Unlikely intersections and equidistribution with a dynamical perspective

by

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Abstract

In this thesis we investigate generalizations of a theorem by Masser and Zannier concerning torsion specializations of sections in a fibered product of two elliptic surfaces.

We consider the Weierstrass family of elliptic curves $E_t : y^2 = x^3 + t$ and points $P_t(a) = (a, \sqrt{a^3+t}) \in E_t$ parametrized by non-zero $t \in \overline{\mathbb{Q}}_2$, where $a \in \overline{\mathbb{Q}}_2$. Given $\alpha, \beta \in \overline{\mathbb{Q}}_2$ such that $\frac{\alpha}{\beta} \in \mathbb{Q}$, we provide an explicit description for the set of parameters $t = \lambda$ such that $P_\lambda(\alpha)$ and $P_\lambda(\beta)$ are simultaneously torsion for E_λ. In particular, we prove that the aforementioned set is empty unless $\frac{\alpha}{\beta} \in \{-2, -\frac{1}{2}\}$. Furthermore, we show that this set is empty even when $\frac{\alpha}{\beta} \notin \mathbb{Q}$ provided that α and β have distinct $2-$adic absolute values and the ramification index $e(\mathbb{Q}_2(\frac{\alpha}{\beta}) \mid \mathbb{Q}_2)$ is coprime with 6. Our methods are dynamical. Using our techniques, we derive a recent result of Stoll concerning the Legendre family of elliptic curves $E_t : y^2 = x(x-1)(x-t)$, which itself strengthened earlier work of Masser and Zannier by establishing, as a special case, that there is no parameter $t = \lambda \in \mathbb{C} \setminus \{0, 1\}$ such that the points with x–coordinates a and b are both torsion in E_λ, provided a, b have distinct reduction modulo 2.

We also consider an extension of Masser and Zannier's theorem in the spirit of Bogomolov's conjecture. Let $\pi : E \to B$ be an elliptic surface defined over a number field K, where B is a smooth projective curve, and let $P : B \to E$ be a section defined over K with canonical height $\hat{h}_E(P) \neq 0$. We use Silverman's results concerning the variation of the Néron-Tate height in elliptic surfaces, together with complex-dynamical arguments to show that the function $t \mapsto \hat{h}_{E_t}(P_t)$ satisfies the hypothesis of Thuillier and Yuan's equidistribution theorems. Thus, we obtain the equidistribution of points $t \in B(\overline{K})$ where P_t is torsion. Finally, combined with

Masser and Zannier's theorems, we prove the Bogomolov-type extension of their theorem. More precisely, we show that there is a positive lower bound on the height $\hat{h}_{A_t}(P_t)$, after excluding finitely many points $t \in B$, for any 'non- special' section P of a family of abelian varieties $A \to B$ that split as a product of elliptic curves.

Lay summary

The topic of this thesis lies at the intersection of diophantine geometry and arithmetic dynamics. Diophantine geometry, a subject with a rich history, studies interesting solutions to systems of polynomial equations. Arithmetic dynamics, a relatively young field in contrast, studies questions arising from the iteration of self-maps with number-theoretic significance. Many questions in diophantine geometry can be cast in the context of arithmetic dynamics, thereby giving rise to new and exciting research directions. Conversely, by lending the tools of complex dynamics to the number theorist's disposal, these reformulations provide new insights into old questions. In this thesis, we apply the dynamical perspective to generalize results by Masser and Zannier in the context of elliptic curves.

Preface

This thesis is adapted from two of the author's manuscripts. Chapter 3 is a transcript of the author's article [Ma] published in Journal of Number Theory. Chapter 4 is a transcript of the author's preprint [DM], currently submitted for publication, obtained in joint work with Laura DeMarco.

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To Giulini

Chapter 1

Introduction

This thesis is motivated by a recurring theme in arithmetic geometry known as 'unlikely intersections'. Broadly speaking, it predicts that arithmetic objects cannot intersect more than dimensional reasons suggest, unless there is an underlying structural reason. A prototypical result in this theme, conjectured by Lang and proved by Ihara, Serre and Tate independently [La1], is the following: If X is an irreducible plane curve which contains infinitely many points with both coordinates roots of unity, then X has the form $y^n = \zeta x^m$ for a root of unity ζ and integers n, m not both equal to zero.

Lang's conjecture has inspired many questions and generalizations; see [Za]. In fact, it was in part motivated by questions raised independently by Manin and Mumford in the setting of abelian varieties. A generalization of Manin and Mumford's questions, known as the Manin-Mumford conjecture, was first proved by Raynaud [Ra1, Ra2]. He showed that if X is a subvariety of an abelian variety A which contains a Zariski dense set of torsion points, then X must be *special*. That is, X must be a translate of an abelian subvariety of A by a torsion point.

In the case that A is defined over $\overline{\mathbb{Q}}$, a more general conjecture was formulated by Bogomolov, who asked whether the Manin-Mumford conjecture is still true upon replacing the torsion points by points of arbitrarily small Néron-Tate height. The Néron-Tate height (see §4.1.1 and [HS]) associates a non-negative real number $\hat{h}_A(P)$ to each point $P \in A(\overline{\mathbb{Q}})$. This number measures roughly the arithmetic complexity of P and equals 0 precisely when P is torsion. Bogomolov's conjecture was established by Ullmo [Ul] for curves embedded in their Jacobian and Zhang [Zh4] for arbitrary abelian varieties. Zhang further established an analogous statement in the setting of tori [Zh1, Zh2], generalizing Lang's conjecture.

Further generalizations of the Manin-Mumford conjecture were considered by Bombieri, Masser and Zannier [BMZ] in the setting of tori and in the more general setting of semiabelian schemes by Pink [Pi] and Zilber [Zi]. A special case of a variant of Pink's conjecture [Pi, Conjecture 6.2] for a curve inside a non-simple abelian surface scheme is addressed by Masser and Zannier [MZ1, MZ2, MZ3]. They prove the following.

Theorem 1.0.1 (Masser-Zannier 2010, 2012, 2014)**.** *Let B be a projective smooth algebraic curve and suppose that $A \to B$ is a family of abelian varieties which is isogenous to the fibered product of two elliptic surfaces. Let $P : B \to A$ be a section and assume that A, B and P are defined over $\mathbb{C}$. We denote by $T(P)$ the set of parameters $t \in B(\mathbb{C})$ such that the specialization P_t of the section $P : B \to A$ is torsion in the abelian variety A_t. Then, the following are equivalent.*

1. *The set $T(P)$ is infinite.*
2. *The section $P : B \to A$ is special (see Definition 1.0.7).*

In this thesis we investigate generalizations of Masser-Zannier's theorem. More specifically, we are interested in the following questions.

1. Are there instances in which one can describe precisely the set $T(P)$ from Theorem 1.0.1?
2. When A, B and P are defined over $\overline{\mathbb{Q}}$, can we extend Theorem 1.0.1 in the spirit of Bogomolov's conjecture? That is, can we replace the set $T(P)$ with the set of parameters t corresponding to points P_t of small Néron-Tate height in A_t?

Note that an elliptic surface $E \to B$ over a projective curve may alternatively be viewed as a family of elliptic curves E_t parametrized by $t \in B$ with finitely many exceptions corresponding to the singular fibers. Moreover, we may consider the generic fiber of $E \to B$, which is an elliptic curve over $\mathbb{C}(B)$ denoted by E. Masser and Zannier laid the groundwork towards establishing Theorem 1.0.1 in [MZ1], where they considered the Legendre family of elliptic curves

$$E_t : y^2 = x(x-1)(x-t),$$

parametrized by $t \in \mathbb{C} \setminus \{0,1\}$ and the two sections corresponding to points

$$P_t = (2, \sqrt{2(2-t)}) \text{ and } Q_t = (3, \sqrt{6(3-t)}) \in E_t.$$

They proved that the set of parameters $t = \lambda$ such that both P_λ and Q_λ are torsion in E_λ is finite. Later, in [MZ2], they generalized their result to sections corresponding to points $P_t, Q_t \in E_t$ with x–coordinates in $\overline{\mathbb{C}(t)}$. They showed that there are at most finitely many specializations $t = \lambda$ such that both P_λ and Q_λ are torsion for E_λ unless there exist $n, m \in \mathbb{Z}$, not both equal to 0, such that

$$[m]P_t = [n]Q_t \text{ for all } t \in \mathbb{C} \setminus \{0,1\}.$$

In the latter case, we say that the section (P,Q) of the fibered square of the Legendre elliptic surface over an appropriate curve is *special*. They concluded the proof of Theorem 1.0.1 in [MZ3] by proving similar finiteness results for any fibered product of two elliptic surfaces.

Chapter 3 of this thesis concerns our progress towards answering Question 1. Our work is inspired by a recent result of Stoll [St2], who proved that in the case of the Legendre family of elliptic curves, given two sections with $x-$coordinates $\alpha \in \overline{\mathbb{Q}}$ and $\beta \in \overline{\mathbb{Q}}$ that have different reductions 'modulo 2', the only possible parameters $t = \lambda$ such that both are torsion for E_λ are α and β. This result improves upon the result in [MZ1]. Stoll's approach involves a careful analysis of the 2–adic behavior of the $n-$th reduced division polynomial of E_t. He also obtains a partial result towards the characterization of the set of parameters (μ, λ) for which three different points are torsion in $E_{\mu,\lambda} : y^2 = x^3 + \mu x + \lambda$, assuming λ is integral at 2. To this end, he describes the 2–adic behavior of the n–th reduced division polynomial of $E_{\mu,\lambda}$; see [St2, Proposition 14]. However, when $\mu = 0$ this description does not give precise information on the parameters λ for which a point with constant $x-$coordinate is torsion for $E_{0,\lambda} : y^2 = x^3 + \lambda$. It is primarily this situation that we address in this thesis.

We are mainly interested in the Weierstrass family of elliptic curves

$$E_t : y^2 = x^3 + t,$$

parametrized by $t \in \mathbb{C}_2 \setminus \{0\}$. Here $\mathbb{C}_2$ denotes the completion of $\overline{\mathbb{Q}}_2$ with respect to the 2−adic absolute value. Letting $T(\alpha)$ denote the set of all parameters $t = \lambda \in \mathbb{C}_2$ such that $(\alpha, \sqrt{\alpha^3 + \lambda})$ is torsion in E_λ, we establish the following.

Theorem 1.0.2. *If $\alpha, \beta \in \overline{\mathbb{Q}}_2 \setminus \{0\}$ are such that $\frac{\alpha}{\beta} \in \mathbb{Q} \setminus \{-2, -\frac{1}{2}\}$, then $T(\alpha) \cap T(\beta) = \emptyset$. Moreover, for all $a \in \overline{\mathbb{Q}}_2 \setminus \{0\}$ we have $T(a) \cap T(-2a) = \{-a^3\}$.*

In order to derive Theorem 1.0.2, we study the 2–adic absolute values of the elements in $T(\alpha)$. Our methods are dynamical; we work with an associated family of Lattès maps on $\mathbb{P}^1$, taking a quotient of the multiplication by 2 map on E_t. With this approach, in Theorem 3.4.4 and Corollary 3.4.6, we present an alternative proof of Stoll's result [St2, Theorem 3] concerning the Legendre family of elliptic curves. Furthermore, our method applies to other families of rational maps on $\mathbb{P}^1$, which we illustrate with the non-Lattès example $f_t(z) = \frac{z^d + t}{pz}$ for integer $d \geq 2$ and prime $p \in \mathbb{Z}$.

Theorem 1.0.3. *Let $p \in \mathbb{Z}$ be a prime and consider the natural reduction map $\rho : \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\overline{\mathbb{F}_p})$. For $d \in \mathbb{Z}_{\geq 2}$, let $f_t(z) = \frac{z^d + t}{pz}$. If $\alpha, \beta \in \mathbb{C}_p \setminus \{0\}$ are such that $\rho(\alpha^d) \neq \rho(\beta^d)$, then there is no parameter $t = \lambda \in \mathbb{C}_p$ for which α and β are both preperiodic for f_λ.*

To place our results in the appropriate context, we highlight some key features of earlier work. Masser and Zannier's original approach in [MZ1, MZ2, MZ3] involved a key recent result by Pila and Zannier [PZ] and relied strongly on the existence of the analytic uniformization map for an elliptic curve. Zannier further pointed out a dynamical reformulation of the question based on the fact that for each $t = \lambda$, a point with x–coordinate a is torsion for E_λ if and only if a is a preperiodic point for the Lattès map induced by the multiplication by 2 map in E_λ (see 2.1.2 for the definition of a Lattès map). Using this reformulation and the equidistribution results of [BR1, CL1, FRL, Th, Yu], DeMarco, Wang and Ye

[DWY1] gave an alternative proof of Masser-Zannier's theorem in the case of the Legendre elliptic surface and sections corresponding to points with constant x–coordinates. In fact their proof works more generally to points of small canonical height in the spirit of Bogomolov's conjecture. Thus, DeMarco, Wang and Ye have provided a first result towards Question 2.

Further, motivated by Masser-Zannier's results and replacing the family of Lattès maps by other families of rational maps, many interesting results concerning the finiteness of the set of parameters such that both a and b are preperiodic for a 1–parameter family of rational maps have appeared; see [BD1, BD2, DWY2, FG1, FG2, GH, GHT1, GHT2, GKN, GKNY, GY, MY].

As opposed to the aforementioned approaches, our method, as outlined next, is much simpler. We consider the family of Lattès maps $f_t(z) = \frac{z^4-8tz}{4(z^3+t)}$, parametrized by $t \in \mathbb{C}_2 \setminus \{0\}$, induced by the duplication map on $E_t : y^2 = x^3 + t$. In §3.2.1, using the fact that for each $t = \lambda \in \mathbb{C}_2 \setminus \{0\}$ and $\alpha \in \mathbb{C}_2$ we have

$$P_\lambda(\alpha) := (\alpha, \sqrt{\alpha^3+\lambda}) \text{ is torsion for } E_\lambda \Longleftrightarrow$$
$$\alpha \text{ is a preperiodic point for } f_\lambda(z) = \frac{z^4 - 8\lambda z}{4(z^3+\lambda)},$$

we provide a relation between the 2–adic absolute values of λ and α such that $P_\lambda(\alpha)$ is torsion for E_λ; see Theorem 3.2.5. This relation strongly depends on whether 0 or ∞, which are both 2–adically attracting fixed points for all f_λ, belong in the orbit of α under iteration by f_λ. Furthermore, it is a useful step towards finding pairs $(\alpha, \beta) \in \mathbb{C}_2^2$ such that $T(\alpha) \cap T(\beta) = \emptyset$, which is what we consider subsequently. Using Theorem 3.2.5, we establish results of this flavour in Corollary 3.2.16. As a special case, we get that if $\alpha, \beta \in \mathbb{Q}_2^{unr}$ have distinct 2–adic absolute values and $\frac{\alpha^3}{\beta^3} \notin \{-8, -\frac{1}{8}\}$, then $T(\alpha) \cap T(\beta) = \emptyset$. An important property of this family of elliptic curves and corresponding Lattès maps that aids in the proof of Theorem 3.2.5, is that they are isotrivial (see Definition 3.2.1 and Example 3.2.3).

In §3.3, building on the results obtained in §3.2.1, we proceed to establish Theorem 1.0.2. More precisely, we use Lemma 3.3.4 to reduce the question to proving the coprimality of certain families of polynomials, which can be done by elementary means.

In §3.4, to further demonstrate the efficacy of our approach, as well as the fact that our method does not rely on isotriviality in general, we apply our techniques to give a shorter proof of Stoll's result. As a by-product of our method, in Theorem 3.4.4 and Corollary 3.4.6, we obtain a slight strengthening of Stoll's original result, [St1, Corollary 4]. [1] We conclude Chapter 3 with a discussion on other families of maps which are not Lattès in §3.5, where we establish Theorem 1.0.3.

