DYNAMIC BEHAVIOR OF FLUID FILM BEARINGS, APPLICATIONS IN THE FLEXIBLE ROTOR INSTABILITY ANALYSIS

by

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Abstract

Fluid film bearings are widely used in many different industrial applications such as high-speed rotating machinery. Many of such rotating machinery suffer from excessive sub-synchronous whirl motion when rotating speed of the shaft exceeds "threshold speed of instability". In this situation the rotor-bearing system experiences sub-synchronous whirling known as "oil whirl/whip" which is the most common type of rotor instability. Existing nonlinear stability models fall short in predicting the nature of sub-synchronous instabilities in flexible rotor supported by journal bearings. In this thesis, linear and non-linear stability of a flexible rotor-bearing system supported on short and long journal bearings is studied for both laminar and turbulent operating conditions.

The turbulent pressure distribution and forces are calculated analytically from the modified Reynolds equation based on two turbulent models. Hopf bifurcation theory was utilized to estimate the local stability of periodic solutions near bifurcating operating points. The shaft stiffness was found to play an important role in bifurcating regions on the stable boundaries. It was found that for shafts supported on short journal bearings with shaft stiffness above a critical value, the dangerous subcritical region can be eliminated from a range of operating conditions with high static load. By increasing the Reynolds number, under shear effect assumption, stable operating region expands at high Sommerfeld numbers. The results presented have been verified by published outcomes in the open literature.

It was found that, for a specific rotor bearing system having a stiffness lower than the critical stiffness of the shaft, there exist two transition system characteristic numbers α_1 and α_2 . The operating system undergoes supercritical bifurcation for the rotor bearing system with intermediate system characteristic numbers ($\alpha_1 < \alpha < \alpha_2$); hence, to avoid hysteresis phenomenon in a rotor bearing system fluid film viscosity shall be maintained within the range of $\mu'_1 < \mu < \mu'_2$, where μ'_1 and μ'_2 correspond to system characteristic numbers α_1 and α_2 respectively. Stable operating region of flexible shafts supported journal bearings were shown to squeeze in size slightly by increasing the oil inlet pressure at low Sommerfeld numbers. Fluid film pressure distribution was found to be a strong function of oil inlet position.

Lay Summary

Multistage compressors are uses extensively in high pressure natural gas operations and liquid natural gas industries. These compressors frequently suffer from high vibrations and sub-synchronous whirl. This large vibration sub-synchronous conditions may be detrimental to the rotor-bearing system and it is normally evident at frequencies different from the shaft rotating frequency. This phenomenon is crucial for the safe operation of these multistage compressors as well as other turbomachinery. For these machines, there exists a "threshold speed of instability" after which the rotor bearing system becomes unstable in oil whirl/whip that is characterized by sub-synchronous large vibrations.

Local instability of a journal system is characterized by an operating condition that is closeenough to an operating equilibrium point. Although the load-displacement response of a journal system is inherently nonlinear, such local instabilities may be described (can be accurately estimated) by linearized bearing coefficients and parameters.

Preface

The following journal articles and articles in the proceedings of professional conferences have been published from the research work presented in this dissertation. Professor Mohamed S. Gadala and Dr. Mohammad Miraskari extensively helped with all aspects of the research work.

Chapter 2 is collected from the data presented in the following articles:

- Hemmati, F., Miraskari, M., & Gadala, M. S. (2017). "Dynamic analysis of short and long journal bearings in laminar and turbulent regimes, application in critical shaft stiffness determination." Applied Mathematical Modelling, 48, 451-475.
- Hemmati, F., Gadala, M. S., "Stability analysis of fluid film bearings under laminar and turbulent regimes." 2nd International Conference on Design and Production Engineering & Mechatronics, Automation and Smart Materials, November 13-14 (2017), Paris, France.

I derived analytical solutions for turbulent forces and generated results for optimized shaft stiffness for short and long bearings utilizing nonlinear stability analysis (Hopf Bifurcation theory). The manuscript was written in consultation with Mohamed S. Gadala. Some modifications were suggested by Mohammad Miraskari.

 Miraskari, M., Hemmati, F., Alqaradawi, M. Y., & Gadala, M. S. (2017). "Linear stability analysis of finite length journal bearings in laminar and turbulent regimes." Proceedings of the Institution of Mechanical Engineers, Part J: Journal of Engineering Tribology, 1350650117691697.

The finite length turbulent coefficients were compared against with turbulent short bearing coefficients which I derived analytically. Mohammad Miraskari derived the governing equations for finite length bearings, solved the appropriate equations, and extracted the results. The manuscript was written in consultation with Mohamed S. Gadala

Chapter 3 and 4 is collected from the data presented in the following articles:

• Hemmati, F., Gadala, M. S., "Flexible rotor-bearing system design considering fluid-film shear effect, application in safe operating region determination." International Conference

on Mechanical, Electric and Industrial Engineering (MEIE2018), May 26-28 (2018), Hangzhou, China (Keynote Conference).

I derived analytical solutions for turbulent forces of flexible rotor bearing system considering shear effect. I concluded that bearing performance is strongly dependent on lubricant viscosity, in these chapters, safe operating region for lubricant viscosity was calculated and presented. The manuscript was written in consultation with Mohamed S. Gadala.

- Miraskari, M., Hemmati, F., & Gadala, M. S. (2018). "Nonlinear Dynamics of Flexible Rotors Supported on Journal Bearings—Part I: Analytical Bearing Model." Journal of Tribology, 140(2), 021704.
- Miraskari, M., Hemmati, F., & Gadala, M. S. (2018). "Nonlinear Dynamics of Flexible Rotors Supported on Journal Bearings—Part II: Numerical Bearing Model." Journal of Tribology, 140(2), 021705.

I was the lead investigator for the project located in Chapter 5 where I was responsible for optimizing the inlet position and magnitude of fluid film for turbulent journal bearing applications. I derived equations and provided practical solutions for grooved journal bearings supported on flexible shafts. The results were compared with existing experimental data within literature. The work was done under the supervision of Mohamed S. Gadala.

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List of Symbols

L	Bearing length
М	Reduced mass of the rotor (kg)
т	Rotor central unbalance mass (kg)
r	Unbalance mass eccentricity radius (m)
С	Radial clearance (m)
R	Journal radius (m)
ε	Eccentricity ratio
0	Centre of bearing chamber
<i>O_j</i>	Centre of the journal inside the chamber
8	Gravitational constant (m/s^2)
ω	Shaft rotating speed (rad/s)
$\overline{\omega}$	Dimensionless shaft rotating speed ($\omega \sqrt{M/K_s}$)
$\overline{\omega}_s$	Dimensionless critical speed of the shaft
ω_w	Whirl rotating speed (rad/s)
$k_{ heta}$	Circumferential turbulent coefficient
k _z	Longitudinal turbulent coefficient
ρ	Fluid film density (kg/m^3)
W	Load per bearing $(\frac{Mg}{2})$
Γ	Stability parameter $\left(\frac{c}{w}M\omega_s^2\right)$
\overline{X}_i	Dimensionless coordinate in horizontal and vertical directions $(\frac{X_1}{c})$
\overline{K}_{ij}	Cartesian-coordinate dimensionless stiffness coefficients $(\pi(C/R)^3/\mu\omega L) K_{ij}$
S_z	Non-dimensional shaft stiffness coefficient, $\left(\frac{C}{W}\right)K_s$

$\bar{F}_{r,t}$	Non-dimensional forces, $F_{r,t}\left(\frac{\pi C^2}{\mu\omega LR^3}\right)$
F_R	Radial fluid force (N)
F_T	Tangential fluid force (N)
F_{x}	Radial fluid force in x direction (N)
F_y	Radial fluid force in y direction (N)
F_x^t	Total force on bearing in x direction (N)
F_y^t	Total force on bearing in y direction (N)
S	Sommerfeld number $(\frac{\mu\omega LR^3}{\pi WC^2})$
K _s	Shaft stiffness (N/m)
Ω	Whirl frequency ratio
Re	Mean Reynolds number $(\frac{\rho R \omega C}{\mu})$
Re*	Local Reynolds number $(\frac{\rho R \omega h}{\mu})$
μ	Fluid film (lubricant) viscosity (Pa.s)
\overline{k}_{ij}	Polar dimensionless stiffness coefficients $(\pi(C/R)^3/\mu\omega L) k_{ij}$
\bar{c}_{ij}	Polar dimensionless damping coefficients ($\pi(C/R)^3/\mu L$) c_{ij}
θ	Circumferential coordinate
φ	Attitude angle
h	Oil film thickness, $C(1 + \epsilon \cos(\theta))$
Р	Oil film pressure (Pa)
\bar{P}	Dimensionless oil film pressure $\left(\left(\frac{C}{D}\right)^2 \frac{8\pi}{\mu\omega}P\right)$
a _i , â _i	Turbulent constants
\bar{C}_{ij}	Cartesian-coordinate dimensionless damping coefficients ($\pi(C/R)^3/\mu L$) C_{ij}

$$\gamma$$
Dimensionless unbalance moment $\left(\frac{m}{M}\frac{r}{c}\right)$ HBTHopf Bifurcation Theory α Dimensionless Rotor Bearing Load, $\alpha = \frac{2\sqrt{2}\pi S(L/D)^2}{\sqrt{\Gamma_c}}$ $\overline{\Omega}_k$ Dimensionless threshold of instability, $\overline{\Omega}_k = \sqrt{\frac{\Gamma_c}{2}}$

 $\overline{\Omega}_k$

 K_k Shaft Dimensionless Parameter, $S_z/2$

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I wish to give my wholehearted thanks to my parents and I am also indebted to them for their unwavering support, loving and care, and their persistent encouragement throughout my M.A.Sc and Ph.D.

Dedication

To My Lovely Parents

Chapter 1: Introduction

1.1 Background and Motivation

Condition monitoring is the process of monitoring the current condition and predicting the future condition of machines while in operation. Condition monitoring of heavy rotating machinery and equipment such as turbines, compressors and generators, is gaining importance in various industries since it keeps the plant in healthy condition for maximum production; helps in detecting faults at initial stages; avoids serious accidents and damage; and reduces downtime. The original dynamic problem of rotor bearing systems was associated with approaching the excitation frequency of the shaft to the natural frequency of the system. This problem was first addressed and analyzed by Rankine [1] concluding rotor bearing machines become unstable after first critical speed of the system. Later it was proved that rotor bearing models could become stable at speeds higher than first critical speed of the system through inclusion of Coriolis force in the equations of motion [2].

In the field of rotor bearing system analysis, the importance of fluid film bearings was not recognized until the work of Reynolds in 1886 that provided general mathematical equations of fluid films within a bearing chamber. In 1919, Jeffcott [3] provided mathematical proofs that rotor bearing systems can operate at supercritical speeds. The Jeffcott model is illustrated in Figure 1.1. In 1920s the major issue of fluid film bearing instabilities (oil whip/whirl) was first addressed by Newkirk and Kimball [4, 5].



Figure 1.1. Jeffcott rotor bearing system.

1.1.1 Fluid-Film Bearings

The first studies of rotor supported bearings under full hydrodynamic conditions were performed by F.A. von Pauli in 1849 and by G.A. Hirn in 1854 [6]. In 1883 a Russian researcher, Nikilay Petroff, concluded that hydronic phenomenon in fluid film bearings could cause friction. His resulting power loss (friction loss) equation has continued to provide a foundation in this field. Evidence of experimental pressure generation within fluid film journal bearings was reported by Beauchamp Tower in 1883 [6]. Based on Tower's outcomes, Osborne Reynolds in 1886 developed a mathematical expression for pressure profile within a journal bearing chamber that has become the fundamental expression of hydrodynamic analysis of bearing performance. A schematic of a journal bearing and a journal bearing chamber is shown in Figure 1.2.

Since Reynolds equation is a second order partial differential equation, it's direct solution was found to be cumbersome at that time; thus, Arnold Sommerfeld [7] in 1902 developed a simplified direct integration for infinitely long journal bearings. In 1949 Cameron and Wood [8] provided an extremely efficient extension of Reynolds equation for finite-length journal bearings through a relaxation procedure carried out with a mechanical desktop calculator. Intial studies related to numerical solutions of the Reynolds equation for journal bearing performance analysis was done by Oscar Pinkus [9], Albert Raimondi [10], and John Boyd [11].

Newkirk and Taylor [12] were among the first researches to relate the bearing induced instability to the oil film bearings. They realized that at a speed twice the critical speed of a rotor bearing system, machine's instability is due to sub-synchronous whirl motion; however, they could not successfully explain the phenomenon. Fundamental characteristics of journal bearing oil whip phenomenon was properly addressed by Hori [13] in 1959, through shaft stability analysis under induced pressures of journal bearings.



Figure 1.2. Schematic of a complete set of a journal and a journal bearing chamber.

1.1.2 Rotor Bearing System Models

In 1933 D.M Smith [14] provided a more advanced bearing model compare to Jeffcott system rotor bearing system. He went far beyond Jeffcott's work and concluded that, even though, damping in bearing supports improve system stability at speeds below the critical speed, unstable boundaries observed to happen earlier at speeds above critical speed of the system. A simplified approach for flexible rotor bearing critical speed calculation was introduced in 1945 by Prohl [15] through discretizing a rotor into series of lumped masses at different locations known as "transfer matrix" method. Many researchers modeled flexible rotor bearing systems as circular, flexible beams with two concentrated masses located at bearing points. They considered bearings as rigid points until the work of Hagg and Sankey [16] in 1956 that considered journal bearings as radial springs and dampers.

Raimondi and Boyd in 1958 [11] were first attempted to solve Reynolds equation numerically to calculate pressure profile within journal bearing chambers. Later in 1966, J.W Lund [17] defined linearized fluid-film bearing force coefficients and he was the first to introduce bearing stiffness and damping coefficients (including direct and cross-coupled coefficients). A modified Jeffcott model with journal forces modeled as spring and dampers is shown in Figure 1.3.

Even though most of the rotor bearing systems being used before the 1940s were generally operating at low speeds, the development of gas turbines for aircraft propulsion, as well as the need for centrifugal machines for petrochemical gas compression, brought many more

applications at a variety of speeds in turbomachinery. Design of multistage compressors became important as the demand for high pressure injection of natural gas in underground became necessary and the major cause of failure of those compressors was due to violent subsynchronous whirl motion [2].



Figure 1.3. Schematic of a complete set of a journal and a journal bearing chamber [18].

Major catastrophic failures in rotor bearing system, especially industrial compressors, were lead to development of computer simulations/models of rotor dynamic instability. The major breakthrough was done by Lund in 1974 [19] developing an algorithm as an extension to Myklestad-Prohl method (which only calculated the critical speeds). His mathematical model included damping and cross coupled force coefficients in the rotor bearing system design. He also calculated the damped eigenvalues of the rotor bearing system.

1.1.3 Previous Research on the Fluid Film Bearing Instability Analysis

There are numerous studies on the journal bearing stability analysis. A substantial portion of the existing research in the literature is concerned with only linear stability analysis of rotor bearing system under laminar fluid flow assumption. However, the number of studies concerned with nonlinear stability analysis of flexible rotor bearing systems under turbulent flow considering cavitation and shear force fluid film effect is limited.

1.2 Literature Review

1.2.1 Stability Analysis with Bifurcations in Dynamic Response of Journal Bearings

While stable and unstable regions of rigid and flexible shafts supported on journal bearings is widely studied in the literature, they fall short on both utilizing the precise pressure distributions including all the external effects, e.g., turbulence, cavitation, shaft stiffness and damping, rotary inertia and gyroscopic moments. Since the precise identification of the threshold speed of instability is very sensitive to the external forces acting on the system, one cannot simply rely on laminar fluid film bearing on a rigid shaft. Proper implementation of actual pressure distribution along with other forces involved in a rotating system such as rotary inertia, the gyroscopic forces, rotor weight, rotor stiffness and damping, the presence of unbalance are essential to achieve a correct estimate of the threshold speed ω_s .

When the rotor operating speed exceeds the threshold speed of instability, the whirl amplitude can become unbounded either gradually (supercritical bifurcation) or suddenly (subcritical bifurcation) [20]. In the case of subcritical bifurcation; a small perturbation from the equilibrium point may lead to instability even when the rotor speed is under the threshold limit. Thus, subcritical bifurcations are more dangerous than the supercritical bifurcations and proper considerations are needed in design stage to assure the operation within the supercritical region [20].

In the classical paper by Newkirk and Taylor [21] on dynamic effects of oil whip phenomenon, it was reported that at running speeds of lower than the threshold speed, violent rotor whipping can occur due to small shock applied to the system. This was later confirmed by Hori [22] who did extensive research on how an earthquake (a shock) can affect the occurrence of the oil whip [23, 24]. As an attempt to explain the shock effects on occurrence of oil whip, Khonsari and Chang [25] introduced the concept of stability envelope (R_s which its shape and size depends on the operating conditions) as a distinct region of stability within a closed boundary that encircles the steady-state equilibrium position. Otherwise, the orbit will settle into a steady-state equilibrium position. If released from outside the R_s , the orbit will grow larger and larger until the system reaches the so-called whip condition where the orbit extends to the clearance circle of the rotor-bearing system, endangering the system's operation. To find the stability envelope, Khonsari and Chang [25] proposed a trial and error method using bisection method at each specified attitude angle of the journal. An alternative method for obtaining the stability envelope was proposed by Wang and Khonsari [26] for short journal bearings by obtaining the periodic solutions based on the Hopf bifurcation theory.

Noah and Sundararajan [20] applied the Hopf bifurcation theory to a rotor supported on finite length journal bearings. They used impedance descriptions originally proposed by Childs et al. [27, 28] to obtain approximate analytic expressions for the fluid film forces in journal bearings. It was shown that in the case of supercritical bifurcations, there is a gradual transition from stability to instability. On the other hand; in the case of subcritical bifurcation, stable journal orbits may exist above and below the threshold limit and no gradual transition from stability to instability occurs. Ding et al. [29] applied the Hopf bifurcation theory to investigate the balanced and unbalanced dynamics of a symmetric rotor-seal system based on Muszynska's [30] nonlinear seal model. Implementing algebraic criterion proposed by Poor [31], they showed that only supercritical region exists for perfectly balanced systems.

Wang and Khonsari [32] showed that, for short bearings, turbulence tends to deteriorate the stability of the rotor-bearing system and as the Reynolds number increases in the bearing fluid film, the stability threshold speed decreases especially with small Sommerfeld numbers. Wang and Khonsari [33], based on the application of Hopf bifurcation theory, showed that the type of bifurcation in rotor-bearing systems can change from subcritical to supercritical through changing the oil viscosity. They also reported that rotor stiffness has a pronounce influence on system's stability threshold speed and its bifurcation type and should not be neglected.

The threshold speed analysis in linear theory based on short or long bearing approximations, suggests unstable operation after passing the threshold speed. While oil whirl does occur after passing the threshold speed, it is not unstable, and the system can still be pushed to the higher operating speeds which requires nonlinear stability analysis due to nonlinearity of fluid film forces. Another objective of this research is to accurately measure the stability boundaries of a rotor-dynamic system supported on two journal bearings without solving the complete Reynolds equation, considering turbulence along with cavitation effects, and to validate the theories with experimental measurements.

1.2.2 Importance of Turbulence and Fluid Film Rupture (Cavitation) in Journal Bearing Pressure Distributions

Cavitation bubbles may come from within the lubricating film, or be fed by the environment, and they represent a rupture in the continuity of the liquid film. This discontinuity at the bubble interface presents a challenge in the context of the solution to the Reynolds equation. The solution offered by Sommerfeld [34], did not take into account film rupture and solved for a full film around the circumference demonstrated in Figure 1.4(a). This approach allowed sub-atmospheric and even negative pressures, no matter how low they were [35]. Gümbel [36], was first to report that for a steadily loaded bearing operating at constant angular velocity, the rupture originates in the immediate vicinity of the film's minimum clearance, at a predetermined pressure p_{cav} and remains constant at that value for the entire divergent region, Figure 1.4(b). He did not account for film reformation, nor did his approach respect mass continuity. Because the circumferential pressure distribution allows only the positive pressure region, the Gümbel condition is also known as the 'half-Sommerfeld' condition (Figure 1.4(b)).

A better alternative to the 'half-Sommerfeld' condition was offered by Swift [37] and Stieber [38], shown in Figure 1.4(c). Swift [37] in 1932 stated that a zero derivative of the pressure is an appropriate condition for marking the inception of cavitation and considered it to be a 'stability condition'. Stieber [38] in 1933 published a full solution for a 360° journal bearing considering cavitation for a zero tensile strength lubricant (Floberg later offered different approaches for the zero and non-zero tensile stress in lubricant's cavitation [39]).

Stieber, like Swift, assumed a zero-pressure gradient at the start of the cavitation zone, but considered it to be a continuity condition. These two forefathers of cavitation modelling, while approaching it from different angles, came to the same conclusion regarding the inception and development of cavitation. The cavitation zone formation conditions known as the Swift–Stieber conditions which is demonstrated in Figure 1.4(c) are:

$$\frac{\partial P}{\partial \theta} = 0, \qquad p = p_{cav}$$
(1.1)

Note that these conditions consider the entire cavitation zone at $p = p_{cav}$ and do not make allowance for the existence of sub-cavitation pressures nor for variations in the pressure inside the cavitation zone. A constant cavity pressure imposes a zero-pressure gradient within the cavity.

The pressure gradient calculated in the fluid has to interface with the cavity pressure gradient along the boundary, where at certain locations there can be a relatively steep slope (e.g. the point of cavity inception). According to Floberg [39], these boundary conditions work well at moderate loads. According to Brewe et al. [40], while the Swift-Stieber conditions work reasonably well for the establishment of the film rupture, they do not predict the reformation of the film satisfactorily; nor do they seem to give satisfactory results in the case of dynamically loaded bearings [40, 41].



Figure 1.4. Circumferential pressure development based on fluid film theories: (a) Sommerfeld, (b) Gümbel, (c) Swift-Stieber, and (d) JFO and Floberg [42].

The more elusive problem in dealing with film rupture has been respecting mass continuity within the confines of the Reynolds equation. Gümbel's condition [36] did not respect the continuity of mass while the more advanced Swift-Stieber model [37, 38] accounts for the entire mass of liquid transported in between (see Figure 1.5) the gas cavities, but does not take into account the gas cavities' number, size, and volume. Unlike Swift-Stieber, the separation models of Coyne and Elrod [43, 44], Mori et al. [45], Hopkins [46], and Birkhoff and Hays [47] provide

for the fluid transport above and beneath the bubble. All these models use continuity of mass to define the cavity interface. However, in spite of their success, none of them were able to handle well the situation of a moving liquid/gas boundary where mass was conserved both in the cavitation region as well as on the boundary.

Floberg [39], Jakobson and Floberg [48], Olsson [49], and Floberg [50] again assumed, similarly to Swift and Stieber, a striated flow (shown in Figure 1.5), where the liquid was transported in between gas cavities that extended fully across the clearance between the stationary and the moving surfaces. These contributions, made by the respective authors independent of each other, are referred to as the Jakobson, Floberg and Olsson (JFO) cavitation theory [40, 51] demonstrated in Figure 1.4(d). JFO model includes sub-cavity pressure and mass conservation across the entire bearing based on Reynolds equation, in agreement with experiments; however, it is difficult to implement in analytical and numerical schemes.



Figure 1.5. Film rupture striated gas and liquid streams (Swift-Stieber and JFO) boundary conditions.

Instability analysis of large diameter bearings cannot be accurately achieved when using classical lubrication theory (i.e. Reynolds equation). The reason for this is mostly because of two underlying assumptions to derive Reynolds equation: the assumption of laminar flow and the phenomenon of film rupture. Liquid cavitation in fluid film bearings is not only important because its onset and extent determine the load capacity of a fluid film bearing, but also because vapor cavitation collapse (implosion) can cause severe surface material damage. Furthermore, in dynamically loaded bearings, the appearance of cavitation largely influences the rotor dynamic stability of a rotor-bearing system and its maximum whirl amplitude of vibration.

To correctly predict the bearing performance required for precise identification of instability phenomenon, both the flow regime (being laminar or turbulent) as well as the cavitation effect need to be taken into account. In turbulent flow, separation effect contributes to the film rupture (cavitation) which causes negative pressure in fluid film pressure distribution. The higher rate of separation in turbulent flow can lead to higher amplitude of negative pressure and hence lower load capacity of the journal. On the other hand, higher momentum in turbulent flow leads to increasing load capacity of the journal. There are thus two competing effects both influencing the oil pressure and the result may only be predicted by means of simultaneously considering the turbulent effects as well as precisely calculating boundary conditions in the Reynolds equation (corrected for turbulent motion).

While stability of rotor-bearing systems is studied in the literature, the bulk of the work is based on laminar short and long bearing approximations in which analytical and semi-analytical expressions exist. A complete rotor bearing system consisting of a flexible shaft supported on short/long journal bearings with considerations of both cavitation and flow regime (being laminar or turbulent) has not been studied yet and is the main goal of this thesis.

1.2.3 Short and Long Bearing Pressure Calculations

The first step towards characterization of the oil whirl instability is to identify the external forces exerted on the shaft at the journal bearings supports. To do so, the oil induced pressure needs to be obtained by solving the Reynolds partial differential equation (Equation (1.2)).

$$\frac{1}{R^2} \frac{\partial}{\partial \theta} \left(\frac{h^3}{k_{\theta} \mu} \frac{\partial P}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(\frac{h^3}{k_z \mu} \frac{\partial P}{\partial z} \right) = \frac{\omega}{2} \frac{\partial h}{\partial \theta} + \frac{\partial h}{\partial t}$$
(1.2)

If only the first term in the LHS of Equation (1.2) is kept; an approximation which holds only for journal to length ratio of $L/D \ge 2$; solving the resulting ODE yields the long bearing approximation pressures. Keeping the second term only, which holds for $L/D \le 0.5$, yields the pressure based on the short bearing approximation. Since in most industrial applications, short and long bearing analytical forces provide close approximations to finite length bearings, this project provides a thorough investigation on precisely measuring bearing forces, including turbulence and cavitation, to have a better approximation of stability margins.

1.2.4 Dynamic Stiffness and Damping Coefficients

Even though nonlinear investigations of stable whirl orbits are proven to be vital for precise stability analysis, the well-developed area of linear stability analysis based on realistic stiffness and damping coefficients may still be useful to study system lateral vibrations. The dynamic coefficients-based vibration analysis is significantly less time expensive and is capable of providing some estimates of stability boundaries. The linearized coefficients are evaluated at the equilibrium position of the journal and are commonly calculated either by the infinitesimal approach of Lund [52] or the finite perturbation approach as in Qiu and Tieu [53].

