# The Kähler-Ricci flow on non-compact manifolds

by

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# Abstract

We first study the general theory of Kähler-Ricci flow on non-compact complex manifolds. By using a parabolic Schwarz lemma and a local scalar curvature estimate, we prove a general existence theorem for Kähler metrics lying in the $C^\infty_{loc}$-closure of complete bounded curvature Kähler metrics that are uniformly equivalent to a fixed background metric. In particular we do not assume any curvature bounds. Next, we compare the maximal existence time of two complete bounded curvature solutions by using the equivalence of the initial metrics and using this, we also estimate the maximal existence time of a complete bounded curvature solution in terms of the curvature bound of a background metric. We also prove a uniqueness theorem for Kähler-Ricci flow which slightly improves the result of [13] in the Kähler case.

We apply the above results to study the Kähler-Ricci flow on some specific non-compact complex manifolds. We first study the Kähler-Ricci flow on $\mathbb{C}^n$. By applying our general existence theorem and existence time estimate, we show that any complete non-negatively curved $U(n)$-invariant Kähler metric admits a long-time $U(n)$-invariant solution to the Kähler-Ricci flow, and the solution converges to the standard Euclidean metric after rescaling.

Then we study the Kähler-Ricci flow on a quasi-projective manifold $\overline{M}\setminus D$. By modifying the approximation theorem of [1] and applying a general existence theorem of Lott-Zhang [23], we construct a Kähler-Ricci flow solution starting from certain smooth Kähler metrics. In particular, if the metric is the restriction of a smooth Kähler metric in the ambient space $\overline{M}$, then the solution instantaneously becomes complete and has cusp singularity at $D$. We also produce a solution starting from some complete metrics that may not have bounded curvature, and the solution is likewise complete with cusp singularity for positive time. On the other hand, if the initial data has bounded curvature and is asymptotic to the standard cusp model at $D$ in a certain sense, we find the maximal existence time of the corresponding complete bounded curvature solution to the Kähler-Ricci flow.

# Lay Summary

The Ricci flow was introduced by Richard Hamilton in 1982 to solve the famous Poincaré conjecture in mathematics, and it is a geometric process that deforms the shape of a given space according to how the space is curved. Hamilton proved that some nicely curved 3 dimensional spaces deform to a sphere along the Ricci flow. Similar results have been proved for the Ricci flow since then. On the other hand, some spaces cannot be deformed at all due to their roughness, and this thesis discusses conditions under which a space can be deformed along the Ricci flow, and when is there a unique way to carry out the deformation. We also apply the Ricci flow to spaces with certain symmetries. For example, we prove that certain rotationally symmetric spaces will be flattened along the Ricci flow, and we relate this to a well known conjecture in geometry.

# Preface

All of the work presented in this thesis was conducted as I was a PhD student in the Mathematics Department of University of British Columbia.

Materials in Chapter 2 and 3 are from [10], [11] and [12], these are joint works with Professor Albert Chau and Professor Luen-Fai Tam and they are all published. In [10], I contributed the parabolic Schwarz computations and the construction of background $U(n)$-invariant metrics. In [11], all authors contributed equally. In [12], I made contributions to the existence of long-time solution to $U(n)$-invariant Kähler-Ricci flow. Some arguments in [12] are replaced by new original arguments in this thesis.

Materials in Chapter 4 are from [9], which is a joint work with Professor Albert Chau and Doctor Liangming Shen, it has been posted on arXiv. In this work, all authors contributed equally.

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# Acknowledgments

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# Chapter 1

# Introduction

In this thesis, we study the Kähler-Ricci flow on non-compact complex manifolds. Let $(M,g)$ be a n dimensional Kähler manifold, a smooth family of Kähler metrics $g(t)$ is said to satisfy the Kähler-Ricci flow on $M \times [0,T)$ starting from $g$ if

$$\begin{cases} \dfrac{\partial}{\partial t} g_{i\bar{j}}(t) = -R_{i\bar{j}}(g(t)) \\ g(0) = g, \end{cases} \tag{1.0.1}$$

on $M \times [0,T)$, where $R_{i\bar{j}}(g(t))$ is the Ricci curvature of $g(t)$. Any Kähler metric can also be represented by a Kähler form, if $\omega$ is the Kähler form of $g$ then the equation (1.0.1) is equivalent to

$$\begin{cases} \dfrac{\partial}{\partial t} \omega(t) = -\mathrm{Ric}(\omega(t)) \\ \omega(0) = \omega, \end{cases} \tag{1.0.2}$$

where $\mathrm{Ric}(\omega(t)) = -\sqrt{-1}\partial\bar{\partial}\log\omega^n(t)$ is the Ricci form of $\omega(t)$. In the following content, we will use Greek alphabets to denote a Kähler form while Roman alphabets will be used to denote a Kähler metric.

The thesis is divided into three parts. In the first part, we study the general theory of Kähler-Ricci flow on non-compact complex manifolds. We prove an existence theorem for initial metrics that can be approximated by complete Kähler metrics of bounded curvature satisfying some $C^0$ conditions. We also prove an existence time estimate and a uniqueness theorem for complete bounded curvature solutions.

In the second part, we study the Kähler-Ricci flow on $\mathbb{C}^n$ starting from a complete $U(n)$-invariant Kähler metric. We show that if the $U(n)$-invariant Kähler metric has non-negative holomorphic bisectional curvature, then the Kähler-Ricci flow has a corresponding long-time bounded curvature solution to the Kähler-Ricci flow. And the solution we obtain converges to the standard Euclidean metric after rescaling.

In the third part, we study the Kähler-Ricci flow on quasi-projective manifold. A Kähler manifold $M$ is called a quasi-projective manifold if $M = \overline{M} \setminus D$, where $\overline{M}$ is compact Kähler and $D \subset \overline{M}$ is a divisor with normal crossings. We will construct a solution to Kähler-Ricci flow starting from certain initial metrics on $M$ and the solution we construct becomes instantaneously complete and with a cusp singularity at $D$. We also estimate the existence time of our solution.

## 1.1 General Theorems

One of the most fundamental questions in the study of Kähler-Ricci flow is the existence of solutions. Unlike the Kähler-Ricci flow on compact Kähler manifolds, the existence problem on non-compact Kähler manifolds is not a direct consequence of general theory. The most classical result which is due to Shi( [26, 27]) states that if $(M,g)$ is a complete Kähler manifold with bounded curvature, then there is a complete bounded curvature solution to (1.0.1). Shi's solution has been used extensively in the application of Ricci flow on non-compact Riemannian manifolds. Chen-Zhu [13] showed that Shi's solution is unique among complete bounded curvature solutions while Lott-Zhang [23] gave an analytic characterization of the maximal existence time of the solution. As a result, given a complete bounded curvature solution, we can discuss the maximal existence time of the corresponding complete bounded curvature solution to Kähler-Ricci flow. There are other more general existence results concerning complete Kähler metrics on non-compact manifolds in the literature, but they often require rather specific curvature assumptions and their uniqueness and existence time are often not known.

In case the initial metric has unbounded curvature, we prove that if it can be approximated by certain complete bounded curvature metrics, then we can construct a solution to 1.0.1:

**Theorem 1.1.1.** *Let $g_0$ be a complete continuous Hermitian metric on a non-compact complex manifold $M^n$. Suppose there exists a sequence $\{h_{k,0}\}$ of smooth complete Kähler metrics with bounded curvature on $M$ converging uniformly on compact subsets to $g_0$ and another complete Kähler metric $\hat{g}$ on $M$ with bounded curvature and holomorphic sectional curvature bounded from above by $K \geq 0$ such that*

*(i)* $\frac{1}{C}\hat{g} \leq h_{k,0} \leq C\hat{g}$ *for some $C$ independent of $k$;*

*(ii)* $h_k$ *has bounded curvature for every $k$.*

*Let $T = 1/(2nCK)$ if $K > 0$, otherwise let $T = \infty$. Then the Kähler-Ricci flow* (1.0.1) *has a smooth solution $g(t)$ on $M \times (0,T)$ such that*

*(a)* $(1/(nC) - 2Kt)\hat{g} \leq g(t) \leq B(t)\hat{g}$ *on $M \times (0,T)$ for some positive continuous function $B(t)$ depending only on $C$, $\hat{g}$ and $n$.*

*(b)* $g(t)$ *has bounded curvature for $t > 0$. More precisely, for any $0 < T' < T$ and for any $l \geq 0$ there exists a constant $C_l$ depending only on $C$, $l$, $T'$, $\hat{g}$ and the dimension $n$ such that*

$$sup_M |\nabla^l \mathrm{Rm}(g(t))|^2_{g(t)} \leq \frac{C_l}{t^{l+2}},$$

*(c)* $g(t)$ *converges uniformly on compact subsets to $g_0$ as $t \to 0$.*

*Moreover, if $g_0$ is smooth and $\{h_{k,0}\}$ converges smoothly and uniformly on compact subsets of $M$, then $g(t)$ extends to a smooth solution on $M \times [0,T)$ with $g(0) = g_0$.*

The proof of this theorem is based on estimating the local equivalence between the evolving metric and the fixed background metric $\hat{g}$. This is achieved by using a parabolic Schwarz lemma, an Aubin-Yau trace estimate and a local scalar curvature estimate. After establishing the equivalence, the higher order estimates follow from Evans-Krylov theory. Furthermore, our proof to this theorem actually shows that if $h$ is a complete bounded curvature Kähler metric such that $\frac{1}{C}\hat{g} \leq h \leq C\hat{g}$ for some constant $C$, then the maximal existence time of Shi's solution satisfies $T \geq \frac{1}{2nCK}$ if $K > 0$ and $T = \infty$ otherwise. That is, we can estimate the existence time of a complete bounded curvature solution by the curvature bound of an equivalent complete bounded curvature Kähler metric.

On the other hand, we will also prove a theorem which compares the maximal existence time of two complete bounded curvature metrics using their $C^0$ data.

**Theorem 1.1.2.** *If $g$ and $h$ are two equivalent complete Kähler metrics of bounded curvature and $C$ is a constant such that $g \leq Ch$, then $T(g) \leq CT(h)$. Here $T(\cdot)$ is the maximal existence time of the complete bounded curvature solution to the Kähler-Ricci flow starting from the given metric.*

In particular, our result implies that if $g$ is a complete Kähler metric of bounded curvature with $T(g) = \infty$, then this property is shared by all complete bounded curvature Kähler metrics equivalent to $g$.

We also prove a theorem about the uniqueness:

**Theorem 1.1.3.** *Let $(M^n, \widehat{g})$ be a complete non-compact Kähler manifold. Suppose there is an exhaustion function $\zeta > 0$ on $(M^n, \widehat{g})$ with $\lim_{x\to\infty} \zeta(x) = \infty$ such that $|\partial\bar{\partial}\zeta|_{\widehat{g}}$ and $|\widehat{\nabla}\zeta|_{\widehat{g}}$ are bounded.*

*Let $g_1(x,t)$ and $g_2(x,t)$ be two solutions of the Kähler-Ricci flow (1.0.1) on $M \times [0,T]$ with the same initial data $g_0(x) = g_1(x,0) = g_2(x,0)$. Suppose there is a positive function $\sigma$ with $\lim_{x\to\infty} \log\sigma(x)/\log\zeta(x) = 0$ such that the following conditions hold for all $(x,t) \in M \times [0,T]$:*

*(i)*

$$\widehat{g}(x) \leq \zeta(x) g_1(x,t); \;\; \widehat{g}(x) \leq \zeta(x) g_2(x,t),$$

*(ii)*

$$-\sigma(x) \leq \frac{\det((g_1)_{i\bar{j}}(x,t))}{\det((g_2)_{i\bar{j}}(x,t))} \leq \sigma(x).$$

*Then $g_1 \equiv g_2$ on $M \times [0,T]$.*

This theorem implies that if $g_1$ and $g_2$ are uniformly equivalent to $\widehat{g}$ on $M \times [0,T]$, then $g_1 \equiv g_2$. This slightly improves the uniqueness result of Chen-Zhu in the Kähler case.

## 1.2 Non-negatively curved $U(n)$-invariant metrics

The result in Chapter 3 is related to Yau's uniformization conjecture for non-compact Kähler manifolds: Let $(M^n, g)$ be a complete non-compact Kähler manifold with positive holomorphic bisectional curvature, then $M$ is biholomorphic to $\mathbb{C}^n$. This is a long standing conjecture and there have been several partial results so far, in particular using the Kähler-Ricci flow. For example, Chau-Tam [8] used Kähler-Ricci flow to prove that a complete non-compact Kähler manifold with bounded non-negative holomorphic bisectional curvature and maximal volume growth is biholomorphic to $\mathbb{C}^n$. Recently, this result was generalized by Liu [21] by removing the bounded curvature condition. Liu did not use Kähler-Ricci flow in his proof, but his result was re-proved by Lee-Tam [20] using the Kähler-Ricci flow.

The $U(n)$-invariant Kähler metrics on $\mathbb{C}^n$ were first studied by Wu-Zheng [33] in an attempt to give examples of positively curved compete Kähler metrics on $\mathbb{C}^n$. Prior to their works, there were only three examples of positively curved complete Kähler metrics on $\mathbb{C}^n$ and they are all $U(n)$-invariant ( [6, 7, 19]). In Wu-Zheng's work, they gave a very convenient parametrization of the complete $U(n)$-invariant metrics with non-negative bisectional curvature, in particular illustrating that these generically had unbounded curvature. Motivated by these examples, Yang-Zheng [35] showed that if $g(t)$ is a complete $U(n)$-invariant solution to the Kähler-Ricci flow with initial metric having non-negative holormophic bisectional curvature, then the solution will also have non-negative holomorphic bisectional curvature for all time. Furthermore the asymptotic volume ratio remains constant along the flow. They also used the construction of Cabezas-Rivas and Wilking [2] to produce a short time solution for complete $U(n)$-invariant initial metric with non-negative sectional curvature satisfying some technical assumptions. Using a different approach, we prove that:

**Theorem 1.2.1.** *Let $g_0$ be a complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ with non-negative holomorphic bisectional curvature. Then*

*(i) the Kähler-Ricci flow* (1.0.1) *has a unique smooth long-time $U(n)$-invariant solution $g(t)$ which is equivalent to $g_0$ and has bounded non-negative bisectional curvature;*

*(ii)* $g(t)$ *converges, after rescaling at the origin, to the standard Euclidean metric on* $\mathbb{C}^n$.

Here $g(t)$ is equivalent to $g_0$ means that for all $T < \infty$, there exists constant $C$ such that $\frac{1}{C}g_0 \le g(t) \le Cg_0$ for all $t \in [0,T]$. Our result does not assume any volume growth rate on the initial metrics but yet the solution converges to the standard Euclidean metric in a certain sense. This showed that the Kähler-Ricci flow may still be used to attack Yau's conjecture even if we do not assume any volume growth conditions.

## 1.3 Kähler-Ricci flow on quasi-projective manifolds

In Chapter 4, we study the Kähler-Ricci flow on quasi-projective manifolds. A Kähler manifold $M$ is called a quasi-projective manifold if $M = \overline{M} \setminus D$, where $\overline{M}$ is Kähler and $D \subset \overline{M}$ is a divisor with normal crossings. $M$ is clearly a non-compact manifold, given any Kähler metric on $M$, we can define different notions of singularity at $D$. A metric on $M$ is said to have cusp singularities at $D$ if it is equivalent to a Carlson-Griffiths type form $\eta - i\partial\bar{\partial}\log\log^2|S|^2$ where $\eta$ is a Kähler form on $\overline{M}$ and $S$ is the holomorphic section of $[D]$ that vanishes on $D$. The Carlson-Griffiths form is analogous to the Poincaŕe metric on the punctured disk, with the divisor corresponding to the origin. In particular, cusp metrics are complete on $M$ with possibly unbounded curvature. In Lott-Zhang [23], they considered bounded curvature cusp metrics satisfying certain asymptotics conditions at $D$, and showed that their maximal existence time for (1.0.1) is bounded above by

$$T_{[\omega_0]} := \sup\{T : [\eta] + T(c_1(K_{\overline{M}}) + c_1(\mathscr{O}_D)) \in \mathscr{K}_{\overline{M}}\} \tag{1.3.1}$$

where $\mathscr{K}_{\overline{M}}$ is the Kähler cone of $\overline{M}$.

The main theorems in Chapter 4 are:

**Theorem 1.3.1.** *Let* $\omega_0 = \eta + i\partial\bar{\partial}\varphi_0$ *be a smooth Kähler metric on M, and let* $T_{[\omega_0]}$ *be given by (1.3.1).*

*(a) If* $\varphi_0 \in L^\infty(\overline{M}) \bigcap C^\infty(M) \bigcap PSH(\overline{M},\eta)$ *and* $\omega_0 \ge c\eta$ *for some constant* $c > 0$.

*Then* (1.0.1) *has a unique smooth solution* $\omega(t)$ *on* $M \times [0, T_{[\omega_0]})$ *where*

$$c_1(t)\hat{\omega} \le \omega_t \le c_2(t)\hat{\omega} \tag{1.3.2}$$

*for all* $t \in (0, T_{[\omega_0]})$ *and some positive functions* $c_i(t)$*.*

(b) *Let* $\varphi_0 \in C^\infty(M) \bigcap PSH(\overline{M}, \eta)$ *have zero Lelong number and* $\omega_0 \ge c\hat{\omega}$ *for some* $c > 0$*. Then the Kähler -Ricci flow* (1.0.1) *has a smooth solution* $\omega(t)$ *on* $M \times [0, T_{[\omega_0]})$ *and*

$$\omega(t) \ge (\frac{1}{n} - \frac{4\hat{K}t}{c})\hat{\omega} \tag{1.3.3}$$

*for all* $t \le \frac{c}{4n\hat{K}}$*.*

As a particular case of part (a), if $\omega_0$ is the restriction of a Kähler metric on $\overline{M}$ or has conical singularities at $D$, the solution instantaneously becomes complete and is equivalent to a cusp metric for $t > 0$. On the other hand, the initial metric in (b) is complete with possibly unbounded curvature, and there the solution is likewise complete for a uniform amount of time.

There have been several earlier works about solutions to the Kähler-Ricci flow with conical singularities at $D$ (see for example [14], [22], [25]). These works established the characterization of the existence time in terms of the cohomology of $\overline{M}$, and also produced an instantaneously conical solution starting from initial data that is restriction from smooth metrics on $\overline{M}$. Our results can be viewed as a cuspidal versions of these results.

In addition to Theorem 1.3.1, we will also discuss the maximal existence time of a complete bounded curvature solution starting from a Kähler metric which has bounded curvature and is asymptotic to a cusp model in a certain sense.

# Chapter 2

# General Theorems

## 2.1 Background

The Ricci flow was first introduced by Hamilton [18] to prove that a closed 3 dimensional Riemannian manifold with positive Ricci curvature is diffeomorphic to a spherical space form or its quotient. Since then the Ricci flow has been applied to many problems in geometry, it was first applied to Kähler manifolds by Cao in [4] who used the flow to re-prove the Calabi-Yau theorem.

The behavior of Ricci flow on compact Kähler manifolds has been well studied. Suppose that $(M, g)$ is a closed Kähler manifold, then it is known that a short-time solution $g(t)$ to the Ricci flow exists and the solution remains Kähler . The existence follows from the fact that on Kähler manifolds the Ricci flow is a parabolic system of partial differential equations. In fact by applying the $\partial\bar{\partial}$-lemma, it can be shown here the Ricci flow is equivalent to a parabolic Monge-Ampère equation for a scalar function. With the aid of this equivalence, the uniqueness follows readily by a maximum principle argument. The maximal existence time of the solution was characterized by Tian-Zhang [31] in terms of the cohomology class of the initial metric:

**Theorem 2.1.1** (Tian-Zhang [31])**.** *Let $(M, \omega)$ be a closed Kähler manifold, suppose that $\omega(t)$ solves the Kähler-Ricci flow. Then the maximal existence time of the* *solution is given by*

$$T_{[\omega]} = \sup\{t > 0 : [\omega] - tc_1(M) > 0\}.$$

Here we say a cohomology class $[\alpha] > 0$ if there is a Kähler form inside the class $[\alpha]$, and $c_1(M)$ is the first Chern class of the manifold, which is equal to the class $[\mathrm{Ric}(\omega)]$ for arbitrary Kähler metric $\omega$ on $M$.

When the Kähler manifold $(M, g)$ is non-compact, we do not expect the Ricci flow to have a solution unless $g$ satisfies some extra conditions. And even if it exists, the solution $g(t)$ may not be Kähler or even Hermitian for $t > 0$. In case $g(t)$ solves the Ricci flow:

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -Ric(g(t)) \\ g(0) = g. \end{cases} \tag{1.0.1}$$

and $g(t)$ is Kähler for all $t$, we will call it a solution to Kähler-Ricci flow. The research of Kähler-Ricci flow on non-compact manifolds was pioneered by Shi ([26, 27]), and he proved the short time existence of Kähler-Ricci flow on complete non-compact Kähler manifolds with bounded curvature:

**Theorem 2.1.2.** *Let $(M^n, g_0)$ be a complete non-compact Kähler manifold with curvature bounded by a constant K. Then for some $0 < T \leq \infty$ depending only on K and the dimension n, there exists a smooth solution $g(t)$ to* (1.0.1) *on $M \times [0, T)$ with $g(0) = g_0$ such that*

*(i)* *$g(t)$ is Kähler and equivalent to $g_0$ for all $t \in [0, T)$;*

*(ii)* *$g(t)$ has uniformly bounded curvature on $M \times [0, T']$ for all $0 < T' < T$. More precisely, for any $l \geq 0$ there exists a constant $C_l$ depending only on l, $T'$, $g_0$ and the dimension n such that*

$$sup_M |\nabla^l \mathrm{Rm}(g(t))|^2_{g(t)} \leq \frac{C_l}{t^l},$$

*on $M \times [0, T']$.*

*(iv)* *If $T < \infty$ and $\lim_{t \to T} \sup_M |Rm(x, t)| < \infty$, then $g(t)$ extends to a smooth solution to*

(1.0.1) *on* $M \times [0, T_1)$ *for some* $T_1 > T$ *so that (ii) is still true with* $T$ *replaced by* $T_1$.

From now on, given a bounded curvature complete $g_0$, we will refer to a solution $g(t)$ to (1.0.1) satisfying (i), (ii) as Shi's solution to (1.0.1). In fact, Chen-Zhu [13] showed that Shi's solution is unique. More precisely,

**Theorem 2.1.3** (Chen-Zhu [13])**.** *Let* $(M, g_0)$ *be a complete non-compact Kähler manifold of bounded curvature, if* $g_1(t)$, $g_2(t)$ *are solutions to Kähler-Ricci flow on* $M \times [0, T]$ *such that* $g_i(t)$ $(i = 1, 2)$ *are complete, have uniformly bounded curvature and* $g_i(0) = g_0$*, then* $g_1(t) = g_2(t)$.

*Remark* 2.1.1. Chen-Zhu's uniqueness theorem was originally proved for general real Ricci flow, the version we stated here is the restriction to the Kähler case.

Lott-Zhang [23] also found a similar existence time characterization for Shi's solution like Theorem 2.1.1, although it is no longer described by the cohomology class of $\overline{M}$:

**Theorem 2.1.4** (Lott-Zhang [23])**.** *Let* $g_0$ *be a complete Kähler metric with bounded curvature on a non-compact manifold* $M$*. Then the maximal existence time* $T_{g_0}$ *of a bounded curvature solution is equal to the supremum of the numbers* $T$ *for which there is a bounded function* $F_T \in C^\infty(M)$ *such that*

*(i)* $g_0 - T\mathrm{Ric}(g_0) + \sqrt{-1}\partial\bar{\partial}F_T \geq c_T g_0$ *for some* $c_T > 0$

*(ii)* $|F_T|$ *and the quantities* $|\nabla^l \mathrm{Rm}(g_0)|_{g_0}$, $|\nabla^l \partial\bar{\partial}F_T|_{g_0}$*, for* $0 \leq l \leq 2$*, are uniformly bounded on* $M$.

*Moreover if* $T$ *satisfies (i) and (ii), then for any* $T' < T$*, there is a constant* $C$ *depending* $T'$, $c_T$*, and the bound on the quantities in (ii) such that*

$$C^{-1} g_0 \leq g(t) \leq C g_0$$

*on* $M \times [0, T']$.

In the following sections, we will construct a solution to (1.0.1) for a complete Kähler metric $g_0$ that can be approximated by a sequence of complete bounded curvature Kähler metrics satisfying some $C^0$ assumptions. Each of sequence members admits a solution to Kähler-Ricci flow by Shi's Theorem 2.1.2, then by establishing some $C^0$ estimates, we can use Evans-Krylov theory to conclude that the sequence of solutions converges to a limit solution starting from $g_0$. We will also prove a uniqueness result of Kähler-Ricci flow which can be applied to unbounded curvature metrics satisfying certain conditions.

## 2.2 $C^0$ estimates in terms of background metrics

Before we prove the existence theorems, we need to establish some $C^0$ estimates for solutions of Kähler-Ricci flow in terms of a background metric, these estimates allow us to use the Evans-Krylov theory to conclude that our approximating sequence converges.

In the following, $\hat{\nabla}$ always denotes the covariant derivative of $\hat{g}$.

**Lemma 2.2.1.** *Let $h(t)$ be a solution to* (1.0.1) *on $M^n \times [0,T_0)$ with $h(0) = h_0$ such that $h(t)$ has uniformly bounded curvature on $M \times [0,T']$ for all $0 < T' < T_0$. Let $\hat{g}$ be another complete Kähler metric on $M$ with bounded curvature such that the holomorphic bisectional curvature bounded above by $K \geq 0$. Let $T = \frac{1}{2nK}$ if $K > 0$, otherwise let $T = \infty$.*

*(i) Suppose $h_0 \geq \hat{g}$. Then $h(t) \geq \left(\frac{1}{n} - 2Kt\right)\hat{g}$ on $M \times [0, \min\{T_0, T\})$.*

*(ii) Suppose in addition to (i) we have $h_0 \leq C\hat{g}$, that is, suppose $\hat{g} \leq h_0 \leq C\hat{g}$, then*

$$(1 - w(t))\hat{g} \leq h(t) \leq (1 + w(t))\hat{g}$$

*on $M \times [0, \min\{T_0, T\})$,*

*where $w(t) = \sqrt{v_2(t)(v_1(t) + v_2(t) - 2n)}$,*

$$v_1(t) = \frac{1}{\frac{1}{n} - 2Kt} \ , v_2(t) = nCe^{-2\kappa v_1(t)t}$$

*and $\kappa$ is a lower bound on the bisectional curvature of $\hat{g}$. In particular, we have $w(0) = n\sqrt{C(C-1)}$.*

*Proof.* (i) Let $\phi(t) := \mathrm{tr}_{h(t)}\hat{g} = h(t)^{i\bar{j}}\hat{g}_{i\bar{j}}$. Let $\Box = \frac{\partial}{\partial t} - \Delta$, where $\Delta$ is the Laplacian with respect to $h(t)$. Then as in [28], we can calculate in a normal coordinate relative to $h(t)$ and use (1.0.1) to get

$$\begin{aligned}\Box\phi =&((h_t)^{i\bar{j}}\hat{g}_{i\bar{j}}) - h^{k\bar{l}}(h^{i\bar{j}}\hat{g}_{i\bar{j}})_{k\bar{l}} \\ =&(R^{i\bar{j}}\hat{g}_{i\bar{j}}) - (R^{i\bar{j}}\hat{g}_{i\bar{j}}) + h^{k\bar{l}}h^{i\bar{j}}\widehat{R}_{i\bar{j}k\bar{l}} - \widehat{g}^{p\bar{q}}h^{k\bar{l}}h^{i\bar{j}}\partial_k\widehat{g}_{i\bar{q}}\partial_{\bar{l}}\widehat{g}_{p\bar{j}} \\ &\le 2K\phi^2.\end{aligned} \tag{2.2.1}$$

Now $v_1(t)$ is the positive solution to the ODE

$$\frac{dv_1(t)}{dt} = 2Kv_1^2(t);\ \ v_1(0) = n$$

for $t \in [0,T)$. Let $S \in (0, \min\{T_0,T\})$ be fixed. Since $h(t)$ has uniformly bounded curvature on $M \times [0,S]$ we have $h(t) \ge C_1 h_0 \ge C_1\widehat{g}$ for some $C_1 > 0$ and hence $\phi$ is a bounded function on $M \times [0,S]$. Moreover, $v_1$ is also a bounded function on $M \times [0,S]$. Let $A = \sup_{M\times[0,S]}(\phi + v_1)$. Then on $M \times [0,S]$

$$\begin{aligned}&\Box\left(e^{-(2AK+1)t}(\phi - v_1)\right) \\ &\qquad \le e^{-(2AK+1)t}\quad \left[2K(\phi^2 - v_1^2) - (2AK+1)(\phi - v_1)\right] \\ &\qquad = e^{-(2AK+1)t}\quad \left[2K(\phi + v_1) - (2AK+1)\right](\phi - v_1)\end{aligned}$$

which is nonpositive at the points where $\phi - v_1 \ge 0$. Using the fact that $h(t)$ has uniformly bounded curvature on $M \times [0,S]$ and the fact that $e^{-(2AK+1)t}(\phi - v_1) \le 0$ at $t = 0$, which is uniformly bounded on $M \times [0,S]$, we conclude that $e^{-(2AK+1)t}(\phi - v_1) \le 0$ and thus $(\phi - v_1) \le 0$ on $M \times [0,S]$ by the maximum principle, see [24, Theorem 1.2] for example. This proves (i).