Chapter 4 of this thesis concerns the Bogomolov-type extension of Masser-Zannier's theorem, as in Question 2. Suppose $E \to B$ is an elliptic surface defined over a number field K. Here, B is a smooth projective curve and for all but finitely many $t \in B(\overline{K})$, the fibers E_t are smooth elliptic curves. We let $\hat{h}_E$ denote the Néron-Tate canonical height of E viewed as an elliptic curve over the function field $k = \overline{K}(B)$ and we let $\hat{h}_{E_t}$ denote the canonical height on the smooth fibers E_t; see §4.1.1 for a definition of the Néron-Tate height.

Suppose that $P : B \to E$ is a section defined over K for which $\hat{h}_E(P) \neq 0$. In particular, the latter yields that the points $P_t \in E_t$ are not torsion for all t; [De1]. Tate [Ta] showed that the function

$$t \mapsto \hat{h}_{E_t}(P_t)$$

is a Weil height on $B(\overline{K})$, up to a bounded error term. More precisely, there exists a divisor $D_P \in \mathrm{Pic}(B) \otimes \mathbb{Q}$ of degree equal to $\hat{h}_E(P)$ so that as t varies in $B(\overline{K})$, we have

$$\hat{h}_{E_t}(P_t) = h_{D_P}(t) + O_P(1), \tag{1.1}$$

where h_{D_P} is a Weil height on $B(\overline{K})$ associated to D_P. The implicit constant in the error term $O_P(1)$ depends on the section P but is independent of t. In a series of three articles [Si2, Si4, Si5], Silverman refined statement (1.1) by analyzing the Néron decomposition of the canonical height on the fibers as a sum of local heights

$$\hat{h}_{E_t}(P_t) = \sum_{v \in M_K} n_v \hat{\lambda}_{E_t,v}(P_t).$$

[1] While our article [Ma] was under review, Stoll, in [St2, Proposition 7], obtained a stronger result that covers our Theorem 3.4.4 and Corollary 3.4.6 by incorporating elements of our approach in his proof.

Here, M_K denotes the set of places of the number field K, and n_v are the integers appearing in the product formula $\prod_{v\in M_K} |x|_v^{n_v} = 1$ for all $x \in K^*$. We refer the reader to §4.1.2 for the definition of the local heights and to §4.1.3 for a more precise statement of Silverman's results.

Before presenting our results, we briefly describe the history behind Tate's and Silverman's results. Tate's work is motivated by a conjecture of Silverman, who proved a weaker result valid for arbitrary families of abelian varieties; [Si1]. In the special case of elliptic surfaces, [Si1] yields that

$$\lim_{h_B(t)\to\infty} \frac{\hat{h}_{E_t}(P_t)}{h_B(t)} = \hat{h}_E(P),$$

where h_B is a Weil height on $B(\overline{K})$ corresponding to a divisor of degree equal to 1. We should also point out that, under an assumption on the Néron model of the generic fiber, Lang [La2, §12] has extended Tate's results on families of abelian varieties and Call [Ca1] has proved analogous results for the local heights. Green [Gr] extended Tate's result to arbitrary families of abelian varieties. Further, replacing the family of Lattès maps by an arbitrary family of rational maps, dynamical analogues of [Ta] and [Si1] have been studied; see [CS, GHT2, GM, I13, MY].

In this thesis, we explain how Silverman's conclusions about the local functions $\hat{\lambda}_{E_t,v}(P_t)$ are precisely the input needed to show that $t \mapsto \hat{h}_{E_t}(P_t)$ is a "good" height function on the base curve B, from the point of view of equidistribution. Combining his work with methods from complex dynamics, as in [DWY1], and the inequalities of Zhang on successive minima [Zh5, Zh2], we prove that our height function on B satisfies the hypotheses of the equidistribution theorems of Thuillier and Yuan for points of small height on curves [CL1, Th, Yu]; refer to §4.2 for the statement of this result and an introduction to the relevant terminology. More precisely, in §4.4 we prove the following.

Theorem 1.0.4. *Let K be a number field and $k = K(B)$ for a smooth projective curve B defined over K. Fix any elliptic surface $E \to B$ defined over K and point $P \in E(k)$ satisfying $\hat{h}_E(P) \neq 0$. Then, the function*

$$h_P(t) := \hat{h}_{E_t}(P_t)$$

for t with smooth fibers, is the restriction of a height function on $B(\overline{K})$ induced from an ample continuous, semipositive adelic metrized line bundle $\overline{\mathscr{L}}$, satisfying

$$h_P(B) := c_1(\overline{\mathscr{L}})^2/(2c_1(\mathscr{L})) = 0.$$

Theorem 1.0.4 implies that our height function on B satisfies the hypotheses of the equidistribution theorems of Thuillier and Yuan for points of small height on curves [CL1, Th, Yu], and we deduce the following:

Corollary 1.0.5. *Let K be a number field and $k = K(B)$ for a smooth projective curve B defined over K. Fix any elliptic surface $E \to B$ defined over K and point $P \in E(k)$ satisfying $\hat{h}_E(P) \neq 0$. There is a collection of probability measures $\mu_P = \{\mu_{P,v} : v \in M_K\}$ on the Berkovich analytifications B_v^{an} such that for any infinite, non-repeating sequence of $t_n \in B(\overline{K})$ with*

$$\hat{h}_{E_{t_n}}(P_{t_n}) \to 0$$

as $n \to \infty$, the discrete measures

$$\frac{1}{|\operatorname{Gal}(\overline{K}/K)\cdot t_n|} \sum_{t \in \operatorname{Gal}(\overline{K}/K)\cdot t_n} \delta_t$$

converge weakly on B_v^{an} to the measure $\mu_{P,v}$ at each place v of K.

Remark 1.0.6. The measures $\mu_{P,v}$ of Corollary 1.0.5 are not difficult to describe, at least at the archimedean places. At each archimedean place v, there is a canonical positive $(1,1)$-current T_v on the surface $E(\mathbb{C})$ with continuous potentials away from the singular fibers, which restricts to the Haar measure on each smooth fiber $E_t(\mathbb{C})$. The measure $\mu_{P,v}$ on $B(\mathbb{C})$ is just the pull-back of this current by the section P. Moreover, at every place, the measure $\mu_{P,v}$ is the Laplacian of the local height function $\hat{\lambda}_{E_t,v}(P_t)$, away from its singularities. We give more details about (and a dynamical perpective on) the construction of the current T_v in §4.3.

As a consequence of Theorem 1.0.4, and combined with Masser and Zannier's Theorem 1.0.1 from [MZ1, MZ2, MZ3], we obtain the so-called Bogomolov extension of their theorem and provide a positive answer to Question 2. Fix an integer

$m \geq 2$ and suppose that $E_i \to B$ is an elliptic surface over a curve B, defined over $\overline{\mathbb{Q}}$, for $i = 1, \ldots, m$.

Definition 1.0.7. Consider a section $P = (P_1, P_2, \ldots, P_m)$ of the fiber product $A = E_1 \times_B \cdots \times_B E_m$ defined over $\overline{\mathbb{Q}}$. We say that the section P is **special** if

- for each $i = 1, \ldots, m$, either P_i is torsion on E_i or $\hat{h}_{E_i}(P_i) \neq 0$; and
- for any pair $i, j \in \{1, \ldots, m\}$ such that neither P_i nor P_j is torsion, there are nonzero group homomorphisms $\phi : E_i \to E_j$ and $\psi : E_j \to E_j$ so that $\phi(P_i) = \psi(P_j)$.

If a family of abelian surfaces $A \to B$ is isogenous to a fiber product (after performing a base change $B' \to B$ if needed), we say that a section of A is special if it is special on the fiber product.

If for example $m = 2$ and the elliptic surfaces $E_i \to B$ are not isotrivial[2], a section $P = (P_1, P_2) : B \to E_1 \times_B E_2$ is special if there are group homomorphisms $a : E_1 \to E_2$ and $b : E_2 \to E_2$, not both trivial, such that

$$a(P_1) = b(P_2).$$

It is well known that a special section will always pass through infinitely many torsion points in the fibers $A_t = E_{1,t} \times \cdots \times E_{m,t}$. That is, there are infinitely many $t \in B(\overline{\mathbb{Q}})$ for which

$$\hat{h}_{E_{1,t}}(P_1(t)) = \cdots = \hat{h}_{E_{m,t}}(P_2(t)) = 0.$$

For a proof see [Za, Chapter 3] or, for dynamical proofs, see [De1]. The converse statement is also true, but it is much more difficult to prove. Indeed, this is precisely Masser and Zannier's theorem stated as Theorem 1.0.1 earlier. In §4.5 we extend Masser-Zannier's result from points of height 0 to points of small height.

Theorem 1.0.8. *Let B be a quasiprojective smooth algebraic curve defined over $\overline{\mathbb{Q}}$. Suppose $A \to B$ is a family of abelian varieties of relative dimension $m \geq 2$*

[2] We say that an elliptic surface $E \to B$ is isotrivial if there exists a finite cover $B' \to B$ such that $E \times_B B' \to B'$ is a constant elliptic surface.

defined over $\overline{\mathbb{Q}}$ which is isogeneous to a fibered product of $m \geq 2$ elliptic surfaces. Let $\mathscr{L}$ be a line bundle on A which restricts to an ample and symmetric line bundle on each fiber A_t, and let $\hat{h}_t$ be the induced Néron-Tate canonical height on A_t, for each $t \in B(\overline{\mathbb{Q}})$. For each non-special section $P : B \to A$ defined over $\overline{\mathbb{Q}}$, there is a constant $c = c(\mathscr{L}, P) > 0$ so that the set

$$\{t \in B(\overline{\mathbb{Q}}) : \hat{h}_t(P_t) < c\}$$

is finite.

If $A \to B$ is isotrivial, then Theorem 1.0.8 is a special case of the Bogomolov Conjecture, proved by Ullmo and Zhang [Zh4, Ul]. A key ingredient in their proofs is the equidistribution theorem of Szpiro, Ullmo, and Zhang [SUZ]. In his 1998 ICM lecture notes [Zh3], Zhang presented a conjecture about geometrically simple families of abelian varieties, which, stated in its most basic form, asserts the following.

Conjecture 1.0.9 (Zhang)**.** *Let B be a quasiprojective smooth algebraic curve defined over $\overline{\mathbb{Q}}$. Suppose $A \to B$ is a non-isotrivial family of abelian varieties defined over $\overline{\mathbb{Q}}$ with a simple generic fiber of dimension > 1. Let $\mathscr{L}$ be a line bundle on A which restricts to an ample and symmetric line bundle on each fiber A_t and let $\hat{h}_t$ be the induced Néron-Tate canonical height on A_t, for each $t \in B(\overline{\mathbb{Q}})$. For each non-torsion section $P : B \to A$ defined over $\overline{\mathbb{Q}}$, there is a constant $c = c(\mathscr{L}, P) > 0$ so that the set*

$$\{t \in B(\overline{\mathbb{Q}}) : \hat{h}_t(P_t) < c\}$$

is finite.

When the dimension of the fibers A_t is equal to 2, the finiteness of $\{t \in B(\overline{\mathbb{Q}}) : \hat{h}_t(P_t) = 0\}$ for sections as in Conjecture 1.0.9 was established recently by Masser and Zannier in [MZ4]. It is well known that the conclusion of Conjecture 1.0.9 can fail to hold if A is not simple and certainly fails if it is a family of elliptic curves, as mentioned above. However, the results of Masser and Zannier [MZ2, MZ3] suggested a formulation of Zhang's conjecture for the non-simple case when A splits as a product of elliptic curves; this is what we proved in our Theorem 1.0.8.

Finally, we point out that Theorem 1.0.4, Corollary 1.0.5, and Theorem 1.0.8 were obtained in the special case of the Legendre family $E_t = \{y^2 = x(x-1)(x-t)\}$ over $B = \mathbb{P}^1$ and the abelian variety $A_t = E_t \times E_t$, for sections P with x–coordinates in $\overline{\mathbb{Q}}(t)$ in [DWY1], using methods from complex dynamical systems, without appealing to Silverman and Tate's results on the height function. Moreover, restricting further to sections P with constant x–coordinate (in $\mathbb{P}^1(\overline{\mathbb{Q}})$), Theorem 1.0.8 was obtained without relying on the theorems of Masser and Zannier and gave an alternate proof of their result. This includes the special case treated by Masser and Zannier in their article [MZ1]. Note that for sections with constant x-coordinate, the hypothesis on P (that $\hat{h}_E(P) \neq 0$) is equivalent to asking that $x(P) \neq 0, 1, \infty$ [DWY1, Proposition 1.4].

Chapter 2

Preliminaries and notation

In this Chapter we introduce some background notions that will be used throughout this thesis.

2.1 Preperiodic points and Lattès maps

Arithmetic dynamics study number-theoretic questions arising from the iteration of maps $\phi : X \to X$, where X is a set endowed with some arithmetic structure. For us, the set X is usually $\mathbb{P}^1$, and the map ϕ is a morphism. To this end, we use the $n-$th iterates of the map, denoted by

$$\phi^n = \underbrace{\phi \circ \ldots \circ \phi}_{\text{n times}}$$

and study orbits of points $x \in X$ under the action of ϕ, denoted by

$$\mathscr{O}_\phi(x) = \{\phi^n(x) \ : \ n \in \mathbb{N}\}.$$

We remark that throughout this thesis, we assume that

$$0 \in \mathbb{N}.$$

Numerous important notions in diophantine geometry possess dynamical analogues. Consider, for instance, elliptic curves and their torsion points. It turns out that these torsion points can be understood as *preperiodic points* of certain rational maps known as *Lattès maps*, introduced in 1918, which play an exceptional role in complex dynamics.

Definition 2.1.1. Let $\phi : X \to X$. A point $x \in X$ is a **periodic** point of ϕ if there is an $n \in \mathbb{N}_{\geq 1}$ such that $\phi^n(x) = x$, and is a **preperiodic** point of ϕ if there are distinct $n, m \in \mathbb{N}$ satisfying $\phi^n(x) = \phi^m(x)$. Equivalently, a point $x \in X$ is a preperiodic point of ϕ if its orbit $\mathscr{O}_\phi(x)$ is a finite set. We denote the set of periodic points of ϕ by $\mathrm{Per}(\phi)$ and the set of preperiodic points of ϕ by $\mathrm{PrePer}(\phi)$.

Definition 2.1.2. [Si6, Section 6.4] A rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$ is called a **Lattès map** if there are an elliptic curve E, a morphism $\psi : E \to E$, and a finite separable covering $\pi : E \to \mathbb{P}^1$ such that the following diagram is commutative.

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E \\ \downarrow\pi & & \downarrow\pi \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

For a survey on the various remarkable properties of Lattès maps, we refer the reader to [Mi] and [Si6, Chapter 6]. The analogy between torsion points of an elliptic curve and preperiodic points of its Lattès map is made explicit in the following proposition.

Proposition 2.1.3. *[Si6, Proposition 6.44] Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a Lattès map corresponding to an elliptic curve E. Then* $\mathrm{PrePer}(\phi) = \pi(E_{tors})$.

Example 2.1.4. A Lattès map induced by the multiplication by 2 on $E : y^2 = x(x-1)(x-3)$, denoted by f, is defined by the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{[2]} & E \\ \downarrow\pi & & \downarrow\pi \\ \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1, \end{array}$$

where π is the projection onto the x–coordinate. More precisely, we have

$$\begin{aligned} &f : \mathbb{P}^1 \to \mathbb{P}^1 \\ &[X:Y] \mapsto [(X^2 - 3Y^2)^2 : 4X(X-Y)(X-3Y)Y]. \end{aligned}$$

Notice that for instance the point $P = (3,0) \in E$, which is a $2-$torsion point of E, corresponds to $[3:1] \in \mathbb{P}^1$, which is a preperiodic point of f. Indeed, $f([3:1]) = [1:0]$ so that $f^n([3:1]) = [1:0]$ for all $n \geq 1$.

2.2 Absolute values and height

Of great interest in arithmetic dynamics are dynamical systems defined over a number field K, for example rational functions $\phi \in K(z)$. Moreover, often considering ϕ as rational map defined over a local field provides important information about its dynamics.