1.3 Research Objectives

The main objectives of this thesis are as follows:

- Accurate estimation of the stability boundaries of the rotor bearing systems through characterization of the oil induced instability phenomenon in rotating machines supported on oil film journal bearings.
- To study the rotor bearing instability for a comprehensive model under industrial conditions of variant viscous lubricant under turbulent regimes of flow of a short and long journal bearings through Analytical, and Experimental Analysis (for comparison).
- To study the influence of fluid shear force on the linear and nonlinear stability analysis of a flexible rotor bearing system.
- Precise estimation of rotor bearing hysteresis phenomenon and design recommendation for optimal fluid film viscosity (oil inlet temperature).
- To study the phenomenon of liquid cavitation in dynamically loaded fluid film bearings, considering turbulent effects, and the selecting the most adequate boundary conditions at the inception and reformation boundaries of cavitation zone.
- Studying the response of a short/long cylindrical journal bearing, considering turbulence and cavitation, operating close to the critical speed of stability boundary.

- To study the effect of oil inlet position and magnitude on the stability boundary and providing optimal design parameters.
- To improve the safe operating region of flexible rotor bearings supported on fluid film journals by optimization of oil inlet position.

1.4 Scope of the Present Work

Investigation of flexible rotor bearing stability analysis is of particular interest in current studies. This thesis touches on four main areas of fluid film bearing instability phenomenon. These areas are the fluid film turbulence effect, fluid film shear force effect, precise estimation of rotor bearing hysteresis phenomenon, and the effect of oil inlet pressure and position on the rotor bearing stability boundary considering cavitation effect. A schematic diagram explaining the project relevant areas is presented in Figure 1.6.



Figure 1.6 Schematic diagram of principal areas of a flexible rotor bearing stability analysis.

In a completely balanced rotor-bearing system rotating at low rotational speeds, the only mode of motion is the rotor spinning at its designed operating speed. At the equilibrium position, the journal bearing produced pressure within journal bearing chamber can support the weight of the shaft. This point is referred to as the static equilibrium position or the steady state point of the rotor bearing system. If the rotor bearing system is not disturbed, the system remains at its steady state operating point. A small unbalance force however, will push the shaft to orbit around the steady state point. Even though any other system harmonics beyond the desired spinning is generally undesirable, small orbits are usually endured and do not endanger the safe operation of the system. Upon increasing the rotating speed of the shaft, and after crossing threshold speed of instability, high amplitude vibrations can suddenly occur even for a fully balanced rotor. This occurrence of high amplitude vibrations is known as "oil whirl/whip" phenomenon and needs to be accurately predicted prior to its occurrence.

This thesis is presented in five (5) chapters,

Chapter 2 is devoted to derivation of an analytical expression to accurately estimate the turbulent short and long bearing forces and turbulent dynamic coefficients for the whole range of operating conditions. These coefficients and forces are then used as basis of linear and nonlinear stability analysis of a flexible rotor-bearing system. In this chapter short and long bearing theory for the dynamic characteristic problems of turbulent fluid-film journal bearings on a flexible shaft is described. The dynamic oil film forces are analytically obtained utilizing Constantinescu and Ng-Pan-Elrod turbulent models under the short and long bearing assumptions with Gümbel's boundary condition. The closed form expressions for eight spring and damping coefficients are derived by linearizing the oil film forces are determined by the linear stability criterion. For nonlinear stability analysis, Hopf bifurcation theory is used to find the local stability of periodic solutions near bifurcating operating points. Critical shaft stiffness has also been identified in this chapter for both short and long bearing assumptions.

Chapter 3 is concerned with drag force effects on the threshold speed of instability of a flexible rotor bearing system under turbulent assumption for the entire operating condition. Nonlinear stability analysis, using Hopf bifurcation theory, has been utilized to obtain bifurcation profiles for short journal bearings. Experimental verifications presented to confirm the effectiveness of the proposed mathematical mode. Safe operating range of dimensionless unbalance moment (γ) is also presented.

Chapter 4 presents application of Hopf bifurcation theory (HBT) to find the local stability of periodic solutions near bifurcation operating points. In this chapter, it is shown that, there is a good agreement between predicated bifurcation profile, utilizing proposed analytical method, and experimental results provided by Wang and Khonsari [54]. Hysteresis phenomenon can also be effectively predicted utilizing proposed method. The fluid film oil viscosity is found to play

an important role in bifurcating regions on the stable boundaries. Safe operating fluid film viscosity range is proposed for rotor bearing system design purposes.

Chapter 5 is devoted to implementation of cavitation boundary condition under turbulent flow assumption. In order to correctly predict the bearing performance, the flow regime (being laminar or turbulent) and cavitation region need to be taken into account. Therefore, it is important to determine the turbulence and cavitation effects on the dynamic fluid forces and operating stability margin of journal bearings for reliable operation of rotor-bearing systems. The purpose of this chapter is first to derive an analytical expression to accurately estimate the turbulent forces and turbulent dynamic coefficients, including cavitation effect, for the whole range of operating conditions. These coefficients and forces are then used as basis of linear stability analysis of a flexible rotor-bearing system. The dynamic oil film forces are analytically obtained utilizing Ng-Pan-Elrod turbulent model under long bearing assumptions with Reynolds–Floberg–Jakobsson (RFJ) boundary condition. The closed form expressions for eight spring and damping coefficients are derived by linearizing the oil film forces around the steady-state equilibrium position of the journal center, and the whirl onset velocities are determined by the linear stability criterion. By having a complete rotor bearing model, oil inlet pressure and position influences on the rotor bearing instability have been analyzed thoroughly.

Chapter 6 presents the summary and conclusions. This chapter also provides suggestions for future work on this subject.

Chapter 2: Dynamic Analysis Journal Bearings in Laminar and Turbulent Regimes: Application in Critical Shaft Stiffness Determination

2.1 Introduction

The demand for high pressure injection of natural gas in underground has led to the design of multistage compressors. In 1970's many of such compressors were suffering from violent sub-synchronous whirl [55], a form of self-excited instability. In turbo-machinery, instabilities are characterized by whirling of the rotor bearing systems at frequencies other than the rotating frequency of the shaft. While large amplitudes of sub-synchronous vibration do not occur frequently, they can appear at certain operating conditions and can lead to high amplitude and destructive vibrations. If the operating speed of rotors exceeds a threshold speed known as "threshold speed of instability", the rotor-bearing system becomes unstable in oil whirl/whip which is characterized by sub-synchronous whirling [56]. Linearized stiffness and damping coefficients (dynamic coefficients) are used as basis for stability analysis of the rotor bearing systems. In the linearized analysis, bearing dynamic coefficients are evaluated at the equilibrium position of the journal.

While the load-displacement curve of a journal bearing is evidently nonlinear, the bearing behavior and stability can be characterized by means of linear dynamic coefficients if certain conditions are met. It is known that the "local" stability of a non-linear system and its linearized counterpart are essentially the same even though their stability type could be different. By local, one means that the current operating condition is close enough to an operating equilibrium point, a condition that is also vital for validity of utilizing linearized bearing coefficients to represent the journal force in rotor-bearing analysis. Choy et al. [57] calculated nonlinear bearing stiffness coefficients and showed that for displacements sufficiently far away from the equilibrium position, oil film forces will exhibit nonlinearities. San-Andres and Santiago [58] experimentally determined the dynamic coefficients under high dynamic loads with large orbital motion of up to 50% of the bearing clearance. Their results are in good agreement with analytical linearized coefficients. Meruane and Pascual [59] estimated the linear and nonlinear bearing coefficients under large orbital motion and even during oil whirl. They showed that the linearized analytical coefficients agree reasonably with linear coefficients estimated

numerically considering a nonlinear model and under large orbital motion. Small variation between the linear and nonlinear model were reported provided that the operating speed is kept below the instability threshold speed. Muzakkir et al. [60] concluded, based on experimental results, that high viscosity lubricants for heavily loaded slow-speed journal bearings, in laminar regime flow, would improve bearing stability; however, no stability boundary region were provided. Lahmar et al. and Singh et al. [61, 62] provided pressure distribution and bearing coefficients for thrust and compliant journal bearings under laminar flow assumption and no discussion were provided at high velocities were fluid film behaves turbulent.

All of the abovementioned papers were based on the declaration that fluid-film flow remains laminar. Due to the high demand of large bearings running at high rotating speeds and also using low kinematic viscosity of lubrications, the fluid film region is turbulence. Instability analysis of large bearings cannot be accurately achieved when using classical lubrication theory (i.e. Reynolds equation). The reason for this is mostly because of two underlying assumptions to derive Reynolds equation: the assumption of laminar flow and constant viscosity in the oil film [63]. To correctly predict the bearing performance required for precise identification of instability phenomenon, the flow regime (being laminar or turbulent) needs to be taken into account. Therefore, it is important to determine the turbulent effects on the dynamic fluid forces and operating stability margin of journal bearings for reliable operation of rotor-bearing systems. Research dealing with the effect of turbulence on performance of fluid-film journal bearings has been primarily limited to steady-state, with the exception of a series of notable contributions by Hashimoto et al. [64, 65] and Wang et al. [32] who investigated dynamic effects using analytical methods for short bearing theory. However, their simplifying assumptions added to the Ng-Pan-Elrod model in order to analytically derive the short bearing forces leads to high errors at large eccentricity ratios and high Reynolds numbers. Cai-Wan Chang-Jian et al. [66, 67] investigated the dynamics of a flexible rotor bearing system supported by two turbulent journal bearings; however, their treatment in deriving the analytical turbulent force components leads into erroneous results.

The purpose of this chapter is first to derive an analytical expression to accurately estimate the turbulent short and long bearing forces and turbulent dynamic coefficients for the whole range of operating conditions. These coefficients and forces are then used as basis of linear and

nonlinear stability analysis of a flexible rotor-bearing system. In this chapter short and long bearing theory for the dynamic characteristic problems of turbulent fluid-film journal bearings on a flexible shaft is described. The dynamic oil film forces are analytically obtained utilizing Constantinescu and Ng-Pan-Elrod turbulent models under the short and long bearing assumptions with Gümbel's boundary condition. The closed form expressions for eight spring and damping coefficients are derived by linearizing the oil film forces around the steady-state equilibrium position of the journal center, and the whirl onset velocities are determined by the linear stability criterion.

For nonlinear stability analysis, Hopf bifurcation theory is used to find the local stability of periodic solutions near bifurcating operating points. Results show minor difference between laminar and turbulent stability boundaries at high Sommerfeld numbers, while significant reduction in the size of the stable region was observed at low Sommerfeld numbers. This reduction in the size of stable region is further increased at high Reynolds numbers. The shaft stiffness was found to play an important role in bifurcating regions on the stable boundaries. The results predicted based on the flexible rotor model have been verified by results provided by Khonsari and Wang [33]. The critical shaft stiffness was introduced for the first time, at the point of transferring two bifurcation regions to three bifurcation regions. This critical shaft stiffness was shown to be a strong function of Reynolds at low Reynolds numbers for both short and long bearing supported shafts. The stability envelope in subcritical region of operation was calculated based on the Hopf bifurcation theory and was shown to reduce in size as the flow regime is transitioned from laminar to turbulent.

2.2 Governing Equations and Turbulent Lubrication Models

Figure 2.1 (a) shows a schematic of a journal bearing inside the bearing clearance. A simple rotor-bearing system can be represented as a concentrated mass supported on journal bearings modeled as direct and cross coupled spring and dampers as shown in Figure 2.1 (b) (cross coupled coefficients are not shown).


Figure 2.1. (a) Schematic of a journal bearing, and (b) journal bearings fluid film model using linear stiffness and damping coefficients

According to Figure 2.1 (a), under static equilibrium, the weight of the rotor and journal is in balance with the bearing force such that $\vec{F} + \vec{W} = 0$ and hence relative to the static equilibrium state, the x and y components of the dynamic deviation of bearing force upon the rotor can be expressed as follows:

$$F_{x} + W_{x} = F_{x}^{t} = \frac{\partial F_{x}^{t}}{\partial x}x + \frac{\partial F_{x}^{t}}{\partial \dot{x}}\dot{x} + \frac{\partial F_{x}^{t}}{\partial y}y + \frac{\partial F_{x}^{t}}{\partial \dot{y}}\dot{y} + (higher order terms)$$

$$F_{y} + W_{y} = F_{y}^{t} = \frac{\partial F_{y}^{t}}{\partial x}x + \frac{\partial F_{y}^{t}}{\partial \dot{x}}\dot{x} + \frac{\partial F_{y}^{t}}{\partial y}y + \frac{\partial F_{y}^{t}}{\partial \dot{y}}\dot{y} + (higher order terms)$$
(2.1)

It is convenient to write Equation (2.1) in the following matrix form:

$$\begin{cases} F_x^t \\ F_y^t \end{cases} = - \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} \begin{cases} x \\ y \end{cases} - \begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix} \begin{cases} \dot{x} \\ \dot{y} \end{cases}$$
(2.2)

Where $k_{ij} \equiv -(\partial F_i^t / \partial x_j)$ and $c_{ij} \equiv -(\partial F_i^t / \partial \dot{x}_j)$ are the eight bearing stiffness and damping coefficients as proposed by Lund [68]. The state of the flow in fluid-film journal bearings can be judged by its Reynolds number. Turbulence comes into an effect at mean Reynolds number (\overline{Re}) of approximately 2000 and higher [69]. For unsteady, incompressible turbulent flows in the oil film journal bearings as shown in Figure 2.1, the Reynolds equation in the cylindrical coordinate system can be written as:

$$\frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\frac{h^3}{k_{\theta}\mu}\frac{\partial P}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(\frac{h^3}{k_{z}\mu}\frac{\partial P}{\partial z}\right) = \frac{\omega}{2}\frac{\partial h}{\partial\theta} + \frac{\partial h}{\partial t}$$
(2.3)

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where h is the fluid film thickness and k_{θ} and k_z are the turbulence coefficients in circumferential and longitudinal directions respectively. In this chapter two turbulence methods are employed. First is the Constantinescu's model based on the Prandtl mixing length hypothesis [70, 71]; second, is the Ng-Pan-Elrod model based on eddy viscosity [70, 72]. Frene et al. [73] mentioned Constantinescu's approach is valid for $2000 \le \overline{Re} \le 10^5$. Durany et al. [74] stated that Constantinescu's approach is effective for $1000 \le \overline{Re} \le 10^5$. Furthermore, Constantinescu mentioned his results are in good agreement with Fuller's experiment for $2000 \le \overline{Re} \le 5 \times 10^4$ in his paper [71]. The turbulence coefficients along with their constants are presented in Equation (2.5) and Table 2.1.

$$h = C(1 + \epsilon Cos(\theta)) \tag{2.4}$$

$$\begin{cases} k_z = 12 + a_1 (Re^*)^{a_2} \\ k_\theta = 12 + \hat{a}_1 (Re^*)^{\hat{a}_2} \end{cases}$$
(2.5)

where $Re^* = \frac{\rho R \omega h}{\mu} = \frac{\rho R \omega C(1 + \epsilon Cos(\theta))}{\mu}$ is the local Reynolds number and varies with θ . Under conventional laminar flow assumption, k_z and k_{θ} approach 12. The local Reynolds number adds an extra nonlinearity term to the classical Reynolds equation which makes it impossible to derive an exact analytical expression for short and long bearing approximations. Within the next section, assumptions to analytically derive accurate fluid-film forces for short and long bearing approximations are explained in detail.

Constants	Turbulent Models	
	Constantinescu Model	Ng-Pan-Elrod Model
a_1	0.0198	0.0044
<i>a</i> ₂	0.741	0.96
\hat{a}_1	0.026	0.0136
\hat{a}_2	0.8265	0.9

Table 2.1. Turbulent constants of Constantinescu and Ng-Pan-Elrod model

It has to be mentioned that, the turbulence coefficients should be considered in the Reynolds equation when the mean Reynolds value (\overline{Re}) is above 2000. The variation of k_{θ} and k_z with Re^* is shown in Figure 2.2. As it can be seen from Figure 2.2, the discrepancy between Constantinescu's approach and Ng-Pan-Elrod model increases by increasing the local Reynolds

number. This reason for the difference lies in Constantinescu's turbulent model, where buffer zone is left untreated and non-planar flows have not been carefully modeled. Also accurate measurement of the mixing length constant for fluid-film bearings with small clearances is a big challenge in Constantinescu's model while most of the mentioned problems are addressed in Ng-Pan Elrod turbulent formulation [70].



Figure 2.2. Variation of turbulence constants k_{θ} and k_z with local Reynolds number.

2.3 Analytical Derivation of Dynamic Turbulent Fluid-Film Forces

2.3.1 Turbulent Short Bearing Theory Formulation

From Equations (2.3) and (2.4), and considering $\dot{\theta} = -\dot{\phi}$ [75], turbulent Reynolds equation may be written as:

$$\frac{1}{R^2} \frac{\partial}{\partial \theta} \left(\frac{h^3}{k_{\theta} \mu} \frac{\partial P}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(\frac{h^3}{k_z \mu} \frac{\partial P}{\partial z} \right) = 0.5C \left((2\dot{\varphi} - \omega) \epsilon Sin(\theta) + 2\dot{\epsilon} Cos(\theta) \right)$$
(2.6)

By short bearing approximation $(L/D) \le 0.5$, it may be assumed axial pressure flow dominates circumferential pressure flow [76]. Since the first term in Reynolds' equation disappears, updated turbulence Reynolds' equation for turbulent short bearing theory can be written as:

$$\frac{\partial^2 P}{\partial z^2} = \frac{k_z \mu C}{2h^3} \left((2\dot{\varphi} - \omega)\epsilon Sin(\theta) + 2\dot{\epsilon}Cos(\theta) \right)$$
(2.7)

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By applying the boundary conditions $(z = \pm \frac{L}{2} \rightarrow P = 0)$, turbulence short bearing pressure may be written as:

$$P = \frac{k_z \mu C}{16h^3} \left((2\dot{\varphi} - \omega)\epsilon Sin(\theta) + 2\dot{\epsilon}Cos(\theta) \right) (4z^2 - L^2)$$
(2.8)

The comparison of non-dimensional static pressure distribution at the journal mid-span (z = 0) for laminar and turbulent models at two eccentricity ratios are shown in Figure 2.3 from 0 to π , since Gümbel boundary condition is used to produce bearing forces.



Figure 2.3. The comparison of analytically obtained non-dimensional pressure distributions at mid-span of a short journal bearing with L/D = 0.5 for laminar and turbulent flows at (a) $\epsilon = 0.5$, and (b) $\epsilon = 0.8$

where the dimensionless pressure $\overline{P} = P 8\pi (C/D)^2/\mu\omega$. The corresponding pressure distributions are evaluated for two different cases. First for relatively low eccentricity ratio of $\epsilon = 0.5$ and high eccentricity ratio of $\epsilon = 0.8$, as shown in Figure 2.3(a) and (b) respectively. Results indicate that, as Reynolds number increases at higher loading conditions, the predicted pressure distributions of the modified Reynolds equation deviates significantly from the pressure as predicted by classical Reynolds equation (Laminar bearing theory). As it is shown in Figure 2.3(a) and (b), Ng-Pan-Elrod fluid film turbulence model is more conservative compare to Constantinescu's model due to its higher evaluation of fluid-film pressure. The resulting fluid film turbulent forces acting on journal bearing in radial and tangential directions (rotating system of coordinates) can be found through integration of Equation (2.8) over the area of journal bearing considering Gümbel (π film) boundary condition.

$$F_{R} = \int_{0}^{\pi} Rd\theta \int_{-\frac{L}{2}}^{\frac{L}{2}} PCos(\theta)dz$$

$$F_{t} = \int_{0}^{\pi} Rd\theta \int_{-\frac{L}{2}}^{\frac{L}{2}} PSin(\theta)dz$$
(2.9)

By substituting Equation (2.8) into Equation (2.9), force components in polar coordinates may be derived as follow:

$$F_{R} = -\frac{L^{3}\mu RC}{24} \int_{0}^{\pi} \frac{k_{z}}{h^{3}} ((2\dot{\varphi} - \omega)\epsilon Sin(\theta) + 2\dot{\epsilon}Cos(\theta))Cos(\theta)d\theta$$

$$F_{t} = -\frac{L^{3}\mu RC}{24} \int_{0}^{\pi} \frac{k_{z}}{h^{3}} ((2\dot{\varphi} - \omega)\epsilon Sin(\theta) + 2\dot{\epsilon}Cos(\theta))Sin(\theta)d\theta$$
(2.10)

By substituting the turbulence coefficients from Equation (2.5) in Equation (2.10), and expanding the terms into laminar and turbulent parts, the following equation may be derived:

$$F_{R} = -\frac{L^{3}\mu R}{24C^{2}} \left(\int_{0}^{\pi} \frac{12Cos(\theta)}{(1+\epsilon Cos(\theta))^{3}} ((2\dot{\varphi}-\omega)\epsilon Sin(\theta)+2\dot{\epsilon}Cos(\theta))d\theta - \int_{0}^{\pi} \frac{a_{1}(\overline{Re}(1+\epsilon Cos(\theta)))^{a_{2}}Cos(\theta)}{(1+\epsilon Cos(\theta))^{3}} ((2\dot{\varphi}-\omega)\epsilon Sin(\theta) + 2\dot{\epsilon}Cos(\theta))d\theta \right)$$

$$F_{t} = -\frac{L^{3}\mu R}{24C^{2}} \left(\int_{0}^{\pi} \frac{12Sin(\theta)}{(1+\epsilon Cos(\theta))^{3}} ((2\dot{\varphi}-\omega)\epsilon Sin(\theta)+2\dot{\epsilon}Cos(\theta))d\theta - \frac{1}{24C^{2}} \right) d\theta$$

$$(2.11)$$

$$-\int_{0}^{\pi} \frac{a_{1}(\overline{Re}(1+\epsilon Cos(\theta)))^{a_{2}}Sin(\theta)}{(1+\epsilon Cos(\theta))^{3}} ((2\dot{\varphi}-\omega)\epsilon Sin(\theta)$$
$$+2\dot{\epsilon}Cos(\theta))d\theta \end{pmatrix}$$

where $\overline{Re} = \frac{\rho R \omega C}{\mu}$ is defined as the mean Reynolds number. Therefore, the above equations can be simplified as,

$$\begin{split} F_{R} &= -\frac{L^{3}\mu R}{24C^{2}} \Biggl\{ \Biggl((\omega - 2\phi) \frac{24\epsilon^{2}}{(1 - \epsilon^{2})^{2}} + \frac{12\pi(1 + 2\epsilon^{2})\dot{\epsilon}}{(1 - \epsilon^{2})^{5/2}} \Biggr) \\ &+ a_{1}(\overline{Re})^{a_{2}} \Biggl((\omega \\ &- 2\phi) \frac{(1 + \epsilon)^{a_{2} - 2}(1 - (a_{2} - 2)\epsilon) - (1 - \epsilon)^{a_{2} - 2}(1 + (a_{2} - 2)\epsilon)}{(a_{2} - 2)(a_{2} - 1)\epsilon} \\ &+ 2\dot{\epsilon} \int_{0}^{\pi} \frac{Cos(\theta)^{2}}{(1 + \epsilon Cos(\theta))^{3 - a_{2}}} d\theta \Biggr) \Biggr\} \end{split}$$
(2.12)
$$F_{t} &= \frac{L^{3}\mu R}{24C^{2}} \Biggl\{ \Biggl((\omega - 2\phi) \frac{6\pi\epsilon}{(1 - \epsilon^{2})^{3/2}} + \frac{48\epsilon\dot{\epsilon}}{(1 - \epsilon^{2})^{2}} \Biggr) \\ &+ a_{1}(\overline{Re})^{a_{2}} \Biggl((\omega - 2\dot{\phi})\epsilon \int_{0}^{\pi} \frac{Sin(\theta)^{2}}{(1 + \epsilon Cos(\theta))^{3 - a_{2}}} d\theta \\ &- 2\dot{\epsilon} \frac{(1 + \epsilon)^{a_{2} - 2}((a_{2} - 2)\epsilon - 1) + (1 - \epsilon)^{a_{2} - 2}((a_{2} - 2)\epsilon + 1)}{(a_{2} - 2)(a_{2} - 1)\epsilon^{2}} \Biggr) \Biggr\} \end{split}$$

