New solutions to local and non-local elliptic equations

by

Hon To Hardy Chan

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Examining Committee:

Juncheng Wei, Mathematics
Co-supervisor

Nassif Ghoussoub, Mathematics
Co-supervisor

Supervisory Committee Member

Ailana Fraser, Mathematics
University Examiner

Gordon Semenoff, Physics
University Examiner

Additional Supervisory Committee Members:

Jingyi Chen, Mathematics
Supervisory Committee Member

Michael Ward, Mathematics
Supervisory Committee Member

Abstract

We obtain a few existence results for elliptic equations.

We develop in Chapter 2 a new infinite dimensional gluing scheme for fractional elliptic equations in the mildly non-local setting. Here it is applied to the catenoid. As a consequence of this method, a counter-example to a fractional analogue of De Giorgi conjecture can be obtained [51].

Then, in Chapter 3, we construct singular solutions to the fractional Yamabe problem using conformal geometry. Fractional order ordinary differential equations are studied.

Finally, in Chapter 4, we obtain the existence to a suitably perturbed doubly-critical Hardy–Schrödinger equation in a bounded domain in the hyperbolic space.

Lay Summary

Jointly with my collaborators we prove that certain equations that involve calculus do have solutions. We use two methods in finding the solutions — by looking at energy levels, or by gluing pieces together.

Preface

All the materials are adapted from the author's research articles [7] (joint work with Weiwei Ao, Azahara DelaTorre, Marco A. Fontelos, Maria del Mar González and Juncheng Wei), [49] (joint work with Yong Liu and Juncheng Wei) and [48] (joint work with Nassif Ghoussoub, Saikat Mazumdar, Shaya Shakerian and Luiz Fernando de Oliveira Faria). These works are put on arXiv (respectively arXiv:1802.07973, arXiv:1711.03215 and arXiv:1710.01271) and are ready for submission. They are under review and have not yet been accepted by any journal.

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Chapter 1

Introduction

The study of elliptic partial differential equations arises in many subjects including notably physics, geometry, biology and applied modeling. Solutions can be considered as the steady-states in reaction-diffusion systems. From the mathematical point of view, the fundamental issues are the existence, regularity, uniqueness, symmetry, and other qualitative properties. In the subsequent chapters we will be dealing with non-local versions of

- the *Allen–Cahn equation*
$$-\Delta u = u - u^3$$
in phase transitions;
- the *Lane–Emden equation*
$$-\Delta u = u^p$$
in astrophysics;

and a local but geometric version of

- the *Schrödinger equation*
$$-\Delta u + Vu = u^p$$
in quantum mechanics.

Non-local equations have attracted a great deal of interest in the last decade. A non-local diffusion term, usually as a model given by a fractional Laplacian, accounts for long range interactions. Intrinsic difficulties arise from the fact that the fractional Laplacian is in fact an integro-differential operator. They are in many cases, but not always, overcome by the Caffarelli–Silvestre extension [43], an equivalent *local* problem in a space with one extra dimension, where classical techniques may be applied. Since then a huge amount of effort has been made in the study of fractional order equations.

Posed by E. De Giorgi [64] in 1979, the conjecture that all bounded entire solutions of the Allen–Cahn equation are one-dimensional at least in dimensions $n \leq 8$, has been almost completely settled: by Ghoussoub–Gui [103] for $n = 2$, Ambrosio–Cabré [12] for $n = 3$, Savin [154] for $4 \leq n \leq 8$ under a mild limit assumption, and del Pino–Kowalczyk–Wei [67] who constructed a counter-example for $n \geq 9$.

Its fractional analogue for $s \in [\frac{1}{2}, 1)$ (having taken into consideration the Γ-convergence result [157]), namely the one-dimensional symmetry of bounded solutions of

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n,$$

has also received considerable attention in low dimensions. Positive results have been obtained in low dimensions by Sire–Valdinoci [165] and Cabré–Sire [37] for $n = 2$ and $s \in (0,1)$, Cabré–Cinti [29, 30] for $n = 3$ and $s \in [\frac{1}{2}, 1)$, Savin [155, 156] for $4 \leq n \leq 8$ and $s \in [\frac{1}{2}, 1)$ again under a limit assumption, and recently Figalli–Serra [94] for $n = 4$ and $s = \frac{1}{2}$.

In order to give a counter-example in high dimensions $n \geq 9$, in Chapter 2 we develop a new infinite dimensional gluing method for fractional elliptic equations. As a model problem, we construct a solution of the fractional Allen–Cahn equation vanishing on a rotationally symmetric surface which resembles a catenoid and has sub-linear growth at infinity. The crux of the analysis is the fine expansion of the fractional Laplacian in Fermi coordinates and the splitting of the inner problem. Via the argument of Jerison–Monneau [120], this leads to counter-examples to De Giorgi Conjecture for the fractional Allen–Cahn equation [51], a work that is in progress.

The Yamabe problem asks to find a constant curvature metric in a given conformal class [180]. This was proved by Trudinger [171] (who also discovered a critical error in Yamabe's proof), Aubin [16] and Schoen [162]. The fractional Yamabe problem, in which a constant fractional curvature is prescribed, takes the form

$$(-\Delta)^s u = u^{\frac{n+2s}{n-2s}} \quad \text{in } \mathbb{R}^n.$$

We consider in Chapter 3 the problem of constructing solutions that are singular at a given smooth sub-manifold, for which we establish the classical gluing method of Mazzeo and Pacard [132] for the scalar curvature in the fractional setting.

From the way infinite dimensional gluing methods were developed, their local nature is apparent – the tangential and normal variables on the hypersurface are separated. While similar technical estimates are needed in the localization by cut-off functions, it is essential to analyze the model linearized operator, where conformal geometry and non-Euclidean harmonic analysis are used. Moreover, the existence of a radial fast-decaying solution needs to be established by a blow-up argument together with a bifurcation method.

The Hardy–Schrödinger operator, whether local or non-local, has a potential that is homogeneous to the Laplacian. Such operator is already seen as the linearization of the singular solution in the fractional Yamabe problem in Chapter 3, where the infinitely many complex indicial roots are computed. In fact, variational problems involving such operator have their own interests.

We study in Chapter 4 the existence of extremals of a non-linear elliptic Hardy–Schrödinger equation in the hyperbolic space. The loss of compactness due to the scaling invariance gives rise to interesting concentration phenomena. Inspired by the recent analysis of Ghoussoub–Robert [105, 106], we obtain sufficient conditions for the attainability of the best constant of Hardy–Sobolev inequalities in terms of the linear perturbation or the mass of the domain.

The essential observation in this work is that, in the radial setting, solutions of the hyperbolic Hardy–Sobolev equation are classified explicitly in terms of the fundamental solutions of the Laplace–Beltrami operator. With this it remains to generalize [105, 106] to include singular perturbations.

To conclude, a strong connection between the fields of partial differential equations and geometry is seen from the geometric quantities that come into play in all the above problems. These results point to similar problems in more general settings, or even parabolic equations. Moreover, the gluing method devised opens up a new area of constructing solutions for non-local equations.

Chapter 2

Fractional Gluing on the Catenoid

2.1 Introduction

2.1.1 The Allen–Cahn equation

In this chapter we are concerned with the fractional Allen–Cahn equation, which takes the form

$$(-\Delta)^s u + f(u) = 0 \quad \text{in } \mathbb{R}^n \tag{2.1}$$

where $f(u) = u^3 - u = W'(u)$ is a typical example that $W(u) = \left(\frac{1-u^2}{2}\right)^2$ is a bi-stable, balanced double-well potential.

In the classical case when $s = 1$, such equation arises in the phase transition phenomenon [11, 45]. Let us consider, in a bounded domain Ω, a rescaled form of the equation (2.1),

$$-\varepsilon^2 \Delta u_\varepsilon + f(u_\varepsilon) = 0 \quad \text{in } \Omega.$$

This is the Euler–Lagrange equation of the energy functional

$$J_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx.$$

The constant solutions $u = \pm 1$ corresponds to the stable phases. For any subset $S \in \Omega$, we see that the discontinuous function $u_S = \chi_S - \chi_{\Omega \setminus S}$ minimize the potential energy, the second term in $J_\varepsilon(u)$. The gradient term, or the kinetic energy, is inserted to penalize unnecessary forming of the interface ∂S.

Using Γ-convergence, Modica [140] proved that any family of minimizers (u_ε) of J_ε with uniformly bounded energy has to converge to some u_S in certain sense, where ∂S has minimal perimeter. Caffarelli and Córdoba [39] proved that the level sets $\{u_\varepsilon = \lambda\}$ in fact converge locally uniformly to the interface.

Observing that the scaling $v_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ solves

$$-\Delta v_\varepsilon + f(v_\varepsilon) = 0 \quad \text{in } \varepsilon^{-1}\Omega,$$

which formally tends as $\varepsilon \to 0$ to (2.1), the intuition is that $v_\varepsilon(x)$ should resemble the one-dimensional solution $\tilde{w}(z) = \tanh \frac{z}{\sqrt{2}}$ where z is the normal coordinate on the interface M, an asymptotically flat minimal surface. Indeed, we have that

$$J_\varepsilon(v_\varepsilon) \approx \text{Area}(M) \int_{\mathbb{R}} \left(\frac{1}{2} \tilde{w}'(z)^2 + W(\tilde{w}(z)) \right) dz.$$

Thus a classification of solutions of (2.1) was conjectured by E. De Giorgi [64].

Conjecture 2.1.1. *Let $s = 1$. At least for $n \leq 8$, all bounded solutions to* (2.1) *monotone in one direction must be one-dimensional, i.e. $u(x) = w(x_1)$ up to translation and rotation.*

It has been proven for $n = 2$ by Ghoussoub and Gui [103], $n = 3$ by Ambrosio and Cabré [12], and for $4 \leq n \leq 8$ under an extra mild assumption by Savin [154]. In higher dimensions $n \geq 9$, a counter-example has been constructed by del Pino, Kowalczyk and Wei [67]. See also [35, 104, 120].

2.1.2 The fractional case and non-local minimal surfaces

While Conjecture 2.1.1 is almost completely settled, a recent and intense interest arises in the study of the fractional non-local equations. A typical non-local diffusion term is the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$, which is defined as a pseudo-differential operator with symbol $|\xi|^{2s}$, or equivalently by a singular integral formula

$$(-\Delta)^s u(x_0) = C_{n,s} \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x_0)-u(x)}{|x_0-x|^{n+2s}}\, dx, \qquad C_{n,s} = \frac{2^{2s} s \Gamma\left(\frac{n+2s}{2}\right)}{\Gamma(1-s)\pi^{\frac{n}{2}}},$$

for locally C^2 functions with at most mild growth at infinity. Caffarelli and Silvestre [43] formulated a local extension problem where the fractional Laplacian is realized as a Dirichlet-to-Neumann map. This extension theorem was generalized by Chang and González [52] in the setting of conformal geometry. Expositions to the fractional Laplacian can be found in [2, 28, 73, 109].

In a parallel line of thought, Γ-convergence results have been obtained by Ambrosio, De Philippis and Martinazzi [13], González [108], and Savin and Valdinoci [157]. The latter authors also proved the uniform convergence of level sets [160]. Owing to the varying strength of the non-locality, the energy

$$J_\varepsilon(u) = \varepsilon^{2s} \|u\|_{H^s(\Omega)} + \int_\Omega W(u)\, dx$$

Γ-converges (under a suitable rescaling) to the classical perimeter functional when $s \in [\frac{1}{2}, 1)$, and to a non-local perimeter when $s \in (0, \frac{1}{2})$.

A singularly perturbed version of (2.1) was studied by Millot and Sire [138] for the critical parameter $s = \frac{1}{2}$, and also by these two authors and Wang [139] in the case $s \in (0, \frac{1}{2})$.

In the highly non-local case $s \in (0, \frac{1}{2})$, the corresponding non-local minimal surface was first studied by Caffarelli, Roquejoffre and Savin [41].

Concerning regularity, Savin and Valdinoci [159] proved that any non-local minimal surface is locally $C^{1,\alpha}$ except for a singular set of Hausdorff dimension $n-3$. Caffarelli and Valdinoci [44] showed that in the asymptotic case $s \to (1/2)^-$, in accordance to the classical minimal surface theory, any s-minimal cone is a hyperplane for $n \leq 7$ and any s-minimal surface is locally a $C^{1,\alpha}$ graph except for a singular set of codimension at least 8. Recently Cabré, Cinti and Serra [31] classified stable s-minimal cones in $\mathbb{R}^3$ when s is close to $(1/2)^-$. Barrios, Figalli and Valdinoci [17] improved the regularity of $C^{1,\alpha}$ s-minimal surfaces to C^∞. Graphical properties and boundary stickiness behaviors were investigated by Dipierro, Savin and Valdinoci [79, 80].

Non-trivial examples of such non-local minimal surface were constructed by Dávila, del Pino and Wei [63] at the limit $s \to (1/2)^-$, as an analog to the catenoid. Note that the non-local catenoid they constructed is eventually linear, as opposed to logarithmic, at infinity; a similar effect is seen in the construction in the present chapter.

Strong interests are also seen in a fractional version of De Giorgi Conjecture.

Conjecture 2.1.2. *Bounded monotone entire solutions to* (2.1) *must be one dimensional, at least for dimensions* $n \leq 8$.

In the rest of this chapter we will focus on the mildly non-local regime $s \in [\frac{1}{2}, 1)$. Positive results have been obtained: $n = 2$ by Sire and Valdinoci [165] and by Cabré and Sire [37], $n = 3$ by Cabré and Cinti [30] (see also Cabré and Solà-Morales [38]), $n = 4$ and $s = \frac{1}{2}$ by Figalli and Serra [94], and the remaining cases for $n \leq 8$ by Savin [155] under an additional mild assumption. A natural question is whether or not Savin's result is *optimal*. In a forthcoming paper [51], we will construct global minimizers in dimension 8 and give counter-examples to Conjecture 2.1.2 for $n \geq 9$ and $s \in (\frac{1}{2}, 1)$.

Some work related to Conjecture 2.1.2 involving more general operators includes [27, 34, 81, 90, 158]. For similar results in elliptic systems, the readers are referred to [20, 21, 74, 87–89, 91, 174, 175] for the local case, and [25, 77, 92, 176] under the fractional setting.

The construction of solution by gluing for non-local equations is a relatively new subject. Du, Gui, Sire and Wei [82] proved the existence of multi-layered solutions of (2.1) when $n = 1$. Other work involves the fractional Schrödinger equation [54, 62], the fractional Yamabe problem [15] and non-local Delaunay surfaces [58].

For general existence theorems for non-local equations, the readers may consult, among others, [53, 55, 95, 96, 116, 141, 143, 145, 146, 167, 170, 177, 178] as well as the references therein. Related questions on the fractional Allen–Cahn equations, non-local isoperimetric problems and non-local free boundary problems are also widely studied in [24, 42, 69, 70, 72, 75, 78, 93, 125, 127]. See also the expository articles [1, 100, 172].

Despite similar appearance, (2.1) for $s \in (0,1)$ is different from that for $s = 1$ in a number of striking ways. Firstly, the non-local nature disallows the use of local Fermi coordinates. Secondly, the one-dimensional solution $w(z)$ only has an algebraic decay of order $2s$ at infinity, in contrast to the exponential decay when $s = 1$. Thirdly, the fractional Laplacian is a strongly coupled operator and hence it is impossible to "integrate by parts" in lower dimensions. Finally the inner-outer gluing using cut-off functions no longer work due to the nonlocality of the fractional operator.

The purpose of this chapter is to establish a *new* gluing approach for fractional elliptic equations for constructing solutions with a layer over higher-dimensional sub-manifolds. In particular, in the second part [51] we will apply it to partially answer Conjecture 2.1.2. To overcome the aforementioned difficulties, the main tool is an expansion of the fractional Laplacian in the Fermi coordinates, a refinement of the computations already seen in [50], supplemented by technical integral calculations. This can be considered *fractional Fermi coordinates*. When applying an infinite dimensional Lyapunov–Schmidt reduction, the orthogonality condition is to be expressed in the extension. The essential difference from the classical case [68] is that the inner problem is subdivided into many pieces of size $R = o(\varepsilon^{-1})$, where ε is the scaling parameter, so that the manifold is nearly flat on each piece. In this way, in terms of the Fermi normal coordinates, the equations can be well approximated by a model problem.

2.1.3 A brief description

We define an approximate solution $u^*(x)$ using the one-dimensional profile in the tubular neighborhood of $M_\varepsilon = \{|x_n| = F_\varepsilon(|x'|)\}$, namely $u^*(x) = w(z)$ where z is the normal coordinate and F_ε is close to the catenoid $\varepsilon^{-1}\cosh^{-1}(\varepsilon|x'|)$ near the origin. In contrast to the classical case we take into account the non-local interactions near infinity and define $u^*(x) = w(z_+) + w(z_-) + 1$ where $z_\pm$ are the signed distances to the upper and lower leaves $M_\varepsilon^\pm = \{x_n = \pm F_\varepsilon(|x'|)\}$. As hinted in Corollary 2.6.3,

$F_\varepsilon(r) \sim r^{\frac{2}{2s+1}}$ as $r \to +\infty$. The parts of u^* are glued to the constant solutions ± 1 smoothly to the regions where the Fermi coordinates are not well-defined.

We look for a real solution of the form $u = u^* + \varphi$, where φ is small and satisfies

$$(-\Delta)^s \varphi + f'(u^*)\varphi = g. \tag{2.2}$$

Our new idea is to localize the error in the near interface into many pieces of diameter $R = o(\varepsilon^{-1})$ for another parameter R which is to be taken large. At each piece the hypersurface is well-approximated by some tangent hyperplane. Therefore, using Fermi coordinates, it suffices to study the model problem where $u^*(x)$ is replaced by $w(z)$ in (2.2).

As opposed to the local case $s = 1$, an integration by parts is not available for the fractional Laplacian in the z-direction, unless $n = 1$. So we develop a linear theory using the Caffarelli–Silvestre local extension [43].

Finally we will solve a non-local, non-linear reduced equation which takes the form

$$\begin{cases} H[F_\varepsilon] = O(\varepsilon^{2s-1}) & \text{for } 1 < r \leq r_0, \\ H[F_\varepsilon] = \dfrac{C\varepsilon^{2s-1}}{F_\varepsilon^{2s}}(1 + o(1)) & \text{for } r > r_0, \end{cases}$$

where $H[F_\varepsilon]$ denotes the mean curvature of the surface described by F_ε. (Note that the surface is adjusted far away through the nonlocal interactions of the leafs. A similar phenomenon has been observed in Agudelo, del Pino and Wei [10] for $s = 1$ and dimensions ≥ 4.) A solution of the desired form can be obtained using the contraction mapping principle, justifying the *a priori* assumptions on F_ε.

In this setting, our main result can be stated as follows.

Theorem 2.1.3. *Let* $1/2 < s < 1$ *and* $n = 3$*. For all sufficiently small* $\varepsilon > 0$*, there exists a rotationally symmetric solution* u *to* (2.1) *with the zero level set* $M_\varepsilon = \left\{(x', x_3) \in \mathbb{R}^3 : |x_3| = F_\varepsilon(|x'|)\right\}$*, where*

$$F_\varepsilon(r) \sim \begin{cases} \varepsilon^{-1}\cosh^{-1}(\varepsilon r) & \textit{for } r \leq r_\varepsilon, \\ r^{\frac{2}{2s+1}} & \textit{for } r \geq \delta_0 |\log \varepsilon| r_\varepsilon, \end{cases}$$

where $r_\varepsilon = \left(\frac{|\log\varepsilon|}{\varepsilon}\right)^{\frac{2s-1}{2}}$ *and* $\delta_0 > 0$ *is a small fixed constant.*

In a forthcoming paper [51], together with Juan Dávila and Manuel del Pino, we will construct similarly a global minimizer on the Simons' cone. Via the Jerison–Monneau program [120], this provides counter-examples to the De Giorgi conjecture for fractional Allen–Cahn equation in dimensions $n \geq 9$ for $s \in (\frac{1}{2}, 1)$.

Remark 2.1.4. Our approach depends crucially on the assumption $s \in (\frac{1}{2}, 1)$. Firstly, it is only in this regime that the local mean curvature *alone* appears in the error estimate. A related issue is also seen in the choice of those parameters between 0 and (a factor times) $2s-1$. Secondly, it gives the L^2 integrability of an integral involving the kernel w_z in the extension. It will be interesting to see whether this gluing method will work in the case $s = \frac{1}{2}$ under suitable modifications.

On the other hand, we do not know how to deal with other pseudo-differential operators which cannot be realized locally.

This chapter is organized as follows. We outline the argument with key results in Section 2.2. In Section 2.3 we compute the error using an expansion of the fractional Laplacian in the Fermi coordinates. In Section 2.4 we develop a linear theory and then the gluing reduction is carried out in Section 2.5. Finally in Section 2.6 we solve the reduced equation.

2.2 Outline of the construction

2.2.1 Notations and the approximate solution

Let

- $s \in (\frac{1}{2}, 1)$, $\alpha \in (0, 2s-1)$, $\tau \in \left(1, 1 + \frac{\alpha}{2s}\right)$,
- M be an approximation to the catenoid defined by the function F,

$$M = \left\{(x', x_n) : |x_n| = F(\left|x'\right|),\ \left|x'\right| \geq 1\right\},$$

- $\varepsilon > 0$ be the scaling parameter in

$$M_\varepsilon = \varepsilon^{-1}M = \left\{x_n = F_\varepsilon(\left|x'\right|) = \varepsilon^{-1}F(\varepsilon\left|x'\right|)\right\},$$

- z be the normal coordinate direction in the Fermi coordinates of the rescaled manifold, i.e. signed distance to the M_ε, with $z > 0$ for $x_n > F(\varepsilon|x'|) > 0$,

- y_+, z_+ be respectively the projection onto and signed distance (increasing in x_n) from the upper leaf

$$M_\varepsilon^+ = \left\{x_n = F_\varepsilon(\left|x'\right|)\right\},$$

- y_-, z_- be respectively the projection onto and signed distance (decreasing in x_n) to the lower leaf

$$M_\varepsilon^- = \left\{x_n = -F_\varepsilon(\left|x'\right|)\right\},$$

- $\bar{\delta} > 0$ be a small fixed constant so that the Fermi coordinates near M_ε is defined for $|z| \leq \frac{8\bar{\delta}}{\varepsilon}$,

- $\bar{R} > 0$ be a large fixed constant,

- R_0 be the width of the tubular neighborhood of M_ε where Fermi coordinates are used, see (2.3),

- R_1 be the radius of the cylinder from which the main contribution of $(-\Delta)^s$ is obtained, see Proposition 2.2.1,

- $R_2 > \frac{4\bar{R}}{\varepsilon}$ be the radius of the inner gluing region (i.e. threshold of the end, see Section 2.2.3),

- $u_o^*(x) = \operatorname{sign}(x_n - F_\varepsilon(|x'|))$ for $x_n > 0$ and is extended continuously (i.e. $u_o^*(x) = +1$ for $|x'| \leq \varepsilon^{-1}$),

- $\eta : \mathbb{R} \to [0,1]$ be a cut-off with $\eta = 1$ on $(-\infty, 1]$ and $\eta = 0$ on $[2, +\infty)$,

- $\chi : \mathbb{R} \to [0,1]$ be a cut-off with $\chi = 0$ on $(-\infty, 0]$ and $\chi = 1$ on $[1, +\infty)$,

- $\|\kappa\|_\alpha$ ($0 \leq \alpha < 1$) be the Hölder norm of the curvature, see Lemma 2.3.6,

- $\langle x\rangle = \sqrt{1+|x|^2}$.

Define the approximate solution

$$u^*(x) = \eta\left(\frac{\varepsilon|z|}{\bar{\delta}R_0(|x'|)}\right)\left(w(z)+\chi\left(|x'|-\frac{\bar{R}}{\varepsilon}\right)(w(z_+)+w(z_-)+1-w(z))\right) \\ +\left(1-\eta\left(\frac{\varepsilon|z|}{\bar{\delta}R_0(|x'|)}\right)\right)u_o^*(x), \tag{2.3}$$

where

$$R_0 = R_0(|x'|) = 1+\chi\left(|x'|-\bar{R}\right)\left(F_\varepsilon^{2s}(|x'|)-1\right).$$

Roughly,

- $u^*(x) = +1$ for large $|z|$, small $|x'|$ and large $|x_n|$,
- $u^*(x) = -1$ for large $|z|$, large $|x'|$ and small $|x_n|$,
- $u^*(x) = w(z)$ for small $|z|$ and small $|x'|$,
- $u^*(x) = w(z_+)+w(z_-)+1$ for small $|z|$ and large $|x'|$.

The main contributions of $(-\Delta)^s$ come from the inner region with certain radius. We choose such radius that joins a small constant times ε^{-1} to a power of F_ε as $|x'|$ increases. More precisely, let us set

$$R_1 = R_1(|x'|) = \eta\left(|x'|-\frac{2\bar{R}}{\varepsilon}+2\right)\frac{\bar{\delta}}{\varepsilon}+\left(1-\eta\left(|x'|-\frac{2\bar{R}}{\varepsilon}+2\right)\right)F_\varepsilon^{\tau}(|x'|), \tag{2.4}$$

where $\tau\in\left(1,1+\frac{\alpha}{2s}\right)$. We remark that the factor 2 is inserted to make sure that $u^*(x) = w(z_+)+w(z_-)-1$ in the whole ball of radius $F_\varepsilon^{\tau}(|x'|)$ where the main order terms of $(-\Delta)^s u^*$ are obtained.

2.2.2 The error

Denote the error by $S(u^*) = (-\Delta)^s u^* + (u^*)^3 - u^*$. In a tubular neighborhood where the Fermi coordinates are well-defined, write $x = y+z\nu(y)$ where $y = y(|x'|) =$

$(|x'|, F_\varepsilon(|x'|)) \in M_\varepsilon$ and $\nu(y) = \nu(y(|x'|)) = \dfrac{(-DF_\varepsilon(|x'|), 1)}{\sqrt{1+|DF_\varepsilon(|x'|)|^2}}$ be the unit normal pointing up in the upper half space (and down in the lower half).

Proposition 2.2.1. *Let $x = \mathrm{y} + z\nu(\mathrm{y}) \in \mathbb{R}^n$. If $|z| \le R_1$, where R_1 as in* (2.4)*, then we have*

$$S(u^*)(x) = \begin{cases} c_H(z) H_{M_\varepsilon}(\mathrm{y}) + O(\varepsilon^{2s}), & \text{for } \dfrac{1}{\varepsilon} \le r \le \dfrac{4\bar{R}}{\varepsilon}, \\ c_H(z_+) H_{M_\varepsilon^+}(\mathrm{y}_+) + c_H(z_-) H_{M_\varepsilon^-}(\mathrm{y}_-) & \\ \quad +3(w(z_+)+w(z_-))(1+w(z_+))(1+w(z_-)) & \\ \quad +O\left(F_\varepsilon^{-2s\tau}\right), & \text{for } r \ge \dfrac{4\bar{R}}{\varepsilon}. \end{cases}$$

The proof is given in Section 2.3.

2.2.3 The gluing reduction

We look for a solution of (2.1) of the form $u = u^* + \varphi$ so that

$$(-\Delta)^s \varphi + f'(u^*)\varphi = S(u^*) + N(\varphi) \quad \text{ in } \mathbb{R}^n,$$

where $N(\varphi) = f(u^* + \varphi) - f(u^*) - f'(u^*)\varphi$. Consider the partition of unity

$$1 = \tilde{\eta}_o + \tilde{\eta}_+ + \tilde{\eta}_- + \sum_{i=1}^{\bar{i}} \tilde{\eta}_i,$$

where the support of each $\tilde{\eta}_i$ is a region of radius R centered at some $\mathrm{y}_i \in M_\varepsilon$, and $\tilde{\eta}_\pm$ are supported on a tubular neighborhood of the ends of $M_\varepsilon^\pm$ respectively. It will be convenient to denote $\mathscr{I} = \{1, \dots, \bar{i}\}$ and $\mathscr{J} = \mathscr{I} \cup \{+,-\}$. For $j \in \mathscr{J}$, let ζ_j be cut-off functions such that the sets $\left\{\zeta_j = 1\right\}$ include $\operatorname{supp} \tilde{\eta}_j$, with comparable spacing that is to be made precise. We decompose

$$\varphi = \phi_o + \zeta_+ \phi_+ + \zeta_- \phi_- + \sum_{i=1}^{\bar{i}} \zeta_i \phi_i = \phi_o + \sum_{j \in \mathscr{J}} \zeta_j \phi_j,$$

in which

- ϕ_o solves for the contribution of the error away from the interface (support of $\tilde{\eta}_o$),
- $\phi_\pm$ solves for that in the far interfaces near $M_\varepsilon^\pm$ (support of $\tilde{\eta}_\pm$),
- ϕ_i solves for that in a compact region near the manifold (support of $\tilde{\eta}_i$).

In the following we write $\Delta_{(y,z)} = \Delta_y + \partial_{zz}$.We consider the approximate linear operators

$$\begin{cases} L_o = (-\Delta)^s + 2 & \text{for } \phi_o, \\ L = (-\Delta_{(y,z)})^s + f'(w) & \text{for } \phi_j, \quad j \in \mathscr{J}. \end{cases}$$

Notice that w is not exactly the approximate solution in the far interface. We rearrange the equation as

$$(-\Delta)^s \left(\phi_o + \sum_{j\in\mathscr{J}} \zeta_j \phi_j \right) + f'(u^*) \left(\phi_o + \sum_{j\in\mathscr{J}} \zeta_j \phi_j \right) = S(u^*) + N(\varphi),$$

$$\begin{aligned} L_o\phi_o + \zeta_+ L\phi_+ + \zeta_- L\phi_- + \sum_{i=1}^{\bar{i}} \zeta_i L\phi_i & \\ = \left(\tilde{\eta}_o + \tilde{\eta}_+ + \tilde{\eta}_- + \sum_{i=1}^{\bar{i}} \tilde{\eta}_i \right) & \\ \cdot \Bigg(S(u^*) + N(\varphi) + (2 - f'(u^*))\phi_o - \sum_{j\in\mathscr{J}} [(-\Delta_{(y,z)})^s, \zeta_j]\phi_j & \\ + \sum_{j\in\mathscr{J}} \zeta_j (f'(w_j) - f'(u^*))\phi_j - \sum_{j\in\mathscr{J}} ((-\Delta_x)^s - (-\Delta_{(y,z)})^s)(\zeta_j\phi_j) \Bigg), & \quad (2.5) \end{aligned}$$

where $[(-\Delta_{(y,z)})^s, \zeta_j]\phi_j = (-\Delta_{(y,z)})^s(\zeta_j\phi_j) - \zeta_j(-\Delta_{(y,z)})^s\phi_j$, and the summands in the last term means

$$(-\Delta_x)^s(\zeta_j\phi_j)(Y_j(y) + z\nu(Y_j(y))) - (-\Delta_{(y,z)})^s(\bar{\eta}_j\bar{\zeta}\bar{\phi}(y,z))$$

for $\zeta_j = \bar{\eta}_j(y)\bar{\zeta}(z)$ and $\phi_j(Y_j(y) + z\nu(Y_j(y))) = \bar{\phi}_j(y,z)$ with a chart $\mathrm{y} = Y_j(y)$ of M_ε. In fact, for $j \in \mathscr{J}$ one can parameterize M_ε locally by a graph over a tangent hyperplane, and for $j \in \{+,-\}$ one uses the natural graph $M_\varepsilon^{\pm} = \{(y, \pm F_\varepsilon(|y|)) : |y| \geq R_2\}$.

Let us denote the last bracket of the right hand side of (2.5) by $\mathscr{G}$. Since $\tilde{\eta}_j = \zeta_j \tilde{\eta}_j$, we will have solved (2.5) if we get a solution to the system

$$\begin{cases} L_o \phi_o = \tilde{\eta}_o \mathscr{G} & \text{for } x \in \mathbb{R}^n, \\ L\bar{\phi}_+ = \tilde{\eta}_+ \mathscr{G} & \text{for } (y,z) \in \mathbb{R}^{n-1} \times \mathbb{R}, \\ L\bar{\phi}_- = \tilde{\eta}_- \mathscr{G} & \text{for } (y,z) \in \mathbb{R}^{n-1} \times \mathbb{R}, \\ L\bar{\phi}_i = \tilde{\eta}_i \mathscr{G} & \text{for } (y,z) \in \mathbb{R}^{n-1} \times \mathbb{R}, \end{cases}$$

for all $i \in \mathscr{I}$. Except the outer problem with $L_o = (-\Delta)^s + 2$, the linear operator L in all the other equations has a kernel w' and so we will use an infinite dimensional Lyapunov–Schmidt reduction procedure.

From now on we consider the product cut-off functions, defined in the Fermi coordinates (y, z) where $\mathrm{y} = Y(y)$ is given by a chart of M_ε,

$$\tilde{\eta}_j(x) = \eta_j(\mathrm{y})\zeta(z), \quad \text{for } j \in \mathscr{J}.$$

The diameters of $\zeta(z)$ and $\eta_i(\mathrm{y})$ are of order R, a parameter which we choose to be large (before fixing ε). We may assume, without loss of generality, that for $i \in \mathscr{I}$, $\eta_i(\mathrm{y})$ is centered at $\mathrm{y}_i \in M_\varepsilon$, $B_R(\mathrm{y}_i) \subset \{\tilde{\eta}_i = 1\} \subset \operatorname{supp} \tilde{\eta}_i \subset B_{2R}(\mathrm{y}_i)$, $|D\tilde{\eta}_i| = O(R^{-1})$, and $\frac{|\mathrm{y}_{i_1} - \mathrm{y}_{i_2}|}{R} \geq c > 0$ for any $i_1, i_2 \in \mathscr{I}$.

We define the *projection* orthogonal to the kernels $w'(z)$,

$$\Pi g(y,z) = g(y,z) - c(y) w'(z), \quad c(y) = \frac{\displaystyle\int_{\mathbb{R}} \zeta(\tilde{z}) g(y,\tilde{z}) w'(\tilde{z})\, d\tilde{z}}{\displaystyle\int_{\mathbb{R}} \zeta(\tilde{z}) w'(\tilde{z})^2\, d\tilde{z}}.$$

Note that in the region of integration $|z| \leq 2R < \bar{\delta}\varepsilon^{-1}$ the Fermi coordinates are well-defined, and that the projection is independent of $j \in \mathscr{J}$.

We define the norm

$$\|\phi\|_{\mu,\sigma} = \sup_{(y,z) \in \mathbb{R}^n} \langle y \rangle^\mu \langle z \rangle^\sigma |\phi(y,z)|,$$

where $\langle y\rangle = \sqrt{1+|y|^2}$. Motivated by Proposition 2.2.1 and Lemma 2.4.6, for each $i \in \mathscr{I}$ we expect the decay

$$\left\|\bar{\phi}_i(y,z)\right\|_{\mu,\sigma} \le CR^{\mu+\sigma} \langle \mathrm{y}_i\rangle^{-\frac{4s}{2s+1}}.$$

So we define

$$\|\phi_i\|_{i,\mu,\sigma} = \langle \mathrm{y}_i\rangle^{\theta} \left\|\bar{\phi}_i\right\|_{\mu,\sigma} = \langle \mathrm{y}_i\rangle^{\theta} \sup_{(y,z)\in\mathbb{R}^n} \langle y\rangle^{\mu} \langle z\rangle^{\sigma} \left|\bar{\phi}_i(y,z)\right|,$$

with $1 < \theta < 1 + \frac{2s-1}{2s+1} = \frac{4s}{2s+1} < 2s$. At the ends $M_\varepsilon^{\pm}$ where $r \ge R_2$ we have, for $\mu < \frac{4s}{2s+1} - \theta$,

$$\left\|\bar{\phi}_{\pm}(y,z)\right\|_{\mu,\sigma} \le CR_2^{-\left(\frac{4s}{2s+1}-\mu\right)}.$$

This suggests

$$\|\phi_{\pm}\|_{\pm,\mu,\sigma} = R_2^{\theta} \left\|\bar{\phi}_{\pm}\right\|_{\mu,\sigma} = R_2^{\theta} \sup_{(y,z)\in\mathbb{R}^n} \langle y\rangle^{\mu} \langle z\rangle^{\sigma} \left|\bar{\phi}_{\pm}(y,z)\right|,$$

with $0 < \theta < \frac{2s-1}{2s+1} - \mu$. Therefore for $j \in \mathscr{J}$, we consider the Banach spaces

$$X_j = \left\{\phi_j : \left\|\phi_j\right\|_{j,\mu,\sigma} < \tilde{C}\delta\right\},$$

where, under the constraint $R \le |\log\varepsilon|$, $\delta = \delta(R,\varepsilon) = R^{\mu+\sigma}\varepsilon^{\frac{4s}{2s+1}-\theta}$ with $1 < \theta < 1 + \frac{2s-1}{2s+1} = \frac{4s}{2s+1}$. For the other parameters we take $0 < \mu < \frac{4s}{2s+1} - \theta < \theta$ sufficiently small and R_2 sufficiently large, so that $R_2^{\mu}\delta$ is small and $2-2s < \sigma < 2s-\mu$. The decay of order $\sigma > 2-2s$ in the z-direction will be required in the orthogonality condition (2.21). That $R_2^{\mu}\delta$ is small will be used in the inner gluing reduction. The condition $\sigma+\mu < 2s$ ensures that the contribution of the term $(2-f'(u^*))\phi_o$ is small compared to $S(u^*)$.

We will first solve the outer equation for ϕ_o. Let us write

$$M_{\varepsilon,R} = \{\mathrm{y} + z\nu(\mathrm{y}) : \mathrm{y} \in M_\varepsilon \text{ and } |z| < R\}$$

for the tubular neighborhood of M_ε with width R.

Proposition 2.2.2. *Consider*

$$\|\phi_o\|_\theta = \sup_{(x',x_n)\in\mathbb{R}^n} \langle x'\rangle^\theta \langle \operatorname{dist}(x, M_{\varepsilon,R})\rangle^{2s} |\phi_o(x)|,$$

$$X_o = \left\{\phi_o : \|\phi_o\|_\theta \le \tilde{C}\varepsilon^\theta\right\}.$$

If $\phi_j \in X_j$ for all $j \in \mathscr{J}$ with $\sup_{j\in\mathscr{J}} \left\|\phi_j\right\|_{j,\mu,\sigma} \le 1$, then there exists a unique solution $\phi_o = \Phi_o((\phi_j)_{j\in\mathscr{J}})$ to

$$L_o\phi_o = \tilde{\eta}_o\mathscr{G} = \tilde{\eta}_o\Bigg(S(u^*) + N(\varphi) + (2 - f'(u^*))\phi_o - \sum_{j\in\mathscr{J}} [(-\Delta_{(y,z)})^s, \zeta_j]\phi_j$$
$$+ \sum_{j\in\mathscr{J}} \zeta_j(f'(w_j) - f'(u^*))\phi_j - \sum_{j\in\mathscr{J}} ((-\Delta_x)^s - (-\Delta_{(y,z)})^s)(\zeta_j\phi_j)\Bigg) \quad \text{in } \mathbb{R}^n \tag{2.6}$$

in X_o such that for any pairs $(\phi_j)_{j\in\mathscr{J}}$ and $(\psi_j)_{j\in\mathscr{J}}$ in the respective X_j with $\sup_{j\in\mathscr{J}} \left\|\phi_j\right\|_{j,\mu,\sigma} \le 1$,

$$\left\|\Phi_o((\phi_j)_{j\in\mathscr{J}}) - \Phi_o((\psi_j)_{j\in\mathscr{J}})\right\|_\theta \le C\varepsilon^\theta \sup_{j\in\mathscr{J}} \left\|\phi_j - \psi_j\right\|_{j,\mu,\sigma}. \tag{2.7}$$

The proof is carried out in Section 2.5.2.

Then the equations

$$L\bar{\phi}_j(y,z) = \eta_j(y)\zeta(z)\mathscr{G}(y,z)$$

are solved in two steps: (1) eliminating the part of error orthogonal to the kernels, i.e.

$$L\bar{\phi}_j(y,z) = \eta_j(y)\zeta(z)\Pi\mathscr{G}(y,z); \tag{2.8}$$

and (2) adjust $F_\varepsilon(r)$ such that $c(y) = 0$, i.e. to solve the reduced equation

$$\int_{\mathbb{R}} \zeta(z)\mathscr{G}(y,z)w'(z)\,dz = 0. \tag{2.9}$$

Using the linear theory in Section 2.4, step (1) is proved in the following

Proposition 2.2.3. *Suppose $\mu \leq \theta$. Then there exists a unique solution $(\phi_j)_{j\in\mathscr{J}}$, $\phi_j \in X_j$, to the system*

$$L\bar{\phi}_j = \tilde{\eta}_j \Pi\mathscr{G} = \eta_j \zeta \Pi\Bigg(S(u^*) + N(\varphi) + (2 - f'(u^*))\phi_o - \sum_{j\in\mathscr{J}} [(-\Delta_{(y,z)})^s, \zeta_j]\phi_j$$
$$+ \sum_{j\in\mathscr{J}} \zeta_j (f'(w_j) - f'(u^*))\phi_j - \sum_{j\in\mathscr{J}} ((-\Delta_x)^s - (-\Delta_{(y,z)})^s)(\zeta_j\phi_j)\Bigg) \quad (2.10)$$

for $(y,z) \in \mathbb{R}^n$.

The proof is given in Section 2.5.3.

Step (2) is outlined in the next subsection.

2.2.4 Projection of error and the reduced equation

As shown above, the error is to be projected onto w_j' weighted with a cut-off function ζ supported on $[-2R, 2R]$. In fact we have

Proposition 2.2.4 (The reduced equation)**.** *In terms of the rescaled function $F(r) = \varepsilon F_\varepsilon(\varepsilon^{-1}r)$ and its inverse $r = G(\mathtt{z})$ where $G : [0,+\infty) \to [1,+\infty)$,* (2.9) *is equivalent to the system*

$$\begin{cases} H_M(G(\mathtt{z}),\mathtt{z}) = \left(\dfrac{G'(\mathtt{z})}{\sqrt{1+G'(\mathtt{z})^2}}\right)' - \dfrac{1}{G(\mathtt{z})\sqrt{1+G'(\mathtt{z})^2}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad = N_1[F] \quad \text{for } 0 \leq \mathtt{z} \leq \mathtt{z}_1, \\ H_M(r,F(r)) = \dfrac{1}{r}\left(\dfrac{rF'(r)}{\sqrt{1+F'(r)^2}}\right)' = N_1[F] \qquad \text{for } r_1 \leq r \leq 4\bar{R}, \\ F''(r) + \dfrac{F'(r)}{r} - \dfrac{\bar{C}_0\varepsilon^{2s-1}}{F^{2s}(r)} = N_2[F] \qquad \text{for } r \geq 4\bar{R}, \end{cases} \quad (2.11)$$

subject to the boundary conditions

$$\begin{cases} G(0) = 1 \\ G'(0) = 0 \\ F(r_1) = \mathtt{z}_1 \\ F'(r_1) = \dfrac{1}{G'(\mathtt{z}_1)}, \end{cases} \tag{2.12}$$

where $\mathtt{z}_1 = F(r_1) = O(1)$*,* $N_1[F] = O(\varepsilon^{2s-1})$ *and* $N_2[F] = o\left(\dfrac{\varepsilon^{2s-1}}{F_0^{2s}(r)}\right)$*, with* F_0 *as in Corollary 2.6.3. Moreover,* N_1 *and* N_2 *have a Lipschitz dependence on* F*.*

This is proved in Section 2.6.1.

The equation (2.11)–(2.12) is to be solved in a space with weighted Hölder norms allowing sub-linear growth. More precisely, for any $\alpha \in (0,1)$, $\gamma \in \mathbb{R}$ we define the norms

$$\begin{aligned} \|\phi\|_* = \sup_{[r_1,+\infty)} r^{\gamma-2}|\phi(r)| + \sup_{[r_1,+\infty)} r^{\gamma-1}\left|\phi'(r)\right| + \sup_{[r_1,+\infty)} r^{\gamma}\left|\phi''(r)\right| \\ + \sup_{r\neq\rho \text{ in } [r_1,+\infty)} \min\{r,\rho\}^{\gamma+\alpha} \frac{|\phi''(r)-\phi''(\rho)|}{|r-\rho|^{\alpha}} \end{aligned} \tag{2.13}$$

and

$$\|h\|_{**} = \sup_{r\in[1,+\infty)} r^{\gamma}|h(r)| + \sup_{r\neq\rho \text{ in } [1,+\infty)} \min\{r,\rho\}^{\gamma+\alpha} \frac{|h(r)-h(\rho)|}{|r-\rho|^{\alpha}}. \tag{2.14}$$

Proposition 2.2.5. *There exists a solution to* (2.11) *in the space*

$$X_* = \left\{ (G,F) \in C^{2,\alpha}([0,\mathtt{z}_1]) \times C^{2,\alpha}_{\text{loc}}([r_1,+\infty)) : \begin{array}{l} \|G\|_{C^{2,\alpha}([0,\mathtt{z}_1])} < +\infty, \\ \|F\|_* < +\infty, \\ (2.12) \text{ holds} \end{array} \right\}.$$

The proof is contained in Section 2.6.

2.3 Computation of the error: Fermi coordinates expansion

We prove the following

Proposition 2.3.1 (Expansion in Fermi coordinates)**.** *Suppose* $0<\alpha<2s-1$ *and* $F_\varepsilon \in C^{2,\alpha}_{\mathrm{loc}}([1,+\infty))$. *Let* $x_0 = \mathrm{y}_0 + z_0\nu(\mathrm{y}_0)$ *where* $\mathrm{y}_0 = (x', F_\varepsilon(|x'|))$ *is the projection of* x_0 *onto* M_ε*, and* $u_0(x) = w(z)$*. Then for any* $\tau \in \left(1, 1+\frac{\alpha}{2s}\right)$ *and* $|z_0| \le R_1$*, we have*

$$(-\Delta)^s u_0(x_0) = w(z_0) - w(z_0)^3 + c_H(z_0) H_{M_\varepsilon}(\mathrm{y}_0) + N_1[F]$$

where

$$c_H(z_0) = C_{1,s}\int_{\mathbb{R}} \frac{w(z_0)-w(z)}{|z_0-z|^{1+2s}}(z_0-z)\,dz,$$

$$R_1 = R_1(|x'|) = \eta\left(|x'| - \frac{2\bar{R}}{\varepsilon}+2\right)\frac{\bar{\delta}}{\varepsilon} + \left(1-\eta\left(|x'|-\frac{2\bar{R}}{\varepsilon}+2\right)\right)F_\varepsilon^\tau(|x'|),$$

and $N_1[F] = O\left(R_1^{-2s}\right)$ *is finite in the norm* $\|\cdot\|_{**}$.

Remark 2.3.2. $c_H(z_0)$ is even in z_0. Also

$$c_H(z_0) = \frac{C_{1,s}}{2s-1}\int_{\mathbb{R}} \frac{w'(z)}{|z_0-z|^{2s-1}}\,dz \sim \langle z_0\rangle^{-(2s-1)}.$$

This implies Proposition 2.2.1. A proof is given at the end of this section.

A similar computation gives the decay in $r=|x'|$ away from the interface.

Corollary 2.3.3. *Suppose* $x_0 = \mathrm{y}_0 + z_0\nu(\mathrm{y}_0)$, $\mathrm{y}_0 = (x_0', F_\varepsilon(r_0))$ *and* $z_0 \ge c r_0^{\frac{2}{2s+1}}$.

$$(-\Delta)^s u^*(x_0) = O\left(r_0^{-\frac{4s}{2s+1}}\right) \quad \text{as } r_0 \to +\infty.$$

Proof. Take a ball around x_0 of radius of order $r_0^{\frac{2}{2s+1}}$. In the inner region one uses the closeness to $+1$ of the approximate solution u^*. □

For more general functions one has a less precise expansion. On compact sets, we have

Corollary 2.3.4. *Let $u_1(x) = \phi(y,z)$ in a neighborhood of $x_0 = \mathrm{y}_0 + z_0\nu(\mathrm{y}_0)$ where $|y_0|, |z_0| \le 4R = o(\varepsilon^{-1})$, and $u_1 = 0$ outside a ball of radius $8R$. Then*

$$(-\Delta_x)^s u_1(x_0) = (-\Delta_{(y,z)})^s \phi(y_0,z_0) \cdot (1 + O(R\|\kappa\|_0)) \\ + O\left(R_1^{-2s}\left(|\phi(y_0,z_0)| + \sup_{|(y_0-y,z_0-z)|\ge R_1} |\phi(y,z)|\right)\right).$$

Proof. The lower order terms contain either $\kappa_i|z_0|$ or $\kappa_i|y_0|$, where $i = 1$ or 2. □

At the ends of the catenoidal surface we need the following

Corollary 2.3.5. *Let $u_1(x) = \phi(y,z)$ in a neighborhood of $x_0 = y_0 + z_0\nu(y_0)$ where $|y_0| \ge R_2$, $|z_0| \le 4R = o(\varepsilon^{-1})$, and $u_1 = 0$ when $z \ge 8R$. Then*

$$(-\Delta_x)^s u_1(x_0) = (-\Delta_{(y,z)})^s \phi(y_0,z_0) \cdot \left(1 + O\left(F_\varepsilon^{-(2s-\tau)}\right)\right) \\ + O\left(F_\varepsilon^{-2s\tau}\left(|\phi(y_0,z_0)| + \sup_{|(y_0-y,z_0-z)|\ge F_\varepsilon^\tau} |\phi(y,z)|\right)\right).$$

To prove Proposition 2.3.1, we consider M_ε as a graph in a neighborhood of y_0 over its tangent hyperplane and use the Fermi coordinates. Suppose (y_1,y_2,z) is an orthonormal basis of the tangent plane of M_ε at y_0. Write

$$C_{R_1} = \left\{(y,z) \in \mathbb{R}^2 \times \mathbb{R} : |y| \le R_1,\, |z| \le R_1\right\}.$$

Then there exists a smooth function $g : B_{R_1}(0) \to \mathbb{R}$ such that, in the (y,z) coordinates,

$$M_\varepsilon \cap C_{R_1} = \left\{(y, g(y)) \in \mathbb{R}^3 : |y| \le R_1\right\}.$$

Then $g(0) = 0$, $Dg(0) = 0$ and $\Delta g(0) = 2H_{M_\varepsilon}(x_0)$. We may also assume that $\partial_{y_1y_2} g(0) = 0$. We denote the principal curvatures at y by $\kappa_i(y)$ so that $\kappa_i(0) = \partial_{y_iy_i} g(0)$.

We state a few lemmata whose non-trivial proofs are postponed to the end of this section.

Lemma 2.3.6 (Local expansions)**.** *Let* $|y| \le R_1$*. For* $i = 1,2$ *we have*

$$|\kappa_i(y) - \kappa_i(0)| \lesssim \|\kappa_i\|_{C^\alpha(B_{2R_1}(|x'|))} |y|^\alpha \lesssim \left\|F_\varepsilon^{-2s}\right\|_{C^\alpha(B_1(|x'|))} |y|^\alpha$$
$$\lesssim \begin{cases} \varepsilon^{2s+\alpha}|y|^\alpha & \textit{for all } |x'| \le \dfrac{2\bar{R}}{\varepsilon}, \\ \dfrac{F_\varepsilon^{-2s}(|x'|)}{|x'|^\alpha}|y|^\alpha & \textit{for all } |x'| \ge \dfrac{\bar{R}}{\varepsilon}. \end{cases}$$

The quantity $\|F_\varepsilon\|_{C^{2,\alpha}(B_{R_1}(|x'|)} \lesssim \left\|F_\varepsilon^{-2s}\right\|_{C^\alpha(B_1(|x'|))}$ *will be used repeatedly and will be simply denoted by* $\|\kappa\|_\alpha$*, as a function of* $|x'|$*, for any* $0 \le \alpha < 1$*. We have*

$$\begin{aligned} g(y) &= \frac{1}{2}\sum_{i=1}^{2} \kappa_i(0)y_i^2 + O\left(\|\kappa\|_\alpha |y|^{2+\alpha}\right), \\ Dg(y)\cdot y &= \sum_{i=1}^{2} \kappa_i(0)y_i^2 + O\left(\|\kappa\|_\alpha |y|^{2+\alpha}\right), \\ |Dg(y)|^2 &= O\left(\|\kappa\|_0^2 |y|^2\right). \end{aligned}$$

In particular,

$$\begin{aligned} g(y) - Dg(y)\cdot y &= -\frac{1}{2}\sum_{i=1}^{2} \kappa_i(0)y_i^2 + O\left(\|\kappa\|_\alpha |y|^{2+\alpha}\right) = O(\|\kappa\|_0 |y|^2), \\ \sqrt{1+|Dg(y)|^2} - 1 &= O\left(\|\kappa\|_0^2 |y|^2\right), \\ 1 - \frac{1}{\sqrt{1+|Dg(y)|^2}} &= O\left(\|\kappa\|_0^2 |y|^2\right), \\ g(y)^2 &= O\left(\|\kappa\|_0^2 |y|^4\right). \end{aligned}$$

Lemma 2.3.7 (The change of variable)**.** *Let* $|y|, |z|, |z_0| \le R_1$*. Under the Fermi change of variable* $x = \Phi(y,z) = y + z\nu(y)$*, the Jacobian determinant*

$$J(y,z) = \sqrt{1+|Dg(y)|^2}(1+\kappa_1(y)z)(1+\kappa_2(y)z)$$

satisfies

$$J(y,z) = 1 + (\kappa_1(0) + \kappa_2(0))z + O\left(\|\kappa\|_\alpha |y|^\alpha |z|\right) + O\left(\|\kappa\|_0^2 (|y|^2 + |z|^2)\right),$$

and the kernel $|x_0 - x|^{-3-2s}$ *has an expansion*

$$\begin{aligned}|x_0 - x|^{-3-2s} = |(y, z_0 - z)|^{-3-2s} \Bigg[1 + \frac{3+2s}{2}(z_0 + z)\sum_{i=1}^{2} \kappa_i(0) \frac{y_i^2}{|(y, z_0 - z)|^2} \\ + O\left(\frac{\|\kappa\|_\alpha |y|^{2+\alpha}(|z| + |z_0|)}{|(y, z_0 - z)|^2}\right) + O\left(\frac{\|\kappa\|_0^2 |y|^2(|y|^2 + |z|^2 + |z_0|^2)}{|(y, z_0 - z)|^2}\right)\Bigg].\end{aligned}$$

Lemma 2.3.8 (Reducing the kernel)**.** *There hold*

$$\begin{aligned} C_{3,s}\int_{\mathbb{R}^2} \frac{1}{|(y, z_0 - z)|^{3+2s}}\,dy &= C_{1,s}\frac{1}{|z_0 - z|^{1+2s}}, \\ C_{3,s}\int_{\mathbb{R}^2} \frac{y_i^2}{|(y, z_0 - z)|^{5+2s}}\,dy &= \frac{1}{3+2s} C_{1,s}\frac{1}{|z_0 - z|^{1+2s}} \quad \textit{for } i = 1, 2, \\ \int_{\mathbb{R}^2} \frac{|y|^\alpha}{|(y, z_0 - z)|^{3+2s}}\,dy &= C\frac{1}{|z_0 - z|^{1+2s-\alpha}}. \end{aligned}$$

Proof of Proposition 2.3.1. The main contribution of the fractional Laplacian comes from the local term which we compute in Fermi coordinates $\Phi(y,z) = y + z\nu(y)$,

$$\begin{aligned}(-\Delta)^s u_0(x_0) &= C_{3,s}\int_{\Phi(C_{R_1})} \frac{u_0(x_0) - u_0(x)}{|x - x_0|^{3+2s}}\,dx + O(R_1^{-2s}) \\ &= C_{3,s}\iint_{C_{R_1}} \frac{w(z_0) - w(z)}{|\Phi(y_0, z_0) - \Phi(y,z)|^{3+2s}} J(y,z)\,dydz + O(R_1^{-2s}).\end{aligned}$$

By Lemma 2.3.7 we have

$$J(y,z) = 1 + (\kappa_1(0) + \kappa_2(0))z + O\left(\|\kappa\|_\alpha |y|^\alpha |z|\right) + O\left(\|\kappa\|_0^2 (|y|^2 + |z|^2)\right),$$

$$
\begin{aligned}
&\frac{1}{|\Phi(y_0,z_0)-\Phi(y,z)|^{3+2s}}\\
&\qquad = \frac{1}{|(y,z_0-z)|^{3+2s}}\left[1+\frac{3+2s}{2}(z_0+z)\sum_{i=1}^{2}\kappa_i(0)\frac{y_i^2}{|(y,z_0-z)|^2}\right.\\
&\qquad\quad \left. +O\left(\frac{\|\kappa\|_\alpha |y|^{2+\alpha}(|z|+|z_0|)}{|(y,z_0-z)|^2}\right)+O\left(\frac{\|\kappa\|_0^2|y|^2(|y|^2+|z|^2+|z_0|^2)}{|(y,z_0-z)|^2}\right)\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{J(y,z)}{|\Phi(y_0,z_0)-\Phi(y,z)|^{3+2s}}\\
&= \frac{1}{|(y,z_0-z)|^{3+2s}}\\
&\quad \cdot\left[1+(\kappa_1(0)+\kappa_2(0))z+O\left(\|\kappa\|_\alpha |y|^\alpha |z|\right)+O\left(\|\kappa\|_0^2(|y|^2+|z|^2)\right)\right]\\
&\quad \cdot\left[1+\frac{3+2s}{2}(z_0+z)\sum_{i=1}^{2}\kappa_i(0)\frac{y_i^2}{|(y,z_0-z)|^2}\right.\\
&\qquad \left. +O\left(\frac{\|\kappa\|_\alpha |y|^{2+\alpha}(|z|+|z_0|)}{|(y,z_0-z)|^2}\right)+O\left(\frac{\|\kappa\|_0^2|y|^2(|y|^2+|z|^2+|z_0|^2)}{|(y,z_0-z)|^2}\right)\right]\\
&= \frac{1}{|(y,z_0-z)|^{3+2s}}\left[1+(\kappa_1(0)+\kappa_2(0))z+\frac{3+2s}{2}(z_0+z)\sum_{i=1}^{2}\kappa_i(0)\frac{y_i^2}{|(y,z_0-z)|^2}\right.\\
&\qquad \left. +O\left(\|\kappa\|_\alpha |y|^\alpha(|z|+|z_0|)\right)+O\left(\|\kappa\|_0^2(|y|^2+|z|^2+|z_0|^2)\right)\right].
\end{aligned}
$$

We have

$$
\begin{aligned}
&(-\Delta)^s u_0(x_0)\\
&= C_{3,s} \iint_{C_{R_1}} \frac{w(z_0)-w(z)}{|\Phi(y_0,z_0)-\Phi(y,z)|^{3+2s}} J(y,z)\, dydz + O(R_1^{-2s})\\
&= C_{3,s} \iint_{C_{R_1}} \frac{w(z_0)-w(z)}{|(y,z_0-z)|^{3+2s}} \Bigg[1 + (\kappa_1(0)+\kappa_2(0))z\\
&\qquad + \frac{3+2s}{2}(z_0+z)\sum_{i=1}^{2} \kappa_i(0) \frac{y_i^2}{|(y,z_0-z)|^2}\\
&\qquad + O\left(\|\kappa\|_\alpha |y|^\alpha (|z|+|z_0|)\right) + O\Big(\|\kappa\|_0^2 (|y|^2+|z|^2+|z_0|^2)\Big) \Bigg]\\
&= I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= C_{3,s} \iint_{C_{R_1}} \frac{w(z_0)-w(z)}{|(y,z_0-z)|^{3+2s}}\, dydz,\\
I_2 &= C_{3,s}(\kappa_1(0)+\kappa_2(0)) \iint_{C_{R_1}} \frac{w(z_0)-w(z)}{|(y,z_0-z)|^{3+2s}} z\, dydz,\\
I_3 &= C_{3,s} \frac{3+2s}{2} \sum_{i=1}^{2} \kappa_i(0) \iint_{C_{R_1}} \frac{w(z_0)-w(z)}{|(y,z_0-z)|^{5+2s}} (z_0+z) y_i^2\, dydz,\\
I_4 &= O(\|\kappa\|_\alpha) \iint_{C_{R_1}} \frac{\left| w(z_0)-w(z)-\chi_{B_1^1(z_0)}(z) w'(z_0)(z_0-z)\right|}{|(y,z_0-z)|^{3+2s}} |y|^\alpha\\
&\qquad \cdot (|z|+|z_0|)\, dydz,\\
I_5 &= O\left(\|\kappa\|_0^2\right) \iint_{C_{R_1}} \frac{\left| w(z_0)-w(z)-\chi_{B_1^1(z_0)}(z) w'(z_0)(z_0-z)\right|}{|(y,z_0-z)|^{3+2s}}\\
&\qquad \cdot (|y|^2+|z|^2+|z_0|^2)\, dydz.
\end{aligned}
$$

In the last terms I_4 and I_5, the linear odd term near the origin has been added to eliminate the principal value before the integrals are estimated by their absolute values. One may obtain more explicit expressions by extending the domain and using Lemma 2.3.8 as follows. I_1 resembles the fractional Laplacian of the one-dimensional solution.

$$\begin{aligned}
I_1 &= C_{3,s}\iint_{\mathbb{R}^3}\frac{w(z_0)-w(z)}{|(y,z_0-z)|^{3+2s}}\,dydz - C_{3,s}\iint_{\mathbb{R}^3\setminus C_{R_1}}\frac{w(z_0)-w(z)}{|(y,z_0-z)|^{3+2s}}\,dydz\\
&= C_{3,s}\int_{\mathbb{R}}(w(z_0)-w(z))\int_{\mathbb{R}^2}\frac{1}{|(y,z_0-z)|^{3+2s}}\,dydz + O\left(\int_{R_1}^{\infty}\rho^{-3-2s}\rho^2\,d\rho\right)\\
&= C_{1,s}\int_{\mathbb{R}}\frac{w(z_0)-w(z)}{|z_0-z|^{1+2s}}\,dz + O\left(R_1^{-2s}\right)\\
&= w(z_0)-w(z_0)^3 + O\left(R_1^{-2s}\right).
\end{aligned}$$

Hereafter $\rho=\sqrt{|y|^2+|z_0-z|^2}$. I_2 and I_3 are of the next order where we see the mean curvature.

$$\begin{aligned}
I_2 &= -C_{3,s}\sum_{i=1}^{2}\kappa_i(0)\iint_{C_{R_1}}\frac{w(z_0)-w(z)}{|(y,z_0-z)|^{3+2s}}z\,dydz\\
&= -C_{3,s}\sum_{i=1}^{2}\kappa_i(0)\iint_{\mathbb{R}^3}\frac{w(z_0)-w(z)}{|(y,z_0-z)|^{3+2s}}z\,dydz\\
&\qquad - C_{3,s}\sum_{i=1}^{2}\kappa_i(0)\iint_{\mathbb{R}^3\setminus C_{R_1}}\frac{w(z_0)-w(z)}{|(y,z_0-z)|^{3+2s}}(z_0+(z-z_0))\,dydz\\
&= -C_{1,s}\sum_{i=1}^{2}\kappa_i(0)\int_{\mathbb{R}}\frac{w(z_0)-w(z)}{|z_0-z|^{1+2s}}z\,dz\\
&\qquad + O\left(\|\kappa\|_0|z_0|\int_{R_1}^{\infty}\frac{1}{\rho^{3+2s}}\rho^2\,d\rho\right) + O\left(\|\kappa\|_0\int_{R_1}^{\infty}\frac{\rho}{\rho^{3+2s}}\rho^2\,d\rho\right)\\
&= -2\left(C_{1,s}\int_{\mathbb{R}}\frac{w(z_0)-w(z)}{|z_0-z|^{1+2s}}z\,dz\right)H_{M_\varepsilon}(\mathrm{y}_0) + O\left(\|\kappa\|_0 R_1^{-2s}(|z_0|+R_1)\right).
\end{aligned}$$

Also,

$$
\begin{aligned}
I_3 &= C_{3,s}\frac{3+2s}{2}\sum_{i=1}^{2}\kappa_i(0)\iint_{\mathbb{R}^3}\frac{w(z_0)-w(z)}{|(y,z_0-z)|^{5+2s}}(z_0+z)y_i^2\,dydz \\
&\quad + O(\|\kappa\|_0)\iint_{\mathbb{R}^3\setminus C_{R_1}}\frac{w(z_0)-w(z)}{|(y,z_0-z)|^{5+2s}}(2z_0-(z_0-z))y_i^2\,dydz \\
&= C_{1,s}\frac{1}{2}\sum_{i=1}^{2}\kappa_i(0)\int_{\mathbb{R}}\frac{w(z_0)-w(z)}{|z_0-z|^{1+2s}}(z_0+z)\,dz \\
&\quad + O\left(\|\kappa\|_0|z_0|\int_{R_1}^{\infty}\frac{\rho^2}{\rho^{5+2s}}\rho^2\,d\rho\right)+O\left(\|\kappa\|_0\int_{R_1}^{\infty}\frac{\rho^3}{\rho^{5+2s}}\rho^2\,d\rho\right) \\
&= \left(C_{1,s}\int_{\mathbb{R}}\frac{w(z_0)-w(z)}{|z_0-z|^{1+2s}}(z_0+z)\,dz\right)H_{M_\varepsilon}(y_0)+O\left(\|\kappa\|_0R_1^{-2s}(|z_0|+R_1)\right).
\end{aligned}
$$

The remainder terms I_4 and I_5 are estimated as follows.

$$
\begin{aligned}
I_4 &= O(\|\kappa\|_\alpha)\iint_{C_{R_1}}\frac{\left|w(z_0)-w(z)-\chi_{B_1^1(z_0)}(z)w'(z_0)(z_0-z)\right|}{|(y,z_0-z)|^{3+2s}}|y|^\alpha(|z|+|z_0|)\,dydz \\
&= O(\|\kappa\|_\alpha)\int_{\mathbb{R}}\left|w(z_0)-w(z)+\chi_{B_1^1(0)}(z)w'(z_0)(z_0-z)\right| \\
&\qquad \cdot\int_{\mathbb{R}^2}\frac{|y|^\alpha(|z_0-z|+|z_0|)}{\left(|y|^2+|z_0-z|^2\right)^{\frac{3+2s}{2}}}\,dydz \\
&\qquad + O\left(\|\kappa\|_\alpha(|z|+|z_0|)\int_{R_1}^{\infty}\frac{\rho^\alpha}{\rho^{3+2s}}\rho^2\,d\rho\right) \\
&= O(\|\kappa\|_\alpha)\left(\int_{\mathbb{R}}\frac{\left|w(z_0)-w(z)+\chi_{B_1^1(0)}(z)w'(z_0)(z_0-z)\right|}{|z_0-z|^{1+2s-\alpha}}(|z_0-z|+|z_0|)\right)dz \\
&\qquad + O\left(\|\kappa\|_\alpha R_1^{-2s+\alpha}(|z|+|z_0|)\right) \\
&= O\left(\|\kappa\|_\alpha(1+R_1^{-2s+\alpha}(|z|+|z_0|))\right).
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= O\left(\|\kappa\|_0^2\right) \iint_{C_{R_1}} \frac{\left|w(z_0)-w(z)-\chi_{B_1^1(z_0)}(z)w'(z_0)(z_0-z)\right|}{|(y,z_0-z)|^{3+2s}} \\
&\quad \cdot (|y|^2+|z|^2+|z_0|^2)\,dydz \\
&= O\left(\|\kappa\|_0^2\right)\left(1+\int_1^{R_1}\frac{\rho^2+|z_0|^2}{\rho^{3+2s}}\rho^2\,d\rho\right) \\
&= O\left(\|\kappa\|_0^2\,(1+R_1^{2-2s}+R_1^{-2s}|z_0|^2)\right).
\end{aligned}
$$

In conclusion, we have, since $|z_0|\le R_1$ and $\alpha<2s-1$,

$$
\begin{aligned}
(-\Delta)^s u_0(x_0) &= w(z_0)-w(z_0)^3+\left(C_{1,s}\int_{\mathbb{R}}\frac{w(z_0)-w(z)}{|z_0-z|^{1+2s}}(z_0-z)\,dz\right)H_{M_\varepsilon}(y_0) \\
&\quad + O\left(R_1^{-2s}\left(1+\|\kappa\|_0 R_1+\|\kappa\|_\alpha R_1^{2s}+\|\kappa\|_0^2R_1^2\right)\right) \\
&= w(z_0)-w(z_0)^3+c_H(z_0)H_{M_\varepsilon}(y_0)+O(R_1^{-2s}),
\end{aligned}
$$

the last line following from the estimate

$$
\begin{aligned}
\|\kappa\|_\alpha R_1^{2s} &\lesssim \begin{cases} \varepsilon^\alpha & \text{for } |x'|\le \dfrac{2\bar R}{\varepsilon} \\ \dfrac{F_\varepsilon^{2s(\tau-1)}}{|x'|^\alpha} & \text{for } |x'|\ge \dfrac{\bar R}{\varepsilon} \end{cases} \\
&\lesssim \begin{cases} \varepsilon^\alpha & \text{for } |x'|\le \dfrac{2\bar R}{\varepsilon} \\ \varepsilon^{\alpha-2s(\tau-1)}(\varepsilon|x'|)^{-2s(\tau-1)(1-\frac{2}{2s+1})} & \text{for } |x'|\ge \dfrac{\bar R}{\varepsilon} \end{cases} \\
&\lesssim \varepsilon^{\alpha-2s(\tau-1)}.
\end{aligned}
$$

The finiteness of the remainder in the norm $\|\cdot\|_{**}$ is a tedious but straightforward computation. As an example, the difference of the exterior error with two radii F_ε^τ and G_ε^τ is controlled by

$$
\begin{aligned}
&\left|\int_{\Phi(C^c_{F_\varepsilon^\tau})}\frac{u_0(x_0)-u_0(x)}{|x-x_0|^{3+2s}}\,dx-\int_{\Phi(C^c_{G_\varepsilon^\tau})}\frac{u_0(x_0)-u_0(x)}{|x-x_0|^{3+2s}}\,dx\right| \\
&=\left|\iint_{C_{G_\varepsilon^\tau}\setminus C_{F_\varepsilon^\tau}}\frac{w(z_0)-w(z)}{|\Phi(y_0,z_0)-\Phi(y,z)|^{3+2s}}J(y,z)\,dydz\right|.
\end{aligned}
$$

Following the computations in the above proof, a typical term would be

$$O\left(G_\varepsilon^{-2s\tau} - F_\varepsilon^{-2s\tau}\right) = O\left(r^{-\frac{2(2s\tau+1)}{2s+1}} |F_\varepsilon - G_\varepsilon|\right),$$

which implies Lipschitz continuity with decay in r.

$\square$

Similarly we prove the expansion at the end.

Proof of Corollary 2.3.5. We recall that a tubular neighborhood of an end of M_ε^+ are parameterized by

$$x = \mathrm{y} + z\nu(\mathrm{y}) = (y, F_\varepsilon(r)) + z\frac{\left(-F_\varepsilon'(r)\frac{y}{r}, 1\right)}{\sqrt{1+F_\varepsilon'(r)^2}} \quad \text{for } r = |y| > r_0,\ |z| < \frac{\bar{\delta}}{\varepsilon},$$

where $r = |y|$. In place of Lemma 2.3.7 we have for $|z| \leq F_\varepsilon^\tau(r)$ with $1 < \tau < \frac{2s+1}{2}$,

$$\begin{aligned}
J(y,z) &= \left(1 + O\left(F_\varepsilon'(r)^2\right)\right)\left(1 + O\left(F_\varepsilon''(r)F_\varepsilon^\tau(r)\right)\right)^2 \\
&= \left(1 + O\left(F_\varepsilon^{-(2s-1)}(r)\right)\right)\left(1 + O\left(F_\varepsilon^{-(2s-\tau)}(r)\right)\right)^2 \\
&= 1 + O\left(F_\varepsilon^{-(2s-\tau)}(r)\right), \\
|x - x_0|^2 &= \left(|y_0 - y|^2 + |z_0 - z|^2\right)\left(1 + O\left(F_\varepsilon^\tau(r)F_\varepsilon''(r)\right)\right) \\
&= \left(|y_0 - y|^2 + |z_0 - z|^2\right)\left(1 + O\left(F_\varepsilon^{-(2s-\tau)}\right)\right).
\end{aligned}$$

The result follows by the same proof as in Proposition 2.3.1. $\square$

We now give a proof of the error estimate stated in Section 2.2.

Proof of Proposition 2.2.1. Using the Fermi coordinates expansion of the fractional Laplacian (Proposition 2.3.1), we have, in an expanding neighborhood of M_ε, the following estimates on the error:

- For $\frac{1}{\varepsilon} \leq |x'| \leq \frac{2\bar{R}}{\varepsilon}$ and $|z| \leq \frac{\bar{\delta}}{\varepsilon}$,

$$S(u^*)(x) = c_H(z)H_{M_\varepsilon}(\mathrm{y}) + O\left(\varepsilon^{2s}\right).$$

- For $|x'| \geq \dfrac{4\bar{R}}{\varepsilon}$ and $|z| \leq F_\varepsilon^\tau(|x'|)$,

$$\begin{aligned}
S(u^*)(x) &= (-\Delta)^s(w(z_+)+w(z_-)+1)+f(w(z_+)+w(z_-)-1) \\
&\quad + O\left(F_\varepsilon^{-2s\tau}\right) \\
&= f(w(z_+)+w(z_-)+1)-f(w(z_+))-f(w(z_-)) \\
&\quad + c_H(z_+)H_{M_\varepsilon^+}(\mathrm{y}_+)+c_H(z_-)H_{M_\varepsilon^-}(\mathrm{y}_-)+O\left(F_\varepsilon^{-2s\tau}\right) \\
&= 3(w(z_+)+w(z_-))(1+w(z_+))(1+w(z_-)) \\
&\quad + c_H(z_+)H_{M_\varepsilon^+}(\mathrm{y}_+)+c_H(z_-)H_{M_\varepsilon^-}(\mathrm{y}_-)+O\left(F_\varepsilon^{-2s\tau}\right).
\end{aligned}$$

- For $\dfrac{2\bar{R}}{\varepsilon} \leq |x'| \leq \dfrac{4\bar{R}}{\varepsilon}$, $x_n > 0$ and $|z| \leq R_1(|x'|)$,

$$\begin{aligned}
S(u^*)(x) &= (-\Delta)^s w(z_+) + (-\Delta)^s\left(\left(1-\eta\left(\left|x'\right|-\frac{\bar{R}}{\varepsilon}\right)(w(z_-)+1)\right)\right) \\
&\quad + f\left(w(z_+)+\left(1-\eta\left(\left|x'\right|-\frac{\bar{R}}{\varepsilon}\right)(w(z_-)+1)\right)\right) \\
&= c_H(z_+)H_{M_\varepsilon}(\mathrm{y}_+)+O(\varepsilon^{2s}).
\end{aligned}$$

  Here the second term is small because of the smallness of the cut-off error up to two derivatives.

- For $\dfrac{2\bar{R}}{\varepsilon} \leq |x'| \leq \dfrac{4\bar{R}}{\varepsilon}$, $x_n < 0$ and $|z| \leq R_1(|x'|)$, we have similarly

$$S(u^*)(x) = c_H(z_-)H_{M_\varepsilon}(\mathrm{y}_-)+O(\varepsilon^{2s}).$$

This completes the proof. □

Proof of Lemma 2.3.7. Referring to Lemma 2.3.6 and keeping in mind that $\|\kappa\|_0 R_1 = o(1)$, for the Jacobian determinant we have

$$\begin{aligned}
J(y,z) &= 1 + (\kappa_1(0) + \kappa_2(0))z + ((\kappa_1+\kappa_2)(y) - (\kappa_1+\kappa_2)(0))z \\
&\quad + \left(\sqrt{1+|Dg(y)|^2} - 1\right)(1 + (\kappa_1(y)+\kappa_2(y))z + \kappa_1(y)\kappa_2(y)z^2) \\
&= 1 + (\kappa_1(0)+\kappa_2(0))z + O\left(\|\kappa\|_\alpha |y|^\alpha |z|\right) + O\left(\|\kappa\|_0^2 |z|^2\right) \\
&\quad + O\left(\|\kappa\|_0^2 |y|^2\right)(1 + O(\|\kappa\|_0 |z|))^2 \\
&= 1 + (\kappa_1(0)+\kappa_2(0))z + O\left(\|\kappa\|_\alpha |y|^\alpha |z|\right) + O\left(\|\kappa\|_0^2 (|y|^2 + |z|^2)\right).
\end{aligned}$$

To expand the kernel we first consider

$$x_0 - x = (y, g(y)) - (0, z_0) + z\frac{(-Dg(y), 1)}{\sqrt{1+|Dg(y)|^2}},$$

$$\begin{aligned}
&|x_0 - x|^2 \\
&= |y|^2 + g(y)^2 + z^2 + z_0^2 - \frac{2zz_0}{\sqrt{1+|Dg(y)|^2}} + \frac{2z(g(y) - Dg(y)\cdot y)}{\sqrt{1+Dg(y)^2}} - 2z_0 g(y) \\
&= |y|^2 + |z_0 - z|^2 + 2z(g(y) - Dg(y)\cdot y) - 2z_0 g(y) \\
&\quad + g(y)^2 + (2zz_0 - 2z(g(y) - Dg(y)\cdot y))\left(1 - \frac{1}{\sqrt{1+|Dg(y)|^2}}\right) \\
&= |(y, z_0 - z)|^2 - (z_0 + z)\sum_{i=1}^{2} \kappa_i(0) y_i^2 + O\left(\|\kappa\|_\alpha |y|^{2+\alpha}(|z| + |z_0|)\right) \\
&\quad + O\left(\|\kappa\|_0^2 |y|^4\right) + O\left(\|\kappa\|_0^2 |y|^2 |z| \left(|z_0| + \|\kappa\|_0 |y|^2\right)\right) \\
&= |(y, z_0 - z)|^2 - (z_0 + z)\sum_{i=1}^{2} \kappa_i(0) y_i^2 \\
&\quad + O\left(\|\kappa\|_\alpha |y|^{2+\alpha}(|z| + |z_0|)\right) + O\left(\|\kappa\|_0^2 |y|^2 (|y|^2 + |z||z_0|)\right).
\end{aligned}$$

By binomial theorem,

$$
\begin{aligned}
&|x_0-x|^{-3-2s}\\
&=|(y,z_0-z)|^{-3-2s}\Bigg[1+\frac{3+2s}{2}(z_0+z)\sum_{i=1}^{2}\kappa_i(0)\frac{y_i^2}{|(y,z_0-z)|^2}\\
&\quad+O\left(\frac{\|\kappa\|_\alpha |y|^{2+\alpha}(|z|+|z_0|)}{|(y,z_0-z)|^2}\right)+O\left(\frac{\|\kappa\|_0^2 |y|^2(|y|^2+|z||z_0|)}{|(y,z_0-z)|^2}\right)\\
&\quad+O\left(\frac{\|\kappa\|_0^2 |y|^4(|z_0|^2+|z|^2)}{|(y,z_0-z)|^4}\right)\Bigg]\\
&=|(y,z_0-z)|^{-3-2s}\Bigg[1+\frac{3+2s}{2}(z_0+z)\sum_{i=1}^{2}\kappa_i(0)\frac{y_i^2}{|(y,z_0-z)|^2}\\
&\quad+O\left(\frac{\|\kappa\|_\alpha |y|^{2+\alpha}(|z|+|z_0|)}{|(y,z_0-z)|^2}\right)+O\left(\frac{\|\kappa\|_0^2 |y|^2(|y|^2+|z|^2+|z_0|^2)}{|(y,z_0-z)|^2}\right)\Bigg].
\end{aligned}
$$

□

Proof of Lemma 2.3.8. The first and third equalities follow by the change of variable $y=|z_0-z|\tilde{y}$. To prove the second one, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\frac{y_i^2}{|(y,z_0-z)|^{5+2s}}\,dy\\
&=\frac{1}{2}\int_{\mathbb{R}^2}\frac{\left(|y|^2+|z_0-z|^2\right)-|z_0-z|^2}{\left(|y|^2+|z_0-z|^2\right)^{\frac{5+2s}{2}}}\,dy\\
&=\frac{1}{2}\int_{\mathbb{R}^2}\frac{dy}{\left(|y|^2+|z_0-z|^2\right)^{\frac{3+2s}{2}}}-\frac{1}{2}|z_0-z|^2\int_{\mathbb{R}^2}\frac{dy}{\left(|y|^2+|z_0-z|^2\right)^{\frac{5+2s}{2}}}\\
&=\frac{1}{2}\frac{C_{1,s}}{C_{3,s}}\frac{1}{|z_0-z|^{1+2s}}-\frac{1}{2}\frac{C_{3,s}}{C_{5,s}}\frac{|z_0-z|^2}{|z_0-z|^{3+2s}}\\
&=\frac{1}{2}\frac{C_{1,s}}{C_{3,s}}\left(1-\frac{C_{3,s}^2}{C_{1,s}C_{5,s}}\right)\frac{1}{|z_0-z|^{1+2s}}.
\end{aligned}
$$

Recalling that

$$C_{n,s} = \frac{2^{2s}s}{\Gamma(1-s)} \frac{\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}},$$

we have

$$1 - \frac{C_{3,s}^2}{C_{1,s}C_{5,s}} = 1 - \frac{\Gamma\left(\frac{3+2s}{2}\right)^2}{\Gamma\left(\frac{1+2s}{2}\right)\Gamma\left(\frac{5+2s}{2}\right)} = 1 - \frac{1+2s}{3+2s} = \frac{2}{3+2s}$$

and hence

$$\int_{\mathbb{R}^2} \frac{y_i^2}{|(y, z_0 - z)|^{5+2s}}\, dy = \frac{1}{3+2s} \frac{C_{1,s}}{C_{3,s}} \frac{1}{|z_0 - z|^{1+2s}}.$$

□

2.4 Linear theory

In this section we use a different notation. We write $w = w(z,t)$ for the layer in the extension and $\underline{w}(z)$ for its trace.

2.4.1 Non-degeneracy of one-dimensional solution

Consider the linearized equation of $(-\Delta)^s u + f(u) = 0$ at $\underline{w}$, the one-dimensional solution, namely

$$(-\Delta)^s \phi + f'(\underline{w})\phi = 0 \quad \text{for } (y,z) \in \mathbb{R}^n, \tag{2.15}$$

or the equivalent extension problem (here $a = 1 - 2s$)

$$\begin{cases} \nabla \cdot (t^a \nabla \phi) = 0 & \text{for } (y,z,t) \in \mathbb{R}^{n+1}_+ \\ t^a \dfrac{\partial \phi}{\partial \nu} + f'(w)\phi = 0 & \text{for } (y,z) \in \mathbb{R}^n. \end{cases} \tag{2.16}$$

Given $\xi \in \mathbb{R}^{n-1}$, we define on

$$X = H^1(\mathbb{R}^2_+, t^a)$$

the bilinear form

$$(u,v)_X = \int_{\mathbb{R}^2_+} t^a \left(\nabla u \cdot \nabla v + |\xi|^2 uv \right) dzdt + \int_{\mathbb{R}} f'(w) uv \, dz.$$

Lemma 2.4.1 (An inner product)**.** *Suppose* $\xi \neq 0$. *Then* $(\cdot,\cdot)_X$ *defines an inner product on* X.

