On properties of projection operators associated with convex sets

by

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Abstract

The aim of this research work is twofold. On the one hand, under mild assumptions, we give an explicit formula for projection operator associated with the intersection of a cone and a ball/sphere; nevertheless, this formula plays a role in determining copositivity of matrices. On the other hand, we provide a complete answer to the question “When is the sum of projectors a projector?,” which allows us to fill a gap in the current literature.
Lay summary

Given a set $C$, one often does not have an explicit formula for points in $C$ that are closest (in distance) to a given point—especially when $C$ is the intersection or the sum of sets. On the one hand, we provide formulae for determining points in the intersection of a cone and a ball/sphere that a closest to a given point under mild assumptions. These formulae have a potential to be useful in optimization problems where a constraint is given. Indeed, one of these formulae was a key tool in Lange’s algorithm to determine copositivity of matrices. On the other hand, we analyze carefully a case where the closest point onto the sum of convex sets of a given point can be expressed as a sum of individual closest points, which is not well understood in the current literature of Convex Analysis.
Preface

This thesis is based on the paper [7] (Chapter 4) and the submitted manuscript [6] (Chapter 5) by Heinz Bauschke, Minh Bui, and Xianfu Wang.

For the aforementioned work, each author contributed equally.
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## Glossary of notation and symbols

### Real line

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<tr>
<td>$\mathbb{R}$</td>
<td>The set of real numbers</td>
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<td>$\mathbb{N}$</td>
<td>The set of positive integers ${0, 1, 2, \ldots}$</td>
<td>3</td>
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<tr>
<td>$\mathbb{R}^+$</td>
<td>The set of positive real number ${\xi \in \mathbb{R} \mid \xi \geq 0}$</td>
<td>3</td>
</tr>
<tr>
<td>$\mathbb{R}^{++}$</td>
<td>The set of strictly positive real number ${\xi \in \mathbb{R} \mid \xi &gt; 0}$</td>
<td>3</td>
</tr>
<tr>
<td>$\mathbb{R}^-$</td>
<td>The set of negative real number ${\xi \in \mathbb{R} \mid \xi \leq 0}$</td>
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</tr>
<tr>
<td>$\mathbb{R}^{--}$</td>
<td>The set of strictly negative real number ${\xi \in \mathbb{R} \mid \xi &lt; 0}$</td>
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### Hilbert spaces

<table>
<thead>
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<th>Symbol</th>
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<td>$\mathcal{H}$</td>
<td>Real Hilbert space</td>
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<tr>
<td>$\langle \cdot</td>
<td>\cdot \rangle$</td>
<td>Scalar product</td>
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<td>$| \cdot |_2$</td>
<td>Norm</td>
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<td>$\text{Id}$</td>
<td>The identity operator on $\mathcal{H}$</td>
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</tr>
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<td>$B(x; \rho)$</td>
<td>Closed ball with center $x$ and radius $\rho$</td>
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<td>$S(x; \rho)$</td>
<td>Sphere with center $x$ and radius $\rho$</td>
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<td>$\mathbb{R}^N$</td>
<td>The standard $N$-dimensional Euclidean space</td>
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<td>The positive orthant in $\mathbb{R}^N$</td>
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<td>$S^N$</td>
<td>Space of real symmetric matrices of size $N \times N$</td>
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<td>$S^N_+$</td>
<td>Set of real $N \times N$ symmetric positive semidefinite matrices</td>
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<td>$U^N$</td>
<td>Set of real $N \times N$ orthogonal matrices</td>
<td>49</td>
</tr>
</tbody>
</table>
Glossary of notation and symbols

Sets

- $2^X$: Power set of a set $X$ .................................................. 3
- $\prod_{i \in I} C_i$: Cartesian product of $(C_i)_{i \in I}$ ..................... 8
- $\sum_{i \in I} C_i$: Minkowski sum of a finite family of sets $(C_i)_{i \in I}$ 13
- $\overline{C}$: Closure of a set $C$ .................................................. 3
- $C^\perp$: Orthogonal complement of a set $C$ .......................... 3
- $\text{span } C$: Linear span of a set $C$ ......................................... 79
- $\text{conv } C$: Convex hull of a set $C$ .......................................... 8
- $\overline{\text{conv }} C$: Closed convex hull of a set $C$ .............. 13
- $\text{cone } C$: Conical hull of a set $C$ .......................................... 12
- $\text{cone } C$: Closed conical hull of a set $C$ ...................... 12
- $\text{C}^\circ$: Polar cone of a set $C$ ............................................. 13
- $\text{pos } C$: Positive span of a set $C$ ........................................ 13
- $\text{rec } C$: Recession cone of a set $C$ ...................................... 17

Operators and functions

- $\text{ran } A$: Range of an operator $A: \mathcal{H} \to 2^\mathcal{H}$ ........ 22
- $\overline{\text{ran }} A$: Closure of the range of an operator $A: \mathcal{H} \to 2^\mathcal{H}$ 22
- $\text{gra } A$: Graph of an operator $A: \mathcal{H} \to 2^\mathcal{H}$ .......... 21
- $A^{-1}$: Inverse of an operator $A: \mathcal{H} \to 2^\mathcal{H}$ .............. 21
- $\text{Fix } T$: Set of fixed points of an operator $T: \mathcal{H} \to \mathcal{H}$ 59
- $\text{dom } f$: Domain of a function $f: \mathcal{H} \to [-\infty, +\infty]$ 18
- $\overline{\text{dom } f}$: Closure of the domain of a function $f: \mathcal{H} \to [-\infty, +\infty]$ 18
- $\Gamma_0(\mathcal{H})$: Set of proper lower semicontinuous convex functions from $\mathcal{H}$ to $]-\infty, +\infty[$ ...................................................... 18
- $f \boxplus g$: Infimal convolution of functions $f$ and $g$ ............. 20
- $f \boxdot g$: Exact infimal convolution of functions $f$ and $g$ ...... 20
- $i_C$: Indicator function of a set $C$ ........................................... 13
- $\sigma_C$: Support function of a set $C$ .................................. 13
- $d_C$: Distance function to a set $C$ ........................................ 3
- $P_C$: Set-valued projector associated with a set $C$ .............. 9
- $\text{Prox}_f$: Proximity operator of a function $f$ ................. 20
- $\partial f$: Subdifferential of a function $f$ ................................. 21
- $\nabla f$: Gradient operator of a function $f$ ............................ 7
- $f^*$: (Fenchel) Conjugate of a function $f$ .......................... 19
- $f^{**}$: Biconjugate of a function $f$ ................................. 19
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Dành cho Mẹ

Con xin dânh tặng luận văn này cho Mẹ.
Cảm ơn Mẹ, vì tất cả những gì Mẹ đã dânh cho con.
Chapter 1

What is this thesis about?

The notion of projection operator in a Hilbert space is fundamental to the development of Optimization Theory, and in turn, one of the basic problems is to compute the projection of a given point onto a set. Recently, in [28, Example 5.5.2], Lange presented a closed form for the projector associated with the intersection of the orthant and the unit sphere and used it for an algorithm on determining copositivity of a matrix; however, this projection has the potential to be useful in other settings where a priori constraints are present (e.g., positivity and energy). This gives rise to the nontrivial problem of projecting onto the intersection of a cone and a sphere (both are centered at the origin); indeed, the projection onto the intersection of two sets generally does not allow for a closed form even when the individual projection operators have explicit descriptions. In Chapter 4, we systematically analyze the projection onto the intersection of a cone with either a ball or a sphere in the general case. Several cases are provided where the projector is available explicitly. Various examples based on finitely generated cones (as a consequence, we recover Lange’s result), the Lorentz cone, and the cone of positive semidefinite matrices are presented. The usefulness of our formulae is illustrated by numerical experiments for determining copositivity of real symmetric matrices. Having done a systematic study on the projection onto the intersection of sets, we now turn to the the projection onto the Minkowski sum of closed convex sets. This projector is generally not equal to the sum of individual projectors as one may expect. However, if that sum is a projector, we shall see in Proposition 5.1 that the aforementioned equality holds. Therefore, one may wish to analyze the question “When is the sum of projectors a projector?” If all the sets are subspaces, an answer to this question dates back to, at least, Halmos [21]. In 1971, Zarantonello [39] extended Halmos’s result to the case of cones. The general case is, however, not well understood and creates a gap in the literature of Convex Analysis. In Chapter 5, we provide a complete answer to this question in the general case. Our results unify and extend the case of linear subspaces and Zarantonello’s results for projectors...
Chapter 1. What is this thesis about?

onto cones. We then establish the partial sum property for projectors onto convex cones, which is a connection with the work [4], and we also present various examples as well as a detailed analysis in the univariate case. For the sake of self-containedness, basic concepts and well-known results in Real Analysis, Convex Analysis, and Monotone Operator Theory that are used in Chapters 4 and 5 are provided in Chapters 2 and 3. The results in this thesis are based on [7] and [6], except for facts with explicit references.
Chapter 2

Elements of real analysis

In this chapter, we collect a few results and notation of Real Analysis that will be used later. We assume that the reader is familiar with basic Real Analysis; in particular, real Hilbert spaces.

2.1 Notation and symbols

Throughout this thesis,

\[ \mathcal{H} \text{ is a real Hilbert space} \] (2.1)
equipped with a scalar product \( \langle \cdot | \cdot \rangle \). The corresponding norm is denoted by \( \| \cdot \| \), i.e.,

\[ (\forall x \in \mathcal{H}) \quad \| x \| := \sqrt{\langle x | x \rangle}, \] (2.2)

and the identity operator on \( \mathcal{H} \) is denoted by \( \text{Id} \).

The notation used in this thesis is standard and mainly follows [8]. We set \( \mathbb{N} := \{0, 1, 2, \ldots \} \), \( \mathbb{R}_+ := [0, +\infty[ \), \( \mathbb{R}_{++} := ]0, +\infty[ \), \( \mathbb{R}_- := ]-\infty, 0] \), and \( \mathbb{R}_{--} := ]-\infty, 0[ \). Let \( C \) be a subset of \( \mathcal{H} \). Then the orthogonal complement of \( C \) is

\[ C^\perp := \{ u \in \mathcal{H} \mid (\forall x \in C) \langle x | u \rangle = 0 \}, \] (2.3)

and the closure of \( C \) is denoted by \( \overline{C} \). Next, the distance function to \( C \) is

\[ d_C : \mathcal{H} \to [0, +\infty] : x \mapsto \inf\| x - C \|. \] (2.4)

(Note that \( d_C \equiv +\infty \) when \( C \) is empty.) Now let \( x \in \mathcal{H} \) and \( \rho \in \mathbb{R}_{++} \). The closed ball and sphere with center \( x \) and radius \( \rho \) are \( B(x; \rho) := \{ y \in \mathcal{H} \mid \| x - y \| \leq \rho \} \) and \( S(x; \rho) := \{ y \in \mathcal{H} \mid \| x - y \| = \rho \} \), respectively.

Finally, let \( A : \mathcal{H} \to 2^\mathcal{H} \) be such that \( \text{dom } A := \{ x \in \mathcal{H} \mid Ax \neq \emptyset \} \neq \emptyset \). Assume that, for every \( x \in \text{dom } A \), \( Ax \) is a singleton. Then, for every \( x \in \text{dom } A \), without any confusion, we shall identify \( Ax \) with its unique element.
2.2. Identities and inequalities

Fact 2.1 (Cauchy–Schwarz) Let \( x \) and \( y \) be in \( \mathcal{H} \). Then
\[
|\langle x | y \rangle| \leq \|x\|\|y\|,
\]
and equality holds if and only if there exists \( \alpha \in \mathbb{R}_+ \) such that \( x = \alpha y \) or \( y = \alpha x \).

Lemma 2.2 Let \( \{\alpha_i\}_{i \in I} \) be a finite subset of \( \mathbb{R} \) such that
\[
(\forall i \in I)(\forall j \in I) \quad \alpha_i \alpha_j = 0 \quad (2.6)
\]
and that
\[
\sum_{i \in I} \alpha_i = 1. \quad (2.7)
\]
Then there exists \( i \in I \) such that \( \alpha_i = 1 \) and \((\forall j \in I \setminus \{i\})\) \( \alpha_j = 0 \).

Proof. Suppose that there exist \( i \) and \( j \) in \( I \) such that \( i \neq j \), that \( \alpha_i \neq 0 \), and that \( \alpha_j \neq 0 \). Then \( \alpha_i \alpha_j \neq 0 \), which violates (2.6). Hence, \( \{\alpha_i\}_{i \in I} \) contains at most one nonzero number. On the other hand, by (2.7), \( \{\alpha_i\}_{i \in I} \) must contain at least one nonzero number. Altogether, we conclude that there exists \( i \in I \) such that \( \alpha_i \neq 0 \) and \((\forall j \in I \setminus \{i\})\) \( \alpha_j = 0 \). Consequently, it follows from (2.7) that \( \alpha_i = 1 \), as claimed. \( \blacksquare \)

Lemma 2.3 Let \( x \in \mathcal{H} \), let \( (x_i)_{i \in I} \) be a finite family in \( \mathcal{H} \), and let \( (\alpha_i)_{i \in I} \) be a family in \( \mathbb{R} \) such that \( \sum_{i \in I} \alpha_i = 1 \). Then the following hold:

(i) \[
\|\sum_{i \in I} \alpha_i x_i\|^2 + \sum_{(i,j) \in I \times I} \|x_i - x_j\|^2 \leq \sum_{i \in I} \alpha_i \|x_i\|^2.
\]

(ii) Suppose that \( I = \{1, \ldots, m\} \), where \( m \in \{2, 3, \ldots\} \). Then
\[
\left\| x - \sum_{i \in I} x_i \right\|^2 = \sum_{i \in I} \|x - x_i\|^2 - (m - 1)\|x\|^2 + 2 \sum_{(i,j) \in I \times I} \langle x_i | x_j \rangle. \quad (2.8)
\]

(iii) Set \( x := \sum_{i \in I} \alpha_i x_i \) and \( \beta := \|x\| \). Suppose that
\[
(\forall i \in I) \quad \|x_i\| = \beta, \quad (2.9)
\]
that
\[
(\forall i \in I) \quad \alpha_i \geq 0, \quad (2.10)
\]
2.2. Identities and inequalities

and that the vectors \((x_i)_{i \in I}\) are pairwise distinct, i.e.,

\[
(\forall i \in I)(\forall j \in I) \quad i \neq j \Rightarrow x_i \neq x_j.
\] (2.11)

Then \((\exists i \in I) x = x_i\).

Proof. (i): See [8, Lemma 2.14(ii)].

(ii): We shall proceed by induction on \(m\). The claim is true for \(m = 2\) due to

\[
\|x - (x_1 + x_2)\|^2 = \|x - x_1\|^2 - 2\langle x - x_1 | x_2 \rangle + \|x_2\|^2 \quad (2.12a)
\]

\[
= \|x - x_1\|^2 + (\|x\|^2 - 2\langle x | x_2 \rangle + \|x_2\|^2) \quad (2.12b)
\]

\[
- \|x\|^2 + 2\langle x_1 | x_2 \rangle \quad (2.12c)
\]

\[
= \|x - x_1\|^2 + \|x - x_2\|^2 - \|x\|^2 + 2\langle x_1 | x_2 \rangle. \quad (2.12d)
\]

In turn, assume that \(m \in \{3, 4, \ldots\}\) and that the result holds for families containing \(m - 1\) vectors; furthermore, set \(J := \{1, \ldots, m - 1\}\). We then infer from the induction hypothesis that

\[
\left\| x - \sum_{i \in I} x_i \right\|^2 = \left\| \left( x - \sum_{i \in J} x_i \right) - x_m \right\|^2 \quad (2.13a)
\]

\[
= \left\| x - \sum_{i \in J} x_i \right\|^2 + \|x_m\|^2 - 2\left\langle x - \sum_{i \in J} x_i | x_m \right\rangle \quad (2.13b)
\]

\[
= \sum_{i \in J} \|x - x_i\|^2 - ((m - 1) - 1)\|x\|^2
\]

\[
+ 2 \sum_{(i,j) \in J \times I \atop i < j} \langle x_i | x_j \rangle + \|x_m\|^2 - 2\left\langle x | x_m \right\rangle
\]

\[
+ 2 \sum_{i \in J} \langle x_i | x_m \rangle \quad (2.13c)
\]

\[
= \sum_{i \in J} \|x - x_i\|^2 + (\|x_m\|^2 - 2\langle x | x_m \rangle + \|x\|^2)
\]

\[
- (m - 1)\|x\|^2 + 2 \sum_{(i,j) \in J \times I \atop i < j} \langle x_i | x_j \rangle \quad (2.13d)
\]

\[
= \sum_{i \in I} \|x - x_i\|^2 - (m - 1)\|x\|^2 + 2 \sum_{(i,j) \in I \times I \atop i < j} \langle x_i | x_j \rangle, \quad (2.13e)
\]
2.3. Miscellaneous results

which yields the desired assertion.

(iii): Since \( \sum_{i \in I} a_i = 1 \), we deduce from (i) and (2.9) that

\[
\beta^2 + \sum_{(i,j) \in I \times I} a_i a_j \| x_i - x_j \|^2 / 2 = \sum_{i \in I} a_i \beta^2 = \beta^2,
\]

which yields \( \sum_{(i,j) \in I \times I} a_i a_j \| x_i - x_j \|^2 = 0 \) or, equivalently, by (2.10),

\[
(\forall i \in I)(\forall j \in I) \quad a_i a_j \| x_i - x_j \|^2 = 0.
\]

Thus, we get from (2.11) and (2.15) that \( (\forall i \in I)(\forall j \in I) \ i \neq j \Rightarrow \| x_i - x_j \| \neq 0 \Rightarrow a_i a_j = 0 \), and because \( \sum_{i \in I} a_i = 1 \), Lemma 2.2 guarantees the existence of \( i \in I \) such that \( a_i = 1 \) and \( (\forall j \in I \setminus \{i\}) \ a_j = 0 \). Consequently, it follows from the very definition of \( x \) that \( x = x_i \), as desired. \( \blacksquare \)

Lemma 2.4 Let \( \alpha \) be in \( \mathbb{R} \), let \( \beta \) be in \( \mathbb{R}_+ \), and let \( x = (x, \xi) \in H \oplus \mathbb{R} \). Set \( S_{\alpha, \beta} := S(0; \beta) \times \{ \alpha \} \). Then \( \max(x \mid S_{\alpha, \beta}) = \beta \| x \| + \xi \alpha \).

Proof. We shall assume that \( x \neq 0 \), since otherwise \( \langle x \mid S_{\alpha, \beta} \rangle = \{ \xi \alpha \} \) and the assertion is clear. Now, for every \( y = (y, \alpha) \in S_{\alpha, \beta} \), since \( \| y \| = \beta \), the Cauchy–Schwarz inequality yields

\[
\langle x \mid y \rangle = \langle x \mid y \rangle + \xi \alpha \leq \| x \| \| y \| + \xi \alpha = \beta \| x \| + \xi \alpha.
\]

Hence \( \sup \langle x \mid S_{\alpha, \beta} \rangle \leq \beta \| x \| + \xi \alpha \). Consequently, because \( (\beta x / \| x \|, \alpha) \in S_{\alpha, \beta} \) and

\[
\left\langle x \left| \left( \frac{\beta x}{\| x \|}, \alpha \right) \right. \right\rangle = \left\langle x \left| \frac{\beta x}{\| x \|} \right. \right\rangle + \xi \alpha = \beta \| x \| + \xi \alpha,
\]

we obtain the conclusion. \( \blacksquare \)

2.3 Miscellaneous results

Lemma 2.5 Let \( C \) and \( D \) be subsets of \( H \). Then \( \overline{C + D} = \overline{C} + \overline{D} \).

Proof. Since clearly \( \overline{C + D} \subseteq \overline{C} + \overline{D} \), it suffices to show that \( \overline{C + D} \subseteq \overline{C} + \overline{D} \). To this end, take \( x \in \overline{C + D} \). Then, on the one hand, there exist sequences \( (c_n)_{n \in \mathbb{N}} \) in \( \overline{C} \) and \( (d_n)_{n \in \mathbb{N}} \) in \( D \) such that \( \lim_n (c_n + d_n) = x \). On the other hand, for every \( n \in \mathbb{N} \), because \( c_n \in \overline{C} \), there exists \( e_n \in C \) such that \( \| c_n - e_n \| < 1 / (n + 1) \); hence, we obtain a sequence \( (e_n)_{n \in \mathbb{N}} \) in \( C \) that
satisfies \( \lim_{n} \|c_n - e_n\| = 0 \). Altogether, \( e_n + d_n = (e_n - c_n) + (c_n + d_n) \to 0 + x = x \), and since \( (e_n + d_n)_{n \in \mathbb{N}} \) is a sequence in \( C + D \), it follows that \( x \in C + D \), as announced. ■

The following is a variant of [40, Lemma 6.1]. We provide a proof for completeness.

**Lemma 2.6** Let \( f : \mathcal{H} \to \mathbb{R} \) be Gâteaux differentiable on \( \mathcal{H} \), and suppose that \( \nabla f \) is positively homogeneous. Then

\[
(\forall x \in \mathcal{H}) \quad f(x) = \frac{1}{2} \langle x \mid \nabla f(x) \rangle + f(0). \tag{2.18}
\]

**Proof.** By assumption

\[
(\forall x \in \mathcal{H})(\forall \lambda \in \mathbb{R}^+) \quad \nabla f(\lambda x) = \lambda \nabla f(x). \tag{2.19}
\]

Now fix \( x \in \mathcal{H} \), and set \( \phi : \mathbb{R} \to \mathbb{R} : t \mapsto f(tx) \). Then, since \( f \) is Gâteaux differentiable, we see that

\[
(\forall t \in \mathbb{R}) \quad \lim_{0 \neq \alpha \to 0} \frac{\phi(t + \alpha) - \phi(t)}{\alpha} = \lim_{0 \neq \alpha \to 0} \frac{f((t + \alpha)x) - f(tx)}{\alpha} \tag{2.20a}
\]

\[
= \lim_{0 \neq \alpha \to 0} \frac{f(tx + \alpha x) - f(tx)}{\alpha} \tag{2.20b}
\]

\[
= \langle x \mid \nabla f(tx) \rangle. \tag{2.20c}
\]

Hence, \( \phi \) is differentiable on \( \mathbb{R} \) and, in view of (2.19)&(2.20c), \( (\forall t \in \mathbb{R}^+) \phi'(t) = t \langle x \mid \nabla f(x) \rangle \). Consequently,

\[
f(x) - f(0) = \phi(1) - \phi(0) = \int_{0}^{1} \phi'(t)dt \tag{2.21a}
\]

\[
= \int_{0}^{1} t \langle x \mid \nabla f(x) \rangle dt \tag{2.21b}
\]

\[
= \frac{1}{2} \langle x \mid \nabla f(x) \rangle, \tag{2.21c}
\]

from which we obtain the conclusion. ■
Chapter 3

Elements of convex analysis
and monotone operator theory

This chapter presents known results in Convex Analysis and Monotone Operator Theory that will be useful in the sequel. The main reference for facts here is [8], and for basic Convex Analysis, the reader is referred to, e.g., [34, 23, 24, 38].

3.1 Convex sets

Definition 3.1 Let \( C \) be a subset of \( H \). Then \( C \) is convex if

\[
(\forall \alpha \in ]0, 1[) \quad (1 - \alpha)C + \alpha C = C
\]

or, equivalently,

\[
(\forall x \in C)(\forall y \in C) \quad ]x, y[ \subseteq C.
\]

3.1 Convex sets

Example 3.2 The following sets are convex:

(i) Open and closed balls corresponding to a norm on \( H \).

(ii) Affine subspaces.

(iii) Half-spaces.

(iv) The intersection of a family of convex sets.

(v) Let \((H_i)_{i \in I}\) be a finite family of real Hilbert spaces and, for every \( i \in I \), let \( C_i \) be a convex subset of \( H_i \). Then \( \bigcap_{i \in I} C_i \) is a convex subset of \( \bigoplus_{i \in I} H_i \).

Definition 3.3 Let \( C \) be a subset of \( H \). Then the convex hull of \( C \), in symbol, \( \text{conv} C \), is the intersection of all the convex subsets of \( H \) that contain \( C \);
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in other words, $\text{conv} \ C$ is the smallest convex subset of $\mathcal{H}$ containing $C$. Likewise, the closed convex hull of $C$, in symbol, $\overline{\text{conv}} \ C$, is the smallest closed convex subset of $\mathcal{H}$ that contains $C$.

**Example 3.4** Let $\rho \in \mathbb{R}_{++}$, and set $C := S(0; \rho)$. Then $\text{conv} \ C = B(0; \rho)$.

**Proof.** Since $B(0; \rho)$ is convex and $C \subseteq B(0; \rho)$, we obtain $\text{conv} \ C \subseteq B(0; \rho)$. Conversely, take $x \in B(0; \rho)$, and we consider the following two conceivable cases:

(a) $x = 0$: Fix $y \in C$. Then clearly $-y \in C$ and $x = 0 = (1/2)y + (1/2)(-y) \in \text{conv} \ C$.

(b) $x \neq 0$: Set $x_+ := (\rho/\|x\|)x$, $x_- := (\rho/\|x\|)(-x)$, and $\alpha := (1 + \|x\|/\rho)/2$. Then $\{x_+, x_-\} \subseteq C$, and because $\|x\| \leq \rho$, we have $\alpha \in [0, 1]$. Thus, since it is readily verified that $x = \alpha x_+ + (1 - \alpha)x_-$, we get $x \in \text{conv} \ C$.

Hence, $x \in \text{conv} \ C$ in both cases, which completes the proof. ■

**Fact 3.5** Let $C$ be a subset of $\mathcal{H}$. Then

$$\text{conv} \ C = \left\{ \sum_{i \in I} \alpha_i x_i \ \middle| \ I \text{ is finite}, \ \{\alpha_i\}_{i \in I} \subseteq [0, 1], \ \sum_{i \in I} \alpha_i = 1, \ \{x_i\}_{i \in I} \subseteq C \right\},$$

i.e., $\text{conv} \ C$ is the set of all convex combinations of elements in $C$.

**Proof.** See, e.g., [34, Theorem 2.3]. ■

**Fact 3.6** Let $I$ be a nonempty finite set, and let $(\mathcal{H}_i)_{i \in I}$ be a family of real Hilbert spaces. In addition, for every $i \in I$, let $C_i$ be a subset of $\mathcal{H}_i$. Then

$$\text{conv} \left( \bigotimes_{i \in I} C_i \right) = \bigotimes_{i \in I} \text{conv} \ C_i.$$  \hfill (3.4)

**Definition 3.7** Let $C$ be a nonempty subset of $\mathcal{H}$, let $x \in \mathcal{H}$, and let $p \in \mathcal{H}$. Then $p$ is a projection of $x$ onto $C$ if $\|x - p\| = d_C(x)$. Moreover, the projector (or projection operator) onto $C$ is defined by

$$P_C : \mathcal{H} \to 2^\mathcal{H} : x \mapsto \{p \in C \mid \|x - p\| = d_C(x)\}.$$  \hfill (3.5)

**Proposition 3.8** Let $C$ be a nonempty subset of $\mathcal{H}$, and let $x$ and $y$ be in $\mathcal{H}$. Then $P_{y+C}x = y + P_C(x - y)$. 

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Proof. Let us first show that
\[
y + P_C(x - y) \subseteq P_{y+C}x.
\] (3.6)
Towards this goal, take \( p \in P_C(x - y) \). Then, on the one hand, clearly \( y + p \in y + C \). On the other hand, because \( p \in P_C(x - y) \), we have
\[
(\forall z \in C) \| x - (y + p) \| = \| (x - y) - p \| \leq \| (x - y) - z \| = \| x - (y + z) \|.
\]
Altogether, \( y + p \in P_{y+C}x \), as announced in (3.6). Consequently, replacing \( (C, x, y) \) in (3.6) by \( (y + C, x - y, -y) \), it follows that \( -y + P_{y+C}(x) = -y + P_{y+C}((x - y) - (-y)) \subseteq P_{(y+C)-y}(x - y) = P_C(x - y) \), which completes the proof. \( \blacksquare \)

We now turn to projectors onto subsets of spheres.

