Index bounds and existence results for minimal surfaces and harmonic maps

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Abstract

In this work, we focus on three problems. First, we give a relationship between the eigenvalues of the Hodge Laplacian and the eigenvalues of the Jacobi operator for a free boundary minimal hypersurface of a Euclidean convex body. We then use this relationship to obtain new index bounds for such minimal hypersurfaces in terms of their topology. In particular, we show that the index of a free boundary minimal surface in a convex domain in $\mathbb{R}^3$ tends to infinity as its genus or the number of boundary components tends to infinity. Second, we consider the relationship between the kth normalized eigenvalue of the Dirichlet-to-Neumann map (the kth Steklov eigenvalue) and the geometry of rotationally symmetric Möbius bands. More specifically, we look at the problem of finding a metric that maximizes the kth Steklov eigenvalue among all rotationally symmetric metrics on the Möbius band. We show that such a metric can always be found and that it is realized by the induced metric on a free boundary minimal Möbius band in $\mathbb{B}^4$. Third, we consider the existence problem for harmonic maps into CAT(1) spaces. If Σ is a compact Riemann surface, X is a compact locally CAT(1) space and $\varphi : \Sigma \to X$ is a continuous finite energy map, we use the technique of harmonic replacement to prove that either there exists a harmonic map $u : \Sigma \to X$ homotopic to φ or there exists a conformal harmonic map $v : \mathbb{S}^2 \to X$. To complete the argument, we prove compactness for energy minimizers and a removable singularity theorem for conformal harmonic maps.

Lay Summary

A minimal surface is a surface that locally minimizes area. Free boundary minimal surfaces of a ball are a special class of minimal surfaces that meet the boundary of the ball orthogonally. A minimal surface may not have the smallest area; the area could decrease by perturbing the surface in certain directions. First, we relate the surface's topology to the number of directions in which perturbations yield decreases in area.

A Möbius band is constructed by twisting one end of a strip of paper 180° and gluing the ends of the paper together. Second, we construct examples of free boundary minimal Möbius bands.

The energy of a map between two spaces measures the amount a map stretches the original space. Third, we show that one can always find a smallest-energy map between a surface and a space with a notion of distance whose curvature cannot be too large.

Preface

This thesis is based on three previous works, two of which have been published in academic journals, and the other of which is in preparation.

The material presented in Chapter 2 and Appendix A is based on the paper "Index bounds for free boundary minimal hypersurfaces of convex bodies" [55] appearing in the journal *Proceedings of the American Mathematical Society*, Volume 145 (2017), pages 2467-2480. I chose this problem under the guidance of my supervisor, Ailana Fraser, and was responsible for all aspects of this work.

The material in Chapter 3 is based on a recent project "Free boundary minimal Möbius bands in $\mathbb{B}^4$", which is currently in preparation to be submitted to an academic journal. Again, this problem was chosen under the guidance of my supervisor, and I was responsible for all aspects of this work.

The material presented in Chapter 4 and Appendix B is based on the paper "Existence of harmonic maps into CAT(1) spaces" [6] which will appear in the journal *Communications in Analysis and Geometry*. This was a joint work with Christine Breiner, Ailana Fraser, Lan-Hsuan Huang, Chikako Mese and Yingying Zhang.

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Dedication

To my friends and family.

Chapter 1

Introduction

This work is devoted to problems related to free boundary minimal surfaces and harmonic maps. Minimal surfaces are critical points of the area functional and satisfy certain nonlinear partial differential equations. Examples of minimal surfaces include soap films, helicoids (the geometric shape of DNA and double-spiral staircases), as well as catenoids (minimal surfaces obtained by rotating catenaries about their directrices). While important in geometry, they also have significant applications in other fields of mathematics and, in fact, played a crucial role in the celebrated proof of the Poincaré conjecture. In addition, minimal surface theory also has physical applications in fluid interface problems and deep connections to fundamental questions in general relativity.

In addition to problems in minimal surface theory, we look at problems concerning the existence of harmonic maps into singular spaces. A natural notion of energy for a map between geometric spaces is defined by measuring the total stretch of the map at each point of the domain and then integrating it over the domain. Harmonic maps are critical points of the energy functional. They can be seen as both a generalization of harmonic functions in complex analysis and a higher dimensional analogue of parameterized geodesics in Riemannian geometry. In the absence of a totally geodesic map, a harmonic map is perhaps the most natural way to map one given geometric space into another. The theory of harmonic maps has proven to have important applications; for example, the existence theory for harmonic two-spheres played a role in the proof of a generalization of the classical sphere theorem to pointwise quarter-pinching. Other important applications of harmonic maps include those in rigidity problems and in Teichmüller theory amongst others.

More specifically, we focus on three problems: finding index bounds for free boundary minimal hypersurfaces of convex bodies, constructing free boundary minimal Möbius bands in the 4-dimensional Euclidean ball, and proving the existence of harmonic maps from a compact Riemann surface into a compact locally CAT(1) space. Here, we outline the problems and state the main results, and outline the layout of the thesis.

1.1 Index bounds for free boundary minimal hypersurfaces of convex bodies

If M is a Riemannian manifold with $\partial M \neq \emptyset$ and Σ is a n-dimensional submanifold with nonempty boundary $\partial\Sigma \subset \partial M$, then Σ is a free boundary minimal submanifold if it is a critical point for the volume functional among all n-dimensional submanifolds whose boundary lie in the boundary of M. It is easy to show that Σ is a free boundary minimal submanifold of M if and only if it has zero mean curvature and if it meets the boundary of M orthogonally. In the case when $M = \mathbb{B}^3$, the simplest example of a free boundary minimal surface is the equatorial disk.

Despite their name, minimal submanifolds do not, in general, minimize volume and instead are saddle points of the volume functional. Roughly speaking, the index of a minimal submanifold measures the degree to which it does not minimize volume and intuitively corresponds to the number of independent directions in which one can perturb the submanifold and decrease its volume.

In [55], we ask whether one can estimate the index of a free boundary minimal hypersurface of a convex body in terms of the hypersurface's topology and dimension. This question, and the approach taken to answer it, was motivated by the work of Savo who, in [56], gave a lower bound on the index of a minimal hypersurface of $\mathbb{S}^{n+1}$ in terms of the hypersurface's topology and dimension. In particular, if M^n is a minimal hypersurface of $\mathbb{S}^{n+1}$ with first Betti number $\beta_1(M)$, Savo showed that

$$\operatorname{Ind}(M) \geq \frac{\beta_1(M)}{\binom{n+2}{2}} + n + 2.$$

To obtain this result, Savo first found a relationship between the eigenvalues of the Jacobi operator J and the eigenvalues of the Hodge Laplacian Δ_1 on one-forms. Namely, if $\lambda_j(J)$ is the jth eigenvalue of the Jacobi operator and $\lambda_j(\Delta_1)$ is the jth eigenvalue of the Hodge Laplacian, then Savo showed that

$$\lambda_j(J) \leq \lambda_{m(j)}(\Delta_1) - 2(n-1),$$

where $m(j) = \binom{n+2}{2}(j-1) + 1$.

We prove the analogous result for free boundary minimal hypersurfaces of convex bodies by analyzing the Hodge Laplacian on one-forms. Unlike the case of minimal hypersurfaces of $\mathbb{S}^{n+1}$, however, here our hypersurfaces have boundary which forces us to analyze boundary value problems for the Hodge Laplacian. Specifically, we analyze the relationship between the eigenvalues of the Jacobi operator and the eigenvalues of the Hodge Laplacian on one-forms with absolute boundary conditions and obtain the following theorem.

Theorem 1.1.1. *Let M^n be an orientable free boundary minimal hypersurface of a convex body in $\mathbb{R}^{n+1}$ with Jacobi operator J. Then, for all positive integers j, one has that*

$$\lambda_j(J) \leq \lambda_{m(j)}(\Delta_1),$$

where $m(j) = \binom{n+1}{2}(j-1)+1$ and $\lambda_{m(j)}(\Delta_1)$ is the $m(j)$th eigenvalue of the Laplacian eigenvalue problem with absolute boundary conditions.

We are then able to use this to get a lower bound for the index of free boundary minimal hypersurfaces of convex bodies in terms of the topology and dimension of the surface. In particular, if we let $\beta_a^1 = \dim H_a^1(M)$ be the first absolute Betti number of M, we get the following index estimate.

Theorem 1.1.2. (Index Bound) *If M is an orientable free boundary minimal hy-*

persurface of a convex body in $\mathbb{R}^{n+1}$*, then*

$$\mathrm{Ind}(M) \geq \left\lfloor \frac{\beta_a^1 + \binom{n+1}{2} - 1}{\binom{n+1}{2}} \right\rfloor .$$

In the special case of a free boundary minimal surface of a convex body in $\mathbb{R}^3$ with genus g and k boundary components, this reduces to

$$\mathrm{Ind}(M) \geq \left\lfloor \frac{2g + k + 1}{3} \right\rfloor .$$

This result provides new index bounds for free boundary minimal surfaces of $\mathbb{B}^3$ with large topology. In particular, it shows that $\mathrm{Ind}(M) \geq 4$ when $2g + k \geq 11$ and $\mathrm{Ind}(M)$ tends to infinity as the genus or the number of boundary components tends to infinity. This was obtained simultaneously, but independently and through different methods, by Ambrozio, Carlotto and Sharp [3].

1.2 Free boundary minimal Möbius bands in $\mathbb{B}^4$

We are also interested finding explicit constructions of free boundary minimal submanifolds. Here, we focus our attention on constructing free boundary minimal Möbius bands in $\mathbb{B}^4$. Our construction is somewhat indirect and is motivated by the works of Fraser and Schoen [24, 26] and Fan, Tam and Yu [21].

In [26], Fraser and Schoen provide a connection between metrics that maximize the kth Steklov eigenvalue on a surface and the geometry of that surface. More specifically, they show that a metric that maximizes the kth Steklov eigenvalue arises geometrically as the induced metric on a free boundary minimal surface of a Euclidean ball by showing that one can construct a conformal minimal immersion into a ball from the eigenfunctions corresponding to the Steklov eigenvalue. As a consequence, if one can find the metrics that maximize the Steklov eigenvalues, then one has existence of free boundary minimal surfaces of balls.

This problem, however, is quite difficult to solve and in general does not yield explicit solutions. As an alternative, in previous work Fraser and Schoen [24], and later Fan, Tam and Yu [21], consider the more specialized problem of maximizing Steklov eigenvalues over all rotationally symmetric metrics on the annulus. In this more specialized setting, the result of Fraser and Schoen [26] no longer guarantees that the maximizing metric, if it exists, arises as the metric on a free boundary minimal surface of a Euclidean ball. However, restricting their attention to this smaller class of metrics allowed them in [24] to solve the problem explicitly through the method of separation of variables. Fraser and Schoen showed that there is a metric that maximizes the first Steklov eigenvalue and that this metric is the induced metric on the critical catenoid, a free boundary minimal surface of $\mathbb{B}^3$. Fan, Tam and Yu considered the same problem for higher Steklov eigenvalues and showed that, except in the case of the 2nd Steklov eigenvalue, whose supremum cannot be achieved, there is a metric that maximizes the kth Steklov eigenvalue and that it corresponds to the induced metric on a free boundary minimal surface of $\mathbb{B}^3$ or $\mathbb{B}^4$. More specifically, they show that the metrics that maximize the Steklov eigenvalues are the metrics induced on either the n-critical catenoid or the so-called n-critical Möbius band. This provided explicit constructions of new free boundary minimal surfaces in $\mathbb{B}^3$ and $\mathbb{B}^4$. Further, Fan, Tam and Yu conjectured that the supremum of the 2nd Steklov eigenvalue can never be achieved for any surface.

Motivated by these works, we provide constructions of free boundary minimal Möbius bands in $\mathbb{B}^4$ by explicitly finding metrics that maximize Steklov eigenvalues among all rotationally symmetric metrics on the Möbius band. In particular, we obtain the following theorem.

Theorem 1.2.1. *For all $n \geq 1$, the maximum of the nth Steklov eigenvalue among all rotationally symmetric metrics on the Möbius band is achieved by the metric on a free boundary minimal Möbius band in $\mathbb{B}^4$ given explicitly by the immersion*

$$\Phi(t,\theta) = \frac{1}{R_n}(2n\sinh(t)\cos(\theta), 2n\sinh(t)\sin(\theta), \cosh(2nt)\cos(2n\theta), \cosh(2nt)\sin(2n\theta)),$$

where $R_n = \sqrt{4n^2\sinh^2(T_{n,1}) + \cosh^2(2nT_{n,1})}$ and $(t,\theta) \in [-T_{n,1}, T_{n,1}] \times S^1/\sim$.

In particular, Theorem 1.2.1 shows that the supremum of the 2nd Steklov eigenvalue can be achieved and that the conjecture of Fan, Tam and Yu is false.

1.3 Existence of harmonic maps into singular spaces

Another topic closely related to minimal surfaces is the theory of harmonic maps from two-dimensional domains. The focus of our third problem is on obtaining existence results for harmonic maps into singular spaces. In the smooth setting, the celebrated work of Sacks and Uhlenbeck [53] developed an existence theory for harmonic maps from surfaces into compact Riemannian manifolds; see also the related works of Lemaire [44], Sacks-Uhlenbeck [54], and Schoen-Yau [58]. In chapter 4, we extend the Sacks-Uhlenbeck existence theory to the case where the target is a metric space with an upper curvature bound.

For some applications, it is important to consider harmonic maps when the smooth Riemannian target is replaced by a singular space. The seminal works of Gromov-Schoen [28] and Korevaar-Schoen [40] consider harmonic maps from a Riemannian domain into a non-Riemannian target. In particular, they generalized the classical notion of the energy of a map in order to define the notion of a harmonic map. As one can not use variational methods to obtain an Euler-Lagrange equation for the energy functional in the singular setting, here, a harmonic map is defined to be a map that is locally energy minimizing. Further exploration of harmonic map theory in the singular setting includes works of Jost [33], J. Chen [8], Eells-Fuglede [18] and Daskalopoulos-Mese [12].

The classical notion of curvature also needs to be generalized in the singular setting. In the smooth setting, if M is a Riemannian manifold with sectional curvature bounded above by κ and M_κ is the model space with constant sectional curvature κ, then Toponogov's Theorem allows us to compare the lengths of geodesics in geodesic triangles in M and the corresponding geodesic triangles in M_κ (see Figure 1.1). In the singular setting, one uses this idea in reverse to define the notion of a metic space with curvature bounded above by κ.

The above mentioned works all assume non-positivity of curvature (NPC) in the

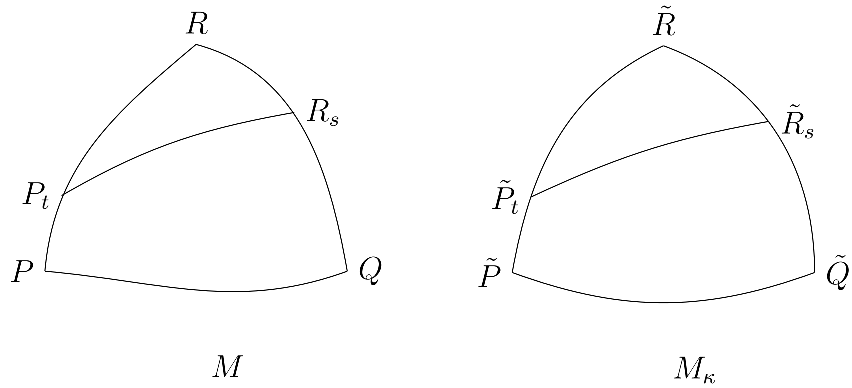

Figure 1.1: An illustration of a geodesic triangle in M (left) and a comparison triangle in the model space M_κ (right). Toponogov's Theorem implies that $d(P_t, R_s) \leq d(\tilde{P}_t, \tilde{R}_s)$.

target space, and this curvature condition is heavily used. Without the assumption of non-positive curvature, the existence problem for harmonic maps becomes more complicated and, in general, is not well understood even in the smooth setting.

In chapter 4, we investigate the existence theory for harmonic maps in the case when the target curvature is bounded above by a constant that is not necessarily 0. In this direction, we mention the local existence result of Serbinowski [60] for harmonic maps from Riemannian manifold domains. Our third problem specifically focuses on obtaining existence results for harmonic maps when the domain is a compact Riemann surface and the target is a compact locally CAT(1) space, that is, a complete metric space with curvature bounded above by 1 in the sense outlined above. We obtain the following theorem.

Theorem 1.3.1. *Let Σ be a compact Riemann surface, X a compact locally CAT(1) space and $\varphi \in C^0 \cap W^{1,2}(\Sigma, X)$. Then either there exists a harmonic map $u : \Sigma \to X$ homotopic to φ or a nontrivial conformal harmonic map $v : \mathbb{S}^2 \to X$.*

This provides a generalization of the Sacks and Uhlenbeck existence result to the case of metric space targets. The method used by Sacks and Uhlenbeck is not accessible in the singular setting as it depends on a priori estimates stemming from the Euler-Lagrange equation of their perturbed energy functional. In the singular setting, one can no longer use variational methods to obtain an Euler-Lagrange equation. To circumnavigate this, we exploit the local convexity of the target CAT(1) space.

1.4 Layout

The focus of chapter 2 the proof of Theorem 1.1.2. We introduce the problem by providing an overview of free boundary minimal submanifolds, the Morse index of a minimal submanifold and the Hodge Laplacian. We also provide all of the calculations needed to prove Theorem 1.1.2.

In chapter 3, we focus on proving Theorem 1.2.1. We introduce the Dirichlet-to-Neumann map and the Steklov eigenvalue problem, explicitly calculate the eigenvalues and eigenfunctions for rotationally symmetric metrics on the Möbius band, and prove a series of lemmas to determine which metric maximizes the k-th eigenvalue. We conclude the chapter by proving Theorem 1.2.1.

Chapter 4 is devoted to proving Theorem 1.3.1. We outline both the definition of energy and harmonicity for maps into metric spaces and CAT(1) spaces. We then prove compactness of energy minimizing maps into CAT(1) spaces, and prove a removable singularity theorem. We then prove Theorem 1.3.1 using a local harmonic replacement construction.

Appendix A is an appendix to chapter 2. Here, we explicitly calculate the first absolute Betti number for a surface of genus g with k boundary components.

Appendix B is an appendix to chapter 4. Here, we provide all of the details of the quadrilateral estimates in CAT(1) spaces, local energy convexity, and the local existence and uniqueness results needed throughout chapter 4.

Chapter 2

Index Bounds for Free Boundary Minimal Surfaces of Convex Bodies

2.1 Introduction

In this chapter we look at the problem of obtaining lower bounds on the index of free boundary minimal surfaces of convex bodies in terms of their topology. Index estimates for minimal surfaces are generally difficult to obtain, and there are few minimal surfaces for which the index is explicitly known. However, index bounds can help in the classification of minimal surfaces, especially when the topology is explicitly represented in the bounds, and have applications in understanding the relationships between the curvature and topology of Riemannian manifolds. Moreover, minimal surfaces whose index is known have proven to be useful in other problems; Urbano's [68] index characterization of the Clifford torus as being the unique minimal surface in $\mathbb{S}^3$ of index 5 was recently used by Marques and Neves [46] in their celebrated proof of the longstanding Willmore Conjecture. In [56], Savo was able to obtain index bounds for minimal hypersurfaces in $\mathbb{S}^n$ in terms of their topology making use of the Laplacian on 1-forms. His work has inspired the approach taken here.

2.1.1 Free Boundary Minimal Hypersurfaces in Convex Bodies

A submanifold M of a compact Riemannian manifold $\overline{M}$ with boundary $\partial M \subset \partial\overline{M}$ is said to be a *free boundary minimal submanifold* in $\overline{M}$ if it is a critical point for the volume functional among submanifolds with boundary in $\partial\overline{M}$. That is, M is a free boundary minimal submanifold of $\overline{M}$ if for every admissible variation M_t of M, $\frac{d}{dt}\mathrm{Vol}(M_t)\big|_{t=0} = 0$. The first variation formula for a variation M_t of M with variation field V is given by,

$$\frac{d}{dt}\mathrm{Vol}(M_t)\big|_{t=0} = -\int_M \langle V, H\rangle dV + \int_{\partial M} \langle V, \eta\rangle dA,$$

where η is the outward unit conormal vector field. It follows that M is a free boundary minimal submanifold of $\overline{M}$ if and only if $H \equiv 0$ and η is orthogonal to $T(\partial\overline{M})$, *i.e.*, M meets $\partial\overline{M}$ orthogonally.

Throughout, we will focus our attention on free boundary minimal hypersurfaces M^n properly immersed in convex bodies B^{n+1}. Here, a *convex body* is a smooth, compact, connected $(n+1)$-dimensional submanifold of $\mathbb{R}^{n+1}$ for which the scalar second fundamental form of the boundary satisfies $h^{\partial B}(U, U) < 0$ (with respect to the outward pointing normal vector) for all vectors U tangent to ∂B.

We will also place some attention on the special case when $B = \mathbb{B}$, the Euclidean ball, as there are more existence results for free boundary minimal hypersurfaces of Euclidean balls. Free boundary minimal hypersurfaces of Euclidean balls have also been shown to have an alternative characterization: in [24], Fraser and Schoen showed that if Σ^k is a properly immersed submanifold of the Euclidean unit ball $\mathbb{B}^{n+1}$, then Σ is a free boundary minimal submanifold if and only if the coordinate functions of the immersion are (Steklov) eigenfunctions of the Dirichlet-to-Neumann map with (Steklov) eigenvalue 1. Furthermore, free boundary minimal surfaces in $\mathbb{B}^{n+1}$ are extremal surfaces for the Steklov eigenvalue problem.

2.1.2 The Index of a Minimal Hypersurface

Suppose that $M^n \subset B^{n+1}$ is a free boundary minimal hypersurface and that N is a smooth unit normal vector field. Then, for a normal variation with variation field uN, the second variation formula is

$$\frac{d^2}{dt^2}\mathrm{Vol}(M_t)\Big|_{t=0} = \int_M \left(\|\nabla u\|^2 - \|A\|^2 u^2\right) dV + \int_{\partial M} h^{\partial B}(N,N) u^2 dA.$$

Let J denote the *Jacobi operator* (also called the *stability operator*),

$$J = \Delta - \|A\|^2,$$

and let Q denote the associated symmetric bilinear form,

$$\begin{aligned} Q(u) &= \int_M \left[\|\nabla u\|^2 - \|A\|^2 u^2\right] dV + \int_{\partial M} h^{\partial B}(N,N) u^2 \, dA \\ &= \int_M u \cdot Ju \, dV + \int_{\partial M} \left(\frac{\partial u}{\partial \eta} + h^{\partial B}(N,N)u\right) u \, dA. \end{aligned}$$

We say that $\lambda(J)$ is an eigenvalue of J with eigenfunction $u \in C^\infty(M)$ if

$$\begin{cases} Ju = \lambda u \quad \text{on } M, \\ \frac{\partial u}{\partial \eta} + h^{\partial B}(N,N)u = 0 \quad \text{on } \partial M. \end{cases}$$

The (*Morse*) *index* of a minimal hypersurface is the maximal dimension of a subspace of $C^\infty(M)$ on which the second variation is negative.

A free boundary minimal hypersurface is said to be *stable* if it has index 0. For free boundary minimal hypersurfaces in B^{n+1}, there are none which are stable. This is easy to see since if we use the variation with variation field $1 \cdot N$, then we get that

$$Q(1) = -\int_M \|A\|^2 \, dV + \int_{\partial M} \left(0 + h^{\partial B}(N,N)\right) \cdot 1 \, dA < 0.$$

Hence, any free boundary minimal hypersurface in B^{n+1} has index at least 1.

It is well known that the equatorial disk has index 1. Moreover, it is the only free boundary minimal hypersurface on $\mathbb{B}^{n+1}$ to have index 1. To see this, note that, by the above argument, the index is at least 1. Now, suppose $\text{Ind}(D) > 1$. Then there is a two-dimensional subspace $\mathcal{S}$ of normal variations containing the variation $1 \cdot N$ on which the second variation of area is negative. Let $V \in \mathcal{S}$ be a normal variation orthogonal to $1 \cdot N$, i.e. $\int_M \langle V, N \rangle \, dx_1 dx_2 = 0$. Then, V has zero average in the sense that $\int_M V dx_1 dx_2 = 0$. Now, consider the constraint that the surface must divide the volume of the ball in half. Subject to this additional constraint, any equatorial disk is area minimizing. However, a variation with zero average preserves the constraint. Hence, the second variation of area for this variation must be non-negative, a contradiction.

One can also show that, if M is not the equatorial disk, then $\text{Ind}(M) \geq 3$ (see Theorem 3.1 in [26])

2.1.3 Examples and Existence Results

For general convex bodies different from $\mathbb{B}^n$, little is known about the existence of free boundary minimal submanifolds. In [64], Struwe showed the existence of a (possibly branched) immersed free boundary minimal disk in any domain in $\mathbb{R}^3$ diffeomorphic to $\mathbb{B}^3$, and Grüter and Jost [30] showed that there is an embedded free boundary minimal disk in any convex body in $\mathbb{R}^3$. Jost [32] was also able to show that any convex body in $\mathbb{R}^3$ actually contains at least three embedded free boundary minimal disks. More recently, Maximo, Nunes and Smith [47] showed that any convex body in $\mathbb{R}^3$ contains a minimal annulus. By the above argument, we know that any free boundary minimal hypersurface of a convex body has index at least one. However, little else is known regarding the existence and index of minimal surfaces of greater topological complexity in convex bodies.

If we focus on free boundary minimal submanifolds of Euclidean balls, then more is known. The simplest examples of free boundary minimal submanifolds in $\mathbb{B}^{n+1}$ are the equatorial k-planes $D^k \subset \mathbb{B}^{n+1}$. By [50] and [27], any simply connected free boundary minimal surface in $\mathbb{B}^n$ must be a flat equatorial disk, and it is well known that the equatorial disk has index 1 (see p. 3741 in [23]). In fact, it is the only free boundary minimal surface of $\mathbb{B}^3$ to have index 1. However, there are now many examples of free boundary minimal surfaces of different topological type. The critical catenoid, a minimal surface with genus 0 and 2 boundary components, is an explicit example of such a surface. In [26], Fraser and Schoen proved the existence of free boundary minimal surfaces in $\mathbb{B}^3$ with genus 0 and k boundary components for any $k > 0$. Using gluing techniques, in [22] Folha, Pacard and Zolotareva gave an independent construction of free boundary minimal surfaces in $\mathbb{B}^3$ of genus 0 with k boundary components for k large. They were also able to use the same techniques to construct a genus 1 free boundary minimal surface with k boundary components for k large. Examples of free boundary minimal surfaces in $\mathbb{B}^3$ with any sufficiently large genus and 3 boundary components have also been constructed. Specifically, Kapouleas and M. Li [36] constructed such surfaces by using gluing techniques to glue an equatorial disk to a critical catenoid, and Ketover [39] used variational methods to construct such surfaces. Furthermore, Kapouleas and Wiygul [37] used gluing techniques to construct free boundary minimal surfaces with one boundary component and genus g for sufficiently large g. Less is known about the index of such surfaces. By the above argument, the equatorial disk has index 1 and Devyver [15], Smith and Zhou [63] and Tran [67] have independently shown that the critical catenoid has index 4. If M is not an equatorial disk, then Tran also showed that $\mathrm{Ind}(M) \geq 4$.

In this chapter, we give a relationship between the eigenvalues of the Jacobi operator and the eigenvalues of the Laplacian on 1-forms and, as a corollary, obtain new index bounds for orientable free boundary minimal hypersurfaces of convex bodies. More specifically, our first main result is:

Theorem. 1.1.1 *Let M^n be an orientable free boundary minimal hypersurface of a convex body in $\mathbb{R}^{n+1}$ with Jacobi operator J. Then, for all positive integers j, one has that*

$$\lambda_j(J) \leq \lambda_{m(j)}(\Delta_1),$$

where $m(j) = \binom{n+1}{2}(j-1)+1$ *and* $\lambda_{m(j)}(\Delta_1)$ *is the* $m(j)$*th eigenvalue of the Laplacian eigenvalue problem with absolute boundary conditions.*

Let $\beta^1_a = \dim H^1_a(M)$ be the first absolute Betti number of M. Our second main result is:

Theorem. 1.1.2 (Index Bound) *If* M *is an orientable free boundary minimal hypersurface of a convex body in* $\mathbb{R}^{n+1}$*, then*

$$\mathrm{Ind}(M) \geq \left\lfloor \frac{\beta^1_a + \binom{n+1}{2} - 1}{\binom{n+1}{2}} \right\rfloor.$$

Corollary 2.1.1. *If* M *is an orientable free boundary minimal surface of a convex body in* $\mathbb{R}^3$ *with genus* g *and* k *boundary components, then*

$$\mathrm{Ind}(M) \geq \left\lfloor \frac{2g + k + 1}{3} \right\rfloor.$$

Corollary 2.1.1 provides new index bounds for free boundary minimal surfaces of $\mathbb{B}^3$ with large topology. In particular, it shows that $\mathrm{Ind}(M) \geq 4$ when $2g + k \geq 11$ and $\mathrm{Ind}(M)$ tends to infinity as the genus or the number of boundary components tends to infinity. Corollary 2.1.1 was obtained simultaneously, but independently, by Ambrozio, Carlotto and Sharp [3]. In particular, they use different methods to obtain similar Morse index estimates for free boundary minimal hypersurfaces of strictly mean convex domains of Euclidean spaces.

The remainder of the chapter is structured as follows: In the second section, we outline the basic notation and conventions that we will use throughout the chapter and give a brief introduction to the Hodge Laplacian on p-forms. Here, we define the Hodge Laplacian on p-forms and then focus on the special case when $p = 1$. We also introduce the two main boundary conditions for the eigenvalue problem of the Laplacian for 1-forms on a manifold with boundary. In the third section, we provide several preliminary calculations that will ultimately allow us to see how the Jacobi operator acts on specific test functions, which will be needed to prove our main results. We give the proofs of our two main results in the fourth section.

2.2 Notation and Conventions

Let M^n be an orientable free boundary minimally immersed hypersurface in a convex Euclidean domain B^{n+1} ($\partial M \neq \emptyset$). Throughout, we will let N be a unit normal vector field on M.

Let D denote the Levi-Civita connection on $\mathbb{R}^{n+1}$ and ∇ the Levi-Civita connection on M. We will let A denote the second fundamental form of $M \subset B$, and S the associated shape operator. That is, for $X, Y \in \Gamma(TM)$,

$$\begin{aligned} A(X,Y) &= (D_X Y)^N = \langle D_X Y, N\rangle \cdot N \\ S(X) &= -(D_X N)^T, \end{aligned}$$

so that $\langle A(X,Y), N\rangle = \langle S(X), Y\rangle$

In this setting, the Gauss equation tells us that, for any $X, Y, Z, W \in \Gamma(TM)$,

$$0 = R_{\mathbb{R}^{n+1}}(X,Y,Z,W) = \langle A(X,W), A(Y,Z)\rangle - \langle A(X,Z), A(Y,W)\rangle, \tag{2.1}$$

and the Codazzi equation tells us that, for any $X, Y, Z \in \Gamma(TM)$,

$$(\nabla_X A)(Y,Z) - (\nabla_Y A)(X,Z) = (R_{\mathbb{R}^{n+1}}(X,Y)Z)^N = 0, \tag{2.2}$$

where

$$(\nabla_X A)(Y,Z) = (D_X A(Y,Z))^N - A(\nabla_X Y, Z) - A(Y, \nabla_X Z).$$

For any parallel vector field $\overline{V}$ in $\mathbb{R}^{n+1}$, we have the orthogonal decomposition

$$\overline{V} = V + V^N,$$

where $V \in TM$ is the orthogonal projection of $\overline{V}$ onto M and $V^N = \langle \overline{V}, N\rangle \cdot N \in NM$. Since parallel vector fields on $\mathbb{R}^{n+1}$ and their orthogonal projections onto M will be used throughout, we introduce the following vector spaces:

$$\overline{\mathcal{P}} = \{\text{parallel vector fields on } \mathbb{R}^{n+1}\},$$
$$\mathcal{P} = \{\text{vector fields on } M \text{ which are orthogonal projections of elements of } \overline{\mathcal{P}}\}.$$

Throughout, we will let Δ_p denote the Hodge Laplacian on p-forms (though the p will usually be dropped for convenience) and we will let $\nabla^*\nabla$ denote the rough Laplacian on vector fields. So, if ω is a p-form on M and ξ is a vector field on M, then

$$\Delta_p\omega = (d\delta + \delta d)\omega$$
$$\nabla^*\nabla\xi = -\sum_{j=1}^{n}\left(\nabla_{e_j}\nabla_{e_j}\xi - \nabla_{\nabla_{e_j}e_j}\xi\right),$$

where d is the exterior derivative, δ is the codifferential, and $\{e_1, \ldots, e_n\}$ is any local orthonormal frame of TM. Recall that a vector field X is dual to a 1-form θ if and only if $\langle X, Y\rangle = \theta(Y)$ for all $Y \in \Gamma(TM)$. If ξ is the vector field dual to ω, then one can also define the Hodge Laplacian of ξ, denoted $\Delta\xi$, to be the vector field dual to the 1-form $\Delta_1\omega$. The Bochner formula relates the two Laplacians:

$$\Delta\xi = \nabla^*\nabla\xi + \mathrm{Ric}(\xi),$$

where Ric is seen as a symmetric endomorphism of TM.