Since there are no explicit solutions for $\int_0^{\pi} \frac{(1+\epsilon Cos(\theta))^{a_2}Sin(\theta)^2}{(1+\epsilon Cos(\theta))^3} d\theta$ and $\int_0^{\pi} \frac{(1+\epsilon Cos(\theta))^{a_2}Cos(\theta)^2}{(1+\epsilon Cos(\theta))^3} d\theta$, these integrations can be accurately approximated by the following

expressions:

$$\int_{0}^{\pi} \frac{\cos(\theta)^{2}}{(1+\epsilon\cos(\theta))^{3-a_{2}}} d\theta$$

$$\approx \int_{0}^{\pi} \frac{(1+a_{3}\epsilon\cos(\theta))\cos(\theta)^{2}}{(1+\epsilon\cos(\theta))^{3}} d\theta$$

$$= \pi \left(\frac{a_{3}}{\epsilon^{2}} \frac{2\epsilon^{4} + \epsilon^{2} - a_{3}(6\epsilon^{4} - 5\epsilon^{2} + 2)}{2\epsilon^{2}(1-\epsilon^{2})^{5/2}}\right)$$

$$\int_{0}^{\pi} \frac{\sin(\theta)^{2}}{(1+\epsilon\cos(\theta))^{3-a_{2}}} d\theta$$

$$\approx \int_{0}^{\pi} \frac{(1+a_{4}\epsilon\cos(\theta))\sin(\theta)^{2}}{(1+\epsilon\cos(\theta))^{3}} d\theta$$

$$= -\pi \frac{2a_{4}(1-\epsilon^{2})^{3/2} - \epsilon^{2} + a_{4}(3\epsilon^{2} - 2)}{2\epsilon^{2}(1-\epsilon^{2})^{3/2}}$$
(2.13)

where a_3 , and a_4 are constants which can be found through optimization and curve fitting to $\int_0^{\pi} \frac{\cos(\theta)^2}{(1+\epsilon \cos(\theta))^{3-a_2}} d\theta \quad \text{and} \quad \int_0^{\pi} \frac{\sin(\theta)^2}{(1+\epsilon \cos(\theta))^{3-a_2}} d\theta, \text{ respectively. Constants } a_3, \text{ and } a_4 \text{ for}$ Constantinescu's model are 0.9481 and 0.9589 and for Ng-Pan-Elrod model are found to be 0.9945 and 0.9747, respectively. Thus, the complete form of analytical turbulence forces for short bearing theory can be written as:

$$\begin{split} F_{R} &= -\frac{L^{3}\mu R}{24C^{2}} \Biggl\{ \Biggl((\omega - 2\phi) \frac{24\epsilon^{2}}{(1 - \epsilon^{2})^{2}} + \frac{12\pi(1 + 2\epsilon^{2})\dot{\epsilon}}{(1 - \epsilon^{2})^{5/2}} \Biggr) \\ &+ a_{1}(\overline{Re})^{a_{2}} \Biggl((\omega \\ &- 2\phi) \frac{(1 + \epsilon)^{a_{2} - 2}(1 - (a_{2} - 2)\epsilon) - (1 - \epsilon)^{a_{2} - 2}(1 + (a_{2} - 2)\epsilon)}{(a_{2} - 2)(a_{2} - 1)\epsilon} \\ &+ (\pi \dot{\epsilon}) \frac{2a_{3}(1 - \epsilon^{2})^{5/2} + 2\epsilon^{4} + \epsilon^{2} - a_{3}(6\epsilon^{4} - 5\epsilon^{2} + 2)}{\epsilon^{2}(1 - \epsilon^{2})^{5/2}} \Biggr) \Biggr\}$$
(2.14)
$$F_{t} = \frac{L^{3}\mu R}{24C^{2}} \Biggl\{ \Biggl((\omega - 2\phi) \frac{6\pi\epsilon}{(1 - \epsilon^{2})^{3/2}} + \frac{48\epsilon\dot{\epsilon}}{(1 - \epsilon^{2})^{2}} \Biggr) \\ &- a_{1}(\overline{Re})^{a_{2}} \Biggl(\pi(\omega - 2\phi) \frac{2a_{4}(1 - \epsilon^{2})^{3/2} - \epsilon^{2} + a_{4}(3\epsilon^{2} - 2)}{2\epsilon(1 - \epsilon^{2})^{3/2}} \\ &+ 2\dot{\epsilon} \frac{(1 + \epsilon)^{a_{2} - 2} \Bigl((a_{2} - 2)\epsilon - 1 \Bigr) + (1 - \epsilon)^{a_{2} - 2} \Bigl((a_{2} - 2)\epsilon + 1)}{(a_{2} - 2)(a_{2} - 1)\epsilon^{2}} \Biggr) \Biggr\} \end{split}$$

Having the force components in R and T directions by means of Equation (2.14), the nondimensional bearing stiffness and damping coefficients in R-T coordinate system are calculated as follows [32]:

$$\bar{k}_{ij} = (\pi (C/R)^3 / \mu \omega L) \begin{pmatrix} -\frac{\partial F_R}{C\partial \epsilon} & -\frac{\partial F_R}{C\epsilon \partial \varphi} + \frac{F_T}{C\epsilon} \\ -\frac{\partial F_T}{C\partial \epsilon} & -\frac{\partial F_T}{C\epsilon \partial \varphi} - \frac{F_R}{C\epsilon} \end{pmatrix}$$

$$\bar{c}_{ij} = (\pi (C/R)^3 / \mu L) \begin{pmatrix} -\frac{\partial F_R}{C\partial \dot{\epsilon}} & -\frac{\partial F_R}{C\epsilon \partial \dot{\varphi}} \\ -\frac{\partial F_T}{C\partial \dot{\epsilon}} & -\frac{\partial F_T}{C\epsilon \partial \dot{\varphi}} \end{pmatrix}$$
(2.15)

It should be noted that journal bearing forces in a polar coordinate system are not functions of the attitude angle of the journal position; hence, $\frac{\partial F_R}{\partial \varphi} = \frac{\partial F_T}{\partial \varphi} = 0$. The non-dimensional form of turbulent stiffness and damping coefficients in polar coordinate system are tabulated in Table A.1 for short bearings. Having calculated the coefficients in the *R*-*T* (rotating) coordinate system, a simple coordinate transformation with angle of rotation being the journal bearing attitude angle φ , will yield the coefficients in *X*-*Y* (stationary) coordinate system. Having the confidents in stationary coordinate system will help to validate turbulent coefficients experimentally for future purposes.

$$M = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}, \ \overline{K} = M \overline{k} M^T, \ \overline{C} = M \overline{C} M^T$$
(2.16)

where *M* is the transformation matrix, \overline{K} and \overline{C} are the non-dimensional stiffness and damping coefficients in Cartesian (stationary) coordinate system, respectively. The dimensionless analytically calculated values for stiffness and damping of a short journal bearing coefficients in both laminar and turbulent regimes at different Reynolds numbers for a length to diameter ratio of $\frac{L}{D} = 0.5$ are plotted against their calculated Sommerfeld number ($S = \frac{\mu \omega LR^3}{\pi WC^2}$) and are shown in Figure 2.4 and Figure 2.5, respectively.

The Reynolds dependence of the dynamic stiffness coefficients in turbulent flow can be observed in Figure 2.4. Although variation of the direct stiffness coefficient in y direction by increasing the Reynolds number is very small, the general trend for direct stiffness coefficient indicates that as the Reynolds number increases, the stiffness decreases for almost the whole range of Sommerfeld-loading conditions, Figure 2.4(d). By comparing the results of direct turbulent stiffnesses for a fixed Reynolds number based on the two examined turbulent models, one can see that stiffness predictions based on the Constantinescue's model (dashed lines) are higher than those predicted by the Ng-Pan-Elrod model (solid lines). The same trend, however, does not hold for the case of cross-coupled coefficients where trend-lines of dynamic stiffness cross each other at moderate Sommerfeld numbers and don't hold a consistent trend compared to the direct stiffness coefficients Figure 2.4(b) and (c). The coefficients move to the top and left (higher values and lower Sommerfeld number) by increasing \overline{Re} . In both cases (\overline{K}_{xy} and $abs(\overline{K}_{yx})$), at low Sommerfeld numbers, the stiffness predictions of the Constantinescue's

model remain higher compared to those of the Ng-Pan-Elrod model while this trend switches at high Sommerfeld values. Similarly, the higher calculated values of laminar stiffness at low Sommerfeld numbers will change to lower predictions at high Sommerfeld numbers compared to the turbulent values.



Figure 2.4. Stiffness coefficients comparison of laminar and turbulent short journal bearings (L/D = 0.5), (a) K_{xx} , (b) K_{xy} , (c) $abs(K_{yx})$, and (d) K_{yy} .



Figure 2.5. Damping coefficients comparison of laminar and turbulent short journal bearings (L/D = 0.5), (a) C_{xx} , (b) C_{xy} , (c) C_{yx} , and (d) C_{yy} .

At high Sommerfeld numbers (moderate to low loading) the turbulent cross-coupled stiffness coefficients are generally higher compared to laminar stiffness coefficients and this suggests that a shaft supported on turbulent bearings could potentially be more stable at low loading which itself is the range where system is more susceptible to oil-whirl instability; however, this is not the case and will be discussed in more detail within the context of the next section. Similar

treatment was carried out to calculate the damping coefficients of turbulent journal bearings. The calculated bearing damping coefficients in turbulent flow are compared to the calculated damping coefficients in the laminar region and are plotted against their corresponding Sommerfeld numbers which are plotted in Figure 2.5.

Although, direct damping coefficients increase by increasing the Reynolds number, crosscoupled coefficients remain steady. The results presented in Figure 2.4, Figure 2.5, and Table A.1 do not fully agree with the results presented by Capone et al. [77], Wang et al. [32], and Hashimoto et al. [65] possibly because previous methods relied on inaccurate turbulent force calculations.

2.3.2 Turbulent Long Bearing Theory Formulation

By long bearing approximation $(L/D) \ge 2$, it can be assumed longitudinal pressure distribution is constant $((\partial P/\partial \theta) \gg (\partial P/\partial z))$; thus, $\frac{\partial P}{\partial z} = 0$ and updated turbulence Reynolds' equation for long bearing theory can be written as:

$$\frac{\partial}{\partial\theta} \left(\frac{h^3}{k_{\theta}} \frac{\partial P}{\partial\theta} \right) = \frac{\mu C R^2}{2} \left((2\dot{\varphi} - \omega) \epsilon Sin(\theta) + 2\dot{\epsilon} Cos(\theta) \right)$$
(2.17)

The above equation can be integrated over θ to obtain the following equation:

$$\begin{aligned} \frac{\partial P}{\partial \theta} &= \frac{k_{\theta}}{h^3} \left(\frac{\mu C R^2}{2} \left((\omega - 2\dot{\phi}) \epsilon Cos(\theta) + 2\dot{\epsilon} Sin(\theta) \right) + c_1 \right) \\ &= \frac{12 + \hat{a}_1 (\overline{Re})^{\hat{a}_2} \left((1 + \epsilon Cos(\theta)) \right)^{\hat{a}_2}}{\left(C (1 + \epsilon Cos(\theta)) \right)^3} \left(\frac{\mu C R^2}{2} \left((\omega - 2\dot{\phi}) \epsilon Cos(\theta) \right) \end{aligned}$$
(2.18)
$$&+ 2\dot{\epsilon} Sin(\theta) + c_1 \end{aligned}$$

In order to be able to integrate the above equation to find an analytical expression for long bearing pressure, it is assumed $((1 + \epsilon Cos(\theta)))^{\hat{a}_2} \approx (1 + \hat{a}_3 \epsilon Cos(\theta))$. The constant \hat{a}_3 may be found by curve fitting via optimization, which is calculated to be 0.8437, and 0.91 for Constantinescu and Ng-Pan-Elrod theories, respectively. By applying the aforementioned

assumption along with applying Gümbel boundary condition, the non-dimensional form of the turbulent long bearing pressure can be written as:

 \overline{P}

$$=\frac{\pi\epsilon(\epsilon\sin(2\theta)(2\hat{a}_{3}^{2}H^{2}\epsilon^{2} - (H+12)(\hat{a}_{3}H + H+12)) + 2\sin(\theta)(\hat{a}_{3}H\epsilon^{2}(\hat{a}_{3}H + H+12) - 2(H+12)^{2}))}{2(\epsilon^{2}((3\hat{a}_{3} - 1)H - 12) - 2(H+12))(\epsilon\cos(\theta) + 1)^{2}}$$
(2.19)

where $H = \hat{a}_1(\overline{Re})^{\hat{a}_2}$. By substituting Equation (2.19) into Equation (2.9), analytical expression of polar turbulent forces for a long journal bearing can be derived as:

$$F_{r} = \frac{1}{16C^{2}\epsilon^{2}(\epsilon^{2}-1)^{2}(\pi\epsilon^{2}((3\hat{a}_{3}-1)H-12)-2\pi(H+12))} (8R^{3}\mu L(\sqrt{1-\epsilon^{2}}\dot{\epsilon}(\hat{a}_{3}^{2}H^{2}\epsilon^{2}(\pi^{2}(6(1-\epsilon^{2})^{3/2}(\epsilon^{2}-1))^{2}(\pi\epsilon^{2}(3\hat{a}_{3}-1)H-12)-2\pi(H+12)) (R^{2}((1-\epsilon^{2})^{3/2}(\epsilon^{2}+2)+3\epsilon^{4}+2\epsilon^{2}-2)-16\epsilon^{4}) + (H+12)^{2}(\pi^{2}(\epsilon^{2}+2)-16)\epsilon^{2}) + 2\pi\epsilon(\epsilon^{2}-1)(\hat{a}_{3}^{2}H^{2}(2\epsilon^{2}-3)\epsilon^{3}-\hat{a}_{3}H(\epsilon^{2}-1)\tanh^{-1}(\epsilon)(\epsilon^{2}((3\hat{a}_{3}-1)H-12)-2(H+12)) + 2\hat{a}_{3}H(H+12)\epsilon - (H+12)^{2}\epsilon^{3})(\omega-2\dot{\varphi})))$$

$$F_{t} = \frac{8R^{3}\mu L}{64C^{2}(1-\epsilon^{2})^{3/2}(\epsilon^{4}(3\hat{a}_{3}H-H-12)-2(H+12)\epsilon^{2})} (4\pi\epsilon(\epsilon^{2}-1)(\hat{a}_{3}^{2}H^{2}\epsilon^{2}(-3\sqrt{1-\epsilon^{2}}-2\epsilon^{2}+3)-\hat{a}_{2}H(H+12)(2-\sqrt{1-\epsilon^{2}}(\epsilon^{2}+2)) + (H+12)^{2}\epsilon^{2})(\omega-2\dot{\varphi})$$

$$(2.20)$$

$$+ 16\sqrt{1 - \epsilon^{2}}\dot{\epsilon}(\hat{a}_{3}^{2}H^{2}(2\epsilon^{2} - 3)\epsilon^{3} - \hat{a}_{3}H(\epsilon^{2} - 1)\tanh^{-1}(\epsilon)(\epsilon^{2}((3\hat{a}_{3} - 1)H - 12))$$

$$- 2(H + 12)) + 2\hat{a}_{3}H(H + 12)\epsilon - (H + 12)^{2}\epsilon^{3})$$

The same approach may be applied to derive the analytical stiffness and damping coefficients (non-dimensional form) for a turbulent long journal bearing which are tabulated in Table A.2. According to Table A.2, turbulent coefficients for a long journal bearing are not functions of length to diameter ratio (L/D). The variation of non-dimensional stiffness and damping coefficients with respect to their corresponding Sommerfeld numbers in Cartesian coordinate system are plotted in Figure 2.6 and Figure 2.7, respectively. According to Figure 2.6 and Figure 2.7, turbulence has a huge effect on long journal bearings compare to short journal bearings.

It should be noted that, the discrepancy between the coefficients calculated based on Constantinescu and Ng-Pan-Elrod models is very small due to small difference between their circumferential turbulence coefficients (k_{θ}), shown in Figure 2.2.



Figure 2.6. Stiffness coefficients comparison of laminar and turbulent long journal bearings, (a) K_{xx} , (b) K_{xy} , (c) $abs(K_{yx})$, and (d) K_{yy} .



Figure 2.7. Damping coefficients comparison of laminar and turbulent long journal bearings, (a) C_{xx} , (b) C_{xy} , (c) C_{yx} , and (d) C_{yy} .

2.4 Equations of Motions and Linear Stability Analysis of Short and Long Turbulent Journal Bearings

In an ideal and fully balanced rotor-bearing system, the shaft spins at its designed operating speed while the induced pressures inside end journals can support the weight of the shaft and hence there exists and equilibrium point where the journal center finds and stays there during the operation. This point is referred to as the static equilibrium position. If the rotor bearing system is undisturbed, the rotor will remain in its equilibrium position. For small disturbances, for instance small unbalance force, the shaft will no longer stays at the static equilibrium point and orbits via a closed elliptic curve around its equilibrium point. These are the harmonic solutions to a stable dynamic system operating in its stable region. If the operating speed of the shaft exceeds the so-called threshold speed of instability ω_s , no stable harmonic solution exists and the journal orbit will spiral outward towards an either bounded stable limit cycle (with orbits generally much greater than the stable closed harmonic orbits) or towards the point where metal to metal contact occurs.

This change of the dynamical behavior of the system is called a bifurcation and is known in rotor dynamics terminology as the oil whirl-whip phenomenon. It should be noted that oil whirl and rotor whirling will happen for both fully balanced and unbalanced rotor with the exception that the threshold speed of a balanced system is higher than the threshold speed of an unbalanced system. The transition (bifurcation) to high amplitude vibration, whether to a stable limit cycle or towards the bearing clearance, can be either gradual (supercritical bifurcation) or sudden (subcritical bifurcation). Bifurcation towards the limit cycle is generally considered as an unstable operating condition while it is not necessarily unstable. The reason for this is that the majority of the stability analysis available in the literature are based on linear analysis and from the viewpoint of linear analysis there is no difference between unstable and stable limit cycles (they are both considered as unstable).

Even though the bifurcation type cannot be evaluated via linear analysis, the stability margin can indeed be identified by simple linear analysis. This is because that local stability of nonlinear and linearized systems is essentially the same. Linearized analysis will be used here to identify the stability margins of the system. To do this, one needs to find the threshold speed of instability for a given rotor-bearing system at any applied static load to the shaft. If the operating speed of the rotor is kept below this threshold speed, the system remains in its stable region and hence the operation is considered as safe. The analysis here is following from [63, 78] with minor corrections.

Figure 2.8 illustrates a centrally loaded rotor-bearing system along with its coordinates used in this thesis. In Figure 2.8, O_j is the center of the journal bearing, O_M is the center of the central mass, and W is the load per bearing. The coordinates (x_1, y_1) and (x_2, y_2) are utilized to denote the dynamic positions of the journal bearings and central mass, respectively. Referring to Figure 2.8, to analyze a flexible, center-loaded and perfectly balanced rotor symmetrically supported by two identical fluid-film journal bearings, the following assumptions are made:

- Deflection of the flexible shaft is small to allow the use of linear beam theory.
- The mass of the shaft and rotor torque of the midpoint mass are negligible.
- The rotor mass is lumped at the midpoint.
- Axial and torsional vibrations of the lumped mass are negligible.
- Gyroscopic effect of the shaft, disk and journal bearings are negligible.



Figure 2.8. Schematic of a flexible rotor supported on journal bearings. O_j and O_M correspond to the geometric center of the journal and the central disc respectively.

It should be noted that, K_s is the effective rotor stiffness which can be approximated as $192EI/L_s^3$. Where, L_s is the length of the shaft, *E* is shaft's Young's modulus, and *I* is its second moment of area. According to the Newton's second law, the equations of motion for the central disk and journal bearings may be written as follows:

$$Central \ Disk \to \begin{cases} M\ddot{x}_2 + K_s(x_2 - x_1) = 0\\ M\ddot{y}_2 + K_s(y_2 - y_1) = 0 \end{cases}$$

$$Journal \ Bearings \to \begin{cases} -2(K_{xx}x_1 + K_{xy}y_1 + C_{xx}\dot{x}_1 + C_{xy}\dot{y}_1) + K_s(x_2 - x_1) = 0\\ -2(K_{yx}x_1 + K_{yy}y_1 + C_{yx}\dot{x}_1 + C_{yy}\dot{y}_1) + K_s(y_2 - y_1) = 0 \end{cases}$$
(2.21)

If the system is running at the threshold speed of ω_s , then it can be assumed both journal bearings and central mass *M* will undergo whirl motion at frequency of ω_w ; hence, their motions may be written as follows:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} e^{i\omega_W t}, \qquad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} e^{i\omega_W t}$$
(2.22)

By making use of the derivations in Appendix B, the stability margins of short and long journal bearings for different non-dimensional stiffness coefficients of the shaft $(\frac{C K_S}{W})$ are calculated. Stability parameter $\overline{\Gamma} = \frac{C}{W} M \omega_s^2 = (\frac{C K_S}{W}) \overline{\omega}_s^2 = S_z \overline{\omega}_s^2$ is plotted vs the Sommerfeld number as shown in Figure 2.9 and Figure 2.10 for short and long bearings respectively and the stable and unstable regions of operation are indicated.

From the stability plots of Figure 2.9 and Figure 2.10 it can be seen that as the non-dimensional shaft stiffness increases, the stable operating margin expands towards higher threshold speeds and hence the rotor-bearing system becomes stable at higher rotating spin speeds of the shaft. This trend remains valid through the whole loading range of both short and long bearings. By comparing Figure 2.9 and Figure 2.10, it may be concluded that, at a same shaft flexibility and Sommerfeld number, short bearings are more stable compare to long bearings. By comparing the stability of turbulent bearings to the laminar bearings and holding the non-dimensional shaft stiffness constant as in Figure 2.9 and Figure 2.10, it is clear that the turbulent curves are shifted to the left of the laminar curve, indicating that the unstable region has grown in size.



Figure 2.9. Turbulent effects on the stability parameter Γ of a flexible shaft supported on short length journal bearings (L/D = 0.5) at non-dimensional shaft stiffness coefficients of, (a) $CK_s/W = 0.1$, (b) $CK_s/W = 1$, (c) $CK_s/W = 10$, and (d) $CK_s/W = 100$.

As the static load increases (low Sommerfeld number), there is a limit of $S \cong 0.01$ and $S \cong 0.001$ for short and long bearings, respectively in which further increase of the static load will not affect the stability of the system and system will remain stable at all operating speeds for mean Reynolds numbers up to 50,000. This suggests that as the static loading increases on the shaft, the turbulent journal bearings are progressively less stable compared to the bearing which is operating in the laminar region of oil film flow. At high Sommerfeld region $S \ge 0.2$ (low load) the stability curves of laminar and turbulent bearings are very much alike for both short and long bearings. This destabilizing effect of turbulent bearings increases by the Reynolds number.



Figure 2.10. Turbulent effects on the stability parameter Γ of a flexible shaft supported on long length journal bearings at non-dimensional shaft stiffness coefficients of, (a) $CK_{s'}W = 0.1$, (b) $CK_{s'}W = 1$, (c) $CK_{s'}W = 10$, and (d) $CK_{s'}W = 100$.

At each Reynolds and at each non-dimensional shaft stiffness, the stability curves of the Constantinescu's turbulent model are positioned above the curves obtained based on the Ng-Pan-Elrod turbulent model suggesting that the Ng-Pan-Elrod turbulent treatment is more conservative when used as basis for design of the rotor-bearing system. To illustrate the expansion of the unstable operating region and shortcoming of the laminar theory to accurately predict the stable regions of the rotor-bearing system at high Reynolds numbers, the journal trajectories at different operating points should be calculated. To do this, the set of governing equations of motion for the journal and the shaft are to be numerically calculated to obtain the trajectory of the journal center. This is done first by writing the force balance at the journal center and mass center demonstrated in Figure 2.8, along with weight of the shaft.

$$M\ddot{x}_{2} + K_{s}(x_{2} - x_{1}) - Mg = 0$$

$$M\ddot{y}_{2} + K_{s}(y_{2} - y_{1}) = 0$$

$$2(F_{r}Cos(\varphi) - F_{t}Sin(\varphi)) + K_{s}(x_{2} - x_{1}) = 0$$

$$2(F_{r}Sin(\varphi) + F_{t}Cos(\varphi)) + K_{s}(y_{2} - y_{1}) = 0$$
(2.23)

Normalizing Equation (2.23), utilizing $\omega x'_i = \dot{x}_i$, $\omega y'_i = \dot{y}_i$, $\bar{x}_i = x_i/C$, $\bar{y}_i = y_i/C$, $\bar{F}_r = x_i/C$ $F_r\left(\frac{\pi C^2}{\mu\omega LR^3}\right)$, and $\overline{F}_t = F_t\left(\frac{\pi C^2}{\mu\omega LR^3}\right)$, the nonlinear dimensionless forms of the equations of motion

may be written as:

$$\bar{x}_{2}^{\prime\prime} + \frac{S_{z}}{\bar{\Gamma}}(\bar{x}_{2} - \bar{x}_{1}) - \frac{2}{\bar{\Gamma}} = 0$$

$$\bar{y}_{2}^{\prime\prime} + \frac{S_{z}}{\bar{\Gamma}}(\bar{y}_{2} - \bar{y}_{1}) = 0$$

$$2\bar{F}_{r} = -\frac{S_{z}}{S} \left((\bar{x}_{2} - \bar{x}_{1}) \frac{\bar{x}_{1}}{\sqrt{\bar{x}_{1}^{2} + \bar{y}_{1}^{2}}} + (\bar{y}_{2} - \bar{y}_{1}) \frac{\bar{y}_{1}}{\sqrt{\bar{x}_{1}^{2} + \bar{y}_{1}^{2}}} \right)$$

$$2\bar{F}_{t} = \frac{S_{z}}{S} \left((\bar{x}_{2} - \bar{x}_{1}) \frac{\bar{y}_{1}}{\sqrt{\bar{x}_{1}^{2} + \bar{y}_{1}^{2}}} - (\bar{y}_{2} - \bar{y}_{1}) \frac{\bar{x}_{1}}{\sqrt{\bar{x}_{1}^{2} + \bar{y}_{1}^{2}}} \right)$$

$$(2.25)$$

where $S_z = \left(\frac{C}{W}\right) K_s$ is the non-dimensional form of shaft stiffness coefficient and S is the Sommerfeld number. It shall be noted that trajectories can be solely obtained based on the linearized coefficients provided that the journal position remains close to its steady state position. This in return can be very beneficiary since the computational cost of obtaining trajectories based on the linearized coefficients is much cheaper than obtaining the journal force and hence the trajectories at each time step by directly solving the Reynolds lubrication equation.

System of Equation (2.21) can be used as basis for calculation of the journal trajectories for much more complicated systems provided that the dynamic coefficients are available (either previously calculated or experimentally obtained). The system of coupled nonlinear ODEs of Equations (2.24) and (2.25) are integrated numerically by an explicit scheme in MATLAB to obtain the trajectories of the journal and the disk center. Non-dimensional journal trajectories, plot of $\bar{x}_1 vs \bar{y}_1$, at three different operating points were calculated based on both laminar and turbulent models (Constantinescu, Ng-Pan-Elrod models at $\overline{Re} = 10,000$). Trajectories of the selected operating points, shown in Figure 2.11, are calculated for a short journal bearing $(L/D = 0.5, S_z = \left(\frac{C}{W}\right)K_s = 10, \overline{\Gamma} = 15)$ and demonstrated in Figure 2.12. For all the cases shown in Figure 2.12, the journal center trajectory is started at the position near by the center of the bearing, that are $\overline{x_1} = 0.1/\sqrt{2}, \overline{y_1} = 0.1/\sqrt{2}$, and trajectories are followed in time.



Figure 2.11. Selected operating points under laminar and turbulent short journal bearings (L/D = 0.5).

The first operating point is selected such that it lies in the unstable region of both laminar and turbulent curves at S = 0.2, $\overline{\Gamma} = 15$. Similar results were obtained where journal centre grows fast in time from the turbulent models, Figure 2.12(d) and (g). The next operating point S = 0.07, $\overline{\Gamma} = 15$ lies between the two different turbulent theories and on the stable region of the laminar curve. As expected, the laminar based trajectory along with Constantinescu hypothesis, Figure 2.12(b), and (e) can still find their steady state point while the turbulent trajectory based on Ng-Pan-Elrod method, Figure 2.12(h), start to grow and hence becomes unstable.

This again indicates that stable regions obtained based on the laminar theory can become unstable at higher Reynolds numbers. Both laminar and the two different turbulent trajectories find their static equilibrium points at S = 0.05, $\overline{\Gamma} = 15$ with different operating speeds, Figure 2.12(c), (f), and (i). As it can be seen from Figure 2.10, Figure 2.11, and Figure 2.12, Ng-Pan-



Elrod turbulent model is more conservative than Constantinescu's model and has been utilized for nonlinear stability analysis at high Reynolds numbers where fluid regime is turbulent.

