraulic model to describe transport of the fluids along the flowpath. We assume that a sequence of \( K \) fluids is pumped into the casing, each characterized as a Herschel-Bulkley fluid. For simplicity only the last stage (fluid \( K \)) is a foamed slurry. The flow path is parameterized by \( \hat{z} \), from
\[ \hat{z} = \hat{z}_c = 0 \] at the inflow at surface, down the casing to \( \hat{z}_{bh} \) at bottom hole, and returning upwards along the annulus to \( \hat{z}_a \) at the exit of the annulus. To simplify, we will only consider onshore wells and assume that \( \hat{z}_c \) and \( \hat{z}_a \) are at the same height on the surface; hence \( \hat{z}_a - \hat{z}_c = 2\hat{z}_{bh} \). To simplify the model we neglect temperature variation along the flow path. Wellbore heat transfer is usually convectively dominated, so this approximation degenerates for deep wells where conduction has time to act. Thus, the assumption is that the nitrogen is injected at a specified surface pressure \( \hat{p}_c \) (and temperature), at the top of the casing.

The fluids are modelled as a locally homogeneous mixture, with \( c_j \) representing the volume fraction of fluid \( j \). The mixture moves with a single velocity \( \hat{u} \) (no slip between the constituents or phases). Mass conservation is modeled as

\[ \frac{\partial}{\partial t}(\hat{\rho}c_j) + \hat{\nabla} \cdot (\hat{\rho}c_j \hat{u}) = 0, \quad j = 1, 2, \ldots, K, \tag{B.10} \]

where \( \hat{\rho} \) and \( c_j \) are density and volume fraction of fluid \( j \), respectively. These equations may also be written in terms of mass fraction \( Y_j \) of fluid \( j \):

\[ \frac{\partial}{\partial t}(\hat{\rho}Y_j) + \hat{\nabla} \cdot (\hat{\rho}Y_j \hat{u}) = 0, \quad j = 1, 2, \ldots, K. \tag{B.11} \]

Here

\[ \hat{\rho} = \sum_{j=1}^K \hat{\rho}_j c_j = \frac{1}{\sum_{j=1}^K Y_j / \hat{\rho}_j}. \tag{B.12} \]

On summing (B.11) over \( j \) we find:

\[ \frac{\partial \hat{\rho}}{\partial t} + \hat{\nabla} \cdot (\hat{\rho} \hat{u}) = 0, \tag{B.13} \]

representing mass conservation of the total mixture. On subtracting (B.13) from (B.11) we find that

\[ \frac{\partial}{\partial t}Y_j + \hat{u} \cdot \hat{\nabla}Y_j = 0, \tag{B.14} \]

i.e. each mass fraction is advected with the flow. This may be repeated specifically
for the slurry mass conservation equation, with density $\hat{\rho}_K$, to show that:

$$\frac{\partial}{\partial t} Y_g + \hat{\mathbf{u}} \cdot \hat{\nabla} Y_g = 0,$$

(B.15)

for the gas mass fraction.

The cross-sectional area, at distance $\hat{z}$ along the flow path, is denoted $\hat{A}(\hat{z})$. In a one-dimensional model we assume that the mass (or volume) fractions vary primarily in the $\hat{z}$-direction. On averaging across the flow path, we find:

$$\frac{\partial}{\partial t} (\hat{A} \hat{\rho} Y_j) + \frac{\partial}{\partial \hat{z}} (\hat{A} \hat{\rho} Y_j \hat{\mathbf{w}}) = 0, \quad j = 1, 2, ..., K,$$

(B.16)

or advectively:

$$\frac{\partial}{\partial t} Y_j + \hat{\mathbf{w}} \frac{\partial}{\partial \hat{z}} Y_j = 0, \quad j = 1, 2, ..., K, \text{ and } g$$

(B.17)

and on summing:

$$\frac{\partial}{\partial t} (\hat{A} \hat{\rho}) + \frac{\partial}{\partial \hat{z}} (\hat{A} \hat{\rho} \hat{\mathbf{w}}) = 0.$$

(B.18)

Here $\hat{\mathbf{w}}$ is the averaged velocity in the $\hat{z}$-direction.

We note that the mixture density $\hat{\rho}$ is not an independent variable, being wholly defined by the mass fractions and densities, via (B.12). The mass fractions evolve according to (B.17) and the densities of the constituents are all constant, with the exception of the gas density $\hat{\rho}_g$. The latter is defined via the pressure (and temperature, neglected here). Thus, the pressure and averaged velocity are the 2 main independent variables of our 1D model.

The pressure is calculated from a steady one-dimensional averaged axial momentum equation:

$$- \frac{\partial}{\partial \hat{z}} (\hat{A} \hat{\rho}) - \hat{S}_w + \hat{A} \hat{\rho} \hat{\mathbf{g}} \cos \beta = 0,$$

(B.19)

where $\beta$ is the angle of inclination, between the $\hat{z}$-axis and vertical direction. $\hat{S}(\hat{z})$ is the length of (wetted) perimeter at $\hat{z}$ and $\hat{\kappa}_w$ is the wall shear stress. Equation (B.19) assumes a steady flow with slow axial variation in both the mean velocity and the axial stresses. The wall shear stress is expressed via the closures explained in §B.2.3: primarily it is a function of the local flow velocity, mixture density and
(within the foamed slurry), also the gas fraction and hence pressure.

The velocity is calculated from (B.18), but we first simplify this by neglecting the time derivative. The first term in (B.18) is concerned with density changes that are due to temporal changes in either the pressure or the mixture mass fractions. Changes in $\dot{\rho}$ due to the mixture mass fractions come from advection, and are accounted for by (B.17). Temporal changes in pressure are assessed in chapter 3 of [16] as part of a discussion of when flows may be considered approximately solenoidal. Generally speaking this term is small relative to the divergence terms, provided that the Mach number is small and that any forced oscillations in the flow are slow. Cementing flows involve velocities in the range $0.1 - 3 \text{m/s}$, so that the Mach number is small. Steady flow rates are imposed for long periods of time, so that oscillations are mainly only due to the pump and felt near the inflow. On neglecting the time derivative, (B.18) is approximated by

$$\frac{\partial}{\partial z}(\hat{A} \dot{\rho} \hat{w}) = 0.$$  \hspace{1cm} (B.20)

Note that we cannot neglect the gradients in $\dot{\rho}$ along the well, as this is the chief way in which compressibility influences the velocity field. As a guide to this, foamed cements are typically denser than water, for which the static pressure increases by approximately 1 atmosphere every 10 metres. Thus, in wells with depth $500 - 1000\text{m}$ we can have considerable compression of the gas phase, according to (B.3).

Equation (B.20) allows simple calculation of the averaged velocity. We see that the total mass flow rate across each cross-section is constant in our hydraulic model, given by the mass flow rate at surface: $\hat{M} = \hat{M}_g + \hat{M}_c$. For given pressure and mass fractions, the density is determined and hence the mean axial velocity is calculated from the mass flow rate:

$$\hat{w} = \frac{\hat{M}}{\hat{\rho} \hat{A}}.$$ \hspace{1cm} (B.21)

Note that although the frictional pressures (wall shear stress terms) introduce the additional complexity of coupling the momentum balance with pressure, velocity and gas fraction, for normal ranges of fluid rheologies and flow rates wall shear
are typically $< 30\text{Pa}$, meaning that the frictional contribution to the total pressure drop is generally $\lesssim 10\%$ of the static pressure in any vertical section. Thus, as an initial approximation, for many wells these terms may be neglected or evaluated in a semi-implicit way when computing the pressure. However, these terms become important in horizontal wells and in all wells in situations of rapid expansion and/or unsteady flow (e.g. U-tubing, between casing and annulus if no check-valve is fitted), so our modelling approach is preliminary.

**Boundary conditions & solution method**

Two modes of operational control are common in foamed cement placement.

**Mode 1:** Constant nitrogen to base slurry volumetric flowrate ratio, on surface at specified temperature and pressure. Effectively this fixes the inflow density and hence the mass fraction $Y_g$. Two different ways of applying this are common.

**Mode 1A:** the pressure at the top of the casing is specified (inflow pressure).

**Mode 1B:** the pressure at the top of the annulus is fixed (back pressure) and that at the casing is then adjusted iteratively to satisfy the outflow pressure constraint.

This mode can result in a large density gradient in the annulus and is typically implemented while holding back pressure at the top of the annulus to prevent quick depressurization of foam as it rises.

**Mode 2:** Constant density in the annulus at the end of placement. This is satisfied only approximately (e.g. $\approx 100 \text{kg/m}^3$ tolerance) and is achieved by adjusting the foam quality on surface in a sequence of constant steps. Depending on the well we can end up with 10-20 pump stages with different nitrogen flowrates: greater homogeneity of the foam density results from more stages. Here back pressure is either not applied or is held constant. The problem is to design the pump schedule, say in terms of steps in $Y_g$ at fixed surface temperature and pressure.
Although the above two modes are different operationally, the computational method of solving the 1D model is similar. Mode 2 is however complex to design the pump schedule and consequently we focus on mode 1.

At any time step, we assume that we specify at \( \hat{z}_c \): an inflow pressure \( \hat{P}_c(\hat{t}) \), inflow mass fractions of each constituent, \( Y_{c,j}(\hat{t}) \), and the total mass flow rate \( \hat{M}(\hat{t}) \). These are calculated from the specified flow rate ratios and other pump schedule. Assuming the \( Y_j \) are known at the current timestep, the momentum balance is integrated in the positive \( \hat{z} \)-direction, using a modified Euler finite difference scheme. This computes the pressure, density and mean velocity (from (B.21)) everywhere along the flow path at the current timestep. In the case that the pressure is fixed at the outflow (back pressure imposed), we iteratively change the inflow pressure \( \hat{P}_c(\hat{t}) \) to satisfy the outflow condition. The transport equations (B.17) are hyperbolic and are solved using an FCT scheme [252] to give the new \( Y_j \) at the next timestep.

**Example flows**

We now present 3 illustrative examples, which we will follow throughout the paper. For each of these we consider only mode 1 operational/boundary conditions. The well geometry is fixed as a 500m deep vertical (\( \beta = 0 \)) borehole, with casing inner, casing outer and hole diameters: 18.42, 20.32, 25.40 cm, respectively (i.e. 7.25, 8 & 10 inches). The pipe is initially full of a drilling mud (displaced fluid) with properties: \( \hat{\rho}_1 = 1400 \text{ kg/m}^3, \hat{\kappa}_1 = 0.02 \text{ Pa.s}^{\kappa_1}, n_1 = 1, \hat{\tau}_{Y,1} = 4 \text{ Pa} \). The mud is displaced by a Nitrogen-foamed cement (displacing fluid). The properties of the pure cement slurry are: \( \hat{\rho}_2 = 1800 \text{ kg/m}^3, \hat{\kappa}_2 = 0.04 \text{ Pa.s}^{\kappa_2}, n_2 = 1, \hat{\tau}_{Y,2} = 8 \text{ Pa} \). The inlet temperature is fixed at 300K. and the following conditions are used at surface.

**Example (i):** \( \hat{p}_c = 20 \text{ atm, } \hat{Q}_g = 120 \text{ l/min, } \hat{Q}_c = 180 \text{ l/min,} \)

**Example (ii):** \( \hat{p}_c = 40 \text{ atm, } \hat{Q}_g = 120 \text{ l/min, } \hat{Q}_c = 180 \text{ l/min,} \)

**Example (iii):** \( \hat{p}_c = 20 \text{ atm, } \hat{Q}_g = 180 \text{ l/min, } \hat{Q}_c = 120 \text{ l/min.} \)

The focus of these examples is on the effects of density and flow rate during displacement, to explore the effect of compressibility. The mud rheology is not ex-
xtreme and provided the annulus were adequately centralized we would not expect any difficulty to displace this well. Although the fluid properties are realistic, we have restricted to only 2 fluids (mud and cement). Also we retain $n = 1$ throughout for our examples, although the model is valid for varying $n$. Compressibility does not influence the power law index in our rheology closure and taking $n = 1$ has no effects on the qualitative conclusions drawn in the paper.

The results are shown in Figures B.2-B.4. Each figure plots density, quality, velocity and pressure in the form of spatiotemporal colourmaps showing evolution as the cement-mud interface travels down the casing and back up the annulus. We observe the faster speeds of the interface in the annulus, due primarily to smaller cross-sectional area. The foam velocity decreases with depth within the casing and increases in the annulus, as it decompresses. The discontinuity in velocity at fixed time is due to change in cross-sectional area (pipe to annulus). The discontinuity at fixed position is due to the interface passing (and caused by the jump in density). We see a slight asymmetry in the pressure with respect to distance from bottom-hole, at both start and end of the examples (when the flow loop is full of a single fluid). This asymmetry is due to frictional pressure losses.

Comparing examples (i) and (ii), both have the same surface quality at $\hat{z}_c$, but
Figure B.3: Results from 1D displacement model with parameters from example (ii)

Figure B.4: Results from 1D displacement model with parameters from example (iii)
the entire system is operating at increased pressure in example (ii). Thus, the effects of pressure variations are generally less pronounced in example (ii): frictional pressure effects are smaller and the variation in quality, foam density and velocity are smaller.

Comparing examples (i) and (iii), we have increased the inflow quality from 40% to 60%. With the relatively short well the density in the foam remains below that of the mud throughout the simulation, and the interface is density unstable throughout the annulus. Frictional pressure effects are comparable to example (i), except close to the top of the annulus. Here the higher quality (and the smaller area) leads to high velocities. We will analyze mechanical instability more closely in §B.2.6.

### B.2.5 Analytical Model

If the frictional pressures are neglected, we arrive at a model that can be integrated analytically. This is advantageous for both analysis and for identifying limitations on the foamed cementing process; see e.g. [155]. Since we neglect the frictional terms, this hydrostatic model is equally valid for either inside the casing or in the annulus. To simplify further, we also assume a uniform cross-sectional area in each geometry (straightforwardly extended to piecewise constant wellbore geometries) at constant inclination $\beta$. We give only the solution for $K = 2$ fluid stages, (but easily extended to $K > 2$).

First, let’s assume that the foamed slurry (fluid 2) occupies $[\hat{z}_c, \hat{z}_i)$ with the drilling mud occupying $[\hat{z}_i, \hat{z}_a]$, and that the pressure $\hat{p}_c$ is imposed. For any $\hat{z} \in [\hat{z}_c, \hat{z}_i)$, we find the pressure by integrating the momentum equations, rearranged as follows:

$$\frac{d\hat{p}}{dz} = \rho_2 \hat{g} \cos \beta, \quad \Rightarrow \quad \frac{d\hat{p}}{\rho_2} = \hat{g} \cos \beta dz.$$  \hspace{1cm} (B.22)

The gas mass fraction is fixed in the foamed slurry, and using (B.4) we find:

$$\frac{Y_g \hat{k}\hat{T}}{M} \ln(\hat{p}(\hat{z})/\hat{p}_c) + \frac{1 - Y_g}{\hat{p}_c} (\hat{p}(\hat{z}) - \hat{p}_c) = \begin{cases} \hat{g} \cos \beta (\hat{z} - \hat{z}_c), & \hat{z} \leq \hat{z}_{bh}, \\ \hat{g} \cos \beta (2\hat{z}_{bh} - \hat{z} - \hat{z}_c), & \hat{z} > \hat{z}_{bh}. \end{cases}$$  \hspace{1cm} (B.23)

This implicit relation is inverted numerically at each $\hat{z}$, when needed, or simply at
the interface position $\hat{z}_i$. The right-hand side of (B.23) changes at $\hat{z}_{bh}$ due to the reversal in direction of gravitational acceleration as we return up the annulus. For $\hat{z} \in [\hat{z}_i, \hat{z}_a]$, the density is constant and

$$
\hat{\rho}(\hat{z}) = \hat{\rho}_i + \begin{cases}
\hat{\rho}_1 g \cos \beta (\hat{z} - \hat{z}_i), & \hat{z} \leq \hat{z}_{bh}, \\
\hat{\rho}_1 g \cos \beta (2\hat{z}_{bh} - \hat{z} - \hat{z}_i), & \hat{z} \geq \hat{z}_{bh} \geq \hat{z}_i \\
\hat{\rho}_1 g \cos \beta (\hat{z}_i - \hat{z}), & \hat{z} \geq \hat{z}_i > \hat{z}_{bh}.
\end{cases}
$$

This gives the annulus pressure at the exit as:

$$
\hat{p}_a = \hat{\rho}_i + \hat{\rho}_1 g \cos \beta (\hat{z}_i - \hat{z}_a).
$$

If $\hat{z}_i = \hat{z}_a$, then evidently $\hat{p}_a = \hat{\rho}_i$, and from (B.23), if $2\hat{z}_{bh} = \hat{z}_a - \hat{z}_c$, we find: $\hat{\rho}_i = \hat{\rho}_c$, with the physical meaning that the foam decompresses exactly the same in the annulus as it compresses inside the casing.