To get a bound on the index of M, we will consider the following eigenvalue problem defined by the *absolute* boundary conditions:

$$\begin{cases} J_1\omega &= \lambda\omega, \\ i^*\iota_\eta\omega &= i^*\iota_\eta d\omega = 0, \end{cases}$$

where i is the inclusion $\partial M \hookrightarrow M$, ι_η denotes interior multiplication by η and J_1 is the Jacobi operator on 1-forms defined by $J_1 = \Delta_1 - \|A\|^2$. We will often drop the subscripts for convenience. These absolute boundary conditions are a generalization of Neumann boundary conditions for functions. We say that ω is *tangential* on ∂M if $i^* \iota_\eta \omega = 0$, *i.e.*, ω vanishes whenever its argument is normal to the boundary of M. So, if ω satisfies the absolute boundary conditions, then both ω and $d\omega$ are tangential ($d\omega$ is tangential whenever one of its arguments is normal to ∂M).

We define the following space of harmonic 1-forms

$$\mathcal{H}^1_N(M) = \{\omega \in \Omega^1(M) \mid \Delta\omega = 0,\ \omega \text{ satisfies the absolute boundary conditions}\},$$

and note that $\beta^1_a = \dim H^1_a(M) = \dim \mathcal{H}^1_N(M)$, where $H^1_a(M)$ is the first absolute cohomology group of M.

2.3 Preliminary Calculations

The calculations done here are analogous to those done by Savo in [56] for the case of a minimal hypersurface in $\mathbb{S}^{n+1}$. In $\mathbb{S}^{n+1}$, a hypersurface has two normal vectors (one tangent to the sphere and one normal to both the sphere and the hypersurface) whereas a free boundary minimal hypersurface of a convex body B^n just has one. The absence of a second normal vector simplifies many of the preliminary calculations. However, a minimal hypersurface in $\mathbb{S}^{n+1}$ has no boundary, so the main barrier in modifying the approach of Savo to this free boundary setting is presence of boundary terms. To deal with these boundary terms, we extend a result of Ros [52] to arbitrary dimensions.

Lemma 2.3.1. *Let $\overline{V} \in \overline{\mathcal{P}}$ and let $V \in \mathcal{P}$ be its orthogonal projection onto M. Let A and S be the second fundamental form and shape operator (respectively) of the immersion $\phi : M \to B^n$. Then*

(a) $\nabla\langle \overline{V}, N\rangle = -S(V)$.

(b) $\Delta\langle \overline{V}, N\rangle = |S|^2\langle \overline{V}, N\rangle$.

Proof. To show (a), take any $X \in \Gamma(TM)$. Then we have that

$$\begin{aligned}
\langle \nabla \langle \overline{V}, V \rangle, X \rangle &= d\left(\langle \overline{V}, N \rangle\right)(X) \\
&= X(\langle \overline{V}, N \rangle) \\
&= \langle D_X \overline{V}, N \rangle + \langle \overline{V}, D_X N \rangle \\
&= \langle \overline{V}, D_X N \rangle,
\end{aligned}$$

since $\overline{V}$ is parallel. Now, since $\langle N, D_X N \rangle = \frac{1}{2} X\left(\|N\|^2\right) \equiv 0$ and $[X, V]$ is tangent to M, we have that

$$\begin{aligned}
\langle \overline{V}, D_X N \rangle &= \langle \overline{V}, N \rangle \cdot \langle N, D_X N \rangle + \langle V, D_X N \rangle \\
&= -\langle D_X V, N \rangle \\
&= -\langle D_V X + [X, V], N \rangle \\
&= -\langle D_V X, N \rangle \\
&= \langle X, (D_V N)^T \rangle.
\end{aligned}$$

Hence, $\nabla \langle \overline{V}, N \rangle = -S(V)$.

For (b), let $\{e_1, \ldots e_n\}$ denote normal coordinate vector fields centred at a point $p \in M$. Then (at p),

$$\begin{aligned}
-\Delta \langle \overline{V}, N \rangle &= \sum_{i=1}^{n} \langle \nabla_{e_i} \nabla \langle \overline{V}, N \rangle, e_i \rangle \\
&= \sum_{i=1}^{n} \langle \nabla_{e_i} \left(D_V N\right)^T, e_i \rangle \\
&= \sum_{i=1}^{n} e_i \langle D_V N, e_i \rangle - \langle (D_V N)^T, \nabla_{e_i} e_i \rangle \\
&= -\sum_{i=1}^{n} e_i \langle N, D_V e_i \rangle
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{n} e_i\langle N, A(V, e_i)\rangle \\
&= -\sum_{i=1}^{n} \langle D_{e_i}N, A(V, e_i)\rangle + \langle N, D_{e_i}A(V, e_i)\rangle.
\end{aligned}
$$

Since $D_{e_i}N$ has no normal component, and A is symmetric, we have that

$$\Delta\langle \overline{V}, N\rangle = \sum_{i=1}^{n} \langle N, (D_{e_i}A(e_i, V))^N\rangle.$$

Now,

$$
\begin{aligned}
(D_{e_i}A(e_i, V))^N &= (\nabla_{e_i}A)(e_i, V) + A(\nabla_{e_i}V, e_i) + A(V, \nabla_{e_i}e_i) \\
&= (\nabla_{e_i}A)(e_i, V) + A(\nabla_{e_i}V, e_i),
\end{aligned}
$$

and from the Codazzi equation (2.2)

$$
\begin{aligned}
(\nabla_{e_i}A)(e_i, V) &= (\nabla_V A)(e_i, e_i) \\
&= (D_V A(e_i, e_i))^N - A(\nabla_V e_i, e_i) - A(e_i, \nabla_V e_i).
\end{aligned}
$$

So,

$$(D_{e_i}A(e_i, V))^N = (D_V A(e_i, e_i))^N - 2A(e_i, \nabla_V e_i) + A(\nabla_{e_i}V, e_i).$$

Now

$$A(e_i, \nabla_{e_i}V) = (D_{e_i}\nabla_V e_i)^N,$$

and, at p,

$$
\begin{aligned}
\langle D_{e_i}\nabla_V e_i, N\rangle &= -\langle \nabla_V e_i, D_{e_i}N\rangle \\
&= \sum_{i=1}^{n} \langle V, e_j\rangle\langle \nabla_{e_j}e_i, D_{e_i}N\rangle = 0
\end{aligned}
$$

Moreover, since M is minimal,

$$\sum_{i=1}^{n} \left(D_V A(e_i, e_i)\right)^N = \left(D_V \left(\sum_{i=1}^{n} A(e_i, e_i)\right)\right) = 0.$$

Therefore,

$$\begin{aligned}
\Delta\langle \overline{V}, N\rangle &= \sum_{i=1}^{n} \langle N, \left(D_{e_i} A(e_i, V)\right)^N\rangle \\
&= \sum_{i=1}^{n} \langle N, A(\nabla_{e_i} V, e_i)\rangle \\
&= \sum_{i=1}^{n} \langle N, A(e_i, \nabla_{e_i} V)\rangle \\
&= \sum_{i=1}^{n} \langle N, D_{e_i} \nabla_{e_i} V\rangle \\
&= -\sum_{i=1}^{n} \langle \left(D_{e_i} N\right)^T, \nabla_{e_i} V\rangle.
\end{aligned}$$

□

Lemma 2.3.2. *For any vector field $\xi \in \Gamma(TM)$ and any $\overline{V} \in \overline{P}$ with orthogonal projection V,*

(a) $\Delta\xi = \nabla^*\nabla\xi - S^2(\xi)$.

(b) $\nabla^*\nabla V = S^2(V),\ \Delta V = 0$.

Proof. Let $\{e_1, \ldots, e_n\}$ be local normal coordinate vector fields centred at a point

$p \in M$. Then, at p,

$$\begin{aligned}
\mathrm{Ric}(\xi) &= \sum_{i,j=1}^{n} \mathrm{Ric}(g^{ij} e_i, \xi) e_j \\
&= \sum_{i=1}^{n} \mathrm{Ric}(e_i, \xi) e_i \\
&= \sum_{i,k=1}^{n} R_M(e_k, e_i, \xi, e_k) e_i
\end{aligned}$$

Using the minimality of M and the Gauss equation (2.1), we have that

$$\begin{aligned}
\sum_{k=1}^{n} R_M(e_k, e_i, \xi, e_k) &= \sum_{k=1}^{n} \langle A(e_k, e_k), A(e_i, \xi)\rangle - \langle A(e_k, \xi), A(e_i, e_k)\rangle \\
&= -\sum_{k=1}^{n} \langle A(e_k, \xi), A(e_i, e_k)\rangle.
\end{aligned}$$

Now

$$\begin{aligned}
-\sum_{k=1}^{n} \langle A(e_k, \xi), A(e_i, e_k)\rangle &= -\sum_{k=1}^{n} \langle D_\xi e_k, N\rangle \langle D_{e_i} e_k, N\rangle \\
&= -\sum_{k=1}^{n} \langle e_k, D_\xi N\rangle \langle e_k, D_{e_i} N\rangle \\
&= -\langle (D_\xi N)^T, D_{e_i} N\rangle \\
&= \langle D_{e_i} (D_\xi N)^T, N\rangle \\
&= \langle D_{(D_\xi N)^T} e_i + [e_i, (D_\xi N)^T], N\rangle \\
&= -\langle e_i, \left(D_{(D_\xi N)^T} N\right)^T\rangle \\
&= -\langle e_i, S^2(\xi)\rangle.
\end{aligned}$$

Therefore, $\mathrm{Ric}(\xi) = -S^2(\xi)$, and (a) follows from the Bochner formula.

To see that $\nabla^*\nabla V = S^2(V)$, we'll first show that $\nabla^*\nabla N = 0$ in the sense that if $\{e_1, \ldots, e_n\}$ are again local normal coordinate vector fields centred at $p \in M$, then, at p,

$$\sum_{i=1}^{n} \left(D_{e_i}(D_{e_i}N)^T\right)^T = 0.$$

Since, $D_{e_i}N$ is tangential,

$$\begin{aligned}
\sum_{i=1}^{n} \left(D_{e_i}(D_{e_i}N)^T\right)^T &= \sum_{i=1}^{n} (D_{e_i}D_{e_i}N)^T \\
&= \sum_{i,j=1}^{n} \langle D_{e_i}D_{e_i}N, e_j\rangle e_j \\
&= \sum_{i,j=1}^{n} (e_i\langle D_{e_i}N, e_j\rangle - \langle D_{e_i}N, \nabla_{e_i}e_j\rangle)e_j \\
&= -\sum_{i,j=1}^{n} e_i\langle N, D_{e_i}e_j\rangle e_j \\
&= -\sum_{i,j=1}^{n} e_i\langle N, A(e_j, e_i)\rangle e_j \\
&= -\sum_{i,j=1}^{n} \left(\langle D_{e_i}N, A(e_j, e_i)\rangle + \langle N, D_{e_i}(A(e_j, e_i))\rangle\right) e_j \\
&= -\sum_{i,j=1}^{n} \langle N, D_{e_i}(A(e_j, e_i))\rangle e_j.
\end{aligned}$$

Now, using the Codazzi equation (2.2), we have that

$$\begin{aligned}
(D_{e_i}(A(e_j, e_i)))^N &= (\nabla_{e_i}A)(e_j, e_i) + A(\nabla_{e_i}e_j, e_i) + A(e_j, \nabla_{e_i}e_i) \\
&= (\nabla_{e_i}A)(e_j, e_i) \\
&= \left(\nabla_{e_j}A\right)(e_i, e_i) \\
&= \left(D_{e_j}(A(e_i, e_i))\right)^N - 2A(\nabla_{e_j}e_i, e_i) \\
&= \left(D_{e_j}(A(e_i, e_i))\right)^N.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{n} \left(D_{e_i}(D_{e_i}N)^T\right)^T &= -\sum_{i,j=1}^{n} \langle N, D_{e_i}(A(e_j, e_i))\rangle e_j \\
&= -\sum_{i,j=1}^{n} \langle N, D_{e_j}(A(e_i, e_i))\rangle e_j \\
&= -\sum_{j=1}^{n} \left\langle N, D_{e_j}\left(\sum_{i=1}^{n} A(e_i, e_i)\right)\right\rangle e_j = 0,
\end{aligned}$$

again using the minimality of M.

Now, if we write $V = \overline{V} - \langle \overline{V}, N\rangle N$, then we can use this calculation and the fact that $\overline{V}$ is parallel to help us calculate $\nabla^*\nabla V$.

$$\begin{aligned}
\nabla^*\nabla V &= \sum_{i=1}^{n} \left(D_{e_i}(D_{e_i}\overline{V})^T\right)^T - \left(D_{e_i}(D_{e_i}(\langle \overline{V}, N\rangle N))^T\right)^T \\
&= -\sum_{i=1}^{n} \left(D_{e_i}(e_i(\langle \overline{V}, N\rangle)N + \langle \overline{V}, N\rangle D_{e_i}N)^T\right)^T \\
&= -\sum_{i=1}^{n} \left(D_{e_i}(\langle \overline{V}, N\rangle D_{e_i}N)\right)^T \\
&= -\sum_{i=1}^{n} \left(e_i(\langle \overline{V}, N\rangle)D_{e_i}N + \langle \overline{V}, N\rangle D_{e_i}D_{e_i}N\right)^T \\
&= -\left(\sum_{i=1}^{n} e_i(\langle \overline{V}, N\rangle)D_{e_i}N\right) - \langle \overline{V}, N\rangle \nabla^*\nabla N \\
&= -\sum_{i=1}^{n} \langle V, D_{e_i}N\rangle D_{e_i}N \\
&= \sum_{i=1}^{n} \langle D_{e_i}V, N\rangle D_{e_i}N
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \langle D_V e_i + [e_i, V], N\rangle D_{e_i} N \\
&= -\sum_{i=1}^{n} \langle e_i, D_V N\rangle D_{e_i} N \\
&= D_{D_V N} N = S^2(V).
\end{aligned}
$$

The fact that $\Delta V = 0$ now follows from (a). □

Lemma 2.3.3. *Let $\overline{V}, \overline{W} \in \overline{\mathcal{P}}$ and let $V, W \in \mathcal{P}$ be their orthogonal projections onto M. Then, for any $\xi \in \Gamma(TM)$,*

(a) $\Delta\langle V, \xi\rangle = 2\langle S(V), S(\xi)\rangle + \langle V, \Delta\xi\rangle - 2\langle \overline{V}, N\rangle\langle S, \nabla\xi\rangle$.

(b) $\langle \nabla\langle \overline{V}, N\rangle, \nabla\langle W, \xi\rangle\rangle = -\langle \overline{W}, N\rangle\langle S(V), S(\xi)\rangle - \langle W, \nabla_{S(V)}\xi\rangle$.

(c) $$\begin{aligned}\Delta(\langle \overline{V}, N\rangle\langle W, \xi\rangle) = {} & |S|^2\langle \overline{V}, N\rangle\langle W, \xi\rangle + 2(\langle \overline{W}, N\rangle\langle S(V), S(\xi)\rangle + \langle W, \nabla_{S(V)}\xi\rangle) \\ & + \langle \overline{V}, N\rangle(2\langle S(W), S(\xi)\rangle + \langle W, \Delta\xi\rangle - 2\langle \overline{W}, N\rangle\langle S, \nabla\xi\rangle).\end{aligned}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be local normal coordinate vector fields centred at a point $p \in M$. Then, at p,

$$
\begin{aligned}
\Delta\langle V, \xi\rangle &= -\sum_{i=1}^{n} e_i e_i \langle V, \xi\rangle \\
&= -\sum_{i=1}^{n} e_i \left(\langle \nabla_{e_i} V, \xi\rangle + \langle V, \nabla_{e_i}\xi\rangle\right) \\
&= -\sum_{i=1}^{n} \langle \nabla_{e_i}\nabla_{e_i} V, \xi\rangle + 2\langle \nabla_{e_i} V, \nabla_{e_i}\xi\rangle + \langle V, \nabla_{e_i}\nabla_{e_i}\xi\rangle \\
&= \langle \nabla^*\nabla V, \xi\rangle - 2\langle \nabla V, \nabla\xi\rangle + \langle V, \nabla^*\nabla\xi\rangle,
\end{aligned}
$$

where $\langle \nabla V, \nabla\xi\rangle = \sum_{i=1}^{n}\langle \nabla_{e_i} V, \nabla_{e_i}\xi\rangle$. From Lemma 2.3.2 we have that $\nabla^*\nabla V =$

$S^2(V)$ and $\nabla^*\nabla\xi = \Delta\xi + S^2(\xi)$. We also have that

$$\begin{aligned}
\langle S^2(V), \xi\rangle &= \langle D_{D_V N} N, \xi\rangle \\
&= -\langle N, D_{D_V N}\xi\rangle \\
&= -\langle N, D_\xi D_V N + [D_V N, \xi]\rangle \\
&= \langle D_\xi N, D_V N\rangle = \langle S(\xi), S(V)\rangle,
\end{aligned}$$

and, similarly, $\langle V, S^2(\xi)\rangle = \langle S(V), S(\xi)\rangle$. Therefore,

$$\Delta\langle V, \xi\rangle = 2\langle S(V), S(\xi)\rangle + \langle V, \Delta\xi\rangle - 2\langle \nabla V, \nabla\xi\rangle.$$

Finally,

$$\begin{aligned}
\langle \nabla_{e_i} V, \nabla_{e_i}\xi\rangle &= \langle D_{e_i} V, \nabla_{e_i}\xi\rangle \\
&= \langle D_{e_i}(\overline{V} - \langle \overline{V}, N\rangle N), \nabla_{e_i}\xi\rangle \\
&= -\langle \overline{V}, N\rangle\langle D_{e_i} N, \nabla_{e_i}\xi\rangle \\
&= \langle \overline{V}, N\rangle\langle S(e_i), \nabla_{e_i}\xi\rangle.
\end{aligned}$$

Hence, summing over i gives us that

$$\Delta\langle V, \xi\rangle = 2\langle S(V), S(\xi)\rangle + \langle V, \Delta\xi\rangle - 2\langle \overline{V}, N\rangle\langle S, \nabla\xi\rangle.$$

From Lemma 2.3.1(a) we know that $\nabla\langle \overline{V}, N\rangle = -S(V)$, so we just need to calculate $\nabla\langle W, \xi\rangle$. First, notice that for any vector field X on M, since $\overline{W}$ is parallel,

$$\begin{aligned}
\nabla_X W &= \big(D_X(\overline{W} - \langle \overline{W}, N\rangle N)\big)^T \\
&= -\big(X(\langle \overline{W}, N\rangle)N + \langle \overline{W}, N\rangle D_X N\big)^T = \langle \overline{W}, N\rangle S(X)
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle \nabla\langle W, \xi\rangle, X\rangle = X(\langle W, \xi\rangle) &= \langle \nabla_X W, \xi\rangle + \langle W, \nabla_X\xi\rangle \\
&= \langle \overline{W}, N\rangle\langle S(X), \xi\rangle + \langle W, \nabla_X\xi\rangle.
\end{aligned}$$

So, for $X = -S(V)$ $(= \nabla\langle \overline{V}, N\rangle)$, we have that

$$\begin{aligned}\langle \nabla\langle \overline{V}, N\rangle, \nabla\langle W, \xi\rangle\rangle &= -\langle \overline{W}, N\rangle\langle S^2(V), \xi\rangle - \langle W, \nabla_{S(V)}\xi\rangle \\ &= -\langle \overline{W}, N\rangle\langle S(V), S(\xi)\rangle - \langle W, \nabla_{S(V)}\xi\rangle.\end{aligned}$$

Now (c) follows from (a) and (b) and Lemma 2.3.1(b). $\square$

Let $\overline{\mathcal{U}} = \{\overline{V} \in \overline{\mathcal{P}} \mid \|V\| \equiv 1\}$. Then $\mathcal{U}$ can naturally be identified with S^n if we endow it with the measure $\mu = \frac{n+1}{\mathrm{Vol}(S^n)} dv_{S^n}$.

Lemma 2.3.4. *For any $\overline{X}, \overline{Y} \in \mathbb{R}^{n+1}$,*

$$\int_{\overline{\mathcal{U}}} \langle \overline{V}, \overline{X}\rangle\langle \overline{V}, \overline{Y}\rangle \, d\overline{V} = \langle \overline{X}, \overline{Y}\rangle.$$

The proof of Lemma 2.3.4 follows from a direct, but tedious, calculation after changing to spherical coordinates and repeatedly applying the integral identity

$$\int \sin^m x \, dx = -\frac{1}{m} \sin^{m-1} x \cos x + \frac{m-1}{m} \int \sin^{m-2} x \, dx.$$

The following lemma was originally proved by Ros [52] for free boundary minimal surfaces in a smooth domain in $\mathbb{R}^3$. Here, we extend his proof to obtain the analogous result for free boundary minimal hypersurfaces in smooth domains in $\mathbb{R}^n$.

Lemma 2.3.5. *Suppose ξ is a vector field on M dual to a 1-form ω which satisfies the absolute boundary conditions. Then, at a point $p \in \partial M$,*

$$\langle \nabla_\eta \xi, \xi\rangle = h^{\partial B}(N, N)\|\xi\|^2.$$

Proof. Let η be the (outward pointing) conormal vector along ∂M. Then, since ω satisfies the absolute boundary conditions on ∂M, at p we have that

$$\begin{aligned}\omega(\eta) &= 0, \\ d\omega(\eta, t) &= \eta(\omega(t)) - t(\omega(\eta)) - \omega([\eta, t]) = 0,\end{aligned}$$

for any vector $t \in T_p(\partial M)$. In particular, if ξ is the vector field dual to ω, then the first condition implies that $\xi_p \in T_p(\partial M)$, and so the second condition implies that $d\omega(\eta, \xi) = 0$ at p. Now,

$$\langle \xi, \nabla_\eta \xi \rangle = \eta\langle \xi, \xi \rangle - \langle \xi, \nabla_\eta \xi \rangle = (\nabla_\eta \omega)(\xi),$$

and we claim that $(\nabla_\eta \omega)(\xi) = (\nabla_\xi \omega)(\eta)$. To see this, note that, by definition,

$$(\nabla_\xi \omega)(\eta) - (\nabla_\eta \omega)(\xi) = \xi(\omega(\eta)) - \omega(\nabla_\xi \eta) - \eta(\omega(\xi)) + \omega(\nabla_\eta \xi).$$

However,

$$\omega(\nabla_\xi \eta) - \omega(\nabla_\eta \xi) = \omega(\nabla_\xi \eta - \nabla_\eta \xi) = \omega([\xi, \eta]),$$

and, since $d\omega(\eta, \xi) = 0$, $\omega([\eta, \xi]) = \eta(\omega(\xi)) - \xi(\omega(\eta))$. Therefore

$$(\nabla_\xi \omega)(\eta) - (\nabla_\eta \omega)(\xi) = \xi(\omega(\eta)) - \eta(\omega(\xi)) + \omega([\eta, \xi]) = 0.$$

So,

$$\langle \xi, \nabla_\eta \xi \rangle = (\nabla_\eta \omega)(\xi) = (\nabla_\xi \omega)(\eta).$$

Now, since ξ is tangent to ∂M and $\omega(\eta) = 0$ on ∂M,

$$(\nabla_\xi \omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_\xi \eta) = \langle \nabla_\xi \xi, \eta \rangle = h^{\partial B}(\xi, \xi).$$

Hence,

$$\langle \nabla_\eta \xi, \xi \rangle = \langle \xi, \nabla_\eta \xi \rangle = h^{\partial B}(\xi, \xi) = h^{\partial B}(N, N)\|\xi\|^2.$$

□

2.4 Proofs of Main Theorems

2.4.1 Eigenvalue Relationship

Theorem. 1.1.1 *Let M^n be an orientable free boundary minimal hypersurface of a* *convex body in $\mathbb{R}^{n+1}$ with Jacobi operator J. Then, for all positive integers j, one has that*

$$\lambda_j(J) \leq \lambda_{m(j)}(\Delta_1),$$

where $m(j) = \binom{n+1}{2}(j-1)+1$ and $\lambda_{m(j)}(\Delta_1)$ is the $m(j)$th eigenvalue of the Laplacian eigenvalue problem with absolute boundary conditions.

Lemma 2.4.1. *For $\overline{V}, \overline{W} \in \overline{\mathcal{P}}$, let*

$$X_{V,W} = \langle \overline{V}, N \rangle W - \langle \overline{W}, N \rangle V.$$

Let ξ be any vector field on M and consider the function $u = \langle X_{V,W}, \xi \rangle$. Then

$$Ju = \langle X_{V,W}, \Delta \xi \rangle + 2v,$$

where v is the smooth function

$$v = \langle \nabla_{S(V)} \xi, W \rangle - \langle \nabla_{S(W)} \xi, V \rangle.$$

Proof of Lemma 2.4.1. Since $u = \langle X_{V,W}, \xi \rangle = \langle \overline{V}, N \rangle \langle W, \xi \rangle - \langle \overline{W}, N \rangle \langle V, \xi \rangle$, from part (c) of Lemma 2.3.3, (after some cancellations) we get that

$$\Delta u = |S|^2 u + \langle X_{V,W}, \Delta \xi \rangle + 2v,$$

and so $Ju = \langle X_{V,W}, \Delta \xi \rangle + 2v$. $\square$

Proof of Theorem 1.1.1. Let $\{\phi_1, \phi_2, \ldots,\}$ be an orthonormal basis for $L^2(M)$ given by eigenfunctions of J, where ϕ_i is an eigenfunction associated to $\lambda_i(J)$. Let $V^m(\Delta_1) = \bigoplus_{i=1}^m E^N_{\lambda_i(\Delta_1)}$, where $E^N_{\lambda_i(\Delta_1)}$ is the space of eigenforms of Δ_1 associated with $\lambda_1(\Delta_1)$ with absolute boundary conditions. We want to find $\omega \in V^m(\Delta_1)$, $\omega \not\equiv 0$, for which

$$\int_M \langle X_{V,W}, \xi \rangle \phi_i dV = 0, \tag{2.3}$$

for $i = 1, \ldots, j-1$ and for all $\overline{V}, \overline{W} \in \overline{\mathcal{P}}$, where ξ is the vector field dual to ω. Since $X_{V,W}$ is a skew-symmetric bilinear function of $\overline{V}, \overline{W}$, and since $\dim \overline{\mathcal{P}} = \dim \mathbb{R}^{n+1} = n+1$, there are $\binom{n+1}{2}$ equations that need to be satisfied in (2.3) for each i, and therefore $\binom{n+1}{2}(j-1)$ homogeneous linear equations in total. So, if $m(j) = \binom{n+1}{2}(j-1)+1$, then we're guaranteed that there is a $\omega \in V^{m(j)}(\Delta_1)$, $\omega \not\equiv 0$, whose dual vector field satisfies (2.3) for all V, W and for $i = 1, \ldots j-1$. From the min-max principle and Lemma 2.4.1 we have that,

$$\begin{aligned}
\lambda_j(J) \int_M u^2 \, dV &\leq \int_M uJu \, dV + \int_{\partial M} \left(\frac{\partial u}{\partial \eta} + h^{\partial B}(N,N)u \right) u \, dA \\
&= \int_M u \langle X_{V,W}, \Delta \xi \rangle \, dV + 2 \int_M uv \, dV + \int_{\partial M} \left(\frac{\partial u}{\partial \eta} + h^{\partial B}(N,N)u \right) u \, dA.
\end{aligned} \tag{2.4}$$

In addition,

$$\begin{aligned}
\frac{\partial u}{\partial \eta} &= \eta \left(\langle \overline{V}, N \rangle \langle W, \xi \rangle - \langle \overline{W}, N \rangle \langle \overline{V}, \xi \rangle \right) \\
&= \langle \overline{V}, D_\eta N \rangle \langle W, \xi \rangle + \langle \overline{V}, N \rangle \left(\langle D_\eta \overline{W}, \xi \rangle + \langle \overline{W}, D_\eta \xi \rangle \right) \\
&\quad - \langle \overline{W}, D_\eta N \rangle \langle V, \xi \rangle + \langle \overline{W}, N \rangle \left(\langle D_\eta \overline{V}, \xi \rangle + \langle \overline{V}, D_\eta \xi \rangle \right).
\end{aligned}$$

We'll now use an integration technique that exploits Lemma 2.3.4 to help us simplify (2.4). We'll then apply Lemma 2.3.5 to get the claimed eigenvalue relationship.

Using the product metric on $\overline{\mathcal{U}} \times \overline{\mathcal{U}}$, Lemma 2.3.4 implies that (pointwise)

$$\begin{aligned}
&\int_{\overline{\mathcal{U}} \times \overline{\mathcal{U}}} u^2 \, d\overline{V} d\overline{W} = 2\|\xi\|^2, \\
&\int_{\overline{\mathcal{U}} \times \overline{\mathcal{U}}} u \langle X_{V,W}, \Delta \xi \rangle \, d\overline{V} d\overline{W} = 2 \langle \xi, \Delta \xi \rangle, \\
&\int_{\overline{\mathcal{U}} \times \overline{\mathcal{U}}} uv \, d\overline{V} d\overline{W} = 0,
\end{aligned}$$

$$\int_{\overline{\mathcal{U}}\times\overline{\mathcal{U}}} u\langle \overline{V}, D_\eta N\rangle\langle \overline{W},\xi\rangle \, d\overline{V} d\overline{W} = 0,$$
$$\int_{\overline{\mathcal{U}}\times\overline{\mathcal{U}}} u\langle \overline{V}, N\rangle\langle \overline{W},D_\eta\xi\rangle \, d\overline{V} d\overline{W} = \langle \xi, D_\eta \xi\rangle = \frac{1}{2}\eta(\|\xi\|^2).$$

Therefore, integrating (2.4) over $\overline{\mathcal{U}}\times\overline{\mathcal{U}}$ yields

$$2\lambda_j(J)\int_M \|\xi\|^2\,dV \le 2\int_M \langle \xi, \Delta\xi\rangle\,dV + \int_{\partial M}\left(\eta(\|\xi\|^2) + 2h^{\partial B}(N,N)\|\xi\|^2\right)dA.$$

From Lemma 2.3.5 we know that $\eta(\|\xi\|^2) = 2h^{\partial B}(N,N)\|\xi\|^2$ on ∂M, since ξ is the dual vector field of a 1-form satisfying the absolute boundary conditions. Moreover, since ξ is the dual vector field to a linear combination of eigenforms of Δ_1, it now follows that

$$2\lambda_j(J)\int_M \|\xi\|^2\,dV \le 2\lambda_{m(j)}(\Delta_1)\int_M \|\xi\|^2\,dV + 4\int_{\partial M} h^{\partial B}(N,N)\|\xi\|^2\,dA.$$

Since $h^{\partial B}(U,U) < 0$ for any vector tangent to ∂B, we get that

$$2\lambda_j(J)\int_M \|\xi\|^2\,dV \le 2\lambda_{m(j)}(\Delta_1)\int_M \|\xi\|^2\,dV.$$

Now, since $\omega \not\equiv 0$, we can divide both sides by the $L^2(M)$-norm of ξ to get

$$\lambda_j(J) \le \lambda_{m(j)}(\Delta_1).$$

$\square$

Remark 2.4.2. We note that when $m(j) \le \dim \mathcal{H}^1_N(M)$, *i.e.* when ω is a linear combination of harmonic forms and therefore a harmonic form itself, we actually get the strict inequality $\lambda_j(J) < \lambda_{m(j)}(\Delta_1) = 0$. This follows from the fact that $\omega \not\equiv 0$ implies that $\omega|_{\partial M} \not\equiv 0$ (see Theorem 3.4.4 on p.131 of [59]), and so we get the strict inequality $4\int_{\partial M} h^{\partial M}(N,N)\|\xi\|^2\,dA < 0$.