(ii) Let $\psi(t) := \mathrm{tr}_{\hat{g}}h(t)$. For any fixed $S \in [0, min\{T_h,T\})$, as in [4] we calculate in a normal coordinate relative to $\hat{g}$ and use (1.0.1) to get that on $M \times [0,S)$:

$$
\begin{aligned}
\Box\psi =&(\hat{g}^{i\bar{j}}(h_t)_{i\bar{j}}) - h^{k\bar{l}}(\hat{g}^{i\bar{j}}h_{i\bar{j}})_{k\bar{l}} \\
=&-(\hat{g}^{i\bar{j}}R_{i\bar{j}}) - h^{k\bar{l}}(\hat{R}^{i\bar{j}}_{k\bar{l}}h_{i\bar{j}}) + (\hat{g}^{i\bar{j}}R_{i\bar{j}}) - \hat{g}^{i\bar{j}}h^{p\bar{q}}h^{k\bar{l}}\partial_i h_{p\bar{l}}\partial_{\bar{j}}h_{k\bar{q}} \\
=&-h^{k\bar{l}}h_{i\bar{j}}\widehat{R}^{\bar{j}i}_{k\bar{l}} - \widehat{g}^{i\bar{j}}h^{p\bar{q}}h^{k\bar{l}}\partial_i h_{p\bar{l}}\partial_{\bar{j}}h_{k\bar{q}} \\
\leq& -2\kappa v_1(t)\psi \\
\leq& -2\kappa v_1(S)\psi
\end{aligned}
\tag{2.2.2}
$$

by (i). Let $w_S(t) = nCe^{-2cv_1(S)t}$ be the solution to the ODE

$$
\frac{dw_S(t)}{dt} = -2cv_1(S)w_S(t);\;\; w_S(0) = nC.
$$

Then arguing as above, we have $\psi \leq w_S$ on $M^n \times [0,S]$. In particular, we get $\psi(S) \leq w_S(S)$ for every $S \in [0, min\{T_0, T\})$.

So far, we have $\phi(t) \leq v_1(t)$, and $\psi(t) \leq v_2(t)$ on $M \times [0, min\{T_0, T\})$ where $v_1, v_2$ are as in the statement of the Lemma. Now we follow an idea from [26]. At any point in $(p,t) \in M \times [0, min\{T_0, T\})$, let $\lambda_i' s$ be the eigenvalues of $h$ with respect to $\hat{g}$, and calculate at $(p,t)$

$$
\begin{aligned}
\sum_{i=1}^{n} \frac{1}{\lambda_i}(1-\lambda_i)^2 =& \sum_{i=1}^{n} \frac{1}{\lambda_i} + \lambda_i - 2 \\
=& \phi + \psi - 2n \\
\leq& v_1(t) + v_2(t) - 2n
\end{aligned}
\tag{2.2.3}
$$

and thus for any fixed $i$ we have

$$
-w(t) \leq \lambda_i - 1 \leq w(t) \tag{2.2.4}
$$

where $w(t) = \sqrt{v_2(t)(v_1(t) + v_2(t) - 2n)}$. The conclusion in (ii) then follows.

□

The following lemma basically says that if a local solution $h(t)$ to (1.0.1) is a priori uniformly equivalent to a fixed metric $\hat{g}$ in space time, and close to $\hat{g}$ at time $t=0$, then it remains close to $\hat{g}$ in a uniform space time region. Note that in contrast to Lemma 2.2.1, the a priori assumption here is on $h(t)$ for all $t$.

**Lemma 2.2.2.** *Let $h(t)$ be a smooth solution to* (1.0.1) *on $B(1)\times[0,T)$ with $h(0)=h_0$ where $B(1)$ is the unit Euclidean ball in $\mathbb{C}^n$. Let $\hat{g}$ be a smooth Kähler metric on $B(1)$. Suppose*

$$N^{-1}\hat{g}\le h(t)\le N\hat{g} \tag{2.2.5}$$

*on $B(1)\times[0,T)$ for some $N>0$, and that*

$$\hat{g}\le h_0\le C\hat{g} \tag{2.2.6}$$

*on $B(1)$. Then there exists a positive continuous function $a(t):[0,T)\to\mathbb{R}$ depending only on $\hat{g},N,C$ and $n$ such that*

$$\frac{(1-a(t))}{C}h_0\le h\le(1+a(t))h_0 \tag{2.2.7}$$

*on $B(1/2)\times[0,T)$, where $a(0)=n\sqrt{C(C-1)}$.*

*Proof.* As in the previous Lemma, let $\phi=\mathrm{tr}_h\hat{g}$, $\psi=\mathrm{tr}_{\hat{g}}h$ on $B(1)\times[0,T_0)$. Choose some smooth non-negative cutoff function on $\eta:B(1)\to\mathbb{R}$ satisfying $\eta|_{B(1/2)}=1$, $\eta|_{(B(3/4))^c}=0$, $|\hat{\nabla}\eta|^2\le C_1\eta$, $|\partial\bar{\partial}\eta|_{\hat{g}}\le C_2$ on $B(1)$ for some constants $C_1,C_2$ depending only on $\hat{g}$. Using the fact that $h(t)\ge N^{-1}\hat{g}$, we have

$$|\nabla\eta|^2=h^{i\bar{j}}\eta_i\eta_{\bar{j}}\le N|\hat{\nabla}\eta|^2\le NC_1,$$

and

$$|\Delta\eta|=\left|h^{i\bar{j}}\eta_{i\bar{j}}\right|\le N|\partial\bar{\partial}\eta|_{\hat{g}}\le NC_2.$$

Now we consider the function $\eta\phi$ on $B(1)\times[0,T)$. Then in $B(1)\times[0,T)$ at the point where $\eta > 0$, as in the proof of Lemma 2.2.1 (i) we obtain

$$\begin{aligned}(\partial_t - \Delta)(\eta\phi) &= \eta(\partial_t - \Delta)\phi - 2 < \nabla\eta, \nabla\phi > -\phi\Delta\eta \\ &= \eta(\partial_t - \Delta)\phi - 2\frac{< \nabla\eta, \nabla(\eta\phi) >}{\eta} + \frac{2|\nabla\eta|^2}{\eta}\phi - \phi\Delta\eta \\ &\leq \eta C_3\phi^2 - 2\frac{< \nabla\eta, \nabla(\eta\phi) >}{\eta} + 2NC_1\phi + NC_2\phi \\ &\leq C_4 - 2\frac{< \nabla\eta, \nabla(\eta\phi) >}{\eta}\end{aligned} \tag{2.2.8}$$

where the constants $C_3, C_4$ depend only on $\hat{g}, N, C$ and $n$, where we have used the assumption (2.2.7). Since $\eta\phi$ is zero outside $B(3/4)$, applying the maximum principle to $\eta\phi - C_4 t$ one can conclude that

$$\eta\phi \leq n + C_4 t =: \tilde{v_1}(t)$$

on $B(1) \times [0, T)$.

Now consider the function $\eta\psi$ on $B(1) \times [0, T)$. Using the proof of Lemma 2.2.1 (ii) and estimating as above we obtain

$$\eta\psi \leq nC + C_5 t =: \tilde{v_2}(t) \tag{2.2.9}$$

on $B(1) \times [0, T)$ for som constants $C_5$ depending only on $\hat{g}, N, C$ and $n$.

Now at any point in $(p, t) \in B(1/2) \times [0, T)$, let $\lambda_i' s$ be the eigenvalues of $h$ with respect to $\hat{g}$. Then as in the proof of Lemma 2.2.1 (ii) we get that at $(p, t)$

$$-\tilde{w}(t) \leq \lambda_i - 1 \leq \tilde{w}(t) \tag{2.2.10}$$

where $\tilde{w}(t) = \sqrt{\tilde{v_2}(t)(\tilde{v_1}(t) + \tilde{v_2}(t) - 2n)}$. Since $\tilde{v_1}(0) = n$ and $\tilde{v_2}(0) = nC$, the lemma follows easily from this. □

In contrast to the previous lemma, in the following lemmas we only assume a lower bound on a solution $h(x, t)$ to (1.0.1).

**Lemma 2.2.3.** *Let $h(x, t)$ be a smooth solution to* (1.0.1) *on $M \times [0, T)$ with $h(0) = h_0$. Let $p \in M$. Suppose there is a positive continuous function $\alpha(t) : [0, T) \to \mathbb{R}$*

*such that*

$$h(t) \geq \alpha(t)\hat{g}.$$

*where $\hat{g}$ is a complete Kähler metric with bounded curvature. Then, there exists a positive continuous function $\beta(r,t) : [1,\infty) \times [0,T) \to \mathbb{R}$ depending only on $\hat{g}$ the upper bound of $\mathrm{tr}_{\hat{g}} h_0$ in $B_{\hat{g}}(p,2r)$, the lower bound of scalar curvature $R(0)$ of $h(0)$ in $B_{\hat{g}}(p,2r)$, $\alpha(t)$ and the dimension $n$ such that for $r \geq 1$*

$$h(t) \leq \beta(r,t)\hat{g}.$$

*in $B_{\hat{g}}(p,r) \times [0,T)$.*

*Proof.* Let $d(x)$ be the distance with respect to $\hat{g}$ from $x$ to a fixed point $p \in M$. Since $\hat{g}$ has bounded curvature, by [27] there exists a smooth positive function $\rho(x)$ satisfying $d(x)+1 \leq \rho(x) \leq d(x)+C$ on $M$ for some $C > 0$, with $|\hat{\nabla}\rho|$, $|\hat{\nabla}^2\rho|$ are bounded on $M$. Hence without loss of generality, we may assume for simplicity that $d(x)$ is in fact smooth with $|\hat{\nabla}d|, |\hat{\nabla}^2 d|$ bounded on $M$.

Let $\phi(s)$ be smooth function on $\mathbb{R}$ such that $\phi = 1$ for $s \leq 1$ and is zero for $s \geq 2$. Moreover, we assume $\phi' \leq 0$, $(\phi')^2/\phi \leq C_1$, $|\phi''| \leq C_2$. Let $R$ be the scalar curvature of $h(t)$. Then

$$\left(\frac{\partial}{\partial t} - \Delta\right) R \geq \frac{1}{n} R^2. \tag{2.2.11}$$

on $M \times [0,T)$. Let $\varphi(x) = \phi(d(x)/r)$. Then $\varphi(x) = 0$ if $d(x) \geq 2r$. Fix some $T' < T$. Then as in the proof of the previous lemma, we compute

$$\begin{aligned}
|\nabla\varphi|^2 =& \frac{1}{r^2}(\phi')^2|\nabla d|^2 \\
=& \frac{1}{r^2}(\phi')^2 h^{i\bar{j}} d_i d_{\bar{j}} \\
\leq& \frac{1}{r^2\alpha(t)}(\phi')^2 \hat{g}^{i\bar{j}} d_i d_{\bar{j}} \\
\leq& \frac{C_3}{r^2}(\phi')^2
\end{aligned}$$

on $B(2r)\times[0,T']$ for some constant $C_3$ depending only on $T',\alpha(t)$ and $\hat{g}$. Similarly,

$$
\begin{aligned}
|\Delta\varphi| =&|\frac{1}{r}\phi'\Delta d+\frac{1}{r^2}\phi''|\nabla d|^2|\\
\leq& C_4(\frac{1}{r}+\frac{1}{r^2})
\end{aligned}
$$

on $B(2r)\times[0,T']$ where $C_4$ depends on $C_1,C_2,T',\alpha(t)$ and $\hat{g}$.

Now

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)(\varphi R) =&\varphi\left(\frac{\partial}{\partial t}-\Delta\right)R-R\Delta\varphi-2\langle\nabla R,\nabla\varphi\rangle\\
\geq&\frac{1}{n}\varphi R^2-C_5|R|-2\langle\nabla R,\nabla\varphi\rangle
\end{aligned}
\tag{2.2.12}
$$

on $B(2r)\times[0,T']$ where $C_5$ depends only on $C_4$ and $r$. Suppose the infimum of $\varphi R$ on $B(2r)\times[0,T']$ is attained at $t=0$, then $R\geq\min\{0,\inf_{B_{\hat{g}}(p,2r)}R(h_0)\}$ on $B_{\hat{g}}(r)$. Suppose instead that $\varphi R$ attains a negative minimum at some $(x,t)\in B(2r)\times[0,T']$ where $t>0$. Then at $(x,t)$, $\nabla R=-\frac{R\nabla\varphi}{\phi}$. Hence at this point,

$$
0\geq\frac{1}{n}\varphi R^2-C_6|R| \tag{2.2.13}
$$

where $C_6$ depends only on $C_6,C_3$ and $r$. Hence

$$
\varphi^2|R|\leq nC_6.
$$

on $B(2r)\times[0,T']$ and we conclude that $R\geq -C_7$ on $B_{\hat{g}}(p,r)\times[0,T']$ for some $C_7$ depending only on $T',\hat{g},r,\alpha(t)$. On the other hand,

$$
\frac{\partial}{\partial t}\log\left(\frac{\det(h_{\alpha\bar{\beta}})(t)}{\det(h_{\alpha\bar{\beta}}(0))}\right)=-R\leq C_7.
$$

So

$$
\frac{\det(h_{\alpha\bar{\beta}})(t)}{\det(\hat{g}_{\alpha\bar{\beta}})}\leq e^{C_7t}\frac{\det(h_{\alpha\bar{\beta}})(0)}{\det(\hat{g}_{\alpha\bar{\beta}})}.
$$

on $B_{\hat{g}}(p,r)\times[0,T']$. Let $\lambda_i$ be eigenvalues of $h(t)$ with respect to $\hat{g}$. By assumption, $\lambda_i(x,T')\geq\alpha(T')$ for each $i$ and $x\in B_{\hat{g}}(p,r)$, and the above inequality then implies

$\lambda_i(x,T') \le \beta(r,T')$ for some $\beta(r,T')$ depending only on the those constants listed in the Lemma. Moreover, it is not hard to see that $\beta(r,T')$ can be chosen to depend continuously on $r,T'$ as $\alpha(t)$ is continuous. The Lemma follows as $T'$ was chosen arbitrarily. $\square$

*Remark* 2.2.1. The completeness assumption of $\hat{g}$ is only used to ensure the existence of an exhaustion function $\rho$ with bounded gradient and Hessian, it could be dropped if such $\rho$ could be constructed independently.

*Remark* 2.2.2. Given only a local solution $h(t)$ to (1.0.1) on $B(1)\times[0,T)$ where $B(1)$ is the unit ball on $\mathbb{C}^n$, it is not hard to see from its proof that the conclusion of Lemma 2.2.3 will hold in $B(r)\times[0,T)$ for all $r \le 1/2$.

## 2.3 General existence theorems

We will now prove the main general existence Theorems for (1.0.1) using the estimates in the previous section. Theorems 2.3.1 and 2.3.2 provide general existence Theorems for (1.0.1) when the initial Kähler metric is realized as a limit of a sequence of Kähler metrics satisfying certain properties. In fact, our initial metric may have unbounded curvature or may even be only Hermitian continuous with curvature undefined. When the initial metric is only Hermitian, the convergence of the solution $g(t)$ at time zero has to be understood in the $C^0$ sense.

In the following, we say that a sequence of smooth metrics $h_k$ converge uniformly (resp. smoothly) to a metric $g$ on a set $U$, if $h_k$ converge to $g$ in the $C^0$ (resp. $C^k$ for all $k$) norm on $U$.

**Theorem 2.3.1.** *Let $g_0$ be a complete continuous Hermitian metric on a noncompact complex manifold $M^n$. Suppose there exists a sequence $\{h_{k,0}\}$ of smooth complete Kähler metrics with bounded curvature on $M$ converging uniformly on compact subsets to $g_0$ and another complete Kähler metric $\widehat{g}$ on $M$ with bounded curvature and holomorphic bisectional curvature bounded from above by $K \ge 0$ such that*

*(i)* $h_{k,0} \ge \hat{g}$ *for all $k$;*

*(ii) for every k, the Kähler-Ricci flow* (1.0.1) *has smooth solution $h_k(t)$ with initial data $h_{k,0}$ on $M \times [0,T')$ for some $T' > 0$ independent of k such that the curvature of $h_k(t)$ is uniformly bounded on $M \times [0,T_1]$ for all $0 < T_1 < T'$;*

*(iii) The scalar curvature $R_k$ of $h_{k,0}$ satisfies: for any $r > 0$, there exists a constant $C_r > 0$ such that $R_k \geq -C_r$ on $B_{\hat{g}}(p,r)$ for some fixed point $p \in M$ and all k.*

*Let $T = \min\{T', \frac{1}{2nK}\}$ if $K > 0$, otherwise let $T = T'$. Then the Kähler-Ricci flow* (1.0.1) *has a complete smooth solution $g(t)$ on $M \times (0,T)$ which extends continuously to $M \times [0,T)$ with $g(0) = g_0$ and satisfies $g(t) \geq (1/n - 2nK)\hat{g}$ on $M \times (0,T)$.*

*Moreover, if $g_0$ is smooth and $\{h_{k,0}\}$ converges smoothly and uniformly on compact subsets of M, then $g(t)$ extends to a smooth solution to* (1.0.1) *on $M \times [0,T)$ with $g(0) = g_0$.*

*Proof.* By Lemma 2.2.1, we have

$$h_k(t) \geq \left(\frac{1}{n} - 2Kt\right)\hat{g} \tag{2.3.1}$$

as long as $t < T_0 = 1/(2nK)$. By Theorem 2.1.2, let $\hat{g}(t)$ be the solution Kähler-Ricci flow in the theorem with initial condition $\hat{g}$. Then for any $1 > \varepsilon > 0$ small, choose $0 < t_0$ small enough so that $(1-\varepsilon)\widehat{g}(t_0) \leq \widehat{g} \leq (1+\varepsilon)\widehat{g}(t_0)$. Then we have

$$h_k(t) \geq \left(\frac{1}{n} - 2Kt\right)(1-\varepsilon)\hat{g}(t_0) \tag{2.3.2}$$

and $\hat{g}(t_0)$ has bounded geometry of infinite order. By Lemma 2.2.3, for there is a positive continuous function $\beta(r,t) : [1,\infty) \times [0,T_0) \to \mathbb{R}$ such that for $r \geq 1$

$$h_k(t) \leq \beta(r,t)\hat{g}(t_0). \tag{2.3.3}$$

in $\hat{B}(p,r) \times [0,T)$ where $T = \min\{T', \frac{1}{2nK}\}$ and $p \in M$ is a fixed point. We conclude from Theorem A.1.1 (i), that passing to some subsequence, the $h_k(t)$'s converge to a solution $g(t)$ of Kähler-Ricci on $M \times (0,T)$ so that (2.3.1) is true. Moreover, if $g_0$ is smooth and $\{h_k\}$ converges smoothly and uniformly to $g_0$ on compact sets, then we see from Theorem A.1.1 (ii) that in fact $g(t)$ extends to a smooth solution on $M \times [0,T)$ such that $g(0) = g_0$.

We now prove $g(t)$ converge uniformly on compact set to $g_0$ as $t \to 0$ when $g_0$ is only assumed to be continuous. Fix any $x \in M$ and a local biholomorphism $\phi : B(1) \to M$ where $B(1)$ is the open unit ball in $\mathbb{C}^n$, and $\phi(0) = x$. Consider the pullbacks $\phi^* h_k(t)$, $\phi^* h_k = \phi^* h_k(0)$, $\phi^* \hat{g}$, which by abuse of notation we will simply denote by $h_k(t)$, $h_k$, $\hat{g}$, respectively, for the remainder of proof. In particular, $h_k(t)$ solves Kähler-Ricci flow (1.0.1) on $B(1) \times [0, T)$.

Now by our hypothesis on the convergence of $h_k$, given any $\delta > 0$ we may find $k_0$ such that $|h_{k_0,0} - g_0|_{\hat{g}} \leq \delta$ and

$$(1-\delta) h_{k_0,0} \leq h_{k,0} \leq (1+\delta) h_{k_0,0} \tag{2.3.4}$$

for all $k \geq k_0$. On the other hand, by (2.3.2) and (2.3.3) we can find $N > 0$ such that

$$N^{-1} h_{k_0,0} \leq h_k(t) \leq N h_{k_0,0} \tag{2.3.5}$$

in $B(1) \times [0, T/2)$ for all $k \geq k_0$. Then by Lemma 2.2.2, there exists a continuous function $a(t)$ depending on $N, h_{k_0}$ and $\delta$ such that

$$(1 - a(t)) \frac{(1-\delta)^2}{(1+\delta)} h_{k_0,0} \leq h_k(t) \leq (1 + a(t))(1+\delta) h_{k_0,0}$$

in $B(\frac{1}{2}) \times [0, T/2)$ with $a(0) = n\sqrt{C(C-1)}$, with $C = (1+\delta)/(1-\delta)$. Note that $a(t)$ is independent of $k$. Letting $k \to \infty$ gives

$$(1 - a(t)) \frac{(1-\delta)^2}{(1+\delta)} h_{k_0,0} \leq g(t) \leq (1 + a(t))(1+\delta) h_{k_0,0} \tag{2.3.6}$$

in $B(\frac{1}{2}) \times (0, T/2)$. We then get

$$\begin{aligned}
&\limsup_{t \to 0} |g(t) - g_0|_{\hat{g}} \\
&\quad \leq \limsup_{t \to 0} \left( |g(t) - h_{k_0,0}|_{\hat{g}} + |h_{k_0,0} - g_0|_{\hat{g}} \right) \\
&\quad \leq \left[ \left| 1 - (1 - a(0)) \frac{(1-\delta)^2}{(1+\delta)} \right| + |(1 + a(0))(1+\delta) - 1| \right] |h_{k_0,0}|_{\hat{g}} \\
&\qquad + \delta |h_{k_0,0}|_{\hat{g}}
\end{aligned}$$

uniformly on $B(\frac{1}{2})$. Then letting $\delta \to 0$ above, and using the fact that $a(0) \to 0$ as $\delta \to 0$, and (2.3.2) and (2.3.3) we conclude that

$$\limsup_{t\to 0} |g(t) - g_0|_{\hat{g}} = 0.$$

uniformly on $B(\frac{1}{2})$. Hence $g(t)$ converge to $g_0$ uniformly on compact sets as $t \to 0$.

□

We do not have any bound on the curvature of the solution $g(t)$ in the previous theorem. Also in the previous theorem, we assume that the Kähler-Ricci flow (1.0.1) has solution with initial condition $h_{k,0}$ on a fixed time interval independent of $k$. We want to remove this assumption and obtain curvature bound for the solutions. In order to do this, we assume $h_{k,0}$ also has an uniform upper bound.

**Theorem 2.3.2.** *Let $g_0$ be a complete continuous Hermitian metric on a noncompact complex manifold $M^n$. Suppose there exists a sequence $\{h_{k,0}\}$ of smooth complete Kähler metrics with bounded curvature on $M$ converging uniformly on compact subsets to $g_0$ and another complete Kähler metric $\widehat{g}$ on $M$ with bounded curvature and holomorphic sectional curvature bounded from above by $K \geq 0$ such that*

*(i) $C^{-1}\hat{g} \leq h_{k,0} \leq C\hat{g}$ for some $C$ independent of $k$;*

*(ii) $h_k$ has bounded curvature for every $k$.*

*Let $T = 1/(2CnK)$ if $K > 0$, otherwise let $T = \infty$. Then the Kähler-Ricci flow* (1.0.1) *has a smooth solution $g(t)$ on $M \times (0,T)$ such that*

*(a) $(1/(nC) - 2Kt)\hat{g} \leq g(t) \leq B(t)\hat{g}$ on $M \times (0,T)$ for some positive continuous function $B(t)$ depending only on $C$, $\hat{g}$ and $n$.*

*(b) $g(t)$ has bounded curvature for $t > 0$. More precisely, for any $0 < T' < T$ and for any $l \geq 0$ there exists a constant $C_l$ depending only on $C$, $l$, $T'$, $\hat{g}$ and the dimension $n$ such that*

$$sup_M |\nabla^l \mathrm{Rm}(g(t))|^2_{g(t)} \leq \frac{C_l}{t^{l+2}},$$

*(c) $g(t)$ converges uniformly on compact subsets to $g_0$ as $t \to 0$.*

*Moreover, if $g_0$ is smooth and $\{h_{k,0}\}$ converges smoothly and uniformly on compact subsets of $M$, then $g(t)$ extends to a smooth solution on $M \times [0,T)$ with $g(0) = g_0$.*

*Proof.* For each $k$, let $h_k(t)$ be the solution to (1.0.1) with initial condition $h_k$ from Theorem 2.1.2 which is defined on $M \times [0,T_k)$ for some $T_k > 0$. We first claim that there is $T > 0$ such that $T_k \geq T$ for all $k$. By Lemma 2.2.1, there is a positive continuous function $B(t) : [0,T) \to \mathbb{R}$ independent of $k$ such that

$$(1/n - 2CKt)\hat{g} \leq h_k(t) \leq B(t)\hat{g}$$

in $M \times [0, \min\{T_k, T\})$ where $T = 1/(2nCK)$. As before, we may assume that $\hat{g}$ has bounded geometry of infinite order. By Theorem A.1.1, we conclude that if $T_k < T$, then $|\mathrm{Rm}(h_k(t))|_{h_k(t)}$ are bounded in $M \times [0,T_k)$. By Theorem 2.1.2, we see that one can extend $h_k(t)$ so that $T_k \geq T$ for all $k$ as claimed. Given upper and lower bounds on $h_k(t)$ as above, we may conclude from Theorem A.1.1, as in the proof of Theorem 2.3.1, that there is a smooth solution to the Kähler-Ricci flow $g(t)$ on $M \times (0,T)$ satisfying condition (a) and (c) from which we conclude, by Theorem A.1.1 (i), that condition (b) is also satisfied.