In what follows, we denote by $M_{\mathbb{Q}}$ the set of all inequivalent absolute values $|\cdot|_p$ of $\mathbb{Q}$ corresponding to primes $p \in \mathbb{N}$ (non-archimedean places) and to the place at $p = \infty$ (archimedean place) and normalized such that for each non-zero $x \in \mathbb{Q}$ the following product formula holds

$$\prod_{p \in M_{\mathbb{Q}}} |x|_p = 1.$$

For any $|\cdot|_p \in M_{\mathbb{Q}}$, we denote the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ by $\mathbb{Q}_p$. Note that the p–adic absolute value of $\mathbb{Q}_p$ extends to an absolute value on its algebraic closure $\overline{\mathbb{Q}}_p$, which we also denote by $|\cdot|_p$. Throughout this thesis, $|\cdot|_p$ is normalized so that for each prime p we have

$$|p|_p = \frac{1}{p}.$$

We denote the completion of $\overline{\mathbb{Q}}_p$ with respect to $|\cdot|_p$ by $\mathbb{C}_p$ and we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Note that for the place at infinity, $\mathbb{C}_\infty$ is the set of complex numbers $\mathbb{C}$ endowed with the classical complex topology.

Using the p-adic absolute values for all p, we can define the Weil height of an element in $\mathbb{P}^1(\overline{\mathbb{Q}})$, which allows us to roughly measure its arithmetic complexity. Let K be a number field. For each $P = [x_0 : x_1] \in \mathbb{P}^1(K)$, we define the logarithmic

Weil height as

$$h([x_0 : x_1]) := \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma:K\hookrightarrow\overline{\mathbb{Q}}} \sum_{p\in M_{\mathbb{Q}}} \log\max\{|\sigma(x_0)|_p, |\sigma(x_1)|_p\}, \tag{2.1}$$

where the first summation runs over all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}}$. Note that the definition of $h([x_0 : x_1])$ is independent of the choice of number field K such that $[x_0 : x_1] \in \mathbb{P}^1(K)$; see [HS]. Furthermore, we identify any $\alpha \in \overline{\mathbb{Q}}$ with $[\alpha : 1] \in \mathbb{P}^1(\overline{\mathbb{Q}})$ and define its height as $h(\alpha) := h([\alpha : 1])$.

Example 2.2.1. If $q = \frac{a}{b} \in \mathbb{Q}$ in lowest terms, we have

$$h(q) = h([a:b]) = \log\max\{|a|_\infty, |b|_\infty\}.$$

Moreover, $h(2^{1/d}) = \frac{1}{d}\log 2$.

An important property of the height is that for $\alpha \in \overline{\mathbb{Q}}$ we have

$$h(\alpha) = 0 \quad \Leftrightarrow \quad \alpha = 0 \text{ or } \alpha \text{ is a root of unity.}$$

This property of the height, known as Kronecker's theorem, can be derived in a straightforward way from the next theorem, which illustrates another crucial property of the Weil height – the so-called *Northcott property*.

Theorem 2.2.2. *[Si6, Theorem 3.7] Let $d \in \mathbb{Z}$ and $c \in \mathbb{R}$. The set*

$$\{\alpha \in \overline{\mathbb{Q}} \ : \ h(\alpha) \le c \text{ and } [\mathbb{Q}(\alpha):\mathbb{Q}] \le d\},$$

is finite.

Chapter 3

Impossible intersections

This Chapter is a transcript of [Ma] consisting of independent work by the author. In §3.1 we establish some additional terminology needed to present our results. Then, in §3.2.5 we establish our results, including Theorems 1.0.2 and 1.0.3.

3.1 Non-archimedean dynamics

Recall our notations.

p	prime or ∞
$\mathbb{Q}_p$	the p–adic rational numbers
$\overline{\mathbb{Q}}_p$	the algebraic closure of $\mathbb{Q}_p$
$\mathbb{C}_p$	the completion of $\overline{\mathbb{Q}}_p$
$\lvert\cdot\rvert_p$	the p–adic absolute value on $\mathbb{C}_p$.

We often consider $\mathbb{P}^1(\mathbb{C}_p)$ and when doing so, we identify $[1:0]$ with ∞ and $\mathbb{P}^1(\mathbb{C}_p)$ with $\mathbb{C}_p \cup \{\infty\}$.

In complex dynamics (when $p = \infty$), we classify the fixed points of a rational map into attracting, repelling and neutral points by looking at the derivatives. This classification indicates whether nearby points move closer to or are getting repelled from the fixed point under repeated applications of the rational map. Analogous classifications exist in non-archimedean dynamics.

Definition 3.1.1. Let $\phi \in \mathbb{C}_p(z)$ and let $\alpha \in \mathbb{C}_p \subset \mathbb{P}^1(\mathbb{C}_p)$ be a periodic point of ϕ of exact period n. The **multiplier** of ϕ at α is defined as $\lambda_\alpha(\phi) = (\phi^n)'(\alpha)$.

Notice that $\lambda_\alpha(\phi)$ is the product of all values of ϕ at each of the n distinct points in the orbit of α. In general, the value of the derivative depends on the choice of coordinates. However, the set of multipliers is invariant under change of coordinates. More specifically, if $f \in \mathrm{PGL}(\mathbb{C}_p, 2)$ and $\phi^f = f^{-1} \circ \phi \circ f$, then we have $\lambda_\alpha(\phi) = \lambda_{f^{-1}(\alpha)}(\phi^f)$. To be compatible with this, if ∞ is a periodic point of ϕ, we define the multiplier at ∞ to be $\lambda_\infty(\phi) := \lambda_{f^{-1}(\infty)}(\phi^f)$ for any $f \in \mathrm{PGL}(\mathbb{C}_p, 2)$.

According to the behaviour of their multipliers, periodic points are called

$$\begin{aligned}
&\text{superattracting if } \lambda_\alpha(\phi) = 0,\\
&p\text{–adically attracting if } |\lambda_\alpha(\phi)|_p < 1,\\
&p\text{–adically neutral if } |\lambda_\alpha(\phi)|_p = 1,\\
&p\text{–adically repelling if } |\lambda_\alpha(\phi)|_p > 1.
\end{aligned}$$

Example 3.1.2. Let $f(z) = z^2 - 2$. The fixed points of f are

$$\alpha = 2, \;\; \beta = -1, \;\; \gamma = \infty.$$

Their corresponding multipliers are

$$\lambda_\alpha(f) = 4, \;\; \lambda_\beta(f) = -2, \;\; \lambda_\gamma(f) = 0.$$

Thus, ∞ is a p–adically superattracting fixed point for any p. On the other hand, α and β are 2–adically atracting fixed points, p–adically neutral fixed points for any prime $p \neq 2$, and repelling with respect to ∞.

The name attracting (respectively repelling) comes from the fact that we can choose a small neighborhood U of an attracting (respectively repelling) fixed point such that points in U move closer to (respectively move further from) α under iteration. Observe that in a neighborhood of α we may write

$$\phi(z) = \lambda_\alpha(\phi)(z-\alpha) + \sum_{n\geq 2} c_n (z-\alpha)^n.$$

An important property of the non-archimedean absolute values that we will repeatedly use is that the ultrametric inequality is almost an equality. More precisely, for primes p we have:

$$\text{If } |a|_p \neq |b|_p, \text{ then } |a+b|_p = \max\{|a|_p, |b|_p\}.$$

3.2 A Weierstrass family: a trichotomy and some impossible intersections

We study the family of elliptic curves

$$E_\lambda : y^2 = x^3 + \lambda, \text{ where } \lambda \in \mathbb{C}_2 \setminus \{0\}.$$

Our approach is dynamical and we use the Lattès maps induced by the multiplication by 2 on E_λ for $\lambda \neq 0$. More precisely, we use the maps $\mathbf{f}_\lambda$ defined by the commutative diagram below.

$$\begin{array}{ccc} E_\lambda & \xrightarrow{[2]} & E_\lambda \\ \downarrow\pi & & \downarrow\pi \\ \mathbb{P}^1 & \xrightarrow{\mathbf{f}_\lambda} & \mathbb{P}^1. \end{array}$$

Here $\pi : E_\lambda \to \mathbb{P}^1$ is the projection map onto the x–coordinate of the elliptic curve E_λ and $\mathbf{f}_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ is given as $\mathbf{f}_\lambda([X:Y]) = [X^4 - 8\lambda XY^3 : 4Y(X^3 + \lambda Y^3)]$. We mainly work with the de-homogenized version of $\mathbf{f}_\lambda$ defined as

$$f_\lambda(z) = \frac{z^4 - 8\lambda z}{4(z^3 + \lambda)},$$

and in this setting we identify ∞ with the point $[1:0] \in \mathbb{P}^1$.

Note that the family of elliptic curves $\{E_\lambda\}_{\{\lambda \in \mathbb{C}_2 \setminus \{0\}\}}$ may also be viewed as a single elliptic curve $E : y^2 = x^3 + t$ defined over the function field $\mathbb{C}_2(t)$, and the family of Lattès maps $\{f_\lambda\}_{\{\lambda \in \mathbb{C}_2 \setminus \{0\}\}}$ may also be viewed as a single rational map $f \in \mathbb{C}_2(t)(z)$. One important property of this elliptic curve E and this rational function f that will aid to the proof of Theorem 3.3.3, is that they are isotrivial.

Definition 3.2.1. Let K be an algebraically closed field. An **elliptic curve** E defined over the function field $K(t)$ is called **isotrivial** if there exists a finite extension L of $K(t)$ and an elliptic curve $\mathscr{E}$ defined over K such that E is $L-$isomorphic to $\mathscr{E}$.

Example 3.2.2. For the elliptic curve $E: y^2 = x^3 + t$ over $\mathbb{C}_2(t)$ and $\mathscr{E}: y^2 = x^3 + 1$ over $\mathbb{C}_2$, we have a $\mathbb{C}_2(t^{1/6})-$isomorphism as follows.

$$\begin{aligned} \mathscr{E} &\to E \\ (x,y) &\mapsto (xt^{1/3}, yt^{1/2}). \end{aligned}$$

Definition 3.2.3. Let K be an algebraically closed field. A **rational function** $\phi \in K(t)(z)$ is called **isotrivial** if there exists a finite extension L of $K(t)$ and a Möbius map $M \in \mathrm{PGL}(2,L)$ such that $M^{-1} \circ \phi \circ M \in K(z)$.

Example 3.2.4. For the rational function $f(z) = \frac{z^4 - 8tz}{4(z^3+t)} \in \mathbb{C}_2(t)(z)$, we have that if $M(z) = t^{1/3}z \in \mathrm{PGL}(2, \mathbb{C}_2(t^{1/3}))$ then

$$M^{-1} \circ f \circ M(z) = f_1(z) = \frac{z^4 - 8z}{4(z^3+1)} \in \mathbb{C}_2(z).$$

This equation reflects the fact that all E_λ for $\lambda \neq 0$ are isomorphic; see [Si6, Theorem 6.46]. Our approach is to exploit the 2–adic dynamics of f_λ. For $\alpha, \lambda \in \mathbb{C}_2$, the orbit of α under the action of f_λ is denoted by $\mathscr{O}_{f_\lambda}(\alpha)$. Moreover, we write $\mathrm{PrePer}(g) \subset \mathbb{C}_2$ for the set of preperiodic points of $g \in \mathbb{C}_2(z)$. We let

$$\begin{aligned} T(\alpha) &= \{\lambda \in \mathbb{C}_2 \setminus \{0\} \ : \ (\alpha, \sqrt{\alpha^3 + \lambda}) \in (E_\lambda)_{tors}\} \\ &= \{\lambda \in \mathbb{C}_2 \setminus \{0\} \ : \ \alpha \text{ is preperiodic for } f_\lambda\}. \end{aligned}$$

Finally, we note that $0, \infty$ are persistently preperiodic points for the family of rational maps f_λ, $\lambda \in \mathbb{C}_2 \setminus \{0\}$. Thus, in what follows, we assume that

$$\alpha \neq 0, \infty.$$

3.2.1 A trichotomy towards impossible intersections

In this section we establish the following result.

Theorem 3.2.5. *Let $\lambda \in T(\alpha)$. Then either $\lambda \in \left\{-\alpha^3, \frac{\alpha^3}{8}\right\}$ or*

$$|\lambda|_2 \in \{4|\alpha|_2^3, 4^{1-(1/4)^m}|\alpha|_2^3, 2^{2+(1/4)^m}|\alpha|_2^3 \,:\, m \in \mathbb{N}_{\geq 1}\}.$$

Moreover, exactly one of the following is true.

1. $|\lambda|_2 = 4|\alpha|_2^3 \iff 0, \infty \notin \mathscr{O}_{f_\lambda}(\alpha)$.
2. $|\lambda|_2 = 4^{1-(1/4)^m}|\alpha|_2^3$ *for some* $m \in \mathbb{N}_{\geq 1}$*, or* $\lambda = -\alpha^3 \iff \infty \in \mathscr{O}_{f_\lambda}(\alpha)$.
3. $|\lambda|_2 = 2^{2+(1/4)^m}|\alpha|_2^3$ *for some* $m \in \mathbb{N}_{\geq 1}$*, or* $\lambda = \frac{\alpha^3}{8} \iff 0 \in \mathscr{O}_{f_\lambda}(\alpha)$.

The isotriviality of $f(z) = \frac{z^4-8tz}{4(z^3+t)} \in \mathbb{C}_2(t)(z)$ will play an important role in the proof of this theorem. We find it worthwhile to point out that if $L(z) = (4t)^{1/3}z \in \mathrm{PGL}(2, \mathbb{C}_2(t^{1/3}))$, then

$$L^{-1} \circ f \circ L(z) = \frac{z^4 - 2z}{4z^3 + 1} = g(z) \in \mathbb{C}_2(z). \tag{3.1}$$

The map g here is the Lattès map corresponding to the multiplication by 2 on the elliptic curve $y^2 = x^3 + \frac{1}{4}$, and has the property that it exhibits 2–adic good reduction (see [Si6, Section 2.5] for definition).

For the rest of this section, we write

$$g(z) = \frac{z^4 - 2z}{4z^3 + 1} \in \mathbb{C}_2(z) \text{ as in (3.1).}$$

Thoerem 3.2.5 will be a consequence of the following proposition.

Proposition 3.2.6. *Let $w \in \mathrm{Preper}(g) \setminus \{0, \infty\}$. Then, exactly one of the following holds.*

1. $|w|_2 = 1 \iff 0, \infty \notin \mathscr{O}_g(w)$.
2. $|w|_2 = |4^{-1/3}|_2^{\frac{1}{4^m}}$ *for some* $m \in \mathbb{N}_{\geq 1}$*, or* $w^3 = -\frac{1}{4} \iff \infty \in \mathscr{O}_g(w)$.
3. $|w|_2 = |2^{1/3}|_2^{\frac{1}{4^m}}$ *for some* $m \in \mathbb{N}_{\geq 1}$*, or* $w^3 = 2 \iff 0 \in \mathscr{O}_g(w)$.

Assume for the moment that the aforementioned proposition holds, for the sake of establishing Theorem 3.2.5.

Proof of Theorem 3.2.5. The proof is a consequence of the isotriviality of $f \in \mathbb{C}_2(t)(z)$ and Proposition 3.2.6 as follows. In view of (3.1), we get that for all $n \in \mathbb{N}$ and $\lambda, \alpha \in \mathbb{C}_2$ we have

$$f_\lambda^n(\alpha) = (4\lambda)^{1/3} g^n\left(\frac{\alpha}{(4\lambda)^{1/3}}\right).$$

This implies $\mathscr{O}_{f_\lambda}(\alpha) = (4\lambda)^{1/3}\mathscr{O}_g\left(\frac{\alpha}{(4\lambda)^{1/3}}\right)$, and thus

$$\lambda \in T(\alpha) \iff \frac{\alpha}{(4\lambda)^{1/3}} \in \mathrm{PrePer}(g).$$

Hence, to find the 2–adic absolute value of the elements of $T(\alpha)$ it suffices to find the 2–adic absolute value of the preperiodic points of g, as in Proposition 3.2.6. $\square$

We now return to the proof of Proposition 3.2.6. For this purpose, we will need the following lemmas, which exploit the fact that 0 and ∞ are both 2–adically attracting fixed points of the map g with multipliers -2 and 4 respectively, as we can see in the following remark.