Lemma 3.9 Let \( C \) be a nonempty subset of \( \mathcal{H} \) consisting of vectors of equal norm, let \( x \in \mathcal{H} \), and let \( p \in C \). Then the following hold:

(i) \( p \in P_Cx \iff \langle x \mid p \rangle = \max \langle x \mid C \rangle \).

(ii) \( P_Cx \neq \emptyset \) if and only if \( \langle x \mid \cdot \rangle \) achieves its supremum over \( C \).

Proof. (i): Indeed, since \( p \in C \) and \( (\forall y \in C) \| y \| = \| p \| \) by our assumption, we see that
\[
p \in P_Cx \iff (\forall y \in C) \| x - p \|^2 \leq \| x - y \|^2 \tag{3.7a}
\]
\[
\iff (\forall y \in C) -2\langle x \mid p \rangle \leq -2\langle x \mid y \rangle \tag{3.7b}
\]
\[
\iff (\forall y \in C) \langle x \mid y \rangle \leq \langle x \mid p \rangle \tag{3.7c}
\]
\[
\iff \langle x \mid p \rangle = \max \langle x \mid C \rangle, \tag{3.7d}
\]
which verifies the claim.

(ii): This follows from (i). \( \blacksquare \)

The following example provides an instance in which \( P_Cx \neq \emptyset \), where \( C \) and \( x \) are as in Lemma 3.9.

Example 3.10 Consider the setting of Lemma 3.9 and suppose, in addition, that \( C \) is weakly closed. Then \( P_Cx \neq \emptyset \).

Proof. Since, by assumption, \( C \) is bounded and since \( C \) is weakly closed, we deduce that \( C \) is weakly compact (see, for instance, [8, Lemma 2.36]). Therefore, because \( \langle x \mid \cdot \rangle \) is weakly continuous, its supremum over \( C \) is achieved, and the assertion therefore follows from Lemma 3.9(ii). \( \blacksquare \)
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Fact 3.11 (Projection theorem) Let $C$ be a nonempty closed convex subset of $\mathcal{H}$. Then $P_C$ is single-valued and, for every $x \in \mathcal{H}$ and every $p \in \mathcal{H}$,

$$ p = P_C x \iff [ p \in C \quad \text{and} \quad (\forall y \in C) \langle y - p \mid x - p \rangle \leq 0 ]. \quad (3.8) $$

Proof. See, for instance, [8, Theorem 3.16]. ■

Example 3.12 Let $C$ be a nonempty closed interval in $\mathbb{R}$. Then

$$ (\forall \xi \in \mathbb{R}) \quad P_C \xi = \begin{cases} 
\inf C, & \text{if } \xi < \inf C; \\
\xi, & \text{if } \xi \in C; \\
\sup C, & \text{if } \xi > \sup C.
\end{cases} \quad (3.9) $$

Proof. See [8, Example 24.34(i)]. ■

Example 3.13 (Projectors onto Lorentz cones) Let $\alpha$ be in $\mathbb{R}^{++}$ and set

$$ K_\alpha := \{(x, \xi) \in \mathcal{H} \oplus \mathbb{R} \mid \|x\| \leq \alpha \xi\}. \quad (3.10) $$

Then, for every $(x, \xi) \in \mathcal{H} \oplus \mathbb{R}$,

$$ P_{K_\alpha}(x, \xi) = \begin{cases} 
(x, \rho), & \text{if } \|x\| \leq \alpha \xi; \\
(0, 0), & \text{if } \alpha \|x\| \leq -\xi; \\
\frac{\alpha \|x\| + \xi}{\alpha^2 + 1} \left( \frac{\alpha}{\|x\|} x, 1 \right), & \text{otherwise.}
\end{cases} \quad (3.11) $$

Proof. See [5, Theorem 3.3.6]. ■

Example 3.14 Let $u \in \mathcal{H} \setminus \{0\}$, let $\eta \in \mathbb{R}$, and set $C := \{x \in \mathcal{H} \mid \langle x \mid u \rangle = \eta\}$. Then

$$ (\forall x \in \mathcal{H}) \quad P_C x = x + \frac{\eta - \langle x \mid u \rangle}{\|u\|^2} u. \quad (3.12) $$

Proof. See, for instance, [8, Example 3.23]. ■
3.2 Convex sets on the real line

**Proposition 3.15** Let $C$ and $D$ be nonempty intervals in $\mathbb{R}$ such that $C \neq \{0\}$ and $D \neq \{0\}$. Suppose that $C \cap D = \{0\}$. Then exactly one of the following cases occurs:

(i) $C \subseteq \mathbb{R}_-$. Then $D \subseteq \mathbb{R}_+$ and $\max C = \min D = 0$.

(ii) $C \cap \mathbb{R}_{++} \neq \emptyset$. Then $C \subseteq \mathbb{R}_+$, $D \subseteq \mathbb{R}_-$, and $\min C = \max D = 0$.

**Proof.** (i): Since $C \subseteq \mathbb{R}_-$ and $0 \in C$, it follows that $\max C = 0$. Let us now verify that $D \subseteq \mathbb{R}_+$. Assume to the contrary that there exists $\xi \in D \cap \mathbb{R}_{-}$. Then, on the one hand, by the convexity of $D$ and the fact that $0 \in D$, it follows that $[\xi, 0] \subseteq D$. On the other hand, because $\{0\} \neq C \subseteq \mathbb{R}_-$ and $C$ is convex, there exists $\eta \in \mathbb{R}_-$ satisfying $[\eta, 0] \subseteq C$. Altogether, $[\max \{\xi, \eta\}, 0] \subseteq C \cap D$, and we reach a contradiction. Thus, $D \subseteq \mathbb{R}_+$, and since $0 \in D$, we conclude that $\min D = 0$, as desired.

(ii): We first argue by contradiction that $D \subseteq \mathbb{R}_-$. Towards this goal, assume that there exists $\eta \in D \cap \mathbb{R}_{++}$, and take $\xi \in C \cap \mathbb{R}_{++}$. Since $0 \in C$ and $\xi \in C$, the convexity of $C$ yields $[0, \xi] \subseteq C$, and likewise, $[0, \eta] \subseteq D$. Thus, $[0, \min \{\xi, \eta\}] \subseteq C \cap D$, which contradicts our assumption. Hence, $D \subseteq \mathbb{R}_-$. Consequently, by interchanging the roles of $C$ and $D$ in (i), we obtain the conclusion. □

Here is a direct consequence of Proposition 3.15.

**Corollary 3.16** Let $C$ and $D$ be nonempty intervals in $\mathbb{R}$ such that $C \cap D \neq \emptyset$. Then

$$C \cap D = \{0\} \iff CD \subseteq \mathbb{R}_-,$$

where $CD := \{\xi \eta \mid \xi \in C \text{ and } \eta \in D\}$.

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**Definition 3.17** Let $C$ be a subset of $\mathcal{H}$. Then $C$ is a cone if $C = \bigcup_{\lambda \in \mathbb{R}_{++}} \lambda C$. The conical hull of $C$ is the intersection of all the cones in $\mathcal{H}$ containing $C$ and is denoted by $\text{cone } C$; in other words, $\text{cone } C$ is the smallest cone in $\mathcal{H}$ that contains $C$. Finally, the closed conical hull of $C$, in symbol, $\text{cone} C$, is the smallest closed cone in $\mathcal{H}$ containing $C$. 

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For the sake of clarity, let us point out the following.

**Remark 3.18** Let $K$ be a nonempty cone in $\mathcal{H}$. Then $0 \in K$, and if $K \neq \{0\}$, then $(\forall \rho \in \mathbb{R}_+) K \cap S(0; \rho) \neq \emptyset$.

**Fact 3.19** Let $C$ be a subset of $\mathcal{H}$. Then the following hold:

(i) $\text{cone } C = \bigcup_{\lambda \in \mathbb{R}_+} \lambda C$.

(ii) $\overline{\text{cone } C} = \overline{\text{cone } C}$.

(iii) $\text{cone}(\text{conv } C)$ is the smallest convex cone containing $C$.

**Proof.** See, e.g., [8, Proposition 6.2(i)–(iii)].

We shall require the following notation.

**Notation** Let $C$ be a nonempty subset of $\mathcal{H}$. Define its *positive span* by

$$\text{pos } C := \left\{ \sum_{i \in I} \alpha_i x_i \left| \begin{array} {l} I \text{ is finite}, \{\alpha_i\}_{i \in I} \subseteq \mathbb{R}_+, \text{ and } \{x_i\}_{i \in I} \subseteq C \end{array} \right. \right\}. \quad (3.14)$$

We observe that if $C$ is finite, then $\text{pos } C$ coincides\(^1\) with the Minkowski sum of the sets $(\mathbb{R}_+ c)_{c \in C}$, i.e.,

$$\text{pos } C = \sum_{c \in C} \mathbb{R}_+ c. \quad (3.15)$$

**Definition 3.20** Let $C$ be a subset of $\mathcal{H}$. The *polar cone* of $C$ is

$$C^\circ := \{ u \in \mathcal{H} \mid \sup \langle C \mid u \rangle \leq 0 \}, \quad (3.16)$$

the *indicator function* of $C$ is

$$\iota_C : \mathcal{H} \to [-\infty, +\infty] : x \mapsto \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise}, \end{cases} \quad (3.17)$$

and the *support function* of $C$ is

$$\sigma_C : \mathcal{H} \to [-\infty, +\infty] : u \mapsto \sup \langle C \mid u \rangle. \quad (3.18)$$

**Fact 3.21** Let $C$ be a subset of $\mathcal{H}$. Then the following hold:

\(^1\)This readily follows from (3.14).
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(i) Suppose that $C$ is a subspace. Then $C^\circ = C^\perp$.

(ii) $C^\circ$ is a nonempty closed convex cone.

(iii) Suppose that $C$ is a nonempty cone. Then $\sigma_C = t_C^\circ$.

**Lemma 3.22** Let $C$ be a nonempty subset of $\mathcal{H}$, and set $K := \text{pos } C$. Then the following hold:

(i) $x \in K^\circ \iff \sup (x \mid C) \leq 0$.

(ii) $K = \text{cone}(\text{conv}(C \cup \{0\})) = \text{cone}(\{0\} \cup \text{conv } C) = \text{cone}(\text{conv } C) \cup \{0\}$.

(iii) $K$ is the smallest convex cone containing $C \cup \{0\}$.

**Proof.** (i): This follows from the definition of polar cones and (3.14).

(ii): Set $D := \text{cone}(\text{conv}(C \cup \{0\}))$, $E := \text{cone}(\{0\} \cup \text{conv } C)$, and $F := \text{cone}(\text{conv } C) \cup \{0\}$. We shall establish that

$$K \subseteq D \subseteq E = F \subseteq K. \quad (3.19)$$

First, take $x \in K$, say $x = \sum_{i \in I} a_i x_i$, where $I$ is finite, $\{a_i\}_{i \in I} \subseteq \mathbb{R}_+$, and $\{x_i\}_{i \in I} \subseteq C$; in addition, set $\alpha := \sum_{i \in I} a_i$. If $\alpha = 0$, then, because $\{a_i\}_{i \in I} \subseteq \mathbb{R}_+$, we obtain $(\forall i \in I) a_i = 0$ and thus $x = 0 \in D$; otherwise, we have $\alpha > 0$ and $x = \alpha \sum_{i \in I} (a_i/\alpha) x_i \in \text{cone}(\text{conv } C) \subseteq D$. Next, fix $y \in D$. Then Fact 3.19(i) and (3.3) yield the existence of $\lambda \in \mathbb{R}_{++}$, a finite subset $\{\beta_i\}_{i \in J}$ of $\mathbb{R}_+$, and a finite subset $\{x_i\}_{i \in J}$ of $C$ such that $y = \lambda \sum_{i \in J} \beta_i x_i$. In turn, if $\sum_{i \in J} \beta_i = 0$, then $(\forall i \in J) \beta_i = 0$ and so $y = 0 \in E$; otherwise, $\sum_{i \in J} \beta_i > 0$ and, upon setting $\beta := \sum_{i \in I} \beta_i$, we get $y = \lambda \beta \sum_{i \in J} (\beta_i/\beta) x_i \in \text{cone}(\text{conv } C) \subseteq E$. Let us now prove that $E = F$. To do so, we infer from Fact 3.19(i) that

$$E = \bigcup_{\lambda \in \mathbb{R}_{++}} \lambda \{0\} \cup \text{conv } C \quad (3.20a)$$

$$= \{0\} \cup \left( \bigcup_{\lambda \in \mathbb{R}_{++}} \lambda \text{ conv } C \right) \quad (3.20b)$$

$$= \{0\} \cup \text{cone}(\text{conv } C) \quad (3.20c)$$

$$= F. \quad (3.20d)$$

Finally, take $z \in F$. If $z = 0$, then clearly $z \in K$; otherwise, $z \in \text{cone}(\text{conv } C)$ and, by Fact 3.19(i) and (3.3), there exist $\mu \in \mathbb{R}_{++}$, a finite subset $\{\delta_i\}_{i \in T}$
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of \([0,1]\), and a finite subset \(\{x_i\}_{i \in T}\) of \(C\) such that \(z = \mu \sum_{i \in T} \delta_i x_i = \sum_{i \in T} (\mu \delta_i) x_i \in K\). Altogether, (3.19) holds.

(iii): Since \(K = \text{cone}(\text{conv}(C \cup \{0\}))\) by (ii), the conclusion thus follows from Fact 3.19(iii). ■

Example 3.23 (Lorentz cone) Let \(\alpha\) and \(\beta\) be in \(\mathbb{R}^+\), set \(H := H \oplus \mathbb{R}\), set

\[
K_\alpha := \{(x, \xi) \in H \mid \|x\| \leq \alpha \xi\}, \tag{3.21}
\]

and set

\[
C_{a,\beta} := S(0; \beta) \times \{\beta / \alpha\} \subseteq H. \tag{3.22}
\]

Then \(K_\alpha\) is a nonempty closed convex cone in \(H\) and

\[
K_\alpha = \text{pos} C_{a,\beta} = \text{cone}(\text{conv} C_{a,\beta}) \cup \{0\}. \tag{3.23}
\]

Proof. Since \(K_\alpha\) is the epigraph of the function \(\|\cdot\| / \alpha\), which is continuous, convex, and positively homogeneous\(^2\), we deduce from [8, Proposition 10.2] that \(K_\alpha\) is a nonempty closed convex cone in \(H\). Next, let us establish (3.23). In view of Lemma 3.22(ii), it suffices to show that \(K_\alpha = \text{cone}(\text{conv} C_{a,\beta}) \cup \{0\}\). Towards this aim, let us first observe that, due to Fact 3.6 and Example 3.4,

\[
\text{conv} C_{a,\beta} = (\text{conv} S(0; \beta)) \times (\text{conv} \{\beta / \alpha\}) = B(0; \beta) \times \{\beta / \alpha\}. \tag{3.24}
\]

Now set \(K := \text{cone}(\text{conv} C_{a,\beta}) \cup \{0\}\), and take \(x = (x, \xi) \in K_\alpha\). If \(\xi = 0\), then (3.21) yields \(x = 0\) and so \(x = 0 \in K\). Otherwise, \(\xi > 0\) and we get from (3.21) that \(\|\beta (\alpha \xi)^{-1} x\| = \beta (\alpha \xi)^{-1} \|x\| \leq \beta\) or, equivalently, \((\alpha \xi)^{-1} x \in B(0; \beta)\); therefore, it follows from (3.24) that

\[
(x, \xi) = \frac{\alpha \xi}{\beta} \left( \frac{\beta}{\alpha \xi} x, \frac{\beta}{\alpha} \right) \in \frac{\alpha \xi}{\beta} (B(0; \beta) \times \{\beta / \alpha\}) \subseteq \text{cone}(\text{conv} C_{a,\beta}) \subseteq K. \tag{3.25}
\]

Altogether, \(K_\alpha \subseteq K\). Conversely, take \(y \in \text{cone}(\text{conv} C_{a,\beta})\). Then, by Fact 3.19(i) and (3.24), there exist \(\lambda \in \mathbb{R}^+\) and \(y \in B(0; \beta)\) such that

\(A\) function \(f: H \to [-\infty, +\infty]\) is positively homogeneous if \((\forall x \in H)(\forall \lambda \in \mathbb{R}^+) f(\lambda x) = \lambda f(x)\).
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\[ y = \lambda(y, \beta / \alpha) = (\lambda y, \lambda \beta / \alpha). \] In turn, since \( \|\lambda y\| \leq \lambda \beta = \alpha (\lambda \beta / \alpha) \), we obtain \( y \in K_\alpha \). Hence \( \text{cone(conv} \ C_{\alpha, \beta}) \subset K_\alpha \). This and the fact that \( 0 \in K_\alpha \) yield \( K \subset K_\alpha \). Hence \( K_\alpha = K \), as claimed.

**Fact 3.24** Let \( K \) be a nonempty closed convex cone in \( H \), and let \( x \) and \( p \) be in \( H \). Then

\[ p = P_K x \iff \{ p \in K, x - p \perp p, \text{ and } x - p \in K^\circ \}. \tag{3.26} \]

and

\[ (\forall \rho \in \mathbb{R}_+) \ P_K (\rho x) = \rho P_K x. \tag{3.27} \]

**Proof.** See, e.g., [8, Proposition 6.28 and Proposition 29.29]. ■

Let us recall the celebrated Moreau decomposition for cones; see [32].

**Fact 3.25 (Moreau)** Let \( K \) be a nonempty closed convex cone in \( H \), and let \( x \in H \). Then the following hold:

(i) \( x = P_K x + P_{K^\circ} x \).

(ii) \( P_K x \perp P_{K^\circ} x \).

(iii) \( \|x\|^2 = d_K^2(x) + d_{K^\circ}^2(x) \).

**Fact 3.26** Let \( K \) and \( S \) be nonempty closed convex cones in \( H \). Then the following hold:

(i) \( K^{\ominus \ominus} = K \).

(ii) \( (K \cap S)^\ominus = \overline{K^\ominus + S^\ominus} \).

(iii) \( (K + S)^\ominus = K^\ominus \cap S^\ominus \).

**Proof.** See [8, Corollary 6.34, Proposition 6.35, and Proposition 6.27]. ■

**Fact 3.27 (Zarantonello)** Let \( K_1 \) and \( K_2 \) be nonempty closed convex cones in \( H \). Then

\[ K_2 \subseteq K_1 \iff \{ (\forall x \in H) \ \langle P_{K_2} x \mid x \rangle \leq \langle P_{K_1} x \mid x \rangle \}. \tag{3.28} \]

**Proof.** See [39, Lemma 5.6]. ■

**Lemma 3.28** Let \( K \) and \( S \) be nonempty closed convex cones in \( H \). Then the following hold:
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(i) \((\forall x \in \mathcal{H}) \|P_K x\|^2 = \langle x | P_K x \rangle\).

(ii) \((\forall x \in \mathcal{H}) \langle P_K^\circ x | P_S^\circ x \rangle + \|P_K x\|^2 + \|P_S x\|^2 = \|x\|^2 + \langle P_K x | P_S x \rangle\).

Proof. Take \(x \in \mathcal{H}\). (i): We derive from Fact 3.25(i)&(ii) that

\[\|P_K x\|^2 = \langle P_K x | P_K x \rangle = \langle x - P_K^\circ x | P_K x \rangle = \langle x | P_K x \rangle,\]

as claimed. (ii): The Moreau conical decomposition and (i) give

\[\langle P_K^\circ x | P_S^\circ x \rangle = \langle x - P_K^\circ x | x - P_S x \rangle,\]

and the assertion follows. \(\blacksquare\)

Lemma 3.29 Let \(K\) be a nonempty closed convex cone in \(\mathcal{H}\), and let \(x \in \mathcal{H}\). Then the following hold:

(i) \(P_K x \neq 0 \iff x \in \mathcal{H} \setminus K^\circ\).

(ii) Suppose that \(P_K x \neq 0\). Let \(\rho \in \mathbb{R}^+\), and set \(p := (\rho / \|P_K x\|) P_K x\). Then

\[\|x - p\|^2 = \sqrt{d_K^2(x) + (\|P_K x\| - \rho)^2}.\]

Proof. (i): We deduce from Fact 3.25 that \(x \in K^\circ \iff x - P_K^\circ x = 0 \iff P_K x = 0\), and the claim follows.

(ii): Set \(\beta := \rho / \|P_K x\|\). Then, because \(P_K x \perp P_K x\) by Fact 3.24, the Pythagorean identity implies that

\[\|x - p\|^2 = \|x - \beta P_K x\|^2 = \|P_K x\|^2 + (1 - \beta)^2 \|P_K x\|^2 = d_K^2(x) + (1 - \rho / \|P_K x\|)^2 \|P_K x\|^2 = \sqrt{d_K^2(x) + (\|P_K x\| - \rho)^2},\]

and thus (3.30) holds. \(\blacksquare\)

Definition 3.30 Let \(C\) be a subset of \(\mathcal{H}\). The recession cone of \(C\) is

\[\text{rec}C := \{x \in \mathcal{H} \mid x + C \subseteq C\}.\]
3.4 Convex functions

**Fact 3.31** Let $C$ be a nonempty convex subset of $\mathcal{H}$. Then the following hold:

(i) $\text{rec } C$ is a convex cone and $0 \in \text{rec } C$.

(ii) Suppose that $C$ is bounded. Then $\text{rec } C = \{0\}$.

*Proof.* See [8, Proposition 6.49(i) and Corollary 6.52].

**Fact 3.32** Let $C$ be a nonempty closed convex subset of $\mathcal{H}$, and let $x \in \mathcal{H}$. Then the following are equivalent:

(i) $x \in \text{rec } C$.

(ii) There exist sequences $(x_n)_{n \in \mathbb{N}}$ in $C$ and $(\alpha_n)_{n \in \mathbb{N}}$ in $[0, 1]$ such that $\alpha_n \to 0$ and $\alpha_n x_n \to x$.

*Proof.* See [8, Proposition 6.51].

### 3.4 Convex functions

**Definition 3.33** Let $f : \mathcal{H} \to [-\infty, +\infty]$. The *domain* of $f$ is $\text{dom } f := \{x \in \mathcal{H} \mid f(x) < +\infty\}$ with closure $\overline{\text{dom } f}$, and $f$ is *proper* if $\text{dom } f \neq \emptyset$ and $-\infty \notin f(\mathcal{H})$. Furthermore, the set of proper lower semicontinuous convex functions from $\mathcal{H}$ to $[-\infty, +\infty]$ is denoted by $\Gamma_0(\mathcal{H})$.

**Fact 3.34** Let $C$ be a subset of $\mathcal{H}$. Then the following hold:

(i) $C$ is nonempty if and only if $\iota_C$ is proper.

(ii) $C$ is closed if and only if $\iota_C$ is lower semicontinuous.

(iii) $C$ is convex if and only if $\iota_C$ is convex.

**Fact 3.35** Let $f : \mathcal{H} \to [-\infty, +\infty]$ be proper. Suppose that $\text{dom } f$ is open and convex, and that $f$ is Gâteaux differentiable on $\text{dom } f$. Then $f$ is convex if and only if $\nabla f$ is monotone, i.e., $(\forall x \in \text{dom } f)(\forall y \in \text{dom } f)(x - y \mid \nabla f(x) - \nabla f(y)) \geq 0$.

*Proof.* See [8, Proposition 17.7].
3.4. Convex functions

Fact 3.36 Let $f : \mathcal{H} \to ]-\infty, +\infty]$ be convex, and let $x \in \text{dom } f$. Suppose that $f$ is Gâteaux differentiable at $x$. Then $f$ is lower semicontinuous at $x$.

Proof. See [8, Proposition 17.48(i)].

Fact 3.37 Let $C$ be a nonempty closed convex subset of $\mathcal{H}$. Then the following hold:

(i) $d_C^2$ is Fréchet differentiable of $\mathcal{H}$ and $\nabla d_C^2 = 2(\text{Id} - P_C)$.

(ii) Suppose that $C$ is a cone and set $q := (1/2)\|\cdot\|^2$. Then $q \circ P_K$ is Fréchet differentiable on $\mathcal{H}$ and $\nabla (q \circ P_K) = P_K$.

Proof. See [8, Corollary 12.31 and Proposition 12.32].

Definition 3.38 Let $f : \mathcal{H} \to ]-\infty, +\infty]$. The conjugate of $f$ is

$$f^* : \mathcal{H} \to ]-\infty, +\infty] : u \mapsto \sup_{x \in \mathcal{H}} \langle x \mid u \rangle - f(x),$$

and the biconjugate of $f$ is $f^{**} := (f^*)^*$.

Example 3.39 Let $C$ be a nonempty subset of $\mathcal{H}$, and set $f := \iota_C + (1/2)\|\cdot\|^2$. Then $f^* = (1/2)(\|\cdot\|^2 - d_C^2)$.

Proof. See [8, Example 13.5].

Example 3.40 Let $C$ be a nonempty closed convex subset of $\mathcal{H}$. Then

$$(\frac{1}{2}d_C^2)^* = \sigma_C + \frac{1}{2}\|\cdot\|^2.$$ (3.34)

Proof. Let $p = 2$ in [8, Example 13.27(iii)].

Fact 3.41 (Self-conjugacy) Let $f : \mathcal{H} \to ]-\infty, +\infty]$. Then

$$f = f^* \iff f = \frac{1}{2}\|\cdot\|^2.$$ (3.35)

Proof. See [8, Proposition 13.19].

Fact 3.42 (Fenchel–Moreau) Let $f : \mathcal{H} \to ]-\infty, +\infty]$ be proper. Then $f \in \Gamma_0(\mathcal{H})$ if and only if $f = f^{**}$. In this case, $f^*$ belongs to $\Gamma_0(\mathcal{H})$ as well.
3.4. Convex functions

**Fact 3.43** Let \( g : \mathcal{H} \to ]-\infty, +\infty] \) be proper, let \( h \in \Gamma_0(\mathcal{H}) \), and set

\[
 f : \mathcal{H} \to [-\infty, +\infty] : x \mapsto \begin{cases} g(x) - h(x), & \text{if } x \in \text{dom } g; \\ +\infty, & \text{otherwise.} \end{cases} \quad (3.36)
\]

Then

\[
 (\forall u \in \mathcal{H}) \quad f^*(u) = \sup_{v \in \text{dom } h^*} \left( g^*(u + v) - h^*(v) \right). \quad (3.37)
\]

**Proof.** See [8, Proposition 14.19]. ■

**Definition 3.44** Let \( f \in \Gamma_0(\mathcal{H}) \), and let \( x \in \mathcal{H} \). Then \( \text{Prox}_f x \) is the unique point in \( \mathcal{H} \) that satisfies

\[
 \min_{y \in \mathcal{H}} \left( f(y) + \frac{1}{2} \|x - y\|^2 \right) = f(\text{Prox}_f x) + \frac{1}{2} \|x - \text{Prox}_f x\|^2. \quad (3.38)
\]

The operator \( \text{Prox}_f : \mathcal{H} \to \mathcal{H} \) is the proximity operator—or proximal mapping—of \( f \).

**Definition 3.45** Let \( f \) and \( g \) be functions from \( \mathcal{H} \) to \( ]-\infty, +\infty] \). The infimal convolution of \( f \) and \( g \) is

\[
 f \boxdot g : \mathcal{H} \to [\,] : x \mapsto \inf_{y \in \mathcal{H}} \left( f(y) + g(x - y) \right), \quad (3.39)
\]

and it is exact at a point \( x \in \mathcal{H} \) if

\[
 (\exists y \in \mathcal{H}) \quad (f \boxdot g)(x) = f(y) + g(x - y) \in ]-\infty, +\infty]; \quad (3.40)
\]

furthermore, \( f \boxdot g \) is exact if it is exact at every point of its domain, in which case we write \( f \boxdot g \).

**Fact 3.46** Let \( f \) and \( g \) be functions from \( \mathcal{H} \) to \( ]-\infty, +\infty] \). Then the following hold:

(i) Suppose that \( f \leq g \). Then \( g^* \leq f^* \).

(ii) \( (f \boxdot g)^* = f^* + g^* \).