2.4.2 Index Bound

Theorem. 1.1.2 (Index Bound) *If M is an orientable free boundary minimal hypersurface of a convex body in $\mathbb{R}^{n+1}$, then*

$$\mathrm{Ind}(M) \geq \left\lfloor \frac{\beta_a^1 + \binom{n+1}{2} - 1}{\binom{n+1}{2}} \right\rfloor.$$

Proof. Suppose j is such that $m(j) \leq \dim \mathcal{H}_N^1(M) := \beta_a^1$. Then $\lambda_j(J) < \lambda_{m(j)}(\Delta) = 0$, so $\mathrm{Ind}(M) \geq j$. Now, $m(j) = \binom{n+1}{2}(j-1) + 1 \leq \beta_a^1$, so $j \leq \left\lfloor \frac{\beta_a^1 + \binom{n+1}{2} - 1}{\binom{n+1}{2}} \right\rfloor$. Hence, $\mathrm{Ind}(M) \geq \left\lfloor \frac{\beta_a^1 + \binom{n+1}{2} - 1}{\binom{n+1}{2}} \right\rfloor$. □

Corollary. 2.1.1 *If M is an orientable free boundary minimal surface in a convex body in $\mathbb{R}^3$ with genus g and k boundary components, then*

$$\mathrm{Ind}(M) \geq \left\lfloor \frac{2g + k + 1}{3} \right\rfloor.$$

Proof. Since $\beta_a^1 = 2g + k - 1$ for a surface (see Appendix A), this follows directly from Theorem 1.1.2. □

Remark 2.4.3. We note that Corollary 2.1.1 can also be obtained by using the work of Ros. In [52], Ros shows that if ω is a harmonic 1-form and ξ is its dual vector field, then

$$\Delta\xi + \|A\|^2\xi = 2\langle \nabla\omega, A\rangle N,$$

and, if ω satisfies the absolute boundary conditions, then

$$\langle \nabla_\eta \xi, \xi\rangle = h^{\partial B}(N, N)\|\xi\|^2.$$

So, for $\xi = (\xi_1, \xi_2, \xi_3)$, if we use the notation $Q(\xi, \xi) = \sum_{i=1}^3 Q(\xi_i, \xi_i)$ and assume

$\omega \not\equiv 0$,

$$\begin{aligned} Q(\xi,\xi) &= -\int_M \langle \Delta\xi + \|A\|^2\xi, \xi\rangle dV + \int_{\partial M} (\langle \nabla_\eta \xi, \xi\rangle + h^{\partial B}(N,N)\|\xi\|^2) dA \\ &= 2\int_{\partial M} h^{\partial B}(N,N)\|\xi\|^2 dA < 0. \end{aligned}$$

Hence $Q(X,X) < 0$, and we get that $\dim \mathcal{H}^1_N(M) - 3 \cdot \mathrm{Ind}(M) = (2g+k-1) - 3 \cdot \mathrm{Ind}(M) \leq 0$, or $\mathrm{Ind}(M) \geq \lceil \frac{(2g+k-1)}{3} \rceil = \lfloor \frac{2g+k+1}{3} \rfloor$.

Chapter 3

Constructing Free Boundary Minimal Möbius Bands in $\mathbb{B}^4$

3.1 Introduction

In this chapter we look at the problem of constructing free boundary minimal Möbius bands in $\mathbb{B}^4$ by solving an extremal eigenvalue problem. Though there are some existence results (see 2.1.3 for an outline of currently known examples) for free boundary minimal surfaces in $\mathbb{B}^3$, explicit constructions are less common. As mentioned in 2.1.3, extremal eigenvalue techniques, gluing techniques and min-max techniques have been used to successfully construct free boundary minimal surfaces with specific topology in $\mathbb{B}^3$. Here we take the approach inspired by the work of Fraser and Schoen [24, 26], and Fan, Tam and Yu [21] in which we use eigenfunctions that maximize the Steklov eigenvalues for rotationally symmetric metrics to construct immersed free boundary minimal Möbius bands in $\mathbb{B}^4$.

3.1.1 The Dirichlet-to-Neumann Map and Steklov Eigenvalue Problem

In [25], Fraser and Schoen showed that there is a connection between Steklov eigenvalue problems on surfaces with boundary and free boundary minimal surfaces in the unit ball. In particular, they showed that metrics that maximize the kth Steklov eigenvalue on surfaces with boundary arise from the metrics on free boundary minimal surfaces in a Euclidean ball. If (Σ, g) is a compact Riemannian manifold with boundary, the Steklov eigenvalue problem is:

$$\begin{cases} \Delta_g u = 0 & \text{on } \Sigma \\ \frac{\partial u}{\partial \eta} = \sigma u & \text{on } \partial\Sigma, \end{cases}$$

where η is the outward unit normal vector to $\partial\Sigma$, $\sigma \in \mathbb{R}$, and $u \in C^\infty(\Sigma)$. Steklov eigenvalues are eigenvalues of the Dirichlet-to-Neumann map, which sends a given smooth function on the boundary to the normal derivative of its harmonic extension to the interior. That is, if $u \in C^\infty(\partial\Sigma)$ and if $\bar{u} \in C^\infty(\Sigma)$ is its harmonic extension, then the Dirichlet-to-Neumann map is the map $L : C^\infty(\partial\Sigma) \to C^\infty(\Sigma)$ defined by

$$L(u) = \frac{\partial \bar{u}}{\partial \eta}.$$

The Dirichlet-to-Neumann map is a non-negative, self-adjoint operator with discrete spectrum

$$\sigma_0 = 0 < \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_k \leq \ldots \to \infty.$$

The first nonzero Steklov eigenvalue of L can be characterized variationally as

$$\sigma_1 = \inf_{\int_{\partial\Sigma} u = 0} \frac{\int_\Sigma |\nabla u|^2 dv_\Sigma}{\int_{\partial\Sigma} u^2 dv_{\partial\Sigma}}$$

and in general,

$$\sigma_k = \inf \left\{ \frac{\int_\Sigma |\nabla u|^2 dv_\Sigma}{\int_{\partial\Sigma} u^2 dv_{\partial\Sigma}} \ : \ \int_{\partial\Sigma} u\phi_j = 0 \text{ for } j = 0, 1, 2, \ldots, k-1. \right\},$$

where ϕ_j is an eigenfunction corresponding to the eigenvalue σ_j, for $j = 1, 2, \ldots, k-1$.

3.1.2 Extremal Steklov Eigenvalue Problem and Free Boundary Minimal Surfaces

A classical result by Weinstock [69] shows that, on a simply-connected planar domain, the maximum of the first normalized Steklov eigenvalue is achieved by the round disk in the Euclidean plane. In [26] Fraser and Schoen proved the existence of a metric that maximizes the first normalized eigenvalue on any surface of genus zero, and showed that it is realized by the induced metric on a free boundary minimal surface in $\mathbb{B}^3$. In the case of the annulus, the surface of genus zero with 2 boundary components, they characterized the maximizing metric as the induced metric on the critical catenoid. They also proved the existence of a maximizing metric on the Möbius band, and characterized it as the induced metric on the critical Möbius band, the surface obtained by suitably scaling the embedding

$$\phi(t, \theta) = (2\sinh(t)\cos(\theta), 2\sinh(t)\sin(\theta), \cosh(2t)\cos(2\theta), \cosh(2t)\sin(2\theta))$$

to lie in the unit ball, where $(t, \theta) \in [-T_0, T_0] \times \mathbb{S}^1$ and T_0 is the unique positive solution of $\coth(t) = 2\tanh(2t)$.

In the special case of rotationally symmetric metrics on the annulus and Möbius band, in [24, 26], Fraser and Schoen explicitly calculated the eigenvalues and eigenfunctions of the Dirichlet-to-Neumann map and showed that the critical catenoid and critical Möbius band maximize the first normalized eigenvalue among all rotationally symmetric metrics. Motivated by the work of Fraser and Schoen, in [21], Fan, Tam and Yu considered the problem of maximizing the kth normalized Steklov eigenvalue over all rotationally symmetric metrics on a cylinder. They showed that, except for the 2nd normalized Steklov eigenvalue, the maximum is achieved by either the n-critical catenoid or the so-called n-Möbius band. However, they also showed that the supremum of the 2nd normalized Steklov eigenvalue can not be achieved. Girouard and Polterovich proved that for simply-connected planar domains, the supremum of the second normalized Steklov eigenvalue is 4π and can not be achieved. This led Fan, Tam and Yu to conjecture that the supremum of the second normalized Steklov eigenvalue can never be achieved.

In this chapter, we consider the problem of maximizing the kth normalized Steklov eigenvalue on the Möbius band over all rotationally symmetric metrics. In particular, we show that this problem is solvable for all k, i.e. for each k, among all rotationally symmetric metrics on the Möbius band, there is a metric that maximizes the kth normalized Steklov eigenvalue and it is achieved by a free boundary minimal Möbius band in $\mathbb{B}^4$. Specifically, our main result is:

Theorem. 1.2.1 *For all $n \geq 1$, the maximum of the nth Steklov eigenvalue among all rotationally symmetric metrics on the Möbius band is achieved by the metric on a free boundary minimal Möbius band in $\mathbb{B}^4$ given explicitly by the immersion*

$$\Phi(t,\theta) = \frac{1}{R_n}(n\sinh(t)\cos(\theta), n\sinh(t)\sin(\theta), \cosh(nt)\cos(n\theta), \cosh(nt)\sin(n\theta)),$$

where $R_n = \sqrt{n^2\sinh^2(T_{n,1}) + \cosh^2(nT_{n,1})}$ *and* $(t,\theta) \in [-T_{n,1}, T_{n,1}] \times S^1/\sim$.

In particular, this provides a counterexample to Fan, Tam and Yu's conjecture.

In [38], Karpukhin *et al.* showed that, for $k > 1$, the supremum of the kth normalized eigenvalue of the Laplacian on a sphere cannot be achieved. This, together with the result of Girouard and Polterovich, could suggest that, in general, the supremum of higher normalized Steklov eigenvalues might not be achievable. The results of chapter 3, which show that the supremum of the kth normalized Steklov eigenvalue among rotationally symmetric metrics on the Möbius band is achievable, are interesting in that they could suggest that, for the Möbius band, the supremum of the kth normalized Steklov eigenvalue among all metrics might actually be achievable. Based on the case when $k = 1$, one might expect that when maximizing metrics exist, the maximizing metrics are rotationally symmetric.

The remainder of the chapter is structured as follows: In the second section, we introduce the Steklov eigenvalue problem for rotationally symmetric metrics on the Möbius band. Here, we prove a series of lemmas needed to find the rotationally symmetric metric on the Möbius band that maximizes the kth normalized Steklov eigenvalue. In the third section we prove that there is a metric that maximizes the kth normalized Steklov eigenvalue, and we use the corresponding eigenfunctions to construct a free boundary minimal surface and prove the main theorem.

3.2 The Steklov eigenvalue problem for rotationally symmetric metrics on the Möbius band

Let Σ be a Möbius band, *i.e.* $\Sigma = [-T, T] \times S^1/\sim$, where $(t, \theta) \sim (t', \theta')$ if $t' = -t$ and $\theta' = \theta + \pi$. From [26] we know that the critical Möbius band maximizes the first normalized Steklov eigenvalue over all smooth metrics on the Möbius band. In general, from [25] we know that a metric on Σ that maximizes the kth Steklov eigenvalue among all smooth metrics on Σ arises as the induced metric on a free boundary minimal Möbius band in $\mathbb{B}^4$. However, solving this optimization problem is, in general, quite difficult. Here we investigate the simpler problem of finding a rotationally symmetric metric on Σ that maximizes the kth Steklov eigenvalue among all rotationally symmetric metrics on Σ. That is, we consider metrics of the form

$$g = f(t)^2(dt^2 + d\theta^2),$$

where $f : [-T, T] \to \mathbb{R}$ is a smooth function satisfying $f(t) = f(-t)$. Let $\eta = \frac{1}{f(T)}\frac{\partial}{\partial t}$ be the outward unit conormal on $\partial\Sigma$. Our goal is to maximize the kth nonzero normalized eigenvalues $\tilde{\sigma}_k(T) = \sigma_k(T)L_g(\partial\Sigma) = 2\pi f(T)\sigma_k(T)$.

If $u(t, \theta)$ is Steklov eigenfunction on Σ, $u(t, \theta)$ is a harmonic map (with respect to the flat metric) satisfying the boundary conditions $u(t, \theta) = u(-t, \theta + \pi)$ and is an eigenfunction of the Dirichlet-to-Neumann map. We may use the method of separation of variables to get $u(t,\theta) = \alpha(t)\beta(\theta)$, with $\alpha(t) = \alpha(-t)$ and $\beta(\theta) = \beta(\theta+\pi)$ and

$$\frac{\alpha''(t)}{\alpha(t)} = -\frac{\beta''(\theta)}{\beta(\theta)} = k^2.$$

If $k = 0$, then $\alpha(t) = A + Bt$ and $\beta(t) = C + D\theta$. Since $\alpha(t) = \alpha(-t)$ and $\beta(\theta) = \beta(\theta+\pi)$, we must have that $2Bt = 0$ and $C = C + D\pi$, so $B = D = 0$. Thus, $\alpha(t) = A$ and $\beta(\theta) = C$, so u is constant. However, since u is an eigenfunction of the Dirichlet-to-Neumann map, on $\partial\Sigma$

$$\frac{d}{d\eta}u = \sigma u \Rightarrow \frac{1}{f(T)}\alpha'(T) = \sigma\alpha(T)\beta(\theta),$$

so $\sigma = 0$.

If $k \neq 0$, then it is easy to show that

$$\alpha(t) = A_k \sinh(kt) + B_k \cosh(kt) \quad \text{and} \quad \beta(\theta) = C_k \sin(k\theta) + D_k \cos(k\theta).$$

So, since $\alpha(t)\beta(\theta) = \alpha(-t)\beta(\theta+\pi)$,

$$\begin{aligned}
&(A_k \sinh(kt) + B_k \cosh(kt))\,(C_k \sin(k\theta) + D_k \cos(k\theta)) \\
&= (-A_k \sinh(kt) + B_k \cosh(kt))\,(C_k \sin(k\theta)\cos(k\pi) + D_k \cos(k\theta)\cos(k\pi)) \\
&= (-A_k \sinh(kt) + B_k \cosh(kt))\,\big(C_k \sin(k\theta)(-1)^k + D_k \cos(k\theta)(-1)^k\big)
\end{aligned}$$

So,

$$\begin{aligned}
A_k \sinh(kt)(1 + (-1)^k)\,&(C_k \sin(k\theta) + D_k \cos(k\theta)) \\
&+ B_k \cosh(kt)(1 - (-1)^k)\,(C_k \sin(k\theta) + D_k \cos(k\theta)) = 0.
\end{aligned}$$

If k is even, then we have that

$$2A_k \sinh(kt)\,(C_k \sin(k\theta) + D_k \cos(k\theta)) = 0,$$

so either $A_k = 0$ or $C_k = D_k = 0$. However, if $C_k = D_k = 0$, then $\beta(\theta) \equiv 0$. This means that $u \equiv 0$, which is not possible since u is an eigenfunction. Thus, $A_k = 0$ and the eigenfunctions are

$$u(t, \theta) = \cosh(kt) \left(C_k \sin(k\theta) + D_k \cos(k\theta) \right).$$

for some constants C_k, D_k. Now, since u is an eigenfunction of the Dirichlet-to-Neumann map, on $\partial\Sigma$, $\frac{\partial}{\partial\eta} u = \sigma_k u$, and so

$$\frac{1}{f(T)} k \sinh(kT) \left(C_k \sin(k\theta) + D_k \cos(k\theta) \right) = \sigma_k \cosh(kt) \left(C_k \sin(k\theta) + D_k \cos(k\theta) \right).$$

Hence,

$$\sigma_k(T) = \frac{k}{f(T)} \tanh(kT).$$

If k is odd, then we have that

$$2B_k \cosh(kt) \left(C_k \sin(k\theta) + D_k \cos(k\theta) \right) = 0,$$

so, similarly to the previous case, we conclude that $B_k = 0$. Thus, the eigenfunctions are

$$u(t, \theta) = \sinh(kt) \left(C_k \sin(k\theta) + D_k \cos(k\theta) \right).$$

for some constants C_k, D_k. Now, again, since u is an eigenfunction of the Dirichlet-to-Neumann map, on $\partial\Sigma$ we get that

$$\frac{1}{f(T)} k \cosh(kT) = \sigma_k \sinh(kT),$$

so

$$\sigma_k(T) = \frac{k}{f(T)} \coth(kT).$$

Thus, the nonzero eigenvalues of the Dirichlet-to-Neumann map are

$$\lambda_k(T) = \frac{2k}{f(T)} \tanh(2kT), \quad \text{and} \quad \mu_k(T) = \frac{(2k-1)}{f(T)} \coth((2k-1)T),$$

$k = 1, 2, \ldots$, and the normalized eigenvalues are

$$\tilde{\lambda}_k(T) = 4\pi k \tanh(2kT), \quad \text{and} \quad \tilde{\mu}_k(T) = 2\pi(2k-1)\coth((2k-1)T),$$

$k = 1, 2, \ldots$.

Lemma 3.2.1. *Let $k, l \geq 1$. Then*

(i) $\tilde{\lambda}_k < \tilde{\lambda}_{k+1}$, $\tilde{\mu}_l < \tilde{\mu}_{l+1}$. *Furthermore,* $\tilde{\lambda}_n < \tilde{\mu}_{n+1}$ *for* $n \geq 1$, *and each* $\tilde{\lambda}_k$ *and* $\tilde{\mu}_l$ *has multiplicity 2.*

(ii) $\tilde{\lambda}_k(T)$ *is monotone increasing in* T *and* $\tilde{\mu}_l(T)$ *is monotone decreasing in* T.

(iii) $\tilde{\lambda}_k(\infty) := \lim_{T\to\infty} \tilde{\lambda}_k(T) = 4\pi k$ *and* $\tilde{\mu}_l(\infty) := \lim_{T\to\infty} \tilde{\mu}_l(T) = 2\pi(2l-1)$.

Proof. First, (i) and (iii) are clear by direct calculation. Now, (ii) follows from the fact that

$$\frac{d\tilde{\lambda}_k}{dT} = 8\pi k^2 \operatorname{sech}^2(2kT) > 0 \quad \text{and} \quad \frac{d\tilde{\mu}_l}{dT} = -2\pi(2l-1)^2 \operatorname{csch}^2((2l-1)T) < 0.$$

□

Lemma 3.2.2. *There exists $T > 0$ such that $\tilde{\lambda}_k(T) = \tilde{\mu}_l(T)$ if and only if $l \leq k$. Moreover, T is unique if it exists.*

Proof. Let $F_{k,l}(T) = \tilde{\lambda}_k(T) - \tilde{\mu}_l(T) = 2\pi\left(2k\tanh(2kT) - (2l-1)\coth\left((2l-1)T\right)\right)$. Then $F_{k,l}(T)$ is continuous on $(0, \infty)$ and

$$\lim_{T\to 0} F_{k,l}(T) = -\infty \quad \text{and} \quad \lim_{T\to\infty} F_{k,l}(T) = 2\pi(2k - (2l-1)).$$

Thus $\lim_{T\to\infty} F_{k,l}(T) > 0$ if and only if $l \leq k$. Furthermore, $F_{k,l}(T)$ is monotone increasing on $(0, \infty)$ since $\tilde{\lambda}_k(T)$ is monotone increasing and $\tilde{\mu}_l(T)$ is monotone decreasing. Hence there exists a unique $T > 0$ for which $\tilde{\lambda}_k(T) = \tilde{\mu}_l(T)$ if and only if $l \leq k$. □

Definition 3.2.3. For $l \leq k$ let $T_{k,l}$ be the unique positive number such that

$$\tilde{\lambda}_k(T_{k,l}) = \tilde{\mu}_l(T_{k,l}).$$

Lemma 3.2.4. *For $l \leq k$, $T_{k,l}$ is decreasing in k and increasing in l.*

Proof. Since $\tilde{\lambda}_k(T) < \tilde{\lambda}_{k+1}(T)$, we have that

$$\tilde{\mu}_l(T_{k,l}) = \tilde{\lambda}_k(T_{k,l}) < \tilde{\lambda}_{k+1}(T_{k,l}).$$

Hence, $F_{k+1,l}(T_{k,l}) > 0$, where $F_{k,l}$ is as in the proof of Lemma 3.2.2, and, again, $\lim_{T\to 0} F_{k+1,l}(T) = -\infty$. Hence $T_{k+1,l} < T_{k,l}$. Similarly, if $l+1 \leq k$,

$$\tilde{\lambda}_k(T_{k,l}) = \tilde{\mu}_l(T_{k,l}) < \tilde{\mu}_{l+1}(T_{k,l}),$$

and so $F_{k,l+1}(T_{k,l}) < 0$. Since $\lim_{t\to\infty} F_{k,l+1}(T) > 0$, it follows that $T_{k,l} < T_{k,l+1}$. □

For fixed $k > 0$, let $s = \lfloor \frac{k}{2} \rfloor$. By Lemma 3.2.4, if $k \geq 2$, we see that we can decompose $(0,\infty)$ as

$$(0,\infty) = (0, T_{k-1,1}) \cup \left(\bigcup_{j=2}^{s} [T_{k-j+1,j-1}, T_{k-j,j}) \right) \cup [T_{k-s,s}, \infty).$$

Note that if $k = 1$, then $s = 0$ and we do not decompose $(0,\infty)$.

Lemma 3.2.5. *For $k \geq 1$,*

$$\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) \leq \begin{cases} \tilde{\lambda}_k(T_{k,1}) & \text{if } T \in (0, T_{k-1,1}) \\ \tilde{\lambda}_{k-j+1}(T_{k-j+1,j}) & \text{if } T \in [T_{k-j+1,j-1}, T_{k-j,j}),\ 2 \leq j \leq s \\ \tilde{\lambda}_{k/2}(\infty) & \text{if } T \in [T_{k-s,s}, \infty),\ s = \frac{k}{2},\ k \text{ even} \\ \tilde{\lambda}_{(k+1)/2}(T_{(k+1)/2,(k+1)/2}) & \text{if } T \in [T_{k-s,s}, \infty),\ s = \frac{k-1}{2},\ k \text{ odd} \end{cases}.$$

Proof. First suppose $T \in (0, T_{k-1,1})$. Then, since $\tilde{\lambda}_{k-1}(T)$ is increasing in T and

$\tilde{\mu}_1(T)$ is decreasing in T by Lemma 3.2.1, we have that

$$\tilde{\lambda}_{k-1}(T) < \tilde{\lambda}_{k-1}(T_{k-1,1}) = \tilde{\mu}_1(T_{k-1,1}) < \tilde{\mu}_1(T).$$

Since each $\tilde{\lambda}_n(T)$ and each $\tilde{\mu}_n(T)$ have multiplicity two, either $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\lambda}_k(T)$ or $\sigma_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\mu}_1(T)$. Now, since $T_{k,l}$ is decreasing in k by Lemma 3.2.4, $0 < T_{k,1} < T_{k-1,1}$. So, if $T \leq T_{k,1}$,

$$\tilde{\lambda}_k(T) \leq \tilde{\lambda}_k(T_{k,1}) = \tilde{\mu}_1(T_{k,1}) \leq \tilde{\mu}_1(T)$$

and so $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\lambda}_k(T) \leq \tilde{\lambda}_k(T_{k,1})$. Otherwise, $T_{k,1} < T < T_{k-1,1}$, so

$$\tilde{\lambda}_k(T) > \tilde{\lambda}_k(T_{k,1}) = \tilde{\mu}_1(T_{k,1}) > \tilde{\mu}_1(T),$$

and $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\mu}_1(T) < \tilde{\mu}_1(T_{k,1}) = \tilde{\lambda}_k(T_{k,1})$. Hence, in either case, $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) \leq \tilde{\lambda}_k(T_{k,1})$.

Now, if $2 \leq j \leq s$ and $T \in [T_{k-j+1,j-1}, T_{k-j,j})$, then

$$\tilde{\lambda}_{k-j+1}(T) > \tilde{\lambda}_{k-j+1}(T_{k-j+1,j-1}) = \tilde{\mu}_{j-1}(T_{k-j+1,j-1}) > \tilde{\mu}_{j-1}(T)$$

and

$$\tilde{\lambda}_{k-j}(T) < \tilde{\lambda}_{k-j}(T_{k-j,j}) = \tilde{\mu}_j(T_{k-j,j}) < \tilde{\mu}_j(T).$$

So, either $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\lambda}_{k-j+1}(T)$ or $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\mu}_j(T)$. Again, since $T_{k,l}$ is decreasing in k and increasing in l by Lemma 3.2.4, $T_{k-j+1,j-1} < T_{k-j+1,j} < T_{k-j,j}$. So, if $T_{k-j+1,j-1} \leq T \leq T_{k-j+1,j}$, then

$$\tilde{\lambda}_{k-j+1}(T) \leq \tilde{\lambda}_{k-j+1}(T_{k-j+1,j}) = \tilde{\mu}_j(T_{k-j+1,j}) \leq \tilde{\mu}_j(T)$$

and so $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\lambda}_{k-j+1}(T) \leq \tilde{\lambda}_{k-j+1}(T_{k-j+1,j})$. Otherwise $T_{k-j+1,j} < T < T_{k-j,j}$, so

$$\tilde{\lambda}_{k-j+1}(T) > \tilde{\lambda}_{k-j+1}(T_{k-j+1,j}) = \tilde{\mu}_j(T_{k-j+1,j}) > \tilde{\mu}_j(T)$$

and $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\mu}_j(T) \leq \tilde{\lambda}_{k-j+1}(T_{k-j+1,j})$. Hence, in either case, $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) \leq \tilde{\lambda}_{k-j+1}(T_{k-j+1,j})$.

If $T \in [T_{k-s,s}, \infty)$, $s = \frac{k}{2}$, k even ($T_{k-s,s} = T_{k/2,k/2}$), then

$$\tilde{\lambda}_{k/2}(T) \geq \tilde{\lambda}_{k/2}(T_{k/2,k/2}) = \tilde{\mu}_{k/2}(T_{k/2,k/2}) \geq \tilde{\mu}_{k/2}(T).$$

Furthermore, $T_{k/2,k/2+1}$ is undefined by Lemma 3.2.2, so

$$\tilde{\lambda}_{k/2}(T) < \tilde{\mu}_{k/2+1}(T) \quad \forall\, T > 0.$$

Hence, $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\lambda}_{k/2}(T) < \tilde{\lambda}_{k/2}(\infty)$.

Finally, if $T \in [T_{k-s,s}, \infty)$, $s = \frac{k-1}{2}$, k odd ($T_{k-s,s} = T_{(k+1)/2,(k-1)/2}$), then

$$\tilde{\lambda}_{(k+1)/2}(T) \geq \tilde{\lambda}_{(k+1)/2}(T_{(k+1)/2,(k-1)/2}) = \tilde{\mu}_{(k-1)/2}(T_{(k+1)/2,(k-1)/2}) \geq \tilde{\mu}_{(k-1)/2}(T).$$

Furthermore, $T_{(k-1)/2,(k+1)/2}$ is undefined by Lemma 3.2.2, so

$$\tilde{\lambda}_{(k-1)/2}(T) < \tilde{\mu}_{(k+1)/2}(T) \quad \forall T > 0.$$

So, either $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\lambda}_{(k+1)/2}(T)$ or $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\mu}_{(k+1)/2}(T)$. Now, $T_{(k+1)/2,(k+1)/2}$ is defined by Lemma 3.2.2 and $T_{(k+1)/2,(k-1)/2} < T_{(k+1)/2,(k+1)/2}$ by Lemma 3.2.4. If $T_{(k+1)/2,(k-1)/2} \leq T \leq T_{(k+1)/2,(k+1)/2}$, then

$$\tilde{\lambda}_{k+1/2}(T) \leq \tilde{\lambda}_{(k+1)/2}(T_{(k+1)/2,(k+1)/2}) = \tilde{\mu}_{(k+1)/2}(T_{(k+1)/2,(k+1)/2}) \leq \tilde{\mu}_{(k+1)/2}(T)$$

and so $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\lambda}_{(k+1)/2}(T) \leq \tilde{\lambda}_{(k+1)/2}(T_{(k+1)/2,(k+1)/2})$. Otherwise $T > T_{(k+1)/2,(k+1)/2}$ and

$$\tilde{\lambda}_{k+1/2}(T) > \tilde{\lambda}_{(k+1)/2}(T_{(k+1)/2,(k+1)/2}) = \tilde{\mu}_{(k+1)/2}(T_{(k+1)/2,(k+1)/2}) > \tilde{\mu}_{(k+1)/2}(T).$$

So, $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) = \tilde{\mu}_{(k+1)/2}(T) < \tilde{\mu}_{(k+1)/2}(T_{(k+1)/2,(k+1)/2})$ and $\tilde{\mu}_{(k+1)/2}(T_{(k+1)/2,(k+1)/2}) = \tilde{\lambda}_{(k+1)/2}(T_{(k+1)/2,(k+1)/2})$. Thus, in either case, $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) \leq \tilde{\mu}_{(k+1)/2}(T_{(k+1)/2,(k+1)/2})$. $\square$

Lemma 3.2.6. *Let*

$$f(t) = \sinh(t)\cosh(t) - t \quad \textit{and} \quad g(t) = \frac{\sinh(t)\cosh(t) - t}{t^2}.$$

Then $f(t) > 0$ and $g'(t) > 0$ for all $t > 0$.

Proof. We have that

$$\begin{aligned} f'(t) &= \cosh^2(t) + \sinh^2(t) - 1 \\ &= 2\sinh^2(t) \end{aligned}$$

which is positive for $t > 0$. Since $f(0) = 0$, it follows that $f(t) > 0$ for $t > 0$.

Now

$$\begin{aligned} g'(t) = \frac{f'(t)t^2 - 2tf(t)}{t^4} &= \frac{2t(\sinh^2(t) + 1) - 2\sinh(t)\cosh(t)}{t^3} \\ &= \frac{2t\cosh^2(t) - 2\sinh(t)\cosh(t)}{t^3}. \end{aligned}$$

Since $\cosh^2(t) = \frac{1}{4}(e^{2t} + 2 + e^{-2t})$ and $\cosh(t)\sinh(t) = \frac{1}{4}(e^{2t} - e^{-2t})$, we get that

$$\begin{aligned} g'(t) &= \frac{(2t-2)e^{2t} + (2t+2)e^{-2t} + 4t}{4t^3} \\ &= \frac{1}{4t^3}\left[\left((2t-2)\sum_{k=0}^{\infty}\frac{(2t)^k}{k!}\right) + \left((2t+2)\sum_{k=0}^{\infty}(-1)^k\frac{(2t)^k}{k!}\right) + 4t\right] \\ &= \frac{1}{4t^3}\left[\left(\sum_{k=0}^{\infty}\frac{(2t)^{k+1}}{k!}\right) - 2\left(\sum_{k=0}^{\infty}\frac{(2t)^k}{k!}\right) + \left(\sum_{k=0}^{\infty}\frac{(-1)^k(2t)^{k+1}}{k!}\right)\right. \\ &\qquad\qquad \left. + 2\left(\sum_{k=0}^{\infty}\frac{(-1)^k(2t)^k}{(k!)}\right) + 4t\right] \\ &= \frac{1}{4t^3}\left[2\left(\sum_{k=0}^{\infty}\frac{(2t)^{2k+1}}{(2k)!}\right) - 4\left(\sum_{k=0}^{\infty}\frac{(2t)^{2k+1}}{(2k+1)!}\right) + 4t\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4t^3}\left[\sum_{k=1}^{\infty}\left(\frac{2}{(2k)!} - \frac{4}{(2k+1)!}\right)(2t)^{2k+1}\right] \\
&= \frac{1}{4}\sum_{k=1}^{\infty}\frac{2(2k-1)}{(2k+1)!}(2t)^{2k-2} \\
&= \frac{1}{4}\sum_{k=0}^{\infty}\frac{2(2k+1)}{(2k+3)!}(2t)^{2k}
\end{aligned}
$$

Since all of the coefficients are positive, it follows that $g'(t) > 0$ for $t > 0$. $\square$

Lemma 3.2.7. *Let $x(a, b)$ be the unique positive solution of*

$$a \tanh(ax) = b \coth(bx)$$

for $a \geq b > 0$. Let

$$u(a, b) = a \tanh(ax(a, b)).$$

Then $u(a, b) < u(a + c, b - c)$ for $a \geq b > c > 0$.