□

## 2.4 A uniqueness theorem and an existence time estimate

In this section we will discuss some results on existence time and uniqueness of the complete bounded curvature solutions and uniqueness of solutions in general. We first prove a uniqueness theorem which could be applied to solutions that may not have bounded curvature:

**Theorem 2.4.1.** *Let $(M^n, \widehat{g})$ be a complete non-compact Kähler manifold. Suppose there is an exhaustion function $\zeta > 0$ on $(M^n, \widehat{g})$ with $\lim_{x \to \infty} \zeta(x) = \infty$ such that $|\partial\bar{\partial}\zeta|_{\widehat{g}}$ and $|\widehat{\nabla}\zeta|_{\widehat{g}}$ are bounded.*

*Let $g_1(x,t)$ and $g_2(x,t)$ be two solutions of the Kähler-Ricci flow (1.0.1) on $M \times [0,T]$ with the same initial data $g_0(x) = g_1(x,0) = g_2(x,0)$. Suppose there is* *a positive function* $\sigma$ *with* $\lim_{x\to\infty}\log\sigma(x)/\log\zeta(x)=0$ *such that the following conditions hold for all* $(x,t)\in M\times[0,T]$*:*

*(i)*

$$\widehat{g}(x)\le\zeta(x)g_1(x,t);\;\;\widehat{g}(x)\le\zeta(x)g_2(x,t),\tag{2.4.1}$$

*(ii)*

$$-\sigma(x)\le\frac{\det((g_1)_{i\bar{j}}(x,t))}{\det((g_2)_{i\bar{j}}(x,t))}\le\sigma(x).$$

*Then* $g_1\equiv g_2$ *on* $M\times[0,T]$*. In particular, if* $g_1$ *and* $g_2$ *are uniformly equivalent to* $\widehat{g}$ *on* $M\times[0,T]$*, then* $g_1\equiv g_2$*.*

*Proof.* By adding a positive constant to $\zeta$ we may assume that $\eta:=\log\zeta>1$. Then

$$\eta_{i\bar{j}}=\frac{\zeta_{i\bar{j}}}{\zeta}-\frac{\zeta_i\zeta_{\bar{j}}}{\zeta^2}.$$

Since $|\partial\bar{\partial}\eta|_{\widehat{g}}$ and $|\widehat{\nabla}\eta|_{\widehat{g}}$ are uniformly bounded, there is $c_1>0$ such that

$$|\partial\bar{\partial}\eta|_{\widehat{g}}\le\frac{c_1}{\zeta}$$

on $M$. Let $h(s,t)(x)=sg_1(x,t)+(1-s)g_2(x,t)$, $0\le s\le 1$. By (i), we have $\widehat{g}(x)\le\zeta(x)h(s,t)(x)$ for all $(x,t)\in M\times[0,T]$ and for all $s$. Let $(x,t)$ be fixed and diagonalize $\partial\bar{\partial}\eta$ with respect to $\widehat{g}$ at $x$. Then $|\eta_{i\bar{i}}|\le\frac{c_1}{\zeta}$. On the other hand,

$$\Delta_{h(s,t)}\eta=(h(s,t))^{i\bar{j}}\eta_{i\bar{j}}=(h(s,t))^{i\bar{i}}\eta_{i\bar{i}}\le n\zeta\cdot\frac{c_1}{\zeta}=nc_1.\tag{2.4.2}$$

Let

$$w(x,t)=\int_0^t\left(\log\frac{\det((g_1)_{i\bar{j}}(x,s))}{\det((g_2)_{i\bar{j}}(x,s))}\right)ds.$$

Then

$$w_{i\bar{j}}(x,t)=\int_0^t\big((R_1)_{i\bar{j}}(x,s)-(R_2)_{i\bar{j}}(x,s)\big)\,ds=-(g_1)_{i\bar{j}}(x,t)+(g_2)_{i\bar{j}}(x,t)$$

where $(R_k)_{i\bar{j}}$ is the Ricci tensor of $g_k$, $k=1,2$. Here we have used the Kähler-Ricci flow and the fact that $g_1=g_2$ at $t=0$. Hence in order to prove the proposition, it is sufficient to prove that $w \equiv 0$. Now

$$
\begin{aligned}
\frac{\partial}{\partial t} w(x,t) &= \int_0^1 \frac{\partial}{\partial s} \log\det(h_{i\bar{j}}(s,t)(x)))ds \\
&= \int_0^1 \Delta_{h(t,s)} w(x,t)ds.
\end{aligned}
\tag{2.4.3}
$$

Let $W(x,t) = e^{At}\eta$ where $A = nc_1 + 1$. By (2.4.2),

$$
\begin{aligned}
\frac{\partial}{\partial t} W(x,t) - \int_0^1 \Delta_{h(t,s)} W(x,t)ds \geq & e^{At}(A\eta - nc_1) \\
\geq & e^{At}\eta
\end{aligned}
$$

where we have used the fact that $\eta > 1$. For any $\varepsilon > 0$,

$$
\frac{\partial}{\partial t}(\varepsilon W - w)(x,t) - \int_0^1 \Delta_{h(t,s)}(\varepsilon W - w)(x,t)ds \geq e^{At}\eta
$$

By (ii), $\lim_{x\to\infty}(\varepsilon W - w)(x,t) = \infty$ uniformly in $t$. By the maximum principle, we conclude that $w \leq \varepsilon W$. Letting $\varepsilon \to 0$ gives $w \leq 0$. Similarly, one can prove that $-w \leq 0$ and hence $w \equiv 0$ on $M \times [0,T]$. This completes the proof of the proposition. □

As a corollary, we have

**Corollary 2.4.1.** *Let $(M^n, \hat{g})$ be a complete non-compact Kähler manifold with bounded curvature. Let $g_1(x,t)$ and $g_2(x,t)$ be two solutions of the Kähler-Ricci flow* (1.0.1) *on $M \times [0,T]$ with the same initial data $g_0(x) = g_1(x,0) = g_2(x,0)$. Suppose there is a constant C such that:*

$$
C^{-1}\hat{g} \leq g_1(t), g_2(t) \leq C\hat{g}
$$

*on $M \times [0,T]$. Then $g_1 \equiv g_2$ on $M \times [0,T]$.*

By using Theorem 2.1.4, we can compare the maximal existence time of the Kähler-Ricci flow solutons by comparing the $C^0$ data of the initial metric. For convenience, if $g$ is a complete Kähler metric with bounded curvature, we let $T_g$ be the maximal existence time of Shi's solution.

**Theorem 2.4.2.** *Let $M$ be a non-compact complex manifold, $g_0$ and $h_0$ be complete Kähler metrics with bounded curvature. If $g_0$ and $h_0$ are equivalent and $g_0 \geq ah_0$ then $T_{g_0} \geq aT_{h_0}$,*

*Proof.* First observe that if $\lambda > 0$ is a constant, then $T_{\lambda h_0} = \lambda T_{h_0}$. Hence without loss of generality, we may assume that $a = 1$. Also, we assume without loss of generality that $g_0, h_0$ have bounded geometry of order infinity. For if not, we let $g(t), h(t)$ are the corresponding solutions as in Theorem 2.1.2, then we first prove the Theorem for $g(\varepsilon), h(\varepsilon)$ for arbitrary small $\varepsilon$ and then let $\varepsilon \to 0$ and use the fact that $g(\varepsilon), h(\varepsilon)$ converge uniformly to $g_0, h_0$ (respectively) on $M$, and $T_{g(\varepsilon)} = T_g - \varepsilon$ and $T_{h(\varepsilon)} = T_h - \varepsilon$ by Theorem 2.4.1. Note that by Theorem A.1.1, all the covariant derivatives of $g(\varepsilon)$ with respect to $h_0$ are bounded. So we may assume in addition that all the covariant derivatives of $g_0$ with respect to $h_0$ are bounded.

Now for any $0 < T < T_{h_0}$, we have

$$g_0 - T\mathrm{Ric}(g_0) = h_0 - T\mathrm{Ric}(h_0) + T(\mathrm{Ric}(g_0) - \mathrm{Ric}(h_0)) + (g_0 - h_0).$$

By Theorem 2.1.4, there is a smooth bounded function $f$ with bounded covariant derivatives with respect to $h_0$ such that

$$h_0 - T\mathrm{Ric}(h_0) + \sqrt{-1}\partial\bar{\partial} f \geq C_1 h_0$$

for some $C_1 > 0$. Then letting $F = \log \frac{\omega_0^n}{\eta^n}$ where $\omega_0$ and $\eta$ are the Kähler forms of $g_0$ and $h_0$ respectively, gives

$$\begin{aligned} g_0 - T\mathrm{Ric}(g_0) + \sqrt{-1}\partial\bar{\partial}(f + TF) \geq & C_1 h_0 + (g_0 - h_0) \\ \geq & C_2 g_0 \end{aligned}$$

for some $C_2 > 0$ because $g_0 \geq h_0$ and $g_0$ is uniformly equivalent to $h_0$. From the facts that $g_0, h_0$ are equivalent and that all the covariant derivatives of $g_0$ with respect to $h_0$ are bounded, we may conclude that all the covariant derivatives of $f$ are bounded with respect to $g_0$ as well. We may also conclude from these facts that $F$ and $|\nabla^l \partial\bar{\partial} F|_{g_0}$ are uniformly bounded for $0 \leq l \leq 2$ where we have used that $\sqrt{-1}\partial\bar{\partial} F = -\mathrm{Ric}(g_0) + \mathrm{Ric}(h_0)$. By Theorem 2.1.4, we conclude that $T \leq T_{g_0}$.

From this the result follows. □

By the theorem, we have the following monotonicity and continuity of $T_g$. Namely

**Corollary 2.4.2.** *Let $M^n$ be a non-compact complex manifold.*

*(i) Let $g_0 \geq h_0$ be complete uniformly equivalent Kähler metrics on $M$ with bounded curvature. Then $T_{g_0} \geq T_{h_0}$. In particular, if $T_{h_0} = \infty$, then $T_{g_0} = \infty$.*

*(ii) Let $\mathscr{K}$ be the set of complete Kähler metrics on $M$ with bounded curvature. Then $T_g$ is continuous on $\mathscr{K}$ with respect to the $C^0$ norm in the following sense: Let $g_0 \in \mathscr{K}$. Then for $h_0 \in \mathscr{K}$, $T_{h_0} \to T_{g_0}$ as $||h_0 - g_0||_{g_0} \to 0$.*

As a corollary, instead of the $C^0$ data, we can also estimate the existence by the upper bound of the Ricci curvature of another equivalent Kähler metric.

**Corollary 2.4.3.** *Let $(M,h)$ be a complete Kähler manifold having bounded curvature with Ricci curvature bounded above by $K$. If $g$ is equivalent to $h$ and $g \geq h$. then $T_g \geq \frac{1}{K}$.*

*Proof.* Since $\text{Ric}(h) \leq Kh$, so $h - t\text{Ric}(h) = (1 - Kt)h$. By Theorem 2.1.4, $T_h \geq \frac{1}{K}$. By Theorem 2.4.2, $T_g \geq \frac{1}{K}$. □

*Remark* 2.4.1. As an example of application, let $(M,h)$ be a complete Kähler manifold with non-positive bisectional curvature (e.g. an Euclidean space or a Poincaré disk), assuming that $g$ is equivalent to $h$, since $K = 0$ we have $T_g = \infty$.

# Chapter 3

# $U(n)$-invariant Kähler metrics

In this chapter we will study the Kähler-Ricci flow starting from $U(n)$-invariant Kähler metrics, the main theorem is

**Theorem 3.0.1.** *Let $g_0$ be a complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ with non-negative bisectional curvature. Then*

*(i) the Kähler-Ricci flow* (1.0.1) *has a unique smooth longtime $U(n)$-invariant solution $g(t)$ which is equivalent to $g_0$ and has bounded non-negative bisectional curvature for all $t > 0$;*

*(ii) $g(t)$ converges, after rescaling at the origin, to the standard Euclidean metric on $\mathbb{C}^n$.*

Here we say that the solution $g(t)$ converges to $\tilde{g}$ after rescaling at a point $p$ if for some $V \in T_pM$, the metrics $\frac{1}{|V|_t^2} g(t)$ converges to $\tilde{g}$ smoothly and uniformly on compact subsets on $M$ where $|V|_t^2 = g_t(V,V)$.

The content is organized as follows: We first collect some results about $U(n)$-invariant Kähler metrics. These results correlate $U(n)$-invariant metrics with non-negative holomorphic bisectional curvature with non-decreasing real valued functions. Then we will construct a short-time $U(n)$-invariant Kähler-Ricci flow solution starting from any non-negatively curved $U(n)$-invariant metric. After that, we will show that the short-time solution can be extended to a long-time solution and the solution is unique in a certain class. Finally we will show that the long-time solution converges to the standard Euclidean metric after rescaling.

## 3.1 Background materials

Let $g$ be a $U(n)$-invariant Kähler metric. Then there is a smooth $p \in C^\infty[0,\infty)$ such that the Kähler form of the metric satisfies $\omega = i\partial\bar{\partial}p(r)$ where $r = |z|^2$. In the standard coordinate on $\mathbb{C}^n$, $g_{i\bar{j}} = f(r)\delta_{ij} + f'(r)\bar{z}_i z_j$ where $f(r) = p'(r)$. Let $h = (rf)'$, then at the point $(z_1, 0, \cdots, 0)$,

$$g_{1\bar{1}} = h, g_{i\bar{i}} = f \text{ and } g_{i\bar{j}} = 0 \text{ for all } i \neq j.$$

Due to $U(n)$-invariance, this basically describes the metric on the whole of $\mathbb{C}^n$. Using this observation, Wu-Zheng [33] parametrized the $U(n)$-invariant Kähler metrics by a constant $C$ and a smooth function $\xi : [0,\infty) \to \mathbb{R}$ as follows:

**Theorem 3.1.1.** *(a)* ([WU-ZHENG] [33]) *Every smooth $U(n)$ invariant Kähler metric $g$ is generated by a function $\xi : [0,\infty) \to \mathbb{R}$ with $\xi(0) = 0$ such that if*

$$h_\xi(r) := Ce^{\int_0^r -\frac{\xi(s)}{s}ds}; \quad f_\xi(r) := \frac{1}{r}\int_0^r h_\xi(s)ds$$

*where $h_\xi(0) = C > 0$ and $f_\xi(0) = h_\xi(0)$, where $r = |z|^2$, then*

$$g_{i\bar{j}} = f_\xi(r)\delta_{ij} + f'_\xi(r)\overline{z_i}z_j.$$

*where $g_{i\bar{j}}$ are the components of $g$ in the standard coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{C}^n$. Moreover $g$ is complete if and only if*

$$\int_0^\infty \frac{\sqrt{h}_\xi(s)}{\sqrt{s}}ds = \infty.$$

*(b)* ([WU-ZHENG] [33]) *Let $h = h_\xi$, $f = f_\xi$. At the point $z = (z_1, 0, \ldots, 0)$, relative to the orthonormal frame $e_1 = \frac{1}{\sqrt{h}}\partial_{z_1}, e_i = \frac{1}{\sqrt{f}}\partial_{z_i}, i \geq 2$, with respect to $g$, the curvature tensor has components*

$$A = R_{1\bar{1}1\bar{1}} = \frac{\xi'}{h},$$

$$B = R_{1\bar{1}i\bar{i}} = \frac{1}{(rf(r))^2} \int_0^r \xi'(s) \left( \int_0^t h(s)ds \right) dt,$$
$$C = R_{i\bar{i}i\bar{i}} = 2R_{i\bar{i}j\bar{j}} = \frac{2}{(rf(r))^2} \int_0^r h(s)\xi(s)dt,$$

*where $2 \le i \ne j \le n$ and these are the only non-zero components of the curvature tensor at $z$ except those obtained from $A, B$ or $C$ by the symmetric properties of the curvature tensor.*

*(c)* ([WU-ZHENG] [33], YANG [34]) *$g$ has positive (nonnegative) bisectional curvature if and only if $\xi' > 0$ ($\xi' \ge 0$). In particular, if $g$ has nonnegative bisectional curvature and is complete, then $\xi \le 1$.*

*Remark* 3.1.1. If $g_1$ and $g_2$ are two smooth $U(n)$ invariant Kähler metrics on $\mathbb{C}^n$ generated by $\xi_1, \xi_2$ respectively, and if the corresponding functions $h_{\xi_1}$, $h_{\xi_2}$ satisfy $h_{\xi_1} \ge h_{\xi_2}$, then $g_1 \ge g_2$. Conversely, if $g_1 \ge g_2$, then $h_{\xi_1} \ge h_{\xi_2}$. This can be seen by comparing the metrics at the points $(a, 0, \dots, 0)$.

Using Theorem 3.1.1 (a) and (b), we can find a condition for which the curvature of $g$ is uniformly bounded.

**Lemma 3.1.1.** *Let $g$ be a complete $U(n)$ invariant Kähler metric on $\mathbb{C}^n$ generated by $\xi$. If $\left|\frac{\xi'}{h}\right|$ is uniformly bounded, then the curvature of $g$ is uniformly bounded.*

*Proof.* It is sufficient to prove that the holomorphic bisectional curvature is uniformly bounded under the assumption that $\left|\frac{\xi'}{h}\right|$ is uniformly bounded by $c$, say. By Theorem 3.1.1, in the notations of the theorem it is sufficient to prove that $|A|, |B|, |C|$ are uniformly bounded. It is obviously $|A| \le c$. Now

$$\begin{aligned} |B| \le & \frac{1}{r^2 f^2} \int_0^r ch(t)dt \left( \int_0^t h(s)ds \right) dt \\ \le & \frac{c}{r^2 f^2} \left( \int_0^r h(t)dt \right)^2 \\ = & c \end{aligned}$$

because $h > 0$ and $rf(r) = \int_0^r h(t)dt$. Similarly, since

$$|\xi(r)| \le \int_0^r |\xi'(t)|dt \le c \int_0^r h(t)dt,$$

we have

$$|C| \le 2c.$$

□

Theorem 3.1.1 (c) showed that any complete $U(n)$-invariant Kähler metric with non-negative bisectional curvature corresponds to a non-decreasing $\xi$ with $\xi(\infty) = \lim_{r\to\infty} \xi(r) \le 1$. In fact, the volume growth rate can also be described by $\xi$. Chen-Zhu [13] proved that if $(M,g)$ is complete non-compact with non-negative bisectional curvature, then for all $x_0 \in M$, there exists $c > 0$ such that

$$\frac{1}{c}s^n \le V(s) \le cs^{2n}, \quad \forall s > 1,$$

where $V(s)$ be the volume of $B_g(x_0,s)$.

This means that complete non-negatively curved Kähler manifolds have volume growth rate between half-Euclidean and Euclidean. In case the volume growth rate is half-Euclidean (i.e. $V(s)$ is asymptotic to $s^n$), the space is called a cigar; if the volume growth rate is Euclidean (i.e. $V(s)$ is asymptotic to $s^{2n}$), the space is called a conoid. In the $U(n)$-invariant case, conoids and cigars satisfy the following conditions:

**Theorem 3.1.2** (Wu-Zheng, [33]). *The metric $\omega$ is a conoid if the corresponding $\xi$ satisfies $\xi(\infty) < 1$. It is a cigar if $\int_1^\infty \frac{1-\xi}{r}dr < \infty$.*

Using the construction of Cabezas-Rivas and Wilking [2], Yang-Zheng [35] proved the short time existence of the Kähler-Ricci flow for complete non-collapsed $U(n)$-invariant metric with non-negative sectional curvature. Their solution is $U(n)$-invariant when some technical assumptions on initial data is satisfied, and when the solution is $U(n)$ invariant, they proved the following theorem which we will use later:

**Theorem 3.1.3** (Yang-Zheng, [35]). *Let $g(t), t \in [0,T]$ be a complete solution of the Kähler-Ricci flow on $\mathbb{C}^n$ with $U(n)$-symmetry. If $g(0)$ has non-negative holomorphic bisectional curvature, so does $g(t)$ for all $t \in [0,T]$.*

In the following content, by rescaling the initial metric if necessary, we will assume that the constant $C$ of $g_0$ in Theorem 3.1.1(a) is equal to 1.

## 3.2 Short time existence of $U(n)$-invariant Kähler-Ricci flow

We will use Theorem 2.3.2 to prove short time existence. We first prove a Proposition:

**Proposition 3.2.1.** *Assume that $g_0$ is a complete $U(n)$-invariant metric with non-negative bisectional curvature $\xi$, there exists a complete $U(n)$ invariant metric $\hat{g}$ with bounded curvature and*

$$c^{-1}\hat{g} \leq g_0 \leq c\hat{g} \tag{3.2.1}$$

*on $\mathbb{C}^n$ for some constant $c > 0$.*

*Proof.* We will prove the Theorem by constructing a $\hat{\xi}$ satisfying certain properties, the constant $C$ of $\widehat{g}$ in Theorem 3.1.1(a) is taken to be 1. Let $\xi$ and $\hat{\xi}$ correspond to $g_0$ and $\hat{g}$ respectively, if $\xi \equiv 0$, then $g_0$ is Euclidean and so the statement is true by taking $\widehat{g} = g_0$.

Suppose now $\xi \neq 0$, we consider two cases:

*Case 1:* If $\int_1^r \frac{\xi-1}{t}dt \geq -C$ for some constant $C$. Let $\hat{\xi}$ be a smooth function on $[0,\infty)$ such that $\hat{\xi}(r) = 1$ for all sufficiently large $r$, it is clear that it generates a complete bounded curvature $\widehat{g}$. By the definition of $h$, for all $r \geq 0$, we have

$$\frac{\hat{h}(r)}{h(r)} = \exp(\int_0^r \frac{\xi(t) - \hat{\xi}(t)}{t}dt).$$

Because $\int_1^r \frac{\xi-1}{t}dt \geq -C$, there exists $C'$ such that $\int_0^r \frac{\xi(t)-\hat{\xi}(t)}{t}dt \geq -C'$ and therefore $h(r) \leq C_2\hat{h}(r)$ for some constant $C_2$. On the other hand, because $\xi \leq 1$ and $\hat{\xi}(r) = 1$ for all sufficiently large $r$, therefore, $h(r) \geq C_1\hat{h}(r)$ for some $C_1$. Since $C_1\hat{h}(r) \leq h(r) \leq C_2\hat{h}(r)$ for all $r$, by Remark 3.1.1, we have $C_1\hat{g} \leq g_0 \leq C_2\hat{g}$ as claimed.

*Case 2:* If $\int_1^r \frac{\xi-1}{t}dt$ is not bounded from below. By the assumption $\xi \neq 0$ and $\xi$ is non-decreasing, $\int_1^r \frac{\xi}{t}dt$ is not bounded from above. We want to construct $\hat{\xi}$ by oscillating between 0 and 1 at a suitable rate so that the corresponding metric $\widehat{g}$ is complete with bounded curvature and is equivalent to $g$. More precisely, we will find $\hat{\xi}$ and $1 \leq a_0 < a_1 < a_2 \cdots \to \infty$ such that $\hat{\xi}$ generates a complete $U(n)$ metric $\widehat{g}$ such that

$$\int_{a_{2i}}^{a_{2(i+1)}} \frac{\xi - \hat{\xi}}{t} dt = 0 \tag{3.2.2}$$

for all $i \geq 0$;

$$\left| \int_{a_{2i}}^{r} \frac{\xi - \hat{\xi}}{t} dt \right| \leq c_1 \tag{3.2.3}$$

for some $c_1$ for all $i \geq 0$ and for all $r \in [a_{2i}, a_{2(i+1)})$; and

$$\left| \frac{\hat{\xi}'(r)}{\hat{h}(r)} \right| \leq c_2 \tag{3.2.4}$$

for some $c_2$ for all $r \geq 0$. Then by Lemma 3.1.1, Theorem 3.1.1 (a), we can conclude that $\widehat{g}$ satisfies the conditions of the Proposition.

Fix a smooth function $\rho$ on $\mathbb{R}$, such that

$$\rho(t) = \begin{cases} 1, & \text{if } t \leq 1+\varepsilon; \\ 0, & \text{if } t \geq 3-\varepsilon, \end{cases}$$

and $\rho' \leq 0$, where $\varepsilon > 0$ is small enough so that $1+\varepsilon < 3-\varepsilon$. Then $0 \leq \rho \leq 1$.

Let $\hat{\xi}$ be a smooth function on $[0,1]$ with $\hat{\xi}(0) = 0$ and $\hat{\xi}(r) = 1$ near $r = 1$ such that $0 \leq \hat{\xi} \leq 1$. We are going to find $a_i$ and $\hat{\xi}(r)$ on $[a_i, a_{i+1}]$ inductively. Let $a_0 = 1$.

$$\int_{a_0}^{3a_0} \frac{\xi - \rho(\frac{t}{a_0})}{t} dt \leq \int_{a_0}^{3a_0} \frac{1 - \rho(\frac{t}{a_0})}{t} dt \leq \log 3.$$

Since $\int_{3a_0}^{r} \frac{\xi}{t} dt$ is not bounded from above, there is a *first* $a_1 > 3a_0$ such that

$$\int_{a_0}^{3a_0} \frac{\xi - \rho}{t} dt + \int_{3a_0}^{a_1} \frac{\xi}{t} dt = c_3$$

where $c_3 = \log 3 + 1$. On the other hand,

$$\int_{a_1}^{3a_1} \frac{\xi - (1 - \rho(\frac{t}{a_1}))}{t} dt \geq -\log 3.$$

Since $\int_{3a_1}^{r} \frac{\xi-1}{t}dt$ is not bounded from below, there exists a *first* $a_2 > 3a_1$, such that

$$\int_{a_1}^{3a_1} \frac{\xi-(1-\rho(\frac{t}{a_1}))}{t}dt + \int_{3a_1}^{a_2} \frac{\xi-1}{t}dt = -c_3$$

Define

$$\hat{\xi}(r) = \begin{cases} \rho(\frac{r}{a_0}), & \text{if } a_0 \le r \le 3a_0; \\ 0, & \text{if } 3a_0 < r \le a_1; \\ 1-\rho(\frac{r}{a_1}), & \text{if } a_1 < r \le 3a_1; \\ 1, & \text{if } 3a_1 < r \le a_2. \end{cases}$$

It is easy to see that $\hat{\xi}$ is smooth on $[0,a_2]$ with $\hat{\xi}(r) = 1$ near $a_2$. Moreover, $0 \le \hat{\xi} \le 1$ on $[1,a_2]$, and

$$\int_{a_0}^{a_2} \frac{\xi-\hat{\xi}}{t}dt = 0.$$

so (3.2.2) is true for $i = 0$. It is easy to see that

$$|\xi'| \le \frac{c_4}{r}$$

where $c_4 = 3\max|\rho'|$.