**Proof.** (i): See [8, Proposition 13.16(ii)]. (ii): See [8, Proposition 13.24(i)]. ■

**Fact 3.47 (Moreau)** Let \( f \in \Gamma_0(\mathcal{H}) \) and set \( q := (1/2)\|\cdot\|^2 \). Then

\[
 (f \boxdot q) + (f^* \boxdot q) = q \quad \text{and} \quad \text{Prox}_f + \text{Prox}_{f^*} = \text{Id}. \quad (3.41)
\]
3.5 Subdifferential operators

Definition 3.48 Let \( f: \mathcal{H} \to ]-\infty, +\infty[ \) be proper. The subdifferential of \( f \) is

\[
\partial f: \mathcal{H} \to 2^{\mathcal{H}}: x \mapsto \{ u \in \mathcal{H} \mid (\forall y \in \mathcal{H}) \langle y - x \mid u \rangle + f(x) \leq f(y) \}. \tag{3.42}
\]

Fact 3.49 Let \( f \in \Gamma_0(\mathcal{H}) \). Then the following hold:

(i) \( \partial f^* = (\partial f)^{-1} \).

(ii) \( \text{dom} \partial f \) is a dense subset of \( \text{dom} f \).

Proof. See, e.g., [8, Corollary 16.30 and Corollary 16.39]. \( \blacksquare \)

Fact 3.50 Let \( f: \mathcal{H} \to ]-\infty, +\infty[ \) be proper and convex, and let \( x \in \text{dom} f \). Suppose that \( f \) is Gâteaux differentiable at \( x \). Then the following hold:

(i) \( \partial f(x) = \{ \nabla f(x) \} \).

(ii) \( f^*(\nabla f(x)) = \langle x \mid \nabla f(x) \rangle - f(x) \).

Proof. See [8, Proposition 17.31(i) and Proposition 17.35]. \( \blacksquare \)

3.6 Monotone operators

Definition 3.51 Let \( A: \mathcal{H} \to 2^{\mathcal{H}} \). Then \( A \) is monotone if

\[
(\forall (x,u) \in \text{gra} A)(\forall (y,v) \in \text{gra} A) \quad \langle x - y \mid u - v \rangle \geq 0. \tag{3.43}
\]

Furthermore, \( A \) is maximally monotone if it is monotone and there is no monotone operator \( B: \mathcal{H} \to 2^{\mathcal{H}} \) such that \( \text{gra} A \) is a proper subset of \( \text{gra} B \).

Example 3.52 Let \( C \) be a nonempty subset of \( \mathcal{H} \). Then \( P_C \) is monotone.

Proof. See [8, Example 20.12]. \( \blacksquare \)

Fact 3.53 Let \( T: \mathcal{H} \to \mathcal{H} \) be nonexpansive, i.e., Lipschitz continuous with constant 1. Then \( \text{Id} - T \) is monotone.

Proof. See [8, Example 20.7]. \( \blacksquare \)
3.6. Monotone operators

Fact 3.54 Let $T: \mathcal{H} \to \mathcal{H}$ be linear and continuous. Then $L^*L$ is monotone, where $L^*: \mathcal{H} \to \mathcal{H}$ is the adjoint of $L$.

Proof. See [8, Example 20.16].

Fact 3.55 Let $T: \mathcal{H} \to \mathcal{H}$ be monotone and continuous. Then $T$ is maximally monotone.

Proof. See [8, Corollary 20.28].

Example 3.56 Let $C$ be a nonempty closed convex subset of $\mathcal{H}$. Then $P_C$ is maximally monotone.

Proof. See [8, Example 20.32].

Definition 3.57 Let $A: \mathcal{H} \to 2^{\mathcal{H}}$ be monotone. Then $A$ is $3^*$ monotone if
\[
(\forall (x, u) \in \text{dom } A \times \text{ran } A) \inf_{(y,v) \in \text{gra } A} \langle x - y | u - v \rangle > -\infty. \tag{3.44}
\]

Example 3.58 Let $C$ be a nonempty closed convex subset of $\mathcal{H}$. Then $P_C$ is $3^*$ monotone.

Proof. Combine [8, Proposition 4.16 and Example 25.20(ii)].

Fact 3.59 Let $A$ and $B$ be $3^*$ monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$. Then $A + B$ is $3^*$ monotone.

Proof. See [8, Proposition 25.22(i)].

Fact 3.60 (Brézis–Haraux) Let $A$ and $B$ be $3^*$ monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ such that $A + B$ is maximally monotone. Then
\[
\overline{\text{ran}}(A + B) = \overline{\text{ran}} A + \overline{\text{ran}} B. \tag{3.45}
\]

Proof. See, e.g., [8, Theorem 25.24].
Chapter 4

Projecting onto the intersection of a cone and a sphere

4.1 Overview

Let $K$ and $S$ be subsets of $\mathcal{H}$. Only in rare cases is it possible to obtain a “closed form” for $P_{K\cap S}$ in terms of $P_K$ and $P_S$: e.g., when $K$ and $S$ are either both half-spaces (Haugazeau; see [22] and also [8, Corollary 29.25]) or both subspaces (Anderson–Duffin; see [1, Theorem 8] and also [8, Corollary 25.38]). Inspired by Example 5.5.2 in the recent and charming book [28], our aim in this chapter is to systematically study the case when $K$ is a closed convex cone and $S$ is either the (convex) unit ball or (nonconvex) unit sphere centered at the origin. We obtain formulae describing the full (possibly set-valued) projector and also discuss nonpolyhedral cones such as the Lorentz cone or the cone of positive semidefinite matrices. We also revisit Lange’s copositivity example and tackle it with other algorithms that appear to perform quite well.

We conclude this short introductory section with some comments on the organization of this chapter. In Section 4.2, we provide various results on cones and conical hulls. Cones that are finitely generated and corresponding projectors are investigated in Section 4.3. Our main results are presented in Section 4.4 (cone intersected with ball) and Section 4.5 (cone intersected with sphere), respectively. Additional examples are provided in Section 4.6. In the final Section 4.7, we put the theory to good use and offer new algorithmic approaches to determine copositivity.

4.2 Cones and conical hulls revisited

In general, for subsets $C$ and $D$ of $\mathcal{H}$, $\overline{C \cap D} \neq \overline{C} \cap \overline{D}$. However, the following result provides an interesting instance where taking intersections...
and closures commutes.

**Proposition 4.1** Let $K$ be a nonempty cone in $\mathcal{H}$, and let $\rho \in \mathbb{R}_{++}$. Then the following hold:

(i) $\overline{K \cap S(0;\rho)} = \overline{K \cap S(0;\rho)}$.

(ii) $\overline{K \cap B(0;\rho)} = \overline{K \cap B(0;\rho)}$.

**Proof.** We assume that

(K \neq \{0\}, \quad (4.1))

since otherwise the assertions are clear.

(i): Since we obviously have $\overline{K \cap S(0;\rho)} \subseteq \overline{K \cap S(0;\rho)}$, it suffices to verify that $\overline{K \cap S(0;\rho)} \subseteq \overline{K \cap S(0;\rho)}$. To do so, take $x \in \overline{K \cap S(0;\rho)}$. Then, because $x \in K$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $K$ such that

$x_n \to x$. \quad (4.2)

In turn, by the continuity of $\| \cdot \|$ and the fact that $x \in S(0;\rho)$,

$\|x\| \to \|x\| = \rho \in \mathbb{R}_{++}, \quad (4.3)$

and therefore, we can assume without loss of generality that $(\forall n \in \mathbb{N}) \|x_n\| \neq 0$. Hence, for every $n \in \mathbb{N}$, since $x_n \in K$ and $\|\rho x_n/\|x_n\|| = \rho$, the assumption that $K$ is a cone implies that $\rho x_n/\|x_n\||$ lies in $K \cap S(0;\rho)$. Thus, $(\rho x_n/\|x_n\||)_{n \in \mathbb{N}}$ is a sequence in $K \cap S(0;\rho)$; moreover, (4.2) and (4.3) assert that $\rho x_n/\|x_n\|| \to \rho x/\rho = x$. Consequently, $x \in \overline{K \cap S(0;\rho)}$, as announced.

(ii): First, it is clear that $\overline{K \cap B(0;\rho)} \subseteq \overline{K \cap B(0;\rho)}$. Conversely, fix $x \in \overline{K \cap B(0;\rho)}$, and we shall consider two conceivable cases:

(a) $x = 0$: By (4.1), there exists

$y \in K \setminus \{0\}$. \quad (4.4)

In turn, set

$(\forall n \in \mathbb{N}) \quad y_n := \frac{\rho}{(n+1)\|y\|} y. \quad (4.5)$

Then, for every $n \in \mathbb{N}$, since $K$ is a cone, (4.4) and (4.5) assert that $y_n \in K$ and thus, since $\|y_n\| = \rho/(n+1) \leq \rho$ by (4.5), we deduce that $y_n \in K \cap B(0;\rho)$. Hence, because $y_n = \rho y/[(n+1)\|y\|] \to 0 = x$, we infer that $x \in \overline{K \cap B(0;\rho)}$. \quad \(\square\)
4.2. Cones and conical hulls revisited

(b) \(x \neq 0\): Since \(x \in \overline{K}\), there is a sequence \((x_n)_{n \in \mathbb{N}}\) in \(K\) such that

\[x_n \to x.\]  \hspace{1cm} (4.6)

In turn, by the continuity of \(\| \cdot \|\),

\[\|x_n\| \to \|x\| \in \mathbb{R}^+,\] \hspace{1cm} (4.7)

and we can therefore assume that \((\forall n \in \mathbb{N}) \|x_n\| \neq 0\). Now set

\[(\forall n \in \mathbb{N}) \quad y_n := \frac{\|x\|}{\|x_n\|} x_n.\] \hspace{1cm} (4.8)

For every \(n \in \mathbb{N}\), because \(x_n \in K\) and \(\|x\| \leq \rho\), the assumption that \(K\) is a cone and (4.8) yield \(y_n \in K \cap B(0; \rho)\). Consequently, since \(y_n \to \|x\|/\|x\| = x\) due to (4.6) and (4.7), we obtain \(x \in \overline{K} \cap B(0; \rho)\).

To sum up, in both cases, we have \(x \in \overline{K} \cap B(0; \rho)\), and the conclusion follows.

Here is an improvement of [8, Corollary 6.53].

**Proposition 4.2** Let \(C\) be a nonempty closed convex set in \(\mathcal{H}\). Suppose that there exists a nonempty closed subset \(D\) of \(C\) such that \(0 \not\in D\) and that one of the following holds:

(a) \((\text{cone } D) \cup \{0\} = (\text{cone } C) \cup \{0\}\).

(b) \(\text{cone } D = \overline{\text{cone } C}\).

Then the following hold:

(i) \((\text{cone } C) \cup (\text{rec } C) = \overline{\text{cone } C}\).

(ii) Suppose that \(\text{rec } C = \{0\}\). Then \(\text{cone}(C \cup \{0\})\) is closed.

**Proof.** Let us first show that (a)\(\Rightarrow\)(b). To establish this, assume that (a) holds. Then, since \(0 \in \overline{\text{cone } D}\) and \(0 \in \overline{\text{cone } C}\) due to Remark 3.18, we infer from Fact 3.19(ii) that \(\overline{\text{cone } D} = \overline{\text{cone } D \cup \{0\}} = (\text{cone } D) \cup \{0\} = (\text{cone } C) \cup \{0\} = \overline{\text{cone } C \cup \{0\}} = \overline{\text{cone } C}\), which verifies the claim. Thus, it is enough to assume that (b) holds and to show that (i)\&(ii) hold.

(i): Clearly \(\text{cone } C \subseteq \overline{\text{cone } C}\). We now prove that \(\text{rec } C \subseteq \overline{\text{cone } C}\). To this end, take \(x \in \text{rec } C\). Then Fact 3.32 ensures the existence of sequences
4.2. Cones and conical hulls revisited

$(x_n)_{n \in \mathbb{N}}$ in $C$ and $(\alpha_n)_{n \in \mathbb{N}}$ in $[0, 1]$ such that $\alpha_n x_n \to x$. Hence, because $\{\alpha_n x_n\}_{n \in \mathbb{N}} \subseteq \text{cone } C$ by Fact 3.19(i), we deduce from Fact 3.19(ii) that $x \in \text{cone } C = \text{cl } C$. Thus $(\text{cone } C) \cup \text{rec } C \subseteq \text{cone } C$. Conversely, fix $y \in \text{cone } C = \text{cl } D$. It then follows from Fact 3.19(ii) that $y \in \text{cone } D$, and therefore, in view of Fact 3.19(i), there exist sequences $(\beta_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^+$ and $(y_n)_{n \in \mathbb{N}}$ in $D$ such that

$$\beta_n y_n \to y.$$ (4.9)

After passing to subsequences and relabeling if necessary, we assume that

$$\beta_n \to \beta \in [0, +\infty).$$ (4.10)

In turn, let us establish that $\beta \in \mathbb{R}^+$ by contradiction: assume that $\beta = +\infty$. Then it follows from (4.9) that $\|y_n\| = (1/\beta_n)\|\beta_n y_n\| \to 0$ or, equivalently, $y_n \to 0$. Hence, since $\{y_n\}_{n \in \mathbb{N}} \subseteq D$, the closedness of $D$ asserts that $0 \in D$, which violates our assumption. Therefore $\beta \in \mathbb{R}^+$, and this leads to two conceivable cases:

(a) $\beta = 0$: Then, by (4.10), we can assume without loss of generality that $\{\beta_n\}_{n \in \mathbb{N}} \subseteq [0, 1]$. In turn, since $\{y_n\}_{n \in \mathbb{N}} \subseteq D \subseteq C$, we infer from (4.9)&(4.10) and Fact 3.32 that $y \in \text{rec } C$.

(b) $\beta > 0$: Then, in view of (4.9)&(4.10), $y_n = (1/\beta_n)(\beta_n y_n) \to y/\beta$. Therefore, because $\{y_n\}_{n \in \mathbb{N}} \subseteq C$ and $C$ is closed, we obtain $y/\beta \in C$. Consequently, $y \in \beta C \subseteq \text{cone } C$.

To sum up, $(\text{cone } C) \cup \text{rec } C = \text{cl } C$, as announced.

(ii): Since $C = \text{conv } C$ by the convexity of $C$, we derive from (i) and Lemma 3.22(ii) that $\text{cone } C = (\text{cone } C) \cup \{0\} = \text{cone } (C \cup \{0\})$, which guarantees that $\text{cone } (C \cup \{0\})$ is closed. $\blacksquare$

**Corollary 4.3** Let $C$ be a nonempty subset of $\mathcal{H}$, and set $K := \text{pos } C$. Suppose that $0 \notin \text{conv } C$ and that $\text{conv } C$ is weakly compact. Then $K$ is the smallest closed convex cone containing $C \cup \{0\}$.

**Proof.** According to Lemma 3.22(iii), it suffices to verify that $K$ is closed. Since $\text{conv } C$ is weakly compact, it is weakly closed and bounded. In turn, on the one hand, since $\text{conv } C$ is convex and weakly closed, we derive from [8, Theorem 3.34] that $\text{conv } C$ is closed. On the other hand, the boundedness of $\text{conv } C$ guarantees that $\text{rec } (\text{conv } C) = \{0\}$. Altogether, because $K = \text{cone } \{0\} \cup \text{conv } C$ due to Lemma 3.22(ii) and because $0 \notin \text{conv } C$, applying Proposition 4.2(ii) to $\text{conv } C$ (with the subset $D$—as in the setting of Proposition 4.2—being $\text{conv } C$) yields the closedness of $K$, as required. $\blacksquare$
4.2. Cones and conical hulls revisited

The following two examples provide instances in which the assumption of Proposition 4.2 holds.

Example 4.4 Let $C$ be a nonempty closed convex subset of $\mathcal{H}$ such that $C \smallsetminus \{0\} \neq \emptyset$. Suppose that there exists $\rho \in \mathbb{R}_{++}$ satisfying

\[(\text{cone } C) \cap S(0; \rho) \subseteq C, \tag{4.11}\]

and set $D := (\text{cone } C) \cap S(0; \rho)$. Then the following hold:

(i) $D$ is a nonempty closed subset of $C$ and $0 \notin D$.

(ii) $(\text{cone } D) \cup \{0\} = (\text{cone } C) \cup \{0\}$.

Proof. (i): The closedness of $D$ is clear. Next, since $0 \notin S(0; \rho)$, we have $0 \notin D$. In turn, since $C \smallsetminus \{0\} \neq \emptyset$, we see that $\emptyset \neq \text{cone } C \neq \{0\}$, and since $\text{cone } C$ is a cone, Remark 3.18 yields $D \neq \emptyset$. Finally, it follows from Fact 3.19(ii), Proposition 4.1(i) (applied to $\text{cone } C$), (4.11), and the closedness of $C$ that $D = \text{cone } C \cap S(0; \rho) = (\text{cone } C) \cap S(0; \rho) \subseteq C = C$, as claimed.

(ii): Because $D \subseteq C$, we get $(\text{cone } D) \cup \{0\} \subseteq (\text{cone } C) \cup \{0\}$. Conversely, take $x \in \text{cone } C$. We then deduce from Fact 3.19(i) the existence of $\lambda \in \mathbb{R}_{++}$ and $y \in C$ such that $x = \lambda y$. If $y = 0$, then $x = 0 \in (\text{cone } D) \cup \{0\}$. Otherwise, $\|y\| \neq 0$ and, since $\rho y/\|y\| \in (\text{cone } C) \cap S(0; \rho) \subseteq D$, we obtain $x = \lambda y = (\lambda \|y\|/\rho)(\rho y/\|y\|) \in \text{cone } D$. Therefore $(\text{cone } C) \cup \{0\} \subseteq (\text{cone } D) \cup \{0\}$, and the conclusion follows. ■

Before we present a new proof of the well-known fact that finitely generated cones are closed (see [34, Theorem 19.1, Corollary 2.6.2 and the remarks following Corollary 2.6.3]), we make a few comments.

Remark 4.5 Let $\{x_i\}_{i \in I}$ be a finite subset of $\mathcal{H}$, and set $C := \text{conv}\{x_i\}_{i \in I}$.

(i) Since $C = \text{conv} \cup_{i \in I} \{x_i\}$, [8, Proposition 3.39(i)] implies that $C$ is compact, and so it is closed and bounded. In turn, the boundedness of $C$ and Fact 3.31(ii) give rec $C = \{0\}$.

(ii) The geometric interpretation of the proof of Example 4.6 is as follows. If $y$ lies in $C \smallsetminus \{0\}$, then the ray $\mathbb{R}_+ y$ must intersect a “face” of $C$ that does not contain 0.

(iii) Example 4.6 illustrates that the assumption of Proposition 4.2 is mild and covers the case of finitely generated cones.
4.2. Cones and conical hulls revisited

Example 4.6 Let \( \{x_i\}_{i \in I} \) be a finite subset of \( \mathcal{H} \) and set

\[
K := \sum_{i \in I} \mathbb{R}_+ x_i. \tag{4.12}
\]

Then \( K \) is the smallest closed convex cone containing \( \{x_i\}_{i \in I} \cup \{0\} \).

Proof. We derive from (3.15) and Lemma 3.22(iii) that \( K \) is the smallest convex cone in \( \mathcal{H} \) containing \( \{x_i\}_{i \in I} \cup \{0\} \). Therefore, it suffices to establish the closedness of \( K \). Towards this goal, we first infer from Lemma 3.22(ii) (applied to \( \{x_i\}_{i \in I} \)) that

\[
K = \text{cone}(\{0\} \cup \text{conv}\{x_i\}_{i \in I}). \tag{4.13}
\]

Furthermore, we assume that

\[
\{x_i\}_{i \in I} \setminus \{0\} \neq \emptyset, \tag{4.14}
\]

because otherwise the claim is trivial. In turn, set \( C := \text{conv}\{x_i\}_{i \in I'} \)

\[
I := \{\emptyset \neq J \subseteq I \mid 0 \notin \text{conv}\{x_i\}_{i \in J}\}, \tag{4.15}
\]

and

\[
D := \bigcup_{J \in I} \text{conv}\{x_i\}_{i \in J} \subseteq C. \tag{4.16}
\]

Then, by (4.14), \( I \) is nonempty,\(^3\) and thus, \( 0 \notin D \neq \emptyset \). Moreover, \( D \) is closed as a finite union of closed sets, namely \( (\text{conv}\{x_i\}_{i \in J})_{J \in I} \). We now claim that

\[
(\text{cone } D) \cup \{0\} = (\text{cone } C) \cup \{0\}. \tag{4.17}
\]

To do so, it suffices to verify that \( (\text{cone } C) \cup \{0\} \subseteq (\text{cone } D) \cup \{0\} \). Take \( x \in (\text{cone } C) \setminus \{0\} \). Then Fact 3.19(i) ensures the existence of \( \lambda \in \mathbb{R}_{++} \) and

\[
y \in C \setminus \{0\}
\]

such that \( x = \lambda y \). Since \( y \in C = \text{conv}\{x_i\}_{i \in I} \), there exist a nonempty subset \( J \) of \( I \) and

\[
\{\alpha_i\}_{i \in J} \subseteq [0,1] \tag{4.19}
\]

such that \( \sum_{i \in J} \alpha_i = 1 \) and \( y = \sum_{i \in J} \alpha_i x_i \). If \( J \in I \), then
\( y \in \text{conv}\{x_i\}_{i \in J} \subseteq D \) and hence \( x = \lambda y \in \text{cone } D \). Otherwise, \( 0 \in \text{conv}\{x_i\}_{i \in J} \), and there thus

\(^3\)We just need to pick a nonzero element \( x_j \) of \( \{x_i\}_{i \in I} \) and set \( J := \{j\} \).
exists $\{\beta_i\}_{i \in J} \subseteq [0, 1]$ such that $\sum_{i \in J} \beta_i x_i = 0$ and $J_+ := \{i \in J \mid \beta_i > 0\} \neq \emptyset$. In turn, set
\[
\gamma := \min_{i \in J_+} \frac{\alpha_i}{\beta_i},
\]
and $(\forall i \in J) \delta_i := \alpha_i - \gamma \beta_i$. By (4.19) and (4.20),
\[
(\forall i \in J) \quad \delta_i \geq 0. \tag{4.21}
\]
Now fix $j \in J_+$ such that $\alpha_j/\beta_j = \gamma$. Then we get $\delta_j = 0$ as well as $J \setminus \{j\} \neq \emptyset$ (since otherwise, $J = \{j\}$ and $y = \alpha_j x_j = \gamma \beta_j x_j = 0$, which is absurd), and hence,
\[
y = y - \gamma 0 = \sum_{i \in J} \alpha_i x_i - \gamma \sum_{i \in J} \beta_i x_i = \sum_{i \in J \setminus \{j\}} \delta_i x_i. \tag{4.22}
\]
Therefore, in view of (4.21), (4.22), and (4.18), we must have $\sum_{i \in J \setminus \{j\}} \delta_i > 0$. In turn, if $J \setminus \{j\} \in \mathcal{I}$, then set $\delta := \sum_{i \in J \setminus \{j\}} \delta_i$ and observe that $y = \delta \sum_{i \in J \setminus \{j\}} (\delta_i/\delta) x_i \in \text{cone} D$, which yields $x = \lambda y \in \text{cone} D$. Otherwise, we reapply the procedure to $y = \sum_{i \in J \setminus \{j\}} \delta_i x_i$ recursively until $y$ can be written as $y = \sum_{i \in J'} \gamma_i x_i$, where $J' \in \mathcal{I}$ and $\{\gamma_i\}_{i \in J'} \subseteq \mathbb{R}_+$ satisfying $\mu := \sum_{i \in J'} \gamma_i > 0$. Consequently, $y = \mu \sum_{i \in J'} (\gamma_i/\mu) x_i \in \text{cone} D$, from which we deduce that $x = \lambda y \in \text{cone} D$. Thus (4.17) holds, and since rec $C = \{0\}$ (see Remark 4.5(i)), it follows from Proposition 4.2(ii) (applied to $C = \text{conv} \{x_i\}_{i \in I}$) and (4.13) that $K$ is closed, as desired. \[\square\]

### 4.3 Projectors onto cones generated by orthonormal sets

We start with a conical version of [8, Example 3.10].

**Theorem 4.7** Let $\{e_i\}_{i \in I}$ be a nonempty finite orthonormal subset of $\mathcal{H}$, set
\[
K := \sum_{i \in I} \mathbb{R}_+ e_i, \tag{4.23}
\]
and let $x \in \mathcal{H}$. Then $K$ is a nonempty closed convex cone in $\mathcal{H}$,
\[
P_K x = \sum_{i \in I} \max\{\langle x | e_i \rangle, 0\} e_i \quad \text{and} \quad d_K(x) = \sqrt{\|x\|^2 - \sum_{i \in I} (\max\{\langle x | e_i \rangle, 0\})^2}. \tag{4.24}
\]
4.3. Projectors onto cones generated by orthonormal sets

Proof. We first infer from Example 4.6 that $K$ is a nonempty closed convex cone. Thus, it is enough to verify (4.24). To this end, set

\[
\forall i \in I \quad \alpha_i := \max \{ \langle x | e_i \rangle, 0 \} \in \mathbb{R}_+
\]

(4.25)

and

\[
p := \sum_{i \in I} \alpha_i e_i.
\]

(4.26)

Then, by (4.25)&(4.26)&(4.23), we have $p \in K$, and by assumption, we get

\[
\|p\|^2 = \left\| \sum_{i \in I} \alpha_i e_i \right\|^2 = \sum_{i \in I} \alpha_i^2.
\]

(4.27)

Furthermore, (4.25) implies that

\[
\forall i \in I \quad [\alpha_i = \langle x | e_i \rangle \text{ or } \alpha_i = 0 \] \iff \alpha_i (\langle x | e_i \rangle - \alpha_i) = 0 \iff \alpha_i \langle x | e_i \rangle = \alpha_i^2,
\]

(4.28)

and therefore, we get from (4.26) that

\[
\langle x | p \rangle = \left\langle x \left| \sum_{i \in I} \alpha_i e_i \right. \right\rangle = \sum_{i \in I} \alpha_i \langle x | e_i \rangle = \sum_{i \in I} \alpha_i^2.
\]

(4.29)

In turn, on the one hand, (4.27) and (4.29) yield $\langle x - p | p \rangle = \langle x | p \rangle - \|p\|^2 = 0$. On the other hand, invoking (4.26), (4.25), and our hypothesis, we deduce that

\[
\forall i \in I \quad \langle x - p | e_i \rangle = \langle x | e_i \rangle - \left( \sum_{j \in I} \alpha_j e_j \right) e_i = \langle x | e_i \rangle - \alpha_i \leq 0,
\]

(4.30)

and hence, by (4.23), $x - p \in K^\ominus$. Altogether, we conclude that $P_K x = p = \sum_{i \in I} \max \{ \langle x | e_i \rangle, 0 \} e_i$ via Fact 3.24. Consequently, (4.27)&(4.29)&(4.25) give

\[
d^2_K(x) = \|x - p\|^2
\]

(4.31a)

\[
= \|x\|^2 - 2\langle x | p \rangle + \|p\|^2
\]

(4.31b)

\[
= \|x\|^2 - \sum_{i \in I} \alpha_i^2
\]

(4.31c)

\[
= \|x\|^2 - \sum_{i \in I} (\max \{ \langle x | e_i \rangle, 0 \})^2,
\]

(4.31d)

which completes the proof. □
4.3. Projectors onto cones generated by orthonormal sets

Remark 4.8 Here are a few comments concerning Theorem 4.7.

(i) In the setting of Theorem 4.7, suppose that \( \{e_i\}_{i \in I} \) is a singleton, say \( e \). Then \( K = \mathbb{R}_+ e \) is a ray and (4.24) becomes

\[
P_K x = \max\{\langle x \mid e \rangle, 0\} e \quad \text{and} \quad d_K(x) = \sqrt{\|x\|^2 - (\max\{\langle x \mid e \rangle, 0\})^2},
\]

which is precisely the formula for projectors onto rays (see, e.g., [8, Example 29.31]).