Proof. Clearly $(u,u)_X < \infty$ for any $u \in X$. For $R > 0$, denote $B_R^+ = B_R(0) \cap \mathbb{R}^2_+$ and its boundary in $\mathbb{R}^2_+$ by ∂B_R^+. It suffices to prove that

$$\int_{B_R^+} t^a |\nabla u|^2 \, dzdt + \int_{\partial B_R^+} f'(w) u^2 \, dz = \int_{B_R^+} t^a w_z^2 \left| \nabla \left(\frac{u}{w_z} \right) \right|^2 dzdt. \tag{2.17}$$

Since the right hand side is non-negative, the result follows as we take $R \to +\infty$. To check the above equality, we compute

$$\begin{aligned}
&\int_{B_R^+} t^a w_z^2 \left| \nabla \left(\frac{u}{w_z} \right) \right|^2 dzdt \\
= &\int_{B_R^+} t^a \left| \nabla u - \frac{u}{w_z} \nabla w_z \right|^2 dzdt \\
= &\int_{B_R^+} t^a |\nabla u|^2 \, dzdt + \int_{B_R^+} t^a \frac{u^2}{w_z^2} |\nabla w_z|^2 \, dzdt - \int_{B_R^+} t^a \nabla (u^2) \cdot \frac{\nabla w_z}{w_z} \, dzdt.
\end{aligned}$$

Since $\nabla \cdot (t^a \nabla w_z) = 0$ in $\mathbb{R}^2_+$, we can integrate the last integral by parts as

$$\begin{aligned}
-\int_{B_R^+} t^a \nabla (u^2) \cdot \frac{\nabla w_z}{w_z} \, dzdt &= -\int_{\partial B_R^+} u^2 \frac{t^a \partial_\nu w_z}{w_z} \, dz + \int_{B_R^+} u^2 \nabla \cdot \left(t^a \frac{\nabla w_z}{w_z} \right) dzdt \\
&= \int_{\partial B_R^+} u^2 \frac{f'(w) w_z}{w_z} \, dz + \int_{B_R^+} t^a u^2 \nabla w_z \cdot \nabla \cdot \frac{1}{w_z} \, dzdt \\
&= \int_{\partial B_R^+} f'(w) u^2 \, dz - \int_{B_R^+} t^a \frac{u^2}{w_z^2} |\nabla w_z|^2 \, dzdt.
\end{aligned}$$

Therefore, (2.17) holds and the proof is complete. □

Lemma 2.4.2 (Solvability of the linear equation)**.** *Suppose $\xi \neq 0$. For any $g \in C_c^\infty(\overline{\mathbb{R}_+^2})$ and $h \in C_c^\infty(\mathbb{R})$, there exists a unique $u \in X$ of*

$$\begin{cases} -\nabla \cdot (t^a \nabla u) + t^a |\xi|^2 u = g & \text{in } \mathbb{R}_+^2 \\ t^a \dfrac{\partial u}{\partial \nu} + f'(w) u = h & \text{on } \partial \mathbb{R}_+^2. \end{cases} \tag{2.18}$$

Proof. This equation has the weak formulation

$$\begin{aligned} (u,v)_X = \int_{\mathbb{R}_+^2} t^a \left(\nabla u \cdot \nabla v + |\xi|^2 uv \right) dzdt + \int_{\mathbb{R}} f'(w) uv\, dz \\ = \int_{\mathbb{R}_+^2} gv\, dzdt + \int_{\mathbb{R}} hv\, dz. \end{aligned}$$

By Riesz representation theorem, there is a unique solution $u \in X$. □

Lemma 2.4.3 (Non-degeneracy in one dimension [82, Lemma 4.2])**.** *Let $\underline{w}(z)$ be the unique increasing solution of*

$$(-\partial_{zz})^s \underline{w} + f(\underline{w}) = 0 \quad \text{in } \mathbb{R}.$$

If $\phi(z)$ is a bounded solution of

$$(-\partial_{zz})^s \phi + f'(\underline{w}) \phi = 0 \quad \text{in } \mathbb{R},$$

then $\phi(z) = C\underline{w}'(z)$.

Lemma 2.4.4 (Non-degeneracy in higher dimensions)**.** *Let $\phi(y,z,t)$ be a bounded solution of*

$$\begin{cases} \nabla_{(y,z,t)} \cdot (t^a \nabla_{(y,z,t)} \phi) = t^a \left(\partial_{tt} + \dfrac{a}{t} \partial_t + \partial_{zz} + \Delta_y \right) \phi = 0 & \text{in } \mathbb{R}_+^{n+1} \\ t^a \dfrac{\partial \phi}{\partial \nu} + f'(w) \phi = 0 & \text{on } \partial \mathbb{R}_+^{n+1}, \end{cases} \tag{2.19}$$

where $w(z,t)$ is the one-dimensional solution so that

$$\begin{cases} \nabla_{(z,t)} \cdot (t^a \nabla_{(z,t)} w_z) = t^a \left(\partial_{tt} + \dfrac{a}{t} \partial_t + \partial_{zz} \right) w_z = 0 & \text{in } \mathbb{R}^2_+ \\ t^a \dfrac{\partial w_z}{\partial \nu} + f'(w) w_z = 0 & \text{on } \partial \mathbb{R}^2_+. \end{cases}$$

Then $\phi(y,z,t) = c w_z(z,t)$ for some constant c.

Proof. For each $(z,t) \in \mathbb{R}^2_+$, let $\psi(\xi,z,t)$ be a smooth function in ξ rapidly decreasing as $|\xi| \to +\infty$. The Fourier transform $\hat{\phi}(\xi,z,t)$ of $\phi(y,z,t)$ in the y-variable, which is the distribution defined by

$$\langle \hat{\phi}(\cdot,z,t), \mu \rangle_{\mathbb{R}^{n-1}} = \langle \phi(\cdot,z,t), \hat{\mu} \rangle_{\mathbb{R}^{n-1}} = \int_{\mathbb{R}^{n-1}} \phi(\xi,z,t) \hat{\mu}(\xi)\, d\xi$$

for any smooth rapidly decreasing function μ, satisfies

$$\begin{aligned} \int_{\mathbb{R}^{n+1}_+} &\left(-\nabla \cdot (t^a \nabla \psi) + t^a |\xi|^2 \psi \right) \hat{\phi}(\xi,z,t)\, d\xi dz dt \\ &= \int_{\mathbb{R}^n} \left(-f'(w)\psi + t^a \psi_t|_{t=0} \right) \hat{\phi}(\xi,z,0)\, d\xi dz. \end{aligned}$$

Let $\mu \in C_c^\infty(\mathbb{R}^{n-1})$, $\varphi_+ \in C_c^\infty(\overline{\mathbb{R}^2_+})$ and $\varphi_0 \in C_c^\infty(\mathbb{R})$ such that

$$0 \notin \operatorname{supp}(\mu).$$

By Lemma 2.4.2, for any $\xi \neq 0$ we can solve the equation

$$\begin{cases} -\nabla \cdot (t^a \nabla \psi) + t^a |\xi|^2 \psi = \mu(\xi) \varphi_+(z,t) & \text{in } \mathbb{R}^2_+ \\ t^a \dfrac{\partial \psi}{\partial \nu} + f'(w) \psi = \mu(\xi) \varphi_0(z) & \text{on } \partial \mathbb{R}^2_+ \end{cases}$$

uniquely for $\psi(\xi,\cdot,\cdot) \in X$ such that

$$\psi(\xi,z,t) = 0 \quad \text{if} \quad \xi \notin \operatorname{supp}(\mu).$$

In particular, $\psi(\cdot,z,t)$ is rapidly decreasing for any $(z,t)\in\mathbb{R}^2_+$. This implies

$$\int_{\mathbb{R}^2_+}\langle\hat{\phi}(\cdot,z,t),\mu\rangle_{\mathbb{R}^{n-1}}\varphi_+(z,t)\,dzdt=\int_{\mathbb{R}}\langle\hat{\phi}(\cdot,z,0),\mu\rangle_{\mathbb{R}^{n-1}}\varphi_0(z)\,dz$$

for any $\varphi_+\in C_c^\infty(\overline{\mathbb{R}^2_+})$ and $\varphi_0\in C_c^\infty(\mathbb{R})$. In other words, whenever $0\notin\operatorname{supp}(\mu)$, we have

$$\langle\hat{\phi}(\cdot,z,t),\mu\rangle_{\mathbb{R}^{n-1}}=0\quad\text{ for all }(z,t)\in\overline{\mathbb{R}^2_+}.$$

Such distribution with $\operatorname{supp}(\hat{\phi}(\cdot,z,t))\subset\{0\}$ is characterized as a linear combination of derivatives up to a finite order of Dirac masses at zero, namely

$$\hat{\phi}(\xi,z,t)=\sum_{j=0}^N a_j(z,t)\delta_0^{(j)}(\xi),$$

for some integer $N\geq 0$. Taking inverse Fourier transform, we see that $\phi(y,z,t)$ is a polynomial in y with coefficients depending on (z,t). Since we assumed that ϕ is bounded, it is a zeroth order polynomial, i.e. ϕ is independent of y. Now the trace $\phi(z,0)$ solves

$$(-\Delta)^s\phi+f'(\underline{w})\phi=0\quad\text{ in }\mathbb{R}.$$

By Lemma 2.4.3,

$$\phi(z,t)=Cw_z(z,t)$$

for some constant $C\in\mathbb{R}$. This completes the proof. $\square$

2.4.2 A priori estimates

Consider the equation

$$(-\Delta)^s\phi(y,z)+f'(w(z))\phi(y,z)=g(y,z)\quad\text{ for }(y,z)\in\mathbb{R}^n. \tag{2.20}$$

Let $\langle y\rangle=\sqrt{1+|y|^2}$ and define the norm

$$\|\phi\|_{\mu,\sigma}=\sup_{(y,z)\in\mathbb{R}^n}\langle y\rangle^{\mu}\langle z\rangle^{\sigma}|\phi(y,z)|$$

for $0\leq\mu<n-1+2s$ and $2-2s<\sigma<1+2s$ such that $\mu+\sigma<n+2s$.

Lemma 2.4.5 (Decay in z)**.** *Let $\phi \in L^\infty(\mathbb{R}^n)$ and $\|g\|_{0,\sigma} < +\infty$. Then we have*

$$\|\phi\|_{0,\sigma} \le C.$$

With the decay established, the following orthogonality condition (2.21) is well-defined.

Lemma 2.4.6 (*A priori* estimate in y, z)**.** *Let $\phi \in L^\infty(\mathbb{R}^n)$ and $\|g\|_{\mu,\sigma} < +\infty$. If the s-harmonic extension $\phi(t,y,z)$ is orthogonal to $w_z(t,z)$ in $\mathbb{R}^{n+1}_+$, namely,*

$$\iint_{\mathbb{R}^2_+} t^a \phi w_z \, dt dz = 0, \tag{2.21}$$

then we have

$$\|\phi\|_{\mu,\sigma} \le C \|g\|_{\mu,\sigma}.$$

Before we give the proof, we estimate some integrals which arise from the product rule

$$\begin{aligned}(-\Delta)^s (uv)(x_0) &= u(x_0)(-\Delta)^s v(x_0) + C_{n,s} \int_{\mathbb{R}^n} \frac{u(x_0)-u(x)}{|x_0-x|^{n+2s}} v(x)\, dx \\ &= u(x_0)(-\Delta)^s v(x_0) + v(x_0)(-\Delta)^s u(x_0) - (u,v)_s(x_0),\end{aligned}$$

where

$$(u,v)_s(x_0) = C_{n,s} \int_{\mathbb{R}^n} \frac{(u(x_0)-u(x))(v(x_0)-v(x))}{|x_0-x|^{n+2s}}\, dx.$$

Lemma 2.4.7 (Decay estimates)**.** *Suppose $\phi(y,z)$ is a bounded function.*

1. *As $|y| \to +\infty$,*
$$(-\Delta)^s \langle y \rangle^{-\mu} = O\left(\langle y \rangle^{-2s-\min\{\mu,n-1\}}\right),$$
$$(\phi, \langle y \rangle^{-\mu})_s = O\left(\langle y \rangle^{-2s-\min\{\mu,n-1\}}\right).$$

2. *As $|z| \to +\infty$,*
$$(-\Delta)^s \langle z \rangle^{-\sigma} = O\left(\langle z \rangle^{-2s-\min\{\sigma,1\}}\right),$$
$$(\phi, \langle z \rangle^{-\sigma})_s = O\left(\langle z \rangle^{-2s-\min\{\sigma,1\}}\right).$$

3. *As* $\min\{|y|,|z|\} \to +\infty$,

$$
\begin{aligned}
(\langle y\rangle^{-\mu}, \langle z\rangle^{-\sigma})_s = O&\Big(|(y,z)|^{-n-2s}(|y|^{n-1-\mu}+1)(|z|^{1-\sigma}+1)\Big)\\
&+O\Big(|y|^{-n-2s}(|y|^{n-1-\mu}+1)|z|^{-\sigma-2}\min\{|y|,|z|\}^3\Big)\\
&+O\Big(|y|^{-\mu-2}|z|^{-n-2s}(|z|^{1-\sigma}+1)\min\{|y|,|z|\}^{n+1}\Big)\\
&+O\Big(|z|^{-\sigma}(|y|+|z|)^{-(n-1+2s)}(|y|^{n-1-\mu}+1)\Big)\\
&+O\Big(|y|^{-\mu}(|y|+|z|)^{-1-2s}(|z|^{1-\sigma}+1)\Big)\\
&+O\Big(|y|^{-\mu}|z|^{-\sigma}(|y|+|z|)^{-2s}\Big).
\end{aligned}
$$

In particular, if $\mu < n-1+2s$ *and* $\sigma < 1+2s$*, then*

$$(\langle y\rangle^{-\mu}, \langle z\rangle^{-\sigma})_s = o\left(|y|^{-\mu}|z|^{-\sigma}\right) \quad \text{as } \min\{|y|,|z|\} \to +\infty.$$

4. *Suppose* $\mu < n-1+2s$ *and* $\sigma < 1+2s$*. As* $\min\{|y|,|z|\} \to +\infty$,

$$(-\Delta)^s\left(\langle y\rangle^{-\mu}\langle z\rangle^{-\sigma}\right) = o\left(|y|^{-\mu}|z|^{-\sigma}\right),$$

$$(\phi, \langle y\rangle^{-\mu}\langle z\rangle^{-\sigma})_s = o\left(|y|^{-\mu}|z|^{-\sigma}\right).$$

5. *Suppose* $\eta_{\mathtt{R}}(y) = \eta\left(\frac{|y|}{\mathtt{R}}\right)$ *where* η *is a smooth cut-off function as in* (2.25)*, and* $\phi(y,z) \le C\langle z\rangle^{-\sigma}$*. For all sufficiently large* $\mathtt{R} > 0$*, we have*

$$|[(-\Delta)^s, \eta_{\mathtt{R}}]\phi(y,z)| \le C\left(\langle z\rangle^{-1} + \langle z\rangle^{-\sigma}\right)\max\{|y|,\mathtt{R}\}^{-2s}. \tag{2.22}$$

Let us assume the validity of Lemma 2.4.7 for the moment.

Proof of Lemma 2.4.5. It follows from Lemma 2.4.7(2) and a maximum principle [50]. □

Proof of Lemma 2.4.6. We will first establish the *a priori* estimate assuming that $\|\phi\|_{\mu,\sigma} < +\infty$. We use a blow-up argument. Suppose on the contrary that there exist a sequence $\phi_m(y,z)$ and $h_m(y,z)$ such that

$$(-\Delta)^s \phi_m + f'(w)\phi_m = g_m \quad \text{for } (y,z) \in \mathbb{R}^n$$

and

$$\|\phi_m\|_{\mu,\sigma} = 1 \quad \text{and} \quad \|g_m\|_{\mu,\sigma} \to 0 \quad \text{as } m \to +\infty.$$

Then there exist a sequence of points $(y_m, z_m) \in \mathbb{R}^n$ such that

$$\phi_m(y_m, z_m) \langle y_m \rangle^{\mu} \langle z_m \rangle^{\sigma} \geq \frac{1}{2}. \tag{2.23}$$

We consider four cases.

1. y_m, z_m bounded:

   Since ϕ_m is bounded and $g_m \to 0$ in $L^\infty(\mathbb{R}^n)$, by elliptic estimates and passing to a subsequence, we may assume that ϕ_m converges uniformly in compact subsets of $\mathbb{R}^n$ to a function ϕ_0 which satisfies

   $$(-\Delta)^s \phi_0 + f'(w)\phi_0 = 0, \quad \text{in } \mathbb{R}^n$$

   and, by (2.21),
   $$\iint_{\mathbb{R}^2_+} t^a \phi_0 w_z \, dt dz = 0.$$

   By the non-degeneracy of w' (Lemma 2.4.4), we necessarily have $\phi_0(y,z) = Cw'(z)$. However, the orthogonality condition yields $C = 0$, i.e. $\phi_0 \equiv 0$. This contradicts (2.23).

2. y_m bounded, $|z_m| \to \infty$:

   We consider $\tilde{\phi}_m(y,z) = \langle z_m + z \rangle^{\sigma} \phi_m(y, z_m + z)$, which satisfies in $\mathbb{R}^n$

   $$\begin{aligned} \langle z_m + z \rangle^{-\sigma} (-\Delta)^s \tilde{\phi}_m(y,z) + \tilde{\phi}_m(y,z)(-\Delta)^s \langle z_m + z \rangle^{-\sigma} \\ - \left(\tilde{\phi}_m(y,z), \langle z_m + z \rangle^{\sigma}\right)_s \\ + f'(w(z_m + z)) \langle z_m + z \rangle^{-\sigma} \tilde{\phi}_m(y,z) = g_m(y, z_m + z), \end{aligned}$$

or

$$(-\Delta)^s \tilde{\phi}_m + \left(f'(w(z_m+z)) + \frac{(-\Delta)^s \langle z_m+z\rangle^{-\sigma}}{\langle z_m+z\rangle^{-\sigma}} \right) \tilde{\phi}_m$$
$$= g_m + \frac{\left(\tilde{\phi}_m(y,z), \langle z_m+z\rangle^{\sigma}\right)_s}{\langle z_m+z\rangle^{-\sigma}}.$$

Using Lemma 2.4.7(2), the limiting equation is

$$(-\Delta)^s \tilde{\phi}_0 + 2\tilde{\phi}_0 = 0 \quad \text{in } \mathbb{R}^n.$$

Thus $\tilde{\phi}_0 = 0$, contradicting (2.23).

3. $|y_m| \to \infty$, z_m bounded:

   We define $\tilde{\phi}_m(y,z) = \langle y_m+y\rangle^{\mu} \phi_m(y_m+y,z)$, which satisfies

$$(-\Delta)^s \tilde{\phi}_m(y,z) + \left(f'(w(z)) + \frac{(-\Delta)^s \left(\langle y_m+y\rangle^{-\mu}\right)}{\langle y_m+y\rangle^{-\mu}} \right) \tilde{\phi}_m(y,z)$$
$$= g_m(y_m+y,z) + \frac{\left(\tilde{\phi}_m(y,z), \langle y_m+y\rangle^{-\mu}\right)_s}{\langle y_m+y\rangle^{-\mu}} \quad \text{in } \mathbb{R}^n.$$

   By Lemma 2.4.7(1), the subsequential limit $\tilde{\phi}_0$ satisfies

$$(-\Delta)^s \tilde{\phi}_0 + f'(w)\tilde{\phi}_0 = 0 \quad \text{in } \mathbb{R}^n.$$

   This leads to a contradiction as in case (1).

4. $|y_m|, |z_m| \to \infty$:

   This is similar to case (2). In fact for $\tilde{\phi}_m(y,z) = \langle y_m+y\rangle^{\mu} \langle z_m+z\rangle^{\sigma} \phi_m(y_m+$

$y, z_m + z)$, we have

$$
\begin{aligned}
&(-\Delta)^s \tilde{\phi}_m(y,z) \\
&\quad + \left(f'(w(z_m+z)) + \frac{(-\Delta)^s \left(\langle y_m + y \rangle^{-\mu} \langle z_m + z \rangle^{-\sigma} \right)}{\langle y_m + y \rangle^{-\mu} \langle z_m + z \rangle^{-\sigma}} \right) \tilde{\phi}_m(y,z) \\
&\quad = g_m(y_m+y, z_m+z) + \frac{\left(\tilde{\phi}_m(y,z), \langle y_m + y \rangle^{-\mu} \langle z_m + z \rangle^{\sigma} \right)_s}{\langle y_m + y \rangle^{-\mu} \langle z_m + z \rangle^{-\sigma}} \quad \text{in } \mathbb{R}^n.
\end{aligned}
$$

In the limiting situation $\tilde{\phi}_m \to \tilde{\phi}_0$, by Lemma 2.4.7(4),

$$(-\Delta)^s \tilde{\phi}_0 + 2\tilde{\phi}_0 = 0 \quad \text{in } \mathbb{R}^n,$$

forcing $\tilde{\phi}_0 = 0$ which contradicts (2.23).

We conclude that

$$\|\phi\|_{\mu,\sigma} \le C \|g\|_{\mu,\sigma} \quad \text{provided} \quad \|\phi\|_{\mu,\sigma} < +\infty. \tag{2.24}$$

Now we will remove the condition $\|\phi\|_{\mu,\sigma} < +\infty$. By Lemma 2.4.5, we know that $\|\phi\|_{0,\sigma} < +\infty$. Let $\eta : [0,+\infty) \to [0,1]$ be a smooth cut-off function such that

$$\eta = 1 \text{ on } [0,1] \quad \text{and} \quad \eta = 0 \text{ on } [2,+\infty). \tag{2.25}$$

Write $\eta_{\mathtt{R}}(y) = \eta\left(\frac{|y|}{\mathtt{R}}\right)$. We apply the above derived *a priori* estimate to $\psi(y,z) = \eta_{\mathtt{R}}(y)\phi(y,z)$, which satisfies

$$(-\Delta)^s \psi + f'(w)\psi = \eta_{\mathtt{R}} g + \phi(-\Delta)^s \eta_{\mathtt{R}} - (\eta_{\mathtt{R}}, \phi)_s. \tag{2.26}$$

It is clear that $\|\eta_{\mathtt{R}} g\|_{\mu,\sigma} \le \|g\|_{\mu,\sigma}$ and $\|\phi(-\Delta)^s \eta_{\mathtt{R}}\|_{\mu,\sigma} \le C\mathtt{R}^{-2s}$ because of the estimate $(-\Delta)^s \eta(|y|) \le C \langle y \rangle^{-(n-1+2s)}$. By Lemma 2.4.7(5),

$$|[(-\Delta)^s, \eta_{\mathtt{R}}]\phi(y_0,z_0)| \le C\left(|z_0|^{-1} + |z_0|^{-\sigma}\right) \max\{|y_0|, \mathtt{R}\}^{-2s}.$$

For $\sigma < 1$ and $0 \le \mu < 2s$, this yields

$$\|[(-\Delta)^s, \eta_{\mathtt{R}}]\phi\|_{\mu,\sigma} \le C\mathtt{R}^{-(2s-\mu)}.$$

Therefore, (2.24) and (2.26) give

$$\|\eta_{\mathtt{R}}\phi\|_{\mu,\sigma} \le C\|g\|_{\mu,\sigma} + C\mathtt{R}^{-2s} + C\mathtt{R}^{-(2s-\mu)}.$$

Letting $\mathtt{R} \to +\infty$, we arrive at

$$\|\phi\|_{\mu,\sigma} \le C\|g\|_{\mu,\sigma},$$

as desired. $\square$

Proof of Lemma 2.4.7. We will only prove the statements regarding the fractional Laplacian of the explicit function. The associated assertion concerning the inner product with ϕ will follow from the same proof using its boundedness, since all the terms are estimated in absolute value.

1. We have

$$\begin{aligned}
(-\Delta_{(y,z)})^s(\langle y\rangle^{-\mu})|_{y=y_0} &= (-\Delta_y)^s \langle y\rangle^{\mu}|_{y=y_0} \\
&= C_{n-1,s}\int_{\mathbb{R}^{n-1}} \frac{\langle y_0\rangle^{-\mu} - \langle y\rangle^{-\mu}}{|y_0-y|^{n-1+2s}}\,dy \\
&\equiv I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where

$$I_1 = C_{n-1,s} \int_{B_{\frac{|y_0|}{2}}(y_0)} \frac{\langle y_0 \rangle^{-\mu} - \langle y \rangle^{-\mu} - D \langle y \rangle^{-\mu} |_{y=y_0} (y_0 - y)}{|y_0 - y|^{n-1+2s}} \, dy,$$

$$I_2 = C_{n-1,s} \int_{B_1(0)} \frac{\langle y_0 \rangle^{-\mu} - \langle y \rangle^{-\mu}}{|y_0 - y|^{n-1+2s}} \, dy,$$

$$I_3 = C_{n-1,s} \int_{B_{\frac{|y_0|}{2}}(0) \setminus B_1(0)} \frac{\langle y_0 \rangle^{-\mu} - \langle y \rangle^{-\mu}}{|y_0 - y|^{n-1+2s}} \, dy,$$

$$I_4 = C_{n-1,s} \int_{\mathbb{R}^{n-1} \setminus \left(B_{\frac{|y_0|}{2}}(y_0) \cup B_{\frac{|y_0|}{2}}(0) \right)} \frac{\langle y_0 \rangle^{-\mu} - \langle y \rangle^{-\mu}}{|y_0 - y|^{n-1+2s}} \, dy.$$

If $|y_0| \le 1$, it is simple to get boundedness since $\langle y\rangle^{-\mu}$ is smooth and bounded. For $|y_0| \ge 1$, we compute

$$
\begin{aligned}
|I_1| &\lesssim \int_{B_{\frac{|y_0|}{2}}(y_0)} \frac{\left|D^2 \langle y\rangle^{-\mu}|_{y=y_0}[y_0-y]^2\right|}{|y_0-y|^{n-1+2s}}\,dy \\
&\lesssim |y_0|^{-\mu-2} \int_0^{\frac{|y_0|}{2}} \frac{\rho^2}{\rho^{1+2s}}\,d\rho \\
&\lesssim |y_0|^{-(\mu+2s)}, \\
|I_2| &\lesssim \int_{B_1(0)} \frac{1}{|y_0|^{n-1+2s}}\,dy \\
&\lesssim |y_0|^{-(n-1+2s)}, \\
|I_3| &\lesssim |y_0|^{-(n-1+2s)} \int_{B_{\frac{|y_0|}{2}}(0)\setminus B_1(0)} \left(\langle y_0\rangle^{-\mu} + |y|^{-\mu}\right) dy \\
&\lesssim |y_0|^{-(n-1+2s)} \int_1^{\frac{|y_0|}{2}} \left(\langle y_0\rangle^{-\mu} + \rho^{-\mu}\right) \rho^{n-2}\,d\rho \\
&\lesssim |y_0|^{-(n-1+2s)} \left(\langle y_0\rangle^{-\mu} (|y_0|^{n-1}-1) + |y_0|^{-\mu+n-1} - 1\right) \\
&\lesssim |y_0|^{-(\mu+2s)} + |y_0|^{-(n-1+2s)}, \\
|I_4| &\lesssim |y_0|^{-\mu} \int_{\mathbb{R}^{n-1}\setminus\left(B_{\frac{|y_0|}{2}}(y_0)\cup B_{\frac{|y_0|}{2}}(0)\right)} \frac{1}{|y_0-y|^{n-1+2s}}\,dy \\
&\lesssim |y_0|^{-\mu} \int_{\frac{|y_0|}{2}}^{\infty} \frac{1}{\rho^{1+2s}}\,d\rho \\
&\lesssim |y_0|^{-(\mu+2s)}.
\end{aligned}
$$

2. This follows from the same proof as (1).

3. We divide $\mathbb{R}^{n-1}\times\mathbb{R}$ into 14 regions in terms of the relative size of $|y|, |z|$ with respect to $|y_0|, |z_0|$ which tend to infinity. We will consider such distance "small" if $|y| < 1$ and "intermediate" if $1 < |y| < \frac{|y_0|}{2}$, similarly for z. Once the non-decaying part of $\langle y\rangle^{-\mu}, \langle z\rangle^{-\sigma}$ are excluded, the remaining parts can be either treated radially where we consider (y_0, z_0) as the origin, or reduced

to the one-dimensional case. More precisely, we write

$$\begin{aligned}(\langle y\rangle^{-\mu},\langle z\rangle^{-\sigma})_s(y_0,z_0) &= C_{n,s}\iint_{\mathbb{R}^n}\frac{\left(\langle y\rangle^{-\mu}-\langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma}-\langle z_0\rangle^{-\sigma}\right)}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz\\ &\equiv \sum_{\substack{1\le i,j\le 4\\ \min\{i,j\}\le 2}} I_{ij}+I^{sing}+I^{rest},\end{aligned}$$

where

$$I_{11}=C_{n,s}\iint_{|y|<1,\,|z|<1}\frac{\left(\langle y\rangle^{-\mu}-\langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma}-\langle z_0\rangle^{-\sigma}\right)}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz,$$

$$I_{12}=C_{n,s}\iint_{|y|<1,\,1<|z|<\frac{|z_0|}{2}}\frac{\left(\langle y\rangle^{-\mu}-\langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma}-\langle z_0\rangle^{-\sigma}\right)}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz,$$

$$I_{13}=C_{n,s}\iint_{|y|<1,\,|z-z_0|<\frac{|z_0|}{2}}\frac{\left(\langle y\rangle^{-\mu}-\langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma}-\langle z_0\rangle^{-\sigma}\right)}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz,$$

$$I_{14}=C_{n,s}\iint_{|y|<1,\,\min\{|z|,|z-z_0|\}>\frac{|z_0|}{2}}\frac{\left(\langle y\rangle^{-\mu}-\langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma}-\langle z_0\rangle^{-\sigma}\right)}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz,$$

$$I_{21}=C_{n,s}\iint_{1<|y|<\frac{|y_0|}{2},\,|z|<1}\frac{\left(\langle y\rangle^{-\mu}-\langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma}-\langle z_0\rangle^{-\sigma}\right)}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz,$$

$$I_{22}=C_{n,s}\iint_{1<|y|<\frac{|y_0|}{2},\,1<|z|<\frac{|z_0|}{2}}\frac{\left(\langle y\rangle^{-\mu}-\langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma}-\langle z_0\rangle^{-\sigma}\right)}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz,$$

$$I_{23}=C_{n,s}\iint_{1<|y|<\frac{|y_0|}{2},\,|z-z_0|<\frac{|z_0|}{2}}\frac{\left(\langle y\rangle^{-\mu}-\langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma}-\langle z_0\rangle^{-\sigma}\right)}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz,$$

$$\begin{aligned}I_{24}=C_{n,s}\iint_{1<|y|<\frac{|y_0|}{2},\,\min\{|z|,|z-z_0|\}>\frac{|z_0|}{2}}&\\ &\frac{\left(\langle y\rangle^{-\mu}-\langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma}-\langle z_0\rangle^{-\sigma}\right)}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz,\end{aligned}$$

$$I_{31} = C_{n,s} \iint_{|y-y_0|<\frac{|y_0|}{2},\, |z|<1} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right)}{|(y-y_0, z-z_0)|^{n+2s}}\, dydz,$$

$$I_{32} = C_{n,s} \iint_{|y-y_0|<\frac{|y_0|}{2},\, 1<|z|<\frac{|z_0|}{2}} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right)}{|(y-y_0, z-z_0)|^{n+2s}}\, dydz,$$

$$I_{41} = C_{n,s} \iint_{\min\{|y|,|y-y_0|\}>\frac{|y_0|}{2},\, |z|<1} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right)}{|(y-y_0, z-z_0)|^{n+2s}}\, dydz,$$

$$I_{42} = C_{n,s} \iint_{\min\{|y|,|y-y_0|\}>\frac{|y_0|}{2},\, 1<|z|<\frac{|z_0|}{2}} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right)}{|(y-y_0, z-z_0)|^{n+2s}}\, dydz,$$

$$I^{sing} = C_{n,s} \iint_{|y|>\frac{|y_0|}{2},\, |z|>\frac{|z_0|}{2},\, |(y-y_0,z-z_0)|<\frac{|y_0|+|z_0|}{2}} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right)}{|(y-y_0, z-z_0)|^{n+2s}}\, dydz,$$

$$I^{rest} = C_{n,s} \iint_{|y|>\frac{|y_0|}{2},\, |z|>\frac{|z_0|}{2},\, |(y-y_0,z-z_0)|>\frac{|y_0|+|z_0|}{2}} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right)}{|(y-y_0, z-z_0)|^{n+2s}}\, dydz.$$

We will estimate these integrals one by one. In the unit cylinder we have

$$\begin{aligned} |I_{11}| &\lesssim \frac{1}{|(y_0,z_0)|^{n+2s}} \iint_{|y|<1,\, |z|<1} dydz \\ &\lesssim |(y_0,z_0)|^{-n-2s}. \end{aligned}$$

On a thin strip near the origin,

$$\begin{aligned} |I_{12}| &\lesssim \frac{1}{|(y_0,z_0)|^{n+2s}} \iint_{|y|<1,\, 1<|z|<\frac{|z_0|}{2}} \left(|z|^{-\sigma} + \langle z_0\rangle^{-\sigma}\right) dydz \\ &\lesssim |(y_0,z_0)|^{-n-2s} \left(|z_0|^{1-\sigma} + 1\right). \end{aligned}$$

Similarly

$$\begin{aligned} |I_{21}| &\lesssim \frac{1}{|(y_0,z_0)|^{n+2s}} \iint_{1<|y|<\frac{|y_0|}{2},\, |z|<1} \left(|y|^{-\mu} + \langle y_0\rangle^{-\mu}\right) dydz \\ &\lesssim |(y_0,z_0)|^{-n-2s} \left(|y_0|^{n-1-\mu} + 1\right), \end{aligned}$$

and in the intermediate rectangle,

$$
\begin{aligned}
|I_{22}| &\lesssim \iint_{1<|y|<\frac{|y_0|}{2},\, 1<|z|<\frac{|z_0|}{2}} \left(|y|^{-\mu} + \langle y_0\rangle^{-\mu}\right)\left(|z|^{-\sigma} + \langle z_0\rangle^{-\sigma}\right) dydz \\
&\lesssim |(y_0,z_0)|^{-n-2s}\left(|y_0|^{n-1-\mu}+1\right)\left(|z_0|^{1-\sigma}+1\right).
\end{aligned}
$$

The integral on a thin strip afar is more involved. We first integrate the z variable by a change of variable $z = z_0 + |y_0 - y|\zeta$.

$$
\begin{aligned}
I_{13} &= C_{n,s}\iint_{|y|<1,\, |z-z_0|<\frac{|z_0|}{2}} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}\right)}{|(y-y_0, z-z_0)|^{n+2s}} \\
&\qquad \left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma} - D\langle z\rangle^{-\sigma}|_{z_0}(z-z_0)\right) dydz \\
&= C_{n,s}\iint_{|y|<1,\, |z-z_0|<\frac{|z_0|}{2}} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}\right)}{|(y-y_0, z-z_0)|^{n+2s}} \\
&\qquad (z-z_0)^2\left(\int_0^1 (1-t)D^2\langle z\rangle^{-\sigma}|_{z_0+t(z-z_0)}\, dt\right) dydz \\
&= C_{n,s}\int_{|y|<1} \frac{\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}}{|y-y_0|^{n-3+2s}} \\
&\qquad \int_{|\zeta|<\frac{|z_0|}{2|y-y_0|}} \left(\int_0^1 (1-t)D^2\langle z\rangle^{-\sigma}|_{z_0+t|y-y_0|\zeta}\, dt\right) \frac{\zeta^2\, d\zeta}{(1+\zeta^2)^{\frac{n+2s}{2}}}\, dy.
\end{aligned}
$$

Observing that in this regime $|y-y_0| \sim |y_0|$ and that

$$
\int_0^T \frac{t^2}{(1+t^2)^{\frac{n+2s}{2}}}\, dt \lesssim \min\left\{T^3, 1\right\},
$$

we have

$$
\begin{aligned}
|I_{13}| &\lesssim \int_{|y|<1} \frac{1}{|y-y_0|^{n-3+2s}} |z_0|^{-\sigma-2} \min\left\{\left(\frac{|z_0|}{|y-y_0|}\right)^3, 1\right\} dy \\
&\lesssim |y_0|^{-n-2s}|z_0|^{-\sigma-2}\min\{|y_0|, |z_0|\}^3.
\end{aligned}
$$

Similarly, changing $y = y_0 + |z - z_0|\eta$, we have

$$
\begin{aligned}
I_{31} &= C_{n,s} \iint_{|y-y_0|<\frac{|y_0|}{2},\,|z|<1} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu} - D\langle y\rangle^{-\mu}|_{y_0}\cdot(y-y_0)\right)}{|(y-y_0, z-z_0)|^{n+2s}} \\
&\qquad \left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right) dydz \\
&= C_{n,s} \iint_{|y-y_0|<\frac{|y_0|}{2},\,|z|<1} \frac{\left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right)}{|(y-y_0, z-z_0)|^{n+2s}} \\
&\qquad \cdot \left(\sum_{i,j=1}^{n-1} \int_0^1 (1-t)\partial_{ij}\langle y\rangle^{-\mu}|_{y_0+t(y-y_0)}\, dt\right)(y-y_0)_i (y-y_0)_j\, dydz \\
&= \sum_{i,j=1}^{n-1} \int_{|z|<1} \frac{\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}}{|z-z_0|^{2s-1}} \\
&\qquad \int_{|\eta|<\frac{|y_0|}{2|z-z_0|}} \left(\int_0^1 (1-t)\partial_{ij}\langle y\rangle^{-\mu}|_{y_0+t|z-z_0|\eta}\, dt\right) \frac{\eta_i\eta_j\, d\eta}{|(\eta,1)|^{n+2s}}\, dz.
\end{aligned}
$$

The t-integral is controlled by $\langle y_0\rangle^{-\mu-2}$ since $\left|y_0 + t|z-z_0|\eta\right| < \frac{|y_0|}{2}$. Then using

$$
\begin{aligned}
\int_{|\eta|<\eta_0} \frac{|\eta_i||\eta_j|}{\left(|\eta|^2+1\right)^{\frac{n+2s}{2}}}\, d\eta &\lesssim \int_0^{\eta_0} \frac{\rho^2\rho^{n-2}}{(\rho^2+1)^{\frac{n+2s}{2}}}\, d\rho \\
&\lesssim \min\left\{\eta_0^{n+1}, 1\right\},
\end{aligned}
$$

(noting that here we again require $s > 1/2$) we have

$$
\begin{aligned}
|I_{31}| &\lesssim \sum_{i,j=1}^{n-1} \int_{|z|<1} \frac{1}{|z-z_0|^{2s-1}} \langle y_0\rangle^{-\mu-2} \min\left\{\left(\frac{|y_0|}{|z-z_0|}\right)^{n+1}, 1\right\} dz \\
&\lesssim |z_0|^{-n-2s} \langle y_0\rangle^{-\mu-2} \min\{|y_0|, |z_0|\}^{n+1}.
\end{aligned}
$$

Next we deal with the y-intermediate, z-far regions, namely I_{23}. The treatment is similar to that of I_{13} except that we need to integrate in y. We have, as above,

$$
\begin{aligned}
I_{23} &= C_{n,s} \iint_{1<|y|<\frac{|y_0|}{2},\, |z-z_0|<\frac{|z_0|}{2}} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}\right)}{|(y-y_0, z-z_0)|^{n+2s}} \\
&\qquad \left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma} - D\langle z\rangle^{-\sigma}|_{z_0}(z-z_0)\right) dydz \\
&= C_{n,s} \int_{1<|y|<\frac{|y_0|}{2}} \frac{\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}}{|y-y_0|^{n-3+2s}} \\
&\qquad \int_{|\zeta|<\frac{|z_0|}{2|y-y_0|}} \left(\int_0^1 (1-t) D^2 \langle z\rangle^{-\sigma}|_{z_0+t|y-y_0|\zeta}\, dt\right) \frac{\zeta^2\, d\zeta}{(1+\zeta^2)^{\frac{n+2s}{2}}}\, dy.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|I_{23}| &\lesssim \int_{1<|y|<\frac{|y_0|}{2}} \frac{|y|^{-\mu} + \langle y_0\rangle^{-\mu}}{|y-y_0|^{n-3+2s}} |z_0|^{-\sigma-2} \min\left\{\left(\frac{|z_0|}{|y-y_0|}\right)^3, 1\right\} dy \\
&\lesssim |y_0|^{-n-2s} |z_0|^{-\sigma-2} \min\{|y_0|, |z_0|\}^3 \int_{1<|y|<\frac{|y_0|}{2}} \left(|y|^{-\mu} + \langle y_0\rangle^{-\mu}\right) dy \\
&\lesssim |y_0|^{-n-2s} |z_0|^{-\sigma-2} \min\{|y_0|, |z_0|\}^3 \left(|y_0|^{n-1-\mu} + 1\right).
\end{aligned}
$$

Similarly, we estimate

$$
\begin{aligned}
I_{32} &= C_{n,s} \iint_{|y-y_0|<\frac{|y_0|}{2},\, 1<|z|<\frac{|z_0|}{2}} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu} - D\langle y\rangle^{-\mu}|_{y_0} \cdot (y-y_0)\right)}{|(y-y_0, z-z_0)|^{n+2s}} \\
&\qquad \left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right) dydz \\
&= \sum_{i,j=1}^{n-1} \int_{1<|z|<\frac{|z_0|}{2}} \frac{\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}}{|z-z_0|^{2s-1}} \\
&\qquad \int_{|\eta|<\frac{|y_0|}{2|z-z_0|}} \left(\int_0^1 (1-t)\partial_{ij} \langle y\rangle^{-\mu}|_{y_0+t|z-z_0|\eta}\, dt\right) \frac{\eta_i \eta_j\, d\eta}{|(\eta, 1)|^{n+2s}}\, dz,
\end{aligned}
$$

which yields

$$
\begin{aligned}
|I_{32}| &\lesssim \sum_{i,j=1}^{n-1} \int_{1<|z|<\frac{|z_0|}{2}} \frac{|z|^{-\sigma} + \langle z_0\rangle^{-\sigma}}{|z-z_0|^{2s-1}} \langle y_0\rangle^{-\mu-2} \min\left\{ \left(\frac{|y_0|}{|z-z_0|} \right)^{n+1}, 1 \right\} dz \\
&\lesssim |z_0|^{-n-2s} |y_0|^{-\mu-2} \min\{|y_0|, |z_0|\}^{n+1} \int_{1<|z|<\frac{|z_0|}{2}} \left(|z|^{-\sigma} + \langle z_0\rangle^{-\sigma} \right) dz \\
&\lesssim |z_0|^{-n-2s} |y_0|^{-\mu-2} \min\{|y_0|, |z_0|\}^{n+1} \left(|z_0|^{1-\sigma} + 1 \right).
\end{aligned}
$$

We consider the remaining part of the small strip, namely I_{14} and I_{41}. Using the change of variable $z = z_0 + |y_0|\zeta$, we have

$$
I_{14} = C_{n,s} \iint_{|y|<1,\, \min\{|z|,|z-z_0|\} > \frac{|z_0|}{2}} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu} \right) \left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma} \right)}{|(y-y_0, z-z_0)|^{n+2s}} \, dydz,
$$

$$
\begin{aligned}
|I_{14}| &\lesssim \langle z_0\rangle^{-\sigma} \iint_{|y|<1,\, \min\{|z|,|z-z_0|\} > \frac{|z_0|}{2}} \frac{1}{|(y_0, z-z_0)|^{n+2s}} \, dydz \\
&\lesssim \langle z_0\rangle^{-\sigma} \int_{\min\{|z|,|z-z_0|\} > \frac{|z_0|}{2}} \frac{1}{|(y_0, z-z_0)|^{n+2s}} \, dz \\
&\lesssim \langle z_0\rangle^{-\sigma} \frac{1}{|y_0|^{n-1+2s}} \int_{|\zeta| > \frac{|z_0|}{2|y_0|},\, \left|\zeta - \frac{z_0}{|y_0|}\right| > \frac{|z_0|}{2|y_0|}} \frac{1}{|(1,\zeta)|^{n+2s}} \, d\zeta \\
&\lesssim \langle z_0\rangle^{-\sigma} |y_0|^{-(n-1+2s)} \int_{\frac{|z_0|}{2|y_0|}}^{\infty} \frac{d\zeta}{(1+\zeta^2)^{\frac{n+2s}{2}}} \\
&\lesssim \langle z_0\rangle^{-\sigma} |y_0|^{-(n-1+2s)} \min\left\{ 1, \left(\frac{|z_0|}{|y_0|} \right)^{-(n-1+2s)} \right\} \\
&\lesssim \langle z_0\rangle^{-\sigma} \min\left\{ |y_0|^{-(n-1+2s)}, |z_0|^{-(n-1+2s)} \right\} \\
&\lesssim \langle z_0\rangle^{-\sigma} \left(|y_0| + |z_0| \right)^{-(n-1+2s)}.
\end{aligned}
$$

Similarly, with $y = y_0 + |z_0|\eta$,

$$
I_{41} = C_{n,s} \iint_{\min\{|y|,|y-y_0|\} > \frac{|y_0|}{2},\, |z|<1} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu} \right) \left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma} \right)}{|(y-y_0, z-z_0)|^{n+2s}} \, dydz,
$$

$$
\begin{aligned}
|I_{41}| &\lesssim \langle y_0\rangle^{-\mu} \iint_{\min\{|y|,|y-y_0|\}>\frac{|y_0|}{2},\,|z|<1} \frac{1}{|(y-y_0,z_0)|^{n+2s}}\,dydz \\
&\lesssim \langle y_0\rangle^{-\mu}\,|z_0|^{-(1+2s)} \int_{|\eta|>\frac{|y_0|}{2|z_0|}} \frac{d\eta}{(|\eta|^2+1)^{\frac{n+2s}{2}}} \\
&\lesssim \langle y_0\rangle^{-\mu}\,|z_0|^{-(1+2s)} \int_{\frac{|y_0|}{2|z_0|}}^{\infty} \frac{\rho^{n-2}}{(\rho^2+1)^{\frac{n+2s}{2}}}\,d\rho \\
&\lesssim \langle y_0\rangle^{-\mu}\,|z_0|^{-(1+2s)} \min\left\{\left(\frac{|y_0|}{2|z_0|}\right)^{-(1+2s)},1\right\} \\
&\lesssim \langle y_0\rangle^{-\mu}\,(|y_0|+|z_0|)^{-(1+2s)}.
\end{aligned}
$$

In the remaining intermediate region, we first “integrate” in z by the change of variable $z = z_0 + |y-y_0|\zeta$ as follows.

$$
\begin{aligned}
I_{24} &= C_{n,s} \iint_{1<|y|<\frac{|y_0|}{2},\,\min\{|z|,|z-z_0|\}>\frac{|z_0|}{2}} \\
&\qquad \frac{\left(\langle y\rangle^{-\mu}-\langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma}-\langle z_0\rangle^{-\sigma}\right)}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz, \\
|I_{24}| &\lesssim \langle z_0\rangle^{-\sigma} \iint_{1<|y|<\frac{|y_0|}{2},\,\min\{|z|,|z-z_0|\}>\frac{|z_0|}{2}} \frac{|y|^{-\mu}+\langle y_0\rangle^{-\mu}}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz \\
&\lesssim \langle z_0\rangle^{-\sigma} \int_{1<|y|<\frac{|y_0|}{2}} \frac{|y|^{-\mu}+\langle y_0\rangle^{-\mu}}{|y-y_0|^{n-1+2s}} \\
&\qquad \int_{|\zeta|>\frac{|z_0|}{2|y-y_0|},\,\left|\zeta-\frac{z_0}{|y-y_0|}\right|>\frac{|z_0|}{2|y-y_0|}} \frac{d\zeta}{(1+\zeta^2)^{\frac{n+2s}{2}}}\,dy \\
&\lesssim \langle z_0\rangle^{-\sigma} \int_{1<|y|<\frac{|y_0|}{2}} \frac{|y|^{-\mu}+\langle y_0\rangle^{-\mu}}{|y-y_0|^{n-1+2s}} \\
&\qquad \min\left\{1,\left(\frac{|z_0|}{|y-y_0|}\right)^{-(n-1+2s)}\right\}dy \\
&\lesssim \langle z_0\rangle^{-\sigma} \int_{1<|y|<\frac{|y_0|}{2}} \left(|y|^{-\mu}+\langle y_0\rangle^{-\mu}\right)(|y-y_0|+|z_0|)^{-(n-1+2s)}\,dy \\
&\lesssim \langle z_0\rangle^{-\sigma}\,(|y_0|+|z_0|)^{-(n-1+2s)} \int_{1<|y|<\frac{|y_0|}{2}} \left(|y|^{-\mu}+\langle y_0\rangle^{-\mu}\right)dy \\
&\lesssim |y|^{n-1-\mu}\,\langle z_0\rangle^{-\sigma}\,(|y_0|+|z_0|)^{-(n-1+2s)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_{42} = C_{n,s} \iint_{\min\{|y|,|y-y_0|\}>\frac{|y_0|}{2},\, 1<|z|<\frac{|z_0|}{2}} & \\
\frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right)}{|(y-y_0, z-z_0)|^{n+2s}}\, dydz, & \\
|I_{42}| \lesssim \langle y_0\rangle^{-\mu} \iint_{\min\{|y|,|y-y_0|\}>\frac{|y_0|}{2},\, 1<|z|<\frac{|z_0|}{2}} \frac{|z|^{-\sigma} + \langle z_0\rangle^{-\sigma}}{|(y-y_0, z-z_0)|^{n+2s}}\, dydz & \\
\lesssim \langle y_0\rangle^{-\mu} \int_{1<|z|<\frac{|z_0|}{2}} \frac{|z|^{-\sigma} + \langle z_0\rangle^{-\sigma}}{|z-z_0|^{1+2s}} \int_{|\eta|>\frac{|y_0|}{2|z-z_0|}} \frac{d\eta}{(|\eta|^2+1)^{\frac{n+2s}{2}}}\, dz & \\
\lesssim \langle y_0\rangle^{-\mu} \int_{1<|z|<\frac{|z_0|}{2}} \frac{|z|^{-\sigma} + \langle z_0\rangle^{-\sigma}}{|z-z_0|^{1+2s}} \min\left\{ \left(\frac{|y_0|}{2|z-z_0|}\right)^{-1-2s}, 1 \right\} dz & \\
\lesssim \langle y_0\rangle^{-\mu} |z_0|^{1-\sigma} \left(|y_0| + |z_0|\right)^{-(1+2s)}. &
\end{aligned}
$$

Now we estimate the singular part I^{sing}. The only concern is that if, say, $|y_0| \gg |z_0|$, then the line segment joining z_0 and z may intersect the y-axis. To fix the idea we suppose that $|y_0| \geq |z_0|$. Having all estimates for the integrals in a neighborhood of the axes, one can factor out the decay $\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}$ and obtain integrability by expanding the bracket with y to second order, as follows. For simplicity let us write

$$
\Omega_{sing} = \left\{ (y,z) \in \mathbb{R}^n : |y| > \frac{|y_0|}{2},\, |z| > \frac{|z_0|}{2},\, |(y-y_0, z-z_0)| < \frac{|y_0| + |z_0|}{2} \right\}.
$$

Then

$$
\begin{aligned}
I^{sing} &= C_{n,s} \iint_{\Omega_{sing}} \frac{\left(\langle y\rangle^{-\mu} - \langle y_0\rangle^{-\mu}\right)\left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right)}{|(y-y_0, z-z_0)|^{n+2s}}\, dydz \\
&= C_{n,s} \iint_{\Omega_{sing}} \frac{\left(\langle z\rangle^{-\sigma} - \langle z_0\rangle^{-\sigma}\right)}{|(y-y_0, z-z_0)|^{n+2s}} \\
&\qquad \cdot \left(\sum_{i,j=1}^{n-1} \int_0^1 (1-t) \partial_{ij} \langle y\rangle^{-\mu} \big|_{y_0+t(y-y_0)}\, dt \right) (y-y_0)_i (y-y_0)_j\, dydz.
\end{aligned}
$$

Thus

$$\begin{aligned}\left|I^{sing}\right| &\lesssim \langle z_0\rangle^{-\sigma}\langle y_0\rangle^{-\mu-2}\iint_{\Omega_{sing}}\frac{|y-y_0|^2}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz\\ &\lesssim \langle z_0\rangle^{-\sigma}\langle y_0\rangle^{-\mu-2}\int_0^{\frac{|y_0|+|z_0|}{2}}\frac{\rho^2}{\rho^{1+2s}}\,d\rho\\ &\lesssim \langle y_0\rangle^{-\mu-2s}\langle z_0\rangle^{-\sigma}.\end{aligned}$$

The same argument implies that if $|z_0|\geq|y_0|$ then

$$\left|I^{sing}\right|\lesssim\langle y_0\rangle^{-\mu}\langle z_0\rangle^{-\sigma-2s}.$$

Therefore, we have in general

$$\begin{aligned}\left|I^{sing}\right| &\lesssim \langle y_0\rangle^{-\mu}\langle z_0\rangle^{-\sigma}\max\{|y_0|,|z_0|\}^{-2s}\\ &\lesssim \langle y_0\rangle^{-\mu}\langle z_0\rangle^{-\sigma}(|y_0|+|z_0|)^{-2s}.\end{aligned}$$

Finally, the remaining exterior integral is controlled by

$$\begin{aligned}\left|I^{rest}\right| &\lesssim \langle y_0\rangle^{-\mu}\langle z_0\rangle^{-\sigma}\iint_{|y|>\frac{|y_0|}{2},\,|z|>\frac{|z_0|}{2},\,|(y-y_0,z-z_0)|<\frac{|y_0|+|z_0|}{2}}\\ &\qquad\frac{1}{|(y-y_0,z-z_0)|^{n+2s}}\,dydz\\ &\lesssim \langle y_0\rangle^{-\mu}\langle z_0\rangle^{-\sigma}\int_{\frac{|y_0|+|z_0|}{2}}^{\infty}\frac{d\rho}{\rho^{1+2s}}\\ &\lesssim \langle y_0\rangle^{-\mu}\langle z_0\rangle^{-\sigma}(|y_0|+|z_0|)^{-2s}.\end{aligned}$$

4. This follows from the product rule

$$\begin{aligned}&(-\Delta)^s\left(\langle y\rangle^{-\mu}\langle z\rangle^{-\sigma}\right)\\ &=\langle y\rangle^{-\mu}(-\Delta)^s\langle z\rangle^{-\sigma}+\langle z\rangle^{-\sigma}(-\Delta)^s\langle y\rangle^{-\mu}-(\langle y\rangle^{-\mu},\langle z\rangle^{-\sigma})_s\\ &=\langle y\rangle^{-\mu}\langle z\rangle^{-\sigma}\Big(O(\langle y\rangle^{-2s})+O(\langle z\rangle^{-2s})+o(1)\Big).\end{aligned}$$

5. The s-inner product is computed as follows. We may assume that $1 \le |z_0| \le \frac{\mathtt{R}}{2}$. When $|y_0| \ge 3\mathtt{R}$,

$$
\begin{aligned}
&|[(-\Delta)^s, \eta_{\mathtt{R}}]\phi(y_0, z_0)| \\
&\le C \int_{\mathbb{R}^n} \frac{|-\eta_{\mathtt{R}}(y)|\,\langle z\rangle^{-\sigma}}{|(y_0,z_0)-(y,z)|^{n+2s}}\,dydz \\
&\le C \int_{\mathbb{R}} \int_{|y|\le 2\mathtt{R}} \frac{\langle z\rangle^{-\sigma}}{|(y_0,z_0)-(y,z)|^{n+2s}}\,dydz \\
&\le C\mathtt{R}^{n-1} \int_{\mathbb{R}} \frac{\langle z\rangle^{-\sigma}}{\left(|y_0|^2+|z_0-z|^2\right)^{\frac{n+2s}{2}}}\,dz \\
&\le C\mathtt{R}^{n-1} \left(\int_{|z|\ge \frac{|z_0|}{2}} \frac{\langle z_0\rangle^{-\sigma}}{\left(|y_0|^2+|z_0-z|^2\right)^{\frac{n+2s}{2}}}\,dz \right. \\
&\qquad \left. + \int_{|z|\le \frac{|z_0|}{2}} \frac{\langle z\rangle^{-\sigma}}{\left(|y_0|^2+|z_0|^2\right)^{\frac{n+2s}{2}}}\,dz \right) \\
&\le C\mathtt{R}^{n-1} \left(|z_0|^{-\sigma}|y_0|^{-(n-1+2s)} + (1+|z_0|^{1-\sigma})|(y_0,z_0)|^{-n-2s} \right) \\
&\le C \left(|z_0|^{-\sigma}|y_0|^{-2s} + (|z_0|^{-1}+|z_0|^{-\sigma})|(y_0,z_0)|^{-2s} \right) \\
&\le C \left(|z_0|^{-1}+|z_0|^{-\sigma} \right) |y_0|^{-2s}.
\end{aligned}
$$

When $|y_0| \le \frac{\mathtt{R}}{2}$,

$$
\begin{aligned}
&|[(-\Delta)^s, \eta_{\mathtt{R}}]\phi(y_0,z_0)| \\
&\le C \int_{\mathbb{R}^n} \frac{(1-\eta_{\mathtt{R}}(y))\,\langle z\rangle^{-\sigma}}{|(y_0,z_0)-(y,z)|^{n+2s}}\,dydz \\
&\le C \int_{\mathbb{R}} \int_{|y|\ge \mathtt{R}} \frac{\langle z\rangle^{-\sigma}}{|(y_0,z_0)-(y,z)|^{n+2s}}\,dydz \\
&\le C \int_{\mathbb{R}} \int_{|y|\ge \frac{\mathtt{R}}{2}} \frac{\langle z\rangle^{-\sigma}}{\left(|y|^2+|z_0-z|^2\right)^{\frac{n+2s}{2}}}\,dydz \\
&\le C \int_{\mathbb{R}} \frac{\langle z\rangle^{-\sigma}}{|z_0-z|^{1+2s}} \int_{|\tilde{y}|\ge \frac{\mathtt{R}}{2|z_0-z|}} \frac{d\tilde{y}}{\left(|\tilde{y}|^2+1\right)^{\frac{n+2s}{2}}}\,dz \\
&\le C \int_{\mathbb{R}} \frac{\langle z\rangle^{-\sigma}}{|z_0-z|^{1+2s}} \min\left\{1, \left(\frac{|z_0-z|}{\mathtt{R}}\right)^{1+2s}\right\} dz \\
&\le C \left(\int_{z_0-\mathtt{R}}^{z_0+\mathtt{R}} \langle z\rangle^{-\sigma}\,\mathtt{R}^{-1-2s}\,dz + \int_{|z_0-z|>\mathtt{R}} \frac{\langle z\rangle^{-\sigma}}{|z_0-z|^{1+2s}}\,dz \right) \\
&\le C \left(\mathtt{R}^{-1-2s}(1+\mathtt{R}^{1-\sigma}) + \mathtt{R}^{-\sigma}\mathtt{R}^{-2s}\right) \\
&\le C \left(\mathtt{R}^{-1-2s} + \mathtt{R}^{-\sigma-2s}\right).
\end{aligned}
$$

When $\frac{\mathtt{R}}{2} \le |y_0| \le 3\mathtt{R}$, we have

$$
\partial_{y_iy_j}\eta_{\mathtt{R}} = \frac{1}{\mathtt{R}^2}\eta''\left(\frac{y}{\mathtt{R}}\right)\frac{y_iy_j}{|y|^2} + \frac{1}{\mathtt{R}|y|}\eta'\left(\frac{y}{\mathtt{R}}\right)\left(\delta_{ij} - \frac{y_iy_j}{|y|^2}\right),
$$

which implies that $\left\|D^2\eta_{\mathtt{R}}\right\|_{L^\infty([y_0,y])} \le C\mathtt{R}^{-2}$ for $|y_0-y| \le \frac{y_0}{2}$, where $[y_0,y]$ denotes the line segment joining y_0 and y. Thus

$$
\begin{aligned}
&|[(-\Delta)^s,\eta_{\mathtt{R}}]\phi(y_0,z_0)| \\
&\le C\int_{\mathbb{R}^n} \frac{\left|\eta_{\mathtt{R}}(y_0)-\eta_{\mathtt{R}}(y)+\chi_{\{|y-y_0|<1\}}D\eta_{\mathtt{R}}(y_0)\cdot(y-y_0)\right|\langle z\rangle^{-\sigma}}{|(y_0,z_0)-(y,z)|^{n+2s}}\,dydz \\
&\le C\Bigg(\int_{\mathbb{R}^{n-1}}\int_{|z|\le\frac{|z_0|}{2}} \frac{\left|\eta_{\mathtt{R}}(y_0)-\eta_{\mathtt{R}}(y)+\chi_{\{|y-y_0|<1\}}D\eta_{\mathtt{R}}(y_0)\cdot(y-y_0)\right|}{\left(|y_0-y|^2+|z_0|^2\right)^{\frac{n+2s}{2}}} \\
&\qquad \langle z\rangle^{-\sigma}\,dydz \\
&+\int_{\mathbb{R}^{n-1}}\int_{|z|\ge\frac{|z_0|}{2}} \frac{\left|\eta_{\mathtt{R}}(y_0)-\eta_{\mathtt{R}}(y)+\chi_{\{|y-y_0|<1\}}D\eta_{\mathtt{R}}(y_0)\cdot(y-y_0)\right|\langle z_0\rangle^{-\sigma}}{\left(|y_0-y|^2+|z_0-z|^2\right)^{\frac{n+2s}{2}}}\,dydz\Bigg) \\
&\le C\Bigg((1+|z_0|^{1-\sigma})\int_{\mathbb{R}^{n-1}} \frac{\left|\eta_{\mathtt{R}}(y_0)-\eta_{\mathtt{R}}(y)+\chi_{\{|y-y_0|<1\}}D\eta_{\mathtt{R}}(y_0)\cdot(y-y_0)\right|}{\left(|y_0-y|^2+|z_0|^2\right)^{\frac{n+2s}{2}}}\,dy \\
&\qquad +|z_0|^{\sigma}\int_{\mathbb{R}^{n-1}} \frac{\left|\eta_{\mathtt{R}}(y_0)-\eta_{\mathtt{R}}(y)+\chi_{\{|y-y_0|<1\}}D\eta_{\mathtt{R}}(y_0)\cdot(y-y_0)\right|}{|y_0-y|^{n-1+2s}}\,dy\Bigg) \\
&\le C\left(|z_0|^{-1}+|z_0|^{-\sigma}\right)\Bigg(\int_{|y_0-y|\ge\frac{y_0}{2}} \frac{dy}{|y_0-y|^{n-1+2s}} \\
&\qquad +\int_{|y_0-y|\le\frac{y_0}{2}} \frac{\left\|D^2\eta_{\mathtt{R}}\right\|_{L^\infty([y_0,y])}|y_0-y|^2}{|y_0-y|^{n-1+2s}}\,dy\Bigg) \\
&\le C\left(|z_0|^{-1}+|z_0|^{-\sigma}\right)\left(|y_0|^{-2s}+\mathtt{R}^{-2}|y_0|^{2-2s}\right) \\
&\le C\left(|z_0|^{-1}+|z_0|^{-\sigma}\right)|y_0|^{-2s}.
\end{aligned}
$$

This completes the proof of (2.22).

$\square$

2.4.3 Existence

In order to solve the linearized equation

$$(-\Delta)^s \phi + f'(\underline{w})\phi = g \quad \text{ for } (y,z) \in \mathbb{R}^n,$$

we consider the equivalent problem in the Caffarelli–Slivestre extension [43],

$$\begin{cases} -\nabla \cdot (t^a \nabla \phi) = 0 & \text{for } (t,y,z) \in \mathbb{R}^{n+1}_+ \\ t^a \dfrac{\partial \phi}{\partial \nu} + f'(w)\phi = g & \text{for } (y,z) \in \partial \mathbb{R}^{n+1}_+. \end{cases} \tag{2.27}$$

We will prove the following

Proposition 2.4.8. *Let $\mu, \sigma > 0$ be small. For any g with $\|g\|_{\mu,\sigma} < +\infty$ satisfying*

$$\int_{\mathbb{R}} g(y,z)w'(z)\,dz = 0, \tag{2.28}$$

there exists a unique solution $\phi \in H^1(\mathbb{R}^{n+1}_+, t^a)$ of (2.27) satisfying

$$\iint_{\mathbb{R}^2_+} t^a \phi(t,y,z) w_z(t,z)\,dt dz = 0 \quad \text{for all } y \in \mathbb{R}^{n-1}, \tag{2.29}$$

such that the trace $\phi(0,y,z)$ satisfies $\|\phi\|_{\mu,\sigma} < +\infty$. Moreover,

$$\|\phi\|_{\mu,\sigma} \le C \|g\|_{\mu,\sigma}. \tag{2.30}$$

Let us recall the corresponding known result [82] in one dimension.

Lemma 2.4.9. *Let $n = 1$. For any g with $\int_{\mathbb{R}} gw'\,dz = 0$, there exists a unique solution ϕ to (2.27) satisfying $\iint_{\mathbb{R}^2_+} t^a \phi w_z\,dt dz = 0$ such that*

$$\|\phi\|_{0,\sigma} \le C \|g\|_{0,\sigma}.$$

Proof. This is Proposition 4.1 in [82]. In their notations, take $m = 1$, $\xi_1 = 0$ and $\mu = \sigma$. □

Proof of Proposition 2.4.8. 1. We first assume that $g \in C_c^\infty(\mathbb{R}^n)$. Taking Fourier transform in y, we solve for each $\xi \in \mathbb{R}^{n-1}$ a solution $\hat{\phi}(t,\xi,z)$ to

$$\begin{cases} -\nabla \cdot (t^a \nabla \hat{\phi}) + |\xi|^2 t^a \hat{\phi} = 0 & \text{for } (t,z) \in \mathbb{R}^2_+, \\ t^a \dfrac{\partial \hat{\phi}}{\partial \nu} + f'(w)\hat{\phi} = \hat{g} & \text{for } z \in \partial \mathbb{R}^2_+, \end{cases}$$

with orthogonality condition

$$\iint_{\mathbb{R}^2_+} t^a \hat{\phi}(t,\xi,z) w_z(t,z)\, dt dz = 0 \quad \text{for all } \xi \in \mathbb{R}^{n-1}$$

corresponding to (2.29). One can then obtain a solution for $\xi = 0$ by Lemma 2.4.9 and for $\xi \neq 0$ by Lemma 2.4.2. From the embedding $H^1(\mathbb{R}^2_+, t^a) \hookrightarrow H^s(\mathbb{R})$ [36], we have the estimate

$$\left\| \hat{\phi}(\cdot,\xi,\cdot) \right\|_{H^1(\mathbb{R}^2_+,t^a)} \leq C(\xi) \left\| \hat{g}(\xi,\cdot) \right\|_{L^2(\mathbb{R})}.$$

We claim that the constant can be taken independent of ξ, i.e.

$$\left\| \hat{\phi}(\cdot,\xi,\cdot) \right\|_{H^1(\mathbb{R}^2_+,t^a)} \leq C \left\| \hat{g}(\xi,\cdot) \right\|_{L^2(\mathbb{R})}. \tag{2.31}$$

If this were not true, there would exist sequences $\xi_m \to 0$ (the case $|\xi_m| \to +\infty$ is similar), $\hat{\phi}_m$ and $\hat{g}_m$ such that

$$\left\| \hat{\phi}_m(\cdot,\xi_m,\cdot) \right\|_{H^1(\mathbb{R}^2_+,t^a)} = 1, \quad \left\| \hat{g}_m(\xi_m,\cdot) \right\|_{L^2(\mathbb{R})} = 0, \tag{2.32}$$

$$\begin{cases} -\nabla \cdot (t^a \nabla \hat{\phi}_m) + |\xi_m|^2 t^a \hat{\phi}_m = 0 & \text{for } (t,z) \in \mathbb{R}^2_+, \\ t^a \dfrac{\partial \hat{\phi}_m}{\partial \nu} + f'(w)\hat{\phi}_m = \hat{g}_m & \text{for } z \in \partial \mathbb{R}^2_+, \end{cases}$$

and

$$\iint_{\mathbb{R}^2_+} t^a \hat{\phi}_m(t,\xi_m,z) w_z(t,z)\, dt dz = 0.$$

Elliptic regularity implies that a subsequence of $\hat{\phi}_m(t,\xi_m,z)$ converges locally uniformly in $\mathbb{R}^2_+$ to some $\hat{\phi}_0(t,z)$, which solves weakly

$$\begin{cases} -\nabla\cdot(t^a\nabla\hat{\phi}_0)=0 & \text{for } (t,z)\in\mathbb{R}^2_+ \\ t^a\dfrac{\partial\hat{\phi}_0}{\partial\nu}+f'(w)\hat{\phi}_0=0 & \text{for } z\in\partial\mathbb{R}^2_+. \end{cases}$$

and

$$\iint_{\mathbb{R}^2_+} t^a\hat{\phi}_0(t,z)w_z(t,z)\,dtdz=0 \quad \text{for all } \xi\in\mathbb{R}^{n-1}.$$

By Lemma 2.4.4, we conclude that $\hat{\phi}_0=0$, contradicting (2.32). This proves (2.31).

Integrating over $\xi\in\mathbb{R}^{n-1}$ and using Plancherel's theorem, we obtain a solution ϕ satisfying

$$\|\phi\|_{H^1(\mathbb{R}^{n+1}_+,t^a)}\le C\|g\|_{L^2(\mathbb{R}^n)}.$$

Higher regularity yields, in particular, $\phi\in L^\infty(\mathbb{R}^n)$. Then (2.30) follows from Lemma 2.4.6.

2. In the general case, we solve (2.27) with g replaced by $g_m\in C_c^\infty(\mathbb{R}^n)$ which converges uniformly to g. Then the solution ϕ_m is controlled by

$$\|\phi_m\|_{\mu,\sigma}\le C\|g_m\|_{\mu,\sigma}\le C\|g\|_{\mu,\sigma}.$$

By passing to a subsequence, ϕ_m converges to some ϕ uniformly on compact subsets of $\mathbb{R}^n$, which also satisfies (2.30).

3. The uniqueness follows from the non-degeneracy of w' and the orthogonality condition (2.29).

□

2.4.4 The positive operator

We conclude this section by stating a standard estimate for the operator $(-\Delta)^s+2$.

Lemma 2.4.10. *Consider the equation*

$$(-\Delta)^s u + 2u = g \quad \text{in } \mathbb{R}^n.$$

and $|g(x)| \le C \langle x' \rangle^{-\theta}$ *for all* $x \in \mathbb{R}^n$ *and* $g(x) = 0$ *for* x *in* $M_{\varepsilon,R}$*, a tubular neighborhood of* M_ε *of width* R*. Then the unique solution* $u = ((-\Delta)^s + 2)^{-1} g$ *satisfies the decay estimate*

$$|u(x)| \le C \left\langle x' \right\rangle^{-\theta} \langle \operatorname{dist}(x, M_{\varepsilon,R}) \rangle^{-2s}.$$

Proof. The decay in x' follows from a maximum principle; that in the interface is seen from the Green's function for $(-\Delta)^s + 2$ which has a decay $|x|^{-(n+2s)}$ at infinity [62]. □

2.5 Fractional gluing system

2.5.1 Preliminary estimates

We have the following

Lemma 2.5.1 (Some non-local estimates)**.** *For* $\phi_j \in X_j$*,* $j \in \mathscr{J}$*, the following holds true.*

1. *(commutator at the near interface)*

$$\left|[(-\Delta_{(y,z)})^s, \bar{\eta}\,\bar{\zeta}]\bar{\phi}_i(y,z)\right| \le C \|\phi_i\|_{i,\mu,\sigma} \langle \mathrm{y}_i \rangle^{-\theta} R^n (R + |(y,z)|)^{-n-2s}.$$

   As a result,

$$\sum_{i \in \mathscr{J}} \left|[(-\Delta_{(y,z)})^s, \zeta_i]\phi_i(x)\right| \le Cr^{-\theta} \sup_{i \in \mathscr{J}} \|\phi_i\|_{i,\mu,\sigma} \left(R + \operatorname{dist}\left(x, \operatorname{supp} \sum_{i \in \mathscr{J}} \zeta_i\right)\right)^{-2s}.$$

2. *(commutator at the end)*

$$\left|[(-\Delta_{(y,z)})^s, \bar{\eta}_+\bar{\zeta}]\phi_+(y,z)\right| \le C \|\phi_+\|_{+,\mu,\sigma} R_2^{-\theta} \langle y \rangle^{-\mu} \langle z \rangle^{-1-2s},$$

and similarly for ϕ_-.

3. *(linearization at u^*)*

$$\sum_{j\in\mathscr{J}} \left|\zeta_j(f'(w_j)-f'(u^*))\phi_j\right|$$
$$\leq C\sup_{j\in\mathscr{J}}\left\|\phi_j\right\|_{j,\mu,\sigma}\left(\sum_{i\in\mathscr{I}}\zeta_i R^{\mu+\sigma}\left\langle \mathrm{y}_i\right\rangle^{-\theta-\frac{4s}{2s+1}}+(\zeta_++\zeta_-)R_2^{-\theta}\left\langle y\right\rangle^{-\mu}\right).$$

4. *(change of coordinates around the near interface)*

$$\sum_{i\in\mathscr{I}}\left|((-\Delta_x)^s-(-\Delta_{(y,z)})^s)(\zeta_i\phi_i)(x)\right|$$
$$\leq CR^{n+1+\mu+\sigma}\varepsilon\left\|\bar{\phi}_i\right\|_{i,\mu,\sigma}$$
$$\left(\sum_{i\in\mathscr{I}}\zeta_i\left\langle \mathrm{y}_i\right\rangle^{-\theta}+\varepsilon^{\theta}\left\langle \mathrm{dist}\left(x,\mathrm{supp}\sum_{i\in\mathscr{I}}\zeta_i\right)\right\rangle^{-2s}\right).$$

5. *(change of coordinates around the end)*

$$\left|((-\Delta_x)^s-(-\Delta_{(y,z)})^s)(\zeta_+\phi_+)(x)\right|$$
$$\leq Cr^{-\frac{2(2s-\tau)}{2s+1}}\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma}R_2^{-\theta}\left\langle y\right\rangle^{-\mu}\left\langle z\right\rangle^{-1-2s},$$

and similarly for ϕ_-.

In particular, all these terms are dominated by $S(u^*)$.

Proof of Lemma 2.5.1. 1. (a) Since $\phi_i\in X_i$, we have for $|(y_0,z_0)|\geq 3R$,

$$\begin{aligned}
&\left|[(-\Delta_{(y,z)})^s,\bar{\eta}\bar{\zeta}]\bar{\phi}_i(y_0,z_0)\right|\\
&\leq C\left\|\phi_i\right\|_{i,\mu,\sigma}\left|\int_{|(y,z)|\leq 2R}\frac{-\bar{\eta}(y)\bar{\zeta}(z)}{\left|(y_0,z_0)\right|^{n+2s}}R^{\mu+\sigma}\left\langle \mathrm{y}_i\right\rangle^{-\theta}\left\langle y\right\rangle^{-\mu}\left\langle z\right\rangle^{-\sigma}\,dydz\right|\\
&\leq C\left\|\phi_i\right\|_{i,\mu,\sigma}R^{\mu+\sigma}\left\langle \mathrm{y}_i\right\rangle^{-\theta}\left|(y_0,z_0)\right|^{-n-2s}\int_{|(y,z)|\leq 2R}\left\langle y\right\rangle^{-\mu}\left\langle z\right\rangle^{-\sigma}\,dydz\\
&\leq C\left\|\phi_i\right\|_{i,\mu,\sigma}R^{\mu+\sigma}(1+R^{1-\sigma})(1+R^{n-1-\mu})\left\langle \mathrm{y}_i\right\rangle^{-\theta}\left|(y_0,z_0)\right|^{-n-2s}\\
&\leq CR^{n}\left|(y_0,z_0)\right|^{-n-2s}\left\|\phi_i\right\|_{i,\mu,\sigma}\left\langle \mathrm{y}_i\right\rangle^{-\theta}\qquad\text{for }\sigma<1,\ \mu<n-1.
\end{aligned}$$

(b) For $\frac{R}{2} \leq |(y_0, z_0)| \leq 3R$,

$$
\begin{aligned}
&\left|[(-\Delta_{(y,z)})^s, \bar{\eta}\bar{\zeta}]\bar{\phi}_i(y_0, z_0)\right| \\
&\leq C \int_{|y_0-y|<\frac{R}{4}} \int_{|z_0-z|<\frac{R}{4}} \frac{R^{-2}\left(|y_0-y|^2+|z_0-z|^2\right)}{\left(|y_0-y|^2+|z_0-z|^2\right)^{\frac{n+2s}{2}}} \\
&\qquad R^{\mu+\sigma} \|\phi_i\|_{i,\mu,\sigma} \langle \mathrm{y}_i \rangle^{-\theta} \langle y \rangle^{-\mu} \langle z \rangle^{-\sigma} \, dydz \\
&\qquad + C \int_{|y_0-y|>\frac{R}{4}} \int_{|z_0-z|>\frac{R}{4}} \frac{1}{\left(|y_0-y|^2+|z_0-z|^2\right)^{\frac{n+2s}{2}}} \\
&\qquad R^{\mu+\sigma} \|\phi_i\|_{i,\mu,\sigma} \langle \mathrm{y}_i \rangle^{-\theta} \langle y \rangle^{-\mu} \langle z \rangle^{-\sigma} \, dydz \\
&\leq CR^{-2s} \|\phi_i\|_{i,\mu,\sigma} \langle \mathrm{y}_i \rangle^{-\theta}.
\end{aligned}
$$

(c) For $0 \leq |(y_0, z_0)| \leq \frac{R}{2}$,

$$
\begin{aligned}
&\left|[(-\Delta_{(y,z)})^s, \bar{\eta}\bar{\zeta}]\bar{\phi}_i(y_0, z_0)\right| \\
&\leq C \|\phi_i\|_{i,\mu,\sigma} \int_{|(y,z)|\geq R} \frac{1-\bar{\eta}(y)\bar{\zeta}(z)}{|(y-y_0, z-z_0)|^{n+2s}} \\
&\qquad R^{\mu+\sigma} \langle \mathrm{y}_i \rangle^{-\theta} \langle y \rangle^{-\mu} \langle z \rangle^{-\sigma} \, dydz \\
&\leq CR^{-2s} \|\phi_i\|_{i,\mu,\sigma} \langle \mathrm{y}_i \rangle^{-\theta}.
\end{aligned}
$$

2. We consider different cases according to the values of the cut-off functions $\bar{\eta}_+(y)$ and $\bar{\zeta}(z)$.

(a) When $\bar{\eta}_+(y_0)\bar{\zeta}(z_0)=0$ with $|y_0|\geq 2R_2$ and $|z_0|\geq 3R$,

$$
\begin{aligned}
&\left|[(-\Delta_{(y,z)})^s,\bar{\eta}_+\bar{\zeta}]\phi_+(y_0,z_0)\right|\\
&\leq C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma}R_2^{-\theta}\int_{|y|>R_2}\int_{|z|<2R}\frac{\langle y\rangle^{-\mu}\langle z\rangle^{-\sigma}}{|(y_0,z_0)-(y,z)|^{n+2s}}\,dydz\\
&\leq C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma}R_2^{-\theta}(1+R^{1-\sigma})\int_{|y|>R_2}\frac{\langle y\rangle^{-\mu}}{\left(|y_0-y|^2+|z_0|^2\right)^{\frac{n+2s}{2}}}\,dy\\
&\leq C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma}R_2^{-\theta}(1+R^{1-\sigma})\left(\int_{R_2<|y|\leq\frac{|y_0|}{2}}\frac{\langle y\rangle^{-\mu}}{\left(|y_0|^2+|z_0|^2\right)^{\frac{n+2s}{2}}}\,dy\right.\\
&\qquad\qquad\left.+\int_{|y|\geq\frac{|y_0|}{2}}\frac{\langle y_0\rangle^{-\mu}}{\left(|y_0-y|^2+|z_0|^2\right)^{\frac{n+2s}{2}}}\,dy\right)\\
&\leq C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma}R_2^{-\theta}(1+R^{1-\sigma})\left(\frac{|y_0|^{n-1-\mu}}{|(y_0,z_0)|^{n+2s}}+\frac{\langle y_0\rangle^{-\mu}}{|z_0|^{1+2s}}\right)\\
&\leq C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma}R_2^{-\theta}(1+R^{1-\sigma})\langle y_0\rangle^{-\mu}\langle z_0\rangle^{-1-2s}.
\end{aligned}
$$

(b) When $\bar{\eta}_+(y_0)\bar{\zeta}(z_0)=0$ with $|y_0|\leq 2R_2$ and $|z_0|\geq 3R$,

$$
\begin{aligned}
&\left|[(-\Delta_{(y,z)})^s,\bar{\eta}_+\bar{\zeta}]\phi_+(y_0,z_0)\right|\\
&\leq C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma}R_2^{-\theta-\mu}(1+R^{1-\sigma})\int_{|y|>R_2}\frac{dy}{\left(|y_0-y|^2+|z_0|^2\right)^{\frac{n+2s}{2}}}\\
&\leq C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma}R_2^{-\theta-\mu}(1+R^{1-\sigma})|z_0|^{-1-2s}.
\end{aligned}
$$

(c) When $\bar{\eta}_+(y_0)\bar{\zeta}(z_0)=0$ with $|y_0|\leq R_2-2R$,

$$
\begin{aligned}
&\left|[(-\Delta_{(y,z)})^s,\bar{\eta}_+\bar{\zeta}]\phi_+(y_0,z_0)\right|\\
&\leq C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma}R_2^{-\theta}\int_{|y|>R_2}\int_{|z|<2R}\frac{\langle y\rangle^{-\mu}\langle z\rangle^{-\sigma}}{|(y_0,z_0)-(y,z)|^{n+2s}}\,dydz\\
&\leq C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma}R_2^{-\theta-\mu}\int_{|z|<2R}\langle z\rangle^{-\sigma}\min\left\{\frac{1}{|z_0-z|^{1+2s}},\frac{1}{R^{1+2s}}\right\}dz\\
&\leq C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma}R_2^{-\theta-\mu}(1+R^{1-\sigma})\langle z_0\rangle^{-1-2s}.
\end{aligned}
$$

(d) When $0 \le \bar{\eta}_+(y_0)\bar{\zeta}(z_0) \le 1$ with $|y_0| \ge R_2 - 2R$ and $0 \le |z_0| \le 3R$,

$$
\begin{aligned}
&\left|[(-\Delta_{(y,z)})^s, \bar{\eta}_+\bar{\zeta}]\phi_+(y_0,z_0)\right| \\
&\le C\int_{|y_0-y|<R}\int_{|z_0-z|<R} \frac{R^{-2}\left(|y_0-y|^2+|z_0-z|^2\right)}{\left(|y_0-y|^2+|z_0-z|^2\right)^{\frac{n+2s}{2}}} \\
&\qquad \left\|\bar{\phi}_+\right\|_{+,\mu,\sigma} R_2^{-\theta} \langle y\rangle^{-\mu} \langle z\rangle^{-\sigma}\, dydz \\
&\quad + C\int_{|y_0-y|>R}\int_{|z_0-z|>R} \frac{1}{\left(|y_0-y|^2+|z_0-z|^2\right)^{\frac{n+2s}{2}}} \\
&\qquad \left\|\bar{\phi}_+\right\|_{+,\mu,\sigma} R_2^{-\theta} \langle y\rangle^{-\mu} \langle z\rangle^{-\sigma}\, dydz \\
&\le CR^{-2s}\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma} R_2^{-\theta} \langle y_0\rangle^{-\mu} \\
&\quad + C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma} R_2^{-\theta}\int_{|y_0-y|>R} \frac{\langle y\rangle^{-\mu}}{|y_0-y|^{n-1+2s}}\, dy \\
&\le C\left\|\bar{\phi}_+\right\|_{+,\mu,\sigma} R_2^{-\theta} |y_0|^{-\mu}.
\end{aligned}
$$

3. For the localized inner terms,

$$
\begin{aligned}
\sum_{i\in\mathscr{I}} \left|\zeta_i(f'(w)-f'(u^*))\phi_i\right| &\le C\left\|\phi_i\right\|_{i,\mu,\sigma} \zeta_i F_\varepsilon^{2s} R^{\mu+\sigma} \langle \mathrm{y}_i\rangle^{-\theta} \\
&\le C\left\|\phi_i\right\|_{i,\mu,\sigma} \sum_{i\in\mathscr{I}} \zeta_i R^{\mu+\sigma} \langle \mathrm{y}_i\rangle^{-\theta-\frac{4s}{2s+1}}.
\end{aligned}
$$

The two terms at the ends are controlled by

$$
\left|\zeta_\pm(f'(w)-f'(u^*))\phi_\pm\right| \le C\left\|\phi_\pm\right\|_{\pm,\mu,\sigma} \zeta_\pm R^{\sigma} R_2^{-(\theta-\mu)} \langle y\rangle^{-\mu}.
$$

By summing up we obtain the desired estimate.