For $a_0 \le r \le a_1$, by the definition of $a_1$ we have

$$\int_{a_0}^{r} \frac{\xi-\hat{\xi}}{t}dt \le c_3.$$

For $a_1 < r \le a_2$,

$$\begin{aligned}\int_{a_0}^{r} \frac{\xi-\hat{\xi}}{t}dt &= \left(\int_{a_0}^{a_1} + \int_{a_1}^{r}\right) \frac{\xi-\hat{\xi}}{t}dt \\ &\le c_3 + \int_{a_1}^{r} \frac{1-\hat{\xi}}{t}dt \\ &\le c_3 + \log 3.\end{aligned}$$

Hence for $a_0 \le r \le a_2$,

$$\int_{a_0}^{r} \frac{\xi-\hat{\xi}}{t}dt \le 2c_3.$$

Similarly, one can prove that

$$\int_{a_0}^{r} \frac{\xi - \hat{\xi}}{t} dt \geq -2c_3.$$

To summarize, we have find $\hat{\xi}(r)$ and $a_0 < a_1 < a_2$ such that $\hat{\xi}$ is smooth and defined on $[0, a_2]$ with $\leq \hat{\xi} \leq 1$ on $[a_0, a_2]$, satisfying (3.2.2) with $i = 0$, (3.2.3) with $i = 0$, $c_1 = 2c_3$, and $|\hat{\xi}'| \leq \frac{c_4}{r}$ on $[a_0, a_2]$. Moreover, $\hat{\xi}(r) = 1$ near $r = a_2$.

From the above construction, it is easy to see that one can continue and find $a_2 < a_3 < a_4 \cdots \to \infty$ and $\hat{\xi}$ with $0 \leq \hat{\xi}(r) \leq 1$ for $r \geq a_0$, satisfying (3.2.2) and (3.2.3) with $c_1 = 2c_3$, and $|\hat{\xi}'| \leq \frac{c_4}{r}$ on $[a_0, \infty)$.

Since $\hat{\xi} \leq 1$,

$$\hat{h}(r) \geq c_5 \exp(-\int_1^r \frac{1}{t} dt) \geq \frac{c_5}{r}$$

for some $c_5 > 0$ for all $r \geq 1$. Combing with the fact that $|\hat{\xi}'| \leq \frac{c_4}{r}$ on $[a_0, \infty)$, we conclude that (3.2.4) is also true. This completes the proof. □

We also need the following Lemma:

**Lemma 3.2.1.** *Let $g_0$ be a complete $U(n)$-invariant metric with non-negative bisectional curvature and let $\hat{g}$ be a complete bounded curvature $U(n)$-invariant metric equivalent to $g_0$. Then we can find a sequence of complete $U(n)$-invariant metric $h_k$ such that*

1. *$h_k$ converges to $g_0$ in $C^\infty_{loc}$-sense;*
2. *there exists a uniform constant $C$ such that $\frac{1}{C}\widehat{g} \leq h_k \leq C\widehat{g}$ and*
3. *$h_k$ has bounded curvature for all $k$,*

*Proof.* Choose $\delta_k > 0$ and smooth functions $\eta_k : (-\infty, \infty) \to \mathbb{R}$ satisfying

$$\eta_k(r) : \begin{cases} = 1 & \text{if } -\infty < r \leq k \\ 0 < \eta_k(r) < 1 & \text{if } k < r < k + \delta_k \\ = 0 & \text{if } k + \delta_k \leq r < \infty. \end{cases} \tag{3.2.5}$$

and

$$\int_{k}^{k+\delta_k} \left| \frac{(\xi - \hat{\xi})}{t} \right| dt \le 1 \tag{3.2.6}$$

for all $k$. Let $\{\xi_k\} : [0,\infty) \to \infty$ be defined by

$$\xi_k(r) = \eta_k \xi + (1-\eta_k)\hat{\xi}.$$

Then each $\xi_k$ generates a $U(n)$ invariant Kähler metric $h_k$ and for all $r \ge 0$,

$$\begin{aligned}\int_0^r \frac{\xi_k(t) - \hat{\xi}(t)}{t} dt &= \int_0^r \frac{\eta_k(\xi - \hat{\xi})}{t} dt \\ &= \begin{cases} \int_0^r \frac{\xi - \hat{\xi}}{t} dt, & \text{if } r \le k; \\ \int_0^k \frac{\xi - \hat{\xi}}{t} dt + \alpha_k, & \text{if } r > k \end{cases}\end{aligned}$$

where

$$|\alpha_k| \le \int_k^{k+\delta_k} \left| \frac{\xi - \hat{\xi}}{t} \right| dt \le 1.$$

By Remark 3.1.1, since $g$ and $\widehat{g}$ are equivalent, there exists constant $C$ such that $|\int_0^r \frac{\xi - \hat{\xi}}{t} dt| \le C$. Combining the above inequalities and use Remark 3.1.1 again, we have

$$c_2^{-1} \hat{g} \le h_k \le c_2 \hat{g} \tag{3.2.7}$$

for some constant $c_2 > 0$, for all $k$. Since $\xi_k = 1(r)$ for all sufficiently large $r$, $h_k$ is complete with bounded curvature. And because $\xi_k = \xi$ (and hence $h_k = g_0$) on the set $\{r < k\}$, we have $h_k$ converges to $g_0$ in $C^\infty_{loc}$-sense. □

Now we are ready to prove a short time existence theorem for $U(n)$-invariant Kähler-Ricci flow:

**Theorem 3.2.1.** *Let $g_0$ be a complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ with non-negative bisectional curvature. Then for some $T > 0$ the Kähler-Ricci flow* (1.0.1) *has a complete smooth $U(n)$-invariant solution $g(t)$ on $\mathbb{C}^n \times [0,T)$ with $g(0) = g_0$. Moreover, for every $l \ge 0$ there exists a constant $c_l$ depending only on* *such that*

$$\sup_{p\in\mathbb{C}^n} \|\nabla^l \mathrm{Rm}(p,t)\|_t^2 \leq \frac{c_l}{t^{l+2}} \tag{3.2.8}$$

*on* $\mathbb{C}^n \times (0,T)$.

*Proof.* Let $\hat{g}$ be the $U(n)$ invariant Kähler metric with bounded curvature generated by $\hat{\xi}$ defined in Proposition 3.2.1, so that

$$c_1^{-1}\hat{g} \leq g_0 \leq c_1\hat{g} \tag{3.2.9}$$

for some $c_1 > 0$ as in Proposition 3.2.1. Also let $h_k$ be the $U(n)$ invariant Kähler metric with bounded curvature generated by $\xi_k$ defined in Lemma 3.2.1. In particular, $h_k$ is complete with bounded curvature, converges in $C^\infty_{loc}$ to $g_0$ and

$$c_2^{-1}\hat{g} \leq h_k \leq c_2\hat{g} \tag{3.2.10}$$

for some constant $c_2 > 0$, for all $k$. Now recall that the curvature of $\hat{g}$ is bounded by a constant $K$ as in Proposition 3.2.1, and thus by Theorem 2.1.2 we may assume without loss of generality that $\hat{g}$ has bounded geometry of order infinity. By Theorem 2.3.2, there is a solution $g(t)$ of the Kähler-Ricci flow with initial condition $g_0$ on $M \times [0,T)$ for some $T > 0$ so that

$$||\mathrm{Rm}(g(t))||^2_{g(t)} \leq \frac{c_3}{t}$$

for some $c_3 > 0$ and for all $0 < t < T$. The estimates for $||\nabla^l \mathrm{Rm}||$ for each $l \geq 0$ then follows from the general results of [26].

Recall that in Theorem 2.3.2, $g(t)$ is obtained as the limit of $h_k(t)$, since $h_k(t)$ is $U(n)$-invariant, so $g(t)$ is also $U(n)$-invariant. □

Combining Theorem 3.2.1 with Theorem 3.1.3, we have

**Corollary 3.2.1.** *Let $g_0$ be a complete $U(n)$-invariant metric with non-negative bisectional curvature, there exists a complete $U(n)$-invariant metric $g_1$ with bounded non-negative bisectional curvature such that $g_1$ and $g_0$ are equivalent.*

*Proof.* As shown in the proof of Theorem 3.2.1, there is a background metric $\widehat{g}$ which is complete with bounded curvature and a short time $U(n)$-invariant solution

$g(t)$ on $[0,T]$ such that $\frac{1}{c_2}\widehat{g} \leq g(t) \leq c_2\widehat{g}$ for all $t \in [0,T]$. It suffices to take $g_1$ to be $g(\varepsilon)$ where $0 < \varepsilon \leq T$, the curvature bound follows from Theorem 3.2.1 and the non-negativity of bisectional curvature is a consequence of Theorem 3.1.3. $\square$

## 3.3 Long time solution

The main theorem of this section is :

**Theorem 3.3.1.** *Let $g_0$ be a complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ with non-negative holomorphic bisectional curvature. Then the Kähler-Ricci flow* (1.0.1) *has a unique long time $U(n)$-invariant solution $g(t)$ which is equivalent to $g_0$ and has bounded non-negative bisectional curvature for all $t > 0$.*

Recall that $g(t)$ and $g_0$ are equivalent if for all $T < \infty$, there exists constant $C$ such that $\frac{1}{C}g_0 \leq g(t) \leq Cg_0$ for all $t \in [0,T]$. The uniqueness is an immediate consequence. In fact, suppose $g_1(t)$ and $g_2(t)$ satisfy Theorem 3.3.1, by Proposition 3.2.1, there is a complete bounded curvature $\widehat{g}$ equivalent to $g_0$, hence $\widehat{g}$ is equivalent to both $g_1(t)$ and $g_2(t)$. The uniqueness then follows from Corollary 2.4.1 immediately.

In Theorem 3.2.1, we already obtained a short-time solution $g(t)$ satisfying the conditions and by the uniqueness argument above, we only need to extend it to a long time solution which is equivalent to $g_0$. And replacing $g_0$ by $g(\varepsilon)$ if necessary, we can assume $g_0$ has bounded curvature.

We will prove the theorem by considering two cases:

**Case 1: When $g_0$ is a cigar (i.e. $g_0$ has half-Euclidean volume growth).**

*Proof.* Let $h$ be Cao's cigar constructed in [6]. Recall that it is a complete $U(n)$-invariant metric which has positive bisectional curvature everywhere and the scalar curvature decays like $\frac{1}{\rho}$, where $\rho$ is the distance to the origin with respect to $h$. It is a soliton so that there is a Ricci flow solution $h(t)$ on $[0,\infty)$ such that $h(t)$ is isometric to $h(0)$ for all $t \in [0,\infty)$. Since $h$ has positive bisectional curvature and its scalar curvature decays linearly, $h(t)$ is a complete bounded curvature long time solution to Ricci flow. Suppose $\xi$ and $\xi_h$ corresponds to $g_0$ and $h$ respectively, by

Theorem 3.1.2 and the fact that $\xi$, $\xi_h$ are both non-decreasing, we have

$$\left|\int_0^r \frac{\xi-\xi_h}{t}dt\right| \le C_0 + \int_1^\infty \frac{1-\xi}{t}dt + \int_1^\infty \frac{1-\xi_h}{t}dt \le C_1$$

where $C_0, C_1$ are constants independent of $r$. In particular, $g_0$ and $h$ are equivalent, so by Corollary 2.4.2 (i), $g_0$ admits a long-time complete bounded curvature solution $g(t)$. And since $g(t)$ has bounded curvature, it must be equivalent to $g_0$. To see that $g(t)$ is $U(n)$-invariant, we observe that $g(t)$ coincides with the short time $U(n)$-invariant solution obtained in Theorem 3.2.1 due to the uniqueness theorem. Let $T$ be the supremum of time such that $g(t)$ remains $U(n)$-invariant on $[0,T)$ and suppose that $T<\infty$. Since $g(t)$ is a long time bounded curvature solution, $g(t)$ converges in $C^\infty_{loc}$ to $g(T)$ as $t\to T-$, this forces $g(T)$ to be $U(n)$-invariant. By Theorem 3.1.3, $g(t)$ has non-negative bisectional curvature for all $t\in[0,T]$. By applying the short time existence theorem and uniqueness theorem to $g(T)$, we conclude that the $U(n)$ symmetry is also preserved beyond time $T$, a contradiction.

□

**Case 2: When $g_0$ is not a cigar.**

In this case, either $\xi(\infty)=\beta<1$ or $\xi(\infty)=1$ with $\int_1^\infty \frac{1-\xi}{t}dt=\infty$. We first construct background metrics for both cases:

**Proposition 3.3.1.** *Let $g$ be a smooth $U(n)$-invariant metric with bounded curvature generated by $\xi$ and $g$ is not a cigar. Then given any $\varepsilon>0$ there exists $\widetilde{g}$ satisfying*

*(a) The curvature of $\widetilde{g}$ is bounded by a constant independent of $\varepsilon$.*

*(b) $(1/\varepsilon)\widetilde{g}\le g\le C\widetilde{g}$ for some constant $C$.*

*Proof.* Suppose first that $\xi(\infty)=\beta<1$. For each $k\ge1$, consider the linear automorphism of $\mathbb{C}^n$ given by $\phi_k(z)=z/\sqrt{k}$ and consider the $U(n)$-invariant Kähler metric $g_k:=\phi_k^*g$ on $\mathbb{C}^n$. Consider the functions $h_k(r),\xi_k(r)$ and $h(r),\xi(r)$ etc corresponding to $g_k$ and $g$. Then for each $k\ge1$ we have

1. $h_k(r)=(1/k)h(r/k)$
2. $\xi_k(r)=\xi(r/k)$

3. The curvature of $g_k$ is bounded by a constant independent of $k$ because $g_k$ is isometric to $g_0$.

Now

$$\frac{h_k(r)}{h(r)} = \frac{1}{k}\frac{h(\frac{r}{k})}{h(r)} = \frac{1}{k}\exp(\int_{\frac{r}{k}}^{r}\frac{\xi(s)}{s}ds) \tag{3.3.1}$$

and thus by $0 \le \xi \le \beta$ we have

$$k^{-1} \le \frac{h_k(r)}{h(r)} \le k^{(\beta-1)}. \tag{3.3.2}$$

By Remark 3.1.1, for any $k \ge 1$ we have

$$k^{(1-\beta)}g_k \le g \le kg_k. \tag{3.3.3}$$

Thus the Proposition follows in this case by the fact that $\beta < 1$.

Suppose now $\int_1^\infty \frac{1-\xi}{t}dt = \infty$. Let $\varepsilon > 0$ be given. Let $\widehat{g}$ be any $U(n)$-invariant non-negative bisectional curvature metric with $\widehat{h}(0) = 1$ and generated by some $\widehat{\xi}$ with $\widehat{\xi}(r) = 1$ for $r \ge 1$. Let the curvature of $\widehat{g}$ be bounded by $\widehat{K}$. For each $k \ge 1$ define the pullbacks $\widehat{g}_k := \phi_k^*(\widehat{g})$ as before. Let $\widehat{h}_k(r), \widehat{\xi}_k(r)$ and $\widehat{h}(r)$ etc corresponding to $\widehat{g}_k$ and $\widehat{g}$. Then properties (1) and (2) (3) above still hold, but with $h, \xi, h_k, \xi_k$ replaced with $\widehat{h}, \widehat{\xi}, \widehat{h}_k, \widehat{\xi}_k$.

STEP 1: First note that by (3.3.1) (applied to $\hat{h}(r)$) and the fact that $\hat{\xi} \le 1$, we see that $h_k(r)$ is non-increasing in $k$ for all $r$. Now fix $\varepsilon > 0$, by $\int_1^\infty \frac{1-\xi}{t}dt = \infty$ there is $r_0 > 0$ such that if $r \ge r_0$, then for $k \ge 1$

$$\begin{aligned} h(r) =& \widehat{h}(r)\frac{h(r)}{\widehat{h}(r)} \\ \ge& \widehat{h}_k(r)\exp\left(\int_0^r \frac{\widehat{\xi}(s) - \xi(s)}{s}ds\right) \\ \ge& \frac{1}{\varepsilon}\widehat{h}_k(r) \end{aligned}$$

where we have used the fact that $\widehat{\xi}(r) = 1$ for $r \ge 1$. On the other hand, by (3.3.1) one can see this is true for $r \le r_0$ if $k$ is large enough depending on $r_0$. Hence by

Remark 3.1.1 we can find $k > 1$ depending on $\varepsilon$ such that,

$$\frac{1}{\varepsilon}\hat{g}_k \leq g$$

on $\mathbb{C}^n$.

STEP 2: Define

$$\widetilde{\xi}_k(r) := \hat{\xi}_k(r) + o_k(r)$$

where $o_k(r) : [0,\infty) \to \mathbb{R}$ is a non-positive smooth function with $|o_k| \leq \frac{1}{k}$ to be chosen. Let $\widetilde{h}_k(0) = 1/k$ and consider the corresponding metric $\widetilde{g}_k$.

*Claim 1*: There exists a constant $R_k > k$ and a smooth function $o_k(r)$ which is 0 on $[0, R_k]$ and satisfies:

$$|o_k'(r)| \leq \frac{4}{kr}$$

and

$$|\int_{R_k}^{r} \frac{\widetilde{\xi}_k(s) - \xi(s)}{s} ds| = \left| \int_{R_k}^{r} \frac{1 + o_k(s) - \xi(s)}{s} ds \right| \leq 1 + 2\log 2 \qquad (3.3.4)$$

for $r \geq R_k$.

We may choose $R_k > k$ such that $1 - 1/k \leq \xi(r) \leq 1$ on $[R_k, \infty)$. The construction of $o_k(r)$ follows from the construction in the proof of Proposition 3.2.1. We first choose a smooth non-increasing function $\rho(r) : [0,\infty) \to \mathbb{R}$ with $\rho = 0$ on $[0,1]$, $\rho = 1/k$ on $[2,\infty)$ and $0 \leq \rho' \leq 2/k$. Let

$$I(r) := \int_{2R_k}^{r} \frac{1 + o_k(r) - \xi(s)}{s} ds.$$

(note that $\widetilde{\xi}_k(r) = 1 + o_k(r)$ for $r \geq R_k$). For any positive sequence $\{r_i\}$ such that $r_0 := R_k$ and $2r_i < r_{i+1}$, define $o_k(r) := 0$ if $r \in [0, r_0)$, $o_k(r) := \rho(r/r_0)$ if $r \in [r_0, 2r_0)$, and

$$o_k(r):\begin{cases} = 1/k & \text{if } r\in[2r_0,r_1]\\ = 1/k-2\rho(r/r_1) & \text{if } r\in[r_1,2r_1]\\ = -1/k & \text{if } r\in[2r_1,r_2]\\ = -1/k+2\rho(r/r_2) & \text{if } r\in[r_2,2r_2]\\ = 1/k & \text{if } r\in[2r_2,r_3]\\ = 1/k-2\rho(r/r_3) & \text{if } r\in[r_3,2r_3]\\ \dots \end{cases} \tag{3.3.5}$$

Now for $r\in[2r_0,\infty)$ we have $1-1/k\le\xi(r)\le 1$, and as long as $k\ge 2$ we have $|1+o_k(r)-\xi(r)|\le 1$ as well. The definition of $I(r)$ then gives the following for all $i\ge 0$

$$I(r)=I(2r_i)+\int_{2r_i}^{r}\frac{(1+(-1)^i/k)-\xi(s)}{s}ds$$

for $r\in[2r_i,r_{i+1})$,

$$I(r_{i+1})-\log 2\le I(r)\le I(r_{i+1})+\log 2$$

for $r\in[r_{i+1},2r_{i+1})$.

By the fact $\xi(r)\to 1$, we may choose the $r_i$'s to be the smallest numbers with $I(r_1)=1$, $I(r_2)=-1$, $I(r_3)=1$,..etc, and the estimates above give

$$-1-\log 2\le I(r)\le 1+\log 2 \tag{3.3.6}$$

for all $r\in[2R_k,\infty)$. The integral bound in the claim follows from (2.7) and the fact that $|1+o_k(r)-\xi(r)|\le 1$ for $r\in[R_k,2R_k]$.

Finally, we also have $|o_k'(r)|=\frac{2}{r_i}\rho'(\frac{r}{r_i})\le\frac{4}{kr_i}\le\frac{4}{kr}$ for all $r\in[r_i,2r_i]$.

*Claim 2*: Let $o_k(r)$ be as in Claim 1. Then $(1/4e\varepsilon)\widetilde{g}_k\le g\le C_k\widetilde{g}_k$ for some $C_k$ and the curvature of $\widetilde{g}_k$ is bounded depending only on $\widehat{g}$.

To prove the first part of the claim, when $r\le R_k$ we have $\widetilde{g}_k(r)=g_k(r)$, and so we only have to consider when $r \geq R_k$. In this case, we have $C_k \geq h(r)/\widetilde{h}_k(r) = (h(R_k)/\widetilde{h}_k(R_k))e^{\int_{R_k}^{r} \frac{1+o_k(s)-\xi(s)}{s} ds} \geq \frac{1}{2e\varepsilon}$ for some $C_k$ where we have used Step 1 and Claim 1.

To prove the second part of the claim, note that for $r \geq R_k$ we have $|\widetilde{\xi}_k'(r)| = |o_k'(r)| \leq 4/kr$ and

$$
\begin{aligned}
\widetilde{h}_k(r) &= \widetilde{h}_k(1)\exp\left(-\int_1^r \frac{\widetilde{\xi}_k(s)}{s}ds\right) \\
&= \widetilde{h}_k(1)\exp\left(-\int_1^{R_k} \frac{\widetilde{\xi}_k(s)}{s}ds - \int_{R_k}^r \frac{\widetilde{\xi}_k(s)}{s}ds\right) \\
&\geq \widetilde{h}_k(1)\frac{1}{R_k}\exp\left(-\int_{R_k}^r \left(\frac{\widetilde{\xi}_k(s)-\xi(s)}{s} + \frac{\xi(s)-1}{s} + \frac{1}{s}\right)ds\right) \qquad (3.3.7) \\
&\geq \widetilde{h}_k(1)\frac{1}{R_k}\frac{R_k}{4e^{1+\delta}r} \\
&= h_k(1)\frac{1}{4e^{1+\delta}r} \\
&\geq \frac{\widehat{h}(1)}{k}\frac{1}{4e^{1+\delta}r}
\end{aligned}
$$

where in the third line we have used that $0 \leq \widetilde{\xi}_k \leq 1$ by definition, and in the fourth line we have used Claim 1 and $\int_1^\infty \frac{1-\xi}{t}dt = \infty$, so that $\int_{R_k}^r \frac{1-\xi(s)}{s}ds \geq -\delta$.

Thus $|\widetilde{\xi}_k'(r)/\widetilde{h}_k(r)| \leq 16e^{1+\delta}/\widehat{h}(1)$ for $r \geq R_k$. Since $\widetilde{g}_k(r) = g_k(r)$ for $r \leq R_k$, by the previous computations of $\xi_k$ and $h_k$ we have $|\widetilde{\xi}_k'(r)/\widetilde{h}_k(r)| = |\xi_k'(r)/h_k(r)| = |\hat{\xi}'(\frac{r}{k})/\hat{h}(\frac{r}{k})|$. As $\hat{\xi}(r) = 1$ if $r \geq 1$, so $|\widetilde{\xi}_k'(r)/\widetilde{h}_k(r)|$ is bounded unformly independent of $k$ for $r \leq R_k$ . As a result, we conclude that the curvature of $\widetilde{g}_k$ is bounded by a constant independent of $k$ by Lemma 3.1.1.

□

*Proof of Case 2.* By Proposition 3.3.1, for all $\varepsilon > 0$, there is a complete $U(n)$-invariant metric $\widehat{g}_\varepsilon$ with curvature bounded by $K$ with $K$ being independent of $\varepsilon$ such that

$$\frac{1}{\varepsilon}\widehat{g}_\varepsilon \leq g_0 \leq c(\varepsilon)\widehat{g}_\varepsilon$$

for some constant $c(\varepsilon)$ which may depend on $\varepsilon$. By Theorem 2.3.2, there exists a Kähler-Ricci flow $g(t)$ on $[0,T_\varepsilon)$, where $T_\varepsilon = \frac{1}{2nK\varepsilon}$, such that $g(t)$ is uniformly equivalent to $g_0$ for all $t \in [0,T')$ for any $T' < T_\varepsilon$ and has uniformly bounded curvature on $(\delta,T')$ for all $0<\delta<T'<T_\varepsilon$. Let $\varepsilon \to 0$ and use Theorem 2.4.1, one may conclude the theorem is true. Since $g(t)$ is a bounded curvature solution, the $U(n)$ symmetry and the non-negativity of bisectional curvature can be proved using the same argument as in Case 1.

□

## 3.4 Bounding the scalar curvature

Starting from $g_0$ which is $U(n)$-invariant with non-negative bisectional curvature, we have proved the long time existence of Kähler-Ricci flow in the last section, but before we prove the convergence of the flow, we will prove that the scalar curvature is uniformly bounded on compact subset along the flow, this is an important property that will be used when proving the convergence. First we need the following localized version of Ni-Tam's estimate [24, Theorem 2.1].

**Lemma 3.4.1.** *Let $(M^n, g(t))$ be complete non-compact solution of the Kähler-Ricci flow* (1.0.1) *on $M\times[0,T)$ with bounded nonnegative bisectional curvature. Let*

$$F(x,t) = \log\left(\frac{\det(g_{i\bar{j}}(x,t))}{\det(g_{i\bar{j}}(x,0))}\right)$$

*and for any $\rho>0$, let $\mathfrak{m}(\rho,x,t) = \inf_{y\in B_0(x,\rho)} F(y,t)$. Then there is $c>0$ depending only on $n$ such that for any $x_0 \in M$ and for all $\rho,t>0$*

$$\begin{aligned}-F(x_0,t) \leq c\Bigg[&\left(1+\frac{t(1-\mathfrak{m}(\rho,x_0,t))}{\rho^2}\right)\int_0^{2\rho} sk(x_0,s)ds \\ &- \frac{t\mathfrak{m}(\rho,x_0,t))(1-\mathfrak{m}(\rho,x_0,t))}{\rho^2}\Bigg]\end{aligned} \tag{3.4.1}$$

*where $B_0(x,\rho)$ is the geodesic ball with respect to $g_0 = g(0)$, and*

$$k(x,s) := \fint_{B(x,s)} R_0(y)dV_0$$

*is the average of the scalar curvature $R_0$ of $g_0$ over $B_0(x,s)$.*

*Proof.* By [24, (2.6)], if $G_\rho$ is the positive Green's function on $B_0(x_0,\rho)$ with Dirichlet boundary value, then

$$\begin{aligned}
&\int_{B_0(x_0,\rho)} G_\rho(x_0,y)\left(1-e^{F(y,t)}\right)dV_0\\
&\le t\int_{B_0(x_0,\rho)} G_\rho(x_0,y)R_0(y)dV_0+\int_0^t\int_{B_0(x_0,\rho)} G_\rho(x_0,y)\Delta_0(-F(y,s))dV_0ds\\
&=: I+II.
\end{aligned} \tag{3.4.2}$$

As in [24, p.126],

$$II \le -t\mathfrak{m}(\rho,x_0,t). \tag{3.4.3}$$

As in [24, (2.8)] there is a constant $c_1$ depending only on $n$ such that

$$\begin{aligned}
&\rho^2 ⨍_{B_0(x_0,\frac{1}{5}\rho)}(-F(y,t))dV_0\\
&\quad\le c_1 t(1-\mathfrak{m}(\rho,x_0,t))\left(\int_{B_0(x_0,\rho)} G_\rho(x_0,y)R_0(y)dV_0-\mathfrak{m}(\rho,x_0,t)\right).
\end{aligned}$$

Using the fact that $\Delta_0(-F)\ge -R_0$ and [24, Lemma 2.1], we obtain

$$\begin{aligned}
&-F(x_0,t)\le\\
&\int_{B_0(x_0,\frac{1}{5}\rho)} G_\rho(x_0,y)R_0(y)dV_0\\
&+c_2\rho^{-2}t(1-\mathfrak{m}(\rho,x_0,t))\left(\int_{B_0(x_0,\rho)} G_\rho(x_0,y)R_0(y)dV_0-\mathfrak{m}(\rho,x_0,t)\right)
\end{aligned} \tag{3.4.4}$$

for some $c_2$ depending only on $n$. As in [24, p.127], we get the result. □

We also need the following:

**Lemma 3.4.2.** *Let $g(t)$ be the complete $U(n)$-invariant solution of the Kähler-Ricci flow* (1.0.1) *with nonnegative bisectional curvature. Let*

$$F(r,t) := F(z,t) = \log\left(\frac{\det(g_{i\bar{j}}(z,t))}{\det(g_{i\bar{j}}(z,0))}\right)$$

*where* $r = |z|^2$. *Then for* $r \geq 1$ *and for all* $t$

$$F(r,t) \geq -c - n\log r + F(1,t)$$

*for some constant* $c > 0$ *depending only on* $g(0)$. *If in addition the generating function* $\xi$ *of* $g_0$ *satisfies:*

$$\lim_{r\to\infty}\int_1^r \frac{1-\xi(s)}{s}ds = b < \infty, \tag{3.4.5}$$

*then for* $r \geq 1$ *and for all* $t$

$$F(r,t) \geq -c + nF(1,t)$$

*for some constant* $c > 0$ *depending only on* $g(0)$.