(ii) Consider the setting of Theorem 4.7. Suppose that \( N \) is a strictly positive integer, that \( I = \{1, \ldots, N\} \), that \( H = \mathbb{R}^N \), and that \( (e_i)_{i \in I} \) is the canonical orthonormal basis of \( H \). Then \( K = \mathbb{R}^N_+ \) is the positive orthant in \( H \). Now take \( x = (\xi_i)_{i \in I} \in H \). In the light of (4.24), since \( (\forall i \in I) \langle x \mid e_i \rangle = \xi_i \), we retrieve the well-known formula

\[
P_K x = (\max\{\xi_i, 0\})_{i \in I}; \quad \text{(4.33)}
\]

see, for instance, [8, Example 6.29]. Moreover, upon setting \( I_- := \{i \in I \mid \xi_i < 0\} \), we derive from (4.24) that

\[
d_K(x) = \sqrt{\|x\|^2 - \sum_{i \in I} (\max\{\xi_i, 0\})^2}, \quad \text{(4.34a)}
\]

\[
= \sqrt{\sum_{i \in I} \xi_i^2 - \sum_{i \in I \setminus I_-} \xi_i^2} \quad \text{(4.34b)}
\]

\[
= \sqrt{\sum_{i \in I \setminus I_-} \xi_i^2} \quad \text{(4.34c)}
\]

with the convention that \( \sum_{i \in \emptyset} \xi_i^2 = 0 \).

Corollary 4.9 Let \( \{e_i\}_{i \in I} \) be a nonempty finite orthonormal subset of \( H \). Set

\[
K := \{y \in H \mid (\forall i \in I) \langle y \mid e_i \rangle \leq 0\}, \quad \text{(4.35)}
\]

and let \( x \in H \). Then \( K \) is a nonempty closed convex cone in \( H \),

\[
P_K x = x - \sum_{i \in I} \max\{\langle x \mid e_i \rangle, 0\} e_i, \quad \text{and} \quad d_K(x) = \sqrt{\sum_{i \in I} (\max\{\langle x \mid e_i \rangle, 0\})^2}.
\]

(4.36)

Proof. Since

\[
K = \bigcap_{i \in I} \{e_i\}^\oplus, \quad \text{(4.37)}
\]
we see that \( K \) is a nonempty closed convex cone. Next, by (4.37), Fact 3.26(iii) implies that \( K = \bigcap_{i \in I} (\mathbb{R}_+ + e_i)^\ominus = (\sum_{i \in I} \mathbb{R}_+ + e_i)^\ominus \), and since \( \sum_{i \in I} \mathbb{R}_+ e_i \) is a nonempty closed convex cone by Example 4.6, taking the polar cones and invoking Fact 3.26(ii) yield \( K^\ominus = (\sum_{i \in I} \mathbb{R}_+ + e_i)^\ominus = \sum_{i \in I} \mathbb{R}_+ e_i \). Hence, according to Moreau’s theorem (Fact 3.25) and Theorem 4.7, we conclude that \( P_K x = x - P_{K^\ominus} x = x - \sum_{i \in I} \max\{\langle x | e_i \rangle, 0\} e_i \) and that

\[
\begin{align*}
\frac{d^2}{dK}(x) &= \|x\|^2 - \frac{d^2}{dK^\ominus}(x) \\
&= \|x\|^2 - \left( \|x\|^2 - \sum_{i \in I} (\max\{\langle x | e_i \rangle, 0\})^2 \right) \\
&= \sum_{i \in I} (\max\{\langle x | e_i \rangle, 0\})^2,
\end{align*}
\]

as claimed in (4.36).

4.4 The projector onto the intersection of a cone and a ball

It turns out that the projector onto the intersection of a cone and a ball has a pleasing explicit form.

**Theorem 4.10 (cone intersected with ball)** Let \( K \) be a nonempty closed convex cone in \( \mathcal{H} \), let \( \rho \in \mathbb{R}_+ \), and set \( C := K \cap B(0; \rho) \). Then

\[
\begin{align*}
P_C x &= \max\{\|P_K x\|, \rho\} P_K x, \\
d_C(x) &= \sqrt{d_K^2(x) + \left( \max\{\|P_K x\| - \rho, 0\} \right)^2}.
\end{align*}
\]

**Proof.** Take \( x \in \mathcal{H} \), set \( \beta := \rho / \max\{\|P_K x\|, \rho\} \in \mathbb{R}_+ \), and set \( p := \beta P_K x \). Then, since \( K \) is a cone and \( P_K x \in K \), we get \( p \in K \), and thus, since \( \|p\| = \beta \|P_K x\| = \rho (\|P_K x\| / \max\{\|P_K x\|, \rho\}) \leq \rho \), it follows that \( p \in K \cap B(0; \rho) = C \). Hence, because \( C \) is closed and convex, in the light of (3.8), it remains to verify that \( (\forall y \in C) \langle x - p | y - p \rangle \leq 0 \). To this end, take \( y \in C \), and we consider two alternatives:

(a) \( \|P_K x\| \leq \rho \): Then \( \beta = \rho / \rho = 1 \). It follows that \( p = P_K x \), and so

\[
\|x - p\| = \|x - P_K x\| = d_K(x).
\]

(b) \( \|P_K x\| > \rho \): Then \( \beta \neq 1 \). It follows that \( p \neq P_K x \), and the distance from \( x \) to \( C \) is

\[
d_C(x) = \sqrt{d_K^2(x) + \left( \max\{\|P_K x\| - \rho, 0\} \right)^2}.
\]

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Next, because \( y \in K \), (3.8) asserts that
\[
\langle x - p \mid y - p \rangle = \langle x - P_K x \mid y - P_K x \rangle \leq 0. \tag{4.41}
\]

(b) \( \|P_K x\| > \rho \): Then \( \beta = \rho / \|P_K x\| \in ]0, 1[ \), and so Lemma 3.29(ii) implies that
\[
\|x - p\| = \sqrt{d_K^2(x) + (\|P_K x\| - \rho)^2}. \tag{4.42}
\]
In turn, on the one hand, since \( y \) belongs to the cone \( K \), it follows that \( (1/\beta)y \in K \), from which and (3.8) we deduce that
\[
\langle x - P_K x \mid y - \beta P_K x \rangle = \beta \langle x - P_K x \mid (1/\beta)y - P_K x \rangle \leq 0. \tag{4.43}
\]
On the other hand, because \( y \in B(0; \rho) \) and \( \beta = \rho / \|P_K x\| \), the Cauchy–Schwarz inequality yields
\[
\langle P_K x \mid y - \beta P_K x \rangle = \langle P_K x \mid y \rangle - \rho \|P_K x\| \leq \|P_K x\| \|y\| - \rho \|P_K x\| \leq 0. \tag{4.44}
\]
Altogether, combining (4.43)&(4.44) and using the fact that \( \beta \in ]0, 1[ \), we obtain
\[
\langle x - p \mid y - p \rangle = \langle x - \beta P_K x \mid y - \beta P_K x \rangle \tag{4.45a}
\]
\[
= \langle x - P_K x \mid y - \beta P_K x \rangle + (1 - \beta) \langle P_K x \mid y - \beta P_K x \rangle \tag{4.45b}
\]
\[
\leq 0. \tag{4.45c}
\]
Hence, in both cases, we have \( \langle x - p \mid y - p \rangle \leq 0 \). Thus \( p = P_C x \), and it follows from (4.40)&(4.42) that
\[
d_C(x) = \|x - P_C x\| = \|x - p\| = \sqrt{d_K^2(x) + (\max\{\|P_K x\| - \rho, 0\})^2}, \tag{4.46}
\]
as stated in (4.39).

Here are some easy consequences of Theorem 4.10.

**Example 4.11** In the setting of Theorem 4.10, suppose that \( K = \mathcal{H} \). Then \( C = B(0; \rho) \), \( P_K = \text{Id} \), \( d_K \equiv 0 \), and (4.39) becomes
\[
(\forall x \in \mathcal{H}) \quad P_C x = \frac{\rho}{\max\{\|x\|, \rho\}} x \quad \text{and} \quad d_C(x) = \max\{\|x\| - \rho, 0\}. \tag{4.47}
\]
We thus recover the formula for projectors onto balls. For a rigorous proof of (4.47) without invoking (3.8), see [10, Example 5].
4.4. The projector onto the intersection of a cone and a ball

**Corollary 4.12** Let $K$ be a nonempty closed convex cone in $\mathcal{H}$, let $\rho \in \mathbb{R}^{++}$, and set $C := K \cap B(0; \rho)$. Then $P_C = P_{B(0;\rho)} \circ P_K$.

**Proof.** Combine (4.39) and (4.47). Alternatively, set $f := \iota_{B(0;\rho)}$ and $\kappa := \iota_K$ in the equivalence “(iii)$\iff$(iv)” of [37, Theorem 4]. (Note that $\iota_{B(0;\rho)} + \iota_K = \iota_C$.) ■

**Remark 4.13** In the setting of Corollary 4.12, as we shall see in Example 4.14, $P_C \neq P_K \circ P_{B(0;\rho)}$, i.e., $P_{B(0;\rho)} \circ P_K \neq P_K \circ P_{B(0;\rho)}$, in general.

**Example 4.14** Suppose that $\mathcal{H} = \mathbb{R}^2$. Set $K := \mathbb{R}_+^2$ and $x := (1, -1)$. Then (see also Figure 4.1)

\[
(P_K \circ P_{B(0;1)})x = P_K(P_{B(0;1)}x) \overset{(4.47)}{=} P_K \left( \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}} \right) \overset{(4.33)}{=} \left( \frac{1}{\sqrt{2}}, 0 \right) (4.48)
\]

and

\[
(P_{B(0;1)} \circ P_K)x = P_{B(0;1)}(P_Kx) \overset{(4.33)}{=} P_{B(0;1)}(1, 0) \overset{(4.47)}{=} (1, 0). (4.49)
\]

Hence

\[
P_{B(0;1)} \circ P_K \neq P_K \circ P_{B(0;1)}. (4.50)
\]

**Figure 4.1:** Example 4.14 illustrates that the projectors onto a cone and ball may fail to commute.
4.5. The projector onto the intersection of a cone and a sphere

As will be seen in the next result, Example 4.14 is, however, not a coincidence.

**Corollary 4.15** Let $K$ be a nonempty closed convex cone in $H$, and let $\rho \in \mathbb{R}_{++}$. Then

$$\left( \forall x \in H \right) \left( P_K \circ P_{B(0,\rho)} \right) x = \frac{\rho}{\max\{\|x\|, \rho\}} P_K x.$$  \hspace{1cm} (4.51)

**Proof.** It follows from (4.47) and (3.27) that

$$\left( \forall x \in H \right) \left( P_K \circ P_{B(0,\rho)} \right) x = P_K \left( \frac{\rho}{\max\{\|x\|, \rho\}} x \right) \quad (4.52a)$$

$$= \frac{\rho}{\max\{\|x\|, \rho\}} P_K x, \quad (4.52b)$$

as desired. \hfill \blacksquare

**Remark 4.16** Consider the setting of Corollary 4.15. Using Corollary 4.15, Theorem 4.10, and Corollary 4.12, we deduce that

$$\left( \forall x \in H \right) \left( P_K \circ P_{B(0,\rho)} \right) x = \frac{\max\{\|P_K x\|, \rho\}}{\max\{\|x\|, \rho\}} (P_{B(0,\rho)} \circ P_K) x.$$  \hspace{1cm} (4.53)

4.5 The projector onto the intersection of a cone and a sphere

In this section, which contains our second half of main results, we develop formulae for the projector onto the intersection of a cone and a sphere.

**Lemma 4.17** Let $C$ be a nonempty subset of $H$, let $\beta \in \mathbb{R}_{++}$, and let $u \in \text{pos} C$, say $u = \sum_{i \in I} \alpha_i x_i$, where $\{\alpha_i\}_{i \in I}$ and $\{x_i\}_{i \in I}$ are finite subsets of $\mathbb{R}_+$ and $C$, respectively. Suppose that $\|u\| = \beta$ and that $(\forall y \in C) \|y\| = \beta$. Then the following hold:

(i) $\sum_{i \in I} \alpha_i \geq 1$.

(ii) Let $x \in H$, and set $\kappa := \sup(\langle x \mid C \rangle)$. Suppose that $\kappa \in ]-\infty, 0]$ and that $\kappa \leq \langle x \mid u \rangle$. Then the following hold:

\hspace{1cm} \footnote{Here pos $C$ was defined in (3.14).}
4.5. The projector onto the intersection of a cone and a sphere

(a) $P_{C}x \neq \emptyset$ and $\langle x \mid u \rangle = \max \langle x \mid C \rangle = \kappa$.

(b) $u \in S(0; \beta) \cap \text{cone}(\text{conv } P_{C}x)$.

(c) Suppose that $\kappa < 0$. Then $u \in P_{C}x$.

Proof. (i): Since, by assumption, $(\forall i \in I) \|x_i\| = \beta$, it follows from the triangle inequality that

$$\beta = \|u\| = \left\| \sum_{i \in I} \alpha_i x_i \right\| \leq \sum_{i \in I} \alpha_i \|x_i\| = \beta \sum_{i \in I} \alpha_i.$$  (4.54)

Therefore, because $\beta > 0$, we obtain $\sum_{i \in I} \alpha_i \geq 1$.

(ii): Let us first establish that $(\forall i \in I) \quad x_i \in C \setminus P_{C}x \Rightarrow \alpha_i = 0$  (4.55)

by contradiction: assume that there exists $i_0 \in I$ such that

$$x_{i_0} \in C \setminus P_{C}x$$  (4.56)

but that

$$\alpha_{i_0} > 0.$$  (4.57)

Then, because the vectors in $C$ are of equal norm, we deduce from Lemma 3.9(i) and (4.56) that $\langle x \mid x_{i_0} \rangle < \sup \langle x \mid C \rangle = \kappa$, and so, by (4.57), $\alpha_{i_0} \langle x \mid x_{i_0} \rangle < \alpha_{i_0} \kappa$. Hence, since

$$\left\{ \begin{array}{l} \kappa \leq 0, \\
\kappa \leq \langle x \mid u \rangle, \\
(\forall i \in I) \quad 0 \leq \alpha_i \quad \text{and} \quad \langle x \mid x_i \rangle \leq \kappa,
\end{array} \right.$$  (4.58)

it follows from (i) that

$$\kappa \leq \langle x \mid u \rangle = \sum_{i \in I} \alpha_i \langle x \mid x_i \rangle = \alpha_{i_0} \langle x \mid x_{i_0} \rangle + \sum_{i \in I \setminus \{i_0\}} \alpha_i \langle x \mid x_i \rangle < \alpha_{i_0} \kappa + \sum_{i \in I \setminus \{i_0\}} \alpha_i \kappa = \kappa \sum_{i \in I} \alpha_i \leq \kappa,$$  (4.59)
and we thus arrive at a contradiction, namely $\kappa < \kappa$. Therefore, (4.55) holds.

(ii)(a): If $P_C x$ were empty, then (4.55) would yield $(\forall i \in I) \ a_i = 0$ and it would follow that $u = 0$ or, equivalently, $\beta = \|u\| = 0$, which is absurd. Thus $P_C x \neq \emptyset$, and so Lemma 3.9(i) implies that $\kappa = \max \langle x | C \rangle$. Furthermore, we infer from (4.58) and (i) that

$$\kappa \leq \langle x | u \rangle = \sum_{i \in I} a_i \langle x | x_i \rangle \leq \sum_{i \in I} a_i \kappa = \kappa \sum_{i \in I} a_i \leq \kappa,$$

(4.60)

and the latter assertion follows.

(ii)(b): In the remainder, since $u \neq 0$, appealing to (4.55), we assume without loss of generality that

$$(\forall i \in I) \ x_i \in P_C x \quad (4.61)$$

and that $(\forall i \in I)(\forall j \in I) \ i \neq j \Rightarrow x_i \neq x_j$. Hence, upon setting $\alpha := \sum_{i \in I} a_i \geq 1$, we deduce from (4.61) that

$$u = \alpha \sum_{i \in I} \frac{a_i}{\alpha} x_i \in \alpha \conv P_C x \subseteq \conv (\conv P_C x). \quad (4.62)$$

Consequently, since $\|u\| = \beta$, the claim follows.

(ii)(c): Invoking Lemma 3.9(i) and (4.61), we get $(\forall i \in I) \ \langle x | x_i \rangle = \max \langle x | C \rangle = \kappa$. Thus, by (ii)(a), $\kappa = \langle x | u \rangle = \sum_{i \in I} a_i \langle x | x_i \rangle = \kappa \sum_{i \in I} a_i$, and since $\kappa \neq 0$, it follows that $\sum_{i \in I} a_i = 1$. To summarize, we have

$$\begin{cases} u = \sum_{i \in I} a_i x_i, \\
(\forall i \in I) \ |x_i| = |u| = \beta, \\
\{a_i\}_{i \in I} \subseteq \mathbb{R}_+ \text{ satisfying } \sum_{i \in I} a_i = 1, \\
(\forall i \in I)(\forall j \in I) \ i \neq j \Rightarrow x_i \neq x_j. \end{cases} \quad (4.63)$$

Lemma 2.3(iii) and (4.61) therefore imply that $(\exists i \in I) \ u = x_i \in P_C x$, as desired. \[\blacksquare\]

The following example shows that the conclusion of Lemma 4.17(ii)(c) fails if the assumption that $u \in \text{pos} \ C$ is omitted.

**Example 4.18** Suppose that $\mathcal{H} = \mathbb{R}^3$ and that $(e_1, e_2, e_3)$ is the canonical orthonormal basis of $\mathcal{H}$. Set $C := \{e_1, e_2\}$, $x := (\mathbf{1}, -1, 0)$, and $u := (1/2, 1/2, \sqrt{2}/2)$. Then $u$ is not a conical combination of elements of $C$ and, as in the assumption of Lemma 4.17, $(\beta, \kappa) = (1, -1)$. Moreover, a simple computation gives $\|u\| = 1$, $\langle x | u \rangle = -1 = \kappa$, and $\|x - e_1\| = \|x - e_2\| = \sqrt{5}$. Hence, $P_C x = C \neq \emptyset$ while $u \notin C$.  

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4.5. The projector onto the intersection of a cone and a sphere

**Theorem 4.19** Let $K$ be a nonempty closed convex cone in $\mathcal{H}$, let $\rho \in \mathbb{R}_{++}$, and set $C := K \cap S(0; \rho)$. Suppose that $K \neq \{0\}$. Then the following hold:

(i) For every $x \in K^\perp$, $P_C x = C$ and $d_C(x) = \sqrt{\|x\|^2 + \rho^2}$.

(ii) For every $x \in \mathcal{H} \setminus K^\ominus$, $P_C x = \{\rho/\|P_K x\|\} P_K x$ and $d_C(x) = \sqrt{d_K^2(x) + (\|P_K x\| - \rho)^2}$.

**Proof.** We first observe that, by assumption and Remark 3.18, $C \neq \emptyset$.

(i): Fix $x \in K^\perp$. Then, for every $y \in C = K \cap S(0; \rho)$, since $x \perp y$ and $\|y\| = \rho$, we get $\|x - y\|^2 = \|x\|^2 + \|y\|^2 = \|x\|^2 + \rho^2$. It follows that $d_C(x) = \sqrt{\|x\|^2 + \rho^2}$ and that $P_C x = C$, as desired.

(ii): First, by the very definition of $C$, we see that $C$ consists of vectors of equal norm. (4.64)

Now take $x \in \mathcal{H} \setminus K^\ominus$; set $5 \alpha := \rho/\|P_K x\| \in \mathbb{R}_{++}$ and $p := \alpha P_K x$. (4.65)

Then, because $P_K x$ belongs to the cone $K$, we obtain $p \in K$, and because

$$\|p\| = \frac{\rho}{\|P_K x\|} P_K x = \rho,$$

it follows that $p \in K \cap S(0; \rho) = C$. (4.67)

Next, fix $y \in C$. Since $y \in C \subseteq K$ and $K$ is a cone, we have $\alpha^{-1}y \in K$. Therefore, since $\|y\| = \rho$, we derive from (4.65), (3.8), and (4.66) that

$$\langle x | p \rangle - \langle x | y \rangle = \langle x | p - y \rangle$$

$$= \langle x - P_K x | p - y \rangle + \langle P_K x | p - y \rangle$$

$$= \langle x - P_K x | \alpha P_K x - y \rangle + \langle \alpha^{-1} p | p - y \rangle$$

$$= \alpha \langle x - P_K x | P_K x - \alpha^{-1} y \rangle + \alpha^{-1} \langle p | p - y \rangle$$

$$\geq 0 \text{ by } (3.8)$$

$$\geq (2\alpha)^{-1}(\|p\|^2 + \|p - y\|^2 - \|y\|^2)$$

$$= (2\alpha)^{-1}(\rho^2 + \|p - y\|^2 - \rho^2).$$

---

5Due to Lemma 3.29(i), we have $P_K x \neq 0$. 

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To summarize, we have shown that \((\forall y \in C) \ y \neq p \Rightarrow \langle x \mid y \rangle < \langle x \mid p \rangle\). Combining this, (4.67), and (4.64), we infer from Lemma 3.9(i) that \(P_C x = \{p\}\). This and Lemma 3.29(ii) yield the latter assertion, and the proof is complete. \[\blacksquare\]

Let us provide some examples.

**Corollary 4.20 (Projections onto circles)** Let \(V\) be a nonzero closed linear subspace of \(H\), let \(\rho \in \mathbb{R}_{++}\), and set \(C := V \cap S(0; \rho)\). Then

\[
(\forall x \in H) \quad P_C x = \begin{cases} 
C, & \text{if } x \in V^\perp; \\
\left\{ \frac{\rho}{\|P_V x\|} P_V x \right\}, & \text{otherwise.}
\end{cases} \tag{4.69}
\]

*Proof.* Combine Theorem 4.19 and the fact that \(V^\ominus = V^\perp\). \[\blacksquare\]

**Remark 4.21** Letting \(V = H\) in Corollary 4.20, we see that \(C = S(0; \rho)\), that \(V^\perp = \{0\}\), that \(P_V = \text{Id}\), and that (4.69) becomes

\[
(\forall x \in H) \quad P_C x = \begin{cases} 
C, & \text{if } x = 0; \\
\left\{ \frac{\rho}{\|x\|} x \right\}, & \text{otherwise.}
\end{cases} \tag{4.70}
\]

Hence, we recover the well-known formula for projectors onto spheres.

**Example 4.22** Let \(\alpha \in \mathbb{R}\) and \(\beta \in \mathbb{R}_{++}\), and set

\[
S_{\alpha, \beta} := S(0; \beta) \times \{\alpha\}. \tag{4.71}
\]

Then

\[
(\forall x = (x, \xi) \in H) \quad P_{S_{\alpha, \beta}} x = \begin{cases} 
S_{\alpha, \beta}, & \text{if } x = 0; \\
\left\{ \left( \frac{\beta}{\|x\|} x, \alpha \right) \right\}, & \text{otherwise.}
\end{cases} \tag{4.72}
\]

*Proof.* Set \(V := H \times \{0\}\), which is a nonzero closed linear subspace of \(H\). Let us first observe that

\[
V = \{x = (x, \xi) \in H \mid \langle x \mid (0, 1) \rangle = 0\} = \{(0, 1)\}^\perp, \tag{4.73}
\]
4.5. The projector onto the intersection of a cone and a sphere

and thus,

\[(\forall x = (x, \xi) \in \mathcal{H}) \quad x \in V^\perp \iff x \in \mathbb{R}(0, 1) \iff x = 0. \quad (4.74)\]

Moreover, it is straightforward to verify that

\[S_{0, \beta} = V \cap S(0; \beta). \quad (4.75)\]

Now fix \(x = (x, \xi) \in \mathcal{H}. \) Then, appealing to (3.12) and (4.73), we see that \(P_V x = (x, 0), \) Combining this, (4.75), and (4.74), we deduce from Corollary 4.20 that

\[P_{S_{0, \beta}} x = P_{S_{0, \beta}}(x, \xi) = \begin{cases} 
S_{0, \beta}, & \text{if } x = 0; \\
\left\{ \frac{\beta}{\|P_V x\|} P_V x \right\}, & \text{otherwise}
\end{cases} \quad (4.76a)\]

\[= \begin{cases} 
S_{0, \beta}, & \text{if } x = 0; \\
\left\{ \left( \frac{\beta}{\|x\|} x, 0 \right) \right\}, & \text{otherwise.}
\end{cases} \quad (4.76b)\]

Consequently, since\(^6\) \(S_{\alpha, \beta} = (0, \alpha) + S_{0, \beta}, \) we derive from (4.76b) (applied to the point \((x, \xi - \alpha)\)) and Proposition 3.8 that

\[P_{S_{\alpha, \beta}} x = (0, \alpha) + P_{S_{0, \beta}}(x - (0, \alpha)) \quad (4.77a)\]

\[= (0, \alpha) + P_{S_{0, \beta}}(x, \xi - \alpha) \quad (4.77b)\]

\[= \begin{cases} 
(0, \alpha) + S_{0, \beta}, & \text{if } x = 0; \\
(0, \alpha) + \left\{ \left( \frac{\beta}{\|x\|} x, 0 \right) \right\}, & \text{otherwise}
\end{cases} \quad (4.77c)\]

\[= \begin{cases} 
S_{\alpha, \beta}, & \text{if } x = 0; \\
\left\{ \left( \frac{\beta}{\|x\|} x, \alpha \right) \right\}, & \text{otherwise,}
\end{cases} \quad (4.77d)\]

as announced in (4.72). □

Next, we turn to the more complicated case when the point to be projected belongs to the polar cone.

\[\text{□}\]

\(^6\)As the reader can easily verify.
4.5. The projector onto the intersection of a cone and a sphere

**Theorem 4.23** Let $K$ be a convex cone in $\mathcal{H}$ such that $K \setminus \{0\} \neq \emptyset$, let $\rho$ be in $\mathbb{R}_{++}$, and let $x \in K^\circ$. Suppose that there exists a nonempty subset $C$ of $K$ such that

$$\forall y \in C \quad \|y\| = \rho$$

and that

$$K = \text{pos} \ C.$$  \hfill (4.79)

Set

$$D := K \cap S(0; \rho) \quad \text{and} \quad \kappa := \sup \langle x \mid C \rangle.$$  \hfill (4.80)

Then the following hold:

(i) Suppose that $P_C x = \emptyset$. Then $P_D x = \emptyset$.

(ii) Suppose that $P_C x \neq \emptyset$, and set $E := S(0; \rho) \cap \text{cone}(\text{conv } P_C x)$. Then the following hold:

(a) $P_C x \subseteq P_D x \subseteq E$ and $\max \langle x \mid D \rangle = \max \langle x \mid C \rangle$.

(b) Suppose that $\kappa < 0$. Then $P_D x = P_C x$.

(c) Suppose that $\kappa = 0$. Then $P_D x = E$.

(iii) $P_C x \neq \emptyset \iff P_D x \neq \emptyset$.

**Proof.** We start with a few observations. First, since $K \neq \{0\}$ by assumption, it follows from Remark 3.18 that $D \neq \emptyset$. Next, in view of (4.78) and the assumption that $C \subseteq K$, we have

$$C \subseteq D.$$  \hfill (4.81)

In turn, because $x \in K^\circ$, we get from (4.79) and Lemma 3.22(i) that

$$\kappa \leq 0.$$  \hfill (4.82)

Finally, by the very definition of $D$, we see that

the vectors in $D$ are of equal norm.  \hfill (4.83)

(i): We prove the contrapositive and therefore assume that there exists

$$u \in P_D x.$$  \hfill (4.84)

Then, by (4.81), (4.83), (4.84), and Lemma 3.9(i), we obtain

$$\kappa = \sup \langle x \mid C \rangle \leq \sup \langle x \mid D \rangle = \langle x \mid u \rangle.$$  \hfill (4.85)
In turn, combining (4.78), (4.82), (4.85), and the fact that \( u \in D = (\text{pos } C) \cap S(0; \rho) \), we infer from Lemma 4.17(ii)(a) that \( P_Cx \neq \emptyset \).