4. By using Corollary 2.3.4 and (2.10), we have in the Fermi coordinates,

$$
\begin{aligned}
&\left|((-\Delta_x)^s-(-\Delta_{(y,z)})^s)(\zeta_i\phi_i)(x)\right|\\
&\leq CR\varepsilon\left|(-\Delta_{(y,z)})^s(\bar\eta\bar\zeta\bar\phi_i)(y,z)\right|+C\varepsilon^{2s}\left|(\bar\eta\bar\zeta\bar\phi_i)(y,z)\right|\\
&\leq CR\varepsilon\left(\bar\eta(y)\bar\zeta(z)\left|(-\Delta_{(y,z)})^s\bar\phi_i(y,z)\right|+\left|[(-\Delta_{(y,z)})^s,\bar\eta\bar\zeta]\bar\phi_i(y,z)\right|\right)\\
&\qquad+C\varepsilon^{2s}(\bar\eta\bar\zeta\bar\phi_i)(y,z)\\
&\leq CR\varepsilon\Big(\bar\eta(y)\bar\zeta(z)R^{\mu+\sigma}\left\|\bar\phi_i\right\|_{i,\mu,\sigma}\langle \mathrm{y}_i\rangle^{-\theta}\langle y\rangle^{-\mu}\langle z\rangle^{-\sigma}\\
&\qquad+\left\|\bar\phi_i\right\|_{i,\mu,\sigma}\langle \mathrm{y}_i\rangle^{-\theta}R^n(R+|(y,z)|)^{-n-2s}\Big)\\
&\leq CR^{n+1+\mu+\sigma}\varepsilon\left\|\bar\phi_i\right\|_{i,\mu,\sigma}\langle \mathrm{y}_i\rangle^{-\theta}\left(\bar\eta(y)\bar\zeta(z)+(R+|(y,z)|)^{-n-2s}\right).
\end{aligned}
$$

Going back to the x-coordinates and summing up over $i\in\mathscr{I}$, we have

$$
\begin{aligned}
&\sum_{i\in\mathscr{I}}\left|((-\Delta_x)^s-(-\Delta_{(y,z)})^s)(\zeta_i\phi_i)(x)\right|\\
&\leq CR^{n+1+\mu+\sigma}\varepsilon\left\|\bar\phi_i\right\|_{i,\mu,\sigma}\\
&\qquad\cdot\left(\sum_{i\in\mathscr{I}}\zeta_i\langle \mathrm{y}_i\rangle^{-\theta}+\varepsilon^{\theta}\left\langle \operatorname{dist}\left(x,\operatorname{supp}\sum_{i\in\mathscr{I}}\zeta_i\right)\right\rangle^{-2s}\right).
\end{aligned}
$$

5. Similarly, using Corollary 2.3.5 and (2.10),

$$
\begin{aligned}
&\left|((-\Delta_x)^s-(-\Delta_{(y,z)})^s)(\zeta_+\phi_+)(x)\right|\\
&\leq Cr^{-\frac{2(2s-\tau)}{2s+1}}\left|(-\Delta_{(y,z)})^s(\bar\eta_+\bar\zeta\bar\phi_+)(y,z)\right|+Cr^{-\frac{4s\tau}{2s+1}}\left|(\bar\eta_+\bar\zeta\bar\phi_+)(y,z)\right|\\
&\leq Cr^{-\frac{2(2s-\tau)}{2s+1}}\left(\bar\eta_+(y)\bar\zeta(z)\left|(-\Delta_{(y,z)})^s\bar\phi_+(y,z)\right|+\left|[(-\Delta_{(y,z)})^s,\bar\eta_+\bar\zeta]\bar\phi_+(y,z)\right|\right)\\
&\qquad+Cr^{-\frac{4s\tau}{2s+1}}(\bar\eta_+\bar\zeta\bar\phi_+)(y,z)\\
&\leq Cr^{-\frac{2(2s-\tau)}{2s+1}}\Big(\bar\eta_+(y)\bar\zeta(z)\left\|\bar\phi_+\right\|_{+,\mu,\sigma}R_2^{-\theta}\langle y\rangle^{-\mu}\\
&\qquad+\left\|\bar\phi_+\right\|_{+,\mu,\sigma}R_2^{-\theta}\langle y\rangle^{-\mu}\langle z\rangle^{-1-2s}\Big)\\
&\leq Cr^{-\frac{2(2s-\tau)}{2s+1}}\left\|\bar\phi_+\right\|_{+,\mu,\sigma}R_2^{-\theta}\langle y\rangle^{-\mu}\langle z\rangle^{-1-2s}.
\end{aligned}
$$

$\square$

2.5.2 The outer problem: Proof of Proposition 2.2.2

We give a proof of Proposition 2.2.2 and solve ϕ_o in terms of $(\phi_j)_{j\in\mathscr{J}}$.

Proof of Proposition 2.2.2. We solve it by a fixed point argument. By Corollary 2.3.3 and Lemma 2.5.1, the right hand side $g_o = g_o(\phi_o)$ of (2.6) satisfies $g_o = 0$ in $M_{\varepsilon,R}$ and

$$\begin{aligned}
\|g_o\|_\theta &\le C\varepsilon^\theta + \|\tilde{\eta}_o N(\varphi)\|_\theta + \left\|\tilde{\eta}_o(2-f'(u^*))\phi_o\right\|_\theta \\
&\le C\varepsilon^\theta + \|\phi_o\|_{L^\infty(\mathbb{R}^n)}\|\phi_o\|_\theta + CR^{-2s}\|\phi_o\|_\theta,
\end{aligned}$$

so that by Lemma 2.4.10,

$$\left\|((-\Delta)^s+2)^{-1}g_o\right\|_\theta \le \left(C+\tilde{C}^2\varepsilon^\theta+\tilde{C}R^{-2s}\right)\varepsilon^\theta \le \tilde{C}\varepsilon^\theta.$$

Next we check that for $\phi_o, \psi_o \in X_o$, $g_o(\phi_o) - g_o(\psi_o) = 0$ in $M_{\varepsilon,R}$ as well as

$$\begin{aligned}
\|g_o(\phi_o)-g_o(\psi_o)\|_\theta &\le \left\|N\left(\phi_o+\sum_{j\in\mathscr{J}}\zeta_j\phi_j\right)-N\left(\psi_o+\sum_{j\in\mathscr{J}}\zeta_j\phi_j\right)\right\|_\theta \\
&\quad + \left\|\tilde{\eta}_o(2-f'(u^*))(\phi_o-\psi_o)\right\|_\theta \\
&\le C(\varepsilon^\theta+R^{-2s})\|\phi_o-\psi_o\|_\theta.
\end{aligned}$$

Hence

$$\left\|((-\Delta)^s+2)^{-1}(g_o(\phi_o)-g_o(\psi_o))\right\|_\theta \le C(\varepsilon^\theta+R^{-2s})\|\phi_o-\psi_o\|_\theta.$$

By contraction mapping principle, there is a unique solution $\phi_o = \Phi_o((\phi_j)_{j\in\mathscr{J}})$. The Lipschitz continuity of Φ_o with respect to $(\phi_j)_{j\in\mathscr{J}}$ can be obtained by taking a difference. □

2.5.3 The inner problem: Proof of Proposition 2.2.3

Here we solve the inner problem for $(\phi_j)_{j\in\mathscr{J}}$, with the solution of the outer problem $\phi_o = \Phi_o((\phi_j)_{j\in\mathscr{J}})$ plugged in.

Proof of Proposition 2.2.3. Let us denote the right hand side of (2.10) by g_j. We notice that the norms can can be estimated without the projection (up to a constant). Indeed, for any function $\bar{h}$ with $\|\bar{h}\|_{\mu,\sigma} < +\infty$,

$$\left\| \left(\int_{-2R}^{2R} \bar{\zeta}(t) \bar{h}(y,t) w'(t)\, dt \right) w'(z) \right\|_{\mu,\sigma} \leq C \|\bar{h}\|_{\mu,\sigma} \sup_{z\in\mathbb{R}} \langle z \rangle^{-1-2s+\sigma} \\ \leq C \|\bar{h}\|_{\mu,\sigma}.$$

Then, keeping in mind that a barred function denotes the corresponding one in Fermi coordinates, we have

$$\begin{aligned} \|\tilde{\eta}_i S(u^*)\|_{i,\mu,\sigma} &\leq \langle \mathrm{y}_i \rangle^{\theta} \sup_{|y|,|z|\leq 2R} \langle y \rangle^{\mu} \langle z \rangle^{\sigma} \cdot \langle \mathrm{y}_i \rangle^{-\frac{4s}{2s+1}} \langle z \rangle^{-(2s-1)} \\ &\leq CR^{\mu} \langle \mathrm{y}_i \rangle^{-\left(\frac{4s}{2s+1} - \theta\right)} \\ &\leq C\delta, \end{aligned}$$

$$\begin{aligned} &\left\| \tilde{\eta}_i (2 - f'(u^*)) \Phi_o((\phi_j)_{j\in\mathscr{J}}) \right\|_{i,\mu,\sigma} \\ &\leq \left\| \tilde{\eta}_i \Phi_o((\phi_j)_{j\in\mathscr{J}}) \right\|_{i,\mu,\sigma} \\ &\leq \langle \mathrm{y}_i \rangle^{\theta} \sup_{|y|,|z|\leq 2R} \langle y \rangle^{\mu} \langle z \rangle^{\sigma} \cdot \left| \overline{\Phi_o((\phi_j)_{j\in\mathscr{J}})}(y,z) \right| \\ &\leq \langle \mathrm{y}_i \rangle^{\theta} \sup_{|y|,|z|\leq 2R} \langle y \rangle^{\mu} \langle z \rangle^{\sigma} \cdot \langle \mathrm{y}_i \rangle^{-\theta} \left\| \overline{\Phi_o((\phi_j)_{j\in\mathscr{J}})} \right\|_{\theta} \\ &\leq CR^{\mu+\sigma} \varepsilon^{\theta} \sup_{j\in\mathscr{J}} \|\phi_j\|_{j,\mu,\sigma} \\ &\leq CR^{\mu+\sigma} \varepsilon^{\theta} \tilde{C} \delta, \end{aligned}$$

and

$$
\begin{aligned}
&\left\|\tilde{\eta}_i N\left(\Phi_o((\phi_j)_{j\in\mathscr{J}})+\sum_{j\in\mathscr{J}}\zeta_j\phi_j\right)\right\|_{i,\mu,\sigma}\\
&\le C\langle \mathrm{y}_i\rangle^{\theta}\sup_{|y|,|z|\le 2R}\langle y\rangle^{\mu}\langle z\rangle^{\sigma}\left|\overline{\Phi_o((\phi_j)_{j\in\mathscr{J}})}(y,z)+\sum_{\substack{j\in\mathscr{J}\\ \operatorname{supp}\tilde{\eta}_i\cap\operatorname{supp}\zeta_j\neq\emptyset}}\bar{\eta}_j\bar{\zeta}\bar{\phi}_j(y,z)\right|^2\\
&\le CR^{\mu+\sigma}\langle \mathrm{y}_i\rangle^{\theta}\sup_{|y|,|z|\le 2R}\left(\langle \mathrm{y}_i\rangle^{-2\theta}\left(\sup_{j\in\mathscr{J}}\|\phi_j\|_{j,\mu,\sigma}\right)^2\right.\\
&\qquad\left.+\sum_{\substack{j\in\mathscr{J}\\ \operatorname{supp}\tilde{\eta}_i\cap\operatorname{supp}\zeta_j\neq\emptyset}}\langle \mathrm{y}_j\rangle^{-2\theta}\left(\sup_{j\in\mathscr{J}}\|\phi_j\|_{j,\mu,\sigma}\right)^2\right)\\
&\le CR^{\mu+\sigma}\langle \mathrm{y}_i\rangle^{-\theta}\tilde{C}\delta\sup_{j\in\mathscr{J}}\|\phi_j\|_{j,\mu,\sigma}\\
&\le CR^{\mu+\sigma}\varepsilon^{\theta}\tilde{C}^2\delta^2.
\end{aligned}
$$

Using Lemma 2.5.1 and estimating as in the proof of Proposition 2.2.2, we have for all $i\in\mathscr{I}$,

$$
\|g_i\|_{i,\mu,\sigma}\le C\delta(1+R^{\mu+\sigma}\varepsilon^{\theta}\tilde{C}+R^{\mu+\sigma}\varepsilon^{\theta}\tilde{C}\delta+o(1)).
$$

Now we estimate the functions $\phi_\pm$ at the ends. We have similarly

$$
\begin{aligned}
\|\tilde{\eta}_+S(u^*)\|_{+,\mu,\sigma}&\le CR_2^{\theta}\sup_{y\ge R_2,z\le 2R}\langle y\rangle^{\mu}\langle z\rangle^{\sigma}\langle y\rangle^{-\frac{4s}{2s+1}}\langle z\rangle^{-(2s-1)}\\
&\le CR_2^{-\left(\frac{4s}{2s+1}-\mu-\theta\right)}\\
&\le C\delta\qquad\text{for } R_2 \text{ chosen large enough,}
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\tilde{\eta}_{+}(2-f'(u^*))\Phi_o((\phi_j)_{j\in\mathscr{J}})\right\|_{+,\mu,\sigma}\\
&\leq CR_2^{\theta}\sup_{y\geq R_2,z\leq 2R}\langle y\rangle^{\mu}\langle z\rangle^{\sigma}\left|\overline{\Phi_o((\phi_j)_{j\in\mathscr{J}})}(y,z)\right|\\
&\leq CR^{\sigma}R_2^{\theta}\sup_{y\geq R_2,z\leq 2R}\langle y\rangle^{\mu}\cdot\langle y\rangle^{-\theta}\varepsilon^{\theta}\sup_{j\in\mathscr{J}}\left\|\phi_j\right\|_{j,\mu,\sigma}\\
&\leq CR_2^{\mu}\varepsilon^{\theta}\tilde{C}\delta \qquad (\text{since } \mu\leq\theta)\\
&\leq C\tilde{C}\varepsilon^{\frac{\theta}{2}}\delta \qquad \text{for } \mu \text{ chosen small enough,}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\tilde{\eta}_{+}N\left(\Phi_o((\phi_j)_{j\in\mathscr{J}})+\sum_{j\in\mathscr{J}}\zeta_j\phi_j\right)\right\|_{+,\mu,\sigma}\\
&\leq CR_2^{\theta}\sup_{y\geq R_2,z\leq 2R}\langle y\rangle^{\mu}\langle z\rangle^{\sigma}\left|\overline{\Phi_o((\phi_j)_{j\in\mathscr{J}})}(y,z)+\sum_{\substack{j\in\mathscr{J}\\ \operatorname{supp}\tilde{\eta}_{+}\cap\operatorname{supp}\zeta_j\neq\emptyset}}\bar{\eta}_j\bar{\zeta}\bar{\phi}_j(y,z)\right|^2\\
&\leq CR^{\sigma}\sup_{y\geq R_2,z\leq 2R}\langle y\rangle^{\mu}\left(\langle y\rangle^{-2\theta}\left(\sup_{j\in\mathscr{J}}\left\|\phi_j\right\|_{j,\mu,\sigma}\right)^2\right.\\
&\qquad\left.+\sum_{\substack{j\in\mathscr{J}\\ \operatorname{supp}\tilde{\eta}_{+}\cap\operatorname{supp}\zeta_j\neq\emptyset}}\left\langle \mathrm{y}_j\right\rangle^{-2\theta}\bar{\eta}_j\left(\sup_{j\in\mathscr{J}}\left\|\phi_j\right\|_{j,\mu,\sigma}\right)^2\right)\\
&\leq CR^{\sigma}\left(R_2^{-\theta}+\sum_{\substack{j\in\mathscr{J}\\ \operatorname{supp}\tilde{\eta}_{+}\cap\operatorname{supp}\zeta_j\neq\emptyset}}\left\langle \mathrm{y}_j\right\rangle^{-\theta}\right)\left(\sup_{j\in\mathscr{J}}\left\|\phi_j\right\|_{j,\mu,\sigma}\right)^2\\
&\leq CR^{\sigma}\varepsilon^{\theta}\tilde{C}\delta\left(\sup_{j\in\mathscr{J}}\left\|\phi_j\right\|_{j,\mu,\sigma}\right)\\
&\leq CR^{\sigma}\varepsilon^{\theta}\tilde{C}^2\delta^2.
\end{aligned}
$$

Putting together these estimates together with the non-local terms yields, using the linear theory (Proposition 2.4.8 and Lemma 2.4.6),

$$\begin{aligned}\sup_{j\in\mathscr{J}}\left\|L^{-1}g_j\right\|_{j,\mu,\sigma} &\leq C\sup_{j\in\mathscr{J}}\left\|g_j\right\|_{j,\mu,\sigma}\\ &\leq C\delta(1+o(1))\\ &\leq \tilde{C}\delta.\end{aligned}$$

It suffices to check the Lipschitz continuity with respect to $\phi_j \in X_j$. Suppose $\phi_j, \psi_j \in X_j$. Using (2.7), we have for instance

$$\begin{aligned}&\langle \mathrm{y}_i\rangle^{\theta}\sup_{|y|,|z|\leq 2R}\langle y\rangle^{\mu}\langle z\rangle^{\sigma}\Bigg(\left|\overline{\Phi_o((\phi_j)_{j\in\mathscr{J}})}(y,z)-\overline{\Phi_o((\psi_j)_{j\in\mathscr{J}})}(y,z)\right|\\ &\qquad +N\Bigg(\Phi_o((\phi_j)_{j\in\mathscr{J}})+\sum_{j\in\mathscr{J}}\zeta_j\phi_j\Bigg)-N\Bigg(\Phi_o((\psi_j)_{j\in\mathscr{J}})+\sum_{j\in\mathscr{J}}\zeta_j\psi_j\Bigg)\Bigg)\\ &\leq CR^{\mu+\sigma}\sup_{|y|,|z|\leq 2R}\Bigg((1+\delta)\left\|\overline{\Phi_o((\phi_j)_{j\in\mathscr{J}})}(y,z)-\overline{\Phi_o((\psi_j)_{j\in\mathscr{J}})}(y,z)\right\|_{\theta}\\ &\qquad\qquad +\delta\langle \mathrm{y}_i\rangle^{\theta}\sum_{\substack{j\in\mathscr{J}\\ \operatorname{supp}\tilde{\eta}_i\cap\operatorname{supp}\zeta_j\neq\emptyset}}\bar{\eta}_j\bar{\zeta}\left|\bar{\phi}_j-\bar{\psi}_j\right|(y,z)\Bigg)\\ &\leq CR^{\mu+\sigma}\delta\sup_{j\in\mathscr{J}}\left\|\phi_j-\psi_j\right\|_{j,\mu,\sigma},\end{aligned}$$

and

$$
\begin{aligned}
&R_2^{\theta} \sup_{|y|\geq R_2,\,|z|\leq 2R} \langle y\rangle^{\mu} \langle z\rangle^{\sigma} \left(\left| \overline{\Phi_o((\phi_j)_{j\in\mathscr{J}})}(y,z) - \overline{\Phi_o((\psi_j)_{j\in\mathscr{J}})}(y,z) \right| \right. \\
&\qquad \left. + N\left(\Phi_o((\phi_j)_{j\in\mathscr{J}}) + \sum_{j\in\mathscr{J}} \zeta_j \phi_j \right) - N\left(\Phi_o((\psi_j)_{j\in\mathscr{J}}) + \sum_{j\in\mathscr{J}} \zeta_j \psi_j \right) \right) \\
&\leq CR^{\sigma} R_2^{\theta} \\
&\qquad \sup_{|y|\geq R_2,\,|z|\leq 2R} \left((1+\delta) \langle y\rangle^{\mu-\theta} \left\| \overline{\Phi_o((\phi_j)_{j\in\mathscr{J}})}(y,z) - \overline{\Phi_o((\psi_j)_{j\in\mathscr{J}})}(y,z) \right\|_{\theta} \right. \\
&\qquad\qquad \left. + \delta \langle y\rangle^{\mu} \sum_{\substack{j\in\mathscr{J} \\ \operatorname{supp}\tilde{\eta}_i \cap \operatorname{supp}\zeta_j \neq \emptyset}} \bar{\eta}_j \bar{\zeta} \left| \bar{\phi}_j - \bar{\psi}_j \right| (y,z) \right) \\
&\leq CR^{\sigma} R_2^{\mu} \delta \sup_{j\in\mathscr{J}} \left\| \phi_j - \psi_j \right\|_{j,\mu,\sigma}.
\end{aligned}
$$

Therefore

$$
\sup_{j\in\mathscr{J}} \left\| L^{-1} g_j((\phi_j)_{j\in\mathscr{J}}) - L^{-1} g_j((\psi_j)_{j\in\mathscr{J}}) \right\|_{j,\mu,\sigma} \leq o(1) \sup_{j\in\mathscr{J}} \left\| \phi_j - \psi_j \right\|_{j,\mu,\sigma}
$$

and $(\phi_k)_{k\in\mathscr{J}} \mapsto L^{-1} g_j((\phi_k)_{k\in\mathscr{J}})$ defines a contraction mapping on the product space endowed with the supremum norm for suitably chosen parameters R, R_2 large and ε, μ small. This concludes the proof. $\square$

2.6 The reduced equation

2.6.1 Form of the equation: Proof of Proposition 2.2.4

Proof of Proposition 2.2.4. Recalling Proposition 2.2.1, in the near and intermediate regions $r \in \left[\frac{1}{\varepsilon}, \frac{4\bar{R}}{\varepsilon}\right]$,

$$\Pi S(u^*)(r) = \bar{C} H_{M_\varepsilon}(r) + O(\varepsilon^{2s}),$$

where

$$\bar{C} = \int_{-2R}^{2R} c_H(z)\zeta(z)w'(z)\,dz.$$

For the far region $r \geq \frac{4\bar{R}}{\varepsilon}$, let us assume that $x_n > 0$ to fix the idea. Denote by $\Pi_\pm$ the projections onto the kernels $w'_\pm(z)$ of the upper and lower leaves respectively, where $w_\pm(z) = w(z_\pm)$. Then $z_- = -2F_\varepsilon(r)(1+o(1)) - z_+$ and so from the asymptotic behavior $w(z) \sim_{z\to+\infty} 1 - \frac{c_w}{z^{2s}}$, we have

$$\begin{aligned}
&\Pi_+ 3(w(z_+)+w(z_-))(1+w(z_+))(1+w(z_-))(r)\\
&= \int_{-2R}^{2R} 3(w(z) + w(-2F_\varepsilon(r)(1+o(1)) - z))\\
&\quad \cdot (1+w(z))(1+w(-2F_\varepsilon(r)(1+o(1)) - z))\zeta(z)w'(z)\,dz\\
&= -\frac{\bar{C}_\pm}{F_\varepsilon^{2s}(r)}(1+o(1)),
\end{aligned}$$

where

$$\bar{C}_\pm = \int_{-2R}^{2R} 3c_w(1-w(z)^2)\zeta(z)w'(z)\,dz.$$

Similarly this is also true for the projection onto $w'_-(z)$ with the same coefficient $\bar{C}_\pm(r)$,

$$\Pi_- 3(w(z_+)+w(z_-))(1+w(z_+))(1+w(z_-))(r) = -\frac{\bar{C}_\pm(r)}{F_\varepsilon^{2s}(r)}(1+o(1)).$$

The other projections are estimated as follows.

$$\begin{aligned}\Pi_+ c_H(z_+)H_{M_\varepsilon}(\mathtt{y}_+) &= \int_{-2R}^{2R} c_H(z)\zeta(z)w'(z)\,dz\cdot H_{M_\varepsilon}(\mathtt{y}_+) = \bar{C}H_{M_\varepsilon}(\mathtt{y}_+),\\ \Pi_+ c_H(z_-)H_{M_\varepsilon}(\mathtt{y}_-)(r) &= \int_{-2R}^{2R} c_H(2F_\varepsilon(r)(1+o(1))-z)\zeta(z)w'(z)\,dz\cdot H_{M_\varepsilon}(\mathtt{y}_-)\\ &= O\left(F_\varepsilon^{-(2s-1)}\cdot F_\varepsilon^{-2s}\right)\\ &= O\left(F_\varepsilon^{-(4s-1)}\right),\end{aligned}$$

$$\begin{aligned}\Pi_- c_H(z_-)H_{M_\varepsilon}(\mathtt{y}_-) &= \bar{C}H_{M_\varepsilon}(\mathtt{y}_-),\\ \Pi_- c_H(z_+)H_{M_\varepsilon}(\mathtt{y}_+) &= O\left(F_\varepsilon^{-(4s-1)}\right).\end{aligned}$$

We conclude that for $r \geq \frac{4\bar{R}}{\varepsilon}$,

$$\Pi_\pm S(u^*)(r) = \bar{C}H_{M_\varepsilon}(\mathtt{y}) - \frac{\bar{C}_\pm(r)}{F_\varepsilon^{2s}(r)}(1+o(1)).$$

Taking into account the quadratically small term and the solution of the outer problem, the reduced equation reads

$$\begin{cases}\bar{C}H[F_\varepsilon](r) = O(\varepsilon^{2s}) & \text{for } \dfrac{1}{\varepsilon} \leq r \leq \dfrac{4\bar{R}}{\varepsilon},\\ \bar{C}H[F_\varepsilon](r) = \dfrac{\bar{C}_\pm}{F_\varepsilon^{2s}(r)}(1+o(1)) & \text{for } r \geq \dfrac{4\bar{R}}{\varepsilon}.\end{cases}$$

By a scaling $F_\varepsilon(r) = \varepsilon^{-1}F(\varepsilon r)$, it suffices to solve

$$\begin{cases}\dfrac{1}{r}\left(\dfrac{rF'(r)}{\sqrt{1+F'(r)^2}}\right)' = O(\varepsilon^{2s-1}) & \text{for } 1 \leq r \leq 4\bar{R},\\ \dfrac{1}{r}\left(\dfrac{rF'(r)}{\sqrt{1+F'(r)^2}}\right)' = \dfrac{\bar{C}_0\varepsilon^{2s-1}}{F^{2s}(r)}(1+o(1)) & \text{for } r \geq 4\bar{R}.\end{cases}$$

For large enough r one may approximate the mean curvature by $\Delta F = \frac{1}{r}(rF')'$. Hence, we arrive at

$$
\begin{cases}
\dfrac{1}{r}\left(\dfrac{rF'(r)}{\sqrt{1+F'(r)^2}}\right)' = O(\varepsilon^{2s-1}) & \text{for } 1 \le r \le 4\bar{R}, \\
F''(r) + \dfrac{F'(r)}{r} = \dfrac{\bar{C}_0 \varepsilon^{2s-1}}{F^{2s}(r)}(1+o(1)) & \text{for } r \ge 4\bar{R}.
\end{cases}
$$

Then the inverse G of F is introduced to deal with the singularity at $r = 1$ in the usual coordinates. Finally, the Lipschitz dependence of the error follows directly from the previously involved computations. $\square$

2.6.2 Initial approximation

In this section we study an ODE which is similar to the one in [63]. The reduced equation for $F_\varepsilon : [\varepsilon^{-1}, +\infty) \to [0, +\infty)$ can be approximated by

$$
F_\varepsilon''(r) + \frac{F_\varepsilon'(r)}{r} = \frac{1}{F_\varepsilon^{2s}(r)}, \quad \text{for all } r \text{ large.}
$$

Under the scaling $F_\varepsilon(r) = \varepsilon^{-1} F(\varepsilon r)$, the equation for $F : [1, +\infty) \to [0, +\infty)$ is

$$
F''(r) + \frac{F'(r)}{r} = \frac{\varepsilon^{2s-1}}{F^{2s}(r)}, \quad \text{for all } r \text{ large.}
$$

For r small, we approximate F by the catenoid. More precisely, let $f_C(r) = \log(r + \sqrt{r^2-1})$, $r = |x'| \ge 1$, $r_\varepsilon = \left(\frac{|\log \varepsilon|}{\varepsilon}\right)^{\frac{2s-1}{2}}$, and consider the Cauchy problem

$$
\begin{cases}
f_\varepsilon'' + \dfrac{f_\varepsilon'}{r} = \dfrac{\varepsilon^{2s-1}}{f_\varepsilon^{2s}} & \text{for } r > r_\varepsilon, \\
f_\varepsilon(r_\varepsilon) = f_C(r_\varepsilon) = \dfrac{2s-1}{2}(|\log \varepsilon| + \log|\log \varepsilon|) + \log 2 + O\left(r_\varepsilon^{-2}\right), \\
f_\varepsilon'(r_\varepsilon) = f_C(r_\varepsilon) = r_\varepsilon^{-1}\left(1 + O\left(r_\varepsilon^{-2}\right)\right).
\end{cases}
$$

Then an approximation F_0 to F can be defined by

$$
F_0(r) = f_C(r) + \chi(r - r_\varepsilon)(f_\varepsilon(r) - f_C(r)), \quad r \ge 1,
$$

where $\chi : \mathbb{R} \to [0,1]$ is a smooth cut-off function with

$$\chi = 0 \quad \text{on } (-\infty, 0] \quad \text{and} \quad \chi = 1 \quad \text{on } [1, +\infty). \tag{2.33}$$

Note that $f_\varepsilon'(r) \geq 0$ for all $r \geq r_\varepsilon$.

Lemma 2.6.1 (Estimates near initial value)**.** *For $r_\varepsilon \leq r \leq |\log\varepsilon| r_\varepsilon$, we have*

$$\begin{aligned} \frac{1}{2}|\log\varepsilon| \leq f_\varepsilon(r) &\leq C|\log\varepsilon|, \\ f_\varepsilon'(r) &\leq C r_\varepsilon^{-1}, \\ \left|f_\varepsilon''(r)\right| &\leq \frac{1}{r^2} + \frac{C}{|\log\varepsilon| r_\varepsilon^2}. \end{aligned}$$

In fact the last inequality holds for all $r \geq r_\varepsilon$.

Proof. It is more convenient to write

$$f_\varepsilon(r) = |\log\varepsilon| \tilde{f}_\varepsilon\left(r_\varepsilon^{-1} r\right)$$

so that $\tilde{f}_\varepsilon$ satisfies

$$\begin{cases} \tilde{f}_\varepsilon'' + \dfrac{\tilde{f}_\varepsilon'}{r} = \dfrac{1}{|\log\varepsilon| \tilde{f}_\varepsilon^{2s}}, \quad \text{for } r > 1, \\ \tilde{f}_\varepsilon(1) = \dfrac{2s-1}{2} + \dfrac{2s-1}{2} \dfrac{\log|\log\varepsilon|}{|\log\varepsilon|} + \dfrac{\log 2}{|\log\varepsilon|} + O\left(\dfrac{\varepsilon^{2s-1}}{|\log\varepsilon|^{2s}}\right), \\ \tilde{f}_\varepsilon'(1) = \dfrac{1}{|\log\varepsilon|} + O\left(\dfrac{\varepsilon^{2s-1}}{|\log\varepsilon|^{2s}}\right). \end{cases}$$

To obtain a bound for the first derivative, we integrate once to obtain

$$r\tilde{f}_\varepsilon'(r) - \tilde{f}_\varepsilon'(1) = \frac{1}{|\log\varepsilon|^2} \int_1^r \frac{\tilde{r}}{\tilde{f}_\varepsilon(\tilde{r})^{2s}}\, d\tilde{r} \quad \text{for } r \geq 1.$$

By the monotonicity of f_ε, hence $\tilde{f}_\varepsilon$, we have

$$\begin{aligned}\tilde{f}'_\varepsilon(r) &\le \frac{1}{r}\left(\tilde{f}'_\varepsilon(1) + \frac{1}{2|\log\varepsilon|^2 \tilde{f}_\varepsilon(1)^{2s}} r^2\right) \\ &\le \frac{1}{r|\log\varepsilon|} + \frac{Cr}{|\log\varepsilon|^2}\end{aligned}$$

for $r \ge 1$. In particular,

$$\tilde{f}'_\varepsilon(r) \le \frac{C}{|\log\varepsilon|} \quad \text{for } 1 \le r \le |\log\varepsilon|.$$

This also implies

$$\tilde{f}_\varepsilon(r) \le C \quad \text{for } 1 \le r \le |\log\varepsilon|.$$

From the equation we obtain an estimate for $\tilde{f}''_\varepsilon$ by

$$\begin{aligned}\left|\tilde{f}''_\varepsilon(r)\right| &\le \frac{1}{r}\tilde{f}'_\varepsilon(r) + \frac{1}{|\log\varepsilon|^2 \tilde{f}^{2s}_\varepsilon} \\ &\le \frac{1}{r^2|\log\varepsilon|} + \frac{C}{|\log\varepsilon|^2},\end{aligned}$$

for all $r \ge 1$. $\square$

To study the behavior of $f_\varepsilon(r)$ near infinity, we write

$$f_\varepsilon(r) = |\log\varepsilon| g_\varepsilon\left(\frac{r}{|\log\varepsilon| r_\varepsilon}\right).$$

Then $g_\varepsilon(r)$ satisfies

$$\begin{cases} g''_\varepsilon + \dfrac{g'_\varepsilon}{r} = \dfrac{1}{g^{2s}_\varepsilon}, \quad \text{for } r \ge \dfrac{1}{|\log\varepsilon|}, \\ g_\varepsilon\left(\dfrac{1}{|\log\varepsilon|}\right) = \dfrac{2s-1}{2} + \dfrac{2s-1}{2}\dfrac{\log|\log\varepsilon|}{|\log\varepsilon|} + \dfrac{\log 2}{|\log\varepsilon|} + O\left(\dfrac{\varepsilon^{2s-1}}{|\log\varepsilon|^{2s}}\right), \\ g'_\varepsilon\left(\dfrac{1}{|\log\varepsilon|}\right) = 1 + O\left(\dfrac{\varepsilon^{2s-1}}{|\log\varepsilon|^{2s}}\right). \end{cases} \tag{2.34}$$

Lemma 2.6.2 (Long-term behavior)**.** *For any fixed $\delta_0 > 0$, there exists $C > 0$ such that for all $r \geq \delta_0$,*

$$\begin{aligned}\left| g_\varepsilon(r) - r^{\frac{2}{2s+1}} \right| &\leq C r^{-\frac{2s-1}{2s+1}},\\ \left| g_\varepsilon'(r) - \frac{2}{2s+1} r^{-\frac{2s-1}{2s+1}} \right| &\leq C r^{-\frac{4s}{2s+1}},\\ \left| g_\varepsilon''(r) \right| &\leq C r^{-\frac{4s}{2s+1}}.\end{aligned}$$

Proof. Consider the change of variable of Emden–Fowler type,

$$g_\varepsilon(r) = r^{\frac{2}{2s+1}} \tilde{h}_\varepsilon(t), \quad t = \log r \geq -\log|\log \varepsilon|.$$

Then $\tilde{h}_\varepsilon(t) > 0$ solves

$$\tilde{h}_\varepsilon'' + 2\frac{2}{2s+1}\tilde{h}_\varepsilon' + \left(\frac{2}{2s+1}\right)^2 \tilde{h}_\varepsilon = \frac{1}{\tilde{h}_\varepsilon^{2s}} \quad \text{ for } t \geq -\log|\log \varepsilon|.$$

The function h_ε defined by $\tilde{h}_\varepsilon(t) = \left(\frac{2s+1}{2}\right)^{\frac{2}{2s+1}} h_\varepsilon\left(\frac{2}{2s+1}t\right)$ satisfies

$$h_\varepsilon'' + 2h_\varepsilon' + h_\varepsilon = \frac{1}{h_\varepsilon^{2s}} \quad \text{ for } t \geq -\frac{2s+1}{2}\log|\log \varepsilon|. \tag{2.35}$$

We will first prove a uniform bound for h_ε with its derivative using a Hamiltonian

$$G_\varepsilon(t) = \frac{1}{2}(h_\varepsilon')^2 + \frac{1}{2}\left(h_\varepsilon^2 - 1\right) + \frac{1}{2s-1}\left(\frac{1}{h_\varepsilon^{(2s-1)}} - 1\right),$$

which satisfies

$$G_\varepsilon'(t) = -2(h_\varepsilon')^2 \leq 0. \tag{2.36}$$

By Lemma 2.6.1, we have

$$\begin{aligned}h_\varepsilon(0) &= O(\tilde{h}_\varepsilon(0)) = O(g_\varepsilon(1)) = O(1),\\ h_\varepsilon'(0) &= O(\tilde{h}_\varepsilon'(0)) = O\left(g_\varepsilon'(1) - \frac{2}{2s+1}g_\varepsilon(1)\right) = O(1).\end{aligned}$$

Therefore, $G_\varepsilon(0) = O(1)$ as $\varepsilon \to 0$ and by (2.36), $G_\varepsilon(t) \leq C$ for all $t \geq 0$ and $\varepsilon > 0$ small. This implies that for some uniform constant $C_1 > 0$,

$$0 < C_1^{-1} \leq h_\varepsilon(t) \leq C_1 < +\infty \quad \text{and} \quad \left|h'_\varepsilon(t)\right| \leq C_1, \quad \text{for all } t \geq 0. \tag{2.37}$$

In fact, (2.36) implies

$$\int_0^t h'_\varepsilon(\tilde{t})^2 \, d\tilde{t} = 2G_\varepsilon(0) - 2G_\varepsilon(t) \leq 2G_\varepsilon(0) \leq C,$$

with C independent of ε and t, hence

$$\int_0^\infty h'_\varepsilon(\tilde{t})^2 \, d\tilde{t} \leq C,$$

uniform in small $\varepsilon > 0$. In particular, $|h'_\varepsilon(t)| \to 0$ as $t \to \infty$. We claim that the convergence is uniform and exponential. Indeed, let us define the Hamiltonian

$$G_{1,\varepsilon} = \frac{1}{2}(h''_\varepsilon)^2 + \frac{1}{2}(h'_\varepsilon)^2 \left(1 + \frac{2s}{h_\varepsilon^{2s+1}}\right)$$

for the linearized equation

$$h'''_\varepsilon + 2h''_\varepsilon + \left(1 + \frac{2s}{h_\varepsilon^{2s+1}}\right) h'_\varepsilon = 0.$$

We have

$$G'_{1,\varepsilon} = -2(h''_\varepsilon)^2 - s(2s+1)\frac{h'^3_\varepsilon}{h_\varepsilon^{2s+2}}.$$

By the uniform bounds in (2.37), if we choose $2C_2 = s(2s+1)C_1^{2s+3} + 1$, then $\tilde{G}_\varepsilon = C_2 G_\varepsilon + G_{1,\varepsilon}$ satisfies

$$\tilde{G}'_\varepsilon \leq -(h''_\varepsilon)^2 - (h'_\varepsilon)^2.$$

Using (2.37) and the vanishing of the zeroth order term together with its derivative at $h_\varepsilon = 1$, we have

$$\begin{aligned}
\tilde{G}_\varepsilon &= C_2\left(\frac{1}{2}(h'_\varepsilon)^2 + \frac{1}{2}\left(h_\varepsilon^2 - 1\right) + \frac{1}{2s-1}\left(\frac{1}{h_\varepsilon^{2s-1}} - 1\right)\right) \\
&\quad + \frac{1}{2}(h''_\varepsilon)^2 + \frac{1}{2}(h'_\varepsilon)^2\left(1 + \frac{2s}{h_\varepsilon^{2s+1}}\right) \\
&\leq C\left((h''_\varepsilon)^2 + (h'_\varepsilon)^2 + \left(h_\varepsilon - \frac{1}{h_\varepsilon^{2s}}\right)^2\right) \\
&\leq -C\tilde{G}'_\varepsilon.
\end{aligned}$$

It follows that for some constants $C, \delta_0 > 0$ independent of $\varepsilon > 0$ small,

$$\tilde{G}_\varepsilon(t) \leq Ce^{-\delta_0 t} \quad \text{for all } t \geq 0$$

and, in particular,

$$|h_\varepsilon(t) - 1| + \left|h'_\varepsilon(t)\right| \leq Ce^{-\frac{\delta_0}{2}t}, \quad \text{for all } t \geq 0.$$

This implies that after a fixed t_1 independent of ε, the point $(h_\varepsilon(t_1), h'_\varepsilon(t_1))$ is sufficiently close to $(1,0)$. Let

$$\begin{aligned}
v_1 &= h_\varepsilon \\
v_2 &= h'_\varepsilon + h_\varepsilon.
\end{aligned}$$

Then (2.35) is equivalent to

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \begin{pmatrix} -v_1 + v_2 \\ v_1^{-2s} - v_2 \end{pmatrix}. \tag{2.38}$$

For t_1 large the point $(v_1(t_1), v_2(t_1))$ is sufficiently close to $(1,1)$ which is a hyperbolic equilibrium point of (2.38). Now the linearization of (2.38), namely

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \begin{pmatrix} -1 & 1 \\ -2s & -1 \end{pmatrix}\begin{pmatrix} v_1 - 1 \\ v_2 - 1 \end{pmatrix},$$

has eigenvalues $-1 \pm i\sqrt{2s}$. By applying a C^1 conjugacy we obtain

$$|(v_1(t), v_2(t)) - (1,1)| \leq Ce^{-t} \quad \text{ for all } t \geq t_1.$$

This implies in turn

$$|h_\varepsilon(t) - 1| + \left|h'_\varepsilon(t)\right| \leq Ce^{-t} \quad \text{ for all } t \geq 0,$$

$$\left|\tilde{h}_\varepsilon(t) - 1\right| + \left|\tilde{h}'_\varepsilon(t)\right| \leq Ce^{-t} \quad \text{ for all } t \geq 0,$$

and for any fixed $r_0 > 0$, there exists $C > 0$ such that for all $r \geq r_0$,

$$\left|g_\varepsilon(r) - r^{\frac{2}{2s+1}}\right| \leq Cr^{-\frac{2s-1}{2s+1}} \quad \text{ and } \quad \left|g'_\varepsilon(r) - \frac{2}{2s+1} r^{-\frac{2s-1}{2s+1}}\right| \leq Cr^{-\frac{4s}{2s+1}}$$

and, in view of (2.34),

$$\left|g''_\varepsilon(r)\right| \leq Cr^{-\frac{4s}{2s+1}}.$$

□

Corollary 2.6.3 (Properties of the initial approximation)**.** *We have the following properties of F_0.*

- *For* $1 \leq r \leq r_\varepsilon$, $F_0(r) = f_C(r) = \log(r + \sqrt{r^2 - 1})$ *and*

$$\begin{aligned} F_0(r) &= \log(2r) + O(r^{-2}), \\ F'_0(r) &= \frac{1}{\sqrt{r^2-1}} = \frac{1}{r} + O(r^{-3}), \\ F''_0(r) &= -\frac{1}{r^2} + O(r^{-4}), \\ F'''_0(r) &= \frac{2}{r^3} + O(r^{-5}). \end{aligned}$$

- *For $r_\varepsilon \le r \le \delta_0 |\log\varepsilon| r_\varepsilon$ where $\delta_0 > 0$ is fixed,*

$$\begin{aligned}\frac{1}{2}|\log\varepsilon| \le F_0(r) &\le C|\log\varepsilon|,\\ F_0'(r) &\le C r_\varepsilon^{-1},\\ \left|F_0''(r)\right| &\le C\left(\frac{1}{r^2}+\frac{1}{|\log\varepsilon| r_\varepsilon^2}\right),\\ \left|F_0'''(r)\right| &\le C r_\varepsilon^{-1}\left(\frac{1}{r^2}+\frac{1}{|\log\varepsilon| r_\varepsilon^2}\right).\end{aligned}$$

- *For $r \ge \delta_0|\log\varepsilon| r_\varepsilon$, $F_0(r) = f_\varepsilon(r)$ and*

$$\begin{aligned}F_0(r) &= \varepsilon^{\frac{2s-1}{2s+1}} r^{\frac{2}{2s+1}} + O\left(\varepsilon^{-\frac{(2s-1)^2}{2(2s+1)}} |\log\varepsilon|^{\frac{2s+1}{2}} r^{-\frac{2s-1}{2s+1}}\right),\\ F_0'(r) &= \frac{2}{2s+1}\varepsilon^{\frac{2s-1}{2s+1}} r^{-\frac{2s-1}{2s+1}} + O\left(\varepsilon^{-\frac{(2s-1)^2}{2(2s+1)}} |\log\varepsilon|^{\frac{2s+1}{2}} r^{-\frac{4s}{2s+1}}\right),\\ F_0''(r) &= O\left(\varepsilon^{\frac{2s-1}{2s+1}} r^{-\frac{4s}{2s+1}}\right),\\ F_0'''(r) &= O\left(\varepsilon^{\frac{2s-1}{2s+1}} r^{-\frac{6s+1}{2s+1}}\right).\end{aligned}$$

Proof. They follow from Lemmata 2.6.1 and 2.6.2. For the third derivative, we differentiate the equation and use the estimates for the lower order derivatives. □

2.6.3 The linearization

Now we build a right inverse for the linearized operator

$$\mathscr{L}_0(\phi)(r) = (1-\chi_\varepsilon(r))\frac{1}{r}\left(\frac{r\phi'}{(1+F_0'(r)^2)^{\frac{3}{2}}}\right)' + \chi_\varepsilon(r)\left(\phi'' + \frac{\phi'}{r} + \frac{2s\varepsilon^{2s-1}}{F_0(r)^{2s+1}}\phi\right),$$

where χ_ε is any family of smooth cut-off functions with $\chi_\varepsilon(r) = 0$ for $1 \le r \le r_\varepsilon$ and $\chi_\varepsilon(r) = 1$ for $r \ge \delta_0|\log\varepsilon| r_\varepsilon$ where $\delta_0 > 0$ is a sufficiently small number. The goal is to solve

$$\mathscr{L}_0(\phi)(r) = h(r) \quad \text{for } r \ge 1, \tag{2.39}$$

in a weighted function space which allows the expected sub-linear growth. Let us recall the norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ defined in (2.13) and (2.14).

Proposition 2.6.4. *Let $\gamma \leq 2 + \frac{2s-1}{2s+1}$. For all sufficiently small $\delta_0, \varepsilon > 0$, there exists $C > 0$ such that for all h with $\|h\|_{**} < +\infty$, there exists a solution $\phi = T(h)$ of* (2.39) *with $\|\phi\|_* < +\infty$ that defines a linear operator T of h such that*

$$\|\phi\|_* \leq C\|h\|_{**}$$

and $\phi(1) = 0$.

We start with an estimate of the kernels of the linearized equation in the far region, namely

$$Z'' + \frac{Z'}{r} + \frac{2s\varepsilon^{2s-1}}{f_\varepsilon(r)^{2s+1}} Z = 0, \quad \text{for } r \geq \delta_0 |\log\varepsilon| r_\varepsilon. \tag{2.40}$$

Lemma 2.6.5. *There are two linearly independent solutions Z_1, Z_2 of* (2.40) *so that for $i = 1, 2$, we have*

$$|Z_i(r)| \leq C\left(\frac{r}{r_\varepsilon|\log\varepsilon|}\right)^{-\frac{2s-1}{2s+1}} \quad \textit{and} \quad |Z_i'(r)| \leq \frac{C}{r_\varepsilon|\log\varepsilon|}\left(\frac{r}{r_\varepsilon|\log\varepsilon|}\right)^{-\frac{2s-1}{2s+1}}$$

for $r \geq \delta_0|\log\varepsilon| r_\varepsilon$ where $\delta_0 > 0$ is fixed and $r_\varepsilon = \left(\frac{|\log\varepsilon|}{\varepsilon}\right)^{\frac{2s-1}{2}}$.

Proof. We will show that the elements $\tilde{Z}_i$ of the kernel of the linearization around g_ε, which solve

$$\tilde{Z}'' + \frac{\tilde{Z}'}{r} + \frac{2s}{g_\varepsilon(r)^{2s+1}} \tilde{Z} = 0 \quad \text{for } r \geq \frac{1}{|\log\varepsilon|}, \tag{2.41}$$

satisfies

$$\left|\tilde{Z}_i(r)\right| \leq Cr^{-\frac{2s-1}{2s+1}} \quad \text{and} \quad \left|\tilde{Z}_i'(r)\right| \leq Cr^{-\frac{2s-1}{2s+1}} \quad \text{for all } r \geq \delta_0$$

for $i = 1, 2$; the result then follows by setting $Z_i(r) = \tilde{Z}_i\left(\frac{r}{r_\varepsilon|\log\varepsilon|}\right)$.

A first kernel $\tilde{Z}_1$ can be obtained from the scaling invariance $g_{\varepsilon,\lambda}(r) = \lambda^{-\frac{2}{2s+1}} g_\varepsilon(\lambda r)$ of (2.34), giving

$$\tilde{Z}_1(r) = r g_\varepsilon'(r) - \frac{2}{2s+1} g_\varepsilon(r).$$

Then for $\tilde{Z}_2$ we solve (2.41) with the initial conditions

$$\tilde{Z}_2(\delta_0) = -\frac{\tilde{Z}_1'(\delta_0)}{\delta_0\left(\tilde{Z}_1(\delta_0)^2 + \tilde{Z}_1'(\delta_0)^2\right)}, \qquad \tilde{Z}_2'(\delta_0) = \frac{\tilde{Z}_1(\delta_0)}{\delta_0\left(\tilde{Z}_1(\delta_0)^2 + \tilde{Z}_1'(\delta_0)^2\right)}$$

for a fixed $\delta_0 > 0$. In particular the Wrońskian $\tilde{W} = \tilde{Z}_1\tilde{Z}_2' - \tilde{Z}_1'\tilde{Z}_2$ is computed exactly as

$$\tilde{W}(r) = \frac{\delta_0 \tilde{W}(\delta_0)}{r} = \frac{1}{r} \quad \text{for all } r > \frac{1}{|\log\varepsilon|}. \tag{2.42}$$

As in the proof of Lemma 2.6.2, we write $t = \log r$ and consider the Emden–Fowler change of variable $\tilde{Z}(r) = r^{\frac{2}{2s+1}}\tilde{v}(t)$ followed by a re-normalization $\tilde{v}(t) = \left(\frac{2}{2s+1}\right)^{-\frac{2}{2s+1}} v\left(\frac{2}{2s+1}t\right)$ which yield respectively

$$\tilde{v}'' + 2\frac{2}{2s+1}\tilde{v}' + \left(\left(\frac{2}{2s+1}\right)^2 + \frac{2s}{\tilde{h}_\varepsilon^{2s+1}}\right)\tilde{v} = 0, \quad \text{for } t \geq -\log|\log\varepsilon|,$$
$$v'' + 2v' + (1+2s)v = 2s\left(1 - \frac{1}{h_\varepsilon^{2s+1}}\right)v, \quad \text{for } t \geq -\frac{2s+1}{2}\log|\log\varepsilon|.$$

From this point we may express $v_2(t)$, and hence $\tilde{Z}_2(r)$, as a perturbation of the linear combination of the kernels

$$e^{-t}\cos(\sqrt{2s}\,t) \quad \text{and} \quad e^{-t}\sin(\sqrt{2s}\,t).$$

□

Now we show the existence of the right inverse.

Proof of Proposition 2.6.4. We sketch the argument by obtaining a solution in a weighted L^∞ space. The general case follows similarly.

1. Note that we will need to control ϕ up to two derivatives in the intermediate region. For this purpose, for any $\gamma \in \mathbb{R}$ and any interval $I \subseteq [r_1, +\infty)$ we define the norm

$$\|\phi\|_{\gamma,I} = \sup_I r^{\gamma-2}|\phi(r)| + \sup_I r^{\gamma-1}\left|\phi'(r)\right| + \sup_I r^{\gamma}\left|\phi''(r)\right|.$$

By solving the linearized mean curvature equation in the inner region using the variation of parameters formula, we obtain the estimate

$$\|\phi\|_{\gamma,[r_1,r_\varepsilon]} \leq C\,\|r^\gamma h\|_{L^\infty([1,+\infty))},$$

which in particular gives a bound for ϕ together with its derivatives at r_ε.

2. In the intermediate region we write the equation as

$$\phi'' + \frac{\phi'}{r} = h - \tilde{h}, \quad r_\varepsilon \leq r \leq \tilde{r}_\varepsilon,$$

where

$$\tilde{r}_\varepsilon = \delta_0 |\log \varepsilon| r_\varepsilon,$$

and

$$\begin{aligned}\tilde{h}(r) &= \chi_\varepsilon(r)\frac{2s\varepsilon^{2s-1}}{F_0'(r)^{2s+1}}\phi(r)\\ &+ (1-\chi_\varepsilon(r))\left(\left(1 - \frac{1}{(1+F_0'(r)^2)^{\frac{3}{2}}}\right)\left(\phi'' + \frac{\phi'}{r}\right) + \frac{3F_0'(r)F_0''(r)}{(1+F_0'(r)^2)^{\frac{3}{2}}}\phi'\right)\end{aligned}$$

is small. Again we integrate to obtain

$$\begin{aligned}\phi(r) &= \phi(r_\varepsilon) + r_\varepsilon\phi'(r_\varepsilon)\log\frac{r}{r_\varepsilon} + \int_{r_\varepsilon}^{r}\frac{1}{t}\int_{r_\varepsilon}^{t}\tau(h(t)-\tilde{h}(t))\,d\tau\,dt,\\ \phi'(r) &= \frac{r_\varepsilon\phi'(r_\varepsilon)}{r} + \frac{1}{r}\int_{r_\varepsilon}^{r} t(h(t)-\tilde{h}(t))\,dt,\\ \phi''(r) &= -\frac{r_\varepsilon\phi'(r_\varepsilon)}{r^2} + h(r) - \tilde{h}(r) - \frac{1}{r^2}\int_{r_\varepsilon}^{r} t(h(t)-\tilde{h}(t))\,dt.\end{aligned}$$

Using Corollary 2.6.3 we have, for small enough δ_0 and ε,

$$\begin{aligned}\left\|r^\gamma \tilde{h}\right\|_{L^\infty([r_\varepsilon,\tilde{r}_\varepsilon])} &\le C\frac{\varepsilon^{2s-1}}{|\log\varepsilon|^{2s+1}}r^2\|\phi\|_{\gamma,[r_\varepsilon,\tilde{r}_\varepsilon]}+C\left(\frac{\varepsilon}{|\log\varepsilon|}\right)^{2s-1}\|\phi\|_{\gamma,[r_\varepsilon,\tilde{r}_\varepsilon]}\\ &\quad +C\left(\frac{\varepsilon}{|\log\varepsilon|}\right)^{\frac{2s-1}{2}}\left(\frac{1}{r^2}+\frac{\varepsilon^{2s-1}}{|\log\varepsilon|^{2s}}\right)r\|\phi\|_{\gamma,[r_\varepsilon,\tilde{r}_\varepsilon]}\\ &\le C\left({\delta_0}^2+\delta_0\left(\frac{\varepsilon}{|\log\varepsilon|}\right)^{\frac{2s-1}{2}}|\log\varepsilon|\right)\|\phi\|_{\gamma,[r_\varepsilon,\tilde{r}_\varepsilon]}\\ &\le \delta_0\|\phi\|_{\gamma,[r_\varepsilon,\tilde{r}_\varepsilon]}.\end{aligned}$$

This implies

$$\|\phi\|_{\gamma,[r_\varepsilon,\tilde{r}_\varepsilon]}\le C\|r^\gamma h\|_{L^\infty([1,+\infty))}+\delta_0\|\phi\|_{\gamma,[r_\varepsilon,\tilde{r}_\varepsilon]},$$

or

$$\|\phi\|_{\gamma,[r_\varepsilon,\tilde{r}_\varepsilon]}\le C\|r^\gamma h\|_{L^\infty([1,+\infty))} \tag{2.43}$$

which is the desired estimate.

3. In the outer region, we need to solve

$$\phi''+\frac{\phi'}{r}+\frac{2s\varepsilon^{2s-1}}{f_\varepsilon^{2s+1}}\phi=h,\quad r>\tilde{r}_\varepsilon.$$

In terms of the kernels Z_i given in Lemma 2.6.5, the Wrońskian $W=Z_1Z_2'-Z_1'Z_2$ is given by

$$W(r)=\frac{1}{r_\varepsilon|\log\varepsilon|}\tilde{W}\left(\frac{r}{r_\varepsilon|\log\varepsilon|}\right)=\frac{1}{r} \tag{2.44}$$

using (2.42). Using the variation of parameters formula, we may write

$$\phi(r)=c_1Z_1(r)+c_2Z_2(r)+\phi_0(r),$$

where

$$\phi_0(r) = -Z_1(r)\int_{\tilde{r}_\varepsilon}^{r} \rho Z_2(\rho)h(\rho)\,d\rho + Z_2(r)\int_{\tilde{r}_\varepsilon}^{r} \rho Z_1(\rho)h(\rho)\,d\rho$$

and the constants c_i are determined by

$$\begin{aligned}
\phi(\tilde{r}_\varepsilon) &= c_1 Z_1(\tilde{r}_\varepsilon) + c_2 Z_2(\tilde{r}_\varepsilon),\\
\phi'(\tilde{r}_\varepsilon) &= c_1 Z_1'(\tilde{r}_\varepsilon) + c_2 Z_2'(\tilde{r}_\varepsilon).
\end{aligned}$$

By Lemma 2.6.5, (2.44) and (2.43), we readily check that for $i = 1,2$,

$$\begin{aligned}
|\phi_0(r)| &\le C\left(\frac{r}{\tilde{r}_\varepsilon}\right)^{-\frac{2s-1}{2s+1}} \int_{\tilde{r}_\varepsilon}^{r} \rho \left(\frac{\rho}{\tilde{r}_\varepsilon}\right)^{-\frac{2s-1}{2s+1}} \rho^{-\gamma} \|r^\gamma h\|_{L^\infty([1,+\infty))}\, d\rho\\
&\le C r^{2-\gamma} \|r^\gamma h\|_{L^\infty([1,+\infty))},\\
|c_i| &\le C r_1 \left(\frac{C}{r_1} r^{2-\gamma} \|r^\gamma h\|_{L^\infty([1,+\infty))} + C r_1^{1-\gamma} \|r^\gamma h\|_{L^\infty([1,+\infty))}\right)\\
&\le C \tilde{r}_\varepsilon^{2-\gamma} \|r^\gamma h\|_{L^\infty([1,+\infty))},\\
|c_i||Z_i(r)| &\le C\left(\frac{r}{\tilde{r}_\varepsilon}\right)^{-\frac{2s-1}{2s+1}-(2-\gamma)} r^{2-\gamma} \|r^\gamma h\|_{L^\infty([1,+\infty))}\\
&\le C r^{2-\gamma} \|r^\gamma h\|_{L^\infty([1,+\infty))} \quad \text{since } \gamma \le 2 + \frac{2s-1}{2s+1},
\end{aligned}$$

from which we conclude

$$\left\| r^{\gamma-2}\phi \right\|_{L^\infty([\tilde{r}_\varepsilon,+\infty))} \le C \|r^\gamma h\|_{L^\infty([1,+\infty))}.$$

□

2.6.4 The perturbation argument: Proof of Proposition 2.2.5

We solve the reduced equation

$$\mathscr{L}(F) = \mathscr{N}_1[F] \quad \text{for } r \ge 1, \tag{2.45}$$

using the knowledge of the initial approximation F_0 and the linearized operator $\mathscr{L}_0$ at F_0 obtained in Sections 2.6.2 and 2.6.3 respectively. We look for a solution $F = F_0 + \phi$. Then ϕ satisfies

$$\mathscr{L}_0\phi = A[\phi] = \mathscr{N}_1[F_0+\phi] - \mathscr{L}(F_0) - \mathscr{N}_2[\phi],$$

where $\mathscr{N}_2[\phi] = \mathscr{L}(F_0+\phi) - \mathscr{L}(F_0) - \mathscr{L}'(F_0)\phi$ and $\phi(0) = 0$. In terms of the operator T defined in Proposition 2.6.4, we can write it in the form

$$\phi = T\left(A[\phi]\right). \tag{2.46}$$

We apply a standard argument using contraction mapping principle as in [63]. First we note that the approximation $\mathscr{L}(F_0)$ is small and compactly supported in the intermediate region. The non-linear terms in $A[\phi]$ are also small in the norm $\|\cdot\|_{**}$. Hence $T(A[\phi])$ defines a contraction mapping in the space X_*. The details are left to the interested readers.

Chapter 3

Fractional Yamabe Problem

3.1 Introduction

We construct singular solutions to the following non-local semilinear problem

$$(-\Delta_{\mathbb{R}^n})^\gamma u = u^p \text{ in } \mathbb{R}^n, \quad u > 0, \tag{3.1}$$

for $\gamma \in (0,1)$, $n \geq 2$, where the fractional Laplacian is defined by

$$(-\Delta_{\mathbb{R}^n})^\gamma u(z) = k_{n,\gamma} P.V. \int_{\mathbb{R}^n} \frac{u(z)-u(\tilde{z})}{|z-\tilde{z}|^{n+2\gamma}}\, d\tilde{z}, \quad \text{for} \quad k_{n,\gamma} = \pi^{-n/2} 2^{2\gamma} \frac{\Gamma\left(\frac{n}{2}+\gamma\right)}{\Gamma(1-\gamma)} \gamma. \tag{3.2}$$

Equation (3.1) for the critical power $p = \frac{n+2\gamma}{n-2\gamma}$ corresponds to the fractional Yamabe problem in conformal geometry, which asks to find a constant fractional curvature metric in a given conformal class (see [86, 111, 112, 122, 129]). In particular, for $\gamma = 1$ the fractional curvature coincides with the scalar curvature modulo a multiplicative constant, so (3.1) reduces to the classical Yamabe problem. However, classical methods for local equations do not generally work here and one needs to develop new ideas.

Non-local equations have attracted a great deal of interest in the community since they are of central importance in many fields, from the points of view of both pure analysis and applied modeling. By the substantial effort made in the past decade by many authors, we have learned that non-local elliptic equations do enjoy good PDE properties such as uniqueness, regularity and maximum principle. However, not so much is known when it comes to the study of an integro-differential equation such as (3.1) from an ODE perspective since most of the ODE theory relies on local properties and phase-plane analysis; our first achievement is the development of a suitable theory for the fractional order ODE (3.6), that arises when studying radial singular solutions to (3.1).

On the one hand, we construct singular radial solutions for (3.1) directly with a completely different argument. On the other hand, using ideas from conformal geometry and scattering theory we replace phase-plane analysis by a global study to obtain that solutions of the nonlocal ODE (3.6) do have a behavior similar in spirit to a classical second-order autonomous ODE, and initiate the study of a non-local phase portrait. In particular, we show that a linear non-local ODE has a two-dimensional kernel. This is surprising since this non-local ODE has an infinite number of indicial roots at the origin and at infinity, which is very different from the local case where the solution to a homogeneous linear second order problem can be written as a linear combination of two particular solutions and thus, its asymptotic behavior is governed by two pairs of indicial roots.

Then, with these tools at hand, we arrive at our second accomplishment: to develop a Mazzeo-Pacard gluing program [132] for the construction of singular solutions to (3.1) in the non-local setting. This gluing method is indeed local by definition; so one needs to rethink the theory from a fresh perspective in order to adapt it for such non-local equation. More precisely, the program relies on the fact that the linearization to (3.1) has good properties. In the classical case, this linearization has been well studied applying microlocal analysis (see [130], for instance), and it reduces to the understanding of a second order ODE with two regular singular points. In the fractional case this is obviously not possible. Instead, we use conformal geometry, complex analysis and some non-Euclidean harmonic analysis coming from representation theory in order to provide a new proof.

Thus conformal geometry is the central core in this chapter, but we provide an interdisciplinary approach in order to approach the following analytical problem:

Theorem 3.1.1. *Let $\Sigma = \bigcup_{i=1}^{K} \Sigma_i$ be a disjoint union of smooth, compact submanifolds Σ_i without boundary of dimensions k_i, $i = 1, \ldots, K$. Assume, in addition* *to* $n-k_i \geq 2$, *that*

$$\frac{n-k_i}{n-k_i-2\gamma} < p < \frac{n-k_i+2\gamma}{n-k_i-2\gamma},$$

or equivalently,

$$n-\frac{2p\gamma+2\gamma}{p-1} < k_i < n-\frac{2p\gamma}{p-1}$$

for all $i=1,\ldots,K$. *Then there exists a positive solution for the problem*

$$(-\Delta_{\mathbb{R}^n})^\gamma u = u^p \text{ in } \mathbb{R}^n \setminus \Sigma \tag{3.3}$$

that blows up exactly at Σ.

As a consequence of the previous theorem we obtain:

Corollary 3.1.2. *Assume that the dimensions* k_i *satisfy*

$$0 < k_i < \frac{n-2\gamma}{2}. \tag{3.4}$$

Then there exists a positive solution to the fractional Yamabe equation

$$(-\Delta_{\mathbb{R}^n})^\gamma u = u^{\frac{n+2\gamma}{n-2\gamma}} \text{ in } \mathbb{R}^n \setminus \Sigma \tag{3.5}$$

that blows up exactly at Σ.

The dimension estimate (3.4) is sharp in some sense. Indeed, it was proved by González, Mazzeo and Sire [110] that, if such u blows up at a smooth sub-manifold of dimension k and is polyharmonic, then k must satisfy the restriction

$$\Gamma\left(\frac{n}{4}-\frac{k}{2}+\frac{\gamma}{2}\right) \Big/ \Gamma\left(\frac{n}{4}-\frac{k}{2}-\frac{\gamma}{2}\right) > 0,$$

which in particular, includes (3.4). Here, and for the rest of the chapter, Γ denotes the Gamma function. In addition, the asymptotic behavior of solutions to (3.5) when the singular set has fractional capacity zero has been considered in [121].

Let us describe our methods in detail. First, note that it is enough to let Σ be a single sub-manifold of dimension k, and we will restrict to this case for the rest of the chapter. We denote $N=n-k$.

The first step is to construct the building block, i.e, a solution to (3.3) in $\mathbb{R}^n \setminus \mathbb{R}^k$ that blows up exactly at $\mathbb{R}^k$. For this, we write $\mathbb{R}^n \setminus \mathbb{R}^k = (\mathbb{R}^N \setminus \{0\}) \times \mathbb{R}^k$, parameterized with coordinates $z = (x, y)$, $x \in \mathbb{R}^N \setminus \{0\}$, $y \in \mathbb{R}^k$, and construct a solution u_1 that only depends on the radial variable $r = |x|$. Then u_1 is also a radial solution to

$$(-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma} u^p \quad \text{in } \mathbb{R}^N \setminus \{0\}, \quad u > 0.$$

We write $u = r^{-\frac{2\gamma}{p-1}} v$, $r = e^{-t}$. Then, in the radially symmetric case, this equation can be written as the integro-differential ODE

$$P.V. \int_{\mathbb{R}} K(t-t')[v(t) - v(t')]\, dt' + A_{N,p,\gamma} v(t) = A_{N,p,\gamma} v^p \quad \text{in } \mathbb{R}, \quad v > 0, \tag{3.6}$$

where the kernel K is given precisely in (3.80). However, in addition to having the right blow up rate at the origin, u_1 must decay fast as $r \to \infty$ in order to perform the Mazzeo-Pacard gluing argument later. The existence of such fast-decaying singular solutions in the case of $\gamma = 1$ is an easy consequence of phase-plane analysis as (3.6) is reduced to a second order autonomous ODE (see Proposition 1 of [132]). The analogue in the fractional case turns out to be quite non-trivial. To show the existence, we first use Kelvin transform to reduce our problem for entire solutions to a supercritical one (3.13). Then we consider an auxiliary non-local problem (3.14), for which we show that the minimal solution w_λ is unique using Schaaf's argument as in [83] and a fractional Pohožaev identity [149]. A blow up argument, together with a Crandall-Rabinowitz bifurcation scheme yields the existence of this u_1. This is the content of Section 3.2.

Then, in Section 3.3, we exploit the conformal properties of the equation to produce a geometric interpretation for (3.3) in terms of scattering theory and conformally covariant operators. Singular solutions for the standard fractional Laplacian in $\mathbb{R}^n \setminus \mathbb{R}^k$ can be better understood by considering the conformal metric g_k from (3.45), that is the product of a sphere $\mathbb{S}^{N-1}$ and a half-space $\mathbb{H}^{k+1}$. Inspired by the arguments by DelaTorre and González [66], our point of view is to rewrite the well known extension problem in $\mathbb{R}^{n+1}_+$ for the fractional Laplacian in $\mathbb{R}^n$ due to [43], as a different, but equivalent, extension problem and to consider the corresponding Dirichlet-to-Neumann operator $P^{g_k}_\gamma$, defined in $\mathbb{S}^{N-1} \times \mathbb{H}^{k+1}$. Here $\mathbb{R}^{n+1}_+$ is replaced by anti-de Sitter (AdS) space, but the arguments run in parallel.

This $P_\gamma^{g_k}$ turns out to be a conjugate operator for $(-\Delta_{\mathbb{R}^n})^\gamma$, (see (3.46)), and behaves well when the nonlinearity in (3.3) is the conformal power. However, the problem (3.3) is not conformal for a general p, so we need to perform a further conjugation (3.57) and to consider the new operator $\tilde{P}_\gamma^{g_k}$. Then the original equation (3.3) in $\mathbb{R}^n \setminus \mathbb{R}^k$ is equivalent to

$$\tilde{P}_\gamma^{g_k}(v) = v^p \quad \text{in } \mathbb{S}^{N-1} \times \mathbb{H}^{k+1}, \quad v = r^{\frac{2\gamma}{p-1}} u, \quad v > 0 \text{ and smooth.} \tag{3.7}$$

Rather miraculously, both $P_\gamma^{g_k}$ and $\tilde{P}_\gamma^{g_k}$ diagonalize under the spherical harmonic decomposition of $\mathbb{S}^{N-1}$. In fact, they can be understood as pseudo-differential operators on hyperbolic space $\mathbb{H}^{k+1}$, and we calculate their symbols in Theorem 3.3.5 and Proposition 3.3.6, respectively, under the Fourier-Helgason transform (to be denoted by $\widehat{\cdot}$) on the hyperbolic space understood as the symmetric space $M = G/K$ for $G = SO(1, k+1)$ and $K = SO(k+1)$ (see the Appendix for a short introduction to the subject). This is an original approach that yields new results even in the classical case $\gamma = 1$, simplifying some of the arguments in [132]. The precise knowledge of their symbols allows, as a consequence, for the development of the linear theory for our problem, as we will comment below.

Section 3.4 collects these ideas in order to develop new methods for the study of the non-local ODE (3.6), which is precisely the projection of equation (3.7) for $k = 0$, $n = N$, over the zero-eigenspace when projecting over spherical harmonics of $\mathbb{S}^{N-1}$. The advantage of shifting from u to v is that we obtain a new equation that behaves very similarly to a second order autonomous ODE. This includes the existence of a Hamiltonian quantity along trajectories.

Moreover, one can take the spherical harmonic decomposition of $\mathbb{S}^{N-1}$ and consider all projections $m = 0, 1, \ldots$. In Proposition 3.4.2 we are able to write every projected equation as an integro-differential equation very similar to the $m = 0$ projection (3.6). This formulation immediately yields regularity and maximum principles for the solution of (3.7) following the arguments in [65].

Now, to continue with the proof of Theorem 3.1.1, one takes the fast decaying solution in $\mathbb{R}^n \setminus \mathbb{R}^k$ we have just constructed and, after some rescaling by ε, glues it to the background Euclidean space in order to have a global approximate solution $\bar{u}_\varepsilon$ in $\mathbb{R}^n \setminus \Sigma$. Even though the fractional Laplacian is a non-local operator, one is able to perform this gluing just by carefully estimating the tail terms that appear in the integrals after localizaton. This is done in Section 3.5.1 and, more precisely, Lemma 3.5.7, where we show that the error we generate when approximating a true solution by $\bar{u}_\varepsilon$, given by

$$f_\varepsilon := (-\Delta_{\mathbb{R}^n})^\gamma \bar{u}_\varepsilon - \bar{u}_\varepsilon^p,$$

is indeed small in suitable weighted Hölder spaces.

Once we have an approximate solution, we define the linearized operator around it,

$$L_\varepsilon \phi := (-\Delta_{\mathbb{R}^n})^\gamma \phi - p\bar{u}_\varepsilon^{p-1}\phi.$$

The general scheme of Mazzeo-Pacard's method is to set $u = \bar{u}_\varepsilon + \phi$ for an unknown perturbation ϕ and to rewrite equation (3.3) as

$$L_\varepsilon(\phi) + Q_\varepsilon(\phi) + f_\varepsilon = 0,$$

where Q_ε contains the remaining nonlinear terms. If L_ε is invertible, then we can write

$$\phi = (L_\varepsilon)^{-1}(-Q_\varepsilon(\phi) - f_\varepsilon),$$

and a standard fixed point argument for small ε will yield the existence of such ϕ, thus completing the proof of Theorem 3.1.1 (see Section 3.9).

Thus, a central argument here is the study of the linear theory for L_ε and, in particular, the analysis of its indicial roots, injectivity and Fredholm properties. However, while the behaviour of a second order ODE is governed by two boundary conditions (or behavior at the singular points using Frobenius method), this may not be true in general for a non-local operator.

We first consider the model operator $\mathscr{L}_1$ defined in (3.124) for an isolated singularity at the origin. Near the singularity $\mathscr{L}_1$ behaves like

$$(-\Delta_{\mathbb{R}^N})^\gamma - \frac{\kappa}{r^{2\gamma}} \tag{3.8}$$

or, after conjugation, like $P_\gamma^{g_0} - \kappa$, which is a fractional Laplacian operator with a Hardy potential of critical type.

The central core of the linear theory deals with the operator (3.8). In Section 3.6 we perform a delicate study of the Green's function by inverting its Fourier symbol Θ_γ^m (see (3.38)). This requires a very careful analysis of the poles of the symbol, in both the stable and unstable cases. Contrary to the local case $\gamma = 1$, in which there are only two indicial roots for each projection m, here we find an infinite sequence for each m. But in any case, these are controlled. It is also interesting to observe that, even though we have a non-local operator, the first pair of indicial roots governs the asymptotic behavior of the operator and thus, its kernel is two-dimensional in some sense (see, for instance, Proposition 3.6.10 for a precise statement).

Then, in Section 3.7 we complete the calculation of the indicial roots (see Lemma 3.7.1). Next, we show the injectivity for $\mathscr{L}_1$ in weighted Hölder spaces, and an *a priori* estimate (Lemma 3.7.4) yields the injectivity for L_ε.

In addition, in Section 3.8 we work with weighted Hilbert spaces and we prove Fredholm properties for L_ε in the spirit of the results by Mazzeo [130, 131] for edge type operators by constructing a suitable parametrix with compact remainder. The difficulty lies precisely in the fact that we are working with a non-local operator, so the localization with a cut-off is the non-trivial step. However, by working with suitable weighted spaces we are able to localize the problem near the singularity; indeed, the tail terms are small. Then we conclude that L_ε must be surjective by purely functional analysis reasoning. Finally, we construct a right inverse for L_ε, with norm uniformly bounded independently of ε, and this concludes the proof of Theorem 3.1.1.

The Appendix contains some well known results on special functions and the Fourier-Helgason transform.

As a byproduct of the proof of Theorem 3.1.1, we will obtain the existence of solutions with isolated singularities in the subcritical regime (note the shift from n to N in the spatial dimension, which will fit better our purposes).

Theorem 3.1.3. *Let $\gamma \in (0,1)$, $N \geq 2$ and*

$$\frac{N}{N-2\gamma} < p < \frac{N+2\gamma}{N-2\gamma}. \tag{3.9}$$

Let Σ be a finite number of points, $\Sigma = \{q_1, \ldots, q_K\}$. Then equation

$$(-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma} u^p \text{ in } \mathbb{R}^N \setminus \Sigma$$

has positive solutions that blow up exactly at Σ.

Remark 3.1.4. The constant $A_{N,p,\gamma}$ is chosen so that the model function $u_\gamma(x) = |x|^{-\frac{2\gamma}{p-1}}$ is a singular solution to (3.3) that blows up exactly at the origin. In particular,

$$A_{N,p,\gamma} = \Lambda\left(\tfrac{N-2\gamma}{2} - \tfrac{2\gamma}{p-1}\right) \quad \text{for} \quad \Lambda(\alpha) = 2^{2\gamma} \frac{\Gamma\left(\frac{N+2\gamma+2\alpha}{4}\right)\Gamma\left(\frac{N+2\gamma-2\alpha}{4}\right)}{\Gamma\left(\frac{N-2\gamma-2\alpha}{4}\right)\Gamma\left(\frac{N-2\gamma+2\alpha}{4}\right)}. \qquad (3.10)$$

Note that, for the critical exponent $p = \frac{N+2\gamma}{N-2\gamma}$, the constant $A_{N,p,\gamma}$ coincides with $\Lambda_{N,\gamma} = \Lambda(0)$, the sharp constant in the fractional Hardy inequality in $\mathbb{R}^N$. Its precise value is given in (3.44).

Let us make some comments on the bibliography. First, the problem of uniqueness and non-degeneracy for some fractional ODE has been considered in [61, 97, 98], for instance.

The construction of singular solutions in the range of exponents for which the problem is stable, i.e., $\frac{N}{N-2\gamma} < p < p_1$ for $p_1 < \frac{N+2\gamma}{N-2\gamma}$ defined in (3.12), was studied in the previous paper by Ao, the author, González and Wei [8]. In addition, for the critical case $p = \frac{N+2\gamma}{N-2\gamma}$, solutions with a finite number of isolated singularities were obtained in the article by Ao, DelaTorre, González and Wei [9] using a gluing method. The difficulty there was the presence of a non-trivial kernel for the linearized operator. With all these results, together with Theorem 3.1.1, we have successfully developed a complete fractional Mazzeo-Pacard program for the construction of singular solutions of the fractional Yamabe problem.

Gluing methods for fractional problems are starting to be developed. A finite dimensional reduction has been applied in [62] to construct standing-wave solutions to a fractional nonlinear Schrödinger equation and in [82] to construct layered solutions for a fractional inhomogeneous Allen-Cahn equation.

The next development came in [9] for the fractional Yamabe problem with isolated singularities, that we have just mentioned. There the model for an isolated singularity is a Delaunay-type metric (see also [134, 135, 164] for the construction of constant mean curvature surfaces with Delaunay ends and [133, 136] for the scalar curvature case). However, in order to have enough freedom parameters at the perturbation step, for the non-local gluing in [9] the authors replace the Delaunay-type solution by a bubble tower (an infinite, but countable, sum of bubbles). As a consequence, the reduction method becomes infinite dimensional. Nevertheless, it can still be treated with the tools available in the finite dimensional case and one reduces the PDE to an infinite dimensional Toda type system. The most recent works related to gluing are [49, 51] for the construction of counterexamples to the fractional De Giorgi conjecture. This reduction is fully infinite dimensional.

For the fractional De Giorgi conjecture with $\gamma \in [\frac{1}{2}, 1)$ we refer to [30, 38, 155] and the most recent striking paper [94]. Related to this conjecture, in the case $\gamma \in (0, \frac{1}{2})$ there exists a notion of non-local mean curvature for hypersurfaces in $\mathbb{R}^n$, see [41] and the survey [172]. Much effort has been made regarding regularity [17, 31, 44] and various qualitative properties [79, 80]. More recent work on stability of non-local minimal surfaces can be found in [56]. Delaunay surfaces for this curvature have been constructed in [32, 33]. After the appearance of [58], Cabré has pointed out that this paper also constructs Delaunay surfaces with constant nonlocal mean curvature.

3.2 The fast decaying solution

We aim to construct a fast-decay singular solution to the fractional Lane–Emden equation

$$(-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma} u^p \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{3.11}$$

for $\gamma \in (0,1)$ and p in the range (3.9).

We consider the exponent $p_1 = p_1(N,\gamma) \in (\frac{N}{N-2\gamma}, \frac{N+2\gamma}{N-2\gamma})$ defined below by (3.12) such that the singular solution $u_\gamma(x) = |x|^{-\frac{2\gamma}{p-1}}$ is stable if and only if $\frac{N}{N-2\gamma} < p < p_1$. In the notation of Remark 3.1.4, p_1 as defined as the root of

$$pA_{N,p,\gamma} = \Lambda(0). \tag{3.12}$$

The main result in this section is:

Proposition 3.2.1. *For any $\varepsilon \in (0,\infty)$ there exists a fast-decay entire singular solution u_ε of* (3.11) *such that*

$$u_\varepsilon(x) \sim \begin{cases} O\left(|x|^{-\frac{2\gamma}{p-1}}\right) & \text{as } |x| \to 0, \\ \varepsilon |x|^{-(N-2\gamma)} & \text{as } |x| \to \infty. \end{cases}$$

The proof in the stable case $\frac{N}{N-2\gamma} < p < p_1 < \frac{N+2\gamma}{N-2\gamma}$ is already contained in the paper [8], so we will assume for the rest of the section that we are in the unstable regime

$$\frac{N}{N-2\gamma} < p_1 \leq p < \frac{N+2\gamma}{N-2\gamma}.$$

We first prove uniqueness of minimal solutions for the non-local problem (3.14) using Schaaf's argument and a fractional Pohožaev identity obtained by Ros-Oton and Serra (Proposition 3.2.2 below). Then we perform a blow-up argument on an unbounded bifurcation branch. An application of Kelvin's transform yields an entire solution of the Lane–Emden equation with the desired asymptotics.

Set $A = A_{N,p,\gamma}$. Note that the Kelvin transform $w(x) = |x|^{-(N-2\gamma)} u\left(\frac{x}{|x|^2}\right)$ of u satisfies

$$(-\Delta)^\gamma w(x) = A|x|^\beta w^p(x), \tag{3.13}$$

where $\beta =: p(N-2\gamma) - (N+2\gamma) \in (-2\gamma, 0)$.

Consider the following non-local Dirichlet problem in the unit ball $B_1 = B_1(0) \subset \mathbb{R}^N$,

$$\begin{cases} (-\Delta)^\gamma w(x) = \lambda |x|^\beta A(1+w(x))^p & \text{in } B_1, \\ w = 0 & \text{in } \mathbb{R}^N \setminus B_1. \end{cases} \tag{3.14}$$

Since $(-\Delta)^\gamma |x|^{\beta+2\gamma} = c_0 |x|^\beta$ and $(-\Delta)^\gamma (1-|x|^2)_+^\gamma = c_1$ for some positive constants c_0 and c_1, we have that $|x|^{\beta+2\gamma} + (1-|x|^2)_+^\gamma$ is a positive super-solution for small λ. Thus there exists a minimal radial solution $w_\lambda(r)$. Moreover, it is bounded, radially non-increasing for fixed $\lambda \in (0,\lambda^*)$ and non-decreasing in λ. We will show that w_λ is the unique solution of (3.14) for all small λ.

Proposition 3.2.2. *There exists a small $\lambda_0 > 0$ depending only on $N \geq 2$ and $\gamma \in (0,1)$ such that for any $0 < \lambda < \lambda_0$, w_λ is the unique solution to* (3.14) *among the class*

$$\tilde{\mathscr{C}}^2_\gamma(\mathbb{R}^N) = \left\{ w \in \mathscr{C}^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|w(x)|}{(1+|x|)^{N+2\gamma}}\, dx < \infty \right\}.$$

The idea of the proof follows from [83] and similar arguments can be found in [161], [117] and [118].

3.2.1 Useful inequalities

The first ingredient is the Pohožaev identity for the fractional Laplacian. Such identities for integro-differential operators have been recently studied in [149], [151] and [114].