*Proof.* Consider the functions $\xi(r,t), h(r,t), f(r,t)$ corresponding $g(r,t)$. Then $0 \leq \xi(r,t) \leq 1$ since $g(r,t)$ has non-negative bisectional curvature, and by Theorem 3.1.1 we then get $0 \leq h(r,t), f(r,t) \leq 1$. Thus for $r \geq 1$ we have

$$f(r,t) = \frac{1}{r}\int_0^r h(s,t)ds \geq \frac{1}{r}\int_0^1 h(s,t)dt = \frac{1}{r}f(1,t).$$

$$h(r,t) = h(1,t)\exp\left(\int_1^r -\frac{\xi(s)}{s}ds\right) \geq \frac{1}{r}h(1,t),$$

and using the formula $\det(g_{i\bar{j}}(r,t)) = h(r,t)f^{n-1}(r,t)$ we then get

$$\begin{aligned}\frac{\det(g_{i\bar{j}}(r,t))}{\det(g_{i\bar{j}}(r,0))} \geq & \det(g_{i\bar{j}}(r,t)) \\ \geq & \frac{1}{r^n}h(1,t)f^{n-1}(1,t) \\ = & \frac{1}{r^n}\det(g_{i\bar{j}}(1,t)) \\ = & \frac{1}{r^n}\frac{\det(g_{i\bar{j}}(1,t))}{\det(g_{i\bar{j}}(1,0))}\cdot\det(g_{i\bar{j}}(1,0)).\end{aligned}$$

From this, it is easy to see the first result follows. Now suppose the generating function $\xi$ of $g_0$ also satisfies (3.4.5). Then for $r \geq 1$,

$$
\begin{aligned}
\frac{h(r,t)}{h(r,0)} =& \frac{h(1,t)\exp\left(-\int_1^r \frac{\xi(s,t)}{s}ds\right)}{h(1,0)\exp\left(-\int_1^r \frac{\xi(s,0)}{s}ds\right)} \\
\geq & \frac{h(1,t)}{h(1,0)}\exp\left(\int_1^r \frac{\xi(s,0)-1}{s}ds\right) \\
\geq & c_1\frac{h(1,t)}{h(1,0)}
\end{aligned}
$$

for some constant $c_1 > 0$ independent of $r,t$, provided $r \geq 1$. Also for $r \geq 1$

$$
\begin{aligned}
\frac{f(r,t)}{f(r,0)} =& \frac{\int_0^r h(s,t)ds}{\int_0^r h(s,0)ds} \\
=& \frac{\int_0^1 h(s,t)ds + \int_1^r h(s,t)ds}{\int_0^1 h(s,0)ds + \int_1^r h(s,0)ds} \\
\geq & \frac{h(1,t) + c_1\frac{h(1,t)}{h(1,0)}\int_1^r h(s,0)ds}{\int_0^1 h(s,0)ds + \int_1^r h(s,0)ds} \\
\geq & c_2\frac{h(1,t)}{h(1,0)}
\end{aligned}
$$

for some constant $c_2 > 0$ independent of $r,t$, provided $r \geq 1$.

Hence for $r \geq 1$, at the point $|z|^2 = r$

$$
\begin{aligned}
\frac{\det(g_{i\bar{j}}(z,t))}{\det(g_{i\bar{j}}(z,0))} =& \frac{h(r,t)f^{n-1}(r,t)}{h(r,0)f^{n-1}(r,0)} \\
\geq & c_3\left(\frac{h(1,t)}{h(1,0)}\right)^n \\
\geq & c_3\left(\frac{h(1,t)f^{n-1}(1,t)}{h(1,0)f^{n-1}(1,0)}\right)^n \\
=& c_3\left(\frac{\det(g_{i\bar{j}}(1,t))}{\det(g_{i\bar{j}}(1,0))}\right)^n
\end{aligned}
$$

for some $c_3 > 0$, where we have used the fact that $f(r,t) \leq f(r,0)$. From this the second result follows. □

And now we prove a lemma about bounding the scalar curvature on compact subsets.

**Lemma 3.4.3.** *Let $g(t)$ be as in Theorem 3.3.1. Then for any $\alpha > 0$ the curvature of $g(t)$ is uniformly bounded in $D(\alpha) \times [0,\infty)$, where $D(\alpha) = \{|z|^2 < \alpha\}$.*

*Proof.* Let

$$k(z,\rho) = \frac{1}{V_0(z,\rho)} \int_{B_0(z,\rho)} R(0)$$

be the average of the scalar curvature $R(0)$ of $g_0$ over the geodesic ball $B_0(z,\rho)$ with respect to $g_0$. Let

$$k(\rho) = \sup_{|z|\le 1} k(z,\rho).$$

By [33, Theorem 7], there is a constant $c_1$ such that

$$k(\rho) \le \frac{c_1}{1+\rho}. \tag{3.4.6}$$

Suppose $|z|^2 = r$, then the distance $\rho(z) = \rho(r)$ from $z$ to the origin satisfies

$$\rho(z) = \rho(r) = \frac{1}{2}\int_0^r \frac{\sqrt{h}}{\sqrt{s}} ds \ge c_2 \log r \tag{3.4.7}$$

for some constant $c_2 > 0$ for all $r \ge 1$. Let

$$F(z,t) = \log \frac{\det(g_{i\bar{j}}(z,t))}{\det(g_{i\bar{j}}(z,0))}$$

and let $\mathfrak{m}(\rho,t) = \inf_{z\in\mathbb{C}^n,\rho(z)\le\rho} F(z,t)$. Fix $r_0 > 1$ and let $\rho_0 = \rho(r_0)$. Denote $-\mathfrak{m}(\rho_0,t)$ by $\eta(t)$. By Lemmas 3.4.1 and 3.4.2, there exist positive constants $c_3, c_4$ independent of $t$ and $\rho$ such that

$$\begin{aligned}
\eta(t) \le & c_4 \left[ \left(1 + \frac{t(1-\mathfrak{m}(\rho+\rho_0,t))}{\rho^2}\right) K(\rho) - \frac{t\mathfrak{m}(\rho+\rho_0,t)(1-\mathfrak{m}(\rho+\rho_0,t))}{\rho^2} \right] \\
\le & c_4 \left[ \left(1 + \frac{t(1+c_1+\log\widetilde{r}(\rho)+\eta(t))}{\rho^2}\right) K(\rho) \right. \\
& \left. + \frac{t(c_3+\log\widetilde{r}(\rho)+\eta(t))(1+c_1+\log\widetilde{r}(\rho)+\eta(t)}{\rho^2} \right]
\end{aligned}$$

where

$$K(\rho) = \int_0^{2\rho} sk(s)ds$$

and $\widetilde{r}(\rho)$ is such that,

$$\rho + \rho_0 = \frac{1}{2}\int_0^{\widetilde{r}(\rho)} \frac{\sqrt{h}}{\sqrt{s}} ds.$$

By (3.4.6) and (3.4.7), there is a constant $c_5$ independent of $\rho$ and $t$, such that

$$1+\eta(t) \le c_5\left(\rho + \frac{t(1+\eta(t))}{\rho} + \frac{t(1+\eta(t))^2}{\rho^2} + t\right).$$

Let $\rho^2 = 2c_5 t(1+\eta(t))$, then

$$1+\eta(t) \le c_6\left(t^{\frac{1}{2}}(1+\eta(t))^{\frac{1}{2}} + t\right)$$

for some constant $c_6 > 0$ independent of $t$. From this we conclude that $\eta(t) \le c_7(1+t)$ for some constant $c_7$ independent of $t$. Hence

$$-F(z,t) \le c_7(1+t)$$

for all $z$ with $|z|^2 \le r_0$, which implies

$$\int_t^{2t} R(z,s)ds \le \int_0^{2t} R(z,s)ds = -F(z,t) \le c_7(1+t).$$

On the other hand, by the Li-Yau-Hamilton inequality [5], $sR(z,s) \ge tR(z,t)$ for $s \ge t$. Hence we have

$$R(z,t) \le c_8$$

for all $t$ and for all $z$ with $|z|^2 \le r_0$. This completes the proof of the lemma. □

## 3.5 Convergence after rescaling

**Theorem 3.5.1.** *Let $g(t)$ be a complete longtime $U(n)$-invariant solution of the Kähler-Ricci flow* (1.0.1) *with bounded non-negative bisectional curvature, and assume $g(0)$ also has bounded curvature. Then $g(t)$ converges, after rescaling at the origin, to the standard Euclidean metric on $\mathbb{C}^n$.*

In order to prove the theorem, we need the following lemmas.

**Lemma 3.5.1.** *Let $g(t)$ be as in Theorem 3.5.1. Suppose the curvature of $g(t)$ is uniformly bounded by $c_1$, in $D(\alpha)\times[0,\infty)$, where $D(\alpha)=\{|z|^2<\alpha\}$. Then there is a constant $c_2$ depending only on $c_1$ and $\alpha$ such that*

$$h(r,t)\le h(0,t)\le c_2 h(r,t);\;\; f(r,t)\le f(0,t)\le c_2 f(r,t) \tag{3.5.1}$$

*for all $0<r<\alpha$ and for all $t$.*

*Proof.* By Remark 3.1.1 and the fact that $g(t)$ has nonnegative bisectional curvature, we have $h(0,t)\ge h(r,t)$ and $f(0,t)\ge f(r,t)$ for all $r>0$. On the other hand, we have $A(r,t),B(r,t),C(r,t)\le c_1$ in $D(\alpha)\times[0,\infty)$ by hypothesis. Thus $C=-\frac{2f_r}{f^2}$ gives

$$\frac{1}{f(r,t)}-\frac{1}{f(0,t)}\le\frac{c_1\alpha}{2}$$

in $D(\alpha)\times[0,\infty)$, and by $f(0,t)\le f(0,0)=h(0,0)=1$ we get

$$f(0,t)\le\left(\frac{c_1\alpha}{2}+1\right)f(r,t).$$

Also, $A=\frac{\xi_r(r,t)}{h(r,t)}\le c_1$ gives

$$\xi_r(r,t)\le c_1h(r,t)\le c_1h(r,0)\le c_1h(0,0)=c_1$$

in $D(\alpha)\times[0,\infty)$, and thus $\xi(r,t)\le c_1 r$, giving

$$h(r,t)=h(0,t)\exp\left(\int_0^r\frac{-\xi(s,t)}{s}ds\right)\ge\exp(-c_1\alpha)h(0,t).$$

This completes the proof of the lemma. □

*Proof of Theorem 3.5.1.* Let $a(t)=h(0,t)$. We claim that the curvature of $\frac{1}{a(t)}g(x,t)$ converges to 0 uniformly on compact sets. Note that $\frac{1}{a(t)}g(x,t)$ has nonnegative bisectional curvature. Let $R(z,t)$ be the scalar curvature of $g(t)$ at $z\in\mathbb{C}^n$. Suppose first that $\lim_{t\to\infty}R(0,t)=0$. Then by the Li-Yau-Hamilton inequality [5], we conclude that $\lim_{t\to\infty}R(z,t)=0$ uniform on compact sets. Since $a(t)\le a(0)=h(0,0)=1$, the claim is true in this case.

Suppose on the other hand that there exist $k \to \infty$ and $c_1 > 0$ such that $R(0,t_k) \geq c_1$ for all $k$. We may assume that $t_{k+1} \geq t_k + 1$. By the Li-Yau-Hamilton inequality again, there is $c_2 > 0$ such that $R(0,t_k+s) \geq c_2$ for all $k$ and for all $0 \leq s \leq 1$. By the $U(n)$ symmetry, the Ricci tensor of $g(t)$ at the origin is $\mathrm{Ric} = \frac{R}{n}g$, using the Kähler-Ricci flow equation, we have

$$h(0,t_{k+1}) \leq h(0,t_k+1) \leq e^{-c_3}h(0,t_k)$$

for some $c_3 > 0$ for all $k$. Hence $h(0,t_k) \to 0$ as $k \to \infty$. Since $h(0,t)$ is nonincreasing, we have $\lim_{t\to\infty} a(t) = \lim_{t\to\infty} h(0,t) = 0$. On the other hand, the curvature of $g(t)$ is uniformly bounded on compact sets by Lemma 3.4.3. Thus our claim is true in this case as well.

Consider any sequence $t_k \to \infty$. Let $a_k = h(0,t_k)$ and let $\widetilde{g}_k(x,t) = \frac{1}{a_k}g(x,a_kt + t_k)$. Then $\widetilde{g}_k(t)$ is a $U(n)$-invariant solution to the Kähler-Ricci flow on $\mathbb{C}^n \times [-\frac{t_k}{a_k},\infty)$. Note that $-t_k/a_k \leq -t_k$ because $a_k \leq 1$. By Lemmas 3.5.1, 3.4.3, for any $\alpha > 0$, $\widetilde{g}_k(x,0)$ is uniformly equivalent to the standard Euclidean metric $g_e$ on $D(\alpha)$ (with respect to $k$). By the claim above and the Li-Yau-Hamilton inequality [5], the curvature of the metrics $\widetilde{g}_k(x,t)$ approach zero uniformly (with respect to $k$) on compact subsets of $\mathbb{C}^n \times (-\infty,0]$.

In particular, we conclude that $\widetilde{g}_k(t)$ is uniformly equivalent to $g_e$ in $D(\alpha)$ provided $-1 \leq t \leq 0$, and thus by Theorem A.1.1, we have for any $m \geq 0$, there is a $c_4$ depending on $\alpha$ such that

$$|\nabla_e^m \widetilde{g}_k(0)| \leq c_4$$

on $D(\frac{\alpha}{2})$, where $\nabla_e$ is the derivative with respect to the standard Euclidean metric. From this it is easy to conclude the subsequence convergence of $\widetilde{g}_k(0)$ uniformly and smoothly on compact subsets of $\mathbb{C}^n$ to a flat $U(n)$-invariant Kähler metric $g_\infty$, generated by some $\xi_\infty$ say. Since the curvature is zero, we have $\xi_\infty' \equiv 0$ and thus $\xi_\infty \equiv 0$. Moreover, at the origin $(g_\infty)_{i\bar{j}} = \delta_{ij}$. Hence $h_\infty(0) = 1$ which implies that $(g_\infty)_{i\bar{j}} = \delta_{ij}$ everywhere. From this the Theorem follows as $t_k$ was chosen arbitrarily.

□

# Chapter 4

# Quasi-projective manifolds

## 4.1 Introduction

In this chapter, we discuss the Kähler-Ricci flow on quasi-projective manifolds. A complex manifold $M$ is said to be quasi-projective if $M = \bar{M} \setminus D$ for some compact Kähler manifold $\bar{M}$ and $D \subset \bar{M}$ is a divisor with simple normal crossing. With this structure, we can define different notions of singularity for Kähler metrics on $M$. We are interested in cusplike merics on $M$, which are metrics equivalent to the standard complete local model

$$\frac{idz^1 \wedge dz^{\bar{1}}}{|z^1|^2 \log^2 |z^1|^2} + i \sum_{j=2}^{n} dz^j \wedge dz^{\bar{j}},$$

where $(z^i)$ is a local holomorphic coordinate in which $D = \{z^1 = 0\}$. As an example, let $\eta$ be a Kähler form on $\bar{M}$ and let $S$ be a holomorphic section of $\mathscr{O}_D$ that vanishes precisely on $D$, a Kähler metric $\hat{\omega}$ on $M$ is said to be a Carlson-Griffiths form if

$$\begin{aligned} \hat{\omega} &= \eta - i\partial\bar{\partial} \log \log^2 ||S||_h^2 \\ &= \eta - 2i \frac{\partial\bar{\partial} \log ||S||_h^2}{\log ||S||_h^2} + 2i \frac{\partial \log ||S||_h^2}{\log ||S||_h^2} \wedge \frac{\bar{\partial} \log ||S||_h^2}{\log ||S||_h^2} \end{aligned} \tag{4.1.1}$$

for some Hermitian metric $h$ on $\mathscr{O}_D$. They are cusplike metrices inroduced in [3], below are some facts about Carlson-Griffiths forms and we refer to [3] and [17] for more details and explanations:

1. $\hat{\omega}$ has bounded geometry of infinite order.
2. $-\log\log^2\|S\|_h^2$ is bounded above and in $L^1(\overline{M})$.
3. $\log\frac{\hat{\omega}^n\|S\|_h^2\log^2\|S\|_h^2}{\Omega}$ is bounded on $M$ where $\Omega$ is any smooth volume form on $\overline{M}$.

In particular (2) implies that $\hat{\omega}$ is a well defined current on $\overline{M}$. (see for example [23] (§8, example 8.15).

In [23] the authors showed that if $\omega_0$ is cusplike with so called superstandard spatial asymptotics at $D$, then a bounded curvature cusplike solution $\omega(t)$ to (1.0.1) exists on $M\times[0,T_{[\omega_0]})$, having the same asymptotics for all $t$, where

$$T_{[\omega_0]} := \sup\{T : [\eta]+T(c_1(K_{\overline{M}})+c_1(\mathscr{O}_D))\in\mathscr{K}_{\overline{M}}\} \tag{4.1.2}$$

and $\mathscr{K}_{\overline{M}}$ is the Kähler cone of $\overline{M}$. They also showed under a weaker condition called standard spatial asymptotics, a similar bounded curvature solution exists on $M\times[0,T)$, for some maximal $T$ where $T\leq T_{[\omega_0]}$. A main point here is that the maximal existence time $T_{[\omega_0]}$ depends only on the cohomology class of the initial form $\omega_0$. One example of metric having superstandard spatial asymptotics is $\eta-i\partial\bar{\partial}\log\log^2|S|^2+i\partial\bar{\partial}\varphi$ where $\varphi\in C^\infty(\overline{M})$ and these metrics will play a crucial role in our discussion.

In this chapter, we will construct a solution to (1.0.1) when

$$\omega_0=\eta-i\partial\bar{\partial}\log\log^2|S|^2+i\partial\bar{\partial}\varphi$$

for general class of potentials $\varphi$. Assuming that $\omega_0$ is only bounded below on $M$ by a cusplike metric, and has zero Lelong number (see the Appendix A.2 for the definition and the related propostions we are going to use in this chapter), we will construct a solution to (1.0.1) on $M\times[0,T_{[\omega_0]})$ which is bounded below by a cusplike metric on some definite positive time subinterval of $[0,T_{[\omega_0]})$ (see Theorem

4.1.2 and below for details). In particular, $\omega_0$ may have unbounded curvature on $M$ here. On the other hand, in cases when $\omega_0$ may be incomplete on $M$, including when $\omega_0$ is smooth on $\overline{M}$, we can still construct solutions on $M \times [0, T_{[\omega_0]})$ which is cusplike on $M$ for all positive times (see Theorem 4.1.1). In these cases $\omega_0$ becomes instantaneously complete on $M$ under (1.0.1). We now describe our results in more details below.

We first consider the case $\omega_0 \geq c\eta$ on $M$ for some $c > 0$ where $\varphi_0$ is bounded and smooth on $M$. In particular, $\omega_0$ is typically incomplete on $M$ here. Our main result here is

**Theorem 4.1.1.** *Let $\varphi_0 \in L^\infty(\overline{M}) \bigcap C^\infty(M) \bigcap Psh(\overline{M}, \eta)$ such that*

$$\omega_0 = \eta + i\partial\bar{\partial}\varphi_0 \geq c\eta \tag{4.1.3}$$

*for some constant $c > 0$. Let $T_{[\omega_0]}$ be as in* (4.1.2)*. Then* (1.0.1) *has a unique smooth solution $\omega(t)$ on $M \times [0, T_{[\omega_0]})$ where*

$$c_1(t)\hat{\omega} \leq \omega_t \leq c_2(t)\hat{\omega} \tag{4.1.4}$$

*for all $t \in (0, T_{[\omega_0]})$ ,and some positive functions $c_i(t)$ and Carlson-Griffiths form on $\hat{\omega}$ on $M$.*

*Also, for any Hermitian metric $h$ on $\mathscr{O}_D$ and volume form $\Omega$ on $\overline{M}$,* (4.1.6) *and* (4.1.7) *hold on $M \times [0, T_{[\omega_0]})$ for some $\varphi(t)$ which is bounded on $M$ for each $t \in [0, T_{[\omega_0]})$.*

*Remark* 4.1.1. Theorem 4.1.1 includes as a special case, when $\omega_0$ has conical singularities at $D$ or is in fact smooth on $\overline{M}$.

Next we consider when $\omega_0$ is a complete metric on $M$ in which case $\varphi_0$ may be unbounded on $M$. Our first result here is

**Theorem 4.1.2.** *Let $\varphi_0 \in C^\infty(M) \bigcap Psh(\overline{M}, \eta)$ have zero Lelong number such that*

$$\omega_0 = \eta + i\partial\bar{\partial}\varphi_0 \geq c\hat{\omega}$$

*for some $c > 0$ and Carlson-Griffiths form $\hat{\omega}$ on $M$. Let $T_{[\omega_0]}$ be as in* (4.1.2)*. Then* *the Kähler -Ricci flow* (1.0.1) *has a smooth solution* $\omega(t)$ *on* $M \times [0, T_{[\omega_0]})$ *and*

$$\omega(t) \geq (\frac{1}{n} - \frac{4\hat{K}t}{c})\hat{\omega} \tag{4.1.5}$$

*for all* $t \leq \frac{c}{4n\hat{K}}$ *where* $\hat{K}$ *is a non-negative upper bound on the bisectional curvatures of* $\hat{\omega}$. *Moreover,*

*(1) For any hermitian metric* $h$ *on* $\mathscr{O}_D$ *and volume form* $\Omega$ *on* $\overline{M}$, (4.1.6) *and* (4.1.7) *hold on* $M \times [0, T_{[\omega_0]})$ *where* $\varphi(t) \leq c(t)$ *on* $M \times [0, T_{[\omega_0]})$ *for some continuous function* $c(t)$.

*(2) Suppose further that* $\omega_0$ *is cusplike and* $-C \log\log^2 \|S\|_h^2 \leq \varphi_0$ *on* $M$ *for some constant* $C > 0$ *and Hermitian metric* $h$. *Then for any* $0 < T < T_{[\omega_0]}$, (4.1.6) *and* (4.1.7) *hold on* $M \times [0, T]$ *where* $-c \log\log^2 \|S\|_{h'}^2 \leq \varphi(t) \leq c$ *on* $M \times [0, T]$ *for some constant* $c > 0$ *and Hermitian metric* $h'$.

*Remark* 4.1.2. It can be proved that there is a unique solution $g(t)$ satisfying (4.1.5) and (1) on the time interval $[0, \frac{c}{4n\hat{K}})$, though it is not known if our solution is unique on the whole time interval $[0, T_{[\omega_0]})$.

*Remark* 4.1.3. In Theorem 4.1.1 and 4.1.2 above, we only considered the case of a single smooth divisor $D \subset \overline{M}$. On the other hand, straight forward extensions of our definitions and techniques allow us also to consider the case of some collection of simple normal crossing divisors $D_1, .., D_k$ in which case we can have similar statements.

Note also that Theorem 4.1.2 leaves open the possibility that the solution may exist beyond $t = T_{[\omega_0]}$. Note also, in Theorem 4.1.2 $\omega_0$ is complete while the solution may not be complete for all positive times. Meanwhile in Theorem 4.1.1, $\omega_0$ may be incomplete while the solution is complete for all positive times. This seems counterintuitive, and is a result of the stronger a priori estimates in the case $\varphi_0$ is bounded. On the other hand, (2) says cupslikeness is preserved at the potential level for all times in some sense. If we assume $\omega_0$ above in fact has bounded curvature and is sufficiently asymptotic to the standard model at $D$ in a sense, the following Theorem says the solution is indeed cusplike for all times, and $[0, T_\omega)$ is indeed a maximal time interval.

**Theorem 4.1.3.** *Let $\eta$ be a smooth Kähler form on $\overline{M}$ and $\hat{\omega} = \eta - i\partial\bar{\partial}\log\log^2 \|S\|^2$ be a Carlson-Griffiths form on $M$. Let $\omega_0 = \hat{\omega} + i\partial\bar{\partial}\varphi$ be a smooth complete bounded curvature Kähler metric on $M$ such that $\frac{\varphi}{\log\log^2 \|S\|^2} \to 0$, $\frac{|d\varphi|_{\hat{\omega}}}{\log\log^2 \|S\|^2} \to 0$ and $|\omega_0 - \hat{\omega}|_{\hat{\omega}} \to 0$ as $\|S\| \to 0$. Let $T_{[\omega_0]}$ be as in* (4.1.2). *Then* (1.0.1) *has a unique smooth maximal bounded curvature solution $\omega(t)$ on $M \times [0, T_{[\omega_0]})$.*

In the follow discussion, we will study (1.0.1) through an associated parabolic Monge Ampère equation set up as follows. For any Hermitian metric $h$ on $\mathscr{O}_D$, and volume form $\Omega$ on $\overline{M}$, consider a solution $\varphi(t)$ to the parabolic Monge Ampère equation

$$\begin{cases} \partial_t \varphi(t) = \log \dfrac{\|S\|_h^2 \log^2 \|S\|_h^2 (\theta_t + i\partial\bar{\partial}\varphi(t))^n}{\Omega}; \\ \varphi(0) = \varphi_0. \end{cases} \tag{4.1.6}$$

$$\theta_t := \eta + t\chi; \quad \chi := -\mathrm{Ric}(\Omega) + \Theta_h - i\partial\bar{\partial}\log\log^2 \|S\|_h^2$$

and the associated family of Kähler metrics

$$\omega(t) := \theta_t + i\partial\bar{\partial}\varphi(t) \tag{4.1.7}$$

on $M \times [0, T)$ for some $T$. Here $\mathrm{Ric}(\Omega) = -i\partial\bar{\partial}\log\Omega$ and $\Theta_h$ represents the curvature form of the metric $h$ on $\mathscr{O}_D$.

It follows that $\omega(t)$ solves (1.0.1) on $M \times [0, T)$, and conversely, if $\omega(t)$ solves (1.0.1) on $M \times [0, T)$ then (4.1.7) holds for some solution $\varphi(t)$ to (4.1.6) (see the derivation of (4.2.2)). The equation (4.1.6) is different from the parabolic Monge-Ampère equations considered in the earlier works mentioned above in the appearance of the $\|S\|_h^2 \log \|S\|_h^2$ term in the numerator of the right hand side. This term will be useful in establishing the cusplike-ness of our solutions for positive times.