Figure B.5 shows the results of our analytical model for the example (i), which is to be compared with Figure B.2. At the start and finish of the simulation, we can see the symmetry of pressure, density and quality, with respect to distance, as expected with no frictional pressures present. This is used to validate the numerical method used for computing the 1D model, (i.e. by setting the frictional pressure terms to zero in the 1D model). A more quantitative comparison is made in Figure B.6. Here the left panel shows the contour of density along the flow path ($\hat{z}_c \rightarrow \hat{z}_{bh} \rightarrow \hat{z}_a$) as the interface position ($\hat{z}_i$) progresses. The pressure loss due to viscous friction is taken into account in the left panel. The right panel shows the error in computing the density, had we not taken into account the frictional pressure loss. The error caused by ignoring the friction is at worst about 5% and is only relatively significant near the top of annulus on the return to the surface. Evidently, these comparisons suggest that neglecting frictional pressure is a reasonable first approximation for this type of well and cementing parameters.
B.2.6 Mechanical stability limitation

In a typical cementing operation the cement slurry is separated from the other fluids by rubberized plugs, as it is pumped down inside the casing. These minimize the risk of mixing and contamination on the way down the well. As the pressure increases with depth we have seen that the foam density also increases and this is a mechanically stable situation. On the other hand, in the annulus the fluid stages are not separated. As the cement-mud (or cement-spacer) interface advances along the
Inflow quality is 0.4

Inflow quality is 0.6

Figure B.7: Density of the foam at the interface, for different \( \hat{p}_c = \hat{p}_a \) using the analytical model

annulus, there is also the possibility that the displacing foamed cement slurry may have a density below that of the fluid above it. This situation is mechanically unstable and may promote buoyancy-induced mixing. Equally, depending on the foam rheology and that of the drilling mud, there may also result a negative viscosity hierarchy - also conducive to instability.

We shall explore some of these possibilities in the annular displacement model developed below in §B.3. However, for now we can give a conservative prediction of this mechanically unstable situation by using the analytical model. Note that neglecting the frictional pressures means that the actual pressures along the flowpath are higher (for a fixed outflow pressure) than those calculated neglecting friction. Hence the foamed cement slurry is less dense in the analytical model than in the frictional 1D model: density unstable configurations in the annulus of the analytical model are a necessary condition for the same to exist in the frictional model.

For the same well as in examples (i)-(iii), Figure B.7 plots the foam density evaluated at the interface position for different \( \hat{p}_c = \hat{p}_a \). The dark contour illustrates the mud density. As we have seen, for the lower surface pressures imposed the foam compresses more down hole and higher densities are achieved downhole. There are clearly competing objectives here: on the one hand one motivation for using the foamed cement is having lower densities downhole, but on the other hand there is a risk of density-driven instability in the annulus.
B.2.7 Positive flow and U-tubing

Other constraints come from the hydraulics. Neglecting friction, the pressure at the top of the casing $\hat{p}_c$ must overcome the resisting pressures, which are $\hat{p}_a$ and the net static pressure, i.e. in order to flow in the positive direction we require:

$$\hat{p}_c > \hat{p}_a - \int_{\hat{z}_c}^{\hat{z}_a} \hat{\rho} \hat{g} \cos \beta \, d\hat{z}.$$  \hfill (B.26)

If (B.26) is not satisfied there is a net flow driven from the annulus back towards the casing. In practice a check valve may prevent this flow from occurring. If instead (B.26) is satisfied the difference in (driving-resisting) pressures is positive and will ordinarily be balanced by the frictional pressures, i.e. the flow moves at a speed such that the frictional pressure balances.

It can also happen that the static pressure terms in (B.26) are in such imbalance that the equilibrium flow rate exceeds the capacity of the pump. In this case the fluids are in free-fall (perhaps only temporarily) and $\hat{p}_c$ will drop. This occurs in other pumping operations where there can result a pressure imbalance (U-tubing). These situations represent a physical limitation of our simple 1D steady hydraulics model. The model can be modified readily to account for these dynamic features, retaining the acceleration terms in the momentum balance and possibly the time derivative in the mixture mass conservation equation. Note that an additional complication results here in that a loss of pressure may rapidly decompress the foam, which is an evident risk. For the examples presented here we simply avoid regimes where U-tubing occurs.

B.3 Modeling annular displacement flows

We now turn to the main focus of the paper: to develop a two-dimensional model for the placement of foamed cement in the annulus. Here we stick to laminar flow regimes and the model is thus an extended version of that initially proposed by Bittleston et al. [24]. Much of our model derivation and simplification to a Hele-Shaw model follows the scaling arguments in [24, 147], and we are consequently
brief in this regard. For simplicity we model only half of the annulus, assuming symmetry at wide ($\phi = 0$) and narrow ($\phi = 1$) sides; see Figure B.1.

A cylindrical coordinate system ($\hat{r}, \theta, \hat{\xi}$) is used to describe the well geometry: $\hat{\xi}$ measures distance along the central axis of the casing $\hat{r} = 0$ which is assumed to be inclined to the vertical with angle $\beta(\hat{\xi})$. In the local coordinate system the following Navier Stokes system is satisfied by the velocity $\hat{u} = (\hat{u}, \hat{v}, \hat{w})$ and pressure $\hat{p}$. The dimensional momentum equations are\(^1\):

\[
\begin{align*}
\left[ \frac{\partial}{\partial \hat{t}} + \hat{u} \cdot \hat{\nabla} \right] (\hat{p} \hat{u}) &= \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} [\hat{r} \hat{u} \hat{r}] + \frac{1}{\hat{r}} \frac{\partial}{\partial \theta} \hat{v} \hat{r} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{\xi}} \hat{u} \hat{\xi} - \frac{\hat{\xi} \hat{\theta}}{\hat{r}} - \frac{\partial \hat{p}}{\partial \hat{r}} + \hat{\rho} \hat{g}, \tag{B.27} \\
\left[ \frac{\partial}{\partial \hat{t}} + \hat{u} \cdot \hat{\nabla} \right] (\hat{p} \hat{v}) &= \frac{1}{\hat{r}^2} \frac{\partial}{\partial \hat{r}^2} \hat{r} \hat{v} \hat{r} + \frac{1}{\hat{r}} \frac{\partial}{\partial \theta} \hat{v} \hat{\theta} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{\xi}} \hat{v} \hat{\xi} - \frac{1}{\hat{r}} \frac{\partial \hat{p}}{\partial \theta} + \hat{\rho} \hat{g} \hat{\theta}, \tag{B.28} \\
\left[ \frac{\partial}{\partial \hat{t}} + \hat{u} \cdot \hat{\nabla} \right] (\hat{p} \hat{w}) &= \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \hat{r} \hat{w} \hat{r} + \frac{1}{\hat{r}} \frac{\partial}{\partial \theta} \hat{w} \hat{\theta} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{\xi}} \hat{w} \hat{\xi} - \frac{\partial \hat{p}}{\partial \hat{\xi}} + \hat{\rho} \hat{g} \hat{\xi}, \tag{B.29}
\end{align*}
\]

where the components of the gravity acceleration vector are:

\[
\begin{align*}
\hat{g}_r &= -\hat{g} \sin \beta(\hat{\xi}) \cos \theta & \hat{g}_\theta &= \hat{g} \sin \beta(\hat{\xi}) \sin \theta & \hat{g}_\xi &= -\hat{g} \cos \beta(\hat{\xi}), \tag{B.30}
\end{align*}
\]

and $\hat{g} = 9.81 \text{ m/s}^2$. Note, we have retained the earlier assumptions regarding modelling the foam as a compressible fluid with no slip between phases. Each fluid in the sequence of $K$ fluids is characterized as a Herschel-Bulkley fluid, with the foam rheology described as in §B.2.2. The fluids combine locally to form a miscible mixture defined through the mass fractions $Y_k$ of each constituent and thus we have a single velocity and pressure field. Mass conservation for the total mixture and individual components are described by (B.13)-(B.15). Temperature variations are ignored.

**B.3.1 Scaled and simplified model**

The local cross-section of the well is an eccentric annulus with inner radius $\hat{r}_i(\hat{\xi})$ (outer radius of the inserted casing) and outer radius $\hat{r}_o(\hat{\xi})$ (equal to the radius of the wellbore or previous casing). We define the (local) mean radius and mean

\(^1\)As earlier, we adopt the notation of showing dimensional parameters with a “hat” (e.g. $\hat{p}$) and dimensionless parameters without (e.g. $p$).
half-gap width:

\[ \hat{r}_a(\hat{\xi}) \equiv \frac{1}{2}[\hat{r}_o(\hat{\xi}) + \hat{r}_i(\hat{\xi})], \quad \hat{d}(\hat{\xi}) \equiv \frac{1}{2}[\hat{r}_o(\hat{\xi}) - \hat{r}_i(\hat{\xi})], \]  

(B.31)

and then average along the flow path:

\[ \hat{r}_a^* = \frac{1}{\hat{Z}} \int_{\hat{\xi}_{bh}}^{\hat{\xi}_{tz}} \hat{r}_a(\hat{\xi}) d\hat{\xi}, \quad \hat{Z} = \hat{\xi}_{tz} - \hat{\xi}_{bh}. \]  

(B.32)

Here \( \hat{\xi}_{bh} \) denotes the bottomhole and \( \hat{\xi}_{tz} \) is the top of the zone of interest. The \( \hat{\xi} \) coordinate is curvilinear along the flowpath (annulus axis), measured upwards from bottom hole. Local and mean aspect ratios are defined by:

\[ \delta(\hat{\xi}) = \frac{\hat{d}(\hat{\xi})}{\hat{r}_a(\hat{\xi})}, \quad \delta^* = \frac{1}{\hat{Z}} \int_{\hat{\xi}_{bh}}^{\hat{\xi}_{tz}} \delta(\hat{\xi}) d\hat{\xi}. \]  

(B.33)

Using these parameters we may define the dimensionless (local) average radius and eccentricity of the annulus as:

\[ r_a(\hat{\xi}) = \frac{\hat{r}_a(\hat{\xi})}{\hat{r}_a^*}, \quad e(\hat{\xi}) = \frac{\hat{e}(\hat{\xi})}{2\hat{d}(\hat{\xi})}. \]  

(B.34)

Dimensionless radial, azimuthal, and axial coordinates are defined by:

\[ y = \frac{\hat{r} - r_a(\hat{\xi})}{\hat{r}_a^*}, \quad \phi = \frac{\theta}{\pi}, \quad \xi = \frac{\hat{\xi}}{\hat{r}_a^*}. \]  

(B.35)

In terms of these coordinates, the external (wellbore) and internal (casing) walls are defined by \( y = \pm H(\phi, \xi) \), where:

\[ H(\phi, \xi) = \frac{\delta(\hat{\xi}) r_a(\hat{\xi})(1 + e(\hat{\xi}) \cos \pi \phi)}{\delta^*} \]  

(B.36)

to leading order in \( \delta^*/\pi \).

The parameter \( \delta^*/\pi \) is representative of the ratio of annular gap width to circumference of the annulus. In a typical well the mean annular gap remains constant while the diameter of casings decreases with depth, so \( \delta^*/\pi \) may range from 0.02
to 0.06 along a typical well.

The narrow gap assumption, $\delta^*/\pi \ll 1$, is made below and used extensively to simplify and retain only the leading order effects. The following assumptions are either implicit in this or stated explicitly: (i) axial variations in the cross-section geometry and inclination are very slow. (ii) $2 \dot{a}(\xi) > \dot{e}(\xi)$, which means the casing does not touch the well wall, i.e. the eccentricity $e(\xi) \in [0, 1)$. (iii) the wide side of the annulus is on the upper side of the well.

**Scaled momentum equations**

To develop scales we consider that a pump schedule is imposed at the top of the casing. Via a one-dimensional model (as described in the previous section) this determines the inflowing fluid(s) at bottomhole. Given that foamed slurry is compressible we specify the pump schedule as a mass flow rate into the well: $\dot{M}_{\text{pump}}(\hat{t})$. We define a reference cross-sectional area of the annulus

$$\hat{A}_0 = 4 \pi \delta^* (\hat{r}_a)^2,$$

and take as reference density the in-situ mud density $\hat{\rho}_1$.

The dimensionless pump schedule is

$$\dot{M}(t) = \frac{\dot{M}_{\text{pump}}(\hat{t})}{M_0} : \dot{M}_0 = \max_i \dot{M}_{\text{pump}}(\hat{t}),$$

and the velocity scale is:

$$\hat{W} = \frac{\dot{M}_0}{\rho_1 \hat{A}_0} = \frac{\max_i \dot{M}_{\text{pump}}(\hat{t})}{4 \pi \delta^* [\hat{r}_a]^2 \hat{\rho}_1}.$$  

All azimuthal and axial velocities are scaled with $\hat{W}$, and radial velocities are scaled with $\hat{W} \delta^*/\pi$. The scaled time is the axial length-scale divided by the reference velocity:

$$\hat{t} = \hat{t}_0 \hat{t}, \quad \hat{t}_0 = \frac{\pi \hat{r}_a^2}{\hat{W}}.$$  

Given that $\delta_0/\pi \ll 1$, we may assume that the dominant components of mean
velocity will be in the \((\phi, \xi)\)-directions, scaling approximately with \(\dot{W}\). These velocities will give rise to viscous stresses, due to shear in the narrow gap. Using the rheology, gap size and \(\dot{W}\) we can develop an estimate for the size of shear stresses, say \(\dot{\tau}_0\), which we use in the scaling below. Having determined the shear stress scale \(\dot{\tau}_0\) the relative sizes of other components of the deviatoric stress follow from the velocity and length-scales. For the pressure, we subtract off the bottom hole pressure and a static component due to the 1st fluid, and then scale the remaining pressure to balance the viscous stresses at leading order:

\[
\hat{\rho} = \hat{\rho}_{bh}(\hat{f}) + \hat{\rho}_1 \dot{g} \cdot \mathbf{x} + (\pi \hat{\xi}_0 / \delta_0)p. \tag{B.41}
\]

We substitute into the momentum equations (B.27), (B.28) & (B.29) and divide through by the largest dimensional scales, to give the following.

\[
O\left(\frac{\delta_0^3 \hat{\rho} \dot{W}^2}{\pi^3 \hat{\xi}_0}\right)_{IT} = -\frac{\partial p}{\partial y} + O\left(\frac{\delta_0 \rho - 1}{\pi Fr^2}\right)_{BT} + O\left(\frac{\delta_0^2}{\pi^2}\right)_{ST} + O\left(\frac{\delta_0}{\pi}\right)_{CT}, \tag{B.42}
\]

\[
O\left(\frac{\delta_0 \hat{\rho} \dot{W}^2}{\pi \hat{\xi}_0}\right)_{IT} = -\frac{1}{r_a} \frac{\partial p}{\partial \phi} + \frac{(\rho - 1) \sin \beta \sin \pi \phi}{Fr^2} \frac{\partial}{\partial y} \tau_{\phi y} + O\left(\frac{\delta_0^2}{\pi^2}\right)_{ST} + O\left(\frac{\delta_0}{\pi}\right)_{CT}, \tag{B.43}
\]

\[
O\left(\frac{\delta_0 \hat{\rho} \dot{W}^2}{\pi \hat{\xi}_0}\right)_{IT} = -\frac{\partial p}{\partial \xi} - \frac{(\rho - 1) \cos \beta}{Fr^2} + \frac{\partial}{\partial y} \tau_{\xi y} + O\left(\frac{\delta_0^2}{\pi^2}\right)_{ST} + O\left(\frac{\delta_0}{\pi}\right)_{CT}, \tag{B.43}
\]

where \(\rho\) is the scaled density and \(Fr\) is the Froude number:

\[
Fr = \sqrt{\frac{\delta_0}{\hat{\rho}_1 \hat{g} \hat{\xi}_0 r_a \hat{\rho}_a \hat{\xi}_0}}.
\]

Our leading order model is the narrow gap limit, \(\delta_0 / \pi \to 0\), with other parameters fixed. The lower order terms that vanish in this limit are identified by the underbraces as follows: \(IT\) - inertial terms; \(BT\) - buoyancy terms; \(ST\) - stress terms; \(CT\) - terms associated with neglect of curvature. It is interesting to note that the leading order momentum balance is not affected by the compressibility, except through the
definition of the density and rheology in terms of the gas fraction (quality).