Theorem 3.2.3 (Proposition 1.12 in [149])**.** *Let Ω be a bounded $\mathscr{C}^{1,1}$ domain, $f \in \mathscr{C}^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$, u be a bounded solution of*

$$\begin{cases} (-\Delta)^\gamma u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.15}$$

and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Then

$$u/\delta^\gamma|_\Omega \in \mathscr{C}^\alpha(\overline{\Omega}) \qquad \text{for some } \alpha \in (0,1),$$

and there holds

$$\int_\Omega \left(F(x,u) + \frac{1}{N} x \cdot \nabla_x F(x,u) - \frac{N-2\gamma}{2N} u f(x,u) \right) dx \\ = \frac{\Gamma(1+\gamma)^2}{2N} \int_{\partial\Omega} \left(\frac{u}{\delta^\gamma} \right)^2 (x \cdot \nu)\, d\sigma$$

where $F(x,t) = \int_0^t f(x,\tau)\, d\tau$ and ν is the unit outward normal to $\partial\Omega$ at x.

Using integration by parts (see, for instance, (1.5) in [149]), it is clear that

$$\int_\Omega u f(x,u)\, dx = \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\gamma}{2}} u \right|^2 dx,$$

which yields our fundamental inequality:

Corollary 3.2.4. *Under the assumptions of Theorem 3.2.3, we have for any star-shaped domain* Ω *and any* $\sigma \in \mathbb{R}$*,*

$$\int_{\Omega} \left(F(x,u) + \frac{1}{N} x \cdot \nabla_x F(x,u) - \sigma u f(x,u) \right) dx \geq \left(\frac{N-2\gamma}{2N} - \sigma \right) \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\gamma}{2}} u \right|^2 dx \tag{3.16}$$

The second ingredient is the fractional Hardy–Sobolev inequality which, via Hölder inequality, is an interpolation of fractional Hardy inequality and fractional Sobolev inequality:

Theorem 3.2.5 (Lemma 2.1 in [107])**.** *Assume that* $0 \leq \alpha < 2\gamma < \min\{2, N\}$*. Then there exists a constant c such that*

$$c \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\gamma}{2}} u \right|^2 dx \geq \left(\int_{\mathbb{R}^N} |x|^{-\alpha} |u|^{\frac{2(N-\alpha)}{N-2\gamma}} \right)^{\frac{N-2\gamma}{N-\alpha}}. \tag{3.17}$$

3.2.2 Proof of Proposition 3.2.2

We are now in a position to prove the uniqueness of solutions of (3.14) with small parameter.

Proof. Suppose w and w_λ are solutions to (3.14). Then $u = w - w_\lambda$ is a positive solution to the Dirichlet problem

$$\begin{cases} (-\Delta)^\gamma u = \lambda A |x|^\beta g_\lambda(x,u) & \text{in } B_1(0), \\ u = 0 & \text{in } \mathbb{R}^N \setminus B_1(0), \end{cases}$$

where $g_\lambda(x,u) = (1 + w_\lambda(x) + u)^p - (1 + w_\lambda(x))^p \geq 0$ for $u \geq 0$. Denoting

$$G_\lambda(x,u) = \int_0^u g_\lambda(x,t)\, dt,$$

we apply (3.16) with $f(x,u) = \lambda A|x|^{\beta} g_{\lambda}(x,u)$ over $\Omega = B_1$ to obtain

$$\begin{aligned}
&\left(\frac{N-2\gamma}{2N} - \sigma\right) \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\gamma}{2}} u\right|^2 dx \\
&\leq \lambda A \int_{B_1} \left(|x|^{\beta} G_{\lambda}(x,u) + \frac{1}{N} x \cdot \nabla_x \left(|x|^{\beta} G_{\lambda}(x,u)\right) - \sigma |x|^{\beta} u g_{\lambda}(x,u) \right) dx \\
&= \lambda A \int_{B_1} |x|^{\beta} \left(\left(1 + \frac{\beta}{N}\right) G_{\lambda}(x,u) + \frac{1}{N} x \cdot \nabla_x G_{\lambda}(x,u) - \sigma u g_{\lambda}(x,u) \right) dx.
\end{aligned} \tag{3.18}$$

Note that

$$G_{\lambda}(x,u) = u^2 \int_0^1 \int_0^1 pt(1 + w_{\lambda}(x) + \tau t u)^{p-1} \, d\tau dt \tag{3.19}$$

and

$$\nabla_x G_{\lambda}(x,u) = u^2 \int_0^1 \int_0^1 p(p-1)t(1 + w_{\lambda}(x) + \tau t u)^{p-2} \, d\tau dt \cdot \nabla w_{\lambda}(x).$$

Since w_{λ} is radially decreasing, $x \cdot \nabla w_{\lambda}(x) \leq 0$ and hence $x \cdot \nabla_x G_{\lambda}(x,u) \leq 0$. Then (3.18) becomes

$$\begin{aligned}
&\left(\frac{N-2\gamma}{2N} - \sigma\right) \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\gamma}{2}} u\right|^2 dx \\
&\qquad \leq \lambda A \int_{B_1} |x|^{\beta} \left(\left(1 + \frac{\beta}{N}\right) G_{\lambda}(x,u) - \sigma u g_{\lambda}(x,u) \right) dx.
\end{aligned} \tag{3.20}$$

Now, since for any $\lambda \in \left[0, \frac{\lambda^*}{2}\right]$ and any $x \in B_1$,

$$\begin{aligned}
&\lim_{t \to \infty} \frac{G_{\lambda}(x,t)}{t g_{\lambda}(x,t)} \\
&= \lim_{t \to \infty} \frac{\frac{1}{p+1}\left((1 + w_{\lambda}(x) + t)^{p+1} - (1 + w_{\lambda}(x))^{p+1}\right) - (1 + w_{\lambda}(x))^p t}{t\left((1 + w_{\lambda}(x) + t)^p - (1 + w_{\lambda}(x))^p\right)} \\
&= \frac{1}{p+1},
\end{aligned}$$

we deduce that for any $\varepsilon > 0$ there exists an $M = M(\varepsilon) > 0$ such that

$$G_{\lambda}(x,t) \leq \frac{1+\varepsilon}{p+1} u g_{\lambda}(x,t)$$

whenever $t \geq M$. From this we estimate the tail of the right hand side of (3.20) as

$$\begin{aligned}
&\int_{B_1 \cap \{u \geq M\}} |x|^\beta \left(\left(1+\frac{\beta}{N}\right) G_\lambda(x,u) - \sigma u g_\lambda(x,u) \right) dx \\
&\qquad \leq \int_{B_1 \cap \{u \geq M\}} |x|^\beta \left(\left(1+\frac{\beta}{N}\right) \frac{1+\varepsilon}{p+1} - \sigma \right) u g_\lambda(x,u)\, dx.
\end{aligned}$$

We wish to choose ε and σ such that

$$\left(1+\frac{\beta}{N}\right)\frac{1+\varepsilon}{p+1} < \sigma < \frac{N-2\gamma}{2N},$$

so that the above integral is non-positive. Indeed we observe that

$$\begin{aligned}
\left(\frac{N+\beta}{N}\right)\frac{1}{p+1} - \frac{N-2\gamma}{2N} &= \frac{2(p(N-2\gamma)-2\gamma)-(N-2\gamma)(p+1)}{2N(p+1)} \\
&= \frac{(p-1)(N-2\gamma)-4\gamma}{2N(p+1)} \\
&< 0
\end{aligned}$$

as $p-1 \in \left(\frac{2\gamma}{N-2\gamma}, \frac{4\gamma}{N-2\gamma}\right)$. Then there exists a small $\varepsilon > 0$ such that

$$\left(1+\frac{\beta}{N}\right)\frac{1+\varepsilon}{p+1} < \frac{N-2\gamma}{2N},$$

from which the existence of such σ follows. With this choice of ε and σ, (3.20) gives

$$\begin{aligned}
&\left(\frac{N-2\gamma}{2N} - \sigma\right) \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\gamma}{2}} u\right|^2 dx \\
&\leq \lambda A \int_{B_1 \cap \{u < M\}} |x|^\beta \left(\left(1+\frac{\beta}{N}\right) G_\lambda(x,u) - \sigma u g_\lambda(x,u) \right) dx \\
&\leq \lambda A \left(1+\frac{\beta}{N}\right) \int_{B_1 \cap \{u < M\}} |x|^\beta G_\lambda(x,u)\, dx.
\end{aligned}$$

Recalling the expression (3.19) for $G_\lambda(x,u)$, we have

$$\left(\frac{1}{2} - \frac{\sigma}{N}\right) \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\gamma}{2}} u\right|^2 dx \leq \lambda A C_M \int_{B_1 \cap \{u < M\}} |x|^\beta u^2\, dx,$$

where

$$C_M = \frac{p}{2}\left(1 + w_{\frac{\lambda^*}{2}}(0) + M\right)^{p-1} \tag{3.21}$$

by the monotonicity properties of w_λ.

On the other hand, since $p > \frac{N}{N-2\gamma}$,

$$-\beta = -p(N-2\gamma) + (N+2\gamma) = 2\gamma - (N-2\gamma)\left(p - \frac{N}{N-2\gamma}\right) < 2\gamma,$$

and thus the fractional Hardy–Sobolev inequality (3.17) implies

$$c\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{\gamma}{2}}u\right|^2 dx \geq \left(\int_{\mathbb{R}^N}|x|^\beta u^{2\eta}\,dx\right)^{\frac{1}{\eta}} = \left(\int_{B_1}|x|^\beta u^{2\eta}\,dx\right)^{\frac{1}{\eta}},$$

where

$$\eta = \frac{N+\beta}{N-2\gamma} = \frac{p(N-2\gamma)-2\gamma}{N-2\gamma} = 1 + \left(p - \frac{N}{N-2\gamma}\right) > 1.$$

Hence,

$$\left(\int_{B_1}|x|^\beta u^{2\gamma}\,dx\right)^{\frac{1}{\gamma}} \leq \frac{2N}{N-2\gamma}cC_M\lambda A\int_{B_1}|x|^\beta u^2\,dx.$$

However, by Hölder's inequality, we have

$$\begin{aligned}\int_{B_1}|x|^\beta u^2\,dx &= \int_{B_1}|x|^{\frac{\beta}{\eta}}u^2\cdot|x|^{\beta\left(1-\frac{1}{\eta}\right)}\,dx \leq \left(\int_{B_1}|x|^\beta u^{2\eta}\,dx\right)^{\frac{1}{\eta}}\left(\int_{B_1}|x|^\beta\,dx\right)^{1-\frac{1}{\eta}}\\ &\leq (N+\beta)^{-\frac{N+2\gamma}{N+\beta}}\left(\int_{B_1}|x|^\beta u^{2\eta}\,dx\right)^{\frac{1}{\eta}}.\end{aligned}$$

Therefore, we have

$$\left(\int_{B_1}|x|^\beta u^{2\eta}\,dx\right)^{\frac{1}{\eta}} \leq \frac{2NcAC_M}{(N-2\gamma)(N+\beta)^{\frac{N+2\gamma}{N+\beta}}}\lambda\left(\int_{B_1}|x|^\beta u^{2\eta}\,dx\right)^{\frac{1}{\eta}},$$

which forces $u \equiv 0$ for any

$$\lambda < \lambda_0 = \left(\frac{2NcAC_M}{(N-2\gamma)(N+\beta)^{\frac{N+2\gamma}{N+\beta}}}\right)^{-1}. \tag{3.22}$$

$\square$

3.2.3 Existence of a fast-decay singular solution

Consider the space of twice differentiable, positive and radially decreasing functions supported in the unit ball,

$$E = \left\{ w \in \mathscr{C}^2(\mathbb{R}^n) : w(x) = \tilde{w}(|x|),\ \tilde{w}' \leq 0,\ w > 0 \text{ in } B_1 \text{ and } w \equiv 0 \text{ in } \mathbb{R}^N \setminus B_1 \right\}.$$

We begin with an *a priori* estimate followed by a generic existence result for the non-local ODE (3.14), from which a bifurcation argument follows.

Lemma 3.2.6 (Uniform bound)**.** *There exists a universal constant $C_0 = C_0(N,\gamma,p,\lambda^*)$ such that for any function $w \in E$ solving* (3.14) *and for any $x \in B_{1/2}(0) \setminus \{0\}$,*

$$w(x) \leq C_0 |x|^{-\frac{\beta+2\gamma}{p-1}} = C_0 |x|^{-\frac{p(N-2\gamma)-N}{p-1}}.$$

Proof. Using the Green's function for the Dirichlet problem in the unit ball ([148]), we have

$$w(x) = \int_{B_1} G(x,y) \lambda A |y|^{\beta} (1 + w(y))^p \, dy,$$

where

$$G(x,y) = C(N,\gamma) \frac{1}{|x-y|^{N-2\gamma}} \int_0^{r_0(x,y)} \frac{r^{\gamma-1}}{(r+1)^{\frac{N}{2}}} \, dr$$

with

$$r_0(x,y) = \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}.$$

Here $C(N,\gamma)$ is some normalizing constant. Let

$$y \in B_{\frac{|x|}{4}}\left(\frac{3x}{4}\right) \subset B_{\frac{|x|}{2}}(x) \cap B_{|x|}(0) \subset B_1(0).$$

From $y \in B_{\frac{|x|}{2}}(x)$, we have

$$|x-y| \leq \frac{|x|}{2} \leq \frac{1}{4} \quad \text{and} \quad |y| \leq \frac{3|x|}{2} \leq \frac{3}{4}$$

and so

$$r_0(x,y) \geq \frac{\left(1-\frac{1}{4}\right)\left(1-\frac{9}{16}\right)}{\frac{1}{16}} \geq \frac{21}{4} > 5.$$

On the other hand, since $y \in B_{|x|}(0)$ and u is radially non-increasing, we have

$$|y|^\beta \geq |x|^\beta \quad \text{and} \quad u(y) \geq u(x).$$

Therefore, we may conclude

$$G(x,y) \geq C(N,\gamma)\left(\frac{2}{|x|}\right)^{N-2\gamma} \int_0^5 \frac{r^{\gamma-1}}{(r+1)^{\frac{N}{2}}}\, dr$$

and

$$\begin{aligned}
w(x) &\geq A\int_{B_{\frac{|x|}{4}}\left(\frac{3x}{4}\right)} C(N,\gamma)\frac{2^{N-2\gamma}}{|x|^{N-2\gamma}}\left(\int_0^5 \frac{r^{\gamma-1}}{(r+1)^{\frac{N}{2}}}\, dr\right)\lambda_0|x|^\beta w(x)^p\, dy \\
&\geq C(N,\gamma)A2^{N-2\gamma}\left(\int_0^5 \frac{r^{\gamma-1}}{(r+1)^{\frac{N}{2}}}\, dr\right)\lambda_0 \cdot \frac{|x|^\beta}{|x|^{N-2\gamma}} w(x)^p \cdot |B_1|\left(\frac{|x|}{4}\right)^N \\
&\geq C_0^{-(p-1)}|x|^{\beta+2\gamma}w(x)^p,
\end{aligned}$$

where

$$C_0^{-(p-1)} = \frac{C(N,\gamma)|B_1|A\lambda_0}{2^{N+2\gamma}}\int_0^5 \frac{r^{\gamma-1}}{(r+1)^{\frac{N}{2}}}\, dr.$$

The inequality clearly rearranges to

$$w(x) \leq C_0|x|^{-\frac{\beta+2\gamma}{p-1}},$$

as desired. The dependence of the constant C_0 follows from (3.22) and (3.21). □

Lemma 3.2.7 (Existence)**.** *For any $\lambda \in (0,+\infty)$ the non-local Dirichlet problem* (3.14) *has a positive solution.*

Proof. We use Schauder fixed point theorem. Let us denote the Gagliardo norm by

$$[u]^2_{H^\gamma(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\gamma}}\, dxdy.$$

Define the operator T by

$$Tw(x) = \begin{cases} \int_{B_1(0)} G(x,y)\lambda A|y|^{\beta}(1+w(y))^p\,dy & \text{for } x \in B_1(0) \\ 0 & \text{for } x \in \mathbb{R}^N \setminus B_1(0), \end{cases}$$

where G is the Green's function as in the proof of Lemma 3.2.6.

Suppose that $w \in L^2(B_1(0))$. We first observe that the right hand side of (3.14) is in $L^{\frac{2N}{N+2\gamma}}(B_1(0))$, where $\frac{2N}{N+2\gamma}$ is the conjugate of the critical Sobolev exponent $2^*(N,\gamma) = \frac{2N}{N-2\gamma}$. Indeed, by Lemma 3.2.6, we have

$$|x|^{\beta}(1+w^p) \le |x|^{\beta} + |x|^{-2\gamma}w \le |x|^{-N+\frac{2\gamma}{p-1}}$$

and the integrability follows from

$$\left(-N+\frac{2\gamma}{p-1}\right)\frac{2N}{N+2\gamma}+N = \frac{N(N-2\gamma)}{(p-1)(N+2\gamma)}\left(\frac{4\gamma}{N-2\gamma}-(p-1)\right) > 0.$$

Using Hölder inequality and fractional Sobolev inequality (see, for instance, [73]), we have

$$\begin{aligned} C_1(N,\gamma)^{-1}\,\|Tw\|^2_{L^{2^*(N,\gamma)}(B_1(0))} \le [Tw]^2_{H^{\gamma}(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} Tw(x)(-\Delta)^{\gamma}Tw(x)\,dx \\ &= \int_{B_1(0)} Tw(x)\cdot\lambda A|x|^{\beta}(1+w(x))^p\,dx \\ &\le C_2(N,\gamma,p,\lambda)\,\|Tw\|_{L^{2^*(N,\gamma)}(B_1(0))} \\ &\le C_3(N,\gamma,p,\lambda)[Tw]_{H^{\gamma}(\mathbb{R}^N)}. \end{aligned}$$

This implies the existence of $\bar{C} > 0$ such that

$$\|Tw\|_{L^2(B_1(0))} \le \bar{C},$$

i.e. $T : \mathscr{B} \to \mathscr{B}$ with

$$\mathscr{B} = \left\{u \in L^2(B_1(0)) : \|u\|_{L^2(B_1(0))} \le \bar{C}\right\},$$

as well as

$$[Tw]_{H^\gamma(B_1(0))} \le [Tw]_{H^\gamma(\mathbb{R}^N)} \le \bar{C},$$

hence the compactness of T via the Sobolev embedding. By Schauder fixed point theorem, there exists a weak solution $w \in L^2(B_1(0))$. It remains to apply elliptic regularity. $\square$

Lemma 3.2.8 (Bifurcation)**.** *There exists a sequence of solutions* (λ_j, w_j) *of* (3.14) *in* $(0,\lambda^*] \times E$ *such that*

$$\lim_{j\to\infty} \lambda_j = \lambda_\infty \in [\lambda_0, \lambda^*] \quad \textit{and} \quad \lim_{t\to\infty} \|w_j\|_{L^\infty} = \infty,$$

where λ_0 *is given in Proposition 3.2.2.*

Proof. Consider the continuation

$$\mathscr{C} = \{(\lambda(t), w(t)) : t \ge 0\}$$

of the branch of minimal solutions $\{(\lambda, w_\lambda) : 0 \le \lambda < \lambda^*\}$, where $(\lambda(0), w(0)) = (0,0)$. By Proposition 3.2.2, we see that $\mathscr{C} \subset (\lambda_0, \lambda^*] \times E$. Moreover, since $w_\lambda > 0$ in B_1, we also have $w > 0$ in B_1 for any $(\lambda, w) \in \mathscr{C}$. If $\mathscr{C}$ were bounded, then Lemma 3.2.7 would give a contradiction around $\lim_{t\to\infty}(\lambda(t), w(t))$. Therefore, $\mathscr{C}$ is unbounded and the existence of the desired sequence of pairs (λ_j, w_j) follows. $\square$

We are ready to establish the existence of a fast-decay singular solution.

Proof of Proposition 3.2.1. Let (λ_j, w_j) be as in Lemma 3.2.8. By Lemma 3.2.6,

$$w_j(x) \le C_0 |x|^{-\frac{\beta+2\gamma}{p-1}}.$$

Define

$$m_j = \|w_j\|_{L^\infty(B_1)} = w_j(0) \quad \text{and} \quad R_j = m_j^{\frac{p-1}{\beta+2\gamma}} = m_j^{\frac{p-1}{p(N-2\gamma)-N}}$$

so that $m_j, R_j \to \infty$ as $j \to \infty$. Set also

$$W_j(x) = \lambda^{\frac{1}{p-1}} m_j^{-1} w_{\lambda_j}\Big(\frac{x}{R_j}\Big).$$

Then $0 \le W_j \le 1$ and W_j is a bounded solution to

$$\begin{cases} (-\Delta)^\gamma W_j = m_j^{p-1} R_j^{-\beta-2\gamma} A|x|^\beta \left(\lambda_j^{\frac{1}{p-1}} m_j^{-1} + W_j\right)^p & \text{in } B_{R_j}(0), \\ W_j = 0 & \text{in } \mathbb{R}^N \setminus B_{R_j}(0), \end{cases}$$

that is,

$$\begin{cases} (-\Delta)^\gamma W_j = A|x|^\beta \left(\lambda_j^{\frac{1}{p-1}} m_j^{-1} + W_j\right)^p & \text{in } B_{R_j}(0), \\ W_j = 0 & \text{in } \mathbb{R}^N \setminus B_{R_j}(0). \end{cases}$$

In $B_{R_j}(0)$, $W_j(x)$ has the upper bound

$$\begin{aligned} W_j(x) &\le \lambda^{\frac{1}{p-1}} m_j^{-1} \cdot C_0 \left(\frac{x}{R_j}\right)^{-\frac{\beta+2\gamma}{p-1}} \le C_0 \left(\lambda_0^{\frac{1}{p-1}} + (\lambda^*)^{\frac{1}{p-1}}\right) |x|^{-\frac{\beta+2\gamma}{p-1}} \\ &= C_1 |x|^{-\frac{\beta+2\gamma}{p-1}} = C_1 |x|^{\frac{2\gamma}{p-1} - (N-2\gamma)}. \end{aligned} \tag{3.23}$$

Note that $|x|^\beta \in L^q(B_{R_j}(0))$ for any $\frac{N}{2\gamma} < q < \frac{N}{-\beta}$. Hence, for such q, by the regularity result in [150], $W_j \in \mathscr{C}^\eta_{loc}(\mathbb{R}^N)$ for $\eta = \min\left\{\gamma, 2\gamma - \frac{N}{q}\right\} \in (0,1)$. Therefore, by passing to a subsequence, W_j converges uniformly on compact sets of $\mathbb{R}^N$ to a radially symmetric and non-increasing function w which satisfies

$$\begin{cases} (-\Delta)^\gamma w = A|x|^\beta w^p & \text{in } \mathbb{R}^N, \\ w(0) = 1, \\ w(x) \le C_1 |x|^{\frac{2\gamma}{p-1} - (n-2\gamma)}, \end{cases}$$

in view of (3.23).

Now the family of rescaled solutions $w_\varepsilon(x) = \varepsilon w\left(\varepsilon^{\frac{p-1}{\beta+2\gamma}} x\right)$ solves

$$\begin{cases} (-\Delta)^\gamma w_\varepsilon = A|x|^\beta w_\varepsilon^p & \text{in } \mathbb{R}^N, \\ w_\varepsilon(0) = \varepsilon, \\ w_\varepsilon(x) \le C_1 |x|^{\frac{2\gamma}{p-1} - (N-2\gamma)} & \text{in } \mathbb{R}^N \setminus \{0\}. \end{cases}$$

Finally, its Kelvin transform $u_\varepsilon(x) = |x|^{-(N-2\gamma)} w_\varepsilon\left(\frac{x}{|x|^2}\right)$ satisfies

$$\begin{cases} (-\Delta)^\gamma u_\varepsilon = A u_\varepsilon^p & \text{in } \mathbb{R}^N \setminus \{0\}, \\ u_\varepsilon(x) \le C_1 |x|^{-\frac{2\gamma}{p-1}} & \text{in } \mathbb{R}^N \setminus \{0\}, \\ u_\varepsilon(x) \sim \varepsilon |x|^{-(N-2\gamma)} & \text{as } |x| \to \infty, \end{cases}$$

as desired. □

3.3 The conformal fractional Laplacian in the presence of k-dimensional singularities

3.3.1 A quick review on the conformal fractional Laplacian

Here we review some basic facts on the conformal fractional Laplacian that will be needed in the next sections (see [52, 109] for the precise definitions and details).

If (X, g^+) is a $(n+1)$-dimensional conformally compact Einstein manifold (which, in particular, includes the hyperbolic space), one can define a one-parameter family of operators P_γ of order 2γ on its conformal infinity $M^n = \partial_\infty X^{n+1}$. P_γ is known as the conformal fractional Laplacian and it can be understood as a Dirichlet-to-Neumann operator on M. In the particular case that X is the hyperbolic space $\mathbb{H}^{n+1}$, whose conformal infinity is $M = \mathbb{R}^n$ with the Euclidean metric, P_γ coincides with the standard fractional Laplacian $(-\Delta_{\mathbb{R}^n})^\gamma$.

Let us explain this definition in detail. It is known that, having fixed a metric g_0 in the conformal infinity M, it is possible to write the metric g^+ in the normal form $g^+ = \rho^{-2}(d\rho^2 + g_\rho)$ in a tubular neighborhood $M \times (0, \delta]$. Here g_ρ is a one-parameter family of metrics on M satisfying $g_\rho|_{\rho=0} = g_0$ and ρ is a defining function in $\overline{X}$ for the boundary M (i.e., ρ is a non-degenerate function such that $\rho > 0$ in X and $\rho = 0$ on M).

Fix $\gamma \in (0, n/2)$ not an integer such that $n/2 + \gamma$ does not belong to the set of L^2-eigenvalues of $-\Delta_{g^+}$. Assume also that the first eigenvalue for $-\Delta_{g^+}$ satisfies $\lambda_1(-\Delta_{g^+}) > n^2/4 - \gamma^2$. It is well known from scattering theory [113, 115] that, given $w \in \mathscr{C}^\infty(M)$, the eigenvalue problem

$$-\Delta_{g^+}\mathscr{W} - \left(\tfrac{n^2}{2} - \gamma^2\right)\mathscr{W} = 0 \text{ in } X, \tag{3.24}$$

has a unique solution with the asymptotic expansion

$$\mathscr{W} = \mathscr{W}_1\rho^{\frac{n}{2}-\gamma} + \mathscr{W}_2\rho^{\frac{n}{2}+\gamma}, \quad \mathscr{W}_1, \mathscr{W}_2 \in \mathscr{C}^\infty(\overline{X}) \tag{3.25}$$

and Dirichlet condition on M

$$\mathscr{W}_1|_{\rho=0} = w. \tag{3.26}$$

The conformal fractional Laplacian (or scattering operator, depending on the normalization constant) on (M, g_0) is defined taking the Neumann data

$$P_\gamma^{g_0} w = d_\gamma \mathscr{W}_2|_{\rho=0}, \quad \text{where } d_\gamma = 2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(-\gamma)}, \tag{3.27}$$

and the fractional curvature as $Q_\gamma^{g_0} := P_\gamma^{g_0}(1)$.

$P_\gamma^{g_0}$ is a self-adjoint pseudodifferential operator of order 2γ on M with the same principal symbol as $(-\Delta_M)^\gamma$. In the case that the order of the operator is an even integer we recover the conformally invariant GJMS operators on M. In addition, for any $\gamma \in (0, \frac{n}{2})$, the operator is conformal. Indeed,

$$P_\gamma^{g_w} f = w^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^{g_0}(wf), \quad \forall f \in \mathscr{C}^\infty(M), \tag{3.28}$$

for a change of metric

$$g_w := w^{\frac{4}{n-2\gamma}} g_0, \; w > 0.$$

Moreover, (3.28) yields the Q_γ curvature equation

$$P_\gamma^{g_0}(w) = w^{\frac{n+2\gamma}{n-2\gamma}} Q_\gamma^{g_w}.$$

Explicit formulas for P_γ are not known in general. The formula for the cylinder will be given in Section 3.3.2, and it is one of the main ingredients for the linear theory arguments of Section 3.7.

The extension (3.24) takes a more familiar form under a conformal change of metric.

Proposition 3.3.1 ([52])**.** *Fix $\gamma \in (0,1)$ and $\bar{g} = \rho^2 g^+$. Let $\mathscr{W}$ be the solution to the scattering problem* (3.24)-(3.25) *with Dirichlet data* (3.26) *set to w. Then $W = \rho^{-n/2+\gamma}\mathscr{W}$ is the unique solution to the extension problem*

$$\begin{cases} -\operatorname{div}\left(\rho^{1-2\gamma}\nabla W\right) + E_{\bar{g}}(\rho)W = 0 & \text{in } (X,\bar{g}), \\ W|_{\rho=0} = w & \text{on } M, \end{cases} \tag{3.29}$$

where the derivatives are taken with respect to the metric $\bar{g}$, and the zero-th order term is given by

$$\begin{aligned} E_{\bar{g}}(\rho) &= -\Delta_{\bar{g}}(\rho^{\frac{1-2\gamma}{2}})\rho^{\frac{1-2\gamma}{2}} + \left(\gamma^2 - \tfrac{1}{4}\right)\rho^{-(1+2\gamma)} + \tfrac{n-1}{4n} R_{\bar{g}}\rho^{1-2\gamma} \\ &= \rho^{-\frac{n}{2}-\gamma-1}\left\{-\Delta_{g^+} - \left(\tfrac{n^2}{4} - \gamma^2\right)\right\}\left(\rho^{\frac{n}{2}-\gamma}\right). \end{aligned} \tag{3.30}$$

Moreover, we recover the conformal fractional Laplacian as

$$P_\gamma^{g_0} w = -\tilde{d}_\gamma \lim_{\rho\to 0} \rho^{1-2\gamma}\partial_\rho W,$$

where

$$\tilde{d}_\gamma = -\frac{d_\gamma}{2\gamma} = -\frac{2^{2\gamma-1}\Gamma(\gamma)}{\gamma\Gamma(-\gamma)}. \tag{3.31}$$

We also recall the following result, which allows us to rewrite (3.29) as a pure divergence equation with no zeroth order term. The more general statement can be found in Lemma 3.3.7, and it will be useful in the calculation of the Hamiltonian from Section 3.4.2.

Proposition 3.3.2 ([47, 52])**.** *Fix $\gamma \in (0,1)$. Let $\mathscr{W}^0$ be the solution to* (3.24)-(3.25) *with Dirichlet data* (3.26) *given by $w \equiv 1$, and set $\rho^* = (\mathscr{W}^0)^{\frac{1}{n/2-\gamma}}$. The function ρ^* is a defining function of M in X such that, if we define the metric $\bar{g}^* = (\rho^*)^2 g^+$, then $E_{\bar{g}^*}(\rho^*) \equiv 0$. Moreover, ρ^* has the asymptotic expansion near the conformal infinity*

$$\rho^*(\rho) = \rho\left[1 + \frac{Q_\gamma^{g_0}}{(n/2-\gamma)d_\gamma}\rho^{2\gamma} + O(\rho^2)\right].$$

By construction, if W^* is the solution to

$$\begin{cases} -\operatorname{div}\left((\rho^*)^{1-2\gamma}\nabla W^*\right) = 0 & \text{in } (X, \bar{g}^*), \\ W^* = w & \text{on } (M, g_0), \end{cases}$$

with respect to the metric $\bar{g}^*$, then

$$P_\gamma^{g_0} w = -\tilde{d}_\gamma \lim_{\rho^* \to 0} (\rho^*)^{1-2\gamma} \partial_{\rho^*} W^* + w Q_\gamma^{g_0}.$$

Remark 3.3.3. In the particular case that $X = \mathbb{R}^{n+1}_+ = \{(x, \ell) : x \in \mathbb{R}^n, \ell > 0\}$ is hyperbolic space $\mathbb{H}^{n+1}$ with the metric $g^+ = \frac{d\ell^2 + |dx|^2}{\ell^2}$ and $M = \mathbb{R}^n$, this is just the construction for the fractional Laplacian $(-\Delta_{\mathbb{R}^n})^\gamma$ as a Dirichlet-to-Neumann operator for a degenerate elliptic extension problem from [43]. Indeed, let U be the solution to

$$\begin{cases} \partial_{\ell\ell} U + \dfrac{1-2\gamma}{\ell} \partial_\ell U + \Delta_{\mathbb{R}^n} U = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ U|_{\ell=0} = u & \text{on } \mathbb{R}^n, \end{cases} \tag{3.32}$$

then

$$(-\Delta_{\mathbb{R}^n})^\gamma u = -\tilde{d}_\gamma \lim_{\ell \to 0} \ell^{1-2\gamma} \partial_\ell U. \tag{3.33}$$

From now on, (X, g^+) will be fixed to be hyperbolic space with its standard metric. Our point of view in this chapter is to rewrite this extension problem (3.32)-(3.33) using different coordinates for the hyperbolic metric in X, such as (3.39).

3.3.2 An isolated singularity

Before we go to the general problem, let us look at positive solutions to

$$(-\Delta_{\mathbb{R}^N})^\gamma u = \Lambda_{N,\gamma} u^{\frac{N+2\gamma}{N-2\gamma}} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\} \tag{3.34}$$

that have an isolated singularity at the origin. It is known ([40]) that such solutions have the asymptotic behavior near the origin like $r^{-\frac{N-2\gamma}{2}}$, for $r = |x|$. Thus it is natural to write

$$u = r^{-\frac{N-2\gamma}{2}} w. \tag{3.35}$$

Note that the power of the nonlinearity in the right hand side of (3.34) is chosen so that the equation has good conformal properties. Indeed, let $r = e^{-t}$ and $\theta \in \mathbb{S}^{N-1}$ and write the Euclidean metric in $\mathbb{R}^N$ as

$$|dx|^2 = dr^2 + r^2 g_{\mathbb{S}^{N-1}}$$

in polar coordinates. We use conformal geometry to rewrite equation (3.34). For this, consider the conformal change

$$g_0 := \frac{1}{r^2}|dx|^2 = dt^2 + g_{\mathbb{S}^{N-1}},$$

which is a complete metric defined on the cylinder $M_0 := \mathbb{R} \times \mathbb{S}^{N-1}$. The advantage of using g_0 as a background metric instead of the Euclidean one on $\mathbb{R}^N$ is the following: since the two metrics are conformally related, any conformal change may be rewritten as

$$\tilde{g} = u^{\frac{4}{N-2\gamma}}|dx|^2 = w^{\frac{4}{N-2\gamma}} g_0,$$

where we have used relation (3.35). Then, looking at the conformal transformation property (3.28) for the conformal fractional Laplacian P_γ, it is clear that

$$P_\gamma^{g_0}(w) = r^{\frac{N+2\gamma}{2}} P_\gamma^{|dx|^2}(r^{-\frac{N-2\gamma}{2}} w) = r^{\frac{N+2\gamma}{2}} (-\Delta_{\mathbb{R}^N})^\gamma u, \tag{3.36}$$

and thus equation (3.34) is equivalent to

$$P_\gamma^{g_0}(w) = \Lambda_{N,\gamma} w^{\frac{N+2\gamma}{N-2\gamma}} \quad \text{in} \quad \mathbb{R} \times \mathbb{S}^{N-1}.$$

The operator $P_\gamma^{g_0}$ on $\mathbb{R} \times \mathbb{S}^{N-1}$ is explicit. Indeed, in [66] the authors calculate its principal symbol using the spherical harmonic decomposition for $\mathbb{S}^{N-1}$. With some abuse of notation, let μ_m, $m = 0, 1, 2, \ldots$ be the eigenvalues of $\Delta_{\mathbb{S}^{N-1}}$, repeated according to multiplicity (this is, $\mu_0 = 0$, $\mu_1, \ldots, \mu_N = N-1, \ldots$). Then any function on $\mathbb{R} \times \mathbb{S}^{N-1}$ may be decomposed as $\sum_m w_m(t) E_m$, where $\{E_m(\theta)\}$ is the corresponding basis of eigenfunctions. The operator $P_\gamma^{g_0}$ diagonalizes under such eigenspace decomposition, and moreover, it is possible to calculate the Fourier symbol of each projection. Let

$$\hat{w}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi \cdot t} w(t)\, dt \tag{3.37}$$

be our normalization for the one-dimensional Fourier transform.

Proposition 3.3.4 ([66]). *Fix $\gamma \in (0, \frac{N}{2})$ and let P_γ^m be the projection of the operator $P_\gamma^{g_0}$ over each eigenspace $\langle E_m \rangle$. Then*

$$\widehat{P_\gamma^m(w_m)} = \Theta_\gamma^m(\xi)\, \widehat{w_m},$$

and this Fourier symbol is given by

$$\Theta_\gamma^m(\xi) = 2^{2\gamma} \frac{\left|\Gamma\left(\frac{1}{2} + \frac{\gamma}{2} + \frac{1}{2}\sqrt{(\frac{N}{2}-1)^2 + \mu_m} + \frac{\xi}{2} i\right)\right|^2}{\left|\Gamma\left(\frac{1}{2} - \frac{\gamma}{2} + \frac{1}{2}\sqrt{(\frac{N}{2}-1)^2 + \mu_m} + \frac{\xi}{2} i\right)\right|^2}. \tag{3.38}$$

Proof. Let us give some ideas in the proof because they will be needed in the next subsections. It is inspired in the calculation of the Fourier symbol for the conformal fractional Laplacian on the sphere $\mathbb{S}^n$ (see the survey [109], for instance). The method is, using spherical harmonics, to reduce the scattering equation (3.24) to an ODE. For this, we go back to the scattering theory definition for the fractional Laplacian and use different coordinates for the hyperbolic metric g^+. More precisely,

$$g^+ = \rho^{-2}\left\{\rho^2 + \left(1 + \frac{\rho^2}{4}\right)^2 dt^2 + \left(1 - \frac{\rho^2}{4}\right)^2 g_{\mathbb{S}^{N-1}}\right\}, \quad \bar{g} = \rho^2 g^+, \tag{3.39}$$

where $\rho \in (0,2)$, $t \in \mathbb{R}$. The conformal infinity $\{\rho = 0\}$ is precisely the cylinder $(\mathbb{R} \times \mathbb{S}^{N-1}, g_0)$. Actually, for the particular calculation here it is better to use the new variable $\sigma = -\log(\rho/2)$, and write

$$g^+ = d\sigma^2 + (\cosh\sigma)^2 dt^2 + (\sinh\sigma)^2 g_{\mathbb{S}^{N-1}}. \tag{3.40}$$

Using this metric, the scattering equation (3.24) is

$$\partial_{\sigma\sigma}\mathscr{W}+R(\sigma)\partial_\sigma\mathscr{W}+(\cosh\sigma)^{-2}\partial_{tt}\mathscr{W}+(\sinh\sigma)^{-2}\Delta_{\mathbb{S}^{N-1}}\mathscr{W}+\left(\tfrac{N^2}{4}-\gamma^2\right)\mathscr{W}=0, \tag{3.41}$$

where $\mathscr{W}=\mathscr{W}(\sigma,t,\theta)$, $\sigma\in(0,\infty)$, $t\in\mathbb{R}$, $\theta\in\mathbb{S}^{N-1}$, and

$$R(\sigma)=\frac{\partial_\sigma\left(\cosh\sigma\sinh^{N-1}\sigma\right)}{\cosh\sigma\sinh^{N-1}\sigma}.$$

After projection over spherical harmonics, and Fourier transform in t, the solution to equation (3.41) maybe written as

$$\widehat{\mathscr{W}_m}=\widehat{w_m}\,\varphi(\tau),$$

where we have used the change of variable $\tau=\tanh(\sigma)$ and $\varphi:=\varphi^{(m)}$ is a solution to the boundary value problem

$$\begin{cases}(1-\tau^2)\partial_{\tau\tau}\varphi+\left(\frac{N-1}{\tau}-\tau\right)\partial_\tau\varphi+\Big[-\mu_m\frac{1}{\tau^2}+(\frac{n^2}{4}-\gamma^2)\frac{1}{1-\tau^2}-\xi^2\Big]\varphi=0,\\ \text{has the expansion (3.25) with } w\equiv 1 \text{ near the conformal infinity } \{\tau=1\},\\ \varphi \text{ is regular at } \tau=0.\end{cases}$$

This is an ODE that can be explicitly solved in terms of hypergeometric functions, and indeed,

$$\begin{aligned}\varphi(\tau)=&(1+\tau)^{\frac{N}{4}-\frac{\gamma}{2}}(1-\tau)^{\frac{N}{4}-\frac{\gamma}{2}}\tau^{1-\frac{N}{2}+\sqrt{\left(\frac{N}{2}-1\right)^2+\mu_m}}\\ &\cdot\ {}_2\mathrm{F}_1(a,b;a+b-c+1;1-\tau^2)\\ &+S(1+\tau)^{\frac{N}{4}+\frac{\gamma}{2}}(1-\tau)^{\frac{N}{4}+\frac{\gamma}{2}}\tau^{1-\frac{N}{2}+\sqrt{\left(\frac{N}{2}-1\right)^2+\mu_m}}\\ &\cdot\ {}_2\mathrm{F}_1(c-a,c-b;c-a-b+1;1-\tau^2),\end{aligned} \tag{3.42}$$

where

$$S(\xi)=\frac{\Gamma(-\gamma)}{\Gamma(\gamma)}\frac{\left|\Gamma\left(\frac12+\frac{\gamma}{2}+\frac12\sqrt{(\frac{N}{2}-1)^2+\mu_m}+\frac{\xi}{2}i\right)\right|^2}{\left|\Gamma\left(\frac12-\frac{\gamma}{2}+\frac12\sqrt{(\frac{N}{2}-1)^2+\mu_m}+\frac{\xi}{2}i\right)\right|^2},$$

and

$$
\begin{aligned}
a &= \tfrac{-\gamma}{2} + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\tfrac{N}{2}-1)^2+\mu_m} + i\tfrac{\xi}{2},\\
b &= \tfrac{-\gamma}{2} + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\tfrac{N}{2}-1)^2+\mu_m} - i\tfrac{\xi}{2},\\
c &= 1+\sqrt{(\tfrac{N}{2}-1)^2+\mu_m}.
\end{aligned}
$$

The Proposition follows by looking at the Neumann condition in the expansion (3.25). □

The interest of this proposition will become clear in Section 3.7, where we calculate the indicial roots for the linearized problem. It is also the crucial ingredient in the calculation of the Green's function for the fractional Laplacian with Hardy potential in Section 3.6.

We finally recall the fractional Hardy's inequality in $\mathbb{R}^N$ ([18, 99, 124, 179])

$$
\int_{\mathbb{R}^N} u(-\Delta_{\mathbb{R}^N})^\gamma u\,dx \geq \Lambda_{N,\gamma}\int_{\mathbb{R}^N}\frac{u^2}{r^{2\gamma}}\,dx, \tag{3.43}
$$

where $\Lambda_{N,\gamma}$ is the Hardy constant given by

$$
\Lambda_{N,\gamma} = 2^{2\gamma}\frac{\Gamma^2(\frac{N+2\gamma}{4})}{\Gamma^2(\frac{N-2\gamma}{4})} = \Theta_\gamma^0(0). \tag{3.44}
$$

Under the conjugation (3.35), inequality (3.43) is written as

$$
\int_{\mathbb{R}\times\mathbb{S}^{N-1}} wP_\gamma^{g_0}w\,dt\,d\theta \geq \Lambda_{N,\gamma}\int_{\mathbb{R}\times\mathbb{S}^{N-1}} w^2\,dt\,d\theta.
$$

3.3.3 The full symbol

Now we consider the singular Yamabe problem (3.5) in $\mathbb{R}^n\setminus\mathbb{R}^k$. This particular case is important because it is the model for a general higher dimensional singularity (see [121]).

As in the introduction, set $N := n-k$. We define the coordinates $z=(x,y)$, $x\in\mathbb{R}^N$, $y\in\mathbb{R}^k$ in the product space $\mathbb{R}^n\setminus\mathbb{R}^k = (\mathbb{R}^{n-k}\setminus\{0\})\times\mathbb{R}^k$. Sometimes we will consider polar coordinates for x, which are

$$r = |x| = \mathrm{dist}(\cdot, \mathbb{R}^k) \in \mathbb{R}_+, \ \theta \in \mathbb{S}^{N-1}.$$

We write the Euclidean metric in $\mathbb{R}^n$ as

$$|dz|^2 = |dx|^2 + |dy|^2 = dr^2 + r^2 g_{\mathbb{S}^{N-1}} + |dy|^2.$$

Our model manifold M is going to be given by the conformal change

$$g_k := \frac{1}{r^2}|dz|^2 = g_{\mathbb{S}^{N-1}} + \frac{dr^2 + |dy|^2}{r^2} = g_{\mathbb{S}^{N-1}} + g_{\mathbb{H}^{k+1}}, \tag{3.45}$$

which is a complete metric, singular along $\mathbb{R}^k$. In particular, $M := \mathbb{S}^{N-1} \times \mathbb{H}^{k+1}$. As in the previous case, any conformal change may be rewritten as

$$\tilde{g} = u^{\frac{4}{n-2\gamma}}|dz|^2 = w^{\frac{4}{n-2\gamma}} g_k,$$

where we have used relation

$$u = r^{-\frac{n-2\gamma}{2}} w,$$

so we may just use g_k as our background metric. As a consequence, arguing as in the previous subsection, the conformal transformation property (3.28) for the conformal fractional Laplacian yields that

$$P_\gamma^{g_k}(w) = r^{\frac{n+2\gamma}{2}} P_\gamma^{|dz|^2}(r^{-\frac{n-2\gamma}{2}} w) = r^{\frac{n+2\gamma}{2}} (-\Delta_{\mathbb{R}^n})^\gamma u, \tag{3.46}$$

and thus the original Yamabe problem (3.5) is equivalent to the following:

$$P_\gamma^{g_k}(w) = \Lambda_{n,\gamma} w^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{on} \quad M.$$

Moreover, the expression for $P_\gamma^{g_k}$ in the metric g_k (with respect to the standard extension to hyperbolic space $X = \mathbb{H}^{n+1}$) is explicit, and this is the statement of the following theorem. For our purposes, it will be more convenient to write the standard hyperbolic metric as

$$g^+ = \rho^{-2}\left\{d\rho^2 + \left(1+\tfrac{\rho^2}{4}\right)^2 g_{\mathbb{H}^{k+1}} + \left(1-\tfrac{\rho^2}{4}\right)^2 g_{\mathbb{S}^{N-1}}\right\}, \tag{3.47}$$

for $\rho \in (0,2)$, so its conformal infinity $\{\rho = 0\}$ is precisely (M, g_k).

Consider the spherical harmonic decomposition for $\mathbb{S}^{N-1}$ as in Section 3.3.2. Then any function w on M may be decomposed as $w = \sum_m w_m E_m$, where $w_m = w_m(\zeta)$ for $\zeta \in \mathbb{H}^{k+1}$. We show that the operator $P_\gamma^{g_k}$ diagonalizes under such eigenspace decomposition, and moreover, it is possible to calculate the Fourier symbol for each projection. Let $\widehat{\cdot}$ denote the Fourier-Helgason transform on $\mathbb{H}^{k+1}$, as described in the Appendix (section 3.11).

Theorem 3.3.5. *Fix $\gamma \in (0, \frac{n}{2})$ and let P_γ^m be the projection of the operator $P_\gamma^{g_k}$ over each eigenspace $\langle E_m \rangle$. Then*

$$\widehat{P_\gamma^m(w_m)} = \Theta_\gamma^m(\xi)\,\widehat{w_m},$$

and this Fourier symbol is given by

$$\Theta_\gamma^m(\lambda) = 2^{2\gamma}\frac{\left|\Gamma\left(\frac{1}{2}+\frac{\gamma}{2}+\frac{1}{2}\sqrt{(\frac{N}{2}-1)^2+\mu_m}+\frac{\lambda}{2}i\right)\right|^2}{\left|\Gamma\left(\frac{1}{2}-\frac{\gamma}{2}+\frac{1}{2}\sqrt{(\frac{N}{2}-1)^2+\mu_m}+\frac{\lambda}{2}i\right)\right|^2}. \tag{3.48}$$

Proof. We follow the arguments in Proposition 3.3.4, however, the additional ingredient here is to use Fourier-Helgason transform to handle the extra term $\Delta_{\mathbb{H}^{k+1}}$ that will appear.

For the calculations below it is better to use the new variable

$$\sigma = -\log(\rho/2), \quad \rho \in (0,2),$$

and to rewrite the hyperbolic metric in $\mathbb{H}^{n+1}$ from (3.47) as

$$g^+ = d\sigma^2 + (\cosh\sigma)^2 g_{\mathbb{H}^{k+1}} + (\sinh\sigma)^2 g_{\mathbb{S}^{N-1}},$$

for the variables $\sigma \in (0,\infty)$, $\zeta \in \mathbb{H}^{k+1}$ and $\theta \in \mathbb{S}^{N-1}$. The conformal infinity is now $\{\sigma = +\infty\}$ and the scattering equation (3.24) is written as

$$\partial_{\sigma\sigma}\mathscr{W} + R(\sigma)\partial_\sigma \mathscr{W} + (\cosh\sigma)^{-2}\Delta_{\mathbb{H}^{k+1}}\mathscr{W} + (\sinh\sigma)^{-2}\Delta_{\mathbb{S}^{N-1}}\mathscr{W} + \left(\tfrac{n^2}{4} - \gamma^2\right)\mathscr{W} = 0, \tag{3.49}$$

where $\mathscr{W} = \mathscr{W}(\sigma,\zeta,\theta)$, and

$$R(\sigma) = \frac{\partial_\sigma\left((\cosh\sigma)^{k+1}(\sinh\sigma)^{N-1}\right)}{(\cosh\sigma)^{k+1}(\sinh\sigma)^{N-1}}.$$

The change of variable

$$\tau = \tanh(\sigma), \tag{3.50}$$

transforms equation (3.49) into

$$\begin{aligned}(1-\tau^2)^2\partial_{\tau\tau}\mathscr{W} + \left(\tfrac{n-k-1}{\tau} + (k-1)\tau\right)(1-\tau^2)\partial_\tau\mathscr{W} + (1-\tau^2)\Delta_{\mathbb{H}^{k+1}}\mathscr{W}& \\ + \left(\tfrac{1}{\tau^2} - 1\right)\Delta_{\mathbb{S}^{N-1}}\mathscr{W} + \left(\tfrac{n^2}{4} - \gamma^2\right)\mathscr{W} &= 0.\end{aligned}$$

Now we project onto spherical harmonics. This is, let $\mathscr{W}_m(\tau,\zeta)$ be the projection of $\mathscr{W}$ over the eigenspace $\langle E_m\rangle$. Then each $\mathscr{W}_m$ satisfies

$$(1-\tau^2)\partial_{\tau\tau}\mathscr{W}_m + \left(\tfrac{n-k-1}{\tau} + (k-1)\tau\right)\partial_\tau\mathscr{W}_m + \Delta_{\mathbb{H}^{k+1}}\mathscr{W}_m - \mu_m\tfrac{1}{\tau^2}\mathscr{W}_m + \frac{\frac{n^2}{4}-\gamma^2}{1-\tau^2}\mathscr{W}_m = 0. \tag{3.51}$$

Taking the Fourier-Helgason transform in $\mathbb{H}^{k+1}$ we obtain

$$\begin{aligned}(1-\tau^2)\partial_{\tau\tau}\widehat{\mathscr{W}_m} + \left(\tfrac{n-k-1}{\tau} + (k-1)\tau\right)\partial_\tau\widehat{\mathscr{W}_m}& \\ + \Big[-\mu_m\tfrac{1}{\tau^2} + (\tfrac{n^2}{4} - \gamma^2)\tfrac{1}{1-\tau^2} - (\lambda^2 + \tfrac{k^2}{4})\Big]\widehat{\mathscr{W}_m} &= 0\end{aligned}$$

for $\widehat{\mathscr{W}_m} = \widehat{\mathscr{W}_m}(\lambda,\omega)$. Fixed $m = 0,1,\ldots$, $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{S}^k$, we know that

$$\widehat{\mathscr{W}_m} = \widehat{w_m}\,\varphi_k^\lambda(\tau),$$

where $\varphi := \varphi_k^\lambda(\tau)$ is the solution to the following boundary value problem:

$$\begin{cases} (1-\tau^2)\partial_{\tau\tau}\varphi + \left(\frac{n-k-1}{\tau} + (k-1)\tau\right)\partial_\tau\varphi + \Big[-\mu_m \frac{1}{\tau^2} \\ \qquad +(\frac{n^2}{4}-\gamma^2)\frac{1}{1-\tau^2} - (\lambda^2 + \frac{k^2}{4})\Big]\varphi = 0, \\ \text{has the expansion (3.25) with } w \equiv 1 \text{ near the conformal infinity } \{\tau = 1\}, \\ \varphi \text{ is regular at } \tau = 0. \end{cases} \tag{3.52}$$

This is an ODE in τ that has only regular singular points, and can be explicitly solved. Indeed, from the first equation in (3.52) we obtain

$$\begin{aligned} \varphi(\tau) =& A(1-\tau^2)^{\frac{n}{4}-\frac{\gamma}{2}}\tau^{1-\frac{n}{2}+\frac{k}{2}+\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}}\ {}_2\mathrm{F}_1(a,b;c;\tau^2) \\ &+B(1-\tau^2)^{\frac{n}{4}-\frac{\gamma}{2}}\tau^{1-\frac{n}{2}-\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}}\ {}_2\mathrm{F}_1(\tilde{a},\tilde{b};\tilde{c};,\tau^2), \end{aligned} \tag{3.53}$$

for any real constants A, B, where

$$\begin{aligned} a &= \tfrac{-\gamma}{2} + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m} + i\tfrac{\lambda}{2}, \\ \tilde{a} &= \tfrac{-\gamma}{2} + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m} + i\tfrac{\lambda}{2}, \\ b &= \tfrac{-\gamma}{2} + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m} - i\tfrac{\lambda}{2}, \\ \tilde{b} &= \tfrac{-\gamma}{2} + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m} - i\tfrac{\lambda}{2}, \\ c &= 1 + \sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m}, \\ \tilde{c} &= 1 - \sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m}, \end{aligned}$$

and ${}_2\mathrm{F}_1$ denotes the standard hypergeometric function described in Lemma 3.10.1. Note that we can write λ instead of $|\lambda|$ in the arguments of the hypergeometric functions because $a = \overline{b}$, $\tilde{a} = \overline{\tilde{b}}$ and property (3.163).

The regularity at the origin $\tau = 0$ implies $B = 0$ in (3.53). Moreover, using (3.162) we can write

$$\varphi(\tau) = A\Big[\alpha(1-\tau^2)^{\frac{n}{4}-\frac{\gamma}{2}}\tau^{1-\frac{n}{2}+\frac{k}{2}+\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}}\ {}_2\mathrm{F}_1(a,b;a+b-c+1;1-\tau^2)$$
$$+\beta(1-\tau^2)^{\frac{n}{4}+\frac{\gamma}{2}}\tau^{1-\frac{n}{2}+\frac{k}{2}+\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}}\ {}_2\mathrm{F}_1(c-a,c-b;c-a-b+1;1-\tau^2)\Big],$$

where

$$\alpha = \frac{\Gamma\Big(1+\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}\Big)\Gamma(\gamma)}{\Gamma\Big(\frac{1}{2}+\frac{\gamma}{2}+\frac{1}{2}\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}-i\frac{\lambda}{2}\Big)\Gamma\Big(\frac{1}{2}+\frac{\gamma}{2}+\frac{1}{2}\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}+i\frac{\lambda}{2}\Big)},$$
$$\beta = \frac{\Gamma\Big(1+\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}\Big)\Gamma(-\gamma)}{\Gamma\Big(\frac{1}{2}-\frac{\gamma}{2}+\frac{1}{2}\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}+i\frac{\lambda}{2}\Big)\Gamma\Big(\frac{1}{2}-\frac{\gamma}{2}+\frac{1}{2}\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}-i\frac{\lambda}{2}\Big)}.$$

Note that our changes of variable give

$$\tau = \tanh(\sigma) = \frac{4-\rho^2}{4+\rho^2} = 1-\frac{1}{2}\rho^2+\cdots, \tag{3.54}$$

which yields, as $\rho \to 0$,

$$\varphi(\rho) \sim A\Big[\alpha\rho^{\frac{n}{2}-\gamma}+\beta\rho^{\frac{n}{2}+\gamma}+\ldots\Big].$$

Here we have used (3.161) for the hypergeometric function.

Looking at the expansion for the scattering solution (3.25) and the definition of the conformal fractional Laplacian (3.27), we must have

$$A = \alpha^{-1}, \quad \text{and} \quad \Theta_\gamma^m(\lambda) = d_\gamma\beta\alpha^{-1}. \tag{3.55}$$

Property (3.165) yields (3.48) and completes the proof of Theorem 3.3.5. □

3.3.4 Conjugation

We now go back to the discussion in Section 3.3.2 for an isolated singularity but we allow any subcritical power $p \in (\frac{N}{N-2\gamma}, \frac{N+2\gamma}{N-2\gamma})$ in the right hand side of (3.34);

this is,

$$(-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma} u^p \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{3.56}$$

This equation does not have good conformal properties. But, given $u \in \mathscr{C}^\infty(\mathbb{R}^N \setminus \{0\})$, we can consider

$$u = r^{-\frac{N-2\gamma}{2}} w = r^{-\frac{2\gamma}{p-1}} v, \quad r = e^{-t},$$

and define the conjugate operator

$$\tilde{P}_\gamma^{g_0}(v) := r^{-\frac{N-2\gamma}{2}+\frac{2\gamma}{p-1}} P_\gamma^{g_0}\big(r^{\frac{N-2\gamma}{2}-\frac{2\gamma}{p-1}} v\big) = r^{\frac{2\gamma}{p-1}p}(-\Delta_{\mathbb{R}^N})^\gamma u. \tag{3.57}$$

Then problem (3.56) is equivalent to

$$\tilde{P}_\gamma^{g_0}(v) = A_{N,p,\gamma} v^p \quad \text{in } \mathbb{R} \times \mathbb{S}^{N-1},$$

for some $v = v(t,\theta)$ smooth, $t \in \mathbb{R}$, $\theta \in \mathbb{S}^{N-1}$.

This $\tilde{P}_\gamma^{g_k}$ can then be seen from the perspective of scattering theory, and thus be characterized as a Dirichlet-to-Neumann operator for a special extension problem in Proposition 3.3.9, as inspired by the paper of Chang and González [52]. Note the Neumann condition (3.76), which differs from the one of the standard fractional Laplacian.

In the notation of Section 3.3.2, we set $X = \mathbb{H}^{N+1}$ with the metric given by (3.39). Its conformal infinity is $M = \mathbb{R} \times \mathbb{S}^{N-1}$ with the metric g_0. We would like to repeat the arguments of Section 3.3 for the conjugate operator $\tilde{P}_\gamma^{g_0}$. But this operator does not have good conformal properties. In any case, we are able to define a new eigenvalue problem that replaces (3.24)-(3.25).

More precisely, let $\mathscr{W}$ be the unique solution to the scattering problem (3.24)-(3.25) with Dirichlet data (3.26) set to w. We define the function $\mathscr{V}$ by the following relation

$$r^{Q_0}\mathscr{W} = \mathscr{V}, \quad Q_0 := -\tfrac{N-2\gamma}{2} + \tfrac{2\gamma}{p-1}, \tag{3.58}$$

Substituting into (3.24), the new scattering problem is

$$-\Delta_{g^+}\mathscr{V} + \left(\tfrac{4+\rho^2}{4\rho}\right)^{-2}\left[-2Q_0\,\partial_t V - Q_0^2\mathscr{V}\right] - \left(\tfrac{N^4}{2} - \gamma^2\right)\mathscr{V} = 0 \text{ in } X, \tag{3.59}$$

Moreover, if we set

$$\mathscr{V} = \rho^{\frac{N}{2}-\gamma}\mathscr{V}_1 + \rho^{\frac{N}{2}+\gamma}\mathscr{V}_2, \tag{3.60}$$

the Dirichlet condition (3.26) will turn into

$$\mathscr{V}_1|_{\rho=0} = v, \tag{3.61}$$

and the Neumann one (3.27) into

$$d_\gamma \mathscr{V}_2|_{\rho=0} = \tilde{P}_\gamma^{g_0}(v). \tag{3.62}$$

The following proposition is the analogous to Proposition 3.3.4 for $\tilde{P}_\gamma^{g_0}$:

Proposition 3.3.6. *Fix $\gamma \in (0, \frac{n}{2})$ and let $\tilde{P}_\gamma^m$ be the projection of the operator $\tilde{P}_\gamma^{g_0}$ over each eigenspace $\langle E_m \rangle$. Then*

$$\widehat{\tilde{P}_\gamma^m(v_m)} = \tilde{\Theta}_\gamma^m(\xi)\widehat{v_m},$$

and this Fourier symbol is given by

$$\begin{aligned}\tilde{\Theta}_\gamma^m(\xi) = 2^{2\gamma}\Gamma\Big(&\frac{1}{2}+\frac{\gamma}{2}+\frac{\sqrt{(\frac{N}{2}-1)^2+\mu_m}}{2}+\frac{1}{2}(Q_0+\xi i)\Big)\\ \cdot\Gamma\Big(&\frac{1}{2}+\frac{\gamma}{2}+\frac{\sqrt{(\frac{N}{2}-1)^2+\mu_m}}{2}-\frac{1}{2}(Q_0+\xi i)\Big)\\ \cdot\Gamma\Big(&\frac{1}{2}-\frac{\gamma}{2}+\frac{\sqrt{(\frac{N}{2}-1)^2+\mu_m}}{2}+\frac{1}{2}(Q_0+\xi i)\Big)^{-1}\\ \cdot\Gamma\Big(&\frac{1}{2}-\frac{\gamma}{2}+\frac{\sqrt{(\frac{N}{2}-1)^2+\mu_m}}{2}-\frac{1}{2}(Q_0+\xi i)\Big)^{-1}.\end{aligned} \tag{3.63}$$

Proof. We write the hyperbolic metric as (3.40) using the change of variable $\sigma = -\log(\rho/2)$. The scattering equation for $\mathscr{W}$ is (3.49) in the particular case $k=0$, $n=N$, and thus, we follow the arguments in the proof of Theorem 3.3.5. Set $r = e^{-t}$ and project over spherical harmonics as in (3.51), which yields

$$\partial_{\sigma\sigma}\mathscr{W}_m + R(\sigma)\partial_\sigma\mathscr{W}_m + (\cosh\sigma)^{-2}\partial_{tt}\mathscr{W}_m - (\sinh\sigma)^{-2}\mu_m\mathscr{W}_m + \big(\tfrac{N^2}{4}-\gamma^2\big)\mathscr{W}_m = 0 \tag{3.64}$$

for

$$R(\sigma) = \frac{\partial_\sigma(\cosh\sigma \sinh^{N-1}\sigma)}{\cosh\sigma \sinh^{N-1}\sigma}.$$

Recall the relation (3.58) and rewrite the extension equation (3.64) in terms of each projection $\mathscr{V}_m$ of $\mathscr{V}$. This gives

$$\begin{aligned}\partial_{\sigma\sigma}\mathscr{V}_m + R(\sigma)\partial_\sigma\mathscr{V}_m + (\cosh\sigma)^{-2}\{\partial_{tt}\mathscr{V}_m + 2Q_0\partial_t\mathscr{V}_m + Q_0^2\mathscr{V}_m\} \\ -(\sinh\sigma)^{-2}\mu_m\mathscr{V}_m + \left(\tfrac{N^2}{4} - \gamma^2\right)\mathscr{V}_m = 0.\end{aligned} \tag{3.65}$$

Now we use the change of variable (3.50), and take Fourier transform (3.37) with respect to the variable t. Then

$$\begin{aligned}(1-\tau^2)\partial_{\tau\tau}\widehat{\mathscr{V}_m} + \left(\tfrac{N-1}{\tau} - \tau\right)\partial_\tau\widehat{\mathscr{V}_m} + \Big[-\mu_m\tfrac{1}{\tau^2} \\ + \left(\tfrac{N^2}{4} - \gamma^2\right)\tfrac{1}{1-\tau^2} - (\xi - iQ_0)^2\Big]\widehat{\mathscr{V}_m} = 0.\end{aligned} \tag{3.66}$$

The Fourier symbol (3.63) is obtained following the same steps as in the proof of Theorem 3.3.5. Note that the only difference is the coefficient of $\widehat{\mathscr{V}_m}$ in (3.66).

We note here than an alternative way to calculate the symbol is by taking Fourier transform in relation $\tilde{P}_\gamma^{g_0}(v) = e^{-Q_0 t}P_\gamma^{g_0}(w)$, as follows:

$$\widehat{\tilde{P}_\gamma^m v_m(t)} = \widehat{P_\gamma^m w_m}(\xi - iQ_0) = \Theta_\gamma^m(\xi - iQ_0)\hat{w}_m(\xi - iQ_0) = \Theta_\gamma^m(\xi - iQ_0)\hat{v}_m(\xi).$$

Thus $\tilde{\Theta}_\gamma^m(\xi) = \Theta_\gamma^m(\xi - iQ_0)$, as desired. □

Now we turn to Proposition 3.3.2, and we show that there exists a very special defining function adapted to $\mathscr{V}$.

Lemma 3.3.7. *Let $\gamma \in (0,1)$. There exists a new defining function ρ^* such that, if we define the metric $\bar{g}^* = (\rho^*)^2 g^+$, then*

$$E_{\bar{g}^*}(\rho^*) = (\rho^*)^{-(1+2\gamma)}\left(\tfrac{4\rho}{4+\rho^2}\right)^2 Q_0^2,$$

where $E_{\bar{g}^*}(\rho^*)$ is defined in (3.30). *The precise expression for ρ^* is*

$$\rho^*(\rho) = \left[\alpha^{-1}\left(\tfrac{4\rho}{4+\rho^2}\right)^{\frac{N-2\gamma}{2}} {}_2\mathrm{F}_1\left(\tfrac{\gamma}{p-1}, \tfrac{N-2\gamma}{2} - \tfrac{\gamma}{p-1}; \tfrac{N}{2}; \left(\tfrac{4-\rho^2}{4+\rho^2}\right)^2\right)\right]^{2/(N-2\gamma)}, \tag{3.67}$$

$\rho \in (0,2)$*, where*

$$\alpha = \frac{\Gamma(\frac{N}{2})\Gamma(\gamma)}{\Gamma\big(\gamma + \frac{\gamma}{p-1}\big)\Gamma\big(\frac{N}{2} - \frac{\gamma}{p-1}\big)}.$$

The function ρ^* is strictly monotone with respect to ρ, and in particular, $\rho^* \in (0,\rho_0^*)$ for

$$\rho_0^* := \rho^*(2) = \alpha^{\frac{-2}{N-2\gamma}}. \tag{3.68}$$

Moreover, it has the asymptotic expansion near the conformal infinity

$$\rho^*(\rho) = \rho\left[1 + O(\rho^{2\gamma}) + O(\rho^2)\right]. \tag{3.69}$$

Proof. The proof follows Lemma 4.5 in [52]. The scattering equation (3.24) for $\mathscr{W}$ is modified to (3.59) when we substitute (3.58), but the additional terms do not affect the overall result. Then we know that, given $v \equiv 1$ on M, (3.59) has a unique solution $\mathscr{V}^0$ with the asymptotic expansion

$$\mathscr{V}^0 = \mathscr{V}_1^0 \rho^{\frac{N}{2}-\gamma} + \mathscr{V}_2^0 \rho^{\frac{N}{2}+\gamma}, \quad \mathscr{V}_1^0, \mathscr{V}_2^0 \in \mathscr{C}^\infty(\overline{X})$$

and Dirichlet condition on $M = \mathbb{R} \times \mathbb{S}^{N-1}$

$$\mathscr{V}_1^0|_{\rho=0} = 1. \tag{3.70}$$

Actually, from the proof of Proposition 3.3.6 and the modifications of Proposition 3.3.4 we do obtain an explicit formula for such $\mathscr{V}^0$. Indeed, from (3.53) and (3.55) for $k=0$, $n=N$, $m=0$, replacing $i\lambda$ by Q_0, we arrive at

$$\mathscr{V}^0(\tau) = \varphi(\tau) = \alpha^{-1}(1-\tau^2)^{\frac{N}{4}-\frac{\gamma}{2}}\ {}_2\mathrm{F}_1\Big(\tfrac{\gamma}{p-1}, \tfrac{N-2\gamma}{2} - \tfrac{\gamma}{p-1}; \tfrac{N}{2}; \tau^2\Big).$$

Finally, substitute in the relation between τ and ρ from (3.54) and set

$$\rho^*(\rho) = (\mathscr{V}^0)^{\frac{1}{N/2-\gamma}}(\rho). \tag{3.71}$$

Then, recalling (3.30), for this ρ^* we have

$$E_{\bar{g}^*}(\rho^*) = (\rho^*)^{-\frac{N}{2}-\gamma-1}\Big\{-\Delta_{g^+} - \big(\tfrac{N^2}{4} - \gamma^2\big)\Big\}(\mathscr{V}^0) = (\rho^*)^{-(1+2\gamma)}\big(\tfrac{4\rho}{4+\rho^2}\big)^2 Q_0^2,$$

as desired. Here we have used the scattering equation for $\mathscr{V}^0$ from (3.59) and the fact that $\mathscr{V}^0$ does not depend on the variable t.

To show monotonicity, denote $\eta := \left(\frac{4-\rho^2}{4+\rho^2}\right)^2$ for $\eta \in (0,1)$. It is enough to check that

$$f(\eta) := (1-\eta)^{\frac{N-2\gamma}{4}} \ {}_2\mathrm{F}_1\left(\tfrac{\gamma}{p-1}, \tfrac{N-2\gamma}{2} - \tfrac{\gamma}{p-1}; \tfrac{N}{2}; \eta\right)$$

is monotone with respect to η. From properties (3.163) and (3.164) of the Hypergeometric function and the possible values for p in (3.9) we can assert that

$$\begin{aligned}
&\frac{d}{d\eta} f(\eta) \\
&= \frac{d}{d\eta}\left((1-\eta)^{-\frac{N-2\gamma}{4}+\frac{\gamma}{p-1}}(1-\eta)^{\frac{N-2\gamma}{2}-\frac{\gamma}{p-1}} \ {}_2\mathrm{F}_1\left(\tfrac{N-2\gamma}{2} - \tfrac{\gamma}{p-1}, \tfrac{\gamma}{p-1}; \tfrac{N}{2}; \eta\right)\right) \\
&= \left(\tfrac{N-2\gamma}{4} - \tfrac{\gamma}{p-1}\right)(1-\eta)^{-\frac{N-2\gamma}{4}+\frac{\gamma}{p-1}-1}(1-\eta)^{\frac{N-2\gamma}{2}-\frac{\gamma}{p-1}} \\
&\qquad \cdot \ {}_2\mathrm{F}_1\left(\tfrac{N-2\gamma}{2} - \tfrac{\gamma}{p-1}, \tfrac{\gamma}{p-1}; \tfrac{N}{2}; \eta\right) \\
&\quad - \tfrac{2}{N}\left(\tfrac{N-2\gamma}{2} - \tfrac{\gamma}{p-1}\right)\left(\tfrac{N}{2} - \tfrac{\gamma}{p-1}\right)(1-\eta)^{\frac{N-2\gamma}{2}-\frac{\gamma}{p-1}-1} \\
&\qquad {}_2\mathrm{F}_1\left(\tfrac{N-2\gamma}{2} - \tfrac{\gamma}{p-1}+1, \tfrac{\gamma}{p-1}; \tfrac{N}{2}+1; \eta\right) \\
&< 0.
\end{aligned}$$

□

Remark 3.3.8. For the Neumann condition, note that, by construction,

$$\tilde{P}_\gamma^{g_0}(1) = d_\gamma \mathscr{V}_2^0|_{\rho=0}, \tag{3.72}$$

while from (3.57) and the definition of $A_{N,p,\gamma}$ from (3.10),

$$\tilde{P}_\gamma^{g_0}(1) = r^{\frac{2\gamma}{p-1}p}(-\Delta_{\mathbb{R}^N})^\gamma (r^{-\frac{2\gamma}{p-1}}) = A_{N,p,\gamma}.$$

The last result in this section shows that the scattering problem for $\mathscr{V}$ (3.59) can be transformed into a new extension problem as in Proposition 3.3.2, and whose Dirichlet-to-Neumann operator is precisely $\tilde{P}_\gamma^{g_0}$. For this we will introduce the new metric on $\mathbb{R}^N \setminus \{0\}$

$$\bar{g}^* = (\rho^*)^2 g^+, \tag{3.73}$$

where ρ^* is the defining function defined in (3.67), and let us denote

$$V^* = (\rho^*)^{-(N/2-\gamma)}\mathscr{V}. \tag{3.74}$$

Proposition 3.3.9. *Let v be a smooth function on $M = \mathbb{R}\times\mathbb{S}^{N-1}$. The extension problem*

$$\begin{cases} -\operatorname{div}_{\bar{g}^*}((\rho^*)^{1-2\gamma}\nabla_{\bar{g}^*}V^*) - (\rho^*)^{-(1+2\gamma)}\left(\frac{4\rho}{4+\rho^2}\right)^2 2Q_0\,\partial_t V^* = 0 & \text{in } (X,\bar{g}^*), \\ V^*|_{\rho=0} = v & \text{on } (M,g_0), \end{cases} \tag{3.75}$$

has a unique solution V^*. Moreover, for its Neumann data,

$$\tilde{P}^{g_0}_\gamma(v) = -\tilde{d}_\gamma \lim_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\partial_{\rho^*}(V^*) + A_{N,p,\gamma}v. \tag{3.76}$$

Proof. The original scattering equation (3.24)-(3.25) was rewritten in terms of $\mathscr{V}$ (recall (3.58)) as (3.59)-(3.60) with Dirichlet condition $\mathscr{V}_1|_{\rho=0} = v$. Let us rewrite this equation into the more familiar form of Proposition 3.3.2. We follow the arguments in [52]; the difference comes from some additional terms that appear when changing to $\mathscr{V}$.

First use the definition of the classical conformal Laplacian for g^+ (that has constant scalar curvature $R_{g^+} = -N(N+1)$),

$$P_1^{g^+} = -\Delta_{g^+} - \tfrac{N^2-1}{4},$$

and the conformal property of this operator (3.28) to assure that

$$P_1^{g^+}(\mathscr{V}) = (\rho^*)^{\frac{N+3}{2}} P_1^{\bar{g}}((\rho^*)^{-\frac{N-1}{2}}\mathscr{V}).$$

Using (3.74) we can rewrite equation (3.59) in terms of V^* as

$$P_1^{\bar{g}}((\rho^*)^{\frac{1-2\gamma}{2}}V^*) + (\rho^*)^{\frac{-3-2\gamma}{2}}\left\{\left(\tfrac{4+\rho^2}{4\rho}\right)^{-2}\left(-2Q_0\,\partial_t V^* - Q_0^2 V^*\right) + \left(\gamma^2 - \tfrac{1}{4}\right)V^*\right\} = 0$$

or, equivalently, using that for $\rho := \rho^{\frac{1-2\gamma}{2}}$,

$$\rho \Delta_{\bar g^*}(\rho V) = \mathrm{div}_{\bar g^*}(\rho^2 \nabla_{\bar g^*} V) + \rho V \Delta_{\bar g^*}(\rho),$$

we have

$$\begin{aligned} &- \mathrm{div}_{\bar g^*}((\rho^*)^{1-2\gamma} \nabla_{\bar g^*} V^*) + E_{\bar g^*}(\rho^*) V^* \\ &\qquad + (\rho^*)^{-(1+2\gamma)} \left(\tfrac{4+\rho^2}{2\rho}\right)^{-2} \left(-2Q_0\, \partial_t V^* - Q_0^2 V^*\right) = 0, \end{aligned}$$

with $E_{\bar g^*}(\rho^*)$ defined as in (3.30). Finally, note that the defining function ρ^* was chosen as in Lemma 3.3.7. This yields (3.75).

For the boundary conditions, let us recall the asymptotics (3.69). The Dirichlet condition follows directly from (3.26) and the asymptotics. For the Neumann condition, we recall the definition of ρ^* from (3.71), so

$$V^* = (\rho^*)^{-\frac{N}{2}+\gamma} \mathscr{V} = \frac{\mathscr{V}}{\mathscr{V}^0} = \frac{\mathscr{V}_1 + \rho^{2\gamma} \mathscr{V}_2}{\mathscr{V}_1^0 + \rho^{2\gamma} \mathscr{V}_2^0},$$

and thus

$$-\tilde d_\gamma \lim_{\rho \to 0} \rho^{1-2\gamma} \partial_\rho V^* = d_\gamma \left(\mathscr{V}_2 \mathscr{V}_1^0 - \mathscr{V}_1 \mathscr{V}_2^0\right)\big|_{\rho=0} = \tilde P_\gamma^{g_0} v - A_{N,p,\gamma} v,$$

where we have used (3.61) and (3.62) for $\mathscr{V}$, and (3.70) and (3.72) for $\mathscr{V}^0$. This completes the proof of the Proposition. □

3.4 New ODE methods for non-local equations

In this section we use the conformal properties developed in the previous section to study positive singular solutions to equation

$$(-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma} u^p \text{ in } \mathbb{R}^N \setminus \{0\}. \tag{3.77}$$

The first idea is, in the notation of Section 3.3.4, to set $v = r^{\frac{2\gamma}{p-1}} u$ and rewrite this equation as

$$\tilde P_\gamma^{g_0}(v) = A_{N,p,\gamma} v^p, \quad \text{in } \mathbb{R} \times \mathbb{S}^{N-1}, \tag{3.78}$$

and to consider the projection over spherical harmonics in $\mathbb{S}^{N-1}$,

$$\tilde{P}_\gamma^m(v_m) = A_{N,p,\gamma}(v_m)^p, \quad \text{for } v = v(t),$$

While in Proposition 3.3.6 we calculated the Fourier symbol for $\tilde{P}_\gamma^m$, now we will write it as an integro-differential operator for a well behaved convolution kernel. The advantage of this formulation is that immediately yields regularity for v_m as in [65].

Now we look at the $m = 0$ projection, which corresponds to finding radially symmetric singular solutions to (3.77). This is a non-local ODE for $u = u(r)$. In the second part of the section we define a suitable Hamiltonian quantity in conformal coordinates in the spirit a classical second order ODE.

3.4.1 The kernel

We consider first the projection $m = 0$. Following the argument in [65], one can use polar coordinates to rewrite $\tilde{P}_\gamma^0$ as an integro-differential operator with a new convolution kernel. Indeed, polar coordinates $x = (r, \theta)$ and $\bar{x} = (\bar{r}, \bar{\theta})$ in the definition of the fractional Laplacian (3.2) give

$$(-\Delta_{\mathbb{R}^N})^\gamma u(x) = k_{N,\gamma} P.V. \int_0^\infty \int_{\mathbb{S}^{N-1}} \frac{r^{-\frac{2\gamma}{p-1}} v(r) - \bar{r}^{-\frac{2\gamma}{p-1}} v(\bar{r})}{|r^2 + \bar{r}^2 + 2r\bar{r}\langle \theta, \bar{\theta} \rangle|^{\frac{N+2\gamma}{2}}} \bar{r}^{N-1} \, d\bar{r} \, d\bar{\theta}.$$

After the substitutions $\bar{r} = rs$ and $v(r) = (1 - s^{-\frac{2\gamma}{p-1}}) v(r) + s^{-\frac{2\gamma}{p-1}} v(r)$, and recalling the definition for $\tilde{P}_\gamma^0$ from (3.57) we have

$$\tilde{P}_\gamma^0(v) = k_{N,\gamma} P.V. \int_0^\infty \int_{\mathbb{S}^{N-1}} \frac{s^{-\frac{2\gamma}{p-1}+N-1}(v(r) - v(rs))}{|1 + s^2 - 2s\langle \theta, \bar{\theta} \rangle|^{\frac{N+2\gamma}{2}}} \, ds \, d\bar{\theta} + Cv(r),$$

where

$$C = k_{N,\gamma} P.V. \int_0^\infty \int_{\mathbb{S}^{N-1}} \frac{(1 - s^{-\frac{2\gamma}{p-1}}) s^{N-1}}{|1 + s^2 - 2s\langle \theta, \bar{\theta} \rangle|^{\frac{N+2\gamma}{2}}} \, ds \, d\bar{\theta}.$$

Using the fact that $v \equiv 1$ is a solution, one gets that $C = A_{N,p,\gamma}$. Finally, the change of variables $r = e^{-t}$, $\bar{r} = e^{-t'}$ yields

$$\tilde{P}_\gamma^0(v)(t) = P.V. \int_{\mathbb{R}} \tilde{\mathscr{K}_0}(t-t')[v(t)-v(t')]\,dt' + A_{N,p,\gamma}v(t) \tag{3.79}$$

for the convolution kernel

$$\begin{aligned}\tilde{\mathscr{K}_0}(t) &= \int_{\mathbb{S}^{N-1}} \frac{k_{N,\gamma}e^{-(\frac{2\gamma}{p-1}-N)t}}{|1+e^{2t}-2e^t\langle\theta,\bar{\theta}\rangle|^{\frac{N+2\gamma}{2}}}\,d\bar{\theta}\\ &= c\,e^{-(\frac{2\gamma}{p-1}-\frac{N-2\gamma}{2})t}\int_0^\pi \frac{(\sin\phi_1)^{N-2}}{(\cosh t-\cos\phi_1)^{\frac{N+2\gamma}{2}}}\,d\phi_1,\end{aligned}$$

where ϕ_1 is the angle between θ and $\bar{\theta}$ in spherical coordinates, and c is a positive constant that only depends on N and γ. From here we have the explicit expression

$$\tilde{\mathscr{K}_0}(t) = c\,e^{-(\frac{2\gamma p}{p-1})t}\ {}_2\mathrm{F}_1\left(\tfrac{N+2\gamma}{2}, 1+\gamma; \tfrac{N}{2}; e^{-2t}\right), \tag{3.80}$$

for a different constant c.

As in [65], one can calculate its asymptotic behavior, and we refer to this paper for details:

Lemma 3.4.1. *The kernel $\tilde{\mathscr{K}_0}(t)$ is decaying as $t \to \pm\infty$. More precisely,*

$$\tilde{\mathscr{K}_0}(t) \sim \begin{cases} |t|^{-1-2\gamma} & \text{as } |t| \to 0,\\ e^{-(N-\frac{2\gamma}{p-1})|t|} & \text{as } t \to -\infty,\\ e^{-\frac{2p\gamma}{p-1}|t|} & \text{as } t \to +\infty.\end{cases}$$

The main result in this section is that one obtains a formula analogous to (3.79) for any projection $\tilde{P}_\gamma^m$. However, we have not been able to use the previous argument and instead, we develop a new approach using conformal geometry and the special defining function ρ^* from Proposition 3.3.2.