Remark 3.2.7. *For $z \in D(0,1) := \{z \in \mathbb{C}_2 \,:\, |z|_2 < 1\}$, we may write*

$$g(z) = -2z - \frac{9z}{4}\sum_{n\geq 1}(-4z^3)^n.$$

Moreover, for $\phi(z) = \frac{1}{g(1/z)} = \frac{z^4+4z}{1-2z^3} \in \mathbb{C}_2(z)$ and $z \in D(0,1)$ we have

$$\phi(z) = 4z + \frac{9z}{2}\sum_{n\geq 1}(2z^3)^n.$$

Lemma 3.2.8. *If $w \in D(0,1)$, then as $n \to \infty$ both $|g^n(w)|_2 \to 0$ and $|\phi^n(w)|_2 \to 0$. In particular,*

- *if $w \in \mathrm{PrePer}(g)$, then $g^m(w) = 0$ for some $m \in \mathbb{N}$.*
- *if $w \in \mathrm{PrePer}(\phi)$, then $\phi^k(w) = 0$ for some $k \in \mathbb{N}$.*

Proof. In view of Remark 3.2.7, we have that if $w \in D(0,1)$, then

$$|g(w)|_2 \leq \max\left\{\frac{|w|_2}{2}, |w|_2^4\right\} \text{ and } |\phi(w)|_2 \leq \max\left\{\frac{|w|_2}{4}, |w|_2^4\right\}.$$

Thus, we infer that as $n \to \infty$ both $|g^n(w)|_2 \to 0$ and $|\phi^n(w)|_2 \to 0$. The rest of the statement now follows. □

Remark 3.2.9. *Lemma 3.2.8 follows from a more general fact about maps $f \in K(z)$ with good reduction and having an attracting fixed point, where K is a local field, see [Be1, Lemma 2.3]. It implies that if $a \in \overline{K}$ is an attracting fixed point of f, then all the preperiodic points of f that lie in the residue class of a must map to a.*

Lemma 3.2.10. *Let $n \in \mathbb{N}$ and $w \in D(0,1)$. Then the following hold.*

- *If $|g(w)|_2 = |2^{1/3}|_2^{\frac{1}{4^n}}$, then $|w|_2^4 = |2^{1/3}|_2^{\frac{1}{4^n}}$.*
- *If $|\phi(w)|_2 = |4^{1/3}|_2^{\frac{1}{4^n}}$, then $|w|_2^4 = |4^{1/3}|_2^{\frac{1}{4^n}}$.*

Furthermore, if $n \in \mathbb{N}$ is the smallest integer such that $g^{n+1}(w) = 0$, then $|w|_2^{4^n} = |2^{1/3}|_2$. Similarly, if $n \in \mathbb{N}$ is the smallest integer such that $\phi^{n+1}(w) = 0$, then $|w|_2^{4^n} = |4^{1/3}|_2$.

Proof. Let $n \in \mathbb{N}$ and $w \in D(0,1)$. Using our hypothesis and the Taylor expansion in Remark 3.2.7, we have

$$|g(w)|_2 = \left|-2w - \frac{9w}{4}\sum_{n\geq 1}(-4w^3)^n\right|_2 = |2^{1/3}|_2^{\frac{1}{4^n}}.$$

If $|w^3|_2 \leq |2|_2$, then using the ultrametic inequality we infer that

$$|2^{1/3}|_2^{\frac{1}{4^n}} \leq \max\{|2w|_2, |9w^4|_2\} \leq |2|_2^{4/3},$$

which contradicts the fact that $|2|_2 < 1$. Therefore, we must have $|w^3|_2 > |2|_2$. Another application of the ultrametric inequality now yields that

$$|g(w)|_2 = |w|_2^4 = |2^{1/3}|_2^{\frac{1}{4^n}},$$

as claimed in the statement of the lemma. Now, if $n \in \mathbb{N}$ is the smallest integer such that $g^{n+1}(w) = 0$, then $(g^n(w))^3 = 2$, and hence $|g^n(w)|_2 = |2^{1/3}|_2$. Inductively, we get $|w|_2^{4^n} = |2^{1/3}|_2$.

The case of ϕ is similar. In view of Remark 3.2.7 and our hypothesis we have

$$|\phi(w)|_2 = \left| 4w + \frac{9w}{2} \sum_{n \geq 1} (2w^3)^n \right|_2 = |4^{1/3}|_2^{\frac{1}{4^n}}.$$

If $|w^3|_2 \leq |4|_2$, the ultrametric inequality yields $|4^{1/3}|_2^{\frac{1}{4^n}} \leq \max\{|4w|_2, |9w^4|_2\} \leq |4|_2^{4/3}$, contradicting the fact that $|4|_2 < 1$. Thus, $|w^3|_2 > |4|_2$ and $|\phi(w)|_2 = |w|_2^4 = |4^{1/3}|_2^{\frac{1}{4^n}}$. Now, if $n \in \mathbb{N}$ is the smallest integer such that $\phi^{n+1}(w) = 0$, then $(\phi^n(w))^3 = -4$, and hence $|\phi^n(w)|_2 = |4^{1/3}|_2$. Inductively, we get $|w|_2^{4^n} = |4^{1/3}|_2$. This finishes the proof of the lemma. □

We can now piece together Lemmas 3.2.8 and 3.2.10 to prove Proposition 2.7.

Proof of Proposition 3.2.6. Let $w \in \mathrm{PrePer}(g)$. We consider the cases $|w|_2 = 1$, $|w|_2 < 1$ and $|w|_2 > 1$ separately.

If $|w|_2 = 1$, then the ultrametric inequality yields that $|g^n(w)|_2 = 1$ for all $n \in \mathbb{N}$ and in particular $0, \infty \notin \mathscr{O}_g(w)$.

If $|w|_2 > 1$, then $z = \frac{1}{w} \in \mathrm{PrePer}(\phi) \cap D(0,1)$ and hence Lemma 3.2.8 yields $\phi^{m+1}(z) = 0$ for some $m \in \mathbb{N}$. This, by using Lemma 3.2.10, implies $|z|_2^{4^m} = |4^{1/3}|_2$ and hence $|w|_2 = |4^{-1/3}|_2^{\frac{1}{4^m}}$. If in particular we have $m = 0$, then we immediately get $w^3 = -\frac{1}{4}$.

Finally, assume $|w|_2 < 1$. By Lemma 3.2.8 we have that $g^{m+1}(w) = 0$ for some $m \in \mathbb{N}$. Lemma 3.2.10 now yields $|w|_2 = |2^{1/3}|_2^{\frac{1}{4^m}}$. In the case $m = 0$ we get $w^3 = 2$. The proposition is now established. □

We conclude this section with some related remarks.

Remark 3.2.11. We find it worthwhile to mention that g has three other 2–adically attracting fixed points, namely -1, $-\xi$, and $-\xi^2$ for a cube root of unity ξ, each with multiplier -2. These points give additional information about the preperiodic points of g of a flavor similar to the cases of 0 and ∞. More specifically, for $w \in \mathrm{PrePer}(g)$, the following hold.

(i) If $w \in D(-1,1)$, then there exists an $m \in \mathbb{N}$ such that $g^{m+1}(w) = -1$. For the smallest such m we have the equality $|w+1|_2 = |2^{1/3}|_2^{\frac{1}{4^m}}$.

(ii) If $w \in D(-\xi,1)$, then there exists an $m \in \mathbb{N}$ such that $g^{m+1}(w) = -\xi$. For the smallest such m we have the equality $|w+\xi|_2 = |2^{1/3}|_2^{\frac{1}{4^m}}$.

(iii) If $w \in D(-\xi^2,1)$, then there exists an $m \in \mathbb{N}$ such that $g^{m+1}(w) = -\xi^2$. For the smallest such m we have the equality $|w+\xi^2|_2 = |2^{1/3}|_2^{\frac{1}{4^m}}$.

The proof follows along the same lines as Proposition 3.2.6. We will briefly sketch the case of -1. For $z \in D(-1,1)$, we have

$$|g(z)+1|_2 = |-2(z+1) - 6(z+1)^2 - 16(z+1)^3 - 43(z+1)^4 + R(z+1)|_2,$$

where $|R(z+1)|_2 \leq |z+1|_2^5$. We infer that if $w \in \mathrm{PrePer}(g) \cap D(-1,1)$, then there exists a smallest $m \in \mathbb{N}$ such that $g^{m+1}(w) = -1$, which in turn yields $|w+1|_2^4 = |2^{1/3}|_2^{\frac{1}{4^n}}$. For the latter, notice that if $|g(w)+1|_2 \geq |2^{1/3}|_2$ then the ultrametric inequality yields $|g(w)+1|_2 = |w+1|_2^4$.

In particular, points (i), (ii) and (iii) provide additional information when $w \in \mathrm{PrePer}(g)$ is such that $|w|_2 = 1$ and $|w^3+1|_2 < 1$. Note that for this case our insight in Proposition 3.2.6 is not explicit. However, they do not provide explicit information for all $w \in \mathrm{PrePer}(g)$ with $|w|_2 = 1$ as there are many such $w \in \mathbb{C}_2$ which additionally satisfy $|w^3+1|_2 = 1$. In fact, all periodic points w of g except its fixed points $0, -1, -\xi, -\xi^2$ and ∞ satisfy

$$|w|_2 = |w+1|_2 = |w+\xi|_2 = |w+\xi^2|_2 = 1. \tag{3.2}$$

This follows by our observation in Remark 3.2.9, as otherwise the orbit of w under the action of g meets a fixed point of g, contradicting the periodicity of w.

Remark 3.2.12. We expect that the information about the 2–adic behavior of the preperiodic points of g in Proposition 3.2.6 and Remark 3.2.11 can be refined by considering other periodic points of g besides its fixed points. First notice that g has infinitely many periodic points. One way to see this is to recall that g is the Lattès map corresponding to the duplication map on $E: y^2 = x^3 + \frac{1}{4}$ and hence its periodic points are the x–coordinates of the points in $\cup_{n \in \mathbb{N}_{\geq 1}} E[2^n - 1] \cup E[2^n + 1]$. Moreover, all periodic points of g are 2–adically attracting; if z_n is a periodic point of g of exact period $n \geq 2$, then its multiplier is $\lambda_{z_n}(g) = g'(z_n) \cdot g'(g(z_n)) \cdots g'(g^{n-1}(z_n))$. By using (3.2) and the ultrametric inequality we get that $|g'(z_n)|_2 < 1$ and $|\lambda_{z_n}(g)|_2 < 1$. Hence, by considering the Taylor expansion of $g^n(z) - z_n$ in a 2–adic neighborhood of z_n, we expect to derive information about the values $|w - z_n|_2$ for $w \in \mathrm{PrePer}(g)$ that are 2–adically close to z_n.

Remark 3.2.13. The only $\mathbb{Q}_2$–preperiodic points of $g \in \mathbb{C}_2(z)$ are the $\mathbb{Q}_2$–fixed points of g, that is 0, ∞, and -1. To see this note that if $z \in \mathbb{Q}_2 \cap \mathrm{PrePer}(g) \setminus \{0, \infty, -1\}$ then either $|z|_2 < 1$ or $|z|_2 > 1$ or $|z+1|_2 < 1$, in which case Lemma 3.2.10 and Remark 3.2.11 yield that there exists an $n \in \mathbb{N}$ such that $|z|_2 = |2^{1/3}|_2^{\frac{1}{4^n}}$ or $|z|_2 = |4^{-1/3}|_2^{\frac{1}{4^n}}$ or $|z+1|_2 = |2^{1/3}|_2^{\frac{1}{4^n}}$ respectively, contradicting the fact that $z \in \mathbb{Q}_2$.

Remark 3.2.14. Observe that all the absolute values for $\lambda \in T(\alpha)$ that appear in Theorem 3.2.5 do indeed occur, from which it immediately follows that $T(\alpha)$ is an infinite set. To see this, it suffices to prove that all absolute values that appear in Proposition 3.2.6 for preperiodic points of g do indeed occur. As we have seen, -1 is a fixed point of g of absolute value 1. Let $n \in \mathbb{N}$. To find $w \in \mathrm{PrePer(g)}$ such that $|w|_2 = |2^{1/3}|_2^{\frac{1}{4^n}}$, in view of Lemma 3.2.10, it suffices to find $w \in \mathbb{C}_2$ such that $g^{m+1}(w) = 0$ and $g^m(w) \neq 0$. This can be achieved for $w \in \mathbb{C}_2$ satisfying $(g^m(w))^3 = 2$. Analogously, to find $w \in \mathrm{PrePer(g)}$ such that $|w|_2 = |4^{-1/3}|_2^{\frac{1}{4^n}}$, by Lemma 3.2.10, it suffices to find $z \in \mathbb{C}_2$ such that $\phi^{m+1}(z) = 0$ and $\phi^m(z) \neq 0$. This can be achieved for $z \in \mathbb{C}_2$ satisfying $(\phi^m(z))^3 = -4$.

3.2.2 Some applications

Let $\alpha, \beta \in \mathbb{C}_2$. Assuming the existence of $\lambda \in T(\alpha) \cap T(\beta)$, Theorem 3.2.5 allows us to compute an explicit list for the possible values of $\frac{|\alpha|_2}{|\beta|_2}$.

Corollary 3.2.15. *Assume that $T(\alpha) \cap T(\beta) \neq \emptyset$ and let*

$$X = \left\{1, 2^{\frac{1}{4^r}}, 2^{\frac{1}{3 \cdot 4^r}}, 2^{\frac{1}{3}\left(\frac{1}{4^r} - \frac{1}{4^s}\right)}, 2^{\frac{2}{3 \cdot 4^r}}, 2^{\frac{2}{3}\left(\frac{1}{4^r} - \frac{1}{4^s}\right)}, 4^{\frac{1}{3 \cdot 4^r}} 2^{\frac{1}{3 \cdot 4^s}} \;:\; r, s \in \mathbb{N}, r \neq s\right\}.$$

Then we have that either $\frac{|\alpha|_2}{|\beta|_2} \in X$ or $\frac{|\beta|_2}{|\alpha|_2} \in X$. Moreover, $\frac{|\alpha|_2}{|\beta|_2} = \frac{1}{2}$ or $\frac{|\alpha|_2}{|\beta|_2} = 2$ if and only if $\frac{\alpha^3}{\beta^3} = -8$ or $\frac{\alpha^3}{\beta^3} = -\frac{1}{8}$ respectively.

Proof. The proof follows immediately from Theorem 3.2.5. $\square$

A consequence now is that $T(\alpha)\cap T(\beta)=\emptyset$, for all α,β that 'disagree' with our list in Corollary 3.2.15. More specifically, we get the following theorem.

Theorem 3.2.16. *If $\alpha,\beta\in\overline{\mathbb{Q}}_2$ satisfy* $\gcd\left(6,e(\mathbb{Q}_2(\frac{\alpha}{\beta})|\mathbb{Q}_2)\right)=1$, $\left|\frac{\alpha}{\beta}\right|_2\neq 1$ *and* $\frac{\alpha^3}{\beta^3}\notin\{-8,-\frac{1}{8}\}$*, then we have that $T(\alpha)\cap T(\beta)=\emptyset$. Moreover, $T(a)\cap T(-2a)=\{-a^3\}$ for all $a\in\mathbb{C}_2\setminus\{0\}$.*

Proof. The proof follows combining the fact that for any $c\in\overline{\mathbb{Q}}_2$ if $e:=e(\mathbb{Q}_2(c)|\mathbb{Q}_2)$, then $|c|\in 2^{\frac{\mathbb{Z}}{e}}$ with Corollary 3.2.15. To see that $T(a)\cap T(-2a)=\{-a^3\}$, note that by Theorem 3.2.5 we get that if $\lambda\in T(a)\cap T(-2a)$ then $\lambda=-a^3$, in which case we have $f_{-a^3}(a)=\infty$ and $f_{-a^3}(-2a)=0$. $\square$

3.3 A Weierstrass family: more impossible intersections

We use our notation as in §3.2.5. Theorem 3.2.16 raises the question whether we could describe $T(\alpha)\cap T(\beta)$ for $\alpha,\beta\in\mathbb{C}_2$ with equal 2–adic absolute values. In this section, towards partially answering this question, we aim to prove Theorem 1.0.2, which asserts that if we restrict our attention to $\alpha,\beta\in\overline{\mathbb{Q}}_2$ satisfying $\frac{\alpha}{\beta}\in\mathbb{Q}$, then there are no parameters λ such that both α,β are preperiodic for f_λ, unless $\frac{\alpha}{\beta}\in\{-2,-\frac{1}{2}\}$.