Proof. Differentiating the first equation with respect to a yields

$$\tanh(ax) + a\text{sech}^2(ax)\left(x + a\frac{\partial x}{\partial a}\right) = -b^2\text{csch}^2(bx) \cdot \frac{\partial x}{\partial a}$$

and so

$$\frac{\partial x}{\partial a} = \frac{-\tanh(ax) - ax\text{sech}^2(ax)}{a^2\text{sech}^2(ax) + b^2\text{csch}^2(bx)} < 0.$$

Similarly,

$$\frac{\partial x}{\partial b} = \frac{\coth(bx) - bx\text{csch}^2(bx)}{a^2\text{sech}^2(ax) + b^2\text{csch}^2(bx)} = \frac{\sinh(bx)\cosh(bx) - bx}{\sinh^2(bx)(a^2\text{sech}^2(ax) + b^2\text{csch}^2(bx))} > 0,$$

where we have used Lemma 3.2.6 to conclude its sign.

Now, since $u(a, b) = b \coth(bx(a, b))$ and $\frac{\partial x}{\partial a} < 0$,

$$\frac{\partial u}{\partial a} = -b^2\text{csch}^2(bx) \cdot \frac{\partial x}{\partial a} > 0.$$

Similarly,

$$\frac{\partial u}{\partial b} = a^2 \operatorname{sech}^2(ax) \cdot \frac{\partial x}{\partial b} > 0.$$

Hence,

$$\frac{\left(\frac{\partial u}{\partial a}\right)}{\left(\frac{\partial u}{\partial b}\right)} = \frac{b^2(\sinh(ax)\cosh(ax) + ax)}{a^2(\sinh(bx)\cosh(bx) - bx)} > \frac{b^2(\sinh(ax)\cosh(ax) - ax)}{a^2(\sinh(bx)\cosh(bx) - bx)} \geq 1$$

by Lemma 3.2.6 since $a \geq b$. Note that the inequality is strict when $a > b$. Thus, for $f(t) = u(a+t, b-t)$,

$$f'(t) = \frac{\partial u}{\partial a} - \frac{\partial u}{\partial b},$$

and so $f'(t) > 0$ for $t > 0$. Hence $u(a, b) < u(a+c, b-c)$ for $a \geq b > c > 0$. $\square$

Corollary 3.2.1. *For $k \geq l > c > 0$ we have that*

$$\tilde{\lambda}_k(T_{k,l}) < \tilde{\lambda}_{k+c}(T_{k+c,l-c}).$$

Proof. By Lemma 3.2.7, for $k \geq l > c > 0$ we have that $u(2k, 2l-1) < u(2k + 2c, 2l-1-2c)$. Hence $\tilde{\lambda}_k(T_{k,l}) < \tilde{\lambda}_{k+c}(T_{k+c,l-c})$. $\square$

In particular, this tells us that

$$\tilde{\lambda}_{k-j+1}(T_{k-j+1,j}) < \tilde{\lambda}_k(T_{k,1}),$$

for $2 \leq j < s$ and, when k is odd,

$$\tilde{\lambda}_{(k+1)/2}(T_{(k+1)/2,(k+1)/2}) < \tilde{\lambda}_k(T_{k,1}).$$

So, when k is odd, by Lemma 3.2.5 we have that $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) \leq \tilde{\lambda}_k(T_{k,1})$, and when k is even, $\tilde{\sigma}_{2k-1}(T) = \tilde{\sigma}_{2k}(T) \leq \max(\tilde{\lambda}_{k/2}(\infty), \tilde{\lambda}_k(T_{k,1}))$.

Lemma 3.2.8. *For $k \geq 2$ even,*

$$\tilde{\lambda}_{k/2}(\infty) < \tilde{\lambda}_k(T_{k,1})$$

Proof. First note that $\tilde{\lambda}_{k/2}(\infty) = 2\pi k$ and $\tilde{\lambda}_k(T_{k,1}) = 4\pi k \tanh(2kT_{k,1})$, so, if we let $k = 2n$ with $n \geq 1$, then we need to show that $8\pi n \tanh(4nT_{2n,1}) > 4\pi n$. Now

$$\tanh(4nT) = \frac{e^{4nT} - e^{-4nT}}{e^{4nT} + e^{-4nT}} = \frac{1}{2} \quad \Leftrightarrow \quad T = \frac{\log(3)}{8n}.$$

Since $2\pi \coth(T_{2n,1}) = 8\pi n \tanh(4nT_{2n,1})$, if $\coth\left(\frac{\log(3)}{8n}\right) > 2n = 4n \tanh\left(4n\frac{\log(3)}{8n}\right)$, then it would follow that $2\pi \coth(T_{2n,1}) = 8\pi n \tanh(4nT_{2n,1}) > 4\pi n$. By direct calculation, we have that

$$\coth\left(\frac{\log(3)}{8n}\right) = \frac{3^{1/8n} + \frac{1}{3^{1/8n}}}{3^{1/8n} - \frac{1}{3^{1/8n}}} = \frac{3^{1/4n} + 1}{3^{1/4n} - 1},$$

so $\coth\left(\frac{\log(3)}{8n}\right) > 2n$ is equivalent to

$$\frac{3^{1/4n} + 1}{3^{1/4n} - 1} > 2n \Leftrightarrow \frac{3^{1/4n} + 1}{2n(3^{1/4n} - 1)} > 1.$$

Let

$$f(n) = \frac{3^{1/4n} + 1}{2n(3^{1/4n} - 1)}.$$

Then, thinking of n as a positive real number,

$$f'(n) =$$

$$\frac{\left(-\frac{\log(3)}{4n^2} \cdot 3^{1/4n}\right) \cdot \left(2n \cdot (3^{1/4n} - 1)\right) - \left(3^{1/4n} + 1\right) \cdot \left(2\left(3^{1/4n} - 1\right) + 2n\left(-\frac{\log(3)}{4n^2} \cdot 3^{1/4n}\right)\right)}{4n^2\left(3^{1/4n} - 1\right)^2}$$

$$= \frac{-\frac{\log(3)}{2n} \cdot 3^{1/2n} + \frac{\log(3)}{2n} \cdot 3^{1/4n} - 2 \cdot 3^{1/2n} + 2 + \frac{\log(3)}{2n} \cdot 3^{1/2n} + \frac{\log(3)}{2n} \cdot 3^{1/4n}}{4n^2\left(3^{1/4n} - 1\right)^2}$$

$$= \frac{\frac{\log(3)}{n} \cdot 3^{1/4n} + 2 \cdot (1 - 3^{1/2n})}{4n^2\left(3^{1/4n} - 1\right)^2}.$$

Since the denominator is always positive, we will focus on the numerator. Using

Taylor series we have that

$$\begin{aligned}\frac{\log(3)}{n}\cdot 3^{1/4n}+2\cdot(1-3^{1/2n}) &= \frac{\log(3)}{n}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\log(3)}{4n}\right)^k\right)-2\left(\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{\log(3)}{2n}\right)^k\right)\\ &= \left(\sum_{k=0}^{\infty}\frac{4}{k!}\left(\frac{\log(3)}{4n}\right)^{k+1}\right)-\left(\sum_{k=1}^{\infty}\frac{2^{k+1}}{k!}\left(\frac{\log(3)}{4n}\right)^k\right)\\ &= \left(\sum_{k=1}^{\infty}\frac{4}{(k-1)!}\left(\frac{\log(3)}{4n}\right)^k\right)-\left(\sum_{k=1}^{\infty}\frac{2^{k+1}}{k!}\left(\frac{\log(3)}{4n}\right)^k\right)\\ &= \sum_{k=1}^{\infty}\left(\frac{4}{(k-1)!}-\frac{2^{k+1}}{k!}\right)\cdot\left(\frac{\log(3)}{4n}\right)^k.\end{aligned}$$

So, since $\frac{4}{(k-1)!}=\frac{2^{k+1}}{k!}$ for $k=1,2$, and $\frac{4}{(k-1)!}<\frac{2^{k+1}}{k!}$ for all $k\geq 3$, it follows that $f'(n)<0$ for all $n>0$ and so $f(n)$ is monotone decreasing.

Now,

$$\lim_{n\to\infty}2n\cdot(3^{1/4n}-1)=\lim_{n\to\infty}\frac{3^{1/4n}-1}{\frac{1}{2n}}=\lim_{n\to\infty}\frac{-\frac{\log(3)}{4n^2}\cdot 3^{1/4n}}{-\frac{1}{2n^2}}=\frac{\log(3)}{2},$$

so

$$\lim_{n\to\infty}f(n)=\frac{4}{\log(3)}>1.$$

Thus, $f(n)$ is bounded below by 1 and so

$$\coth\left(\frac{\log(3)}{8n}\right)=\frac{3^{1/4n}+1}{3^{1/4n}-1}>2n.$$

Therefore, $8\pi n\tanh(4nT_{2n,1})=2\pi\coth(T_{2n,1})>4\pi n$, and so $\tilde{\lambda}_{k/2}(\infty)<\tilde{\lambda}_k(T_{k,1})$. $\square$

3.3 Free boundary minimal Möbius bands in $\mathbb{B}^4$

Here, using the results from the previous section, we first show that we can always find a rotationally symmetric metric that maximizes the kth Steklov eigenvalue. We then use the eigenfunctions corresponding to these maximal Steklov eigenvalues to get constructions of free boundary minimal Möbius bands in $\mathbb{B}^4$.

Theorem 3.3.1. *Let $k \geq 1$ and $M_k = \sup_{T>0}(\tilde{\sigma}_k(T))$. Then $M_{2k-1} = M_{2k} = \tilde{\lambda}_k(T_{k,1})$, and is attained precisely when $T = T_{k,1}$.*

Proof. This follows directly from Lemma 3.2.1 and Lemma 3.2.8. $\square$

Consider the immersed surface in $\mathbb{R}^4$ given by

$$\begin{cases} x(t,\theta) = 2n\sinh(t)\cos(\theta) \\ y(t,\theta) = 2n\sinh(t)\sin(\theta) \\ z(t,\theta) = \cosh(2nt)\cos(2n\theta) \\ w(t,\theta) = \cosh(2nt)\sin(2n\theta) \end{cases}$$

for $t \in [-T_{n,1}, T_{n,1}]$. Now, since the coordinate functions are Steklov eigenfunctions, they are harmonic extensions of their restriction to $\partial\Sigma$. Furthermore, if we let $\Phi(t,\theta) = (x(t,\theta), y(t,\theta), z(t,\theta), w(t,\theta))$, then

$$\frac{\partial \Phi}{\partial t} = (n\cosh(t)\cos(\theta), n\cosh(t)\sin(\theta), n\sinh(nt)\cos(n\theta), n\sinh(nt)\sin(n\theta)),$$
$$\frac{\partial \Phi}{\partial \theta} = (-n\sinh(t)\sin(\theta), n\sinh(t)\cos(\theta), -n\cosh(nt)\sin(n\theta), n\cosh(nt)\cos(n\theta)).$$

It follows that $\dfrac{\partial \Phi}{\partial t}\cdot\dfrac{\partial \Phi}{\partial \theta} = 0$ and $\left|\dfrac{\partial \Phi}{\partial t}\right| = \left|\dfrac{\partial \Phi}{\partial \theta}\right| = n^2\left(\sinh^2(t) + \cosh^2(nt)\right)$. Hence, Φ is also conformal and so we see that the immersion defined by the coordinate functions is a minimal immersion. Moreover, since $|\Phi|$ is constant on $\partial\Sigma$, it follows from the maximum principle that Φ defines a surface contained in a ball centred at the origin of radius $\sqrt{4n^2\sinh^2(T_{n,1}) + \cosh^2(2nT_{n,1})}$. To obtain a free boundary minimal Möbius band in $\mathbb{B}^4$, we scale the portion of this immersed surface inside the ball centred at the origin of radius $\sqrt{4n^2\sinh^2(T_{n,1}) + \cosh^2(2nT_{n,1})}$ to lie in $\mathbb{B}^4$. This yields the following:

Theorem. 1.2.1 *For all $n \geq 1$, the maximum of the nth Steklov eigenvalue among all rotationally symmetric metrics on the Möbius band is achieved by the metric on* *a free boundary minimal Möbius band in* $\mathbb{B}^4$ *given explicitly by the immersion*

$$\Phi(t,\theta) = \frac{1}{R_n}(2n\sinh(t)\cos(\theta), 2n\sinh(t)\sin(\theta), \cosh(2nt)\cos(2n\theta), \cosh(2nt)\sin(2n\theta)),$$

where $R_n = \sqrt{4n^2\sinh^2(T_{n,1}) + \cosh^2(2nT_{n,1})}$ *and* $(t,\theta) \in [-T_{n,1}, T_{n,1}] \times S^1/\sim$.

Chapter 4

Existence of Harmonic Maps into CAT(1) Spaces

4.1 Introduction

In this chapter we prove an existence result for harmonic maps from compact Riemann surfaces into complete metric spaces with an upper curvature bound. The theory of harmonic maps has proven to have important applications; for example, the existence theory for harmonic two-spheres of Sacks and Uhlenbeck [53] was extended by Micallef and Moore [49] and used to prove a generalization of the classical sphere theorem to pointwise quarter-pinching. Other important applications of harmonic maps include those in rigidity problems (for example, [62], [10], [28]) and in Teichmüller theory (for example, [70], [16], [14]) amongst others.

For some of the above mentioned applications, it has been necessary to consider harmonic maps when the smooth Riemannian target is replaced by a singular space. The seminal works of Gromov-Schoen [28] and Korevaar-Schoen [40] consider harmonic maps from a Riemannian domain into a non-Riemannian target. Further exploration of harmonic map theory in the singular setting includes works of Jost [33], J. Chen [8], Eells-Fuglede [18] and Daskalopoulos-Mese [12]. However, all of the above mentioned works assume non-positivity of curvature (NPC) in the target space.

When the curvature of the target space is allowed to be positive, the existence problem for harmonic maps becomes more complicated, and in many ways, more interesting. Although the general problem is not well understood, a breakthrough was achieved in the case of two-dimensional domains by Sacks and Uhlenbeck [53]. Indeed, they discovered a "bubbling phenomena" for harmonic maps; more specifically, they prove the following dichotomy: given a finite energy map from a Riemann surface into a compact Riemannian manifold, either there exists a harmonic map homotopic to the given map or there exists a branched minimal immersion of the 2-sphere. We also mention the related works of Lemaire [44], Sacks-Uhlenbeck [54], and Schoen-Yau [58].

The goal of this chapter is to provide a generalization of the Sacks and Uhlenbeck existence result to the case of metric space targets. We specifically look at the setting in which the target is a CAT(1) space, i.e. a complete metric space with curvature bounded above by 1 in the sense of Alexandrov. The method used by Sacks and Uhlenbeck is not accessible in the singular setting as it depends on a priori estimates stemming from the Euler-Lagrange equation of their perturbed energy functional and, in the singular setting, one can no longer use variational methods to obtain an Euler-Lagrange equation. Here, we develop an alternative method that instead exploits the local convexity of the target CAT(1) space.

Our original motivation for considering the existence problem in the singular setting was to develop an approach to the non-smooth uniformization problem of finding a conformal (or more generally, a quasisymmetric) parameterization of a metric space homeomorphic to the 2-sphere, via harmonic map methods. We expect to able to use an application of our theorem to solve the non-smooth uniformization problem in the special case when the metric space in question has an additional property that it is locally CAT(1).

Before stating our result precisely, we first describe the setting of our problem in more detail by outlining harmonic maps with singular targets and CAT(1) spaces.

4.1.1 Harmonic maps

Let (M^m, g) and (N^n, h) be two Riemannian manifolds, and let $u : M \to N$. Then there is a natural notion of the *energy* of u, $E(u)$, which roughly measures the amount the map u stretches M. More precisely, if $\{e_i\}_{i=1}^m$ is a local orthonormal frame for T_xM, then the *energy density* at $x \in M$ is

$$\begin{aligned} |du_x|^2 = \mathrm{Tr}_g(u^*h) &= \sum_{i=1}^m |du_x(e_i)|^2 \\ &= g^{\alpha\beta}(x)h_{ij}(u(x))\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}, \end{aligned}$$

and the energy of u is defined to be

$$E(u) = \int_M |du|^2 d\mu_g.$$

A harmonic map is then a critical point of the energy functional.

When the target space is no longer a smooth manifold but simply a complete metric space, Gromov and Schoen [28] and Korevaar and Schoen [40] developed a Sobolev space theory for maps into metric spaces and harmonic maps theory into complete metric spaces with non-positive curvature in the sense of Alexandrov. If (Ω, g) is a Riemannian domain and (X, d) is a complete metric space, then a map $u : \Omega \to X$ is in $L^2(\Omega, X)$ if u is a Borel measurable function with separable range and for some $P \in X$,

$$\int_\Omega d^2(u(x), P)d\mu_g < \infty.$$

To define the Sobolev space $W^{1,2}(\Omega, X) \subset L^2(\Omega, X)$, we need to define the energy of a map $u : \Omega \to X$ when X is a complete metric space. We first define the *ϵ-approximate energy density* $e^u_\epsilon : \Omega \to \mathbb{R}$ by

$$e^u_\epsilon(x) = \int_{S(x,\epsilon)} \frac{d^2(u(x), u(y))}{\epsilon^2} \cdot \frac{d\sigma_{x,\epsilon}(y)}{\epsilon^{n-1}},$$

where $\sigma_{x,\epsilon}$ is the induced measure on the ϵ-sphere $S(x, \epsilon)$ centred at x. The ϵ-

approximating energy $E^u_\epsilon : C_c(\Omega) \to \mathbb{R}$ is then

$$^{d}E^u_\epsilon(\phi) = \int_\Omega \phi e^u_\epsilon d\mu_g.$$

We will often suppress the superscript d when the context is clear. An L^2 map $u : \Omega \to X$ is said to have finite energy if

$$E(u) = \sup_{\phi \in C_c(\Omega),\ 0 \le \phi \le 1} \limsup_{\epsilon \to 0} E^u_\epsilon(\phi) < \infty,$$

and the Sobolev space $W^{1,2}(\Omega, X)$ is defined to the be the subset of $L^2(\Omega, X)$ consisting of finite energy maps. In the case that u has finite energy, there is an *energy density* function $|\nabla u|^2(x)$ such that

$$e^u_\epsilon(x) d\mu_g \rightharpoonup |\nabla u|^2(x) d\mu_g.$$

For $u \in W^{1,2}(\Omega, X)$ and a smooth vector field $V \in \Gamma(\Omega)$, there is a *directional energy density* function $|u_*(V)|^2(x) \in L^1(\Omega)$ such that

$$|u_*(V)|^2(x) = \lim_{\epsilon \to 0} \frac{d^2(u(x), u(\exp_x(\epsilon V)))}{\epsilon^2} \quad \text{for a.e. } x \in \Omega,$$

the energy density is given by

$$|\nabla u|^2(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1} \subset T_x\Omega} |u_*(V)|^2(x) d\sigma,$$

and the energy of u is

$$E(u) = \int_\Omega |\nabla u|^2 d\mu.$$

Given two finite energy maps u and v, the distance $d(u, v) : \Omega \to \mathbb{R}^+$ between them belongs to the Sobolev space $W^{1,2}(\Omega)$. Therefore one can make the following definition: $u = v$ on $\partial\Omega$ if $d(u, v) \in W^{1,2}_0(\Omega)$. A finite energy map $u : \Omega \to X$ is *energy minimizing* if

$$E(u) = \inf\{E(v) : v \in W^{1,2}(\Omega, X),\ v = u \text{ on } \partial\Omega\}.$$

Given $h \in W^{1,2}(\Omega, X)$, we define

$$W^{1,2}_h(\Omega, X) = \{f \in W^{1,2}(\Omega, X) \,:\, h = f \text{ on } \partial\Omega\}.$$

Definition 4.1.1. We say that a map $u : \Omega \to X$ is *harmonic* if it is locally energy minimizing with locally finite energy; precisely, for every $p \in \Omega$, there exist $r > 0$, $\rho > 0$ and $P \in X$ such that $h = u\big|_{B_r(p)}$ has finite energy and minimizes energy among all maps in $W^{1,2}_h(B_r(p), \overline{\mathcal{B}_\rho(P)})$, where $B_r(p)$ is the geodesic ball in Ω of radius r centred at p and $\mathcal{B}_\rho(P)$ is the geodesic ball in X of radius ρ centred at P.

We refer the reader to [40] for further details and background.

4.1.2 CAT(1) Spaces

Roughly speaking, a CAT(1) space is a complete metric space with curvature bounded above by 1 in the sense of triangle comparison.

A complete metric space (X, d), is a *geodesic space* if for each $P, Q \in X$, there exists a curve γ_{PQ} such that the length of γ_{PQ} is exactly $d(P, Q)$. We call γ_{PQ} a *geodesic* between P and Q.

Remark 4.1.2. For ease of notation, we will often denote $d(P, Q)$ by d_{PQ}.

We determine a weak notion of an upper sectional curvature bound on X by using comparison triangles. Given any three points $P, Q, R \in X$ such that $d_{PQ} + d_{QR} + d_{RS} < 2\pi$, the *geodesic triangle* $\triangle PQR$ is the triangle in X with sides given by the geodesics $\gamma_{PQ}, \gamma_{QR}, \gamma_{RS}$.

Let $\triangle\tilde{P}\tilde{Q}\tilde{R}$ denote a geodesic triangle on the standard sphere $\mathbb{S}^2$ such that $d_{PQ} = d_{\tilde{P}\tilde{Q}}$, $d_{QR} = d_{\tilde{Q}\tilde{R}}$ and $d_{RP} = d_{\tilde{R}\tilde{P}}$. We call $\triangle\tilde{P}\tilde{Q}\tilde{R}$ a *comparison triangle* for the geodesic triangle $\triangle PQR$. Note that a comparison triangle is convex since the perimeter of the geodesic triangle is less than 2π.

Given a geodesic space (X, d) and a geodesic γ_{PQ} with $d_{PQ} < \pi$, for $\tau \in [0, 1]$ let $(1 - \tau)P + \tau Q$ denote the point on γ_{PQ} at distance τd_{PQ} from P. That is,

$$d((1-\tau)P + \tau Q, P) = \tau d_{PQ}.$$

Definition 4.1.3. Let (X, d) be a complete geodesic space. Then X is a *CAT(1) space* if, given any geodesic triangle $\triangle PQR$ (with perimeter less than 2π) and a comparison triangle $\triangle \tilde{P}\tilde{Q}\tilde{R}$ in $\mathbb{S}^2$,

$$d_{P_t R_s} \leq d_{\tilde{P}_t \tilde{R}_s} \tag{4.1}$$

where

$$P_t = (1-t)P + tQ,\ R_s = (1-s)R + sQ,$$
$$\tilde{P}_t = (1-t)\tilde{P} + t\tilde{Q},\ \tilde{R}_s = (1-s)\tilde{R} + s\tilde{Q}.$$

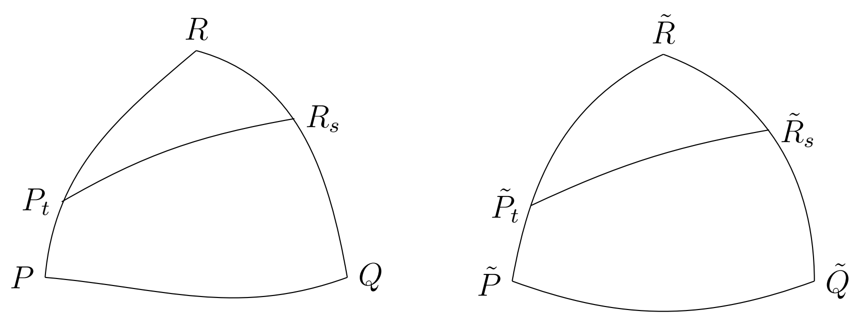

Figure 4.1: An illustration of a triangle in a geodesic space (X, d) (left) and a comparison triangle in $\mathbb{S}^2$ (right). If the geodesics connecting the sides of the triangle in (X, d) are shorter than the corresponding geodesics for the comparison triangle in $\mathbb{S}^2$, then (X, d) is called a CAT(1) space.

The simplest examples of CAT(1) spaces are the complete Riemannian manifolds with curvature bounded above by 1. In particular, $\mathbb{S}^2$ is a CAT(1) space. However, there are many examples of CAT(1) spaces other than Riemannian manifolds.

A metric space (X, d) is said to be *locally CAT(1)* if every point of X has a geodesically convex CAT(1) neighbourhood. Note that for a compact locally CAT(1) space, there exists a radius $r(X) > 0$ such that for all $y \in X$, $\overline{B_{r(X)}(y)}$ is a compact CAT(1) space. We refer the reader to Section 2.2 of [7] for further background on CAT(1) spaces.

4.1.3 Main results and outline

The goal of this chapter is to prove a result analogous to the existence result of Sacks and Uhlenbeck when the target space is a compact CAT(1) space. More specifically, we obtain the following theorem.

Theorem. 1.3.1 *Let Σ be a compact Riemann surface, X a compact locally CAT(1) space and $\varphi \in C^0 \cap W^{1,2}(\Sigma, X)$. Then either there exists a harmonic map $u : \Sigma \to X$ homotopic to φ or a nontrivial conformal harmonic map $v : \mathbb{S}^2 \to X$.*

Sacks and Uhlenbeck used the perturbed energy method in the proof of Theorem 1.3.1 for Riemannian manifolds. In doing so, they rely heavily on a priori estimates procured from the Euler-Lagrange equation of the perturbed energy functional. One of the difficulties in working in the singular setting is that, because of the lack of local coordinates, one does not have a P.D.E. derived from a variational principle (e.g. harmonic map equation). In order to prove results in the singular setting, we cannot rely on P.D.E. methods. To this end, we use a 2-dimensional generalization of the Birkhoff curve shortening method [4], [5]. This local replacement process can be thought of as a discrete gradient flow. This idea was used by Jost [33] to give an alternative proof of the Sacks-Uhlenbeck theorem in the smooth setting. More recently, in studying width and proving finite time extinction of the Ricci flow, Colding-Minicozzi [9] further developed the local replacement argument and proved a new convexity result for harmonic maps and continuity of harmonic replacement. However, even these arguments rely on the harmonic map equation and hence do not translate to our case. The main accomplishment of our method is to eliminate the need for a P.D.E. by using the local convexity properties of the target CAT(1) space. (The necessary convexity properties of a CAT(1) space are given in Appendix B.)

For clarity, we provide a brief outline of the harmonic replacement construction. Given $\varphi : \Sigma \to X$, we set $\varphi = u_0^0$ and inductively construct a sequence of energy decreasing maps u_n^l where $n \in \mathbb{N} \cup \{0\}$, $l \in \{0, \ldots, \Lambda\}$, and Λ depends on the geometry of Σ. The sequence is constructed inductively as follows. Given the map u_n^0, we determine the largest radius, r_n, in the domain on which we can apply the existence and regularity of Dirichlet solutions (see Lemma 4.2.1) for this map. Given a suitable cover of Σ by balls of this radius, we consider Λ subsets of this cover such that every subset consists of non-intersecting balls. The maps $u_n^l : \Sigma \to X$, $l \in \{1, \ldots, \Lambda\}$ are determined by replacing u_n^{l-1} by its Dirichlet solution on balls in the l-th subset of the covering and leaving the remainder of the map unchanged. We then set $u_{n+1}^0 := u_n^\Lambda$ to continue by induction. There are now two possibilities, depending on $\liminf r_n = r$. If $r > 0$, we demonstrate that the sequence we constructed is equicontinuous and has a unique limit that is necessarily homotopic to φ. Compactness for minimizers (Lemma 4.2.2) then implies that the limit map is harmonic. If $r = 0$, then bubbling occurs. That is, after an appropriate rescaling of the original sequence, the new sequence is an equicontinuous family of harmonic maps from domains exhausting $\mathbb{C}$. As in the previous case, this sequence converges on compact sets to a limit harmonic map from $\mathbb{C}$ to X. We extend this map to $\mathbb{S}^2$ by a removable singularity theorem developed in section 4.3.

We now give an outline of the chapter. In section 4.2, we introduce some notation and provide the results that are necessary in order to perform harmonic replacement and obtain a harmonic limit map. In particular, we state the existence and regularity results for Dirichlet solutions and prove compactness of energy minimizing maps into a CAT(1) space. In section 4.3, we prove our removable singularity theorem. Namely, in Theorem 4.3.6 we prove that any conformal harmonic map from a punctured surface into a CAT(1) space extends as a locally Lipschitz harmonic map on the surface. This theorem extends to CAT(1) spaces the removable singularity theorem of Sacks-Uhlenbeck [53] for a finite energy harmonic map into a Riemannian manifold, provided the map is conformal. The proof relies on two key ideas. First, for harmonic maps u_0 and u_1 into a CAT(1) space, while $d^2(u_0, u_1)$ is not subharmonic, a more complicated weak differential inequality holds if the maps are into a sufficiently small ball (Theorem B.2.4 in Appendix B.2, [60]). Using this inequality, we prove a local removable singularity theorem for harmonic maps into a small ball. The second key idea, Theorem 4.3.4, is a monotonicity of the area in extrinsic balls in the target space, for conformal harmonic maps from a surface to a CAT(1) space. This theorem extends the classical monotonicity of area for minimal surfaces in Riemannian manifolds to metric space targets. The proof relies on the fact that the distance function from a point in a CAT(1) space is almost convex on a small ball. In application, the monotonicity is used to show that a conformal harmonic map defined on $\Sigma \backslash \{p\}$ is continuous across p. Then the local removable singularity theorem can be applied at some small scale. Section 4.4 contains the harmonic replacement construction outlined above and the proof of the main theorem, Theorem 1.3.1. Note that we give complete proofs of several difficult estimates for quadrilaterals in a CAT(1) space in Appendix B.1. The estimates are stated in the unpublished thesis [60] without proof. We apply these estimates in Appendix B.2 to give complete proofs of some energy convexity, existence, uniqueness, and subharmonicity results (also stated in [60]) that are used throughout this chapter.

4.2 Preliminary results

Throughout this chapter we let (Ω, g) denote a Lipschitz Riemannian domain and (X, d) a locally CAT(1) space. We denote a geodesic ball in Ω of radius r centred at $p \in \Omega$ by $B_r(p)$ and a geodesic ball in X of radius ρ centred at $P \in X$ by $\mathcal{B}_\rho(P)$. The following results will be used in the proof of the main theorem, Theorem 1.3.1.

Lemma 4.2.1 (Existence, Uniqueness and Regularity of the Dirichlet solution)**.** *For any finite energy map $h : \Omega \to \overline{\mathcal{B}_\rho(P)} \subset X$, where $\rho \in (0, \min\{r(X), \frac{\pi}{4}\})$, the Dirichlet solution exists. That is, there exists a unique element ${}^{Dir}h \in W^{1,2}_h(\Omega, \overline{\mathcal{B}_\rho(P)})$ that minimizes energy among all maps in $W^{1,2}_h(\Omega, \overline{\mathcal{B}_\rho(P)})$. Moreover, if ${}^{Dir}h(\partial\Omega) \subset \overline{\mathcal{B}_\sigma(P)}$ for some $\sigma \in (0, \rho)$, then $\overline{{}^{Dir}h(\Omega)} \subset \overline{\mathcal{B}_\sigma(P)}$. Finally, the solution ${}^{Dir}h$ is locally Lipschitz continuous with Lipschitz constant depending only on the total energy of the map and the metric on the domain.*

For further details see Lemma B.2.2 in Appendix B.2, [60], and [7].