Set $Q_0 = \frac{2\gamma}{p-1} - \frac{N-2\gamma}{2}$. In the notation of Proposition 3.3.6 we have:

Proposition 3.4.2. *For the m-th projection of the operator $\tilde{P}_\gamma^{g_0}$ we have the expression*

$$\tilde{P}_\gamma^m(v_m)(t) = \int_{\mathbb{R}} \tilde{\mathscr{K}}_m(t-t')[v_m(t)-v_m(t')]\,dt' + A_{N,p,\gamma} v_m(t),$$

for a convolution kernel $\tilde{\mathscr{K}}_m$ on $\mathbb{R}$ with the asymptotic behavior

$$\tilde{\mathscr{K}}_m(t) \sim \begin{cases} |t|^{-1-2\gamma} & \text{as } |t|\to 0, \\ e^{-\left(1+\gamma+\sqrt{(\frac{N-2}{2})^2+\mu_m}+Q_0\right)t} & \text{as } t\to+\infty, \\ e^{\left(1+\gamma+\sqrt{(\frac{N-2}{2})^2+\mu_m}-Q_0\right)t} & \text{as } t\to-\infty. \end{cases}$$

Proof. We first consider the case that $p=\frac{N+2\gamma}{N-2\gamma}$ so that $Q_0=0$, and look at the operator $P_\gamma^{g_0}(w)$ from Proposition 3.3.4. Let ρ^* be the new defining function from Proposition 3.3.2 and write a new extension problem for w in the corresponding metric $\bar{g}^*$. In this particular case, we can use (3.67) to write

$$\rho^*(\rho) = \left[\alpha^{-1}\left(\tfrac{4\rho}{4+\rho^2}\right)^{\frac{N-2\gamma}{2}} {}_2\mathrm{F}_1\left(\tfrac{N-2\gamma}{4},\tfrac{N-2\gamma}{4},\tfrac{N}{2},\left(\tfrac{4-\rho^2}{4+\rho^2}\right)^2\right)\right]^{\frac{2}{N-2\gamma}}, \quad \alpha = \frac{\Gamma(\frac{N}{2})\Gamma(\gamma)}{\Gamma(\frac{N}{4}+\frac{\gamma}{2})^2}.$$

The extension problem for $\bar{g}^*$ is

$$\begin{cases} -\operatorname{div}_{\bar{g}^*}((\rho^*)^{1-2\gamma}\nabla_{\bar{g}^*}W^*) = 0 & \text{in } (X,\bar{g}^*), \\ W^*|_{\rho=0} = w & \text{on } (M,g_0); \end{cases}$$

notice that it does not have a zero-th order term. Moreover, for the Neumann data,

$$P_\gamma^{g_0}(w) = -\tilde{d}_\gamma \lim_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\partial_{\rho^*}(W^*) + \Lambda_{N,\gamma} w.$$

From the proof of Proposition 3.3.2 we know that $W^*=(\rho^*)^{-(N/2-\gamma)}\mathscr{W}$, where $\mathscr{W}$ is the solution to (3.41). Taking the projection over spherical harmonics, and arguing as in the proof of Proposition 3.3.4, we have that $\widehat{\mathscr{W}_m}(\tau,\xi)=\widehat{w_m}(\xi)\,\varphi(\tau)$, and $\varphi=\varphi_\xi^{(m)}$ is given in (3.42). Let us undo all the changes of variable, but let us keep the notation $\varphi(\rho^*)=\varphi_\xi^{(m)}(\tau)$.

Taking the inverse Fourier transform, we obtain a Poisson formula

$$W_m^*(\rho^*,t) = \int_{\mathbb{R}} \mathscr{P}_m(\rho^*,t-t')w_m(t')\,dt',$$

where

$$\mathscr{P}_m(\rho^*,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} (\rho^*)^{-(N/2-\gamma)}\varphi(\rho^*)e^{i\xi t}\,d\xi.$$

Note that, by construction, $\int_{\mathbb{R}} \mathscr{P}_m(\rho^*,t)\,dt = 1$ for all ρ^*. Now we calculate

$$\begin{aligned}\lim_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\partial_{\rho^*}(W_m^*) &= \lim_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\frac{W_m^*(\rho^*,t)-W_m^*(0,t)}{\rho^*}\\ &= \lim_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\int_{\mathbb{R}}\frac{\mathscr{P}_m(\rho^*,t-t')}{\rho^*}[w_m(t')-w_m(t)]\,dt'.\end{aligned}$$

This implies that

$$P_\gamma^m(w_m)(t) = \int_{\mathbb{R}} \mathscr{K}_m(t-t')[w_m(t)-w_m(t')]\,dt' + \Lambda_{N,\gamma}w_m(t), \tag{3.81}$$

where the convolution kernel is defined as

$$\mathscr{K}_m(t) = \tilde{d}_\gamma \lim_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\frac{\mathscr{P}_m(\rho^*,t)}{\rho^*}.$$

If we calculate this limit, the precise expression for φ from (3.42) yields that

$$\mathscr{K}_m(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\Theta_\gamma^m(\xi)-\Lambda_{N,\gamma})e^{i\xi t}\,d\xi, \quad \mathscr{K}_m(-t) = \mathscr{K}_m(t).$$

which, of course, agrees with Proposition 3.3.4.

The asymptotic behavior for the kernel follows from the arguments in Section 3.6, for instance. In particular, the limit as $t\to 0$ is an easy calculation since Stirling's formula implies that $\Theta_\gamma^m(\xi)\sim|\xi|^{2\gamma}$ as $\xi\to\infty$. For the limit as $|t|\to\infty$ we use that the first pole of the symbol happens at $\pm i(1+\gamma+\sqrt{(\frac{N-2}{2})^2+\mu_m})$ so it extends analytically to a strip that contains the real axis. We have:

$$\mathscr{K}_m(t)\sim\begin{cases}|t|^{-1-2\gamma} & \text{as } |t|\to 0,\\ e^{-\left(1+\gamma+\sqrt{(\frac{N-2}{2})^2+\mu_m}\right)|t|} & \text{as } t\to\pm\infty.\end{cases}$$

Now we move on to $\tilde{P}_\gamma^{g_0}(v)$, whose symbol is calculated in Proposition 3.3.6. Recall that under the change $w(t) = r^{-Q_0}v(t)$, we have

$$\tilde{P}_\gamma^{g_0}(v) = e^{-Q_0 t} P_\gamma^{g_0}(e^{Q_0 t} v).$$

From (3.81), if we split $e^{Q_0 t} v_m(t) = (e^{Q_0 t} - e^{Q_0 t'})v_m(t) + e^{Q_0 t'} v_m(t)$, then

$$\tilde{P}_\gamma^{g_0}(v)(t) = Cv(t) + \int_{\mathbb{R}} \tilde{\mathscr{K}}_m(t-t')(v_m(t) - v_m(t'))\, dt'$$

for the kernel

$$\tilde{\mathscr{K}}_m(t) = \mathscr{K}_m(t)e^{-Q_0 t} = \frac{1}{\sqrt{2\pi}} e^{-Q_0 t} \int_{\mathbb{R}} (\Theta_\gamma^m(\xi) - \Lambda_{N,\gamma}) e^{i\xi t}\, d\xi,$$

and the constant

$$C = \Lambda_{N,\gamma} + \int_{\mathbb{R}} \mathscr{K}_m(t-t')(1 - e^{Q_0(t'-t)})\, dt'.$$

We have not attempted a direct calculation for the constant C. Instead, by noting that $v \equiv 1$ is an exact solution to the equation $\tilde{P}_\gamma^{g_0}(v) = A_{N,p,\gamma} v^p$, we have that $C = A_{N,p,\gamma}$, and this completes the proof of the proposition. $\square$

3.4.2 The Hamiltonian along trajectories

Now we concentrate on positive radial solutions to (3.78). These satisfy

$$\tilde{P}_\gamma^0(v) = A_{N,p,\gamma} v^p, \quad v = v(t). \tag{3.82}$$

We prove the existence of a Hamiltonian type quantity for (3.82), decreasing along trajectories when p is in the subcritical range, while this Hamiltonian remains constant in t for critical p. Monotonicity formulas for non-local equations in the form of a Hamiltonian have been known for some time ([36, 38, 98]). Our main innovation is that our formula (3.83) gives a precise analogue of the ODE local case (see Proposition 1 in [132], and the notes [163]), and hints what the phase portrait for v should be in the non-local setting. We hope to return to this problem elsewhere.

Theorem 3.4.3. *Fix $\gamma \in (0,1)$ and $p \in (\frac{N}{N-2\gamma}, \frac{N+2\gamma}{N-2\gamma})$. Let $v = v(t)$ be a solution to* (3.82) *and set V^* its extension from Proposition 3.3.9. Then, the Hamiltonian quantity*

$$\begin{aligned} H^*_\gamma(t) =& \frac{A_{N,p,\gamma}}{\tilde{d}_\gamma}\left(-\frac{1}{2}v^2 + \frac{1}{p+1}v^{p+1}\right) \\ &+ \frac{1}{2}\int_0^{\rho_0^*} (\rho^*)^{1-2\gamma}\left\{-e_1^*(\partial_{\rho^*}V^*)^2 + e_2^*(\partial_t V^*)^2\right\} d\rho^* \\ =&: H_1(t) + H_2(t) \end{aligned} \tag{3.83}$$

is decreasing with respect to t. In addition, if $p = \frac{N+2\gamma}{N-2\gamma}$, then $H^*_\gamma(t)$ is constant along trajectories.

Here we write, using Lemma 3.3.7, ρ as a function of ρ^*, and

$$\begin{aligned} e^* &= \left(\tfrac{\rho^*}{\rho}\right)^2\left(1+\tfrac{\rho^2}{4}\right)\left(1-\tfrac{\rho^2}{4}\right)^{N-1}, \\ e_1^* &= \left(\tfrac{\rho^*}{\rho}\right)^{-2} e^*, \\ e_2^* &= \left(\tfrac{\rho^*}{\rho}\right)^{-2}\left(1+\tfrac{\rho^2}{4}\right)^{-2} e^*. \end{aligned} \tag{3.84}$$

The constants $A_{N,p,\gamma}$ and $\tilde{d}_\gamma$ are given in (3.10) *and* (3.31)*, respectively.*

Proof. In the notation of Proposition 3.3.9, let v be a function on $M = \mathbb{R}\times\mathbb{S}^{N-1}$ only depending on the variable $t \in \mathbb{R}$, and let V^* be the corresponding solution to the extension problem (3.75). Then $V^* = V^*(\rho, t)$. Use that

$$\begin{aligned} \operatorname{div}_{\bar{g}^*}((\rho^*)^{1-2\gamma}\nabla_{\bar{g}^*}V^*) =& \frac{1}{e^*}\partial_{\rho^*}\left(e^*(\rho^*)^{-(1+2\gamma)}\rho^2\partial_{\rho^*}V^*\right) \\ &+ (\rho^*)^{1-2\gamma}\left(\tfrac{\rho^*}{\rho}\right)^{-2}\left(1+\tfrac{\rho^2}{4}\right)^{-2}\partial_{tt}V^*, \end{aligned}$$

where $e^* = |\sqrt{\bar{g}^*}|$ is given in (3.84), so equation (3.75) reads

$$\begin{aligned} -\partial_{\rho^*}\left(e^*\rho^2(\rho^*)^{-(1+2\gamma)}\partial_{\rho^*}V^*\right) - (\rho^*)^{1-2\gamma}e^*\left(\tfrac{\rho^*}{\rho}\right)^{-2}\left(1+\tfrac{\rho^2}{4}\right)^{-2}\partial_{tt}V^* \\ -(\rho^*)^{-(1+2\gamma)}e^*\left(\tfrac{4+\rho^2}{4\rho}\right)^{-2} 2\left(-\tfrac{N-2\gamma}{2} + \tfrac{2\gamma}{p-1}\right)\partial_t V^* = 0. \end{aligned}$$

We follow the same steps as in [66]: multiply this equation by $\partial_t V^*$ and integrate with respect to $\rho^* \in (0, \rho_0^*)$, where ρ_0^* is given in (3.68). Using integration by parts in the first term, the regularity of the function V^* at ρ_0^*, and the fact that $\frac{1}{2}\partial_t\left[(\partial_t V^*)^2\right] = \partial_{tt}V^*\partial_t V^*$ and $\frac{1}{2}\partial_t\left[(\partial_{\rho^*}V^*)^2\right] = \partial_{t\rho^*}(V^*)\partial_{\rho^*}V^*$, it holds

$$\begin{aligned}
&\lim_{\rho^*\to 0}\left(\partial_t(V^*)e^*(\rho^*)^{-(1+2\gamma)}\rho^2\partial_{\rho^*}V^*\right)\\
&+\int_0^{\rho_0^*}\left[\tfrac{1}{2}e^*(\rho^*)^{-(1+2\gamma)}\rho^2\partial_t\left[(\partial_{\rho^*}V^*)^2\right]\right]d\rho^*\\
&-\int_0^{\rho_0^*}\left[\tfrac{1}{2}(\rho^*)^{1-2\gamma}e^*\left(\tfrac{\rho}{\rho^*}\right)^2\left(1+\tfrac{\rho^2}{4}\right)^{-2}\partial_t\left[(\partial_t V^*)^2\right]\right]d\rho^*\\
&-\int_0^{\rho_0^*}\left[(\rho^*)^{-(1+2\gamma)}e^*\left(\tfrac{4\rho}{4+\rho^2}\right)^2 2\left(-\tfrac{N-2\gamma}{2}+\tfrac{2\gamma}{p-1}\right)[\partial_t V^*]^2\right]d\rho^*\\
&=0.
\end{aligned}$$

But, for the limit as $\rho^*\to 0$, we may use (3.76) and (3.82) to obtain

$$\begin{aligned}
&\tilde{d}_\gamma\lim_{\rho^*\to 0}\left((\rho^*)^{-(1+2\gamma)}\rho^2 e^*\partial_t V^*\partial_{\rho^*}V^*\right)\\
&=\left[-\tilde{P}_\gamma^{g_0}v + A_{N,p,\gamma}v\right]\partial_t v = A_{N,p,\gamma}(v-v^p)\partial_t v\\
&=A_{N,p,\gamma}\partial_t\left(\tfrac{1}{2}v^2-\tfrac{1}{p+1}v^{p+1}\right).
\end{aligned}$$

Then, for $H(t)$ defined as in (3.83), we have

$$\begin{aligned}
&\partial_t\left[H(t)\right]\\
&=-2\int_0^{\rho_0^*}\left[(\rho^*)^{1-2\gamma}e^*\left(\tfrac{\rho}{\rho^*}\right)^{-2}\left(1+\tfrac{\rho^2}{4}\right)^{-2}\left(-\tfrac{N-2\gamma}{2}+\tfrac{2\gamma}{p-1}\right)[\partial_t V^*]^2\right]d\rho^*\\
&\leq 0,
\end{aligned}$$

which proves the result. $\square$

3.5 The approximate solution

3.5.1 Function spaces

In this section we define the weighted Hölder space $\mathscr{C}^{2,\alpha}_{\mu,\nu}(\mathbb{R}^n \setminus \Sigma)$ tailored for this problem, following the notations and definitions in Section 3 of [133]. Intuitively, these spaces consist of functions which are products of powers of the distance to Σ with functions whose Hölder norms are invariant under homothetic transformations centered at an arbitrary point on Σ.

Despite the non-local setting, the local Fermi coordinates are still in use around each component Σ_i of Σ. When Σ_i is a point, these are simply polar coordinates around it. In case Σ_i is a higher dimensional sub-manifold, let $\mathscr{T}^i_\sigma$ be the tubular neighbourhood of radius σ around Σ_i. It is well known that $\mathscr{T}^i_\sigma$ is a disk bundle over Σ_i; more precisely, it is diffeomorphic to the bundle of radius σ in the normal bundle $\mathscr{N}\Sigma_i$. The Fermi coordinates will be constructed as coordinates in the normal bundle transferred to $\mathscr{T}^i_\sigma$ via such diffeomorphism. Let r be the distance to Σ_i, which is well defined and smooth away from Σ_i for small σ. Let also y be a local coordinate system on Σ_i and θ the angular variable on the sphere in each normal space $\mathscr{N}_y\Sigma_i$. We denote by $B^{\mathscr{N}}_\sigma$ the ball of radius σ in $\mathscr{N}_y\Sigma_i$. Finally we let x denote the rectangular coordinate in these normal spaces, so that $r = |x|$, $\theta = \frac{x}{|x|}$.

Let u be a function in this tubular neighbourhood and define

$$\|u\|^{\mathscr{T}^i_\sigma}_{0,\alpha,0} = \sup_{z\in\mathscr{T}^i_\sigma} |u| + \sup_{z,\tilde{z}\in\mathscr{T}^i_\sigma} \frac{(r+\tilde{r})^\alpha |u(z)-u(\tilde{z})|}{|r-\tilde{r}|^\alpha + |y-\tilde{y}|^\alpha + (r+\tilde{r})^\alpha |\theta-\tilde{\theta}|^\alpha},$$

where $z, \tilde{z}$ are two points in $\mathscr{T}^i_\sigma$ and $(r,\theta,y), (\tilde{r},\tilde{\theta},\tilde{y})$ are their Fermi coordinates.

We fix a $R > 0$ be large enough such that $\Sigma \subset B_{\frac{R}{2}}(0)$ in $\mathbb{R}^n$. Hereafter the letter z is reserved to denote a point in $\mathbb{R}^n \setminus \Sigma$. For notational convenience let us also fix a positive function $\rho \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \Sigma)$ that is equal to the polar distance r in each $\mathscr{T}^i_\sigma$, and to $|z|$ in $\mathbb{R}^n \setminus B_R(0)$.

Definition 3.5.1. *The space* $\mathscr{C}_0^{l,\alpha}(\mathbb{R}^n \setminus \Sigma)$ *is defined to be the set of all* $u \in \mathscr{C}^{l,\alpha}(\mathbb{R}^n \setminus \Sigma)$ *for which the norm*

$$\|u\|_{l,\alpha,0} = \|u\|_{\mathscr{C}^{l,\alpha}(\Sigma^c_{\sigma/2})} + \sum_{i=1}^{K}\sum_{j=0}^{l} \|\nabla^j u\|_{\mathscr{C}^{0,\alpha}(\mathscr{T}^i_\sigma)}$$

is finite. Here $\Sigma^c_{\sigma/2} = \mathbb{R}^n \setminus \bigcup_{i=1}^K \mathscr{T}^i_{\sigma/2}$.

Let us define a weighted Hölder space for functions having different behaviors near Σ and at ∞. With $R > 0$ fixed, for any $\mu, \nu \in \mathbb{R}$ we set

$$\mathscr{C}^{l,\alpha}_{\mu}(B_R \setminus \Sigma) = \{u = \rho^\mu \bar{u} : \bar{u} \in \mathscr{C}_0^{l,\alpha}(B_R \setminus \Sigma)\},$$
$$\mathscr{C}^{l,\alpha}_{\nu}(\mathbb{R}^n \setminus B_R) = \{u = \rho^\nu \bar{u} : \bar{u} \in \mathscr{C}_0^{l,\alpha}(\mathbb{R}^n \setminus B_R)\},$$

and thus we can define:

Definition 3.5.2. *The space* $\mathscr{C}^{l,\alpha}_{\mu,\nu}(\mathbb{R}^n \setminus \Sigma)$ *consists of all functions u for which the norm*

$$\|u\|_{\mathscr{C}^{l,\alpha}_{\mu,\nu}} = \sup_{B_R \setminus \Sigma} \|\rho^{-\mu} u\|_{l,\alpha,0} + \sup_{\mathbb{R}^n \setminus B_R} \|\rho^{-\nu} u\|_{l,\alpha,0}$$

is finite. The spaces $\mathscr{C}^{l,\alpha}_{\mu,\nu}(\mathbb{R}^N \setminus \{0\})$ *and* $\mathscr{C}^{l,\alpha}_{\mu,\nu}(\mathbb{R}^n \setminus \mathbb{R}^k)$ *are defined similarly, in terms of the (global) Fermi coordinates* (r,θ) *or* (r,θ,y) *and the weights* r^μ, r^ν.

Remark 3.5.3. From the definition of $\mathscr{C}^{l,\alpha}_{\mu,\nu}$, functions in this space are allowed to blow up like ρ^μ near Σ_i and decay like ρ^ν at ∞. Moreover, near Σ_i, their derivatives with respect to up to l-fold products of the vector fields $r\partial_r, r\partial_y, \partial_\theta$ blow up no faster than ρ^μ while at ∞, their derivatives with respect to up to l-fold products of the vector fields $|z|\partial_i$ decay at least like ρ^ν.

Remark 3.5.4. As it is customary in the analysis of fractional order operators, we write many times, with some abuse of notation, $\mathscr{C}^{2\gamma+\alpha}_{\mu,\nu}$.

3.5.2 Approximate solution with isolated singularities

Let $\Sigma = \{q_1, \cdots, q_K\}$ be a prescribed set of singular points. In the next paragraphs we construct an approximate solution to

$$(-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma} u^p \text{ in } \mathbb{R}^N \setminus \Sigma,$$

and check that it is indeed a good approximation in certain weighted spaces.

Let u_1 be the fast decaying solution to (3.77) that we constructed in Proposition 3.2.1. Now consider the following rescaling

$$u_\varepsilon(x) = \varepsilon^{-\frac{2\gamma}{p-1}} u_1\left(\frac{x}{\varepsilon}\right) \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}. \tag{3.85}$$

Choose χ_d to be a smooth cut-off function such that $\chi_d = 1$ if $|x| \leq d$ and $\chi_d(x) = 0$ for $|x| \geq 2d$, where $d > 0$ is a positive constant such that $d < d_0 = \inf_{i \neq j}\{\text{dist}(q_i, q_j)/2\}$. Let $\bar{\varepsilon} = \{\varepsilon_1, \cdots, \varepsilon_K\}$ be a K-tuple of dilation parameters satisfying $c\varepsilon \leq \varepsilon_i \leq \varepsilon < 1$ for $i = 1, \ldots, K$. Now define our approximate solution by

$$\bar{u}_\varepsilon(x) = \sum_{i=1}^{K} \chi_d(x - q_i) u_{\varepsilon_i}(x - q_i).$$

Set also

$$f_\varepsilon := (-\Delta_x)^\gamma \bar{u}_\varepsilon - A_{N,p,\gamma} \bar{u}_\varepsilon^p. \tag{3.86}$$

For the rest of the section, we consider the spaces $\mathscr{C}^{0,\alpha}_{\tilde{\mu},\tilde{\nu}}$, where

$$-\frac{2\gamma}{p-1} < \tilde{\mu} < 2\gamma \quad \text{and} \quad -(n - 2\gamma) < \tilde{\nu}. \tag{3.87}$$

Lemma 3.5.5. *There exists a constant C, depending on $d, \tilde{\mu}, \tilde{\nu}$ only, such that*

$$\|f_\varepsilon\|_{\mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma, \tilde{\nu}-2\gamma}} \leq C \varepsilon^{N - \frac{2p\gamma}{p-1}}. \tag{3.88}$$

Proof. Using the definition of $(-\Delta)^\gamma$ in $\mathbb{R}^N$, one has

$$\begin{aligned}
&(-\Delta_x)^\gamma (\chi_i u_{\varepsilon_i})(x - q_i) \\
&= k_{N,\gamma} P.V. \int_{\mathbb{R}^N} \frac{\chi_i(x - q_i) u_{\varepsilon_i}(x - q_i) - \chi_i(\tilde{x} - q_i) u_{\varepsilon_i}(\tilde{x} - q_i)}{|x - \tilde{x}|^{N+2\gamma}} \, d\tilde{x} \\
&= \chi_i(x - q_i)(-\Delta_x)^\gamma u_{\varepsilon_i}(x - q_i) \\
&\quad + k_{N,\gamma} P.V. \int_{\mathbb{R}^N} \frac{(\chi_i(x - q_i) - \chi_i(\tilde{x} - q_i)) u_{\varepsilon_i}(\tilde{x} - q_i)}{|x - \tilde{x}|^{N+2\gamma}} \, d\tilde{x}
\end{aligned}$$

for each $i = 1, \dots, K$. Using the equation (3.11) satisfied by u_{ε_i} we have

$$
\begin{aligned}
f_\varepsilon(x) &= A_{N,p,\gamma} \sum_{i=1}^{K} (\chi_i - \chi_i^p) u_{\varepsilon_i}^p (x - q_i) \\
&\quad + k_{N,\gamma} \sum_{i=1}^{K} P.V. \int_{\mathbb{R}^N} \frac{(\chi_i(x - q_i) - \chi_i(\tilde{x} - q_i)) u_{\varepsilon_i}(\tilde{x} - q_i)}{|x - \tilde{x}|^{N+2\gamma}} \, d\tilde{x} \\
&=: I_1 + k_{N,\gamma} I_2.
\end{aligned}
$$

Let us look first at the term I_1. It vanishes unless $|x - q_i| \in [d, 2d]$ for some $i = 1, \dots, K$. But then, one knows from the asymptotic behaviour of u_{ε_i} that

$$
u_{\varepsilon_i}(x) = O\Big(\varepsilon_i^{-\frac{2\gamma}{p-1}} \Big| \frac{x - q_i}{\varepsilon_i} \Big|^{-(N-2\gamma)} \Big) = O(\varepsilon^{N-2\gamma-\frac{2\gamma}{p-1}}) |x - q_i|^{-(N-2\gamma)},
$$

so one has

$$
I_1(x) \le C \varepsilon^{N-2\gamma-\frac{2\gamma}{p-1}} \quad \text{if } |x - q_i| \in [d, 2d].
$$

For the second term $I_2 = I_2(x)$, we fix $i = 1, \dots, K$, and divide it into three cases: $x \in B_{d/2}(q_i)$, $x \in B_{2d}(q_i) \setminus B_{d/2}(q_i)$ and $x \in \mathbb{R}^N \setminus B_{2d}(q_i)$. In the first case, $x \in B_{d/2}(q_i)$, without loss of generality, assume that $q_i = 0$, so

$$
\begin{aligned}
I_2(x) &= P.V. \int_{\mathbb{R}^N} \frac{(\chi_i(x) - \chi_i(\tilde{x})) u_{\varepsilon_i}(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \, d\tilde{x} \\
&= P.V \Big[\int_{B_d(0)} \cdots + \int_{B_{2d} \setminus B_d(0)} \cdots + \int_{\mathbb{R}^N \setminus B_{2d}(0)} \cdots \Big] \\
&\lesssim \int_{\{d < |\tilde{x}| < 2d\}} \frac{u_{\varepsilon_i}(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma-2}} \, d\tilde{x} + \int_{\{|\tilde{x}| > 2d\}} \frac{u_{\varepsilon_i}(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \, d\tilde{x}.
\end{aligned}
$$

Hereafter "$\cdots$" carries its obvious meaning, replacing the previously written integrand. Using that $|x - \tilde{x}| \ge \frac{1}{2} |\tilde{x}|$ for $|\tilde{x}| > 2d$ when $|x| < \frac{d}{2}$ we easily estimate

$$
I_2(x) \le O(\varepsilon^{N-\frac{2p\gamma}{p-1}}) + \int_{2d}^{\infty} \frac{u_{\varepsilon_i}(r)}{r^{1+2\gamma}} \, dr = O(\varepsilon^{N-\frac{2p\gamma}{p-1}}).
$$

Next, if $x \in B_{2d}(q_i) \setminus B_{d/2}(q_i)$,

$$\begin{aligned}
I_2(x) &= P.V.\Big[\int_{B_{d/4}(q_i)} \cdots + \int_{B_{2d}(q_i)\setminus B_{d/4}(q_i)} \cdots + \int_{\mathbb{R}^N\setminus B_{2d}(q_i)} \cdots\Big] \\
&= O\Big(\int_0^{\frac{d}{4\varepsilon}} \varepsilon^{N-\frac{2p\gamma}{p-1}} u_1(\tilde{x})\, d\tilde{x}\Big) + O\Big(\int_{B_{2d}(q_i)\setminus B_{d/4}(q_i)} \frac{\varepsilon^{N-\frac{2p\gamma}{p-1}}}{|x-\tilde{x}|^{N+2\gamma-2}}\, d\tilde{x}\Big) \\
&\quad + O\Big(\int_{\mathbb{R}^N\setminus B_{2d}(q_i)} \frac{u_{\varepsilon_i}(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\, d\tilde{x}\Big) \\
&= O(\varepsilon^{N-\frac{2p\gamma}{p-1}}).
\end{aligned}$$

Finally, if $x \in \mathbb{R}^N \setminus B_{2d}(q_i)$,

$$\begin{aligned}
I_2(x) &= P.V.\Big[\int_{B_d(q_i)} \cdots + \int_{B_{2d}\setminus B_d(q_i)} \cdots + \int_{\mathbb{R}^N\setminus B_{2d}(q_i)} \cdots\Big] \\
&= O(\varepsilon^{N-\frac{2p\gamma}{p-1}} |x|^{-(N+2\gamma)}).
\end{aligned}$$

Combining all the estimates above we get a $\mathscr{C}^0_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}$ bound for a pair of weights satisfying (3.87). But passing to $\mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}$ is analogous and thus we obtain (3.88). $\square$

3.5.3 Approximate solution in general case

First note that our ODE argument for u_1 also yields a fast decaying positive solution to the general problem

$$(-\Delta_{\mathbb{R}^n})^\gamma u = A_{N,p,\gamma} u^p \text{ in } \mathbb{R}^n \setminus \mathbb{R}^k. \tag{3.89}$$

that is singular along $\mathbb{R}^k$. Recall that we have set $N = n-k$.

Indeed, define $\tilde{u}_1(x,y) := u_1(x)$, where $z = (x,y) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$, and use the Lemma below. For this reason, many times we will use indistinctly the notations $u_1(z)$ and $u_1(x)$. Moreover, after a straightforward rescaling, the constant $A_{N,p,\gamma}$ may be taken to be one.

Lemma 3.5.6. *If u is defined on $\mathbb{R}^N$ and we set $\tilde{u}(z) := u(x)$ in $\mathbb{R}^n$ in the notation above, then*

$$(-\Delta_{\mathbb{R}^n})^\gamma \tilde{u} = (-\Delta_{\mathbb{R}^N})^\gamma u.$$

Proof. We compute, first evaluating the y-integral,

$$\begin{aligned}(-\Delta_{\mathbb{R}^n})^\gamma \tilde{u}(z) &= k_{n,\gamma} P.V. \int_{\mathbb{R}^n} \frac{\tilde{u}(z)-\tilde{u}(\tilde{z})}{|z-\tilde{z}|^{n+2\gamma}}\, d\tilde{z} \\ &= k_{n,\gamma} P.V. \int_{\mathbb{R}^k}\int_{\mathbb{R}^N} \frac{u(x)-u(\tilde{x})}{[|x-\tilde{x}|^2+|y-\tilde{y}|^2]^{\frac{n+2\gamma}{2}}}\, d\tilde{x}\, d\tilde{y} \\ &= k_{n,\gamma} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\, d\tilde{x} \int_{\mathbb{R}^k} \frac{1}{(1+|\tilde{y}|^2)^{\frac{n+2\gamma}{2}}}\, d\tilde{y} \\ &= (-\Delta_{\mathbb{R}^N})^\gamma u(x).\end{aligned}$$

Here we have used

$$k_{n,\gamma} \int_{\mathbb{R}^k} \frac{1}{(1+|\tilde{y}|^2)^{\frac{n+2\gamma}{2}}}\, d\tilde{y} = k_{N,\gamma}. \tag{3.90}$$

(See Lemma A.1 and Corollary A.1 in [50]). □

Now we turn to the construction of an approximate solution for (3.3). Let Σ be a k-dimensional compact sub-manifold in $\mathbb{R}^n$. We shall use local Fermi coordinates around Σ, as defined in Section 3.5.1. Let $\mathscr{T}_\sigma$ be the tubular neighbourhood of radius σ around Σ. For a point $z \in \mathscr{T}_\sigma$, denote it by $z=(x,y) \in \mathscr{N}\Sigma \times \Sigma$ where $\mathscr{N}\Sigma$ is the normal bundle of Σ. Let B a ball in $\mathscr{N}\Sigma$. We identify $\mathscr{T}_\sigma$ with $B \times \Sigma$. In these coordinates, the Euclidean metric is written as (see, for instance, [137])

$$|dz|^2 = \begin{pmatrix} |dx|^2 & O(r) \\ O(r) & g_\Sigma + O(r) \end{pmatrix},$$

where $|dx|^2$ is the standard flat metric in B and g_Σ the metric in Σ. The volume form reduces to

$$dz = dx\sqrt{\det g_\Sigma} + O(r).$$

In the ball B we use standard polar coordinates $r>0$, $\theta \in \mathbb{S}^{N-1}$. In addition, near each $q \in \Sigma$, we will consider normal coordinates for g_Σ centered at q. A neighborhood of $\Sigma \ni q$ is then identified with a neighborhood of $\mathbb{R}^k \ni 0$ with the metric

$$g_\Sigma = |dy|^2 + O(|y|^2),$$

which yields the volume form

$$dz = dx\,dy\,(1+O(r)+O(|y|^2)). \tag{3.91}$$

Note that Σ is compact, so we can cover it by a finite number of small balls B.

As in the isolated singularity case, we define an approximate solution as follows:

$$\bar{u}_\varepsilon(x,y) = \chi_d(x)u_\varepsilon(x)$$

where χ_R is a cut-off function such that $\chi_d = 1$ if $|x| \le d$ and $\chi_d(x) = 0$ for $|x| \ge 2d$. In the following we always assume $d < \frac{\sigma}{2}$. Let

$$f_\varepsilon := (-\Delta_{\mathbb{R}^n})^\gamma \bar{u}_\varepsilon - \bar{u}_\varepsilon^p.$$

Lemma 3.5.7. *Assume, in addition to* (3.87)*, that* $-\frac{2\gamma}{p-1} < \tilde{\mu} < \min\{\gamma - \frac{2\gamma}{p-1}, \frac{1}{2} - \frac{2\gamma}{p-1}\}$*. Then there exists a positive constant C depending only on $d, \tilde{\mu}, \tilde{\nu}$ but independent of ε such that for $\varepsilon \ll 1$,*

$$\|f_\varepsilon\|_{\mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}} \le C\varepsilon^q, \tag{3.92}$$

where $q = \min\{\frac{(p-3)\gamma}{p-1} - \tilde{\mu}, \frac{1}{2} - \gamma + \frac{(p-3)\gamma}{p-1} - \tilde{\mu}, N - \frac{2p\gamma}{p-1}\} > 0.$

Proof. Let us fix a point $z = (x,y) \in \mathscr{T}_\sigma$, i.e. $|x| < \sigma$. By the definition of the fractional Laplacian,

$$\begin{aligned}(-\Delta_z)^\gamma \bar{u}_\varepsilon(z) &= k_{n,\gamma} P.V. \int_{\mathbb{R}^n} \frac{\bar{u}_\varepsilon(z) - \bar{u}_\varepsilon(\tilde{z})}{|z-\tilde{z}|^{n+2\gamma}}\, d\tilde{z} \\ &= k_{n,\gamma}\Big[P.V. \int_{\mathscr{T}_\sigma} \cdots + \int_{\mathscr{T}_\sigma^c} \cdots\Big] =: I_1 + I_2.\end{aligned}$$

Note that in this neighborhood we can write $\bar{u}_\varepsilon(z) := \bar{u}_\varepsilon(x)$.

For I_2, since $\bar{u}_\varepsilon(\tilde{x}) = 0$ when $\tilde{z} = (\tilde{x}, \tilde{y}) \in \mathscr{T}_\sigma^c$, one has

$$I_2 = k_{n,\gamma} \int_{\mathscr{T}_\sigma^c} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|z-\tilde{z}|^{n+2\gamma}}\, d\tilde{z} = \bar{u}_\varepsilon(x)\, k_{n,\gamma} \int_{\mathscr{T}_\sigma^c} \frac{1}{|z-\tilde{z}|^{n+2\gamma}}\, d\tilde{z} \le C\bar{u}_\varepsilon(x),$$

so $I_2 = O(1)\bar{u}_\varepsilon(x)$ (the precise constant depends on σ). Next, for I_1, use normal coordinates $\tilde{y}$ in Σ centered at y in a neighborhood $\{|y-\tilde{y}| < \sigma_1\}$ for some σ_1 small but fixed. The constants will also depend on this σ_1. We have

$$\begin{aligned} I_1 &= k_{n,\gamma} P.V. \int_{\mathscr{T}_\sigma} \frac{\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x})}{|z-\tilde{z}|^{n+2\gamma}}\, d\tilde{z} \\ &= k_{n,\gamma} P.V. \Big[\int_{\{|y-\tilde{y}|\le |x|^\beta\}\cap \mathscr{T}_\sigma} \cdots + \int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}\cap \mathscr{T}_\sigma} \cdots + \int_{\{|y-\tilde{y}|>\sigma_1\}\cap \mathscr{T}_\sigma} \cdots \Big] \\ &=: k_{n,\gamma}[I_{11}+I_{12}+I_{13}], \end{aligned}$$

where $\beta \in (0,1)$ is to be determined later. The main term will be I_{11}; let us calculate the other two. First, for I_{12} we recall the expansion of the volume form (3.91), and approximate $|z-\tilde{z}|^2 = |x-\tilde{x}|^2 + |y-\tilde{y}|^2$ and $d\tilde{z} = d\tilde{x}\, d\tilde{y}$ modulo lower order perturbations. Then

$$\begin{aligned} I_{12} &= \int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}\cap \mathscr{T}_\sigma} \frac{\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x})}{|z-\tilde{z}|^{n+2\gamma}}\, d\tilde{z} \\ &= \int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}} \int_{\{|x-\tilde{x}|\le |x|^\beta\}\cap \mathscr{T}_\sigma} \cdots + \int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}} \int_{\{|x-\tilde{x}| > |x|^\beta\}\cap \mathscr{T}_\sigma} \cdots \\ &\lesssim \int_{\{|x-\tilde{x}|\le |x|^\beta\}} (\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x})) \Big(\int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}} \frac{1}{|z-\tilde{z}|^{n+2\gamma}}\, d\tilde{y} \Big) d\tilde{x} \\ &\quad + \int_{\{|x-\tilde{x}| > |x|^\beta\}} \frac{\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}} \Big(\int_{\big\{|\hat{y}| > \frac{|x|^\beta}{|x-\tilde{x}|}\big\}} \frac{1}{(1+|\hat{y}|^2)^{\frac{n+2\gamma}{2}}}\, d\hat{y} \Big) d\tilde{x}. \end{aligned}$$

We estimate the above integrals in $d\tilde{y}$. For instance, for the first term, we have used that

$$\begin{aligned} \int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}} \frac{1}{|z-\tilde{z}|^{n+2\gamma}}\, d\tilde{y} &\le \int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}} \frac{1}{|y-\tilde{y}|^{n+2\gamma}}\, d\tilde{y} \\ &\lesssim \int_{\{|y| \ge |x|^\beta\}} \frac{1}{|y|^{n+2\gamma}}\, dy \\ &\lesssim \int_{|x|^\beta}^\infty \frac{r^{k-1}}{r^{n+2\gamma}}\, dr \lesssim |x|^{-\beta(N+2\gamma)}, \end{aligned}$$

which yields,

$$I_{12} \lesssim \int_{\{|x-\tilde{x}|\le |x|^\beta\}} (\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x}))|x|^{-\beta(N+2\gamma)}\, d\tilde{x} + \int_{\{|x-\tilde{x}| > |x|^\beta\}} \frac{\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\, d\tilde{x}.$$

Now, since $|x-\tilde{x}|>|x|^{\beta}$ implies that $|\tilde{x}|>c_0|x|^{\beta}$ and $|x-\tilde{x}|\sim|\tilde{x}|$ for some $c_0>0$ independent of $|x|$ small,

$$I_{12}\lesssim |x|^{-2\beta\gamma}\bar{u}_\varepsilon(x)+|x|^{-\beta(N+2\gamma)}\int_{\{|x-\tilde{x}|\leq|x|^{\beta}\}}\bar{u}_\varepsilon(\tilde{x})\,d\tilde{x}+\int_{\{|\tilde{x}|>c_0|x|^{\beta}\}}\frac{\bar{u}_\varepsilon(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\,d\tilde{x}$$
$$+\int_{\{|\tilde{x}|\geq c_0|x|^{\beta}\}}\frac{\bar{u}_\varepsilon(x)}{|\tilde{x}|^{N+2\gamma}}.$$

We conclude, using the definition of $\bar{u}_\varepsilon$ and the rescaling (3.85), that

$$I_{12}\lesssim |x|^{-2\beta\gamma}\bar{u}_\varepsilon(x)+\varepsilon^{N-\frac{2\gamma}{p-1}}|x|^{-\beta(N+2\gamma)}\int_{\left\{|\tilde{x}|\leq\frac{|x|^{\beta}}{\varepsilon}\right\}}u_1(\tilde{x})\,d\tilde{x}$$
$$+\varepsilon^{-\frac{2p\gamma}{p-1}}\int_{\left\{|\tilde{x}|>\frac{|x|^{\beta}}{\varepsilon}\right\}}\frac{u_1(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\,d\tilde{x}+|x|^{-2\gamma\beta}\bar{u}_\varepsilon(x).$$

For I_{13}, one has

$$I_{13}\leq C\int_{\mathscr{T}_\sigma}|\bar{u}_\varepsilon(x)|+|\bar{u}_\varepsilon(\tilde{x})|d\tilde{x}\leq C(\bar{u}_\varepsilon(x)+\varepsilon^{N-\frac{2p\gamma}{p-1}}(1+|x|)^{-(N-2\gamma)}).$$

We look now into the main term I_{11}, for which we need to be more precise,

$$\begin{aligned}
I_{11}&=P.V.\int_{\{|y-\tilde{y}|\leq|x|^{\beta}\}}\int_{\{|\tilde{x}|<\sigma\}}\frac{\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x})}{|z-\tilde{z}|^{n+2\gamma}}\,d\tilde{z}\\
&=P.V.\Big[\int_{\{|y-\tilde{y}|\leq|x|^{\beta}\}}\int_{\{|x-\tilde{x}|\leq|x|^{\beta}\}\cap\{|\tilde{x}|<\sigma\}}\cdots\\
&\qquad+\int_{\{|y-\tilde{y}|\leq|x|^{\beta}\}}\int_{\{|x-\tilde{x}|>|x|^{\beta}\}\cap\{|\tilde{x}|<\sigma\}}\cdots\Big]\\
&=:I_{111}+I_{112}.
\end{aligned}$$

Let us estimate these two integrals. First, since for $|x|$ small, $|x-\tilde{x}|<|x|^{\beta}$ implies that $|\tilde{x}|<\sigma$, we have

$$
\begin{aligned}
k_{n,\gamma} I_{111} &= k_{n,\gamma} P.V. \int_{\{|y-\tilde{y}|\le|x|^{\beta}\}} \int_{\{|x-\tilde{x}|\le|x|^{\beta}\}} \frac{\bar{u}_{\varepsilon}(x)-\bar{u}_{\varepsilon}(\tilde{x})}{|z-\tilde{z}|^{n+2\gamma}}\, d\tilde{z} \\
&= k_{n,\gamma} P.V. \int_{\{|y-\tilde{y}|\le|x|^{\beta}\}} \int_{\{|x-\tilde{x}|\le|x|^{\beta}\}} \frac{\bar{u}_{\varepsilon}(x)-\bar{u}_{\varepsilon}(\tilde{x})}{[|x-\tilde{x}|^2+|y-\tilde{y}|^2]^{\frac{n+2\gamma}{2}}} \\
&\qquad \cdot (1+O(|\tilde{x}|)+O(|y-\tilde{y}|))\, d\tilde{x}\, d\tilde{y} \\
&= (1+O(|x|^{\beta})) k_{n,\gamma} P.V. \int_{\{|x-\tilde{x}|\le|x|^{\beta}\}} \frac{\bar{u}_{\varepsilon}(x)-\bar{u}_{\varepsilon}(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}} \\
&\qquad \cdot \int_{\{|y|\le\frac{|x|^{\beta}}{|x-\tilde{x}|}\}} \frac{1}{(1+|y|^2)^{\frac{n+2\gamma}{2}}}\, dy\, d\tilde{x} \\
&= (1+O(|x|^{\beta})) k_{n,\gamma} P.V. \int_{\left\{|x-\tilde{x}|\le|x|^{\beta}\right\}} \frac{\bar{u}_{\varepsilon}(x)-\bar{u}_{\varepsilon}(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}} \\
&\qquad \cdot \Big[\int_{\mathbb{R}^k} \frac{1}{(1+|y|^2)^{\frac{n+2\gamma}{2}}}\, dy - \int_{\left\{|y|>\frac{|x|^{\beta}}{|x-\tilde{x}|}\right\}} \frac{1}{(1+|y|^2)^{\frac{n+2\gamma}{2}}}\, dy\Big]\, d\tilde{x}.
\end{aligned}
$$

Recall relation (3.90), then

$$
\begin{aligned}
k_{n,\gamma} I_{111} &= (1+O(|x|^{\beta})) k_{N,\gamma} P.V. \int_{\{|x-\tilde{x}|\le|x|^{\beta}\}} \frac{\bar{u}_{\varepsilon}(x)-\bar{u}_{\varepsilon}(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\, d\tilde{x} \\
&\quad + O(1) \int_{\{|x-\tilde{x}|\le|x|^{\beta}\}} \frac{\bar{u}_{\varepsilon}(x)-\bar{u}_{\varepsilon}(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}} \Big(\frac{|x|^{\beta}}{|x-\tilde{x}|}\Big)^{-(N+2\gamma)}\, d\tilde{x} \\
&= (1+O(|x|^{\beta})) k_{N,\gamma} P.V. \int_{\mathbb{R}^N} \frac{\bar{u}_{\varepsilon}(x)-\bar{u}_{\varepsilon}(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\, d\tilde{x} \\
&\quad + O(1) \int_{\{|x-\tilde{x}|>|x|^{\beta}\}} \frac{\bar{u}_{\varepsilon}(x)-\bar{u}_{\varepsilon}(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\, d\tilde{x} \\
&\quad + O(1) \int_{\{|x-\tilde{x}|\le|x|^{\beta}\}} \frac{\bar{u}_{\varepsilon}(x)-\bar{u}_{\varepsilon}(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}} \Big(\frac{|x|^{\beta}}{|x-\tilde{x}|}\Big)^{-(N+2\gamma)}\, d\tilde{x}.
\end{aligned}
$$

Using the definition of the fractional Laplacian in $\mathbb{R}^N$,

$$
\begin{aligned}
k_{n,\gamma} I_{111} = & (1+O(|x|^\beta))(-\Delta_x)^\gamma \bar{u}_\varepsilon(x) \\
& + O(1) \int_{\{|x-\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\, d\tilde{x} \\
& + O(1)|x|^{-\beta(N+2\gamma)} \int_{\{|x-\tilde{x}|\le|x|^\beta\}} (\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x}))\, d\tilde{x}.
\end{aligned}
$$

Now we use a similar argument to that of I_{12}, which yields

$$
\begin{aligned}
\int_{\{|x-\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\, d\tilde{x} & \lesssim \int_{\{|\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(x)}{|\tilde{x}|^{N+2\gamma}}\, d\tilde{x} + \int_{\{|\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\, d\tilde{x} \\
& \lesssim |x|^{-2\beta\gamma}\bar{u}_\varepsilon(x) + \varepsilon^{-\frac{2p\gamma}{p-1}} \int_{\left\{\frac{|x|^\beta}{\varepsilon}<|\tilde{x}|<\frac{\sigma}{\varepsilon}\right\}} \frac{u_1(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\, d\tilde{x}
\end{aligned}
$$

and also,

$$
\begin{aligned}
& |x|^{-\beta(N+2\gamma)} \int_{\{|x-\tilde{x}|\le|x|^\beta\}} (\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x}))\, d\tilde{x} \\
& \qquad \lesssim |x|^{-\beta(N+2\gamma)} \left[|x|^{\beta N} \bar{u}_\varepsilon(x) + \varepsilon^{N-\frac{2\gamma}{p-1}} \int_{\{|\tilde{x}|\le\frac{|x|^\beta}{\varepsilon}\}} u_1(\tilde{x})\, d\tilde{x} \right] \\
& \qquad \lesssim |x|^{-2\beta\gamma}\bar{u}_\varepsilon(x) + \varepsilon^{N-\frac{2\gamma}{p-1}} |x|^{-\beta(N+2\gamma)} \int_{\{|\tilde{x}|\le\frac{|x|^\beta}{\varepsilon}\}} u_1(\tilde{x})\, d\tilde{x}.
\end{aligned}
$$

In conclusion, one has

$$
\begin{aligned}
k_{n,\gamma} I_{111} = & (1+O(|x|^\beta))(-\Delta_x)^\gamma \bar{u}_\varepsilon(x) + O(1) \Big[|x|^{-2\beta\gamma}\bar{u}_\varepsilon(x) \\
& + \varepsilon^{-\frac{2p\gamma}{p-1}} \int_{\left\{\frac{|x|^\beta}{\varepsilon}\le|\tilde{x}|<\frac{\sigma}{\varepsilon}\right\}} \frac{u_1(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\, d\tilde{x} + \varepsilon^{N-\frac{2\gamma}{p-1}} |x|^{-\beta(N+2\gamma)} \int_{\{|\tilde{x}|\le\frac{|x|^\beta}{\varepsilon}\}} u_1(\tilde{x})\, d\tilde{x} \Big].
\end{aligned}
$$

Next, for I_{112} we calculate similarly

$$\begin{aligned}
I_{112} &\lesssim \int_{\{|y-\tilde{y}|\leq|x|^\beta\}}\int_{\{|x-\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x})}{[|x-\tilde{x}|^2+|y-\tilde{y}|^2]^{\frac{n+2\gamma}{2}}}\,d\tilde{x}\,d\tilde{y}\\
&= \int_{\{|x-\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}} \int_{\{|y|\leq\frac{|x|^\beta}{|x-\tilde{x}|}\}} \frac{1}{(1+|y|^2)^{\frac{n+2\gamma}{2}}}\,dy\,d\tilde{x}\\
&= \int_{\{|x-\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(x)-\bar{u}_\varepsilon(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}} \Big(\frac{|x|^\beta}{|x-\tilde{x}|}\Big)^k d\tilde{x}\\
&= |x|^{-2\beta\gamma}\bar{u}_\varepsilon(x) + \varepsilon^{-k-\frac{2p\gamma}{p-1}}|x|^{\beta k}\int_{\{|\tilde{x}|>\frac{|x|^\beta}{\varepsilon}\}} \frac{u_1(\tilde{x})}{|\tilde{x}|^{n+2\gamma}}\,d\tilde{x}.
\end{aligned}$$

Combining the estimates for I_{111}, I_{112} and I_{12}, I_{13} we obtain

$$\begin{aligned}
&(-\Delta_z)^\gamma \bar{u}_\varepsilon(x)\\
&= (1+O(|x|^\beta))(-\Delta_x)^\gamma\bar{u}_\varepsilon(x) + O(1)\Big[|x|^{-2\beta\gamma}\bar{u}_\varepsilon(x)\\
&\quad + \varepsilon^{-\frac{2p\gamma}{p-1}}\int_{\{\frac{|x|^\beta}{\varepsilon}<|\tilde{x}|<\frac{\sigma}{\varepsilon}\}} \frac{u_1(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}d\tilde{x} + \varepsilon^{N-\frac{2\gamma}{p-1}}|x|^{-\beta(N+2\gamma)}\int_{\{|\tilde{x}|<\frac{|x|^\beta}{\varepsilon}\}} u_1(\tilde{x})\,d\tilde{x}\\
&\quad + \varepsilon^{-k-\frac{2p\gamma}{p-1}}|x|^{\beta k}\int_{\{|\tilde{x}|>\frac{|x|^\beta}{\varepsilon}\}} \frac{u_1(\tilde{x})}{|\tilde{x}|^{n+2\gamma}}\,d\tilde{x} + O(\varepsilon^{N-\frac{2p\gamma}{p-1}}(1+|x|)^{-(N-2\gamma)})\Big]\\
&= (1+O(|x|^\beta))(-\Delta_x)^\gamma\bar{u}_\varepsilon(x) + O(1)|x|^{-2\beta\gamma}\bar{u}_\varepsilon(x) + \mathscr{R}_1.
\end{aligned}$$

In order to estimate $\mathscr{R}_1$ we use the asymptotic behavior of $u_1(x)$ at 0 and ∞. By direct computation one sees that

$$\mathscr{R}_1(x) = \begin{cases} |x|^{-\beta\frac{2p\gamma}{p-1}}, & \text{if } |x|^\beta<\varepsilon,\\ \varepsilon^{N-\frac{2p\gamma}{p-1}}|x|^{-\beta N}, & \text{if } |x|^\beta>\varepsilon. \end{cases}$$

The choice $\beta=\frac{1}{2}$ yields that $\mathscr{R}_1 = O(|x|^{-\gamma})\bar{u}_\varepsilon(x)$, and thus

$$(-\Delta_z)^\gamma\bar{u}_\varepsilon(z) = (1+O(|x|^\beta))(-\Delta_x)^\gamma\bar{u}_\varepsilon(x) + O(1)|x|^{-\gamma}\bar{u}_\varepsilon(x).$$

Finally, recall that $\bar{u}_\varepsilon(x) = \chi_d(x)u_\varepsilon(x)$, then by the estimates in the previous subsection (3.88), one has

$$|f_\varepsilon(z)| \lesssim |x|^{\frac{1}{2}}|(-\Delta_x)^\gamma\bar{u}_\varepsilon(x)| + |x|^{-\gamma}\bar{u}_\varepsilon(x) + \mathscr{E}, \tag{3.93}$$

where the weighted norm of $\mathscr{E}$ can be bounded by $\varepsilon^{N-\frac{2p\gamma}{p-1}}$.

For $z \in \mathbb{R}^n \setminus \mathscr{T}_{\frac{\sigma}{2}}$, the estimate is similar to the isolated singularity case, we omit the details here. Then we may conclude

$$\|f_\varepsilon\|_{\mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}} \leq C\varepsilon^q,$$

where $q = \min\left\{\frac{(p-3)\gamma}{p-1} - \tilde{\mu}, \frac{1}{2} - \gamma + \frac{(p-3)\gamma}{p-1} - \tilde{\mu}, N - \frac{2p\gamma}{p-1}\right\}$, and it is positive if $-\frac{2\gamma}{p-1} < \tilde{\mu} < \min\left\{\gamma - \frac{2\gamma}{p-1}, \frac{1}{2} - \frac{2\gamma}{p-1}\right\}$. □

Remark 3.5.8. In general, in terms of the local Fermi coordinates (x,y) around a fixed $z_0 = (0,0) \in \Sigma$, for $u \in \mathscr{C}^{\alpha+2\gamma}_{\tilde{\mu},\tilde{\nu}}(\mathbb{R}^n \setminus \Sigma)$, one has the following estimate:

$$(-\Delta)^\gamma u = (-\Delta_{\mathbb{R}^n \setminus \mathbb{R}^k})^\gamma u(x,y) + |x|^\tau \|u\|_*$$

for $|x| \ll 1, |y| \ll 1$, and some $\tau > \tilde{\mu} - 2\gamma$. Indeed, similar to the estimates in Lemma 3.5.7, except the main term in I_{111}, in the estimates, it suffices to control the terms $u(\tilde{x})$ by $\|u\|_* |x|^{\tilde{\mu}}$.

3.6 Hardy type operators with fractional Laplacian

Here we give a formula for the Green's function for the Hardy type operator in $\mathbb{R}^N$,

$$L\phi := (-\Delta_{\mathbb{R}^N})^\gamma \phi - \frac{\kappa}{r^{2\gamma}}\phi, \tag{3.94}$$

where $\kappa \in \mathbb{R}$. In the notation of Section 3.3.2, after the conjugation (3.36) we may study the equivalent operator

$$\tilde{\mathscr{L}} w := e^{-\frac{N+2\gamma}{2}t} \mathscr{L}(e^{\frac{N-2\gamma}{2}t} w) = P_\gamma^{g_0} w - \kappa w \quad \text{on } \mathbb{R} \times \mathbb{S}^{N-1}$$

for $\phi = e^{\frac{N-2\gamma}{2}} w$. Consider the projections over spherical harmonics: for $m = 0, 1 \ldots$, let w_m be a solution to

$$\tilde{\mathscr{L}}_m w := P_\gamma^m w_m - \kappa w_m = h_m \quad \text{on } \mathbb{R}. \tag{3.95}$$

Recall Proposition 3.3.4. Then, in Fourier variables, equation (3.95) simply becomes

$$(\Theta^m_\gamma(\xi)-\kappa)\hat{w}_m = \hat{h}_m.$$

The behavior of this equation depends on the zeroes of the symbol $\Theta^m_\gamma(\xi)-\kappa$. In any case, we can formally write

$$w_m(t) = \int_{\mathbb{R}} \frac{1}{\Theta^m_\gamma(\xi)-\kappa}\hat{h}_m(\xi)e^{i\xi t}\,d\xi = \int_{\mathbb{R}} h_m(s)\mathscr{G}_m(t-t')\,dt', \tag{3.96}$$

where the Green's function for the problem is given by

$$\mathscr{G}_m(t) = \int_{\mathbb{R}} e^{i\xi t}\frac{1}{\Theta^m_\gamma(\xi)-\kappa}\,d\xi.$$

Let us make this statement rigorous in the stable case (this is, below the Hardy constant (3.44)):

Theorem 3.6.1. *Let $0 \le \kappa < \Lambda_{N,\gamma}$ and fix $m = 0,1,\ldots$. Assume that the right hand side h_m in* (3.95) *satisfies*

$$h_m(t) = \begin{cases} O(e^{-\delta t}) & \text{as } t \to +\infty, \\ O(e^{\delta_0 t}) & \text{as } t \to -\infty, \end{cases} \tag{3.97}$$

for some real constants δ, δ_0. It holds:

i. The function $\frac{1}{\Theta^m_\gamma(z)-\kappa}$ is meromorphic in $z \in \mathbb{C}$. Its poles are located at points of the form $\tau_j \pm i\sigma_j$ and $-\tau_j \pm i\sigma_j$, for $j = 0,1,\ldots$. In addition, $\tau_0 = 0$, and $\tau_j = 0$ for j large enough. For such j, σ_j is an increasing sequence with no accumulation points.

ii. If $\delta > 0$ and $\delta_0 \ge 0$, then a particular solution of (3.95) *can be written as*

$$w_m(t) = \int_{\mathbb{R}} h_m(t')\mathscr{G}_m(t-t')\,dt' \tag{3.98}$$

where

$$\mathscr{G}_m(t) = d_0 e^{-\sigma_0|t|} + \sum_{j=1}^{\infty} d_j e^{-\sigma_j|t|}\cos(\tau_j|t|) \tag{3.99}$$

for some constants d_j, $j = 0,1,\ldots$. Moreover, $\mathscr{G}_m$ is an even $\mathscr{C}^\infty$ function when $t \neq 0$ and

$$w_m(t) = O(e^{-\delta t}) \quad as \quad t \to +\infty, \quad w_m(t) = O(e^{\delta_0 t}) \quad as \quad t \to -\infty. \tag{3.100}$$

iii. *Now assume only that $\delta + \delta_0 \geq 0$. If $\sigma_J < \delta < \sigma_{J+1}$ (and thus $\delta_0 > -\sigma_{J+1}$), then a particular solution is*

$$w_m(t) = \int_{\mathbb{R}} h_m(t')\tilde{\mathscr{G}}_m(t-t')\,dt'$$

where

$$\tilde{\mathscr{G}}_m(t) = \sum_{j=J+1}^{\infty} d_j e^{-\sigma_j|t|}\cos(\tau_j|t|). \tag{3.101}$$

Moreover, $\tilde{\mathscr{G}}_m$ is an even $\mathscr{C}^\infty$ function when $t \neq 0$ and the same conclusion as in (3.100) *holds.*

Remark 3.6.2. All solutions of the homogeneous problem $\tilde{\mathscr{L}}_m w = 0$ are of the form

$$w(t) = C_0^- e^{-\sigma_0 t} + C_0^+ \sum_{j=1}^{\infty} C_j^- e^{-\sigma_j t}\cos\tau_j t + \sum_{j=1}^{\infty} C_j^+ e^{+\sigma_j t}\cos\tau_j t$$

for some real constants C_j^-, C_j^+, $j = 0,1,\ldots$. Thus we can see that the only solution to (3.95), in both the cases *ii.* and *iii.*, with decay as in (3.100) is precisely w_m.

We also look at the case when κ leaves the stability regime. In order to simplify the presentation, we only consider the projection $m = 0$ and the equation

$$\tilde{\mathscr{L}}_0 w = h. \tag{3.102}$$

In addition, we assume that only the first pole leaves the stability regime, which happens if $\Lambda_{N,\gamma} < \kappa < \Lambda'_{N,\gamma}$ for some $\Lambda'_{N,\gamma}$. Then, in addition to the poles above, we will have two real poles τ_0 and $-\tau_0$. Some study regarding $\Lambda'_{N,\gamma}$ will be given in the next section but we are not interested in its explicit formula.

Proposition 3.6.3. *Let $\Lambda_{N,\gamma} < \kappa < \Lambda'_{N,\gamma}$. Assume that h decays like $O(e^{-\delta t})$ as $t \to \infty$, and $O(e^{\delta_0 t})$ as $t \to -\infty$ for some real constants δ, δ_0. It holds:*

i. The function $\frac{1}{\Theta^0_\gamma(z)-\kappa}$ *is meromorphic in* $z\in\mathbb{C}$*. Its poles are located at points of the form* $\tau_j \pm i\sigma_j$ *and* $-\tau_j \pm i\sigma_j$*, for* $j=0,1,\ldots$*. In addition,* $\sigma_0=0$*, and* $\tau_j=0$ *for* j *large enough. For such* j*,* σ_j *is an increasing sequence with no accumulation points.*

ii. If $\delta>0$*,* $\delta_0\geq 0$*, then a solution of* (3.102) *can be written as*

$$w_0(t)=\int_{\mathbb{R}} h(t')\mathscr{G}_0(t-t')\,dt', \tag{3.103}$$

where

$$\mathscr{G}_0(t)=d_0\sin(\tau_0 t)\chi_{(-\infty,0)}(t)+\sum_{j=1}^{\infty} d_j e^{-\sigma_j|t|}\cos(\tau_j t)$$

for some constants d_j*,* $j=0,1,\ldots$*. Moreover,* $\mathscr{G}_0$ *is an even* $\mathscr{C}^\infty$ *function when* $t\neq 0$ *and we have the same decay as in* (3.100)*.*

iii. The analogous statements to Theorem 3.6.1, iii.*, and Remark 3.6.2 hold.*

Further study of fractional non-linear equations with critical Hardy potential has been done in [3, 76], for instance.

Define

$$A_m=\tfrac{1}{2}+\tfrac{\gamma}{2}+\tfrac{1}{2}\sqrt{\left(\tfrac{N}{2}-1\right)^2+\mu_m},\quad B_m=\tfrac{1}{2}-\tfrac{\gamma}{2}+\tfrac{1}{2}\sqrt{\left(\tfrac{N}{2}-1\right)^2+\mu_m}. \tag{3.104}$$

and observe that the symbol

$$\Theta^m_\gamma(\xi)=2^{2\gamma}\frac{\left|\Gamma\left(A_m+\frac{\xi}{2}i\right)\right|^2}{\left|\Gamma\left(B_m+\frac{\xi}{2}i\right)\right|^2}=2^{2\gamma}\frac{\Gamma\left(A_m+\frac{\xi}{2}i\right)\Gamma\left(A_m-\frac{\xi}{2}i\right)}{\Gamma\left(B_m+\frac{\xi}{2}i\right)\Gamma\left(B_m-\frac{\xi}{2}i\right)}$$

can be extended meromorphically to the complex plane, which will be denoted by

$$\Theta_m(z):=2^{2\gamma}\frac{\Gamma\left(A_m+\frac{z}{2}i\right)\Gamma\left(A_m-\frac{z}{2}i\right)}{\Gamma\left(B_m+\frac{z}{2}i\right)\Gamma\left(B_m-\frac{z}{2}i\right)},$$

for $z\in\mathbb{C}$.

Remark 3.6.4. It is interesting to observe that

$$\Theta_m(z)=\Theta_m(-z).$$

Moreover, thanks to Stirling formula (expression 6.1.37 in [4])

$$\Gamma(z) \sim e^{-z} z^{z-\frac{1}{2}} (2\pi)^{\frac{1}{2}}, \quad \text{as} \quad |z| \to \infty \text{ in } |\arg z| < \pi, \tag{3.105}$$

one may check that for $\xi \in \mathbb{R}$,

$$\Theta_m(\xi) \sim |m + \xi i|^{2\gamma}, \quad \text{as} \quad |\xi| \to \infty, \tag{3.106}$$

and this limit is uniform in m. Here the symbol $\sim$ means that one can bound one quantity, above and below, by constant times the other. This also shows that, for fixed m, the behavior at infinity is the same as the one for the standard fractional Laplacian $(-\Delta)^\gamma$.

The following proposition uses this idea to study the behavior as $|t| \to 0$. Recall that the Green's function for the fractional Laplacian $(-\Delta_{\mathbb{R}})^\gamma$ in one space dimension is precisely

$$G(t) = |t|^{-(1-2\gamma)}.$$

We will prove that $\mathscr{G}_m$ has a similar behavior.

Proposition 3.6.5. *Let $\gamma \in (0, 1/2)$. Then*

$$\lim_{|t| \to 0} \frac{\mathscr{G}_m(t)}{|t|^{-(1-2\gamma)}} = c$$

for some positive constant c.

Proof. Indeed, recalling (3.106), we have

$$\begin{aligned}
\lim_{|t|\to 0} \frac{\int_{\mathbb{R}} \frac{1}{\Theta_m(\xi) - \lambda} e^{i\xi t} \, d\xi}{|t|^{-(1-2\gamma)}} &= \lim_{|t|\to 0} \int_{\mathbb{R}} \frac{1}{|t|^{2\gamma} \left[\Theta_m\left(\frac{\zeta}{t}\right) - \lambda\right]} e^{i\zeta} \, d\zeta \\
&= \lim_{t \to 0} \left[\int_{\{|\zeta| > t^\delta\}} \ldots + \int_{\{|\zeta| \le t^\delta\}} \ldots \right] =: \lim_{t \to 0} [I_1 + I_2]
\end{aligned}$$

for some $2\gamma < \delta < 1$.

For I_1, we use Stirling's formula (3.105) to estimate

$$I_1 \sim \int_{\{|\zeta| > t^\delta\}} \frac{\cos(\zeta)}{|\zeta|^{2\gamma}} \, d\zeta \to c \quad \text{as} \quad t \to 0,$$

while for I_2,

$$|I_2| \leq \int_{\{|\zeta| \leq t^\delta\}} \frac{1}{|t|^{2\gamma}\big[\Theta_m(0) - \lambda\big]}\, d\zeta \to 0 \quad \text{as} \quad t \to 0,$$

as desired. $\square$

Lemma 3.6.6. *Define the function* $\Phi(x,\xi) = 2^{2\gamma} \frac{\left|\Gamma\left(A_m + x + \frac{\xi}{2} i\right)\right|^2}{\left|\Gamma\left(B_m + x + \frac{\xi}{2} i\right)\right|^2}$. *Then:*

i. *Fixed* $x > B_m$, $\Psi(x,\xi)$ *is a (strictly) increasing function of* $\xi > 0$.

ii. $\Psi(x,0)$ *is a (strictly) increasing function of* $x > 0$.

Proof. As in [66], section 7, one calculates using (3.169),

$$\begin{aligned}\partial_\xi(\log\Theta_m(\xi)) &= \operatorname{Im}\left\{\psi\left(B_m + x + \tfrac{\xi}{2} i)\right) - \psi\left(A_m + x + \tfrac{\xi}{2} i\right)\right\} \\ &= c\, \operatorname{Im} \sum_{l=0}^{\infty} \left(\frac{1}{l + A_m + x + \frac{\xi}{2} i} - \frac{1}{l + B_m + x + \frac{\xi}{2} i} \right) > 0,\end{aligned}$$

as claimed. A similar argument yields the monotonicity in x. $\square$

Now we give the proof of Theorem 3.6.1. Before we consider the general case, let us study first when $\kappa = 0$, for which $\mathscr{G}_m$ can be computed almost explicitly. Fix $m = 0, 1, \ldots$. The poles of the function $\frac{1}{\Theta_m(z)}$ happen at points $z \in \mathbb{C}$ such that

$$\pm \frac{z}{2} i + B_m = -j, \quad \text{for } j \in \mathbb{N} \cup \{0\},$$

i.e, at points $\{\pm i \sigma_j\}$ for

$$\sigma_j := 2(B_m + j), \quad j = 0, 1, \ldots. \tag{3.107}$$

Then the integral in (3.101) can be computed in terms of the usual residue formula. Define the region in the complex plane

$$\Omega = \{z \in \mathbb{C} : |z| < R, \operatorname{Im} z > 0\}. \tag{3.108}$$

A standard contour integration along $\partial\Omega$ gives, as $R \to \infty$, that

$$\mathscr{G}_m(t) = 2\pi i \sum_{j=0}^{\infty} \operatorname{Res}\left(e^{izt}\frac{1}{\Theta_m(z)}, i\sigma_j\right) = 2\pi i \sum_{j=0}^{\infty} e^{-\sigma_j t} c_j, \tag{3.109}$$

where $c_j = c_j(m)$ is the residue of the function $\frac{1}{\Theta_m(z)}$ at the pole $i\sigma_j$. This argument is valid as long as the integral in the upper semicircle tends to zero as $R \to \infty$. This happens when $t > 0$ since $|e^{izt}| = e^{-t\operatorname{Im} z}$. For $t < 0$, we need to modify the contour of integration to $\Omega = \{z \in \mathbb{C} : |z| < R, \operatorname{Im} z < 0\}$, and we have that, for $t < 0$,

$$\mathscr{G}_m(t) = 2\pi i \sum_{j=0}^{\infty} c_j e^{\sigma_j t},$$

which of course gives that $\mathscr{G}_m$ is an even function in t. In any case $\mathscr{G}_m$ is exponentially decaying as $|t| \to \infty$ with speed given by the first pole $|\sigma_0| = 2B_m$.

In addition, recalling the formula for the residues of the Gamma function from (3.168), we have that

$$\begin{aligned} c_j &= \frac{1}{2^{2\gamma}} \frac{\Gamma(2B_m + j)}{\Gamma(A_m - B_m - j)\Gamma(A_m + B_m + j)} \lim_{z \to i\sigma_j} \Gamma(B_m + \tfrac{z}{2} i)(z - i\sigma_j) \\ &= \frac{2}{2^{2\gamma}} \frac{\Gamma\big(1 - \gamma + \sqrt{(\frac{n}{2}-1)^2} + j\big)}{\Gamma(\gamma - j)\Gamma(\big(\gamma + \sqrt{(\frac{n}{2}-1)^2} + j\big)} \frac{-i(-1)^j}{j!} \end{aligned}$$

for $j \geq 1$, which yields the (uniform) convergence of the series (3.99) by Stirling's formula (3.105).

Now take a general $0 < \kappa < \Lambda_{n,\gamma}$. The function $e^{izt}\frac{1}{\Theta_m(z)-\kappa}$ is meromorphic in the complex plane $\mathbb{C}$. Moreover, if z is a root of $\Theta_m(z) = \kappa$, so are $-z$, $\bar{z}$ and $-\bar{z}$.

Let us check then that there are no poles on the real line. Indeed, the first statement in Lemma 3.6.6 implies that is enough to show that

$$\Theta_m(0) - \kappa > 0.$$

But again, from the second statement of the lemma, $\Theta_m(0) > \Theta_0(0)$, so we only need to look at the case $m = 0$. Finally, just note that $\Theta_0(0) = \Lambda_{N,\gamma} > \kappa$.

Next, we look for poles on the imaginary axis. For $\sigma > 0$, $\Theta_m(i\sigma) = \Psi(-\sigma, 0)$ and this function is (strictly) decreasing in σ. Moreover, $\Psi(0,0) = \Theta_m(0) = \Lambda_{N,\gamma} > \kappa$. Let $\sigma_0 \in (0, +\infty]$ be the first point where $\Theta_m(i\sigma_0) = \kappa$. Then $\pm i\sigma_0$ are poles on the imaginary axis. Moreover, the first statement of Lemma 3.6.6 shows that there are no other poles in the strip $\{z : |\mathrm{Im}(z)| \leq \sigma_0\}$.

Denote the rest of the poles by $z_j := \tau_j + i\sigma_j$, $\tau_j - i\sigma_j$, $-\tau_j + i\sigma_j$ and $-\tau_j - i\sigma_j$, $j = 1, 2, \ldots$. Here we take $\sigma_j > \sigma_0 > 0$, $\tau_j \geq 0$. A detailed study of the poles is given in the Section 3.6.4. In particular, for large j, all poles lie there on the imaginary axis, and their asymptotic behavior is similar to that of (3.107).

Now we can complete the proof of statement *ii.* of Theorem 3.6.1. Since we have shown that there is a spectral gap σ_0 from the real line, it is possible to modify the contour of integration in (3.109) to prove a similar residue formula: for $t > 0$,

$$
\begin{aligned}
&\mathscr{G}_m(t) \\
&= 2\pi i \,\mathrm{Res}\left(e^{izt}\frac{1}{\Theta_m(z) - \kappa}, i\sigma_0\right) \\
&+ 2\pi i \sum_{j=1}^{\infty}\left[\mathrm{Res}\left(e^{izt}\frac{1}{\Theta_m(z) - \kappa}, \tau_j + i\sigma_j\right) + \mathrm{Res}\left(e^{izt}\frac{1}{\Theta_m(z) - \kappa}, -\tau_j + i\sigma_j\right)\right] \\
&= 2\pi i c_0 e^{-\sigma_0 t} + 4\pi i \sum_{j=1}^{\infty} c_j e^{-\sigma_j t}\cos(\tau_j t),
\end{aligned}
$$

and for $t < 0$ it is defined evenly. Here $c_j = c_j(m)$ is the residue of the function $\frac{1}{\Theta_m(z) - \kappa}$ at the point $\tau_j + i\sigma_j$; it can be easily shown that c_j is purely imaginary. Moreover, the asymptotic behavior for this residue is calculated in (3.121); indeed, $c_j \sim C j^{-2\gamma}$. The convergence of the series is guaranteed.

Next, we turn to the proof the decay statement (3.100). The main idea is to control the asymptotic behavior of a multipole expansion according to the location of the poles. We start with a simple lemma:

Lemma 3.6.7. *If $f_1(t) = O(e^{-a|t|})$ as $t \to \infty$, $f_2(t) = O(e^{-a_+ t})$ as $t \to +\infty$ and $f_2(t) = O(e^{a_- t})$ as $t \to -\infty$ for some $a, a_+ > 0$, $a_- > -a$, then*

$$f_1 * f_2(t) = O(e^{-\min\{a, a_+\}t}) \quad \text{as} \quad t \to +\infty.$$

Proof. Indeed, for $t > 0$,

$$
\begin{aligned}
|f_1 * f_2(t)| &= \Big| \int_{\mathbb{R}} f_1(t-t') f_2(t')\, dt' \Big| \\
&\lesssim \int_{-\infty}^{0} e^{-a(t-t')} e^{a_- t'}\, dt' + \int_0^t e^{-a(t-t')} e^{-a_+ t'}\, ds + \int_t^{+\infty} e^{a(t-t')} e^{-a_+ t'}\, dt'.
\end{aligned}
$$

The lemma follows by straightforward computations. $\square$

Remark 3.6.8. It is interesting to observe that a_- is not involved in the decay as $t \to +\infty$. Moreover, by reversing the role of t and $-t$, it is possible to obtain the analogous statement for $t \to -\infty$ with the obvious modifications.

Assume that $\delta_0 \geq 0$ and that $\sigma_J < \delta \leq \sigma_{J+1}$ for some $J \geq 0$. Let us use the previous lemma to estimate, for $t > 0$,

$$
\begin{aligned}
&\Big| \int_{\mathbb{R}} \Big[\mathscr{G}_m(t-t') - \sum_{j=0}^{J} d_j e^{-\sigma_j |t-t'|} \cos(\tau_j(t-t')) \Big] h(t')\, dt' \Big| \\
&\qquad\qquad \leq \int_{\mathbb{R}} O(e^{-\sigma_{J+1}|t-t'|}) |h(t')|\, dt' = O(e^{-\delta t}).
\end{aligned}
\tag{3.110}
$$

Let us now look at the term $e^{-\sigma_j |t-t'|} \cos(\tau_j(t-t'))$, $j = 0, \dots, J$, inside the integral. Lemma (3.6.7) would yield an asymptotic behavior $e^{-\sigma_j t}$ as $t \to +\infty$. We will provide an additional argument to improve this behavior, by showing a further cancelation. Indeed, calculate

$$
\begin{aligned}
&\varphi_j(t) \\
:= &\int_{\mathbb{R}} e^{-\sigma_j |t-t'|} \cos(\tau_j(t-t')) h(t')\, dt' \\
= &\int_{-\infty}^{t} e^{-\sigma_j (t-t')} \cos(\tau_j(t-t')) h(t')\, dt' + \int_t^{+\infty} e^{\sigma_j (t-t')} \cos(\tau_j(t-t')) h(t')\, dt'.
\end{aligned}
\tag{3.111}
$$

The first integral in the right hand side above can be rewritten using that, by Fredholm theory, the following compatibility condition must be satisfied:

$$
0 = \int_{\mathbb{R}} e^{\sigma_j t'} \cos(\tau_j(t-t')) h(t')\, dt' = \int_{-\infty}^{t} \dots + \int_t^{+\infty} \dots, \tag{3.112}
$$

and this is rigorous because our growth assumptions on h. Thus (3.111) is reduced to

$$\varphi_j(t) = \int_t^{+\infty} \left[-e^{-\sigma_j(t-t')} + e^{\sigma_j(t-t')}\right] \cos(\tau_j(t-t'))h(t')\,dt'.$$

This is the standard variation of constants formula to produce a particular solution for the second order ODE

$$\varphi_j''(t) = \sigma_j^2 \varphi_j(t) - 2\sigma_j h(t),$$

and in particular shows that $\varphi_j(t)$ decays like $h(t)$ as $t \to +\infty$, which is $O(e^{-\delta t})$.

In addition, for the case $0 < \delta \leq \sigma_0$,

$$\Big| \int_{\mathbb{R}} \mathscr{G}_m(t-t')h(t')\,dt' \Big| \leq \int_{\mathbb{R}} O(e^{-\sigma_0|t-t'|})|h(t')|\,dt' = O(e^{-\delta t}).$$

Finally, reversing $t \to +\infty$ and $t \to -\infty$ yields the proof of statement *ii.* in Theorem 3.6.1.

Now we give the proof of statement *iii.*, which is similar to the above, but with weaker assumptions on δ_0. Fix $m = 0, 1, \ldots$, and drop the subindex m for simplicity.

Assume, as in the previous case, that $\sigma_J < \delta < \sigma_{J+1}$. Here we only have that $\delta_0 \geq -\sigma_{J+1}$. Then the integrals in (3.112) are not finite and the argument for $j = 0, \ldots, J$ does not work. Instead, we change our Fourier transform to integrate on a different horizontal line $\mathbb{R} + i\vartheta$. This is, for $w = w(t)$, set

$$\widetilde{w}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+i\vartheta} e^{-i\zeta t}\, w(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i(\xi+i\vartheta)t}\, w(t)\,dt = \hat{w}(\xi + i\vartheta),$$

whose inverse Fourier transform is

$$w(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+i\vartheta} e^{i\zeta t}\, \widetilde{w}(\zeta)\,d\zeta,$$

Moreover, in the new variable $\zeta = \xi + i\vartheta$ we have that

$$\widetilde{h}(\zeta) = \widetilde{P^{(m)}(w)}(\zeta) = (\Theta_m(\zeta) - \kappa)\widetilde{w}(\zeta).$$

Inverting this symbol we obtain a particular solution

$$w(t) = \int_{\mathbb{R}+i\vartheta} \frac{1}{\Theta_m(\zeta)-\kappa}\widetilde{h}(\zeta)e^{i\zeta t}\,d\zeta = \int_{\mathbb{R}+i\vartheta} h(t')\tilde{\mathscr{G}}(t-t')\,dt',$$

for

$$\tilde{\mathscr{G}}(t) = \int_{\mathbb{R}+i\vartheta} e^{i\zeta t}\frac{1}{\Theta_m(\zeta)-\kappa}\,d\zeta.$$

Replacing the contour of integration from (3.108) to $\partial\Omega_\rho$ for

$$\Omega_\rho = \{z \in \mathbb{C} : |z| < R, \operatorname{Im} z > \rho\}$$

yields, as $R \to \infty$, that

$$\tilde{\mathscr{G}}(t) = 2\pi i \sum_{j=J+1}^{\infty} \operatorname{Res}\left(e^{izt}\frac{1}{\Theta_m(z)-\kappa}, i\sigma_j\right) = 2\pi i \sum_{j=J+1}^{\infty} e^{-\sigma_j t}c_j,$$

where, as above, c_j is the residue of the function $\frac{1}{\Theta_m(z)-\kappa}$ at the pole $\tau_j + i\sigma_j$.

Assume that $\delta + \delta_0 > 0$, and take $\vartheta \in (-\delta_0, \delta)$. Then the growth hypothesis on h from (3.97) imply that $\widetilde{h}(\zeta)$ is well defined. If $\delta_0 + \delta = 0$, taking $\vartheta = \delta$, we can still justify this argument by understanding the Fourier transform in terms of distributions. Moreover, we have the expansion as $t \to +\infty$,

$$\left|\int_{\mathbb{R}} \tilde{\mathscr{G}}(t-t')h(t')\,dt'\right| \le \int_{\mathbb{R}} O(e^{-\sigma_{J+1}|t-t'|})|h(t')|\,dt' = O(e^{-\delta t}).$$

That is, the problematic terms in (3.110) do not appear any longer, and we have found a different particular solution w_m.

This completes the proof of Theorem 3.6.1.

$\square$

3.6.1 Beyond the stability regime

Now we look at the proof of Proposition 3.6.3. As we have mentioned, in order to simplify the presentation, we only consider the projection $m = 0$. Let $\Lambda_{N,\gamma} < \kappa < \Lambda'_{N,\gamma}$ be the region where we have exactly two real poles at τ_0 and $-\tau_0$, for $\tau > 0$. For this, just note that, for real $\xi > 0$, Lemma 3.6.6 shows that $\Theta_0(\xi)$ is an increasing function in ξ, and it is even. Denote the rest of the poles as in the previous subsection, for $j = 1, 2, \ldots$.

We proceed as in the proof of Theorem 3.6.1 and write

$$w_0(t) = \int_{\mathbb{R}} \frac{1}{\Theta_0(\xi) - \kappa} \hat{h}(\xi) e^{i\xi t}\, d\xi = \int_{\mathbb{R}} h(t') \mathscr{G}_0(t - t')\, dt'. \tag{3.113}$$

In this case we can still invert the operator, but one needs to regularize the contour integration in order to account for the real poles in order to give sense to the integral in (3.113). Indeed, for $\varepsilon > 0$ small, let us calculate

$$\mathscr{G}_0^{\varepsilon}(t) = \int_{\mathbb{R}} e^{i\xi t} \frac{1}{\Theta_0(\xi - \varepsilon i) - \kappa}\, d\xi.$$

The poles are now $\tau_0 + \varepsilon i$ and $\tau_0 - \varepsilon i$. Define the region $\Omega = \{z \in \mathbb{C} : |z - (\tau + \varepsilon i)| < R, \operatorname{Re} z > 0\}$. A standard contour integration along $\partial\Omega$ gives, as $R \to \infty$, that for $t > 0$,

$$\mathscr{G}_0^{\varepsilon}(t) = 2\pi i c_0^{\varepsilon} e^{i(\tau_0 + \varepsilon i)t} + 4\pi i \sum_{j=1}^{\infty} e^{-\sigma_j^{\varepsilon} t} \cos(\tau_j^{\varepsilon} t) c_j^{\varepsilon},$$

where

$$c_0^{\varepsilon} = \operatorname{Res}\left(\frac{1}{\Theta_0(z - \varepsilon i) - \kappa}, \tau_0 + \varepsilon i \right).$$

Taking the limit $\varepsilon \to 0$,

$$\mathscr{G}_0^{\varepsilon}(t) \to \mathscr{G}_0(t) = 2\pi i c_0 e^{i\tau_0 t} + 4\pi i \sum_{j=1}^{\infty} c_j e^{-\sigma_j t} \cos(\tau_j t),$$

for $t > 0$, and extended evenly to the real line.

Let us simplify this formula. Using Fredholm theory, to have a solution of equation (3.102), h must satisfy the compatibility condition

$$\begin{aligned} 0 = e^{-i\tau_0 t} \int_{\mathbb{R}} h(t') e^{i\tau_0 t'}\, dt' &= \int_{\mathbb{R}} h(t - t') e^{-i\tau_0 t'}\, dt' \\ &= \int_0^{+\infty} h(t - t') e^{-i\tau_0 t'}\, dt' + \int_{-\infty}^0 h(t - t') e^{-i\tau_0 t'}\, dt'. \end{aligned}$$

Substitute this expression into the formula below

$$
\begin{aligned}
&\int_0^{+\infty} h(t-t')e^{-i\tau_0 t'}\,dt' + \int_{-\infty}^{0} h(t-t')e^{i\tau_0 t'}\,dt' \\
&\qquad = \int_{-\infty}^{0} h(t-t')e^{-i\tau_0 t'} - \int_{-\infty}^{0} h(t-t')e^{i\tau_0 t'}\,dt' \\
&\qquad = \int_t^{+\infty} h(t')\sin(\tau_0(t-t'))\,dt'.
\end{aligned}
$$

Arguing as in the proof of Theorem 3.6.1 we obtain *ii*. The only difference with the stable case is that the $j=0$ term in the summation in formula (3.110) needs to be replaced by

$$\int_t^{+\infty} \sin(\tau_0(t-t'))h(t')\,dt'.$$

A similar argument yields *iii.* too.