(ii)(a): Let us first prove that \( P_Cx \subseteq P_Dx \) and that \( \max \langle x \mid D \rangle = \max \langle x \mid C \rangle \). To this end, take \( u \in P_Cx \) and \( y \in D \). Then, because \( y \in D \subseteq \text{pos } C \), there exist finite sets \( \{a_i\}_{i \in I} \subseteq \mathbb{R}_+ \) and \( \{x_i\}_{i \in I} \subseteq C \) such that \( y = \sum_{i \in I} a_i x_i \). In turn, on the one hand, since \( \|y\| = \rho \), we infer from (4.78) and Lemma 4.17(ii)(b) that \( \max \langle y \mid u \rangle = \rho \). On the other hand, since \( u \in P_Cx \), it follows from (4.78) and Lemma 3.9(i) that

\[
\langle x \mid u \rangle = \max \langle x \mid C \rangle = \kappa. \tag{4.86}
\]

So altogether, since \( (\forall i \in I) \ x_i \in C \), using (4.82), we see that

\[
\langle x \mid y \rangle = \sum_{i \in I} a_i \langle x \mid x_i \rangle \leq \sum_{i \in I} a_i \kappa = \kappa \sum_{i \in I} a_i \leq \kappa = \langle x \mid u \rangle. \tag{4.87}
\]

Therefore, since \( u \in C \subseteq D \) by (4.81), we derive from (4.87) and (4.86) that

\[
\max \langle x \mid D \rangle = \langle x \mid u \rangle = \max \langle x \mid C \rangle. \tag{4.88}
\]

Also, appealing to (4.88) and (4.83), we get from Lemma 3.9(i) that \( u \in P_Dx \), as desired. It now remains to establish the inclusion \( P_Dx \subseteq E \). To do so, fix \( v \in P_Dx \). Then, in view of (4.83), Lemma 3.9(i) and (4.88)&(4.86)&(4.82) assert that

\[
\langle x \mid v \rangle = \max \langle x \mid D \rangle = \max \langle x \mid C \rangle \leq 0. \tag{4.89}
\]

Thus, since

\[
v \in D = (\text{pos } C) \cap S(0; \rho), \tag{4.90}
\]

it follows from (4.78) and Lemma 4.17(ii)(b) that \( v \in S(0; \rho) \cap \text{cone}(\text{conv } P_Cx) = E \), as claimed.

(ii)(b): Consider the element \( v \in P_Dx \) of the proof of (ii)(a). Combining (4.78)&(4.89)&(4.90) and the assumption that \( \kappa < 0 \), we derive from Lemma 4.17(ii)(c) that \( v \in P_Cx \), and hence, \( P_Dx \subseteq P_Cx \). Consequently, since \( P_Cx \subseteq P_Dx \) by (ii)(a), the assertion follows.

(ii)(c): According to (ii)(a), it suffices to show that \( E \subseteq P_Dx \). Towards this end, take \( w \in E \) and \( y \in D \). By the very definition of \( E \), there exist finite sets \( \{\beta_j\}_{j \in J} \subseteq \mathbb{R}_{++} \) and \( \{x_j\}_{j \in J} \subseteq P_Cx \) such that \( w = \sum_{j \in J} \beta_j x_j \). In turn, since \( \{x_j\}_{j \in J} \subseteq P_Cx \), we get from (4.78) and Lemma 3.9(i) that

\[
(\forall j \in J) \ \langle x \mid x_j \rangle = \kappa = 0, \tag{4.91}
\]

from which and (4.88) it follows that

\[
\langle x \mid w \rangle = \sum_{j \in J} \beta_j \langle x \mid x_j \rangle = 0 = \kappa = \max \langle x \mid D \rangle.
\]
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Consequently, since \( w \in E \subseteq D \) by the very definitions of \( E \) and \( D \), invoking (4.83) and Lemma 3.9(i) once more, we conclude that \( w \in P_D x \), as required.

(iii): Combine (i) and (ii)(a). ■

We are now ready for the main result of this section which provides a formula for the projector of a finitely generated cone and a sphere.

**Corollary 4.24 (cone intersected with sphere)** Let \( \{x_i\}_{i \in I} \) be a nonempty finite subset of \( \mathcal{H} \), let \( \rho \in \mathbb{R}_{++} \), and let \( x \in \mathcal{H} \). Set

\[
\begin{aligned}
K &:= \sum_{i \in I} \mathbb{R}_+ x_i, \\
C &:= K \cap S(0; \rho), \\
\kappa &:= \max_{i \in I} \langle x | x_i \rangle, \\
I(x) &:= \{ i \in I | \langle x | x_i \rangle = \kappa \}.
\end{aligned}
\]  

(4.92)

Suppose that \( (\forall i \in I) \| x_i \| = \rho \). Then

\[
P_C x = \begin{cases} 
\left\{ \frac{\rho}{\| P_K x \|} P_K x \right\}, & \text{if } \kappa > 0; \\
S(0; \rho) \cap \text{cone} \left( \text{conv} \{ x_i \}_{i \in I(x)} \right), & \text{if } \kappa = 0; \\
\{ x_i \}_{i \in I(x)}, & \text{if } \kappa < 0.
\end{cases}
\]  

(4.93)

**Proof.** Set \( X := \{ x_i \}_{i \in I} \). First, it follows from Example 4.6 that \( K \) is a nonempty closed convex cone. In addition, Lemma 3.22(i) (applied to \( \{ x_i \}_{i \in I} \)) implies that

\[
x \in K^\ominus \iff \kappa = \max_{i \in I} \langle x | x_i \rangle \leq 0.
\]  

(4.94)

Next, due to our assumption, Lemma 3.9(i) yields

\[
P_X x = \{ x_i \}_{i \in I(x)} \neq \emptyset.
\]  

(4.95)

Let us now identify \( P_C x \) in each of the following conceivable cases:

(a) \( \kappa > 0 \): Then, by (4.94), we have \( x \in \mathcal{H} \setminus K^\ominus \), and hence, Theorem 4.19(ii) asserts that \( P_C x = \{ (\rho/\| P_K x \|) P_K x \} \).

(b) \( \kappa = 0 \): Using Theorem 4.23(ii)(c) (with the set \( C \) being \( X = \{ x_i \}_{i \in I} \) and (4.95), we obtain \( P_C x = S(0; \rho) \cap \text{cone} \left( \text{conv} \{ x_i \}_{i \in I(x)} \right) \).

(c) \( \kappa < 0 \): Invoking Theorem 4.23(ii)(b) and (4.95), we immediately have \( P_C x = \{ x_i \}_{i \in I(x)} \). ■
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Remark 4.25 Consider the setting of Corollary 4.24. Since \( \{ x_i \}_{i \in I(x)} \subseteq S(0, \rho) \cap \text{cone}(\text{conv}\{ x_i \}_{i \in I(x)}) \) by the assumption that \( \| x_i \| = \rho \), we see that

\[
s: \mathcal{H} \rightarrow \mathcal{H} : x \mapsto \begin{cases} \frac{\rho}{\| P_K x \|} P_K x, & \text{if } \max_{i \in I} \langle x_i \rangle > 0; \\ s(x) \in \{ x_i \}_{i \in I(x)}, & \text{otherwise} \end{cases}
\]  

(4.96)

is a selection of \( P_C \).

Example 4.26 Consider the setting of Theorem 4.7. Set

\[
\begin{align*}
  C &= K \cap S(0; 1), \\
  \kappa &= \max_{i \in I} \langle x_i | e_i \rangle, \\
  I(x) &= \{ i \in I | \langle x_i | e_i \rangle = \kappa \}, \\
  \lambda &= \sqrt{\sum_{i \in I} (\max\{ \langle x_i | e_i \rangle, 0 \})^2}.
\end{align*}
\]

(4.97)

Then

\[
P_C x = \begin{cases} \left\{ \lambda^{-1} \sum_{i \in I} \max\{ \langle x_i | e_i \rangle, 0 \} e_i \right\} & \text{if } \kappa > 0; \\
  \left\{ \sum_{i \in I(x)} a_i e_i \middle| \{ a_i \}_{i \in I(x)} \subseteq \mathbb{R}_+, \sum_{i \in I(x)} a_i^2 = 1 \right\} & \text{if } \kappa = 0; \\
  \{ e_i \}_{i \in I(x)} & \text{if } \kappa < 0. \end{cases}
\]

(4.98)

Proof. Since \( P_K x = \sum_{i \in I} \max\{ \langle x_i | e_i \rangle, 0 \} e_i \) by (4.24), we obtain

\[
\| P_K x \|^2 = \left\| \sum_{i \in I} \max\{ \langle x_i | e_i \rangle, 0 \} e_i \right\|^2 = \sum_{i \in I} (\max\{ \langle x_i | e_i \rangle, 0 \})^2 = \lambda^2. 
\]

(4.99)

Next, let us show that

\[
S(0; 1) \cap \text{cone}\left( \text{conv}\{ e_i \}_{i \in I(x)} \right) = \left\{ \sum_{i \in I(x)} a_i e_i \middle| \{ a_i \}_{i \in I(x)} \subseteq \mathbb{R}_+, \sum_{i \in I(x)} a_i^2 = 1 \right\}.
\]

(4.100)

To this end, denote the set on the right-hand side of (4.100) by \( D \). Take \( y \in S(0; 1) \cap \text{cone}(\text{conv}\{ e_i \}_{i \in I(x)}) \). Then there exist \( \lambda \in \mathbb{R}_{++} \) and \( \{ a_i \}_{i \in I(x)} \subseteq \mathbb{R}_+ \) such that

\[
y = \lambda \sum_{i \in I(x)} a_i e_i = \sum_{i \in I(x)} (\lambda a_i) e_i.
\]
Furthermore, since \( \{e_i\}_{i \in I(x)} \) is an orthonormal set, we get
\[
1 = \|y\|^2 = \| \sum_{i \in I(x)} (\lambda e_i) e_i \|^2 = \sum_{i \in I(x)} (\lambda e_i)^2.
\]
Hence \( y \in D \). Conversely, fix \( z \in D \), say \( z = \sum_{i \in I(x)} (x) e_i \), where \( \{\beta_i\}_{i \in I(x)} \subseteq \mathbb{R}_+ \) satisfying \( \sum_{i \in I(x)} \beta_i^2 = 1 \), and set \( \beta := \sum_{i \in I(x)} (\beta_i / \beta) e_i \in \text{cone} \{e_i\}_{i \in I(x)} \). In turn, because
\[
\|z\|^2 = \sum_{i \in I(x)} \beta_i^2 = 1,
\]
It follows that \( z \in S(0; 1) \cap \text{cone} \{e_i\}_{i \in I(x)} \). Thus (4.100) holds. Consequently, using (4.24)&(4.99)&(4.100), we obtain (4.98) via Corollary 4.24.

The following nice result was mentioned in [28, Example 5.5.2 and Problem 5.6.14].

**Example 4.27 (Lange)** Suppose that \( \mathcal{H} = \mathbb{R}^N \), that \( I = \{1, \ldots , N\} \), and that \( (e_i)_{i \in I} \) is the canonical orthonormal basis of \( \mathcal{H} \). Set
\[
K := \mathbb{R}^N_+ \quad \text{and} \quad C := K \cap S(0; 1).
\]
Now let \( x = (\xi_i)_{i \in I} \in \mathcal{H} \); set \( \kappa := \max_{i \in I} \xi_i \), \( I(x) := \{i \in I \mid \xi_i = \kappa\} \), and \( x_+ := (\max\{\xi_i, 0\})_{i \in I} \). Then
\[
P_C x = \begin{cases} 
\left\{ \frac{1}{\|x_+\|} x_+ \right\}, & \text{if } \kappa > 0; \\
\left\{ \sum_{i \in I(x)} a_i e_i \mid \{a_i\}_{i \in I(x)} \subseteq \mathbb{R}_+, \sum_{i \in I(x)} a_i^2 = 1 \right\}, & \text{if } \kappa = 0; \\
\{e_i\}_{i \in I(x)} \setminus \{e_i\}_{i \in I(x)}, & \text{if } \kappa < 0.
\end{cases}
\]

**Proof.** Because \( (\forall i \in I) \langle x | e_i \rangle = \xi_i \) and \( \|x_+\|^2 = \sum_{i \in I} (\max\{\xi_i, 0\})^2 \), (4.102) therefore follows from Example 4.26.

### 4.6 Further examples

In this section, we provide further examples based on the Lorentz cone and on the cone of positive semidefinite matrices.

**Example 4.28** Let \( a \) and \( \rho \) be in \( \mathbb{R}_+ \), let
\[
K_a = \{(x, \xi) \in \mathcal{H} \oplus \mathbb{R} \mid \|x\| \leq a \xi\}
\]

(4.103)
be the Lorentz cone of parameter \(\alpha\) of Example 3.23, set \(C := K_\alpha \cap S(0; \rho)\), and let \(x = (x, \xi) \in \mathcal{H}\). Then

\[
P_{C}x = \begin{cases} 
  \Big\{ \frac{\rho}{\|x\|}x \Big\}, & \text{if } \|x\| \leq \alpha \xi \text{ and } \xi > 0; \\
  \Big\{ \frac{\rho}{\sqrt{1 + \alpha^2}} \left( \frac{\alpha x}{\|x\|} \right) \Big\}, & \text{if } \|x\| > \max\{\alpha \xi, -\xi / \alpha\} \\
  S(0; \beta) \times \{\beta / \alpha\}, & \text{if } x = 0 \text{ and } \xi < 0; \\
  C, & \text{if } (x, \xi) = (0, 0).
\end{cases}
\] (4.104)

Proof. Set

\[
\beta := \frac{\rho \alpha}{(1 + \alpha^2)^{1/2}} \in \mathbb{R}_{++},
\] (4.105)

\(C_{a, \beta} := S(0; \beta) \times \{\beta / \alpha\}\), and \(\kappa := \max\langle x \mid C_{a, \beta} \rangle\). Then it is readily verified that

\[
(\forall y \in C_{a, \beta}) \quad \|y\| = \rho,
\] (4.106)

and due to Lemma 2.4,

\[
\kappa = \beta \|x\| + \xi \beta / \alpha.
\] (4.107)

Furthermore, by Example 3.23,

\[
K_\alpha = \text{pos} \ C_{a, \beta} = \text{cone}(\text{conv} \ C_{a, \beta}) \cup \{0\},
\] (4.108)

and by Example 4.22 (applied to \(C_{a, \beta}\)), we have

\[
\emptyset \neq P_{C_{a, \beta}}x = \begin{cases} 
  C_{a, \beta}, & \text{if } x = 0; \\
  \Big\{ \left( \frac{\beta}{\|x\|}, \frac{\beta}{\alpha} \right) \Big\}, & \text{otherwise}.
\end{cases}
\] (4.109)

Let us now identify \(P_{C}x\) in the following conceivable cases:

(a) \(\|x\| > -\xi / \alpha\): Then \(\kappa > 0\) by (4.107), and so by (4.108) and Lemma 3.22(i), \(x \in \mathcal{H} \setminus K_\alpha^\circ\). In turn, it follows from Theorem 4.19(ii) (applied to \(C = K_\alpha \cap S(0; \rho)\)) that

\[
P_{C}x = \left\{ \frac{\rho}{\|P_{K_\alpha}x\|} P_{K_\alpha}x \right\}.
\] (4.110)

To evaluate \(P_{C}x\) further, we consider two subcases:
4.6. Further examples

(a.1) $\|x\| \leq \alpha \tilde{\xi}$: Then $x \in K_\alpha$ by (4.103), and so $P_{K_\alpha}x = x$, which yields $P_{C}x = \{ (\rho/\|x\|)x \}$. 

(a.2) $\|x\| > \alpha \tilde{\xi}$: Then, according to Example 3.13, 

$$P_{K_\alpha}x = P_{K_\alpha}(x, \xi) = \frac{\alpha \|x\| + \xi}{1 + \alpha^2} \left( \frac{\alpha x}{\|x\|}, 1 \right),$$

and since $\alpha \|x\| + \xi > 0$, it follows that 

$$\|P_{K_\alpha}x\| = \frac{\alpha \|x\| + \xi}{1 + \alpha^2} \left( \frac{\alpha x}{\|x\|}, 1 \right) = \frac{\alpha \|x\| + \xi}{1 + \alpha^2} \sqrt{\frac{\alpha^2 \|x\|^2}{\|x\|^2} + 1}$$

Hence, combining (4.110)&(4.111)&(4.112), we get 

$$P_{C}x = \left\{ \frac{\rho}{\sqrt{1 + \alpha^2}} \left( \frac{\alpha x}{\|x\|}, 1 \right) \right\}.$$ 

(b) $\|x\| = -\tilde{\xi}/\alpha$: Then $\kappa = 0$ by (4.107), and invoking (4.106)&(4.108)&(4.109), Theorem 4.23(ii)(c) asserts that 

$$P_{C}x = S(0; \rho) \cap \text{cone}(\text{conv } P_{C_\alpha, \beta}x).$$

We consider two subcases:

(b.1) $x = 0$: Then $\tilde{\xi} = 0$ and so $x = (x, \tilde{\xi}) = 0$. Moreover, due to (4.109), $P_{C_\alpha, \beta}x = C_\alpha, \beta$. Therefore, by (4.108) and (4.114), 

$$C = K_\alpha \cap S(0; \rho)$$

$$= (\text{cone}(\text{conv } C_\alpha, \beta) \cup \{0\}) \cap S(0; \rho)$$

$$= \text{cone}(\text{conv } C_\alpha, \beta) \cap S(0; \rho)$$

$$= \text{cone}(\text{conv } P_{C_\alpha, \beta}x) \cap S(0; \rho)$$

(b.2) $x \neq 0$: Then (4.109) yields $P_{C_\alpha, \beta}x = \{ (\beta x/\|x\|, \beta/\alpha) \}$. In turn, since $\| (\beta x/\|x\|, \beta/\alpha) \| = \rho$ by (4.105) and a simple computation, we obtain from (4.114) and Fact 3.19(i) that 

$$P_{C}x = S(0; \rho) \cap \text{cone}(\text{conv } P_{C_\alpha, \beta}x)$$
4.6. Further examples

\[ = S(0; \rho) \cap \left( \mathbb{R}_{++} \left( \frac{\beta x}{\|x\|}, \frac{\beta}{\alpha} \right) \right) \]  
(4.116b)

\[ = \left\{ \left( \frac{\beta x}{\|x\|}, \frac{\beta}{\alpha} \right) \right\} \]  
(4.116c)

\[ = \left\{ \frac{\beta}{\alpha} \left( \frac{ax}{\|x\|}, 1 \right) \right\} \]  
(4.116d)

\[ = \left\{ \frac{\rho}{\sqrt{1 + \alpha^2}} \left( \frac{ax}{\|x\|}, 1 \right) \right\}. \]  
(4.116e)

(c) \( \|x\| < -\xi/\alpha \): Then \( \kappa < 0 \) by (4.107), and so, in view of (4.106)&(4.108)&(4.109), we deduce from Theorem 4.23(ii)(b) that \( P_{C}x = P_{C_{a,\beta}}x \). Hence, by (4.109) and (4.105), we get

\[ P_{C}x = \begin{cases} 
C_{a,\beta}, & \text{if } x = 0; \\
\left\{ \left( \frac{\beta}{\|x\|}, \frac{\beta}{\alpha} \right) \right\}, & \text{if } x \neq 0
\end{cases} \]  
(4.117a)

\[ = \begin{cases} 
C_{a,\beta}, & \text{if } x = 0; \\
\left\{ \frac{\rho}{\sqrt{1 + \alpha^2}} \left( \frac{ax}{\|x\|}, 1 \right) \right\}, & \text{if } x \neq 0.
\end{cases} \]  
(4.117b)

To sum up, we have shown that

\[ P_{C}x = \begin{cases} 
\left\{ \frac{\rho}{\|x\|} x \right\}, & \text{if } -\xi/\alpha < \|x\| \leq a\xi; \\
\left\{ \frac{\rho}{\sqrt{1 + \alpha^2}} \left( \frac{ax}{\|x\|}, 1 \right) \right\}, & \text{if } \|x\| > \max\{a\xi, -\xi/\alpha\} \\
C_{a,\beta}, & \text{if } x = 0 \text{ and } 0 < -\xi; \\
C, & \text{if } (x, \xi) = (0, 0)
\end{cases} \]  
(4.118a)

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4.6. Further examples

\[
\begin{cases}
\left\{ \frac{\rho}{\|x\|} x \right\}, & \text{if } \|x\| \leq a\xi \text{ and } \xi > 0; \\
\left\{ \frac{\rho}{\sqrt{1 + a^2 \|x\|^2}} \left( \frac{\alpha x}{\|x\|} , 1 \right) \right\}, & \text{if } \|x\| > \max\{a\xi, -\xi/a\} \\
S(0; \beta) \times \{\beta/\alpha\}, & \text{if } x = 0 \text{ and } \xi < 0; \\
C, & \text{if } (x, \xi) = (0, 0),
\end{cases}
\]

as announced in (4.104).

In the remaining of this section, \(N\) is a strictly positive integer, and suppose that \(\mathcal{H} = S^N\) is the Hilbert space of real symmetric matrices endowed with the scalar product \(\langle \cdot, \cdot \rangle\): \((A, B) \mapsto \text{tr}(AB)\), where \(\text{tr}\) is the trace function; the associated norm is the Frobenius norm \(\| \cdot \|_F\). The closed convex cone of positive semidefinite symmetric matrix in \(\mathcal{H}\) is denoted by \(S^+_N\), and the set of orthogonal matrices of size \(N \times N\) is \(U_N = \{U \in \mathbb{R}^{N \times N} \mid UU^T = \text{Id}\}\), where \(\text{Id}\) is the identity matrix of \(\mathbb{R}^{N \times N}\). Next, for every \(x = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N\), set \(x_+ := (\max\{\xi_i, 0\})_{1 \leq i \leq N}\) and define \(\text{Diag} x\) to be the diagonal matrix whose, starting from the upper left corner, diagonal entries are \(\xi_1, \ldots, \xi_N\). Now, for every \(A \in \mathcal{H}\), the eigenvalues of \(A\) (not necessarily distinct) are denoted by \((\lambda_i(A))_{1 \leq i \leq N}\) with the convention that \(\lambda_1(A) \geq \cdots \geq \lambda_N(A)\). In turn, the mapping \(\lambda: \mathcal{H} \to \mathbb{R}^N: A \mapsto (\lambda_1(A), \ldots, \lambda_N(A))\) is well defined. Finally, the Euclidean scalar product and norm of \(\mathbb{R}^N\) are respectively denoted by \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\).

**Remark 4.29** Let \(A \in \mathcal{H}\), \(U \in U_N\), and \(x \in \mathbb{R}^N\). Then it is straightforward to verify that

\[
\|UAU^T\|_F = \|A\|_F = \|\lambda(A)\|
\]

and that

\[
\|U(\text{Diag} x)U^T\|_F = \|\text{Diag} x\|_F = \|x\|.
\]

**Lemma 4.30** Set \(K := S^+_N\). Let \(A \in \mathcal{H}\), and let \(U \in U_N\) be such that \(A = U(\text{Diag} \lambda(A))U^T\). Then \(P_KA = U(\text{Diag}(\lambda(A))_+)U^T\) and \(\|P_KA\|_F = \|\lambda(A)_+\|\).

**Proof.** It is well known that \(P_KA = U(\text{Diag}(\lambda(A))_+)U^T\) (see, e.g., [29, Theorem A1] or [8, Example 29.32]). In turn, since \(U \in U_N\), it follows from Remark 4.29 that \(\|P_KA\|_F = \|\lambda(A)_+\|\).
4.6. Further examples

Fact 4.31 (Theobald) (See [35].) Let $A$ and $B$ be in $\mathcal{H}$. Then the following hold:

(i) $\langle A \mid B \rangle \leq \langle \lambda(A) \mid \lambda(B) \rangle$.
(ii) $\langle A \mid B \rangle = \langle \lambda(A) \mid \lambda(B) \rangle$ if and only if there exists $U \in \mathbb{U}^N$ such that $A = U(\text{Diag } \lambda(A))U^T$ and $B = U(\text{Diag } \lambda(B))U^T$.

Lemma 4.32 Let $\rho \in \mathbb{R}_{++}$, and set

$$ C_\rho := \left\{ A \in S_+^N \mid \text{rank } A = 1 \text{ and } \| A \|_F = \rho \right\}. \quad (4.121) $$

Then the following hold:

(i) $S_+^N = \text{pos } C_\rho$.
(ii) $C_\rho = \{ A \in \mathcal{H} \mid (\exists U \in \mathbb{U}^N) \ A = U(\text{Diag}(\rho, 0, \ldots, 0))U^T \}$.
(iii) Let $A \in \mathcal{H}$. Then $\max \langle A \mid C_\rho \rangle = \rho \lambda_1(A)$ and

$$ P_{C_\rho} A 
\quad = \left\{ U(\text{Diag}(\rho, 0, \ldots, 0))U^T \mid U \in \mathbb{U}^N, A = U(\text{Diag } \lambda(A))U^T \right\} \quad (4.122a) $$

$$ \neq \emptyset. \quad (4.122b) $$

Proof. (i): Set $I := \{1, \ldots, N\}$, and let $(e_i)_{i \in I}$ be the canonical orthonormal basis of $\mathbb{R}^N$. First, since $C_\rho \cup \{0\} \subseteq S_+^N$ and $S_+^N$ is a convex cone, we infer from Lemma 3.22(iii) that $\text{pos } C_\rho \subseteq S_+^N$. Conversely, take $A \in S_+^N$, and let $U \in \mathbb{U}^N$ be such that $A = U(\text{Diag } \lambda(A))U^T$; in addition, set $(\forall i \in I) D_i := \text{Diag}(\rho e_i) \in S_+^N$. Then, for every $i \in I$, since rank $D_i = 1$ and $\| UD_i U^T \|_F = \| D_i \|_F = \| \rho e_i \| = \rho$, we get from (4.121) that $UD_i U^T \in C_\rho$. In turn, because $\{\lambda_i(A)\}_{i \in I} \subseteq \mathbb{R}_+$ and

$$ A = U(\text{Diag } \lambda(A))U^T \quad (4.123a) $$

$$ = U \left( \sum_{i \in I} \text{Diag} (\lambda_i(A) e_i) \right) U^T \quad (4.123b) $$

$$ = U \left( \sum_{i \in I} \frac{\lambda_i(A)}{\rho} D_i \right) U^T \quad (4.123c) $$

$$ = \sum_{i \in I} \frac{\lambda_i(A)}{\rho} (UD_i U^T), \quad (4.123d) $$

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we deduce that $A \in \text{pos } C_\rho$. Hence, $S^N_+ = \text{pos } C_\rho$.

(ii): Recall that, if $A$ is a matrix of rank $r$ in $S^N_+$, then
\[
\lambda_1(A) \geq \cdots \geq \lambda_r(A) > \lambda_{r+1}(A) = \cdots = \lambda_N(A) = 0. \tag{4.124}
\]

Now set $D := \{A \in \mathcal{H} \mid (\exists U \in \mathbb{U}^N) A = U(\text{Diag}(\rho, 0, \ldots, 0)) U^T\}$. First, take $A \in C_\rho$, and let $U \in \mathbb{U}^N$ be such that $A = U(\text{Diag} \lambda(A)) U^T$. Then, since rank $A = 1$ and $A \in S^N_+$, it follows from (4.124) that $\lambda(A) = (\lambda_1(A), 0, \ldots, 0) \land \lambda_1(A) > 0$; therefore, because $\|A\|_F = \rho$, we obtain $\rho = \|A\|_F = \|\lambda(A)\| = \lambda_1(A)$. Hence, $A = U(\text{Diag}(\lambda_1(A), 0, \ldots, 0)) U^T = U(\text{Diag}(\rho, 0, \ldots, 0)) U^T$, which yields $A \in D$. Conversely, take $B \in D$, say $B = V(\text{Diag}(\rho, 0, \ldots, 0)) V^T$, where $V \in \mathbb{U}^N$. Then, since $\rho > 0$, we have $B \in S^N_+$. Next, on the one hand, because $V$ is nonsingular and $\rho \neq 0$, we have rank $B = \text{rank } \text{Diag}(\rho, 0, \ldots, 0) = 1$. On the other hand, since $V \in \mathbb{U}^N$, it follows that $\|B\|_F = \|V(\text{Diag}(\rho, 0, \ldots, 0)) V^T\|_F = \|(\rho, 0, \ldots, 0)\| = \rho$. Altogether, $B \in C_\rho$, which completes the proof.