**Constitutive laws**

The dimensionless density is simply:

\[ \rho = \frac{1}{\sum_{j=1}^{K} \frac{Y_j}{\rho_j}}, \quad \rho_j = \frac{\hat{\rho}_j}{\hat{\rho}_1}, \]  

(B.44)

For \( j = K \) the foamed slurry has density

\[ \rho_K = \alpha \rho_g + (1 - \alpha) \rho_c = \frac{1}{\frac{Y_g}{\rho_g} + \frac{1 - Y_g}{\rho_c}}, \]  

(B.45)

where \( \rho_c = \hat{\rho}_c/\hat{\rho}_1 \) and \( \rho_g = \hat{\rho}_g/\hat{\rho}_1 \) is pressure dependent, following (B.3).

Each individual fluid \( j \) is characterized by \((\hat{\tau}_{Y,j}, \hat{\kappa}_j, n_j)\), indicating yield stress, consistency and power-law index of fluid \( j \) in the pumped sequence. The narrow annulus resembles locally a plane channel and so, following [147] we define a strain rate relevant to laminar flow through a plane channel:

\[ \hat{\dot{\gamma}}_0 = \frac{\hat{\dot{W}}}{\delta \hat{r}^2}, \]  

(B.46)

which we use to define \( \hat{\tau}_0 \):

\[ \hat{\tau}_0 = \max_{j=1,...,K} [\hat{\tau}_{Y,j} + \hat{\kappa}_j \hat{\dot{\gamma}}_0]. \]  

(B.47)

The stress scale is then used to scale the rheological parameters:

\[ \tau_{j,Y} = \frac{\hat{\tau}_{j,Y}}{\hat{\tau}_0}, \quad \kappa_k = \frac{\hat{\kappa}_k \hat{\dot{\gamma}}_0}{\hat{\tau}_0}. \]  

(B.48)
Narrow gap approximation

The dimensionless continuity equation for our system is:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y} (\rho u) + \frac{1}{r_a} \frac{\partial}{\partial \phi} (\rho v) + \frac{\partial}{\partial \xi} (\rho w) + O\left(\frac{\delta_0}{\pi}\right)_{CT} = 0. \tag{B.49}
\]

As before in our 1D model, we note that \( \rho \) is not a primary variable, but defined by the pressure and mass fractions. The temporal variations with pressure may be considered small provided the Mach number is small and no high frequency oscillations are imposed. The changes in \( \rho \) with the mass fractions are accounted for by advection and we consequently simplify by ignoring the time derivative above. We also neglect the curvature terms (marked \( CT \)), taking \( \delta_0/\pi \rightarrow 0 \). We then integrate across the gap between \( y = \pm H \):

\[
\int_{-H}^{H} \frac{1}{r_a} \frac{\partial}{\partial \phi} (\rho v) \, dy + \int_{-H}^{H} \frac{\partial}{\partial \xi} (\rho w) \, dy = \frac{1}{r_a} \frac{\partial}{\partial \phi} \int_{-H}^{H} \rho v \, dy + \frac{\partial}{\partial \xi} \int_{-H}^{H} \rho w \, dy = 0. \tag{B.50}
\]

We have used no slip to eliminate the boundary terms above. We have seen that to leading order there is no pressure variation across the annular gap (B.42) and now assume that the constituent mass fractions are approximated to leading order by their gap-averaged values. Therefore, the density is simply evaluated with the (gap-averaged) pressure and mass fractions. We see that the above simplifications, in retaining the density under the integral sign, keep the leading order effects of the compression/expansion on the velocity field.

The mass conservation equation is satisfied by using a stream function. Here however, we differ from [24, 147] in that our stream function \( \Psi_M \) is based on the mass flux (not volume flux as is usual):

\[
- \frac{\partial \Psi_M}{\partial \xi} = \int_{-H}^{H} \rho v \, dy = 2H \rho \bar{v}, \quad \frac{1}{r_a} \frac{\partial \Psi_M}{\partial \phi} = \int_{-H}^{H} \rho w \, dy = 2H \rho \bar{w}, \tag{B.51}
\]

where \( (\bar{v}, \bar{w}) \) are the gap-averaged azimuthal and axial velocity components. We have

\[
(-\bar{v}, \bar{w}) = \frac{1}{2H \rho} \nabla_a \Psi_M, \tag{B.52}
\]

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where $\nabla_a$ is the azimuthal gradient. This and the azimuthal divergence operator are defined as:

$$\nabla_a X = \left( \frac{1}{r_a(\xi)} \frac{\partial X}{\partial \phi}, \frac{\partial X}{\partial \xi} \right), \quad \nabla_a \cdot X = \frac{1}{r_a(\xi)} \frac{\partial X}{\partial \phi} + \frac{\partial X}{\partial \xi}.$$

Scaling and averaging of the individual component mass conservation equations leads to:

$$\frac{\partial}{\partial t} Y_j + (\bar{v}, \bar{w}) \nabla_a Y_j = 0, \quad j = 1, 2, \ldots, K, g, \quad (B.53)$$

again to leading order in $\delta^*/\pi$.

Finally, taking $\delta^*/\pi \to 0$, the momentum equations become:

$$0 = -\frac{\partial p}{\partial y}, \quad (B.54)$$

$$0 = -\frac{1}{r_a} \frac{\partial p}{\partial \phi} + \frac{(\rho - 1) \sin \beta \sin \pi \phi}{Fr^2} + \frac{\partial}{\partial y} \tau_{\phi y}, \quad (B.55)$$

$$0 = -\frac{\partial p}{\partial \xi} - \frac{(\rho - 1) \cos \beta}{Fr^2} + \frac{\partial}{\partial y} \tau_{\xi y}. \quad (B.56)$$

### B.3.2 Compressible Hele-Shaw model

The equations (B.54)-(B.56) can now be treated as in [24, 147] to derive a compressible Hele-Shaw model, which is geometrically equivalent to unwrapping the narrow annulus into a two-dimensional channel of varying width; see Figure B.1.

To remove the $y$-dependency we use the condition of symmetry: $(\tau_{\phi y}, \tau_{\xi y}) = (0, 0)$ at $y = 0$, and integrate (B.55) & (B.56):

$$(\tau_{\phi y}, \tau_{\xi y}) = y \left( \frac{1}{r_a} \frac{\partial p}{\partial \phi} - \frac{(\bar{\rho} - 1) \sin \beta \sin \pi \phi}{Fr^2}, \frac{\partial p}{\partial \xi} + \frac{(\rho - 1) \cos \beta}{Fr^2} \right). \quad (B.57)$$

As in [24, 147] we may deduce that the direction of flow is down the modified pressure gradient. The flow direction, say $e_s$, is also given by the gap-averaged velocity, and hence stream function:

$$e_s = \frac{1}{|\nabla_a \Psi_M|} \left( -\frac{\partial \Psi_M}{\partial \xi}, \frac{1}{r_a} \frac{\partial \Psi_M}{\partial \phi} \right). \quad (B.58)$$
By changing coordinates so that locally \((s, n)\) are in the direction of flow, (B.57) can be reduced to a single component in the direction of \(e_s\):

\[
\tau_{sy} = -\frac{y}{H} \tau_w. \tag{B.59}
\]

Here \(\tau_w\) is the dimensionless wall shear stress, defined as follows:

\[
\tau_w = H \left| \left( \frac{1}{r_a} \frac{\partial p}{\partial \phi} - \frac{(\bar{\rho} - 1) \sin \beta \sin \pi \phi}{F r^2} \frac{\partial p}{\partial \xi} + \frac{(\bar{\rho} - 1) \cos \beta}{F r^2} \right) \right|. \tag{B.60}
\]

Combining (B.58) and (B.60), we have:

\[
(-\frac{\partial \Psi_M}{\partial \xi}, \frac{1}{r_a} \frac{\partial \Psi_M}{\partial \phi})_{|\nabla_a \Psi_M|} = -\frac{H}{\tau_w} \left( \frac{1}{r_a} \frac{\partial p}{\partial \phi} - \frac{(\rho - 1) \sin \beta \sin \pi \phi}{F r^2} \frac{\partial p}{\partial \xi} + \frac{(\rho - 1) \cos \beta}{F r^2} \right). \tag{B.61}
\]

After cross-differentiating to eliminate the pressure, we have:

\[
\nabla_a \cdot [S + b] = 0, \tag{B.62}
\]

where

\[
S = \frac{r_a \tau_w (|\nabla_a \Psi_M|)}{H |\nabla_a \Psi_M|} \nabla_a \Psi_M, \quad b = \frac{r_a (\bar{\rho} - 1)}{F r^2} (\cos \beta, \sin \pi \phi \sin \beta). \tag{B.63}
\]

Remarkably, (B.63) is identical with that for incompressible displacements, as in [147], except that the stream function used is the mass stream function \(\Psi_M\).

**B.3.3 Closure model for \(\tau_w\)**

Here we outline the closure relationship that defines \(\tau_w(|\nabla_a \Psi_M|; \phi, \xi, t)\) locally in the annulus, and hence

\[
S = \frac{r_a \tau_w (|\nabla_a \Psi_M|)}{H |\nabla_a \Psi_M|} \nabla_a \Psi_M.
\]

We adopt the hydraulic framework introduced in [146], where the flow of a Herschel-Bulkley fluid along a narrow channel is studied in laminar, transitional and turbulent regimes. We are only concerned with the laminar regime. Given the shear stress scale \(\hat{\tau}_0\), the dimensional wall shear stress is \(\hat{\tau}_w = \hat{\tau}_0 \tau_w\). The dimensional
mean speed (averaged across the local gap) and the dimensional local annular gap are given as:

\[ \hat{W}_0 = \hat{W} \frac{1}{2\rho H} |\nabla a \Psi_M|, \quad 2\hat{H} = 2H \delta_0 \hat{r}_{a,0}; \quad (B.64) \]

recall that \( \hat{W} \) is the velocity scale for the entire annulus. We also assume that, according to the mass fractions \( (Y_j : j = 1, 2, \cdots, K - 1, g) \) at \( (\phi, \xi, t) \), we may construct the dimensional local mixture density \( \hat{\rho} \) and the rheological parameters \( \hat{\tau}_Y, \hat{\kappa} \) and \( n \). The desired closure can be obtained using the relation between \( \hat{W}_0 \) and \( \hat{\tau}_w \) which was given earlier in equation (B.9).

### B.3.4 Computational method

At every time step, we solve (B.62) to update the mass stream function and subsequently the velocity field. We then move to the next time step by updating the mass fractions through solving (B.53). Note that (B.62) is not directly time-dependent: time dependency only enters indirectly through \( Y_j \) or possible changes in mass flow rate \( \dot{M} \). In the case of incompressible fluids, equation (B.62) is a nonlinear elliptic partial differential equation, which is properly formulated and solved in a variational setting due to non-differentiability associated with resolving regions of zero flow in the annulus. For example, in [147] we have solved this using an augmented Lagrangian method. Here we have the additional complexity of pressure-dependency of the rheology and density. We have not so far analysed the effect of compressibility on the existence of solutions to (B.62), nor how this might affect methods such as the augmented Lagrangian method. Other pressure-dependent systems have been analysed, e.g. [68], and we anticipate that since the frictional effects are anyway small for most foamed cementing flows, development of a theoretical and computational analysis should be possible for practical flow regimes. This is intended as part of our future work.

Here however, we adopt a simpler but approximate method to solve (B.62). Due to the length of the annulus, we assume that the \( \xi \)-derivatives in (B.62) are generally small and neglect them, which leads to:

\[ \frac{\partial}{\partial \phi} [S_\phi + b_\phi] = 0, \]
or in other words:

\[ S_\phi + b_\phi = G(\xi). \]  \hspace{1cm} (B.65)

Physically, \( G \) is the modified pressure gradient and is independent of \( \phi \). Thus, on each annular slice, \( \xi = \xi_i \), we iteratively change \( G \) until the net imposed mass flowrate through the exit section is attained. This slice method was also used in [24] and appears to give a reasonable approximation for vertical wells. Although this may appear to be a convoluted procedure, it is straightforward numerically, quick and robust. Neglect of the \( \xi \)-derivatives in (B.62) may not be valid everywhere in the annulus, in particular as instabilities grow (see the results below), but as with most such methods it allows us to predict onset of potential instability.

The advection equation for the mass fraction (B.53) is solved using the Flux-Corrected Transport (FCT) method [252] to capture any shock behavior. The FCT method has been used in [147] and previously for our incompressible model simulations.

### B.4 Results

To explore different features of our model, we revisit Example (i), introduced earlier in §B.2.4, and compute the two-dimensional flow. Figure B.8 shows the results of this simulation, assuming a concentric annulus. Note here that our 2D simulations shown here and later only concern the displacement flows in the annulus. Figure B.8a shows snapshots of the mud mass fraction \( Y_1 \) at regular time intervals as the displacement progresses along the annulus. Superimposed on the mass fraction contours are the streamlines of the mass streamfunction \( \Psi_M \). As explained, only half the annulus is considered, from wide (W) to narrow (N) side on the horizontal axis. For the majority of the displacement the streamlines are parallel, uniformly spaced and the interface between fluids is horizontal. The flow here is essentially 1D and compares well with the 1D model results, shown earlier in §B.2.4, at least for the earlier part of the run. Note that the density and frictional pressure models are identical.

The difference between the 1D and annular models emerges only later in the flow, as the interface nears the top (see the last panel in Figure B.8a), when a
Figure B.8: 2D simulation of displacement of Example (i) with no eccentricity in the annulus. a) Snapshots of mud mass fraction $Y_1$. White lines show mass streamlines. b) From left to right, contours of mud mass fraction, density, quality, axial velocity and pressure at time $\hat{t} = 1991$ s. Only the annulus is shown, with here $\xi_{th0} = 0$ and $\xi_{tz} = 500$ m.
flow instability emerges close to the interface. The parameters of this example are such that there remains a significant viscosity gradient between foam and mud over the ranges of quality experienced, i.e. the foamed slurry is more viscous. As the displacement progresses the foam density at the interface decreases below that of the mud and the mean velocity consequently increases above that of the mud. This switch in mechanical stability occurs just after the interface has reached half way up the annulus; see Figures B.2 & B.7a. But although mechanically unstable, it appears that a threshold density difference needs to be attained before the instability is triggered.

This is similar to most density-driven porous media miscible displacement instabilities, i.e. the Muskat problem, and of course at its heart we have here a miscible displacement upwards along a uniform vertical Hele-Shaw cell. When the mechanical stability is lost in such systems, the destabilizing buoyancy stress still needs to overcome a stabilizing viscous effect. For example, with two Newtonian fluids in a vertical Hele-Shaw cell, thin penetrating fingers grow ahead of the front when:

$$6\hat{W}_0(\hat{\mu}_1 - \hat{\mu}_2) + (\hat{\rho}_1 - \hat{\rho}_2)\hat{g}\hat{H}^2 > 0,$$

(B.66)

where fluid 1 is displaced upwards by fluid 2 at speed $\hat{W}_0$ and $2\hat{H}$ is the gap width. This makes the threshold explicit. Thus, we expect a similar threshold behaviour here, although more complex due to the rheology.

Figure B.8b shows the other flow variables at the final time of Figure B.8a, we see that the mass fractions, density, quality and velocity have evident perturbations. The pressure field does not vary with $\phi$. We should acknowledge that our imposition of symmetry conditions at the wide and narrow side of the annulus, although convenient for reducing computational time, restricts the azimuthal propagation of this instability unphysically. However, the axial propagation is of more interest. Behind the unstable region we see that the underlying density gradients in the foam are stabilizing, so there is no driving mechanism to propagate upstream of the front.

Ahead of the front the mud density is constant and it appears that fingers should be able to propagate. However, on close inspection of the density subplot it appears that there is a layer of dense mixed fluid that results at the interface, denser than the
mud downstream. This mixed layer removes the physical mechanism for fingers to propagate downstream ahead of the front, hence limiting the unstable region as shown. The origin of this higher density mixed region lies with the interpolation of the density. As we track advection of both $Y_g$ and the individual fluid constituents $Y_1$, $Y_2$, etc., it follows that the each of these mass fractions must change close to the displacement front. Ahead of the displacement front we have no gas fraction, hence $Y_g \to 0$ across the front. Also we have $Y_2 \to 0$ across the front. The combination of these leads to a small band of concentrations over which the mixture density exceeds that of the mud, i.e. because the slurry density is increasing as $Y_g \to 0$. It is hard to see how to rectify this easily, and perhaps also the notion that $Y_2 \to 0$ with fixed gas fraction $Y_g$ may not be correct when the two fluids can mix. We leave this for a future model refinement, as the main point here is the onset of instability which is driven by density differences.