Before we state the main theorem of this section, a couple of remarks are in order.

Remark 3.3.1. The isotriviality of $f(z)\in\mathbb{C}_2(t)(z)$ implies that for all $\alpha,\lambda,z\in\mathbb{C}_2$, we have $f_{\lambda\alpha^3}(\alpha z)=\alpha f_\lambda(z)$. It easily follows that $\mathcal{O}_{f_{\lambda\alpha^3}}(\alpha)=\alpha\mathcal{O}_{f_\lambda}(1)$ and $T(\alpha)=\alpha^3T(1)$.

Remark 3.3.2. From Remark 3.3.1, we get

$$\lambda\in T(\alpha)\cap T(\beta)\Leftrightarrow\frac{\lambda}{\beta^3}\in T(1)\cap T\left(\frac{\alpha}{\beta}\right).$$

In particular, $\#(T(\alpha)\cap T(\beta))=\#\left(T(1)\cap T\left(\frac{\alpha}{\beta}\right)\right)$.

For the following we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_2$. For the reader's convenience, we now restate Theorem 1.0.2.

Theorem 3.3.3. *If $\alpha, \beta \in \overline{\mathbb{Q}}_2 \setminus \{0\}$ are such that $\frac{\alpha}{\beta} \in \mathbb{Q} \setminus \{-2, -\frac{1}{2}\}$, then $T(\alpha) \cap T(\beta) = \emptyset$. Moreover, for all $a \in \overline{\mathbb{Q}}_2 \setminus \{0\}$ we have $T(a) \cap T(-2a) = \{-a^3\}$.*

As we have already seen in Theorem 3.2.16, when $\alpha, \beta \in \overline{\mathbb{Q}}_2$ with $\frac{\alpha}{\beta} \in \mathbb{Q} \setminus \{-2, -\frac{1}{2}\}$ and $|\alpha|_2 \neq |\beta|_2$, we have $T(\alpha) \cap T(\beta) = \emptyset$. Moreover when $a \in \overline{\mathbb{Q}}_2 \setminus \{0\}$, we have $T(a) \cap T(-2a) = \{-a^3\}$. Therefore, to prove Theorem 3.3.3, it suffices to show that $T(\alpha) \cap T(\beta) = \emptyset$ when $|\alpha|_2 = |\beta|_2$. Our strategy will be to first show in Lemma 3.3.4 that if $\lambda \in T(\alpha) \cap T(\beta)$, then either $0 \in \mathscr{O}_{f_\lambda}(\alpha) \cap \mathscr{O}_{f_\lambda}(\beta)$ or $\infty \in \mathscr{O}_{f_\lambda}(\alpha) \cap \mathscr{O}_{f_\lambda}(\beta)$. Then, after proving the coprimality of certain polynomials in Lemmas 3.3.8 and 3.3.9, we will rule out these two cases as well.

Lemma 3.3.4. *Let $\alpha, \beta \in \mathbb{C}_2$ with $|\alpha|_2 = |\beta|_2$ and $|\alpha - \beta|_2 \leq \frac{|\alpha|_2}{2}$. Consider $\lambda \in T(\alpha) \cap T(\beta)$. Then either $0 \in \mathscr{O}_{f_\lambda}(\alpha) \cap \mathscr{O}_{f_\lambda}(\beta)$ or $\infty \in \mathscr{O}_{f_\lambda}(\alpha) \cap \mathscr{O}_{f_\lambda}(\beta)$.*

Proof. Let $\lambda \in T(\alpha) \cap T(\beta)$ and assume that $0, \infty \notin \mathscr{O}_{f_\lambda}(\alpha) \cap \mathscr{O}_{f_\lambda}(\beta)$. We want to show that this leads to a contradiction. In light of Theorem 3.2.5 and the fact that $|\alpha|_2 = |\beta|_2$ we get that $0 \in \mathscr{O}_{f_\lambda}(\alpha)$ (respectively $\infty \in \mathscr{O}_{f_\lambda}(\alpha)$) if and only if $0 \in \mathscr{O}_{f_\lambda}(\beta)$ (respectively $\infty \in \mathscr{O}_{f_\lambda}(\beta)$). Our assumption thus implies that $0, \infty \notin \mathscr{O}_{f_\lambda}(\alpha) \cup \mathscr{O}_{f_\lambda}(\beta)$. For the rest of this proof we will denote $f_\lambda^n(\alpha)$ and $f_\lambda^n(\beta)$ by t_n and u_n respectively.

Since $\lambda \in T(\alpha) \cap T(\beta)$, we know that the sets

$$S = \{t_n \ : \ n \in \mathbb{N}\}, \text{ and } T = \{u_n \ : \ n \in \mathbb{N}\}$$

are finite. Therefore, the set $M = \{|t_n - u_n|_2 \ : \ n \in \mathbb{N}\}$ is also finite.

We claim that

$$|t_n - u_n|_2 = \frac{|\alpha - \beta|_2}{2^n}$$

for all $n \in \mathbb{N}$. This will contradict the fact that M is finite, thus finishing our proof. We will now prove the claim using induction. For $n = 0$, we have $|t_0 - u_0|_2 = |\alpha - \beta|_2$. For the inductive step, assume that the statement holds for some $n \geq 0$.

Then

$$
\begin{aligned}
|t_{n+1}-u_{n+1}|_2 &= 4\left|\frac{t_n^4-8\lambda t_n}{t_n^3+\lambda}-\frac{u_n^4-8\lambda u_n}{u_n^3+\lambda}\right|_2 \\
&= 4\left|\frac{(t_n^4-8\lambda t_n)(u_n^3+\lambda)-(u_n^4-8\lambda u_n)(t_n^3+\lambda)}{(u_n^3+\lambda)(t_n^3+\lambda)}\right|_2 \\
&= \frac{4}{|u_n^3+\lambda|_2|t_n^3+\lambda|_2}\cdot \\
&\quad \left|t_n^3u_n^3(t_n-u_n)-\lambda(u_n^4-t_n^4)+8\lambda t_nu_n(t_n^2-u_n^2)+8\lambda^2(u_n-t_n)\right|_2.
\end{aligned}
$$

By the induction hypothesis, we know that $|t_n-u_n|_2=\frac{|\alpha-\beta|_2}{2^n}\leq\frac{|\alpha|_2}{2^{n+1}}$. Moreover, since $\lambda\in T(\alpha)\cap T(\beta)$ and $0,\infty\notin\mathscr{O}_{f_\lambda}(\alpha)\cup\mathscr{O}_{f_\lambda}(\beta)$, we also have $\lambda\in T(t_n)\cap T(u_n)$ and $0,\infty\notin\mathscr{O}_{f_\lambda}(t_n)\cup\mathscr{O}_{f_\lambda}(u_n)$ for all $n\in\mathbb{N}$. Theorem 3.2.5 now yields that for all $n\in\mathbb{N}$, $|t_n|_2=|u_n|_2=\sqrt[3]{\frac{|\lambda|_2}{4}}=|\alpha|_2$. Therefore, we get that

$$
\begin{aligned}
&|u_n^3+\lambda|_2=|t_n^3+\lambda|_2=4|\alpha|_2^3, \\
&|t_n+u_n|_2=|t_n-u_n+2u_n|_2\leq\frac{|\alpha|_2}{2}, \text{ and} \\
&|t_n^2+u_n^2|_2=|(t_n+u_n)^2-2t_nu_n|_2=\frac{|\alpha|_2^2}{2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|t_n^3u_n^3(t_n-u_n)|_2 &= |\alpha|_2^6|t_n-u_n|_2, \\
|\lambda(u_n^4-t_n^4)|_2 &\leq |\alpha|_2^6|t_n-u_n|_2, \\
|8\lambda t_nu_n(t_n^2-u_n^2)|_2 &\leq \frac{|\alpha|_2^6}{4}|t_n-u_n|_2, \\
|8\lambda^2(t_n-u_n)|_2 &= 2|\alpha|_2^6|t_n-u_n|_2.
\end{aligned}
$$

Thus, using the induction hypothesis, we obtain

$$
|t_{n+1}-u_{n+1}|_2=\frac{1}{2}|t_n-u_n|_2=\frac{|\alpha-\beta|_2}{2^{n+1}}.
$$

This establishes our claim and concludes the proof. □

Remark 3.3.5. We point out here that the condition $\frac{\alpha}{\beta}\in\mathbb{Q}$ in Theorem 3.3.3 has been made to ensure that when α, β have equal 2–adic absolute values, then $|\alpha - \beta|_2 \leq \frac{|\alpha|_2}{2}$ holds. In this case we can apply Lemma 3.3.4.

To proceed with our proof, we need the following definition.

Definition 3.3.6. *Given $a \in \mathbb{C} \setminus \{0\}$, we write $f_t^n(a) = \frac{A_n(a,t)}{B_n(a,t)}$, where $A_n(a,t), B_n(a,t) \in \mathbb{C}[t]$ are polynomials given recursively as*

$$
\begin{aligned}
&A_0(a,t) = a, B_0(a,t) = 1.\\
&A_{n+1}(a,t) = A_n(a,t)^4 - 8tA_n(a,t)B_n(a,t)^3.\\
&B_{n+1}(a,t) = 4B_n(a,t)A_n(a,t)^3 + 4tB_n(a,t)^4.
\end{aligned}
$$

Lemma 3.3.7. *Let $a \in \mathbb{C} \setminus \{0\}$. We have $\gcd(A_n(a,t), B_n(a,t)) = 1$ for all $n \in \mathbb{N}$. Moreover, $\deg_t(A_n(a,t)) = \deg_t(B_n(a,t)) = \frac{4^n - 1}{3}$ for all $n \in \mathbb{N}$.*

Proof. The proof follows from an easy induction. Note that our hypothesis that $a \neq 0$ implies that $A_n(a,0), B_n(a,0) \neq 0$ and thus $t \nmid A_n(a,t), B_n(a,t)$ for all $n \in \mathbb{N}$. $\square$

Our aim next is to prove that when $a, b \in \mathbb{Z}$, the polynomials $A_n(a,t)$ and $A_n(b,t)$ (respectively $B_n(a,t)$ and $B_n(b,t)$) are coprime. This will in turn imply that $0, \infty \notin \mathscr{O}_{f_\lambda}(a) \cap \mathscr{O}_{f_\lambda}(b)$, which combined with view of Remark 3.3.2 is what we need in order to conclude the proof of Theorem 3.3.3, when $\frac{\alpha}{\beta} = \frac{a}{b}$. To achieve this we will first establish a few key lemmas.

Lemma 3.3.8. *For all $a, b \in \mathbb{Z}$ for which there exists a prime $p \neq 2$ such that $p|a$ and $p \nmid b$, we have*

$$\gcd(A_n(a,t), A_n(b,t)) = \gcd(B_n(a,t), B_n(b,t)) = 1, \textit{ for all } n \in \mathbb{N}.$$

Proof. We start by noting that given $g(t) \in \mathbb{Z}[t]$, we denote its reduction modulo p by $\overline{g}(t) \in \mathbb{F}_p[t]$, where p is the same prime as in the statement of the lemma. Moreover, we denote $a_n(a,t) = \frac{A_n(a,t)}{a}$. From the recursion in Definition 3.3.6 we see that $a_n(a,t) \in \mathbb{Z}[t]$. Our strategy is to first establish that

$$\gcd\left(\overline{a_n}(a,t), \overline{A_n}(b,t)\right) = \gcd(\overline{B_n}(a,t), \overline{B_n}(b,t)) = 1. \tag{3.3}$$

Since $a_n(a,t) \in \mathbb{Z}[t]$ and $p|a$, we get that $\overline{A_n}(a,t) = 0$ for all $n \in \mathbb{N}$. Therefore, the recursive relations in Definition 3.3.6 yield that

$$\overline{B_{n+1}}(a,t) = 4t\overline{B_n}(a,t)^4 \text{ and } \overline{B_1}(a,t) = 4t.$$

Thus, we obtain that $\overline{B_n}(a,t) = (4t)^{\frac{4^n-1}{3}}$. Additionally, we have the following equalities.

$$\overline{a_{n+1}}(a,t) = -2(4t)^{4^n}\overline{a_n}(a,t) \text{ and } \overline{a_1}(a,t) = -8t. \tag{3.4}$$

From (3.4), it follows that $\overline{a_n}(a,t) = (-2)^n(4t)^{\frac{4^n-1}{3}}$.

Observe now that, to derive (3.3), it suffices to prove that $\overline{A_n}(b,0), \overline{B_n}(b,0) \neq 0$. Notice that $A_n(b,0) = b^{4^n}$ and $B_n(b,0) = 4^n b^{4^n-1}$. Since $p \neq 2$ and $p \nmid b$, we have established the equality in (3.3). Now note that the degrees of the polynomials $a_n(a,t), A_n(b,t), B_n(a,t)$ and $B_n(b,t)$, as computed in Lemma 3.3.7, are equal to the degrees of their reductions modulo p. This, combined with (3.3) finishes the proof of the lemma. $\square$

We will prove another useful lemma of a similar flavor.

Lemma 3.3.9. *For all* $n \in \mathbb{N}$, $\gcd(A_n(1,t), A_n(-1,t)) = \gcd(B_n(1,t), B_n(-1,t)) = 1$.

Proof. We proceed in a similar fashion to the proof of Lemma 3.3.8, just that this time we reduce polynomials modulo 3. Throughout this proof we write $\overline{g}(t) \in \mathbb{F}_3[t]$ for the reduction modulo 3 of $g(t) \in \mathbb{Z}[t]$. We will show that

$$\gcd\left(\overline{A_n}(1,t), \overline{A_n}(-1,t)\right) = \gcd\left(\overline{B_n}(1,t), \overline{B_n}(-1,t)\right) = 1. \tag{3.5}$$

Inductively, we can prove that $\overline{A_n}(1,t) = \overline{B_n}(1,t)$ for all $n \in \mathbb{N}$. Therefore, using the recursion for $A_n(1,t)$, we get that $\overline{A_{n+1}}(1,t) = \overline{A_n}(1,t)^4(1+t)$. Thus,

$$\overline{A_n}(1,t) = \overline{B_n}(1,t) = (1+t)^{\frac{4^n-1}{3}}.$$

Moreover, inductively we can see that

$$\overline{A_n}(-1,t) = (1-t)^{\frac{4^n-1}{3}},$$
$$\overline{B_n}(-1,t) = -(1-t)^{\frac{4^n-1}{3}}.$$

This establishes (3.5). Since the degrees of the polynomials $A_n(1,t), A_n(-1,t), B_n(1,t)$ and $B_n(-1,t)$, computed in Lemma 3.3.7, are equal to the degrees of their reductions modulo 3, (3.5) yields the lemma. □

We are now ready to prove Theorem 3.3.3. In the following, for a non-zero integer n and a prime p, we write

$$\exp_p(n) = \max\{e \in \mathbb{N} \, : \, p^e | n\}.$$

Proof of Theorem 3.3.3. Let $\alpha, \beta \in \overline{\mathbb{Q}_2}$ be such that $\frac{\alpha}{\beta} \in \mathbb{Q} \setminus \{-2, -\frac{1}{2}\}$. We aim to prove that $T(\alpha) \cap T(\beta) = \emptyset$. In view of Theorem 3.2.16, we have that $T(\alpha) \cap T(\beta) = \emptyset$ if $|\alpha| \neq |\beta|$, unless $\frac{\alpha}{\beta} \notin \{-2, -\frac{1}{2}\}$. Assume now $|\alpha|_2 = |\beta|_2$. We may write $\frac{\alpha}{\beta} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ are coprime and satisfy $|a|_2 = |b|_2 = 1$. By Remark 3.3.2 it suffices to prove that $T(a) \cap T(b) = \emptyset$. Assume to the contrary that $\lambda \in T(a) \cap T(b)$. Now Lemma 3.3.4 and Remark 3.3.5 together imply that either $0 \in \mathscr{O}_{f_\lambda}(a) \cap \mathscr{O}_{f_\lambda}(b)$ or $\infty \in \mathscr{O}_{f_\lambda}(a) \cap \mathscr{O}_{f_\lambda}(b)$.