Lemma 4.2.2 (Compactness for minimizers into CAT(1) space)**.** *Let (X,d) be a CAT(1) space and $B_r \subset \Omega$ a geodesic (and topological) ball of radius $r > 0$ where (Ω, g) is a Riemannian manifold. Let $u_i : B_r \to X$ be a sequence of energy minimizers with $E^{u_i}[B_r] \leq \Lambda$ for some $\Lambda > 0$.*

Suppose that u_i converges uniformly to u on B_r and that there exists $P \in X$ such that $u(B_r) \subset \mathcal{B}_{\rho/2}(P)$ where ρ is as in Lemma 4.2.1. Then u is energy minimizing on $B_{r/2}$.

Proof. We will follow the ideas of the proof of Theorem 3.11 [41]. Rather than prove the bridge principle for CAT(1) spaces, we will modify the argument and appeal directly to the bridge principle for NPC spaces (see Lemma 3.12 [41]).

Since $u_i \to u$ uniformly and $u(B_r) \subset \mathcal{B}_{\rho/2}(P)$, there exists I large such that for all $i \geq I$, $u_i(B_r) \subset \mathcal{B}_\rho(P)$. By Lemma 4.2.1, there exists $c > 0$ depending only on Λ and g such that for all $i \geq I$, $u_i|_{B_{3r/4}}$ is Lipschitz with Lipschitz constant c. It follows that for $t > 0$ small, there exists $C > 0$ depending on c and the dimension of Ω such that

$$E^{u_i}[B_{r/2} \backslash B_{r/2-t}] \leq Ct. \tag{4.2}$$

For $\varepsilon > 0$, increase I if necessary so that for all $i \geq I$ and all $x \in B_{3r/4}$,

$$d^2(u_i(x), u(x)) < \varepsilon. \tag{4.3}$$

For notational ease, let $U_t := B_{r/2-t}$. Let $w_t : U_t \to X$ denote the energy minimizer $w_t := {}^{Dir}u|_{U_t} \in W^{1,2}_u(U_t, X)$, with existence guaranteed by Lemma 4.2.1. Following the argument in the proof of Theorem 3.11 [41], (4.2) and the lower semi-continuity of the energy imply that $\lim_{t\to 0} E^{w_t}[U_t] = E^{w_0}[B_{r/2}]$. Observe that by the lower semi-continuity of energy, Theorem 1.6.1 [40],

$${}^dE^u[B_{r/2}] \leq \liminf_{i\to\infty} {}^dE^{u_i}[B_{r/2}].$$

Thus, it will be enough to show that

$$\limsup_{i\to\infty} {}^dE^{u_i}[B_{r/2}] \le {}^dE^{w_0}[B_{r/2}].$$

Let $v_t : B_{r/2} \to X$ be the map such that $v_t|_{U_t} = w_t$ and $v_t|_{B_{r/2}\setminus U_t} = u$. Given $\delta > 0$, choose $t > 0$ sufficiently small so that

$$ {}^dE^{v_t}[B_{r/2}] < {}^dE^{w_0}[B_{r/2}] + \delta. \tag{4.4}$$

Since v_t is not a competitor for u_i (i.e. $v_t|_{\partial B_{r/2}}$ is not necessarily equal to $u_i|_{\partial B_{r/2}}$), for each i we want to bridge from v_t to u_i for values near $\partial B_{r/2}$. Since we want to exploit a bridging lemma into NPC spaces, rather than bridge between v_t and u_i, we will bridge between their lifted maps in the cone $\mathcal{C}(X)$.

Let $\mathcal{C}(X) := (X \times [0,\infty)/X \times \{0\}, D)$ where

$$D^2([P,x],[Q,y]) = x^2 + y^2 - 2xy\cos\min(d(P,Q),\pi).$$

Then $\mathcal{C}(X)$ is an NPC space and we can identify X with $X \times \{1\} \subset \mathcal{C}(X)$. For any map $f : B_r \to X$, we let $\overline{f} : B_r \to X \times \{1\}$ such that $\overline{f}(x) = [f(x), 1]$. Note that for $f \in W^{1,2}(B_r, \mathcal{B}_\rho(Q))$, since

$$\lim_{P\to Q} \frac{D^2([P,1],[Q,1])}{d^2(P,Q)} = \lim_{P\to Q} \frac{2(1-\cos(d(P,Q)))}{d^2(P,Q)} = 1,$$

it follows that ${}^DE^{\overline{f}}[\Omega] = {}^dE^f[\Omega]$ for $\Omega \subset B_r$.

For each $i \ge I$, and a fixed $s, \rho > 0$ to be chosen later, define the map

$$v_i : \partial U_s \times [0,\rho] \to \mathcal{C}(X)$$

such that

$$v_i(x,z) := \left(1 - \frac{z}{\rho}\right)\overline{v}_t(x) + \frac{z}{\rho}\overline{u}_i(x).$$

The map v_i is a bridge between $\overline{v}_t|_{\partial U_s}$ and $\overline{u}_i|_{\partial U_s}$ in the NPC space $\mathcal{C}(X)$. That is, we are interpolating along geodesics connecting $\overline{v}_t(x), \overline{u}_i(x)$ in the NPC space $\mathcal{C}(X)$ and not along geodesics in X. By [41] (Lemma 3.12) and the equivalence of the energies for a map f and its lift $\overline{f}$,

$$
\begin{aligned}
{}^{D}E^{v_i}[\partial U_s \times [0,\rho]] &\leq \frac{\rho}{2}\left({}^{D}E^{\overline{v}_t}[\partial U_s] + {}^{D}E^{\overline{u}_i}[\partial U_s]\right) + \frac{1}{\rho}\int_{\partial U_s} D^2([v_t,1],[u_i,1])d\sigma \\
&= \frac{\rho}{2}\left({}^{d}E^{v_t}[\partial U_s] + {}^{d}E^{u_i}[\partial U_s]\right) + \frac{1}{\rho}\int_{\partial U_s} D^2([v_t,1],[u_i,1])d\sigma.
\end{aligned}
$$

By (4.2), and since $v_t = u$ on $B_{r/2}\backslash U_t$, for $s \in [2t/3, 3t/4]$ the average values of the tangential energies of v_t and u_i on ∂U_s are bounded above by $Ct/(3t/4 - 2t/3) = 12C$. Moreover, since $u_i(B_{r/2}), v_t(B_{r/2}) \subset \mathcal{B}_\rho(P)$, (4.3) implies that for all $x \in B_{r/2}\backslash U_t$,

$$
D^2(\overline{u}_i(x), \overline{v}_t(x)) = 2(1 - \cos d(u_i(x), v_t(x))) \leq d^2(u_i(x), v_t(x)) < \varepsilon. \tag{4.5}
$$

Thus, there exists $C' > 0$ depending only on g such that for every $s \in [2t/3, 3t/4]$,

$$
\int_{\partial U_s} D^2([v_t,1],[u_i,1])d\sigma < C'\varepsilon.
$$

Note that for each $\varepsilon > 0$, the bound above depends on I but not on t. Now, we first choose an $s \in (2t/3, 3t/4)$ such that ${}^{d}E^{v_t}[\partial U_s] + {}^{d}E^{u_i}[\partial U_s] \leq 24C$. Next, pick $0 < \mu \ll 1$ such that $[s, s+\mu t] \subset [2t/3, 3t/4]$ and $12C\mu t < \delta/2$. For this t, μ, decrease ε if necessary (by increasing I) such that

$$
\begin{aligned}
{}^{D}E^{v_i}[\partial U_s \times [0,\mu t]] &= \frac{\mu t}{2}\left({}^{d}E^{v_t}[\partial U_s] + {}^{d}E^{u_i}[\partial U_s]\right) + \frac{1}{\mu t}\int_{\partial U_s} D^2([v_t,1],[u_i,1])d\sigma \\
&< 24C\mu t/2 + C'\varepsilon/(\mu t) \\
&< \delta.
\end{aligned}
$$

Now, define $\tilde{v}_i : B_{r/2} \to \mathcal{C}(X)$ such that on U_s, $\tilde{v}_i$ is the conformally dilated map of $\overline{v}_t$ so that $\tilde{v}_i|_{\partial U_{s+\mu t}} = \overline{v}_t|_{\partial U_s}$. On $U_s\backslash U_{s+\mu t}$, let $\tilde{v}_i$ be the bridging map v_i, reparametrized in the second factor from $[0, \mu t]$ to $[s, s+\mu t]$. Finally, on $B_{r/2}\backslash U_s$, let $\tilde{v}_i = \overline{u}_i$. Then, for all $i \geq I$,

$$ {}^{D}E^{\tilde{v}_i}[B_{r/2}] \leq {}^{d}E^{v_t}[B_{r/2}] + \delta + {}^{d}E^{u_i}[B_{r/2} \backslash U_s]. \tag{4.6}$$

While the map $\tilde{v}_i$ agrees with $\overline{u}_i$ on $\partial B_{r/2}$, it is not a competitor for u_i into X since $\tilde{v}_i$ maps into $\mathcal{C}(X)$. However, by defining $\underline{v}_i : B_{r/2} \to X$ such that $\tilde{v}_i(x) = [\underline{v}_i(x), h(x)]$, $\underline{v}_i$ is a competitor. Note that for all $x \in \partial U_s$, (4.5) implies that $h(x) \geq 1 - \sqrt{\varepsilon}$. Therefore, on the bridging strip we may estimate the change in energy under the projection map by first observing the pointwise bound

$$\begin{aligned}
D^2(\tilde{v}_i(x), \tilde{v}_i(y)) &= D^2([\underline{v}_i(x), h(x)], [\underline{v}_i(y), h(y)]) \\
&= h(x)^2 + h(y)^2 - 2h(x)h(y)\cos(d(\underline{v}_i(x), \underline{v}_i(y))) \\
&= (h(x) - h(y))^2 + 2h(x)h(y)(1 - \cos(d(\underline{v}_i(x), \underline{v}_i(y)))) \\
&\geq 2(1 - \sqrt{\varepsilon})^2(1 - \cos(d(\underline{v}_i(x), \underline{v}_i(y)))) \\
&= (1 - \sqrt{\varepsilon})^2 D^2([\underline{v}_i(x), 1], [\underline{v}_i(y), 1]).
\end{aligned}$$

Therefore,

$$ {}^{d}E^{\underline{v}_i}[B_{r/2}] = {}^{D}E^{[\underline{v}_i,1]}[B_{r/2}] \leq \left(1 - \sqrt{\varepsilon}\right)^{-2} \, {}^{D}E^{\tilde{v}_i}[B_{r/2}]. \tag{4.7}$$

Since $\underline{v}_i$ is a competitor for u_i on $B_{r/2}$, (4.7), (4.6), (4.4), and (4.2) imply that

$$ {}^{d}E^{u_i}[B_{r/2}] \leq \left(1 - \sqrt{\varepsilon}\right)^{-2} \, {}^{D}E^{\tilde{v}_i}[B_{r/2}] \leq \left(1 - \sqrt{\varepsilon}\right)^{-2} \, \left({}^{d}E^{w_0}[B_{r/2}] + 2\delta + Ct\right)$$

Since for any $\varepsilon, \delta > 0$, by choosing $t > 0$ sufficiently small and $I \in \mathbb{N}$ large enough, the previous estimate holds for all $i \geq I$, the inequality

$$\limsup_{i \to \infty} {}^{d}E^{u_i}[B_{r/2}] \leq {}^{d}E^{w_0}[B_{r/2}]$$

then implies the result.

$\square$

4.3 Monotonicity and removable singularity theorem

We first show the removable singularity theorem for harmonic maps into small balls. Note that the first theorem of this section is true for domains of dimension $n \geq 2$, but all other results require the domain dimension $n = 2$.

Theorem 4.3.1. *Let $u : B_r(p) \setminus \{p\} \to \mathcal{B}_\rho(P) \subset X$ be a finite energy harmonic map, where ρ is as in Lemma 4.2.1 and $\dim(B_r(p)) = n$. Then u can be extended on $B_r(p)$ as the unique energy minimizer among all maps in $W^{1,2}_u(B_r(p), \mathcal{B}_\rho(P))$.*

Proof. Let $v \in W^{1,2}_u(B_r(p), \mathcal{B}_\rho(P))$ minimize the energy. It suffices to show that $u = v$ on $B_r(p) \setminus \{p\}$. Since u is harmonic, there exists a locally finite countable open cover $\{U_i\}$ of $B_r(p) \setminus \{p\}$, and $\rho_i > 0, P_i \in \mathcal{B}_\rho(P)$ such that $u|_{U_i}$ minimizes energy among all maps in $W^{1,2}_u(U_i, \mathcal{B}_{\rho_i}(P_i))$. Let

$$F = \sqrt{\frac{1 - \cos d}{\cos R^u \cos R^v}}$$

where $d(x) = d(u(x), v(x))$ and $R^u = d(u, P), R^v = d(v, P)$. By Theorem B.2.4,

$$\operatorname{div}(\cos R^u \cos R^v \nabla F) \geq 0$$

holds weakly on each U_i. Therefore, for a partition of unity $\{\varphi_i\}$ subordinate to the cover $\{U_i\}$ and for any test function $\eta \in C^\infty_c(B_r(p) \setminus \{p\})$,

$$-\int_{B_r(p)\setminus\{p\}} \nabla\eta \cdot (\cos R^u \cos R^v \nabla F)\, d\mu_g = -\sum_i \int_{U_i} \nabla(\varphi_i \eta) \cdot (\cos R^u \cos R^v \nabla F)\, d\mu_g \geq 0, \tag{4.8}$$

where we use $\sum_i \varphi_i = 1$ and $\sum_i \nabla\varphi_i = 0$.

Using polar coordinates in $B_r(p)$ centered at p, for $0 < \epsilon \ll 1$, we define

$$\phi_\epsilon = \begin{cases} 0 & r \le \epsilon^2 \\ \frac{\log r - \log \epsilon^2}{-\log \epsilon} & \epsilon^2 \le r \le \epsilon \\ 1 & \epsilon \le r \end{cases} .$$

Letting ω_{n-1} denote the volume of the unit $(n-1)$-dimensional sphere, note that

$$\int_{B_r(p)} |\nabla \phi_\epsilon|^2 \, d\mu_g = \frac{\omega_{n-1}}{(\log \epsilon)^2} \int_{\epsilon^2}^{\epsilon} r^{n-3} \, dr + o(\epsilon) \to 0 \qquad \text{as } \epsilon \to 0.$$

Therefore, for $\eta \in C_c^\infty(B_r(p))$,

$$\begin{aligned}
& -\int_{B_r(p)} \phi_\epsilon \nabla \eta \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g \\
& = -\int_{B_r(p)} \nabla(\eta \phi_\epsilon) \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g + \int_{B_r(p)} \eta \nabla \phi_\epsilon \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g \\
& \ge \int_{B_r(p) \setminus \{p\}} \eta \nabla \phi_\epsilon \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g \qquad \text{(by (4.8))} \\
& \ge -\left(\int_{B_r(p) \setminus \{p\}} |\nabla \phi_\epsilon|^2 \, d\mu_g \right)^{\frac{1}{2}} \left(\int_{B_r(p) \setminus \{p\}} \eta^2 |\cos R^u \cos R^v \nabla F|^2 \, d\mu_g \right)^{\frac{1}{2}},
\end{aligned}$$

by Hölder's inequality. The last line converges to zero as $\epsilon \to 0$ because d, R^u, R^v are bounded by the compactness of $\overline{\mathcal{B}_\rho(P)}$ and $\int_{B_r(p) \setminus \{p\}} |\nabla F|^2 \, d\mu_g$ is bounded by energy convexity. We conclude that

$$-\int_{B_r(p)} \nabla \eta \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g = -\lim_{\epsilon \to 0} \int_{B_r(p)} \phi_\epsilon \nabla \eta \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g \ge 0,$$

and hence $\operatorname{div}(\cos R^u \cos R^v \nabla F) \ge 0$ holds weakly on $B_r(p)$.

Since $d(u(x), v(x)) = 0$ on $\partial B_r(p)$, by the maximum principle $d(u(x), v(x)) \equiv 0$ in $B_r(p)$. This implies that $u \equiv v$ is the unique energy minimizer.

□

Remark 4.3.2. Note that Theorem 4.3.1 implies that if $u : \Omega \to \mathcal{B}_\rho(P)$ is harmonic, then u is energy minimizing.

From this point on we assume our domain is of dimension 2. Recall the construction in [40] and [7] of a continuous, symmetric, bilinear, non-negative tensorial operator

$$\pi^u : \Gamma(T\Omega) \times \Gamma(T\Omega) \to L^1(\Omega) \tag{4.9}$$

associated with a $W^{1,2}$-map $u : \Omega \to X$ where $\Gamma(T\Omega)$ is the space of Lipschitz vector fields on Ω defined by

$$\pi^u(Z, W) := \frac{1}{4}|u_*(Z+W)|^2 - \frac{1}{4}|u_*(Z-W)|^2$$

where $|u_*(Z)|^2$ is the directional energy density function (cf. [40, Section 1.8]). This generalizes the notion of the pullback metric for maps into a Riemannian manifold, and hence we shall refer to $\pi = \pi^u$ also as the pullback metric for u.

Definition 4.3.3. If Σ is a Riemann surface, then $u \in W^{1,2}(\Sigma, X)$ is *(weakly) conformal* if

$$\pi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right) = \pi\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\right) \text{ and } \pi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = 0,$$

where $z = x_1 + ix_2$ is a local complex coordinate on Σ.

For a conformal harmonic map $u : \Sigma \to X$ with conformal factor $\lambda = \frac{1}{2}|\nabla u|^2$, and any open sets $S \subset \Sigma$ and $\mathcal{O} \subset X$, define

$$A(u(S) \cap \mathcal{O}) := \int_{u^{-1}(\mathcal{O}) \cap S} \lambda \, d\mu_g,$$

where $d\mu_g$ is the area element of (Σ, g).

Theorem 4.3.4 (Monotonicity)**.** *There exist constants c, C such that if $u : \Sigma \to X$ is a non-constant conformal harmonic map from a Riemann surface Σ into a compact locally CAT(1) space (X, d), then for any $p \in \Sigma$ and $0 < \sigma < \sigma_0 = \min\{\rho, d(u(p), u(\partial\Sigma))\}$, the following function is increasing:*

$$\sigma \mapsto \frac{e^{c\sigma^2} A(u(\Sigma) \cap \mathcal{B}_\sigma(u(p)))}{\sigma^2},$$

and

$$A(u(\Sigma) \cap \mathcal{B}_\sigma(u(p))) \geq C\sigma^2.$$

Proof. Since Σ is locally conformally Euclidean and the energy is conformally invariant, without loss of generality, we may assume that the domain is Euclidean. Fix $p \in \Sigma$ and let $R(x) = d(u(x), u(p))$. Since u is continuous and locally energy minimizing, by [60, Proposition 1.17], [7, Lemma 4.3] we have that the following differential inequality holds weakly on $u^{-1}(\mathcal{B}_\rho(u(p)))$:

$$\frac{1}{2}\Delta R^2 \geq (1 - O(R^2))|\nabla u|^2. \tag{4.10}$$

Let $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ be any smooth nonincreasing function such that $\zeta(t) = 0$ for $t \geq 1$, and let $\zeta_\sigma(t) = \zeta(\frac{t}{\sigma})$. By (4.10), for $\sigma < \sigma_0$ we have

$$\begin{aligned} -\int_\Sigma \nabla R^2 \cdot \nabla(\zeta_\sigma(R))\, dx_1 dx_2 &\geq 2\int_\Sigma \zeta_\sigma(R)\,(1 - O(R^2))|\nabla u|^2\, dx_1 dx_2 \\ &= 4\int_\Sigma \zeta_\sigma(R)\,(1 - O(R^2))\,\lambda\, dx_1 dx_2. \end{aligned}$$

Therefore,

$$\begin{aligned} 2\int_\Sigma \zeta_\sigma(R)\,(1 - O(R^2))\,\lambda\, dx_1 dx_2 &\leq -\int_\Sigma R\,\nabla R \cdot \nabla(\zeta_\sigma(R))\, dx_1 dx_2 \\ &= -\int_\Sigma \frac{R}{\sigma}\,\zeta'\left(\frac{R}{\sigma}\right)|\nabla R|^2\, dx_1 dx_2 \\ &\leq -\int_\Sigma \frac{R}{\sigma}\,\zeta'\left(\frac{R}{\sigma}\right)\frac{1}{2}|\nabla u|^2\, dx_1 dx_2 \\ &= -\int_\Sigma \frac{R}{\sigma}\zeta'\left(\frac{R}{\sigma}\right)\,\lambda\, dx_1 dx_2 \\ &= \int_\Sigma \sigma\,\frac{d}{d\sigma}(\zeta_\sigma(R))\,\lambda\, dx_1 dx_2 \\ &= \sigma\frac{d}{d\sigma}\int_\Sigma \zeta_\sigma(R)\,\lambda\, dx_1 dx_2, \end{aligned}$$

where in the second inequality we have used that $\zeta' \leq 0$ and $|\nabla R|^2 \leq \frac{1}{2}|\nabla u|^2$, since

u is conformal. Set $f(\sigma) = \int_\Sigma \zeta_\sigma(R)\,\lambda\,dx_1 dx_2$. We have shown that

$$2(1 - O(\sigma^2))f(\sigma) \le \sigma f'(\sigma).$$

Integrating this, we conclude that there exist $c > 0$ such that the function

$$\sigma \mapsto \frac{e^{c\sigma^2} f(\sigma)}{\sigma^2} \tag{4.11}$$

is increasing for all $0 < \sigma < \sigma_0$. Approximating the characteristic function of $[-1, 1]$, and letting ζ be the restriction to $\mathbb{R}^+$, it then follows that

$$\frac{e^{c\sigma^2} A(u(\Sigma) \cap \mathcal{B}_\sigma(u(p)))}{\sigma^2}$$

is increasing in σ for $0 < \sigma < \sigma_0$.

Since $\lambda = \frac{1}{2}|\nabla u|^2 \in L^1(\Sigma, \mathbb{R})$,

$$\lim_{r\to 0} \frac{\int_{B_r(x)} \lambda\,dx_1 dx_2}{\pi r^2} = \lambda(x), \qquad \text{a.e. } x \in \Sigma \tag{4.12}$$

by the Lebesgue-Besicovitch Differentiation Theorem. Since u is conformal, for every $\omega \in \mathbb{S}^1$,

$$\lambda(x) = \lim_{t\to 0} \frac{d^2(u(x + t\omega), u(x))}{t^2}, \qquad \text{a.e. } x \in \Sigma \tag{4.13}$$

([40, Theorem 1.9.6 and Theorem 2.3.2]). Since u is locally Lipschitz [7, Theorem 1.2], by an argument as in the proof of Rademacher's Theorem ([20, p. 83-84]),

$$\lambda(x) = \lim_{y\to x} \frac{d^2(u(y), u(x))}{|y - x|^2} \tag{4.14}$$

for almost every $x \in \Sigma$. To see this, choose $\{\omega_k\}_{k=1}^\infty$ to be a countable, dense subset of $\mathbb{S}^1$. Set

$$S_k = \{x \in \Sigma\ : \lim_{t\to 0} \frac{d(u(x + t\omega_k), u(x))}{t} \text{ exists, and is equal to } \sqrt{\lambda(x)}\}$$

for $k = 1, 2, \ldots$ and let

$$S = \cap_{k=1}^{\infty} S_k.$$

Observe that $\mathcal{H}^2(\Sigma \setminus S) = 0$. Fix $x \in S$, and let $\varepsilon > 0$. Choose N sufficiently large such that if $\omega \in \mathbb{S}^1$ then

$$|\omega - \omega_k| < \frac{\varepsilon}{2\mathrm{Lip}(u)}$$

for some $k \in \{1, \ldots, N\}$. Since

$$\lim_{t \to 0} \frac{d(u(x + t\omega_k), u(x))}{t} = \sqrt{\lambda(x)}$$

for $k = 1, \ldots, N$, there exists $\delta > 0$ such that if $|t| < \delta$ then

$$\left| \frac{d(u(x + t\omega_k), u(x))}{t} - \sqrt{\lambda(x)} \right| < \frac{\varepsilon}{2}$$

for $k = 1, \ldots, N$. Consequently, for each $\omega \in \mathbb{S}^1$ there exists $k \in \{1, \ldots, N\}$ such that

$$\begin{aligned}
&\left| \frac{d(u(x + t\omega), u(x))}{t} - \sqrt{\lambda(x)} \right| \\
&\leq \left| \frac{d(u(x + t\omega_k), u(x))}{t} - \sqrt{\lambda(x)} \right| + \left| \frac{d(u(x + t\omega), u(x))}{t} - \frac{d(u(x + t\omega_k), u(x))}{t} \right| \\
&\leq \left| \frac{d(u(x + t\omega_k), u(x))}{t} - \sqrt{\lambda(x)} \right| + \left| \frac{d(u(x + t\omega), u(x + t\omega_k))}{t} \right| \\
&< \frac{\varepsilon}{2} + \mathrm{Lip}(u)|\omega - \omega_k| \\
&< \varepsilon.
\end{aligned}$$

Therefore the limit in (4.14) exists, and (4.14) holds, for almost every $x \in \Sigma$.

The zero set of λ is of Hausdorff dimension zero by [48]. At points where $\lambda(x) \neq 0$ and (4.14) holds, we have that for any $\varepsilon > 0$

$$u(B_{\frac{\sigma}{(1+\varepsilon)\sqrt{\lambda}}}(x)) \subset u(\Sigma) \cap \mathcal{B}_{\sigma}(u(x))$$

if σ is sufficiently small. Therefore by (4.12),

$$\Theta(x) := \lim_{\sigma \to 0} \frac{A(u(\Sigma) \cap \mathcal{B}_\sigma(u(x)))}{\pi \sigma^2} \geq 1, \quad \text{a.e. } x \in \Sigma. \tag{4.15}$$

By the monotonicity of (4.11), $\Theta(x)$ exists for every $x \in \Sigma$, and $\Theta(x)$ is upper semicontinuous since it is a limit of continuous functions (the density at a given radius is a continuous function of x). Therefore, $\Theta(x) \geq 1$ for every $x \in \Sigma$. Together with the monotonicity of (4.11), it follows that

$$A(u(\Sigma) \cap \mathcal{B}_\sigma(u(p))) \geq C\sigma^2$$

for $0 < \sigma < \sigma_0$. □

Remark 4.3.5. Note that if $u : M \to \mathcal{B}_\rho(P)$ is a harmonic map from a compact Riemannian manifold M, then u must be constant. This follows from the maximum principle, since equation (4.10) implies that $R^2(x) = d^2(u(x), P)$ is subharmonic.

For a *conformal* harmonic map from a surface into a Riemannian manifold, continuity follows easily using monotonicity ([57, Theorem 10.4], [29], [33, Theorem 9.3.2]). By Theorem 4.3.4, using this idea we can prove the following removable singularity result for conformal harmonic maps into a CAT(1) space.

Theorem 4.3.6 (Removable singularity)**.** *If $u : \Sigma \setminus \{p\} \to X$ is a conformal harmonic map of finite energy from a Riemann surface Σ into a compact locally CAT(1) space (X, d), then u extends to a locally Lipschitz harmonic map $u : \Sigma \to X$.*

Proof. Let B_r denote $B_r(p)$, the geodesic ball of radius r centered at the point p in Σ, and let $C_r = \partial B_r$ denote the circle of radius r centered at p. By the Courant-Lebesgue Lemma, there exists a sequence $r_i \searrow 0$ so that

$$L_i = L(u(C_{r_i})) := \int_{C_{r_i}} \sqrt{\lambda}\, ds_g \to 0$$

as $i \to \infty$, where ds_g denotes the induced measure on $C_{r_i} = \partial B_{r_i}$ from the metric g on Σ. Since $E(u) < \infty$, $\lambda = \frac{1}{2}|\nabla u|^2$ is an L^1 function and, by the Dominated

Convergence Theorem,

$$A_i = A(u(B_{r_i} \setminus \{p\})) := \int_{B_{r_i} \setminus \{p\}} \lambda \, d\mu_g \to 0$$

as $i \to \infty$.

First we claim that there exists $P \in X$ such that $u(C_{r_i}) \to P$ with respect to the Hausdorff distance as $i \to \infty$. Let $d_{i,j} = d(u(C_{r_i}), u(C_{r_j}))$. Suppose $i < j$ so $r_i > r_j$, and choose $Q \in u(B_{r_i} \setminus \bar{B}_{r_j})$ such that $d(Q, u(C_{r_i}) \cup u(C_{r_j})) \geq d_{i,j}/2$. For $\sigma = \min\{\frac{d_{i,j}}{3}, \frac{\rho}{2}\}$, by monotonicity (Theorem 4.3.4),

$$A(u(B_{r_i} \setminus \bar{B}_{r_j}) \cap \mathcal{B}_\sigma(Q)) \geq C\sigma^2.$$

Since $A(u(B_{r_i} \setminus \bar{B}_{r_j}) \cap \mathcal{B}_\sigma(Q)) \leq A(u(B_{r_i} \setminus \{p\})) = A_i$, it follows that $\sigma \leq c\sqrt{A_i} \to 0$ as $i \to \infty$, and we must have $d_{i,j} \to 0$. Therefore any sequence of points $P_i \in u(C_{r_i})$ is a Cauchy sequence since

$$d(P_i, P_j) \leq d_{i,j} + L_i + L_j \to 0$$

as i, $j \to \infty$. Hence, there exists $P \in X$ independent of the sequence, such that $P_i \to P$.

Finally, we claim that $\lim_{x \to p} u(x) = P$. It follows from this that we may extend u continuously to Σ by defining $u(p) = P$. To prove the claim, consider a sequence $x_i \in \Sigma \setminus \{p\}$ such that $x_i \to p$. We want to show that $u(x_i) \to P$. Suppose $x_i \in B_{r_{j(i)}} \setminus \bar{B}_{r_{j(i)+1}}$ for some $j(i)$, and let $d_i = d(u(x_i), u(C_{r_{j(i)}}) \cup u(C_{r_{j(i)+1}}))$. For $\sigma = \min\{\frac{d_i}{3}, \frac{\rho}{2}\}$, by monotonicity (Theorem 4.3.4),

$$A(u(B_{r_{j(i)}} \setminus \bar{B}_{r_{j(i)+1}}) \cap \mathcal{B}_\sigma(u(x_i))) \geq C\sigma^2.$$

Therefore, $\sigma < c\sqrt{A_{j(i)}} \to 0$ as $i \to \infty$, and we must have $d(u(x_i), u(C_{r_{j(i)}}) \cup u(C_{r_{j(i)+1}})) \to 0$. It follows that $u(x_i) \to P$ and u extends continuously to Σ.

We may now apply Theorem 4.3.1 to show that u is energy minimizing at p. Since u is continuous, there exists $\delta > 0$ such that $u(B_\delta) \subset \mathcal{B}_\rho(Q) \subset X$. By Theorem

4.3.1, u is the unique energy minimizer in $W^{1,2}_u(B_\delta, \mathcal{B}_\rho(Q))$. Hence u is locally energy minimizing on Σ and by [7, Theorem 1.2], u is locally Lipschitz on Σ. $\square$

The following is derived using only domain variations as in [57, Lemma 1.1] (using [40, Theorem 2.3.2] to justify the computations involving change of variables) and is independent of the curvature of the target space (see for example, [28, (2.3) page 193]).

Lemma 4.3.7. *Let $u : \Sigma \to X$ be a harmonic map from a Riemann surface into a locally CAT(1) space. The Hopf differential*

$$\Phi(z) = \left[\pi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right) - \pi\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\right) - 2i\pi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right] dz^2,$$

where $z = x_1 + ix_2$ is a local complex coordinate on Σ and π is the pull-back inner product, is holomorphic.

Corollary 4.3.1. *Let $u : \mathbb{C} \to X$ be a harmonic map of finite energy and (X, d) be a compact locally CAT(1) space. Then u extends to a locally Lipschitz harmonic map $u : \mathbb{S}^2 \to X$.*

Proof. Let $p : \mathbb{S}^2 \setminus \{n\} \to \mathbb{R}^2$ be stereographic projection from the north pole $n \in \mathbb{S}^2$. Set $\hat{u} = u \circ p : \mathbb{S}^2 \setminus \{n\} \to X$. We will show that n is a removable singularity.