## 4.2 Proof of Theorem 4.1.1

Analogous to [16], we will use Theorem A.2.1 to construct approximation solutions to Kähler-Ricci flow from which we will take a limit to obtain the desired solution. Our study here is similar in ways to [23] where the authors also studied cusp type Kähler-Ricci flow solutions. On the other hand, our work here is different from these in the following ways. First, our initial metric is not cusplike (or even complete) as in the case of [23], yet we are looking solutions which are cusplike for positive times. This is one reason that the Monge Ampère equation (4.1.6) we consider is somewhat different than the ones studied in other works. Second, the background Carlson-Griffith metric has only $L^1$ volume form so we cannot approximate $\phi_0$ using the procedure from [29] which is based on Kolodziej's $L^p$ estimate. In the following we will introduce approximation procedures to overcome these difficulties.

We first make the following technical assumptions which we will use throughout the rest of the section.

**Assumption 1.** *Let $\eta, \hat{\omega}$ and $\varphi_0$ be as in Theorem 4.1.1 and $T_{[\omega_0]}$ be as in* (4.1.2). *Fix some $\tilde{T} < T_{[\omega_0]}$ and choose a Hermitian metric $\hat{h}$ on $\mathscr{O}_D$ and smooth volume form $\Omega$ on $\overline{M}$ so that $\|S\|^2_{\hat{h}} < 1$ on $\overline{M}$ and*

1. $\eta + t(-Ric(\Omega) + \Theta_{\hat{h}}) > 0$ *on* $\overline{M} \times [0, \tilde{T}]$
2. $\eta + t(-Ric(\Omega) + \Theta_{\hat{h}} + \frac{2\Theta_{\hat{h}}}{\log \|S\|^2_{\hat{h}}}) > 0$ *on* $M \times [0, \tilde{T}]$

*Finally, we will abbreviate $\|S\|^2_{\hat{h}}$ and $\Theta_{\hat{h}}$ simply by $\|S\|^2$ and $\Theta$ repsectively.*

We first choose $\hat{h}$ so that $\|S\|^2_{\hat{h}} < 1$ on $\overline{M}$. By the definition of $T_{[\omega_0]}$ we can then choose $\Omega$ such that (1) holds for $t = \tilde{T}$. Then, by scaling $\hat{h}$ smaller if necessary, we may also assume the inequality in (2) also hold at $t = \tilde{T}$ (by the smoothness of $\Theta$ on $\overline{M}$. The fact that (1) and (2) holds for all $t \in [0, \tilde{T}]$ then follows by interpolation between $t = 0$ and $t = \tilde{T}$.

### 4.2.1 Proof of Theorem 4.1.1 when $\varphi_0 \in C^\infty(\overline{M})$

In this case, $\omega_0 = \eta + i\partial\bar{\partial}\varphi_0$ is a Kähler form on $\overline{M}$, from Theorem 8.19 in [23] we have

**Lemma 4.2.1.** *For $\varepsilon > 0$ sufficiently small, $\omega_{\varepsilon,0} = \omega_0 - \varepsilon\partial\bar{\partial}\log\log^2\|S\|^2_h$ is a Carlson-Griffiths metric on $M$ and* (1.0.1) *has a bounded curvature solution $\omega_\varepsilon(t)$ on $M \times [0, T_{[\omega_0]})$ with initial data $\omega_{\varepsilon,0}$.*

For $\omega_\varepsilon(t)$ above, we will now derive estimates on compact subsets of $M \times [0, T_{[\omega_0]})$ which are uniform with respect to $\varepsilon$. We will then let $\varepsilon \to 0$ to obtain a limit smooth solution $\omega(t)$ to (1.0.1) on $M \times [0, T_{[\omega_0]})$ with initial data $\omega_0 = \eta + i\partial\bar{\partial}\varphi_0$. We may write

$$\omega_\varepsilon(t) = \theta_{\varepsilon,t} + i\partial\bar{\partial}\varphi_\varepsilon(t) \tag{4.2.1}$$

where $\varphi_\varepsilon(t)$ solves the parabolic Monge Ampère equation on $M \times [0, T_{[\omega_0]})$:

$$\begin{cases} \partial_t \varphi_\varepsilon(t) = \log \dfrac{\|S\|^2 \log\|S\|^2 (\theta_{\varepsilon,t} + i\partial\bar{\partial}\varphi_\varepsilon(t))^n}{\Omega}; \\ \varphi_\varepsilon(0) = \varphi_0. \end{cases} \tag{4.2.2}$$

$$\theta_{\varepsilon,t} := \eta + t\chi - \varepsilon\partial\bar{\partial}\log\log^2\|S\|^2; \quad \chi := -\mathrm{Ric}(\Omega) + \Theta - i\partial\bar{\partial}\log\log^2\|S\|^2$$

Indeed, letting

$$\varphi_\varepsilon(t) = \varphi_0 + \int_0^t \log \frac{\|S\|^2 \log^2\|S\|^2 (\omega_\varepsilon(t))^n}{\Omega}$$

and defining $\theta_{\varepsilon,t}$ as above we see that (4.2.2) is obviously satisfied. On the other hand, we have $\theta_{\varepsilon,0} + i\partial\bar{\partial}\varphi_\varepsilon(0) = \omega_\varepsilon(0)$ while on $M \times [0, T_{[\omega_0]})$

$$\begin{aligned} (\theta_{\varepsilon,t} + i\partial\bar{\partial}\varphi_\varepsilon(t))' =& -\mathrm{Ric}(\Omega) + \Theta - i\partial\bar{\partial}\log\log^2\|S\|^2 \\ &+ i\partial\bar{\partial}(\log\|S\|^2 + \log\log^2\|S\|^2 + \log(\omega_\varepsilon^n(t)) - \log\Omega) \\ =& i\partial\bar{\partial}\log(\omega_\varepsilon^n(t)) \\ =& -Ric(\omega_\varepsilon(t)) \end{aligned} \tag{4.2.3}$$

and it follows from (1.0.1) that (4.2.1) holds on $M \times [0, T_{[\omega_0]})$. Conversely, reversing the process above shows that the metric in (4.2.1) satisfies (1.0.1) for any given solution to (4.2.2).

Note by (2) in Assumption 1 and (4.1.1), for $\varepsilon > 0$ we have

$$c_1(t+\varepsilon)\hat{\omega} \leq \theta_{t,\varepsilon} \leq c_2\hat{\omega}, \tag{4.2.4}$$

for some constants $c_1, c_2 > 0$ independent of $\varepsilon$ for all $t \in [0, \widetilde{T}]$. We will now derive estimates for $\varphi_\varepsilon(t)$ on $M \times [0, \tilde{T}]$ which will be uniform with respect to $\varepsilon$ and will yield uniform $C^\infty_{loc}$ estimates for $\omega_\varepsilon(t)$. These will allow us to let $\varepsilon \to 0$ and $\tilde{T} \to T_{[\omega_0]}$ and obtain a limit solution on $M \times [0, T_{[\omega_0]})$ satisfying the conditions in Theorem 4.1.1.

We begin with the following $C^0$-estimates:

**Lemma 4.2.2.** *On the time interval $(0, \tilde{T}]$, we have $|\varphi_\varepsilon| \leq C$, $|\dot{\varphi}_\varepsilon| \leq \frac{C}{t}$, where $C$ is independent of $\varepsilon$ sufficiently small.*

*Proof.* Fix $\varepsilon > 0$ such that (4.2.4) holds. Since $\theta_{t,\varepsilon}$ is a Carlson-Griffiths form for all $t \in [0, \tilde{T}]$, so the curvature on $M = \overline{M} \setminus D$ is uniformly bounded. Let $\psi := \varphi_\varepsilon - Ct$ for some $C > 0$ to be chosen. Then since $\omega_\varepsilon(t)$ is a bounded curvature solution, $\psi$ is uniformly bounded on $M \times [0, \tilde{T}]$. We first suppose that $\psi$ attains a maximum value on $M \times [0, \tilde{T}]$ at some point $(\bar{x}, \bar{t})$. If $\bar{t} > 0$ then using (4.2.2) we have at $(\bar{x}, \bar{t})$

$$\partial_t \psi_\varepsilon \leq \log \frac{\|S\|^2 \log \|S\|^2 \theta^n_{\bar{t},\varepsilon}}{\Omega} - C < -1 \tag{4.2.5}$$

by choosing $C$ sufficiently large independent of $\varepsilon$. Indeed, such a choice of $C$ exists from the upper bound in (4.2.4) and property (3) of Carlson-Griffiths metrics (see the paragraph after (4.1.1)). But (4.2.5) contradicts the maximality of $\psi$ unless $\bar{t} = 0$ in which case $\psi \leq \sup \psi(0) = \sup \varphi_0$ on $M \times [0, \tilde{T}]$ and thus

$$\varphi_\varepsilon \leq \sup \varphi_0 + C\tilde{T} \tag{4.2.6}$$

on $M \times [0, \tilde{T}]$. Now in general, suppose $\psi$ does not attain a maximal value on $M \times [0, \tilde{T}]$. Since $\omega_\varepsilon(t)$ is a bounded curvature solution, we have $|\partial_t \psi_\varepsilon|$ is bounded on $M \times [0, \widetilde{T}]$, and by the Omori-Yau maximum principal we can find a sequence $x_k \in M$ and $\bar{t} \in [0, \tilde{T}]$ such that $\psi_\varepsilon(x_k, \bar{t}) \to \sup_{M \times [0, \tilde{T}]} \psi_\varepsilon$ and $i\partial\bar{\partial}\psi_\varepsilon(x_k, \bar{t}) \leq \lambda_k \theta_{\bar{t},\varepsilon}$ where $\lambda_k$ decreases to 0 as $k \to \infty$. Combining these with (4.2.2), we may argue as above that for any $\delta$ we have $\partial_t \psi_\varepsilon(x_k, \bar{t}) \leq -1$ for some $C$ independent of $\varepsilon$

and all $k$ sufficeintly large. On the other hand, $|\partial_t^2 \psi_\varepsilon|(x_k,\bar{t})$ is uniformly bounded independent of $k$ using again that $\omega_\varepsilon(t)$ is a bounded curvature solution. These last two facts contradict that $\psi_\varepsilon(x_k,\bar{t}) \to \sup_{M\times[0,\tilde{T}]} \psi_\varepsilon$ unless $\bar{t}=0$ and we conclude again as above that (4.2.6) likewise holds on $M\times[0,\widetilde{T}]$ in this case as well.

Using again that $\omega_\varepsilon(t)$ is a bounded curvature solution, we also have $|\partial_t^2 \psi_\varepsilon|$ is bounded on $M\times[0,\widetilde{T}]$.

By a similar argument, using (4.2.4) and (4.2.2) we may also get

$$\inf \varphi_0 + C' \int_0^t \log(s+\varepsilon)ds \le \varphi_\varepsilon \tag{4.2.7}$$

on $M\times[0,\tilde{T}]$ for some constant $C'>0$ independent of $\varepsilon$. We thus conclude the estimates for $|\varphi_e|$ in the Lemma hold.

Next we will apply the methods in [29] to derive estimates for $\dot{\varphi}_\varepsilon$. We have

$$(\partial_t - \Delta)\dot{\varphi}_\varepsilon = tr_{\omega_\varepsilon}\chi$$

where the $\Delta$ denotes the Laplacian with respect to $\omega_\varepsilon$, and also

$$(\partial_t - \Delta)(t\dot{\varphi}_\varepsilon - \varphi_\varepsilon - nt) = -tr_{\omega_\varepsilon}(\eta - \varepsilon\partial\bar{\partial}\log\log^2\|S\|^2) < 0,$$

where the last inequality comes from the fact that $\eta - \varepsilon\partial\bar{\partial}\log\log^2\|S\|_h^2 > 0$ when $\varepsilon$ is small by (4.1.1). Applying the maximum principle in [27], we conclude the supremum of $(t\dot{\varphi}_e - \varphi_\varepsilon - nt)$ on $M\times(0,\tilde{T}]$, which is indeed finite, is attained when $t=0$ and thus $\dot{\varphi}_\varepsilon \le \frac{\varphi_\varepsilon - \varphi_0}{t} + n$ on $M\times(0,\tilde{T}]$. On the other hand, for sufficiently large $A$ independent of $\varepsilon$ we have

$$\begin{aligned}
&(\partial_t - \Delta)(\dot{\varphi}_e + A\varphi_\varepsilon - n\log t)\\
&= tr_{\omega_\varepsilon}(\chi + A\theta_{t,\varepsilon}) + A\log\frac{\omega_\varepsilon^n\|S\|^2\log^2\|S\|^2}{\Omega} - (An + \frac{n}{t})\\
&\ge \frac{A}{2}tr_{\omega_\varepsilon}\theta_{t,\varepsilon} + A\log\frac{(t+\varepsilon)^n\omega_\varepsilon^n}{\theta_{t,\varepsilon}^n} - \frac{C}{t}\\
&\ge \frac{A}{4}\left(\frac{\theta_{t,\varepsilon}^n}{\omega_\varepsilon^n}\right)^{1/n} - \frac{C}{t}.
\end{aligned} \tag{4.2.8}$$

where in the second inequality we have made use of (4.2.4) and property (3) of

Carlson-Griffiths metrics, and in the third inequality we have again used (4.2.4). Now let $\psi = (\dot{\varphi}_e + A\varphi_\varepsilon - n\log t)$, and assume $\psi$ attains a minimum value on $M\times[0,\tilde{T}]$ at some point $(\bar{x},\bar{t})$. It follows that $\bar{t}>0$, and (4.2.8) then gives $\omega_\varepsilon^n(\bar{x},\bar{t}) \geq c\theta_{t,\varepsilon}^n(\bar{x},\bar{t})\bar{t}^n$ for some $c>0$ independent of $\varepsilon$. From this, (4.2.4), (4.2.2) and property (3) of Carlson-Griffiths metrics we may then have $\psi \geq C\log t - C$ and thus $\dot{\varphi}_e \geq C\log t - C$ on $M\times(0,\tilde{T}]$ for some $C>0$ where we have used (4.2.6) and (4.2.7). In general, $\psi$ is only bounded but may not attain a minimum value on $M\times[0,\tilde{T}]$, though we may argue as before, applying the above estimate along an appropriate space-time sequence obtained by the Omori-Yau maximum principle, to conclude $\dot{\varphi}_e \geq C\log t - C$ on $M\times(0,\tilde{T}]$ for some $C>0$ in this case as well. We thus conclude the estimates for $|\dot{\varphi}_e|$ in the Lemma hold. □

Next we want to derive a Laplacian estimate for $\varphi_e$.

**Lemma 4.2.3.** *for each $t\in(0,\tilde{T}]$ we have $C_1(t)\hat{\omega} \leq \omega_\varepsilon(t) \leq C_2(t)\hat{\omega}$, for constants $C_1(t), C_2(t) > 0$ independent of $\varepsilon$ sufficiently small.*

*Proof.* First recall that $\hat{\omega}$ is complete and has uniformly bounded bisectional curvature. Then the parabolic version of the Chern-Lu inequality (see [28]) gives:

$$(\partial_t - \Delta)\log\, tr_{\omega_\varepsilon}\hat{\omega} \leq C tr_{\omega_\varepsilon}\hat{\omega} + C$$

where the constant $C$ depends on the upper bound of the bisectional curvature of $\hat{\omega}$. For the rest of the proof, $C$ will denote a constant, which may change from line to line, and which is independent of $\varepsilon$. Now by (4.2.4) we may choose $A$ sufficiently large independent of $\varepsilon$ so that

$$\begin{aligned}
&(\partial_t - \Delta)(t\log\, tr_{\omega_\varepsilon}\hat{\omega} - A\varphi_e)\\
&\leq tr_{\omega_\varepsilon}(Ct\hat{\omega} - A\theta_{t,\varepsilon}) + \log\, tr_{\omega_\varepsilon}\hat{\omega} - A\log\frac{\omega_\varepsilon^n\|S\|^2\log^2\|S\|^2}{\Omega} + (An+Ct)\\
&\leq -\frac{A}{2}tr_{\omega_\varepsilon}\theta_{t,\varepsilon} + \log\, tr_{\omega_\varepsilon}\theta_{t,\varepsilon} + A\log\frac{\theta_{t,\varepsilon}^n}{\omega_\varepsilon^n} - CA\log(t+\varepsilon) + C\\
&\leq -\frac{A}{4}tr_{\omega_\varepsilon}\theta_{t,\varepsilon} - CA\log(t+\varepsilon) + C,
\end{aligned} \tag{4.2.9}$$

Now suppose $t\log\, tr_{\omega_\varepsilon}\hat{\omega} - A\varphi_e$ attains a maximum value on $M\times[0,\tilde{T}]$ at some point $(\bar{x},\bar{t})$. Then if $\bar{t}>0$, using (4.2.9) we have at $(\bar{x},\bar{t})$ that $tr_{\omega_\varepsilon}\theta_{t,\varepsilon}\le -C\log t+C$ and so

$$t\log\, tr_{\omega_\varepsilon}\hat{\omega}-A\varphi_e\le Ct(\log\log\frac{1}{t}+\log\frac{1}{t+\varepsilon})+C\le C.$$

In this case it follows that

$$tr_{\omega_\varepsilon}\hat{\omega}\le e^{\frac{C}{t}} \tag{4.2.10}$$

on $M\times[0,\tilde{T}]$ where we have used estimate for $|\varphi_\varepsilon|$ Lemma 4.2.2. On the other hand, if $\bar{t}=0$ we also clearly have (4.2.10). In general, In general, $t\log\, tr_{\omega_\varepsilon}\hat{\omega}-A\varphi_e$ is only bounded but may not attain a maximum value on $M\times[0,\tilde{T}]$, though we may argue as in the proof of Lemma 4.2.2 and apply the above estimates along an appropriate sequence in space-time to likewise conclude that (4.2.10) holds on $M\times[0,\tilde{T}]$ in this case as well.

Finally, for any $t\in(0,\tilde{T}]$, (4.2.2) gives

$$\frac{\omega_\varepsilon^n}{\hat{\omega}^n}=e^{\dot{\varphi}_\varepsilon}\frac{\Omega}{\hat{\omega}^n\|S\|^2\log^2\|S\|^2}\le C_t \tag{4.2.11}$$

on $M$ for some constant $C_t>0$ where we have used the estimate for $|\dot{\varphi}_\varepsilon|$ in Lemma 4.2.2 and property (3) of Carlson-Griffiths metrics. The Lemma follows from (4.2.11) and (4.2.10).

□

*Completion proof of Theorem 4.1.1 when $\varphi_0\in C^\infty(\overline{M})$.* The previous two lemmas and the Evans-Krylov theory applied to (4.2.2) imply that for any $K\subset\subset M$ and $s\in(0,\tilde{T})$ we have

$$\max_K\|\nabla_\eta^k\varphi_\varepsilon(t)\|_\eta\le C_{k,s,K,t}$$

independent of $\varepsilon$ and $t\in(s,\tilde{T}]$ where the norm and covariant derivative here are with respect to $\eta$. Thus for some subsequence $\varepsilon_i\to 0$, $\varphi_{\varepsilon_i}$ will converge locally uniformly to a smooth solution $\varphi$ to (4.1.6) on $M\times(0,\tilde{T})$ which is bounded on $M$ for each $t$. Thus $\omega_{\varepsilon_i}(t)$ converges locally uniformly to a smooth solution $\omega(t)$ to the flow in equation (1.0.1) on $M\times(0,\tilde{T})$ and as $\tilde{T}<T_{[\omega_0]}$ was arbitrary, we may in fact assume the convergence to a solution on $M\times(0,T_{[\omega_0]})$ satisfying the estimates in Lemma 4.2.3.

We now show the limit solution $\omega(t)$ in fact converges smoothly uniformly on compact subsets of $M$ to $\omega_0$ as $t \to 0$, and thus can be extended to a smooth solution to (1.0.1) on $M \times [0, T_{[\omega_0]})$ with initial data $\omega_0$. This basically follows from applying Theorem 2.3.1 to the sequence $\omega_{\varepsilon_i}(t)$, and observing that the completeness of the background metric $\hat{g}$ in that Theorem is in fact not necessary in our case. We describe this in more detail as follows. Consider the family of solutions $\omega_{\varepsilon_i}(t)$ on $M \times [0, T_{[\omega_0]})$. For each $i$ we have $\omega_{\varepsilon_i}(0) \geq c\eta$ on $M$ for some $c > 0$ independent of $i$ and we conclude from the proof of Lemma 2.2.1 that

$$\omega_{\varepsilon_i}(t) \geq c\eta \tag{4.2.12}$$

on $M \times [0, T]$ for some $c, T > 0$ independent of $i$. For this simply observe that in Lemma 2.2.1, the completeness of $\hat{g}$ is never actually used in the proof. Next we choose any smooth non-negative function $\psi : [0, \infty) \to \mathbb{R}$ which is identically zero in some neighborhood of 0 and let $\varphi(t) := \psi(\|S\|^2)$. Then using (4.2.12), as in the proof of Lemma 2.2.3 we may have $\|\nabla_{\varepsilon_i,t}\varphi(t)\|, \|\Delta_{\varepsilon_i,t}\varphi(t)\| \leq C$ on $M$ where the norms here are relative to $\omega_{\varepsilon_i}(t)$ and $C$ is independent of $i$ and $t \in [0, T)$, and by the same proof there we may conclude that $\varphi(t)R_{\varepsilon_i}(t) \geq C$ on $M \times [0, T]$ where $R_\varepsilon(t)$ is the scalar curvature of $\omega_{\varepsilon_i}(t)$ and the constant $C$ is independent of $i$. From this and the fact that $\phi$ was arbitrarily chosen, we can conclude as in Lemma 2.2.3 that for any compact $K \subset\subset M$ we have the upper bound

$$\omega_{\varepsilon_i}(t) \leq c\eta \tag{4.2.13}$$

on $K \times [0, T]$ for some $c$ independent of $i$. The smooth convergence of $\omega_{\varepsilon_i}(t)$ on compact subsets of $M \times [0, T]$ then follows from (4.2.12), (4.2.13) and the Evans-Krylov theory. Thus the limit solution $\omega(t)$ extends smoothly to $M \times [0, T_{[\omega_0]})$ as claimed above.

To complete the proof of the Theorem in this case, it remains only to prove the uniqueness statement which we do in the next sub-section in Proposition 4.2.1. □

### 4.2.2 Proof of Theorem 4.1.1 when $\varphi_0 \in L^\infty(M) \bigcap C^\infty(M)$

By (4.1.3) we can choose some $\varepsilon > 0$ so that in fact we have $\varphi_0 \in Psh(\overline{M}, (1-\varepsilon)\eta)$. Then, using Theorem A.2.1 we may choose a sequence $\{\varphi_j\} \subset C^\infty(\overline{M}) \bigcap Psh(\overline{M}, (1-$

$\varepsilon)\eta)$ so that $\varphi_j \downarrow \varphi_0$ pointwise on $\overline{M}$ and locally smoothly on $M$. In particular, it follows that

1. $|\varphi_j| \leq C$ for all $j$ and some $C$
2. $\omega_j = \eta + i\partial\bar{\partial}\varphi_j \geq \varepsilon\eta$ for all $j$ on $\overline{M}$.

Now for each $j$ we let $\omega_j(t)$ be the solution to (1.0.1) on $M \times [0, T_{[\omega_0]})$ with initial data $\omega_j(0) = \eta + i\partial\bar{\partial}\varphi_j$ constructed in the previous sub-section. Under Assumption 1, let $\varphi_j(t)$ be the corresponding solution to (4.1.6) on $M \times [0, T_{[\omega_0]})$ also as previously constructed. From Lemma 4.2.2 and (1) it follows that $|\varphi_j|(t)$ and $|\dot{\varphi}_j(t)|$ are uniformly bounded on compact subsets of $M \times (0, \tilde{T})$ independently of $j$ where $\tilde{T}$ is from Assumption 1. From Lemma 4.2.3 it further follows that $\omega_j(t)$ is uniformly equivalent to $\hat{\omega}$ on $M$, independent of $j$, on compact intervals of $(0, \tilde{T})$. As $\tilde{T} < T_{[\omega_0]}$ was arbitrary and by applying the arguments in the last sub-section separately for each $j$, we may conclude smooth local estimates for $\omega_j(t)$ on compact subsets of $M \times [0, T_{[\omega_0]})$ which are independent of $j$, and that some subsequence of $\omega_j(t)$ converges to a solution $\omega(t)$ to (1.0.1) on $M \times [0, T_{[\omega_0]})$ satisfying (4.1.4) for all $t \in (0, T_{[\omega_0]})$.

The proof of Theorem 4.1.1 will be complete once we prove uniqueness of such a solution which we do in the Proposition below. First, we note that by (4.1.3) each $\omega_j(t)$ will satisfy the lower bound in (4.2.12) for constants $c, T > 0$ independent of $j$ and thus the limit solution $\omega(t)$ likewise satisfies the lower bound in (4.2.12). We will use this fact in the following proof.

**Proposition 4.2.1.** *Let $\varphi_1(t), \varphi_2(t)$ be two solutions to* (4.1.6) *on $M \times [0, T_{[\omega_0]})$ with initial data $\varphi_1(0) = \varphi_2(0) \in L^\infty(M) \bigcap C^\infty(M)$. Suppose $|\varphi_1(t)|, |\varphi_2(t)|$ are both bounded on $M \times [0, T)$ for every $T < T_{[\omega_0]}$. Then $\varphi_1(t) = \varphi_2(t)$ on $M \times [0, T]$.*

*Proof.* We assume without loss of generality that the solutions $\omega_1(t)$ and $\varphi_1(t)$ are as constructed as in the proof of Theorem 4.1.1 so far. Now for any $T < T_{[\omega_0]}$ we prove that $\varphi_2 \leq \varphi_1$ on $M \times [0, T]$. Let $|\cdot|^2$ be a Hermitian metric such that $\|S\|^2 < 1$ and let $\Theta$ denotes its curvature form. As noted above, $\omega_1(t)$ satisfies the inequality in (4.2.12) on $M \times [0, \varepsilon]$ for some constant $c, \varepsilon > 0$. Thus for all $a > 0$ we can find $C_a \to 0$ as $a \to 0$ such that $\log \frac{(\omega_1(t) + a\Theta)^n}{\omega_1(t)^n} < C_a$ on $M \times [0, T]$. Consider

$\psi = \varphi_2 - \varphi_1 + a\log\|S\|^2 - C_a t$, since $\varphi_2 - \varphi_1$ is a bounded function, $\psi$ attains a maximum on $M\times[0,T]$ at some point $(\bar{x},\bar{t})$. If $\bar{t}>0$, then at $(\bar{x},\bar{t})$ we have

$$0 < \partial_t \psi = \log \frac{(\omega_1(t) + a\Theta + i\partial\bar{\partial}\psi)^n}{\omega_1(t)^n} - C_a < 0,$$

a contradiction. Thus since $\psi(x,0)<0$ we have $\varphi_2 \le \varphi_1 - a\log\|S\|^2 + C_a t$, and letting $a\to 0$ we get $\varphi_2 \le \varphi_1$.

To prove $\varphi_2 \ge \varphi_1$ we argue similarly. Namely, we first choose $C_a \to 0$ as $a\to 0$ so that $\log(\omega_1 - a\Theta)^n/\omega_1^n \ge C_a$. Then we let $\psi = \varphi_2 - \varphi_1 - a\log\|S\|^2 - C_1 t$ and argue as before using the maximum principle that $\psi \ge \min_M \psi(0)$ everywhere then conclude by letting $a\to 0$ that $\varphi_2 \ge \varphi_1$ on $M$.