In Figures B.9 and B.10 we increase the eccentricity of the annulus in the simulation of Figure B.8 to $e = 0.1$ and $e = 0.5$, respectively. We see an interesting effect as the displacement progresses. First, low down in the annulus where the slurry density is high we see a steady displacement front. Even with significant eccentricity a steady front may result for sufficiently positive density and viscosity gradients between fluids; see [179, 181]. This appears true initially, but by the 3rd and 4th time frame of Figure B.9a we begin to see that the density difference is no longer sufficient to displace the mud on the narrow side of the annulus: the front begins to lag behind. To understand intuitively this shift, note that in a narrow gap geometry the buoyancy stress scales with the gap width and thus decreases on the narrow side, where it must still overcome a constant yield stress and generate viscous stresses to move the fluids. As the front moves still higher and the slurry density falls below the mud density, we see that the asymmetry of the flow (between wide and narrow sides) is enhanced by the buoyancy-driven instability. Relative to the mean flow, the narrow-side fluids move backwards down the well and the front elongates along the narrow side. We also observe the onset of density driven fingering in the azimuthal direction, i.e. from the narrow to wide side, as the tail of the displacement extends.

These same effects occur earlier and are more severe for $e = 0.5$. Deep in the annulus the narrow side is not completely displaced and a severe channel of mud
is left behind along the narrow side as the foamed slurry expands up the wide side. There are more azimuthal instabilities behind the front and these penetrate further into the wide side, nearly cutting off the stream of low density slurry.

In order to clarify more clearly that these instabilities result primarily from the changes in foam cement density, we have modified this example. Considering again the example (i) parameters, we now select some representative properties of the cement slurry before the interface becomes mechanically unstable. Here we take the density and rheology values of the foamed slurry when the interface is at
Figure B.10: Same as Fig B.8, except the well has eccentricity of \( e = 0.5 \).
Wide and narrow sides are denoted with W and N.

\[ \hat{\xi} = 150 \text{ m and } \hat{t} = 1004 \text{ s (as marked by the dashed red line in the fourth panel in Figure B.8a).} \]

Using these values we run a simulation in which the displacing cement slurry is no longer compressible, i.e. \( Y_g = 0 \) for all time, but with these same properties: \( \hat{\rho} = 1524 \text{ kg/m}^3, n = 1, \hat{\kappa} = 0.055 \text{ Pa.s and } \hat{\tau}_Y = 7.94 \text{ Pa.} \)

Figure B.11 shows snapshots of the displacement for three values of eccentricity \( e = 0, 0.1 \) and 0.5. For \( e = 0 \) & 0.1 the displacement front advances at the steady mean speed along the annulus, displacing perfectly and with no sign of instability. At \( e = 0.5 \) we also see a steady displacement, but with a very narrow residual channel left
behind on the narrow side of the annulus. Again there is no instability.

We have demonstrated that the changes in density due to foamed cement compressibility lead to buoyancy-driven fingering type instabilities. These occur both in combination with eccentricity effects and in concentric annuli. In general Hele-Shaw type displacement models are effective at predicting instabilities. However, after instability occurs a number of model assumptions (on scaling etc.) typically
break down. Although we expect mixing of the type illustrated, other neglected effects (e.g. inertia, gap-scale effects, dispersion) may all be present in the physical flow, so we should not rely overly on the exact form of solution after instability.

To close this section, we now consider a variation of example (i) in which Mode 1B conditions are implemented instead of Mode 1A, holding a back pressure on the annulus. This has the effect of pressurizing the entire flow path. Thus, the initial inflow gas fraction is also pressurized, which means that the slurry is compressed less as it travels around the flow path compared to example (i). Overall the result is that the compressed foam density at bottom hole is reduced and we retain a lighter weight slurry throughout the annulus. We consider two cases in which the back pressure at the top of the annulus is held at $\hat{p}_a = 20$ atm and $40$ atm, respectively. The simulations are slightly more complex numerically as at each timestep we need to iteratively determine the bottom hole pressure in the annulus, so as to attain the target $\hat{p}_a$.

Figure B.12 shows displacement snapshots of the mud mass fraction and mass flow streamlines, as well as colourmaps of the physical variables at the latest time $\hat{t} = 1843$ s. The annuli are concentric but after the density instability starts we see asymmetric channeling along the annulus as the flow develops. This effect starts deeper in the well when the back pressure is 40 atm and the foam densities are lower (Figure B.13). There remains a threshold for instability, as before, i.e. mechanical instability is necessary but not sufficient. However, we can see from these simulations that the threshold is relatively low for the parameters in these simulations, e.g. $50 - 100$ kg/m$^3$ is sufficient to trigger these instabilities.

### B.5 Conclusions

This paper outlines the development of a mathematical and computational model for miscible displacement of foamed fluids during the primary cementing process. The basis of the model is a miscible mixture model in which the foamed slurry is modelled as a homogeneous compressible fluid, i.e. the foamed slurry is defined by the local mass fraction of gas and there is no slip between gas and liquid cement phases. First we have derived a simple 1D hydraulics-based model for the displacement flow. We have also simplified this model to a purely hydrostatic model
and used the comparison between the two models to show that in many cases the frictional pressure losses can be neglected relative to the hydrostatic pressure.

Although negligible, frictional pressures are important in determining the annular flow, and a model for annular displacement flow is one of the main novel contributions of the paper. The model derived is coupled to the 1D model, which provides input into the annulus. Somewhat surprisingly, the form of the pressure (or streamfunction) equation is near-identical with that for incompressible displacements [24, 147]. The main difference is that, due to the compressibility, we use a
mass streamfunction $\Psi_M$ instead of the usual volumetric streamfunction.

Numerical solution of the annular displacement model shows that a buoyancy-driven instability is triggered at some threshold value of density difference, when the stabilizing viscosity ratio is also overcome. More clearly, as the foamed slurry displaces upwards in the annulus the density falls, eventually dropping below that of the drilling mud. At some distance above this, the instability starts. Although we have not performed a stability analysis for these two fluids, this threshold type of instability is reminiscent of miscible porous media displacement instabilities.

**Figure B.13:** Same as Figure B.8, except a back pressure of $\hat{p}_a = 40$ atm is enforced on top of annulus.
The instability is triggered lower in the annulus for a more eccentric annulus, as the density difference is progressively less effective on the narrow side of the annulus (see our earlier discussion). Thus, we observe a significant residual mud layer/channel emerge on the narrow side of the annulus. By comparison with an incompressible displacement, we are able to identify that the root cause of the instability is the decreasing density of the foamed slurry. Reduction of the effect by application of a back pressure on the annulus in fact had the effect of reducing the density variation in the foamed slurry and hence triggering the density-driven instability lower in the well.

The occurrence of this type of flow instability raises serious questions about the usage of foamed cements. A key motivation for using foamed cements is control of the hydrostatic gradient in the annulus, i.e. low density. If it is necessary to keep the slurry density above the mud density to avoid instabilities, this limits the range of density reduction. Although there are other benefits to using foamed cements (as discussed earlier), we should also note that there are other lightweight cement alternatives and other ways to achieve similar benefits to foam with use of different additives.

Specifically, foamed cement appears problematic from the point of view of hydrodynamic instability. These density-driven instabilities are amplified as the foam expands along the annulus, which self-reinforces the destabilizing mechanism. Although here we have used a cement slurry that is more viscous than the mud, this is not always the case and the uncertainty of foamed cement rheology with pressure (and temperature) means that a viscosity gradient threshold might not be reliable. Even if the foamed cementing operation is undertaken under a controlled pressure, loss of control in this way presents dangers. As in well control, if the annulus is closed/restricted as the foamed slurry rises, the annular pressure will increase. If the outflow is not restricted then the destabilizing expansion will continue, increasing $\alpha$ and eventually leading to coalescence of the gas.

Finally, we should admit the limitations of our model, which is quite preliminary. Including temperature effects and modifying the gas density closure are straightforward improvements. A lot of effort should be made on improving the cement rheology models, to account for a full range of $\alpha$ and to consider e.g. slip and depletion effects at larger $\alpha$. There is a wider theoretical background to develop,
underlying pressure-dependent field equations of type (B.63) which arise from both compressible fluids and $\alpha$-dependent rheology. Although to date we have worked with the stream-function formulation, here and previously [180, 37, 147] the (dual) pressure problem might be a better option for compressible flows. Note too that physically we have other cementing scenarios where compressibility may be important (e.g. deep wells with oil-based drilling muds). These are some future directions for our work.
Appendix C

Effects of geometrical irregularities on displacement flows

I co-authored this paper with A. Rentaria, I. Frigaard, B. Lund, J. Ytrehus and A. Taghipour. Currently, the paper is under revision in the Journal of Petroleum Science and Engineering.

Abstract

Anywhere between 0%-80% of primary cemented wells have integrity failures. Strong local variability suggests both geological and operational factors. One way geology affects cementing is via irregular wellbores, e.g. washouts. Here we study the effects of washouts on mud removal in strongly inclined wellbores, experimentally and via 2-D computational simulations, with the aim of identifying key control parameters. Experiments were performed with 2 fluids with properties representative of drilling mud and cement (or spacer), displaced at constant flow rate through a 10 m long annular flow loop. One downstream section of annulus had an enlarged outer diameter to mimic a wellbore with a washout. In the uniform horizontal sections, the density difference promotes slumping towards the bottom, while eccentricity promotes flow along the wide side of the annulus. At inclinations...
less than horizontal the slumping tendency is reduced. This competition between slumping and eccentricity is affected by the washout. In most of the experiments the mud was removed from the washout, but the simulations showed this is not true for higher yield stress muds than those tested experimentally: residual fluid was left in the washout section. The simulations also provide detailed information on the evolution of the fluid-fluid interfaces as they pass through the annulus and washout.

C.1 Introduction

Primary cementing of highly deviated and horizontal wellbores has long been recognised as technically challenging [128], combining difficulties of hole cleaning, mud conditioning, cement placement and post-placement settling. These problems are compounded by modern drilling techniques allowing penetration through an increasingly wide ranges of formations, including those that are poorly consolidated, for which washout sections occur commonly. Partial blockages, centralizers, washouts, eccentricity and geo-mechanical elastic deformation are all types of geometric irregularity of the annular space along which primary cementing flows occur. That wellbore irregularity can have a serious detrimental effect on primary cementing was noted more than 50 years ago [156], and mobilization of narrow side drilling mud to avoid channelling is a part of every primary cementing design methodology.

This paper is part of a combined experimental/numerical approach studying one specific irregularity: washouts, and focuses on highly deviated wellbores. By washout we mean any (unplanned) enlargement of the wellbore; Figure C.1. These may occur regularly in homogeneously weak formations, e.g. due to regular pauses in drilling penetration. Alternatively, they may be related to local geological features (small cavities, other drilling breaks), due to damage as casing is run in, due disturbance from jetting, etc. Thus, there is no typical washout and actual geometries are only known by caliper measurement. In this sense our study is exploratory and we have not found other studies in the technical or scientific literature. Our results, although novel, in some ways only open a window onto the complexity of these flows and motivate further more detailed study.
Figure C.1: Schematic of highly deviated wellbore with an irregular enlarged section.

The complexity here comes from 2 directions: the washout and the wellbore deviation. Washouts present a generic problem for primary cementing and well integrity. Drilling muds have a yield stress and the flow of such fluids is characterised by regions of unyielded fluid (plugs). In many regular ducts (pipes, concentric annuli) plug regions are found in the centre of the flow. However, irregularity and corners may lead to static plug regions, e.g. the square duct [161], or the eccentric annulus [240, 218]. The washout is a more complex irregularity, consisting of an expansion/contraction of the regular annulus. Mitsoulis and co-workers (e.g. [160]) have studied both planar and axisymmetric expansion flows, which show significant regions of static fluid in the corner after the expansion. Flow of yield stress fluids through an expansion-contraction has been studied experimentally and computationally in de Souza Mendes [53], de Souza Mendes et al. [55], Nassar et al. [165]. In de Souza Mendes et al. [55] Carbopol solutions were pumped through a sudden expansion/contraction, i.e. narrow pipe – wide pipe – narrow pipe. A range of flow rates were studied and the yield surfaces were visualised nicely by a particle seeding arrangement. In the washout context, Roustaei and co-workers[193, 192] systematically studied two-dimensional washout flows, representing a longitudinal section along the annulus, for a single fluid, i.e. during conditioning of the wellbore. They observed firstly, that the fluid tends to self-select its own flowpath, leaving behind regions of static fluid in the washout, which generally grow in extent with the depth of the washout. For sufficiently deep washouts the drilling mud
does not sense the walls of the washout and the mobilised region of mud is independent of the precise washout shape. Secondly, they observed that the intuitive practice of simply increasing the flow rate does not automatically result in mobilizing more mud. Instead, there is a tendency for the more inertial flow to flow along straighter streamlines, by-passing the washout.

These studies, involving one fluid in pre-circulation/conditioning operations, are significant for the well cementing industry. However, there is little analysis of flow removing the mud through these geometries. In Roustaei et al. [194] preliminary results of 2D displacement flow computations are presented, in similar geometries to Roustaei and Frigaard [192], but for simplicity do not consider density differences. Very recently Kragset et al. [135] have presented preliminary results of a 3D computational fluid dynamics simulating the same experiments as here, and work continues in this direction.

The other aspect is the effects of wellbore deviation on laminar displacement flows. For near-horizontal wells these have been studied extensively by Carrasco-Teja and co-workers [37, 35, 36], including casing rotation, and more recently by Tardy and co-workers [226, 225]. In the absence of rotation, horizontal well cementing represents a delicate balance between slumping (buoyancy pushing heavy fluid around to the narrow side) and eccentricity (promoting higher mobility on the wide side). On entering a washout section the relative magnitudes of inertial, buoyant and viscous stresses changes locally and dramatically. These are some of the effects that we reveal below.

### C.2 Methodology

In this paper we use two complementary methods to study displacement flows along an eccentric annular geometry with an enlarged washout section: computational simulation (§C.2.1) and experiments (§C.2.2). As well as using these comparatively, we later use the model to run simulations over extended ranges of parameters.
C.2.1 Simulation model

The displacement flow simulation is based on a two-dimensional model of the annular flow, in which the flow variables are first simplified through scaling (i.e. principally the flow is a shear flow) and then averaged across the annular gap. This results in a two-dimensional 2nd order elliptic partial differential equation for the gap-averaged stream function (or alternatively for the pressure). The different fluids are assumed miscible, represented via scalar concentrations which are advected with the flow and also diffuse. The model equations are derived at length in Maleki and Frigaard [147] and for brevity not repeated here. The methodology is as outlined in Bittleston et al. [24] and this type of 2D model has been used widely in the industry for a range of applications, e.g. Berg et al. [21], Bogaerts et al. [26], Dooply et al. [60].

While effective at reproducing large scale process features, 2D models also have restrictions due to their assumptions, so some caution is needed in comparing against experiments. In particular we can mention the following. (i) These models are based on the dominance of shear-flows at leading order, which relies on the gap being narrow with respect to the annular circumference. (ii) Averaging across the gap means that these models predict only the mean concentrations of fluids and not the transverse distribution. (iii) These models are based on the flows being fully developed locally, i.e. relative to the gap width. We will come back to these points later in the paper.

C.2.2 Experimental description

The experiments were conducted in a 10 m long flow loop located at SINTEF Industry, Trondheim. The inner and outer annulus diameters were 127 mm & 165 mm, respectively (5” & 6.5”). The outer walls were plexiglas, allowing visual access. However, the fluids used were opaque, so that only limited information could be gained in this way. Both the eccentricity of the annulus and its inclination could be altered. The flow loop was equipped with tanks, valves and pumps, as appropriate; the schematic is shown in Figure C.2.

As well as the regular dimensions above, approximately 6 m from the inflow a washout section was introduced, consisting of an angled (50 degree) expansion
to an enlarged outer diameter of 280 mm (11”), constant for a length of 1.66 m, then contracting again to the regular outer diameter for the last $\approx 2$ m of annulus; see Figure C.2. Further detail of the experimental setup is described in Lund et al. [145].

The main quantitative measurements of the flow came from 4 banks of conductivity probes that measured fluid conductivity at up to 8 azimuthal positions around the annulus. These were positioned at 2 m, 4 m, 6 m & 8 m from the inflow, with the last 2 banks thus straddling the enlarged washout section. Differential pressure (DP) measurements were also made at the same intervals along the annulus.