Figure 2.12. Short journal bearing (L/D = 0.5) trajectory at three operating points (a), (b), and (c) under laminar flow, (d), (e) and (f) under Constantinescu's turbulent model (g), (h) and (i) under turbulent Ng-Pan-Elrod regime at Reynolds number of 10,000.

2.5 Application of Hopf Bifurcation Theory in Flexible Rotor Bearing Systems

Hopf bifurcation theorem characterizes local behavior of periodic solution (limit cycle) that bifurcates from a fixed point $\mathbf{x}_{\mathbf{e}}$ when parameter μ is near a critical value μ_c . The Hopf bifurcation theory describes the birth of a periodic solution of a system whose behavior is defined by the ordinary differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$, ($\mathbf{x} \in \mathbb{R}^n$). A Hopf bifurcation occurs, when as the system parameter μ varies, a single complex conjugate pair of eigenvalues of the linearized system equations become purely imaginary (in the process of crossing the imaginary axis) [79]. The conditions that must be met for the Hopf bifurcation theory are [80]:

- (i) Equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$ must have an isolated stationary point at $\mathbf{x} = \mathbf{x}_{\mathbf{e}}(\mu)$;
- (ii) The Jacobian matrix $\left(\mathbf{A}(\mu) = \left(\frac{\partial f_i}{\partial x_j}\right)(\mathbf{x}_{\mathbf{e}}(\mu),\mu); i, j = 1, ..., n\right)$ has exactly a pair of complex conjugate eigenvalues $\lambda_{1,2} = \alpha(\mu) \pm i\beta(\mu)$ such that when $\mu = \mu_c, \alpha(\mu) = 0$ and $\beta(\mu_c) > 0$. However, real part of the rest of the eigenvalues must be negative;
- (iii) f is analytic in x and μ in the neighborhood of $(\mathbf{x}, \mu) = (\mathbf{x}_{e}, \mu_{c})$; and
- (iv) $(d\alpha(\mu)/d\mu)(\mu_c) \neq 0$, where $\alpha(\mu)$ is the real part of the pair of eigenvalues that are continuous at μ_c .

In the above hypotheses, μ_c is called the critical value of μ . If assumption (ii) holds, then assumption (iv) implies that the linear stability of the stationary point $\mathbf{x}_{\mathbf{e}}(\mu)$ will be lost as μ crosses μ_c . Under these conditions, at the onset of bifurcation, the system has a family of periodic solutions. The Hopf bifurcation theorem provides appropriate criteria for the prediction of the existence, shape, and period of the periodic solution [80]. In general (excluding the special case where bifurcation occurs only for $\mu \equiv \mu_c$), periodic solutions exist in exactly one of the cases: either $\mu > \mu_c$ or $\mu < \mu_c$ [81]. The periodic solutions in the case of $\mu > \mu_c$, is called a supercritical bifurcation. If the periodic solutions exist only in the case of $\mu < \mu_c$, then the system is said to undergo a so called subcritical bifurcation. The transition from steady state to limit cycle could be gradual (supercritical bifurcation) or sudden (subcritical bifurcation) and hence it is of absolute importance to identify the bifurcation type of a dynamical system. The equations of motion as given in Equations (2.24) and (2.25) have the suitable form of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \overline{\Gamma})$ and possess a steady state equilibrium position \mathbf{x}_s for the application of the Hopf bifurcation theory. Here, the non-dimensional running speed $\overline{\Gamma}$ is considered as the system parameter when all other parameters of the rotor-bearing system are fixed. According to HBT, if the parameter $\overline{\Gamma}$ in the equations of motion (2.24) and (2.25) becomes greater than some critical value $\overline{\Gamma}_{cr}$, an isolated stationary point $\mathbf{x}_s(\overline{\Gamma})$ will lose its linear stability by having a complex conjugate pair of eigenvalues of the linearized system crossing the imaginary axis of the complex plane [79]. Hence, this critical value $\overline{\Gamma}_{cr}$ is the non-dimensional form of the threshold speed of instability of a rotor-bearing system. To implement the Hopf bifurcation analysis, first, the right hand side of the nonlinear equations of motion (2.24) and (2.25) is expanded in a Taylor series about the static equilibrium position ($\mathbf{x} = \mathbf{x}_s$) as follows;

$$\mathbf{f}(\mathbf{x},\bar{\Gamma}) = \mathbf{f}(\mathbf{x}_{s},\bar{\Gamma}) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_{s},\bar{\Gamma})\Delta\mathbf{x} + \frac{\partial^{2}\mathbf{f}}{\partial \mathbf{x}^{2}}(\mathbf{x}_{s},\bar{\Gamma})\Delta\mathbf{x}^{2} + \frac{\partial^{3}\mathbf{f}}{\partial \mathbf{x}^{3}}(\mathbf{x}_{s},\bar{\Gamma})\Delta\mathbf{x}^{3} + \text{HOT}$$
(2.26)

where $\Delta \mathbf{x}(\bar{\Gamma}) = \mathbf{x}(\bar{\Gamma}) - \mathbf{x}_{\mathbf{s}}(\bar{\Gamma})$ and HOT represents the higher order terms in the Taylor expansion series. The zeroth order terms $\mathbf{f}(\mathbf{x}_{\mathbf{s}}, \bar{\Gamma})$ are used to determine the static equilibrium position. The first order terms $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_{\mathbf{s}}, \bar{\Gamma})$, usually referred to as the Jacobian matrix of the equations of motion (2.24) and (2.25), are used to determine the dynamic performance through the analysis of the eigenvalues. The second order terms $\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2}(\mathbf{x}_{\mathbf{s}}, \bar{\Gamma})$ and third order terms $\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^3}(\mathbf{x}_{\mathbf{s}}, \bar{\Gamma})$ are used to determine the stability of the periodic solutions. Specifically, the stability, the amplitude and the frequency of the periodic solutions are provided by these terms [79]. In this section, the Hopf bifurcation theory is used to predict the instability threshold speed and its bifurcation type at a range of operating conditions i.e. a range of Sommerfeld numbers (S).

Bifurcation type (supercritical or subcritical) and stability boundary of a flexible rotor bearing system supported by short and long journal bearings are identified at different Reynolds numbers. Following the algorithm developed by Hassard et al. [80], a Hopf bifurcation subroutine was implemented using MATLAB to calculate the bifurcation parameters (μ_2 and β_2 [80] which are the Floquet exponents of the periodic solutions that determined their orbital stability) of the system. If $\beta_2 > 0$; periodic solutions exist for $\mu > \mu_{cr}$; and if $\beta_2 < 0$; periodic solutions exist for $\mu < \mu_{cr}$. Moreover, stability of the limit cycles can be determined

from the stability parameter μ_2 . If $\mu_2 < 0$; the solutions are orbital-asymptotically stable; however, if $\mu_2 > 0$; the solutions are orbital-asymptotically unstable. The bifurcation maps for flexible rotor-bearing systems are presented in Figure 2.13 and Figure 2.14 for short and long bearings at different Reynolds numbers.

It should be noted that Ng-Pan-Elrod model was used to derive bifurcation maps in turbulent zones ($\overline{Re} \ge 2000$). Figure 2.13 and Figure 2.14 show that how the dimensionless instability threshold speed and its bifurcation type change with increasing the Sommerfeld number (S) for given dimensionless values of rotor stiffness(S_z). As it can be inferred from these figures, neglecting the rotor stiffness and turbulence effects may lead to erroneous results in the prediction of the instability threshold speed and its bifurcation type as the expansion of the unstable region with decreased shaft stiffness is neglected in stability curves that are obtained based on the rigid rotor assumption.

Any solitary curve in Figure 2.13 and Figure 2.14 provides two pieces of information; (1) whether the system at any operating condition is stable or not for a range of Reynolds numbers; and (2) the bifurcation type at any point along the curve. The boundary line separating the stable and unstable regions is drawn in either solid or dashed lines according to the bifurcation type. If the bifurcation type is supercritical, a boundary line is drawn with a solid line whereas if the bifurcation type is subcritical, it is shown as a dashed line. Based on the linear theory of stability, when the operating speed exceeds the threshold speed of instability, the system is to be rendered unstable and any bifurcation is hence treated the same regardless of its type. However, the transition from stable to unstable could show significant difference depending on the bifurcation type.

To better demonstrate this, two points corresponding to two different operating conditions are chosen along a stability curve. The first point is chosen to fall in the solid part of the curve (i.e. in the supercritical bifurcation region) and the latter in the region marked with dashed lines (i.e. in the subcritical bifurcation region). The journal trajectory at this operating condition is then found by integrating the equations of motion as the operating speed crosses the stability curve and are shown in Figure 2.15(a) and (b). As it can be seen in Figure 2.15, the amplitude of the journal lateral vibrations indeed grow after crossing the stable curve but this increase in

amplitude of lateral vibrations is shown to be gradual as in Figure 2.15(a) or sudden with ever increasing amplitudes as shown in Figure 2.15(b).



Figure 2.13. Stability parameter Γ and its bifurcation type by increasing Sommerfeld number for a flexible shaft supporting on two identical short bearings (L/D = 0.5), (a) Laminar flow, (b) Re = 10,000, (c) Re = 30,000, (d) Re = 50,000.

As mentioned before, the Hopf bifurcation theory can be implemented in determining the bifurcation type as well as providing a local estimate of the limit cycle size (the amplitude of the periodic solutions).

The limit cycle positioning (with respect to the critical point) and size can change depending on the bifurcation type and the magnitude of the bifurcations parameter (operating speed) compared to the critical bifurcation parameter (threshold speed of instability), respectively. Determining the shape and size of the limit cycle in the subcritical operating regions is of particular interest in designing a rotor bearing system. According to linear stability theory for a rotor-bearing system that is operating at speeds below the threshold speed of instability, upon perturbing the system from its steady state operating point, the system finds its equilibrium state regardless of perturbation size (magnitude of the initial conditions).



Figure 2.14. Stability parameter Γ and its bifurcation type by increasing Sommerfeld number for a flexible shaft supporting on two identical long bearings, (a) Laminar flow, (b) Re = 10,000, (c) Re = 30,000, (d) Re = 50,000.



Figure 2.15. Laminar short journal bearing (L/D = 0.5) trajectory at the speeds above the threshold speed of instability when operating at, (a) supercritical bifurcation region, (b) subcritical bifurcation region

Khonsari et al [25] reported that at certain operating conditions (being in the subcritical region), there exists a stability envelope that if the perturbation amplitude is greater than the amplitude of the envelope, the system can render unstable even at speeds below the threshold speed of instability. It was shown later [26] that the size of stability envelope can be precisely estimated by obtaining the size of the limit cycles as predicted in the hope bifurcation theorem [80]:

$$\mathbf{x}(t,\bar{\Gamma}) = \mathbf{x}_{*}(\bar{\Gamma}_{c}) + \sqrt{\frac{\bar{\Gamma} - \bar{\Gamma}_{c}}{\mu_{2}}} \operatorname{Re}\left(\mathbf{e}^{\frac{2\pi i t}{T}} \mathbf{v}_{1}\right) + \mathbf{o}(\bar{\Gamma} - \bar{\Gamma}_{c})$$

$$T(\bar{\Gamma}) = \frac{2\pi}{\omega_{0}} \left(1 + \tau_{2}\left(\frac{\bar{\Gamma} - \bar{\Gamma}_{c}}{\mu_{2}}\right) + \mathbf{o}(\bar{\Gamma} - \bar{\Gamma}_{c})^{2}\right)$$

$$S_{p}(\bar{\Gamma}) = \beta_{2}\left(\frac{\bar{\Gamma} - \bar{\Gamma}_{c}}{\mu_{2}}\right) + \mathbf{o}(\bar{\Gamma} - \bar{\Gamma}_{c})^{2}$$

$$(2.27)$$

where $T(\overline{\Gamma})$ is the period of the periodic solutions and τ_2 is the second exponent in the asymptotic expansion series of $T(\overline{\Gamma})$. $S_p(\overline{\Gamma})$ is the characteristic exponent which determines periodic solution's stability type (subcritical and/or supercritical). Equation (2.27) is used to calculate the boundary (shape) of the periodic solutions at a range of operating speeds near the threshold speed of instability for a rotor-bearing system operating in the subcritical region. The

calculated boundaries are then plotted together to form a cone shape for both laminar and turbulent operating conditions as shown in Figure 2.16(a) and (b).

It shall be noted here that as the running speed crosses the threshold speed of instability, that is found by the means of linear analysis, a qualitative change happens in the dynamic response of the system which is called bifurcation. This bifurcation can have two subtypes: supercritical and/or subcritical. In the supercritical case, Stable periodic solutions are born after crossing the threshold value and gaining more spin speed. This periodic solution is referred to as a limit cycle. It is noteworthy to mention that in supercritical cases, periodic orbits do not exist at speeds below the threshold speed and even when they appear after crossing the threshold speed, their amplitude is small and tolerable. In the subcritical case; however, no stable periodic solution exists after crossing the threshold speed and system can become suddenly unstable. In such cases, unstable periodic orbits exist at speeds below the threshold speed.

The cone shapes that are found in subcritical cases show the amplitudes of such periodic solutions. these regions can also reveal another interesting characteristic of the system: they show the allowable perturbation amplitudes that can be tolerated by the system i.e. if a shock is given to the system with an amplitude confined within the cone, the system will find the steady state pure spinning mode. On the other hand, if the amplitude of the perturbation is big enough, outside the cone boundary, the system becomes unstable even though the operating running speed was below the threshold speed of instability. Such phenomenon cannot be explained by traditional linear stability analysis.

Since the bifurcation type for the selected operating point is subcritical, the unstable limit cycle exists at operating speeds less than the threshold speed of instability. The limit cycle amplitude decreases as the operating speed approaches the critical point and it ceases to exist after crossing the critical point. For a rotor-bearing system operating at speeds below the threshold speed, if the magnitude of an applied perturbation to the system is confined within the cone, the system will oscillate but the lateral vibrations amplitude will decrease, and the system will find its steady state. However; if the perturbation magnitude is such that the journal is positioned outside the cone, the system will become unstable with ever increasing lateral vibration amplitudes even at speeds below the threshold speed of instability. Such phenomenon cannot be predicted by

linear stability theory. To better compare the stability envelope in laminar vs turbulent operating conditions, a two-dimensional cut of the limit cycle cone is obtained from the cones in Figure 2.16(a) and (b) and are re-plotted together in Figure 2.17.

Figure 2.16 (a) and (b) show shapes of the periodic solutions of journal orbit as a function of dimensionless running speed that is close to but below the critical speed, $\bar{\Gamma}_c = 11.02$ and $\bar{\Gamma}_c = 9.11$ for laminar and turbulent regimes respectively. Each cross section of Figure 2.16(a) and (b) at a given dimensionless running speed $\bar{\Gamma}$ is the periodic solution of the journal orbit at this running speed. Figure 16(a) and (b) show that the periodic solutions of journal orbit shrink to a single point as the running speed approaches the critical value $\bar{\Gamma}_c$. Figure 2.17 shows the bifurcation profile, which depicts the amplitude of the periodic solution as a function of system running speed close to but below the critical speed $\bar{\Gamma}_c$. The subcritical bifurcation profile shrinks to a single point as the running speed approaches the critical value, $\bar{\Gamma}_c$. The amplitude of the periodic solution of system running speed close to but below the critical speed $\bar{\Gamma}_c$. The subcritical bifurcation profile shrinks to a single point as the running speed approaches the critical value, $\bar{\Gamma}_c$. The amplitude of the periodic solution of the journal orbit corresponding to a specific running speed $\bar{\Gamma}$ is symmetrical at $\bar{\Gamma}_c$ and is bounded by $\epsilon_s \pm \sqrt{\frac{\bar{\Gamma}-\bar{\Gamma}_c}{\mu_2}}$. Where ϵ_s is the static equilibrium position of journal center at speed $\bar{\Gamma}_c$.



Figure 2.16. Shapes of the periodic solutions of a short journal bearing (L/D = 0.5) at different running speeds considering shaft stiffness $S_z = 10$. (a) Laminar model; (b) Ng-Pan-Elrod turbulent model (Re = 10,000).

From Figure 2.17, it can be easily identified that as the oil flow region within the bearing chamber is transitioned from laminar into turbulent, the stability envelope shrinks in size whereas the whole stable boundary is shifted leftward, further expanding the unstable regions of operation.



Figure 2.17. Comparison of bifurcation profile of laminar vs. Ng-Pan-Elrod turbulent model (Re = 10,000) of a short journal bearing (L/D = 0.5).

Figure 2.13 and Figure 2.14 reveal that rotor stiffness has a pronounced influence on system's instability threshold speed and its bifurcation type. If the dimensionless rotor stiffness (S_z) is below its critical value, three bifurcation regions (by increasing Sommerfeld there are, subcritical, supercritical, and subcritical regions) exist, and if it is greater than its critical value, two bifurcation regions (by increasing Sommerfeld there are, supercritical and subcritical regions) exist. According to Figure 2.13 and Figure 2.14, the critical stiffness of the shaft is a strong function of the Reynolds number. The critical stiffness of a flexible shaft supported on short and long bearings are calculated at different Reynolds numbers using Ng-Pan-Elrod turbulent model and demonstrated in Figure 2.18(a) and (b).

As it is shown in Figure 2.18(a) and (b), the critical shaft stiffness strongly depends on Reynolds number at relatively low values for both short and long bearing supported shafts. For the case of short bearing supported shafts, the critical dimensionless shaft stiffness increases by

increasing the Reynolds number and becomes steady after $\overline{Re} = 70,000$. However, for the case of long bearing supported shafts, the critical dimensionless shaft stiffness decreases by increasing the Reynolds number and becomes steady after $\overline{Re} = 70,000$. It should be noted that, critical dimensionless shaft stiffness does not exist at very low Reynolds numbers ($0 \le \overline{Re} <$ 17,000) for long bearing supported shafts. This suggest that for a shaft supported on long bearings operating at low Reynolds numbers, increasing shaft stiffness cannot suppress the subcritical region at low Sommerfeld numbers.



Figure 2.18. Reynolds dependency of the critical stiffness of a flexible shaft supported on; (a) short journal bearings (L/D = 0.5), and (b) long journal bearings ($L/D \ge 2$).

2.6 Experimental Verification

Khonsari and Wang [33] designed and tested an experimental test rig for a flexible shaft supported on end journal bearings and identified the threshold speed of instability and bifurcation types for a range of bearing parameters. Bearing parameters are tabulated in Table 2.2. Their experimental results for a flexible shaft with non-dimensional shaft stiffness of $S_z = \left(\frac{c}{w}\right)K_s = 4$ is plotted with the numerical predictions based on the mathematical model of this chapter as shown in Figure 2.19. The experimental results are for laminar short bearing assumption as $\overline{Re} \leq 2000$.

As it can be seen from the Figure 2.19, there is good agreement in threshold speed calculation as well as a qualitative agreement on bifurcation types between the numerical predictions and the experimental results. Both results suggest a transition from supercritical to subcritical bifurcation as bearing parameter increases. The difference from the analytical estimates and the experiments is due to simplifications in the mathematical model such as perfectly balanced rotor, neglecting the fluid inertia, short bearing assumption, neglecting temperature dependence of oil viscosity, and shaft gyroscopic effects, as well as the simplifying assumption of concentrated mass at the shaft center.

Tuble 2.2. Specification of the fotor bearing system for experimental verification [55].		
Journal bearing diameter (D)	0.0254 m	
Journal bearing length (<i>L</i>)	0.0127 m	
Span length between two bearings	0.05271 m	
Inside diameter of the hollow shaft	0.0152 m	
Rotor mass (M)	5.4523 kg	
Journal bearing clearance (<i>C</i>)	50.8 ×10 ⁻⁶ m	
Lubricant type	ISO 32	
Inlet pressure	31 kPa	
Inlet temperature	20°C – 100°C	

Table 2.2. Specification of the rotor-bearing system for experimental verification [33].



Figure 2.19. Comparison between experimental and theoretical results at $S_z = 4$.

2.7 Conclusions and Rotor Bearing System Design Guidelines

In the current study, the stability of flexible shaft supported on end journal bearings was studied for a range of operating conditions in both laminar and turbulent flow bearings and for different available turbulent models. The dynamic oil film forces considering the turbulent effects were obtained analytically under the short and long bearing assumption considering the Gümbel boundary condition. The closed form expressions for eight spring and damping coefficients were derived by linearizing the turbulent oil film forces around the steady-state equilibrium position of the journal center for two available turbulent models. The calculated dynamic coefficients were then utilized to obtain the whirl onset velocity for a flexible rotor supported by two identical journal bearings at various Reynolds numbers. Stable operating region of flexible shafts supported on both turbulent and laminar journal bearings were shown to grow in size by increasing the shaft non-dimensional stiffness parameter. It was found that at low load and high Sommerfeld regions $S \ge 0.2$ the stability won't be affected by the type of flow within the bearing chamber. On the other hand, as the static load on shaft increases in magnitude (i.e. lower Sommerfeld numbers), the unstable operating region grows in size and will result in the unstable oil whirl to happen at even higher static loads, a region where the laminar based stability analysis would consider as safe operating condition. Results also showed that predicted threshold speed of instability based on Constantinescu's model are higher than the threshold speeds predicted by the Ng-Pan-Elrod turbulent model and hence the latter model proves to be more conservative in design.

The Hopf bifurcation theory was used to study the orbital stability of periodic solutions near bifurcating points of the rotor bearing system, i.e. when the operating spin speed of the shaft is close to its threshold speed of instability. A Hopf bifurcation subroutine was written in MATLAB to calculate bifurcation parameters. The calculated bifurcation parameters were used to identify subcritical and supercritical regions along the path of whirl onset boundaries (curves separating stable and unstable operating regions). Subcritical regions were found to be more prevalent at high static loads (low Sommerfeld numbers). The width of dangerous subcritical regions was shown to be a function of the shaft stiffness. The stability envelope for subcritical operating regions were found for both laminar and turbulent operating conditions. The smaller the size of the stability envelope, the rotor bearing system is more susceptible to applied external perturbations. It was shown that the size of the stability envelope does indeed shrink as the flow region is transitioned from laminar to turbulent and hence it is crucial to study turbulent effects at high Reynolds numbers to precisely estimate the susceptibility of the rotor bearing system to externally applied perturbations. A critical shaft stiffness value was found at any Reynolds number, beyond which the unstable low Sommerfeld region can be transformed supercritical (hence stable) for a majority of operating conditions. The calculated critical shaft stiffness was found to be a strong function of the Reynolds number at relatively low Reynolds numbers for both short and long bearing supported shafts. However, the Reynolds dependence of the critical shaft stiffness parameter follows different trends for short and long bearings. While the critical stiffness constantly grows in size with increasing Reynolds number in short bearing supported shafts, the critical stiffness for long bearing supported shafts decreases with Reynolds number and it ceases to exist at low Reynolds numbers. This suggest that for a shaft supported on long bearings operating at low Reynolds numbers, increasing shaft stiffness cannot suppress the unstable critical region. It is hence recommended that for shafts under high static loads, better design choice would be a bearing with low length to diameter ratio to avoid the unstable subcritical regions of operation. As bearing performance is strongly dependent on lubricant viscosity and since viscosity of common lubricants strongly depends on the oil temperature, the results of classical theory can be expected to apply only in cases where the temperature raise in lubricant is negligible across the bearing pad. A proper cavitation model needs to be developed to accurately calculate short and long bearing pressure under laminar and turbulent conditions. To correctly predict the bearing performance required for precise identification of instability phenomenon, both the flow regime (being laminar or turbulent), variation of lubricant viscosity, as well as cavitation need to be taken into account and the stability curves found here should be modified to include this effect. This modification is the subject of future work to the current thesis.

Experimental results in the literature are presented confirming the validity of the bifurcation diagram for flexible rotor-bearing system shown in Figure 2.8 and its practicality for design purposes.

Chapter 3: Flexible Rotor-Bearing System Design Considering Fluid Film Shear Effect: Application in Safe Operating Region Determination

3.1 Introduction

Exerted forces on fluid film bearings comprises of two different components. (1) Pressure force components in radial and tangential directions, and (2) Drag force (shear/friction force) components in radial and tangential directions. In most existing literature, related to rotor bearing system supported on journal bearing stability analysis, the effect of fluid film drag force has been neglected.

State of the art argued that shear force magnitude is in the order of journal bearing clearance over radius (*C/R*) times pressure force; however, provided numerical solutions of bearing force parameters based on finite bearing assumption show that at small eccentricity ratios ($\varepsilon \le 0.1$), considering short bearing theory ($L/D \le 0.5$), journal bearings shear force exceeds pressure force [82]. Akers et al. [83] concluded that, for the majority of a journal bearing operating region, inclusion of friction force in fluid film bearing stability analysis expands safe operating region. Utilizing nonlinear stability analysis, Wang and Khonsari [84] showed that the drag force has significant effects on the threshold speed, bifurcation profile, and the size and shape of periodic solutions of a rigid rotor symmetrically supported by two identical laminar journal bearings. They did not expand their analysis for the entire operating region of journal bearings and limited their study based on one operating condition.

This section provides drag force effects on the threshold speed of instability of a flexible rotor bearing system under turbulent flow assumption for the entire operating condition. Nonlinear stability analysis, using Hopf bifurcation theory, has been utilized to obtain bifurcation profiles for short journal bearings.

3.2 Dynamic Fluid Film Forces of a Rotor-Bearing System Considering Shear Effect

Figure 3.1 demonstrates radial and tangential components of friction forces applied on fluid film bearings. Assuming constant oil viscosity throughout the fluid film, using Ng-Pan-Elrod turbulent model and based on short bearing theory with half-Sommerfeld boundary conditions equations of motion can be derived based on the following steps. Turbulent short bearing
pressure forces can be obtained from Equation (2.12). Viscous shear stress at journal surface of short fluid film bearings, under turbulent flow condition, can be obtained as follow [76, 85],



$$D_s = \left(\frac{k_\theta}{12}\right) \frac{R\mu\omega}{h} \tag{3.1}$$

Figure 3.1. Fluid film bearing friction force components in radial and tangential directions.

where *h* is the fluid film thickness and k_{θ} is turbulent shear coefficient in circumferential direction which can be obtained from Equation (2.5). Under laminar flow assumption, k_{θ} approaches 12 ($\overline{Re} \rightarrow 0$). It this section Ng-Pan-Elrod model is utilized as it is proved to provide more conservative results in rotor bearing system design applications (stability analysis).