□

## 4.3 Proof of Theorem 4.1.2

The proof here roughly the same steps as the proof in Theorem 4.1.1. Namely, we construct a suitable approximating family for $\omega_0$, then consider the corresponding family of approximate solutions to (1.0.1) and convergence to a limit solution is proved using the parabolic Monge Ampère equation. One major difference here is that $\omega_0$ is complete on $M$ and we want to preserve this property in our approximation. Another major difference is that $\phi_0$ is no longer bounded on $M$ in general which again makes the estimates more difficult.

We make the following technical assumptions which we will use throughout the rest of the section.

**Assumption 2.** *Let $\eta, \hat{\omega}, \varphi_0$, be as in Theorem 4.1.2. Let $0 < T < \widetilde{T} < T_{[\omega_0]}$ be arbitrary. Choose a Hermitian metric $\hat{h}$ on $\mathscr{O}_D$, a smooth volume form $\Omega$ on $\bar{M}$ and a constant $\hat{\beta} > 0$ such that $\|S\|^2_{\hat{h}} < 1$ on $\overline{M}$ and*

1. *on $M\times[0,\widetilde{T}]$ we have*

$$\eta + t(-\mathrm{Ric}(\Omega) + \Theta_{\hat{h}} + \frac{2\Theta_{\hat{h}}}{\log\|S\|^2_{\hat{h}}}) > 0$$

2. *on $\overline{M}$ we have*
$$(1-c_{\hat\beta})\eta \le \eta - \hat\beta\Theta_{\hat h} \le (1+c_{\hat\beta})\eta$$
*for some $c_{\hat\beta} < \frac{1}{2}$ with $T < (1-c_{\hat\beta})\widetilde{T}$.*

3. $\log\log^2\|S\|^2_{\hat h} > 1$ *on $M$, and $\hat\omega = \eta + i\partial\bar\partial\log\log^2\|S\|^2_{\hat h}$*

*As before we will abbreviate $\|S\|^2_{\hat h}$ and $\Theta_{\hat h}$ simply by $\|S\|^2$ and $\Theta$.*

We first choose some $\hat h$ so that $\|S\|^2_{\hat h} < 1$ on $\overline{M}$. As in the case of Assumption 1, we can find a smooth volume form $\Omega$, then scale $\hat h$ smaller if necessary, so that the inequality in (1) holds for $t = \tilde T$ in which case it must also hold for all $t \in [0, \widetilde{T}]$ by interpolation. A choice of $\hat\beta$ in (2) is justified by the smoothness of $\Theta$ on $\overline{M}$ and by scaling $\hat h$ smaller if necessary. Finally, by scaling $\hat h$ smaller still we may assume the inequality in (3) holds and that $\eta + i\partial\bar\partial\log\log^2\|S\|^2_{\hat h}$ is also a Carlson-Griffiths metric. Thus without loss of generality we may assume $\hat\omega = \eta + i\partial\bar\partial\log\log^2\|S\|^2_{\hat h}$ where $\hat\omega$ is from Theorem 4.1.2.

### 4.3.1 Approximate solutions $\omega_{\alpha,j}(t)$

Recall in Theorem 4.1.2 we have $\varphi_0 \in C^\infty(M)\bigcap Psh(\overline{M},\eta)$ with zero Lelong number such that
$$\omega_0 = \eta + i\partial\bar\partial\varphi_0 \ge 2\delta\hat\omega \tag{4.3.1}$$
on $M$ for some $\delta > 0$. We begin by construct a two parameter family $\varphi_{\alpha,j}$ approximating $\varphi_0$ as $\alpha \to 0$ and $j \to \infty$ so that the metrics $\omega_{\alpha,j}(0) = \eta + i\partial\bar\partial\varphi_{\alpha,j}$ are likewise bounded below for some fixed Carlson-Griffiths metric for all $\alpha$. This uniform lower bound will be key for our later proofs, and this is one reason for our two parameters construction as opposed to a single parameter approximation as in Theorem A.2.1.

**Lemma 4.3.1.** *There exists $\hat\alpha$ such that for all $0 < \alpha \le \hat\alpha$ there exists a sequence $\varphi_{\alpha,j} \in Psh(\overline{M},\eta)$ such that*

1. *$\varphi_{\alpha,j}$ decreases to $\alpha\log\|S\|^2 + \varphi_0$ (as $j \to \infty$) pointwise and smoothly on compact subsets of $M$.*

2. $\psi_{\alpha,j} = \varphi_{\alpha,j} + \delta \log\log^2 \|S\|^2 \in C^\infty(\overline{M}) \cap Psh(\overline{M},(1-\delta)\eta)$

*Proof.* Now $i\partial\bar{\partial}\log\|S\|^2$ coincides with a smooth form on $\overline{M}$ and thus for $\alpha > 0$ sufficiently small we have $-\delta\hat{\omega} \leq \alpha i\partial\bar{\partial}\log\|S\|^2 \leq \delta\hat{\omega}$ on $M$ and it follows from (4.3.1) that

$$(1-\delta)\eta + i\partial\bar{\partial}(\alpha\log\|S\|^2 + \delta\log\log^2\|S\|^2 + \varphi_0) \geq \delta\hat{\omega} \tag{4.3.2}$$

In particular, since $\varphi_0$ has zero Lelong number (see definition A.2.1) the potential on the LHS of (4.3.2) approaches $-\infty$ when approaching $D$ giving

$$\alpha\log\|S\|^2 + \delta\log\log^2\|S\|^2 + \varphi_0 \in Psh(\overline{M},(1-\delta)\eta) \tag{4.3.3}$$

for all sufficiently small $\alpha > 0$. Thus by Theorem A.2.1 there exists $\psi_{\alpha,j} \in C^\infty(\overline{M}) \cap Psh(\overline{M},(1-\delta)\eta)$ decreasing to $\alpha\log\|S\|^2 + \delta\log\log^2\|S\|^2 + \varphi_0$ as $j \to \infty$ pointwise on $M$ and smoothly on compact sets. In particular, we have

$$\eta + i\partial\bar{\partial}(-\delta\log\log^2\|S\|^2 + \psi_{\alpha,j}) = \delta\hat{\omega} + (1-\delta)\eta + i\partial\bar{\partial}\psi_{\alpha,j} > \delta\hat{\omega} \tag{4.3.4}$$

so that $\varphi_{\alpha,j} := -\delta\log\log^2\|S\|^2 + \psi_{\alpha,j} \in Psh(\overline{M},\eta)$ decreases to $\alpha\log\|S\|^2 + \varphi_0$ as $j \to \infty$ pointwise on $M$ and smoothly on compact sets. Thus for all $\alpha > 0$ sufficiently small $\varphi_{\alpha,j}$ satisfies the conclusions in the lemma.

□

**Lemma 4.3.2.** *For each $\varphi_{\alpha,j}$ in Lemma 4.3.1, $\omega_{\alpha,j}(0) = \eta + i\partial\bar{\partial}\varphi_{\alpha,j}$ is complete with bounded curvature on $M$ with lower bound*

$$\omega_{\alpha,j}(0) \geq \delta\hat{\omega} \tag{4.3.5}$$

*and the Kähler Ricci flow* (1.0.1) *has a smooth maximal bounded curvature solution $\omega_{\alpha,j}(t)$ on $M \times [0, T_{[\omega_0]})$ with initial condition $\omega_{\alpha,j}(0) = \eta + i\partial\bar{\partial}\varphi_{\alpha,j}$ where $T_{[\omega_0]}$ is as in definition 4.1.2.*

*Proof.* The lower bound in (4.3.5) follows immediately from (4.3.4). On the other hand we can also write $\eta + i\partial\bar{\partial}\varphi_{\alpha,j} = \eta + i\partial\bar{\partial}(-\delta\log\log^2\|S\|^2 + \psi_{\alpha,j})$ where

$\psi_{\alpha,j} := \varphi_{\alpha,j} + \delta \log\log^2 \|S\|^2 \in C^\infty(\overline{M})$ as in Lemma 4.3.1 (2), and the Lemma then follows from Theorem 8.19 in [23] (see also example 8.15).

□

Consider $\omega_{\alpha,j}(t)$ as in the above Lemma. In the following we will derive local estimates for $\omega_{\alpha,j}(t)$ on $M \times (0,\tilde{T})$ which will be independent of $\alpha, j$. These will ensure $\omega_{\alpha,j}(t)$ converges in $C^\infty_{loc}$ to a solution $\omega(t)$ on $M \times (0,\tilde{T})$ as $\alpha \to 0$ and $j \to \infty$. Then we will show that $\omega_{\alpha,j}(t)$ in fact converges in $C^\infty_{loc}$ on $M \times [0,\tilde{T})$ and the limit solution is complete for a short time. Since $\tilde{T} < T_{[\omega_0]}$ in Assumption 2 was arbitrary, a diagonal argument will provide a solution on $M \times [0,T_{[\omega_0]})$ as in Theorem 4.1.2. We may derive as in (4.2.2) that

$$\omega_{\alpha,j}(t) = \theta_t + i\partial\bar{\partial}\varphi_{\alpha,j}(t) \tag{4.3.6}$$

where $\varphi_{\alpha,j}(t)$ solves the parabolic Monge Ampère equation:

$$\begin{cases} \partial_t \varphi_{\alpha,j}(t) = \log \dfrac{\|S\|^2 \log\|S\|^2 (\theta_t + i\partial\bar{\partial}\varphi_{\alpha,j}(t))^n}{\Omega}; \\ \varphi_{\alpha,j}(0) = \varphi_{\alpha,j}. \end{cases} \tag{4.3.7}$$
$$\theta_t := \eta + t\chi; \quad \chi := -\mathrm{Ric}(\Omega) + \Theta - i\partial\bar{\partial}\log\log^2\|S\|^2$$

on $M \times [0,T_{[\omega_0]})$.

We will derive local uniform bounds of $|\varphi_{\alpha,j}(t)|$ and $|\partial_t \varphi_{\alpha,j}(t)|$ on $K \times [\varepsilon,\tilde{T}]$ where $K \subset M$ is compact and $0 < \varepsilon$ are arbitrary where the bounds will independent of $\alpha, j$. We will use these to derive uniform local trace estimates for $\omega_{\alpha,j}(t)$ which combined with the local Evans-Krylov estimates will yield the desired uniform $C^\infty_{loc}$ estimates.

### 4.3.2 A priori estimates for $\omega_{\alpha,j}(t)$

Recall the choices for $0 < T < \widetilde{T} < T_{[\omega_0]}$ and $\hat{h}, \Omega, \hat{\beta}$ in Assumption 2 and the notation there. Recall also the definitions of $\theta_t$ and $\chi$ from (4.3.7). We fix some $\hat{\alpha}$ from Lemma 4.1 and will always assume that $\alpha \leq \hat{\alpha}$ in the following.

**Local $C^0$ estimates of $\varphi_{\alpha,j}(t)$.**

From Lemma 4.3.1 there is a constant $C$ and constant $K(\hat{\beta})$ such that

$$\frac{\hat{\beta}}{2}\log\|S\|^2 - K_{\hat{\beta}} \leq \varphi_{\alpha,j} < C \tag{4.3.8}$$

on $M$ for all $\alpha \leq \hat{\beta}/2$ and all $j$. The upper bound follows simply from Lemma 4.3.1 (2) and the fact the $\psi_{\alpha,j} \in C^\infty(M)$ is decreasing. On the other hand, the fact that $\varphi_{\alpha,j} \downarrow \alpha \log\|S\|^2 + \varphi_0$, and $\varphi_0$ has zero Lelong number (see definition A.2.1) and that $\|S\|(x) \to 0$ as $x \to D$ in $\overline{M}$ together imply the lower bound in (4.3.8) some constant $K_{\hat{\beta}} > 0$ and any $\alpha \leq \hat{\beta}/2$ and $j$

**Theorem 4.3.1.** *There is a bounded continuous function $U(t)$ on $[0,\widetilde{T}]$ such that $\varphi_{\alpha,j}(t) \leq U(t)$ on $M \times [0,\widetilde{T}]$ for all $\alpha$ and $j$. There is a continuous function $L_{\hat{\beta}}(t)$ on $[0,(1-c_{\hat{\beta}})\widetilde{T}]$ such that $\frac{3}{2}\hat{\beta}\log\|S\|^2 + L_{\hat{\beta}}(t) \leq \varphi_{\alpha,j}(t)$ on $M \times [0,(1-c_{\hat{\beta}})\widetilde{T}]$ for all $\alpha \leq \hat{\beta}/2$ and all $j$.*

We first prove

**Lemma 4.3.3.** *We have*

1. $\frac{\|S\|^2 \log^2\|S\|^2 \theta_t^n}{\Omega} \leq C_1(1+t)$ *for all* $t \in [0,\widetilde{T}]$;
2. $\frac{\|S\|^2 \log^2\|S\|^2 (\theta_t - \hat{\beta}\Theta)^n}{\Omega} \geq C_2 t$ *for all* $t \in [0,(1-c_{\hat{\beta}})\widetilde{T}]$,

*where the constants $C_i > 0$ depend on $\Omega$, $\hat{h}$ and $\widetilde{T}$.*

*Proof.* Now it suffices to show that $C_2 t \leq \frac{\|S\|^2 \log^2\|S\|^2 \theta_t^n}{\Omega} \leq C_1 + C_1 t$ for all $t \in [0,\widetilde{T}]$, where $C$ is a constant depending only on $h$ and $\widetilde{T}$, since by (2) Assumption 2 we have for all $t \in [0,(1-c_{\hat{\beta}})\widetilde{T}]$

$$\begin{aligned}\frac{\|S\|^2 \log^2\|S\|^2 (\theta_t - \hat{\beta}\Theta)^n}{\Omega} &\geq \frac{(1-c_{\hat{\beta}})^n \|S\|^2 \log^2\|S\|^2 \theta^n_{\frac{t}{1-c_{\hat{\beta}}}}}{\Omega} \\ &\geq \frac{1}{2^n}\frac{\|S\|^2 \log^2\|S\|^2 \theta^n_{\frac{t}{1-c_{\hat{\beta}}}}}{\Omega}.\end{aligned}$$

Now $\theta_t = \tilde{\theta}_t + \frac{2ti\partial\|S\|^2\wedge\bar{\partial}\|S\|^2}{\|S\|^4\log^2\|S\|^2}$ where $\tilde{\theta}_t = \eta - t\mathrm{Ric}\Omega + t\Theta + \frac{2t\Theta}{\log\|S\|^2}$. Thus $\theta_t^n = \tilde{\theta}_t^n + n\tilde{\theta}_t^{n-1}\wedge(\frac{2ti\partial\|S\|^2\wedge\bar{\partial}\|S\|^2}{\|S\|^4\log^2\|S\|^2})$ and

$$\|S\|^2\log\|S\|^2\theta_t^n = \|S\|^2\log^2\|S\|^2\tilde{\theta}_t^n + n\tilde{\theta}_t^{n-1}\wedge\frac{2ti\partial\|S\|^2\wedge\bar{\partial}\|S\|^2}{\|S\|^2}$$

From this, the fact that $\frac{2i\partial\|S\|^2\wedge\bar{\partial}\|S\|^2}{\|S\|^2}$ is a continuous positive (1,1) form on $\bar{M}$, and the positivity of $\tilde{\theta}_t$ on $t\in[0,\tilde{T}]$ we conclude $C_2 t \leq \frac{\|S\|^2\log\|S\|^2\theta_t^n}{\Omega} \leq C_1 + C_1 t$ on $M\times[0,\widetilde{T}]$ as claimed. □

*Proof of Theorem 4.3.1.* For all $\alpha \leq \hat{\beta}/2$ and $j$, consider

$$H_\varepsilon = \varphi_{\alpha,j}(t) - \int_0^t \log[C_1(1+t)]dt - \varepsilon t$$

on $M\times[0,\widetilde{T}]$ for any $\varepsilon > 0$ and $C_1$ from Lemma 4.3.3. Since $H_\varepsilon(x,0) = \varphi_{\alpha,j}$ is bounded above by (4.3.8) and $|\partial_t\varphi_{\alpha,j}(t)|$ and hence $|\partial_t H_\varepsilon|$ is bounded on $M\times[0,\widetilde{T}]$ (by (4.3.7) and that $\omega_{\alpha,j}(t)$ is a complete bounded curvature solution to (1.0.1)), it follows $H_\varepsilon$ is bounded above on $M\times[0,\widetilde{T}]$. Now suppose $H_\varepsilon$ attains a maximum value on $M\times[0,\widetilde{T}]$ at $(\bar{x},\bar{t})$. Then if $\bar{t}>0$, using (4.3.7) and Lemma 4.3.3 we have at $(\bar{x},\bar{t})$:

$$\partial_t H_\varepsilon \quad\leq\quad \log\frac{\|S\|^2\log\|S\|^2\theta_{\bar{t}}^n}{\Omega} - \log[C_1(1+\bar{t})] - \varepsilon \leq -\varepsilon$$

which contradics the maximality assumption. Thus $\bar{t}=0$ in which case we may simply take $U(t) = C + \int_0^t \log[C_1(1+t)]dt$ for some $C$ by (4.3.8). In general, if $H_\varepsilon$ does not attain a maximum value on $M\times[0,\widetilde{T}]$ we may argue as in the proof of Lemma 4.2.2 and apply the above estimates along an appropriate sequence in space-time (using the Omori-Yau maximum principle) and likewise take $U(t) = C + \int_0^t \log[C_1(1+t)]dt$ for some $C$ in this case as well.

For the lower bound we take $Q_\varepsilon(x,t) = \varphi_{\alpha,j}(x,t) - \hat{\beta}\log|S(x)|^2 - \int_0^t \log(C_2 t)dt + \varepsilon t$ on $M\times[0,(1-c_{\hat{\beta}})\widetilde{T}]$ for any $\varepsilon>0$ and $C_2$ from Lemma 4.3.3. It follows from (4.3.8), (4.3.7) and that $\omega_{\alpha,j}(t)$ is a bounded curvature solution that $Q_\varepsilon(x,t)\to\infty$

uniformly as $x$ approaches $D$ on $M \times [0, \widetilde{T}]$ and hence $Q_\varepsilon(x,t)$ attains an interior minimum on $M \times [0, (1-c_{\hat{\beta}})\widetilde{T}]$. Using again Lemma 4.3.3 we may argue as above for the upper bound and conclude that $L_{\hat{\beta}}(t)$ can be taken as $-K_{\hat{\beta}} + \int_0^t \log(C_2 t)dt$. $\square$

**Local $C^0$ estimates of $\dot{\varphi}_{\alpha,j}(t)$.**

**Theorem 4.3.2.** *We have $\dot{\varphi}_{\alpha,j}(t) \leq \frac{U(t)-\hat{\beta}\log\|S\|^2+K_{\hat{\beta}}}{t} + n$ on $M \times [0, (1-c_{\hat{\beta}})\widetilde{T}]$ for all $\alpha \leq \hat{\beta}/2$ and all $j$.*

*Proof.* The proof is the same as Proposition 3.1 in [16]. Let $H = t\dot{\varphi}_{\alpha,j}(t) - (\varphi_{\alpha,j}(t) - \varphi_{\alpha,j}) - nt$. Then using (4.3.7) we have $(\partial_t - \Delta)H < 0$, where $\Delta$ is the Laplacian with respect to $\omega_{\alpha,j}(t)$. Also, since that $\omega_{\alpha,j}(t)$ is a bounded curvature solution it follows $H$ is a bounded function on $M \times [0, \widetilde{T}]$, and thus by the maximum principle in [27] we $H \leq \sup_{x \in M} H(x,0) = 0$ on $M \times [0, \widetilde{T}]$. Then combining with Lemma 4.3.3 and (4.3.8) we obtain the theorem. $\square$

**Theorem 4.3.3.** *For all $A > 0$ with $0 < \widetilde{T} - \frac{1}{A}$ , there exists a smooth function $F(\|S\|^2(x), t)$ on $M \times (0, (1-c_{\hat{\beta}})(\widetilde{T} - \frac{1}{A})]$ such that $\dot{\varphi}_{\alpha,j}(x,t) \geq F(\|S\|^2(x), t)$ on $M \times (0, (1-c_{\hat{\beta}})(\widetilde{T} - \frac{1}{A})]$ for all $\alpha \leq 2\hat{\beta}$ and all $j$.*

*Proof.* For all sufficiently small $\varepsilon$, there exists a constant $C > 0$ such that

$$\theta_t - s\varepsilon i\partial\bar{\partial}\log\log^2\|S\|^2 \geq C\hat{\omega} \tag{4.3.9}$$

on $M \times [0, \widetilde{T}]$ for all $1 \leq s \leq 2$

Fix some $A > 0$ with $0 < \widetilde{T} - \frac{1}{A}$ and $\varepsilon$ as above. Let $Q = \dot{\varphi}_{\alpha,j}(t) + A(\varphi_{\alpha,j}(t) - \hat{\beta}\log\|S\|^2 + \varepsilon\log\log^2\|S\|^2) - n\log t$. By our previous bounds, $Q \to \infty$ on $M$ as $t \to 0$ or $\|S\| \to 0$. In fact, from (4.3.7) and that $\omega_{\alpha,j}(t)$ is a bounded curvature solution, $Q(x,t) \to \infty$ uniformly on $M$ as $x$ approaches $D$ for all $t \in [0, (1-c_{\hat{\beta}})(\widetilde{T} - \frac{1}{A})]$. So $Q$ has a minimum on $M \times (0, (1-c_{\hat{\beta}})(\widetilde{T} - \frac{1}{A})]$ at some point $(\bar{x}, \bar{t})$ with $\bar{t} > 0$. Let $\Delta$ be the Laplacian with respect to $\omega_{\alpha,j}(t)$, using (4.3.7), we have

$$(\partial_t - \Delta)(\varphi_{\alpha,j}(t) - \hat{\beta}\log\|S\|^2 + \varepsilon\log\log^2\|S\|^2$$
$$= \dot{\varphi}_{\alpha,j} - n + Tr_{\omega_{\alpha,j}}\left(\theta_t - \hat{\beta}\Theta - \varepsilon i\partial\bar{\partial}\log\log^2\|S\|^2\right)$$

$$(\partial_t - \Delta)\dot{\varphi}_{\alpha,j}(t) = Tr_{\omega_{\alpha,j}}\chi.$$

Then at $(\bar{x},\bar{t})$ we have the following, where we will use $C$ to denote a constant which is independent of $\alpha$, $j$ and which may differ from line to line.

$$\begin{aligned}
0 &\geq (\partial_t - \Delta)Q(\bar{x},\bar{t}) \\
&= ATr_{\omega_{\alpha,j}}(\theta_{\bar{t}+\frac{1}{A}} - \hat{\beta}\Theta - \varepsilon i\partial\bar{\partial}\log\log^2\|S\|^2) + A\dot{\varphi}_{\alpha,j} - nA - \frac{n}{\bar{t}} \\
&\geq A(1-c_{\hat{\beta}})Tr_{\omega_{\alpha,j}}[\theta_{\frac{1}{1-c_{\hat{\beta}}}(\bar{t}+\frac{1}{A})} - \frac{1}{1-c_{\hat{\beta}}}\varepsilon i\partial\bar{\partial}\log\log^2\|S\|^2] + A\dot{\varphi}_{\alpha,j} \\
&\quad -nA - \frac{n}{\bar{t}}. \\
&\geq AC(1-c_{\hat{\beta}})Tr_{\omega_{\alpha,j}}\hat{\omega} + A\log\frac{\|S\|^2\log\|S\|^2\omega^n_{\alpha,j}}{\Omega} - nA - \frac{n}{\bar{t}} \\
&\geq CTr_{\omega_{\alpha,j}\hat{\omega}} + C\log\frac{\omega^n_{\alpha,j}}{\hat{\omega}^n} - \frac{C}{\bar{t}}. \\
&\geq CTr_{\omega_{\alpha,j}}\hat{\omega} - \frac{C}{\bar{t}} \\
&\geq C\left(\frac{\hat{\omega}^n}{\omega^n_{\alpha,j}}\right)^{\frac{1}{n}} - \frac{C}{\bar{t}}.
\end{aligned}$$

where we have used Assumption 2 in the third line, $c_{\hat{\beta}} \leq \frac{1}{2}$ and (4.3.9) in the fourth line, property (3) of Carlson-Griffiths metrics, and the fact $\frac{1}{\lambda} + C\log\lambda$ is bounded below by some constant depending on $C$ in the sixth line. Therefore, at $(\bar{x},\bar{t})$, $\omega^n_{\alpha,j} \geq C\bar{t}^n\hat{\omega}^n$ and so $\dot{\varphi}_{\alpha,j}(\bar{x},\bar{t}) \geq C + n\log\bar{t}$ by (4.3.7) and property (3) of Carlson-Griffiths metrics. Since $\log\log^2\|S\|^2 > 1$ by Assumption 2, we have $Q(\bar{x},\bar{t}) \geq C + A(\varphi_{\alpha,j}(\bar{x},\bar{t}) - \hat{\beta}\log|S(\bar{x})|^2)$. By Theorem 4.3.1, $\varphi_{\alpha,j}(t) - \hat{\beta}\log\|S\|^2 \geq C$ and so $Q(\bar{x},\bar{t}) \geq C$. From this, and the upper bound of $\varphi_{\alpha,j}(t)$ from Theorem 4.3.1, we conclude the lower bound for $\dot{\varphi}_{\alpha,j}(t)$ in the Theorem. □

**Local trace estimates for $\omega_{\alpha,j}(t)$**

Note for all $\alpha$ and $j$, since $\omega_{\alpha,j}(t)$ is a bounded curvature solution on $M\times[0,T_{[\omega_0]})$, so $\omega_{\alpha,j}(t)$ will be uniformly equivalent to $\hat{\omega}$ on any closed subinterval of $[0,T_{[\omega_0]})$. In particular, $Tr_{\hat{\omega}}\omega_{\alpha,j}(t)$ will be a bounded function on $M\times[0,T]$

**Theorem 4.3.4.** *There is a smooth function $G(\|S\|^2(x),t)$ on $M\times(0,(1-c_{\hat{\beta}})\widetilde{T}]$ such that $Tr_{\hat{\omega}}\omega_{\alpha,j}(x,t)\le G(\|S\|^2(x),t)$ for all $2\alpha\le\hat{\beta}$ and all $j$.*

*Proof.* Consider

$$Q(\cdot,t)=t\log Tr_{\hat{\omega}}\omega_{\alpha,j}(t)-B(\varphi_{\alpha,j}(t)-\hat{\beta}\log\|S\|^2+\varepsilon\log\log^2\|S\|^2),$$

where $\varepsilon$ is chosen as in (4.3.9) and $B>0$ is a large constant which will be determined later, independently of $\alpha,j$. Now $Q(x,0)\to-\infty$ as $x$ approaches $D$ from (4.9), and from (4.3.7) and that $\omega_{\alpha,j}(t)$ is a bounded curvature solution, $Q(x,t)\to-\infty$ uniformly as $x$ approaches $D$ for all $t\in[0,(1-c_{\hat{\beta}})\widetilde{T}]$. Hence $Q(x,t)$ attains a maximum on $M\times[0,(1-c_{\hat{\beta}})\widetilde{T}]$ at some point $(\bar{x},\bar{t})$. In the following, $C_i$'s will denote positive constants independent of $\alpha,j$.