3.6.2 A-priori estimates in weighted Sobolev spaces

For $s>0$, we define the norm in $\mathbb{R}\times\mathbb{S}^{N-1}$ given by

$$\|w\|_s^2 = \sum_{m=0}^{\infty} \int_{\mathbb{R}} (1+\xi^2+m^2)^{2s} |\hat{w}_m(\xi)|^2\,d\xi. \tag{3.114}$$

These are homogeneous norms in the variable $r=e^{-t}$, and formulate the Sobolev counterpart to the Hölder norms in $\mathbb{R}^N\setminus\{0\}$ from Section 3.5.1. That is, for $w^*(r) := w(t)$ and s integer we have

$$
\begin{aligned}
\|w\|_0^2 &= \sum_{m=0}^{\infty} \int_0^{\infty} |\tilde{w}_m^*|^2 r^{-1}\,dr, \\
\|w\|_1^2 &= \sum_{m=0}^{\infty} \int_0^{\infty} (|\tilde{w}_m^*|^2 + |\partial_r \tilde{w}_m^*|^2 r^2)\, r^{-1}\,dr, \\
\|w\|_2^2 &= \sum_{m=0}^{\infty} \int_0^{\infty} (|\tilde{w}_m^*|^2 + |\partial_r \tilde{w}_m^*|^2 r^2 + |\partial_{rr} \tilde{w}_m^*| r^4)\, r^{-1}\,dr.
\end{aligned}
$$

One may also give the corresponding weighted norms, for a weight of the type $r^{-\vartheta} = e^{\vartheta t}$. Indeed, one just needs to modify the norm (3.114) to

$$\|w\|_{s,\vartheta}^2 = \sum_{m=0}^{\infty} \int_{\mathbb{R}+i\vartheta} (1+\xi^2+m^2)^{2s} |\hat{w}_m(\xi)|^2 \, d\xi.$$

For instance, in the particular case $s=1$, this is

$$\|w\|_{1,\vartheta}^2 = \sum_{m=0}^{\infty} \int_0^{\infty} (|w_m^*|^2 r^{-2\vartheta} + |\partial_r w_m^*|^2 r^{2-2\vartheta}) \, r^{-1} \, dr.$$

Proposition 3.6.9. *Let $s \geq 2\gamma$, and fix $\vartheta \in \mathbb{R}$ such that the horizontal line $\mathbb{R}+i\vartheta$ does not cross any pole $\tau_j^{(m)} \pm i\sigma_j^{(m)}$, $j = 0,1,\ldots$, $m = 0,1,\ldots$. If w is a solution to*

$$\tilde{\mathscr{L}} w = h \quad \textit{in } \mathbb{R} \times \mathbb{S}^{N-1}$$

of the form (3.98)*, then*

$$\|w\|_{s,\vartheta} \leq C \|h\|_{s-2\gamma,\vartheta}$$

for some constant $C > 0$.

Proof. We project over spherical harmonics $w = \sum_m w_m E_m$, where w_m is a solution to $\tilde{\mathscr{L}}_m w_m = h_m$. Assume, without loss of generality, that $\vartheta = 0$, otherwise replace the Fourier transform $\hat{\cdot}$ by $\tilde{\cdot}$ on a different horizontal line. In particular, $\hat{w}_m(\xi) = (\Theta_m(\xi) - \kappa)^{-1} \hat{h}_m(\xi)$, and we simply estimate

$$\begin{aligned}
\|w\|_s^2 &= \sum_{m=0}^{\infty} \int_{\mathbb{R}} \frac{(1+|\xi|^2+m^2)^{2s}}{|\Theta_m(\xi)-\kappa|^2} |\hat{h}_m(\xi)|^2 \, d\xi \\
&\leq C \sum_{m=0}^{\infty} \int_{\mathbb{R}} (1+|\xi|^2+m^2)^{2s-4\gamma} |\hat{h}_m(\xi)|^2 \, d\xi \\
&= C \|h\|_{s-2\gamma}^2,
\end{aligned}$$

where we have used that

$$\frac{(1+|\xi|^2+m^2)^{2s}}{|\Theta_m(\xi)-\kappa|^2} \leq C(1+|\xi|^2+m^2)^{2s-4\gamma},$$

which follows from (3.106). $\square$

3.6.3 An application to a non-local ODE

The following result is not needed in the proof of the main theorem, but we have decided to include it here because it showcases a classical ODE type behavior for a non-local equation, and it motivates the arguments in Section 3.7.

Assume that we are in the unstable case, i.e., the setting of Proposition 3.6.3.

Proposition 3.6.10. *Let $q>0$ and fix a potential on $\mathbb{R}$ with the asymptotic behavior*

$$V(t)=\begin{cases}\kappa+O(e^{-qt}) & \text{as } t\to+\infty,\\ O(1) & \text{as } t\to-\infty,\end{cases}$$

for $r=e^{-t}$. Then the space of radial solutions to equation

$$(-\Delta)^{\gamma}u-\frac{V}{r^{2\gamma}}u=0 \quad \text{in } \mathbb{R}^N \tag{3.115}$$

that have a bound of the form $|u(r)|\leq Cr^{-\frac{n-2\gamma}{2}}$ is two-dimensional.

Proof. Let u be one of such solutions, and write $w=ur^{\frac{N-2\gamma}{2}}$, $w=w(t)$. By assumption, w is bounded on $\mathbb{R}$. Moreover, w satisfies the equation $P^{(0)}w-Vw=0$, which will be written as

$$\tilde{\mathscr{L}}_0 w=h, \quad \text{for} \quad h:=(V-\kappa)w.$$

Then we have the bounds for h

$$h(t)=\begin{cases}O(e^{-qt}) & \text{as } t\to+\infty,\\ O(1) & \text{as } t\to-\infty,\end{cases}$$

so we take $\delta=q>0$, $\delta_0=0$, and apply Proposition 3.6.3. Then w must be of the form

$$\begin{aligned}w(t)&=w_0(t)+C_0^1\sin(\tau_0 t)+C_0^2\cos(\tau_0 t)\\&\quad+\sum_{j=1}^{\infty}e^{-\sigma_j t}\left[C_j^1\sin(\tau_j t)+C_j^2\cos(\tau_j t)\right]+\sum_{j=1}^{\infty}e^{\sigma_j t}\left[D_j^1\sin(\tau_j t)+D_j^2\cos(\tau_j t)\right]\end{aligned}$$

for some real constants $C_0^1, C_0^2, C_j^1, C_j^2, D_j^1, D_j^2,\ j = 1,2,\ldots,$ and w_0 is given by (3.103). The same proposition yields that w_0 is decaying as $O(e^{-\delta t})$ when $t \to +\infty$, so we must have $D_j^1, D_j^2 = 0$ for $j = 1,2,\ldots$. Moreover, as $t \to -\infty$, v_0 is bounded, which implies that only C_0^1 and C_0^2 survive. Note also that the behavior as $t \to +\infty$ implies that this combination is nontrivial, so this yields a two-dimensional family of bounded solutions.

This argument also implies that any other solution must decay exponentially as $O(e^{-\delta t})$ for $t \to +\infty$ (this is, $C_0^1 = C_0^2 = 0$). Then we can iterate statement *iii.* with $\delta = lq$, $l = 2,3,\ldots$ and $\delta_0 = 0$, to show that w decays faster than any $O(e^{-\delta t})$, $\delta > 0$, as $t \to +\infty$, which gives that $u(r)$ decays faster than any polynomial, this is $|u(r)| = o(|r|^a)$ for every $a \in \mathbb{N}$. Next, we use a unique continuation result for equation (3.115) to show that $u \equiv 0$. In the stable case, unique continuation was proved in [84] using a monotonicity formula, while in the stable case it follows from [152], where Carleman estimates were the crucial ingredient.

Finally we remark that if, in addition, the potential satisfies a monotonicity condition, one can give a direct proof of unique continuation using Theorem 1 from [98]. Note that, however, in [98] the potential is assumed to be smooth at the origin. But one can check that the lack of regularity of the potential at the origin can be handled by the higher order of vanishing of u. $\square$

3.6.4 Technical results

Here we give a more precise calculation of the poles of the function $\frac{1}{\Theta_m(z)-\kappa}$. For this, given $\kappa \in \mathbb{R}$, we aim to solve the equation

$$\frac{\Gamma(\alpha+iz)\Gamma(\alpha-iz)}{\Gamma(\beta+iz)\Gamma(\beta-iz)} - \kappa = 0 \tag{3.116}$$

with $|\alpha-\beta| < 1$ and $\beta < \alpha$.

Lemma 3.6.11. *Let*

$$z = iR + \zeta$$

with $|z| > R_0$ and R_0 sufficiently large. Then the solutions to (3.116) *are contained in balls of radius $\frac{C\kappa\sin((\alpha-\beta)\pi)}{\mathcal{N}^{2(\alpha-\beta)}}$ around the points $z = (\mathcal{N}+\beta)i$, with $\mathcal{N} = [R]$ and C depending solely on α and β.*

Proof. First we note, by using the identity $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, that

$$\begin{aligned}\frac{\Gamma(\alpha - R + i\zeta)}{\Gamma(\beta - R + i\zeta)} &= \frac{\Gamma(1-\beta+R-i\zeta)}{\Gamma(1-\alpha+R-i\zeta)} \frac{\sin(\pi(\beta - R + i\zeta))}{\sin(\pi(\alpha - R + i\zeta))} \\ &= \frac{\Gamma(1-\beta+R-i\zeta)}{\Gamma(1-\alpha+R-i\zeta)} \frac{\sin(\pi(\beta - \delta + i\zeta))}{\sin(\pi(\alpha - \delta + i\zeta))},\end{aligned}$$

where we have denoted

$$\delta = R - [R].$$

Then, Stirling's formula (3.105) yields

$$|\Gamma(1+z)| \sim |z|^{\operatorname{Re} z} e^{-(\operatorname{Im} z)\arg(z)} e^{-\operatorname{Re} z} \sqrt{2\pi}\, |z|^{\frac{1}{2}},$$

which implies

$$\begin{aligned}&\left| \frac{\Gamma(1-\beta+R-i\zeta)\Gamma(\alpha+R-i\zeta)}{\Gamma(1-\alpha+R-i\zeta)\Gamma(\beta+R-i\zeta)} \right| \\ &\qquad \sim (R^2+\zeta^2)^{\alpha-\beta} e^{\zeta\left(\arctan \frac{\zeta}{1-\beta+R} + \arctan \frac{\zeta}{\alpha+R} - \arctan \frac{\zeta}{1-\alpha+R} - \arctan \frac{\zeta}{\beta+R}\right)} e^{-2(\alpha-\beta)}.\end{aligned}$$

Since

$$\begin{aligned}&\arctan \frac{\zeta}{1-\beta+R} + \arctan \frac{\zeta}{\alpha+R} - \arctan \frac{\zeta}{1-\alpha+R} - \arctan \frac{\zeta}{\beta+R} \\ &\quad = \arctan \frac{\frac{\zeta}{1-\beta+R} - \frac{\zeta}{1-\alpha+R}}{1 + \frac{\zeta}{1-\beta+R}\frac{\zeta}{1-\alpha+R}} + \arctan \frac{\frac{\zeta}{\alpha+R} - \frac{\zeta}{\beta+R}}{1 + \frac{\zeta}{\alpha+R}\frac{\zeta}{\beta+R}} \\ &\quad \sim -2\arctan \frac{(\alpha-\beta)\,\zeta}{R^2+\zeta^2} \sim -2\frac{(\alpha-\beta)\,\zeta}{R^2+\zeta^2},\end{aligned}$$

we can estimate, for $R^2+\zeta^2$ sufficiently large,

$$\left| \frac{\Gamma(1-\beta+R-i\zeta)\Gamma(\alpha+R-i\zeta)}{\Gamma(1-\alpha+R-i\zeta)\Gamma(\beta+R-i\zeta)} \right| \sim (R^2+\zeta^2)^{\alpha-\beta} e^{-2\frac{(\alpha-\beta)\zeta^2}{R^2+\zeta^2}} e^{-2(\alpha-\beta)}.$$

Therefore, for $R^2+\zeta^2 > R_0^2$ with R_0 sufficiently large, we have the bound

$$C^{-1}(R^2+\zeta^2)^{\alpha-\beta} \le \left| \frac{\Gamma(1-\beta+R-i\zeta)\Gamma(\alpha+R-i\zeta)}{\Gamma(1-\alpha+R-i\zeta)\Gamma(\beta+R-i\zeta)} \right| \le C(R^2+\zeta^2)^{\alpha-\beta},$$

where C depends only on α and β. Hence,

$$\frac{\kappa}{C(R^2+\zeta^2)^{\alpha-\beta}} \leq \left|\frac{\sin(\pi(\beta-\delta+i\zeta))}{\sin(\pi(\alpha-\delta+i\zeta))}\right| \leq \frac{\kappa}{C^{-1}(R^2+\zeta^2)^{\alpha-\beta}},$$

and by writing

$$\delta - i\zeta = \beta + \tilde{z},$$

we conclude that necessarily

$$|\tilde{z}| \leq \frac{C\kappa \sin\left((\alpha-\beta)\pi\right)}{R^{2(\alpha-\beta)}},$$

which implies that solutions to (3.116) lie at

$$z = iR + \zeta = i[R] + i\beta + i\tilde{z} = [R] + \beta + O\left(\frac{C\kappa \sin\left((\alpha-\beta)\pi\right)}{[R]^{2\,(\alpha-\beta)}}\right),$$

and this proves the Lemma.

$\square$

Next we write

$$z = i(\beta + \mathscr{N}) + \tilde{z}$$

with $\mathscr{N}$ sufficiently large (according to the previous lemma) natural number, and equation (3.116) reads

$$\frac{\Gamma(\alpha-\beta-\mathscr{N}+i\tilde{z})\Gamma(\alpha+\beta+\mathscr{N}-i\tilde{z})}{\Gamma(-\mathscr{N}+i\tilde{z})\Gamma(2\beta+\mathscr{N}-i\tilde{z})} - \kappa = 0. \tag{3.117}$$

Since

$$\Gamma(-\mathscr{N}+i\tilde{z}) = (-1)^{\mathscr{N}-1}\frac{\Gamma(-i\tilde{z})\Gamma(1+i\tilde{z})}{\Gamma(\mathscr{N}+1-i\tilde{z})}$$

and

$$\Gamma(\alpha-\beta-\mathscr{N}+i\tilde{z}) = (-1)^{\mathscr{N}-1}\frac{\Gamma(\beta-\alpha-i\tilde{z})\Gamma(1+\alpha-\beta+i\tilde{z})}{\Gamma(\mathscr{N}+1-\alpha+\beta-i\tilde{z})},$$

we can write

$$\frac{\Gamma(\alpha-\beta-\mathscr{N}+i\tilde{z})\Gamma(\alpha+\beta+\mathscr{N}-i\tilde{z})}{\Gamma(-\mathscr{N}+i\tilde{z})\Gamma(2\beta+\mathscr{N}-i\tilde{z})}$$
$$=\frac{\Gamma(\mathscr{N}+1-i\tilde{z})\Gamma(\alpha+\beta+\mathscr{N}-i\tilde{z})}{\Gamma(\mathscr{N}+1-\alpha+\beta-i\tilde{z})\Gamma(2\beta+\mathscr{N}-i\tilde{z})}\frac{\Gamma(\beta-\alpha-i\tilde{z})\Gamma(1+\alpha-\beta+i\tilde{z})}{\Gamma(1+i\tilde{z})\Gamma(-i\tilde{z})}.$$

Using that

$$\frac{\Gamma(\beta-\alpha-i\tilde{z})\Gamma(1+\alpha-\beta+i\tilde{z})}{\Gamma(1+i\tilde{z})\Gamma(-i\tilde{z})}=\frac{\sin(-\pi i\tilde{z})}{\sin\left(\pi(\beta-\alpha-i\tilde{z})\right)},$$

as well as Stirling's formula (3.105) to estimate

$$\frac{\Gamma(\mathscr{N}+1-i\tilde{z})\Gamma(\alpha+\beta+\mathscr{N}-i\tilde{z})}{\Gamma(\mathscr{N}+1-\alpha+\beta-i\tilde{z})\Gamma(2\beta+\mathscr{N}-i\tilde{z})}$$
$$\sim\frac{(\mathscr{N}-i\tilde{z})^{\mathscr{N}-i\tilde{z}}(\alpha+\beta+\mathscr{N}-1-i\tilde{z})^{\alpha+\beta+\mathscr{N}-1-i\tilde{z}}}{(\mathscr{N}-\alpha+\beta-i\tilde{z})^{\mathscr{N}-\alpha+\beta-i\tilde{z}}(2\beta-1+\mathscr{N}-i\tilde{z})^{2\beta-1+\mathscr{N}-i\tilde{z}}}$$
$$\cdot e^{-2(\alpha-\beta)}\sqrt{\frac{(\mathscr{N}-i\tilde{z})(\alpha+\beta+\mathscr{N}-1-i\tilde{z})}{(\mathscr{N}-\alpha+\beta-i\tilde{z})(2\beta-1+\mathscr{N}-i\tilde{z})}}$$
$$\sim\mathscr{N}^{2(\alpha-\beta)}e^{-2i(\alpha-\beta)\tilde{z}}e^{-2(\alpha-\beta)},$$

we arrive at the relation

$$\frac{\sin(-\pi i\tilde{z})}{\sin\left(\pi(\beta-\alpha-i\tilde{z})\right)}e^{-2i(\alpha-\beta)\tilde{z}}\sim\frac{\kappa}{\mathscr{N}^{2(\alpha-\beta)}e^{-2(\alpha-\beta)}},$$

which implies

$$\tilde{z}\sim\frac{i}{\pi}\frac{\kappa\sin\left(\pi(\beta-\alpha)\right)}{\mathscr{N}^{2(\alpha-\beta)}e^{-2(\alpha-\beta)}}.$$

In fact, it is easy to see from (3.117) and the estimates above that a purely imaginary solution $\tilde{z}$ does exist and a standard fixed point argument in each of the balls in the previous lemma would show that it is unique.

Finally, the half-ball of radius R_0 around the origin in the upper half-plane is a compact set. Since the function at the left hand side of (3.116) is meromorphic, there cannot exist accumulation points of zeros and this necessarily implies that the number of zeros in that half-ball is finite.

We conclude then that the set of solutions to (3.116) consists of a finite number of solutions in a half ball of radius R_0 around the origin in the upper half-plane together with an infinite sequence of roots at the imaginary axis located at

$$z_{\mathcal{N}} = i(\beta + \mathcal{N}) + O\left(\frac{\kappa \sin(\pi(\beta - \alpha))}{\mathcal{N}^{2(\alpha-\beta)} e^{-2(\alpha-\beta)}}\right) \quad \text{for} \quad \mathcal{N} > R_0, \tag{3.118}$$

as desired.

Now we consider the asymptotics for the residues. We define

$$g(z) := \frac{\Gamma(\alpha + iz)\Gamma(\alpha - iz)}{\Gamma(\beta + iz)\Gamma(\beta - iz)} - \kappa.$$

We will estimate the residue of the function $\frac{1}{g(z)}$ at the poles $z_{\mathcal{N}}$ when $\mathcal{N}$ is sufficiently large. Given the fact that the poles are simple and the function $1/g(z)$ is analytic outside its poles, we have

$$\operatorname{Res}\left(\frac{1}{g(z)}, z_{\mathcal{N}}\right) = \lim_{z \to z_{\mathcal{N}}} \left((z - z_{\mathcal{N}})\frac{1}{g(z)}\right) = \frac{1}{g'(z_{\mathcal{N}})}\ .$$

Hence

$$\begin{aligned}
g'(z) &= \frac{d}{dz}\left(\frac{\Gamma(\alpha + iz)\Gamma(\alpha - iz)}{\Gamma(\beta + iz)\Gamma(\beta - iz)}\right) \\
&= -i\frac{\Gamma'(\beta + iz)}{\Gamma^2(\beta + iz)}\left(\frac{\Gamma(\alpha + iz)\Gamma(\alpha - iz)}{\Gamma(\beta - iz)}\right) + \frac{1}{\Gamma(\beta + iz)}\left(\frac{\Gamma(\alpha + iz)\Gamma(\alpha - iz)}{\Gamma(\beta - iz)}\right)' \\
&=: S_1 + S_2.
\end{aligned}$$

Notice that $g(z_{\mathcal{N}}) = 0$ implies that

$$\Gamma(\beta + iz_{\mathcal{N}}) = \frac{\Gamma(\alpha + iz_{\mathcal{N}})\Gamma(\alpha - iz_{\mathcal{N}})}{\kappa\Gamma(\beta - iz_{\mathcal{N}})}$$

and therefore,

$$\begin{aligned} S_1 &= -i\frac{\Gamma'(\beta+iz_{\mathscr{N}})}{\Gamma^2(\beta+iz_{\mathscr{N}})}\left(\frac{\Gamma(\alpha+iz_{\mathscr{N}})\Gamma(\alpha-iz_{\mathscr{N}})}{\Gamma(\beta-iz_{\mathscr{N}})}\right)\\ &= -i\kappa\frac{\Gamma'(\beta+iz_{\mathscr{N}})}{\Gamma(\beta+iz_{\mathscr{N}})}\\ &= -i\kappa\psi(\beta+iz_{\mathscr{N}}), \end{aligned}$$

where $\psi(z)$ is the digamma function. We recall the expansion (3.169),

$$\psi(z) = -\gamma + \sum_{l=0}^{\infty}\left(\tfrac{1}{l+1} - \tfrac{1}{l+z}\right).$$

In this section, γ denotes the Euler constant. Then

$$S_1 = i\kappa\left(\gamma + \sum_{l=0}^{\infty}\left(\frac{1}{l+\beta+iz_{\mathscr{N}}} - \frac{1}{l+1}\right)\right) = \frac{\pi e^{-2(\alpha-\beta)}}{i\sin(\pi(\alpha-\beta))}\mathscr{N}^{2(\alpha-\beta)} + O(1), \tag{3.119}$$

where we have used the asymptotics of $l+\beta+iz_{\mathscr{N}}$ when $l=\mathscr{N}$ from (3.118). Next, using again (3.118) we estimate

$$\begin{aligned} S_2 = &\ \frac{1}{\Gamma(\beta+iz)}\left(\frac{\Gamma(\alpha+iz)\Gamma(\alpha-iz)}{\Gamma(\beta-iz)}\right)'\Bigg|_{z=z_{\mathscr{N}}}\\ &= \kappa i\left(\frac{\Gamma'(\alpha+iz)}{\Gamma(\alpha+iz)} - \frac{\Gamma'(\alpha-iz)}{\Gamma(\alpha-iz)} + \frac{\Gamma'(\beta-iz)}{\Gamma(\beta-iz)}\right)\\ &= \kappa i\Big(\psi(\alpha-\beta-\mathscr{N}+O(\mathscr{N}^{-2(\alpha-\beta)})) - \psi(\alpha+\beta+\mathscr{N}+O(\mathscr{N}^{-2(\alpha-\beta)}))\\ &\qquad + \psi(2\beta+\mathscr{N}+O(\mathscr{N}^{-2(\alpha-\beta)}))\Big). \end{aligned}$$

By using the relations

$$\begin{aligned} &\psi(1-z) - \psi(z) = \pi\cot(\pi z)\\ &\psi(z) \sim \ln(z-\gamma) + 2\gamma, \text{ as } |z|\to\infty,\ \mathrm{Re}\, z > 0, \end{aligned}$$

we conclude, as $\mathscr{N} \to \infty$,

$$\begin{aligned} &\psi(\alpha-\beta-\mathscr{N}+O(\mathscr{N}^{-2(\alpha-\beta)})) \\ =\ &\psi(1-\alpha+\beta+\mathscr{N}+O(\mathscr{N}^{-2(\alpha-\beta)})) \\ &+\pi\cot(\pi(1-\alpha+\beta+\mathscr{N}+O(\mathscr{N}^{-2(\alpha-\beta)}))) \\ =\ &\ln(\mathscr{N})+O(1), \end{aligned}$$

and hence

$$S_2 = i\kappa \ln \mathscr{N} + O(1). \tag{3.120}$$

Putting together (3.119) and (3.120) we find

$$S_1 + S_2 = \frac{\pi e^{-2(\alpha-\beta)}}{i\sin(\pi(\alpha-\beta))}\mathscr{N}^{2(\alpha-\beta)} + i\kappa\ln\mathscr{N} + O(1),$$

and hence

$$\begin{aligned} \operatorname{Res}\left(\frac{1}{g(z)}, z_{\mathscr{N}}\right) &= \frac{i}{\frac{\pi e^{-2(\alpha-\beta)}}{\sin(\pi(\alpha-\beta))}\mathscr{N}^{2(\alpha-\beta)} - \kappa\ln\mathscr{N} + O(1)} \\ &= i\frac{\sin(\pi(\alpha-\beta))e^{2(\alpha-\beta)}}{\pi}\mathscr{N}^{-2(\alpha-\beta)} + O\left(\frac{\ln\mathscr{N}}{\mathscr{N}^{4(\alpha-\beta)}}\right) \end{aligned} \tag{3.121}$$

as $\mathscr{N} \to \infty$.

3.7 Linear theory - injectivity

Let $\bar{u}_\varepsilon$ be the approximate solution from the Section 3.5.1. In this section we consider the linearized operator

$$L_\varepsilon \phi := (-\Delta_{\mathbb{R}^n})^\gamma \phi - p\bar{u}_\varepsilon^{p-1}\phi, \quad \text{in } \mathbb{R}^n \setminus \Sigma, \tag{3.122}$$

where Σ is a sub-manifold of dimension k (or a disjoint union of smooth k-dimensional manifolds), and

$$L_\varepsilon \phi := (-\Delta_{\mathbb{R}^N})^\gamma \phi - pA_{N,p,\gamma}\bar{u}_\varepsilon^{p-1}\phi, \quad \text{in } \mathbb{R}^N \setminus \{q_1, \dots, q_K\}. \tag{3.123}$$

For this, we first need to study the model linearization

$$\mathscr{L}_1\phi := (-\Delta_{\mathbb{R}^N})^\gamma\phi - pA_{N,p,\gamma}u_1^{p-1}\phi = 0 \quad \text{in} \quad \mathbb{R}^N\setminus\{0\}. \tag{3.124}$$

We will show that any solution (in suitable weighted spaces) to this equation must vanish everywhere (from which injectivity in $\mathbb{R}^n\setminus\mathbb{R}^k$ follows easily), and then we will prove injectivity for the operator L_ε.

Let us rewrite (3.124) using conformal properties and the conjugation (3.36). If we define

$$w = r^{\frac{N-2\gamma}{2}}\phi, \tag{3.125}$$

then this equation is equivalent to

$$P_\gamma^{g_0}(w) - Vw = 0, \tag{3.126}$$

for the radial potential

$$V = V(r) = r^{2\gamma}pA_{N,p,\gamma}u_1^{p-1}. \tag{3.127}$$

The asymptotic behavior of this potential is easily calculated using Proposition 3.2.1 and, indeed, for $r = e^{-t}$,

$$V(t) = \begin{cases} pA_{N,p,\gamma} + O(e^{-q_1 t}) & \text{as } t\to+\infty, \\ O(e^{tq_0}) & \text{as } t\to-\infty, \end{cases} \tag{3.128}$$

for $q_0 = (N-2\gamma)(p-1)-2\gamma > 0$.

Let $\gamma\in(0,1)$. By the well known extension theorem for the fractional Laplacian (3.32)-(3.33), equation (3.124) is equivalent to the boundary reaction problem

$$\begin{cases} \partial_{\ell\ell}\Phi + \dfrac{1-2\gamma}{\ell}\partial_\ell\Phi + \Delta_{\mathbb{R}^N}\Phi = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \qquad -\tilde{d}_\gamma \lim\limits_{\ell\to0}\ell^{1-2\gamma}\partial_\ell\Phi = pA_{N,p,\gamma}u_1^{p-1}\Phi & \text{on } \mathbb{R}^N\setminus\{0\}, \end{cases}$$

where $\tilde{d}_\gamma$ is defined in (3.31) and $\Phi|_{\ell=0} = \phi$.

Keeping the notations of Section 3.3.2 for the spherical harmonic decomposition of $\mathbb{S}^{N-1}$, by μ_m we denote the m-th eigenvalue for $-\Delta_{\mathbb{S}^{N-1}}$, repeated according to multiplicity, and by $E_m(\theta)$ the corresponding eigenfunction. Then we can write $\Phi = \sum_{m=0}^{\infty} \Phi_m(r,\ell)E_m(\theta)$, where Φ_m satisfies the following:

$$\begin{cases} \partial_{\ell\ell}\Phi_m + \dfrac{1-2\gamma}{\ell}\partial_\ell\Phi_m + \Delta_{\mathbb{R}^N}\Phi_m - \dfrac{\mu_m}{r^2}\Phi_m = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\tilde{d}_\gamma \lim_{\ell\to 0} \ell^{1-2\gamma}\partial_\ell\Phi_m = pA_{N,p,\gamma}u_1^{p-1}\Phi_m & \text{on } \mathbb{R}^N \setminus \{0\}, \end{cases} \tag{3.129}$$

or equivalently, from (3.126),

$$P_\gamma^m(w) - Vw = 0, \tag{3.130}$$

for $w = w_m = r^{\frac{N-2\gamma}{2}}\phi_m$, $\phi_m = \Phi_m(\cdot,0)$.

3.7.1 Indicial roots

Let us calculate the indicial roots for the model linearized operator defined in (3.124) as $r \to 0$ and as $r \to \infty$. Recalling (3.128), $\mathscr{L}_1$ behaves like the Hardy operator (3.94) with $\kappa = pA_{N,p,\gamma}$ as $r \to 0$ and $\kappa = 0$ as $r \to \infty$. Moreover, we can characterize very precisely the location of the poles in Theorem 3.6.1 and Proposition (3.6.3).

Here we find a crucial difference from the local case $\gamma = 1$, where the Fourier symbol for the m-th projection $\Theta_m(\xi) - \kappa$ is quadratic in ξ, implying that there are only two poles. In contrast, in the non-local case, we have just seen that there exist *infinitely many* poles. Surprisingly, even though $\mathscr{L}_1$ is a non-local operator, its behavior is controlled by just four indicial roots, so we obtain results analogous to the local case.

For the statement of the next result, recall the shift (3.125).

Lemma 3.7.1. *For the operator $\mathscr{L}_1$ we have that, for each fixed mode $m = 0,1,\dots,$*

i. *At $r = \infty$, there exist two sequences of indicial roots*

$$\{\tilde{\sigma}_j^{(m)} \pm i\tilde{\tau}_j^{(m)} - \tfrac{N-2\gamma}{2}\}_{j=0}^{\infty} \quad \textit{and} \quad \{-\tilde{\sigma}_j^{(m)} \pm i\tilde{\tau}_j^{(m)} - \tfrac{N-2\gamma}{2}\}_{j=0}^{\infty}.$$

Moreover,

$$\tilde{\gamma}_m^{\pm} := \pm\tilde{\sigma}_0^{(m)} - \tfrac{N-2\gamma}{2} = -\tfrac{N-2\gamma}{2} \pm \left[1-\gamma+\sqrt{(\tfrac{N-2}{2})^2+\mu_m}\right], \quad m = 0,1,\dots,$$

and $\tilde{\gamma}_m^{+}$ is an increasing sequence (except for multiplicity repetitions).

ii. *At $r = 0$, there exist two sequences of indicial roots*

$$\{\sigma_j^{(m)} \pm i\tau_j^{(m)} - \tfrac{N-2\gamma}{2}\}_{j=0}^{\infty} \quad \textit{and} \quad \{-\sigma_j^{(m)} \pm i\tau_j^{(m)} - \tfrac{N-2\gamma}{2}\}_{j=0}^{\infty}.$$

Moreover,

a) *For the mode $m = 0$, there exists p_1 with $\frac{N}{N-2\gamma} < p_1 < \frac{N+2\gamma}{N-2\gamma}$ (and it is given by (3.12)), such that for $\frac{N}{N-2\gamma} < p < p_1$ (the stable case), the indicial roots $\gamma_0^{\pm} := \pm\sigma_0^{(0)} - \frac{N-2\gamma}{2}$ are real with*

$$-\tfrac{2\gamma}{p-1} < \gamma_0^- < -\tfrac{N-2\gamma}{2} < \gamma_0^+,$$

while if $p_1 < p < \frac{N+2\gamma}{N-2\gamma}$ (the unstable case), then $\gamma_0^{\pm}$ are a pair of complex conjugates with real part $-\frac{N-2\gamma}{2}$ and imaginary part $\pm\tau_0^{(0)}$.

b) *In addition, for all $j \geq 1$,*

$$\sigma_j^{(0)} > \tfrac{N-2\gamma}{2}.$$

c) *For the mode $m = 1$,*

$$\gamma_1^- := -\sigma_0^{(1)} - \tfrac{N-2\gamma}{2} = -\tfrac{2\gamma}{p-1} - 1.$$

Proof. First we consider statement *ii.* and calculate the indicial roots at $r = 0$. Recalling the shift (3.125), let $\mathscr{L}_1$ act on the function $r^{-\frac{N-2\gamma}{2}+\delta}$, and consider instead the operator in (3.126). Because of Proposition 3.3.4, for each $m = 0,1,\dots$, the indicial root $\gamma_m := -\frac{N-2\gamma}{2} + \delta$ satisfies

$$2^{2\gamma}\frac{\Gamma\big(A_m+\frac{\delta}{2}\big)\Gamma\big(A_m-\frac{\delta}{2}\big)}{\Gamma\big(B_m+\frac{\delta}{2}\big)\Gamma\big(B_m-\frac{\delta}{2}\big)} = pA_{N,p,\gamma}, \tag{3.131}$$

where A_m, B_m are defined in (3.104).

Note that if $\delta \in \mathbb{C}$ is a solution, then $-\delta$ and $\pm\overline{\delta}$ are also solutions. Let us write $\frac{\delta}{2} = \alpha + i\beta$, and denote

$$\Phi_m(\alpha,\beta) = 2^{2\gamma} \frac{\Gamma\big(A_m + \frac{\delta}{2}\big)\Gamma\big(A_m - \frac{\delta}{2}\big)}{\Gamma\big(B_m + \frac{\delta}{2}\big)\Gamma\big(B_m - \frac{\delta}{2}\big)}.$$

From the expression, one can see that $\Phi_m(\alpha,0)$ and $\Phi_m(0,\beta)$ are real functions.

We first claim that on the $\alpha\beta$-plane, provided that $|\alpha| \le B_m$, any solution of (3.131) must satisfy $\alpha = 0$ or $\beta = 0$, i.e., δ must be real or purely imaginary. Observing that the right hand side of (3.131) is real and so is $\Phi_m(0,\beta)$ for $\beta \neq 0$, the claim follows from the strict monotonicity of the imaginary part with respect to α, namely

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}\operatorname{Im}(\Phi_m(\alpha,\beta)) \\
&= -\frac{i}{2}\frac{\partial}{\partial\alpha}\Big[\Phi_m(\alpha,\beta) - \Phi_m(\alpha,-\beta)\Big] \\
&= \sum_{j=0}^{\infty} \operatorname{Im}\Big[\tfrac{1}{j+A_m+\alpha-i\beta} + \tfrac{1}{j+A_m-\alpha-i\beta} + \tfrac{1}{j+B_m+\alpha+i\beta} + \tfrac{1}{j+B_m-\alpha+i\beta}\Big] \\
&= \beta \sum_{j=0}^{\infty} \Big[\tfrac{1}{(j+A_m+\alpha)^2+\beta^2} + \tfrac{1}{(j+A_m-\alpha)^2+\beta^2} - \tfrac{1}{(j+B_m+\alpha)^2+\beta^2} - \tfrac{1}{(j+B_m-\alpha)^2+\beta^2}\Big],
\end{aligned} \tag{3.132}$$

the summands being strictly negative since $A_m > B_m$. If $\beta \neq 0$ and $|\alpha| \le B_m$, it is easy to see that the above expression is not zero. This yields the proof of the claim.

Moreover, $\Phi_m(\alpha,0)$ and $\Phi_m(0,\beta)$ are even functions in α, β, respectively. Using the properties of the digamma function again, one can check that

$$\frac{\partial \Phi_m(\alpha,0)}{\partial\alpha} < 0 \text{ for } \alpha > 0 \tag{3.133}$$

and

$$\frac{\partial \Phi_m(0,\beta)}{\partial\beta} > 0 \text{ for } \beta > 0. \tag{3.134}$$

Let us consider now the case $m = 0$. Using the explicit expression for $A_{N,p,\gamma}$ from (3.10), then δ must be a solution of

$$\frac{\Gamma\big(\frac{N}{4}+\frac{\gamma}{2}+\frac{\delta}{2}\big)\Gamma\big(\frac{N}{4}+\frac{\gamma}{2}-\frac{\delta}{2}\big)}{\Gamma\big(\frac{N}{4}-\frac{\gamma}{2}+\frac{\delta}{2}\big)\Gamma\big(\frac{N}{4}-\frac{\gamma}{2}-\frac{\delta}{2}\big)} = p\frac{\Gamma\big(\frac{N}{2}-\frac{\gamma}{p-1}\big)\Gamma\big(\frac{\gamma}{p-1}+\gamma\big)}{\Gamma\big(\frac{\gamma}{p-1}\big)\Gamma\big(\frac{N}{2}-\gamma-\frac{\gamma}{p-1}\big)} =: \lambda(p). \tag{3.135}$$

From the arguments in [8] (see also the definition of p_1 in (3.12)), there exists a unique p_1 satisfying $\frac{N}{N-2\gamma} < p_1 < \frac{N+2\gamma}{N-2\gamma}$ such that $\Phi_0(0,0) = \lambda(p_1)$, and $\Phi_0(0,0) > \lambda(p)$ when $\frac{N}{N-2\gamma} < p < p_1$, and $\Phi_0(0,0) < \lambda(p)$ when $p_1 < p < \frac{N+2\gamma}{N-2\gamma}$.

Assume first that $\frac{N}{N-2\gamma} < p < p_1$ (the stable case). Then from (3.134), we know that there are no indicial roots on the imaginary axis. Next we consider the real axis. Since $\Phi_0(B_0,0) = 0$, by (3.133), there exists an unique root $\alpha^* \in (0,B_0)$ such that $\Phi_0(\pm\alpha^*,0) = \lambda(p)$. We now show that $\alpha^* \in (0, \frac{2\gamma}{p-1} - \frac{N-2\gamma}{2})$. Note that

$$\Phi_0\big(\tfrac{2\gamma}{p-1}-\tfrac{N-2\gamma}{2},0\big) = (1-p)\frac{\Gamma\big(\frac{N}{2}-\frac{\gamma}{p-1}\big)\Gamma\big(\frac{\gamma}{p-1}+\gamma\big)}{\Gamma\big(\frac{\gamma}{p-1}\big)\Gamma\big(\frac{N}{2}-\gamma-\frac{\gamma}{p-1}\big)} < 0.$$

We conclude using the monotonicity of $\Phi_0(\alpha,0)$ in α.

Now we consider the unstable case, i.e., for $p > p_1$. First by (3.133), there are no indicial roots on the real axis. Then by (3.132), in the region $|\alpha| \le B_0$, if a solution exists, then δ must stay in the imaginary axis. Since $\Phi_0(0,\beta)$ is increasing in β and $\lim_{\beta\to\infty}\Phi_0(0,\beta) = +\infty$, we get an unique $\beta^* > 0$ such that $\Phi_0(0,\pm\beta^*) = \lambda(p)$.

In the notation of Section 3.6, we denote all the solutions to (3.135) to be $\sigma_j^{(0)} \pm i\tau_j^{(0)}$ and $-\sigma_j^{(0)} \pm i\tau_j^{(0)}$, such that σ_j is increasing sequence, then from the above argument, one has the following properties:

$$\begin{cases} \sigma_0^{(0)} \in \big(0, \frac{2\gamma}{p-1}-\frac{N-2\gamma}{2}\big), & \tau_0^{(0)} = 0, \qquad \text{for } \frac{N}{N-2\gamma} < p < p_1, \\ \sigma_0^{(0)} = 0, & \tau_0^{(0)} \in (0,\infty), \qquad \text{for } p_1 \le p < \frac{N+2\gamma}{N-2\gamma}, \end{cases}$$

and

$$\sigma_j^{(0)} > 2B_0 = \tfrac{N-2\gamma}{2} \quad \text{for } j \ge 1.$$

For the next mode $m=1$, one can check by direct calculation that $\alpha_1 = \frac{2\gamma}{p-1} + 1 - \frac{N-2\gamma}{2}$ is a solution to (3.131). By the monotonicity (3.133), there are no other real solutions in $(0,\alpha_1)$. This also implies that $\Phi_1(0,0) > \lambda(p)$, by (3.134), there are no solutions in the imaginary axis.

Moreover, using the fact that $\Phi_m(\alpha,0)$ is increasing in m, and $\Phi_m(\pm B_m,0)=0$, we get a sequence of real solutions $\alpha_m \in (0,B_m)$ for $m \geq 1$ that is increasing. Moreover, from (3.132), one also has that in the region $|\alpha| \leq B_m$, all the solutions to (3.131) are real.

Then, denoting the solutions to (3.131) by $\sigma_j^{(m)} \pm i\tau_j^{(m)}$ and $-\sigma_j^{(m)} \pm i\tau_j^{(m)}$ for $m \geq 1$, we conclude that:

$$\sigma_0^{(1)} = \tfrac{2\gamma}{p-1} + 1 - \tfrac{N-2\gamma}{2}, \quad \{\sigma_0^{(m)}\} \text{ is increasing}, \quad \tau_0^{(m)} = 0.$$

We finally consider statement *i.* in the Lemma and look for the indicial roots of $\mathscr{L}_1$ at $r=+\infty$. In this case, δ will satisfy the following equation:

$$2^{2\gamma} \frac{\left|\Gamma\left(\frac{1}{2}+\frac{\gamma}{2}+\frac{1}{2}\sqrt{\left(\frac{N}{2}-1\right)^2+\mu_m}+\frac{\delta}{2}\right)\right|^2}{\left|\Gamma\left(\frac{1}{2}-\frac{\gamma}{2}+\frac{1}{2}\sqrt{\left(\frac{N}{2}-1\right)^2+\mu_m}+\frac{\delta}{2}\right)\right|^2} = 0.$$

For each fixed $m=0,1,\ldots$, the indicial roots occur when

$$\tfrac{1}{2}-\tfrac{\gamma}{2}+\tfrac{1}{2}\sqrt{\left(\tfrac{N}{2}-1\right)^2+\mu_m} \pm \frac{\delta}{2} = j, \quad \text{for } j=0,-1,-2,\ldots,-\infty,$$

or

$$\pm\delta = (1-\gamma) + \sqrt{(\tfrac{N}{2}-1)^2+\mu_m} + 2j, \quad j=0,1,2,\ldots.$$

Thus, the indicial roots for $\mathscr{L}_1$ at $r=+\infty$ are given by

$$-\tfrac{N-2\gamma}{2} \pm \left\{(1-\gamma)+\sqrt{(\tfrac{N}{2}-1)^2+\mu_m}\right\} \pm 2j, \quad j=0,1,\ldots.$$

This finishes the proof of the Lemma. $\square$

3.7.2 Injectivity of $\mathscr{L}_1$ in the weighted space $\mathscr{C}^{2\gamma+\alpha}_{\mu,\nu_1}$

The arguments in this section rely heavily on the results from Section 3.6. We fix

$$\mu > \mathrm{Re}(\gamma_0^+) \geq -\frac{N-2\gamma}{2}, \quad \nu_1 \leq \min\{0,\mu\}. \tag{3.136}$$

Proposition 3.7.2. *Under the hypothesis* (3.136)*, the only solution* $\phi \in \mathscr{C}^{2\gamma+\alpha}_{\mu,\nu_1}(\mathbb{R}^N \setminus \{0\})$ *of the equation* $\mathscr{L}_1\phi = 0$ *is the trivial solution* $\phi \equiv 0$*.*

Proof. We would like to classify solutions to the following equation:

$$(-\Delta_{\mathbb{R}^N})^\gamma \phi = pA_{N,p,\gamma}u_1^{p-1}\phi \text{ in } \mathbb{R}^N \setminus \{0\},$$

or equivalently, (3.129) or (3.130) for each $m = 0,1,\ldots$.

Step 1: the mode $m = 0$. Define the constant $\tau = pA_{N,p,\gamma}$ and rewrite equation (3.126) as

$$P_\gamma^0(w) - \tau w = (V-\tau)w =: h \tag{3.137}$$

for some $w = w(t)$, $h = h(t)$. We use (3.128) and the definition of w to estimate the right hand side,

$$h(t) = \begin{cases} O(e^{-(q_1+\mu+\frac{N-2\gamma}{2})t}) & \text{as } t \to +\infty, \\ O(e^{-(\nu_1+\frac{N-2\gamma}{2})t}) & \text{as } t \to -\infty. \end{cases}$$

We use Theorem 3.6.1 and Proposition 3.6.3. There could be solutions to the homogeneous problem of the form $e^{(\sigma_j \pm i\tau_j)t}$, $e^{(-\sigma_j \pm i\tau_j)t}$. But these are not allowed by the choice of weights μ, ν_1 from (3.136) since $\mu + \frac{N-2\gamma}{2} > \sigma_0^{(0)}$ and $\nu_1 + \frac{N-2\gamma}{2} < \sigma_1^{(0)}$ (for this, recall statements *a)* and *b)* in Lemma 3.7.1).

Now we apply *iii.* of Theorem 3.6.1 (or Proposition 3.6.3) with $\delta = q_1 + \nu + \frac{N-2\gamma}{2} > \mathrm{Re}(\gamma_0^+ + \frac{N-2\gamma}{2}) = \sigma_0^{(0)}$ and $\delta_0 = -(\nu_1 + \frac{N-2\gamma}{2}) > -\sigma_0^{(0)}$. Obviously, $\delta + \delta_0 > 0$. Assume that $\sigma_J < \delta < \sigma_{J+1}$. Then we can find a particular solution w_0 (depending on J) such that

$$w_0(t) = (e^{-\delta t}), \quad \text{as} \quad t \to +\infty,$$

so w will have the same decay.

Now, by the definition of h in (3.137), we can iterate this process with $\delta = lq_1 + \nu + \frac{N-2\gamma}{2}$, $l \geq 2$, and the same δ_0, to obtain better decay when $t \to +\infty$. As a consequence, we have that w decays faster than any $e^{-\delta t}$ as $t \to +\infty$, which when translated to ϕ means that $\phi = o(r^a)$ as $r \to 0$ for every $a \in \mathbb{N}$. The strong unique continuation result of [84] (stable case) and [152] (unstable case) for the operator $P_\gamma^0 - V$ implies that ϕ must vanish everywhere.

Step 2: the modes $m = 1, \ldots, N$. Differentiating the equation (3.11) we get

$$\mathscr{L}_1 \frac{\partial u_1}{\partial x_m} = 0.$$

Since u_1 only depends on r, we have $\frac{\partial u_1}{\partial x_m} = u_1'(r) E_m$, where $E_m = \frac{x_m}{|x|}$. Using the fact that $-\Delta_{\mathbb{S}^{N-1}} E_m = \mu_m E_m$, the extension for $u_1'(r)$ to $\mathbb{R}^{N+1}_+$ solves (3.129) with eigenvalue $N-1$, and $w_1 := r^{\frac{N-2\gamma}{2}} u_1'$ satisfies $P_\gamma^m w - V w = 0$. Note that u_1' decays like $r^{-(N+1-2\gamma)}$ as $r \to \infty$ and blows up like $r^{-\frac{2\gamma}{p-1}-1}$ as $r \to 0$.

We know that also ϕ_m solves (3.130). Assume it decays like $r^{-(N+1-2\gamma)}$ as $r \to \infty$ and blows up like $r^{\gamma_m^+}$ as $r \to 0$. Then we can find a non-trivial combination of u_1' and ϕ_m that decays faster than $r^{-(N+1-2\gamma)}$ at infinity. Since their singularities at 0 cannot cancel, this combination is non-trivial.

Now we claim that no solution to (3.130) can decay faster than $r^{-(N+1-2\gamma)}$ at ∞, which is a contradiction and yields that $\phi_m = 0$ for $m = 1, \ldots, N$.

To show this claim we argue as in Step 1, using the indicial roots at infinity (namely $-(N+1-2\gamma)$ and 1) and interchanging the role of $+\infty$ and $-\infty$ in the decay estimate. Using the facts that the solution decays like r^σ for some $\sigma < -(N-2\gamma+1)$, i.e. $\sigma + \frac{N-2\gamma}{2} < -\frac{N-2\gamma}{2} - 1 = -\sigma_0^{(1)}$ and $\mathrm{Re}(\gamma_m^+) + \frac{N-2\gamma}{2} < \sigma_1^{(1)}$, one can show that the solution is identically zero.

Step 3: the remaining modes $m \geq N+1$. We use an integral estimate involving the first mode which has a sign, as in [59, 60]. We note that, in particular, $\phi_1(r) = -u_1'(r) > 0$, which also implies that its extension Φ_1 is positive. In general, the

γ-harmonic extension Φ_m of ϕ_m satisfies

$$\begin{cases} \operatorname{div}(\ell^{1-2\gamma}\nabla\Phi_m) = \mu_m \dfrac{\ell^{1-2\gamma}}{r^2}\Phi_m & \text{in } \mathbb{R}^{N+1}_+, \\ -\tilde{d}_\gamma \lim\limits_{\ell\to 0} \ell^{1-2\gamma}\partial_\ell \Phi_m = p u_1^{p-1}\phi_m & \text{on } \mathbb{R}^{N+1}_+. \end{cases}$$

We multiply this equation by Φ_1 and the one with $m=1$ by Φ_m. Their difference gives the equality

$$\begin{aligned}(\mu_m-\mu_1)\frac{\ell^{1-2\gamma}}{r^2}\Phi_m\Phi_1 &= \Phi_1 \operatorname{div}(\ell^{1-2\gamma}\nabla\Phi_m) - \Phi_m \operatorname{div}(\ell^{1-2\gamma}\nabla\Phi_1) \\ &= \operatorname{div}(\ell^{1-2\gamma}(\Phi_1\nabla\Phi_m - \Phi_m\nabla\Phi_1)).\end{aligned}$$

Let us integrate over the region where $\Phi_m > 0$. The functions are regular enough near $x=0$ by the restriction (3.136). The boundary $\partial\{\Phi_m>0\}$ is decomposed into a disjoint union of $\partial^0\{\Phi_m>0\}$ and $\partial^+\{\Phi_m>0\}$, on which $\ell=0$ and $\ell>0$, respectively. Hence

$$\begin{aligned} 0 &\le \tilde{d}_\gamma(\mu_m-\mu_1)\int_{\{\Phi_m>0\}} \frac{\Phi_m\Phi_1}{r^2}\,dx d\ell \\ &= \int_{\partial^0\{\Phi_m>0\}} \left(\phi_1 \lim_{\ell\to 0}\ell^{1-2\gamma}\frac{\partial\Phi_m}{\partial\nu} - \phi_m \lim_{\ell\to 0}\ell^{1-2\gamma}\frac{\partial\Phi_1}{\partial\nu}\right) dx \\ &\quad + \int_{\partial^+\{\Phi_m>0\}} \ell^{1-2\gamma}\left(\Phi_1\frac{\partial\Phi_m}{\partial\nu} - \Phi_m\frac{\partial\Phi_1}{\partial\nu}\right) dx d\ell. \end{aligned}$$

The first integral on the right hand side vanishes due to the equations Φ_1 and Φ_m satisfy. Then we observe that on $\partial^+\{\Phi_m>0\}$, one has $\Phi_1>0$, $\dfrac{\partial\Phi_m}{\partial\nu}\le 0$ and $\Phi_m=0$. This forces (using $\mu_m>\mu_1$)

$$\int_{\{\Phi_m>0\}} \frac{\Phi_m\Phi_1}{r^2}\,dx d\ell = 0,$$

which in turn implies $\Phi_m \le 0$. Similarly $\Phi_m\ge 0$ and, therefore, $\Phi_m\equiv 0$ for $m\ge N+1$. This completes the proof of the Proposition 3.7.2. $\square$

3.7.3 Injectivity of $\mathbb{L}_1$ on $\mathscr{C}^{2\gamma+\alpha}_{\mu,\nu_1}$

In the following, we set $N = n-k$ and consider more general equation (3.89). Set

$$\mathbb{L}_1 = (-\Delta_{\mathbb{R}^n})^\gamma - pA_{N,p,\gamma}u_1^{p-1} \text{ in } \mathbb{R}^n \setminus \mathbb{R}^k.$$

Proposition 3.7.3. *Choose the weights μ, ν_1 as in Proposition 3.7.2. The only solution $\phi \in \mathscr{C}^{2\gamma+\alpha}_{\mu,\nu_1}(\mathbb{R}^n \setminus \mathbb{R}^k)$ of the linearized equation $\mathbb{L}_1\phi = 0$ is the trivial solution $\phi \equiv 0$.*

Proof. The idea is to use the results from Section 3.3.3 to reduce $\mathbb{L}_1$ to the simpler $\mathscr{L}_1$, taking into account that u_1 only depends on the variable r but not on z. In the notation of Proposition 3.3.4, define $w = r^{-\frac{N-2\gamma}{2}}\phi$, and w_m its m-th projection over spherical harmonics. Set $\hat{w}_m(\lambda,\omega)$, $\lambda \in \mathbb{R}$, $\omega \in \mathbb{S}^k$ to denote its Fourier-Helgason transform. By observing that the full symbol (3.48), for each fixed ω, coincides with the symbol (3.38), we have reduced our problem to that of Proposition 3.7.2. This completes the proof. $\square$

3.7.4 *A priori* estimates

Now we go back to the linearized operator L_ε from (3.123) for the point singularity case $\mathbb{R}^N \setminus \{q_1,\ldots,q_K\}$, or (3.122) for the general $\mathbb{R}^n \setminus \Sigma$, and consider the equation

$$L_\varepsilon \phi = h. \tag{3.138}$$

For simplicity, we use the following notation for the weighted norms

$$\|\phi\|_* = \|\phi\|_{\mathscr{C}^{2\gamma+\alpha}_{\mu,\nu}}, \quad \|h\|_{**} = \|h\|_{\mathscr{C}^{0,\alpha}_{\mu-2\gamma,\nu-2\gamma}}. \tag{3.139}$$

Moreover, for this subsection, we assume that μ, ν satisfy

$$\mathrm{Re}(\gamma_0^+) < \mu \le 0, \quad -(n-2\gamma) < \nu.$$

For this choice of weights we have the following *a priori* estimate:

Lemma 3.7.4. *Given h with $\|h\|_{**} < \infty$, suppose that ϕ be a solution of* (3.138)*, then there exists a constant C independent of ε such that*

$$\|\phi\|_* \leq C\|h\|_{**}.$$

Proof. We will argue by contradiction. Assume that there exists $\varepsilon_j \to 0$, and a sequence of solutions $\{\phi_j\}$ to $L_{\varepsilon_j}\phi_j = h_j$ such that

$$\|\phi_j\|_* = 1, \quad \text{and} \quad \|h_j\|_{**} \to 0 \text{ as } j \to \infty.$$

In the following we will drop the index j without confusion.

We first consider the point singularity case $\mathbb{R}^N \setminus \Sigma$ for $\Sigma = \{q_1, \ldots, q_K\}$. By Green's representation formula one has

$$\phi(x) = \int_{\mathbb{R}^N} G(x,\tilde{x})[h + pA_{N,p,\gamma}\bar{u}_\varepsilon^{p-1}\phi]\, d\tilde{x} =: I_1 + I_2,$$

where G is the Green's function for the fractional Laplacian $(-\Delta_{\mathbb{R}^N})^\gamma$ given by $G(x,\tilde{x}) = C_{N,\gamma}|x-\tilde{x}|^{-(N-2\gamma)}$ for some normalization constant $C_{N,\gamma}$. In the first step, let $x \in \mathbb{R}^N \setminus \bigcup_i B_\sigma(q_i)$. In this case

$$\begin{aligned}
I_1 &\lesssim \int_{\mathbb{R}^N} |x-\tilde{x}|^{-(N-2\gamma)} h(\tilde{x})\, d\tilde{x} \\
&= \int_{\{\mathrm{dist}(\tilde{x},\Sigma) < \frac{\sigma}{2}\}} \cdots + \int_{\{\frac{\sigma}{2} < \mathrm{dist}(\tilde{x},\Sigma) < \frac{|x|}{2}\}} \cdots \\
&\quad + \int_{\{\frac{|x|}{2} < \mathrm{dist}(\tilde{x},\Sigma) < 2|x|\}} \cdots + \int_{\{\mathrm{dist}(\tilde{x},\Sigma) > 2|x|\}} \cdots \\
&\leq C\|h\|_{**}(|x|^{-(N-2\gamma)} + |x|^\nu) \\
&\leq C\|h\|_{**}|x|^\nu,
\end{aligned}$$

because of our restriction of ν. Moreover,

$$\begin{aligned}
I_2 &= \int G(x,\tilde{x}) p\bar{u}_\varepsilon(\tilde{x})^{p-1}\phi(\tilde{x})\, d\tilde{x} \\
&= \int_{\{\mathrm{dist}(\tilde{x},\Sigma) < \varepsilon\}} \cdots + \int_{\{\frac{\sigma}{2} > \mathrm{dist}(\tilde{x},\Sigma) > \varepsilon\}} \cdots + \int_{\{\mathrm{dist}(\tilde{x},\Sigma) > \frac{\sigma}{2}\}} \cdots =: I_{21} + I_{22} + I_{23}.
\end{aligned}$$

Calculate

$$
\begin{aligned}
I_{21} &\le \int_{\{\operatorname{dist}(\tilde{x},\Sigma)<\varepsilon\}} |x-\tilde{x}|^{-(N-2\gamma)}\rho(\tilde{x})^{-2\gamma}\phi \\
&\le \|\phi\|_* \int_{\{\operatorname{dist}(\tilde{x},\Sigma)<\varepsilon\}} |x-\tilde{x}|^{-(N-2\gamma)}\rho(\tilde{x})^{\mu-2\gamma}\,d\tilde{x}, \\
&\le C\varepsilon^{N+\mu-2\gamma}\|\phi\|_*\rho(x)^{-(N-2\gamma)},
\end{aligned}
$$

where ρ is the weight function defined in Section 3.5.1, and

$$
\begin{aligned}
I_{22} &= \int_{\{\operatorname{dist}(\tilde{x},\Sigma)>\frac{\sigma}{2}\}} |x-\tilde{x}|^{-(N-2\gamma)}\varepsilon^{N(p-1)-2p\gamma}\rho(\tilde{x})^{-(N-2\gamma)(p-1)}\phi\,d\tilde{x} \\
&\le \|\phi\|_* \int_{\{R>\operatorname{dist}(\tilde{x},\Sigma)>\frac{\sigma}{2}\}} |x-\tilde{x}|^{-(N-2\gamma)}\varepsilon^{N(p-1)-2p\gamma}\rho(\tilde{x})^{\mu-(N-2\gamma)(p-1)}\,d\tilde{x} \\
&\quad + \|\phi\|_* \int_{\{\operatorname{dist}(\tilde{x},\Sigma)>R\}} |x-\tilde{x}|^{-(N-2\gamma)}\varepsilon^{N(p-1)-2p\gamma}\rho(\tilde{x})^{\nu-(N-2\gamma)(p-1)}\Big]\,d\tilde{x} \\
&\lesssim \varepsilon^{N(p-1)-2p\gamma}\rho(x)^{-(N-2\gamma)}\|\phi\|_*.
\end{aligned}
$$

Next for I_{23},

$$
\begin{aligned}
I_{23} &\lesssim \int_{\{\varepsilon<\operatorname{dist}(\tilde{x},\Sigma)<\frac{\sigma}{2}\}} |x-\tilde{x}|^{-(N-2\gamma)}\varepsilon^{N(p-1)-2p\gamma}\rho(\tilde{x})^{-(N-2\gamma)(p-1)}\phi\,d\tilde{x} \\
&\le \varepsilon^{N(p-1)-2p\gamma}\rho(x)^{-(N-2\gamma)}\|\phi\|_* \int_{\{\varepsilon<|\tilde{x}|<\frac{\sigma}{2}\}} |\tilde{x}|^{\mu-(N-2\gamma)(p-1)}\,d\tilde{x} \\
&\lesssim (\varepsilon^{N(p-1)-2p\gamma}+\varepsilon^{\mu-2\gamma+N})\|\phi\|_*\rho(x)^{-(N-2\gamma)}.
\end{aligned}
$$

Combining the above estimates, one has

$$
\begin{aligned}
I_2 &\le C(\varepsilon^{N(p-1)-2p\gamma}+\varepsilon^{\mu-2\gamma+N})\rho(x)^{-(N-2\gamma)}\|\phi\|_* \\
&\lesssim (\varepsilon^{N(p-1)-2p\gamma}+\varepsilon^{\mu-2\gamma+N})\rho(x)^{\nu}\|\phi\|_*,
\end{aligned}
$$

and thus

$$
\sup_{\{\operatorname{dist}(x,\Sigma)>\sigma\}} \{\rho(x)^{-\nu}|\phi|\} \le C(\|h\|_{**}+o(1)\|\phi\|_*),
$$

which implies, because our initial assumptions on ϕ, that there exists q_i such that

$$\sup_{\{|x-q_i|<\sigma\}} |x-q_i|^{-\mu}|\phi| \geq \frac{1}{2}. \tag{3.140}$$

In the second step we study the region $|x-q_i|<\sigma$. Without loss of generality, assume $q_i=0$. Recall that we are writing $\phi = I_1+I_2$. On the one hand,

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^N} G(x,\tilde{x})h(\tilde{x})\,d\tilde{x} \\
&= \int_{\{|\tilde{x}|>2\sigma\}} \cdots + \int_{\{|\tilde{x}|<\frac{|x|}{2}\}} \cdots + \int_{\{\frac{|x|}{2}<|\tilde{x}|<2|x|\}} \cdots + \int_{\{2|x|<|\tilde{x}|<2\sigma\}} \cdots \\
&\leq c\|h\|_{**}\Big[\int_{\{|\tilde{x}|>2\sigma\}} |x-\tilde{x}|^{-(N-2\gamma)}|\tilde{x}|^{\nu-2\gamma}\,d\tilde{x} \\
&\qquad + \int_{\{|\tilde{x}|<\frac{|x|}{2}\}} |x-\tilde{x}|^{-(N-2\gamma)}|\tilde{x}|^{\mu-2\gamma}\,d\tilde{x} \\
&\qquad + \int_{\{\frac{|x|}{2}<|\tilde{x}|<2|x|\}} |x-\tilde{x}|^{-(N-2\gamma)}|\tilde{x}|^{\mu-2\gamma}\,d\tilde{x} \\
&\qquad + \int_{\{2|x|<|\tilde{x}|<2\sigma\}} |x-\tilde{x}|^{-(N-2\gamma)}|\tilde{x}|^{\mu-2\gamma}\,d\tilde{x}\Big] \\
&\leq C\|h\|_{**}|x|^{\mu}.
\end{aligned}$$

On the other hand, for I_2, recall that ϕ is a solution to

$$(-\Delta_{\mathbb{R}^N})^{\gamma}\phi - pA_{N,p,\gamma}\bar{u}_{\varepsilon}^{p-1}\phi = h.$$

Define $\bar{\phi}(\tilde{x}) = \varepsilon^{-\mu}\phi(\varepsilon\tilde{x})$, then $\bar{\phi}$ satisfies

$$(-\Delta_{\mathbb{R}^N})^{\gamma}\bar{\phi} - pA_{N,p,\gamma}u_1^{p-1}\bar{\phi} = \varepsilon^{2\gamma-\mu}h(\varepsilon\tilde{x}).$$

By the assumption that $\|h\|_{**}\to 0$, one has that the right hand side tends to 0 as $j\to\infty$. Since $|\bar{\phi}(\tilde{x})| \leq C\|\phi\|_*|\tilde{x}|^{\mu}$ locally uniformly, and by regularity theory, $\bar{\phi}\in\mathscr{C}^{\eta}_{loc}(\mathbb{R}^N\setminus\{0\})$ for some $\eta\in(0,1)$, thus passing to a subsequence, $\bar{\phi}\to\phi_\infty$ locally uniformly in any compact set, where $\phi_\infty\in\mathscr{C}^{\alpha+2\gamma}_{\mu,\mu}(\mathbb{R}^N\setminus\{0\})$ is a solution of

$$(-\Delta_{\mathbb{R}^N})^{\gamma}\phi_\infty - pA_{N,p,\gamma}u_1^{p-1}\phi_\infty = 0 \quad \text{in } \mathbb{R}^N\setminus\{0\} \tag{3.141}$$

(to handle the non-locality we may pass to the extension in a standard way). Since $\mu \leq 0$, it satisfies the condition in Proposition 3.7.2, from which we get that $\phi_\infty \equiv 0$, so $\bar{\phi} \to 0$.

Now we go back to the calculation of I_2. Here we use the change of variable $x = \varepsilon x_1$.

$$\begin{aligned} I_2 &= \int_{\{|\tilde{x}|<\sigma\}} p|x-\tilde{x}|^{-(N-2\gamma)}\bar{u}_\varepsilon^{p-1}\phi\, d\tilde{x} = \varepsilon^\mu \int_{\{|\tilde{x}|<\frac{\sigma}{\varepsilon}\}} |x_1-\tilde{x}|^{-(N-2\gamma)}u_1^{p-1}(\tilde{x})\bar{\phi}(\tilde{x})\, d\tilde{x} \\ &= \varepsilon^\mu \Big[\int_{\{|\tilde{x}|<\frac{1}{R}\}} \cdots + \int_{\{\frac{1}{R}<|\tilde{x}|<R\}} \cdots + \int_{\{R<|\tilde{x}|<\frac{\sigma}{\varepsilon}\}} \cdots \Big] =: J_1 + J_2 + J_3, \end{aligned}$$

for some positive constant R large enough to be determined later. For J_1, fix x, when $\varepsilon \to 0$ one has

$$\begin{aligned} J_1 &= \varepsilon^\mu \int_{\{|\tilde{x}|<\frac{1}{R}\}} |x_1-\tilde{x}|^{-(N-2\gamma)}u_1^{p-1}(\tilde{x})\bar{\phi}(\tilde{x})\, d\tilde{x} \\ &\leq \varepsilon^\mu \|\phi\|_* |x_1|^{-(N-2\gamma)} \int_{\{|\tilde{x}|<\frac{1}{R}\}} |\tilde{x}|^{\mu-2\gamma}\, d\tilde{x} \\ &\leq CR^{-(N-2\gamma+\mu)}\|\phi\|_*|x|^\mu. \end{aligned}$$

For J_2 we use the fact that in this region $\bar{\phi} \to 0$, so

$$\begin{aligned} J_2 &= \varepsilon^\mu \int_{\{\frac{1}{R}<|\tilde{x}|<R\}} |x_1-\tilde{x}|^{-(N-2\gamma)}u_1^{p-1}(\tilde{x})\bar{\phi}(\tilde{x})\, d\tilde{x} \\ &= o(1)\varepsilon^\mu \int_{\{\frac{1}{R}<|\tilde{x}|<R\}} \frac{1}{|x_1-\tilde{x}|^{N-2\gamma}}\, d\tilde{x} = o(1)\varepsilon^\mu |x_1|^{-(N-2\gamma)} = o(1)|x|^\mu, \end{aligned}$$

and finally,

$$\begin{aligned} J_3 &= \varepsilon^\mu \int_{\{R<|\tilde{x}|<\frac{\sigma}{\varepsilon}\}} |x_1-\tilde{x}|^{-(N-2\gamma)}u_1^{p-1}(\tilde{x})\bar{\phi}(\tilde{x})\, d\tilde{x} \\ &= \varepsilon^\mu \|\phi\|_* \Big[\int_{\{R<|\tilde{x}|<\frac{|x_1|}{2}\}} |x_1-\tilde{x}|^{-(N-2\gamma)}|\tilde{x}|^\mu u_1^{p-1}\, d\tilde{x} \\ &\quad + \int_{\{\frac{|x_1|}{2}<|\tilde{x}|<2|x_1|\}} |x_1-\tilde{x}|^{-(N-2\gamma)}u_1^{p-1}|\tilde{x}|^\mu\, d\tilde{x} \\ &\quad + \int_{\{2|x_1|<|\tilde{x}|<\frac{\sigma}{\varepsilon}\}} |x_1-\tilde{x}|^{-(N-2\gamma)}u_1^{p-1}|\tilde{x}|^\mu\, d\tilde{x}\Big] \\ &\leq C\varepsilon^\mu |x_1|^\mu \|\phi\|_* |x_1|^{-\tau} \leq o(1)\|\phi\|_*|x|^\mu \end{aligned}$$

for some $\tau > 0$.

Combining all the above estimates, one has

$$||x|^{-\mu} I_1| \leq o(1)(\|\phi\|_* + 1),$$

which implies

$$\|\phi\|_* \leq o(1)\|\phi\|_* + o(1) + \|h\|_{**} = o(1).$$

This is a contradiction with (3.140).

For the more general case $\mathbb{R}^n \setminus \Sigma$ when Σ is a smooth k-dimensional submanifold, the argument is similar as above, the only difference is that one arrives to the analogous to (3.141) in the estimate for I_2 near Σ:

$$(-\Delta_{\mathbb{R}^n})^\gamma \phi_\infty - p u_1^{p-1} \phi_\infty = 0 \quad \text{in } \mathbb{R}^n \setminus \mathbb{R}^k.$$

After the obvious rescaling by the constant $A_{N,p,\gamma}$, where $N = n - k$, one uses Remark 3.5.8 and the injectivity result in Proposition 3.7.3 instead of the one in Proposition 3.7.2. This completes the proof of Lemma 3.7.4.

□

3.8 Fredholm properties - surjectivity

Our analysis here follows closely the one in [142] for the local case. These lecture notes are available online but, unfortunately, yet to be published.

For the rest of the chapter, we will take the pair of dual weights $\mu, \tilde{\mu}$ such that $\mu + \tilde{\mu} = -(N - 2\gamma)$ and $\nu + \tilde{\nu} = -(n - 2\gamma)$ satisfying

$$\begin{aligned} &-\frac{2\gamma}{p-1} < \tilde{\mu} < \mathrm{Re}(\gamma_0^-) \leq -\frac{N-2\gamma}{2} \leq \mathrm{Re}(\gamma_0^-) < \mu < 0, \\ &-(n-2\gamma) < \tilde{\nu} < 0. \end{aligned} \tag{3.142}$$

In order to consider the invertibility of the linear operators (3.122) and (3.123), defined in the spaces

$$L_\varepsilon : \mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}} \to \mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma},$$

it is simpler to consider the conjugate operator

$$\tilde{L}_\varepsilon(w) := f_1^{-1} L_\varepsilon(f_2 w), \quad \tilde{L}_\varepsilon : \mathscr{C}^{2\alpha+\gamma}_{\tilde{\mu}+\frac{N-2\gamma}{2},\tilde{\nu}+\frac{n-2\gamma}{2}} \to \mathscr{C}^{2\alpha+\gamma}_{\tilde{\mu}+\frac{N-2\gamma}{2},\tilde{\nu}+\frac{n-2\gamma}{2}}, \tag{3.143}$$

where f_2 is a weight $\rho^{-\frac{n-2\gamma}{2}}$ near infinity, and $\rho^{-\frac{N-2\gamma}{2}}$ near the singular set Σ, while f_1 is $\rho^{-\frac{n+2\gamma}{2}}$ near infinity, and $\rho^{-\frac{N+2\gamma}{2}}$ near the singular set. Recall that ρ is defined in Section 3.5.1. This conjugate operator is better behaved in weighted Hilbert spaces and simplifies the notation in the proof of Fredholm properties.

3.8.1 Fredholm properties

Fredholm properties for extension problems related to this type of operators were considered in [130, 131].

In the notation of Section 3.5.1, and following the paper [132], we define the weighted Lebesgue space $L^2_{\delta,\vartheta}(\mathbb{R}^n\setminus\Sigma)$. These are L^2_{loc} functions for which the norm

$$\begin{aligned}\|\phi\|^2_{L^2_{\delta,\vartheta}(\mathbb{R}^n\setminus\Sigma)} = \int_{\mathbb{R}^n\setminus B_R} |\phi|^2\rho^{-2\gamma-2\vartheta}\,dz + \int_{B_R\setminus\mathscr{T}_\sigma}|\phi|^2\,dz \\ + \int_{\mathscr{T}_\sigma}|\phi|^2\rho^{N-1-2\gamma-2\delta}\,drdyd\theta\end{aligned} \tag{3.144}$$

is finite. Here $drdyd\theta$ denotes the corresponding measure in Fermi coordinates $r>0$, $y\in\Sigma$, $\theta\in\mathbb{S}^{N-1}$. One defines accordingly, for $\gamma>0$, weighted Sobolev spaces $W^{2\gamma,2}_{\delta,\vartheta}$ with respect to the vector fields from Remark 3.5.3 (see [130] for the precise definitions).

The seemingly unusual normalization in the integrals in (3.144) is explained by the change of variable $w=f_2\phi$. Indeed,

$$\begin{aligned}\|w\|^2_{L^2_{\delta,\vartheta}} = \int_{-\infty}^{-\log R}\int_{\mathbb{S}^{n-1}}|w|^2e^{2\vartheta t}\,d\tilde{t}d\tilde{\theta}dr + \int_{\{\text{dist}(\cdot,\Sigma)>\sigma,|z|<R\}}|w|^2\,dz \\ + \int_{-\log\sigma}^{+\infty}\int_\Sigma\int_{\mathbb{S}^{N-1}}|w|^2e^{2\delta t}\,dtdyd\theta.\end{aligned}$$

We have

Lemma 3.8.1. *For the choice of parameters*

$$-\delta < \tilde{\mu} + \tfrac{N-2\gamma}{2}, \quad -\vartheta > \tilde{\nu} + \tfrac{n-2\gamma}{2},$$

we have the continuous inclusions

$$\mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}(\mathbb{R}^n \setminus \Sigma) \hookrightarrow L^2_{-\delta,-\vartheta}(\mathbb{R}^n \setminus \Sigma).$$

The spaces $L^2_{\delta,\vartheta}$ and $L^2_{-\delta,-\vartheta}$ are dual with respect to the natural pairing

$$\langle \phi_1, \phi_2 \rangle_* = \int_{\mathbb{R}^n} \phi_1 \phi_2,$$

for $\phi_1 \in L^2_{\delta,\vartheta}$, $\phi_2 \in L^2_{-\delta,-\vartheta}$. Now, let $\tilde{L}_\varepsilon$ be the operator defined in (3.143). It is a densely defined, closed graph operator (this is a consequence of elliptic estimates). Then, relative to this pairing, the adjoint of

$$\tilde{L}_\varepsilon : L^2_{-\delta,-\vartheta} \to L^2_{-\delta-2\gamma,-\vartheta-2\gamma} \tag{3.145}$$

is precisely

$$(\tilde{L}_\varepsilon)^* = \tilde{L}_\varepsilon : L^2_{\delta+2\gamma,\vartheta+2\gamma} \to L^2_{\delta,\vartheta}. \tag{3.146}$$

Now we fix μ, $\tilde{\mu}$, ν, $\tilde{\nu}$ as in (3.142), and choose $-\delta < 0$ slightly smaller than $\tilde{\mu} + \frac{N-2\gamma}{2}$ and $-\vartheta < 0$ just slightly larger than $\tilde{\nu} + \frac{n-2\gamma}{2}$ so that, in particular,

$$\begin{aligned} &-\tfrac{2\gamma}{p-1} + \tfrac{N-2\gamma}{2} < -\delta < \tilde{\mu} + \tfrac{N-2\gamma}{2} < 0 < \mu + \tfrac{N-2\gamma}{2} < \delta < \tfrac{N-2\gamma}{2}, \\ &-\tfrac{n-2\gamma}{2} < \tilde{\nu} + \tfrac{n-2\gamma}{2} < -\vartheta < 0 < \vartheta < \tfrac{n-2\gamma}{2}, \end{aligned} \tag{3.147}$$

and we have the inclusions from Lemma 3.8.1. In addition, we can choose δ, ϑ different from the corresponding indicial roots. Higher order regularity is guaranteed by the results in Section 3.6.2. We will show:

Proposition 3.8.2. *Let $\delta \in (-\frac{N+2\gamma}{2} - 2\gamma, \frac{N-2\gamma}{2})$ and $\vartheta \in (-\frac{n-2\gamma}{2}, \frac{n-2\gamma}{2})$ be real numbers satisfying* (3.147).

Assume that $w \in L^2_{\delta,\vartheta}$ is a solution to

$$\tilde{L}_\varepsilon w = \tilde{h} \text{ on } \mathbb{R}^n \setminus \Sigma$$

for $\tilde{h} \in L^2_{\delta,\vartheta}$*. Then we have the* a priori *estimate*

$$\|w\|_{L^2_{\delta,\vartheta}} \lesssim \|\tilde{h}\|_{L^2_{\delta,\vartheta}} + \|w\|_{L^2(\mathscr{K})}, \tag{3.148}$$

where $\mathscr{K}$ *is a compact set in* $\mathbb{R}^n \setminus \Sigma$*. Translating back to the original operator* L_ε*, if* ϕ *is a solution to* $L_\varepsilon \phi = h$ *in* $\mathbb{R} \setminus \Sigma$*, then* (3.148) *is rewritten as*

$$\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}} \lesssim \|h\|_{L^2_{\delta,\vartheta}} + \|\phi\|_{L^2(\mathscr{K})}. \tag{3.149}$$

As a consequence, L_ε *has good Fredholm properties. The same is true for the linear operators from* (3.145) *and* (3.146)*.*

Proof. The proof here goes by subtracting suitable parametrices near Σ and near infinity thanks to Theorem 3.6.1. Then the remainder is a compact operator. For simplicity we set $\varepsilon = 1$.

We first consider the point singularity case, i.e., $k = 0$, $n = N$.

Step 1: (Localization) Let us study how the operator $L_1 : L^2_{\delta+2\gamma,\vartheta+2\gamma} \to L^2_{\delta,\vartheta}$ is affected by localization, so that it is enough to work with functions supported near infinity and near the singular set.

In the first step, assume that the singularity happens only at $r = \infty$ (but not at $r = 0$). We would like to patch a suitable parametrix at $r = \infty$. Let χ be a cut-off function such that $\chi = 1$ in $\mathbb{R}^N \setminus B_R$, $\chi = 0$ in $B_{R/2}$. Let $\mathscr{K} = B_{2R}$, and set

$$h_1 := L_1(\chi\phi) = \chi L_1 \phi + [L_1, \chi]\phi,$$

where $[\cdot,\cdot]$ denotes the commutator operator. Contrary to the local case, the commutator term does not have compact support, but can still give good estimates in weighted Lebesgue spaces by carefully controlling the the tail terms. Let

$$\begin{aligned} I(x) := [L_1, \chi]\phi(x) &= (-\Delta_{\mathbb{R}^N})^\gamma(\chi\phi)(x) - \chi(x)(-\Delta_{\mathbb{R}^N})^\gamma\phi(x) \\ &= k_{N,\gamma}\int_{\mathbb{R}^N} \frac{\chi(x)-\chi(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\phi(\tilde{x})\,d\tilde{x}. \end{aligned}$$

Let us bound this integral in $L^2_{\delta,\vartheta}$.

We first consider the case $|x| \ll 1$. Note that

$$\begin{aligned} I(x) &= \int_{B_{2R}\setminus B_{R/2}} \frac{\chi(x)-\chi(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\phi(\tilde{x})\,d\tilde{x} + \int_{\mathbb{R}^N\setminus B_{2R}} \frac{\phi(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\,d\tilde{x} \\ &\le CR^{\frac{2\vartheta+2\gamma-N}{2}}\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}} + \|\phi\|_{L^2(\mathscr{K})}. \end{aligned}$$

We have that $\|I\|_{L^2_\delta(B_1)}$ can be bounded by $o(1)\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}} + \|\phi\|_{L^2(\mathscr{K})}$ for large R if $\vartheta < \frac{N-2\gamma}{2}$ and $\delta < \frac{N-2\gamma}{2}$.

Next, for $|x| \ge 2R$, we need to add the weight at infinity and calculate $\|I\|_{L^2_\vartheta(\mathbb{R}^N\setminus B_{2R})}$. But

$$\begin{aligned} I(x) &= \int_{B_{R/2}} \frac{\phi(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\,d\tilde{x} + \int_{B_{2R}\setminus B_{R/2}} \frac{\chi(x)-\chi(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\phi(\tilde{x})\,d\tilde{x} \\ &\le C\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}} R^{\frac{2\vartheta+6\gamma+N}{2}}|x|^{-(N+2\gamma)} \text{ if } \delta > -\tfrac{N+2\gamma}{2} - 2\gamma. \end{aligned}$$

One has that $\|I\|_{L^2_\vartheta(\mathbb{R}^N\setminus B_R)} = o(1)\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}}$ if $\vartheta > -\frac{N+2\gamma}{2} - 2\gamma$.

Now let $1 \le |x| < 2R$, and calculate $\|I\|_{L^2(B_{2R}\setminus B_1)}$. Again, we split

$$\begin{aligned} I(x) &= \int_{B_{R/2}} \frac{\chi(x)\phi(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\,d\tilde{x} + \int_{B_{2R}\setminus B_{R/2}} \frac{\chi(x)-\chi(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\phi(\tilde{x})\,d\tilde{x} \\ &\quad + \int_{\mathbb{R}^N\setminus B_{2R}} \frac{(\chi(x)-1)\phi(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\,d\tilde{x} \\ &=: I_{21} + I_{22} + I_{23}. \end{aligned}$$

Similar to the above estimates, we can get that the L^2 norm can be bounded by

$$\|\phi\|_{L^2(\mathscr{K})} + o(1)\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}}$$

if $\vartheta < \frac{N-2\gamma}{2}$. Thus we have shown that

$$\|h_1\|_{L^2_{\delta,\vartheta}} \lesssim \|h\|_{L^2_{\delta,\vartheta}} + \|\phi\|_{L^2(\mathscr{K})} + o(1)\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}},$$

so localization does not worsen estimate (3.149).

In addition, the localization at $r=0$ is similar. One just needs to interchange the role of r and $1/r$, and ϑ by δ. Similar to the estimates in Step 5 below, one can get that the error caused by the localization can be bounded by $\|\phi\|_{L^2(\mathscr{K})} + o(1)\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}}$ if $-\frac{N+2\gamma}{2} - 2\gamma < \vartheta < \frac{N-2\gamma}{2}$, $-\frac{N+2\gamma}{2} - 2\gamma < \delta < \frac{N-2\gamma}{2}$.

Step 2: (The model operator) After localization around one of the singular points, say $q_1 = 0$, the operator L_1 can be approximated by the model operator $\mathscr{L}_1$ from (3.124), or by its conjugate given in (3.126). Moreover, recalling the notation (3.127) for the potential term and its asymptotics (3.128), it is enough to show that

$$\|w\|_{L^2_\delta} \lesssim \|\tilde{h}\|_{L^2_\delta}, \tag{3.150}$$

if $w = w(t,\theta)$ is a solution of

$$P_\gamma^{g_0} w - \kappa w = \tilde{h}, \quad t \in \mathbb{R}, \theta \in \mathbb{S}^{N-1}, \tag{3.151}$$

that has compact support in $t \in (0,\infty)$. Here we have denoted $\kappa = pA_{N,p,\gamma}$.