(iii): First, it follows from (ii) that
\[
(\forall B \in \mathcal{H}) \quad B \in C_\rho \iff \lambda(B) = (\rho, 0, \ldots, 0). \tag{4.125}
\]

Next, denote the right-hand set of (4.122) by $D$. Then, by (ii), $\emptyset \neq D \subseteq C_\rho$. Now, for every $B \in C_\rho$, since $\lambda(B) = (\rho, 0, \ldots, 0)$, we infer from Fact 4.31(i) that $\langle A \mid B \rangle \leq \langle \lambda(A) \mid \lambda(B) \rangle = \rho \lambda_1(A)$. Thus, $\sup \langle A \mid C_\rho \rangle \leq \rho \lambda_1(A)$. Furthermore, by (4.125), Fact 4.31(iii), and the very definition of $D$, we see that, for every $B \in C_\rho$,
\[
\langle A \mid B \rangle = \rho \lambda_1(A) \iff \langle A \mid B \rangle = \langle \lambda(A) \mid \lambda(B) \rangle \quad \tag{4.126a}
\]
\[
\quad \iff (\exists U \in \mathbb{U}^N) \begin{cases} A = U(\text{Diag}(\lambda(A))) U^T, \\ B = U(\text{Diag}(\lambda(B))) U^T \end{cases} \tag{4.126b}
\]
\[
\quad \iff (\exists U \in \mathbb{U}^N) \begin{cases} A = U(\text{Diag}(\lambda(A))) U^T, \\ B = U(\text{Diag}(\rho, 0, \ldots, 0)) U^T \end{cases} \tag{4.126c}
\]
\[
\quad \iff B \in D. \tag{4.126d}
\]

Therefore, because $D \neq \emptyset$, we deduce that $\max(A \mid C_\rho) = \rho \lambda_1(A)$ and $\forall B \in C_\rho \quad \langle A \mid B \rangle = \max(A \mid C_\rho) \iff B \in D$. Consequently, since the matrices in $C_\rho$ are of equal norm by (4.121), we derive from Lemma 3.9(i) that $P_{C_\rho} A = D$, as desired. \hfill \blacksquare

**Example 4.33** Set $K := S^N_+$, let $\rho \in \mathbb{R}_{++}$, and set $C := K \cap S(0; \rho)$. In addition, let $A \in \mathcal{H}$, and let $U \in \mathbb{U}^N$ be such that $A = U(\text{Diag} \lambda(A)) U^T$;
4.6. Further examples

set

\[ D := \{ V(\text{Diag}(\rho, 0, \ldots, 0))V^T \mid V \in \mathbb{U}^N, A = V(\text{Diag}(A))V^T \} \]  \hspace{1cm} (4.127)

and

\[ E := S(0; \rho) \cap \text{cone}(\text{conv } D). \]  \hspace{1cm} (4.128)

Then

\[
P_C A = \begin{cases} 
\left\{ \frac{\rho}{\|\text{Diag}(A)\|} U(\text{Diag}(\lambda(A)))U^T \right\}, & \text{if } \lambda_1(A) > 0; \\
E, & \text{if } \lambda_1(A) = 0; \\
D, & \text{if } \lambda_1(A) < 0.
\end{cases} \]  \hspace{1cm} (4.129)

Proof. Set

\[
C_\rho := \left\{ B \in S^N_+ \mid \text{rank } B = 1 \text{ and } \|B\|_F = \rho \right\}. \]  \hspace{1cm} (4.130)

It then follows from Lemma 4.32(iii) that

\[
\max \langle A \mid C_\rho \rangle = \rho \lambda_1(A) \quad \text{and} \quad P_{C_\rho} A = D. \]  \hspace{1cm} (4.131)

Let us now consider all conceivable cases:

(a) \( \lambda_1(A) > 0 \): Then \( \max \langle A \mid C_\rho \rangle > 0 \), and thus, by Lemma 4.32(i) and Lemma 3.22(i), we obtain \( A \in \mathcal{H} \setminus K^\ominus \). Therefore, since \( \{0\} \neq K \) is a nonempty closed convex cone, we infer from Theorem 4.19(ii) and Lemma 4.30 that

\[
P_C A = \left\{ \frac{\rho}{\|P_K A\|_F} P_K A \right\} = \left\{ \frac{\rho}{\|\text{Diag}(\lambda(A))\|} U(\text{Diag}(\lambda(A)))U^T \right\}. \]  \hspace{1cm} (4.132)

(b) \( \lambda_1(A) \leq 0 \): Then \( \max \langle A \mid C_\rho \rangle \leq 0 \). Since \( (\forall B \in C_\rho) \|B\|_F = \rho \) and, by Lemma 4.32(i), \( K = \text{pos } C_\rho \), it follows from Theorem 4.23(ii)(b)&(ii)(c) and (4.131) that

\[
P_C A = \begin{cases} 
P_{C_\rho} A, & \text{if } \max \langle A \mid C_\rho \rangle < 0; \\
S(0; \rho) \cap \text{cone}(\text{conv } P_{C_\rho} A), & \text{if } \max \langle A \mid C_\rho \rangle = 0
\end{cases} \]  \hspace{1cm} (4.133a)

\[ = \begin{cases} 
D, & \text{if } \lambda_1(A) < 0; \\
E, & \text{if } \lambda_1(A) = 0, \\
D, & \text{if } \lambda_1(A) < 0.
\end{cases} \]  \hspace{1cm} (4.133b)

which completes the proof. ■
4.7. Copositive matrices: a numerical experiment

**Remark 4.34** Consider the setting of Example 4.33. Since
\[ U(\text{Diag}(\rho, 0, \ldots, 0))U^T \in D \subseteq E, \] (4.134)
the mapping
\[ A \mapsto \begin{cases} \frac{\rho}{\|\lambda(A)\|_{+}} U(\text{Diag}(\lambda(A))_{+})U^T, & \text{if } \lambda_1(A) > 0; \\ U(\text{Diag}(\rho, 0, \ldots, 0))U^T, & \text{otherwise} \end{cases} \] (4.135)
is a selection of \( P_C \).

**4.7 Copositive matrices: a numerical experiment**

In this final section, \( N \) is a strictly positive integer and \( M \) is a symmetric matrix in \( \mathbb{R}^{N \times N} \). Recall that \( M \) is copositive if \( (\forall x \in \mathbb{R}_+^N) \langle x \mid Mx \rangle \geq 0; \) or, equivalently,
\[ \mu(M) := \min_{x \in \mathbb{R}_+^N \cap S(0;1)} \frac{1}{2} \langle x \mid Mx \rangle \geq 0. \] (4.136)
For further information on copositive matrices, we refer the reader to the surveys [18, 25] and references therein. In view of (4.136), testing copositivity of \( M \) amounts to
\[ \minimize_{x \in \mathbb{R}_+^N \cap S(0;1)} \frac{1}{2} \langle x \mid Mx \rangle. \] (4.137)
Now, set \( C := \mathbb{R}_+^N \cap S(0;1) \), set \( f : \mathbb{R}^N \to \mathbb{R} : x \mapsto (1/2) \langle x \mid Mx \rangle \), and set \( g := i_C \) which is the indicator function of \( C \). Note that neither \( f \) nor \( g \) is convex; however, \( \nabla f \) is Lipschitz continuous with the operator norm \( \|M\| \) (computed as the largest singular value of \( M \)) being a suitable Lipschitz constant. The projection onto \( C \) is computed using (4.102). In turn, (4.137) can be written as
\[ \minimize_{x \in \mathbb{R}^N} f(x) + g(x). \] (4.138)
To solve this problem, we compared the **Fast Iterative Shrinkage-Thresholding Algorithm (FISTA)** (see [13]), the **Projected Gradient Method (PGM)** (see [2, 14]), the algorithm presented in [28, Example 5.5.2] by Lange, the **Douglas–Rachford Algorithm (DRA)** variant presented in [30] by Li and Pong, and the regular DRA for solving (4.138) when \( N \in \{2, 3, 4\} \). For each \( N \in \{2, 3, 4\} \), using the copositivity criteria for matrices of order up to four

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4.7. Copositive matrices: a numerical experiment

(see, e.g., [19, 33]), we randomly generate 100 copositive matrices (group A) together with 100 non-copositive (group B) ones with i.i.d. Gaussian entries and run the mentioned algorithms on those matrices. For each algorithm, if \((x_n)_{n \in \mathbb{N}}\) is the sequence generated, then we terminate the algorithm when

\[
\frac{\|x_n - x_{n-1}\|}{\max\{\|x_{n-1}\|, 1\}} < 10^{-8}. \tag{4.139}
\]

The maximum allowable number of iterations is 1000. For each matrix \(M\) in group A (respectively, group B), we declare success if \(\mu(M) \geq 0\) (respectively, \(\mu(M) < 0\)). We also record the average of the number of iterations until success of each algorithm. The results, obtained using \texttt{Matlab}, are reported in Table 4.1.

<table>
<thead>
<tr>
<th>Size</th>
<th>Copositive</th>
<th>FISTA</th>
<th>PGM</th>
<th>Lange</th>
<th>Li–Pong</th>
<th>DR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>succ</td>
<td>iter</td>
<td>succ</td>
<td>iter</td>
<td>succ</td>
</tr>
<tr>
<td>2 x 2</td>
<td>Yes</td>
<td>100</td>
<td>5</td>
<td>100</td>
<td>12</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>97</td>
<td>15</td>
<td>99</td>
<td>12</td>
<td>91</td>
</tr>
<tr>
<td>3 x 3</td>
<td>Yes</td>
<td>100</td>
<td>27</td>
<td>100</td>
<td>24</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>96</td>
<td>30</td>
<td>98</td>
<td>24</td>
<td>86</td>
</tr>
<tr>
<td>4 x 4</td>
<td>Yes</td>
<td>88</td>
<td>33</td>
<td>87</td>
<td>30</td>
<td>91</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>97</td>
<td>45</td>
<td>94</td>
<td>41</td>
<td>91</td>
</tr>
</tbody>
</table>

Table 4.1: Detecting whether a matrix is copositive using a variety of algorithms. Here “iter” means “the average of the numbers of iterations.”

Finally, let us apply the algorithms to the well-known Horn matrix

\[
H := \begin{bmatrix}
1 & -1 & 1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1
\end{bmatrix}, \tag{4.140}
\]

which is copositive with \(\mu(H) = 0\) (see [20, Equation (3.5)]). For each algorithm, we record the number of iterations and the value of \(f\) at the point that the algorithm is terminated. The results are recorded in Table 4.2.
4.7. Copositive matrices: a numerical experiment

<table>
<thead>
<tr>
<th></th>
<th>FISTA</th>
<th>PGM</th>
<th>Lange</th>
<th>Li–Pong</th>
<th>DR</th>
</tr>
</thead>
<tbody>
<tr>
<td>fval</td>
<td>3.5230e−17</td>
<td>2.8297e−20</td>
<td>2.9979e−07</td>
<td>1.4912e−14</td>
<td>0.0584</td>
</tr>
<tr>
<td>iter</td>
<td>11</td>
<td>10</td>
<td>95</td>
<td>170</td>
<td>13</td>
</tr>
</tbody>
</table>

Table 4.2: Detecting copositivity of the Horn matrix.

We acknowledge that these algorithms might get stuck at points that are not solutions and that the outcome might depend on the starting points; moreover, a detailed complexity analysis is absent. There are thus various research opportunities to improve the current results. Nonetheless, our preliminary results indicate that FISTA and PGM are potentially significant contenders for numerically testing copositivity.
Chapter 5

On the sum of projectors onto convex sets

5.1 Overview

We assume that
\[(C_i)_{i \in I}\] is a finite family of nonempty closed convex subsets of \(H\) \hspace{1cm} (5.1)
with corresponding projectors
\[(P_{C_i})_{i \in I} \hspace{1cm} (5.2)\]

In this chapter, we analyze carefully the question: When is the sum of projectors also a projector? (In view of Proposition 5.1(iii), an affirmative answer to this question requires the sum \(\sum_{i \in I} C_i\) to be closed. This happens, for instance, when each set is bounded.) It is known that, in the case of linear subspaces, \(\sum_{i \in I} P_{C_i}\) is a projector onto a closed linear subspace if and only if the sets \((C_i)_{i \in I}\) are pairwise orthogonal; see [21, Theorem 2, p. 46]. This question is also of interest in Quantum Mechanics [27, p. 50].

In 1971, Zarantonello [39] answered this question in the case of convex cones: \(\sum_{i \in I} P_{C_i}\) is a projector if and only if the projectors \((P_{C_i})_{i \in I}\) are pairwise orthogonal in the sense that, for every \((i, j) \in I \times I\) with \(i \neq j\), we have \((\forall x \in H) \langle P_{C_i}x \mid P_{C_j}x \rangle = 0\). However, the question remains open in the general convex case. One goal of this chapter is to provide necessary and sufficient conditions for \(\sum_{i \in I} P_{C_i}\) to be a projector without any further assumption on the sets \((C_i)_{i \in I}\). This allows us to unify the two aforementioned results and make a connection with the work [4] where it was proved that, if the sum of a family of proximity operators is a proximity operator, then every partial sum remains a proximity operator. Interestingly, we shall see that this property is still valid in the class of projectors onto convex cones; in other words, if a finite sum of projectors onto convex cones is a projector, then so are its partial sums. Nevertheless,
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this result fails outside the world of convex cones. Our main results are summarized as follows:

- We provide a characterization of projectors onto convex sets (Theorem 5.6). This result is a pillar of this chapter and a variant of [39, Theorem 4.1]. Furthermore, we also partially answer an open question by Zarantonello regarding [39, Theorem 4.1].

- Theorem 5.12 provides a necessary and sufficient condition (without any additional assumptions on the underlying sets) under which $\sum_{i \in I} P_{C_i}$ is a projector.

- We present the partial sum property (see [4, Theorem 4.2]) for projectors onto convex cones in Theorem 5.27, whose proof is based on Theorem 5.23 and [4, Theorem 4.2]. We also recover [39, Theorems 5.3 and 5.5].

This chapter is organized as follows. In Section 5.2, we collect miscellaneous results that will be used in the sequel. Our main results are presented in Section 5.3: Theorem 5.6 provides a characterization of projectors, which is a variant of [39, Theorem 4.1] and allows us to recover the classical characterization of orthogonal projectors; see, e.g., [36, Theorem 4.29]. In turn, we shall use Theorem 5.6 to establish a necessary and sufficient condition for a finite sum of projectors to be a projector (Theorem 5.12). In Section 5.4, we show that this condition covers the result obtained by Zarantonello ([39, Theorem 5.5]) and the case of linear subspaces. Furthermore, we provide Theorem 5.23 and Theorem 5.27 to illustrate the connection between our work and [4, 39]. The one-dimensional case is the topic of Section 5.5, where all the pairs $(C, D)$ of nonempty closed convex subsets of $\mathbb{R}$ satisfying $P_{C} + P_{D} = P_{C+D}$ are explicitly determined. Finally, we turn to a generalization of the classical result [21, Theorem 2, p. 46] in Section 5.6. Various examples are given to illustrate the necessity of our assumptions.

5.2 Auxiliary results

Hereinafter, it is convenient to set

\[ q := \frac{1}{2} \| \cdot \|^2, \] (5.3)

and note that $\nabla q = \text{Id}$ is the identity operator on $\mathcal{H}$. 57
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In the finite-dimensional case, Proposition 5.1(ii) can be deduced from [11, Theorem 3.15]. Furthermore, let us point out that Proposition 5.1(iii) generalizes Zarantonello’s [39, Theorem 5.4].

**Proposition 5.1** Let \( m \geq 2 \) be an integer, set \( I := \{1, \ldots, m\} \), and let \((C_i)_{i \in I}\) be a family of nonempty closed convex subsets of \( \mathcal{H} \). Then the following hold:

(i) For every \( x \in \mathcal{H} \),

\[
\frac{1}{2} \left\| x - \sum_{i \in I} P_{C_i} x \right\|^2 = \sum_{i \in I} \frac{1}{2} d_{C_i}^2(x) - (m - 1)q(x) + \sum_{i \neq j} (\langle P_{C_i} x \mid P_{C_j} x \rangle).
\]  

(ii) \( \text{ran} \sum_{i \in I} P_{C_i} = \sum_{i \in I} C_i \).

(iii) Suppose that there exists a closed convex set \( D \) such that \( \sum_{i \in I} P_{C_i} = P_D \). Then \( \sum_{i \in I} C_i \) is closed and \( D = \sum_{i \in I} C_i \).

**Proof.** For brevity, set \( (\forall i \in I) \ P_i := P_{C_i} \) and \( d_i := d_{C_i} \).

(i): Apply Lemma 2.3(ii) to \( (x_i)_{i \in I} = (P_i x)_{i \in I} \) and note that \((\forall i \in I) \left\| x - P_i x \right\| = d_i(x)\).

(ii): Let us verify the claim by induction on \( m \). Assume first that \( m = 2 \). On the one hand, for every \( i \in \{1, 2\} \), Example 3.58 asserts that \( P_i \) is \( 3^* \) monotone. On the other hand, since \( P_1 \) and \( P_2 \) are monotone and continuous, so is \( P_1 + P_2 \), and therefore \( P_1 + P_2 \) is maximally monotone by Fact 3.55. Altogether, the Brézis–Haraux theorem (Fact 3.60) implies that \( \text{ran}(P_1 + P_2) = \text{ran} P_1 + \text{ran} P_2 = C_1 + C_2 \). Now assume that \( m \geq 3 \), that the conclusion holds for families containing \( m - 1 \) sets, and set \( T := \sum_{i=1}^{m-1} P_i \), which is clearly monotone and continuous. Then, since \( \{P_i\}_{i \in I} \) are \( 3^* \) monotone by Example 3.58, we infer from Fact 3.59 that \( T = \sum_{i=1}^{m-1} P_i \) is \( 3^* \) monotone. However, since \( T \) and \( P_m \) are monotone and continuous, so is their sum \( T + P_m \), and thus \( T + P_m \) is maximally monotone by Fact 3.55. Therefore, because \( P_m \) is \( 3^* \) monotone by Example 3.58, we derive from the Brézis–Haraux theorem (Fact 3.60) and Lemma 2.5 that \( \text{ran} \sum_{i \in I} P_i = \text{ran}(T + P_m) = \text{ran} T + \text{ran} P_m = \text{ran} T + \text{ran} P_m \). Consequently, by the induction hypothesis and Lemma 2.5,

\[
\text{ran} \sum_{i \in I} P_i = \text{ran} T + \text{ran} P_m = \sum_{i=1}^{m-1} C_i + C_m = \sum_{i=1}^{m-1} C_i + C_m = \sum_{i \in I} C_i,
\]  

(5.5)

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which concludes the induction argument. Alternatively, apply [12, Lemma 3.1(ii)] with \((A_i)_{i \in I} = (P_i)_{i \in I}\).

(iii): It follows from (ii) and our assumption that \(\sum_{i \in I} C_i \subseteq \sum_{i \in I} C_i = \text{ran} \sum_{i \in I} P_i = D = \text{ran} P = \text{ran} \sum_{i \in I} P_i \subseteq \sum_{i \in I} C_i\). Thus, we conclude that \(\sum_{i \in I} C_i = D\) and that \(\sum_{i \in I} C_i\) is closed.

Recall from [31, pp. 89–90] that, if \(f : H \rightarrow \mathbb{R}_{\geq 0}\), then the Fréchet subdifferential of \(f\) is

\[
\hat{\partial} f : H \rightarrow 2^H : x \mapsto \left\{ u \in H \mid \lim_{x \neq y \rightarrow x} \frac{f(y) - f(x) - \langle u, y - x \rangle}{\|y - x\|} \geq 0 \right\}. \tag{5.6}
\]

Lemma 5.2 Let \(f : H \rightarrow \mathbb{R}\) and \(z \in H\). Suppose that \(f\) is Fréchet differentiable on \(H\). Then

\[
(\forall \epsilon \in \mathbb{R}_{++}) \quad \hat{\partial} (f + \epsilon d_z) (z) = \nabla f(z) + \epsilon B(0; 1). \tag{5.7}
\]

Proof. Fix \(\epsilon \in \mathbb{R}_{++}\). Since \(f\) is Fréchet differentiable and \(d_z\) is convex, we derive from [31, Proposition 1.107 and Theorem 1.93] that \(\hat{\partial} (g + \epsilon d_z) (z) = \nabla g(z) + \hat{\partial} d_z (z) = \nabla g(z) + \epsilon \hat{\partial} d_z (z)\). Hence, in view of [8, Example 16.62] (applied to \(C = \{z\}\)), (5.7) follows.

5.3 Main results

Proposition 5.3 Let \(T : H \rightarrow H\), and set \(f := q \circ (\text{Id} - T)\). Suppose that \(f\) is Fréchet differentiable on \(H\) with \(\nabla f = \text{Id} - T\). Then \(\text{Fix} T \neq \emptyset\).

Proof. Let us proceed by contradiction and therefore assume that \(\text{Fix} T = \emptyset\). Then clearly \(\forall x \in H\) \(f(x) > 0\). Hence, because \(f : H \rightarrow \mathbb{R}_{++}\) is Fréchet differentiable with \(\nabla f = \text{Id} - T\) and \(\sqrt{\cdot} : \mathbb{R}_{++} \rightarrow \mathbb{R}\) is Fréchet differentiable, we deduce from [17, Theorem 5.1.11(b)] that \(g := \sqrt{\cdot} \circ (2f)\) is Fréchet differentiable on \(H\) (thus continuous) and

\[
(\forall x \in H) \quad \nabla g(x) = \frac{2 \nabla f(x)}{2 \sqrt{2f(x)}} = \frac{x - Tx}{\|x - Tx\|}. \tag{5.8}
\]

Now let \(\epsilon \in [0, 1]\). Since \(g\) is bounded below and continuous, Ekeland’s variational principle (see, e.g., [8, Theorem 1.46(iii)]) applied to \(g\) and \((\alpha, \beta) = (\epsilon^2, \epsilon)\) yields the existence of \(z \in H\) such that \((\forall x \in H \setminus \)
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\( \{z\} \) \( g(z) + \varepsilon d_{\{z\}}(z) = g(z) < g(x) + \varepsilon d_{\{z\}}(x) \). This guarantees that \( z \) is the unique minimizer of \( g + \varepsilon d_{\{z\}} \). Thus, [31, Proposition 1.114], Lemma 5.2, and (5.8) imply that

\[
0 \in \hat{\partial}(g + \varepsilon d_{\{z\}})(z) = \nabla g(z) + \varepsilon B(0; 1) = \frac{z - Tz}{\|z - Tz\|} + \varepsilon B(0; 1), \quad (5.9)
\]

which is absurd since \( \varepsilon \in ]0, 1[ \) and \( \|(z - Tz)/\|z - Tz\|\| = 1 \).

\[\blacksquare\]

**Remark 5.4** Consider the setting and the assumption of Proposition 5.3. Zarantonello established in the proof of [39, Theorem 4.1] that, if (in addition to our assumption) \( T \) is Lipschitz continuous, then \( \text{Fix } T \neq \emptyset \). However, we do not need the Lipschitz continuity of \( T \) in our proof.

**Remark 5.5** Consider the setting and assumption of Proposition 5.3 and suppose, in addition, that \( \nabla f \) is continuous. Then we obtain an alternative proof as follows. Suppose to the contrary that \( \text{Fix } T = \emptyset \). Then \( g := \sqrt{\cdot} \circ (2f) \) is continuously Fréchet differentiable on \( \mathcal{H} \) (hence continuous) with

\[
(\forall x \in \mathcal{H}) \quad \nabla g(x) = \frac{x - Tx}{\|x - Tx\|}, \quad (5.10)
\]

Fix \( \varepsilon \in ]0, 1[ \). Since \( g \) is bounded below and continuous, Ekeland’s variational principle implies that there exists \( z \in \mathcal{H} \) such that \((\forall x \in \mathcal{H} \setminus \{z\}) g(z) + \varepsilon d_{\{z\}}(z) = g(z) < g(x) + \varepsilon d_{\{z\}}(x) \). Thus, \( z \) is a minimizer of \( g + \varepsilon d_{\{z\}}(z) \). Therefore, because \( d_{\{z\}} \) is convex, in view of [38, Theorem 3.2.4(iii)&(vi)&(ii)] and [8, Example 16.62], we see that

\[
0 \in \nabla g(z) + \varepsilon \partial d_{\{z\}}(z) = \nabla g(z) + \varepsilon B(0; 1) = \frac{z - Tz}{\|z - Tz\|} + \varepsilon B(0; 1), \quad (5.11)
\]

which contradicts the fact that \( \varepsilon \in ]0, 1[ \).

In [39], Zarantonello provided a necessary and sufficient condition in terms of a differential equation for an operator on \( \mathcal{H} \) to be a projector. The proof there, however, is not within the scope of Convex Analysis. He also conjectured (see the paragraph after [39, Corollary 2, p. 306]) that the Fréchet differentiability of the operator \( P \) in [39, Theorem 4.1] can be replaced by the Gâteaux one. By assuming the monotonicity of \( P \) instead of the Lipschitz continuity, we provide below an affirmative answer. The next result, which plays a crucial role in determining whether a sum of projectors is a projector (see Theorem 5.12 below), is a variant of [39, Theorem 4.1] with a proof rooted in Convex Analysis.

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Theorem 5.6 (Characterization theorem) Let $T : \mathcal{H} \to \mathcal{H}$, and set $f := q \circ (\text{Id} - T)$. Then the following are equivalent:

(i) There exists a nonempty closed convex subset $C$ of $\mathcal{H}$ such that $T = P_C$.

(ii) $T$ is monotone, $f$ is Gâteaux differentiable on $\mathcal{H}$, and $\nabla f = \text{Id} - T$.

If (i) or (ii) holds, then $\text{ran} \ T$ is closed and convex, $T = P_{\text{ran} \ T}$, and $f = (1/2) d^2_{\text{ran} \ T}$ is Fréchet differentiable on $\mathcal{H}$.

Proof. “(i)$\Rightarrow$(ii)”: First, clearly $\text{ran} \ T = \text{ran} \ P_C = C$ is closed and convex. Next, it follows from Example 3.56 that $T = P_C$ is monotone. In turn, since $f = q \circ (\text{Id} - P_C) = (1/2) d^2_C$, we derive from Fact 3.37(i) that $f$ is Fréchet differentiable (and therefore Gâteaux differentiable) on $\mathcal{H}$ with $\nabla f = \text{Id} - P_C = \text{Id} - T$, as desired.

“(i)$\Leftarrow$(ii)”: Set $g := q - f$. Then, on the one hand, because $q$ and $f$ are Gâteaux differentiable, so is $g$. On the other hand, since $\nabla q = \text{Id}$ and $\nabla f = \text{Id} - T$, we infer that $\nabla g = \nabla (q - f) = \nabla q - \nabla f = T$, which is monotone by assumption. Altogether, Fact 3.35 yields the convexity of $g$. Therefore, since $g$ is Gâteaux differentiable on $\mathcal{H}$, it follows from Fact 3.36 that $g$ is lower semicontinuous on $\mathcal{H}$. To sum up, we have shown that $g = q - f$ belongs to $I_0(\mathcal{H})$ and is Gâteaux differentiable on $\mathcal{H}$ with $\nabla g = T$.