Observe that $0 \in \mathscr{O}_{f_\lambda}(a) \cap \mathscr{O}_{f_\lambda}(b)$ if and only if $A_n(a,\lambda) = A_n(b,\lambda) = 0$ for some $n \in \mathbb{N}$, and $\infty \in \mathscr{O}_{f_\lambda}(a) \cap \mathscr{O}_{f_\lambda}(b)$ if and only if $B_n(a,\lambda) = B_n(b,\lambda) = 0$ for some $n \in \mathbb{N}$. Hence, to prove the theorem it suffices to show that

$$\gcd(A_n(a,t), A_n(b,t)) = \gcd(B_n(a,t), B_n(b,t)) = 1 \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

To this end, we will consider two cases. If $a = -b = 1$, on invoking Lemma 3.3.9 we see that (3.6) follows. If on the other hand there exists a prime p such that $\exp_p(a) \neq \exp_p(b)$, since $|a|_2 = |b|_2 = 1$, we see that $p \neq 2$ and (3.6) again holds true by Lemma 3.3.8. □

An immediate corollary of Theorem 3.3.3 is the following.

Corollary 3.3.10. *If $\alpha \in \mathbb{Q}$ and $\lambda \in T(\alpha)$, the orbit of α under the action of f_λ*

does not contain any rational number other than α *except possibly* 0 *and* ∞*.*

Proof. Let $\alpha \in \mathbb{Q}$ and $\lambda \in T(\alpha)$. Assume that for some $n \in \mathbb{N}$ we have $f_\lambda^n(\alpha) \in \mathbb{Q} \setminus \{0, \infty, \alpha\}$ to derive a contradiction. Observe that $\lambda \in T(\alpha) \cap T(f_\lambda^n(\alpha))$. This observation combined with Theorem 3.3.3 now yields that one of the following is true.

• $f_\lambda^n(\alpha) = -2\alpha$, in which case $\lambda = -\alpha^3$ and $f_\lambda(\alpha) = \infty$ contradicting the fact that $f_\lambda^n(\alpha) \neq \infty$.

• $\alpha = -2f_\lambda^n(\alpha)$, in which case $\lambda = \frac{\alpha^3}{8}$ and $f_\lambda(\alpha) = 0$ contradicting the fact that $f_\lambda^n(\alpha) \neq 0$.

In each case we derived a contradiction, yielding our claim. □

We conclude this section with some observations and further questions.

Remark 3.3.11. Let $h : \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ denote the absolute logarithmic Weil height, as in [Si6, Section 3.1]. As seen from (3.1) we have $\lambda \in T(1) \Leftrightarrow \frac{1}{(4\lambda)^{1/3}} \in \mathrm{PrePer}(g)$. Combining this with [Si6, Theorem 3.12] we see that there is a constant $M > 0$ such that $h(\lambda) < M$ for all $\lambda \in T(1)$. Moreover, by Remark 3.3.1 we have $T(\alpha) \cap T(\beta) \neq \emptyset$ if and only if there exist $\lambda, \mu \in T(1)$ such that $\alpha^3\lambda = \beta^3\mu$. Therefore, $3h(\frac{\alpha}{\beta}) = h(\frac{\mu}{\lambda}) \leq 2M$. Hence, there is an absolute constant $C > 0$ such that $h\left(\frac{\alpha}{\beta}\right) < C$ for all $\alpha, \beta \in \overline{\mathbb{Q}}$ that satisfy $T(\alpha) \cap T(\beta) \neq \emptyset$. In particular, by Northcott's theorem 2.2.2, there exist only finitely many $\frac{\alpha}{\beta}$ of bounded degree such that $T(\alpha) \cap T(\beta) \neq \emptyset$. In Theorem 3.3.3, we show that if $\frac{\alpha}{\beta} \in \mathbb{Q}$, then $T(\alpha) \cap T(\beta) \neq \emptyset$ if and only if $\frac{\alpha}{\beta} \in \{-2, -\frac{1}{2}\}$. This raises the following natural question: Fix $d \in \mathbb{Z}$, what is the best upper bound (depending on d) on the number of $\frac{\alpha}{\beta}$ of degree at most d such that $T(\alpha) \cap T(\beta) \neq \emptyset$?

Finally, it would be interesting to know whether there exist $\alpha, \beta \in \overline{\mathbb{Q}}$ such that $2 \leq \#(T(\alpha) \cap T(\beta)) < +\infty$. We note here that by Remark 3.3.1 if $\alpha^3 = \beta^3$, we have $T(\alpha) \cap T(\beta) = T(\alpha) = T(\beta)$, which by Remark 3.2.14 is an infinite set.

3.4 The Legendre family of elliptic curves

For this section, let $E_\lambda : y^2 = x(x-1)(x-\lambda)$ be the Legendre family of elliptic curves parametrized by $\lambda \in \mathbb{C}_2 \setminus \{0, 1\}$. The Lattès map induced by the multiplication by 2 map on E_λ, is given as

$$f_\lambda(z) = \frac{(z^2-\lambda)^2}{4z(z-1)(z-\lambda)}.$$

Let $\alpha \in \mathbb{C}_2$. We define $T(\alpha)$ as follows.

$$\begin{aligned} T(\alpha) &= \{\lambda \in \mathbb{C}_2 \setminus \{0,1\} \ : \ (\alpha, \sqrt{\alpha(\alpha-1)(\alpha-\lambda)}) \in (E_\lambda)_{tors}\} \\ &= \{\lambda \in \mathbb{C}_2 \setminus \{0,1\} \ : \ \alpha \text{ is preperiodic for } f_\lambda\}. \end{aligned}$$

First, we prove the following easy proposition.

Proposition 3.4.1. *For all $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, we have $f^n_{\frac{1}{\lambda}}(\frac{1}{x}) = \frac{1}{\lambda} f^n_\lambda(x)$. In particular, $\lambda \in T(x)$ if and only if $\frac{1}{\lambda} \in T(\frac{1}{x})$.*

Proof. For $n = 1$, we have $f_{\frac{1}{\lambda}}(\frac{1}{x}) = \frac{1}{\lambda} f_\lambda(x)$. Because $f_{\frac{1}{\lambda}}(\frac{z}{\lambda}) = \frac{1}{\lambda} f_\lambda(z)$, we get the result inductively. Alternatively, one can see that $\lambda \in T(x)$ if and only if $\frac{1}{\lambda} \in T(\frac{1}{x})$ by noting that the affine map $(x,y) \to (\frac{1}{x}, \frac{y}{x^2\sqrt{\lambda}})$ extends to an isomorphism between the elliptic curves E_λ and $E_{\frac{1}{\lambda}}$. $\square$

The following theorem is a restatement of [St1, Theorem 3]. Here we provide a different, shorter proof.

Theorem 3.4.2. *Let $\alpha \in \mathbb{C}_2 \setminus \{0,1\}$ and $\lambda \in T(\alpha) \setminus \{\alpha\}$. If $|\alpha|_2 \leq 1$, then $|\alpha^2 - \lambda|_2 < 1$. If $|\alpha|_2 > 1$, then $|\lambda|_2 > 1$.*

Proof. Consider $\alpha \in \mathbb{C}_2$ with $|\alpha|_2 \leq 1$ and let $\lambda \in T(\alpha) \setminus \{\alpha\}$. First we will show that $|\lambda|_2 \leq 1$. Assume, so as to derive a contradiction, that $|\lambda|_2 > 1$. Then, we have

$$|f_\lambda(\alpha)|_2 = \frac{4|\lambda|_2^2}{|\alpha|_2|\alpha-1|_2|\lambda|_2} \geq 4|\lambda|_2.$$

Inductively, we get $|f^{n+1}_\lambda(\alpha)|_2 = 4|f^n_\lambda(\alpha)|_2$ for all $n \geq 1$. Since $\lambda \in T(\alpha)$, this implies $f_\lambda(\alpha) = \infty$ which in turn contradicts the fact that $\lambda \neq \alpha$. Therefore, we must have $|\lambda|_2 \leq 1$. To prove $|\alpha^2 - \lambda|_2 < 1$, let us assume the opposite and see what happens. Using the fact $|\lambda|_2 \le 1$, we have

$$|f_\lambda(\alpha)|_2 = \frac{4|\alpha^2-\lambda|_2^2}{|\alpha|_2|\alpha-1|_2|\alpha-\lambda|_2} \ge 4.$$

Again, inductively we get $|f_\lambda^{n+1}(\alpha)|_2 = 4|f_\lambda^n(\alpha)|_2$ for all $n \ge 1$. Since $\lambda \in T(\alpha)$, this yields $f_\lambda(\alpha) = \infty$, contradicting the fact that $\lambda \ne \alpha$. Hence, $|\alpha^2 - \lambda|_2 < 1$.

The second part of the statement now follows by the first part and Proposition 3.4.1. $\square$

Therefore, exactly as in [St1, Corollary 4], denoting by ρ the natural reduction map $\mathbb{P}^1(\mathbb{C}_2) \to \mathbb{P}^1(\overline{\mathbb{F}_2})$, we get that

Corollary 3.4.3. *If $\alpha, \beta \in \mathbb{C}_2 \setminus \{0,1\}$ such that $\rho(\alpha) \ne \rho(\beta)$, then $T(\alpha) \cap T(\beta) \subset \{\alpha, \beta\}$.*

For examples of α, β such that $T(\alpha) \cap T(\beta)$ is empty or has exactly one or two elements, we refer the reader to [St1, Example 5].

Now we aim to strengthen this result. In particular we provide an effective description of $T(\alpha) \cap T(\beta)$ even in some cases when $\rho(\alpha) = \rho(\beta)$. To this end, we have the following.

Theorem 3.4.4. *If $\alpha \in \mathbb{C}_2$ satisfies $|\alpha|_2 \le \frac{1}{4}$, then $T(2) \cap T(\alpha) \subset \{\alpha\}$. Moreover, if $|\alpha|_2 < \frac{1}{4}$, then $T(2) \cap T(\alpha) = \emptyset$.*

Proof. Let $\alpha \in \mathbb{C}_2$ be such that $|\alpha|_2 \le \frac{1}{4}$ and $\lambda \in T(2) \cap T(\alpha) \setminus \{2, \alpha\}$. In view of Theorem 3.4.2, we know that $|\lambda|_2 < 1$. We claim, in fact, that the following is true.

Claim 3.4.5. *If $\lambda \in T(2) \setminus \{2\}$, then $|\lambda|_2 = \frac{1}{4}$.*

Proof. Using an argument by contradiction, we will first show that $|\lambda|_2 \ge \frac{1}{4}$. Assume that $|\lambda|_2 < \frac{1}{4}$. Then $|f_\lambda(2)|_2 = \frac{8|4-\lambda|_2^2}{|2-\lambda|_2} = 1$ and therefore $|(f_\lambda(2))^2 - \lambda|_2 = 1$. However, since $\lambda \in T(f_\lambda(2)) \setminus \{f_\lambda(2)\}$, this contradicts Theorem 3.4.2.

A similar argument also shows that $|\lambda|_2 \le \frac{1}{2}$. Indeed, if $|\lambda|_2 > \frac{1}{2}$, then we have $|f_\lambda(2)|_2 = 8|\lambda|_2 > 4$, which contradicts Theorem 3.4.2 since $\lambda \in T(f_\lambda(2)) \setminus \{f_\lambda(2)\}$ and $|\lambda|_2 < 1$.

Now we know that $\frac{1}{4} \le |\lambda|_2 \le \frac{1}{2}$. Assume that $|\lambda|_2 > \frac{1}{4}$, so as to derive a contradiction and establish the claim. Since $|\lambda|_2 \le \frac{1}{2}$, we have $|f_\lambda(2)|_2 = \frac{8|\lambda|_2^2}{|2-\lambda|_2} \ge$

$16|\lambda|_2^2 > 1$, which combined with the fact that $\lambda \in T(f_\lambda(2)) \setminus \{f_\lambda(2)\}$ contradicts Theorem 3.4.2. This yields the claim. $\square$

To finish the proof of the theorem, notice that Claim 3.4.5 yields that $|\lambda|_2 = \frac{1}{4}$. This, combined with our assumption that $|\alpha|_2 \leq \frac{1}{4}$, implies $|f_\lambda(\alpha)|_2 = \frac{4|\lambda|_2^2}{|\alpha|_2|\alpha-\lambda|_2} \geq 4$, which by Theorem 3.4.2 contradicts the fact that $\lambda \in T(f_\lambda(\alpha)) \setminus \{f_\lambda(\alpha)\}$. Therefore, we obtain that $T(2) \cap T(\alpha) \subset \{2, \alpha\}$. We can easily see that $2 \notin T(\alpha)$, since $|\alpha|_2 \leq \frac{1}{4}$. Thus, in fact we get that $T(2) \cap T(\alpha) \subset \{\alpha\}$. If in particular $|\alpha|_2 < \frac{1}{4}$, then by Claim 3.4.5 we have $T(2) \cap T(\alpha) = \emptyset$. $\square$

Combining now Proposition 3.4.1 and Theorem 3.4.4, we get the following.

Corollary 3.4.6. *If $\beta \in \mathbb{C}_2$ satisfies $|\beta|_2 \geq 4$, then $T(\frac{1}{2}) \cap T(\beta) \subset \{\beta\}$. Moreover, if $|\beta|_2 > 4$, then $T(\frac{1}{2}) \cap T(\beta) = \emptyset$.*

3.5 Other families of rational maps

We will now consider

$$f_\lambda(z) = \frac{z^d + \lambda}{pz},$$

where $d \geq 2$ and $p \in \mathbb{Z}$ prime. Our method will give results of flavor similar to the results in Section 3.4 for this family of rational maps. However, we find it worthwhile to mention that the above family is not a Lattès family. To see this note that for all $\lambda \in \mathbb{C}$ the maps f_λ have an attracting fixed point in the topology induced by the standard complex absolute value; when $d > 2$ we have that ∞ is a fixed critical point and when $d = 2$ the points $\pm\sqrt{\frac{\lambda}{p-1}}$ are fixed points with multiplier $\frac{2-p}{p}$. On the other hand, for Lattès maps all periodic points are repelling and dense in $\mathbb{P}^1(\mathbb{C})$, as illustrated by the fact that the Julia set of a Lattès map is the entire Riemann sphere [Si6, Theorem 1.43]. For a definition of the Julia set of a rational map we refer the reader to [Si6].

Recall that $|\cdot|_p$ denotes the p–adic absolute value on $\mathbb{C}_p$, with $|p|_p = \frac{1}{p}$. We write

$$T(\alpha) = \{\lambda \in \mathbb{C}_p \ : \ \alpha \text{ is preperiodic for } f_\lambda\}.$$

We note that 0 is a persistently preperiodic point for the family of rational maps f_λ where $\lambda \in \mathbb{C}_p$. Therefore, in the following we consider $\alpha \neq 0$.

Theorem 3.5.1. *Let $\alpha \in \mathbb{C}_p \setminus \{0\}$ with $|\alpha|_p \leq 1$ and let $\lambda \in T(\alpha)$. Then $|\alpha^d + \lambda|_p < 1$. If on the other hand we have $|\alpha|_p > 1$, then $|\lambda|_p > 1$.*

Proof. Consider $\alpha \in \mathbb{C}_p \setminus \{0\}$ with $|\alpha|_p \leq 1$ and let $\lambda \in T(\alpha)$. First, we will show that $|\lambda|_p \leq 1$. Assume to the contrary that $|\lambda|_p > 1$. Then, we have

$$|f_\lambda(\alpha)|_p = \frac{p|\lambda|_p}{|\alpha|_p} \geq p|\lambda|_p > p.$$

Inductively, we get that $|f_\lambda^{n+1}(\alpha)|_p = p|f_\lambda^n(\alpha)|_p^{d-1}$ for all $n \geq 1$. Since $f_\lambda(\alpha) \neq \infty$ and $d \geq 2$, this contradicts the fact that $\lambda \in T(\alpha)$. Therefore, we must have $|\lambda|_p \leq 1$. Next we prove that $|\alpha^d + \lambda|_p < 1$.