Now project over spherical harmonics, so that $w = \sum_m w_m(t)E_m(\theta)$, and w_m satisfies

$$P_\gamma^m w_m - \kappa w_m = h_m, \quad t \in \mathbb{R}.$$

Our choice of weights (3.147) implies that there are no additional solutions to the homogeneous problem and that we can simply write our solution as (3.96), in Fourier variables. Then

$$\begin{aligned}\int_{\mathbb{R}} e^{2\delta t}|w_m(t)|^2\,dt &= \int_{\mathbb{R}} |w_m(\xi+\delta i)|^2\,d\xi = \int_{\mathbb{R}} \frac{1}{|\Theta_\gamma^m(\xi+\delta i) - \kappa|^2}|\hat{h}_m(\xi+\delta i)|^2\,d\xi \\ &\le C\int_{\mathbb{R}} |\hat{h}_m(\xi+\delta i)|^2\,d\xi = \int_{\mathbb{R}} e^{2\delta t}|h_m(t)|^2\,dt,\end{aligned} \tag{3.152}$$

where we have used (3.106). (note that there are no poles on the $\mathbb{R}+\delta i$ line). Estimate (3.150) follows after taking sum in m and the fact that $\{E_m\}$ is an orthonormal basis.

For the estimate near infinity, we proceed in a similar manner, just approximating the potential by $\tau = 0$.

Step 3: (Compactness) Let $\{w_j\}$ be a sequence of solutions to $\tilde{L}_1 w_j = \tilde{h}_j$ with $\tilde{h}_j \in L^2_{\delta,\vartheta}$. Assume that we have the uniform bound $\|w_j\|_{L^2_{\delta,\vartheta}} \leq C$. Then there exists a subsequence, still denoted by $\{w_j\}$, that is convergent in $L^2_{\delta,\vartheta}$ norm. Indeed, by the regularity properties of the equation, $\|w_j\|_{W^{2\gamma,2}_{\delta,\vartheta}} \leq C$, which in particular, implies a uniform $W^{2\gamma,2}$ in every compact set $\mathscr{K}$. But this is enough to conclude that $\{w_j\}$ has a convergent subsequence in $W^{2\gamma,2}(\mathscr{K})$. Finally, estimate (3.148) implies that this convergence is also true in $L^2_{\delta,\vartheta}$, as we claimed.

Step 4: (Fredholm properties for $\tilde{L}_1$) This is a rather standard argument. First, assume that the kernel is infinite dimensional, and take an orthonormal basis $\{w_j\}$ for this kernel. Then, by the claim in Step 3, we can find a Cauchy subsequence. But, for this,

$$\|w_j - w_{j'}\|^2 = \|w_j\|^2 + \|w_{j'}\|^2 = 2,$$

a contradiction.

Second, we show that the operator has closed range. Let $\{w_j\}$, $\{\tilde{h}_j\}$ be two sequences such that

$$\tilde{L}_1 w_j = \tilde{h}_j \quad \text{and} \quad \tilde{h}_j \to h \text{ in } L^2_{\delta,\vartheta}. \tag{3.153}$$

Since $\operatorname{Ker}\tilde{L}_1$ is closed, we can use the projection theorem to write $w_j = w_j^0 + w_j^1$ for $w_j^0 \in \operatorname{Ker}\tilde{L}_1$ and $w_j^1 \in (\operatorname{Ker}\tilde{L}_1)^\perp$. We have that $\tilde{L}_1 w_j^1 = \tilde{h}_j$.

Now we claim that this sequence is uniformly bounded, i.e., $\|w_j^1\|_{L^2_{\delta,\vartheta}} \leq C$ for every j. By contradiction, assume that $\|w_j^1\|_{L^2_{\delta,\vartheta}} \to \infty$ as $j \to \infty$, and rescale

$$\tilde{w}_j = \frac{w_j^1}{\|w_j^1\|_{L^2_{\delta,\vartheta}}}$$

so that the new sequence has norm one in $L^2_{\delta,\vartheta}$. From the previous remark, there is a convergent subsequence, still denoted by $\{\tilde{w}_j\}$, i.e., $\tilde{w}_j \to \tilde{w}$ in $L^2_{\delta,\vartheta}$. Moreover, we know that $\tilde{L}_1 \tilde{w} = 0$. However, by assumption we have that $w_j^1 \in (\operatorname{Ker}\tilde{L}_1)^\perp$, therefore so does $\tilde{w}$. We conclude that $\tilde{w}$ must vanish identically, which is a contradiction with the fact that $\|w_j^1\|_{L^2_{\delta,\vartheta}} = 1$. The claim is proved.

Now, using the remark in Step 3 again, we know that there exists a convergent subsequence $w_j^1 \to w^1$ in $L^2_{\delta,\vartheta}$. This w^1 must be regular, so we can pass to the limit in (3.153) to conclude that $\tilde{L}_1(w^1) = h$, as desired.

Step 5. Now we consider the case that Σ is a sub-manifold of dimension k, and study the localization near a point in $z_0 \in \Sigma$. In Fermi coordinates $z = (x,y)$, this is a similar estimate to that of (3.92).

First let χ be a cut-off function such that $\chi(r) = 1$ for $r \leq d$ and $\chi(r) = 0$ for $r \geq 2d$. Define $\tilde{\chi}(z) = \chi(\mathrm{dist}(z,\Sigma))$, and consider $\tilde{\phi} = \tilde{\chi}\phi$. Using Fermi coordinates near Σ and around a point $z_0 \in \Sigma$, that can be taken to be $z_0 = (0,0)$ without loss of generality, then, for $z = (x,y)$ satisfying $|x| \ll 1, |y| \ll 1$, by checking the estimates in the proof of Lemma 3.5.7, one can get that

$$(-\Delta_z)^\gamma \tilde{\phi}(z) = (1 + |x|^{\frac{1}{2}})(-\Delta_{\mathbb{R}^n \setminus \mathbb{R}^k})^\gamma \tilde{\phi}(x,y) + |x|^{-\gamma}\tilde{\phi} + \mathscr{R}_2 \tag{3.154}$$

where

$$\begin{aligned}
\mathscr{R}_2 = &\int_\Sigma \int_{\{|x|^\beta < |\tilde{x}| < 2d\}} \frac{\tilde{\phi}}{|\tilde{x}|^{N+2\gamma}}\, d\tilde{x}dy + |x|^{-\beta(N+2\gamma)} \int_\Sigma \int_{\{|\tilde{x}| < |x|^\beta\}} \tilde{\phi}\, d\tilde{x}dy \\
&+ |x|^{\beta k} \int_\Sigma \int_{\{|x|^\beta < |\tilde{x}| < 2d\}} \frac{\tilde{\phi}}{|\tilde{x}|^{n+2\gamma}}\, d\tilde{x}dy \\
\leq\ & \|\tilde{\phi}\|_{L^2_{\delta+2\gamma}} |x|^{-\frac{1}{4}(N-2\gamma-2\delta)},
\end{aligned}$$

where we have used Hölder inequality and that $\beta = \frac{1}{2}$. Here $\|\tilde{\phi}\|_{L^2_{\delta+2\gamma}}$ is the weighted norm near Σ. One can easily check that the L^2_δ norm of $\mathscr{R}_2$ and $|x|^{-\gamma}\tilde{\phi}$ are bounded by $o(1)\|\phi\|_{L_{\delta+2\gamma,\vartheta+2\gamma}}$ for small d if $\delta < \frac{N-2\gamma}{2}$.

Next we consider the effect of the localization. Let

$$I_1(z) = [L_1, \tilde{\chi}]\phi = k_{n,\gamma} \int_{\mathbb{R}^n} \frac{\tilde{\chi}(z) - \tilde{\chi}(\tilde{z})}{|z - \tilde{z}|^{n+2\gamma}} \phi(\tilde{z})\, d\tilde{z}.$$

For $|z| \gg 1$, one has

$$I_1(z) \lesssim |z|^{-(n+2\gamma)} \int_{\mathscr{T}_{2d}} \phi(\tilde{z})\, d\tilde{z} \lesssim d^{\delta + \frac{N}{2} + 3\gamma} \|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}} |z|^{-(n+2\gamma)}.$$

Adding the weight at infinity we get that $\|I_1\|_{L^2_\vartheta(\mathbb{R}^n\setminus B_R)}$ can be bounded by $o(1)\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}}$ for d small and R large whenever $\delta > -\frac{N+2\gamma}{2} - 2\gamma$, $\vartheta > -\frac{n+2\gamma}{2} - 2\gamma$.

For $|x| \ll 1$, one has

$$I_1(z) \lesssim \int_{\mathscr{T}_{2d}\setminus\mathscr{T}_d} \frac{\phi}{|z-\tilde{z}|^{n+2\gamma}}\,d\tilde{z} + \int_{\mathscr{T}_{2d}^c} \frac{\phi}{|z-\tilde{z}|^{n+2\gamma}}\,d\tilde{z}.$$

One can check that the L^2_δ term can be bounded by

$$\begin{aligned}\|I_1\|_{L^2_\delta} &\le C[\|\phi\|_{L^2(\mathscr{K})} + R^{\frac{-n+2\gamma+2\vartheta}{2}}\|\phi\|_{L^2_{\vartheta+2\gamma}(B_R^c)}]\\ &\le C[\|\phi\|_{L^2(\mathscr{K})} + o(1)\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}}]\end{aligned}$$

for d small and R large if $\delta < \frac{N-2\gamma}{2}$, $\vartheta < \frac{n-2\gamma}{2}$.

For $z \in \mathscr{K}$, the estimate goes similarly for $\delta > -\frac{N+2\gamma}{2} - 2\gamma$. In conclusion, we have

$$\|I_1\|_{L^2_{\delta,\vartheta}} \le C\left[\|\phi\|_{L^2(\mathscr{K})} + o(1)\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}}\right]. \tag{3.155}$$

Note that this estimate only uses the values of the function ϕ when $|y| \ll 1$. Indeed, by checking the arguments in Lemma 5.7, the main term of the expansion for the fractional Laplacian in (3.154) comes from I_{11}, i.e. for $|y| \ll 1$. The contribution when $|y| > |x|^\beta$ is included in the remainder term $|x|^\gamma\tilde{\phi} + \mathscr{R}_2$. The localization around the point $z_0 = (0,0)$ is now complete.

Step 6. Next, after localization, we can replace (3.151) by

$$P_\gamma^{g_k} w - \tau w = \tilde{h}, \quad \text{in } \mathbb{S}^{N-1}\times\mathbb{H}^{k+1},$$

and w is supported only near a point $z_0 \in \partial\mathbb{H}^{k+1}$, that can be taken arbitrarily. We first consider the spherical harmonic decomposition for $\mathbb{S}^{N-1}$ and recall the symbol for each projection from Theorem 3.3.5.

The L^2_δ estimate follows similarly as in the case of points, but one uses the Fourier-Helgason transform on hyperbolic space instead of the usual Fourier transform as in Theorem 3.3.5. Note, however, that the hyperbolic metric in (3.45) is written in half-space coordinates as $\frac{dr^2+|dy|^2}{r^2} = dt^2 + e^{2t}|dy|^2$, so in order to account for a weight of the form r^δ one would need to use this transform written in rectangular coordinates. This is well known and comes Kontorovich-Lebedev formulas ([169]). Nevertheless, for our purposes it is more suitable to use this transform in geodesic polar coordinates as it is described in Section 3.11. To account for the weight, we just recall the following relation between two different models for hyperbolic space $\mathbb{H}^{k+1}$, the half space model with metrics $\frac{dr^2+|dy|^2}{r^2}$ and the hyperboloid model with metric $ds^2+\sinh s\, g_{\mathbb{S}^k}$ in geodesic polar coordinates:

$$\cosh s = 1 + \frac{|y|^2+(r-2)^2}{4r}.$$

Since we are working locally near a point $z_0 \in \partial\mathbb{H}^{k+1}$, we can choose $y=0$ in this relation, which yields that $e^{-\delta t} = r^\delta = 2^\delta e^{-\delta s}$. Thus we can use a weight of the form $e^{-\delta s}$ in replacement for $e^{-\delta t}$.

One could redo the theory of Section 3.6 using the Fourier-Helgason transform instead. Indeed, after projection over spherical harmonics, and following (3.174), we can write for $\zeta \in \mathbb{H}^{k+1}$,

$$w_m(\zeta) = \int_{\mathbb{H}^{k+1}} \mathscr{G}(\zeta,\zeta')\bar{h}(\zeta')\,d\zeta',$$

where the Green's function is given by

$$\mathscr{G}(\zeta,\zeta') = \int_{-\infty}^{+\infty} \frac{1}{\Theta_\gamma^m(\lambda)-\tau} k_\lambda(\zeta,\zeta')\,d\lambda.$$

The poles of $\frac{1}{\Theta_\gamma^m(\lambda)-\tau}$ are well characterized; in fact, they coincide with those in the point singularity case.

But instead, we can take one further reduction and consider the projection over spherical harmonics in $\mathbb{S}^k$. That is, in geodesic polar coordinates $\zeta=(s,\varsigma)$, $s>0$, $\varsigma \in \mathbb{S}^k$, we can write $w_m(s,\varsigma) = \sum_j w_{m,j}(s)E_j^{(k)}(\varsigma)$, where $E_j^{(k)}$ are the eigenfunctions for $-\Delta_{\mathbb{S}^k}$. Moreover, note that the symbol (3.48) is radial, so it commutes with this additional projection.

Now we can redo the estimate (3.152), just by taking into account the following facts: first, one also has a simple Plancherel formula (3.171). Second, for a radially symmetric function, the Fourier-Helgason transform takes the form of a simple spherical transform (3.172). Third, the spherical function Φ_λ satisfies (3.173) and we are taking a weight of the form $e^{\delta s}$. Finally, the expression for the symbol (3.48) is the same as in the point singularity case (3.38).

This yields estimate (3.148) from which Fredholm properties follow immediately. $\square$

Remark 3.8.3. We do not claim that our restrictions on δ, ϑ in Proposition 3.8.2 are the sharpest possible (indeed, we chose them in the injectivity region for simplicity), but these are enough for our purposes.

Gathering all restrictions on the weights we obtain:

Corollary 3.8.4. *The operator in* (3.146) *is injective, both in* $\mathbb{R}^N \setminus \{q_1, \ldots, q_K\}$ *and* $\mathbb{R}^n \setminus \bigcup \Sigma_i$*. As a consequence, its adjoint* $(\tilde{L}_\varepsilon^*)^* = \tilde{L}_\varepsilon$ *given in* (3.145) *is surjective.*

Proof. Lemma 3.7.4 shows that, after performing the conjugation, $\tilde{L}_\varepsilon$ is injective in $\mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu}+\frac{N-2\gamma}{2}, \tilde{\nu}+\frac{n-2\gamma}{2}}$. By regularity estimates and our choice of δ, ϑ from (3.147), we immediately obtain injectivity for (3.146). Since, thanks to the Fredholm properties,

$$\operatorname{Ker}(\tilde{L}_\varepsilon^*)^\perp = \operatorname{Rg}(\tilde{L}_\varepsilon),$$

the Corollary follows. $\square$

3.8.2 Uniform estimates

Now we return to the operator L_ε defined in (3.123), the adjoint of

$$L_\varepsilon : L^2_{-\delta,-\vartheta} \to L^2_{-\delta-2\gamma,-\vartheta-2\gamma}$$

is just

$$L_\varepsilon^* : L^2_{\delta+2\gamma,\vartheta+2\gamma} \to L^2_{\delta,\vartheta}.$$

From the above results, one knows that L_ε^* is injective and L_ε is surjective.

Fixing the isomorphisms

$$\pi_{2\delta,2\vartheta} : L^2_{-\delta,-\vartheta} \to L^2_{\delta,\vartheta},$$

we may identify the adjoint L_ε^* as

$$L_\varepsilon^* = \pi_{-2\delta,-2\vartheta} \circ L_\varepsilon \circ \pi_{2\delta,2\vartheta} : L^2_{-\delta+2\gamma,-\vartheta+2\gamma} \to L^2_{-\delta,-\vartheta}.$$

Now we have a new operator

$$\mathbf{L}_\varepsilon = L_\varepsilon \circ L_\varepsilon^* : L_\varepsilon \circ \pi_{-2\delta,-2\vartheta} \circ L_\varepsilon \circ \pi_{2\delta,2\vartheta} : L^2_{-\delta+2\gamma,-\vartheta+2\gamma} \to L^2_{-\delta-2\gamma,-\vartheta-2\gamma}.$$

This map is an isomorphism. Hence there exists a bounded two sided inverse

$$\mathbf{G}_\varepsilon : L^2_{-\delta-2\gamma,-\vartheta-2\gamma} \to L^2_{-\delta+2\gamma,-\vartheta+2\gamma}.$$

Moreover, $G_\varepsilon = L_\varepsilon^* \circ \mathbf{G}_\varepsilon$ is right inverse of L_ε which map into the range of L_ε^*. We will fix our inverse to be this one.

From Corollary 3.8.4

$$\mathbf{G}_\varepsilon : \mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma} \to \mathscr{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{\nu}+2\gamma}$$

and

$$G_\varepsilon : \mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma} \to \mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}$$

are bounded.

We are now in the position to prove uniform surjectivity. It is a consequence of the following two results:

Lemma 3.8.5. *If $u \in \mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}$ and $v \in \mathscr{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{\nu}+2\gamma}$ solve equations $L_\varepsilon u = 0,\ u = L_\varepsilon^* v$, then $u \equiv v \equiv 0$.*

Proof. Suppose u, v satisfy the given system, then one has $L_\varepsilon L_\varepsilon^* v = 0$. Consider $\tilde{w} = \pi_{2\delta,2\vartheta} v$. Multiply the equation by w; integration by parts in $\mathbb{R}^n$ yields

$$0 = \int w L_\varepsilon \circ \pi_{-2\delta,-2\vartheta} \circ L_\varepsilon w = \int \pi_{-2\delta,-2\vartheta} |L_\varepsilon w|^2.$$

Thus $L_\varepsilon w = 0$. Moreover, since $v \in \mathscr{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{\nu}+2\gamma}$, one has $w \in \mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu}+2\gamma+2\delta_{\tilde{\mu}},\tilde{\nu}+2\gamma+2\delta_{\tilde{\nu}}} \hookrightarrow \mathscr{C}^{2\gamma+\alpha}_{\mu',\nu'}$ for some $\mu' > \mathrm{Re}(\gamma_0^+), \nu' > -(n-2\gamma)$, thus by the injectivity property, one has $w \equiv 0$. We conclude then that $u \equiv v \equiv 0$. $\square$

Lemma 3.8.6. *Let G_ε be the bounded inverse of L_ε introduced above, then for ε small, G_ε is uniformly bounded, i.e. for $h \in \mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}$, if $u \in \mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}, v \in \mathscr{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{\nu}+2\gamma}$ solve the system $L_\varepsilon u = h$ and $L^*_\varepsilon v = u$, then one has*

$$\|u\|_{\mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}(\mathbb{R}^n\setminus\Sigma)} \leq C\|h\|_{\mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}(\mathbb{R}^n\setminus\Sigma)}$$

for some $C > 0$ independent of ε small.

Proof. The proof is similar to the proof of Lemma 3.7.4. So we just sketch the proof here and point out the differences. It is by contradiction argument. Assume that there exists $\{\varepsilon^{(n)}\} \to 0$ and a sequence of functions $\{h^{(n)}\}$ and solutions $\{u^{(n)}\}$, $\{v^{(n)}\}$ such that

$$\|u\|_{\mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}(\mathbb{R}^n\setminus\Sigma)} = 1, \quad \|h\|_{\mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}(\mathbb{R}^n\setminus\Sigma)} \to 0,$$

and solve the equation

$$L_\varepsilon u = h,\ L^*_\varepsilon v = u.$$

Here note that, for simplicity, we have dropped the superindex (n). Then using the Green's representation formula, following the argument in Proposition 3.7.4, one can show that

$$\sup_{\{\operatorname{dist}(x,\Sigma)>\sigma\}} \{\rho(x)^{-\tilde{\nu}}|u|\} \leq C(\|h\|_{\mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}} + o(1)\|u\|_{\mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}}),$$

which implies that there exists q_i such that

$$\sup_{\{|x-q_i|<\sigma\}} |x-q_i|^{-\tilde{\mu}}|u| \geq \frac{1}{2}. \tag{3.156}$$

In the second step we study the region $\{|x-q_i| < \sigma\}$. Without loss of generality, assume $q_i = 0$. Define the rescaled function as $\bar{u} = \varepsilon^{-\tilde{\mu}}u(\varepsilon x)$ and similarly for $\bar{v}$ and $\bar{h}$. Similarly to the argument in 3.7.4, $\bar{u}$ will tend to a limit u_∞ that solves

$$(-\Delta)^\gamma u_\infty - pA_{N,p,\gamma}u_1^{p-1}u_\infty = 0 \quad \text{in } \mathbb{R}^N.$$

If we show that this limit vanishes identically, $u_\infty \equiv 0$, then we will reach a contradiction with (3.156).

For this, we wish to show that $\bar{v}$ also tends to a limit. If

$$\|v\|_{\mathscr{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{v}+2\gamma}} \leq C_0 \|u\|_{\mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{v}}}, \tag{3.157}$$

then it is true that the limit exists. If not, we can use the same contradiction argument to show that after some scaling, $\bar{v}$ will tend to a limit $v_\infty \in \mathscr{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{\mu}+2\gamma}$ which solves

$$L_1^* v_\infty = 0.$$

This implies that $v \equiv 0$. This will give a contradiction and yield that (3.157) holds for some constant C_0.

By the above analysis we arrive at the limit problem, in which u_∞, v_∞ solve

$$L_1 u_\infty = 0,\ L_1^* v_\infty = u_\infty \quad \text{in } \mathbb{R}^N.$$

Thus $L_1 L_1^* v_\infty = 0$. Multiply the equation by v_∞ and integrate, one has $L_1^* v_\infty = 0$, which implies that $v_\infty \equiv 0$. So also $u_\infty \equiv 0$. Then, following the argument in Lemma 3.7.4, one can get a contradiction. So the uniform surjectivity holds for all ε small. $\square$

3.9 Conclusion of the proof

If ϕ is a solution to

$$(-\Delta_{\mathbb{R}^n})^\gamma(\bar{u}_\varepsilon + \phi) = |\bar{u}_\varepsilon + \phi|^p \quad \text{in } \mathbb{R}^n \setminus \Sigma,$$

we first show that $\bar{u}_\varepsilon + \phi$ is positive in $\mathbb{R}^n \setminus \Sigma$.

Indeed, for z near Σ, there exists $R > 0$ such that if $\rho(z) < R\varepsilon$, then

$$c_1 \rho(z)^{-\frac{2\gamma}{p-1}} < \bar{u}_\varepsilon < c_2 \rho(z)^{-\frac{2\gamma}{p-1}}$$

for some $c_1, c_2 > 0$. Since $\phi \in \mathscr{C}^{2,\alpha}_{\tilde{\mu},\tilde{v}}$, we have $|\phi| \leq c\rho(z)^{\tilde{\mu}}$. But $\tilde{\mu} > -\frac{2\gamma}{p-1}$, so it follows that $\bar{u}_\varepsilon + \phi > 0$ near Σ. Since $\bar{u}_\varepsilon + \phi \to 0$ as $|z| \to \infty$, by the maximum principle (see for example Lemma 4.13 of [36]), we see that $\bar{u}_\varepsilon + \phi > 0$ in $\mathbb{R}^n \setminus \Sigma$, so it is a positive solution and it is singular at all points of Σ.

Next we will prove the existence of such ϕ. For this, we take an additional restriction on $\tilde{\nu}$

$$-(n-2\gamma) < \tilde{\nu} < -\frac{2\gamma}{p-1}.$$

3.9.1 Solution with isolated singularities ($\mathbb{R}^N \setminus \{q_1, \ldots, q_K\}$)

We first treat the case where Σ is a finite number of points. Recall that equation

$$(-\Delta_{\mathbb{R}^N})^\gamma(\bar{u}_\varepsilon + \phi) = A_{N,p,\gamma}|\bar{u}_\varepsilon + \phi|^p \text{ in } \mathbb{R}^N \setminus \{q_1, \ldots, q_K\}$$

is equivalent to the following:

$$L_\varepsilon(\phi) + Q_\varepsilon(\phi) + f_\varepsilon = 0, \tag{3.158}$$

where f_ε is defined in (3.86), L_ε is the linearized operator from (3.122) and Q_ε contains the remaining higher order terms. Because of Lemma 3.8.6, it is possible to construct a right inverse for L_ε with norm bounded independently of ε. Define

$$F(\phi) := G_\varepsilon[-Q_\varepsilon(\phi) - f_\varepsilon], \tag{3.159}$$

then equation (3.158) is reduced to

$$\phi = F(\phi).$$

Our objective is to show that $F(\phi)$ is a contraction mapping from $\mathscr{B}$ to $\mathscr{B}$, where

$$\mathscr{B} = \{\phi \in \mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}(\mathbb{R}^n \setminus \Sigma) : \|\phi\|_* \leq \beta \varepsilon^{N - \frac{2p\gamma}{p-1}}\}$$

for some large positive β.

In this section, $\|\cdot\|_*$ is the $\mathscr{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}$ norm, and $\|\cdot\|_{**}$ is the $\mathscr{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}$ norm where $\tilde{\mu}, \tilde{\nu}$ are taken as in the surjectivity section.

First we have the following lemma:

Lemma 3.9.1. *We have that, independently of ε small,*

$$\|Q_\varepsilon(\phi_1)-Q_\varepsilon(\phi_2)\|_{**}\le\frac{1}{2l_0}\|\phi_1-\phi_2\|_*$$

for all $\phi_1,\ \phi_2\in\mathscr{B}$, where $l_0=\sup\|G_\varepsilon\|$.

Proof. The estimates here are similar to Lemma 9 in [133]. For completeness, we give here the proof.

With some abuse of notation, in the following paragraphs the notation $\|\cdot\|_*$ and $\|\cdot\|_{**}$ will denote the weighted $\mathscr{C}^0$ norms and not the weighted $\mathscr{C}^\alpha$ norms (for the same weights) that was defined in (3.139).

First we show that there exists $\tau>0$ such that for $\phi\in\mathscr{B}$, we have

$$|\phi(x)|\le\frac{1}{4}\bar{u}_\varepsilon(x)\quad\text{for all}\quad x\in\bigcup_{i=1}^{k}B(q_i,\tau).$$

Indeed, from the asymptotic behaviour of u_1 in Proposition 3.2.1 we know that

$$\begin{aligned}&c_1|x|^{-\frac{2\gamma}{p-1}}<u_{\varepsilon_i}(x)<c_2|x|^{-\frac{2\gamma}{p-1}}\text{ if }|x|<R\varepsilon_i,\\&c_1\varepsilon_i^{N-\frac{2p\gamma}{p-1}}|x|^{-(N-2\gamma)}<u_{\varepsilon_i}(x)<c_2\varepsilon_i^{N-\frac{2p\gamma}{p-1}}|x|^{-(N-2\gamma)}\text{ if }R\varepsilon_i\le|x|<\tau.\end{aligned}$$

The claim follows because $\phi\in\mathscr{B}$ implies that

$$|\phi(x)|<c\beta\varepsilon^{N-\frac{2p\gamma}{p-1}}\rho(x)^{\tilde{\mu}}.$$

Next, since $|\frac{\phi}{\bar{u}_\varepsilon}|\le\frac14$ in $B(q_i,\tau)$, by Taylor' expansion,

$$|Q_\varepsilon(\phi_1)-Q_\varepsilon(\phi_2)|\le c|\bar{u}_\varepsilon|^{p-2}(|\phi_1|+|\phi_2|)|\phi_1-\phi_2|.$$

Thus for $x\in B(q_i,\tau)$, we have

$$\begin{aligned}\rho(x)^{2\gamma-\tilde{\mu}}|Q_\varepsilon(\phi_1)-Q_\varepsilon(\phi_2)|&\le c\rho(x)^{\tilde{\mu}+\frac{2\gamma}{p-1}}(\|\phi_1\|_*+\|\phi_2\|_*)\|\phi_1-\phi_2\|_*\\&\le c\tau^{\tilde{\mu}+\frac{2\gamma}{p-1}}\beta\varepsilon^{N-\frac{2\gamma}{p-1}}\|\phi_1-\phi_2\|_*.\end{aligned}$$

The coefficient in front can be taken as small as desired by choosing ε small. Outside the union of the balls $B(q_i,\tau)$ we use the estimates

$$\bar{u}_\varepsilon(x) \leq c\varepsilon^{N-\frac{2p\gamma}{p-1}}|x|^{-(N-2\gamma)} \quad \text{and} \quad |\phi| \leq c\varepsilon^{N-\frac{2p\gamma}{p-1}}|x|^{\tilde{\nu}},$$

where c depends on τ but not on ε nor ϕ.

For $\rho \geq \tau$ and $|x| \leq R$, we can neglect all factors involving $\rho(x)$, so

$$\begin{aligned}|Q_\varepsilon(\phi_1)-Q_\varepsilon(\phi_2)| &\leq c(|\bar{u}_\varepsilon|^{p-1}+|\phi|^{p-1})|\phi_1-\phi_2| \leq c\varepsilon^{(p-1)(N-\frac{2p\gamma}{p-1})}|\phi_1-\phi_2| \\ &\leq c\varepsilon^{p(N-2\gamma)-N}\|\phi_1-\phi_2\|_*,\end{aligned}$$

for which the coefficient can be as small as desired since $p>\frac{N}{N-2\gamma}$.

Lastly, for $|x|\geq R$, in this region $\bar{u}_\varepsilon=0$, so

$$\begin{aligned}\rho(x)^{2\gamma-\tilde{\nu}}|Q_\varepsilon(\phi_1)-Q_\varepsilon(\phi_2)| &\leq c\rho(x)^{2\gamma-\tilde{\nu}}(\phi_1^{p-1}+\phi_2^{p-1})|\phi_1-\phi_2| \\ &\leq c\rho(x)^{2\gamma-\tilde{\nu}+p\tilde{\nu}}\varepsilon^{N(p-1)-2p\gamma}\|\phi_1-\phi_2\|_*,\end{aligned}$$

and here the coefficient can be also chosen as small as we wish because $\tilde{\nu}<-\frac{2\gamma}{p-1}$ implies that $2\gamma-\tilde{\nu}+p\tilde{\nu}<0$.

Combining all the above estimates, one has

$$\|Q_\varepsilon(\phi_1)-Q_\varepsilon(\phi_2)\|_{**} \leq \frac{1}{2l_0}\|\phi_1-\phi_2\|_*$$

as desired.

Now we go back to the original definition of the norms $\|\cdot\|_*$, $\|\cdot\|_{**}$ from (3.139). For this, we need to estimate the Hölder norm of $Q_\varepsilon(\phi_1)-Q_\varepsilon(\phi_2)$. First in each $B(q_i,\tau)$,

$$\begin{aligned}\nabla Q_\varepsilon(\phi) = p\Big((\bar{u}_\varepsilon+\phi)^{p-1}-\bar{u}_\varepsilon^{p-1}-(p-1)\bar{u}_\varepsilon^{p-1}\phi\Big)\nabla\bar{u}_\varepsilon \\ +p((\bar{u}_\varepsilon+\phi)^{p-1}-\bar{u}_\varepsilon^{p-1})\nabla\phi,\end{aligned}$$

and similarly as before, we can get that

$$\rho(x)^{2\gamma+1-\tilde{\mu}}|\nabla(Q_\varepsilon(\phi_1)-Q_\varepsilon(\phi_2))| \leq c\varepsilon^{N-\frac{2p\gamma}{p-1}}\|\phi_1-\phi_2\|_*.$$

Moreover, for $\tau < \rho(x) < R$,

$$|\nabla(Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2))| \le c\varepsilon^{p(N-2\gamma)-N}\|\phi_1 - \phi_2\|_*.$$

Lastly, for $\rho(x) > R$,

$$\nabla Q_\varepsilon(\phi_1 - \phi_2) = p\phi_1^{p-1}\nabla(\phi_1 - \phi_2) + p\nabla\phi_2(\phi_1^{p-1} - \phi_2^{p-1}),$$

which yields

$$\begin{aligned}
&\rho^{-\tilde{\nu}+2\gamma+1}|\nabla Q_\varepsilon(\phi_1 - \phi_2)| \\
&\le \rho^{2\gamma+1-\tilde{\nu}}\Big[(\|\phi_1\|_* + \|\phi_2\|_*)^{p-1}\rho^{(p-1)(\tilde{\nu}-2\gamma)}\|\phi_1 - \phi_2\|_*\rho^{\tilde{\nu}-2\gamma-1} \\
&\quad + \|\phi_2\|_*\rho^{\tilde{\nu}-2\gamma-1}\|\phi_1^{p-1} - \phi_2^{p-1}\|_*\Big] \\
&\le c\varepsilon^{N-\frac{2p\gamma}{p-1}}\|\phi_1 - \phi_2\|_*.
\end{aligned}$$

This completes the desired estimate for $Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2)$ and concludes the proof of the lemma. $\square$

Recall that $\|f_\varepsilon\|_{**} \le C_0\varepsilon^{N-\frac{2p\gamma}{p-1}}$ for some $C_0 > 0$ from (3.88). Then the lemma above gives an estimate for the map (3.159). Indeed,

$$\begin{aligned}
\|F(\phi)\|_* &\le l_0[\|Q_\varepsilon(\phi)\|_{**} + \|f_\varepsilon\|_{**}] \le l_0\|Q_\varepsilon(\phi)\|_{**} + l_0C_0\varepsilon^{N-\frac{2p\gamma}{p-1}} \\
&\le \frac{1}{2}\|\phi\|_* + l_0C_0\varepsilon^{N-\frac{2p\gamma}{p-1}} \le \beta\varepsilon^{N-\frac{2p\gamma}{p-1}},
\end{aligned}$$

and

$$\|F(\phi_1) - F(\phi_2)\|_* \le l_0\|Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2)\|_{**} \le \frac{1}{2}\|\phi_1 - \phi_2\|_*$$

if we choose $\beta > 2l_0C_0$. So $F(\phi)$ is a contraction mapping from $\mathscr{B}$ to $\mathscr{B}$. This implies the existence of a solution ϕ to (3.158).

3.9.2 The general case $\mathbb{R}^n \setminus \Sigma$, Σ a sub-manifold of dimension k

For the more general case, only minor changes need to be made in the above argument. The most important one comes from Lemma 3.5.7 and it says that the weight parameter μ must now lie in the smaller interval:

$$-\frac{2\gamma}{p-1} < \tilde{\mu} < \min\left\{\gamma - \tfrac{2\gamma}{p-1}, \tfrac{1}{2} - \tfrac{2\gamma}{p-1}, \mathrm{Re}(\gamma_0^-)\right\}. \tag{3.160}$$

In this case, we only need to replace the exponent $N - \frac{2p\gamma}{p-1}$ in the above argument by $q = \min\{\frac{(p-3)\gamma}{p-1} - \tilde{\mu}, \frac{1}{2} - \gamma + \frac{(p-3)\gamma}{p-1} - \tilde{\mu}\}$, then $q > 0$ if $\tilde{\mu}$ is chosen to satisfy (3.160). We get a solution to (3.158), and this concludes the proof of Theorem 3.1.1.

3.10 Some known results on special functions

Lemma 3.10.1. [4, 166] *Let $z \in \mathbb{C}$. The hypergeometric function is defined for $|z| < 1$ by the power series*

$${}_2\mathrm{F}_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}.$$

It is undefined (or infinite) if c equals a non-positive integer. Some properties are

i. The hypergeometric function evaluated at $z = 0$ satisfies

$${}_2\mathrm{F}_1(a+j, b-j; c; 0) = 1;\ j = \pm 1, \pm 2, ... \tag{3.161}$$

ii. If $|arg(1-z)| < \pi$, then

$$\begin{aligned}{}_2\mathrm{F}_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2\mathrm{F}_1\left(a,b;a+b-c+1;1-z\right) \\ &+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\ {}_2\mathrm{F}_1(c-a,c-b;c-a-b+1;1-z).\end{aligned} \tag{3.162}$$

iii. The hypergeometric function is symmetric with respect to first and second arguments, i.e

$$ {}_2\mathrm{F}_1(a,b;c;z) = {}_2\mathrm{F}_1(b,a;c;z). \tag{3.163}$$

iv. Let $m \in \mathbb{N}$. The m-derivative of the hypergeometric function is given by

$$\frac{d^m}{dz^m}\left[(1-z)^{a+m-1}\ {}_2\mathrm{F}_1(a,b;c;z)\right] = \frac{(-1)^m(a)_m(c-b)_m}{(c)_m}(1-z)^{a-1}\ {}_2\mathrm{F}_1(a+m,b;c+m;z). \tag{3.164}$$

Lemma 3.10.2. [4, 166] *Let $z \in \mathbb{C}$. Some well known properties of the Gamma function $\Gamma(z)$ are*

$$\Gamma(\bar{z}) = \overline{\Gamma(z)}, \tag{3.165}$$

$$\Gamma(z+1) = z\Gamma(z), \tag{3.166}$$

$$\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z). \tag{3.167}$$

It is a meromorphic function in $z \in \mathbb{C}$ and its residue at each poles is given by

$$\mathrm{Res}(\Gamma(z),-j) = \frac{(-1)^n}{j!}, \quad j = 0,1,\ldots. \tag{3.168}$$

Let $\psi(z)$ denote the Digamma function defined by

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}.$$

This function has the expansion

$$\psi(z) = \psi(1) + \sum_{l=0}^{\infty}\left(\frac{1}{l+1} - \frac{1}{l+z}\right). \tag{3.169}$$

Let $B(z_1,z_2)$ denote the Beta function defined by

$$B(z_1,z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}.$$

If z_2 is a fixed number and $z_1 > 0$ is big enough, then this function behaves

$$B(z_1,z_2) \sim \Gamma(z_2)(z_1)^{-z_2}.$$

3.11 A review of the Fourier-Helgason transform on Hyperbolic space

Consider hyperbolic space $\mathbb{H}^{k+1}$, parameterized with coordinates ζ. It can be written as a symmetric space of rank one as the quotient $\mathbb{H}^{k+1} \approx \frac{SO(1,k+1)}{SO(k+1)}$. Fourier transform on hyperbolic space is a particular case of the Helgason-Fourier transform on symmetric spaces. Some standard references are [26, 119, 168]; we mostly follow the exposition of Chapter 8 in [102].

Hyperbolic space $\mathbb{H}^{k+1}$ may be defined as the upper branch of a hyperboloid in $\mathbb{R}^{k+2}$ with the metric induced by the Lorentzian metric in $\mathbb{R}^{k+2}$ given by $-d\zeta_0^2 + d\zeta_1^2 + \ldots + d\zeta_{k+1}^2$, i.e., $\mathbb{H}^{k+1} = \{(\zeta_0,\ldots,\zeta_{k+1}) \in \mathbb{R}^{k+2} : \zeta_0^2 - \zeta_1^2 - \ldots - \zeta_{k+1}^2 = 1,\ \zeta_0 > 0\}$, which in polar coordinates may be parameterized as

$$\mathbb{H}^{k+1} = \{\zeta \in \mathbb{R}^{k+2} : \zeta = (\cosh s, \varsigma \sinh s),\ s \geq 0,\ \varsigma \in \mathbb{S}^k\},$$

with the metric $g_{\mathbb{H}^{k+1}} = ds^2 + \sinh^2 s\, g_{\mathbb{S}^k}$. Under these definitions the Laplace-Beltrami operator is given by

$$\Delta_{\mathbb{H}^{k+1}} = \partial_{ss} + k\frac{\cosh s}{\sinh s}\,\partial_s + \frac{1}{\sinh^2 s}\,\Delta_{\mathbb{S}^k},$$

and the volume element is

$$d\mu_\zeta = \sinh^k s\, ds\, d\varsigma.$$

We denote by $[\cdot,\cdot]$ the internal product induced by the Lorentzian metric, i.e.,

$$[\zeta,\zeta'] = \zeta_0\zeta_0' - \zeta_1\zeta_1' - \ldots - \zeta_{k+1}\zeta_{k+1}'.$$

The hyperbolic distance between two arbitrary points is given by $\mathrm{dist}(\zeta,\zeta') = \cosh^{-1}([\zeta,\zeta'])$, and in the particular case that $\zeta = (\cosh s, \varsigma \sinh s)$, $\zeta' = O$,

$$\mathrm{dist}(\zeta, O) = s.$$

The unit sphere $\mathbb{S}^{N-1}$ is identified with the subset $\{\zeta \in \mathbb{R}^{k+2} : [\zeta,\zeta] = 0, \zeta_0 = 1\}$ via the map $b(\varsigma) = (1,\varsigma)$ for $\varsigma \in \mathbb{S}^k$.

Given $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{S}^k$, let $h_{\lambda,\omega}(\zeta)$ be the generalized eigenfunctions of the Laplace-Beltrami operator. This is,

$$\Delta_{\mathbb{H}^{k+1}} h_{\lambda,\omega} = -\left(\lambda^2 + \tfrac{k^2}{4}\right) h_{\lambda,\omega}.$$

These may be explicitly written as

$$h_{\lambda,\omega}(\zeta) = [\zeta, b(\omega)]^{i\lambda - \frac{k}{2}} = (\cosh s - \sinh s \langle \varsigma, \omega\rangle)^{i\lambda - \frac{k}{2}}, \quad \zeta \in \mathbb{H}^{k+1}.$$

In analogy to the Euclidean space, the Fourier transform on $\mathbb{H}^{k+1}$ is defined by

$$\hat{u}(\lambda,\omega) = \int_{\mathbb{H}^{k+1}} u(\zeta)\, h_{\lambda,\omega}(\zeta)\, d\mu_\zeta.$$

Moreover, the following inversion formula holds:

$$u(\zeta) = \int_{-\infty}^{\infty} \int_{\mathbb{S}^k} \bar{h}_{\lambda,\omega}(\zeta) \hat{u}(\lambda,\omega)\, \frac{d\omega\, d\lambda}{|c(\lambda)|^2}, \tag{3.170}$$

where $c(\lambda)$ is the Harish-Chandra coefficient:

$$\frac{1}{|c(\lambda)|^2} = \frac{1}{2(2\pi)^{k+1}} \frac{|\Gamma(i\lambda + (\frac{k}{2})|^2}{|\Gamma(i\lambda)|^2}.$$

There is also a Plancherel formula:

$$\int_{\mathbb{H}^{k+1}} |u(\zeta)|^2\, d\mu_\zeta = \int_{\mathbb{R}\times\mathbb{S}^{N-1}} |\hat{u}(\lambda,\omega)|^2 \frac{d\omega\, d\lambda}{|c(\lambda)|^2}, \tag{3.171}$$

which implies that the Fourier transform extends to an isometry between the Hilbert spaces $L^2(\mathbb{H}^{k+1})$ and $L^2(\mathbb{R}_+ \times \mathbb{S}^k, |c(\lambda)|^{-2} d\lambda\, d\omega)$.

If u is a radial function, then $\hat{u}$ is also radial, and the above formulas simplify. In this setting, it is customary to normalize the measure of $\mathbb{S}^k$ to one in order not to account for multiplicative constants. Thus one defines the spherical Fourier transform as

$$\hat{u}(\lambda) = \int_{\mathbb{H}^{k+1}} u(\zeta)\Phi_{-\lambda}(\zeta)\, d\mu_\zeta, \tag{3.172}$$

where

$$\Phi_\lambda(\zeta) = \int_{\mathbb{S}^k} h_{-\lambda,\omega}(\zeta)\, d\omega$$

is known as the elementary spherical function. In addition, (3.170) reduces to

$$u(\zeta) = \int_{-\infty}^{\infty} \hat{u}(\lambda)\Phi_\lambda(\zeta)\, \frac{d\lambda}{|c(\lambda)|^2}.$$

There are many explicit formulas for $\Phi_\lambda(\zeta)$ (we also write $\Phi_\lambda(s)$, since it is a radial function). In particular, $\Phi_{-\lambda}(s) = \Phi_\lambda(s) = \Phi_\lambda(-s)$, which yields regularity at the origin $s = 0$. Here we are interested in its asymptotic behavior. Indeed,

$$\Phi_\lambda(s) \sim e^{(i\lambda - \frac{k}{2})s} \quad \text{as} \quad s \to +\infty. \tag{3.173}$$

It is also interesting to observe that

$$\widehat{\Delta_{\mathbb{H}^{k+1}} u} = -\left(\lambda^2 + \tfrac{k^2}{4}\right)\hat{u}.$$

We define the convolution operator as

$$u * v(\zeta) = \int_{\mathbb{H}^{k+1}} u(\zeta')v(\tau_\zeta^{-1}\zeta')\, d\mu_{\zeta'},$$

where $\tau_\zeta : \mathbb{H}^{k+1} \to \mathbb{H}^{k+1}$ is an isometry that takes ζ into O. If v is a radial function, then the convolution may be written as

$$u * v(\zeta) = \int_{\mathbb{H}^{k+1}} u(\zeta')v(\mathrm{dist}(\zeta,\zeta'))\, d\mu_{\zeta'},$$

and we have the property

$$\widehat{u * v} = \hat{u}\,\hat{v},$$

in analogy to the usual Fourier transform.

On hyperbolic space there is a well developed theory of Fourier multipliers. In L^2 spaces everything may be written out explicitly. For instance, let $m(\lambda)$ be a multiplier in Fourier variables. A function $\hat{u}(\lambda,\omega)=\hat{m}(\lambda)u_0(\lambda,\omega)$, by the inversion formula for the Fourier transform (3.170) and expression (3.11), may be written as

$$\begin{aligned} u(x) &= \int_{-\infty}^{\infty}\int_{\mathbb{S}^k} m(\lambda)\hat{u}_0(\lambda,\omega)\bar{h}_{\lambda,\omega}(\zeta)\,\frac{d\omega\,d\lambda}{|c(\lambda)|^2} \\ &= \int_{-\infty}^{\infty}\int_{\mathbb{H}^{k+1}} m(\lambda)u_0(\zeta')k_\lambda(\zeta,\zeta')\,d\mu_{\zeta'}\,d\lambda, \end{aligned} \tag{3.174}$$

where we have denoted

$$k_\lambda(\zeta,\zeta')=\frac{1}{|c(\lambda)|^2}\int_{\mathbb{S}^k}\bar{h}_{\lambda,\omega}(\zeta)\,h_{\lambda,\omega}(\zeta')\,d\omega.$$

It is known that k_λ is invariant under isometries, i.e.,

$$k_\lambda(\zeta,\zeta')=k_\lambda(\tau\zeta,\tau\zeta'),$$

for all $\tau\in SO(1,k+1)$, and in particular,

$$k_\lambda(\zeta,\zeta')=k_\lambda(\mathrm{dist}(\zeta,\zeta')),$$

so many times we will simply write $k_\lambda(\rho)$ for $\rho=\mathrm{dist}(\zeta,\zeta')$. We recall the following formulas for k_λ:

Lemma 3.11.1 ([102]). *For $k+1\geq 3$ odd,*

$$k_\lambda(\rho)=c_k\left(\frac{\partial_\rho}{\sinh\rho}\right)^{\frac{k}{2}}(\cos\lambda\rho),$$

and for $k+1\geq 2$ even,

$$k_\lambda(\rho)=c_k\int_\rho^\infty\frac{\sinh\tilde{\rho}}{\sqrt{\cosh\tilde{\rho}-\cosh\rho}}\left(\frac{\partial_{\tilde{\rho}}}{\sinh\tilde{\rho}}\right)^{\frac{k+1}{2}}(\cos\lambda\tilde{\rho})\,d\tilde{\rho}.$$

Chapter 4

Extremals for Hyperbolic Hardy–Schrödinger Operators

4.1 Introduction

Hardy–Schrödinger operators on manifolds are of the form $\Delta_g - V$, where Δ_g is the Laplace–Beltrami operator and V is a potential that has a quadratic singularity at some point of the manifold. For hyperbolic spaces, Carron [46] showed that, just like in the Euclidean case and with the same best constant, the following inequality holds on any Cartan–Hadamard manifold M,

$$\frac{(n-2)^2}{4}\int_M \frac{u^2}{d_g(o,x)^2}\,dv_g \le \int_M |\nabla_g u|^2\,dv_g \qquad \text{for all } u \in C_c^\infty(M),$$

where $d_g(o,x)$ denotes the geodesic distance to a fixed point $o \in M$. There are many other works identifying suitable Hardy potentials, their relationship with the elliptic operator on hand, as well as corresponding energy inequalities [6, 57, 71, 123, 126, 181**?**]. In the Euclidean case, the Hardy potential $V(x) = \frac{1}{|x|^2}$ is distinguished by the fact that $\frac{u^2}{|x|^2}$ has the same homogeneity as $|\nabla u|^2$, but also $\frac{u^{2^*(s)}}{|x|^s}$, where $2^*(s) = \frac{2(n-s)}{n-2}$ and $0 \le s < 2$. In other words, the integrals $\int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\,dx$, $\int_{\mathbb{R}^n} |\nabla u|^2\,dx$ and $\int_{\mathbb{R}^n} \frac{u^{2^*(s)}}{|x|^s}\,dx$ are invariant under the scaling $u(x) \mapsto \lambda^{\frac{n-2}{2}} u(\lambda x)$, $\lambda > 0$, which makes corresponding minimization problem non-compact, hence giving rise to interesting concentration phenomena. In [5], Adimurthi and Sekar use the fundamental solution of a general second order elliptic operator to generate natural candidates and derive Hardy-type inequalities. They also extended their arguments to Riemannian manifolds using the fundamental solution of the p-Laplacian. In [71], Devyver, Fraas and Pinchover study the case of a general linear second order differential operator P on non-compact manifolds. They find a relation between positive supersolutions of the equation $Pu = 0$, Hardy-type inequalities involving P and a weight W, as well as some properties of the spectrum of a corresponding weighted operator.

In this paper, we shall focus on the Poincaré ball model of the hyperbolic space $\mathbb{B}^n$, $n \geq 3$, that is the Euclidean unit ball $B_1(0) := \{x \in \mathbb{R}^n : |x| < 1\}$ endowed with the metric $g_{\mathbb{B}^n} = \left(\frac{2}{1-|x|^2}\right)^2 g_{\text{Eucl}}$. This framework has the added feature of radial symmetry, which plays an important role and contributes to the richness of the structure. In this direction, Sandeep and Tintarev [153] recently came up with several integral inequalities involving weights on $\mathbb{B}^n$ that are invariant under scaling, once restricted to the class of radial functions (see also Li and Wang [126]). As described below, this scaling is given in terms of the fundamental solution of the hyperbolic Laplacian $\Delta_{\mathbb{B}^n} u = \mathrm{div}_{\mathbb{B}^n}(\nabla_{\mathbb{B}^n} u)$. Indeed, let

$$f(r) := \frac{(1-r^2)^{n-2}}{r^{n-1}} \quad \text{and} \quad G(r) := \int_r^1 f(t)\,dt, \tag{4.1}$$

where $r = \sqrt{\sum_{i=1}^n x_i^2}$ denotes the Euclidean distance of a point $x \in B_1(0)$ to the origin. It is known that $\frac{1}{n\omega_{n-1}} G(r)$ is a fundamental solution of the hyperbolic Laplacian $\Delta_{\mathbb{B}^n}$. As usual, the Sobolev space $H^1(\mathbb{B}^n)$ is defined as the completion of $C_c^\infty(\mathbb{B}^n)$ with respect to the norm $\|u\|^2_{H^1(\mathbb{B}^n)} = \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 dv_{g_{\mathbb{B}^n}}$. We denote by $H_r^1(\mathbb{B}^n)$ the subspace of radially symmetric functions. For functions $u \in H_r^1(\mathbb{B}^n)$, we consider the scaling

$$u_\lambda(r) = \lambda^{-\frac{1}{2}} u\left(G^{-1}(\lambda G(r))\right), \qquad \lambda > 0. \tag{4.2}$$

In [153], Sandeep–Tintarev have noted that for any $u \in H_r^1(\mathbb{B}^n)$ and $p \geq 1$, one has the following invariance property:

$$\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u_\lambda|^2 \, dv_{g_{\mathbb{B}^n}} = \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} \quad \text{and} \quad \int_{\mathbb{B}^n} V_p |u_\lambda|^p \, dv_{g_{\mathbb{B}^n}} = \int_{\mathbb{B}^n} V_p |u|^p \, dv_{g_{\mathbb{B}^n}},$$

where

$$V_p(r) := \frac{f(r)^2(1-r^2)^2}{4(n-2)^2 G(r)^{\frac{p+2}{2}}}. \tag{4.3}$$

In other words, the hyperbolic scaling $r \mapsto G^{-1}(\lambda G(r))$ is quite analogous to the Euclidean scaling. Indeed, in that case, by taking $\overline{G}(\rho) = \rho^{2-n}$, we see that $\overline{G}^{-1}(\lambda \overline{G}(\rho)) = \overline{\lambda} = \lambda^{\frac{1}{2-n}}$ for $\rho = |x|$ in $\mathbb{R}^n$. Also, note that $\overline{G}$ is –up to a constant– the fundamental solution of the Euclidean Laplacian Δ in $\mathbb{R}^n$. The weights V_p have the following asymptotic behaviors, for $n \geq 3$ and $p > 1$,

$$V_p(r) = \begin{cases} \dfrac{c_0(n,p)}{r^{n(1-p/2^*)}}(1+o(1)) & \text{as } r \to 0, \\ \dfrac{c_1(n,p)}{(1-r)^{(n-1)(p-2)/2}}(1+o(1)) & \text{as } r \to 1. \end{cases}$$

In particular, for $n \geq 3$, the weight $V_2(r) = \frac{1}{4(n-2)^2}\left(\frac{f(r)(1-r^2)}{G(r)}\right)^2 \sim_{r \to 0} \frac{1}{4r^2}$, and at $r = 1$ has a finite positive value. In other words, the weight V_2 is qualitatively similar to the Euclidean Hardy weight, and Sandeep–Tintarev have indeed established the following Hardy inequality on the hyperbolic space $\mathbb{B}^n$ (Theorem 3.4 of [153]). Also, see [71] where they deal with similar Hardy weights.

$$\frac{(n-2)^2}{4} \int_{\mathbb{B}^n} V_2 |u|^2 \, dv_{g_{\mathbb{B}^n}} \leq \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} \quad \text{for any } u \in H^1(\mathbb{B}^n).$$

They also show in the same paper the following Sobolev inequality, i.e., for some constant $C > 0$.

$$\left(\int_{\mathbb{B}^n} V_{2^*} |u|^{2^*} \, dv_{g_{\mathbb{B}^n}}\right)^{2/2^*} \leq C \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} \quad \text{for any } u \in H^1(\mathbb{B}^n),$$

where $2^* = \frac{2n}{(n-2)}$. By interpolating between these two inequalities taking $0 \leq s \leq 2$, one easily obtain the following Hardy–Sobolev inequality.

Lemma 4.1.1. *If $\gamma < \frac{(n-2)^2}{4}$, then there exists a constant $C > 0$ such that, for any $u \in H^1(\mathbb{B}^n)$,*

$$C\left(\int_{\mathbb{B}^n} V_{2^*(s)}|u|^{2^*(s)}\,dv_{g_{\mathbb{B}^n}}\right)^{2/2^*(s)} \leq \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2\,dv_{g_{\mathbb{B}^n}} - \gamma\int_{\mathbb{B}^n} V_2|u|^2\,dv_{g_{\mathbb{B}^n}},$$

where $2^*(s) := \frac{2(n-s)}{(n-2)}$.

Note that, up to a positive constant, we have $V_{2^*(s)} \sim_{r\to 0} \frac{1}{r^s}$, adding to the analogy with the Euclidean case, where we have for any $u \in H^1(\mathbb{R}^n)$,

$$C\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s}\,dx\right)^{2/2^*(s)} \leq \int_{\mathbb{R}^n} |\nabla u|^2\,dx - \gamma\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2}\,dx.$$

Motivated by the recent progress on the Euclidean Hardy–Schrödinger equation (See for example Ghoussoub–Robert [105, 106], and the references therein), we shall consider the problem of existence of extremals for the corresponding best constant, that is

$$\mu_{\gamma,\lambda}(\Omega) := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla_{\mathbb{B}^n} u|^2\,dv_{g_{\mathbb{B}^n}} - \gamma\int_\Omega V_2|u|^2\,dv_{g_{\mathbb{B}^n}} - \lambda\int_\Omega |u|^2\,dv_{g_{\mathbb{B}^n}}}{\left(\displaystyle\int_\Omega V_{2^*(s)}|u|^{2^*(s)}\,dv_{g_{\mathbb{B}^n}}\right)^{2/2^*(s)}}, \tag{4.4}$$

where $H_0^1(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|u\| = \sqrt{\int_\Omega |\nabla u|^2\,dv_{g_{\mathbb{B}^n}}}$. Similarly to the Euclidean case, and once restricted to radial functions, the general Hardy–Sobolev inequality for the hyperbolic Hardy–Schrödinger operator is invariant under hyperbolic scaling described in (4.2), This invariance makes the corresponding variational problem non-compact and the problem of existence of minimizers quite interesting.

In Proposition 4.3.3, we start by showing that the extremals for the minimization problem (4.4) in the class of radial functions $H_r^1(\mathbb{B}^n)$ can be written explicitly as:

$$U(r) = c\left(G(r)^{-\frac{2-s}{n-2}\alpha_-(\gamma)} + G(r)^{-\frac{2-s}{n-2}\alpha_+(\gamma)}\right)^{-\frac{n-2}{2-s}},$$

where c is a positive constant and $\alpha_\pm(\gamma)$ satisfy

$$\alpha_\pm(\gamma) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\gamma}{(n-2)^2}}.$$

In other words, we show that

$$\mu^{\mathrm{rad}}_{\gamma,0}(\mathbb{B}^n) := \inf_{u\in H^1_r(\mathbb{B}^n)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2\, dv_{g_{\mathbb{B}^n}} - \gamma \int_{\mathbb{B}^n} V_2 |u|^2\, dv_{g_{\mathbb{B}^n}}}{\left(\displaystyle\int_{\mathbb{B}^n} V_{2^*(s)} |u|^{2^*(s)}\, dv_{g_{\mathbb{B}^n}}\right)^{2/2^*(s)}} \tag{4.5}$$

is attained by U.

Note that the radial function $G^\alpha(r)$ is a solution of $-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = 0$ on $\mathbb{B}^n \setminus \{0\}$ if and only if $\alpha = \alpha_\pm(\gamma)$. These solutions have the following asymptotic behavior

$$G(r)^{\alpha_\pm(\gamma)} \sim c(n,\gamma) r^{-\beta_\pm(\gamma)} \text{ as } r \to 0,$$

where

$$\beta_\pm(\gamma) = \frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} - \gamma}.$$

These then yield positive solutions to the equation

$$-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = V_{2^*(s)} u^{2^*(s)-1} \quad \text{in } \mathbb{B}^n.$$

We point out the paper [128] (also see [22, 23, 101]), where the authors considered the counterpart of the Brezis–Nirenberg problem on $\mathbb{B}^n$ ($n \geq 3$), and discuss issues of existence and non-existence for the equation

$$-\Delta_{\mathbb{B}^n} u - \lambda u = u^{2^*-1} \quad \text{in } \mathbb{B}^n,$$

in the absence of a Hardy potential.

Next, we consider the attainability of $\mu_{\gamma,\lambda}(\Omega)$ in subdomains of $\mathbb{B}^n$ without necessarily any symmetry. In other words, we will search for positive solutions for the equation

$$\begin{cases} -\Delta_{\mathbb{B}^n} u - \gamma V_2 u - \lambda u = V_{2^*(s)} u^{2^*(s)-1} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

where Ω is a compact smooth subdomain of $\mathbb{B}^n$ such that $0 \in \Omega$, but $\overline{\Omega}$ does not touch the boundary of $\mathbb{B}^n$ and $\lambda \in \mathbb{R}$. Note that the metric is then smooth on such Ω, and the only singularity we will be dealing with will be coming from the Hardy-type potential V_2 and the Hardy–Sobolev weight $V_{2^*(s)}$, which behaves like $\frac{1}{r^2}$ (resp., $\frac{1}{r^s}$) at the origin. This is analogous to the Euclidean problem on bounded domains considered by Ghoussoub–Robert [105, 106]. We shall therefore rely heavily on their work, at least in dimensions $n \geq 5$. Actually, once we perform a conformal transformation, the equation above reduces to the study of the following type of problems on bounded domains in $\mathbb{R}^n$:

$$\begin{cases} -\Delta v - \left(\frac{\gamma}{|x|^2} + h_{\gamma,\lambda}(x)\right) v = b(x) \frac{v^{2^*(s)-1}}{|x|^s} & \text{in } \Omega \\ v \geq 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $b(x)$ is a positive $C^0(\overline{\Omega})$ function with

$$b(0) = \frac{(n-2)^{\frac{n-s}{n-2}}}{2^{2-s}}, \tag{4.7}$$

$$h_{\gamma,\lambda}(x) = \gamma a(x) + \frac{4\lambda - n(n-2)}{(1-|x|^2)^2},$$

$$a(x) = a(r) = \begin{cases} \frac{4}{r} + 8 + g_3(r) & \text{when } n = 3, \\ \\ 8\log\frac{1}{r} - 4 + g_4(r) & \text{when } n = 4, \\ \\ \frac{4(n-2)}{n-4} + rg_n(r) & \text{when } n \geq 5. \end{cases} \tag{4.8}$$

with $g_n(0) = 0$, for all $n \geq 3$. Ghoussoub–Robert [106] have recently tackled such an equation, but in the case where $h(x)$ and $b(x)$ are constants. We shall explore here the extent of which their techniques could be extended to this setting. To start with, the following regularity result will then follow immediately.

Theorem 4.1.2 (Regularity)**.** *Let $\Omega \Subset \mathbb{B}^n$, $n \geq 3$, and $\gamma < \frac{(n-2)^2}{4}$. If $u \not\equiv 0$ is a non-negative weak solution of the equation* (4.6) *in the hyperbolic Sobolev space $H^1(\Omega)$, then*

$$\lim_{|x|\to 0} \frac{u(x)}{G(|x|)^{\alpha_-}} = K > 0.$$

We also need to define a notion of mass of a domain associated to the operator $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$. We therefore show the following.

Theorem 4.1.3 (The hyperbolic Hardy-singular mass of $\Omega \Subset \mathbb{B}^n$)**.** *Let $0 \in \Omega \Subset \mathbb{B}^n$, $n \geq 3$, and $\gamma < \frac{(n-2)^2}{4}$. Let $\lambda \in \mathbb{R}$ be such that the operator $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$ is coercive. Then, there exists a solution $K_\Omega \in C^\infty\left(\overline{\Omega} \setminus \{0\}\right)$ to the linear problem,*

$$\begin{cases} -\Delta_{\mathbb{B}^n} K_\Omega - \gamma V_2 K_\Omega - \lambda K_\Omega = 0 & \text{in } \Omega \\ K_\Omega \geq 0 & \text{in } \Omega \\ K_\Omega = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

such that $K_\Omega(x) \simeq_{|x|\to 0} c\, G(|x|)^{\alpha_+}$ for some positive constant c. Furthermore,

1. *If $K'_\Omega \in C^\infty\left(\overline{\Omega} \setminus \{0\}\right)$ is another solution of the above linear equation, then there exists a $C > 0$ such that $K'_\Omega = CK_\Omega$.*

2. *If $\gamma > \max\left\{\frac{n(n-4)}{4}, 0\right\}$, then there exists $m^H_{\gamma,\lambda}(\Omega) \in \mathbb{R}$ such that*

$$K_\Omega(x) = G(|x|)^{\alpha_+} + m^H_{\gamma,\lambda}(\Omega) G(|x|)^{\alpha_-} + o\left(G(|x|)^{\alpha_-}\right) \qquad \text{as } x \to 0. \tag{4.10}$$

The constant $m^H_{\gamma,\lambda}(\Omega)$ will be referred to as the **hyperbolic mass** *of the domain Ω associated with the operator $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$.*

And just like the Euclidean case, solutions exist in high dimensions, while the positivity of the "hyperbolic mass"will be needed for low dimensions. More precisely,

Theorem 4.1.4. *Let $\Omega \Subset \mathbb{B}^n$ $(n \geq 3)$ be a smooth domain with $0 \in \Omega$, $0 \leq \gamma < \frac{(n-2)^2}{4}$ and let $\lambda \in \mathbb{R}$ be such that the operator $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$ is coercive. Then, the best constant $\mu_{\gamma,\lambda}(\Omega)$ is attained under the following conditions:*

1. $n \geq 5$, $\gamma \leq \dfrac{n(n-4)}{4}$ *and* $\lambda > \dfrac{n-2}{n-4}\left(\dfrac{n(n-4)}{4} - \gamma\right)$.
2. $n = 4$, $\gamma = 0$ *and* $\lambda > 2$.
3. $n = 3$, $\gamma = 0$ *and* $\lambda > \dfrac{3}{4}$.
4. $n \geq 3$, $\max\left\{\dfrac{n(n-4)}{4}, 0\right\} < \gamma < \dfrac{(n-2)^2}{4}$ *and* $m^H_{\gamma,\lambda}(\Omega) > 0$.

As mentioned above, the above theorem will be proved by using a conformal transformation that reduces the problem to the Euclidean case, already considered by Ghoussoub–Robert [106]. Actually, this leads to the following variation of the problem they considered, where the perturbation can be singular but not as much as the Hardy potential.

Theorem 4.1.5. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$, with $0 \in \Omega$ and $0 \leq \gamma < \frac{(n-2)^2}{4}$. Let $h \in C^1(\overline{\Omega} \setminus \{0\})$ be such that*

$$h(x) = -\mathscr{C}_1 |x|^{-\theta} \log|x| + \tilde{h}(x) \text{ where } \lim_{x \to 0} |x|^{\theta} \tilde{h}(x) = \mathscr{C}_2, \tag{4.11}$$

for some $0 \leq \theta < 2$ and $\mathscr{C}_1, \mathscr{C}_2 \in \mathbb{R}$, and the operator $-\Delta - \left(\frac{\gamma}{|x|^2} + h(x)\right)$ is coercive. Also, assume that $b(x)$ is a non-negative function in $C^0(\overline{\Omega})$ with $b(0) > 0$.

Then the best constant

$$\mu_{\gamma,h}(\Omega) := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \left(|\nabla u|^2 - \left(\frac{\gamma}{|x|^2} + h(x)\right) u^2\right) dx}{\left(\displaystyle\int_\Omega b(x)\frac{|u|^{2^*(s)}}{|x|^s}\, dx\right)^{2/2^*(s)}} \tag{4.12}$$

is attained if one of the following two conditions is satisfied:

1. $\gamma \le \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}$ *and, either* $\mathscr{C}_1 > 0$ *or* $\{\mathscr{C}_1 = 0, \mathscr{C}_2 > 0\}$*;*
2. $\frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4} < \gamma < \frac{(n-2)^2}{4}$ *and* $m_{\gamma,h}(\Omega) > 0$*, where* $m_{\gamma,h}(\Omega)$ *is the mass of the domain* Ω *associated to the operator* $-\Delta - \left(\frac{\gamma}{|x|^2} + h(x)\right)$.

The paper is organized as follows. In Section 2, we introduce the Hardy–Sobolev type inequalities in hyperbolic space. In Section 3, we find the explicit solutions for Hardy–Sobolev equations corresponding to (4.5) on $\mathbb{B}^n$. In section 4, we show that our main equation (4.6) can be transformed into the Hardy–Sobolev type equations in Euclidean space under a conformal transformation. Section 5 is then devoted to establish the existence results for (4.6) on compact submanifolds of $\mathbb{B}^n$ by studying the transformed equations in Euclidean space.

4.2 Hardy–Sobolev type inequalities in hyperbolic space

The starting point of the study of existence of weak solutions of the above problems are the following inequalities which will guarantee that functionals (4.4) and (4.5) are well defined and bounded below on the right function spaces. The Sobolev inequality for hyperbolic space [153] asserts that for $n \ge 3$, there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{B}^n} V_{2^*}|u|^{2^*}\, dv_{g_{\mathbb{B}^n}}\right)^{2/2^*} \le C\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2\, dv_{g_{\mathbb{B}^n}} \quad \text{for all } u \in H^1(\mathbb{B}^n),$$

where $2^* = \frac{2n}{n-2}$ and V_{2^*} is defined in (4.3). The Hardy inequality on $\mathbb{B}^n$ [153] states:

$$\frac{(n-2)^2}{4}\int_{\mathbb{B}^n} V_2|u|^2\, dv_{g_{\mathbb{B}^n}} \le \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2\, dv_{g_{\mathbb{B}^n}} \quad \text{for all } u \in H^1(\mathbb{B}^n).$$

Moreover, just like the Euclidean case, $\frac{(n-2)^2}{4}$ is the best Hardy constant in the above inequality on $\mathbb{B}^n$, i.e.,

$$\gamma_H := \frac{(n-2)^2}{4} = \inf_{u\in H^1(\mathbb{B}^n)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}}}{\displaystyle\int_{\mathbb{B}^n} V_2 |u|^2 \, dv_{g_{\mathbb{B}^n}}}.$$

By interpolating these inequalities via Hölder's inequality, one gets the following Hardy–Sobolev inequalities in hyperbolic space.

Lemma 4.2.1. *Let $2^*(s) = \frac{2(n-s)}{n-2}$ where $0 \le s \le 2$. Then, there exist a positive constant C such that*

$$C \left(\int_{\mathbb{B}^n} V_{2^*(s)} |u|^{2^*(s)} \, dv_{g_{\mathbb{B}^n}} \right)^{2/2^*(s)} \le \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} \tag{4.13}$$

for all $u \in H^1(\mathbb{B}^n)$. If $\gamma < \gamma_H := \frac{(n-2)^2}{4}$, then there exists $C_\gamma > 0$ such that

$$C_\gamma \left(\int_{\mathbb{B}^n} V_{2^*(s)} |u|^{2^*(s)} \, dv_{g_{\mathbb{B}^n}} \right)^{2/2^*(s)} \le \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} - \gamma \int_{\mathbb{B}^n} V_2 |u|^2 \, dv_{g_{\mathbb{B}^n}} \tag{4.14}$$

for all $u \in H^1(\mathbb{B}^n)$.

Proof. Note that for $s = 0$ (resp., $s = 2$) the first inequality is just the Sobolev (resp., the Hardy) inequality in hyperbolic space. We therefore have to only consider the case where $0 < s < 2$ where $2^*(s) > 2$. Note that $2^*(s) = \left(\frac{s}{2}\right) 2 + \left(\frac{2-s}{2}\right) 2^*$, and so

$$\begin{aligned}
V_{2^*(s)} &= \frac{f(r)^2(1-r)^2}{4(n-2)^2 G(r)} \left(\frac{1}{\sqrt{G(r)}} \right)^{2^*(s)} \\
&= \left(\frac{f(r)^2(1-r)^2}{4(n-2)^2 G(r)} \right)^{\frac{s}{2} + \frac{2-s}{2}} \left(\frac{1}{\sqrt{G(r)}} \right)^{(\frac{s}{2})2 + (\frac{2-s}{2})2^*} \\
&= \left(\frac{f(r)^2(1-r)^2}{4(n-2)^2 G(r)} \left(\frac{1}{\sqrt{G(r)}} \right)^2 \right)^{\frac{s}{2}} \left(\frac{f(r)^2(1-r)^2}{4(n-2)^2 G(r)} \left(\frac{1}{\sqrt{G(r)}} \right)^{2^*} \right)^{\frac{2-s}{2}} \\
&= V_2^{\frac{s}{2}} V_{2^*}^{\frac{2-s}{2}}.
\end{aligned}$$

Applying Hölder's inequality with conjugate exponents $\frac{2}{s}$ and $\frac{2}{2-s}$, we obtain

$$\begin{aligned}
\int_{\mathbb{B}^n} V_{2^*(s)}|u|^{2^*(s)}\,dv_{g_{\mathbb{B}^n}} &= \int_{\mathbb{B}^n} \left(|u|^2\right)^{\frac{s}{2}} V_2^{\frac{s}{2}} \cdot \left(|u|^{2^*}\right)^{\frac{2-s}{2}} V_{2^*}^{\frac{2-s}{2}}\,dv_{g_{\mathbb{B}^n}} \\
&\leq \left(\int_{\mathbb{B}^n} V_2|u|^2\,dv_{g_{\mathbb{B}^n}}\right)^{\frac{s}{2}} \left(\int_{\mathbb{B}^n} V_{2^*}|u|^{2^*}\,dv_{g_{\mathbb{B}^n}}\right)^{\frac{2-s}{2}} \\
&\leq C^{-1} \left(\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2\,dv_{g_{\mathbb{B}^n}}\right)^{\frac{s}{2}} \left(\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2\,dv_{g_{\mathbb{B}^n}}\right)^{\frac{2^*}{2}\frac{2-s}{2}} \\
&= C^{-1} \left(\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2\,dv_{g_{\mathbb{B}^n}}\right)^{\frac{2^*(s)}{2}}.
\end{aligned}$$

It follows that for all $u \in H^1(\mathbb{B}^n)$,

$$\frac{\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2\,dv_{g_{\mathbb{B}^n}} - \gamma\int_{\mathbb{B}^n} V_2|u|^2\,dv_{g_{\mathbb{B}^n}}}{\left(\int_{\mathbb{B}^n} V_{2^*(s)}|u|^{2^*(s)}\,dv_{g_{\mathbb{B}^n}}\right)^{2/2^*(s)}} \geq \left(1 - \frac{\gamma}{\gamma_H}\right) \frac{\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2\,dv_{g_{\mathbb{B}^n}}}{\left(\int_{\mathbb{B}^n} V_{2^*(s)}|u|^{2^*(s)}\,dv_{g_{\mathbb{B}^n}}\right)^{2/2^*(s)}}.$$

Hence, (4.13) implies (4.14) whenever $\gamma < \gamma_H := \frac{(n-2)^2}{4}$. □

The best constant $\mu_\gamma(\mathbb{B}^n)$ in inequality (4.14) can therefore be written as:

$$\mu_\gamma(\mathbb{B}^n) = \inf_{u \in H^1(\mathbb{B}^n)\setminus\{0\}} \frac{\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2\,dv_{g_{\mathbb{B}^n}} - \gamma\int_{\mathbb{B}^n} V_2|u|^2\,dv_{g_{\mathbb{B}^n}}}{\left(\int_{\mathbb{B}^n} V_{2^*(s)}|u|^{2^*(s)}\ dv_{\mathbb{B}^n}\right)^{2/2^*(s)}}.$$

Thus, any minimizer of $\mu_\gamma(\mathbb{B}^n)$ satisfies –up to a Lagrange multiplier– the following Euler–Lagrange equation

$$-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = V_{2^*(s)}|u|^{2^*(s)-2}u, \tag{4.15}$$

where $0 \leq s < 2$ and $2^*(s) = \frac{2(n-s)}{n-2}$.

4.3 The explicit solutions for Hardy–Sobolev equations on $\mathbb{B}^n$

We first find the fundamental solutions associated to the Hardy–Schrödinger operator on $\mathbb{B}^n$, that is the solutions for the equation $-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = 0$.

Lemma 4.3.1. *Assume $\gamma < \gamma_H := \frac{(n-2)^2}{4}$. The fundamental solutions of*

$$-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = 0$$

are given by

$$u_\pm(r) = G(r)^{\alpha_\pm(\gamma)} \sim \begin{cases} \left(\dfrac{1}{n-2} r^{2-n}\right)^{\alpha_\pm(\gamma)} & \text{as } r \to 0, \\ \left(\dfrac{2^{n-2}}{n-1}(1-r)^{n-1}\right)^{\alpha_\pm(\gamma)} & \text{as } r \to 1, \end{cases}$$

where

$$\alpha_\pm(\gamma) = \frac{\beta_\pm(\gamma)}{n-2} \quad \text{and} \quad \beta_\pm(\gamma) = \frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} - \gamma}. \tag{4.16}$$

Proof. We look for solutions of the form $u(r) = G(r)^{-\alpha}$. To this end we perform a change of variable $\sigma = G(r)$, $v(\sigma) = u(r)$ to arrive at the Euler-type equation

$$(n-2)^2 v''(\sigma) + \gamma \sigma^{-2} v(\sigma) = 0 \quad \text{in } (0, \infty).$$

It is easy to see that the two solutions are given by $v(\sigma) = \sigma^{\pm}$, or $u(r) = c(n,\gamma) r^{-\beta_\pm}$ where $\alpha_\pm$ and $\beta_\pm$ are as in (4.16). □

Remark 4.3.2. We point out that $u_\pm(r) \sim c(n,\gamma) r^{-\beta_\pm(\gamma)}$ as $r \to 0$.

Proposition 4.3.3. *Let $-\infty < \gamma < \frac{(n-2)^2}{4}$. The equation*

$$-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = V_{2^*(s)} u^{2^*(s)-1} \quad \text{in } \mathbb{B}^n, \tag{4.17}$$

has a family of positive radial solutions which are given by

$$U(G(r)) = c\left(G(r)^{-\frac{2-s}{n-2}\alpha_-(\gamma)} + G(r)^{-\frac{2-s}{n-2}\alpha_+(\gamma)}\right)^{-\frac{n-2}{2-s}}$$
$$= c\left(G(r)^{-\frac{2-s}{(n-2)^2}\beta_-(\gamma)} + G(r)^{-\frac{2-s}{(n-2)^2}\beta_+(\gamma)}\right)^{-\frac{n-2}{2-s}},$$

where c is a positive constant and $\alpha_\pm(\gamma)$ and $\beta_\pm(\gamma)$ satisfy (4.16).

Proof. With the same change of variable $\sigma = G(r)$ and $v(\sigma) = u(r)$ we have

$$(n-2)^2 v''(\sigma) + \gamma\sigma^{-2}v(\sigma) + \sigma^{-\frac{2^*(s)+2}{2}} v^{2^*(s)-1}(\sigma) = 0 \quad \text{in } (0,\infty).$$

Now, set $\sigma = \tau^{2-n}$ and $w(\tau) = v(\sigma)$

$$\tau^{1-n}(\tau^{n-1}w'(\tau))' + \gamma\tau^{-2}w(\tau) + w(\tau)^{2^*(s)-1} = 0 \quad \text{on } (0,\infty).$$

The latter has an explicit solution

$$w(\tau) = c\left(\tau^{\frac{2-s}{n-2}\beta_-(\gamma)} + \tau^{\frac{2-s}{n-2}\beta_+(\gamma)}\right)^{-\frac{n-2}{2-s}},$$

where c is a positive constant. This translates to the explicit formula

$$u(r) = c\left(G(r)^{-\frac{2-s}{n-2}\alpha_-(\gamma)} + G(r)^{-\frac{2-s}{n-2}\alpha_+(\gamma)}\right)^{-\frac{n-2}{2-s}}$$
$$= c\left(G(r)^{-\frac{2-s}{(n-2)^2}\beta_-(\gamma)} + G(r)^{-\frac{2-s}{(n-2)^2}\beta_+(\gamma)}\right)^{-\frac{n-2}{2-s}}.$$

□

Remark 4.3.4. We remark that, in the special case $\gamma = 0$ and $s = 0$, Sandeep–Tintarev [153] proved that the following minimization problem

$$\mu_0(\mathbb{B}^n) = \inf_{u \in H_r^1(\mathbb{B}^n)\setminus\{0\}} \frac{\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}}}{\int_{\mathbb{B}^n} V_{2^*}|u|^{2^*} \, dv_{g_{\mathbb{B}^n}}}$$

is attained.

Remark 4.3.5. The change of variable $\sigma = G(r)$ offers a nice way of viewing the radial aspect of hyperbolic space $\mathbb{B}^n$ in parallel to the one in $\mathbb{R}^n$ in the following sense.

- The scaling $r \mapsto G^{-1}(\lambda G(r))$ for $r = |x|$ in $\mathbb{B}^n$ corresponds to $\sigma \mapsto \lambda\sigma$ in $(0,\infty)$, which in turn corresponds to $\rho \mapsto \overline{\lambda}\rho = \overline{G}^{-1}(\lambda\overline{G}(\rho))$ for $\rho = |x|$ in $\mathbb{R}^n$, once we set $\overline{G}(\rho) = \rho^{2-n}$ and $\overline{\lambda} = \lambda^{\frac{1}{2-n}}$;

- One has a similar correspondence with the scaling-invariant equations: if u solves

$$-\Delta_{\mathbb{B}^n} u - \gamma V_2 u = V_{2^*(s)} u^{2^*(s)-1} \quad \text{in } \mathbb{B}^n,$$

  then

  1. as an ODE, and once we set $v(\sigma) = u(r)$, $\sigma = G(r)$, it is equivalent to

$$-(n-2)^2 v''(\sigma) - \gamma\sigma^{-2} v(\sigma) = \sigma^{-\frac{2^*(s)+2}{2}} v(\sigma)^{2^*(s)-1} \quad \text{on } (0,\infty); \tag{4.18}$$

  2. as a PDE on $\mathbb{R}^n$, and by setting $v(\sigma) = u(\rho)$, $\sigma = \overline{G}(\rho)$, it is in turn equivalent to

$$-\Delta v - \frac{\gamma}{|x|^2} v = \frac{1}{|x|^s} v^{2^*(s)-1} \quad \text{in } \mathbb{R}^n.$$

  This also confirm that the potentials $V_{2^*(s)}$ are the "correct" ones associated to the power $|x|^{-s}$.

- The explicit solution u on $\mathbb{B}^n$ is related to the explicit solution w on $\mathbb{R}^n$ in the following way:

$$u(r) = w\left(G(r)^{-\frac{1}{n-2}}\right).$$

- Under the above setting, it is also easy to see the following integral identities:

$$\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}}^n} = \int_0^\infty v'(\sigma)^2 \, d\sigma$$
$$\int_{\mathbb{B}^n} V_2 u^2 \, dv_{g_{\mathbb{B}}^n} = \frac{1}{(n-2)^2} \int_0^\infty \frac{v^2(\sigma)}{\sigma^2} \, d\sigma$$
$$\int_{\mathbb{B}^n} V_p u^p \, dv_{g_{\mathbb{B}}^n} = \frac{1}{(n-2)^2} \int_0^\infty \frac{v^p(\sigma)}{\sigma^{\frac{p+2}{2}}} \, d\sigma,$$

which, in the same way as above, equal to the corresponding Euclidean integrals.

4.4 The corresponding perturbed Hardy–Schrödinger operator on Euclidean space

We shall see in the next section that after a conformal transformation, the equation (4.6) is transformed into the Euclidean equation

$$\begin{cases} -\Delta u - \left(\frac{\gamma}{|x|^2} + h(x)\right) u = b(x) \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.19}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, $h \in C^1(\overline{\Omega} \setminus \{0\})$ with $\lim\limits_{|x| \to 0} |x|^2 h(x) = 0$ is such that the operator $-\Delta - \left(\frac{\gamma}{|x|^2} + h(x)\right)$ is coercive and $b(x) \in C^0(\overline{\Omega})$ is nonnegative with $b(0) > 0$. The equation (4.19) is the Euler–Lagrange equation for following energy functional on $D^{1,2}(\Omega)$,

$$J^{\Omega}_{\gamma,h}(u) := \frac{\int_\Omega \left(|\nabla u|^2 - \left(\frac{\gamma}{|x|^2} + h(x) \right) u^2 \right) dx}{\left(\int_\Omega b(x) \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{2/2^*(s)}}.$$

Here $D^{1,2}(\Omega)$ – or $H_0^1(\Omega)$ if the domain is bounded – is the completion of $C_c^\infty(\Omega)$ with respect to the norm given by $||u||^2 = \int_\Omega |\nabla u|^2\,dx$. We let

$$\mu_{\gamma,h}(\Omega) := \inf_{u\in D^{1,2}(\Omega)\setminus\{0\}} J^\Omega_{\gamma,h}(u)$$

A standard approach to find minimizers is to compare $\mu_{\gamma,h}(\Omega)$ with $\mu_{\gamma,0}(\mathbb{R}^n)$. It is know that $\mu_{\gamma,0}(\mathbb{R}^n)$ is attained when $\gamma \geq 0$, and minimizers are explicit and take the form

$$\begin{aligned} U_\varepsilon(x) &:= c_{\gamma,s}(n)\cdot \varepsilon^{-\frac{n-2}{2}} U\left(\frac{x}{\varepsilon}\right) \\ &= c_{\gamma,s}(n)\cdot \left(\frac{\varepsilon^{\frac{2-s}{n-2}\cdot\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}{\varepsilon^{\frac{2-s}{n-2}\cdot(\beta_+(\gamma)-\beta_-(\gamma))}|x|^{\frac{(2-s)\beta_-(\gamma)}{n-2}} + |x|^{\frac{(2-s)\beta_+(\gamma)}{n-2}}}\right)^{\frac{n-2}{2-s}} \end{aligned}$$

for $x \in \mathbb{R}^n \setminus \{0\}$, where $\varepsilon > 0$, $c_{\gamma,s}(n) > 0$, and $\beta_\pm(\gamma)$ are defined in (4.16), see [105]. In particular, there exists $\chi > 0$ such that

$$-\Delta U_\varepsilon - \frac{\gamma}{|x|^2}U_\varepsilon = \chi\frac{U_\varepsilon^{2^*(s)-1}}{|x|^s} \text{ in } \mathbb{R}^n\setminus\{0\}. \tag{4.20}$$

We shall start by analyzing the singular solutions and then define the mass of a domain associated to the operator $-\Delta - \left(\frac{\gamma}{|x|^2} + h(x)\right)$.