Under continuity assumption, considering circumferential flow rate remains constant in cavitation region ($\pi \le \theta < 2\pi$ for Gümbel boundary condition), the following statement can be obtained [6],

$$q_{cav} = q \ (at \ h = h_{min}) \tag{3.2}$$

Hence, effective length of fluid film bearing within cavitation region can be calculated as,

$$h \times L_{eff} = h_{min} \times L \to L_{eff} = \frac{L(1-\epsilon)}{1+\epsilon Cos(\theta)}$$
(3.3)

The resulting fluid film turbulent shear forces acting on journal bearing in radial and tangential directions (rotating system of coordinates) can be found through integration of Equation (3.1) over the area of journal bearing considering Gümbel (π film) boundary condition which is provided as follow,

$$D_{R} = -R\left(\int_{0}^{\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}}D_{s}Sin(\theta)dz\,d\theta + \int_{\pi}^{2\pi}\int_{-\frac{L_{eff}}{2}}^{\frac{L_{eff}}{2}}D_{s}Sin(\theta)dz\,d\theta\right)$$

$$D_{t} = R\left(\int_{0}^{\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}}D_{s}Cos(\theta)dz\,d\theta + \int_{\pi}^{2\pi}\int_{-\frac{L_{eff}}{2}}^{\frac{L_{eff}}{2}}D_{s}Cos(\theta)dz\,d\theta\right)$$
(3.4)

In order to be able to integrate the above equation to find an analytical expression for shear force in tangential direction, it is assumed $((1 + \epsilon Cos(\theta)))^{\hat{a}_2} \approx (1 + \hat{a}_3 \epsilon Cos(\theta))$. The constant \hat{a}_3 may be found by curve fitting via optimization, which is calculated to be 0.8437, and 0.91 for Constantinescu and Ng-Pan-Elrod turbulent models, respectively. By applying the aforementioned assumption along with applying Gümbel boundary condition, the nondimensional form of the turbulent short bearing shear forces can be derived as,

$$D_{R} = -\frac{R^{2}\mu\omega L}{12C} \left(\frac{H((1+\epsilon)^{\hat{a}_{2}} - (1-\epsilon)^{\hat{a}_{2}}) + 12\hat{a}_{2}\ln(\frac{1+\epsilon}{1-\epsilon})}{\epsilon\hat{a}_{2}} + \frac{H((1+\epsilon)(1-\epsilon)^{\hat{a}_{2}} + (\epsilon-1)(1+\epsilon)^{\hat{a}_{2}}) + 24\epsilon(1-\hat{a}_{2})}{\epsilon(1+\epsilon)(\hat{a}_{2}-1)} \right)$$

$$D_{t} = \frac{R^{2}\mu\omega L}{12C} \left(\frac{\pi}{\epsilon}\right) \left((12+H(1-\hat{a}_{3}))\left(1-\frac{1}{\sqrt{1-\epsilon^{2}}}\right) \right)$$
(3.5)

$$+\frac{H\hat{a}_{3}(\sqrt{1-\epsilon^{2}}-1)-\epsilon^{2}\left(12+H+H\hat{a}_{3}(\sqrt{1-\epsilon^{2}}-2)\right)}{(1+\epsilon)\sqrt{1-\epsilon^{2}}}\right)$$

where $H = \hat{a}_1(\overline{Re})^{\hat{a}_2}$ and $M = a_1(\overline{Re})^{a_2}$. The total force acting on the journal bearing can be calculated as follow,

$$F_R^t = F_R + D_R$$

$$F_t^t = F_t + D_t$$
(3.6)

where F_R and F_t are analytical short bearing turbulent forces from Equation (2.14) [86]. Having the force components in *R* and *T* directions by means of Equation (3.6), the non-dimensional bearing stiffness and damping coefficients in *R*-*T* coordinate system are calculated using Equation (2.15). Analytical short bearing turbulent coefficients considering shear force effect are tabulated in Table A.3.

The dimensionless analytically calculated values for stiffness and damping of a short journal bearing coefficients in both laminar and turbulent regimes (considering shear effect) at different Reynolds numbers, for a rotor bearing setup mentioned in Table 2.2, are plotted against their calculated Sommerfeld number ($S = \frac{\mu\omega LR^3}{\pi WC^2}$) and are shown in Figure 3.2 and Figure 3.3, respectively.

By comparing the results of direct stiffness coefficients, Figure 3.2 and Figure 2.4, it can be concluded that stiffness coefficients at high Sommerfeld number increase by increasing Reynolds number considering fluid shear effect; however, opposite trend was observed through ignoring shear force effect at high Sommerfeld numbers. Figure 3.3 and Figure 2.5 demonstrate minor changes on damping coefficients when shear effect is included in total force components.

By making use of the derivations in Appendix B, the stability margins of a rotor bearing system, see Table 2.2, for different non-dimensional shaft stiffness coefficients $(S_z = \frac{C K_s}{W})$ are calculated. Stability parameter $\overline{\Gamma} = \frac{C}{W} M \omega_s^2 = \left(\frac{C K_s}{W}\right) \overline{\omega}_s^2 = S_z \overline{\omega}_s^2$ is plotted vs the Sommerfeld number as shown in Figure 3.5.

Assuming shaft stiffness $(\frac{C K_S}{W} = 100)$, Figure 3.4 (a) shows turbulence and shear force have pronounced influence on the steady state eccentricity ratio in the range of $0.008 \le S \le 10$. Figure 3.4 (b) demonstrates that, when Sommerfeld number is greater than 3, whirl frequency ratio approaches a constant number (0.49); hence, turbulent effect becomes negligible.

By comparing Figure 2.9 and Figure 3.5, it can be seen that as the non-dimensional shaft stiffness increases, by increasing the Reynolds number, the stable operating margin expands towards higher threshold speeds at high Sommerfeld numbers ($S \ge 0.6$) and hence the rotor-bearing system becomes stable at higher rotating spin speeds of the shaft; however, by ignoring the shear force effect, stability margin decreases by increasing the Reynolds number.



Figure 3.2. Stiffness coefficients comparison of laminar and turbulent short journal bearings considering shear effect (L/D = 0.5), (a) K_{xx} , (b) K_{xy} , (c) $abs(K_{yx})$, and (d) K_{yy} .



Figure 3.3. Damping coefficients comparison of laminar and turbulent short journal bearings considering shear effect (L/D = 0.5), (a) C_{xx} , (b) C_{xy} , (c) C_{yx} , and (d) C_{yy} .



Figure 3.4. Turbulent and shear force effects on, (a) eccentricity ratio, and (b) whirl frequency ratio.



Figure 3.5. Turbulent and shear effects on the stability parameter Γ of a flexible shaft supported on short length journal bearings (L/D = 0.5) at non-dimensional shaft stiffness coefficients of, (a) $CK_s/W = 0.1$, (b) $CK_s/W = 1$, (c) $CK_s/W = 10$, and (d) $CK_s/W = 100$.

3.3 Equations of Motion of a Flexible Rotor Bearing System: Balanced Response

The dimensionless equations of motion for a rotor-bearing system consisting of a flexible rotor supported by two identical fluid-film journal bearings are presented by Equations (2.24) and (2.25) which can be rewritten in the following nondimensionalize format,

$$\bar{x}_{2}^{\prime\prime} + \frac{S_{z}}{\bar{\Gamma}}(\bar{x} - \epsilon \cos \varphi) - \frac{2}{\bar{\Gamma}} = 0$$

$$\bar{y}_{2}^{\prime\prime} + \frac{S_{z}}{\bar{\Gamma}}(\bar{y} - \epsilon \sin \varphi) = 0$$

$$2\bar{F}_{R}^{t} = \frac{S_{z}}{S}(\epsilon - \bar{x} \cos \varphi - \bar{y} \sin \varphi)$$

$$2\bar{F}_{T}^{t} = \frac{S_{z}}{S}(\bar{x} \sin \varphi - \bar{y} \cos \varphi)$$
(3.8)

where "′" represents $d/(\omega dt)$. $\bar{x} = \frac{x}{c}$ and $\bar{y} = \frac{y}{c}$ provide nondimensionalize shaft center location in Cartesian coordinate. \bar{F}_R^t and \bar{F}_T^t are the total dimensionless force componenets in radial and tangential directions, respectively. Assume $x_1 = \epsilon, x_2 = \varphi, x_3 = \bar{x}, x_4 = \bar{x}', x_5 = \bar{y}, x_6 = \bar{y}'$. To solve the above nonlinear equations of motion, two second-order nonlinear equations of motion Equation (3.7), are converted into four first-order Equations (3.11)-(3.14). Through simultaneously solving two terms from Equation (3.8); Equations (3.9) and (3.10) can be calculated in terms of (x_1, x_2, x_3, x_5) .

$$x_1' = \mathbf{Func}[x_1, x_2, x_3, x_5]$$
(3.9)

$$x_2' = \mathbf{Func}[x_1, x_2, x_3, x_5]$$
(3.10)

$$x'_3 = x_4$$
 (3.11)

$$x'_{4} = -\frac{S_{z}}{\bar{\Gamma}}(x_{3} - x_{1}\cos x_{2}) + \frac{2}{\bar{\Gamma}}$$
(3.12)

$$x_5' = x_6 \tag{3.13}$$

$$x_{6}' = -\frac{S_{z}}{\overline{\Gamma}} (x_{5} - x_{1} \operatorname{Sin} x_{2})$$
(3.14)

The above system of equations as give in Equations (3.9)-(3.14) have the suitable form of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \overline{\Gamma})$ and possess a steady state equilibrium position \mathbf{x}_s for the application of the Hopf bifurcation theory. The steady state equilibrium position \mathbf{x}_s in terms of $(x_{1s} = \epsilon_s, x_{2s} = \varphi_s, x_{3s} = \overline{x}_s, x_{4s} = \overline{x}'_s, x_{5s} = \overline{y}_s, x_{6s} = \overline{y}'_s)$ can be found analytically by forcing $\mathbf{f}(\mathbf{x}, \overline{\Gamma}) = 0$

3.4 Application of Hopf Bifurcation Theory in Flexible Rotor Bearing Systems Considering Shear Force Effect

Consider the rotor-bearing system whose specifications are listed in Table 3.1. This perfectly balanced rotor-bearing system consists of a flexible shaft symmetrically supported by two identical fluid-film journal bearings. Based on the Hopf bifurcation theory explained in the previous chapter (see Chapter 2), the nonlinear stability analysis of a rotor bearing system tabulated in Table 3.1, considering shear force effect, is presented in Figure 3.6.

Figure 3.6 demonstrates significant discrepancies between two stability regions (with and without the consideration of shear effect) at high Sommerfeld numbers ($S \ge 0.4$). The difference between two bifurcation profiles grows in zise by increasing the Sommerfeld number.

Table 3.1. Specification of the rotor-bearing system	
Journal bearing diameter (D)	0.0254 m
Journal bearing length (<i>L</i>)	0.0127 m
Nondimensionalize shaft stiffness (S_z)	10
Average Reynolds number <i>Re</i>	10000
Journal bearing clearance (<i>C</i>)	50.8 ×10 ⁻⁶ m

Bifurcation profile results at Sommerfeld (S = 1) are also demonstrated in Figure 3.7 to compare dimensionless instability threshold speed with and without consideration of shear force effect. Figure 3.7 shows that the periodic solutions of the journal orbit shrinks to a single point as the running speed approaches the critical value $\bar{\Gamma}_c$. It is shown that the dimensionless instability threshold speed $\bar{\Gamma}_{cr}$ increases from 9.04 to 10.09 (11.6% increase) through considering shear force components in the equations of motion.

It should be noted that by converting dimensionless instability threshold speed to a dimensional form ($\omega_{cr} = \sqrt{\frac{g\bar{r}_{cr}}{2c}}$), the instability threshold speed with consideration of the drag force is 9,881.54 rpm and the instability threshold speed neglecting the drag force becomes 8,921.61 rpm (959.93 rpm difference).



Figure 3.6. Nonlinear stability analysis comparison with and without consideration of shear force effect.

The amplitude of the periodic solution of the journal orbit corresponding to a specific running speed $\overline{\Gamma}$ is symmetrical at $\overline{\Gamma}_c$ and is bounded by $\epsilon_s \pm \sqrt{\frac{\overline{\Gamma} - \overline{\Gamma}_c}{\mu_2}}$. Where ϵ_s is the static equilibrium position of journal center at speed $\overline{\Gamma}_c$.

Trajectories of operating point I (S = 3, $\overline{\Gamma} = 10$), shown in Figure 3.6, are calculated for two different cases with and without consideration of the shear force effect and demonstrated in Figure 3.8. As it can be seen from Figure 3.8, stable region obtained based on consideration of shear force effect can become unstable by neglecting the drag force effect in equations of motion of a flexible rotor-bearing system.



Figure 3.7. Bifurcation profile comparison with and without consideration of shear effect at Sommerfeld (S) = 1.



Figure 3.8. Journal bearing trajectories of a balanced rotor-bearing system at operating point I, (a) considering shear effect, (b) neglecting shear effect.

The shear force effect results on the stability of a flexible rotor-bearing system presented above are consistent with the outcomes provided by [84, 87]. Newkirk and Grobel [87] concluded that shear component applying opposite to the direction of rotating speed of the shaft tends to stabilize the rotor bearing system.

It shall be noted that not only the difference between two bifurcation profiles (with and without consideration of shear effect) grows by increasing the Sommerfeld number, but also the discrepancy becomes significant by increasing the Reynolds number. Thus, inclusion the shear force effect in the rotor bearing equations of motion always causes the system to become more stable, especially at high Sommerfeld numbers, by increasing the Reynolds number which is in contradiction with the results obtained in the previous chapter. In Chapter 2, shear force effect was neglected.

3.5 Equations of Motion of a Flexible Rotor Bearing System: Unbalanced Response

The simplified rotor-bearing system consists of a flexible, center-loaded rotor that is symmetrically supported by two identical fluid-film journal bearings. The schematic of the rotor-bearing system is given in Figure 2.8. The equations of motion, considering unbalance force, for the central disk and journal bearings can be written as follows,

$$Central Disk \rightarrow \begin{cases} M\ddot{x}_{2} + K_{s}(x - C\epsilon \cos \varphi) - Mg - mr\omega^{2} \cos \theta = 0\\ M\ddot{y}_{2} + K_{s}(y - C\epsilon \sin \varphi) - mr\omega^{2} \sin \theta = 0 \end{cases}$$

$$Journal Bearings \rightarrow \begin{cases} 2(F_{r} \cos \varphi - F_{t} \sin \varphi) + K_{s}(x - C\epsilon \cos \varphi) = 0\\ 2(F_{r} \sin \varphi + F_{t} \cos \varphi) + K_{s}(y - C\epsilon \sin \varphi) = 0 \end{cases}$$

$$(3.15)$$

Where *M* is the total mass of the rotor-bearing system, *m* is the unbalance mass at the central disk location, and *r* is the unbalance mass eccentricity ratio. Assume $x_1 = \epsilon, x_2 = \varphi, x_3 = \bar{x}, x_4 = \bar{x}', x_5 = \bar{y}, x_6 = \bar{y}', x_7 = \theta$. The dimensionless equations of motion for an unbalanced rotor-bearing system consisting of a flexible rotor supported by two identical fluid-film journal bearings can be written as,

$$x_1' = \mathbf{Func}[x_1, x_2, x_3, x_5]$$
(3.16)

$$x_2' = \mathbf{Func}[x_1, x_2, x_3, x_5]$$
(3.17)

$$x_3' = x_4 \tag{3.18}$$

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$$x'_{4} = -\frac{S_{z}}{\overline{\Gamma}}(x_{3} - x_{1}\cos x_{2}) + \frac{2}{\overline{\Gamma}} + \gamma \cos x_{7}$$
(3.19)

$$x_5' = x_6 (3.20)$$

$$x_{6}' = -\frac{S_{z}}{\bar{\Gamma}}(x_{5} - x_{1}\sin x_{2}) + \gamma \sin x_{7}$$
(3.21)

$$x_7' = 1$$
 (3.22)

where $\gamma = \frac{m}{M} \frac{r}{c}$ can be defined as dimensionless unbalance moment. The operating point *II* (*S* = 3, $\overline{\Gamma} = 5$), as marked in Figure 3.6, is selected to demonstrate the system dynamics at an operating point below threshold speed of instability for two different values of γ . From Figure 3.9, it is evident that, as operating point *II* is in subcritical region, by increasing the dimensionless unbalance moment (γ), stability behavior of an unbalanced rotor bearing system changes from stable to unstable at an operating speed way below threshold speed of instability.

It can be concluded that, through utilizing the proposed method, safe operating rage of dimensionless unbalance moment (γ) in subcritical bifurcation region can be identified for rotor-bearing system design and/or tuning purposes.



Figure 3.9. Journal bearing trajectories of an unbalanced rotor-bearing system at operating point *II* considering shear effect, (a) stable rotor bearing system at $\gamma = 0.2$, (b) unstable rotor bearing system at $\gamma = 0.3$.

3.6 Experimental Verifications

3.6.1 Verification Example I

Khonsari and Wang [33] designed and tested an experimental test rig for a flexible shaft supported on end journal bearings and identified the threshold speed of instability and bifurcation types for a range of bearing parameters. Bearing parameters are tabulated in Table 2.2. Their experimental results for a flexible shaft with non-dimensional shaft stiffness of $S_z = \left(\frac{c}{w}\right)K_s = 4$ is plotted with the numerical predictions based on the mathematical model of this chapter (considering shear force effect) as shown in Figure 3.10. The experimental results are for laminar short bearing assumption as $\overline{Re} \leq 2,000$.



Figure 3.10. Comparison between results reported by Khonsari and Wang [33] and analytical results ($S_z = 4$) considering shear force effect for laminar flow ($Re \le 2,000$).

As it can be seen from the Figure 3.10, there is a good agreement in threshold speed of instability calculation as well as a qualitative agreement on bifurcation types between the numerical predictions and the experimental results. Through comparison between Figure 2.19 and Figure 3.10, it can be concluded that, consideration of shear force effect leads to better estimation of subcritical region for a rotor bearing system, especially at high Sommerfeld numbers. Both

results, experimental and analytical, suggest a transition from supercritical to subcritical bifurcation as bearing parameter (*S*) increases. The difference from the analytical estimates and the experiments is due to simplifications in the mathematical model such as perfectly balanced rotor, neglecting the fluid inertia, short bearing assumption, neglecting temperature dependence of oil viscosity, and shaft gyroscopic effects, as well as the simplifying assumption of concentrated mass at the shaft center.

3.6.2 Verification Example II

Pinkus [88] designed and tested an experimental test rig for a flexible shaft, centrally loaded, supported by two identical journal bearings. Flexible rotor bearing parameters utilized by Pinkus [88] are tabulated in Table 3.2. Their experimental results for a flexible shaft with non-dimensional shaft stiffness of $S_z = \left(\frac{c}{w}\right)K_s = 4$ are tabulated in Table 3.3 for two different oil viscosities ($\mu_1 = 0.065$ Pa.s (25 °C) and $\mu_2 = 0.014$ Pa.s (60 °C)).

Pinkus [88] reported at oil viscosity $\mu = \mu_1$ the rotor bearing system could start to whip at a dimensionless speed below the critical speed of the shaft ($\overline{\Gamma} = 4.035 < \overline{\Gamma}_{cr} = 8.6179$) which indicates subcritical bifurcation region.

Table 3.2. Specification of the rotor-bearing system [88].		
Journal bearing diameter (D)	0.0254 m	
Journal bearing length (L)	0.0127 m	
Journal bearing clearance (<i>C</i>)	63.5 ×10 ⁻⁶ m	
Rotor mass (M)	84.823 kg	

Table 3.3. Experimentally measured instability threshold speed [88].

Oil viscosity μ (at °C)	Sommerfeld (S)	Critical Shaft Speed ($\overline{\Gamma}_{cr}$)	Bifurcation Type
0.065 Pa.s (25 °C)	4.189	8.6179	Subcritical observed
0.014 Pa.s (60 °C)	0.8072	6.8991	Supercritical obs.

Experimental (see Table 3.3) and theoretical comparison results, considering shear effect, is presented in Figure 3.11. In Figure 3.11, experimental subcritical bifurcation result is shown by open square while the solid square denotes supercritical bifurcation. As it can be seen from Figure 3.11, proposed mathematical model perfectly estimates experimental instability

threshold speeds ($\overline{\Gamma}_{cr}$) and rotor bearing system bifurcation type at two different journal bearing oil temperatures.

Based on the Hopf bifurcation theory, even at running speeds lower than shaft critical speed (instability threshold speed $\overline{\Gamma}_{cr}$), unstable periodic solution could exist if the system goes through subcritical bifurcation [86, 89, 90]. Therefore, if the magnitude of the external perturbation is strong enough to position the journal center outside the stability envelope region, the system will become unstable even at speeds below threshold speed of instability of the rotor bearing system. This conclusion matches with experimental outcomes by Pinkus [88].



Figure 3.11. Comparison between results reported by Pinkus [88] and analytical results ($S_z = 4$) considering shear force effect for laminar flow ($Re \le 2,000$).

Linear stability analysis, see Appendix B, is not capable of providing stability type (supercritical/subcritical) and size of the stability envelope. According to nonlinear stability analysis shown in Figure 3.11, by increasing oil temperature from 25°C to 60 °C ($\mu_1 = 0.065$ Pa.s to $\mu_2 = 0.014$ Pa.s (60 °C)), system bifurcation type changes from subcritical to supercritical for the experimental setup reported in Table 3.2. At oil viscosity $\mu_1 = 0.065$,

system bifurcation type is subcritical; thus, large enough transient perturbation could force a stable rotor bearing system to become unstable. This is the main reason Pinkus [88] reported unstable rotor bearing system at a rotating speed 50% lower than instability threshold speed of the shaft. At oil viscosity $\mu_2 = 0.014$ rotor bearing system does not become unstable by applying external perturbation as system bifurcation type is supercritical.

3.6.3 Verification Example III

Hori and Kato [91] studied an experimental test rig for a flexible shaft, centrally loaded rotor system, supported by two identical journal bearings. Flexible rotor bearing specifications employed by Hori and Kato [91] are tabulated in Table 3.4. The typical usage of these bearings are in generators with dimensionless frequency ratio of $\frac{\overline{\Gamma}}{\overline{\omega}_s^2} = 1$. Adams and Guo [92, 93] conducted linear and nonlinear stability analysis for different values of B_p of the rotor bearing system reported in Table 3.4. The instability threshold speeds predicted through nonlinear analysis by Adams and Guo [92, 93] is tabulated in Table 3.5.

Table 3.4. Specification of the rotor-bearing system [92, 93].		
Journal bearing diameter (D)	0.662 m	
Journal bearing length (L)	0.331 m	
Journal bearing clearance (<i>C</i>)	$9.93 \times 10^{-4} \text{ m}$	
Journal bearing viscosity (μ)	0.0056 Pa.s	

Experimentally measured instability threshold speeds, conducted by Adams and Guo [92, 93], along with their bifurcation types for four (4) different data points are tabulated in Table 3.5. It should be noted that, hysteresis phenomenon only exists if rotor bearing system goes through subcritical bifurcation (see Chapter 4).

Table 3.5. Experimentally measured instability threshold speeds along with their bifurcation types reported by Adams and Guo [92, 93].

B_p	$\overline{\Gamma}_{cr}$	Hysteresis	Bifurcation Type
0.5	2.1054	No	Supercritical
0.2	2.9234	Yes	Subcritical
0.1	6.3297	Yes	Subcritical
0.05	14.4507	Yes	Subcritical

where B_p can be calculated by [92, 93],

$$B_p = \frac{\mu L}{\pi M} \left(\frac{R}{C}\right)^3 \sqrt{\frac{C}{g}}$$
(3.23)

Sommerfeld number can be rearranged in the following format,

$$S = \frac{\mu L}{\pi M} \left(\frac{R}{C}\right)^3 \sqrt{\frac{C}{g}} \times \omega \sqrt{\frac{C}{g}}$$
(3.24)

Thus, the relation between S, B_p , and $\overline{\Gamma}$ can be written as,

$$S = B_p \times \sqrt{\frac{\overline{\Gamma}}{2}}$$
(3.25)

where $\overline{\Gamma} = \left(\frac{CK_s}{W}\right)\overline{\omega}_s^2$. Since $\frac{\overline{\Gamma}}{\overline{\omega}_s^2} = 1$ for the flexible rotor bearing system used by Adams and Guo [92, 93], dimensionless rotor shaft stiffness can be calculated as,

$$S_z = \frac{CK_s}{W} = \frac{\overline{\Gamma}}{\overline{\omega}_s^2} = 1 \tag{3.26}$$

Based on Equation (3.25), Sommerfeld numbers for each measured instability threshold speed and its associated B_p can be summarized in Table 3.6.

Table 3.6. Experimentally measured instability threshold speeds and their corresponding Sommerfeld numbers at $S_z = 1$ [92, 93].

B_p	S	$\overline{\Gamma}_{cr}$	Bifurcation Type
0.5	0.5130	2.1054	Supercritical
0.2	0.2418	2.9234	Subcritical
0.1	0.1779	6.3297	Subcritical
0.05	0.1343	14.4507	Subcritical

As it can be seen from Figure 3.12, proposed mathematical model perfectly estimates experimental instability threshold speeds ($\overline{\Gamma}_{cr}$) and rotor bearing system bifurcation type at four different journal bearing operating conditions.

In Figure 3.12, experimental subcritical bifurcation result is shown by open square while the solid square denotes supercritical bifurcation. Figure 3.12 demonstrates the proposed method is accurately estimates rotor bearing system bifurcation types at low Sommerfeld numbers.



Figure 3.12. Comparison between results reported by Adams and Guo [92, 93] and analytical results ($S_z = 1$) considering shear force effect for laminar flow ($Re \le 2,000$).

3.7 Conclusions and Rotor Bearing System Design Guidelines

In this chapter, the stability of flexible shaft supported on end journal bearings was studied for a range of operating conditions in turbulent flow bearings considering shear force effect. The dynamic turbulent oil film forces considering shear effects were obtained analytically under the short bearing assumption considering the Gümbel boundary condition. The closed form expressions for eight spring and damping coefficients were derived by linearizing the oil film forces around the steady-state equilibrium position of the journal center for Ng-Pan-Elrod model considering shear effect. The calculated dynamic coefficients were then utilized to obtain the whirl onset velocity for a flexible rotor supported by two identical journal bearings at various Reynolds numbers. Stable operating region of flexible shafts supported on both turbulent and laminar journal bearings were shown to grow in size by increasing the shaft non-dimensional stiffness parameter.