Let $\varphi = \pi(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) - \pi(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}) - 2i\pi(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$. By Lemma 4.3.7, the Hopf differential $\Phi(z) = \varphi(z)dz^2$ is holomorphic on $\mathbb{C}$. By assumption,

$$E(u) = \int_{\mathbb{R}^2} \left(\|u_*(\frac{\partial}{\partial x_1})\|^2 + \|u_*(\frac{\partial}{\partial x_2})\|^2 \right) dx_1 dx_2 < \infty$$

and therefore

$$\int_{\mathbb{R}^2} |\varphi| \, dx_1 dx_2 \leq 2E(u) < \infty.$$

Thus $|\varphi| \in L^1(\mathbb{C}, \mathbb{R})$ and is subharmonic, and hence $\varphi \equiv 0$ and u is conformal. Then by Theorem 4.3.6, u extends to a locally Lipschitz harmonic map $u : \mathbb{S}^2 \to X$. $\square$

4.4 Harmonic Replacement Construction

In this section we prove the main theorem:

Theorem. 1.3.1 *Let Σ be a compact Riemann surface, X a compact locally CAT(1) space and $\varphi \in C^0 \cap W^{1,2}(\Sigma, X)$. Then either there exists a harmonic map $u : \Sigma \to X$ homotopic to φ or a nontrivial conformal harmonic map $v : \mathbb{S}^2 \to X$.*

Lemma 4.4.1 (Jost's covering lemma, [33] Lemma 9.2.6)**.** *For a compact Riemannian manifold Σ, there exists $\Lambda = \Lambda(\Sigma) \in \mathbb{N}$ with the following property: for any covering*

$$\Sigma \subset \bigcup_{i=1}^{m} B_r(x_i)$$

by open balls, there exists a partition $I^1, \dots I^\Lambda$ of the integers $\{1, \dots, m\}$ such that for any $l \in \{1, \dots, \Lambda\}$ and two distinct elements i_1, i_2 of I^l,

$$B_{2r}(x_{i_1}) \cap B_{2r}(x_{i_2}) = \emptyset.$$

Definition 4.4.2. *For each $k = 0, 1, 2, \dots,$ we fix a covering*

$$\mathcal{O}_k = \{B_{2^{-k}}(x_{k,i})\}_{i=1}^{m_k}$$

of Σ by balls of radius 2^{-k}. Furthermore, let $I_k^1, \dots, I_k^\Lambda$ be the disjoint subsets of $\{1, \dots, m_k\}$ as in Lemma 4.4.1; in other words, for every $l \in \{1, \dots, \Lambda\}$,

$$B_{2^{-k+1}}(x_{k,i_1}) \cap B_{2^{-k+1}}(x_{k,i_2}) = \emptyset, \quad \forall i_1, i_2 \in I_k^l, \quad i_1 \neq i_2. \tag{4.16}$$

By the Vitali Covering Lemma, we can ensure that

$$B_{2^{-k-3}}(x_{k,i_1}) \cap B_{2^{-k-3}}(x_{k,i_2}) = \emptyset, \quad \forall i_1, i_2 \in \{1, \dots, m_k\}, \quad i_1 \neq i_2. \tag{4.17}$$

Let Σ be a compact Riemann surface. By uniformization, we can endow Σ with a Riemannian metric of constant Gaussian curvature $+1$, 0 or -1. Let $\Lambda = \Lambda(\Sigma)$ be as in Lemma 4.4.1 and $\rho = \rho(X) > 0$ be as in Lemma 4.2.1. We inductively define a sequence of numbers

$$\{r_n\} \subset 2^{-\mathbb{N}} := \{1, 2^{-1}, 2^{-2}, \dots\}$$

and a sequence of finite energy maps

$$\{u_n^l : \Sigma \to X\}$$

for $l = 0, \dots, \Lambda$, $n = 1, \dots, \infty$ as follows:

Initial Step 0: Fix $\kappa_0 \in \mathbb{N}$ such that $B_{2^{-\kappa_0}}(x)$ is homeomorphic to a disk for all $x \in \Sigma$. Let $u_0^0 := \varphi \in C^0 \cap W^{1,2}(\Sigma, X)$, and let

$$r_0' = \sup\{r > 0 : \forall\, x \in \Sigma, \exists\, P \in X \text{ such that } u_0^0(B_{2r}(x)) \subset \mathcal{B}_{3^{-\Lambda}\rho}(P)\}$$

and $k_0' > 0$ be such that

$$2^{-k_0'} \le r_0' < 2^{-k_0'+1}.$$

Define

$$r_0 = 2^{-k_0} = \min\{2^{-k_0'}, 2^{-\kappa_0}\},$$

and let

$$\mathcal{O}_{k_0} = \{B_{r_0}(x_{k_0,i})\}_{i=1}^{m_{k_0}} \text{ and } I_{k_0}^1, \dots, I_{k_0}^\Lambda$$

be as in Definition 4.4.2. For $l \in \{1, \dots, \Lambda\}$ inductively define $u_0^l : \Sigma \to X$ from u_0^{l-1} by setting

$$u_0^l = \begin{cases} u_0^{l-1} & \text{in } \Sigma \backslash \bigcup_{i \in I_{k_0}^l} B_{2r_0}(x_{k_0,i}) \\ {}^{Dir}u_0^{l-1} & \text{in } B_{2r_0}(x_{k_0,i}),\ i \in I_{k_0}^l \end{cases}$$

where ${}^{Dir}u_0^{l-1}$ is the unique Dirichlet solution in $W^{1,2}_{u_0^{l-1}}(B_{2r_0}(x_{k_0,i}), \mathcal{B}_\rho(P))$ of Lemma 4.2.1. Here there are two things to check related to the definition of the Dirichlet solution. First, since $B_{2r_0}(x_{k_0,i_1}) \cap B_{2r_0}(x_{k_0,i_2}) = \emptyset$, $\forall\, i_1, i_2 \in I_{k_0}^l$ with $i_1 \neq i_2$ (cf. (4.16)), there is no issue of interaction between solutions at a single step so the map is well-defined if it exists. Second, we claim that $u_0^{l-1}(B_{2r_0}(x_{k_0,i})) \subset \mathcal{B}_{3^{-\Lambda+(l-1)}\rho}(P) \subset \mathcal{B}_\rho(P)$

for some $P \in X$ and thus the Dirichlet solution exists and is unique by Lemma 4.2.1. To verify the claim, first note that for each $i = 1, \ldots, m_{k_0}$ there exists $P \in X$ such that $u_0^1(B_{2r_0}(x_{k_0,i})) \subset \mathcal{B}_{3^{-\Lambda+1}\rho}(P)$. Indeed, if $B_{2r_0}(x_{k_0,i}) \cap B_{2r_0}(x_{k_0,j}) = \emptyset$ for all $j \in I_{k_0}^1$ then $u_0^1 = u_0^0$ on $B_{2r_0}(x_{k_0,i})$ and so $u_0^1(B_{2r_0}(x_{k_0,i})) = u_0^0(B_{2r_0}(x_{k_0,i})) \subset \mathcal{B}_{3^{-\Lambda}\rho}(P)$ for some P. On the other hand, if $B_{2r_0}(x_{k_0,i}) \cap B_{2r_0}(x_{k_0,j}) \neq \emptyset$ for one or more $j \in I_{k_0}^1$, then since $u_0^0(B_{2r_0}(x_{k_0,i})) \subset \mathcal{B}_{3^{-\Lambda}\rho}(P)$ for some P and $u_0^1(B_{2r_0}(x_{k_0,j})) \subset \mathcal{B}_{3^{-\Lambda}\rho}(P_j)$ for some P_j with $\mathcal{B}_{3^{-\Lambda}\rho}(P) \cap \mathcal{B}_{3^{-\Lambda}\rho}(P_j) \neq \emptyset$, it follows that $u_0^1(B_{2r_0}(x_{k_0,i})) \subset \mathcal{B}_{3^{-\Lambda+1}\rho}(P)$. Inductively, we may show that for each $i = 1, \ldots, m_{k_0}$ and $l \in \{1, \ldots, \Lambda\}$ there exists $P \in X$ such that $u_0^{l-1}(B_{2r_0}(x_{k_0,i})) \subset \mathcal{B}_{3^{-\Lambda+(l-1)}\rho}(P)$, as claimed.

Inductive Step n: Having defined

$$r_0, \ldots, r_{n-1} \in 2^{-\mathbb{N}},$$

and

$$u_\nu^0,\, u_\nu^1, \ldots, u_\nu^\Lambda : \Sigma \to X, \quad \nu = 0, 1, \ldots, n-1,$$

we set $u_n^0 = u_{n-1}^\Lambda$ and define

$$r_n \in 2^{-\mathbb{N}} \text{ and } \; u_n^1, \ldots, u_n^\Lambda$$

as follows. Let

$$r_n' = \sup\{r > 0 : \forall\, x \in \Sigma, \exists\, P \in X \text{ such that } u_n^0(B_{2r}(x)) \subset \mathcal{B}_{3^{-\Lambda}\rho}(P)\}$$

and $k_n' \in \mathbb{N}$ be such that

$$2^{-k_n'} \leq r_n' < 2^{-k_n'+1}.$$

Define

$$r_n = 2^{-k_n} = \min\{2^{-k_n'}, 2^{-\kappa_0}\}.$$

Let

$$\mathcal{O}_{k_n} = \{B_{r_n}(x_{k_n,i})\}_{i=1}^{m_{k_n}} \text{ and } I_{k_n}^1, \ldots, I_{k_n}^\Lambda$$

be as in Definition 4.4.2. Having defined $u_n^0, \dots, u_n^{l-1}$, we now define $u_n^l : \Sigma \to X$ by setting

$$u_n^l = \begin{cases} u_n^{l-1} & \text{in } \Sigma \backslash \bigcup_{i \in I_{k_n}^l} B_{2r_n}(x_{k_n,i}) \\ {}^{Dir}u_n^{l-1} & \text{in } B_{2r_n}(x_{k_n,i}),\ i \in I_{k_n}^l \end{cases}$$

where ${}^{Dir}u_n^{l-1}$ is the unique Dirichlet solution in $W^{1,2}_{u_n^{l-1}}(B_{2r_n}(x_{k_n,i}), \mathcal{B}_\rho(P))$ for some P of Lemma 4.2.1.

This completes the inductive construction of the sequence $\{u_n^l\}$. Note that

$$E(u_n^\Lambda) \le \cdots \le E(u_n^0) = E(u_{n-1}^\Lambda), \quad \forall\, n = 1, 2, \dots.$$

Thus, there exists E_0 such that

$$\lim_{n \to \infty} E(u_n^l) = E_0, \quad \forall\, l = 0, \dots, \Lambda. \tag{4.18}$$

We consider the following two cases separately:

CASE 1: $\liminf_{n \to \infty} r_n > 0$.

CASE 2: $\liminf_{n \to \infty} r_n = 0$.

For **CASE 1**, we prove that there exists a harmonic map $u : \Sigma \to X$ homotopic to $\varphi = u_0^0$. We will need the following two claims.

Claim 4.4.3. For any $l \in \{0, \dots \Lambda - 1\}$,

$$\lim_{n \to \infty} ||d(u_n^l, u_n^\Lambda)||_{L^2(\Sigma)} = 0.$$

Proof. Fix $l \in \{0, \dots, \Lambda - 1\}$. For $n \in \mathbb{N}$, $\lambda \in \{l+1, \dots, \Lambda\}$ and $i \in I_{k_n}^\lambda$, we apply Theorem B.2.1 with $u_0 = u_n^{\lambda-1}\big|_{B_{2r_n}(x_{k_n,i})}$, $u_1 = u_n^\lambda\big|_{B_{2r_n}(x_{k_n,i})}$ and $\Omega = B_{2r_n}(x_{k_n,i})$. Let $w : \Sigma \to X$ be the map defined as $w = u_n^\lambda = u_n^{\lambda-1}$ outside $\bigcup_{i \in I_{k_n}^\lambda} B_{2r_n}(x_{k_n,i})$ and the map corresponding to w in Theorem B.2.1 in each $B_{2r_n}(x_{k_n,i})$. Then

$$
\begin{aligned}
&(\cos^8 \rho) \int_{B_{2r_n}(x_{k_n,i})} \left| \nabla \frac{\tan \frac{1}{2} d(u_n^{\lambda-1}, u_n^{\lambda})}{\cos R} \right|^2 d\mu \\
&\quad \le \frac{1}{2} \left(\int_{B_{2r_n}(x_{k_n,i})} |\nabla u_n^{\lambda-1}|^2 d\mu + \int_{B_{2r_n}(x_{k_n,i})} |\nabla u_n^{\lambda}|^2 d\mu \right) - \int_{B_{2r_n}(x_{k_n,i})} |\nabla w|^2 d\mu.
\end{aligned}
$$

Summing over i, using that $w = u_n^{\lambda} = u_n^{\lambda-1}$ outside $\bigcup_{i \in I_{k_n}^{\lambda}} B_{2r_n}(x_{k_n,i})$, and applying the Poincaré inequality, we obtain

$$
\int_{\Sigma} d^2(u_n^{\lambda-1}, u_n^{\lambda}) d\mu \le C \left(\frac{1}{2} E(u_n^{\lambda-1}) + \frac{1}{2} E(u_n^{\lambda}) - E(w) \right),
$$

where here and henceforth C is a constant independent of n. Since u_n^{λ} is harmonic in $\bigcup_{i \in I_{k_n}^{\lambda}} B_{2r_n}(x_{k_n,i})$, we have $E(u_n^{\lambda}) \le E(w)$. Hence

$$
\int_{\Sigma} d^2(u_n^{\lambda-1}, u_n^{\lambda}) d\mu \le C \left(\frac{1}{2} E(u_n^{\lambda-1}) - \frac{1}{2} E(u_n^{\lambda}) \right).
$$

Thus,

$$
\begin{aligned}
\int_{\Sigma} d^2(u_n^{l}, u_n^{\Lambda}) d\mu &\le \int_{\Sigma} \left(\sum_{\lambda=l+1}^{\Lambda} d(u_n^{\lambda-1}, u_n^{\lambda}) \right)^2 d\mu \\
&\le (\Lambda - l)^2 \sum_{\lambda=l+1}^{\Lambda} \int_{\Sigma} d^2(u_n^{\lambda-1}, u_n^{\lambda}) d\mu \\
&\le C \sum_{\lambda=l+1}^{\Lambda} \left(E(u_n^{\lambda-1}) - E(u_n^{\lambda}) \right) \\
&= C \left(E(u_n^{l}) - E(u_n^{\Lambda}) \right).
\end{aligned}
$$

This proves the claim since $\lim_{n\to\infty} \left(E(u_n^{l}) - E(u_n^{\Lambda}) \right) = 0$ by (4.18). □

Claim 4.4.4. *Let $\epsilon > 0$ such that $3^{-\Lambda}\epsilon < \rho$, $l \in \{1, \ldots, \Lambda\}$ and $n \in \mathbb{N}$ be given. If*

$\delta \in (0, r_n)$ is such that

$$\sqrt{\frac{8\pi E(u_0^0)}{\log \delta^{-2}}} \le 3^{-\Lambda}\epsilon, \tag{4.19}$$

then

$$\forall\, x \in \bigcup_{\lambda=1}^{l} \bigcup_{i \in I_{k_n}^{\lambda}} B_{r_n}(x_{k_n,i}), \quad \exists\, P \in X \text{ such that } u_n^l(B_{\delta^\Lambda}(x)) \subset \mathcal{B}_{3\epsilon}(P).$$

In particular, for $l = \Lambda$, $\forall\, x \in \Sigma$, $\exists\, P \in X$ such that $u_n^{\Lambda}(B_{\delta^\Lambda}(x)) \subset \mathcal{B}_{3\epsilon}(P)$.

Proof. Fix ϵ, l, n and let δ be as in (4.19). For $x \in \bigcup_{\lambda=1}^{l} \bigcup_{i \in I_{k_n}^{\lambda}} B_{r_n}(x_{k_n,i})$, there exists $\lambda \in \{1, \ldots, l\}$ such that $x \in B_{r_n}(x_{k_n,i})$ for some $i \in I_{k_n}^{\lambda}$ and hence

$$B_{r_n}(x) \subset B_{2r_n}(x_{k_n,i}).$$

Since u_n^{λ} is harmonic in $B_{2r_n}(x_{k_n,i})$, it is harmonic in $B_{r_n}(x)$. By the Courant-Lebesgue Lemma, there exists

$$R_1(x) \in (\delta^2, \delta)$$

such that

$$u_n^{\lambda}(\partial B_{R_1(x)}(x)) \subset \mathcal{B}_{3^{-\Lambda}\epsilon}(P_1) \text{ for some } P_1 \in X.$$

Since u_n^{λ} is a Dirichlet solution and $3^{-\Lambda}\epsilon < \rho$, by Lemma 4.2.1

$$u_n^{\lambda}(B_{\delta^2}(x)) \subset u_n^{\lambda}(B_{R_1(x)}(x)) \subset \mathcal{B}_{3^{-\Lambda}\epsilon}(P_1).$$

Next, by the Courant-Lebesgue Lemma, there exists

$$R_2(x) \in (\delta^3, \delta^2)$$

such that

$$u_n^{\lambda+1}(\partial B_{R_2(x)}(x)) \subset \mathcal{B}_{3^{-\Lambda}\epsilon}(P_2') \text{ for some } P_2' \in X. \tag{4.20}$$

There are two cases to consider:

Case a. $B_{R_2(x)}(x) \cap \overline{\bigcup_{i \in I_{k_n}^{\lambda+1}} B_{2r_n}(x_{k_n,i})} = \emptyset$. In this case, $u_n^{\lambda+1} = u_n^\lambda$ in $B_{R_2(x)}(x)$. Since u_n^λ is harmonic on this ball,

$$u_n^{\lambda+1}(B_{R_2(x)}(x)) = u_n^\lambda(B_{R_2(x)}(x)) \subset u_n^\lambda(B_{\delta^2}(x)) \subset \mathcal{B}_{3^{-\Lambda}\epsilon}(P_1).$$

In this case we let $P_2 = P_1$.

Case b. $B_{R_2(x)}(x) \cap \bigcup_{i \in I_{k_n}^{\lambda+1}} B_{2r_n}(x_{k_n,i}) \neq \emptyset$. In this case, $u_n^{\lambda+1}$ is only piecewise harmonic on $B_{R_2(x)}(x)$. The regions of harmonicity are of two types. On the region $\Omega := B_{R_2(x)}(x) \backslash \overline{\bigcup_{i \in I_{k_n}^{\lambda+1}} B_{2r_n}(x_{k_n,i})}$, we have $u_n^{\lambda+1} = u_n^\lambda$. As in *Case a*, we conclude that the image of this region is contained in $B_{3^{-\Lambda}\epsilon}(P_1)$. All other regions, which we index Ω_i, have two smooth boundary components, one on the interior of $B_{R_2(x)}(x)$, which we label γ_i, and one on $\partial B_{R_2(x)}(x)$, which we label β_i. By construction $u_n^{\lambda+1} = u_n^\lambda$ on γ_i, thus

$$u_n^{\lambda+1}(\gamma_i) \subset \mathcal{B}_{3^{-\Lambda}\epsilon}(P_1).$$

Moreover, $u_n^{\lambda+1}(\beta_i) \subset \mathcal{B}_{3^{-\Lambda}\varepsilon}(P_2')$ by (4.20). Notice that in this case,

$$\mathcal{B}_{3^{-\Lambda}\epsilon}(P_1) \cap \mathcal{B}_{3^{-\Lambda}\epsilon}(P_2') \neq \emptyset.$$

Thus, by the triangle inequality there exists $P_2 \in X$ such that

$$u_n^{\lambda+1}(\cup_{i \in I_{k_n}^{\lambda+1}} \partial\Omega_i) \subset \mathcal{B}_{3^{-\Lambda+1}\epsilon}(P_2).$$

Since $u_n^{\lambda+1}$ is harmonic on each Ω_i,

$$u_n^{\lambda+1}(\cup_{i \in I_{k_n}^{\lambda+1}} \Omega_i) \subset \mathcal{B}_{3^{-\Lambda+1}\epsilon}(P_2).$$

Since $\overline{B_{R_2(x)}(x)} = \overline{\Omega} \cup \bigcup_{i \in I_{k_n}^{\lambda+1}} \overline{\Omega}_i$,

$$u_n^{\lambda+1}(B_{R_2(x)}(x)) \subset \mathcal{B}_{3^{-\Lambda+1}\epsilon}(P_2).$$

Thus, we have shown that in either *Case a* or *Case b*,

$$u_n^{\lambda+1}(B_{\delta^3}(x)) \subset u_n^{\lambda+1}(B_{R_2(x)}(x)) \subset \mathcal{B}_{3^{-\Lambda+1}\epsilon}(P_2).$$

After iterating this argument for $u_n^{\lambda+2}, \ldots, u_n^l$, we conclude that there exists $P_{l-\lambda+1} \in X$ such that

$$u_n^l(B_{\delta^\Lambda}(x)) \subset u_n^l(B_{\delta^{l-\lambda+2}}(x)) \subset \mathcal{B}_{3^{-\Lambda+l-\lambda}\epsilon}(P_{l-\lambda+1}) \subset \mathcal{B}_{3\epsilon}(P_{l-\lambda+1}).$$

Letting $P = P_{l-\lambda+1}$, we obtain the assertion of Claim 4.4.4. $\square$

Since $\liminf_{n\to\infty} r_n > 0$, there exist $k \in \mathbb{N}$ and an increasing sequence $\{n_j\}_{j=1}^\infty \subset \mathbb{N}$ such that $r_{n_j} = 2^{-k}$ (or equivalently $k_{n_j} = k$). In particular, the covering used for STEP n_j in the inductive construction of $u_{n_j}^0, \ldots, u_{n_j}^\Lambda$ is the same for all $j = 1, 2, \ldots$. Thus, we can use the following notation for simplicity:

$$\mathcal{O} = \mathcal{O}_{k_j}, I^l = I_{k_j}^l,\ B_i = B_{r_{n_j}}(x_{k_{n_j},i}) \text{ and } \ tB_i = B_{tr_{n_j}}(x_{k_{n_j},i}) \text{ for } t \in \mathbb{R}^+.$$

With this notation, Claim 4.4.4 implies that for a fixed $l \in \{1, \ldots, \Lambda\}$,

$$\{u_{n_j}^l\} \text{ is an equicontinuous family of maps on } B^l := \bigcup_{\lambda=1}^{l} \bigcup_{i \in I^\lambda} B_i. \tag{4.21}$$

In particular, $\{u_{n_j}^l\}$ is an equicontinuous family of maps in Σ. By taking a further subsequence if necessary, we can assume that

$$\exists u \in C^0(\Sigma, X) \text{ such that } u_{n_j}^\Lambda \rightrightarrows u. \tag{4.22}$$

We claim that for every $l \in \{1, \ldots, \Lambda\}$,

$$u_{n_j}^l \rightrightarrows u \text{ on } B^l \text{ where } u \text{ is as in (4.22).} \tag{4.23}$$

Indeed, if (4.23) is not true, consider a subsequence of $\{u_{n_j}^l\}$ that does not converge to u. By (4.21), we can assume (by taking a further subsequence if necessary) that

$$\exists\, v : B^l \to X \text{ such that } u^l_{n_j} \rightrightarrows v \neq u|_{B^l}.$$

Combining this with (4.22) and Claim 4.4.3, we conclude that

$$||d(v,u)||_{L^2(B^l)} = \lim_{j\to\infty} ||d(u^l_{n_j}, u^\Lambda_{n_j})||_{L^2(B^l)} \leq \lim_{j\to\infty} ||d(u^l_{n_j}, u^\Lambda_{n_j})||_{L^2(\Sigma)} = 0$$

which in turn implies that $u = v$. This contradiction proves (4.23).

Finally, we are ready to prove the harmonicity of u. For an arbitrary point $x \in \Sigma$, there exists $l \in \{1, \ldots, \Lambda\}$ and $i \in I^l$ such that $x \in B_i$. Since $u^l_{n_j}$ is energy minimizing in B_i and $u^l_{n_j} \rightrightarrows u$ in B_i by (4.23), Lemma 4.2.2 implies that u is energy minimizing in $\frac{1}{2}B_i$.

The map u is homotopic to φ since it is a uniform limit of $u^\Lambda_{n_j}$ each of which is homotopic to φ. This completes the proof for **CASE 1** as u is the desired harmonic map homotopic to φ.

For **CASE 2**, we prove that there exists a non-constant harmonic map $u : \mathbb{S}^2 \to X$.

Recall that we have endowed Σ with a metric g of constant Gaussian curvature that is identically $+1$, 0 or -1. Fix

$$y_* \in \Sigma$$

and a local conformal chart

$$\pi : U \subset \mathbb{C} \to \pi(U) = B_1(y_*) \subset \Sigma$$

such that

$$\pi(0) = y_*$$

and the metric $g = (g_{ij})$ of Σ expressed with respect to this local coordinates satisfies

$$g_{ij}(0) = \delta_{ij}. \tag{4.24}$$

For each n, the definition of r_n implies that we can find $y_n, y_n' \in \Sigma$ with

$$2r_n \leq d_g(y_n, y_n') \leq 4r_n$$

where d_g is the distance function on Σ induced by the metric g, and

$$d(u_n^0(y_n), u_n^0(y_n')) \geq 3^{-\Lambda}\rho.$$

Since Σ is a compact Riemannian surface of constant Gaussian curvature, there exists an isometry $\iota_n : \Sigma \to \Sigma$ such that $\iota_n(y_*) = y_n$. Define the conformal coordinate chart

$$\pi_n : U \subset \mathbb{C} \to \pi_n(U) = B_1(y_n) \subset \Sigma, \qquad \pi_n(z) := \iota_n \circ \pi(z).$$

Thus,

$$\pi_n(0) = y_n.$$

Define the dilatation map

$$\Psi_n : \mathbb{C} \to \mathbb{C}, \quad \Psi_n(z) = r_n z$$

and set $\Omega_n := \Psi_n^{-1} \circ \pi_n^{-1}(B_1(y_n)) \subset \mathbb{C}$ and

$$\tilde{u}_n^l : \Omega_n \to X, \quad \tilde{u}_n^l := u_n^l \circ \pi_n \circ \Psi_n.$$

Since $\liminf_{n\to\infty} r_n = 0$, there exists a subsequence

$$\{r_{n_j}\} \text{ such that } \lim_{j\to\infty} r_{n_j} = 0. \tag{4.25}$$

Thus, $\Omega_{n_j} \nearrow \mathbb{C}$. Furthermore, (4.24) implies that

$$\lim_{j\to\infty} \frac{d_g(y'_{n_j}, y_{n_j})}{|\pi_{n_j}^{-1}(y'_{n_j})|} = 1.$$

Hence, for $z_n = \Psi_n^{-1} \circ \pi_n^{-1}(y'_n)$,

$$2 \le \lim_{j\to\infty} |z_{n_j}| \le 4 \tag{4.26}$$

and

$$d(\tilde{u}^0_{n_j}(z_{n_j}), \tilde{u}^0_{n_j}(0)) = d(u^0_{n_j}(y'_{n_j}), u^0_{n_j}(y_{n_j})) \ge 3^{-\Lambda}\rho. \tag{4.27}$$

Additionally, by the conformal invariance of energy, we have that

$$E(\tilde{u}^l_n) = E(u^l_n\big|_{B_1(y_n)}) \le E(u^0_0). \tag{4.28}$$

For $R > 0$, let

$$D_R := \{z \in \mathbb{C} : |z| < R\}.$$

Since harmonicity is invariant under conformal transformations of the domain, we can follow **CASE 1** (cf. (**??**), (**??**) and (4.22)) and prove that

$$||d(\tilde{u}^\Lambda_{n-1}, \tilde{u}^\Lambda_n)||_{L^2(D_R)} \to 0,$$

$$\{\tilde{u}^\Lambda_n\}_{n=n_R}^\infty \text{ is an equicontinuous family in } D_R$$

for some n_R, and

$$\exists\, \tilde{u}_R : D_R \to X \;\text{ such that }\; \tilde{u}^\Lambda_n \rightrightarrows \tilde{u}_R \text{ in } D_R. \tag{4.29}$$

Below, we will prove harmonicity of $\tilde{u}_R$ by following a similar proof to **CASE 1**. We first need the following lemma.

Lemma 4.4.5. *Let $\mathcal{O}_{k_n}$ be as in Definition 4.4.2. For a fixed $R > 0$, there exists M*

independent of n such that for every $n \in \mathbb{N}$,

$$|\{i : B_{2^{-k_n}}(x_{k_n,i}) \cap (\pi_n \circ \Psi_n(D_R)) \neq \emptyset\}| \leq M.$$

Proof. By (4.24),

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(\pi_n \circ \Psi_n(D_{2R}))}{4\pi R^2 2^{-2k_n}} = 1$$

and

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(B_{2^{-k_n-3}}(x_{n,i}))}{\pi 2^{-2k_n-6}} = 1$$

where Vol is the volume in Σ. Let $J \subset \{1, \ldots, m_{k_n}\}$ be such that

$$J = \{i : B_{2^{-k_n}}(x_{k_n,i}) \cap (\pi_n \circ \Psi_n(D_R)) \neq \emptyset\}.$$

By (4.17), we have that for sufficiently large k_n,

$$\begin{aligned} |J|\pi 2^{-2k_n-6} &\leq 2\sum_{i\in J} \mathrm{Vol}(B_{2^{-k_n-3}}(x_{k_n,i})) \\ &\leq 2\mathrm{Vol}(\pi_n \circ \Psi_n(D_{2R})) \\ &\leq 16\pi R^2 2^{-2k_n}. \end{aligned}$$

Hence $|J| \leq R^2 2^{10}$ and $\{B_{2^{-k_n}}(x_{k_n,i})\}_{i\in J}$ covers D_R. $\square$

For each $B_{2^{-k_n}}(x_{k_n,i}) \in \mathcal{O}_{k_n}$, let

$$\tilde{B}_{n,i} := \Psi_n^{-1} \circ \pi_n^{-1}(B_{2^{-k_n}}(x_{k_n,i}))$$

and

$$2\tilde{B}_{n,i} := \Psi_n^{-1} \circ \pi_n^{-1}(B_{2^{-k_n+1}}(x_{k_n,i}))$$

for notational simplicity. After renumbering, Lemma 4.4.5 implies that there exists $M = M(R)$ such that

$$D_R \subset \bigcup_{i=1}^{M} \tilde{B}_{n,i}.$$

If we write

$$I^l_{k_n}(R) = \{i \in I^l_{k_n} : i \leq M\} \quad \forall\, l = 1, \ldots, \Lambda,$$

then

$$D_R \subset \bigcup_{l=1}^{\Lambda} \bigcup_{i \in I^l_{k_n}(R)} \tilde{B}_{n,i}.$$

Choose a subsequence of (4.25), which we will denote again by $\{n_j\}$, such that

$$\Psi_{n_j}^{-1} \circ \pi_{n_j}^{-1}(x_{k_{n_j},i}) \to \tilde{x}_i \quad \forall\, i \in \{1, \ldots, M\}$$

and such that for each $l = 1, \ldots, \Lambda$, the sets

$$\tilde{I}^l := I^l_{k_{n_j}}(R) = \{i \in I^l_{k_{n_j}} : i \leq M\}$$

are equal for all k_{n_j}. Unlike **CASE 1**, where $B_{r_{n_j}}(x_{k_{n_j},i})$ is the same ball B_i for all j, the sets $\tilde{B}_{n_1,i}, \tilde{B}_{n_2,i}, \ldots$ are not necessarily the same. Since the component functions of the pullback metric $\Psi^*_{n_j} g$ converge uniformly to those of the standard Euclidean metric g_0 on $\mathbb{C}$ by (4.24) and $\tilde{B}_{n_j,i}$ with respect to $\Psi^*_{n_j} g$ is a ball of radius 1, $\tilde{B}_{n_j,i}$ with respect to g_0 is close to being a ball of radius 1 in the following sense: for all $\epsilon > 0$, there exists J large enough such that for all $j \geq J$, $B_{1-\epsilon}(\tilde{x}_i) \subset \tilde{B}_{n_j,i}$ for $i = 1, \ldots, M$. Choose $\epsilon > 0$ sufficiently small such that $D_R \subset \bigcup_{i=1}^M B_{1-\epsilon}(\tilde{x}_i)$. Then choose J as above. Set

$$\tilde{B}_i := \bigcap_{j \geq J} \tilde{B}_{n_j,i} \supset B_{1-\epsilon}(\tilde{x}_i) \quad \text{and} \quad t\tilde{B}_i := \bigcap_{j \geq J} t\tilde{B}_{n_j,i} \text{ for } t \in \mathbb{R}^+.$$

Then

$$D_R \subset \bigcup_{i=1}^{M} \tilde{B}_i = \bigcup_{l=1}^{\Lambda} \bigcup_{i \in \tilde{I}^l(R)} \tilde{B}_i. \tag{4.30}$$

Using (4.30), we can now follow **CASE 1** (cf. (4.23)) to prove that for $l \in \{1, \ldots, \Lambda\}$,

$$\tilde{u}_n^l \rightrightarrows \tilde{u}_R \text{ on } \bigcup_{\lambda=1}^{l} \bigcup_{i \in \tilde{I}^\lambda} \tilde{B}_i \text{ where } \tilde{u}_R \text{ is as in (4.29).} \tag{4.31}$$

Let $x \in D_R$. There exists $l \in \{1, \ldots, \Lambda\}$ and $i \in \tilde{I}^l$ such that $x \in \tilde{B}_i$ by (4.30). Since harmonicity is invariant under conformal transformations of the domain, $\tilde{u}_{n_j}^l$ is a energy minimizing on $2\tilde{B}_{n_j,i}$. Since $\tilde{B}_i \subset \tilde{B}_{n_j,i} \subset 2\tilde{B}_{n_j,i}$ and $\tilde{u}_{n_j}^l \rightrightarrows \tilde{u}_R$ on $\tilde{B}_i$ by (4.31), Lemma 4.2.2 implies that $\tilde{u}_R$ is energy minimizing on $\frac{1}{2}\tilde{B}_i$. Since x is an arbitrary point in D_R, we have shown that $\tilde{u}_R$ is harmonic on D_R.