Assume, for the sake of contradiction, that $\lambda \in T(\alpha)$ and $|\alpha^d + \lambda|_p \geq 1$. Then we have $|f_\lambda(\alpha)|_p \geq p$. Inductively this implies $|f_\lambda^{n+1}(\alpha)|_p = p|f_\lambda^n(\alpha)|_p^{d-1}$ for all $n \geq 1$. Since $\lambda \in T(\alpha)$ and $d \geq 2$, this yields $f_\lambda(\alpha) = \infty$, contradicting the fact that $\alpha \neq 0$. Hence, $|\alpha^d + \lambda|_p < 1$.

We will now prove the second part of the statement. Assume that $|\alpha|_p > 1$ and that $\lambda \in T(\alpha)$. We will see that $|\lambda|_p > 1$. Assume, to the contrary, that $|\lambda|_p \leq 1$. Then, $|f_\lambda(\alpha)|_p = p|\alpha|_p^{d-1}$ and inductively $|f_\lambda^{n+1}(\alpha)|_p = p|f_\lambda^n(\alpha)|_p^{d-1}$ for all $n \in \mathbb{N}$. This, combined with $|\alpha|_p > 1$, contradicts our assumption that $\lambda \in T(\alpha)$. Therefore, $|\lambda|_p > 1$. The proof follows. □

Now Theorem 3.5.1 implies Theorem 1.0.3, which we restate here for the convenience of the reader. Denote by ρ the natural reduction map $\mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\overline{\mathbb{F}_p})$.

Theorem 3.5.2. *If $\alpha, \beta \in \mathbb{C}_p \setminus \{0\}$ are such that $\rho(\alpha^d) \neq \rho(\beta^d)$, then $T(\alpha) \cap T(\beta) = \emptyset$.*

To highlight the difference between the results in this section and in §3.2.1, we find it worthwhile to point out the following.

Remark 3.5.3. Recall that the map $z \mapsto \frac{z^4 - 8tz}{4(z^3+t)} \in \mathbb{C}_2(t)(z)$, which we studied in §3.2.1, has the following properties:

- It is isotrivial.
- It is conjugate to the map $z \mapsto \frac{z^4-2z}{4z^3+1} \in \mathbb{C}_2(z)$ with 2–adic good reduction.

For the map $f(z) = \frac{z^d+t}{pz} \in \mathbb{C}_p(t)(z)$ in this section, these properties are true only when $(d,p) = (2,2)$. To see this, first notice that if $d \geq 3$, then it is not isotrivial. On the other hand, when $d = 2$, the map $f(z) = \frac{z^2+t}{pz} \in \mathbb{C}_p(t)(z)$ is isotrivial. Conjugating by $M(z) = t^{1/2}z$ we get

$$M^{-1} \circ f \circ M(z) = \frac{z^2+1}{pz} \in \mathbb{C}_p(z).$$

If further $p = d = 2$ we may conjugate the map $G(z) := \frac{z^2+1}{2z}$ with $L(z) = 2z-1$ and get

$$L^{-1} \circ G \circ L(z) = \frac{z^2}{2z-1},$$

which has 2–adic good reduction. However, if $p \neq 2$, then $G(z) = \frac{z^2+1}{pz} \in \mathbb{C}_p(z)$ in not $\mathrm{PGL}(2,\mathbb{C}_p)-$conjugate to a map with p–adic good reduction, since it has two p–adically repelling fixed points $\pm\frac{1}{\sqrt{p-1}}$ with multiplier $\frac{2-p}{p}$, [Be2, Theorem].

Chapter 4

From Silverman's work to equidistribution

This Chapter is a transcript of [DM] obtained by the author in collaboration with Laura DeMarco. In §4.1 we present some background material on the Néron-Tate canonical height as well as Silverman's results on the variation of heights [Si4, Si5] that will be essential in our proofs. In §4.2 we present the equidistribution theorem of Chambert-Loir, Thuillier and Yuan [CL1, Th, Yu]. The rest of this Chapter is devoted to the proofs of Theorem 1.0.4, Corollary 1.0.5 and Theorem 1.0.8.

4.1 Preliminaries related to a variation of the canonical height

In what follows, we let $\mathscr{F}$ be a product formula field of characteristic 0. So, there exists a family $M_{\mathscr{F}}$ of non-trivial absolute values on $\mathscr{F}$ and a collection of positive integers n_v for $v \in M_{\mathscr{F}}$ such that the following two conditions are satisfied.

- For every $x \in \mathscr{F}^*$, we have $|x|_v = 1$ for all but finitely many $v \in M_{\mathscr{F}}$.

- For every $x \in \mathscr{F}^*$ the following product formula holds

$$\prod_{v \in M_{\mathscr{F}}} |x|_v^{n_v} = 1.$$

For us, $\mathscr{F}$ will either be a number field or the function field of a smooth projective curve. For each $v \in M_{\mathscr{F}}$, we let $\mathscr{F}_v$ denote the completion of $\mathscr{F}$ with respect to $|\cdot|_v$ and $\mathbb{C}_v$ denote the completion of the algebraic closure of $\mathscr{F}_v$ with respect to $|\cdot|_v$. We fix an embedding of $\overline{\mathscr{F}}$ into $\mathbb{C}_v$.

4.1.1 Néron-Tate heights

Let $E/\mathscr{F}$ be an elliptic curve with origin O. The **Néron-Tate canonical height** associated to $E/\mathscr{F}$ is a function

$$\hat{h}_E : E(\overline{\mathscr{F}}) \to [0,\infty),$$

which can be defined by

$$\hat{h}_E(P) = \frac{1}{2}\lim_{n\to\infty} \frac{h(x([n]P))}{n^2}. \tag{4.1}$$

Here h is the logarithmic Weil height on $\mathbb{P}^1$ and $x : E \to \mathbb{P}^1$ is the degree 2 projection to the x–coordinate. This defines a quadratic form on $E(\overline{\mathscr{F}})$, assigning to each point $P \in E(\overline{\mathscr{F}})$ a value $\hat{h}_E(P)$ which roughly measures the arithmetic complexity of P and vanishes precisely on $E(\overline{\mathscr{F}})_{\text{tors}}$.

In fact, the Nèron-Tate canonical height as defined by (4.1) is the canonical height on E relative to the divisor $(O) \in \mathrm{Div}(E)$. More generally, if A is an abelian variety defined over $\mathscr{F}$ one can associate to each divisor $D \in \mathrm{Div}(A)$ a Weil height $h_{A,D} : A(\overline{\mathscr{F}}) \to \mathbb{R}$ and a canonical height $\hat{h}_{A,D} : A(\overline{\mathscr{F}}) \to \mathbb{R}$ satisfying

$$\hat{h}_{A,D}(P) = h_{A,D}(P) + O(1),$$

for all $P \in A(\overline{\mathscr{F}})$. For the construction of these heights the reader is referred to [HS, Theorems B.3.2, B.5.6]. If we further assume that $D \in \mathrm{Div}(A)$ is a symmetric divisor, that is it satisfies $[-1]^*D \sim D$, then the associated canonical height satisfies the *parallelogram law*

$$\hat{h}_{A,D}(P+Q) + \hat{h}_{A,D}(P-Q) = 2\hat{h}_{A,D}(P) + 2\hat{h}_{A,D}(Q) \quad \text{for all } P,Q \in A(\overline{\mathscr{F}}).$$

From this it follows that the canonical height $\hat{h}_{A,D}$ is a quadratic form [HS, Lemma

B.5.2]. That is, $\hat{h}_{A,D}$ is an even function and the associated pairing

$$< \cdot,\cdot >_{A,D} : A(\overline{\mathscr{F}}) \times A(\overline{\mathscr{F}}) \to \mathbb{R},$$

given by

$$(P,Q) \mapsto \frac{\hat{h}_{A,D}(P+Q) - \hat{h}_{A,D}(P) - \hat{h}_{A,D}(Q)}{2},$$

is symmetric and bilinear [HS, Theorem B.5.1]. If moreover D is an ample divisor, then $\hat{h}_{A,D}$ is non-negative and vanishes precisely on the torsion points of A. Thus, $\hat{h}_{A,D}$ defines a positive-definite quadratic form on $A(\overline{\mathscr{F}}) \big/ A(\overline{\mathscr{F}})_{\mathrm{tors}}$, where $A(\overline{\mathscr{F}})_{\mathrm{tors}}$ denotes the torsion points in $A(\overline{\mathscr{F}})$.

Example 4.1.1. Let $A = E_1 \times E_2$, where E_i for $i = 1,2$ is an elliptic curve defined over $\mathscr{F}$ with origin O_i. Let $D = \{O_1\} \times E_2 + E_1 \times \{O_2\} \in \mathrm{Div}(A)$. Then, for each $P = (P_1, P_2) \in A(\overline{\mathscr{F}})$ we have

$$\hat{h}_{A,D}(P) = \hat{h}_{E_1}(P_1) + \hat{h}_{E_2}(P_2).$$

4.1.2 Local heights

Néron and Tate have shown that the canonical height can be decomposed as a sum of *local heights*, one for each place $v \in M_{\mathscr{F}}$, that are 'almost' quadratic forms. We give Tate's definition of these local heights in the case of $\hat{h}_E(\cdot)$. To this end, let us first choose a Weierstrass form for E over $\mathscr{F}$ as

$$E \ : \ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{4.2}$$

We write Δ for the discriminant of E and O for its origin. We may endow $E(\mathbb{C}_v)$ with a v-adic topology as follows. For each point $P = (x_P, y_P) \in E(\mathbb{C}_v) \setminus \{O\}$ we define a basis of open neighborhoods by

$$U_\varepsilon = \{(z,w) \in E(\mathbb{C}_v) \ : \ |z - x_P|_v < \varepsilon \text{ and } |w - y_P|_v < \varepsilon\},$$

for all $\varepsilon > 0$. For O a basis of open neighborhoods is given by

$$U_\varepsilon = \{(z,w) \in E(\mathbb{C}_v) \;:\; |z|_v > \varepsilon^{-1}\} \cup \{O\},$$

for all $\varepsilon > 0$. The local heights are given as follows.

Proposition 4.1.2. *[Si3, Chapter 6, Theorem 1.1] Let $E/\mathscr{F}$ be an elliptic curve as in* (4.2) *and let $v \in M_{\mathscr{F}}$. Then, there exists a unique function*

$$\hat{\lambda}_{E,v} : E(\mathbb{C}_v) \setminus \{O\} \to \mathbb{R},$$

satisfying the following three properties.

1. *$\hat{\lambda}_{E,v}$ is continuous on $E(\mathbb{C}_v) \setminus \{O\}$ and bounded on the complement of any v-adic neighborhood of O;*
2. *the limit of $\hat{\lambda}_{E,v}(P) - \frac{1}{2}\log|x_P|_v$ exists as $P \xrightarrow{v} O$ in $E(\mathbb{C}_v)$; and*
3. *for all $P = (x_P, y_P) \in E(\mathbb{C}_v)$ with $[2]P \neq O$, we have*

$$\hat{\lambda}_{E,v}([2]P) = 4\hat{\lambda}_{E,v}(P) - \log|2y_P + a_1x_P + a_3|_v + \frac{1}{4}\log|\Delta|_v.$$

Remark 4.1.3. The addition of the term involving the discriminant in the formula

$$\hat{\lambda}_{E,v}([2]P) = 4\hat{\lambda}_{E,v}(P) - \log|2y_P + a_1x_P + a_3|_v + \frac{1}{4}\log|\Delta|_v$$

from the third part of Proposition 4.1.2 ensures that the function $\hat{\lambda}_{E,v}$ is independent of the choice of Weierstrass equation of E over $\mathscr{F}$; see [Si3, Chapter 6, Theorem 1.1].

These functions are the local components of the canonical height called the *local Néron height function* on E associated to v. More specifically, by [Si3, Chapter 6, Theorem 2.1] they satisfy

$$\hat{h}_E(P) = \frac{1}{|\operatorname{Gal}(\overline{\mathscr{F}}/\mathscr{F}) \cdot P|} \sum_{Q \in \operatorname{Gal}(\overline{\mathscr{F}}/\mathscr{F}) \cdot P} \; \sum_{v \in M_{\mathscr{F}}} n_v \, \hat{\lambda}_{E,v}(Q), \tag{4.3}$$

for all $P \in E(\overline{\mathscr{F}}) \setminus \{O\}$. Moreover, there are explicit formulas for the local heights for both archimedean [Si3, Chapter 6, §3] and non-archimedean places of $\mathscr{F}$ [Si3, Chapter 6, §4].

For instance for any non-archimedean place $v \in M_{\mathscr{F}}$ such that the coefficients a_i of our Weierstrass equation (4.2) are v–adic integers, i.e. $|a_i|_v \leq 1$ for $i = 1, \ldots, 6$, and the discriminant Δ is a v–adic unit, i.e. $|\Delta|_v = 1$, we have

$$\hat{\lambda}_{E,v}(Q) = \frac{1}{2} \log \max\{|x_Q|_v, 1\}, \text{ for } Q \in E(\overline{\mathscr{F}}) \setminus \{O\}. \tag{4.4}$$

Hence, for each $P \in E(\overline{\mathscr{F}})$ there are finitely many places with non-zero contribution to the a priori infinite sum from (4.3).

4.1.3 Variation of the canonical height

In this Subsection we present the results of Silverman [Si4, Si5] which play a crucial role in the proof of Theorems 1.0.4 and 1.0.8. First, we fix our notation. Let K be a number field, let B be a smooth projective curve defined over K and $E \to B$ an elliptic surface, also defined over K, with zero section $O : B \to E$. Further, let $P : B \to E$ be a non-torsion section defined over K. Alternatively, we may identify the elliptic surface $E \to B$ with its generic fiber and view it as a single elliptic curve defined over $k = \overline{K}(B)$, written in Weierstrass form as

$$E : y^2 = x^3 + Ax + B \tag{4.5}$$

for $A, B \in \overline{K}(B)$. We write $\Delta \in \overline{K}(B)$ for its discriminant. Then, for each $t \in B(\overline{K})$ such that $\Delta(t) \neq 0$ and $A(t)$ and $B(t)$ are defined, we obtain a non-singular fiber E_t of $E \to B$ by evaluating the Weierstrass equation (4.5) at t. Moreover, our non-torsion section $P : B \to E$ corresponds to a non-torsion point in $E(\overline{K}(B))$, written in our Weierstrass coordinates as (x_P, y_P) for rational functions $x_P, y_P \in \overline{K}(B)$. Note that for each $t \in B(\overline{K})$ such that the fiber E_t is non-singular, we have a point $P_t = (x_P(t), y_P(t)) \in E_t(\overline{K})$.

In what follows, we assume that when viewing P as a point on the elliptic curve

E defined over $k = \overline{K}(B)$, we have

$$\hat{h}_E(P) \neq 0.$$

In particular, P is not a torsion point in $E(\overline{K}(B))$.

Silverman [Si2, Si4, Si5] investigates the function

$$t \mapsto \hat{h}_{E_t}(P_t),$$

which is well defined at all but finitely many $t \in B(\overline{K})$, and its local components $\hat{\lambda}_{E_t,v}(P_t)$. Note that via the embedding of $\overline{K}$ into $\mathbb{C}_v$ for each place $v \in M_K$, we may view $E \to B$ as defined over $\mathbb{C}_v$ and consider the Néron local heights $\hat{\lambda}_{E_t,v}(P_t)$ on the non-singular fibers E_t as functions of $t \in B(\mathbb{C}_v)$.