At high shaft stiffness numbers ($S_z \ge 5$), it was found that at low load and high Sommerfeld regions $S \ge 0.7$ the stability will be affected by the type of flow within the bearing chamber and

stable operating region increase by increasing the Reynolds number, a region where the laminar based stability analysis would consider as un-safe operating condition. Neglecting the shear force effect would result in opposite trend at high Sommerfeld numbers (see Chapter 2). On the other hand, as the static load on shaft increases in magnitude (low Sommerfeld numbers), the unstable operating region grows in size and will result in the unstable oil whirl to happen at even higher static loads.

The Hopf bifurcation theory was used to study the orbital stability of periodic solutions near bifurcating points of the rotor bearing system, i.e. when the operating spin speed of the shaft is close to its threshold speed of instability. It was shown that, considering shear force effect would increase stable operating region, especially when rotor bearing system stability type undergoes subcritical bifurcation. Safe operating range of dimensionless unbalance moment (γ) in subcritical bifurcation region can be identified for rotor-bearing system design purposes.

Three experimental results in the literature are presented confirming the validity of the bifurcation diagram, considering shear effect, for a flexible rotor-bearing system shown in Figure 2.8 and its practicality for design purposes. The instability threshold speed and its predicted bifurcation type using the proposed method could be beneficial at the design stage as well as for troubleshooting purposes of an unstable rotor-bearing system. If the bifurcation type of a rotor bearing system, operating at S^* and $\overline{\Gamma}^*$, is in the subcritical bifurcation region, effort should be made on changing S^* through controlling the oil temperature or even oil grade to shift bifurcation type from subcritical to supercritical. Next chapter is devoted to control of journal bearing oil viscosity.

Chapter 4: Rotor Bearing System Hysteresis Phenomenon: Application in Fluid Film Viscosity Determination

4.1 Introduction

The application of Hopf bifurcation theory (HBT) for a rigid rotor bearing system instability analysis supported by two identical long journal bearings was first introduced by Myers [81]. He presented two concepts of supercritical and subcritical bifurcation regions in connection with rotor bearing stability analysis. A similar analysis utilizing short bearing theory was published by Hollis and Taylor [79]. Sundararajan [94] and Noah et al. [95] conducted nonlinear analysis (HBT) on a more general case, using finite length journal bearings, and discussed the effects of different length/diameter (L/D) ratios on the subcritical bifurcation and supercritical bifurcation regions.

Deepak and Noah [96] provided experimental data verifying the existence of the subcritical bifurcation region of a single disk rotor supported on a short journal bearing. Wang and Khonsari [32, 33] extended the application of HBT to fluid film lubrication at different Reynolds numbers. They tried to explain the effect of oil inlet temperature experimentally on the instability threshold speed [26, 54]; however, no mathematical model was presented and no estimate for stability boundary region was provided.

One of the most important topics that has not received sufficient attention is to accurately predict rotor bearing system hysteresis phenomenon based on a mathematical model. Experimentally observed hysteresis phenomenon was first described by Pinkus [88]. He mentioned that, "when whipping was observed under conditions of decreasing speeds, it was noted that whip persisted down to speeds lower than those at which whip started when the speed was being increased". Hori [97] later on confirmed the results provided by Pinkus [88]; However, hysteresis phenomenon did not receive much attention until Hori [98] presented his thorough discoveries in 1988. Subsequently, Guo and Adams [92, 93] published some trial-and-error simulations and experimental results and tried to explain the nature of the hysteresis phenomenon concluding an elusive unstable intermediate solution exists in the hysteresis loop. Horattas [99] conducted a set of experiments and confirmed the existence of the hysteresis phenomenon along with

partially verification of the results reported by Guo and Adams earlier. Muszynska [100] tried to explain the hysteresis phenomenon qualitatively. According to Muszynska, in the rotor bearing system runup process, the fluid circumferential average velocity ratio, is larger than that in the rundown process.

For rotor bearing nonlinear stability analysis, HBT is used to find the local stability of periodic solutions near bifurcation operating points. In this chapter, it is shown that, there is a good agreement between predicated bifurcation profile, utilizing proposed analytical method, and experimental results provided by Wang and Khonsari [54]. The fluid film oil viscosity is found to play an important role in bifurcating regions on the stable boundaries. Safe operating fluid film viscosity range is proposed for rotor bearing system design purposes.

4.2 Rotor Bearing System Hysteresis Phenomenon

In order to identify a rotor bearing instability threshold speed (oil whirl/ship phenomenon), one can monitor dynamic behavior of the system as system operating speed gradually increases (runup process). Once shaft operating speed (ω_s) passes threshold speed of instability (ω_c) through run-up process, one can monitor dynamic behavior of the system as system operating speed gradually decreases in the run-down process. In certain operating conditions, for the run-down case, oil whip phenomenon disappears at a running speed that is below the threshold speed monitored during the run-up process. This hysteresis phenomenon is, in fact, repeatable and occurs even though all other system parameters including oil viscosity remain unchanged. The characteristics of the hysteresis phenomenon, for subcritical region, and rotor bearing system behavior associated with a perfectly balanced, lightly loaded rotor bearing system without any misalignment are illustrated in Figure 4.1(a). When hysteresis exists, the speed at which the oil whip starts is called the run-up threshold speed (ω_{uc}); and the speed at which the oil whip disappears is called the rundown threshold speed (ω_{dc}). As it is demonstrated by Figure 4.1(a), in subcritical bifurcation region, by increasing the rotating speed of the shaft there is a sudden jump (oil whip phenomenon) in journal bearing amplitude after passing ω_{uc} . After this point, the vibration amplitude suddenly reaches to its maximum value (shaft clearance) instead of ramping up slowly as it is observed in supercritical bifurcation region (see Figure 4.1(b)).

As it is shown in Figure 4.1(a), upon decreasing the rotating speed of the shaft, at point ω_{dc} ($\omega_{dc} \leq \omega_{uc}$), the oil whip suddenly disappears, and the vibration amplitude shrinks to its steady state equilibrium position. The dotted line curve between ω_{dc} and ω_{uc} is called subcritical bifurcation profile. Considering subcritical bifurcation region, if a journal is released from a point inside the bifurcation profile with zero initial velocity, its orbit will converge into the steady-state equilibrium position; however, if released from outside the bifurcation profile with zero initial velocity, its orbit will diverge until it reaches to the clearance of the journal bearing (whip condition). For the supercritical bifurcation region, there is a smooth transition after crossing the instability threshold speed (ω_c). In this case hysteresis does not exist and journal bearing amplitude undergoes whirl motion, hence, stable journal bearing orbital amplitude gradually increases by increasing the rotating speed of the shaft.



Figure 4.1. Perfectly balanced rotor bearing system behavior, (a) hysteresis exist (subcritical bifurcation region), (b) hysteresis does not exist (supercritical bifurcation region).

The stability of the rotor bearing system for any running speed between ω_{dc} and ω_{uc} depends on how large the perturbation amplitude is [54]. If the amplitude of the perturbation is located inside the subcritical bifurcation profile, the system will return to the its equilibrium position. If the perturbation amplitude is outside the subcritical bifurcation profile, the perturbation will trigger oil whip causing the system to become unstable.

The hysteresis phenomenon occurs only in the case of subcritical bifurcation due to existence of unstable periodic solution (i.e., unstable bifurcation profile). The subcritical bifurcation profile determines the profile of the hysteresis loop as shown in Figure 4.1(a).

4.3 Experimental Verifications

Khonsari and Wang [54] utilized the experimental test setup designed by Chauvin [101] for a lightly-loaded rotor bearing system, considering flexible shaft, supported by two identical hydrodynamic journal bearings. The specification of the rotor bearing system are tabulated in Table 4.1. Experimental setup shown in Figure 4.2 is equipped with a heating and cooling system, to control fluid film viscosity, capable of supplying oil with controllable oil inlet temperature from 0 to 180°C. Khonsari and Wang reported the system was found to be always stable at running speeds below 7500 rpm [54].

Since Sommerfeld number is a function of the rotating speed of the shaft (ω_s), in order to accurately measure fluid film viscosity (fluid film temperature) effect on the threshold of instability, the following dimensionless parameters utilized,

$$\alpha = \frac{2\pi (L/D)^2 S}{\overline{\Omega}_k} \tag{4.1}$$

$$\overline{\Omega}_k = \sqrt{\frac{\overline{\Gamma}}{2}} \tag{4.2}$$

$$K_k = S_z/2 \tag{4.3}$$

Using the above mentioned dimensions parameters (Equations (4.1)-(4.3)), assuming parameters reported in Table 4.1 are constant, α varies only by changing the fluid film viscosity. Khonsari and Wang [54] conducted two different experimental tests at fluid film viscosities of 7mPa.s and 16mPa.s. Instability threshold speed determined accurately by ramping up the shaft

speed as slowly as possible until approaching the instability threshold speed (run-up test). After observing complete whip phenomenon, the rotating speed of the shaft decreased (run-down test) until oil whip disappeared. Their experimental results for a flexible shaft with non-dimensional shaft stiffness of $K_k = 4$ is plotted (instability threshold speed vs α) with the numerical predictions based on proposed mathematical model as shown in Figure 4.3. As it can be seen from the Figure 4.3, there is good agreement in threshold speed of instability calculation as well as a qualitative agreement on bifurcation types between the numerical predictions and the experimental results. Both results suggest a transition from supercritical to subcritical bifurcation as by increasing fluid film viscosity. The following two examples demonstrate the effectiveness of the proposed mathematical model in predicting the hysteresis phenomenon and bifurcation profiles.



Figure 4.2. Schematic of the experimental test setup [101].

1		L 1
Journal bearing diameter (D)	0.0254 m	
Journal bearing length (L)	0.0127 m	
Span length between two bearings	0.05271 m	
Inside diameter of the hollow shaft	0.0152 m	
Outer diameter of middle weight	0.0762 m	
Length of the middle weight	0.0127 m	
Rotor mass (M)	5.4523 kg	
Journal bearing clearance (C)	50.8 ×10 ⁻⁶ m	
Lubricant type	ISO 32	
Inlet pressure	31 kPa	
Inlet temperature	$0-180^{\circ}\mathrm{C}$	

Table 4.1. Specifications of the rotor-bearing system for experimental verification [33].



Figure 4.3. Comparison between experimental and theoretical results at $K_k = 2$.

4.3.1 Verification Example I: Subcritical Bifurcation Profile Prediction (Hysteresis Phenomenon)

To determine the subcritical bifurcation profile, fluid film (oil) inlet temperature is set to be 53°C. This corresponds to $\mu = 16$ mPa.s (16×10^{-3} Pa.s) for ISO 32 fluid film lubrication. The comparison between experimental and mathematical hysteresis phenomenon is demonstrated in Figure 4.4. As it is shown in Figure 4.4, upon slowly increasing the running speed of the shaft run-up process), experimental and mathematical oil whip instability are detected at about 7980rpm and 7910rpm respectively (0.9% error). Upon slowly reducing the rotating speed of the shaft (run-down process), experimental and mathematical oil whip instability disappears at about 7760rpm and 7764rpm respectively (0.05% error). Figure 4.4 proves the effectiveness of the proposed method in obtaining hysteresis phenomenon and illustrates a clear difference between run-up and run-down threshold speed of instability.



Figure 4.4. Flexible rotor bearing hysteresis plot, subcritical bifurcation, at oil viscosity 16mPa.s (53°C), (a) experimental results [54], (b) theoretical estimation.

Another interesting observation is that, upon increasing the running speed of the shaft and crossing run-up threshold speed of instability, the vibration amplitude abruptly increases up to the clearance circle for both experimental and mathematical results. Upon decreasing the running speed, after passing the run-down threshold of instability, the vibration amplitude suddenly drops to zero and the journal center remains at its steady state equilibrium position for both mathematical and experimental measurements.

This sudden jump in the vibration amplitude around the instability threshold is one of the key features of the subcritical bifurcation region and its unstable periodic solutions (bifurcation profile).

4.3.2 Verification Example II: Supercritical Bifurcation Profile Prediction

In another example to determine the supercritical bifurcation profile, fluid film (oil) inlet temperature is set to be 80°C. This corresponds to $\mu = 7.0$ mPa.s (7×10^{-3} Pa.s) for ISO 32 fluid film lubrication. It is assumed all other system parameters are identical to Example I. The comparison between experimental and mathematical supercritical bifurcation profile is shown in Figure 4.5. As it is shown in Figure 4.5, upon slowly increasing the running speed of the shaft run-up process), experimental and mathematical oil whirl (not whip) instability are detected at about 7540rpm and 7590rpm respectively ($\approx 0.7\%$ error). Experimental oil whip instability speed observed at rotating speed of 8000rpm; however, mathematical model predication is at a speed greater than 1000rpm. The difference from the analytical oil whip estimate and the experiment is due to simplifications in the mathematical model such as perfectly balanced rotor, neglecting the fluid inertia, simplifying assumption of concentrated mass at the shaft center and taking into consideration that HBT is a local analysis and does not provide accurate global information regarding bifurcation profile; hence, by moving away from the threshold speed of instability accuracy of HBT to predict bifurcation profile (stability envelope) decreases as it is illustrated in Figure 4.5(b).

Figure 4.5 reveals that in this example there is no distinct difference between the run-up process and the run-down process, indicating that the hysteresis phenomenon does not exist. Here, the amplitude of oil whirl gradually ramps up after the system running speed crosses the threshold speed of instability and gradually runs up. Upon decreasing the system running speed, the amplitude of oil whirl gradually decreases from near the clearance circle to a very small value. This gradual change of the vibration amplitude above while close to the instability threshold speed is one of the important features of the supercritical bifurcation and its stable periodic solutions.



Figure 4.5. Flexible rotor bearing supercritical bifurcation plot at oil viscosity 7mPa.s (80°C), (a) experimental results [54], (b) theoretical estimation.

4.4 Relationship between Bifurcation Profile and Fluid Film Viscosity

Equation (2.27) describes the size, period, and stability of the periodic solutions of a rotor bearing system journal orbit. Equation (2.27) also provides the bifurcation profile. It shows how the periodic solution moves either toward (in subcritical bifurcation region) or away (in supercritical bifurcation region) from the threshold of instability ($\overline{\Gamma}_c$). The unstable stability envelope exists only when the bifurcation of a rotor bearing system is subcritical. The subcritical bifurcation profile depicts how the amplitude of the unstable stability envelope changes with the system running speed, which is close to but less than the threshold of instability ($\overline{\Gamma}_c$). As the system running speed increases from the stable state, where the perturbation amplitude is small such as synchronous vibration due to residual rotor unbalance, an occurrence of oil whip would be first detected at ω_{uc} , where the vibration amplitude suddenly jumps to its maximum since the small amplitude of the perturbation crosses and moves outside of the subcritical bifurcation profile. However, upon decreasing the running speed of the shaft from the unstable condition to ω_{dc} , the oil whip disappears, and the vibration amplitude suddenly shrinks to a single point since the amplitude of the perturbation is moving inside of the subcritical bifurcation profile. Thus, the Hysteresis phenomenon can be accurately estimated and defined by subcritical bifurcation profile.

It is important to note that, in real operating conditions, residual rotor unbalance cannot be perfectly eliminated; hence, synchronous whirl of the journal shaft due to the residual unbalance is always present. Relative to the ideal steady-state equilibrium position, the synchronous whirl can be treated as a perturbation. Therefore, in real operating condition, the run-up threshold speed is always slightly less than the ideal instability threshold speed ($\overline{\Gamma}_c$). After verifying the proposed method, HBT is applied to predict the bifurcation profile of the rotor bearing system described in Table 4.1 at eleven (11) different fluid film viscosities while maintaining all other system parameters fixed. All information related to mathematically calculated threshold of instabilities and their associated bifurcation types at each fluid film viscosity is tabulated in Table 4.2. It shall be noted that, α is only function of oil viscosity assuming all other rotor system parameters constant.

Fluid Film Viscosity μ (mPa.s)	α	$\overline{\Omega}_k$	Bifurcation Type
1.5	0.126	2.089	Subcritical
1.9	0.16	1.970	Subcritical
2.39	0.198	1.895	Subcritical
2.41	0.2	1.892	Supercritical
7.0	0.58	1.811	Supercritical
9.7	0.8	1.837	Supercritical
15.7	1.302	1.880	Supercritical
15.8	1.306	1.880	Subcritical
16.0	1.326	1.882	Subcritical
24.1	2	1.910	Subcritical
36.2	3	1.933	Subcritical

Table 4.2. Eleven rotor bearing system bifurcation parameters using proposed mathematical model.

According to Figure 4.3, since the stiffness of the shaft is below its critical value (explained in Chapter 2), there exist two transition oil viscosities from subcritical to supercritical bifurcation. The following section explains thoroughly regarding the influence of the oil viscosity (temperature) on bifurcation profile and rotor bearing system instability behavior.

4.4.1 Fluid Film Viscosity Variation Effect: Case I

Figure 4.6 shows the rotor bearing system bifurcation profiles at different oil viscosities ($1.5 \le \mu \le 7.0$). Examination of Figure 4.6 shows that the amplitude of the periodic solution is a function of the deviation of the shaft rotating speed (ω_s) from the instability threshold speed (ω_c) and the fluid film viscosity (μ). For the oil viscosity (μ) ranging from 1.5mPa.s up to 2.39mPa.s, unstable periodic solutions exist (subcritical bifurcation) for $\omega_s \le \omega_c$. For the oil viscosity (μ) ranging from 2.41mPa.s up to 7.0mPa.s, stable periodic solutions associated with supercritical bifurcation exist for $\omega_s \ge \omega_c$. It can be concluded that, if the operating system dimensionless parameter (α) is located on the left side of the supercritical region, by increasing the fluid film viscosity (decreasing oil inlet temperature) system bifurcation behavior can be changed from subcritical to supercritical. The transition viscosity (μ'_1) from subcritical bifurcation is estimated to be 2.4mPa.s.

The bifurcation profiles presented in Figure 4.6 have the following features in terms of oil viscosity (μ) and shaft rotating speed (ω_s). It is shown than, in Case I, by increasing the oil viscosity (μ) instability threshold speed decreases. In every subcritical bifurcation case with the oil viscosity less than μ'_1 , the amplitude of the unstable periodic solution decreases as the rotating speed of the shaft increases. Considering the supercritical bifurcation cases with the oil viscosity higher than μ'_1 , the amplitude of the stable periodic solution increases up to the clearance circle as the rotating speed of the shaft increases. After obtaining the bifurcation profile corresponding to a specific set of system parameters, the existence and characteristics of hysteresis phenomenon can be easily predicted. If the system undergoes subcritical bifurcation region, hysteresis phenomenon exists, and the hysteresis loop is determined by the subcritical bifurcation profile. If the bifurcation type is supercritical, the hysteresis phenomenon does not exist at all. From the above analysis, it can be concluded that by changing the oil inlet

temperature and/or oil grade, system designer can adjust the bifurcation type of the rotor bearing system through changing fluid film viscosity.



Figure 4.6. Flexible rotor bearing system bifurcation profiles operating at different fluid film viscosities (Case I).

4.4.2 Fluid Film Viscosity Variation Effect: Case II

Figure 4.7 illustrates the rotor bearing system bifurcation profiles at different oil viscosities $(9.7 \le \mu \le 36.2)$. For the oil viscosity (μ) ranging from 9.7mPa.s up to 15.7mPa.s, stable periodic solutions exist (supercritical bifurcation) for $\omega_s \ge \omega_c$. For the oil viscosity (μ) ranging from 15.8mPa.s up to 36.2mPa.s, unstable periodic solutions associated with subcritical bifurcation exist for $\omega_s \le \omega_c$. It can be concluded that, if the operating system dimensionless parameter (α) is located on the right side of the supercritical region, by decreasing the fluid film viscosity (increasing oil inlet temperature) system bifurcation behavior can be transformed from

subcritical to supercritical. In Case II, the transition viscosity (μ'_2) from subcritical bifurcation to supercritical bifurcation is estimated to be 15.75mPa.s. It is shown than, in Case II, by increasing the oil viscosity (μ) instability threshold speed increases which is the opposite trend observed in Case I.

According to Figure 4.6 and Figure 4.7, assuming the system undergoes subcritical bifurcation, it can be seen that for Case I by decreasing oil viscosity and for Case II by increasing oil viscosity, unstable bifurcation profile becomes smaller until it does not cross clearance of the bearing ($\epsilon = 1$). It implies that, at certain operating viscosities, after crossing the threshold speed of instability through run-up test, the rotor bearing system cannot become stable (oil whip never disappears) by decreasing the rotating speed of the shaft. Opposite conclusion can be made for supercritical bifurcation region.



Figure 4.7. Flexible rotor bearing system bifurcation profiles operating at different fluid film viscosities (Case II).

4.5 Conclusions and Rotor Bearing System Design Guidelines

As bearing performance is strongly dependent on lubricant viscosity, in the current study, the bifurcation profile behavior of a flexible shaft supported on end journal bearings was studied for a range of operating viscosities considering shear force effect. The Hopf bifurcation theory (HBT) was utilized to accurately predict hysteresis phenomenon when rotor bearing system undergoes subcritical bifurcation. From the results provided in Chapter 2 and 4, it can be concluded that, the Hopf bifurcation profile depends on, average Reynolds number (\overline{Re}), system characteristic number (α), and dimensionless stiffness of the shaft (K_k). For a specific rotor bearing system having a stiffness lower than the critical stiffness of the shaft, there exist two transition system characteristic numbers α_1 and α_2 as it is shown in Figure 4.3.

In subcritical bifurcation region hysteresis phenomenon exist for operating conditions higher than α_2 and lower than α_1 . The operating system undergoes supercritical bifurcation for the rotor bearing system with intermediate system characteristic numbers ($\alpha_1 < \alpha < \alpha_2$); hence, to avoid hysteresis phenomenon in a rotor bearing system fluid film viscosity shall be maintained within the range of $\mu'_1 < \mu < \mu'_2$, where μ'_1 and μ'_2 correspond to system characteristic numbers α_1 and α_2 respectively. It should be noted that, in subcritical bifurcation region, at certain operating viscosities, if oil whip occurs, the rotor bearing system does not become stable by decreasing the rotating speed of the shaft (run-down process). On the contrary, in supercritical bifurcation region, at certain operating viscosities, oil whip never appears through run-up process. This is due to the fact that, stable and/or unstable bifurcation profiles do not cross the journal bearing clearance at certain oil viscosities.

For the case of large journal bearing clearances, accurate determination of the hysteresis profile requires a complete thermos-hydrodynamic analysis of a fluid-film bearing to assess the temperature as well as viscosity field within the fluid film chamber [102-109].

Chapter 5: Flexible Rotor Bearing System Dynamic Analysis: Application in Optimized Fluid Film Inlet Pressure and Position Determination

5.1 Introduction

An axial groove, widely utilized in industrial applications for plain journal bearings, is for distributing oil over the entire length of the journal cavity to improve lubrication flow and evening out fluid film temperature [110]. From the state of the art, it is well recognized that the fluid film inlet location and supply pressure can have a pronounced effect on the rotor bearing system instability. However, a review of the existing literature shows that the influence of these parameters on the dynamic stability of a flexible turbulent journal bearing, considering accurate cavitation start and end points, has not provided much attention. Reynolds boundary condition is widely used to determine the starting position of the cavitation within bearing chamber.

A more complete study by including the Floberg–Jakobsson's boundary condition was done [111, 112] to establish the reformation of fluid film lubrication when cavitation ends. However, as Dowson D. et al. [113] mentioned "At the rupture boundary the well-known Reynolds" condition is applicable to all but very lightly loaded journal bearing," cavitation may not exist for lightly loaded (high Sommerfeld numbers) or for the cases with high oil inlet pressure. Cavitation existence is dependent of the eccentricity ratio, oil inlet position and oil inlet pressure [114]. Based on the assumption that no cavitation exists at high Sommerfeld numbers, Mori et al. [115] conducted a study, experimentally and theoretically, to identify how the oil inlet position affects on the static and dynamic performance of journal bearings. They concluded that the oil inlet position has a strong influence on both the journal center locus and the threshold of instability. Solving the Reynolds equations with oil inlet pressure equal to the ambient pressure and neglecting the negative pressure term, Lundholm [116] studied the effects of fluid film inlet position on the static and dynamic stability of axially grooved journal bearings. Linearized stiffness and damping coefficients were used to predict the instability threshold speed. Later on, Brindley et al. [117] confirmed the effect of oil inlet position on the instability threshold speed numerically for an infinitely long journal bearing under Gümbel (π film) boundary condition. However, they found influence of oil inlet pressure on the every situation is not easy to summarize as decreasing oil inlet pressure in some situation stabilizes the system [117].

Evidently, the effect of oil inlet pressure on the instability threshold speed remains unclear and requires further in-depth research. Through solving the Reynolds equation through assumption of an appropriate starting position of the cavitation point and the reformation of oil film at the end of cavitation using perturbation method and finite difference method, Zhang [118] calculated the linearized stiffness and damping coefficients for an infinitely long, flexible journal bearing. Based on the linearized stiffness and damping coefficients, Zhang [118] showed that oil inlet pressure and inlet position significantly effects on the instability threshold speed. Costa et al. [119, 120] studied the influence of oil inlet pressure and inlet position on the static performance of an axially grooved journal bearing both experimentally and numerically and concluded that axial groove located at a positive angle can lead to reductions in maximum temperature, peak hydrodynamic pressure, and full-film region. Wang and Khonsari [114, 121] concluded that increasing oil inlet pressure destabilizes a rigid rotor bearing system under laminar flow assumption; however, optimized oil inlet position and its effect on the threshold speed of instability has not been studied thoroughly. Instability analysis under the assumption of a flexible rotor bearing system with consideration of turbulence effect an appropriate starting position of the cavitation and the reformation of oil film at the end of cavitation is provided in this chapter. To correctly predict the bearing performance required for precise identification of instability phenomenon, the flow regime (being laminar or turbulent) and cavitation region need to be taken into account. Therefore, it is important to determine the turbulence and cavitation effects on the dynamic fluid forces and operating stability margin of journal bearings for reliable operation of rotor-bearing systems. The purpose of this chapter is first to derive an analytical expression to accurately estimate the turbulent forces and turbulent dynamic coefficients, including cavitation effect, for the whole range of operating conditions. These coefficients and forces are then used as basis of linear stability analysis of a flexible rotor-bearing system. The dynamic oil film forces are analytically obtained utilizing Ng-Pan-Elrod turbulent model under long bearing assumptions with Reynolds–Floberg–Jakobsson (RFJ) boundary condition. The closed form expressions for eight spring and damping coefficients are derived by linearizing the oil film forces around the steady-state equilibrium position of the journal center, and the whirl onset velocities are determined by the linear stability criterion. By having a complete rotor bearing model, oil inlet pressure and position influences on the rotor bearing instability have been analyzed carefully.