Each experiment proceeded as follows. (i) Start circulation of fluids with displaced fluid circulating in test section and displacing fluid circulating in bypass. (ii) Take sample of displaced fluid. (iii) Flush DP cells & stop circulation of displaced fluid for zero DP measurements. Restart circulation of displaced fluid. (iv) Start data logging. (v) Start displacement experiment. (vi) Switch to automated control of valves at end of experiment. (vii) Continue circulation with displacing fluid and take sample of displacing fluid. The dimensional parameter ranges in our experiments are given in Table C.1.

**Experimental test matrix**

A total of 10 experiments were performed, including 2 identical experiments to verify repeatability of the setup (see Figure C.3). The properties of the fluids used in the experiments varied only slightly (see C.2.2) and the same flow rate was imposed in the regular section for each experiment, equivalent to a mean axial
Table C.1: Parameter range of our experimental study

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{w}_0$</td>
<td>0.5 (m/s)</td>
</tr>
<tr>
<td>$e$</td>
<td>0 &amp; 0.42 (100% &amp; 58% standoff)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>90-60 (deg from vertical)</td>
</tr>
<tr>
<td>$Re$</td>
<td>$O(10^5)$</td>
</tr>
</tbody>
</table>

Figure C.3: Repeatability of the setup. Qualitative comparison of arrival-times measured for each probe in a) Experiments B2 vs. E2: $e=0.42$, 60 degrees from vertical; and b) Experiments F2 vs. J2: $e=0$, 60 degrees from vertical. The term *arrival-time*, defined in §C.3, refers to the time when the displacing fluid is identified by each conductivity probe.

velocity $\hat{w}_0 = 0.5$ m/s. The experiments thus studied the effects of inclination from vertical (90, 80, 70, 60 degrees) for eccentricities of zero (concentric) and 0.42, corresponding to 100% and 58% standoff, respectively. The narrow side is at the bottom of the annulus, as is common due to the effect of the weight of the casing in a highly inclined well.

**Fluid characterization**

A sample of both fluids was taken after the experiment to characterize their rheological behaviour. Measurements were performed with a Fann35 viscometer to resemble the data characterization available in field operation. The viscosity curves
were fitted to the Herschel-Bulkley model, as is widely used for shear-thinning fluids in cementing operations:

\[ \hat{\tau}_i = \hat{\tau}_{Y,i} + \hat{\kappa}_i \hat{\dot{\gamma}}_n^i, \quad i = 1, 2. \]  

(C.1)

Here \( i = 1 \) denotes the displaced fluid, \( i = 2 \) is the displacing fluid; \( \hat{\tau}_{Y,i} \) is the yield stress; \( \hat{\kappa}_i \), the consistency; and, \( n_i \) is the shear-thinning index. The parameters of the Herschel-Bulkley model are reported in Table C.2 and Table C.3 for each experiment. The fluid rheologies are within the ranges found for drilling muds and cement slurries in field applications. However, the differences in fluid rheology are relatively modest. The same rheology was used later in our simulations. The fluid densities were \( \hat{\rho}_1 = 1000\text{kg/m}^3 \) and \( \hat{\rho}_2 = 1100\text{kg/m}^3 \).

**Table C.2:** Rheology of the fluids used in the concentric experiments.

<table>
<thead>
<tr>
<th>Test</th>
<th>Inclination (degrees)</th>
<th>( \hat{\kappa}_1 ) (Pa·s(^n))</th>
<th>( n_1 )</th>
<th>( \hat{\tau}_{Y1} ) (Pa)</th>
<th>( \hat{\kappa}_2 ) (Pa·s(^n))</th>
<th>( n_2 )</th>
<th>( \hat{\tau}_{Y2} ) (Pa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>G2</td>
<td>90</td>
<td>0.47</td>
<td>0.55</td>
<td>2.24</td>
<td>0.99</td>
<td>0.55</td>
<td>0.093</td>
</tr>
<tr>
<td>H2</td>
<td>80</td>
<td>0.47</td>
<td>0.55</td>
<td>2.24</td>
<td>0.99</td>
<td>0.55</td>
<td>0.093</td>
</tr>
<tr>
<td>I2</td>
<td>70</td>
<td>0.47</td>
<td>0.55</td>
<td>2.24</td>
<td>0.99</td>
<td>0.55</td>
<td>0.093</td>
</tr>
<tr>
<td>F2</td>
<td>60</td>
<td>0.49</td>
<td>0.55</td>
<td>4.00</td>
<td>1.32</td>
<td>0.51</td>
<td>0.093</td>
</tr>
<tr>
<td>J2</td>
<td>60</td>
<td>0.49</td>
<td>0.55</td>
<td>4.00</td>
<td>1.32</td>
<td>0.51</td>
<td>0.093</td>
</tr>
</tbody>
</table>

**Table C.3:** Rheology of the fluids used in the eccentric experiments (e=0.42).

<table>
<thead>
<tr>
<th>Test</th>
<th>Inclination (degrees)</th>
<th>( \hat{\kappa}_1 ) (Pa·s(^n))</th>
<th>( n_1 )</th>
<th>( \hat{\tau}_{Y1} ) (Pa)</th>
<th>( \hat{\kappa}_2 ) (Pa·s(^n))</th>
<th>( n_2 )</th>
<th>( \hat{\tau}_{Y2} ) (Pa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A2</td>
<td>90</td>
<td>0.55</td>
<td>0.50</td>
<td>6.42</td>
<td>1.70</td>
<td>0.50</td>
<td>0.093</td>
</tr>
<tr>
<td>D2</td>
<td>80</td>
<td>0.80</td>
<td>0.48</td>
<td>4.50</td>
<td>1.70</td>
<td>0.50</td>
<td>0.093</td>
</tr>
<tr>
<td>C2</td>
<td>70</td>
<td>0.80</td>
<td>0.48</td>
<td>4.50</td>
<td>1.70</td>
<td>0.50</td>
<td>0.093</td>
</tr>
<tr>
<td>E2</td>
<td>60</td>
<td>0.49</td>
<td>0.55</td>
<td>4.00</td>
<td>1.32</td>
<td>0.51</td>
<td>0.093</td>
</tr>
<tr>
<td>B2</td>
<td>60</td>
<td>0.49</td>
<td>0.55</td>
<td>4.00</td>
<td>1.32</td>
<td>0.51</td>
<td>0.093</td>
</tr>
</tbody>
</table>
Measurements

In each bank of conductivity probes, 8 azimuthal positions are possible of which 6 are active; see Figure C.4a. As the flows are laminar we expect approximately symmetric flows (i.e. left-right) and 2 of the 6 probes were intended primarily to give measurements to confirm this.

Each probe consists of 2 electrodes (Figure C.4b & c) of 3 mm diameter separated by a distance 5 mm in the azimuthal (circumferential) direction. The electrodes extend about 5 mm into the annulus, orthogonal to the flow direction, and are effectively measuring the mean conductivity of the fluid that passes between them. Note that the mean gap width in the regular section is 19mm, so the conductivity is that of fluids in the 26.3% of the gap positioned nearest to the outer wall.

Conductivity difference of the 2 fluids is varied by adding salt to the displacing fluid; meanwhile, the low-conductivity displaced fluid is calibrated as zero ($C_1 = 0$). In the experiment, the conductivity signals vary as the displacing front passes through the probes and a steady measurement is achieved.

**Figure C.4:** a) Azimuthal positioning of the conductivity probes, W stands for wide (upper) side of the annulus when is eccentric, and N stands for the narrow (bottom) side; b) Insertion though outer pipe wall; c) Probe extension into flow.
Figure C.5: Example of the conductivity signal recorded in the annulus as the displacing fluid passes at four different axial positions: a) 2 m, b) 4 m, c) 6 m (before washout), and d) 8 m (after washout). Data from case H2: concentric, 80 degrees from vertical.

C.3 Results and Discussion

Examples of the recorded data are shown in Figure C.5 and Figure C.6, for concentric and eccentric cases, respectively. The measured conductivity of the displacing fluid is normalised, $C_2(t) \in [0, 1]$, to give a concentration that can be compared with our computed 2D simulations. In each of these we display data from the 6 probes active in the experiment. Note that the black (solid & broken lines) relate to positions near the top of the annulus, symmetric with respect to the centreline, and similarly the two green curves are from symmetrically positioned probes, but in the lower part of the annulus (see Figure C.4).

Generally speaking, in the 1st two images of each experiment the conductivity measurements show good left-right symmetry, with concentrations increasing as
Figure C.6: Example of the conductivity signal recorded in the annulus as the displacing fluid passes at four different axial positions: a) 2 m, b) 4 m, c) 6 m (before washout), and d) 8 m (after washout). Data from case C2: $e=0.42$, 70 degrees from vertical.

The fluids pass the probes. The conductivity measurements show some noise (not filtered) and some of the lower frequency variations may either indicate unsteady motions, or that a piece of displaced fluid has detached from upstream and is advected downstream. Due to opacity of the fluids, the video images taken are not suitable for quantitative image analysis, but we can observe both these features qualitatively. In Figures C.6a & b, we see that the narrow side conductivity is initially zero and the mud is only displaced later on the narrow side.

The last two images of these experiments illustrate the washout section conductivity measurements (before and after). Here we infer: more unsteady motion, breaking of L-R symmetry, evidence of residual mud/bypassing (Figure C.6c). Eventually however, the conductivities asymptote to their final values, where again L-R symmetry is good. Qualitatively, in the videos we observe more unsteadiness.
and stronger azimuthal currents going through the washout section, \textit{i.e.} in line with the conductivity probe observations.

### C.3.1 Experimental results

One feature evident in the concentric annulus experiment of Figure C.5 is that the fluid appears to pass the probes at the bottom of the annulus quicker than those at the top. This effect is due gravitational slumping of the interface. We see the separation of the vertical segments of the conductivity measurements spaces out as we proceed down the annulus (Figures C.5a to d). On the other hand this trend is reversed in Figure C.6, where we have significant eccentricity: the fluid passes the upper (wide side) probes first. This effect is due to eccentricity, \textit{i.e.} the fluid moves faster on the wide side. The competition between slumping and eccentricity has been studied in detail by Carrasco-Teja et al. [37] and we return to this later.

Evidently there is much detail in the conductivity probe data, which is useful for studying specific features of the flow. However, to get a more global picture of each experiment and to facilitate the comparison with simulations, we condense the main features of the experiment into a plot that indicates the different speeds at which the displacement front passes the probes. For this we first define an arrival-time ($\hat{t}_{\text{arrival}}$) of the displacing fluid at a fixed probe position ($\phi, \hat{\xi}_{\text{probe}}$) as the time when the measured conductivity reaches 20% of the probe’s final conductivity signal. The 20% here is somewhat arbitrary, but is set to avoid any noise, \textit{i.e.} we want a significant presence of displacing fluid at the probe. The ratio of $\hat{\xi}_{\text{probe}}/\hat{t}_{\text{arrival}}$ gives a representative measure of the axial velocity of the displacing fluid front at the probe. We then normalize these with $\hat{v}_0$, which is the mean velocity of the displacement flow in the regular section of the annulus (flow rate divided by cross-sectional area).

Figure C.7 shows this normalized local axial velocity for the full set of concentric experiments as we reduce inclination from horizontal (90) to 60 degrees (Figures C.7a to d). We focus first on the velocities at the first 3 probes. We observe faster velocities towards the lower part of the annulus, approximately double those at the top in the case of the horizontal displacement, but reduced as the inclination decreases. The velocities at the 2nd and 3rd axial positions are practically iden-
tical, which indicates that the displacement has reached a steady axial (slumping) velocity. The first probe is only marginally displaced from the 2nd and 3rd, which suggests that the main development of the axial (slumping) velocity occurs in the first 2 m of the annulus. Note that the difference in velocities means that in these displacements, given a longer regular section, the interface between fluids should continue to stretch and elongate axially. As the interface elongates, the effect of the density difference in driving azimuthal motion should decrease and eventually we might expect to attain a steady displacement, depending on the rheological differences between the fluids; see the analysis in Carrasco-Teja et al. [37]. We note that in all 4 cases the results are consistent, suggesting that any unsteadiness in measured conductivities is not significant.

The fourth probe in each sub-figure of Figure C.7 shows a downwards shift for the other 3 probes. This is a purely volumetric effect, i.e. $\hat{w}_0$ is relevant to the main section, but the washout area is much larger and hence arrival times at the end of the washout are longer. The other noticeable effect is that in contrast to the regular section, the axial velocities are much more uniform at the exit to the washout. The washout appears to reduce the degree of slumping. This appears counterintuitive, as the wider gap would allow for more azimuthal flow, driven by the density differences, but perhaps the effect of the expansion dominates by slowing down all axial velocities volumetrically. The velocity in the upper annulus in experiment G2 (horizontal) is particularly retarded. On the other hand experiments I2 and F2, where the annulus is less inclined have reduce azimuthal driving force, and the velocity after the washout is nearly uniform (Figures C.7c & d).

Before proceeding, we note an interesting feature of the normalized local axial velocity in Figure C.7, before the washout. It may seem paradoxical that the normalized local axial velocities in each of Figures C.7a-d would appear to average to a value $> 1$, i.e. surely the arrival time should be delayed as the probes are measuring fluid close to the wall? Our choice of a 20% threshold sensitivity is arbitrary and selecting a higher sensitivity does extend $\hat{t}_{arrival}$, but not by much. A clearer explanation may come through consideration of a toy model for the arrival time, as follows.

First let us consider the annulus locally as a plane channel and neglect the
Figure C.7: Normalized local axial velocity in half annulus from the experimental data. Concentric experiments: a) G2, horizontal; b) H2, 80 degrees from vertical; c) I2, 70 degrees from vertical; d) F2, 60 degrees from vertical.

rheology. The axial flow of a Newtonian fluid is thus of form:

$$\hat{w}(\hat{y}) = \hat{w}(\phi) \frac{3}{2} \left[ 1 - \left( \frac{\hat{y}}{\hat{H}} \right)^2 \right]$$  \hspace{1cm} (C.2)

with $\hat{w}(\phi)$ the mean axial velocity, $2\hat{H}$ the channel width, and $\hat{y}$ measuring distance from the channel centreline. In a concentric annulus, with no slumping we expect $\hat{w}(\phi) = \hat{w}_0$. The probes stick in a distance $a\hat{H}$ from the upper wall (here $a = 5/9.5 \approx 0.53$). Lets assume that the probes measure only 0 or 1, according to the fluid passing between them, and that the conductivity is then just an average of the
measured conductivity.

Assuming that the displacement is a passive tracer (advecting only), the displacing fluid travels to the tip of the probe at $\hat{\xi}_{\text{probe}}$ at

$$\hat{t}_{\text{arrival}, \alpha} = \frac{\hat{\xi}_{\text{probe}}}{\hat{w}(\phi)} \frac{2}{3 \left[1 - (1 - \alpha)^2\right]}.$$ 

According to our criterion, the probe attains 20% concentration when the first 20% of the length is in contact, i.e. 80% of the probe is free of fluid. This happens at $\hat{t}_{\text{arrival}, 0.8\alpha}$, i.e.

$$\hat{t}_{\text{arrival}} \approx \hat{t}_{\text{arrival}, 0.8\alpha} = \frac{\hat{\xi}_{\text{probe}}}{\hat{w}(\phi)} \frac{2}{3 \left[1 - (1 - 0.8\alpha)^2\right]}.$$  \hfill (C.3)

This gives:

$$\frac{\hat{\xi}_{\text{probe}}}{\hat{t}_{\text{arrival}}\hat{w}_0} \approx \frac{\hat{w}(\phi)}{\hat{w}_0} \frac{3}{2} \left[1 - (1 - 0.8\alpha)^2\right].$$

Considering the data of Figure C.7, we have approximately $\hat{\xi}_{\text{probe}}/(\hat{t}_{\text{arrival}}\hat{w}_0) \approx 1.2$, which corresponds to a probe length of $\alpha = 0.69$. While longer than the actual probe ($\alpha \approx 0.53$) the above analysis is simplistic. In particular, the displacing fluids here are shear-thinning yield stress fluids. These have a velocity profile that is more plug-like, which moves fluid closer to the wall. Hence, $\alpha \approx 0.53$ and $\hat{\xi}_{\text{probe}}/(\hat{t}_{\text{arrival}}\hat{w}_0) \approx 1.2$ are reasonable. The analysis also neglects dispersive effects (reducing $\hat{t}_{\text{arrival}}$) and any local flow disturbance effects due to the probes.