If $\bar{t}>0$, then $0\le(\partial_t-\Delta)Q(\bar{x},\bar{t})$, where $\Delta$ is the Laplacian with respect to $\omega_{\alpha,j}(t)$. Also, we have

$$(\partial_t-\Delta)\log Tr_{\hat{\omega}}\omega_{\alpha,j}(t)\le C_1Tr_{\omega_{\alpha,j}}\hat{\omega}\tag{4.3.10}$$

for some constant $C_1$ depending only on $\hat{\omega}$ (see [28]), so

$$\begin{aligned}(\partial_t-\Delta)t\log Tr_{\hat{\omega}}\omega_{\alpha,j}(t) &= \log Tr_{\hat{\omega}}\omega_{\alpha,j}(t)+t(\partial_t-\Delta)\log Tr_{\hat{\omega}}\omega_{\alpha,j}(t)\\ &\le \log Tr_{\hat{\omega}}\omega_{\alpha,j}(t)+C_1tTr_{\omega_{\alpha,j}(t)}\hat{\omega}.\end{aligned}$$

Using the computations in the proof of Theorem 4.3.3, we have

$$(\partial_t-\Delta)(\varphi_{\alpha,j}(t)-\hat{\beta}\log\|S\|^2+\varepsilon\log\log^2\|S\|^2)\ge\dot{\varphi}_{\alpha,j}-n+(1-c_{\hat{\beta}})C_2Tr_{\omega_{\alpha,j}}\hat{\omega}.$$

Therefore at $(\bar{x},\bar{t})$, using that $c_\beta < 1/2$ from Assumption 2, we have

$$\begin{aligned} 0 \leq (\partial_t - \Delta)Q &\leq \log Tr_{\hat{\omega}}\omega_{\alpha,j} - B\dot{\varphi}_{\alpha,j} + nB + (C_1 t - \frac{1}{2}BC_2)Tr_{\omega_{\alpha,j}}\hat{\omega} \\ &\leq \log Tr_{\hat{\omega}}\omega_{\alpha,j} - Tr_{\omega_{\alpha,j}}\hat{\omega} - B\dot{\varphi}_{\alpha,j} + nB \end{aligned}$$

where in the second line we have assumed a choice $B$, independent of $\alpha, j$ and $\bar{t}$, such that $C_1\widetilde{T} - \frac{1}{2}BC_2 < -1$.

Since $Tr_{\hat{\omega}}\omega_{\alpha,j} \leq (Tr_{\omega_{\alpha,j}}\hat{\omega})^{n-1}\frac{\omega^n_{\alpha,j}}{\hat{\omega}^n}$ and $\dot{\varphi}_{a,j} \geq C_3 \log\frac{\omega^n_{\alpha,j}}{\hat{\omega}^n}$ for some $C_3$ depending only on $h$ and $\hat{\omega}$, putting them into the above expression, we get

$$0 \leq (n-1)\log Tr_{\omega_{\alpha,j}}\hat{\omega} + (1-BC_3)\log\frac{\omega^n_{\alpha,j}}{\hat{\omega}^n} - Tr_{\omega_{\alpha,j}}\hat{\omega} + C_4.$$

Assume further that $BC_3 > 2$, we have

$$\begin{aligned} 0 &\leq -Tr_{\omega_{\alpha,j}}\hat{\omega} + (n-1)\log Tr_{\omega_{\alpha,j}}\hat{\omega} - \log\frac{\omega^n_{\alpha,j}}{\hat{\omega}^n} + C_4 \\ &\leq -\frac{1}{2}Tr_{\omega_{\alpha,j}}\hat{\omega} - \log\frac{\omega^n_{\alpha,j}}{\hat{\omega}^n} + C_4, \end{aligned}$$

we used $-x + C\log x$ is bounded above for $x > 0$ by some constant depending on $C$. Now let $\lambda_i$ be the eigenvalue of $\omega_{\alpha,j}(\bar{x},\bar{t})$ relative to $\hat{\omega}(\bar{x},\bar{t})$ and let $C$ denote a positive constant independent of $\alpha, j$ which may differ from line to line. Then the previous equation says

$$\sum_i (\frac{1}{2\lambda_i} + \log\lambda_i) \leq C$$

and from the fact that the function $1/2x + \log x$ is bounded below for all $x > 0$, we get that $(\frac{1}{2\lambda_i} + \log\lambda_i) \leq C$ for each $i$, thus $Tr_{\hat{\omega}}\omega_{\alpha,j}(\bar{x},\bar{t}) \leq C$. Since $\varphi_{\alpha,j}(t) - \hat{\beta}\log\|S\|^2 \geq C$ and $\varepsilon\log\log^2\|S\|^2 \geq 0$ we conclude $Q(\bar{x},\bar{t}) \leq C$. Thus by our earlier observed upper bound for $Q(x,0)$ we get $Q(x,t) \leq C$ on $M \times [0,(1-c_{\hat{\beta}})\widetilde{T}]$ and the Theorem follows from this and the upper bound in Theorem 4.3.1. □

### 4.3.3 Completion of Proof of Theorem 4.1.2

Now recall our family $\omega_{\alpha,j}(t)$ of solutions to (1.0.1) on $M \times [0, T_{[\omega_0]})$ from Lemma 4.3.2. Recall that we wrote $\omega_{\alpha,j}(t) = \theta_t + i\partial\bar{\partial}\varphi_{\alpha,j}(t)$ where $\varphi_{\alpha,j}(t)$ solves (4.3.7) on $M \times [0, T_{[\omega_0]})$. Also recall the choices made in Assumption 2, and in particular that $0 < T < \widetilde{T} < T_{[\omega_0]}$ was arbitrary.

From the Theorem 4.3.3, 4.3.4 and (4.3.7), for any $\varepsilon > 0$ and compact subsets $K_1 \subset\subset K_2 \subset\subset M$ we may have

$$C_1 \eta \leq \omega_{\alpha,j}(t) \leq C_2 \eta \tag{4.3.11}$$

on $K_2 \times [\varepsilon, T]$ for some constants $C_i$ independent over all $\alpha \leq \hat{\beta}/2$ and all $j$. It follows from this and the estimates from the Evans-Krylov theory (see also [32] for a maximum principle proof of these for (1.0.1)), that for some $\alpha_k \to 0$, $j_k \to \infty$, we have $\omega_{\alpha_k, j_k}(t)$ converges on $K_1 \times [\varepsilon, T]$ smoothly to a limit solution $\omega(t)$ to the flow in equation (1.0.1). As $T < T_{[\omega_0]}$ was chosen arbitrarily, by a diagonal argument we may in fact assume $\omega_{\alpha_k, j_k}(t)$ converges on $M \times (0, T_{[\omega_0]})$, smoothly on compact subsets, to a limit solution $\omega(t)$ to the flow in (1.0.1), while also $\omega_{\alpha_k, j_k}(0) \to \omega_0$ smoothly on compact subsets of $M$. By applying Theorem 2.3.1 to the sequence $\omega_{\alpha_k, j_k}(t)$ on $M \times [0, T_{[\omega_0]})$ and observing the uniform lower bound on $\omega_{\alpha_k, j_k}(0) \geq \delta \hat{\omega}$ in (4.3.5), we see that $\omega_{\alpha_k, j_k}(t)$ actually converges smoothly on $M \times [0, T_{[\omega_0]})$ to a limit solution satisfying (4.1.5). In other words the solution $\omega(t)$ extends smoothly on $M \times [0, T_{[\omega_0]})$ and satisfies (4.1.5).

We now show (1) in the Theorem 4.1.2 is satisfied. Fix any Hermitian metric $h$ on $\mathscr{O}_D$ and smooth volume form $\Omega$ on $\overline{M}$. Then as in our derivation of (4.3.7) we see that $\varphi(t) := \varphi_0 + \int_0^t \log\log \frac{\|S\|_h^2 \log \|S\|_h^2 (\omega(t))^n}{\Omega}$ solves (4.1.6) on $M \times [0, T_{[\omega_0]})$ and (4.1.7). In particular, $\varphi(t) = \lim_{k \to \infty} u_{\alpha_k, j_k}(t)$ where $u_{\alpha_k, j_k}(t) = \varphi_{\alpha_k, j_k} + \int_0^t \log\log \frac{\|S\|_h^2 \log \|S\|_h^2 (\omega_{\alpha_k, j_k}(t))^n}{\Omega}$ and $u_{\alpha_k, j_k}(t)$ solves (4.1.6) on $M \times [0, T_{[\omega_0]})$ with initial data $\varphi_{\alpha_k, j_k}$. To see that the upper bound in (1) holds, note that the estimate in Lemma 4.3.3 (1) in fact holds for any, and hence our, choice of $h$ for some constant $C_1$. Then from the proof of the upper bound in Theorem 4.3.1, there exists a continuous function $U(t)$ such that $u_{\alpha_k, j_k}(t) \leq U(t)$ and hence $\varphi(t) \leq U(t)$ on $M \times [0, T_{[\omega_0]})$. This completes the proof of (1) in the Theorem.

Finally, we show that (2) holds. Let $\varphi_0$ be as in (2). For any choice of $0 < T < \tilde{T} < T_{[\omega_0]}$ and corresponding subsequent choices in Assumption 2, consider solutions $\varphi_{\alpha,j}(t)$ to (4.3.7) on $M \times [0, T)$ constructed in the proof of Theorem 4.1.2

so far. Now if $\varphi_0$ also satisfies the lower bound in (2) then we may replace the estimates in (4.3.8) with

$$\alpha\log\|S\|^2 - C\log\log^2\|S\|^2 \leq \varphi_{\alpha,j} < C \tag{4.3.12}$$

for some $C$ and all $\alpha \leq \hat{\alpha}$ and all $j$ where $\hat{\alpha}$ is from Lemma 4.1.

Now for $\hat{\alpha}$ sufficiently small, observe that the estimate in Lemma 4.3.3 (2) still holds after replacing $\hat{\beta}$ with any $\alpha \leq \hat{\alpha}$. Now repeating the proof of the lower bound in Theorem 4.3.1, but using instead the function

$$Q_\varepsilon(x,t) = \varphi_{\alpha,j}(x,t) - 2\alpha\log\|S(x)\|^2 - \int_0^t \log(C_2 s)ds + \varepsilon t,$$

we may have

$$\varphi_{\alpha,j}(t) \geq -\alpha\log\|S\|^2 - C\log\log\|S\|^2 + \int_0^t \log(C_2 s)ds$$

on $M \times [0,T]$ for all $\alpha \leq \hat{\alpha}$ and all $j$. The a priori estimates we derived previously imply that $\varphi_{\alpha_k,j_k}(t)$ converges smoothly on compact subsets of $M \times [0,T]$ to some $\varphi(t)$ satisfying the bounds in (2). This completes the proof of Theorem 4.1.2 (2).

## 4.4 Proof of Theorem 4.1.3

We begin with the following Theorem from which Theorem 4.1.3 will follow. In the following, for any complete Kähler manifold $(M,\omega)$ with bounded curvature, we use $T(\omega)$ to denote the maximal existence time of a complete bounded curvature solution to the Kähler-Ricci flow (1.0.1) starting from $\omega$. Also, we say $\gamma(x)$ is a distance like function on $(M,\omega)$ if for some $p \in M$ and $C_1, C_2 > 0$ we have $C_1^{-1}d(p,x) \leq \gamma(x) \leq C_1 d(p,\cdot)$ whenever $d(p,x) > C_2$, where $d(p,\cdot)$ is the distance function from $p$ on $(M,\omega)$. We begin by proving the following Theorem:

**Theorem 4.4.1.** *Let $(M,\hat{\omega})$ be a complete Kähler manifold with bounded curvature. Let $\gamma : M \to \mathbb{R}$ be a smooth distance-like function with $|\hat{\nabla}\gamma|_{\hat{\omega}} < C$ and $|i\partial\bar{\partial}\gamma|_{\hat{\omega}} < C$ for some constant $C$. Let $\varphi \in C^\infty(M)$ such that $|\varphi|/\gamma \to 0$ and $|\hat{\nabla}\varphi|_{\hat{\omega}}/\gamma \to 0$ as $\gamma \to 0$. If $\omega = \hat{\omega} + i\partial\bar{\partial}\varphi$ is a complete metric with bounded curvature and satisfies $|\omega - \hat{\omega}|_{\hat{\omega}} \to 0$ as $\gamma \to 0$, then $T(\omega) = T(\hat{\omega})$.*

*Proof.* Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\rho = 1$ on $[0,1]$ and $\rho = 0$ on $[2,\infty)$. Define $\rho_R : M \to \mathbb{R}$ by $\rho_R = \rho(\gamma/R)$ and let $\omega_R = \hat{\omega} + i\partial\bar{\partial}(\rho_R\varphi)$. We claim that if $R$ is sufficiently large then $\omega_R$ is a complete Kähler metric and there exists $C_R \to 1$ as $R \to \infty$ such that $\frac{1}{C_R}\omega \le \omega_R \le C_R\omega$.

We have $\omega_R = \rho_R\omega + (1-\rho_R)\hat{\omega} + 2\mathrm{Re}(i\partial\rho_R \wedge \bar{\partial}\varphi) + i\varphi\partial\bar{\partial}\rho_R$. Since $|\omega - \hat{\omega}|_{\hat{\omega}} \to 0$ as $\gamma \to \infty$, we have $\frac{1}{C_R}\omega \le \rho_R\omega + (1-\rho_R)\hat{\omega} \le C_R\omega$ for some $C_R \to 1$ as $R \to \infty$. Now it suffices to show that $|2\mathrm{Re}(i\partial\rho_R \wedge \bar{\partial}\varphi)|_{\hat{\omega}} \to 0$ and $|i\varphi\partial\bar{\partial}\rho_R|_{\hat{\omega}} \to 0$ uniformly on $M$ as $R \to \infty$.

For any point in $M$, we have

$$
\begin{aligned}
|2\mathrm{Re}(i\partial\rho_R \wedge \bar{\partial}\varphi)|_{\hat{\omega}} &\le |\frac{\rho'(\frac{\gamma}{R})}{R}\partial\gamma \wedge \bar{\partial}\varphi|_{\hat{\omega}} \\
&\le \frac{|\rho'(\frac{\gamma}{R})|}{R}|\hat{\nabla}\gamma|_{\hat{\omega}}|\hat{\nabla}\varphi|_{\hat{\omega}} \\
&\le \frac{C(\max_{\mathbb{R}}|\rho'|)\chi_{\gamma^{-1}[R,2R]}}{R}|\nabla\varphi|_{\hat{\omega}} \\
&\le 2C(\max_{\mathbb{R}}|\rho'|)\chi_{\gamma^{-1}[R,2R]}\frac{|\nabla\varphi|_{\hat{\omega}}}{\gamma}.
\end{aligned}
$$

Because $|\hat{\nabla}\varphi|_{\hat{\omega}}/\gamma \to 0$ as $\gamma \to \infty$, the function on the right hand side converges uniformly to 0 as $R \to \infty$. Similar argument works for

$$
|i\varphi\partial\bar{\partial}\rho_R|_{\hat{\omega}} = |\varphi\rho'(\frac{\gamma}{R})\frac{i\partial\bar{\partial}\gamma}{R} + \varphi\rho''(\frac{\gamma}{R})\frac{i\partial\gamma \wedge \bar{\partial}\gamma}{R^2}|_{\hat{\omega}}.
$$

Therefore, we have a family of complete Kähler metrics $\omega_R$ such that $\frac{1}{C_R}\omega \le \omega_R \le C_R\omega$ with $C_R \to 1$ as $R \to \infty$ and it is clear that $\omega_R$ has bounded curvature. Therefore, by Theorem 2.4.2, we have $\frac{1}{C_R}T(\omega_R) \le T(\omega) \le C_R T(\omega_R)$. On the other hand, since $\rho_R\varphi$ has compact support, by Theorem 4.1 in [23], we have $T(\omega_R) = T(\hat{\omega})$ for all $R$. Therefore, passing the limit $R \to \infty$ we obtain $T(\omega) = T(\hat{\omega})$. □

*Proof of Theorem 4.1.3.* The uniqueness of bounded curvature solutions follows from [13]. Let $p \in M$ and let $d_{\hat{\omega}}(p,\cdot)$ be the distance function to $p$ relative to $\hat{\omega}$.

Let $\gamma(x) := \log\log^2|S(x)|^2$ on $M$. Then from (4.1.1) we may write

$$\hat{\omega} = \bar{\eta} - i\partial\bar{\partial}\gamma(x) = \overline{\eta} - 2\frac{dd^c \log\|S\|_h^2}{\log\|S\|_h^2} + 2i\partial\gamma\wedge\bar{\partial}\gamma.$$

Noting that $\eta$ as well as the numerator of the second term above are smooth forms on $\overline{M}$, we see that for all $x \in M$ sufficiently close to $D$, or equivalently when $\gamma(x)$ is sufficiently large, we have $C^{-1}\gamma(x) \leq d_{\hat{\omega}}(p,x) \leq C\gamma(x)$ and $\|d\gamma(x)\|_{\hat{\omega}} < C$ for some constant $C$. Moreover, for all $x \in M$ sufficiently close to $D$ we also see from above that $-i\partial\bar{\partial}\gamma(x) > 0$, and from this and the first equality above we may conclude that $\|i\partial\bar{\partial}\gamma(x)\|_{\hat{\omega}} \leq C$ for some $C$ independent of $x$. In other words, $\gamma$ satisfies the assumption in Theorem 4.4.1 relative to $\hat{\omega}$, and Theorem follows immediately. □

*Remark* 4.4.1. In Theorem 4.1.3 we can remove the condition on $d\varphi$ if we assume $\omega_0$ has the same *standard spatial asymptotics* as that of $\hat{\omega}$ as defined in [23]. As an example, if $\omega = \hat{\omega} + i\partial\bar{\partial}\log\log\log^2\|S\|^2$ defines a metric, then it has standard spatial asymptotics at $D$ but not superstandard spatial asymptotics (see example 8.12 in [23]) while Theorem 4.1.3 still provides a bounded curvature solution on $M \times [0, T_{[\omega_0]})$.

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# Appendix

## A.1 Interior estimates

Let us first fix some notations and terminology. $(M^n, \hat{g})$ is said to have bounded geometry of infinite order if the curvature tensor and all its covariant derivatives are uniformly bounded on $M$. In particular, the solution $g(t)$ in Theorem 2.1.2 has bounded geometry of infinite order for $t > 0$.

Also, we will denote the geodesic ball with respect to the metric $g$ with center at $p$ and radius $r$ by $B_g(p,r)$. The following theorem can be found in [32].

**Theorem A.1.1.** *Let $(M^n, \hat{g})$ be a complete non-compact Kähler manifold with bounded geometry of infinite order. Let $h(t)$ be a solution of Kähler-Ricci* (1.0.1) *on $M \times [0,T)$ with initial condition $h_0$ which is a complete Kähler metric. For any $x \in M$, suppose there is a constant $N > 0$, such that*

$$N^{-1}\hat{g} \leq h(t) \leq N\hat{g} \tag{A.1.1}$$

*on $B_{\hat{g}}(x,1) \times [0,T)$. Let $\hat{\nabla}$ be the covariant derivative with respect to $\hat{g}$. Then*

*(i)*

$$|\hat{\nabla}^k h|^2_{\hat{g}} \leq \frac{C_k}{t^k}$$

*on $B_{\hat{g}}(x,1/2) \times (0,T)$, for some constant $C_k$ depending only on $k$, $\hat{g}$, $n$, $T$ and $N$.*

*(ii) If we assume $|\hat{\nabla}^k h_0|^2_{\hat{g}}$ is bounded in $B_{\hat{g}}(x,1)$ by $c_k$, for $k \geq 1$, then*

$$|\hat{\nabla}^k h|^2_{\hat{g}} \leq C_k,$$

*on $B_{\hat{g}}(x,1/2)\times[0,T)$ for some constant $C_k$ depending only on k, $c_k$, n, T and N.*

*Proof.* Since $\hat{g}$ has bounded geometry of infinite order, by [30], for any $x\in M$ there exists a local biholomorphism $\phi_x : D\to M$, where $D = D(1)$ is the open unit ball in $\mathbb{C}^n$, satisfying the following in $D$

(a) $\phi_x(0) = x$, $\phi_x(D)\subset\hat{B}(x,1)$, $\phi_x(D)\supset\hat{B}(x,2\delta)$ for some $\delta > 0$ which is independent of $x$.

(b) $C^{-1}\delta_{i\bar{j}}\le(\phi_x^*(\hat{g}))_{i\bar{j}}\le C\delta_{i\bar{j}}$ for some $C$ independent of $x$.

(c) $\left|\dfrac{\partial^l(\phi_x^*(\hat{g}))_{i\bar{j}}}{\partial z^L}\right|\le C_l$ for any $l,i,j$ and multi index $L$ of length $l$ for some constant $C_l$ which is independent of $x$.

Consider $\phi_x^*(h(t))$, which clearly will solve (1.0.1) on $D(1)\times[0,T)$. By the Evans-Krylov theory for fully non-linear elliptic and parabolic equations (see also [32] for a maximum principle proof in the case of Kähler Ricci flow), the result follows. □

## A.2 Plurisubharmonic functions

Let $\overline{M}$ be a compact complex manifold with smooth Kähler metric $\eta$.

**Definition A.2.1.** $\varphi$ is called *plurisubharmonic* on the compact Kähler manifold $(\overline{M},\eta)$, written $\varphi\in Psh(\overline{M},\eta)$, when $\varphi:\overline{M}\to\mathbb{R}$ is upper semi-continuous and bounded above, and for any local holomorphic coordinate domain $U_\alpha$, $\eta_\alpha+\varphi$ is a classical plurisubharmonic function in $U_\alpha$ where $\eta_\alpha$ is a local Kähler potential. In this context $\varphi\in C^\infty(M)\bigcap Psh(\overline{M},\eta)$ is said to have *zero Lelong number* if for any $c>0$ we have

$$\lim_{d(x,D)\to 0}\frac{\varphi(x)}{c\log\|S\|_h}\to 0$$

where $d(\cdot,D)$ is the distance to $D\subset M$ relative to $\eta$.

Let $\varphi\in Psh(\overline{M},\eta))$ be given. By [15], or [1] for a simpler proof in our setting, there exists a decreasing sequence $\varphi_j\in C^\infty(\overline{M})\bigcap Psh(\overline{M},\eta)$ converging pointwise to $\varphi$. By a slight modification of the proof in [1], we may assume this convergence actually holds in $C^\infty_{loc}(M)$ when $\varphi \in C^\infty(M)$. We include the statement and proof of this below for completeness.

**Theorem A.2.1.** *Suppose that $\varphi \in Psh(\overline{M},\eta) \cap C^\infty(M)$. Then there exists a sequence $\varphi_j \in C^\infty(\overline{M})$ with $\varphi_j \downarrow \varphi$ pointwise on $\overline{M}$ and smoothly uniformly on compact subsets of $M$.*

*Proof.* Let $S_j$ be an increasing sequence of open sets exhausting $M$ where each $\overline{S_j}$ is compact. Let $m_j = \min\{\varphi(x) : x \in \overline{S_j}\}$ and define $\varphi_j = \max\{\varphi, m_j\} + \frac{1}{j}$. Then $\varphi_j \in C^0(\overline{M}) \bigcap C^\infty(S_j) \bigcap Psh(\overline{M},\eta)$ with $\varphi_j \downarrow \varphi$ pointwise on $\overline{M}$ and smoothly uniformly on compact subsets. Now for each $j$, suppose there exists a sequence $\varphi_{j,k} \in Psh(\overline{M},\eta) \cap C^\infty(\overline{M})$ with $\varphi_{j,k} \downarrow \varphi_j$ pointwise uniformly on $\overline{M}$ and smoothly uniformly on $S_{j-1}$. Then for any diverging sequence $a_j$, we have $\varphi_{j,a_j} \downarrow \varphi$ pointwise on $\overline{M}$ and smoothly uniformly on compact subsets. Moreover, by the fact $\varphi_j - \varphi_{j+1} \geq \frac{1}{j} - \frac{1}{j+1}$, it is clear that we may choose some sequence $a_j$ with $\varphi_{j,a_j} \downarrow \varphi$.

From the above, it suffices now to prove the following

CLAIM: if $\varphi \in C^0(\overline{M}) \bigcap C^\infty(U) \bigcap Psh(\overline{M},\eta)$ for some open set $U$ and $V$ is a precompact open subset of $U$, there exists a sequence $\varphi_j \in C^\infty(\overline{M}) \bigcap Psh(\overline{M},\eta)$ with $\varphi_j \downarrow \varphi$ pointwise on $\overline{M}$ and smoothly uniformly on $V$.

The claim follows from a slight modification of the proof of the main Theorem in [1]. In [1], an arbitrary open cover $U_\alpha$ of $\overline{M}$ is first chosen, then in each $U_\alpha$ a smooth local approximation $\varphi_{\alpha,\delta}$ of $\varphi$ is constructed through the use of local Kähler potentials and mollification. Then for fixed $\delta$, a global smooth approximation of $\varphi$ on $\overline{M}$ is defined as the pointwise regularized maximum (from [15]) of the $\varphi_{\alpha,\delta}$'s where the maximum is taken over all $\alpha$. Then, by letting $\delta \to 0$, it is shown there exists a sequence $\varphi_j \in C^\infty(\overline{M})$ with $\varphi_j \downarrow \varphi$ pointwise on $\overline{M}$. To have smooth convergence uniformly on $V$ we modify this construction slightly as follows.

First we choose some finite open cover $\{V_\alpha\}_{\alpha=1}^k$ of the compact set $\overline{M}\backslash U$. Moreover, by compactness we may assume for each $\alpha > 0$ we have a proper inclusion of open sets $V_\alpha \subset U_\alpha \subset W_\alpha$ where: $W_\alpha \bigcap \overline{V} = \emptyset$ and $W_\alpha$ is a holomorphic coordinate neighborhood with smooth local Kähler potential $\eta = i\partial\bar{\partial} f_\alpha$. Then letting $U_0 := U$ we take $\{U_\alpha\}_{\alpha=0}^k$ as our open cover of $\overline{M}$.

Next we define local approximations $\varphi_{\alpha,\delta}$ of $\varphi$ on each $U_\alpha$. For $\alpha = 0$, let $g_\alpha$ be a smooth function which is equal to 0 in $U \setminus \cup_{\alpha \neq 0} V_\alpha$ and equals $-1$ outside some compact subset of $U$. For all $\alpha \neq 0$, let $g_\alpha$ be a smooth function in $U_\alpha$ such that $g_\alpha = 0$ in $V_\alpha$ and $g_\alpha = -1$ outside some compact subset of $U_\alpha$. Assume that $i\partial\bar{\partial} g_\alpha \geq -C\omega$ for some $C$ independent of $\alpha$. Now we define $\varphi_{\alpha,\delta}$ on $U_\alpha$ as follows. If $\alpha \neq 0$, then as in [1] we let

$$\varphi_{\alpha,\delta} = u_{\alpha,\delta} - f_\alpha + \frac{\varepsilon}{C} g_\alpha$$

where $u_{\alpha,\delta}$ is a mollification of $u_\alpha := \varphi + f_\alpha \in Psh(U_\alpha)$ in $W_\alpha$ so that $u_{\alpha,\delta} \downarrow u_\alpha$ as $\delta \to 0$. If $\alpha = 0$, then we let

$$\varphi_{0,\delta} = \varphi + \frac{\varepsilon}{C} g_0$$

In both cases, we have $\varphi_{\alpha,\delta} \in Psh(U_\alpha, (1+\varepsilon)\eta)$ and it is non-increasing as $\delta$ decreases.

Now given our open cover $\{U_\alpha\}_{\alpha=0}^{k}$, and local approximations $\varphi_{\alpha,\delta}$ in these, the proof of the claim follows exactly as in [1] involving the regularized maximum (from [15]) of the $\varphi_{\alpha,\delta}$'s where the maximum is taken over all $\alpha$. Then, by letting $\delta \to 0$. We refer to [1] for details of this argument.

$\square$