5.2 Turbulent Analytical Dynamic Force and Pressure Calculations, Considering Cavitation Effect

Fluid film inlet position and pressure effects cannot be studied through short bearing theory assumption as the first term in the Reynolds equation is neglected $\left(\frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\frac{h^3}{k_{\theta}\mu}\frac{\partial P}{\partial\theta}\right)\approx 0\right)$; hence long bearing theory assumption is the only option to derive analytical expression for the forces under turbulent flow considering cavitation effect. According to Figure 5.1, an absolute coordinate system ($\Psi = \theta + \varphi$) has been utilized to locate the oil inlet pressure position on the journal bearing chamber.



Figure 5.1. Schematic of a journal bearing considering fluid film inlet position and pressure.

By long bearing approximation $(L/D) \ge 2$, it can be assumed longitudinal pressure distribution is constant $((\partial P/\partial \theta) \gg (\partial P/\partial z))$; thus, $\frac{\partial P}{\partial z} = 0$. $\frac{\partial}{\partial \theta} \left(\frac{h^3}{k_{\theta}} \frac{\partial P}{\partial \theta}\right) = \frac{\mu C R^2}{2} \left((2\dot{\varphi} - \omega)\epsilon Sin(\theta) + 2\dot{\epsilon}Cos(\theta)\right)$ (5.1)

The above equation can be integrated over θ to obtain the following equation:

$$\frac{\partial P}{\partial \theta} = \frac{k_{\theta}}{h^3} \frac{\mu C R^2}{2} ((\omega - 2\dot{\phi})\epsilon Cos(\theta) + 2\dot{\epsilon}Sin(\theta) + C_1)
= \frac{1}{2} \mu \left(\frac{R}{C}\right)^2 \frac{12 + M(1 + \epsilon Cos(\theta))^{\hat{a}_2}}{(1 + \epsilon Cos(\theta))^3} ((\omega - 2\dot{\phi})\epsilon Cos(\theta)
+ 2\dot{\epsilon}Sin(\theta) + C_1)$$
(5.2)

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where $M = \hat{a}_1(\overline{Re})^{\hat{a}_2}$, $k_{\theta} = 12 + \hat{a}_1(Re^*)^{\hat{a}_2} = 12 + M(1 + \epsilon Cos(\theta))^{\hat{a}_2}$, and $Re^* = \frac{\rho R \omega h}{\mu} = \frac{\rho R \omega C(1 + \epsilon Cos(\theta))}{\mu}$. \hat{a}_1 and \hat{a}_2 can be found from Table 2.1. Reynolds boundary conditions can be

written as follows [114],

$$P = 0 \text{ at } \theta = \theta_s \tag{5.3}$$

$$P = 0 \text{ at } \theta = \theta_c \tag{5.4}$$

$$\frac{\partial P}{\partial \theta} = 0 \text{ at } \theta = \theta_c \tag{5.5}$$

where θ_s and θ_c are cavitation start and end points respectively. Reynolds boundary conditions force fluid film pressure and its gradient to be zero at the circumferential location where fluid film ruptures (conservation of mass continuity at the end point of cavitation). To satisfy the conservation of mass continuity at the start point of cavitation, Jackson and Floberg [48] introduced the following boundary condition,

$$\frac{\partial P}{\partial \theta} \ge 0 \text{ at } \theta = \theta_s \tag{5.6}$$

In order to take into account, the fluid film inlet position and pressure, Zhang [118] introduced an extra boundary condition at the fluid film inlet position,

$$P = P_i \text{ at } \theta = \theta_i \tag{5.7}$$

By applying Equation (5.5) into Equation (5.2), C_1 can be found as follow,

$$C_1 = -(\omega - 2\dot{\varphi})\epsilon\cos\theta_c - 2\dot{\epsilon}\sin\theta_c \tag{5.8}$$

According to Chapter 2, in order to be able to integrate the Equation (5.2) to find an analytical expression for long bearing pressure considering cavitation effect, it is assumed $((1 + \epsilon \cos(\theta)))^{\hat{a}_2} \approx (1 + a_3 \epsilon \cos(\theta))$. The constant a_3 may be found by curve fitting via optimization, which is calculated to be 0.8437, and 0.91 for Constantinescu and Ng-Pan-Elrod theories, respectively. In this chapter Ng-Pan-Elrod turbulent model is merely utilized as it is found to be more conservative than Constantinescu's approach. By applying C_1 from Equation (5.8) into Equation (5.2), the following expression can be determined,

$$\frac{\partial P}{\partial \theta} = \frac{1}{2} \mu \left(\frac{R}{C}\right)^2 \frac{12 + M(1 + a_3 \epsilon Cos(\theta))}{(1 + \epsilon Cos(\theta))^3} \left((\omega - 2\dot{\varphi})\epsilon(\cos\theta - \cos\theta_c) + 2\dot{\epsilon}(\sin\theta - \sin\theta_c)\right)$$
(5.9)

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Integrating Equation (5.9) and using Sommerfeld substitutions, see Appendix C, along with boundary condition provided in Equation (5.7), the following analytical expression for pressure distribution of turbulent long journal bearings considering cavitation effect may be obtained,

$$P = P_{i} + \frac{1}{8C^{2}(\epsilon^{2} - 1)^{2}(\epsilon \cos(\alpha_{c}) - 1)} \mu R^{2}(\epsilon \sqrt{1 - \epsilon^{2}}(\omega - 2\varphi'(t))(4\cos(\alpha_{c})(\alpha_{i}(M(a_{3}\epsilon^{2} - 1)))) + (\alpha_{i}(a_{3} - 1)) - (\alpha_{i}(a_{i})(a_{3} - 1))) + (\alpha_{i}(a_{3} - 1)) - (\alpha_{i})(a_{3} - 1)) + (\alpha_{i}(a_{3} - 1)) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i})) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i})) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i})) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i})) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i})) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i})) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}) - \alpha_{i})) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i})) + (\alpha_{i}(a_{i}) + (\alpha_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}) + (\alpha_{i}(a_{i}) - \alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}))) + (\alpha_{i}(a_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}) + (\alpha_{i}(a_{i}) - (\alpha_{i}) + (\alpha_{i}(a_{i}) + (\alpha_{i}))) + (\alpha_{i}(a_{i}) + (\alpha_{i}) + (\alpha_{i})$$

The above mentioned dynamic pressure distribution is only valid for $\alpha_s \leq \alpha \leq \alpha_c$ since the fluid film pressure only exists in the range of $\theta_s \leq \theta \leq \theta_c$. By applying the boundary conditions described by Equations (5.3) and (5.4) in Equation (5.10) and using Sommerfeld substitutions α_s , α_i , α_c can be determined through solving nonlinear Equations (5.11), (5.12), (5.13), and (5.14). Load angle (φ) can be determined by having analytical expression of fluid film forces in radial and tangential directions.

According to Dowson et al. [113], at small eccentricity ratios or when a rotor bearing system is lightly loaded, at high Sommerfeld numbers, cavitation might not exist. In this situation there exist a periodic boundary condition $P(\alpha_c) = P(\alpha_c - 2\pi)$; thus, Equations (5.11) and (5.12) cannot be utilized. Instead, Equations (5.15), (5.16) and (5.17) shall be used for α_c , α_s , and α_i calculation, respectively.

$$\begin{aligned} \frac{1}{4(e^2-1)^2(e\cos(a_c)-1)} \pi(e\sqrt{1-e^2}(1-2\varphi'(t))(4\cos(a_c)(a_i(M(a_3e^2-1)-12)-e((a_3\\-1)M-12)\sin(a_i)) - 4\sin(a_i)(a_3Me^2+6e\cos(a_i)-M-12) + 2e((a_3\\-1)M-12)a_i+(a_3-1)Me\sin(2a_i)) - 2e'(t)(-6a_3Me^2a_i\sin(a_c)+e^2((a_3\\-1)M-12)(-\cos(a_c-2a_i)) + 2e(M(2a_3e^2+a_3-3)-36)\cos(a_c-a_i)\\-2a_3Me\cos(a_c+a_i) - 4a_3Me^2\cos(a_i) + a_3Me\cos(2a_i) + 24e^2a_i\sin(a_c)\\+24e\cos(a_c+a_i) + 48a_i\sin(a_c) + 2Me^2a_i\sin(a_c) + 2Me\cos(a_c+a_i)\\+24e\cos(a_c+a_i) - 12e\cos(2a_i) + 48e\cos(a_i) - Me\cos(2a_i) + 4Me\cos(a_i))\\+e\sqrt{1-e^2}(2\varphi'(t)-1)((M(4-4a_3e^2)+48)\sin(a_c) + 4a_c(M(a_3e^2-1)\\-12)\cos(a_c) + 2e((a_3-1)M-12)(a_c-\sin(a_c)\cos(a_c))))\\+2e'(t)(2(a_c(e^2(-3a_3M+M+12)+2(M+12))\sin(a_c) + e(M(2a_3e^2+a_3)-36)) + (e^2(-5a_3M+M+12) + 2(M+12))\sin(a_c) + e(M(2a_3e^2+a_3)-36)) + (e^2(-5a_3M+M+12) + 2(M+12))\cos(a_c) - e((a_3-1)M\\-12)2\cos(2a_c))) + \overline{P_i} = 0 \end{aligned}$$

$$\frac{1}{4(e^2-1)^2(e\cos(a_c)-1)} \pi(e\sqrt{1-e^2}(1-2\varphi'(t))(4\cos(a_c)(a_i(M(a_3e^2-1)-12) - e((a_3-1)M\\-12)2\cos(2a_c))) + 2e'(M(2a_3e^2+a_3-3)-36)\cos(a_c-a_i)\\-2a_3Me\cos(a_c+a_i) - 4a_3Me^2\cos(a_i) + a_3Me\cos(2a_i) + 24e^2a_i\sin(a_c) + e^2((a_3\\-1)M-12)(-\cos(a_c-2a_i)) + 2e(M(2a_3e^2+a_3-3)-36)\cos(a_c-a_i)\\-2a_3Me\cos(a_c+a_i) - 4a_3Me^2\cos(a_i) + a_3Me\cos(2a_i) + 24e^2a_i\sin(a_c)\\+24e\cos(a_c+a_i) + 48a_i\sin(a_c) + 2Me^2a_i\sin(a_c) + 2Me\cos(a_i) + 24e\cos(a_c+a_i)\\+4Ma_i\sin(a_c) - 12e\cos(2a_i) + 48\cos(a_i) - Me\cos(2a_i) + 4Me\cos(a_i))\\-e\sqrt{1-e^2}(1-2\varphi'(t))(4\cos(a_c)(a_s(M(a_3e^2-1)-12) - e((a_3-1)M\\-12)\sin(a_i)) - 4\sin(a_3)(a_3Me^2 - M + 6\cos(a_3)-12) + 2e((a_3-1)M\\-12)\sin(a_3)) - 4\sin(a_3)(a_3Me^2 - M + 6\cos(a_i) - 12) - e((a_3-1)M\\-12)\sin(a_3)) - 4\sin(a_3(a_3Me^2 - M + 6\cos(a_i) - 12) - e((a_3-1)M\\-12)(-\cos(a_c-2a_i)) + 2e'(1)(-6a_3Me^2a_i\sin(a_c) + e^2(a_3-1)M\\-12)(-\cos(a_c-2a_i)) + 2e'(M(2a_3e^2+a_3-3) - 36)\cos(a_c-a_i)\\-2a_3Me\cos(a_c+a_i) - 4a_3Me^2\cos(a_i) + a_3Me\cos(2a_i) + 2Me^2(a_3\sin(a_c) + e^2(a_3-1)M\\-12)(-\cos(a_c-2a_3)) + 2e'(M(2a_3e^2+a_3-3) - 36)\cos(a_c-a_3)\\-2a_3Me\cos(a_c+a_a) + 4Ma_i\sin(a_c) + 24e^2a_i\sin(a_c) + 24e^2(a_i\sin(a_c) + 24e^2a_i\sin(a_c))\\+2Me\cos(a_c+a_a) + 4Ma_i\sin(a_c) + 24e^2a_i\sin(a_c) + 24e^2(a_3\sin(a_c) + 2Me^2(a_3\sin(a_c) + 2Me^2(a_3\sin(a_c) + 2Me^2(a_3\sin(a_c) + 2Me^2(a_3\sin(a_c) + 2Me^2(a_3\sin(a_c) + 2Me^2(a_3\sin(a_c) + 2M$$

$$\tan \theta_i = \tan(\Psi_i - \varphi) = \frac{\sin \alpha_s \sqrt{1 - \epsilon^2}}{\cos \alpha_s - \epsilon}$$
(5.13)

$$\tan \varphi = \left| \frac{F_T}{F_R} \right| \tag{5.14}$$

$$\pi^{2} \left(\epsilon \sqrt{1 - \epsilon^{2}} (2\varphi'(t) - 1) (a_{3}M\epsilon(2\epsilon\cos(\alpha_{c}) + 1) - (M + 12)(2\cos(\alpha_{c}) + \epsilon)) + 2((M + 12)(\epsilon^{2} + 2) - 3a_{3}M\epsilon^{2})\sin(\alpha_{c})\epsilon'(t)) = 0$$
(5.15)

$$\alpha_s = \alpha_c - 2\pi \tag{5.16}$$

$$\alpha_{i} = \tan^{-1} \left(\frac{\sin(\Psi_{i} - \varphi)\sqrt{1 - \epsilon^{2}}}{\epsilon + \cos(\Psi_{i} - \varphi)} \right)$$
(5.17)

where the dimensionless pressure $\overline{P} = P 8\pi (C/D)^2/\mu \omega$ and Ψ_i is the absolute location of the fluid film inlet. By utilizing Equations (2.9) and (2.15), dynamic fluid forces and coefficients can be derived analytically. The non-dimensional form of laminar stiffness and damping coefficients in polar coordinate system are tabulated in Table A.4 for long bearings considering cavitation effect.

5.3 Effects of Oil Inlet Circumferential Position and Pressure Magnitude on the Fluid Film Pressure Distribution Profile

The comparison of non-dimensional static pressure distribution, for the whole range of eccentricity ratio ($0 \le \epsilon \le 1$) and in terms of absolute coordinate (Ψ) are shown in Figure 5.2, Figure 5.3, and Figure 5.4. The effects of oil inlet pressure (\overline{P}_i), Reynolds number (\overline{Re}), and oil inlet position (Ψ_i) on the fluid film pressure distribution of a long journal bearing, at $\epsilon = 0.6$, are demonstrated in Figure 5.5, Figure 5.6, and Figure 5.7, respectively.

According to Figure 5.2, it can be seen that, journal bearing cavitation area decreases at low eccentricity ratios. Figure 5.2(b) and (c) show that cavitation appears at eccentricity ratios great than 0.35 and 0.45 for oil inlet pressures of $\overline{P}_i = 5$ and $\overline{P}_i = 10$ (at $\Psi_i = 0$), respectively; however, full 2π fluid film exists at lower eccentricity ratios ($0 \le \epsilon \le 0.35$ at $\overline{P}_i = 5$ and $0 \le \epsilon \le 0.45$ at $\overline{P}_i = 10$). Based on the definition of attitude angle when full 2π fluid film exists $\varphi = \tan^{-1} \left| \frac{F_T}{F_R} \right| = \frac{\pi}{2}$. Thus, fluid film static force in radial direction reaches to zero ($F_R = 0$). This implies the rotor bearing system could become unstable (very sensitive) through any external perturbations in radial (horizonal) direction which happens at small eccentric ratios by increasing the oil inlet pressure.

By comparing the pressure distribution results corresponding to different oil inlet pressure at $\epsilon = 0.6$, shown in Figure 5.5, one can see that fluid film pressure increases slightly by increasing the oil inlet pressure magnitude; however, cavitation region follows an opposite trend which could lead to lower rotor bearing system stability margin.



Figure 5.2. Effect of oil inlet pressure on the pressure distribution of Turbulent (Re = 5,000) long journal bearing considering cavitation effect with oil inlet position of $\Psi_i = 0$, (a) $P_i = 0$, (b) $P_i = 5$, and (c) $P_i = 10$.



Figure 5.3. Effect of Reynolds number (*Re*) on the pressure distribution of long journal bearing considering cavitation effect with oil inlet position of $\Psi_i = 0$ and oil inlet pressure of $P_i = 0$, (a) $Re \le 2,000$ (Laminar Flow), (b) Re = 5,000, and (c) Re = 10,000





Figure 5.4. Effect of oil inlet position (Ψ_i) on the pressure distribution of Turbulent (*Re* = 5,000) long journal bearing considering cavitation effect with oil inlet pressure of $P_i = 0$, (a) $\Psi_i = 45^\circ$, (b) $\Psi_i = 90^\circ$, (c) $\Psi_i = 135^\circ$, and (d) $\Psi_i = 180^\circ$.



Figure 5.5. Effects of oil inlet pressure amplitude on the turbulent fluid film pressure distribution with $\Psi_i = 0^\circ$ and Re = 5,000 at $\epsilon = 0.6$.



Figure 5.6. Effects of Reynolds number (*Re*) on the fluid film pressure distribution considering $\Psi_i = 0^\circ$ and $P_i = 0$ at $\epsilon = 0.6$.



Figure 5.7. Effects of oil inlet position on the turbulent fluid film pressure distribution considering $P_i = 0$ and Re = 5,000 at $\epsilon = 0.6$.

From Figure 5.3 and Figure 5.6 it can be seen that, by increasing the Reynolds number cavitation region exists for the whole range of eccentricity ratio; however, cavitation area shrinks as Reynolds number increases. The effect of Reynolds number on fluid film pressure distribution is found to be significant. Hence, Reynolds number has a pronounced effect on the rotor bearing system instability.

From the fluid film pressure distribution plots of Figure 5.4 and Figure 5.7 it can be seen that as the oil inlet position increases, the cavitation region extends to larger area. Fluid film pressure follows an opposite trend and shrinks to smaller amounts as oil inlet position reaches to $\Psi_i =$ 180°. Based on results provided in Figure 5.2-Figure 5.7, it can be concluded that, Reynolds number and fluid film inlet position have noticeable effect on the fluid film pressure distribution and hence instability of the rotor bearing system. It shall be noted that, by increasing the Reynolds number and fluid film inlet pressure, negative impact may be expected in rotor bearing instability as cavitation region decreases and fluid film pressure distribution increases for the entire range of eccentricity ratio. Positive effect might be expected (on the rotor bearing threshold of instability), to some extent, by increasing the fluid film inlet position as cavitation region increases and fluid film pressure distribution magnitude drops significantly.

5.4 Effects of Oil Inlet Circumferential Position and Pressure Magnitude on the Rotor Bearing Threshold of Instability

By making use of the derivations in Appendix B, the stability margins of a flexible rotor bearing system supported on two identical long journal bearings for different oil inlet pressures and positions are calculated considering non-dimensional stiffness coefficients of the shaft ($S_z = \frac{C K_s}{W} = 10$). Stability parameter $\overline{\Gamma} = \frac{C}{W} M \omega_s^2 = \left(\frac{C K_s}{W}\right) \overline{\omega}_s^2 = S_z \overline{\omega}_s^2$ is plotted vs the Sommerfeld number as shown in Figure 5.8 and Figure 5.9 for different oil inlet pressures and positions, respectively.

From the stability plot of Figure 5.8 it can be seen that, assuming oil inlet position $\Psi_i = 0$, as non-dimensional oil inlet pressure increases, the stable operating margin expands towards lower threshold speeds at low Sommerfeld numbers (heavily loaded rotor bearing systems); hence, the rotor-bearing system becomes stable at lower rotating spin speeds of the shaft. It should be noted that rotor bearing threshold of instability does not drop to zero at low eccentricity ratios (high Sommerfeld numbers) when load angle $\varphi = 90^{\circ}$ and non-dimensional oil inlet pressure is positive ($\bar{P}_i > 0$). This is due to the fact that non-dimensional damping ratio in radial direction exists ($\bar{C}_{rr} \neq 0$) for the whole range of eccentricity ratio and rotor bearing system can become stable. This conclusion is not consistent with the results provided by Wang and Khonsari [121]; however, in good agreement with Zang [118].

According to stability plot of Figure 5.9 it is demonstrated that, assuming non-dimensional oil inlet pressure $\overline{P}_i = 0$, as oil inlet position Ψ_i increases, first $(0 \le \Psi_i \le 135^\circ)$ the stable operating margin expands towards higher threshold speeds at low Sommerfeld numbers (heavily loaded rotor bearing systems), and then $(135^\circ \le \Psi_i \le 180^\circ)$ threshold of instability drops significantly. It shall be noted that, increasing the oil inlet position has negative effect on threshold of instability at high Sommerfeld numbers (lightly-loaded rotor bearing system). Thus, heavily-loaded rotor-bearing systems become stable at higher rotating spin speeds of the shaft if the oil inlet position is less than $\Psi_i \le 135^\circ$. The results provided in this section are in good agreement with numerical calculations presented by Rao et al. [122, 123] for oil inlet positions of $0 \le \Psi_i \le 80^\circ$.

In can be generally concluded that for rotor bearing systems supported on two identical journal bearings, groove angle $\Psi_i = 0^\circ$ becomes important for lightly-loaded systems; however, opposite trend can be expected by increasing the groove angle. Oil inlet position of $\Psi_i = 135^\circ$ has positive effect on the threshold of instability for heavily-loaded systems.



Figure 5.8. Oil inlet pressure (P_i) effects on the stability parameter Γ of a flexible shaft supported on long length laminar journal bearings at non-dimensional shaft stiffness coefficient of $S_z = CK_s/W = 10$ under RFJ boundary condition with $\Psi_i = 0^\circ$.



Figure 5.9. Oil inlet position (Ψ_i) effects on the stability parameter Γ of a flexible shaft supported on long length laminar journal bearings at non-dimensional shaft stiffness coefficient of $S_z = CK_s/W = 10$ under RFJ boundary condition with $P_i = 0$.

5.5 Conclusions and Rotor Bearing System Design Guidelines

In the current study, the stability of flexible shaft supported on end journal bearings was studied for a range of operating conditions in both laminar and turbulent flow bearings and for different oil inlet positions and pressure amplitudes. The dynamic oil film pressure and forces considering the turbulent effects were obtained analytically under long bearing assumption considering the RFJ boundary condition. The closed form expressions for eight spring and damping coefficients were derived by linearizing the oil film forces around the steady-state equilibrium position of the journal center for two available turbulent models. Provided dynamic coefficients were then used to calculate the whirl onset velocity for a flexible rotor supported by two identical journal bearings.

Reynolds number has significant effects on fluid film pressure distribution; hence, high Reynolds numbers could lead to expansion of unstable operating region. Stable operating region of flexible shafts supported journal bearings were shown to squeeze in size slightly by increasing the oil inlet pressure at low Sommerfeld numbers. Fluid film pressure distribution was found to be a strong function of oil inlet position. It was found that stable operating region grows in size, at low Sommerfeld number only, by increasing the oil inlet position in the range of $0 \le \Psi_i \le$ 135°. By moving away from oil inlet angle of $\Psi_i = 135°$ and approaching $\Psi_i = 180°$, significant drop in threshold speed of instability was observed. Thus, including two groove angles at $\Psi_i = 0°$ and 135° might improve the safe operating region of rotor supported journal bearings for the entire range of Sommerfeld number.

Chapter 6: Summary, Conclusions and Future Work

6.1 Summary and Conclusions

The purpose of this thesis is to identify and characterize the oil whirl phenomenon — a destructive form of instability in rotor-bearing systems. To do so, the rotor-bearing system must be represented with a mathematical model that encapsulates the physics of the problem. The most significant step in constructing such a model is to represent the bearing forces that are caused by the shearing of oil in the bearing chamber. This is done by means of, (1) analytically modelling of fluid film forces, and (2) journal bearing dynamic coefficients that act as hypothetical springs and dampers in the dynamical model of the system. In order to calculate the dynamic coefficients, the simplified form of the Navier-Stokes equations for thin-film flows, the Reynolds equation, is solved in its original and perturbed form to find journal induced pressure and its gradients analytically. Proper integration of the calculated pressure gradients and/or forces can then reveal the desirable dynamic coefficients.

Once the mathematical model of the system is constructed, linear and nonlinear stability analyses are carried out to find the bifurcation boundary (i.e. the stability boundary or the threshold speed after-which oil whirl occurs) and its directions/types. It was shown that identifying the bifurcation types for rotors supported on realistic journal bearings with finite length is plausible through accurate rotor bearing system modelling. The proposed method was to construct the bearing induced forces, not only using the linear dynamic coefficients (under laminar flow assumption) as is widely adopted in the literature, but also by shear force effect and inclusion of turbulent effects. Linear and nonlinear rotor-bearing models were then constructed based on the calculated forces and coefficients for short and long journal bearings. Nonlinear model was shown to be suitable (and thus superior) for the calculation of bifurcation types while their linear counterparts fell short in providing a distinction between the bifurcations of the system.

The Hopf bifurcation theory was used (Nonlinear Analysis) to study the orbital stability of periodic solutions near bifurcating points of the rotor bearing system, i.e. when the operating spin speed of the shaft is close to its threshold speed of instability. A Hopf bifurcation subroutine 103

was written in MATLAB to calculate bifurcation parameters. The calculated bifurcation parameters were used to identify subcritical and supercritical regions along the path of whirl onset boundaries (curves separating stable and unstable operating regions). Existing experimental results in the literature are presented confirming the validity of the bifurcation diagram and stability envelopes for flexible rotor-bearing systems. Linear analysis also utilized to provide practical information on oil inlet pressure and location within a rotor bearing chamber.

More detailed findings are presented in the following subsections:

6.1.1 Conclusion on The Effects of Turbulence on The Nonlinear Instability Behavior of Flexible Rotor Bearing Systems

The stability bounds of a rotor-bearing system were presented for a range of operating conditions based on a simplified mathematical model of the system. The simplification is mainly achieved by means of reducing the complicated fluid-solid interaction problem in the rotor-bearing system to the oscillation problem of a mass supported on a flexible and isolated support. Such reduction is made possible by replacing the fluid effects of the bearings by the so-called dynamic bearing coefficients.

The closed form expressions for eight spring and damping coefficients were derived by linearizing the turbulent oil film forces around the steady-state equilibrium position of the journal center for two available turbulent models. The proposed simplified model is based on a flexible shaft supported on identical short and/or long length journal bearings. Dynamic coefficients of the journal bearing were calculated based on the Reynolds equation that is modified for turbulent flows. Two turbulent models were used for stability analysis and the corresponding stability boundaries were compared.