Finally, by the conformal invariance of energy, $E(\tilde{u}_n^l) = E(u_n^l\big|_{B_1(y_n)}) \leq E(u_0^0)$. By the lower semicontinuity of energy and (4.28), we have

$$E(\tilde{u}_R) \leq E(u_0^0). \tag{4.32}$$

By considering a compact exhaustion $\{D_{2^m}\}_{m=1}^{\infty}$ of $\mathbb{C}$ and a diagonalization procedure, we prove the existence of a harmonic map $\tilde{u} : \mathbb{C} \to X$. By (4.32),

$$E(\tilde{u}) \leq E(u_0^0).$$

It follows from (4.26) and (4.27) that $\tilde{u}$ is nonconstant. Thus, **CASE 2** is complete by applying the removable singularity result Corollary 4.3.1.

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Appendix A

The Dimension of the Space of Harmonic 1-Forms with Dirichlet Boundary Condition

It is well-known, we believe, that if M is a surface with boundary $\partial M \neq \emptyset$, genus g and k boundary components, then $\dim \mathcal{H}^1_N(M) = 2g + k - 1$, but this result seems difficult to find in the literature. We give a proof here for completeness. When M is a surface, it follows from Lefschetz duality that $\dim \mathcal{H}^1_N(M) = \dim \mathcal{H}^{2-1}_D(M)$, where $\mathcal{H}^1_D(M)$ is the space of harmonic 1-forms on M which satisfy the *relative* boundary conditions:

$$i^*\omega = i^*\delta\omega = 0,$$

where $i : \partial M \hookrightarrow M$ is the inclusion. So, to prove that $\dim \mathcal{H}^1_N(M) = 2g + k - 1$, we will show that $\dim \mathcal{H}^1_D(M) = 2g + k - 1$.

Lemma A.0.6. *Let M be an orientable surface of genus g with k boundary components. Then $\dim \mathcal{H}^1_D(M) = 2g + k - 1$.*

Proof. Let $\mathcal{E}\mathcal{H}^1_D(M)$ denote the subspace of harmonic fields with Dirichlet boundary conditions which are exact. Then,

$$\mathcal{H}^1_D(M) = \mathcal{E}\mathcal{H}^1_D(M) \oplus \left(\mathcal{E}\mathcal{H}^1_D(M)\right)^{\perp},$$

and $\dim \mathcal{H}^1_D(M) = \dim \mathcal{E}\mathcal{H}^1_D(M) + \dim \left(\mathcal{E}\mathcal{H}^1_D(M)\right)^{\perp}$. We claim that $\dim \mathcal{E}\mathcal{H}^1_D(M) = k-1$ and $\dim \left(\mathcal{E}\mathcal{H}^1_D(M)\right)^{\perp} = 2g$.

For the first claim, if $\omega \in \mathcal{E}\mathcal{H}^1_D(M)$, then there is a function $u \in C^\infty(M)$ for which $\omega = du$. Since ω is a harmonic field with Dirichlet boundary conditions, it follows that u is a harmonic function and is constant on the boundary. If we write the boundary as a disjoint union of k curves, $\partial M = \Gamma_1 \cup \cdots \cup \Gamma_k$, then we get that $u|_{\Gamma_i} = c_i$, for some constant c_i, $i = 1, \ldots k$. Now, the Dirichlet problem

$$\begin{cases} \Delta u = 0 \\ u|_{\Gamma_i} = c_i, \end{cases}$$

has a unique solution for each choice of $(c_1, \ldots, c_k)$ (see pg. 307 of [66]). Let

$$\mathcal{F} = \left\{ u \in C^\infty(M) \;\middle|\; \Delta u = 0, u|_{\Gamma_i} = c_i, i = 1 \ldots k, \sum_{i=1}^{k} c_i = 0 \right\}.$$

It easy to see that the differential $d|_{\mathcal{F}} : \mathcal{F} \to \mathcal{E}\mathcal{H}^1_D(M)$ is linear and bijective, and so $\dim \mathcal{E}\mathcal{H}^1_D(M) = \dim \mathcal{F} = k-1$.

Let $\overline{M}$ be a smooth Riemannian manifold obtained from M by gluing a disk into each of its boundary curves Γ_i. To prove the second claim, we will construct an isomorphism between $\left(\mathcal{E}\mathcal{H}^1_D(M)\right)^{\perp}$ and $H^1(\overline{M})$. The result will then follow from the fact that there are $2g$ cohomology classes of closed forms on $\overline{M}$.

Let $\theta \in \Omega(\overline{M})$ be a closed form. We'll first show that there is a closed form $\tilde{\omega} \in \Omega(\overline{M})$ supported on M which is cohomologous to θ. To see this, let $\tilde{D}_i$, $i = 1, \ldots, k$, be a disk slightly larger than and containing D_i, and let ϕ_i be a smooth cut-off function for which $\phi_i|_{D_i} \equiv 1$ and $\phi_i|_{\overline{M}\setminus\tilde{D}_i} \equiv 0$. Since $\tilde{D}_i$ is simply-connected, $\theta|_{\tilde{D}_i} = df_i$ for some smooth functions f_i. Let $\tilde{\omega} = \theta - \sum_{i=1}^{k} d(\phi_i f_i)$. Then $\tilde{\omega}|_{D_i} \equiv 0$ and $d\tilde{\omega} = 0$, so $\tilde{\omega}$ is a closed form in $\Omega(\overline{M})$ with compact support. Since $\sum_{i=1}^{k} d(\phi_i f_i)$ is exact, it follows that θ and $\tilde{\omega}$ are cohomologous. For simplicity, we will suppress the restriction notation and write $\tilde{\omega}|_M$ by $\tilde{\omega}$. Now, we claim that any closed form $\tilde{\omega} \in \Omega(M)$ with compact support is cohomologous to a form $\omega_0 \in (\mathcal{E}\mathcal{H}_D(M))^{\perp}$. To see this, let u be a solution to the Poisson problem

$$\begin{cases} \Delta u = -\delta\tilde{\omega} \\ u|_{\Gamma_i} = 0 \end{cases},$$

and define $\omega = \tilde{\omega} + du$. Then, ω is harmonic, since $\Delta\omega = \Delta\tilde{\omega} + \Delta du = d\delta\tilde{\omega} + 0 - d\Delta u = 0$. Moreover, $i^*\omega = i^*\tilde{\omega} + d(i^*u) = 0$, so ω satisfies the Dirichlet boundary condition. Now, $\omega = \omega_0 + dv$ for some $\omega_0 \in (\mathcal{E}\mathcal{H}^1_D(M))^\perp$ and $dv \in \mathcal{E}\mathcal{H}^1_D(M)$. Hence, ω_0 is cohomologous to ω, and therefore $\tilde{\omega}$ and θ. Note that ω_0 is unique, *i.e.*, for any closed form $\theta \in \Omega(\overline{M})$, there is a unique $\omega_0 \in (\mathcal{E}\mathcal{H}^1_D(M))^\perp$ for which $\omega_0 \sim \theta$. If $\omega_0^1, \omega_0^2 \in (\mathcal{E}\mathcal{H}^1_D(M))^\perp$ are two such forms, then $\omega_0^1 \sim \theta \sim \omega_0^2$. Hence, $\omega_0^1 - \omega_0^2 = d\zeta$, for some smooth function ζ. However, $\omega_0^1 - \omega_0^2 \in (\mathcal{E}\mathcal{H}^1_D(M))^\perp \subset (\mathcal{E}\Omega(M))^\perp$ and $d\zeta \in \mathcal{E}\Omega(M)$, so it follows that $\omega_0^1 = \omega_0^2$.

Let $\mathcal{L} : H^1(\overline{M}) \to (\mathcal{E}\mathcal{H}^1_D(M))^\perp$ be the map $[\theta] \mapsto \omega_0$ (as above). Note that it follows from the uniqueness of ω_0 that $\mathcal{L}$ is well-defined and linear.

Now, $\mathcal{L}$ is also injective. If $\mathcal{L}([\theta_1]) = \mathcal{L}([\theta_2])$, then $\theta_1 + du_1 = \theta_2 + du_2$, for some smooth functions u_1, u_2, which yields $\theta_1 \sim \theta_2$.

Finally, $\mathcal{L}$ is surjective. Suppose $\omega_0 \in \left(\mathcal{E}\mathcal{H}^1_D(M)\right)^\perp$. Then, since $i^*\omega_0 \equiv 0$,

$$\int_{\partial M} \omega_0 = 0,$$

and it follows that ω_0 is exact in a neighbourhood of each boundary curve, *i.e.*, $\omega_0 = d\psi_i$ in a neighbourhood of Γ_i. Since we can extend each ψ_i smoothly over D_i, we can extend ω_0 to a closed form $\theta \in \overline{M}$. It follows from the well-definedness of $\mathcal{L}$ that $\mathcal{L}$ does not depend on the choice of $\tilde{D}_i$ or ϕ_i, $i = 1, \dots k$. Hence, $\mathcal{L}([\theta]) = \omega_0$. $\square$

Appendix B

Quadrilateral Estimates and Energy Convexity

B.1 Quadrilateral estimates

In this section, we include several estimates for quadrilaterals in a CAT(1) space. The estimates are stated in the unpublished thesis [60] without proof. As the calculations were not obvious, we include our proofs for the convenience of the reader. References to the location of each estimate in [60] are also included.

The first lemma is a result of Reshetnyak which will be essential in later estimates.

Lemma B.1.1 ([51, Lemma 2]). *Let $\Box PQRS$ be a quadrilateral in X. Then the sum of the length of diagonals in $\Box PQRS$ can be estimated as follows:*

$$\begin{aligned} \cos d_{PR} + \cos d_{QS} \geq -\frac{1}{2}(d_{PQ}^2 + d_{RS}^2) + \frac{1}{4}(1+\cos d_{PS})(d_{QR} - d_{PS})^2 \\ + \cos d_{QR} + \cos d_{PS} + \mathrm{Cub}\left(d_{PQ}, d_{RS}, d_{QR} - d_{SP}\right). \end{aligned} \tag{B.1}$$

Proof. It suffices to prove the inequality holds for a quadrilateral $\Box PQRS$ in $\mathbb{S}^2$. By viewing $\mathbb{S}^2$ as a unit sphere in $\mathbb{R}^3$, the points P, Q, R, S determine a quadrilateral in

$\mathbb{R}^3$. Applying the identity for the quadrilateral in $\mathbb{R}^3$ (cf. [40, Corollary 2.1.3]),

$$\overline{PR}^2 + \overline{QS}^2 \leq \overline{PQ}^2 + \overline{QR}^2 + \overline{RS}^2 + \overline{SP}^2 - (\overline{SP} - \overline{QR})^2$$

where $\overline{AB}$ denotes the Euclidean distance between A and B in $\mathbb{R}^3$. To prove this, consider the vectors $A = Q - P, B = R - Q, C = S - R, D = P - S$. Then

$$\begin{aligned}
\overline{PR}^2 + \overline{QS}^2 &= \frac{1}{2}\left(|A+B|^2 + |C+D|^2 + |B+C|^2 + |D+A|^2\right) \\
&= |A|^2 + |B|^2 + |C|^2 + |D|^2 + (A \cdot B + C \cdot B + D \cdot A + D \cdot C) \\
&= |A|^2 + |B|^2 + |C|^2 + |D|^2 - |B+D|^2 \text{ since } A+B+C+D = 0 \\
&\leq |A|^2 + |B|^2 + |C|^2 + |D|^2 - \left||B| - |D|\right|^2 .
\end{aligned}$$

Note that $\overline{AB}^2 = 2 - 2\cos d_{AB}$, we obtain

$$\begin{aligned}
\cos d_{PR} + \cos d_{QS} = &-2 + \cos d_{PQ} + \cos d_{RS} + \cos d_{QR} + \cos d_{PS} \\
&+ \frac{1}{2}\left(\sqrt{2 - 2\cos d_{QR}} - \sqrt{2 - 2\cos d_{SP}}\right)^2 .
\end{aligned}$$

The lemma follows from the following Taylor expansion:

$$\begin{aligned}
-2 + \cos d_{PQ} + \cos d_{RS} &= -\frac{1}{2}d_{PQ}^2 - \frac{1}{2}d_{RS}^2 + O(d_{RS}^4 + d_{PQ}^4) \\
\left(\sqrt{2 - 2\cos d_{QR}} - \sqrt{2 - 2\cos d_{SP}}\right)^2 &= \left(\frac{\sin d_{SP}}{\sqrt{2 - 2\cos d_{SP}}}(d_{QR} - d_{SP}) + O\left((d_{QR} - d_{SP})^2\right)\right)^2 \\
&= \frac{1 + \cos d_{PS}}{2}(d_{QR} - d_{SP})^2 + O\left((d_{QR} - d_{SP})^3\right) .
\end{aligned}$$

□

Lemma B.1.2 ([60, Estimate I, Page 11])**.** *Let $\square PQRS$ be a quadrilateral in the CAT(1) space X. Let $P_{\frac{1}{2}}$ be the mid-point between P and S, and let $Q_{\frac{1}{2}}$ be the mid-point between Q and R. Then*

$$\cos^2\left(\frac{d_{PS}}{2}\right) d^2(Q_{\frac{1}{2}}, P_{\frac{1}{2}}) \leq \frac{1}{2}(d_{PQ}^2 + d_{RS}^2) - \frac{1}{4}(d_{QR} - d_{PS})^2$$

$$+ \operatorname{Cub}\left(d_{PQ}, d_{RS}, d(P_{\frac{1}{2}}, Q_{\frac{1}{2}}), d_{QR} - d_{SP}\right).$$

Proof. As a direct consequence of the law of cosines (see Figure B.1), we have the following inequalities

$$\begin{aligned}
\cos d(Q_{\frac{1}{2}}, P_{\frac{1}{2}}) &\geq \alpha \left(\cos d(Q_{\frac{1}{2}}, S) + \cos d(Q_{\frac{1}{2}}, P)\right) \\
\cos d(Q_{\frac{1}{2}}, S) &\geq \beta \left(\cos d_{RS} + \cos d_{QS}\right) \\
\cos d(Q_{\frac{1}{2}}, P) &\geq \beta \left(\cos d_{RP} + \cos d_{QP}\right)
\end{aligned}$$

where

$$\alpha = \frac{1}{2\cos\left(\frac{d_{PS}}{2}\right)} \quad \text{and} \quad \beta = \frac{1}{2\cos\left(\frac{d_{QR}}{2}\right)}.$$

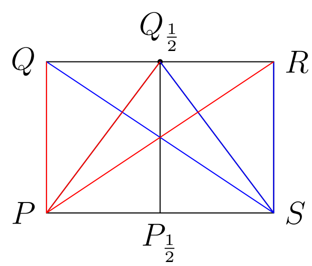


Figure B.1: An illustration of the quadrilateral $\square PQRS$ from Lemma B.1.2.

Combining the above inequalities yields

$$\cos d(Q_{\frac{1}{2}}, P_{\frac{1}{2}}) \geq \alpha\beta \left(\cos d_{RS} + \cos d_{QS} + \cos d_{RP} + \cos d_{QP}\right).$$

We apply (B.1) for the sum of diagonals $\cos d_{QS} + \cos d_{RP}$ and Taylor expansion for $\cos d_{RS}$ and $\cos d_{QP}$. It yields

$$\begin{aligned}
\cos d(Q_{\frac{1}{2}}, P_{\frac{1}{2}}) \geq \alpha\beta\Bigg(& 2 - (d_{PQ}^2 + d_{RS}^2) + \frac{1}{4}(1 + \cos d_{PS})(d_{QR} - d_{PS})^2 + \cos d_{QR} + \cos d_{PS}\Bigg) \\
&+ \operatorname{Cub}\left(d_{PQ}, d_{RS}, d_{QR} - d_{SP}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\beta\left(2 + \cos d_{QR} + \cos d_{PS} + \frac{1}{4}(1+\cos d_{PS})(d_{QR} - d_{PS})^2\right) \\
&\quad - \alpha\beta(d_{PQ}^2 + d_{RS}^2) + \mathrm{Cub}\,(d_{PQ}, d_{RS}, d_{QR} - d_{SP})\,.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&2 + \cos d_{QR} + \cos d_{PS} + \frac{1}{4}(1+\cos d_{PS})(d_{QR} - d_{PS})^2 \\
&= 2(\cos^2\frac{d_{QR}}{2} + \cos^2\frac{d_{PS}}{2}) + \frac{1}{2}\cos^2\frac{d_{PS}}{2}(d_{QR} - d_{PS})^2 \\
&= 2\left(\cos\frac{d_{QR}}{2} - \cos\frac{d_{PS}}{2}\right)^2 + 4\cos\frac{d_{QR}}{2}\cos\frac{d_{PS}}{2} + \frac{1}{2}\cos^2\frac{d_{PS}}{2}(d_{QR} - d_{PS})^2 \\
&= \frac{1}{2}\sin^2\frac{d_{PS}}{2}(d_{QR} - d_{PS})^2 + 4\cos\frac{d_{QR}}{2}\cos\frac{d_{PS}}{2} + \frac{1}{2}\cos^2\frac{d_{PS}}{2}(d_{QR} - d_{PS})^2 \\
&\qquad + O(|d_{QR} - d_{PS}|^3) \\
&= \frac{1}{2}(d_{QR} - d_{PS})^2 + 4\cos\frac{d_{QR}}{2}\cos\frac{d_{PS}}{2} + O(|d_{QR} - d_{PS}|^3).
\end{aligned}
$$

Since $\alpha\beta = \alpha^2 + O(|d_{QR} - d_{PS}|)$, we have

$$
\cos d(Q_{\frac{1}{2}}, P_{\frac{1}{2}}) \geq 1 - \alpha^2(d_{PQ}^2 + d_{RS}^2) + \frac{1}{2}\alpha^2(d_{QR} - d_{PS})^2 + \mathrm{Cub}\,(d_{PQ}, d_{RS}, d_{QR} - d_{SP})\,.
$$

The lemma follows as

$$
\cos d(Q_{\frac{1}{2}}, P_{\frac{1}{2}}) = 1 - \frac{d^2(Q_{\frac{1}{2}}, P_{\frac{1}{2}})}{2} + O(d^4(Q_{\frac{1}{2}}, P_{\frac{1}{2}})).
$$

□

Definition B.1.3. Given a metric space (X, d) and a geodesic γ_{PQ} with $d_{PQ} < \pi$, for $\tau \in [0,1]$ let $(1-\tau)P + \tau Q$ denote the point on γ_{PQ} at distance τd_{PQ} from P. That is

$$
d((1-\tau)P + \tau Q, P) = \tau d_{PQ}.
$$

Lemma B.1.4 (cf. [60, Estimate II, Page 13])**.** *Let ΔPQS be a triangle in the*

CAT(1) space X. For a pair of numbers $0 \le \eta, \eta' \le 1$ define

$$P_{\eta'} = (1-\eta')P + \eta' Q$$
$$S_\eta = (1-\eta)S + \eta Q.$$

Then

$$d^2(P_{\eta'}, S_\eta) \le \frac{\sin^2((1-\eta)d_{QS})}{\sin^2 d_{QS}}(d_{PS}^2 - (d_{QS} - d_{QP})^2) + ((1-\eta)(d_{QS} - d_{QP}) + (\eta' - \eta)d_{QS})^2 + \mathrm{Cub}\left(d_{PS}, d_{QS} - d_{QP}, \eta - \eta'\right).$$

Proof. Again we prove the inequality for a quadrilateral on $\mathbb{S}^2$. Denote $x = d_{QS}$ and $y = d_{QP}$. Denote

$$\alpha_\eta = \frac{\sin(\eta d_{QS})}{\sin d_{QS}} = \frac{\sin(\eta x)}{\sin x}, \qquad \beta_{\eta'} = \frac{\sin(\eta' d_{QP})}{\sin d_{QP}} = \frac{\sin(\eta' y)}{\sin y}.$$

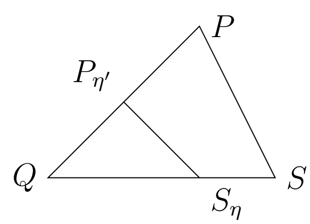


Figure B.2: An illustration of the triangle ΔPQS, and the points S_η and $P_{\eta'}$ from Lemma B.1.4.

By the law of cosines on the sphere (see Figure B.2),

$$\begin{aligned}\cos d_{PS} &= \cos x \cos y + \sin x \sin y \cos\theta = \cos(x-y) + \sin x \sin y(\cos\theta - 1)\\ \cos d(P_{\eta'}, S_\eta) &\ge \cos((1-\eta)x)\cos((1-\eta')y) + \sin((1-\eta)x)\sin((1-\eta')y)\cos\theta\\ &= \cos((1-\eta)x - (1-\eta')y) + \sin((1-\eta)x)\sin((1-\eta')y)(\cos\theta - 1),\end{aligned}$$

where θ denotes the angle $\angle PQS$ on $\mathbb{S}^2$. Substituting the term $(\cos\theta - 1)$ of the second inequality with the one in the first identity, we obtain

$$\begin{aligned}\cos d(P_{\eta'}, S_\eta) &\geq \cos((1-\eta)x - (1-\eta')y) + \alpha_{1-\eta}\beta_{1-\eta'}(\cos d_{PS} - \cos(x-y)) \\ &= \cos\left((1-\eta)(x-y) + (\eta'-\eta)x + (\eta'-\eta)(y-x)\right) \\ &\quad + \alpha_{1-\eta}^2(\cos d_{PS} - \cos(x-y)) \\ &\quad + \alpha_{1-\eta}(\beta_{1-\eta'} - \alpha_{1-\eta})(\cos d_{PS} - \cos(x-y)).\end{aligned}$$

Using the Taylor expansion $\cos a = 1 - \frac{a^2}{2} + O(a^4)$ and $(\beta_{1-\eta'} - \alpha_{1-\eta}) = O(|\eta'-\eta| + |x-y|)$, we derive

$$\begin{aligned}\cos d(P_{\eta'}, S_\eta) \geq 1 - \frac{((1-\eta)(x-y) + (\eta'-\eta)x)^2}{2} + \alpha_{1-\eta}^2\left(-\frac{d_{PS}^2}{2} + \frac{(x-y)^2}{2}\right) \\ + \operatorname{Cub}\left(|\eta'-\eta|, |x-y|, d_{PS}\right).\end{aligned}$$

It implies that

$$\begin{aligned}d^2(P_{\eta'}, S_\eta) \leq \alpha_{1-\eta}^2(d_{PS}^2 - (x-y)^2) + ((1-\eta)(x-y) + (\eta'-\eta)x)^2 \\ + \operatorname{Cub}\left(|\eta'-\eta|, |x-y|, d_{PS}\right).\end{aligned}$$

□

Corollary B.1.1. *Let $u : \Omega \to \mathcal{B}_\rho(Q)$ be a finite energy map and $\eta \in C_C^\infty(\Omega, [0,1])$. Define $\hat{u} : \Omega \to \mathcal{B}_\rho(Q)$ as*

$$\hat{u}(x) = (1-\eta(x))u(x) + \eta(x)Q.$$

Then $\hat{u}$ has finite energy, and for any smooth vector field $W \in \Gamma(\Omega)$ we have

$$|\hat{u}_*(W)|^2 \leq \left(\frac{\sin(1-\eta)R^u}{\sin R^u}\right)^2 (|u_*(W)|^2 - |\nabla_W R^u|^2) + |\nabla_W((1-\eta)R^u)|^2,$$

where $R^u(x) = d(u(x), Q)$.

Note that every error term that appeared in Lemma B.1.4 will converge to the product of an L^1 function and a term that goes to zero. So all error terms vanish when taking limits.

Lemma B.1.5 (cf. [60, Estimate III, page 19])**.** *Let $\square PQRS$ be a quadrilateral in a CAT(1) space X. For $\eta', \eta \in [0,1]$ define*

$$Q_{\eta'} = (1-\eta')Q + \eta' R, \quad P_\eta = (1-\eta)P + \eta S.$$

Then

$$\begin{aligned}
d^2(Q_{\eta'}, P_\eta) &+ d^2(Q_{1-\eta'}, P_{1-\eta}) \\
&\leq \left(1 + 2\eta d_{PS} \tan(\frac{1}{2} d_{PS})\right)(d_{PQ}^2 + d_{RS}^2) \\
&\quad - 2\eta \left(1 + \frac{1}{2} d_{PS} \tan(\frac{1}{2} d_{PS})\right)(d_{QR} - d_{PS})^2 \\
&\quad + 2(2\eta - 1)(\eta' - \eta) d_{PS}(d_{QR} - d_{PS}) \\
&\quad + \eta^2 \mathrm{Quad}(d_{PQ}, d_{RS}, d_{QR} - d_{PS}) + \mathrm{Cub}\left(d_{QR} - d_{PS}, d_{PQ}, d_{RS}, \eta - \eta'\right)
\end{aligned}$$

Proof. For notation simplicity, we denote

$$x = d_{PS}, \qquad y = d_{QR}, \qquad \alpha_\eta = \frac{\sin(\eta x)}{\sin x}, \qquad \beta_{\eta'} = \frac{\sin(\eta' y)}{\sin y}.$$

Apply [60, Definition 1.6] to each of the blue, red, and yellow triangles in Figure B.3 below.

We derive

$$\begin{aligned}
\cos d(Q_{1-\eta'}, P_{1-\eta}) &\geq \alpha_\eta \cos d(Q_{1-\eta'}, S) + \alpha_{1-\eta} \cos d(Q_{1-\eta'}, P) \\
&\geq \alpha_\eta(\beta_{\eta'} \cos d_{SR} + \beta_{1-\eta'} \cos d_{SQ}) + \alpha_{1-\eta}(\beta_{\eta'} \cos d_{PR} + \beta_{1-\eta'} \cos d_{PQ}).
\end{aligned}$$

Compute similarly for $d(Q_{\eta'}, P_\eta)$ for the highlighted triangles below:

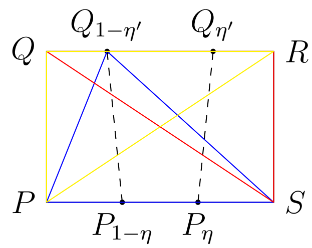
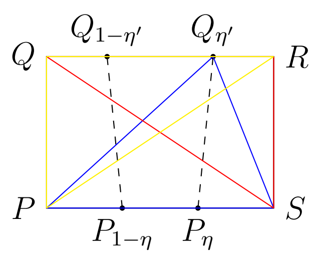


Figure B.3: An illustration of the quadrilateral $\square PQRS$, and the points P_η, $P_{1-\eta}$, $Q_{\eta'}$ and $Q_{1-\eta'}$ from Lemma B.1.5.