(5.12)

Now set $h := g^* - q$ and $C := \text{dom} h$. Let us show that

$$C = \text{ran} \ T \text{ is closed and convex.} \quad (5.13)$$

Indeed, since $h = g^* - q$ and $\text{dom} \ q = \mathcal{H}$, we deduce that $\text{dom} \ h = \text{dom} g^*$. Thus, since $g^* \in I_0(\mathcal{H})$ by (5.12) and Fact 3.42, it follows from Fact 3.49 that $C = \text{dom} \ h = \text{dom} g^* = \text{dom} \partial g^* = \text{dom} (\partial g)^{-1} = \text{ran} \partial g$. However, in the light of (5.12) and Fact 3.50(i), we see that $\text{ran} \partial g = \text{ran} \nabla g = \text{ran} \ T$, and hence $C = \text{dom} g^* = \text{ran} \ T$. Therefore, because $g^*$ is convex, $C = \text{ran} \ T = \text{dom} g^*$ is closed and convex, as announced in (5.13). Next, we shall establish that

$$h = \iota_C. \quad (5.14)$$

Towards this goal, fix $u \in \text{ran} \ T$, say $u = Tx = \nabla g(x)$, where $x \in \mathcal{H}$. Then (5.12), Fact 3.50(ii), and the very definitions of $g$ and $f$ assert that

$$h(u) = g^*(u) - q(u) = g^*(\nabla g(x)) - q(Tx) \quad (5.15a)$$
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\[ = \langle x \mid \nabla g(x) \rangle - g(x) - q(Tx) \] (5.15b)

\[ = \langle x \mid Tx \rangle - q(x) + f(x) - q(Tx) \] (5.15c)

\[ = \langle x \mid Tx \rangle - \frac{1}{2} \|x\|^2 + \frac{1}{2} \|x - Tx\|^2 - \frac{1}{2} \|Tx\|^2 \] (5.15d)

\[ = 0. \] (5.15e)

Hence,

\[ h = 0 \text{ on ran } T. \] (5.16)

Now take \( u \in \text{ran } T \), and let \((u_n)_{n \in \mathbb{N}}\) be a sequence in ran \( T \) converging to \( u \). Since \( f \geq 0 \), we see that \( g = q - f \leq q \), and thus Fact 3.46(i) and Fact 3.41 yield \( g^* \geq q^* = q \), i.e., \( h \geq 0 \). On the one hand, the lower semicontinuity of \( g^* \) and the continuity of \( q \) imply that \( h \) is lower semicontinuous. On the other hand, since \((u_n)_{n \in \mathbb{N}} \subseteq \text{ran } T, (5.16)\) ensures that \((\forall n \in \mathbb{N}) \ h(u_n) = 0. \) Altogether, since \( u_n \to u, \) we get \( 0 \leq h(u) \leq \lim h(u_n) = 0, \) which implies that \( h(u) = 0. \) This and (5.13) guarantee that \( h = 0 \) on \( \text{ran } T = \overline{\text{dom } h}, \) showing that \( h = \iota_{\text{dom } h} = \iota_C, \) as claimed in (5.14). Hence, because \( g \in \Gamma_0(\mathcal{H}) \), the Fenchel–Moreau theorem (Fact 3.42) and Example 3.39 give \( g = g^{**} = (h + q)^* = (\iota_C + q)^* = q - (1/2)d_C^2. \) Consequently, invoking (5.12), (5.13), and Fact 3.37(i), we conclude that \( T = \nabla g = \nabla(q - (1/2)d_C^2) = \text{Id} - (\text{Id} - P_C) = P_C, \) as desired. \( \blacksquare \)

Remark 5.7 Consider the implication “(ii)\( \Rightarrow \) (i)” of Theorem 5.6. If we merely assume that \( T \) is defined on a proper open subset \( D \) of \( \mathcal{H}, \) then, although there may exist a closed set \( C \) such that \( T \) is the restriction to \( D \) of the projector onto \( C, \) the set \( C \) may fail to be convex. An example can be constructed as follows. Suppose that \( \mathcal{H} \neq \{0\}, \) and set

\[ T: \mathcal{H} \setminus \{0\} \to \mathcal{H} : x \mapsto \frac{x}{\|x\|}, \ f := q \circ (\text{Id} - T), \text{ and } C := S(0;1). \] (5.17)

Then clearly \( C \) is a closed nonconvex set and \( T \) is the restriction to \( \mathcal{H} \setminus \{0\} \) of the set-valued projector \( P_C. \) Thus, in the light of Example 3.52, \( T \) is monotone. Next, since \((\forall x \in \mathcal{H} \setminus \{0\}) \ f(x) = (1/2)\|1 - 1/\|x\|\|x\|^2 = (1/2)(\|x\| - 1)^2 = q(x) - \|x\| + 1/2, \text{ we infer that } \) \( f \) is Fréchet differentiable on \( \mathcal{H} \setminus \{0\} \) and

\[ (\forall x \in \mathcal{H} \setminus \{0\}) \quad \nabla f(x) = x - \frac{x}{\|x\|} = x - Tx. \] (5.18)

Open Problem 5.8 We do not know whether the monotonicity of \( T \) can be omitted in Theorem 5.6. Nevertheless, the following remark might be useful in finding counterexamples if one thinks the answer is negative.
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**Remark 5.9 ([16])** Consider the setting of Theorem 5.6 and suppose that \( \mathcal{H} = \mathbb{R}^2 \). Set \( F := \text{Id} - T \) and \((\forall (x, y) \in \mathcal{H}) \) \( F(x, y) := (F_1(x, y), F_2(x, y)) \). Now assume that \( f \) is Fréchet differentiable on \( \mathcal{H} \) with \( \nabla f = \text{Id} - T = F \); in addition, suppose that \( F_1 \) and \( F_2 \) are smooth. Then, since \( (F_1, F_2) = \nabla f \), it follows that \( \partial f / \partial x = F_1 \) and that \( \partial f / \partial y = F_2 \). Hence, due to Schwarz’s theorem,

\[
\frac{\partial F_1}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial F_2}{\partial x}.
\]  

(5.19)

However, because \( \nabla f = F \), a direct computation gives

\[
\begin{align*}
F_1(x, y) &= F_1(x, y) \frac{\partial F_1}{\partial x}(x, y) + F_2(x, y) \frac{\partial F_2}{\partial x}(x, y) \\
&= F_1(x, y) \frac{\partial F_1}{\partial x}(x, y) + F_2(x, y) \frac{\partial F_1}{\partial y}(x, y), \\
F_2(x, y) &= F_1(x, y) \frac{\partial F_1}{\partial y}(x, y) + F_2(x, y) \frac{\partial F_2}{\partial y}(x, y) \\
&= F_1(x, y) \frac{\partial F_2}{\partial x}(x, y) + F_2(x, y) \frac{\partial F_2}{\partial y}(x, y).
\end{align*}
\]

(5.20)

In the first equation of (5.20), one can try to solve for \( F_1 \) in term of \( F_2 \), and vise versa by using the second one. This approach recovers projectors onto linear subspaces of \( \mathbb{R}^2 \) and might suggest a nonmonotone solution of the equation \( \nabla f = \text{Id} - T \). In addition, it is worth noticing that the function \( g: x \mapsto \| x - Tx \| \) satisfies the eikonal equation (see, e.g., [3]), i.e., \((\forall x \in \mathcal{H} \setminus \text{Fix} T) \| \nabla g(x) \| = 1\). This might give us some insights into Open Problem 5.8.

By specializing Theorem 5.6 to positively homogeneous operators on \( \mathcal{H} \), we obtain a characterization for projectors onto closed convex cones.

**Corollary 5.10** Let \( T: \mathcal{H} \to \mathcal{H} \) and set \( f := q \circ T \). Then the following are equivalent:

(i) There exists a nonempty closed convex cone \( K \) such that \( T = P_K \).

(ii) \( T \) is monotone and positively homogeneous, \( f \) is Gâteaux differentiable on \( \mathcal{H} \), and \( \nabla f = T \).

If (i) or (ii) holds, then \( K = \text{ran} T \).
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Proof. “(i)⇒(ii)”: Clearly ran \( T = \text{ran } P_K = K \). Now, it follows from Example 3.56 that \( T = P_K \) is monotone. Next, because \( K \) is a nonempty closed convex cone, (3.27) guarantees that \( T \) is positively homogeneous. In turn, since \( f = q \circ T = q \circ P_K \), Fact 3.37(ii) yield the Gâteaux differentiability of \( f \) and, moreover, \( \nabla f = \nabla (q \circ P_K) = P_K = T \), as desired.

“(i)⇐(ii)”: First, since \( T \) is positively homogeneous, \( \text{ran } T \) is a cone in \( H \).

Now set \( g: H \to \mathbb{R} : x \mapsto (1/2) \langle x | Tx \rangle = (1/2) \langle x | \nabla f(x) \rangle \) and \( h := q \circ (\text{Id} - T) \). Since \( \nabla f = T \) is monotone and positively homogeneous by assumption, Lemma 2.6 ensures that \( g \) is Gâteaux differentiable on \( H \) and \( \nabla g = \nabla f = T \). Thus, because \( h = q - 2g + f \), it follows that \( h \) is Gâteaux differentiable on \( H \) with gradient \( \nabla h = \nabla q - 2\nabla g + \nabla f = \text{Id} - 2T + T = \text{Id} - T \). Consequently, since \( T \) is monotone, we conclude via Theorem 5.6 (applied to \( h \)) and (5.21) that \( \text{ran } T \) is a closed convex cone in \( H \) and that \( T = P_{\text{ran } T} \).

In Corollary 5.10, if \( T \) is a bounded linear operator, then we recover the following characterization of orthogonal projectors. For an alternative proof, which is based on the orthogonal decomposition \( H = V \oplus V^\perp \), where \( V \) is a closed linear subspace of \( H \), see, e.g., [36, Theorem 4.29].

Corollary 5.11 Let \( L: H \to H \). Then the following are equivalent:

(i) There exists a closed linear subspace \( V \) of \( H \) such that \( L = P_V \).

(ii) \( L \in \mathcal{B}(H) \) and \( L = L^* = L^2 \).

(iii) \( L \in \mathcal{B}(H) \) and \( L = L^* L \).

If one of (i)–(iii) holds, then \( V = \text{ran } L \).

Proof. “(i)⇒(ii)”: See, e.g., [8, Corollary 3.24(iii)&(vi)]. Moreover, it is clear that \( \text{ran } L = \text{ran } P_V = V \).

“(ii)⇒(iii)”: Clear.

“(iii)⇒(i)”: On the one hand, because \( L \in \mathcal{B}(H) \), we deduce from Fact 3.54 that \( L = L^* L \) is monotone. On the other hand, since \( L \in \mathcal{B}(H) \), [8, Example 2.60] and our assumption imply that \( q \circ L \) is Fréchet differentiable on \( H \) and \( \nabla (q \circ L) = L^* L = L \). Altogether, because \( L \) is clearly positively homogeneous, we obtain the conclusion via Corollary 5.10.
5.3. Main results

We now establish a necessary and sufficient condition under which a finite sum of projectors is a projector.

**Theorem 5.12 (Sum of projectors)** Let \( m \geq 2 \) be an integer, set \( I := \{1, \ldots, m\} \), and let \( (C_i)_{i \in I} \) be a family of nonempty closed convex subsets of \( \mathcal{H} \). Then there exists a closed convex set \( D \) such that \( \sum_{i \in I} P_{C_i} = P_D \) if and only if

\[
(\exists \gamma \in \mathbb{R})(\forall x \in \mathcal{H}) \sum_{(i,j) \in I \times I \atop i < j} \langle P_{C_i}x | P_{C_j}x \rangle = \gamma; \tag{5.22}
\]

in which case, \( \sum_{i \in I} C_i \) is a closed convex set,

\[
\sum_{i \in I} P_{C_i} = P_{\sum_{i \in I} C_i}, \tag{5.23}
\]

and

\[
d^2_{\sum_{i \in I} C_i} = \sum_{i \in I} d^2_{C_i} - 2(m - 1)q + 2\gamma. \tag{5.24}
\]

**Proof.** For every \( i \in I \), set \( P_i := P_{C_i} \) and \( d_i := d_{C_i} \); in addition, set \( T := \sum_{i \in I} P_i \), which is monotone due to Example 3.56, set \( f := q \circ (\text{Id} - T) \), and set \( g : x \mapsto \sum_{i < j}(P_i x | P_j x) \). By Proposition 5.1(i),

\[
f = \sum_{i \in I} \frac{1}{2} d_i^2 - (m - 1)q + g. \tag{5.25}
\]

Suppose first that there exists a closed convex set \( D \) such that \( T = \sum_{i \in I} P_i = P_D \). Then, in view of Proposition 5.1(iii), it follows that \( \sum_{i \in I} C_i = D \), and hence, we obtain the closedness of \( \sum_{i \in I} C_i \) and (5.23). Now, on the one hand, since clearly

\[
f = q \circ (\text{Id} - P_D) = \frac{1}{2} d_D^2 = \frac{1}{2} d^2_{\sum_{i \in I} C_i}, \tag{5.26}
\]

and by Theorem 5.6 (applied to \( T = P_D \) ), \( f \) Fréchet differentiable on \( \mathcal{H} \) and \( \nabla f = \text{Id} - T \). On the other hand, for every \( i \in I \), in the light of Theorem 5.6 (applied to \( P_i \)), (1/2)\( d_i^2 \) is Fréchet differentiable on \( \mathcal{H} \) and \( \nabla (1/2)d_i^2 = \text{Id} - P_i \). Altogether, (5.25) implies that \( g \) is Fréchet differentiable and \( \text{Id} - T = \nabla f = \sum_{i \in I} (\text{Id} - P_i) - (m - 1)\text{Id} + \nabla g = \text{Id} - \sum_{i \in I} P_i + \nabla g = \text{Id} - T + \nabla g \). Hence, \( \nabla g = 0 \), and so [15, Theorem 3.5] guarantees the existence of \( \gamma \in \mathbb{R} \) such that \( (\forall x \in \mathcal{H}) g(x) = \gamma \). This, (5.25), and (5.26) yield (5.24), as required. Conversely, assume that \( (\exists \gamma \in \mathbb{R})(\forall x \in \mathcal{H}) g(x) = \gamma \). It then follows from (5.25) that \( f = \sum_{i \in I} (1/2)d_i^2 - (m - 1)q + \gamma \). Hence, \( f \) is Fréchet differentiable on \( \mathcal{H} \) and \( \nabla f = \sum_{i \in I} (\text{Id} - P_i) - (m - 1)\text{Id} = \text{Id} - \sum_{i \in I} P_i = \text{Id} - T \). Consequently, because \( T \) is monotone, the conclusion follows from Theorem 5.6. \( \blacksquare \)
5.3. Main results

Corollary 5.13 Let $C$ and $D$ be nonempty closed convex subsets of $\mathcal{H}$. Then the following are equivalent:

(i) There exists a closed convex set $E$ such that $P_C + P_D = P_E$.

(ii) $\exists \gamma \in \mathbb{R} \forall x \in \mathcal{H} \langle P_C x | P_D x \rangle = \gamma$.

If (i) or (ii) holds, then $C + D$ is a closed convex set,

$$P_C + P_D = P_{C+D}, \quad (5.27)$$

and $d_{C+D}^2 = d_C^2 + d_D^2 - 2q + 2\gamma$.

The following simple example shows that the constant $\gamma$ in Corollary 5.13 can take on any value.

Example 5.14 Let $u$ and $v$ be in $\mathcal{H}$, set $C := \{u\}$, and set $D := \{v\}$. Then clearly $P_C + P_D = P_{\{u+v\}} = P_{C+D}$ and $\forall x \in \mathcal{H} \langle P_C x | P_D x \rangle = \langle u | v \rangle$.

As a consequence of Corollary 5.13, a sum of projectors onto orthogonal sets is a projector; see also [9, Proposition 2.6].

Corollary 5.15 Let $C$ and $D$ be nonempty closed convex subsets of $\mathcal{H}$ such that $C \perp D$. Then the following hold:

(i) $C + D$ is a nonempty closed convex set.

(ii) $P_C + P_D = P_{C+D}$.

(iii) $d_{C+D}^2 = d_C^2 + d_D^2 - 2q$.

Proof. Since $\forall x \in \mathcal{H} \langle P_C x | P_D x \rangle = 0$, the conclusions readily follow from Corollary 5.13. \hfill \blacksquare

We now provide an instance where item (ii) of Corollary 5.13 holds, $C \not\subseteq D^\perp$, and neither $C$ nor $D$ is a cone in general.

Example 5.16 Let $K$ be a nonempty closed convex cone in $\mathcal{H}$, let $\rho_1$ and $\rho_2$ be in $\mathbb{R}_{++}$, set $C := K \cap B(0; \rho_1)$, and set $D := K^\ominus \cap B(0; \rho_2)$. It then immediately follows from Theorem 4.10 and Fact 3.25(ii) that, for every $x \in \mathcal{H}$,

$$\langle P_C x | P_D x \rangle = \left\langle \frac{\rho_1}{\max\{\|P_K x\|, \rho_1\}} P_K x \bigg| \frac{\rho_2}{\max\{\|P_{K^\ominus} x\|, \rho_2\}} P_{K^\ominus} x \right\rangle \quad (5.28a)$$

$$= 0. \quad (5.28b)$$
Figure 5.1: A GeoGebra [26] snapshot illustrating the sets \( C \) (yellow) and \( D \) (green) in the setting of Example 5.16.

We next establish a necessary and sufficient condition for \( u + P_C \) to be a projector.

**Example 5.17** Let \( C \) be a nonempty closed convex subset of \( \mathcal{H} \), and let \( u \in \mathcal{H} \). Then, since \((\forall x \in \mathcal{H})\ u = P_{\{u\}}x\), we deduce from Corollary 5.13 that

\[
\begin{align*}
\Rightarrow \quad & u + P_C = P_{\{u\}} + P_C \text{ is a projector onto a closed convex set } \quad (5.29a) \\
\Leftrightarrow \quad & (\exists \gamma \in \mathbb{R})(\forall x \in \mathcal{H}) \langle u | P_C x \rangle = \gamma \quad (5.29b) \\
\Leftrightarrow \quad & (\exists \gamma \in \mathbb{R})(\forall x \in C) \langle u | x \rangle = \gamma \quad (5.29c) \\
\Leftrightarrow \quad & (\forall x \in C)(\forall y \in C) \langle u | x \rangle = \langle u | y \rangle \quad (5.29d) \\
\Leftrightarrow \quad & (\forall x \in C)(\forall y \in C) \langle u | x - y \rangle = 0 \quad (5.29e) \\
\Leftrightarrow \quad & u \in (C - C)^\perp; \quad (5.29f)
\end{align*}
\]
in which case, \( u + P_C = P_{u+C} \) due to Corollary 5.13.

**Remark 5.18** Consider the setting of Example 5.17. Since \( u + P_C \) is monotone, nonexpansive, and a sum of proximity operators, [4, Corollary 2.5] guarantees that \( u + P_C \) is a proximity operator. However, by Example 5.17, it is not a projector unless \( u \in (C - C)^\perp \).

Here is a sufficient, but not necessary, condition for a sum of projectors to be a projector.

**Corollary 5.19** Let \( m \geq 2 \) be an integer, set \( I := \{1, \ldots, m\} \), let \((C_i)_{i \in I}\) be a family of nonempty closed convex subsets of \( \mathcal{H} \), and set \( C := \sum_{i \in I} C_i \). Suppose that, for every \((i, j) \in I \times I \) with \( i < j \), there exists \( \gamma_{ij} \in \mathbb{R} \) such that \( \forall x \in \mathcal{H} \) \( \langle P_{C_i} x | P_{C_j} x \rangle = \gamma_{ij} \). Then \( C \) is a closed convex set and \( \sum_{i \in I} P_{C_i} = P_C \).

**Proof.** Set \((\forall k \in I) D_k := \sum_{i=1}^k C_i\) and let us establish that

\[
(\forall k \in I \setminus \{1\}) \quad D_k \text{ is a closed convex set and } \sum_{i=1}^k P_{C_i} = P_{D_k}. \tag{5.30}
\]

Due to Corollary 5.13, the claim holds if \( k = 2 \), and we therefore assume that, for some \( k \in \{2, \ldots, m - 1\} \), \( D_k \) is a closed convex set and that \( \sum_{i=1}^k P_{C_i} = P_{D_k} \). Then, by our assumption, \( \forall x \in \mathcal{H} \) \( \langle P_{D_k} x | P_{C_{k+1}} x \rangle = \sum_{i=1}^k \langle P_{C_i} x | P_{C_{k+1}} x \rangle = \sum_{i=1}^k \gamma_{ik+1} \), from which and Corollary 5.13 (applied to \( D_k \) and \( C_{k+1} \)) we infer that \( D_{k+1} = D_k + C_{k+1} \) is a closed convex set and, due to the induction hypothesis, \( \sum_{i=1}^{k+1} P_{C_i} = \sum_{i=1}^k P_{C_i} + P_{C_{k+1}} = P_{D_k} + P_{C_{k+1}} = P_{D_k + C_{k+1}} = P_{D_{k+1}} \). Hence, letting \( k = m \) in (5.30) yields the conclusion. \( \blacksquare \)

We now illustrate that the assumption of Corollary 5.19 need not hold when merely \( \sum_{i \in I} P_{C_i} = P_C \).

**Example 5.20** Let \( C \) be a nonempty closed convex subset of \( \mathcal{H} \) such that \( \mathcal{H} \setminus (C - C)^\perp \neq \emptyset \), and suppose that \( u \in \mathcal{H} \setminus (C - C)^\perp \). Then \( P_{\{u\}} + P_{\{-u\}} + P_C = P_C \) is a projector. However, if \( x \mapsto \langle P_{\{u\}} x | P_C x \rangle = \langle u | P_C x \rangle \) were a constant, then it would follow from Corollary 5.13 that \( u + P_C = P_{\{u\}} + P_C \) is a projector, which violates Example 5.17 and the assumption that \( u \not\in (C - C)^\perp \).
5.4 The partial sum property of projectors onto convex cones

In this section, we shall discuss the partial sum property and the connections between our work, Zarantonello’s [39, Theorems 5.5 and 5.3], and the work [4]. We shall need the following two results. Let us provide an instance where the star-difference of two sets (see [23]) can be explicitly determined. Lemma 5.21 was mentioned in [4, Footnote 5] and was also stated implicitly in the proof of [39, Theorem 5.2].

Lemma 5.21 (Star-difference of cones) Let $K_1$ and $K_2$ be nonempty closed convex cones in $H$, and set

$$K := \{ u \in H \mid u + K_2 \subseteq K_1 \}. \tag{5.31}$$

Then the following are equivalent:

(i) $K = K_1$.

(ii) $0 \in K$.

(iii) $K \neq \emptyset$.

(iv) $K_2 \subseteq K_1$.

Proof. The chain of implications “(i)⇒(ii)⇒(iii)” is clear.

“(iii)⇒(iv)”: Fix $u \in K$. Then, since $K_1$ and $K_2$ are cones, we infer that $(\forall \varepsilon \in \mathbb{R}_+) \varepsilon u + K_2 = \varepsilon(u + K_2) \subseteq \varepsilon K_1 = K_1$. In turn, letting $\varepsilon \downarrow 0$ and using the closedness of $K_1$, we obtain $K_2 \subseteq K_1$.

“(iv)⇒(i)”: First, take $u \in K_1$. Since $K_2 \subseteq K_1$ and $K_1$ is a convex cone by assumption, it follows that $u + K_2 \subseteq K_1 + K_1 \subseteq K_1$, and therefore $u \in K$. Conversely, fix $u \in K$. Because $u + K_2 \subseteq K_1$ and $0 \in K_2$, we deduce that $u \in K_1$, which completes the proof. ■

Proposition 5.22 Let $C$ and $D$ be nonempty closed convex subsets of $H$, and set

$$f := \frac{1}{2}d_C^2 + \frac{1}{2}d_D^2 - q \quad \text{and} \quad h := f^* - q. \tag{5.32}$$

Then the following hold:

(i) $(\forall u \in H) \ h(u) = \sup_{v \in D} (\sigma_C(u + v) + \langle u \mid v \rangle)$.
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(ii) Suppose that $C$ and $D$ are cones and $D \subseteq C$. Then $h = \iota_{C \cap D}$.

Proof. (i): Since $D$ is convex, closed, and nonempty, we see that $\iota_D \in \Gamma_0(H)$, and so\(^7\) \((1/2)d_D^2 = \iota_D \square q = \iota_D \diamond q\). In turn, Moreau’s decomposition asserts that $q - (1/2)d_D^2 = q - \iota_D \square q = \iota_D \diamond q$. Thus, (5.32) yields

$$f = \frac{1}{2}d_C^2 - \iota_D^* \square q.$$ \hfill (5.33)

Moreover, since $\iota_D \in \Gamma_0(H)$ and $q^* = q$, Fact 3.46(ii) and the Fenchel–Moreau theorem guarantee that $(\iota_D^* \square q)^* = \iota_D^* + q = \iota_D + q$, which implies that $\text{dom}(\iota_D^* \square q)^* = D$. Consequently, because $\iota_D^* \square q \in \Gamma_0(H)$, Fact 3.43 and Example 3.40 imply that, for every $u \in H$,

$$f^*(u) - q(u) = (\frac{1}{2}d_C^2 - \iota_D^* \square q)^*(u) - q(u) \hfill (5.34a)$$
$$= \sup_{v \in \text{dom}(\iota_D^* \square q)^*} \left( (\frac{1}{2}d_C^2)^*(u + v) - (\iota_D^* \square q)^*(v) \right) - q(u) \hfill (5.34b)$$
$$= \sup_{v \in D} \left( \sigma_C(u + v) + q(u + v) - q(v) \right) - q(u) \hfill (5.34c)$$
$$= \sup_{v \in D} \left( \sigma_C(u + v) + \langle u | v \rangle \right), \hfill (5.34d)$$

as announced.

(ii): First, because $D \subseteq C$, Lemma 5.21 (applied to the pair of closed convex cones $(C, D)$) yields

$$C = \{u \in H \mid u + D \subseteq C\}. \hfill (5.35)$$

Next, we derive from (i) and Fact 3.21(iii) that

$$\forall u \in H : h(u) = \sup_{v \in D} \left( \sigma_C(u + v) + \langle u | v \rangle \right) \hfill (5.36a)$$
$$= \sup_{v \in D} \left( \iota_C(u + v) + \langle u | v \rangle \right). \hfill (5.36b)$$

Now fix $u \in H$, and let us consider two alternatives.

(a) $u \in H \setminus C$: In view of (5.35), there exists $v \in D$ such that $u + v \in H \setminus C$, and therefore, by (5.36b), $h(u) \geq \iota_C(u + v) + \langle u | v \rangle = +\infty$.

\(^7\)Here $\square$ and $\diamond$ denote the infimal convolution and the exact infimal convolution defined in Definition 3.45, respectively.
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(b) \( u \in C^\ominus \): Then, by (5.35), \( u + D \subseteq C^\ominus \). Hence, since \( D \) is a nonempty cone, it follows from (5.36b) and Fact 3.21(iii) that \( h(u) = \sup_{v \in D} \langle u \mid v \rangle = \sigma_D(u) = i_C \circ D(u) = i_{C \cap D^\ominus}(u) \).

Altogether, we obtain the desired conclusion. ■

Here is the first main result of this section. The proof of the implication “(v)⇒(i)” was inspired by [4, Lemma 5.3].

**Theorem 5.23** Let \( K_1 \) and \( K_2 \) be nonempty closed convex cones in \( \mathcal{H} \). Then the following are equivalent:

(i) \( K_1 + K_2 \) is closed and \( P_{K_1} + P_{K_2} = P_{K_1+K_2} \).

(ii) There exists a nonempty closed convex cone \( K \) such that \( P_{K_1} + P_{K_2} = P_K \).

(iii) \( P_{K_1} + P_{K_2} \) is a proximity operator of a function in \( \Gamma_0(\mathcal{H}) \).

(iv) \( P_{K_1} + P_{K_2} \) is nonexpansive.

(v) \( \text{Id} - P_{K_1} - P_{K_2} \) is monotone.