- Stable operating region of flexible shafts supported on both turbulent and laminar journal bearings were shown to grow in size by increasing the shaft non-dimensional stiffness parameter.
- It was found that at low load and high Sommerfeld regions $S \ge 0.2$ the stability won't be affected by the type of flow within the bearing chamber. On the other hand, as the static load

on shaft increases in magnitude (i.e. lower Sommerfeld numbers), the unstable operating region grows in size and will result in the unstable oil whirl to happen at even higher static loads, a region where the laminar based stability analysis would consider as safe operating condition.

- The predicted threshold speed of instability based on Constantinescu's model are higher than the threshold speeds predicted by the Ng-Pan-Elrod turbulent model and hence the latter model proves to be more conservative in design.
- Subcritical regions were found to be more prevalent at high static loads (low Sommerfeld numbers).
- The width of dangerous subcritical regions was shown to be a function of the shaft stiffness.
- The stability envelope for subcritical operating regions were found for both laminar and turbulent operating conditions. The smaller the size of the stability envelope, the rotor bearing system is more susceptible to applied external perturbations.
- A critical shaft stiffness value was found at any Reynolds number, beyond which the unstable low Sommerfeld region can be transformed supercritical (hence stable) for a majority of operating conditions.
- Reynolds dependence of the critical shaft stiffness parameter follows different trends for short and long bearings. While the critical stiffness constantly grows in size with increasing Reynolds number in short bearing supported shafts, the critical stiffness for long bearing supported shafts decreases with Reynolds number and it ceases to exist at low Reynolds numbers. This suggest that for a shaft supported on long bearings operating at low Reynolds numbers, increasing shaft stiffness cannot suppress the unstable critical region. It is hence recommended that for shafts under high static loads, better design choice would be a bearing with low length to diameter ratio to avoid the unstable subcritical regions of operation.

6.1.2 Conclusions on The Shear Force Effect on The Nonlinear Stability Analysis of Flexible Rotor Bearing Systems

Exerted forces on fluid film bearings comprises of two different componenets. (1) Pressure force in radial and tangential directions and, (2) Drag force (shear/friction force) in radial and tangential directions. In most existing literature, related to journal bearing stability analysis, the effect of fluid film drag force have been neglected.

Existing literature argued that shear force is in the order of journal bearing clearance over radius (C/R) times pressure force; however, provided numerical solutions of bearing force parameters based on finite bearing assumption show that at small eccentricity ratios ($\epsilon \le 0.1$), considering short bearing theory $(L/D \le 0.5)$, journal bearings shear force exceeds pressure force. The stability of flexible shaft supported on end journal bearings was studied for a range of operating conditions in turbulent flow bearings considering shear force effect. The dynamic turbulent oil film forces considering shear effects were obtained analytically under the short bearing assumption considering the Gümbel boundary condition. The closed form expressions for eight spring and damping coefficients were derived by linearizing the oil film forces around the steady-state equilibrium position of the journal center for Ng-Pan-Elrod model considering shear effect. The calculated dynamic coefficients were then utilized to obtain the whirl onset velocity for a flexible rotor supported by two identical journal bearings at various Reynolds numbers.

- Stable operating region of flexible shafts supported on both turbulent and laminar journal bearings were shown to grow in size by increasing the shaft non-dimensional stiffness parameter.
- At high shaft stiffness numbers (S_z ≥ 5), it was found that at low load and high Sommerfeld regions S ≥ 0.7 the stability will be affected by the type of flow within the bearing chamber and stable operating region increase by increasing the Reynolds number, a region where the laminar based stability analysis would consider as un-safe operating condition. Neglecting the shear force effect would result in opposite result at high Sommerfeld numbers (existing literature).

- It was shown that, considering shear force effect, would increase stable operating region, especially when rotor bearing system stability type undergoes subcritical bifurcation.
- Safe operating rage of dimensionless unbalance moment (γ) in subcritical bifurcation region can be identified for rotor-bearing system design purposes.
- Three experimental results in the literature are presented confirming the validity of the bifurcation diagram, considering shear effect, for a flexible rotor-bearing system shown in Figure 2.8 and its practicality for design purposes. The instability threshold speed and its predicted bifurcation type using the proposed method could be beneficial at the design stage as well as for troubleshooting purposes of an unstable rotor-bearing system.
- For the case of large journal bearing clearances, accurate determination of the hysteresis profile requires a complete thermos-hydrodynamic analysis of a fluid-film bearing to assess the temperature as well as viscosity field within the fluid film chamber.

6.1.3 Conclusions on The Application of Hopf Bifurcation Theory (HBT) in Journal Bearing Hysteresis Phenomenon Determination

As bearing performance is strongly dependent on lubricant viscosity, in this chapter, the bifurcation profile behavior of a flexible shaft supported on end journal bearings was studied for a range of operating viscosities considering shear force effect. In order to identify a rotor bearing instability threshold speed (oil whirl/ship phenomenon), one can monitor dynamic behavior of the system as system operating speed gradually increases (run-up process). Once instability threshold speed crosses through run-up process, once can monitor dynamic behavior of the system as system operating speed gradually decreases (run-down process). In certain operating conditions, for the run-down case, oil whip phenomenon disappears at a running speed that is below the threshold speed monitored during the run-up process. This hysteresis phenomenon is, in fact, repeatable and occurs even though all other system parameters including oil viscosity remain unchanged. When hysteresis exists, the speed at which the oil whip starts is called the run-up threshold speed (ω_{uc}); and the speed at which the oil whip disappears is called the rundown threshold speed (ω_{dc}).

- Hopf bifurcation depends on, average Reynolds number (*Re*), system characteristic number (α), and dimensionless stiffness of the shaft (*K_k*).
- For a specific rotor bearing system having a stiffness lower than the critical stiffness of the shaft, there exist two transition system characteristic numbers α_1 and α_2 . In subcritical bifurcation region, hysteresis phenomenon, exist for operating conditions higher than α_2 and lower than α_1 .
- The operating system undergoes supercritical bifurcation for the rotor bearing system with intermediate system characteristic numbers ($\alpha_1 < \alpha < \alpha_2$); hence, to avoid hysteresis phenomenon in a rotor bearing system fluid film viscosity shall be maintained within the from $\mu'_1 < \mu < \mu'_2$, where μ_1 and μ_2 correspond to system characteristic numbers α_1 and α_2 respectively.
- In subcritical bifurcation region, at certain operating viscosities, if oil whip occurs, the rotor bearing system does not become stable by decreasing the rotating speed of the shaft (rundown process). On the contrary, in supercritical bifurcation region, at certain operating viscosities, oil whip never appears through run-up process. This is due to the fact that, stable and/or unstable bifurcation profiles do not cross the journal beating clearance at certain oil viscosities.

6.1.4 Conclusions on The Stability Analysis in Optimized Fluid Film Pressure and Position Determination

Instability analysis with consideration of a flexible rotor bearing system including turbulence effect an appropriate starting position of the cavitation and the reformation of oil film at the end of cavitation is studied. Analytical expressions are provided for dynamic fluid film pressure, force and coefficients for the entire range of operating conditions. Fluid film coefficients are then utilized for linear stability analysis of a flexible rotor-bearing system under different oil inlet circumferential positions pressure magnitudes. The dynamic fluid film characteristics are analytically obtained utilizing long bearing assumptions with Reynolds–Floberg–Jakobsson (RFJ) boundary condition. It was concluded that,

- Reynolds number has significant effects on fluid film pressure distribution and high Reynolds numbers could lead to less stable operating region.
- Stable operating region of flexible shafts supported journal bearings were shown to squeeze in size slightly by increasing the oil inlet pressure at low Sommerfeld numbers.
- Fluid film pressure distribution was found to be very sensitive to oil inlet position.
 - 1. For heavily-loaded bearings, it was found that stable operating region grows in size by increasing the oil inlet position in the range of $0 \le \Psi_i \le 135^\circ$.
 - 2. For heavily-loaded bearings, instability threshold speed drops as oil inlet position moves away from $\Psi_i = 135^\circ$ and approaches $\Psi_i = 180^\circ$.
 - 3. Implementing two groove angles at $\Psi_i = 0^\circ$ and 135° might improve that safe operating region of rotor supported journal bearings.

6.2 Limitations and Future Work

It should be noted that, at high rotating speed of the shaft (ω_s), fluid film induced pressure increases; however, due to the fluid film shearing effect and heat generation oil viscosity reduction can be expected which could result in pressure drop within the journal bearing chamber. Through inclusion of fluid film viscosity dependency to heat generation, results providing turbulence influence on the rotor bearing system instability can be affected. This could lead to improving safer operating region of the rotor supported journal bearings. Thus, considering the oil viscosity temperature dependency is the next important step for modifying the stability boundaries and identifying the bifurcation types of the system.

It is also assumed fluid film dynamics is governed by properties of Newtonian fluids. Shearthinning at high Reynolds numbers can affect the stability boundaries of the system and its effects should be included in future studies.

Another assumption throughout this thesis was the bearing chamber is considered to be perfectly rounded. Influence of roundness imperfection the bearing chamber on the instability threshold speed was not studied. It is known that the bearing bushing is susceptible to wear as a result of frequent system shut-down. Journal bearing surface imperfections can have significant influence on the rotor bearing system instability.

The proposed bearing nonlinear algorithm is very well behaved in the rotor bearing system design; however, journal bearings are made in a variety of shapes and incorporating these shapes and cavitation boundaries can introduce convergence difficulties for accurate determination of journal bearing coefficients and hence stability regions. Independent research shall be devoted to thoroughly study such effects on both linear and nonlinear stability of rotor-bearing systems.

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Appendices

Appendix A

	Turbulent Short Bearing Coefficients (Gümbel Boundary Condition)
\bar{k}_{rr}	$\frac{\pi}{6} \Big(\frac{L}{D}\Big)^2 \Big\{ \frac{48\epsilon(1+\epsilon^2)}{(1-\epsilon^2)^3} + \Big(\frac{(1-\epsilon)^{a_2-2}-(1+\epsilon)^{a_2-2}}{(a_2-2)(a_2-1)\epsilon^2} + \frac{(1-\epsilon)^{a_2-3}\big((a_2-2)\epsilon+1\big)-(1+\epsilon)^{a_2-3}\big((a_2-2)\epsilon-1\big)}{(a_2-1)\epsilon} \Big) a_1(\overline{Re})^{a_2} \Big\}$
\bar{k}_{rt}	$-\left(\frac{L}{D}\right)^{2}\left(\frac{\pi}{\epsilon\sqrt{12}}\right)^{2}\left\{\frac{-12\epsilon^{2}}{(1-\epsilon^{2})^{3/2}}+\left(2a_{4}-\frac{\epsilon^{2}+a_{4}(2-3\epsilon^{2})}{(1-\epsilon^{2})^{3/2}}\right)a_{1}(\overline{Re})^{a_{2}}\right\}$
\bar{k}_{tr}	$-\left(\frac{L}{D}\right)^2 \left(\frac{\pi^2}{12}\right) \left\{\frac{12(1+2\epsilon^2)}{(1-\epsilon^2)^{5/2}} + \left(\frac{2a_4(1-\epsilon^2)^{5/2}+2\epsilon^4+\epsilon^2-a_4(6\epsilon^4-5\epsilon^2+2)}{(1-\epsilon^2)^{5/2}\epsilon^2}\right)a_1(\overline{Re})^{a_2}\right\}$
\overline{k}_{tt}	$-\left(\frac{L}{D}\right)^{2}\left(\frac{\pi}{6}\right)\left\{\frac{-24\epsilon}{(1-\epsilon^{2})^{2}}+\left(\frac{(1-\epsilon)^{a_{2}-2}+(1+\epsilon)^{a_{2}-2}}{(a_{2}-1)\epsilon}+\frac{(1-\epsilon)^{a_{2}-2}-(1+\epsilon)^{a_{2}-2}}{(a_{2}-2)(a_{2}-1)\epsilon^{2}}\right)a_{1}(\overline{R}e)^{a_{2}}\right\}$
\bar{c}_{rr}	$\left(\frac{L}{D}\right)^{2} \left(\frac{\pi^{2}}{6}\right) \left\{\frac{12(1+2\epsilon^{2})}{(1-\epsilon^{2})^{5/2}} + \left(\frac{2a_{3}(1-\epsilon^{2})^{5/2}+2\epsilon^{4}+\epsilon^{2}-a_{3}(6\epsilon^{4}-5\epsilon^{2}+2)}{(1-\epsilon^{2})^{5/2}\epsilon^{2}}\right)a_{1}(\overline{Re})^{a_{2}}\right\}$
\bar{c}_{rt}	$-\left(\frac{L}{D}\right)^{2}\left(\frac{\pi}{3}\right)\left\{\frac{24\epsilon}{(1-\epsilon^{2})^{2}}-\left(\frac{(1-\epsilon)^{a_{2}-2}+(1+\epsilon)^{a_{2}-2}}{(a_{2}-1)\epsilon}+\frac{(1-\epsilon)^{a_{2}-2}-(1+\epsilon)^{a_{2}-2}}{(a_{2}-2)(a_{2}-1)\epsilon^{2}}\right)a_{1}(\overline{Re})^{a_{2}}\right\}$
\bar{c}_{tr}	$-\left(\frac{L}{D}\right)^{2}\left(\frac{\pi}{3}\right)\left\{\frac{24\epsilon}{(1-\epsilon^{2})^{2}}-\left(\frac{(1-\epsilon)^{a_{2}-2}+(1+\epsilon)^{a_{2}-2}}{(a_{2}-1)\epsilon}+\frac{(1-\epsilon)^{a_{2}-2}-(1+\epsilon)^{a_{2}-2}}{(a_{2}-2)(a_{2}-1)\epsilon^{2}}\right)a_{1}(\overline{Re})^{a_{2}}\right\}$
\bar{c}_{tt}	$-\left(\frac{L}{D}\right)^{2} \left(\frac{\pi}{\epsilon\sqrt{6}}\right)^{2} \left\{\frac{-12\epsilon^{2}}{(1-\epsilon^{2})^{3/2}} + \left(2a_{4} - \frac{\epsilon^{2} + a_{4}(2-3\epsilon^{2})}{(1-\epsilon^{2})^{3/2}}\right)a_{1}(\overline{Re})^{a_{2}}\right\}$
	$12(\frac{D}{L})^2$
S	$\overline{\pi \sqrt{(\frac{12\pi\epsilon}{(1-\epsilon^2)^{3/2}} - \left(2a_4 - \frac{\epsilon^2 + a_4(2-3\epsilon^2)}{(1-\epsilon^2)^{3/2}}\right)(\frac{\pi}{\epsilon})a_1(\overline{Re})^{a_2})^2 + 4(\frac{24\epsilon^2}{(1-\epsilon^2)^2} + \left(\frac{(1-\epsilon)^{a_2-2} + (1+\epsilon)^{a_2-2}}{(a_2-1)\epsilon} + \frac{(1-\epsilon)^{a_2-2} - (1+\epsilon)^{a_2-2}}{(a_2-2)(a_2-1)\epsilon^2}\right)a_1(\overline{Re})^{a_2})^2}}$
tan(\$\varphi)	$\left(\frac{\pi}{2}\right) \left \frac{\frac{12\epsilon^2}{(1-\epsilon^2)^{3/2}} - \left(2a_4 - \frac{\epsilon^2 + a_4(2-3\epsilon^2)}{(1-\epsilon^2)^{3/2}}\right)a_1(\overline{Re})^{a_2}}{\frac{24\epsilon^3}{(1-\epsilon^2)^2} + \left(\frac{(1-\epsilon)^{a_2-2} + (1+\epsilon)^{a_2-2}}{(a_2-1)} + \frac{(1-\epsilon)^{a_2-2} - (1+\epsilon)^{a_2-2}}{(a_2-2)(a_2-1)\epsilon}\right)a_1(\overline{Re})^{a_2}}\right $

Table A.1. Analytical stiffness and damping coefficients for turbulent short journal bearings



Table A.2. Analytical stiffness and damping coefficients for turbulent long journal bearings

Table A.3. Analytical stiffness and damping coefficients for turbulent short journal bearings considering shear force effect



Table A.4. Analytical stiffness and damping coefficients for laminar long journal bearings considering cavitation effect (RFJ boundary condition)

	Turbulent Long Bearing Coefficients Considering Cavitation Effect (RFJ Boundary Condition)
\bar{k}_{rr}	$\frac{3\pi(2\epsilon^2\cos(\!\!\widetilde{a}_c)-\epsilon^2-1)(\cos(\!\!\widetilde{a}_c)-\cos(\!\!\widetilde{a}_c))^2}{(\epsilon^2-1)^2(\epsilon\cos(\!\!\widetilde{a}_c)-1)^2}$
\bar{k}_{rt}	$\frac{3\pi(2\epsilon \sin(\tilde{\alpha}_c) - \alpha_c + \cos(\tilde{\alpha}_c)(\sin(\tilde{\alpha}_c) - 2(\epsilon\alpha_c - \epsilon\alpha_s + \sin(\tilde{\alpha}_s))) + \sin(\tilde{\alpha}_s)(\cos(\tilde{\alpha}_s) - 2\epsilon) + \alpha_s)}{(1 - \epsilon^2)^{3/2}(1 - \epsilon\cos(\tilde{\alpha}_c))}$
\bar{k}_{tr}	$-\frac{3\pi(2\epsilon(\epsilon^2+2)\sin(\tilde{u}_c)-(2\epsilon^2+1)(a_c-a_c)+\epsilon^2\cos^2(a_c)(-3\epsilon\sin(\tilde{u}_c)+2(2\epsilon^2+1)(a_c-a_c)+6\epsilon\sin(\tilde{u}_c))+\cos(\tilde{u}_c)(1(1-4\epsilon^2)\sin(\tilde{u}_c)+\epsilon(\epsilon^2-4)(a_c-a_c)-\sin(\tilde{u}_c)(3\epsilon^2\cos(\tilde{u}_c)-4\epsilon^4+2\epsilon^2+2))-2\epsilon(\epsilon^2+2)\sin(\tilde{u}_c)+(2\epsilon^2+1)\sin(\tilde{u}_c)\cos(\tilde{u}_c))}{(1-\epsilon^2)^{3/2}(\cos(\tilde{u}_c)-1)^2}$
\bar{k}_{tt}	$-rac{3\pi(\cos(heta_{\ell})-\cos(heta_{\ell}))^2}{(\epsilon^2-1)(\epsilon\cos(heta_{\ell}-1))}$
\bar{c}_{rr}	$-\frac{6\pi(\sin(\tilde{a}_{\epsilon})(\cos(\tilde{a}_{\epsilon}) + \cos(\tilde{a}_{\epsilon}))(\cos(\tilde{a}_{\epsilon}) - 2)) + (1 - \cos(\tilde{a}_{\epsilon}))(a_{\epsilon} - a_{s} + \sin(\tilde{a}_{s})\cos(\tilde{a}_{s})))}{(1 - \epsilon^{2})^{3/2}(1 - \epsilon\cos(\tilde{a}_{\epsilon}))}$
\bar{c}_{rt}	$\frac{6\pi(\cos(\xi t_c)-\cos(\xi t_c))^2}{(\epsilon^2-1)(\epsilon\cos(\xi t_c)-1)}$
\bar{c}_{tr}	$\frac{6\pi(2\epsilon(\cos(\tilde{a}_c)-\epsilon)-\sin^2(a_c)+\sin(\tilde{a}_c)(-\epsilon\cos(\tilde{a}_c)+2\epsilon^2+2)+3\epsilon\sin(\tilde{a}_c)(a_c-a_c)+(\epsilon\cos(\tilde{a}_c)-1)(2\epsilon\cos(\tilde{a}_c)+\sin^2(a_c)))}{(\epsilon^2-1)^2(1-\cos(\tilde{a}_c))}$
\bar{c}_{tt}	$-\frac{6\pi(\sin[\tilde{a}_{x},\cos[\tilde{a}_{x}]+(a_{x}-a_{x})(2\epsilon\cos[\tilde{a}_{x}]+1)-2(\cos[\tilde{a}_{x}]+\epsilon)(\sin[\tilde{a}_{x}]-\sin[\tilde{a}_{x}])-\sin[\tilde{a}_{x}]\cos[\tilde{a}_{x}])}{(1-\epsilon^{2})^{3/2}(1-\epsilon\cos[\tilde{a}_{x}])}$
	1
S	$\sqrt{\frac{9\pi^2\epsilon^2(\cos(t\alpha_c)-\cos(t\alpha_c))^4}{(1-\epsilon^c)^3(1-\cos(t\alpha_c))^2}} + \frac{9\pi^2\epsilon^2(\sin(t\alpha_c)\cos(t\alpha_c)+(\alpha_c-\alpha_c)(2e\cos(t\alpha_c)+1)-2(\cos(t\alpha_c)+\epsilon)(\sin(t\alpha_c)-\sin(t\alpha_c))-\sin(t\alpha_c)\cos(t\alpha_c))^2}{(1-\epsilon^c)^3(1-\cos(t\alpha_c))^2}$
tan∰)	$\frac{\cos(\tilde{a}_{c})\sin(\tilde{a}_{c})-2(\epsilon+\cos(\tilde{a}_{c}))(\sin(\tilde{a}_{c})-\sin(\tilde{a}_{c}))-\cos(\tilde{a}_{c})\sin(\tilde{a}_{c})+(2\epsilon\cos(\tilde{a}_{c})+1)(a_{c}-a_{s})}{\sqrt{1-\epsilon^{2}}(\cos(\tilde{a}_{c})-\cos(\tilde{a}_{c}))^{2}}$

Appendix B

B.1 Linear Stability Formulation of a Flexible Rotor Supported on Two Identical Journal Bearings

By substituting Equation (2.22) into Equation (2.21), the following expression may be obtained:

$$\begin{bmatrix} \frac{K_s M \omega_w^2}{M \omega_w^2 - K_s} + 2K_{xx} + 2i\omega_w C_{xx} & 2K_{xy} + 2i\omega_w C_{xy} \\ 2K_{yx} + 2i\omega_w C_{yx} & \frac{K_s M \omega_w^2}{M \omega_w^2 - K_s} + 2K_{yy} + 2i\omega_w C_{yy} \end{bmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
(B.1)

Using the normalization of $\overline{K}_{ij} = (\pi \left(\frac{c}{R}\right)^3 / \mu \omega L) K_{ij}, \ \overline{C}_{ij} = (\pi \left(\frac{c}{R}\right)^3 / \mu L) C_{ij}, \ (i, j = x, y),$ $\overline{K}_s = (\pi \left(\frac{c}{R}\right)^3 / \mu \omega L) K_s, \ \left(\frac{\overline{X}_1}{\overline{Y}_1}\right) = \frac{1}{c} {X_1 \choose Y_1}, \ \overline{\omega} = \omega \sqrt{M/K_s}, \ \Omega = \omega_w / \omega, \ \gamma = \frac{\overline{K}_s \Omega^2 \overline{\omega}^2}{\Omega^2 \overline{\omega}^2 - 1}, \ \text{the non-dimensional form of Equation (P 1):}$

dimensional form of Equation (B.1):

$$\begin{bmatrix} \gamma + 2\bar{K}_{xx} + 2i\Omega\bar{C}_{xx} & 2\bar{K}_{xy} + 2i\Omega\bar{C}_{xy} \\ 2\bar{K}_{yx} + 2i\Omega\bar{C}_{yx} & \gamma + 2\bar{K}_{yy} + 2i\Omega\bar{C}_{yy} \end{bmatrix} \begin{pmatrix} \bar{X}_1 \\ \bar{Y}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
(B.2)

The condition of non-trivial (non-zero) solution requires that the determinant of the characteristic matrix to be set to zero. By setting the imaginary part of the determinant of Equation (B.2) to zero:

$$\gamma = \frac{2(\bar{K}_{xy}\bar{C}_{yx} + \bar{K}_{yx}\bar{C}_{xy} - \bar{K}_{yy}\bar{C}_{xx} - \bar{K}_{xx}\bar{C}_{yy})}{\bar{C}_{xx} + \bar{C}_{yy}}$$
(B.3)

By setting the real part of the determinant of Equation (B.2) to zero, the whirl frequency ratio can be calculated as:

$$\Omega^{2} = \frac{\gamma^{2} + 2(\bar{K}_{xx} + \bar{K}_{yy})\gamma + 4(\bar{K}_{xx}K_{yy} - \bar{K}_{xy}\bar{K}_{yx})}{4(\bar{C}_{xx}\bar{C}_{yy} - \bar{C}_{xy}\bar{C}_{yx})}$$
(B.4)

Using the definition of non-dimensional parameter γ , the instability threshold speed can be found as:

$$\overline{\omega}_s^2 = \frac{\gamma}{\Omega^2(\gamma - \overline{K}_s)} \tag{B.5}$$

where $\overline{K}_{s} = \left(\pi \left(\frac{C}{R}\right)^{3} / \mu \omega L\right) K_{s} = \frac{1}{s} \frac{C K_{s}}{W}.$

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Appendix C

C.1 Sommerfeld Substitutions [114, 124]

$$d\theta = \frac{\sqrt{1 - \epsilon^2}}{1 - \epsilon \cos \alpha} d\alpha \tag{C.6}$$

$$\cos\theta_s = \frac{\cos\alpha_s - \epsilon}{1 - \epsilon \cos\alpha_s} \tag{C.7}$$

$$\sin \theta_s = \frac{\sin \alpha_s \sqrt{1 - \epsilon^2}}{1 - \epsilon \cos \alpha_s} \tag{C.8}$$

$$\cos\theta_c = \frac{\cos\alpha_c - \epsilon}{1 - \epsilon \cos\alpha_c} \tag{C.9}$$

$$\sin \theta_c = \frac{\sin \alpha_c \sqrt{1 - \epsilon^2}}{1 - \epsilon \cos \alpha_c} \tag{C.10}$$

$$\cos \alpha_i = \frac{\epsilon + \cos \theta_i}{1 - \epsilon \cos \theta_i} = \frac{\epsilon + \cos(\Psi_i - \varphi)}{1 - \epsilon \cos(\Psi_i - \varphi)}$$
(C.11)

$$\sin \alpha_i = \frac{\sin \theta_i \sqrt{1 - \epsilon^2}}{1 + \epsilon \cos \theta_i} = \frac{\sin(\Psi_i - \varphi) \sqrt{1 - \epsilon^2}}{1 + \epsilon \cos(\Psi_i - \varphi)}$$
(C.12)