We now turn to the eccentric annulus displacements and show the normalized local axial velocity for these experiments in Figure C.8. We consider first the 3 sets of conductivity probes before the washout section and then the effects of the washout. In contrast to the concentric experiments, in the regular section of annulus the velocities are larger on the wide (upper) side of the annulus, than on the narrow side. Indeed, on the narrow side of experiments A2 and D2 (horizontal and 80 degrees from vertical) the fluid is barely moving on the narrow side. As the annulus becomes less inclined the narrow side is mobilized. Note that the displacing fluid is denser, so decreasing the inclination from vertical does increase
the axial buoyancy gradient. Since the effect of slumping is reduced, this is the driving force for the displacement on the narrow side.

Unlike the concentric displacements, the axial velocities still appear to be developing in the regular section before the washout. The initial velocities are fastest towards the wide side, but as the displacing fluid passes the second and third probes, the largest velocities are found at the 67.5 degree probe, with velocities at the uppermost (22.5) degree probe now lagging behind. This effect is consistent for all inclinations and is probably a combined effect of azimuthal flow driven by buoyancy gradients, i.e. slumping, competing against the eccentricity effects. Note too that if the wide side were to advance ahead of the main front then the heavy

Figure C.8: Normalized local axial velocity in half annulus from the experimental data. Eccentric experiments (e=0.42): a) A2, horizontal; b) D2, 80 degrees from vertical; c) C2, 70 degrees from vertical; d) E2, 60 degrees from vertical.
displacing fluid would be positioned above the lighter displaced fluid, which is mechanically unstable and would likely result in density driven azimuthal fingering & mixing.

The effects of passing through the enlarged washout section are partly as for the concentric experiments, i.e. volumetric slowing of the front and also some evening out of the velocity azimuthally. In experiments A2 and D2, where it seems that the narrow side is partly blocked, we appear to have a significant slumping effect, from wide to narrow side, which mobilises the narrow side fluids but results in a significantly slower wide side. We should note that the enlarged section is less eccentric than the regular section (due to the increased annular gap), so that the dynamics here should be close to the concentric annulus. For experiments C2 and E2 the velocities are fairly uniform on exiting the washout. In comparison to experiments A2 and D2, we have reduced the slumping effect, but increased axial buoyancy.

C.3.2 Simulation results

We have run simulations to match the experimental parameters as listed in Tables C.2 & C.3. Due to the approximate L-R symmetry in the regular sections of annulus, we have used a half-annulus for our simulations. In general, the results of simulations and experiments are in good qualitative agreement on the main effects, but also reveal some differences.

An example of the simulation output is shown in Figure C.9, for the horizontal concentric annulus (experiment G2). The simulated half annulus is unwrapped: $\phi$ is the azimuthal coordinate, $W$ stands for the wide (upper) side, and $N$ for the narrow side; $\xi$ is the axial coordinate, parallel to the imposed flow direction. The displaced fluid (red) initially occupies the annulus and the displacing fluid (blue) enters at $t = 0$. The interface advances along the regular section, slumping slightly. This induces a disturbance of the streamlines within in the enlarged section, which grows as the displacement front approaches. Before entering the washout, the interface is virtually flat. Once it enters the enlarged section, the displacing fluid slumps significantly to the lower side. As soon as the fluid leaves the enlarged section, the flow is driven back to the upper side leaving a residual channel inside the
enlarged section. This residual channel is slowly washed out as the flow continues. This process is essentially the same observed for the rest of concentric simulations at different inclinations.

To compare with the experiments, we can track the concentration of displacing fluid at any position in the annulus and extract this at the different probe positions. Figure C.10 corresponds to experiment H2, as an example (concentric, 80 degrees). Particularly in the regular part of the annulus, there is very little dispersion of the displacement front, so that the rise from zero concentration to 1 is very rapid. Secondly, unlike the experiments there is very little separation between the concentration curves at different azimuthal positions, at least before the irregular section (Figures C.10a-c), although after the irregular section the slumping is quite evident.

From the simulation concentrations we are able to also compute arrival-times, as for the experiments. Normalized local axial velocities are shown in Figure C.11
Figure C.10: Concentration extracted from the simulation corresponding to Figure C.5: (H2) concentric, 80 degrees from vertical. a) 2 m, b) 4 m, c) 6 m, and d) 8 m (after the irregular the section).

for the concentric simulations (compare with Figure C.7 earlier). The main difference that we note here is that the slumping effect is significantly less than in the experiments. Unlike in the experiments, on passing through the enlarged washout section the interface slumps more than in the regular section. This effect is again reduced by increased inclinations away from horizontal.

The local computed axial velocity from the eccentric annulus simulations (Figure C.12) also shows this significant change in the interface orientation before/after the irregular section. As observed in the experiment (Figure C.8), reducing the inclination from vertical aids to mobilize the fluid in the narrow side of the annulus (see Figure C.12a vs. Figure C.12c). More obvious here is the effect of eccentricity in the increased wide side velocities (reduced narrow side). The interface appears
Figure C.11: Normalized local axial velocity in half annulus from simulations. Concentric simulations: a) G2, horizontal; b) H2, 80 degrees from vertical; c) I2, 70 degrees from vertical; d) F2, 60 degrees from vertical.

to be moving at steady speeds and elongating as it travels along the regular section. The change in inclination does improve the mobility on the narrow side, as with the experiments.

Figure C.13 shows the concentration colourmap and streamlines for the simulation corresponding to the horizontal and eccentric experiment A2, at different times. Initially the displacing fluid travels faster in the wide side, leaving a mud channel in the lower and narrower side of the annulus. This mud channel decreases its size over time as the fluid is mobile. At the entrance of the washout, the heavy fluid entering at the top of the annulus slumps around to the narrow side, but after leaving the irregular section, it goes back towards the wide side leaving a second
mud deposit inside the washout that also decreases its size with time. Note also the significant circulation and mixing within the enlarged section. At the end of the simulation, displaced fluid still remains in the two bypassed sites. Qualitatively similar results are found for the other eccentric simulations.

We note that the residual mud is quite consistent with the conductivity measurements in experiment A2, Figure C.14 shows the detail for this test. At the first two axial positions the arrival time for the fluid in the narrow side is by far larger than in the wide side. At the entrance of the irregular section (see Figure C.14c), the probe located at the narrowest position did not get a signal during the entire experiment span, $C_2(t) = 0$. This position is indeed located in the mud-left zone.
predicted by the simulation. Finally, we note that after the irregular section, the higher flow rates in the wide part of the eccentric annulus is re-established: the fluid travels faster on the wide side and we observe again a residual layer on the narrow side.

Although there are similarities in the effects of eccentricity on the displacement flows, it appears that the biggest discrepancy with the experiments is due to the slumping behaviour. With regard to this, a number of remarks can be made.

• Firstly, as the current experiments were not designed for visualization, we cannot categorically attribute the behaviour to buoyancy driven slumping, i.e. what has been observed is conductivity measured *arrival times* that are different from those expected according to our model.

One speculation is that some experimental artifact may have contributed to this. In particular, there is no special treatment of the inflow of valve system in the experiments, so entry effects are possible and would not be accounted
Figure C.14: Conductivity signal recorded in the annulus as the displacing fluid passes at four different axial positions: a) 2 m, b) 4 m, c) 6 m (before washout), and c) 8 m (after washout). Data from Case A2: eccentric and horizontal.

for in the simulations.

• Secondly, even supposing that there are no such experimental artifacts, the inlet and flow development of eccentric annular flows has recently been studied in detail by Choueiri and Tavoularis [41, 42] who show surprisingly long development lengths. This could be affecting the experimental data, whereas the annular flow models rely on closures that are locally fully developed. Although such effects can be significant on the scale of the 10m experimental annulus used here, they would be less significant in a full-scale well.

• We emphasize that the simulations do show slumping behaviour, just less pronounced than the experiment. Indeed the model underlying the simulation reproduces the results of Carrasco-Teja et al. [37], in which there is
extensive slumping for horizontal wells.

- The annular simulations that we have used evaluate the leading order shear-flow in a narrow annular region, as described at length in Bittleston et al. [24], Carrasco-Teja et al. [37], Maleki and Frigaard [147]. In this model the streamlines follow the modified pressure gradient, which is influenced by both axial and azimuthal buoyancy gradients. Carrasco-Teja et al. [37] study slumping in terms of the buoyancy to viscous stress ratio $\tilde{b}$:

$$
\tilde{b} = \frac{(\rho_2 - \rho_1)2Hg}{\tau_v},
$$

where $\rho_2$ & $\rho_1$ are the heavy (displacing) and light (displaced) fluid densities, $g$ is the gravitational acceleration and $H$ the half gap width. The denominator $\tau_v$ represents a viscous stress scale. For the experiments here the buoyancy stress $(\rho_2 - \rho_1)gd = 9.3$ Pa. The viscous stresses are in the range $7 - 15$ Pa for these experiments (based on representative wall shear strain rates). Thus, we see that $\tilde{b} \sim O(1)$ for these experiments. In Carrasco-Teja et al. [37] excessive slumping occurs only when $\tilde{b} \gg 1$, which is not the case here.

- However, on entering the enlarged section the balance changes significantly: the buoyancy stress scales with $d$, the mean velocity scales with the inverse area and the viscous stress is reduced by the increase in $d$. Hence, $\tilde{b} \sim 10 - 20$ is easily achieved and this is the range of $\tilde{b}$ for which significant slumping begins to be observed in Carrasco-Teja et al. [37] as well as in our simulations here, within the enlarged section. Thus, this aspect is consistent with previous results and analysis.

- If indeed the discrepancy between simulation and experiment is not due to entry/development effects or another artifact, then we must look at what is missing within the model.

A first neglected effect is inertia. Using similar viscous stress scales as above, we can estimate the Reynolds numbers in the annulus to be in the range $Re \approx 20 - 200$, based on local gap variations, indicating significant inertial effects. In a narrow annulus, these terms are neglected under the
assumption that $\varepsilon Re \ll 1$, where $\varepsilon$ is the gap to circumference ratio. Here $\varepsilon \approx 0.04$ in the regular section and $\varepsilon \approx 0.12$ in the enlarged section, meaning that these effects are non-negligible throughout the experimental range.

This exposes a weakness of models such as Bittleston et al. [24], Carrasco-Teja et al. [37], Maleki and Frigaard [147] for horizontal wells. While vertical parts of the well and surface casing have relatively small $\varepsilon$, on reaching the production casings $\varepsilon$ naturally increases due to the smaller casings with near-constant gap width. To correct this type of model at larger $\varepsilon$ requires inclusion of inertial effects. Since the laminar model here is essentially a Hele-Shaw model (*i.e.* a Darcy flow), an obvious answer is to return to inertial remedies for Darcy-like flows, *e.g.* using/adapting the Ergun or Forcheimer equation approaches.

- A second suggested possibility is that assuming the flow is symmetric about the vertical plane may inhibit growth of azimuthal instabilities, as it imposes a zero azimuthal velocity at the top and bottom of the well. Asymmetric instabilities have been observed experimentally in Tehrani et al. [231], in vertical annuli, and simulated in Tardy and Bittleston [226] in horizontal annuli. The model in Tardy and Bittleston [226] is quite similar to that here. We have run a number of the examples above using a version of our code that simulates the full annulus (no symmetry condition anymore), but did not observe any instabilities in the range of the experiments. Even had we observed instabilities, we would be cautious in their interpretation. Models such as Tardy and Bittleston [226] and here may predict the onset of instability, but the unstable flows that are resolved thereafter tend to violate the scaling assumptions on which they are built, *e.g.* inertia is neglected.

- In recent work [225], Tardy has extended the model of [226] to include a better representation of the cross-gap distribution of velocity. This is useful in modifying the description of the laminar transport of fluid concentrations. Adoption of this type of model here could help resolve some of the discrepancy observed.

- A final comment regarding this discrepancy comes from communications
with Kragset & Skadsem, who have been performing 3D simulations on the same set of experiments; see Kragset et al. [135]. Interestingly, they also find that the degree of slumping in the experiments is far greater than in their simulations, i.e. as we find here. As their experiments include inertia (full Navier-Stokes simulations), this suggests that the issue may in fact lie with an experimental artefact.

**Washout conclusions**

A surprising conclusion from the experiments and simulations is that, in some respects, the displacement appears to be improved after the enlarged washout section. This conclusion is based on the observation that the normalised axial velocities after the washout are typically more uniform azimuthally than before. The main cause of this seems to be secondary flows induced by the washout and a reduction in mean velocity as the displacing fluids enter the washout. The increase in gap width means a reduction in eccentricity within the washout, and can allow for more buoyancy dominated slumping. There are, unfortunately, limited experimental data to support the conclusion from the simulation results.

Note however that the axial velocities are not the only aspect of the displacement that is relevant: we have also seen significant mixing, secondary flow and instances of incomplete displacement. Also, after the washout the displacement profile along the regular section recovers to that of the pre-washout profile, i.e. any re-orientation of the interface is temporary on the scale of the well. Finally, we should note that in the experiments, the drilling mud was removed from the washout in most scenarios. This observation is based on readings from the bulk conductivity probe at the exit of the apparatus. However, the experiments were performed with only a moderate difference in fluid properties, compared to field variations. Thus, in the next section we address a wider range situations via simulation: increasing the yield stress, density difference and inclination.

**C.3.3 Parametric studies**

The yield stress in the displaced fluid was increased. A low and high yield stress sequences were run for each set of geometries, concentric/eccentric, and different
inclinations. For the concentric condition, the density difference was also enlarged to enhance the effect of buoyancy.

**Concentric annuli**

Figures C.15-C.18, presents the sequence of concentric annulus displacements with a moderately increased mud yield stress (20 Pa) and a density difference of 30%. The increased density difference results in significant slumping (much more like that observed in the experiments). The irregular section still appears to improve the displacement for the less inclined annuli, straightening the interface although it slumps again afterwards (Figure C.18). In the horizontal case (Figure C.15), the velocity in the enlarged section decreases and allows the upper side fluid to catch up the lower side fluid at the entrance of the irregular section. However, inside the upper part of the enlarged section there remains an unyielded plug of mud which is not removed, *i.e.* the displacing fluid enters the washout, slumps under the residual plug and continues out the other side. When the wellbore is inclined by 80 degrees from vertical (Figure C.16), the slumping is less intense and allows partial shearing/removal of the plug from the washout. At smaller inclinations all the mud is displaced from the washout (Figure C.17, Figure C.18).

In Figures C.19-C.22, the yield stress was increased to 50 Pa, to simulate a poorly conditioned partially dehydrated mud. In the horizontal case (Figure C.19) we see extreme slumping leaving behind a static channel in the regular section as well as through the washout (increasing in size). The mud channel continues beyond the washout. On reducing the inclination from vertical (Figure C.20), slumping and static mud channels still persist in the regular section before the washout. However, the combination of wider gap and increased axial buoyancy force eventually removes the mud from the washout (60 degrees, Figure C.22). However, the 80 (Figure C.20) and 70 (Figure C.21) degree inclinations have very poor displacement of the washout. Note that even for the 60 degree annulus the remaining mud in the regular section can continue to contaminate displacing fluids as they pass.
Figure C.15: Concentration colourmaps and streamlines of simulation for the concentric annulus with moderately increased yield stress. Base case used G2 90 degrees inclined from vertical. Displaced fluid: \( \hat{\rho}_1 = 1000 \text{ kg/m}^3, \hat{\kappa}_1 = 0.5, n_1 = 0.55, \hat{\tau}_{y1} = 20 \text{ Pa} \). Displacing fluid: \( \hat{\rho}_2 = 1300 \text{ kg/m}^3, \hat{\kappa}_2 = 0.98, n_2 = 0.54, \hat{\tau}_{y2} = 0.09 \text{ Pa} \). Dimensionless time in each picture from left to right: \( t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102 \).

Eccentric annuli

We now consider the effects of yield stress on displacement in the eccentric annulus. For the base set of experiments we have observed that a more uniform velocity was found after the washout. However, the irregular section affected the interface differently in eccentric than concentric configurations. In Figures C.23-C.26, we have kept the same parameters as in experiment A2, but increased the yield stress to 20Pa. The eccentricity is equal to 0.42 (58% standoff) and the same low-density difference 10% is used. In the fully horizontal well, the fluid travels only on the upper wide side leaving a mud channel on the narrow side, and a large unyielded plug inside the washout (Figure C.23).
Figure C.16: Concentration colourmaps and streamlines of simulation for the concentric annulus with moderately increased yield stress. Base case used G2 and 80 degrees inclined from vertical. Displaced fluid: \( \hat{\rho}_1 = 1000 \text{ kg/m}^3, \hat{k}_1 = 0.5, n_1 = 0.55, \hat{\tau}_{y1} = 20 \text{ Pa} \). Displacing fluid: \( \hat{\rho}_2 = 1300 \text{ kg/m}^3, \hat{k}_2 = 0.98, n_2 = 0.54, \hat{\tau}_{y2} = 0.09 \text{ Pa} \). Dimensionless time in each picture from left to right: \( t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102 \).