(vi) \( \langle \forall x \in \mathcal{H} \rangle \langle P_{K_1} x \mid P_{K_2} x \rangle = 0. \)

Furthermore, if one of (i)–(vi) holds, then

\[
d_{K_1+K_2}^2 = d_{K_1}^2 + d_{K_2}^2 - 2q = d_{K_1}^2 + d_{K_2}^2 - d_{K_1 \cap K_2}^2. \tag{5.37}
\]

**Proof.** The chain of implications “(i)⇒(ii)⇒(iii)⇒(iv)” is clear, and the implication “(iv)⇒(v)” follows from Fact 3.53. We now assume that (v) holds and establish (i). Towards this end, set

\[
f := \frac{1}{2}d_{K_1}^2 + \frac{1}{2}d_{K_2}^2 - q. \tag{5.38}
\]

and set

\[
h := f^* - q. \tag{5.39}
\]

Let us first establish that \( h = i_{K_1 \cap K_2^\ominus} \). To do so, we derive from the monotonicity of \( \text{Id} - P_{K_1} - P_{K_2} \) and Moreau’s conical decomposition that

\[
\langle \forall x \in \mathcal{H} \rangle \langle x \mid P_{K_1^\ominus} x - P_{K_2} x \rangle = \langle x - 0 \mid (\text{Id} - P_{K_1} - P_{K_2}) x - (\text{Id} - P_{K_1} - P_{K_2}) 0 \rangle \tag{5.40a}
\]

and

\[
\langle \forall x \in \mathcal{H} \rangle \langle x \mid P_{K_1^\ominus} x - P_{K_2} x \rangle = \langle x - 0 \mid (\text{Id} - P_{K_1} - P_{K_2}) x - (\text{Id} - P_{K_1} - P_{K_2}) 0 \rangle \tag{5.40b}
\]
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\[ \geq 0. \]  
\[ (5.40c) \]

Thus, because \( K_1^\circ \) and \( K_2 \) are closed convex cones, Fact 3.27 guarantees that \( K_2 \subseteq K_1^\circ \), from which and Proposition 5.22(ii) we deduce that

\[ h = t_{K_1^\circ \cap K_2^\circ}, \]

as claimed. Next, by Theorem 5.6 (respectively applied to \( P_{K_1} \) and \( P_{K_2} \)), \( f \) is Fréchet differentiable on \( \mathcal{H} \) (hence continuous) and

\[ \nabla f = (\text{Id} - P_{K_1}) + (\text{Id} - P_{K_2}) - \text{Id} = \text{Id} - P_{K_1} - P_{K_2}, \]

which is monotone by assumption. Therefore, in view of Fact 3.35, \( f \) is convex, and so \( f \in \Gamma_0(\mathcal{H}) \). In turn, because \( f^* = h + q = t_{K_1^\circ \cap K_2^\circ} + q \) by (5.39) and (5.41), the Fenchel–Moreau theorem and Example 3.39 yield \( f = f^{**} = (t_{K_1^\circ \cap K_2^\circ} + q)^* = q - (1/2)d_{K_1^\circ \cap K_2^\circ}^2 \). Hence, by (5.42) and Fact 3.37(i), we obtain \( \text{Id} - P_{K_1} - P_{K_2} = \nabla f = \text{Id} - (\text{Id} - P_{K_1^\circ \cap K_2^\circ}) = P_{K_1^\circ \cap K_2^\circ} \). Thus, the Moreau conical decomposition and Fact 3.26(i)&(ii) guarantee that \( P_{K_1} + P_{K_2} = \text{Id} - P_{K_1^\circ \cap K_2^\circ} = P_{K_1^\circ \cap K_2^\circ} = P_{K_1^\circ \cap K_2^\circ} = P_{K_1} + P_{K_2} \). Consequently, Proposition 5.1(iii) asserts that \( K_1 + K_2 \) is closed, and therefore, \( P_{K_1} + P_{K_2} = P_{K_1^\circ \cap K_2^\circ} \), as desired. To summarize, we have shown the equivalences of (i)–(v).

“(i) ⇔ (vi)” Follows from Corollary 5.13 and the fact that \( \langle P_{K_1}^0 | P_{K_2}^0 \rangle = 0 \). Moreover, if (vi) holds, then (5.37) follows from Corollary 5.13 and Fact 3.25(iii).

Replacing one cone by a general convex set may make the implication “(v) ⇒ (i)” of Theorem 5.23 fail, as illustrated by the following example.

**Example 5.24** Let \( K \) be a nonempty closed convex cone in \( \mathcal{H} \), and let \( u \in \mathcal{H} \). Then, by Moreau’s conical decomposition, \( \text{Id} - P_K - P_{\{u\}} = P_{K^\circ} - u \), which is clearly monotone. However, owing to Example 5.17, \( P_{\{u\}} + P_K = u + P_K \) is not a projector provided that \( u \notin (K - K)^\perp \).

Here is an instance where the projector onto the intersection can be expressed in terms of the individual projectors.

**Corollary 5.25** Let \( K_1 \) and \( K_2 \) be nonempty closed convex cones in \( \mathcal{H} \). Then the following are equivalent:

(i) \( P_{K_1 \cap K_2} = P_{K_1} + P_{K_2} - \text{Id}. \)
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(ii) \( P_{K_1} + P_{K_2} - \text{Id} \) is a projector onto a closed convex set.

(iii) \( P_{K_1} + P_{K_2} - \text{Id} \) is monotone.

(iv) \( (\forall x \in \mathcal{H}) \|P_{K_1} x\|^2 + \|P_{K_2} x\|^2 = \|x\|^2 + \langle P_{K_1} x \mid P_{K_2} x \rangle. \)

Proof. We first deduce from the Moreau conical decomposition and Fact 3.26(ii) that

\[
P_{K_1 \cap K_2} = P_{K_1} + P_{K_2} - \text{Id} \iff \text{Id} - P_{(K_1 \cap K_2)^\circ} = P_{K_1} + P_{K_2} - \text{Id} \quad (5.43a)
\]

\[
\iff \text{Id} - P_{\overline{K_1 + K_2}} = P_{K_1} + P_{K_2} - \text{Id} \quad (5.43b)
\]

\[
\iff P_{\overline{K_1 + K_2}} = (\text{Id} - P_{K_1}) + (\text{Id} - P_{K_2}) \quad (5.43c)
\]

\[
\iff P_{\overline{K_1 + K_2}} = P_{K_1^\circ} + P_{K_2^\circ}. \quad (5.43d)
\]

“(i)\(\iff\)(ii)”: Denote by \( \mathcal{C} \) the class of nonempty closed convex cones in \( \mathcal{H} \). Then, because the mapping \( \mathcal{C} \to \mathcal{C} : K \mapsto \overline{K^\circ} \) is bijective due to Fact 3.26(i), we derive from (5.43d), the equivalence “(i)\(\iff\)(ii)” of Theorem 5.23, and the Moreau conical decomposition that

\[
(i) \iff P_{\overline{K_1 + K_2}} = P_{K_1^\circ} + P_{K_2^\circ} \quad (5.44a)
\]

\[
\iff K_1^\circ + K_2^\circ \text{ is closed and } P_{K_1^\circ + K_2^\circ} = P_{K_1^\circ} + P_{K_2^\circ} \quad (5.44b)
\]

\[
\iff (\exists K \in \mathcal{C}) P_K = P_{K_1^\circ} + P_{K_2^\circ} \quad (5.44c)
\]

\[
\iff (\exists K \in \mathcal{C}) \text{Id} - P_{K^\circ} = (\text{Id} - P_{K_1}) + (\text{Id} - P_{K_2}) \quad (5.44d)
\]

\[
\iff (\exists K \in \mathcal{C}) P_{K^\circ} = P_{K_1} + P_{K_2} - \text{Id} \quad (5.44e)
\]

\[
\iff (\exists K \in \mathcal{C}) P_K = P_{K_1} + P_{K_2} - \text{Id} \quad (5.44f)
\]

\[
\iff (\exists K \in \mathcal{C}) \quad (ii), \quad (5.44g)
\]

where the last equivalence follows from the fact that \( P_{K_1} + P_{K_2} - \text{Id} \) is positively homogeneous.

“(i)\(\iff\)(iii)”: Since \( \text{Id} - (P_{K_1^\circ} + P_{K_2^\circ}) = P_{K_1} + P_{K_2} - \text{Id} \) by Moreau’s decomposition, this equivalence is a consequence of (5.43d) and the equivalence “(ii)\(\iff\)(v)” of Theorem 5.23 (applied to \((K_1^\circ, K_2^\circ)\)).

“(i)\(\iff\)(iv)”: This readily follows from (5.43d), the equivalence “(ii)\(\iff\)(vi)” of Theorem 5.23 (applied to \((K_1^\circ, K_2^\circ)\)), and Lemma 3.28(ii). ■

By replacing \((K_1, K_2)\) by \((K_1, K_2^\circ)\) in Corollary 5.25, we provide an alternative proof for [39, Theorem 5.3]. The linear case of Corollary 5.26 goes back at least to Halmos (see [21, Theorem 3, p. 48]).
Corollary 5.26 (Zarantonello) Let $K_1$ and $K_2$ be nonempty closed convex cones in $\mathcal{H}$. Then $P_{K_2}P_{K_1} = P_{K_2}$ if and only if $P_{K_1} - P_{K_2}$ is a projector onto a closed convex set; in which case, $P_{K_1} - P_{K_2} = P_{K_1 \cap K_2^\circ}$.

Proof. First, suppose that $P_{K_2}P_{K_1} = P_{K_2}$. Then, by Fact 3.25(i)&(iii) and Lemma 3.28(i),

$$
(\forall x \in \mathcal{H}) \quad \langle P_{K_1}x, P_{K_2}x \rangle = \langle P_{K_1}x, x \rangle - \langle P_{K_1}x, P_{K_2}P_{K_1}x \rangle
= \|P_{K_1}x\|^2 - \|P_{K_2}P_{K_1}x\|^2
= \|P_{K_1}x\|^2 - \|P_{K_2}x\|^2
= \|P_{K_1}x\|^2 + \|P_{K_2}x\|^2 - \|x\|^2.
$$

(5.45a) (5.45b) (5.45c) (5.45d) (5.45e)

Hence, the equivalence “(i)⇔(iv)” of Corollary 5.25 (applied to $(K_1, K_2^\circ)$) yields $P_{K_1} - P_{K_2} = P_{K_1} + P_{K_2^\circ} - \text{Id} = P_{K_1 \cap K_2^\circ}$, as desired. Conversely, assume that $P_{K_1} - P_{K_2}$ is a projector associated with a closed convex set. Since $P_{K_1} - P_{K_2} = P_{K_1} + P_{K_2^\circ} - \text{Id}$, it follows from the equivalence “(i)⇔(ii)” of Corollary 5.25 (applied to $(K_1, K_2^\circ)$) that

$$
P_{K_1} - P_{K_2} = P_{K_1 \cap K_2^\circ}.
$$

(5.46)

Now take $x \in \mathcal{H}$. On the one hand, because $P_{K_1 \cap K_2^\circ} + P_{K_2} = (P_{K_1} - P_{K_2}) + P_{K_2} = P_{K_1}$ by (5.46), we infer from Theorem 5.23 that $P_{K_1 \cap K_2^\circ}x \perp P_{K_1}x$ or, equivalently, by (5.46), $(P_{K_1}x - P_{K_2}x) \perp P_{K_1}x$. On the other hand, (5.46) implies that $P_{K_1}x - P_{K_2}x \in K_2^\circ$. Altogether, since clearly $P_{K_2}x \in K_2$, (3.26) asserts that $P_{K_2}P_{K_1}x = P_{K_1}x$, and the proof is complete.

The so-called partial sum property, i.e., if a finite sum of proximity operators is a proximity operator, then so is every partial sum, was obtained in [4]. Somewhat surprisingly, as we shall see in the following result, this property is still valid in the class of projectors onto convex cones. The equivalence “(i)⇔(iii)” of the following result was obtained by Zarantonello with a different proof (see [39, Theorem 5.5]).

Theorem 5.27 (Partial sum property for cones) Let $m \geq 2$ be an integer, set $I := \{1, \ldots, m\}$, and let $(K_i)_{i \in I}$ be a family of nonempty closed convex cones in $\mathcal{H}$. Then the following are equivalent:

1. For every $(i, j) \in I \times I$ such that $i \neq j$, we have $(\forall x \in \mathcal{H}) \langle P_{K_i}x, P_{K_j}x \rangle = 0$.  

5.4. The partial sum property of projectors onto convex cones

(ii) \( \sum_{i \in I} K_i \) is closed and \( \sum_{i \in I} P_{K_i} = P_{\sum_{i \in I} K_i} \).

(iii) \( \sum_{i \in I} P_{K_i} \) is a projection onto a closed convex cone in \( \mathcal{H} \).

(iv) \( \sum_{i \in I} P_{K_i} \) is a proximity operator of a function in \( \Gamma_0(\mathcal{H}) \).

(v) For every nonempty subset \( J \) of \( I \), \( \sum_{j \in J} P_{K_j} \) is a proximity operator of a function in \( \Gamma_0(\mathcal{H}) \).

(vi) For every nonempty subset \( J \) of \( I \), \( \sum_{j \in J} K_j \) is closed and \( \sum_{j \in J} P_{K_j} = P_{\sum_{j \in J} K_j} \).

(vii) For every \( (i, j) \in I \times I \) such that \( i \neq j \), we have \( P_{K_i} + P_{K_j} \) is nonexpansive.

(viii) For every \( (i, j) \in I \times I \) such that \( i \neq j \), we have \( \text{Id} - P_{K_i} - P_{K_j} \) is monotone.

Proof. “(i)\( \Rightarrow \) (ii)”: A direct consequence of Corollary 5.19.

“(ii)\( \Rightarrow \) (iii)” and “(iii)\( \Rightarrow \) (iv)”: Clear.

“(iv)\( \Rightarrow \) (v)”: Let \( f \in \Gamma_0(\mathcal{H}) \) be such that \( \sum_{i \in I} P_{K_i} = \text{Prox}_f \). Then, by Moreau’s decomposition, \( \sum_{i \in I} P_{K_i} + \text{Prox}_{f^*} = \text{Prox}_f + \text{Prox}_{f^*} = \text{Id} \). Therefore, since \( \{P_{K_i}\}_{i \in I} \) are proximity operators, the conclusion follows from \([4, \text{Theorem 4.2}]\).

“(v)\( \Rightarrow \) (vii)”: Clear.

“(vii)\( \Rightarrow \) (viii)”: See Fact 3.53.

“(viii)\( \Rightarrow \) (i)”: This is the implication “(v)\( \Rightarrow \) (vi)” of Theorem 5.23.

To sum up, we have shown the equivalence of (i)–(viii) except for (vi).

“(v)\( \Leftrightarrow \) (vi)”: Follows from the equivalence “(ii)\( \Leftrightarrow \) (iv).” ■

As we now illustrate, the partial sum property may, however, fail outside the class of projectors onto convex cones.

**Example 5.28** Suppose that \( \mathcal{H} \neq \{0\} \), let \( w \in \mathcal{H} \setminus \{0\} \), set \( U := \mathbb{R}_+ w \), and set \( V := \mathbb{R}_-(-w) = \mathbb{R}_- w \). Then, appealing to (4.32), we see that \((\forall x \in \mathcal{H}) \langle P_{UX} | P_{UX} \rangle = 0 \). Hence, by Theorem 5.23, \( P_U + P_V = P_{U+V} = P_{\mathbb{R}w} \). Now suppose that \( z \in \mathcal{H} \setminus (U \cup U)^\perp = \mathcal{H} \setminus (\mathbb{R}w)^\perp \). Then clearly \( P_U + P_V + P_{\{z\}} + P_{\{-z\}} = P_{\mathbb{R}w} \) is the projector associated with the line \( \mathbb{R}w \). However, due to Example 5.17, \( P_{\{z\}} + P_U = z + P_U \) is not a projector.

Theorem 5.27 gives us a different perspective of the projection onto cones generated by orthonormal families: It is just the sum of projections onto the generating rays.
Example 5.29 Let \( \{e_i\}_{i \in I} \) be a finite orthonormal subset of \( \mathcal{H} \), set \( (\forall i \in I) \ K_i := \mathbb{R} + e_i \), and set \( K := \sum_{i \in I} K_i \). Then, for every \( (i, j) \in I \times I \) such that \( i \neq j \), since \( K_i \perp K_j \), we see that \( (\forall x \in \mathcal{H}) \ (P_{K_i}x \mid P_{K_j}x) = 0 \). Hence, owing to Theorem 5.27, \( K = \sum_{i \in I} K_i \) is closed and, by using (4.32),

\[
(\forall x \in \mathcal{H}) \quad P_Kx = \sum_{i \in I} P_{K_i}x = \sum_{i \in I} \max\{\langle x \mid e_i \rangle, 0\} e_i. \tag{5.47}
\]

see also Theorem 4.7.

5.5 The one-dimensional case

In this section, we assume that \( \mathcal{H} = \mathbb{R} \) (5.48) and that \( C \) and \( D \) are nonempty closed intervals of \( \mathcal{H} \). (5.49)

The goal of this section is to describe all pairs \( (C, D) \) on the real line such that \( P_C + P_D = P_{C+D} \). We begin with a simple observation.

Remark 5.30 If \( C = \{0\} \) or \( D = \{0\} \), then clearly \( P_C + P_D = P_{C+D} \). Thus, we henceforth assume in this section that

\[
C \neq \{0\} \text{ and } D \neq \{0\}. \tag{5.50}
\]

Here is a sufficient condition under which \( P_C + P_D = P_{C+D} \).

Proposition 5.31 Suppose that \( C \cap D = \{0\} \). Then \( C + D \) is closed and \( P_C + P_C = P_{C+D} \).

Proof. In the light of Proposition 3.15, we may and do assume that \( C \subseteq \mathbb{R}_- \). Then Proposition 3.15(i) implies that \( \max C = \min D = 0 \), and thus (3.9) yields \( (\forall \xi \in \mathcal{H}) \ (P_C\xi \mid P_D\xi) = 0 \). Hence, according to Corollary 5.13, we conclude that \( C + D \) is closed and that \( P_C + P_D = P_{C+D} \). \( \blacksquare \)

The next result classifies all pairs \( (C, D) \) such that \( P_C + P_D = P_{C+D} \). Item (ii) is a partial converse of Proposition 3.15.

Theorem 5.32 (Dichotomy) Suppose that there exists a closed convex set \( E \) such that \( P_C + P_D = P_E \). Then exactly one of the following cases occurs:

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(i) $C$ and $D$ are singletons.

(ii) Neither $C$ nor $D$ is a singleton and $C \cap D = \{0\}$.

Proof. Corollary 5.13 and our assumption guarantee the existence of $\gamma \in \mathbb{R}$ such that
\[
(\forall \xi \in H) \quad (P_C \xi)(P_D \xi) = \langle P_C \xi | P_D \xi \rangle = \gamma. \tag{5.51}
\]

(i): Suppose that $C = \{\pi\}$, where $\pi \neq 0$ due to (5.50). Then, for every $\xi \in D$ and every $\eta \in D$, since $P_C \xi = P_C \eta = \pi$, (5.51) implies that $\pi \xi = \pi \eta$, and because $\pi \neq 0$, it follows that $\xi = \eta$. Therefore, $D$ is a singleton, as required.

(ii): Suppose that $C$ is not a singleton. Let us first show that $D$ is not a singleton by proving the contrapositive. If $D$ is a singleton, then interchanging $C$ and $D$ in (i), we see that $C$ is a singleton. Next, we shall establish that $C \cap D \neq \emptyset$ by contradiction. Assume that $C \cap D = \emptyset$. Then, owing to the separation theorem (see, e.g., [8, Theorem 3.53]), we obtain $\mu \in H \setminus \{0\}$ and $\beta \in \mathbb{R}$ such that
\[
(\forall \xi \in C)(\forall \eta \in D) \quad \xi \mu \leq \beta \leq \eta \mu. \tag{5.52}
\]
Without loss of generality, we may and do assume that
\[
\mu > 0. \tag{5.53}
\]
Then (5.52) asserts that $C$ is bounded above and $D$ is bounded below, and because they are closed, we infer that $\sup C = \max C$ and $\inf D = \min D$. Let us consider the following conceivable cases.

(a) $\max C = 0$: Then $\min D \neq 0$ (otherwise $0 \in C \cap D$, which is absurd). Because $C$ is not a singleton, we can find $\xi_1 \in C$ and $\xi_2 \in C$ such that $\xi_1 \neq \xi_2$. In turn, due to (5.52) and (5.53), $(\forall i \in \{1, 2\}) \xi_i \leq \beta / \mu \leq \min D$, from which and (3.9) we deduce that $P_D \xi_1 = P_D \xi_2 = \min D$. Consequently, since $\{\xi_1, \xi_2\} \subseteq C$, (5.51) implies that $\xi_1 \min D = \langle P_C \xi_1 | P_D \xi_1 \rangle = \langle P_C \xi_2 | P_D \xi_2 \rangle = \xi_2 \min D$, and since $\min D \neq 0$, it follows that $\xi_1 = \xi_2$, which is impossible.

(b) $\max C \neq 0$: Since $D$ is not a singleton, there are $\eta_1 \in D$ and $\eta_2 \in D$ such that $\eta_1 \neq \eta_2$. In turn, we infer from (5.52)&(5.53) that $(\forall i \in \{1, 2\}) \max C \leq \beta / \mu \leq \eta_i$, and therefore (3.9) yields $P_C \eta_1 = P_C \eta_2 = \max C$. Thus, by (5.51) and the fact that $\{\eta_1, \eta_2\} \subseteq D$, we see that $(\max C) \eta_1 = \langle P_C \eta_1 | P_D \eta_1 \rangle = \langle P_C \eta_2 | P_D \eta_2 \rangle = (\max C) \eta_2$. Consequently, since $\max C \neq 0$, it follows that $\eta_1 = \eta_2$, which is absurd.
To summarize, we have shown that \( C \cap D \neq \emptyset \). Let us now verify that \( C \cap D \) is a singleton. To this end, take \( \xi \in C \cap D \) and \( \eta \in C \cap D \), and let \( \varepsilon \in [0, 1] \). On the one hand, by (5.51), we see that

\[
(\forall \xi \in C \cap D) \quad \xi^2 = \gamma. \tag{5.54}
\]

On the other hand, since \( C \cap D \) is convex and \( \varepsilon \in [0, 1] \), \( (1 - \varepsilon)\xi + \varepsilon\eta \in C \cap D \). Altogether, \( \xi^2 = [(1 - \varepsilon)\xi + \varepsilon\eta]^2 \) or, equivalently, \( \varepsilon^2(\xi - \eta)[(2 - \varepsilon)\xi + \varepsilon\eta] = \xi^2 - [(1 - \varepsilon)\xi + \varepsilon\eta]^2 = 0 \). Interchanging \( \xi \) and \( \eta \) yields \( \varepsilon(\eta - \xi)[(2 - \varepsilon)\xi - 2(1 - \varepsilon)\eta] = 0 \), i.e., \( 2\varepsilon(1 - \varepsilon)(\xi - \eta)^2 = 0 \). Therefore, \( \xi = \eta \) and \( C \cap D \) is thus a singleton, say \( C \cap D = \{\pi\} \). It remains to show that \( \pi = 0 \). Since \( C \) is not a singleton, there exists \( \xi \in C \setminus \{\pi\} \). In turn, because \( C \cap D = \{\pi\} \), we derive from [8, Proposition 24.47] (applied to \( (\Omega, \phi) = (D, iC) \)) and (5.51)\&(5.54) that \( \xi \pi = \langle P_{C\xi} | P_{C \cap D\xi} \rangle = \langle P_{C\xi} | P_D(P_{C\xi}) \rangle = \langle P_{C\xi} | P_D\xi \rangle = \gamma = \pi^2 \). Thus, \( \pi(\xi - \pi) = 0 \), and since \( \xi \neq \pi \), it follows that \( \pi = 0 \), which completes the proof. \( \square \)

### 5.6 On a result by Halmos

In this section, we revisit and extend the classical result [21, Theorem 2, p. 46] to the nonlinear case.

**Proposition 5.33** Let \( C \) be a nonempty closed convex subset of \( \mathcal{H} \), and let \( K \) be a nonempty closed convex cone in \( \mathcal{H} \). Suppose that there exits a closed convex set \( D \) such that \( P_C + P_K = P_D \). Then \( C \subseteq K^\circ \).

**Proof.** Our assumption and Corollary 5.13 guarantee the existence of \( \gamma \in \mathbb{R} \) such that \( (\forall x \in \mathcal{H}) \quad \langle P_Cx | P_Kx \rangle = \gamma \). However, because \( P_K0 = 0 \), we infer that \( \gamma = 0 \). Therefore, for every \( x \in C \), it follows from Lemma 3.28(i) that \( \|P_Kx\|^2 = \langle x | P_Kx \rangle = \langle P_Cx | P_Kx \rangle = 0 \); hence, \( P_Kx = 0 \), and Fact 3.25(i) thus implies that \( x = P_{K^\circ}x \in K^\circ \). Consequently, \( C \subseteq K^\circ \), as claimed. \( \square \)

The following example shows that the conclusion of Proposition 5.33 is merely a necessary condition for \( P_C + P_K = P_{C+K} \) even when \( C \) is a cone.

**Example 5.34** Suppose that \( \mathcal{H} = \mathbb{R}^2 \). Set \( u := (1, 0) \) and \( v := (-1, 1) \). Moreover, set \( C := \mathbb{R}_+v \) and \( K := \mathbb{R}_+u \). Then, because \( \langle u | v \rangle = -1 < 0 \), we see that \( C \subseteq K^\circ \). Furthermore, \( C + K \) is a closed cone by Example 4.6. Now set \( x := (1, 1) = v + 2u \in C + K \). According to (4.32), \( P_Cx + P_Kx = (0, 0) + (1, 0) = (1, 0) \neq x = P_{C+K}x \). Therefore, \( P_C + P_K \neq P_{C+K} \).
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We now extend the classical [21, Theorem 2, p. 46] (in the case of two subspaces) by replacing one subspace by a general convex set.

**Corollary 5.35** Let $C$ be a nonempty closed convex subset of $\mathcal{H}$, and let $V$ be a closed linear subspace of $\mathcal{H}$. Then the following are equivalent:

(i) There exists a closed convex set $D$ such that $P_C + P_V = P_D$.

(ii) $C \perp V$.

Moreover, if (i) and (ii) hold, then $D = C + V$ and $P_C + P_V = P_{C+V}$.

**Proof.** “(i)$\Rightarrow$(ii)”: It follows from Corollary 5.13 that $D = C + V$ and that $P_C + P_V = P_{C+V}$. Now, by Proposition 5.33 and Fact 3.21(i), we obtain $C \subseteq V^\ominus = V^\perp$. “(ii)$\Rightarrow$(i)”: Immediate from Corollary 5.15.

However, replacing the subspace $V$ in Corollary 5.35 by cone might not work. The following simple example shows that the implication “(i)$\Rightarrow$(ii)” of Corollary 5.35 may fail even when $C$ and $V$ are cones.

**Example 5.36** Consider the setting of Example 5.28. We have seen that $P_U + P_V = P_{\mathbb{R}w}$. Yet, $U$ is not perpendicular to $V$. In fact, $\text{span } U = \text{span } V = \mathbb{R}w$.

Combining Theorem 5.27, Theorem 5.23, and Corollary 5.35, we obtain the following well-known result; see [21, Theorem 2, p. 46].

**Corollary 5.37** Let $(V_i)_{i \in I}$ be a finite family of closed linear subspaces of $\mathcal{H}$. Then $\sum_{i \in I} P_{V_i}$ is a projector associated with a closed linear subspace if and only if, for every $(i, j) \in I \times I$ with $i \neq j$, we have $V_i \perp V_j$. 
Chapter 6

Conclusion and future work

We have provided formulae for projection onto the intersection of a cone and a ball/sphere under mild assumptions. Our analysis allows us to cover many cases including nonpolyhedral cones, e.g., the ice cream cone and the cone of positive semidefinite matrices; moreover, this analysis may be helpful in future research involving projection onto spheres. Also, we have filled a gap in the literature of Convex Analysis by establishing a complete answer to the question “When is the sum of projectors a projector?.” The provided analysis may give more insights into properties of projection operators associated with convex sets. To conclude this thesis, we list here questions and directions for future research:

**P1** When is a linear combination (in particular, convex combination) of projectors a projector?

**P2** Provide a characterization for proximal mapping that is similar to Theorem 5.6.

**P3** Solve Open Problem 5.8.

**P4** Provide a proof rooted in Convex Analysis for Zarantonello’s [39, Theorem 4.1].
Bibliography


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Séances de l’Académie des Sciences (Séries A), 225 (1962), pp. 238–240. → pages 16