Proposition 4.4.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ such that $0 \in \Omega$ and $\gamma < \frac{(n-2)^2}{4}$. Let $h \in C^1(\overline{\Omega}\setminus\{0\})$ be such that $\lim_{|x|\to 0}|x|^\tau h(x)$ exists and is finite, for some $0 \leq \tau < 2$, and that the operator $-\Delta - \frac{\gamma}{|x|^2} - h(x)$ is coercive. Then*

1. *There exists a solution $K \in C^\infty(\overline{\Omega}\setminus\{0\})$ for the linear problem*

$$\begin{cases} -\Delta K - \left(\frac{\gamma}{|x|^2} + h(x)\right)K = 0 & \text{in } \Omega\setminus\{0\} \\ K > 0 & \text{in } \Omega\setminus\{0\} \\ K = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.21}$$

such that for some $c > 0$,

$$K(x) \simeq_{x\to 0} \frac{c}{|x|^{\beta_+(\gamma)}}. \tag{4.22}$$

Moreover, if $K' \in C^\infty(\overline{\Omega}\setminus\{0\})$ is another solution for the above equation, then there exists $\lambda > 0$ such that $K' = \lambda K$.

2. *Let $\theta = \inf\{\theta' \in [0,2) : \lim\limits_{|x|\to 0} |x|^{\theta'} h(x)$ exists and is finite$\}$. If $\gamma > \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}$, then there exists $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$ such that*

$$K(x) = \frac{c_1}{|x|^{\beta_+(\gamma)}} + \frac{c_2}{|x|^{\beta_-(\gamma)}} + o\left(\frac{1}{|x|^{\beta_-(\gamma)}}\right) \quad \text{as } x \to 0. \tag{4.23}$$

The ratio $\frac{c_2}{c_1}$ is independent of the choice of K. We can therefore define the mass of Ω with respect to the operator $-\Delta - \left(\frac{\gamma}{|x|^2} + h(x)\right)$ as $m_{\gamma,h}(\Omega) := \dfrac{c_2}{c_1}$.

3. *The mass $m_{\gamma,h}(\Omega)$ satisfies the following properties:*
   - *$m_{\gamma,0}(\Omega) < 0$,*
   - *If $h \le h'$ and $h \not\equiv h'$, then $m_{\gamma,h}(\Omega) < m_{\gamma,h'}(\Omega)$,*
   - *If $\Omega' \subset \Omega$, then $m_{\gamma,h}(\Omega') < m_{\gamma,h}(\Omega)$.*

Proof. The proof of (1) and (3) is similar to Proposition 2 and 4 in [106] with only a minor change that accounts for the singularity of h. To illustrate the role of this extra singularity we prove (2). For that, we let $\eta \in C_c^\infty(\Omega)$ be such that $\eta(x) \equiv 1$ around 0. Our first objective is to write $K(x) := \frac{\eta(x)}{|x|^{\beta_+(\gamma)}} + f(x)$ for some $f \in H_0^1(\Omega)$. Note that $\gamma > \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4} \iff \beta_+ - \beta_- < 2 - \theta \iff 2\beta_+ < n - \theta$. Fix θ' such that $\theta < \theta' < \min\left\{\frac{2+\theta}{2}, 2 - (\beta_+(\gamma) - \beta_-(\gamma))\right\}$. Then $\lim\limits_{|x|\to 0} |x|^{\theta'} h(x)$ exists and is finite.

Consider the function

$$g(x) = -\left(-\Delta - \left(\frac{\gamma}{|x|^2} + h(x)\right)\right)(\eta|x|^{-\beta_+(\gamma)}) \quad \text{in } \Omega\setminus\{0\}.$$

Since $\eta(x) \equiv 1$ around 0, we have that

$$|g(x)| \leq \left|\frac{h(x)}{|x|^{\beta_+(\gamma)}}\right| \leq C|x|^{-(\beta_+(\gamma)+\theta')} \quad \text{as } x \to 0. \tag{4.24}$$

Therefore $g \in L^{\frac{2n}{n+2}}(\Omega)$ if $2\beta_+(\gamma) + 2\theta' < n+2$, and this holds since by our assumption $2\beta_+ < n - \theta$ and $2\theta' < 2 + \theta$. Since $L^{\frac{2n}{n+2}}(\Omega) = L^{\frac{2n}{n-2}}(\Omega)' \subset H_0^1(\Omega)'$, there exists $f \in H_0^1(\Omega)$ such that

$$-\Delta f - \left(\frac{\gamma}{|x|^2} + h(x)\right) f = g \qquad \text{in } H_0^1(\Omega).$$

By regularity theory, we have that $f \in C^2(\overline{\Omega} \setminus \{0\})$. We now show that

$$|x|^{\beta_-(\gamma)} f(x) \text{ has a finite limit as } x \to 0. \tag{4.25}$$

Define $K(x) = \frac{\eta(x)}{|x|^{\beta_+(\gamma)}} + f(x)$ for all $x \in \overline{\Omega} \setminus \{0\}$, and note that $K \in C^2(\overline{\Omega} \setminus \{0\})$ and is a solution to

$$-\Delta K - \left(\frac{\gamma}{|x|^2} + h(x)\right) K = 0.$$

Write $g_+(x) := \max\{g(x), 0\}$ and $g_-(x) := \max\{-g(x), 0\}$ so that $g = g_+ - g_-$, and let $f_1, f_2 \in H_0^1(\Omega)$ be weak solutions to

$$-\Delta f_1 - \left(\frac{\gamma}{|x|^2} + h(x)\right) f_1 = g_+ \ \text{ and } \ -\Delta f_2 - \left(\frac{\gamma}{|x|^2} + h(x)\right) f_2 = g_- \text{ in } H_0^1(\Omega). \tag{4.26}$$

In particular, uniqueness, coercivity and the maximum principle yield $f = f_1 - f_2$ and $f_1, f_2 \geq 0$. Assume that $f_1 \not\equiv 0$ so that $f_1 > 0$ in $\Omega \setminus \{0\}$, fix $\alpha > \beta_+(\gamma)$ and $\mu > 0$. Define $u_-(x) := |x|^{-\beta_-(\gamma)} + \mu |x|^{-\alpha}$ for all $x \neq 0$. We then get that there exists a small $\delta > 0$ such that

$$\begin{aligned}
&\left(-\Delta - \left(\frac{\gamma}{|x|^2} + h(x)\right)\right) u_-(x) \\
&= \mu \left(-\Delta - \frac{\gamma}{|x|^2}\right) |x|^{-\alpha} - \mu h(x)|x|^{-\alpha} - h(x)|x|^{-\beta_-(\gamma)} \\
&= \frac{-\mu\left(\alpha - \beta_+(\gamma)\right)\left(\alpha - \beta_-(\gamma)\right) - |x|^2 h(x)\left(|x|^{\alpha - \beta_-(\gamma)} + \mu\right)}{|x|^{\alpha+2}} \\
&< 0 \text{ for } x \in B_\delta(0) \setminus \{0\},
\end{aligned} \tag{4.27}$$

This implies that $u_-(x)$ is a sub-solution on $B_\delta(0) \setminus \{0\}$. Let $C > 0$ be such that $f_1 \geq Cu_-$ on $\partial B_\delta(0)$. Since f_1 and $Cu_- \in H_0^1(\Omega)$ are respectively super-solutions and sub-solutions to $\left(-\Delta - \left(\frac{\gamma}{|x|^2} + h(x)\right)\right) u(x) = 0$, it follows from the comparison principle (via coercivity) that $f_1 > Cu_- > C|x|^{-\beta_-(\gamma)}$ on $B_\delta(0) \setminus \{0\}$. It then follows from (4.24) that

$$g_+(x) \leq |g(x)| \leq C|x|^{-(\beta_+(\gamma)+\theta')} \leq C_1 |x|^{(2-\theta')-(\beta_+(\gamma)-\beta_-(\gamma))} \frac{f_1}{|x|^2}.$$

Then rewriting (4.26) as

$$-\Delta f_1 - \left(\frac{\gamma}{|x|^2} + h(x) + \frac{g_+}{f_1}\right) f_1 = 0$$

yields

$$-\Delta f_1 - \left(\frac{\gamma + O\left(|x|^{(2-\theta')-(\beta_+(\gamma)-\beta_-(\gamma))}\right)}{|x|^2}\right) f_1 = 0.$$

With our choice of θ' we can then conclude by the optimal regularity result in [106, Theorem 8] that $|x|^{\beta_-(\gamma)} f_1$ has a finite limit as $x \to 0$. Similarly one also obtains that $|x|^{\beta_-(\gamma)} f_2$ has a finite limit as $x \to 0$, and therefore (4.25) is verified.

It follows that there exists $c_2 \in \mathbb{R}$ such that

$$K(x) = \frac{1}{|x|^{\beta_+(\gamma)}} + \frac{c_2}{|x|^{\beta_-(\gamma)}} + o\left(\frac{1}{|x|^{\beta_-(\gamma)}}\right) \quad \text{as } x \to 0,$$

which proves the existence of a solution K to the problem with the relevant asymptotic behavior. The uniqueness result yields the conclusion. $\square$

We now proceed with the proof of the existence results, following again [106]. We shall use the following standard sufficient condition for attainability.

Lemma 4.4.2. *Under the assumptions of Theorem 4.1.5, if*

$$\mu_{\gamma,h}(\Omega) := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \left(|\nabla u|^2 - \left(\frac{\gamma}{|x|^2} + h(x)\right) u^2\right) dx}{\left(\displaystyle\int_\Omega b(x)\frac{|u|^{2^*(s)}}{|x|^s}\, dx\right)^{2/2^*(s)}} < \frac{\mu_{\gamma,0}(\mathbb{R}^n)}{b(0)^{2/2^*(s)}},$$

then the infimum $\mu_{\gamma,s}(\Omega)$ is achieved and equation (4.19) *has a solution.*

Proof of Theorem 4.1.5: We will construct a minimizing sequence u_ε in $H_0^1(\Omega)\setminus\{0\}$ for the functional $J_{\gamma,h}^{\Omega}$ in such a way that $\mu_{\gamma,h}(\Omega) < b(0)^{-2/2^*(s)}\mu_{\gamma,0}(\mathbb{R}^n)$. As mentioned above, when $\gamma \geq 0$ the infimum $\mu_{\gamma,0}(\mathbb{R}^n)$ is achieved, up to a constant, by the function

$$U(x) := \frac{1}{\left(|x|^{\frac{(2-s)\beta_-(\gamma)}{n-2}} + |x|^{\frac{(2-s)\beta_+(\gamma)}{n-2}}\right)^{\frac{n-2}{2-s}}} \text{ for } x \in \mathbb{R}^n\setminus\{0\}.$$

In particular, there exists $\chi > 0$ such that

$$-\Delta U - \frac{\gamma}{|x|^2}U = \chi\frac{U^{2^*(s)-1}}{|x|^s} \text{ in } \mathbb{R}^n\setminus\{0\}. \tag{4.28}$$

Define a scaled version of U by

$$U_\varepsilon(x) := \varepsilon^{-\frac{n-2}{2}}U\left(\frac{x}{\varepsilon}\right) = \left(\frac{\varepsilon^{\frac{2-s}{n-2}\cdot\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}{\varepsilon^{\frac{2-s}{n-2}\cdot(\beta_+(\gamma)-\beta_-(\gamma))}|x|^{\frac{(2-s)\beta_-(\gamma)}{n-2}} + |x|^{\frac{(2-s)\beta_+(\gamma)}{n-2}}}\right)^{\frac{n-2}{2-s}} \tag{4.29}$$

for $x \in \mathbb{R}^n\setminus\{0\}$. $\beta_\pm(\gamma)$ are defined in (4.16). In the sequel, we write $\beta_+ := \beta_+(\gamma)$ and $\beta_- := \beta_-(\gamma)$. Consider a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $\eta(x) \equiv 1$ in a neighborhood of 0 contained in Ω.

Case 1: Test-functions for the case when $\gamma \leq \dfrac{(n-2)^2}{4} - \dfrac{(2-\theta)^2}{4}$.

For $\varepsilon > 0$, we consider the test functions $u_\varepsilon \in D^{1,2}(\Omega)$ defined by $u_\varepsilon(x) := \eta(x)U_\varepsilon(x)$ for $x \in \overline{\Omega} \setminus \{0\}$. To estimate $J^{\Omega}_{\gamma,h}(u_\varepsilon)$, we use the bounds on U_ε to obtain

$$\begin{aligned}
\int_\Omega b(x)\frac{u_\varepsilon^{2^*(s)}}{|x|^s}\,dx &= \int_{B_\delta(0)} b(x)\frac{U_\varepsilon^{2^*(s)}}{|x|^s}\,dx + \int_{\Omega\setminus B_\delta(0)} b(x)\frac{u_\varepsilon^{2^*(s)}}{|x|^s}\,dx \\
&= \int_{B_{\varepsilon^{-1}\delta}(0)} b(\varepsilon x)\frac{U^{2^*(s)}}{|x|^s}\,dx + \int_{B_{\varepsilon^{-1}\delta}(0)} b(\varepsilon x)\eta(\varepsilon x)^{2^*(s)}\frac{U^{2^*(s)}}{|x|^s}\,dx \\
&= b(0)\int_{\mathbb{R}^n} \frac{U^{2^*(s)}}{|x|^s}\,dx + O\left(\varepsilon^{\frac{2^*(s)}{2}(\beta_+-\beta_-)}\right).
\end{aligned}$$

Similarly, one also has

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_\varepsilon|^2 - \frac{\gamma}{|x|^2}u_\varepsilon^2\right)dx \\
&= \int_{B_\delta(0)} \left(|\nabla U_\varepsilon|^2 - \frac{\gamma}{|x|^2}U_\varepsilon^2\right)dx + \int_{\Omega\setminus B_\delta(0)} \left(|\nabla u_\varepsilon|^2 - \frac{\gamma}{|x|^2}u_\varepsilon^2\right)dx \\
&= \int_{B_{\varepsilon^{-1}\delta}(0)} \left(|\nabla U|^2 - \frac{\gamma}{|x|^2}U^2\right)dx + O\left(\varepsilon^{\beta_+-\beta_-}\right) \\
&= \int_{\mathbb{R}^n} \left(|\nabla U|^2 - \frac{\gamma}{|x|^2}U^2\right)dx + O\left(\varepsilon^{\beta_+-\beta_-}\right) \\
&= \chi\int_{\mathbb{R}^n} \frac{U^{2^*(s)}}{|x|^s}\,dx + O\left(\varepsilon^{\beta_+-\beta_-}\right).
\end{aligned}$$

Estimating the lower order terms as $\varepsilon \to 0$ gives

$$\int_\Omega \tilde{h}(x)u_\varepsilon^2\,dx = \begin{cases}
\varepsilon^{2-\theta}\left[\mathscr{C}_2\int_{\mathbb{R}^n}\frac{U^2}{|x|^\theta}dx + o(1)\right] & \text{if } \beta_+-\beta_- > 2-\theta, \\[2ex]
\varepsilon^{2-\theta}\log\left(\frac{1}{\varepsilon}\right)\left[\mathscr{C}_2\omega_{n-1} + o(1)\right] & \text{if } \beta_+-\beta_- = 2-\theta, \\[2ex]
O\left(\varepsilon^{\beta_+-\beta_-}\right) & \text{if } \beta_+-\beta_- < 2-\theta.
\end{cases}$$

And

$$-\mathscr{C}_1 \int_\Omega \frac{\log|x|}{|x|^\theta} u_\varepsilon^2 \, dx = \begin{cases} \mathscr{C}_1 \varepsilon^{2-\theta} \log\left(\frac{1}{\varepsilon}\right) \left[\int_{\mathbb{R}^n} \frac{U^2}{|x|^\theta} dx + o(1) \right] \\ \qquad\qquad\qquad\qquad \text{if } \ \beta_+ - \beta_- > 2-\theta, \\ \mathscr{C}_1 \varepsilon^{2-\theta} \left(\log\left(\frac{1}{\varepsilon} \right) \right)^2 \left[\frac{\omega_{n-1}}{2} + o(1) \right] \\ \qquad\qquad\qquad\qquad \text{if } \ \beta_+ - \beta_- = 2-\theta, \\ O\left(\varepsilon^{\beta_+ - \beta_-}\right) \\ \qquad\qquad\qquad\qquad \text{if } \ \beta_+ - \beta_- < 2-\theta. \end{cases}$$

Note that $\beta_+ - \beta_- \geq 2-\theta$ if and only if $\gamma \leq \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}$. Therefore,

$$\int_\Omega h(x) u_\varepsilon^2 \, dx = \begin{cases} \varepsilon^{2-\theta} \int_{\mathbb{R}^n} \frac{U^2}{|x|^\theta} dx \left[\mathscr{C}_1 \log\left(\frac{1}{\varepsilon}\right) (1+o(1)) + \mathscr{C}_2 + o(1) \right] \\ \qquad\qquad\qquad\qquad \text{if } \ \gamma < \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}, \\ \varepsilon^{2-\theta} \log\left(\frac{1}{\varepsilon}\right) \frac{\omega_{n-1}}{2} \left[\mathscr{C}_1 \log\left(\frac{1}{\varepsilon}\right) (1+o(1)) + 2\mathscr{C}_2 + o(1) \right] \\ \qquad\qquad\qquad\qquad \text{if } \ \gamma = \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}. \end{cases}$$

Combining the above estimates, we obtain as $\varepsilon \to 0$,

$$
\begin{aligned}
& J^{\Omega}_{\gamma,h}(u_\varepsilon) \\
&= \frac{\int_\Omega \left(|\nabla u_\varepsilon|^2 - \gamma \frac{u_\varepsilon^2}{|x|^2} - h(x) u_\varepsilon^2\right) dx}{\left(\int_\Omega b(x) \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}\, dx\right)^{2/2^*(s)}} \\
&= \frac{\mu_{\gamma,0}(\mathbb{R}^n)}{b(0)^{2/2^*(s)}} - \begin{cases} \dfrac{\left(\int_{\mathbb{R}^n} \frac{U^2}{|x|^\theta}\, dx\right)\varepsilon^{2-\theta}}{\left(b(0)\int_{\mathbb{R}^n} \frac{U^{2^*(s)}}{|x|^s}\, dx\right)^{2/2^*(s)}} \left[\mathscr{C}_1 \log\left(\frac{1}{\varepsilon}\right)(1+o(1)) + \mathscr{C}_2 + o(1)\right] \\ \qquad\qquad\qquad\qquad \text{if } \gamma < \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}, \\ \\ \dfrac{\omega_{n-1}\varepsilon^{2-\theta}\log\left(\frac{1}{\varepsilon}\right)}{2\left(b(0)\int_{\mathbb{R}^n} \frac{U^{2^*(s)}}{|x|^s}\, dx\right)^{2/2^*(s)}} \left[\mathscr{C}_1 \log\left(\frac{1}{\varepsilon}\right)(1+o(1)) + 2\mathscr{C}_2 + o(1)\right] \\ \qquad\qquad\qquad\qquad \text{if } \gamma = \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}, \end{cases}
\end{aligned}
$$

as long as $\beta_+ - \beta_- \geq 2-\theta$. Thus, for ε sufficiently small, the assumption that either $\mathscr{C}_1 > 0$ or $\mathscr{C}_1 = 0, \mathscr{C}_2 > 0$ guarantees that

$$
\mu_{\gamma,h}(\Omega) \leq J^{\Omega}_{\gamma,h}(u_\varepsilon) < \frac{\mu_{\gamma,0}(\mathbb{R}^n)}{b(0)^{2/2^*(s)}}.
$$

It then follows from Lemma 4.4.2 that $\mu_{\gamma,h}(\Omega)$ is attained.

Case 2: Test-functions for the case when $\dfrac{(n-2)^2}{4} - \dfrac{(2-\theta)^2}{4} < \gamma < \dfrac{(n-2)^2}{4}$.

Here $h(x)$ and θ given by (4.11) satisfy the hypothesis of Proposition (4.4.1). Since $\gamma > \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}$, it follows from (4.23) that there exists $\beta \in D^{1,2}(\Omega)$ such that

$$
\beta(x) \simeq_{x\to 0} \frac{m_{\gamma,h}(\Omega)}{|x|^{\beta_-}}. \tag{4.30}
$$

The function $K(x) := \frac{\eta(x)}{|x|^{\beta_+}} + \beta(x)$ for $x \in \Omega \setminus \{0\}$ satisfies the equation:

$$
\begin{cases} -\Delta K - \left(\frac{\gamma}{|x|^2} + h(x)\right) K = 0 & \text{in } \Omega \setminus \{0\} \\ K > 0 & \text{in } \Omega \setminus \{0\} \\ K = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.31}
$$

Define the test functions

$$u_\varepsilon(x) := \eta(x)U_\varepsilon + \varepsilon^{\frac{\beta_+-\beta_-}{2}}\beta(x) \qquad \text{for } x \in \overline{\Omega}\setminus\{0\}$$

The functions $u_\varepsilon \in D^{1,2}(\Omega)$ for all $\varepsilon > 0$. We estimate $J^{\Omega}_{\gamma,h}(u_\varepsilon)$.

Step 1: Estimates for $\displaystyle\int_\Omega \left(|\nabla u_\varepsilon|^2 - \left(\frac{\gamma}{|x|^2} + h(x)\right) u_\varepsilon^2\right) dx$.

Take $\delta > 0$ small enough such that $\eta(x) = 1$ in $B_\delta(0) \subset \Omega$. We decompose the integral as

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_\varepsilon|^2 - \left(\frac{\gamma}{|x|^2} + h(x)\right) u_\varepsilon^2\right) dx \\
&\qquad = \int_{B_\delta(0)} \left(|\nabla u_\varepsilon|^2 - \left(\frac{\gamma}{|x|^2} + h(x)\right) u_\varepsilon^2\right) dx \\
&\qquad\qquad + \int_{\Omega\setminus B_\delta(0)} \left(|\nabla u_\varepsilon|^2 - \left(\frac{\gamma}{|x|^2} + h(x)\right) u_\varepsilon^2\right) dx.
\end{aligned}$$

By standard elliptic estimates, it follows that $\lim_{\varepsilon\to 0} \frac{u_\varepsilon}{\varepsilon^{\frac{\beta_+-\beta_-}{2}}} = K$ in $C^2_{\mathrm{loc}}(\overline{\Omega}\setminus\{0\})$. Hence

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{\displaystyle\int_{\Omega\setminus B_\delta(0)} \left(|\nabla u_\varepsilon|^2 - \left(\frac{\gamma}{|x|^2} + h(x)\right) u_\varepsilon^2\right) dx}{\varepsilon^{\beta_+-\beta_-}} \\
&= \int_{\Omega\setminus B_\delta(0)} \left(|\nabla K|^2 - \left(\frac{\gamma}{|x|^2} + h(x)\right) K^2\right) dx \\
&= \int_{\Omega\setminus B_\delta(0)} \left(-\Delta K - \left(\frac{\gamma}{|x|^2} + h(x)\right) K\right) K\, dx + \int_{\partial(\Omega\setminus B_\delta(0))} K\partial_\nu K\, d\sigma \\
&= \int_{\partial(\Omega\setminus B_\delta(0))} K\partial_\nu K\, d\sigma = -\int_{\partial B_\delta(0)} K\partial_\nu K\, d\sigma.
\end{aligned}$$

Since $\beta_+ + \beta_- = n-2$, using elliptic estimates, and the definition of K gives us

$$K\partial_\nu K = -\frac{\beta_+}{|x|^{1+2\beta_+}} - (n-2)\frac{m_{\gamma,h}(\Omega)}{|x|^{n-1}} + o\left(\frac{1}{|x|^{n-1}}\right) \quad \text{as } x \to 0.$$

Therefore,

$$\int_{\Omega\setminus B_\delta(0)}\left(|\nabla u_\varepsilon|^2-\left(\frac{\gamma}{|x|^2}+h(x)\right)u_\varepsilon^2\right)dx$$
$$=\varepsilon^{\beta_+-\beta_-}\omega_{n-1}\left(\frac{\beta_+}{\delta^{\beta_+-\beta_-}}+(n-2)m_{\gamma,h}(\Omega)+o_\delta(1)\right)$$

Now, we estimate the term $\int_{B_\delta(0)}\left(|\nabla u_\varepsilon|^2-\left(\frac{\gamma}{|x|^2}+h(x)\right)u_\varepsilon^2\right)dx$.

First, $u_\varepsilon(x)=U_\varepsilon(x)+\varepsilon^{\frac{\beta_+-\beta_-}{2}}\beta(x)$ for $x\in B_\delta(0)$, therefore after integration by parts, we obtain

$$\begin{aligned}
&\int_{B_\delta(0)}\left(|\nabla u_\varepsilon|^2-\left(\frac{\gamma}{|x|^2}+h(x)\right)u_\varepsilon^2\right)dx\\
&=\int_{B_\delta(0)}\left(|\nabla U_\varepsilon|^2-\left(\frac{\gamma}{|x|^2}+h(x)\right)U_\varepsilon^2\right)dx\\
&\quad+2\varepsilon^{\frac{\beta_+-\beta_-}{2}}\int_{B_\delta(0)}\left(\nabla U_\varepsilon\cdot\nabla\beta-\left(\frac{\gamma}{|x|^2}+h(x)\right)U_\varepsilon\beta\right)dx\\
&\quad+\varepsilon^{\beta_+-\beta_-}\int_{B_\delta(0)}\left(|\nabla\beta|^2-\left(\frac{\gamma}{|x|^2}+h(x)\right)\beta^2\right)dx\\
&=\int_{B_\delta(0)}\left(-\Delta U_\varepsilon-\frac{\gamma}{|x|^2}U_\varepsilon\right)U_\varepsilon\,dx+\int_{\partial B_\delta(0)}U_\varepsilon\partial_\nu U_\varepsilon\,d\sigma\\
&\quad-\int_{B_\delta(0)}h(x)U_\varepsilon^2\,dx+2\varepsilon^{\frac{\beta_+-\beta_-}{2}}\int_{B_\delta(0)}\left(-\Delta U_\varepsilon\,dx-\frac{\gamma}{|x|^2}U_\varepsilon\right)\beta\,dx\\
&\quad-2\varepsilon^{\frac{\beta_+-\beta_-}{2}}\int_{B_\delta(0)}h(x)U_\varepsilon\beta\,dx+2\varepsilon^{\frac{\beta_+-\beta_-}{2}}\int_{\partial B_\delta(0)}\beta\partial_\nu U_\varepsilon\,d\sigma\\
&\quad+\varepsilon^{\beta_+-\beta_-}\int_{B_\delta(0)}\left(|\nabla\beta|^2-\left(\frac{\gamma}{|x|^2}+h(x)\right)\beta^2\right)dx.
\end{aligned}$$

We now estimate each of the above terms. First, using equation (4.20) and the expression for U_ε defined as in (4.29), we obtain

$$\begin{aligned}
\int_{B_\delta(0)}\left(-\Delta U_\varepsilon-\frac{\gamma}{|x|^2}U_\varepsilon\right)U_\varepsilon\,dx&=\chi\int_{B_\delta(0)}\frac{U_\varepsilon^{2^*(s)}}{|x|^s}\,dx\\
&=\chi\int_{\mathbb{R}^n}\frac{U^{2^*(s)}}{|x|^s}\,dx+O\left(\varepsilon^{\frac{2^*(s)}{2}(\beta_+-\beta_-)}\right),
\end{aligned}$$

and

$$\int_{\partial B_\delta(0)} U_\varepsilon \partial_\nu U_\varepsilon \, d\sigma = -\beta_+ \omega_{n-1} \frac{\varepsilon^{\beta_+-\beta_-}}{\delta^{\beta_+-\beta_-}} + o_\delta\left(\varepsilon^{\beta_+-\beta_-}\right) \quad \text{as } \varepsilon \to 0.$$

Note that

$$\beta_+ - \beta_- < 2-\theta \iff \gamma > \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4} \implies 2\beta_+ + \theta < n.$$

Therefore,

$$\int_{B_\delta(0)} h(x) U_\varepsilon^2 \, dx = O\left(\varepsilon^{\beta_+-\beta_-} \int_{B_\delta(0)} \frac{1}{|x|^{2\beta_+ + \theta}} dx\right) = o_\delta\left(\varepsilon^{\beta_+-\beta_-}\right) \quad \text{as } \varepsilon \to 0.$$

Again from equation (4.20) and the expression for U and β, we get that

$$\begin{aligned}
&\int_{B_\delta(0)} \left(-\Delta U_\varepsilon \, dx - \frac{\gamma}{|x|^2} U_\varepsilon\right) \beta \, dx \\
&\qquad\qquad\qquad = \varepsilon^{\frac{\beta_+ + \beta_-}{2}} \int_{B_{\varepsilon^{-1}\delta}(0)} \left(-\Delta U \, dx - \frac{\gamma}{|x|^2} U\right) \beta(\varepsilon x) \, dx
\end{aligned}$$

$$\begin{aligned}
&= m_{\gamma,h}(\Omega) \varepsilon^{\frac{\beta_+-\beta_-}{2}} \int_{B_{\varepsilon^{-1}\delta}(0)} \left(-\Delta U \, dx - \frac{\gamma}{|x|^2} U\right) |x|^{-\beta_-} \, dx + o_\delta\left(\varepsilon^{\frac{\beta_+-\beta_-}{2}}\right) \\
&= m_{\gamma,h}(\Omega) \varepsilon^{\frac{\beta_+-\beta_-}{2}} \int_{B_{\varepsilon^{-1}\delta}(0)} \left(-\Delta |x|^{-\beta_-} \, dx - \frac{\gamma}{|x|^2} |x|^{-\beta_-}\right) U \, dx \\
&\quad - m_{\gamma,h}(\Omega) \varepsilon^{\frac{\beta_+-\beta_-}{2}} \int_{\partial B_{\varepsilon^{-1}\delta}(0)} \frac{\partial_\nu U}{|x|^{\beta_-}} \, d\sigma + o_\delta\left(\varepsilon^{\frac{\beta_+-\beta_-}{2}}\right) \\
&= \beta_+ m_{\gamma,h}(\Omega) \omega_{n-1} \varepsilon^{\frac{\beta_+-\beta_-}{2}} + o_\delta\left(\varepsilon^{\frac{\beta_+-\beta_-}{2}}\right).
\end{aligned}$$

Similarly,

$$\int_{\partial B_\delta(0)} \beta \partial_\nu U_\varepsilon \, d\sigma = -\beta_+ m_{\gamma,h}(\Omega) \omega_{n-1} \varepsilon^{\frac{\beta_+-\beta_-}{2}} + o_\delta\left(\varepsilon^{\frac{\beta_+-\beta_-}{2}}\right).$$

Since $\beta_+ + \beta_- + \theta = n - (2-\theta) < n$, we have

$$\begin{aligned}\int_{B_\delta(0)} h(x) U_\varepsilon \beta \, dx &= O\left(\varepsilon^{\frac{\beta_+ - \beta_-}{2}} \int_{B_\delta(0)} \frac{1}{|x|^{\beta_+ + \beta_- + \theta}} \, dx\right) \\ &= o_\delta\left(\varepsilon^{\frac{\beta_+ - \beta_-}{2}}\right).\end{aligned}$$

And, finally

$$\varepsilon^{\beta_+ - \beta_-} \int_{B_\delta(0)} \left(|\nabla \beta|^2 - \left(\frac{\gamma}{|x|^2} + h(x)\right)\beta^2\right) dx = o_\delta(\varepsilon^{\beta_+ - \beta_-}).$$

Combining all the estimates, we get

$$\begin{aligned}&\int_{B_\delta(0)} \left(|\nabla u_\varepsilon|^2 - \left(\frac{\gamma}{|x|^2} + h(x)\right) u_\varepsilon^2\right) dx \\ &\qquad = \chi \int_{\mathbb{R}^n} \frac{U^{2^*(s)}}{|x|^s} \, dx - \beta_+ \omega_{n-1} \frac{\varepsilon^{\beta_+ - \beta_-}}{\delta^{\beta_+ - \beta_-}} + o_\delta(\varepsilon^{\beta_+ - \beta_-}).\end{aligned}$$

So,

$$\begin{aligned}&\int_\Omega \left(|\nabla u_\varepsilon|^2 - \left(\frac{\gamma}{|x|^2} + h(x)\right) u_\varepsilon^2\right) dx \\ &\qquad = \chi \int_{\mathbb{R}^n} \frac{U^{2^*(s)}}{|x|^s} \, dx + \omega_{n-1}(n-2) m_{\gamma,h}(\Omega) \varepsilon^{\beta_+ - \beta_-} + o_\delta(\varepsilon^{\beta_+ - \beta_-}).\end{aligned}$$

Step 2: Estimating $\displaystyle\int_\Omega b(x) \frac{u_\varepsilon^{2^*(s)}}{|x|^s} \, dx$.

One has for $\delta > 0$ small

$$
\begin{aligned}
&\int_\Omega b(x)\frac{u_\varepsilon^{2^*(s)}}{|x|^s}\,dx\\
&=\int_{B_\delta(0)} b(x)\frac{u_\varepsilon^{2^*(s)}}{|x|^s}\,dx+\int_{\Omega\setminus B_\delta(0)} b(x)\frac{u_\varepsilon^{2^*(s)}}{|x|^s}\,dx\\
&=\int_{B_\delta(0)} b(x)\frac{\left(U_\varepsilon(x)+\varepsilon^{\frac{\beta_+-\beta_-}{2}}\beta(x)\right)^{2^*(s)}}{|x|^s}\,dx+o(\varepsilon^{\beta_+-\beta_-})\\
&=\int_{B_\delta(0)} b(x)\frac{U_\varepsilon^{2^*(s)}}{|x|^s}\,dx+\varepsilon^{\frac{\beta_+-\beta_-}{2}}2^*(s)\int_{B_\delta(0)} b(x)\frac{U_\varepsilon^{2^*(s)-1}}{|x|^s}\beta\,dx\\
&\quad+o(\varepsilon^{\beta_+-\beta_-})\\
&=\int_{B_\delta(0)} b(x)\frac{U_\varepsilon^{2^*(s)}}{|x|^s}\,dx+\varepsilon^{\frac{\beta_+-\beta_-}{2}}\frac{2^*(s)}{\chi}\int_{B_\delta(0)} b(x)\left(-\Delta U_\varepsilon\,dx-\frac{\gamma}{|x|^2}U_\varepsilon\right)\beta\,dx\\
&\quad+o(\varepsilon^{\beta_+-\beta_-})\\
&=b(0)\int_{\mathbb{R}^n}\frac{U^{2^*(s)}}{|x|^s}\,dx+\frac{2^*(s)}{\chi}b(0)\beta_+m_{\gamma,\lambda,a}(\Omega)\omega_{n-1}\varepsilon^{\beta_+-\beta_-}+o(\varepsilon^{\beta_+-\beta_-}).
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
&J^\Omega_{\gamma,\lambda,a}(u_\varepsilon) &(4.32)\\
&=\frac{\displaystyle\int_\Omega\left(|\nabla u_\varepsilon|^2-\gamma\frac{u_\varepsilon^2}{|x|^2}-h(x)u_\varepsilon^2\right)dx}{\left(\displaystyle\int_\Omega b(x)\frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}\,dx\right)^{2/2^*(s)}}\\
&=\frac{\mu_{\gamma,0}(\mathbb{R}^n)}{b(0)^{2/2^*(s)}}-m_{\gamma,h}(\Omega)\frac{\omega_{n-1}(\beta_+-\beta_-)}{\left(b(0)\displaystyle\int_{\mathbb{R}^n}\frac{U^{2^*(s)}}{|x|^s}\,dx\right)^{2/2^*(s)}}\varepsilon^{\beta_+-\beta_-}+o(\varepsilon^{\beta_+-\beta_-}).\\
& &(4.33)
\end{aligned}
$$

Therefore, if $m_{\gamma,h}(\Omega) > 0$, we get for ε sufficiently small

$$\mu_{\gamma,h}(\Omega)\le J^\Omega_{\gamma,h}(u_\varepsilon) < \frac{\mu_{\gamma,0}(\mathbb{R}^n)}{b(0)^{2/2^*(s)}}.$$

Then, from Lemma 4.4.2 it follows that $\mu_{\gamma,h}(\Omega)$ is attained. $\square$

Remark 4.4.3. Assume for simplicity that $h(x) = \lambda |x|^{-\theta}$ where $0 \leq \theta < 2$. There is a threshold $\lambda^*(\Omega) \geq 0$ beyond which the infimum $\mu_{\gamma,\lambda}(\Omega)$ is achieved, and below which, it is not. In fact,

$$\lambda^*(\Omega) := \sup\{\lambda : \mu_{\gamma,\lambda}(\Omega) = \mu_{\gamma,0}(\mathbb{R}^n)\}.$$

Performing a blow-up analysis like in [106] one can obtain the following sharp results:

- In high dimensions, that is for $\gamma \leq \frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4}$ one has $\lambda^*(\Omega) = 0$ and the infimum $\mu_{\gamma,\lambda}(\Omega)$ is achieved if and only if $\lambda > \lambda^*(\Omega)$.

- In low dimensions, that is for $\frac{(n-2)^2}{4} - \frac{(2-\theta)^2}{4} < \gamma$, one has $\lambda^*(\Omega) > 0$ and $\mu_{\gamma,\lambda}(\Omega)$ is not achieved for $\lambda < \lambda^*(\Omega)$ and $\mu_{\gamma,\lambda}(\Omega)$ is achieved for $\lambda > \lambda^*(\Omega)$. Moreover under the assumption $\mu_{\gamma,\lambda^*}(\Omega)$ is not achieved, we have that $m_{\gamma,\lambda^*}(\Omega) = 0$, and $\lambda^*(\Omega) = \sup\{\lambda : m_{\gamma,\lambda}(\Omega) \leq 0\}$.

4.5 Existence results for compact submanifolds of $\mathbb{B}^n$

Consider the following Dirichlet boundary value problem in hyperbolic space. Let $\Omega \Subset \mathbb{B}^n$ ($n \geq 3$) be a bounded smooth domain such that $0 \in \Omega$. We consider the Dirichlet boundary value problem:

$$\begin{cases} -\Delta_{\mathbb{B}^n} u - \gamma V_2 u - \lambda u = V_{2^*(s)} u^{2^*(s)-1} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.34}$$

where $\lambda \in \mathbb{R}$, $0 < s < 2$ and $\gamma < \gamma_H := \frac{(n-2)^2}{4}$.

We shall use the conformal transformation $g_{\mathbb{B}^n} = \varphi^{\frac{4}{n-2}} g_{\mathrm{Eucl}}$, where $\varphi = \left(\frac{2}{1-r^2}\right)^{\frac{n-2}{2}}$ to map the problem into $\mathbb{R}^n$. We start by considering the general equation :

$$-\Delta_{\mathbb{B}^n} u - \gamma V_2 u - \lambda u = F(x,u) \quad \text{in } \Omega \Subset \mathbb{B}^n, \tag{4.35}$$

where $F(x,u)$ is a Carathéodory function such that

$$|F(x,u)| \le C|u| \left(1 + \frac{|u|^{2^*(s)-2}}{r^s}\right) \quad \text{for all } x \in \Omega.$$

If u satisfies (4.35), then $v := \varphi u$ satisfies the equation:

$$-\Delta v - \gamma \left(\frac{2}{1-r^2}\right)^2 V_2 v - \left[\lambda - \frac{n(n-2)}{4}\right] \left(\frac{2}{1-r^2}\right)^2 v = \varphi^{\frac{n+2}{n-2}} f\left(x, \frac{v}{\varphi}\right) \quad \text{in } \Omega.$$

On the other hand, we have the following expansion for $\left(\frac{2}{1-r^2}\right)^2 V_2$:

$$\left(\frac{2}{1-r^2}\right)^2 V_2(x) = \frac{1}{(n-2)^2} \left(\frac{f(r)}{G(r)}\right)^2$$

where $f(r)$ and $G(r)$ are given by (4.1). We then obtain that

$$\left(\frac{2}{1-r^2}\right)^2 V_2(x) = \begin{cases} \frac{1}{r^2} + \frac{4}{r} + 8 + g_3(r) & \text{when } n = 3, \\ \\ \frac{1}{r^2} + 8\log\frac{1}{r} - 4 + g_4(r) & \text{when } n = 4, \\ \\ \frac{1}{r^2} + \frac{4(n-2)}{n-4} + r g_n(r) & \text{when } n \ge 5. \end{cases} \tag{4.36}$$

where for all $n \ge 3$, $g_n(0) = 0$ and g_n is $C^0([0,\delta])$ for $\delta < 1$.

This implies that $v := \varphi u$ is a solution to

$$-\Delta v - \frac{\gamma}{r^2} v - \left[\gamma a(x) + \left(\lambda - \frac{n(n-2)}{4}\right)\left(\frac{2}{1-r^2}\right)^2\right] v = \varphi^{\frac{n+2}{n-2}} f\left(x, \frac{v}{\varphi}\right).$$

where $a(x)$ is defined in (4.8). We can therefore state the following lemma:

Lemma 4.5.1. *A non-negative function $u \in H_0^1(\Omega)$ solves* (4.34) *if and only if $v := \varphi u \in H_0^1(\Omega)$ satisfies*

$$\begin{cases} -\Delta v - \left(\frac{\gamma}{|x|^2} + h_{\gamma,\lambda}(x)\right) v = b(x)\frac{v^{2^*(s)-1}}{|x|^s} & \text{in } \Omega \\ v \geq 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.37}$$

where

$$h_{\gamma,\lambda}(x) = \gamma a(x) + \frac{4\lambda - n(n-2)}{(1-|x|^2)^2},$$

$a(x)$ is defined in (4.8)*, and $b(x)$ is a positive function in $C^0(\overline{\Omega})$ with $b(0) = \dfrac{(n-2)^{\frac{n-s}{n-2}}}{2^{2-s}}$. Moreover, the hyperbolic operator $L_\gamma^{\mathbb{B}^n} := -\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$ is coercive if and only if the corresponding Euclidean operator $L_{\gamma,h}^{\mathbb{R}^n} := -\Delta - \left(\dfrac{\gamma}{|x|^2} + h_{\gamma,\lambda}(x)\right)$ is coercive.*

Proof. Note that one has in particular

$$h_{\gamma,\lambda}(x) = h_{\gamma,\lambda}(r) = \begin{cases} \frac{4\gamma}{r} + 8\gamma + \frac{4\lambda-3}{(1-r^2)^2} + \gamma g_3(r) & \text{when } n = 3, \\ \left[8\gamma \log\frac{1}{r} - 4\gamma + 4\lambda - 8\right] \\ \quad + \gamma g_4(r) + (4\lambda - 8)\frac{r^2(2-r^2)}{(1-r^2)^2} & \text{when } n = 4, \\ \frac{4(n-2)}{n-4}\left[\frac{n-4}{n-2}\lambda + \gamma - \frac{n(n-4)}{4}\right] \\ \quad + \gamma r g_n(r) + (4\lambda - n(n-2))\frac{r^2(2-r^2)}{(1-r^2)^2} & \text{when } n \geq 5, \end{cases} \tag{4.38}$$

with $g_n(0) = 0$ and g_n is $C^0([0,\delta])$ for $\delta < 1$, for all $n \geq 3$.

Let $F(x,u) = V_{2^*(s)} u^{2^*(s)-1}$ in (4.35). The above remarks show that $v := \varphi u$ is a solution to (4.37).

For the second part, we first note that the following identities hold:

$$\int_\Omega \left(|\nabla_{\mathbb{B}^n} u|^2 - \frac{n(n-2)}{4}u^2\right)\, dv_{g_{\mathbb{B}^n}} = \int_\Omega |\nabla v|^2\, dx$$

and

$$\int_\Omega u^2 dv_{g_{\mathbb{B}^n}} = \int_\Omega v^2 \left(\frac{2}{1-r^2}\right)^2 dx.$$

If the operator $L_\gamma^{\mathbb{B}^n}$ is coercive, then for any $u \in C^\infty(\Omega)$, we have $\langle L_\gamma^{\mathbb{B}^n} u, u\rangle \geq C\|u\|^2_{H_0^1(\Omega)}$, which means

$$\int_\Omega \left(|\nabla_{\mathbb{B}^n} u|^2 - \gamma V_2 u^2\right) dv_{g_{\mathbb{B}^n}} \geq C \int_\Omega \left(|\nabla_{\mathbb{B}^n} u|^2 + u^2\right) dv_{g_{\mathbb{B}^n}}.$$

The latter then holds if and only if

$$\begin{aligned}
\langle L_{\gamma,\phi}^{\mathbb{R}^n} u, u\rangle &= \int_\Omega \left(|\nabla v|^2 - \left(\frac{2}{1-r^2}\right)^2 \left(\gamma V_2 - \frac{n(n-2)}{4}\right) v^2\right) dx \\
&\geq C \int_\Omega \left(|\nabla v|^2 + \left(\frac{2}{1-r^2}\right)^2 \left(\frac{n(n-2)}{4} + 1\right) v^2\right) dx \\
&\geq C' \int_\Omega \left(|\nabla v|^2 + v^2\right) dx \geq c\|u\|^2_{H_0^1(\Omega)},
\end{aligned}$$

where $v = \varphi u$ is in $C^\infty(\Omega)$. This completes the proof. $\square$

At this point, the proof of Theorems 4.1.2 and 4.1.3 follows verbatim as in the Euclidean case.

One can then use the results obtained in the last section to prove Theorem 4.1.4 stated in the introduction for the hyperbolic space. Indeed, it suffices to consider equation (4.37), where b is a positive function in $C^1(\overline{\Omega})$ satisfying (4.7) and $h_{\gamma,\lambda}$ is given by (4.38).

If $n \geq 5$, then $\lim_{|x|\to 0} h_{\gamma,\lambda}(x) = \frac{4(n-2)}{n-4}\left[\frac{n-4}{n-2}\lambda + \gamma - \frac{n(n-4)}{4}\right]$, which is positive provided

$$\lambda > \frac{n-2}{n-4}\left(\frac{n(n-4)}{4} - \gamma\right).$$

Moreover, since in this case $\theta = 0$, the first alternative in Theorem 4.1.5 holds when $\gamma \leq \frac{(n-2)^2}{4} - 1 = \frac{n(n-4)}{4}$. For $\frac{(n-2)^2}{4} - 1 < \gamma < \frac{(n-2)^2}{4}$, the existence of the extremal is guaranteed by the positivity of the hyperbolic mass $m^H_{\gamma,\lambda}(\Omega)$ associated to the operator $L_\gamma^{\mathbb{B}^n}$, which is a positive multiple of the mass of the corresponding Euclidean operator.

When $n = 4$, we can use the first option in Theorem 4.1.5 using the logarithmic perturbation if

$$\lim_{|x|\to 0}\left(\log\frac{1}{|x|}\right)^{-1} h_{\gamma,\lambda}(x) = 8\gamma > 0$$

and, since $\theta = 0$,

$$\gamma \leq \frac{(4-2)^2}{4} - 1 = 0.$$

This is impossible. In the absence of the dominating term with $\log\frac{1}{|x|}$, i.e. when $\gamma = 0$, we get existence of the extremal if $\lambda > \frac{4(4-2)}{4} = 2$. Otherwise, we require the positivity of the hyperbolic mass $m^H_{\gamma,\lambda}$.

Similarly, if $n = 3$, the threshold for γ with the singular perturbation $\frac{1}{|x|}$ (i.e. $\theta = 1$) is $\gamma \leq \frac{(3-2)^2}{4} - \frac{1}{4} = 0$. In order to use the first option in Theorem 4.1.5, we have to resort to the next term $4\lambda - 3$, which is positive when $\lambda > \frac{3}{4}$, in the case $\gamma = 0$. When $\gamma > 0$ or $\lambda \leq \frac{3}{4}$, one needs that $m^H_{\gamma,\lambda} > 0$.

Bibliography

[1] N. Abatangelo, E. Valdinoci. *A notion of nonlocal curvature.* Numer. Funct. Anal. Optim. 35 (2014), no. 7-9, 793–815. → page 9

[2] N. Abatangelo, E. Valdinoci. Getting acquainted with the fractional Laplacian. Preprint, arXiv:1710.11567v1. → page 7

[3] B. Abdellaoui, M. Medina, I. Peral, A. Primo. *The effect of the Hardy potential in some Calderón-Zygmund properties for the fractional Laplacian.* J. Differential Equations 260 (2016), no. 11, 8160–8206. → page 152

[4] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964. → pages 153, 203, 204

[5] Adimurthi, A. Sekar: *Role of the fundamental solution in Hardy-Sobolev-type inequalities.* Proc. Roy. Soc. Edinburgh Sect. A 136 (2006), no. 6, 1111–1130. → page 210

[6] K. Akutagawa, H. Kumura: *Geometric relative Hardy inequalities and the discrete spectrum of Schrö dinger operators on manifolds.* Calc. Var. Partial Differential Equations 48 (2013), no. 1-2, 67–88. → page 209

[7] W. Ao, H. Chan, A. DelaTorre, M. A. Fontelos, M. d. M. González, J. Wei. *On higher dimensional singularities for the fractional Yamabe problem: a non-local Mazzeo-Pacard program.* Preprint (2018), arXiv:1802.07973. → page v

[8] W. Ao, H. Chan, M. González, J. Wei. *Existence of positive weak solutions for fractional Lane-Emden equations with prescribed singular set.* Preprint. → pages 97, 99, 175

[9] W. Ao, A. DelaTorre, M. González, J. Wei. *A gluing approach for the fractional Yamabe problem with prescribed isolated singularities.* Preprint (2016), arXiv:1609.08903. → pages 97, 98

[10] O. Agudelo, M. del Pino, J. Wei. *Higher dimensional catenoid, Liouville equation and Allen-Cahn equation Liouville's equation.* International Math. Research Note (IMRN) 2016, no. 23, 70517102. → page 10

[11] S. M. Allen, J. W. Cahn. *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening.* Acta Metallurgica 27 (1979), no. 6, 1085–1095. → page 5

[12] L. Ambrosio, X. Cabré. *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi.* J. Amer. Math. Soc. 13 (2000), 725–739. → pages 2, 6

[13] L. Ambrosio, G. De Philippis, L. Martinazzi. *Gamma-convergence of nonlocal perimeter functionals.* Manuscripta Math. 134 (2011), no. 3-4, 377–403. → page 7

[14] W. Ao, H. Chan, J. Wei. *Boundary concentrations on segments for the Lin-Ni-Takagi problem.* To appear in Ann. Sc. Norm. Super. Pisa Cl. Sci. → page ix

[15] W. Ao, A. DelaTorre, M. d. M. González, J. Wei. *A gluing aproach for the fractional Yamabe problem with isolated singularities.* Preprint (2016), arXiv:1609.08903v1. → page 8

[16] T. Aubin. *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire.* J. Math. Pures Appl. (9) 55 (1976), no. 3, 269–296. → page 3

[17] B. Barrios, A. Figalli, E. Valdinoci. *Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces.* Ann. Sc. Norm. Super. Pisa. Cl. Sci. (5) 13 (2014), no. 3, 609–639. → pages 7, 98

[18] W. Beckner. *Pitt's inequality and the uncertainty principle.* Proc. Amer. Math. Soc. 123(6):1897–1905, 1995. → page 117

[19] W. Beckner: *On Hardy-Sobolev embedding.* arXiv:0907.3932v1 (2009).

[20] H. Berestycki, T.-C. Lin, J. Wei, C. Zhao. *On phase-separation models. asymptotics and qualitative properties. Arch. Ration.* Mech. Anal. 208 (2013), no. 1, 163–200. → page 8

[21] H. Berestycki, S. Terracini, K. Wang, J. Wei. *On entire solutions of an elliptic system modeling phase separations.* Adv. Math. 243 (2013), 102–126. → page 8

[22] M. Bhakta, K. Sandeep: *Poincaré-Sobolev equations in the hyperbolic space.* Calc. Var. Partial Differential Equations 44 (2012), no. 1-2, 247–269. → page 213

[23] M. Bonforte, F. Gazzola, G. Grillo, J. L. Vázquez: *Classification of radial solutions to the Emden-Fowler equation on the hyperbolic space.* Calc. Var. Partial Differential Equations 46 (2013), no. 1-2, 375–401. → page 213

[24] L. Brasco, E. Lindgren, E. Parini. *The fractional Cheeger problem.* Interfaces Free Bound. 16 (2014), no. 3, 419–458. → page 9

[25] J. Brasseur, S. Dipierro. *Some monotonicity results for general systems of nonlinear elliptic PDEs.* J. Differential Equations 261 (2016), no. 5, 2854–2880. → page 8

[26] W. Bray. *Aspects of harmonic analysis on real hyperbolic space.* Fourier analysis (Orono, ME, 1992), 77-102, Lecture Notes in Pure and Appl. Math., 157, Dekker, New York, 1994. → page 205

[27] C. Bucur. *A symmetry result in $\mathbb{R}^2$ for global minimizers of a general type of nonlocal energy.* Preprint (2017), arXiv:1708.04924. → page 8

[28] C. Bucur, E. Valdinoci. *Nonlocal diffusion and applications.* Lecture Notes of the Unione Matematica Italiana, 20. Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016. xii+155 pp. → page 7

[29] X. Cabré, E. Cinti. *Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian.* Discrete Contin. Dyn. Syst. 28 (2010), no. 3, 1179–1206. → page 2

[30] X. Cabré, E. Cinti. *Sharp energy estimates for nonlinear fractional diffusion equations.* Calc. Var. Partial Differential Equations 49 (2014), no. 1-2, 233–269. → pages 2, 8, 98

[31] X. Cabré, E. Cinti, J. Serra. *Stable s-minimal cones in $\mathbb{R}^3$ are flat for $s \sim 1$.* Preprint (2017), arXiv:1710.08722. → pages 7, 98

[32] X. Cabré, M. Fall, J. Solà-Morales, T. Weth. *Curves and surfaces with constant mean curvature: meeting Alexandrov and Delaunay.* To appear in *J. Reine Angew. Math.* → page 98

[33] X. Cabré, M. Fall, T. Weth. *Delaunay hypersurfaces with constant nonlocal mean curvature.* J. Math. Pures Appl. (9) 110 (2018), 32–70. → page 98

[34] X. Cabré, J. Serra. *An extension problem for sums of fractional Laplacians and 1-D symmetry of phase transitions.* Nonlinear Anal. 137 (2016), 246–265. → page 8

[35] X. Cabré, J. Terra. *Saddle-shaped solutions of bistable diffusion equations in all of* $\mathbb{R}^{2m}$. J. Eur. Math. Soc. (JEMS) 11 (2009), no. 4, 819–843. → page 6

[36] X. Cabré, Y. Sire. *Nonlinear equations for fractional Laplcains, I: Regularity, maximum principles, and Hamiltonian estimates.* Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), no. 1, 23–53. → pages 60, 134, 199

[37] X. Cabré, Y. Sire. *Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions.* Trans. Amer. Math. Soc. 367 (2015), no. 2, 911–941. → pages 2, 8

[38] X. Cabré, J. Solà-Morales. *Layer solutions in a half-space for boundary reactions.* Comm. Pure Appl. Math. 58 (2005), no. 12, 1678–1732. → pages 8, 98, 134

[39] L. Caffarelli, A. Córdoba. *Uniform convergence of a singular perturbation problem.* Comm. Pure Appl. Math. 48 (1995), 1–12. → page 6

[40] L. Caffarelli, T. Jin, Y. Sire, J. Xiong. *Local analysis of solutions of fractional semi-linear elliptic equations with isolated singularities.* Arch. Ration. Mech. Anal., 213 (2014), n. 1, 245–268. → page 113

[41] L. Caffarelli, J.-M. Roquejoffre, O. Savin. *Nonlocal minimal surfaces.* Comm. Pure Appl. Math. 63 (2010), no. 9, 1111–1144. → pages 7, 98

[42] L. Caffarelli, O. Savin, E. Valdinoci. *Minimziation of a fractional perimeter-Dirichlet integral functional.* Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (2015), no. 4, 901–924. → page 9

[43] L. Caffarelli, L. Silvestre. *An extension problem related to the fractional Laplacian.* Comm. Partial Differential Equations 32 (2007), no. 7-9, 1245–1260. → pages 2, 7, 10, 59, 93, 113

[44] L. Caffarelli, E. Valdinoci. *Regularity properties of nonlocal minimal surfaces via limiting arguments.* Adv. Math. 248 (2013), 843–871. → pages 7, 98

[45] J. Cahn, J. Hilliard. *Free energy of a nonuniform system I. Interfacial free energy.* J. Chem. Phys. 28 (1958), 258–267. → page 5

[46] G. Carron: *Inégalités de Hardy sur les variétés riemanniennes non-compactes.* J. Math. Pures Appl. (9). 76 (1997), no. 10, 883–891. → page 209

[47] J. Case, S.-Y. A. Chang. *On fractional GJMS operators.* Communications on Pure and Applied Mathematics 69 (2016), no. 6, 1017–1061. → page 112

[48] H. Chan, N. Ghoussoub, S. Mazumdar, S. Shakerian, L. F. O. Faria. *Mass and extremals associated with the Hardy-Schrödinger operator on hyperbolic space.* Preprint (2017), arXiv:1710.01271. → page v

[49] H. Chan, Y. Liu, J. Wei. *A gluing construction for fractional elliptic equations. Part I: a model problem on the catenoid.* Preprint (2017), arXiv:1711.03215. → pages v, 98

[50] H. Chan, J. Wei. *Traveling wave solutions for bistable fractional Allen–Cahn equations with a pyramidal front.* J. Differential Equations 262 (2017), no. 9, 4567–4609. → pages 9, 40, 142

[51] H. Chan, J. Dávila, M. del Pino, Y. Liu, J. Wei. *A gluing construction for fractional elliptic equations. Part II: Counterexamples of De Giorgi Conjecture for the fractional Allen–Cahn equation.* In preparation. → pages iii, 2, 8, 9, 11, 98

[52] S.-Y. A. Chang, M.d.M. González. *Fractional Laplacian in conformal geometry.* Adv. Math. 226 (2011), no. 2, 1410–1432. → pages 7, 110, 112, 123, 126, 128

[53] Y.-H. Chen, C. Liu, Y. Zheng. *Existence results for the fractional Nirenberg problem.* J. Funct. Anal. 270 (2016), no. 11, 4043–4086. → page 8

[54] G. Chen, Y. Zheng. *Concentration phenomenon for fractional nonlinear Schrödinger equations.* Commun. Pure Appl. Anal. 13 (2014), no. 6, 2359–2376. → page 8

[55] E. Cinti, J. Dávila, M. del Pino. *Solutions of the fractional Allen-Cahn equation which are invariant under screw motion.* J. Lond. Math. Soc. (2) 94 (2016), no. 1, 295–313. → page 8

[56] E. Cinti, J. Serra, E. Valdinoci. *Quantitative flatness results and BV-estimates for stable nonlocal minimal surfaces.* Preprint. → page 98

[57] L. D'Ambrosio, S. Dipierro: *Hardy inequalities on Riemannian manifolds and applications.* Ann. Inst. H. Poincaré Anal. Non Linéaire. 31 (2014), no. 3, 449–475. → page 209

[58] J. Dávila, M. del Pino, S. Dipierro, E. Valdinoci. *Nonlocal Delaunay surfaces.* Nonlinear Anal. 137 (2016), 357–380. → pages 8, 98

[59] J. Dávila, M. del Pino, M. Musso. *The supercritical Lane-Emden-Fowler equation in exterior domains.* Comm. Partial Differential Equations 32 (2007), no. 7-9, 1225–1254. → page 178

[60] J. Dávila, M. del Pino, M. Musso, J. Wei. *Fast and slow decay solutions for supercritical elliptic problems in exterior domains.* Calc. Var. Partial Differential Equations 32 (2008), no. 4, 453–480. → page 178

[61] J. Dávila, M. del Pino, Y. Sire. *Non degeneracy of the bubble in the critical case for non local equations.* Proc. Amer. Math. Soc. 141 (2013), 3865–3870. → page 97

[62] J. Dávila, M. del Pino, J. Wei. *Concentrating standing waves for the fractional nonlinear Schrödinger equation.* J. Differential Equations 256 (2014), no. 2, 858–892. → pages 8, 62, 97

[63] J. Dávila, M. del Pino, J. Wei. *Nonlocal s-minimal surfaces and Lawson cones.* J. Differential Geom., to appear. → pages 8, 76, 89

[64] E. De Giorgi. *Convergence problems for functionals and operators.* Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131–188, Pitagora, Bologna (1979). → pages 2, 6

[65] A. DelaTorre, M. del Pino, M. d. M. González, J. Wei. *Delaunay-type singular solutions for the fractional Yamabe problem. Math Annalen* 369 (2017) 597–62. → pages 94, 130, 131

[66] A. DelaTorre, M. d. M. González. *Isolated singularities for a semilinear equation for the fractional Laplacian arising in conformal geometry.* Preprint. To appear in Rev. Mat. Iber. → pages 93, 114, 115, 136, 154

[67] M. del Pino, M. Kowalczyk, J. Wei. *On De Giorgi's conjecture in dimension $N \geq 9$.* Ann. of Math. (2) 174 (2011), no. 3, 1485–1569. → pages 2, 6

[68] M. del Pino, M. Kowalczyk, J. Wei. *Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in* $\mathbb{R}^3$. J. Differential Geom. 93 (2013) no. 1, 67–131. → page 9

[69] D. De Silva, J.-M. Roquejoffre. *Regularity in a one-phase free boundary problem for the fractional Laplacian.* Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), no. 3, 335–367. → page 9

[70] D. De Silva, O. Savin. *Regularity of Lipschitz free boundaries for the thin one-phase problem.* J. Eur. Math. Soc. (JEMS) 17 (2015), no. 6, 1293–1326. → page 9

[71] B. Devyver, M. Fraas, Y. Pinchover: *Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon.* J. Funct. Anal. 266 (2014), no. 7, 4422–4489. → pages 209, 210, 211

[72] A. Di Castro, M. Novaga, B. Ruffini, E. Valdinoci. *Nonlocal quantitative isoperimetric inequalities.* Calc. Var. Partial Differential Equations 54 (2015), no. 3, 2421–2464. → page 9

[73] E. Di Nezza, G. Palatucci, E. Valdinoci. *Hitchhiker's guide to the fractional Sobolev spaces.* Bull. Sci. Math. 136 (2012), no. 5, 521–573. → pages 7, 107

[74] S. Dipierro. *Geometric inequalities and symmetry results for elliptic systems.* Discrete Contin. Dyn. Syst. 33 (2013), no. 8, 3473–3496. → page 8

[75] S. Dipierro, A. Figalli, G. Palatucci, E. Valdinoci. *Asymptotics of the s-perimeter as* $s \searrow 0$. Discrete Contin. Dyn. Syst. 33 (2013), no. 7, 2777–2790. → page 9

[76] S. Dipierro, L. Montoro, I. Peral, B. Sciunzi. *Qualitative properties of positive solutions to non-local critical problems involving the Hardy-Leray potential.* Calc. Var. Partial Differential Equations 55 (2016), no. 4, Art. 99, 29 pp. → page 152

[77] S. Dipierro, A. Pinamonti. *A geometric inequality and a symmetry result for elliptic systems involving the fractional Laplacian.* J. Differential Equations 255 (2013), no. 1, 85–119. → page 8

[78] S. Dipierro, O. Savin, E. Valdinoci. *A nonlocal free boundary problem.* SIAM J. Math. Anal. 47 (2015), no. 6, 4559–4605. → page 9

[79] S. Dipierro, O. Savin, E. Valdinoci. *Graph properties for nonlocal minimal surfaces.* Calc. Var. Partial Differential Equations 55 (2016), no. 4, Art. 86, 25 pp. → pages 8, 98

[80] S. Dipierro, O. Savin, E. Valdinoci. *Boundary behavior of nonlocal minimal surfaces.* J. Funct. Anal. 272 (2017), no. 5, 1791–1851. → pages 8, 98

[81] S. Dipierro, J. Serra, E. Valdinoci. *Improvement of flatness for nonlocal phase transitions.* Preprint (2016), arXiv:1611.10105. → page 8

[82] Z. Du, C. Gui, Y. Sire, J. Wei. *Layered solutions for a fractional inhomogeneous Allen-Cahn equation.* NoDEA Nonlinear Differential Equations Appl. 23 (2016), no. 3, Art. 29, 26 pp. → pages 8, 36, 59, 97

[83] P. Esposito, N. Ghoussoub, *Uniqueness of solutions for an elliptic equation modeling MEMS.* Methods Appl. Anal. 15 (2008), no. 3, 341–353. → pages 93, 100

[84] M. Fall, V. Felli. *Unique continuation property and local asymptotics of solutions to fractional elliptic equations.* Comm. Partial Differential Equations 39 (2014), no. 2, 354–397. → pages 164, 178

[85] M.M. Fall, T. Weth. *Nonexistence results for a class of fractional elliptic boundary value problems.* J. Funct. Anal. 263 (2012), no. 8, 2205–2227.

[86] Y. Fang, M. d. M. González. *Asymptotic behavior of Palais-Smale sequences associated with fractional Yamabe type equations.* Pacific J. Math, 278 (2015), no. 2, 369–405. → page 90

[87] A. Farina. *Some symmetry results for entire solutions of an elliptic system arising in phase separation.* Discrete Contin. Dyn. Syst. 34 (2014), no. 6, 2505–2511. → page 8

[88] A. Farina, B. Sciunzi, E. Valdinoci. *Bernstein and De Giorgi type problems. new results via a geometric approach.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7 (2008), no. 4, 741–791.

[89] A. Farina, N. Soave. *Monotonicity and 1-dimensional symmetry for solutions of an elliptic system arising in Bose-Einstein condensation.* Arch. Ration. Mech. Anal. 213 (2014), no. 1, 287–326. → page 8

[90] A. Farina, E. Valdinoci. *Rigidity results for elliptic PDEs with uniform limits: an abstract framework with applications.* Indiana Univ. Math. J. 60 (2011), no. 1, 121–141. → page 8

[91] M. Fazly, N. Ghoussoub. *De Giorgi type results for elliptic systems.* Calc. Var. Partial Differential Equations 47 (2013), no. 3-4, 809–823. → page 8

[92] M. Fazly, Y. Sire. *Symmetry results for fractional elliptic systems and related problems.* Comm. Partial Differential Equations 40 (2015), no. 6, 1070–1095. → page 8

[93] A. Figalli, N. Fusco, F. Maggi, V. Millot, M. Morini. *Isoperimetry and stability properties of balls with respect to nonlocal energies.* Comm. Math. Phys. 336 (2015), no. 1, 441–507. → page 9

[94] A. Figalli, J. Serra. *On stable solutions for boundary reactions: a De Giorgi-type result in dimension* $4+1$. Acta. Math., to appear. → pages 2, 8, 98

[95] G. M. Figueiredo, G. Siciliano. *A multiplicity result via Ljusternick-Schnirelmann category and Morse theory for a fractional Schrödinger equation in* $\mathbb{R}^N$. NoDEA Nonlinear Differential Equations Appl. 23 (2016), no. 2, Art. 12, 22 pp. → page 8

[96] A. Fiscella, E. Valdinoci. *A critical Kirchhoff type problem involving a nonlocal operator.* Nonlinear Anal. 94 (2014), 156–170. → page 8

[97] R. Frank, E. Lenzmann. *Uniqueness of non-linear ground states for fractional Laplacians in* $\mathbb{R}$. Acta Math. 210 (2013), no. 2, 261–318. → page 97

[98] R. Frank, E. Lenzmann, L. Silvestre. *Uniqueness of radial solutions for the fractional Laplacian.* Comm. Pure Appl. Math. 69 (2016), no. 9, 1671–1726. → pages 97, 134, 164

[99] R. L. Frank, E. Lieb and R. Seiringer. *Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators.* J. Amer. Math. Soc. 21(4):925–950, 2008. → page 117

[100] N. Fusco. *The quantitative isoperimetric inequality and related topics.* Bull. Math. Sci. 5 (2015), no. 3, 517–607. → page 9

[101] D. Ganguly, K. Sandeep: *Sign changing solutions of the Brezis-Nirenberg problem in the Hyperbolic space.* Calc. Var. Partial Differential Equations 50 (2014), no. 1-2, 69–91. → page 213

[102] V. Georgiev. *Semilinear hyperbolic equations.* With a preface by Y. Shibata. MSJ Memoirs, 7. Mathematical Society of Japan, Tokyo, 2000. → pages 205, 208

[103] N. Ghoussoub, C. Gui. *On a conjecture of De Giorgi and some related problems.* Math. Ann. 311 (1998), 481–491. → pages 2, 6

[104] N. Ghoussoub, C. Gui. *On De Giorgi's conjecture in dimensions 4 and 5.* Ann. of Math. (2) 157 (2003), no. 1, 313–334. → page 6

[105] N. Ghoussoub, F. Robert: *Sobolev inequalities for the Hardy–Schrödinger operator: Extremals and critical dimensions.* Bull. Math. Sci. 6 (2016), no. 1, 89–144. → pages 3, 212, 214, 224

[106] N. Ghoussoub, F. Robert: *The Hardy-Schrödinger operator with interior singularity: The remaining cases.* Calc. Var. Partial Differential Equations 56 (2017), no. 5, Art. 149, 54 pp. → pages 3, 212, 214, 215, 216, 225, 227, 228, 237

[107] N. Ghoussoub, S. Shakerian, *Borderline variational problems involving fractional Laplacians and critical singularities.* Preprint, arXiv:1503.08193. → page 101

[108] M. d. M. González. *Gamma convergence of an energy functional related to the fractional Laplacian.* Calc. Var. Partial Differential Equations 36 (2009), no. 2, 173–210. → page 7

[109] M. d. M. Gonález. *Recent progress on the fractional Laplacian in conformal geometry.* Preprint (2016), arXiv:1609.08988. → pages 7, 110, 115

[110] M. d. M. González, R. Mazzeo, Y. Sire. *Singular solutions of fractional order conformal Laplacians.* J. Geom. Anal., 22 (2012), no. 3, 845–863. → page 92

[111] M. d. M. González, J. Qing. *Fractional conformal Laplacians and fractional Yamabe problems.* Anal. PDE, 6 (2013), no. 7, 1535–1576. → page 90

[112] M. d. M. González, M. Wang. *Further results on the fractional yamabe problem: the umbilic case.* To appear in Journal of Geometric Analysis. → page 90

[113] C. R. Graham, M. Zworski. *Scattering matrix in conformal geometry.* Invent. Math., 152(1):89–118, 2003. → page 110

[114] G. Grubb. *Integration by parts and Pohozaev identities for space-dependent fractional-order operators.* Preprint, arXiv:1511.03901. → page 100

[115] C. Guillarmou. *Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds.* Duke Math. J., 129(1): 1–37, 2005. → page 110

[116] C. Gui, M. Zhao. *Traveling wave solutions of Allen-Cahn equation with a fractional Laplacian.* Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (2015), no. 4, 785–812. → page 8

[117] Z. Guo, J. Wei. *Infinitely many turning points for an elliptic problem with a singular non-linearity.* J. Lond. Math. Soc. (2) 78 (2008), no. 1, 21–35. → page 100

[118] Z. Guo, J. Wei. *Asymptotic behavior of touch-down solutions and global bifurcations for an elliptic problem with a singular nonlinearity.* Commun. Pure Appl. Anal. 7 (2008), no. 4, 765–786. → page 100

[119] S. Helgason. *Geometric analysis on symmetric spaces.* Second edition. Mathematical Surveys and Monographs, 39. American Mathematical Society, Prnovidence, RI, 2008. xviii+637 pp. → page 205

[120] D. Jerison, R. Monneau. *Towards a counter-example to a conjecture of De Giorgi in high dimensions.* Ann. Mat. Pura Appl. (4) 183 (2004), no. 4, 439–467. → pages 2, 6, 11

[121] T. Jin, O. de Queiroz, Y. Sire, J. Xiong. *On local behavior of singular positive solutions to non-local elliptic equations.* To appear in Calc. Var. Partial Differential Equations. → pages 92, 117

[122] S. Kim, M. Musso, J. Wei. *Existence theorems of the fractional Yamabe problem.* Preprint, arXiv:1603.06617. → page 90

[123] I. Kombe, M. Ozaydin: *Improved Hardy and Rellich inequalities on Riemannian manifolds.* Trans. Amer. Math. Soc. 361 (2009), no. 12, 6191–6203. → page 209

[124] A. Kufner, L. Maligranda, L. Persson. *The prehistory of the Hardy inequality.* Amer. Math. Monthly 113 (2006), no. 8, 715–732. → page 117

[125] H. Knüpfer, C. Muratov. *On an isoperimetric problem with a competing nonlocal term I: The planar case.* Comm. Pure Appl. Math. 66 (2013), no. 7, 1129–1162. → page 9

[126] P. Li, J. Wang: *Weighted Poincaré inequality and rigidity of complete manifolds.* Ann. Sci. Ec. Norm. Super. (4) 39 (2006), no. 6, 921–982. → pages 209, 210

[127] M. Ludwig. *Anisotropic fractional perimeters.* J. Differential Geom. 96 (2014), no. 1, 77–93. → page 9

[128] G. Mancini, K. Sandeep: *On a semilinear equation in* $\mathbb{H}^n$. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7 (2008), no. 4, 635–671. → page 213

[129] M. Mayer, C. B. Ndiaye. *Fractional Yamabe problem on locally flat conformal infinities of Poincare-Einstein manifolds.* Preprint (2017), arXiv:1701.05919. → page 90

[130] R. Mazzeo. *Elliptic theory of differential edge operators I.* Comm. Partial Differential Equations 16 (1991), no. 10, 1615–1664. → pages 91, 96, 186

[131] R. Mazzeo, B. Vertman. *Elliptic theory of differential edge operators, II: Boundary value problems.* Indiana Univ. Math. J. 63 (2014), no. 6, 1911–1955. → pages 96, 186

[132] R. Mazzeo, F. Pacard. *A construction of singular solutions for a semilinear elliptic equation using asymptotic analysis.* J. Differential Geom. 44 (1996), no. 2, 331–370. → pages 3, 91, 93, 94, 134, 186

[133] R. Mazzeo, F. Pacard. *Constant scalar curvature metrices with isolated singularities.* Duke Math. Journal, 99 (1999), no.3, 353-418. → pages 98, 137, 200

[134] R. Mazzeo, F. Pacard. *Constant mean curvature surfaces with Delaunay ends.* Comm. Anal. Geom., 9 (2001), no.1 169–237. → page 98

[135] R. Mazzeo, F. Pacard, D. Pollack. *Connected sums of constant mean curvature surfaces in Euclidean 3 space.* J. Reine Angew. Math., 536 (2001), 115–165. → page 98

[136] R. Mazzeo, D. Pollack, K. Uhlenbeck. *Connected sum constructions for constant scalar curvature metrics.* Topol. Methods Nonlinear Anal., 6 (1995), no. 2, 207–233. → page 98

[137] R. Mazzeo, N. Smale. *Conformally flat metrics of constant positive scalar curvature on subdomains of the sphere.* J. Differential Geom. 34 (1991), no. 3, 581–621. → page 142

[138] V. Millot, Y. Sire. *On a fractional Ginzburg-Landau equation and* $1/2$*-harmonic maps into spheres.* Arch. Ration. Mech. Anal. 215 (2015), no. 1, 125–210. → page 7

[139] V. Millot, Y. Sire, K. Wang. *Asymptotics for the fractional Allen-Cahn equation and stationary nonlocal minimal surfaces.* Preprint (2016), arXiv:1610.07194. → page 7

[140] L. Modica. *Convergence to minimal surfaces problem and global solutions of* $\Delta u = 2(u^3 - u)$. Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 223–244, Pitagora, Bologna (1979). → page 6

[141] G. Molica Bisci, D. Repovš. *On doubly nonlocal fractional elliptic equations.* Atti. Accad. Naz. Lincei. Rend. Lincei Mat. Appl. 26 (2015), no. 2, 161–176. → page 8

[142] F. Pacard. *Connected sum constructions in geometry and nonlinear analysis.* Unpublished notes. Available at http://indico.ictp.it/event/a07155/session/25/contribution/14/material/0/0.pdf. → page 185

[143] D. Pagliardini. *Multiplicity of critical points for the fractional Allen-Cahn energy.* Electron. J. Differential Equations 2016, Paper No. 119, 12 pp. → page 8

[144] Y. Pinchover, K. Tintarev: *A ground state alternative for singular Schrödinger operators.* J. Funct. Anal. 230 (2006), no. 1, 65–77.

[145] P. Pucci, S. Saldi. *Critical stationary Kirchhoff equations in* $\mathbb{R}^N$ *involving nonlocal operators.* Rev. Mat. Iberoam. 32 (2016), no. 1, 1–22. → page 8

[146] H. Qiu, M. Xiang. *Existence of solutions for fractional p-Laplacian problems via Leray-Schauder's nonlinear alternative.* Bound. Value Probl. 2016, Paper No. 83, 8 pp. → page 8

[147] P. Quitnner, P. Souplet: *Superlinear parabolic problems. Blow-up, global existence and steady states.* Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2007.

[148] M. Riesz. *Intégrales de Riemann-Liouville et potentiels.* Acta Sci. Math. Szeged 9 (1): 1–42. → page 105

[149] X. Ros-Oton, J. Serra. *The Pohozaev identity for the fractional Laplacian.* Arch. Rat. Mech. Anal. 213 (2014), 587–628. → pages 93, 100

[150] X. Ros-Oton, J. Serra. *The extremal solution for the fractional Laplacian.* Calc. Var. Partial Differential Equations 50 (2014), 723–750. → page 109

[151] X. Ros-Oton, J. Serra. *Pohozaev identities for anisotropic integro-differential operators.* Preprint, arXiv:1502.01431. → page 100

[152] A. Rüland. *Unique continuation for fractional Schrdinger equations with rough potentials.* Comm. Partial Differential Equations 40 (2015), no. 1, 77–114. → pages 164, 178

[153] K. Sandeep, C. Tintarev: *A subset of Caffarelli–Kohn–Nirenberg inequalities in the hyperbolic space* $\mathbb{H}^N$. Ann. Mat. Pura Appl. (4) 196 (2017), no. 6, 2005–2021. → pages 210, 211, 217, 221

[154] O. Savin. *Regularity of flat level sets in phase transitions.* Ann. of Math., 169 (2009), 41–78. → pages 2, 6

[155] O. Savin. *Rigidity of minimizers in nonlocal phase transitions.* Preprint (2016), arXiv:1610.09295. → pages 2, 8, 98

[156] O. Savin. *Rigidity of minimizers in nonlocal phase transitions II.* Preprint (2016), arXiv:1802.01710. → page 2

[157] O. Savin, E. Valdinoci. Γ*-convergence for nonlocal phase transitions.* Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), no. 4, 479–500. → pages 2, 7

[158] O. Savin, E. Valdinoci. *Some monotonicity results for minimizers in the calculus of variations.* J. Funct. Anal. 264 (2013), no. 10, 2469–2496. → page 8

[159] O. Savin, E. Valdinoci. *Regularity of nonlocal minimal cones in dimension 2.* Calc. Var. Partial Differential Equations 48 (2013), no. 1-2, 33–39. → page 7

[160] O. Savin, E. Valdinoci. *Density estimates for a variational model driven by the Gagliardo norm.* J. Math. Pures Appl. (9) 101 (2014), no. 1, 1–26. → page 7

[161] R. Schaaf. *Uniqueness for semilinear elliptic problems: supercritical growth and domain geometry.* Adv. Differential Equations, 5 (2000), no. 10-12, 1201–1220. → page 100

[162] R. Schoen. *Conformal deformation of a Riemannian metric to constant scalar curvature.* J. Differential Geom. 20 (1984), no. 2, 479–495. → page 3

[163] R. Schoen. Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In *Topics in calculus of variations (Montecatini Terme, 1987)*, volume 1365 of *Lecture Notes in Math.*, pages 120–154. Springer, Berlin, 1989. → page 134

[164] R. Schoen. *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar eqution.* Comm Pure Appl. Math. 41 (1988) 317–392. → page 98

[165] Y. Sire, E. Valdinoci. *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result.* J. Funct. Anal. 256 (2009), no. 6, 1842–1864. → pages 2, 8

[166] S. Y. Slavyanov and W. Lay. *Special functions*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. → pages 203, 204

[167] J. Tan. *Positive solutions for non local elliptic problems.* Discrete Contin. Dyn. Syst. 33 (2013), no. 2, 837–859. → page 8

[168] A. Terras. *Harmonic analysis on symmetric spaces and applications. I.* Springer-Verlag, New York, 1985. → page 205

[169] A. Terras. *Non-Euclidean harmonic analysis.* SIAM Rev. 24 (1982), no. 2, 159–193. → page 194

[170] C. E. Torres Ledesma. *Existence and symmetry result for fractional p-Laplacian in* $\mathbb{R}^n$. Commun. Pure Appl. Anal. 16 (2017), no. 1, 99–113. → page 8

[171] N. S. Trudinger. *Remarks concerning the conformal deformation of Riemannian sturctures on compact manifolds.* Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 265–274. → page 3

[172] E. Valdinoci. *A fractional framework for perimeters and phase transitions.* Milan J. Math. 81 (2013), no. 1, 1–23. → pages 9, 98

[173] J. L. Vazquez: *Fundamental solution and long time behaviour of the porous medium equation in hyperbolic space.* J. Math. Pures Appl. (9) 104 (2015), no. 3, 454–484.

[174] K. Wang. *Harmonic approximation and improvement of flatness in a singular perturbation problem.* Manuscripta Math. 146 (2015), no. 1-2, 281–298. → page 8

[175] K. Wang. *On the De Giorgi type conjecture for an elliptic system modeling phase separation.* Comm. Partial Differential Equations 39 (2014), no. 4, 696–739. → page 8

[176] K. Wang, J. Wei. *On the uniqueness of solutions of a nonlocal elliptic system.* Math. Ann. 365 (2016), no. 1-2, 105–153. → page 8

[177] Y. Wei, X. Su. *Multiplicity of solutions for non-local elliptic equations driven by the fractional Laplacian.* Calc. Var. Partial Differential Equations 52 (2015), no. 1-2, 95–124. → page 8

[178] M. Xiang, B. Zhang, V. D. Rădulescu. *Existence of solutions for a bi-nonlocal fractional p-Kirchhoff type problem.* Comput. Math. Appl. 71 (2016), no. 1, 255–266. → page 8

[179] D. Yafaev. *Sharp constants in the Hardy–Rellich Inequalities.* J. Funct. Anal. 168 (1999), no. 1, 121–144. → page 117

[180] H. Yamabe. *On a deformation of Riemannian structures on compact manifolds.* Osaka Math J. 12 (1960), 21–37.

[181] Q. Yang, D. Su, Y. Kong: *Hardy inequalities on Riemannian manifolds with negative curvature.* Commun. Contemp. Math. 16 (2014), no. 2, 1350043, 24 pp. → page 3
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