In contrast with the observations in the concentric experiments, the less eccentric enlarged section does not seem to aid the displacement. Presumably, the reduced pressure gradient in the enlarged section is insufficient to remove the mud in that zone. With reduced inclination, the washout plug gets slightly smaller and the axial buoyancy effect results progressively in removal of the mud from the washout (Figures C.24-C.26). However, even at the end of simulation a mud layer was left behind in the regular section, both before and after the washout on the narrow side (Figure C.26).

In Figures C.27-C.30, we further increased the yield stress to 50 Pa. The displacing fluid flows in the wide side only, and very rapidly. The region of unsheared displaced fluid is more than 50% of the annulus. The less eccentric irregular section
Figure C.17: Concentration colourmaps and streamlines of simulation for the concentric annulus with moderately increased yield stress. Base case used G2 and 70 degrees inclined from vertical. Displaced fluid: $\hat{\rho}_1 = 1000 \text{ kg/m}^3$, $\hat{\kappa}_1 = 0.5$, $n_1 = 0.55$, $\hat{\tau}_{y1} = 20 \text{ Pa}$. Displacing fluid: $\hat{\rho}_2 = 1300 \text{ kg/m}^3$, $\hat{\kappa}_2 = 0.98$, $n_2 = 0.54$, $\hat{\tau}_{y2} = 0.09 \text{ Pa}$. Dimensionless time in each picture from left to right: $t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102$.

and the inclination do not make any difference in the final result.

C.4 Discussion

As discussed in Carrasco-Teja et al. [37], Kragset et al. [135], horizontal cementing represents a competition between the tendency for the interface to slump to the bottom of the annulus under buoyancy, or to advance along the wider upper side of the well (increased mobility). Interesting in the above respect is that our base parameters are rather modest considered in the wider range of likely fluid properties. Near horizontal wells can be significantly less centralised than those here and density differences of $> 10\%$ are common with typical cements and muds. Horizontal
Figure C.18: Concentration colormaps and streamlines of simulation for the concentric annulus with moderately increased yield stress. Base case used G2 and 60 degrees inclined from vertical. Displaced fluid: \( \hat{\rho}_1 = 1000 \, \text{kg/m}^3, \kappa_1 = 0.5, n_1 = 0.55, \hat{\tau}_{y1} = 20 \, \text{Pa} \). Displacing fluid: \( \hat{\rho}_2 = 1300 \, \text{kg/m}^3, \kappa_2 = 0.98, n_2 = 0.54, \hat{\tau}_{y2} = 0.09 \, \text{Pa} \). Dimensionless time in each picture from left to right: \( t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102 \).

sections are generally harder to ensure good circulation and conditioning of mud in; hence the likelihood of dehydrated fluids with elevated rheological properties. All of these suggest more extreme situations than here, where either slumping or eccentricity may dominate. We have partly explored these effects in sequences of additional simulations: enhanced mud yield stresses and larger density differences, generally with negative consequences from the perspective of fluid displacement.

In terms of the experiments, the flow loop used is full scale in terms of tubular diameters, but has limited length. We do see that steady interface velocities are achieved at different azimuthal positions in the annulus, before the enlarged washout section is encountered. In this sense we do have a developed flow, but in a longer annulus (well) the interface would continue to elongate (along the top
Figure C.19: Concentration colourmaps and streamlines of simulation for the concentric annulus with increased yield stress. Base case used G2 and 90 degrees inclined from vertical. Displaced fluid: $\hat{\rho}_1 = 1000 \text{ kg/m}^3$, $\hat{k}_1 = 0.5$, $n_1 = 0.55$, $\hat{\tau}_y_1 = 50 \text{ Pa}$. Displacing fluid: $\hat{\rho}_2 = 1300 \text{ kg/m}^3$, $\hat{k}_2 = 0.98$, $n_2 = 0.54$, $\hat{\tau}_y_2 = 0.09 \text{ Pa}$. Dimensionless time in each picture from left to right: $t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102$.

or bottom), and thus we are far from understanding experimentally this type of long-time behaviour, which is more in the regime of validity for models such as that used; see e.g. Carrasco-Teja et al. [37]. For intermediate lengths such as here, a better tool would be a 3D computation, particularly as the rheologies are not extreme. Some progress in this direction has been made in Kragset et al. [135].

Regarding the effects of the enlarged washout explicitly, there is some ambiguity in drawing conclusions. First, geometrically we note that any enlargement results in reduced eccentricity. Second, we have observed that the normalised interface velocities (for the experiments and corresponding simulations) are more uniform after the washout than before. Insofar as this is a positive indicator of an effective displacement we must conclude that for cementing displacement flows with modest rheologies and eccentricities in near-horizontal annuli, a washout may
Figure C.20: Concentration colourmaps and streamlines of simulation for the concentric annulus with increased yield stress. Base case used G2 and 80 degrees inclined from vertical. Displaced fluid: $\hat{\rho}_1 = 1000 \text{ kg/m}^3$, $\hat{\kappa}_1 = 0.5$, $n_1 = 0.55$, $\hat{\tau}_y_1 = 50 \text{ Pa}$. Displacing fluid: $\hat{\rho}_2 = 1300 \text{ kg/m}^3$, $\hat{\kappa}_2 = 0.98$, $n_2 = 0.54$, $\hat{\tau}_y_2 = 0.09 \text{ Pa}$. Dimensionless time in each picture from left to right: $t = 0.5$, 15, 29.5, 44, 58.5, 73, 87.5, 102.

have a positive effect on the displacement locally. This was somewhat surprising at the outset of our study.

However, a more detailed exploration of our results reveals a more complex picture. Firstly, note that the displacement front does not indicate what may be left behind the front. For this, the simulations are more revealing than the experiments. Secondly, the larger cross-section means that different features will dominate the flow than in the regular section. Naïvely, the expansion reduces the mean speed (volumetrically) so that bulk inertia is reduced within the washout. Hence, as we have seen, buoyancy/slumping may become dominant. On the other hand, at the entry/exit to the washout, expansion and contraction mean that gradients in the inertial stresses are significant, possibly inducing local mixing and unsteadiness.

Lastly, the above situation is more complex in the presence of yield stress flu-
Figure C.21: Concentration colourmaps and streamlines of simulation for the concentric annulus with increased yield stress. Base case used G2 and 70 degrees inclined from vertical. Displaced fluid: $\dot{\rho}_1 = 1000 \text{ kg/m}^3$, $\dot{k}_1 = 0.5$, $n_1 = 0.55$, $\tau_{y1} = 50 \text{ Pa}$. Displacing fluid: $\dot{\rho}_2 = 1300 \text{ kg/m}^3$, $\dot{k}_2 = 0.98$, $n_2 = 0.54$, $\tau_{y2} = 0.09 \text{ Pa}$. Dimensionless time in each picture from left to right: $t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102$.

Recent studies [193, 192] have shown that in deep washouts, even for modest yield stresses there can be significant residual fluid left and that increasing the flow rate does not always improve this situation. Our simulations here (at elevated yield stresses) have shown that large parts of either the regular or enlarged sections can have stationary mud, either temporarily or permanently. Since the displacing fluid now must flow through a restricted area this further complicates attempts to estimate the effect of the enlargement, e.g. Figures C.19 & C.20 show severely restricted channels through the lower and central part of the washout while Figures C.23 & C.24 have similar constrictions but on the wide side due to eccentricity. In both cases we can say that the washout section is worse than the regular section.
**Figure C.22:** Concentration colourmaps and streamlines of simulation for the concentric annulus with increased yield stress. Base case used G2 and 60 degrees inclined from vertical. Displaced fluid: $\hat{\rho}_1 = 1000 \text{ kg/m}^3$, $\hat{\kappa}_1 = 0.5$, $n_1 = 0.55$, $\hat{\tau}_{y1} = 50 \text{ Pa}$. Displacing fluid: $\hat{\rho}_2 = 1300 \text{ kg/m}^3$, $\hat{\kappa}_2 = 0.98$, $n_2 = 0.54$, $\hat{\tau}_{y2} = 0.09 \text{ Pa}$. Dimensionless time in each picture from left to right: $t = 0.5$, 15, 29.5, 44, 58.5, 73, 87.5, 102.

### C.5 Conclusions

This paper has presented results of a study of eccentricity and inclination effects on displacement flows in near-horizontal annuli, as relevant to primary cementing of near-horizontal wellbores, *i.e.* production casings. We have looked effects in both regular wellbores and in an enlarged washout section. Two comparative methods have been used for the study: laboratory experiments (on the scale of the wellbore) and simulations using a gap-averaged two-dimensional model of the flow.

For the choice of fluid rheologies and density difference, in general the displacement experiments were fairly effective, with the exception of eccentric annuli when horizontal and at 80 degrees. Both experiments and simulations showed a shift from slumping in the concentric annuli to unsteady interface advance in the
Figure C.23: Concentration colourmaps and streamlines of simulation for an eccentric \((e = 0.42)\) annulus with moderately increased yield stress. Base case used A2 and 90 degrees inclined from vertical. Displaced fluid: \(\hat{\rho}_1 = 1000 \text{ kg/m}^3, \hat{\kappa}_1 = 0.5, n_1 = 0.55, \hat{\tau}_{y1} = 20 \text{ Pa}\). Displacing fluid: \(\hat{\rho}_2 = 1100 \text{ kg/m}^3, \hat{\kappa}_2 = 0.98, n_2 = 0.54, \hat{\tau}_{y2} = 0.09 \text{ Pa}\). Dimensionless time in each picture from left to right: \(t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102\).

eccentric annuli. Indeed this delicate balance is captured very well by these parameter ranges.

In terms of a comparative study the simulations tended to under-predict the degree of slumping compared to the experiments. In §C.3.2 we have discussed at length several possible sources for the discrepancy, including inertial effects in the mean flow that are neglected in the narrow-gap approximations models [24, 147], as well as experimental features that are not controlled. These represent key areas for improvement.

In terms of the effects of washout on the displacement efficiency, we may conclude that in situations where a regular annulus displaces effectively an enlarged washout may improve things, but in situations where the displacement is anyway
Figure C.24: Concentration colourmaps and streamlines of simulation for an eccentric ($e = 0.42$) annulus with moderately increased yield stress. Base case used A2 and 80 degrees inclined from vertical. Displaced fluid: $\hat{\rho}_1 = 1000 ~kg/m^3$, $\hat{k}_1 = 0.5$, $n_1 = 0.55$, $\hat{\tau}_{y1} = 20 ~Pa$. Displacing fluid: $\hat{\rho}_2 = 1100 ~kg/m^3$, $\hat{k}_2 = 0.98$, $n_2 = 0.54$, $\hat{\tau}_{y2} = 0.09 ~Pa$. Dimensionless time in each picture from left to right: $t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102$.

Poor it is likely to get worse. We must also view our results in the context of a longer section of wellbore. If isolated, a single washout may not be an issue for well integrity: the main danger of poor displacement is that the residual fluid can contaminate the cement slurry as it passes. However, if due to a weak formation combined with geomechanical loading, further washouts are likely in a horizontal section of wellbore and then we can conceive of communication between defects.
Figure C.25: Concentration colourmaps and streamlines of simulation for an eccentric ($e = 0.42$) annulus with moderately increased yield stress. Base case used A2 and 70 degrees inclined from vertical. Displaced fluid: $\hat{\rho}_1 = 1000 \text{ kg/m}^3$, $\hat{k}_1 = 0.5$, $n_1 = 0.55$, $\hat{\tau}_y1 = 20 \text{ Pa}$. Displacing fluid: $\hat{\rho}_2 = 1100 \text{ kg/m}^3$, $\hat{k}_2 = 0.98$, $n_2 = 0.54$, $\hat{\tau}_y2 = 0.09 \text{ Pa}$. Dimensionless time in each picture from left to right: $t = 0.5$, 15, 29.5, 44, 58.5, 73, 87.5, 102.
Figure C.26: Concentration colourmaps and streamlines of simulation for an eccentric ($e = 0.42$) annulus with moderately increased yield stress. Base case used A2 and 60 degrees inclined from vertical. Displaced fluid: $\hat{\varrho}_1 = 1000 \text{ kg/m}^3$, $\hat{\kappa}_1 = 0.5$, $n_1 = 0.55$, $\hat{\tau}_{y1} = 20 \text{ Pa}$. Displacing fluid: $\hat{\varrho}_2 = 1100 \text{ kg/m}^3$, $\hat{\kappa}_2 = 0.98$, $n_2 = 0.54$, $\hat{\tau}_{y2} = 0.09 \text{ Pa}$. Dimensionless time in each picture from left to right: $t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102.$
Figure C.27: Concentration colourmaps and streamlines of simulation for an eccentric ($e = 0.42$) annulus with increased yield stress. Base case used A2 and 90 degrees inclined from vertical. Displaced fluid: $\hat{\rho}_1 = 1000 \text{ kg/m}^3$, $\hat{\kappa}_1 = 0.5$, $n_1 = 0.55$, $\hat{\tau}_{y1} = 50 \text{ Pa}$. Displacing fluid: $\hat{\rho}_2 = 1100 \text{ kg/m}^3$, $\hat{\kappa}_2 = 0.98$, $n_2 = 0.54$, $\hat{\tau}_{y2} = 0.09 \text{ Pa}$. Dimensionless time in each picture from left to right: $t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102$. 

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Figure C.28: Concentration colourmaps and streamlines of simulation for an eccentric ($e = 0.42$) annulus with increased yield stress. Base case used A2 and 80 degrees inclined from vertical. Displaced fluid: $\hat{\rho}_1 = 1000$ kg/m$^3$, $\hat{\kappa}_1 = 0.5$, $n_1 = 0.55$, $\hat{\tau}_{y1} = 50$ Pa. Displacing fluid: $\hat{\rho}_2 = 1100$ kg/m$^3$, $\hat{\kappa}_2 = 0.98$, $n_2 = 0.54$, $\hat{\tau}_{y2} = 0.09$ Pa. Dimensionless time in each picture from left to right: $t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102$. 
Figure C.29: Concentration colormaps and streamlines of simulation for an eccentric (e = 0.42) annulus with moderately increased yield stress. Base case used A2 and 70 degrees inclined from vertical. Displaced fluid: \( \hat{\rho}_1 = 1000 \text{ kg/m}^3, \hat{\kappa}_1 = 0.5, n_1 = 0.55, \hat{\tau}_{y1} = 50 \text{ Pa} \). Displacing fluid: \( \hat{\rho}_2 = 1100 \text{ kg/m}^3, \hat{\kappa}_2 = 0.98, n_2 = 0.54, \hat{\tau}_{y2} = 0.09 \text{ Pa} \). Dimensionless time in each picture from left to right: \( t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102 \).
Figure C.30: Concentration colourmaps and streamlines of simulation for an eccentric ($e = 0.42$) annulus with increased yield stress. Base case used A2 and 60 degrees inclined from vertical. Displaced fluid: $\hat{\rho}_1 = 1000 \text{ kg/m}^3$, $\hat{k}_1 = 0.5$, $n_1 = 0.55$, $\hat{\tau}_y_1 = 50 \text{ Pa}$. Displacing fluid: $\hat{\rho}_2 = 1100 \text{ kg/m}^3$, $\hat{k}_2 = 0.98$, $n_2 = 0.54$, $\hat{\tau}_y_2 = 0.09 \text{ Pa}$. Dimensionless time in each picture from left to right: $t = 0.5, 15, 29.5, 44, 58.5, 73, 87.5, 102$. 

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Appendix D

Rheological and geometric effects in cementing of irregularly shaped wells

I co-authored this conference paper with A. Rentaria and I. Frigaard. The paper was presented by A. Rentaria in the 2018 Nordic Rheology Conference in Trondheim, Norway.