We derive

$$\begin{aligned}\cos d(Q_{\eta'}, P_\eta) &\geq \alpha_\eta \cos d(Q_{\eta'}, P) + \alpha_{1-\eta} \cos d(Q_{\eta'}, S)\\ &\geq \alpha_\eta(\beta_{\eta'} \cos d_{PQ} + \beta_{1-\eta'} \cos d_{PR}) + \alpha_{1-\eta}(\beta_{\eta'} \cos d_{SQ} + \beta_{1-\eta'} \cos d_{SR}).\end{aligned}$$

Adding the above two inequalities, we obtain

$$\begin{aligned}&\cos d(Q_{1-\eta'}, P_{1-\eta}) + \cos d(Q_{\eta'}, P_\eta)\\ &\geq (\alpha_\eta \beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'})(\cos d_{PQ} + \cos d_{SR}) + (\alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(\cos d_{PR} + \cos d_{SQ}).\end{aligned} \tag{B.2}$$

Applying (B.1) to the term $\cos d_{PR} + \cos d_{SQ}$ and using Taylor expansion, the inequality (B.2) becomes

$$\cos d(Q_{1-\eta'}, P_{1-\eta}) + \cos d(Q_{\eta'}, P_\eta) \geq (\alpha_\eta \beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'}) \left(2 - \frac{d_{PQ}^2}{2} - \frac{d_{SR}^2}{2}\right)$$

$$
\begin{aligned}
&+ (\alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'}) \left(-\frac{1}{2}(d_{PQ}^2 + d_{SR}^2) + \frac{1}{4}(1+\cos d_{PS})(d_{QR} - d_{PS})^2 \right) \\
&+ (\alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'}) \left(\cos d_{QR} + \cos d_{PS}\right) \\
&+ \mathrm{Cub}\left(d_{PQ}, d_{RS}, d_{QR} - d_{SP}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\cos d(Q_{1-\eta'}, P_{1-\eta}) + \cos d(Q_{\eta'}, P_\eta) \\
&\qquad \geq -\frac{1}{2}(\alpha_\eta\beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'} + \alpha_\eta\beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(d_{PQ}^2 + d_{SR}^2) \quad \text{(B.3)} \\
&\qquad + 2(\alpha_\eta\beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'}) + (\alpha_\eta\beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(\cos d_{QR} + \cos d_{PS}) \\
&\qquad \qquad \text{(B.4)} \\
&\qquad + \frac{1}{4}(\alpha_\eta\beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(1 + \cos d_{PS})(d_{QR} - d_{PS})^2 \quad \text{(B.5)} \\
&\qquad + \mathrm{Cub}\left(d_{PQ}, d_{RS}, d_{QR} - d_{SP}\right).
\end{aligned}
$$

We need the following elementary trigonometric identities to compute (B.3), (B.4), (B.5):

$$
\begin{aligned}
\alpha_\eta\beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'} &= \frac{\sin(\eta - \frac{1}{2})x \sin(\eta' - \frac{1}{2})y}{2\sin\frac{1}{2}x \sin\frac{1}{2}y} + \frac{\cos(\eta - \frac{1}{2})x \cos(\eta' - \frac{1}{2})y}{2\cos\frac{1}{2}x \cos\frac{1}{2}y} \\
\alpha_\eta\beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'} &= -\frac{\sin(\eta - \frac{1}{2})x \sin(\eta' - \frac{1}{2})y}{2\sin\frac{1}{2}x \sin\frac{1}{2}y} + \frac{\cos(\eta - \frac{1}{2})x \cos(\eta' - \frac{1}{2})y}{2\cos\frac{1}{2}x \cos\frac{1}{2}y} \\
\left(\frac{\cos(\eta - \frac{1}{2})x}{\cos\frac{1}{2}x}\right)^2 &= 1 + 2\eta x \tan\frac{1}{2}x + O(\eta^2).
\end{aligned}
$$

Noting that

$$
\begin{aligned}
&\alpha_\eta\beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'} + \alpha_\eta\beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'} = \frac{\cos(\eta - \frac{1}{2})x \cos(\eta' - \frac{1}{2})y}{\cos\frac{1}{2}x \cos\frac{1}{2}y} \\
&= \left(\frac{\cos(\eta - \frac{1}{2})x}{\cos\frac{1}{2}x}\right)^2 + O(|\eta - \eta'| + |x - y|)
\end{aligned}
$$

$$= 1 + 2\eta x \tan(\frac{1}{2}x) + O(\eta^2 + |\eta - \eta'| + |x-y|),$$

we obtain for (B.3)

$$\begin{aligned}
&-\frac{1}{2}(\alpha_\eta \beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'} + \alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(d_{PQ}^2 + d_{SR}^2) \\
&= -\frac{1}{2}\left(1 + 2\eta x \tan(\frac{1}{2}x)\right)(d_{PQ}^2 + d_{SR}^2) + O\left((\eta^2 + |\eta - \eta'| + |x-y|)(d_{PQ}^2 + d_{SR}^2)\right).
\end{aligned}$$

Lemma B.1.6. *We can compute* (B.4) *as follows:*

$$\begin{aligned}
&2(\alpha_\eta \beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'}) + (\alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(\cos x + \cos y) \\
&= 2 - \left((\eta - \frac{1}{2})(y-x) + (\eta' - \eta)x\right)^2 + \frac{\sin^2(\eta - \frac{1}{2})x}{4\sin^2\frac{1}{2}x}\cos^2(\frac{1}{2}x)(x-y)^2 \\
&\quad + \frac{\cos^2(\eta - \frac{1}{2})x}{4\cos^2\frac{1}{2}x}\sin^2(\frac{1}{2}x)(x-y)^2 + O(|x-y|^2(|x-y| + |\eta' - \eta|)).
\end{aligned}$$

Proof.

$$\begin{aligned}
2(\alpha_\eta \beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'}) &+ (\alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(\cos x + \cos y) \\
&= \frac{\sin(\eta - \frac{1}{2})x \sin(\eta' - \frac{1}{2})y}{2\sin\frac{1}{2}x \sin\frac{1}{2}y}(2 - \cos x - \cos y) \\
&\quad + \frac{\cos(\eta - \frac{1}{2})x \cos(\eta' - \frac{1}{2})y}{2\cos\frac{1}{2}x \cos\frac{1}{2}y}(2 + \cos x + \cos y).
\end{aligned}$$

Note that

$$\begin{aligned}
2 - \cos x - \cos y &= 2(\sin\frac{1}{2}x)^2 + 2(\sin\frac{1}{2}y)^2 = 2\left(2\sin\frac{1}{2}x\sin\frac{1}{2}y + (\sin\frac{1}{2}x - \sin\frac{1}{2}y)^2\right) \\
&= 4\sin\frac{1}{2}x\sin\frac{1}{2}y + \frac{1}{2}(\cos\frac{1}{2}x)^2(x-y)^2 + O(|x-y|^3) \\
2 + \cos x + \cos y &= 2(\cos\frac{1}{2}x)^2 + 2(\cos\frac{1}{2}y)^2 = 2\left(2\cos\frac{1}{2}x\cos\frac{1}{2}y + (\cos\frac{1}{2}x - \cos\frac{1}{2}y)^2\right) \\
&= 4\cos\frac{1}{2}x\cos\frac{1}{2}y + \frac{1}{2}(\sin\frac{1}{2}x)^2(x-y)^2 + O(|x-y|^3),
\end{aligned}$$

where we apply Taylor expansion in the last equality. Hence we have

$$\begin{aligned}
&2(\alpha_\eta\beta_{\eta'}+\alpha_{1-\eta}\beta_{1-\eta'})+(\alpha_\eta\beta_{1-\eta'}+\alpha_{1-\eta}\beta_{\eta'})(\cos x+\cos y)\\
&=2\left(\sin(\eta-\frac{1}{2})x\sin(\eta'-\frac{1}{2})y+\cos(\eta-\frac{1}{2})x\cos(\eta'-\frac{1}{2})y\right)\\
&\quad+\frac{\sin^2(\eta-\frac{1}{2})x}{4\sin^2\frac{1}{2}x}(\cos\frac{1}{2}x)^2(x-y)^2+\frac{\cos^2(\eta-\frac{1}{2})x}{4\cos^2\frac{1}{2}x}(\sin\frac{1}{2}x)^2(x-y)^2\\
&\quad+O(|x-y|^2(|x-y|+|\eta'-\eta|)).
\end{aligned}$$

Here we use the estimates

$$\frac{\sin(\eta-\frac{1}{2})x\sin(\eta'-\frac{1}{2})y}{2\sin\frac{1}{2}x\sin\frac{1}{2}y}-\frac{\sin^2(\eta-\frac{1}{2})x}{2\sin^2\frac{1}{2}x}=O(|\eta-\eta'|+|x-y|)$$

and

$$\frac{\cos(\eta-\frac{1}{2})x\cos(\eta'-\frac{1}{2})y}{2\cos\frac{1}{2}x\cos\frac{1}{2}y}-\frac{\cos^2(\eta-\frac{1}{2})x}{2\cos^2\frac{1}{2}x}=O(|\eta-\eta'|+|x-y|).$$

Observe that

$$\begin{aligned}
&\left(\sin(\eta-\frac{1}{2})x\sin(\eta'-\frac{1}{2})y+\cos(\eta-\frac{1}{2})x\cos(\eta'-\frac{1}{2})y\right)\\
&\qquad=\cos\left((\eta-\frac{1}{2})(y-x)+(\eta'-\eta)x+(\eta'-\eta)(y-x)\right)
\end{aligned}$$

and use $\cos a=1-\frac{a^2}{2}+O(a^4)$. □

Lemma B.1.7. *Adding the terms in the previous computational lemma that contain $(x-y)^2$ to (B.5), we have the following estimate:*

$$\begin{aligned}
&\frac{1}{4}(\alpha_\eta\beta_{1-\eta'}+\alpha_{1-\eta}\beta_{\eta'})(1+\cos x)(x-y)^2\\
&-(\eta-\frac{1}{2})^2(x-y)^2+\frac{\sin^2(\eta-\frac{1}{2})x}{4\sin^2\frac{1}{2}x}\cos^2(\frac{1}{2}x)(x-y)^2+\frac{\cos^2(\eta-\frac{1}{2})x}{4\cos^2\frac{1}{2}x}\sin^2(\frac{1}{2}x)(x-y)^2\\
&=\eta(1+\frac{1}{2}x\tan\frac{1}{2}x)(x-y)^2+O(|x-y|^2(\eta^2+|x-y|+|\eta-\eta'|)).
\end{aligned}$$

Proof. Noting that $1+\cos x = 2\cos^2(\frac{1}{2}x)$, we have that

$$\begin{aligned}
&\frac{1}{4}(\alpha_\eta\beta_{1-\eta'}+\alpha_{1-\eta}\beta_{\eta'})(1+\cos x)(x-y)^2\\
&=\frac{1}{4}\left(-\left(\frac{\sin(\eta-\frac{1}{2})x}{\sin\frac{1}{2}x}\right)^2+\left(\frac{\cos(\eta-\frac{1}{2})x}{\cos\frac{1}{2}x}\right)^2\right)\cos^2(\frac{1}{2}x)(x-y)^2\\
&\qquad+O(|x-y|^2(|\eta-\eta'|+|x-y|)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{4}(\alpha_\eta\beta_{1-\eta'}+\alpha_{1-\eta}\beta_{\eta'})(1+\cos x)(x-y)^2\\
&-(\eta-\frac{1}{2})^2(x-y)^2+\frac{\sin^2(\eta-\frac{1}{2})x}{4\sin^2\frac{1}{2}x}\cos^2(\frac{1}{2}x)(x-y)^2+\frac{\cos^2(\eta-\frac{1}{2})x}{4\cos^2\frac{1}{2}x}\sin^2(\frac{1}{2}x)(x-y)^2\\
&=\left(\frac{\cos^2(\eta-\frac{1}{2})x}{4\cos^2\frac{1}{2}x}-(\eta-\frac{1}{2})^2\right)(x-y)^2+O(|x-y|^2(|\eta-\eta'|+|x-y|))\\
&=\left(\frac{1}{4}+\frac{1}{2}\eta x\tan\frac{1}{2}x-(-\eta+\frac{1}{4})\right)(x-y)^2+O(|x-y|^2(\eta^2+|\eta-\eta'|+|x-y|)).
\end{aligned}$$

□

Combing the above computations, we have that

$$\begin{aligned}
\cos d(Q_{1-\eta'},P_{1-\eta})+\cos d(Q_{\eta'},P_\eta)\geq 2-\frac{1}{2}&\left(1+2\eta d_{PS}\tan(\frac{1}{2}d_{PS})\right)(d_{PQ}^2+d_{SR}^2)\\
&+\eta(1+\frac{1}{2}d_{PS}\tan\frac{1}{2}d_{PS})(d_{QR}-d_{PS})^2\\
&-(2\eta-1)(\eta'-\eta)d_{PS}(d_{QR}-d_{PS})\\
&+\eta^2\mathrm{Quad}(d_{PQ},d_{RS},d_{QR}-d_{PS})\\
&+\mathrm{Cub}\left(d_{QR}-d_{PS},d_{PQ},d_{RS},\eta'-\eta\right).
\end{aligned}$$

Taylor expansion gives the result. □

Corollary B.1.2. *Given a pair of finite energy maps $u_0,u_1\in W^{1,2}(\Omega,X)$ with* *images* $u_i(\Omega) \subset \mathcal{B}_\rho(Q)$ *and a function* $\eta \in C^1_c(\Omega)$, $0 \leq \eta \leq \frac{1}{2}$, *define the maps*

$$\begin{aligned} u_\eta(x) &= (1-\eta(x))u_0(x) + \eta(x)u_1(x) \\ u_{1-\eta}(x) &= \eta(x)u_0(x) + (1-\eta(x))u_1(x) \\ d(x) &= d(u_0(x), u_1(x)). \end{aligned}$$

Then $u_\eta, u_{1-\eta} \in W^{1,2}(\Omega, X)$ *and*

$$\begin{aligned} |\nabla u_\eta|^2 + |\nabla u_{1-\eta}|^2 \leq (1 + 2\eta d \tan\frac{d}{2})(|\nabla u_0|^2 + |\nabla u_1|^2) \\ - 2\eta(1 + \frac{1}{2}d\tan\frac{d}{2})|\nabla d|^2 - 2d\nabla\eta\cdot\nabla d + \mathrm{Quad}(\eta, |\nabla\eta|). \end{aligned}$$

B.2 Energy Convexity, Existence, Uniqueness, and Subharmonicity

As with the previous section, the results in this section are stated in [60]. Excepting the first theorem, they are stated without proof. As, again, the calculations are non-trivial and tedious, we verify them for the reader.

Theorem B.2.1 ([60, Proposition 1.15]). *Let* $u_0, u_1 : \Omega \to \overline{\mathcal{B}_\rho(O)}$ *be finite energy maps with* $\rho \in (0, \frac{\pi}{2})$. *Denote by*

$$\begin{aligned} d(x) &= d(u_0(x), u_1(x)) \\ R(x) &= d(u_{\frac{1}{2}}(x), O). \end{aligned}$$

Then there exists a continuous function $\eta(x) : \Omega \to [0,1]$ *such that the function* $w : \Omega \to \overline{\mathcal{B}_\rho(O)}$ *defined by*

$$w(x) = (1-\eta(x))u_{\frac{1}{2}}(x) + \eta(x)O$$

is in $W^{1,2}(\Omega, \overline{B_\rho(O)})$ *and satisfies*

$$(\cos^8 \rho) \int_\Omega \left| \nabla \frac{\tan \frac{1}{2} d}{\cos R} \right|^2 d\mu_g \leq \frac{1}{2} \left(\int_\Omega |\nabla u_0|^2 d\mu_g + \int_\Omega |\nabla u_1|^2 d\mu_g \right) - \int_\Omega |\nabla w|^2 d\mu_g.$$

Proof. Once the estimates in Lemma B.1.2 and Lemma B.1.4 are established, we proceed as in [60]. Choose η to satisfy

$$\frac{\sin((1 - \eta(x))R(x))}{\sin R(x)} = \cos \frac{d(x)}{2}.$$

Note that $0 \leq \eta \leq 1$ and η is as smooth as $d(x), R(x)$. It is straightforward to verify that $w \in L^2_h(\Omega, \overline{B_\rho(O)})$.

For $W \in \Gamma(\Omega)$, consider the flow $\epsilon \mapsto x(\epsilon)$ induced by W.

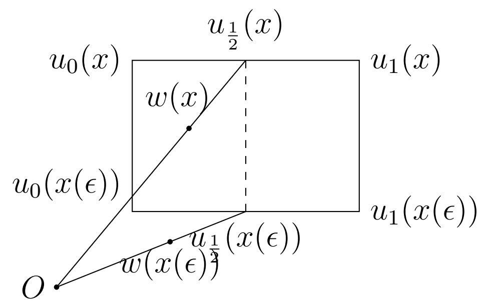


Figure B.4: An illustration of the quadrilateral $\square PQRS$ with $P = u_0(x(\epsilon))$, $Q = u_0(x)$, $R = u_1(x)$ and $S = u_1(x(\epsilon))$ used in the proof of Lemma B.2.1.

Applying Lemma B.1.2 to the quadrilateral determined by $P = u_0(x(\epsilon)), Q = u_0(x), R = u_1(x), S = u_1(x(\epsilon))$ (see Figure B.4), divided by ϵ^2, and integrate the resulting inequality against $f \in C_c^\infty(\Omega)$ and taking $\epsilon \to 0$, we obtain

$$\left(\cos \frac{d(x)}{2} \right)^2 |(u_{\frac{1}{2}})_*(W)|^2 \leq \frac{1}{2} \left(|(u_0)_*(W)|^2 + |(u_1)_*(W)|^2 \right) - \frac{1}{4} |\nabla_W d|^2.$$

Note that the cubic terms vanish in the limit as every cubic term will be the product of an L^1 function and $d(x) - d(x(\epsilon))$ or $d(u_i(x), u_i(x(\epsilon)))$, $i = 0, \frac{1}{2}, 1$.

Applying Lemma B.1.4 to the triangle determined by $Q = O, P = u_{\frac{1}{2}}(x), S = u_{\frac{1}{2}}(x(\epsilon))$ yields

$$\begin{aligned}|(w)_*(W)|^2 &\leq \left(\frac{\sin(1-\eta)R}{\sin R}\right)^2 (|(u_{\frac{1}{2}}) * (W)|^2 - |\nabla_W R|^2) + |\nabla_W((1-\eta)R)|^2 \\ &= \left(\cos\frac{d(x)}{2}\right)^2 (|(u_{\frac{1}{2}})_*(W)|^2 - |\nabla_W R|^2) + |\nabla_W((1-\eta)R)|^2.\end{aligned}$$

The above two inequalities imply

$$\begin{aligned}|w_*(W)|^2 \leq \frac{1}{2}&\left(|(u_0)_*(W)|^2 + |(u_1)_*(W)|^2\right) \\ &- \frac{1}{4}|\nabla_W d|^2 - \left(\cos\frac{d(x)}{2}\right)^2 |\nabla_W R|^2 + |\nabla_W\left((1-\eta)R\right)|^2.\end{aligned}$$

By direct computation,

$$\begin{aligned}&-\frac{1}{4}|\nabla_W d|^2 - \left(\cos\frac{d(x)}{2}\right)^2 |\nabla_W R|^2 + |\nabla_W\left((1-\eta)R\right)|^2 \\ &= -\frac{\cos^4 R(x)\cos^4\frac{d(x)}{2}}{1-\sin^2 R(x)\cos^2\frac{d(x)}{2}}\left|\nabla\frac{\tan\frac{d(x)}{2}}{\cos R(x)}\right|^2.\end{aligned}$$

The lemma follows from estimating

$$\frac{\cos^4 R(x)\cos^4\frac{d(x)}{2}}{1-\sin^2 R(x)\cos^2\frac{d(x)}{2}} \geq \cos^4 R(x)\cos^4\frac{d(x)}{2} \geq \cos^8\rho,$$

dividing the resulting inequality by ϵ^2, integrating over $\mathbb{S}^{n-1}$, letting $\epsilon \to 0$, and then integrating over Ω. $\square$

Theorem B.2.2 (Existence Theorem)**.** *For any $\rho \in (0, \frac{\pi}{4})$ and for any finite energy map $h : \Omega \to \overline{\mathcal{B}_\rho(O)} \subset X$, there exists a unique element ${}^{Dir}h \in W^{1,2}_h(\Omega, \overline{\mathcal{B}_\rho(O)})$ which minimizes energy amongst all maps in $W^{1,2}_h(\Omega, \overline{\mathcal{B}_\rho(O)})$.*

Moreover, for any $\sigma \in (0, \rho)$, if ${}^{Dir}h(\partial\Omega) \subset \overline{\mathcal{B}_\sigma(O)}$ then $\overline{{}^{Dir}h(\Omega)} \subset \overline{\mathcal{B}_\sigma(O)}$.

Proof. Denote by

$$E_0 = \inf\{E(u) : u \in W^{1,2}_h(\Omega, \overline{\mathcal{B}_\rho(O)})\}.$$

Let $u_i \in W^{1,2}(\Omega, \overline{\mathcal{B}_\rho(P)})$ such that $E(u_i) \to E_0$. By Theorem B.2.1, we have that

$$(\cos^8 \rho) \int_\Omega \left| \nabla \frac{\tan \frac{1}{2} d(u_k(x), u_\ell(x))}{\cos R} \right| \, d\mu_g \leq \frac{1}{2} \left(E(u_k) + E(u_\ell) \right) - E(w_{k\ell}),$$

where $w_{k\ell}$ is the interpolation map defined by Theorem B.2.1. The above right hand side goes to 0 as $k, \ell \to \infty$. By the Poincaré inequality,

$$\int_\Omega d(u_k, u_\ell) \, d\mu_g \to 0.$$

Thus the sequence $\{u_k\}$ is Cauchy and $u_k \to u$ for some $u \in W^{1,2}(\Omega, \overline{\mathcal{B}_\rho(O)})$ because $W^{1,2}(\Omega, \overline{\mathcal{B}_\rho(O)})$ is a complete metric space. By trace theory, $u \in W^{1,2}_h(\Omega, \overline{\mathcal{B}_\rho(O)})$. By lower semi-continuity of the energy, $E(u) = E_0$. The energy minimizer is unique by energy convexity.

Finally, since $\rho < \frac{\pi}{4}$, for any $\sigma \in (0, \rho]$, the ball $\mathcal{B}_\sigma(O)$ is geodesically convex. Therefore, the projection map $\pi_\sigma : \overline{\mathcal{B}_\rho(O)} \to \overline{\mathcal{B}_\sigma(O)}$ is well-defined and distance decreasing. Thus, since ${}^{Dir}h(\Omega) \subset \overline{\mathcal{B}_\rho(O)}$, we can prove the final statement by contradiction using the projection map to decrease energy. □

Lemma B.2.3 (cf. [60, (2.5)])**.** *Let $u_0, u_1 : \Omega \to \mathcal{B}_\rho(Q) \subset X$ be finite energy maps (possibly with different boundary values). For any given $\eta \in C^\infty_c(\Omega)$ with $0 \leq \eta < 1/2$, there exists finite energy maps $u_\eta, \hat{u}_\eta \in W^{1,2}_{u_0}(\Omega, \mathcal{B}_\rho(Q))$ and $u_{1-\eta}, \hat{u}_{1-\eta} \in W^{1,2}_{u_1}(\Omega, \mathcal{B}_\rho(Q))$ such that*

$$\begin{aligned}
&|\pi(\hat{u}_\eta)|^2 + |\pi(\hat{u}_{1-\eta})|^2 - |\pi(u_0)|^2 - |\pi(u_1)|^2 \\
&\leq -2 \cos R^{u_\eta} \cos R^{u_{1-\eta}} \nabla \left(\frac{d}{\sin d} \eta F_\eta \right) \cdot \nabla F_\eta + \mathrm{Quad}(\eta, \nabla \eta),
\end{aligned}$$

where

$$d(x) = d(u_0(x), u_1(x))$$

$$R^{u_\eta}(x) = d(u_\eta(x), Q)$$
$$R^{u_{1-\eta}}(x) = d(u_{1-\eta}(x), Q)$$

and

$$F_\eta = \sqrt{\frac{1-\cos d}{\cos R^{u_\eta} \cos R^{u_{1-\eta}}}}.$$

Proof. Let $\eta \in C_c^\infty(\Omega)$ satisfy $0 \leq \eta < 1/2$. For $0 \leq \phi, \psi \leq 1$ that will be determined below, we define the comparison maps

$$\hat{u}_\eta = (1-\phi(x))u_\eta(x) + \phi(x)Q$$
$$\hat{u}_{1-\eta} = (1-\psi(x))u_{1-\eta}(x) + \psi(x)Q,$$

where

$$u_\eta(x) = (1-\eta(x))u_0(x) + \eta(x)u_1(x) \quad \text{and} \quad u_{1-\eta}(x) = \eta(x)u_0(x) + (1-\eta(x))u_1(x).$$

By Corollary B.1.1,

$$\begin{aligned}
|\pi(\hat{u}_\eta)|^2 + |\pi(\hat{u}_{1-\eta})|^2 \leq{} & \left(\frac{\sin(1-\phi)R^{u_\eta}}{\sin R^{u_\eta}}\right)^2 (|\pi(u_\eta)|^2 - |\nabla R^{u_\eta}|^2) + |\nabla((1-\phi)R^{u_\eta})|^2 \\
& + \left(\frac{\sin(1-\psi)R^{u_{1-\eta}}}{\sin R^{u_{1-\eta}}}\right)^2 (|\pi(u_{1-\eta})|^2 - |\nabla R^{u_{1-\eta}}|^2) \\
& + |\nabla((1-\psi)R^{u_{1-\eta}})|^2.
\end{aligned}$$

Define ϕ and ψ so that

$$\frac{\sin^2((1-\phi)R^{u_\eta})}{\sin^2 R^{u_\eta}} = 1 - 2\eta d \tan\frac{d}{2} + O(\eta^2)$$
$$\frac{\sin^2((1-\psi)R^{u_{1-\eta}})}{\sin^2 R^{u_{1-\eta}}} = 1 - 2\eta d \tan\frac{d}{2} + O(\eta^2).$$

Since $\frac{\sin(1-a)\theta}{\sin\theta} = 1 - a\theta\cot\theta + O(a^2)$, we solve

$$\phi = \eta\frac{\tan R^{u_\eta}}{R^{u_\eta}}d\tan\frac{d}{2} \quad \text{and} \quad \psi = \eta\frac{\tan R^{u_{1-\eta}}}{R^{u_{1-\eta}}}d\tan\frac{d}{2}.$$

Note that in particular $u_\eta, \hat{u}_\eta \in W^{1,2}_{u_0}(\Omega, \mathcal{B}_\rho(Q))$ and $u_{1-\eta}, \hat{u}_{1-\eta} \in W^{1,2}_{u_1}(\Omega, \mathcal{B}_\rho(Q))$.

Together with the estimate for $|\pi(u_\eta)|^2 + |\pi(u_{1-\eta})|^2$ in Corollary B.1.2 (which also explains the choice of ϕ and ψ in order to eliminate the coefficient), we have

$$\begin{aligned}
&|\pi(\hat{u}_\eta)|^2 + |\pi(\hat{u}_{1-\eta})|^2 - |\pi(u_0)|^2 - |\pi(u_1)|^2 \\
&\leq -2\eta(1 + \frac{1}{2}d\tan\frac{d}{2})|\nabla d|^2 - 2d\nabla\eta\cdot\nabla d - (1 - 2\eta d\tan\frac{d}{2})(|\nabla R^{u_\eta}|^2 + |\nabla R^{u_{1-\eta}}|^2) \\
&\quad + |\nabla(1 - \eta\frac{\tan R^{u_\eta}}{R^{u_\eta}}d\tan\frac{d}{2})R^{u_\eta}|^2 + |\nabla(1 - \eta\frac{\tan R^{u_{1-\eta}}}{R^{u_{1-\eta}}}d\tan\frac{d}{2})R^{u_{1-\eta}}|^2 + \mathrm{Quad}(\eta, |\nabla\eta|).
\end{aligned}$$

Simplifying the expression and using $1 - \sec^2\theta = -\tan^2\theta$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(|\pi(\hat{u}_\eta)|^2 + |\pi(\hat{u}_{1-\eta})|^2 - |\pi(u_0)|^2 - |\pi(u_1)|^2\right) \\
&\quad \leq \eta\Bigg(- (1 + \frac{1}{2}d\tan\frac{d}{2})|\nabla d|^2 - d\tan\frac{d}{2}(\tan^2 R^{u_\eta}|\nabla R^{u_\eta}|^2 + \tan^2 R^{u_{1-\eta}}|\nabla R^{u_{1-\eta}}|^2) \\
&\qquad - \nabla(d\tan\frac{d}{2})\cdot(\tan R^{u_\eta}\nabla R^{u_\eta} + \tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}})\Bigg) \\
&\qquad + \nabla\eta\cdot\left(-d\nabla d - \tan R^{u_\eta}d\tan\frac{d}{2}\nabla R^{u_\eta} - \tan R^{u_{1-\eta}}d\tan\frac{d}{2}\nabla R^{u_{1-\eta}}\right) + \mathrm{Quad}(\eta, \nabla\eta).
\end{aligned} \tag{B.6}$$

We hope to find a, b, F_η which are functions of d, R^{u_η} and $R^{u_{1-\eta}}$ such that the right hand side above is $\leq a\nabla(b\eta F_\eta)\cdot\nabla F_\eta$.

Since $a\nabla(b\eta F_\eta)\cdot\nabla F_\eta = \eta(ab|\nabla F_\eta|^2 + \frac{a}{2}\nabla b\cdot\nabla F_\eta^2) + \frac{ab}{2}\nabla\eta\cdot\nabla F_\eta^2$, by comparing the terms involving $\nabla\eta$ in (B.6), we solve

$$\begin{aligned}
\frac{ab}{2}\nabla\eta\cdot\nabla F_\eta^2 &= \nabla\eta\cdot\left(-d\nabla d - \tan R^{u_\eta}d\tan\frac{d}{2}\nabla R^{u_\eta} - \tan R^{u_{1-\eta}}d\tan\frac{d}{2}\nabla R^{u_{1-\eta}}\right) \\
&= -d\tan\frac{d}{2}\nabla\eta\cdot\left(\nabla\log\sin^2\frac{d}{2} - \nabla\log\cos R^{u_\eta} - \nabla\log\cos R^{u_{1-\eta}}\right)
\end{aligned}$$

$$= -\frac{d}{\sin d}\cos R^{u_\eta}\cos R^{u_{1-\eta}}\nabla\eta\cdot\nabla\frac{1-\cos d}{\cos R^{u_\eta}\cos R^{u_{1-\eta}}},$$

where we use $2\sin^2\frac{d}{2} = (1-\cos d)$ and $\tan\frac{d}{2} = \frac{1-\cos d}{\sin d}$. It suggests us to choose

$$\frac{ab}{2} = -\frac{d}{\sin d}\cos R^{u_\eta}\cos R^{u_{1-\eta}} \quad \text{and} \quad F_\eta = \sqrt{\frac{1-\cos d}{\cos R^{u_\eta}\cos R^{u_{1-\eta}}}}.$$

We then compute the term $\eta(ab|\nabla F_\eta|^2 + \frac{a}{2}\nabla b\cdot\nabla F_\eta^2)$ for the above choices of a, b, and F_η. For the term $ab|\nabla F_\eta|^2$, we compute

$$\begin{aligned}
ab|\nabla F_\eta|^2 &= -\frac{d}{2\sin d(1-\cos d)}\left|\sin d\nabla d + (1-\cos d)(\tan R^{u_\eta}\nabla R^{u_\eta} + \tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}})\right|^2 \\
&\geq -\left(\frac{d\sin d}{2(1-\cos d)}|\nabla d|^2 + d\nabla d\cdot(\tan R^{u_\eta}\nabla R^{u_\eta} + \tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}})\right. \\
&\qquad \left. +\frac{d(1-\cos d)}{\sin d}(\tan^2 R^{u_\eta}|\nabla R^{u_\eta}|^2 + \tan^2 R^{u_{1-\eta}}|\nabla R^{u_{1-\eta}}|^2)\right),
\end{aligned}$$

where we expand the quadratic term and use the AM-GM inequality to handle the cross term $(\tan R^{u_\eta}\nabla R^{u_\eta})\cdot(\tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}})$. For the term $\frac{a}{2}\nabla b\cdot\nabla F_\eta^2$, we assume $b = b(d)$ and compute:

$$\begin{aligned}
\frac{a}{2}\nabla b\cdot\nabla F_\eta^2 &= \frac{ab}{2}\nabla\log b\cdot\nabla F_\eta^2 \\
&= -d\frac{b'}{b}|\nabla d|^2 - \frac{d(1-\cos d)}{\sin d}\frac{b'}{b}\nabla d\cdot(\tan R^{u_\eta}\nabla R^{u_\eta} + \tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}}).
\end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned}
ab|\nabla F_\eta|^2 + \frac{a}{2}\nabla b\cdot\nabla F_\eta^2 \geq -\Bigg[&\left(\frac{d\sin d}{2(1-\cos d)} + d\frac{b'}{b}\right)|\nabla d|^2 \\
&+\left(d + \frac{d(1-\cos d)}{\sin d}\frac{b'}{b}\right)\nabla d\cdot(\tan R^{u_\eta}\nabla R^{u_\eta} + \tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}}) \\
&+\frac{d(1-\cos d)}{\sin d}(\tan^2 R^{u_\eta}|\nabla R^{u_\eta}|^2 + \tan^2 R^{u_{1-\eta}}|\nabla R^{u_{1-\eta}}|^2)\Bigg].
\end{aligned}$$

Comparing to (B.6), we solve

$$\frac{d\sin d}{2(1-\cos d)}\nabla d + d\nabla\log b = (1+\frac{1}{2}d\tan\frac{d}{2})\nabla d$$
$$d\nabla d + \frac{d(1-\cos d)}{\sin d}\nabla\log b = \nabla(d\tan\frac{d}{2}).$$

which implies that $b = \frac{d}{\sin d}$, and hence $a = -2\cos R^{u_\eta}\cos R^{u_{1-\eta}}$.

$\square$

Theorem B.2.4 (cf. [60, Corollary 2.3])**.** *Let $u_0, u_1 : \Omega \to \mathcal{B}_\rho(P) \subset X$ be a pair of energy minimizing maps (possibly with different boundary values). Let $d(x) = d(u_0(x), u_1(x))$ and $R^{u_i} = d(u_i, P)$. Then the function*

$$F = \sqrt{\frac{1-\cos d}{\cos R^{u_0}\cos R^{u_1}}}$$

satisfies the differential inequality weakly

$$\operatorname{div}(\cos R^{u_0}\cos R^{u_1}\nabla F) \geq 0.$$

Proof. Let $\eta \in C_c^\infty(\Omega)$ with $\eta \geq 0$. For $t > 0$ sufficiently small, we have $0 \leq t\eta < 1/2$. Let $\hat{u}_{t\eta}$ and $\hat{u}_{1-t\eta}$ be the corresponding maps defined as in Lemma B.2.3. Since u_0 and u_1 minimize the energy among maps of the same boundary values, we have

$$\begin{aligned}
0 &\leq \int_\Omega |\pi(\hat{u}_\eta)|^2 + |\pi(\hat{u}_{1-\eta})|^2 - |\pi(u_0)|^2 - |\pi(u_1)|^2 \, d\mu_g \\
&\leq \int_\Omega -2\cos R^{u_{t\eta}}\cos R^{u_{1-t\eta}}\nabla\left(\frac{d}{\sin d}t\eta F_{t\eta}\right)\cdot\nabla F_{t\eta}\, d\mu_g + t^2\mathrm{Quad}(\eta, \nabla\eta).
\end{aligned}$$

Dividing the inequality by t and let $t \to 0$, since $R^{u_{t\eta}} \to R^{u_0}$ and $R^{u_{1-t\eta}} \to R^{u_1}$ and $F_{t\eta} \to F$, we derive

$$0 \leq \int_\Omega -2\cos R^{u_0}\cos R^{u_1}\nabla\left(\frac{d}{\sin d}\eta F\right)\cdot\nabla F\, d\mu_g$$

$$= 2 \int_\Omega \left(\frac{d}{\sin d} \eta F \right) \mathrm{div} \left(\cos R^{u_0} \cos R^{u_1} \nabla F \right) \, d\mu_g.$$

$\square$