Abstract

Large numbers of oil and gas wells, in Canada and worldwide, allow leakage to surface from the reservoir. One common reason is associated with unsuccessful mud removal during the primary cementing operations, i.e. the drilling mud remains stuck in the narrow annular region. Several factors have to be taken into account in order to design a successful cement job; namely, the geometry of the well, the rheology of the fluids and the pumping schedule. In this study, we explore how geometric irregularity can combine with fluid rheology to give a wide range of different behaviours.
D.1 Introduction

This paper looks at both geometric and rheological effects on primary cementing displacements, for simplicity focusing on vertical wells. Primary cementing is a process carried out on every oil and gas well (typically at least twice), in which a steel casing is cemented into a newly drilled borehole. The cylindrical casing is lowered into the well creating an annular space between its outside wall and the borehole inner wall. Casings run many hundreds of metres along the well, penetrating different geological strata. A range of casing diameters are used (larger near the top of the well, smaller in production zones), but with an annular gap that is $2 - 3$ cm wide on average. The well is typically full of drilling mud, which is a shear-thinning yield stress fluid, and this must be replaced with a cement slurry to fill the annular space, where it will harden. The objectives are both to zonally isolate different fluid bearing strata in the formation and to provide mechanical support to the well. Primary cementing proceeds by pumping a sequence of fluids down the inside of the casing to bottom hole, returning upwards in the annulus from the bottom. These fluids are designed (density and rheology) to help with the removal of the drilling fluid. An overview of the cementing processes is given in Nelson and Guillot [167].

The key geometrical factor in the annular displacement is the width of the annular gap and its uniformity. Partly this is influenced by use of centralizers, discussed below, which may reduce eccentricity but also constrict locally, and partly by geological factors. The latter can related to weak formation or casing/pipe connection locations along the well, where there may be washed out sections of the annulus, i.e. enlargements or washouts. Finally, according to the geomechanical stresses and wellbore orientation the borehole may have an elliptical cross-section, rather than circular.

The challenge of primary cementing is easily understood by anyone who has cemented garden paving stones into place or grouted large floor tiles. The annular space in the well is generally eccentric, meaning that to get good coverage of cement (full removal of the mud) we need to remove the drilling mud from an annular gap that may be e.g. 5 mm wide in the narrow part of the annulus, pushing the cement slurry into the gap. Not only is the gap potentially narrow, but it also
extends for many 100’s of metres - unlike the floor tile/garden paver analogy. This is extremely difficult to achieve and over the years considerable effort has been expended to model/simulate the process in order to improve fluid designs. The model used in this work is comprehensively derived by Bittleston et al. [24], although we in fact use a modified version derived later in Maleki and Frigaard [147]. The fluid-fluid displacement problem is simplified from the full Navier-Stokes equations. The derivation uses standard scaling arguments to simplify the momentum balances and the radial dependency is then averaged along the thickness of the annulus, resulting in a two-dimensional model of the bulk fluid motions in azimuthal ($\phi$) and axial ($\xi$) directions. The narrow annulus formed by the space between the formation and the casing is conceptually unwrapped, resembling a Hele-Shaw cell with varying gap width $H(\phi)$ (see Figure D.1). The reduced model consists of a series of first-order conservation equations, for each fluid concentrations pumped, and a quasi-linear Poisson-type equation for the stream function, driven by the pump flow rate and by buoyancy gradients. Rheological effects enter into the non-linearity of the stream funcion equation.

The model that we use has been studied both analytically and computationally, as well as being used in various industrial case studies. For uniform wells the dynamics of displacement are becoming well understood, as reviewed in the next section. If the well is irregular however, there is relatively little study. Some of the

**Figure D.1:** Uniform eccentric wellbore unwrapped into a Hell-Shaw cell. Schematic from Pelipenko and Frigaard [180].
Figure D.2: Colourmaps showing the progression of two uniform eccentric annular displacements ($e = 0.3$): a) $\tau_1 = 20\text{Pa}$, $\rho_2 = 1600\text{kg/m}^3$; b) $\tau_3 = 20\text{Pa}$, $\rho_2 = 1700\text{kg/m}^3$. The length of annulus shown is 500m and time intervals are regularly spaced; W and N denote wide and narrow sides.

earliest experimental studies considered the effects of sudden expansions on the annular geometry [44, 257]. The danger of sudden expansions is to trap drilling fluid, due to its yield stress. Most relevant scientific studies of the fluid mechanics concern single phase flows. Mitsoulis and Huilgol [160] studied both planar and axisymmetric expansion flows with yield stress fluids, showing significant regions of static fluid in the corner after the expansion. This was studied further in expansion-contraction geometries, both experimentally and computationally [55, 53, 165]. Roustaei and Frigaard [191] studied large amplitude wavy walled channel flows numerically, predicting the onset of stationary fluid regions. A more comprehensive study of geometrical variation in Roustaei et al. [193] showed that yield
stress fluid becomes trapped in sharp corners and in the small scale features of the washout walls, as well as filling the deepest parts of the washout as the depth is increased. For sufficiently large yield stress and deep washouts, the actual washout geometry has little effect on the amount of fluid that is mobilized: the flowing fluid “self-selects” its flowing geometry. Considering the effects of (laminar) inertia [192], increasing the Reynolds numbers can result in a reduction in flowing area, i.e. contrary to the industrial intuition that pumping faster is better. Very recently, in Renteria et al. [190] we have been exploring the effects of washout-type irregularity in near-horizontal wells, using a combination of model simulations and lab scale experiments, performed collaboratively.

Here we focus on regular boreholes and explore the effects of varying eccentricity along the well. Eccentricity is controlled via the use of centralizers, which are devices fitted to the outer wall of the casing, designed to exert normal forces when in contact with the borehole wall. A range of centralizers exist and there is no standard geometry/mechanical design. These may be fitted every 9 – 40m along the well. The effectiveness of centralization can be inferred from logging measurements taken after the cement job. Positioning of centralizers is designed using a range of models; see Gorokhova et al. [93] for the state-of-the art. It may seem surprising that even in a vertical section of wellbore the annulus is not fully concentric: the lowest parts of the casing are in compression and the higher parts are in tension. Guillot et al. [100] review current practices, exploring 2 case studies with vertical well sections of around 500m. In both cases the vertical sections, despite frequent spacing of centralisers, show large scale variation in eccentricity (both predicted and measured).

We base our study on a vertical well of length, $L = 500$m, outer and inner radii, \( \hat{r}_o = 0.1122\text{m} \) & \( \hat{r}_i = 0.0889\text{m} \) ($\approx 9 & 7$ inches). We fix the flow rate to give a mean velocity of \( \hat{w} = 0.333\text{m/s} \) (laminar flows only). The axial coordinate, \( \hat{\xi} \), is measured from the bottom of the well. Half of the annulus is modeled, (assuming symmetry about the wide and narrow sides). The azimuthal coordinate \( \phi \) ranges from wide side (W: $\phi = 0$) to narrow side (N: $\phi = 1$).
Figure D.3: Mud displacement for uniform wells: a) $e = 0.3$, $\hat{\tau}_1 = 10\text{Pa}$, $\hat{\rho}_2 = 1600\text{kg/m}^3$; b) $e = 0.3$, $\hat{\tau}_1 = 10\text{Pa}$, $\hat{\rho}_2 = 1700\text{kg/m}^3$; c) $e = 0.6$, $\hat{\tau}_1 = 10\text{Pa}$, $\hat{\rho}_2 = 1600\text{kg/m}^3$; d) $e = 0.6$, $\hat{\tau}_1 = 10\text{Pa}$, $\hat{\rho}_2 = 1700\text{kg/m}^3$; e) $e = 0.6$, $\hat{\tau}_1 = 20\text{Pa}$, $\hat{\rho}_2 = 1600\text{kg/m}^3$; f) $e = 0.6$, $\hat{\tau}_1 = 20\text{Pa}$, $\hat{\rho}_2 = 1700\text{kg/m}^3$. Each snapshot is taken near the end of the displacement.

D.2 Displacements in uniform wells

To illustrate the simplest situations we present results of 8 simulations: 2 sets of displaced fluid yield stress, 2 density differences and 2 uniform eccentricities ($e = 0.3, 0.6$). The displaced fluid 1 (mud) has fixed properties: $\hat{\rho}_1 = 1500\text{kg/m}^3$, $\hat{\kappa}_1 = 0.1\text{Pa.s}$, $n_1 = 0.5$. We consider 2 yield stresses: $\hat{\tau}_1 = 10, 20\text{Pa}$. The displacing fluid 2 (pre-flush or cement slurry) has fixed rheological properties ($\hat{\kappa}_2 = 0.04\text{Pa.s}$, $n_2 = 1$, $\hat{\tau}_2 = 5\text{Pa}$) and we consider 2 densities $\hat{\rho}_2 = 1600, 1700\text{kg/m}^3$.

Figure D.2 shows two computed displacement flows in a modest eccentricity well ($e = 0.3$) with high yield stress mud. Initially the annulus is filled with the displaced fluid (red) representing the mud, then, the displacing fluid (blue) is pumped at a constant flow rate from the bottom. The snapshots are taken at different time intervals, regularly spaced throughout the passage of the front along the well. Particularly in Figure D.2a (with low density difference), the displacing fluid advances mainly in the wide side of the annulus, leaving behind a uniform mud layer in the narrow side. Strategies to avoid this type of unyielded zone in vertical wells have been widely studied since the 1960's[156, 122, 200, 144, 48] culminating in rule-based design systems that can be improved further with mod-
els such as that used here [181]. It is generally accepted that a positive density difference in vertical wells, aids to stabilize the interface preventing the formation of a mud channel in the narrow side. On increasing the fluids’ density difference (see Figure D.2b), the displacement front is flat and removes the narrow side mud much more effectively. Careful inspection however shows that the mud removal is not complete: residual mud partially contaminates the displacing fluid. Note too that secondary flows before/after the advancing front are responsible for dispersing the preflush ahead of the main front.

For the other 6 displacement flows we plot only a single snapshot, taken as the displacement front nears the top of the annulus; see Figure D.3. In terms of rheology, the mud’s yield stress determines the quality of the displacement on the narrow side of the annulus to a large degree, e.g. without a yield stress the static mud channel cannot form. For the smaller yield stress ($\hat{\tau}_y = 10\text{Pa}$) with $e = 0.3$, the displacements are largely effective; see Figure D.3a & b. In Figure D.3a there remains some mixed fluid on the narrow side, comparable to Figure D.2b, illustrating the competition between positive density difference and adverse rheological differences.

When the eccentricity increases (Figure D.3c-f) a bigger density difference is required to effectively displace the mud. Comparing either Figure D.3c with Figure D.3a, or Figure D.3f with Figure D.2b, we see that we have a mud channel at $e = 0.6$, but none at $e = 0.3$. Equally, comparing Figure D.2a with Figure D.3e we see the size of mud channel grows significantly with eccentricity.

Although most of the displacements shown result in poor/incomplete mud removal, the physical trends are clear: smaller eccentricity and yield stress, or larger density difference, all result in better displacements. The conditions under which unsteady/steady displacements and mud channels can arise are formally derived in Pelipenko and Frigaard [181].

### D.3 Displacements in irregular wells

To study a effects of irregularity comparatively we have constructed an annular geometry with a 300m long irregular section. The deepest 50m and the top 150m are uniform (see Figure D.4). In between we construct a sinusoidal variation in eccen-
Eccentricity with amplitude ±0.2 about fixed values \( e = 0.3 \) and \( e = 0.6 \) for the uniform sections, as illustrated. A total of 10 centralizers are placed between the two uniform sections. The distance between centralizers is 30m. In an ideal situation, the centralizer would achieve 100% standoff \((e = 0)\), but under field circumstances this is less likely. Here we assume a constant \( e = e_{\min} \) over the length of the centralizer (here 40cm).

**Figure D.4:** Eccentricity vs. depth: proposed shape for irregular wells; here \( e = 0.3 \) is the uniform value.

The same 8 displacement flows are run for the irregular geometries as for the uniform annuli (i.e. 2 densities, 2 yield stresses and 2 uniform eccentricities). Figure D.5 shows the fluid concentrations near the end of the displacements for \( e = 0.3 \). Overall, a good displacement is achieved in Figure D.5a, with reasonable removal in the uniform sections (comparable to Figure D.3a), but larger patches of mixed fluid located at the points of maximum eccentricity, in the narrow side of the annulus. On increasing the mud yield stress in Figure D.5b, these zones grow substantially, retaining static mud on the narrow side in patches that clearly follow the eccentricity variation. This compares with Figure D.2a for the uniform annulus. The width of static channel in the uniform section is very close in both simulations. For the irregular section, there is more residual mud, but the mud channel is broken periodically at the centralizer positions, which could isolate zones better than the uniform mud channel (although here the removal of the narrow side mud and zonal isolation is clearly precarious).
**Figure D.5:** Mud displacement in the irregular annulus with $\varepsilon = 0.3 \pm 0.2$: 

a) $\hat{\tau}_{y1} = 10\text{Pa}, \hat{\rho}_2 = 1600\text{kg/m}^3$; b) $\hat{\tau}_{y1} = 20\text{Pa}, \hat{\rho}_2 = 1600\text{kg/m}^3$; c) $\hat{\tau}_{y1} = 10\text{Pa}, \hat{\rho}_2 = 1700\text{kg/m}^3$; d) $\hat{\tau}_{y1} = 20\text{Pa}, \hat{\rho}_2 = 1700\text{kg/m}^3$.

**Figure D.6:** Mud displacement in the irregular annulus with $\varepsilon = 0.6 \pm 0.2$: 

a) $\hat{\tau}_{y1} = 10\text{Pa}, \hat{\rho}_2 = 1600\text{kg/m}^3$; b) $\hat{\tau}_{y1} = 20\text{Pa}, \hat{\rho}_2 = 1600\text{kg/m}^3$; c) $\hat{\tau}_{y1} = 10\text{Pa}, \hat{\rho}_2 = 1700\text{kg/m}^3$; d) $\hat{\tau}_{y1} = 20\text{Pa}, \hat{\rho}_2 = 1700\text{kg/m}^3$. 

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Figure D.7: Mud displacement for irregular well with helical geometry: $\hat{\rho}_1 = \hat{\rho}_2 = 1100 \, \text{kg/m}^3$, $\hat{k}_1 = \hat{k}_2 = 0.01$, $n_1 = n_2 = 1$, $\hat{\tau}_{y1} = \hat{\tau}_{y2} = 0 \, \text{Pa}$.

As seen in the uniform cases, increasing the density difference results in a better displacement. Figure D.5c & d show the displacement under same conditions as in Figure D.5 a & b but with the larger density difference. The mud is now removed more effectively, resulting in no mud channel, but still there are patches of contaminated displacing fluid.

The higher eccentricity irregular wellbore is evaluated in Figure D.6 for the same parameters presented in Figure D.5c-f. The eccentricity now varies from $e_{\text{min}} = 0.4$ to $e_{\text{max}} = 0.8$. In this case, there is residual mud in every case (Figure D.6 a-d). Again it appears that the mud channels are wider than for the uniform annuli, but potentially may achieve partial isolation. Notice that increasing the yield stress (Figure D.6b) makes the displacement of the mud, even at the centralizer’s position ($e_{\text{min}}$), quite challenging. In this particular scenario, when the centralizer does not position the casing to give eccentricity $e < e_{\text{min}} = 0.4$, the resulting mud channel is about the same width as that produced in the uniform case without centralization (Figure D.3e). Again, on increasing the density difference in Figure D.6c & d, the displacement is improved and the residual mud is significantly reduced.
D.4 A helical displacement

So far we have studied the effect of an irregular sinusoidal eccentricity along the well. The level of eccentricity changes at different depths while the position of the narrow side is kept fixed. Particularly in a vertical well, the azimuthal position of the narrow and wide side of the casing will not be fixed. Just for a preliminary exploration of this type of effect we have modified our uniform geometry such that the position of the wide side rotates 4 times around the wellbore over the 500m length. Thus, we have a helical eccentric pathway along the well. We now also perform the computations over a full annulus using periodicity conditions in $\phi$.

Due to space limitations only a single example is shown, as an appetizer to the complexities that will arise in a more complete study to follow. Figure D.7 shows a laminar displacement of two Newtonian fluids with identical viscosity (0.01 Pa.s) and density (1100 kg/m$^3$) along the helical channel (with $e = 0.3$). As the fluids are identical here, this is simply a dispersion example. The helical motion of the fluids is evident and the revolving eccentricity appears to result in secondary flows that improve the displacement over that expected in a uniform annulus. It can be expected that more complex rheologies and the introduction of yield stress will lead to unyielded zones and more difficult displacements.

D.5 Conclusions

We have given an overview of recent studies on fluid-fluid displacement in a long thin annulus, covering uniform and irregular geometries. In nominally vertical wells, increasing the density difference in eccentric wells aids the displacement in both, uniform and irregular geometries. However, in presence of a sufficiently high yield stress, neither the use of centralizers nor moderate density difference can prevent the development of a mud channel. The wellbore irregularity leads to a corresponding patterning of narrow side mud channels, which may in marginal cases give a degree of zonal isolation not present in uniform annuli.