In what follows, we define $D_E(P)$ by

$$D_E(P) = \sum_{\gamma \in B(\overline{K})} \hat{\lambda}_{E,\mathrm{ord}_\gamma}(P) \cdot (\gamma). \tag{4.6}$$

Here, $\hat{\lambda}_{E,\mathrm{ord}_\gamma}(P)$ is the local canonical height of the point P on the elliptic curve E over $k = \overline{K}(B)$ at the place ord_γ, for each $\gamma \in B(\overline{K})$. We point out that $D_E(P)$ is actually a $\mathbb{Q}$-divisor on B with degree equal to $\hat{h}_E(P)$. To see this, first note that $\hat{\lambda}_{E,\mathrm{ord}_\gamma}(P)$ is zero for all but finitely many $\gamma \in B(\overline{K})$. Indeed, from (4.4) we get that if $\gamma \in B(\overline{K})$ is such that the fiber E_γ is non-singular, then

$$\hat{\lambda}_{E,\mathrm{ord}_\gamma}(P) = \frac{1}{2} \max\{0, \mathrm{ord}_\gamma x_P^{-1}\}.$$

In other words, $\hat{\lambda}_{E,\mathrm{ord}_\gamma}(P)$ is non-zero if and only if γ is a pole of $x_P \in \overline{K}(B)$. Thus, the support of $D_E(P)$ is a subset of the finite set

$$\{t \in B(\overline{K}) : E_t \text{ is singular}\} \cup \{t \in B(\overline{K}) : P_t = O_t\}.$$

That $D_E(P)$ is a $\mathbb{Q}$-divisor follows from the fact that the local heights $\hat{\lambda}_{E,\mathrm{ord}_\gamma}(P)$ can be viewed as arithmetic intersection numbers on a Néron local model. The reader is referred to [Si3, Chapter III, Theorem 9.3] for a proof that $\hat{h}_E(P) \in \mathbb{Q}$ and to [CS, Section 6, p. 203] and [La2, Chapter 11, Theorem 5.1] for proofs that each local component $\hat{\lambda}_{E,\mathrm{ord}_\gamma}(P)$ is also a rational number. The latter can also been proved by using dynamical methods [DG, Theorem 1.1].

In [Si5, Theorem III.0.2], Silverman considers the difference between the Néron-Tate canonical heights on the fibers and an *analytic Weil height* $h_{B,D_E(P)}(t)$ on B associated to the divisor $D_E(P)$ (see [Si5, §3] for the definition of an analytic Weil height). He shows that the map

$$t \mapsto \hat{h}_{E_t}(P_t) - h_{B,D_E(P)}(t) \tag{4.7}$$

as a function of $t \in B(\overline{K})$ such that E_t is smooth and $P_t \neq O_t$, can be decomposed as a finite sum of well-behaved functions indexed by a finite set of places of K. For instance, away from the parameters corresponding to singular fibers E_t, these functions are real analytic or locally constant when the corresponding place is archimedean or non-archimedean respectively.

His result follows from analogous results [Si4, Theorem II.0.1] and [Si5, Theorem III.0.1] concerning the local components of (4.7) for all places v of K,

$$t \mapsto \hat{\lambda}_{E_t,v}(P_t) - \lambda_{B,D_E(P),v}(t),$$

as a function of $t \in B(\overline{K})$ such that E_t is smooth and $P_t \neq O_t$, where $\lambda_{B,D_E(P)}(t)$ is an analytic Weil local height (see [Si5, §3] for the definition).

Before stating these results, we fix some analytic heights functions on B associated to the divisor $D_E(P)$ defined as in [Si5, §3 Example 1(a)]. Recall that $D_E(P)$ is supported at $\gamma \in B(K)$ such that $\hat{\lambda}_{E,\mathrm{ord}_\gamma}(P) \neq 0$. Let g be the genus of B. For each point $\gamma \in B(K)$, we choose an element ξ_γ of $K(B)$ which has a pole of order $2g+1$ at γ and no other poles. For each non-archimedean place v of K, we set

$$\lambda_{B,D_E(P),v}(t) = \frac{1}{2g+1} \sum_{\gamma \in B(K)} \hat{\lambda}_{E,\mathrm{ord}_\gamma}(P)\ \log^+ |\xi_\gamma(t)|_v, \tag{4.8}$$

for all $t \in B(\mathbb{C}_v) \setminus \operatorname{supp} D_E(P)$. For archimedean places v, we set

$$\lambda_{B,D_E(P),v}(t) = \frac{1}{2(2g+1)} \sum_{\gamma \in B(K)} \hat{\lambda}_{E,\mathrm{ord}_\gamma}(P)\ \log\left(1 + |\xi_\gamma(t)|_v^2\right). \tag{4.9}$$

Then, the corresponding analytic Weil height is given by

$$h_{B,D_E(P)}(t) = \frac{1}{|\operatorname{Gal}(\overline{K}/K)\cdot t|} \sum_{s\in \operatorname{Gal}(\overline{K}/K)\cdot t} \sum_{v\in M_K} n_v \lambda_{B,D_E(P),v}(s),$$

for all $t \in B(\overline{K}) \setminus \operatorname{supp} D_E(P)$. For $t \in \operatorname{supp} D_E(P)$ we may set $h_{B,D_E(P)}(t) = 0$.

Variation of canonical height: quasi triviality and continuity

Silverman proved the following 'quasi triviality' result concerning the local heights.

Theorem 4.1.4. *[Si5, Theorem III.4.1] Let $\lambda_{B,D_E(P),v}$ be the analytic local Weil heights as in* (4.8) *and* (4.9). *There is a finite set S of places so that*

$$\hat{\lambda}_{E_t,v}(P_t) = \lambda_{B,D_E(P),v}(t),$$

for all $t \in B(\overline{K}) \setminus \operatorname{supp} D_E(P)$ and all $v \in M_K \setminus S$.

Note that Theorem 4.1.4 remains true if we choose different functions $\xi_\gamma \in K(B)$ with a pole of order $2g+1$ at γ and no other pole in the definition of the analytic local Weil heights from (4.8) and (4.9). However, the finite set S in Theorem 4.1.4 might change. Fix a point $t_0 \in B(\overline{K})$ and a uniformizer $u \in \overline{K}(B)$ for t_0, and consider the function

$$V_{P,t_0,v}(t) := \hat{\lambda}_{E_t,v}(P_t) + \hat{\lambda}_{E,\operatorname{ord}_{t_0}}(P)\log|u(t)|_v, \tag{4.10}$$

which is not *a priori* defined at t_0. Then, Theorem 4.1.4 implies that

$$V_{P,t_0,v} \equiv 0,$$

for all but finitely many places v in a v–adic neighborhood of each t_0.

Silverman also proved the following 'continuity' result concerning the local heights.

Theorem 4.1.5. *[Si4, Theorem II.0.1] Fix $t_0 \in B(\overline{K})$ and a uniformizer u at t_0. For all $v \in M_K$, there exists a neighborhood $U \subset B(\mathbb{C}_v)$ of t_0 so that the function $V_{P,t_0,v}$ of* (4.10) *extends to a continuous function on U.*

4.2 Preliminaries related to equidistribution

In this Subsection, we introduce the equidistribution result of Chambert-Loir, Thuillier, and Yuan [CL1, Th, Yu] for points of *small height* which is the key ingredient for the proof of Corollary 1.0.5. The height in the preceding statement is associated to an ample, *continuous, semipositive, adelic, metrized* line bundle. In what follows, we will briefly introduce the italicized words. Much of the material here is drawn from Chambert Loir's survey article [CL2], Yuan's survey article [Yu2] and Thuillier's PhD thesis [Th]. We refer to these articles and the references therein for a more comprehensive introduction to this subject. We point out here that result in [CL1, Th, Yu] was initially proved in [BR1, FRL] in the special case of $\mathbb{P}^1$.

Let K be a number field, let X be a smooth projective curve over K with structure sheaf $\mathscr{O}_X$ and let $\mathscr{L}$ be a line bundle on X. The space in which the equidistribution takes place is the *Berkovich analytification* X_v^{an} of X when viewed as a curve over $\mathbb{C}_v$ at a place $v \in M_K$. This is a Hausdorff and locally path-connected space, which contains the totally disconnected space $X(\mathbb{C}_v)$ as a dense subset. For a precise definition the reader is referred to [Be]. See also [BR2] where the case of $X = \mathbb{P}^1$ is discussed in detail. Then, $\mathbb{P}_v^{1,\mathrm{an}}$ has a natural tree structure. If Y is an affine curve, given by $Y = \mathrm{Spec}(A)$ for a finitely generated algebra A over $\mathbb{C}_v$, then one can construct its Berkovich analytification Y_v^{an} as follows. The points z in Y_v^{an} correspond to multiplicative seminorms $|\cdot|_z$ on A which extend $|\cdot|_v$. The space Y_v^{an} is endowed with the weakest topology such that for any $f \in A$, the map

$$\begin{aligned} Y_v^{\mathrm{an}} &\to \mathbb{R} \\ z &\mapsto [f]_z \end{aligned}$$

is continuous. When v is archimedean we get $X_v^{\mathrm{an}} = X(\mathbb{C}_v)$, whereas when v is non-archimedean X_v^{an} is much larger than $X(\mathbb{C}_v)$.

The advantage of working in the v–adic Berkovich analytification of X is that we can develop a theory of harmonic and subharmonic functions on this space for both archimedean and non-archimedean places. Furthermore, this theory in the ultrametric case is analogous to the classical theory over the complex numbers. More precisely, there is a Laplacian operator dd^c on X_v^{an}; see [Th, Chapter 3.24] and

[BR2] for the case of the projective line. When v is archimedean this is the usual Laplacian operator $dd^c := \frac{i}{\pi}\partial\bar{\partial}$. Moreover, just as in the complex case, the subharmonic functions T satisfy $dd^c T \geq 0$ in the non-archimedean setting; [Th, Theorem 3.4.12]. Many important properties of subharmonic functions in the complex setting have direct analogues in the non-archimedean Berkovich analytification; refer to [Th, Chapter 3.1.2]. For instance, the analogue of the maximum principle holds; see [Th, Proposition 3.1.11] and [BR2, Proposition 8.14].

We now introduce the notion of a continuous metric on $\mathscr{L}$.

Definition 4.2.1. A **continuous metric** $||\cdot||_v$ on $\mathscr{L}$ is given by assigning to each open set $U \subset X$ and section $s \in \mathscr{L}(U)$ a function $||s||_{v,U} : U \to \mathbb{R}_{\geq 0}$, so that the following hold.

1. For each open $V \subset U$, $||s||_{v,V}$ is the restriction of $||s||_{v,U}$ on V.

2. For any $f \in \mathscr{O}_X(U)$, $||fs||_v = |f|_v ||s||_v$.

3. If $\varepsilon_U \in \mathscr{L}(U)$ is a local frame, such that multiplication by ε_U yields a trivialization $\mathscr{O}_X|U \xrightarrow{\sim} \mathscr{L}|U$, then $||\varepsilon_U||_v$ does not vanish at any point of U and the function
$$-\log||\varepsilon_U||_v$$
is the restriction on U of a continuous function on the Berkovich analytification U_v^{an}.

Example 4.2.2. Let $v \in M_K$ and $X = \mathbb{P}^1_K$ covered by $U_i = \{z_i \neq 0\}$ for $i = 0,1$, where z_0, z_1 are the homogeneous coordinates. Consider the line bundle $\mathscr{L} := \mathscr{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$ represented by $\{(U_0, 1), (U_1, z_0/z_1)\}$. We have two local frames, on U_0 and U_1 respectively, given by $\varepsilon_0 = 1$ and $\varepsilon_1 = \frac{z_1}{z_0}$. They yield the trivializations
$$\times\varepsilon_0 : \mathscr{O}_{\mathbb{P}^1}|U_0 \xrightarrow{\sim} \mathscr{L}|U_0$$
$$\times\varepsilon_1 : \mathscr{O}_{\mathbb{P}^1}|U_1 \xrightarrow{\sim} \mathscr{L}|U_1.$$

Under this trivialization a section $s \in \mathscr{L}(U)$ corresponds to two regular functions $s_i \in \mathscr{O}_{\mathbb{P}^1}(U \cap U_i)$ satisfying $s_0 = \frac{z_1}{z_0}s_1 \in \mathscr{O}_{\mathbb{P}^1}(U \cap U_0 \cap U_1)$. To give a continuous metric on $\mathscr{L}$ amounts to giving two functions $g_i : U_i \to \mathbb{R}$ that extend to continuous functions on $U_{i,v}^{\text{an}}$ and satisfy

$$g_0 = \log\left|\frac{z_1}{z_0}\right| + g_1 \text{ on } U_0 \cap U_1. \tag{4.11}$$

Then, the metric is given by

$$||s|| = |s_i|e^{-g_i}, \text{ on } U_i.$$

The relation 4.11 guarantees that the axioms in Definition 4.2.1 are satisfied. We can further consider pull-backs of this metric by a morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$. We define the pullback metric on $\phi^*(\mathscr{L})$ by $||\phi^* s|| = ||s|| \circ \phi$, for a section $s \in \mathscr{L}(U)$.

Example 4.2.3 (Fubini-Study metric)**.** Let $|\cdot|$ be an archimedean absolute value on $\mathbb{Q}$, let $X = \mathbb{P}^1$ and $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1}(1)$ and use the notation from Example 4.2.2. If we let

$$g_0 = \log\sqrt{1 + |z_1/z_0|^2}$$
$$g_1 = \log\sqrt{1 + |z_0/z_1|^2},$$

then the corresponding metric is known as the *Fubini-Study metric*. It is denoted by $||\cdot||_{\text{FS}}$. This a continuous metric; in fact it is a smooth metric.

Alternatively, using the fact that $\mathscr{L}$ is generated by its global sections, we may define the Fubini-Study metric as follows. If s_P is a global section of $\mathscr{L}$ represented by the linear form $P = a_0 z_0 + a_1 z_1$, then for any $x = (x_0 : x_1)$, we have

$$||s_P||_{\text{FS}}(x) = \frac{|a_0 x_0 + a_1 x_1|}{\sqrt{|x_0|^2 + |x_1|^2}}.$$

Example 4.2.4 (Weil metric)**.** Let $K = \mathbb{Q}$, $X = \mathbb{P}^1$, $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1}(1)$ and let $v \in M_{\mathbb{Q}}$. We use the notation as in Example 4.2.2. The v-adic Weil metric, denoted by $||\cdot||_{v,W}$, is defined by choosing $g_{1,v} = \log\max\left\{1, \left|\frac{z_0}{z_1}\right|_v\right\}$ (so that $g_{0,v} = \log\max\left\{1, \left|\frac{z_1}{z_0}\right|_v\right\}$).

We then have

$$-\log ||\varepsilon_0||_{\mathrm{W,v}} = g_0$$
$$-\log ||\varepsilon_1||_{\mathrm{W,v}} = g_1.$$

Both g_i are continuous on the corresponding U_i when v is an archimedean place. Further, they extend to continuous functions on $U_{i,v}^{\mathrm{an}}$ when v is non-archimedean, by setting $g_0 = \log\max\left\{1, [T]_{z_0/z_1}\right\}$ and $g_1 = \log\max\left\{1, [T]_{z_1/z_0}\right\}$. Here, $[\cdot]_{z_0/z_1}$ and $[\cdot]_{z_1/z_0}$ are the seminorms on $\mathbb{C}_v[T]$, which correspond to the points z_0/z_1 and z_1/z_0 respectively.

Following [Th, Definition 3.4.17], we can now define the notion of a semipositive metric.

Definition 4.2.5. A continuous metric $||\cdot||_v$ on $\mathscr{L}$ is **semipositive**, if for any local frame $\varepsilon_U \in \mathscr{L}(U)$ on an open $U \subset X$, the function

$$-\log ||\varepsilon_U||_v$$

is the restriction on U of a subharmonic function on the Berkovich analytification U_v^{an}.

Example 4.2.6. It is easy to see that the Fubini-Study metric and the Weil metric are both semipositive at each archimedean place. The Weil metric is semipositive at the non-archimedean places as well; refer to [BR2, Proposition 8.26 (D)].

A priori, the notion of semipositivity from Definition 4.2.5 does not coincide with the one of [Zh5] or [CL2]. There the semipositivity of a continuously metrized line bundle is defined as a uniform limit of "smooth semipositive" metrics at each non-archimedean place. However, Thuillier [Th, Theorem 4.3.3] has established that the subharmonicity of the potentials as in Definition 4.2.5 and [Th, Definition 3.4.17] is enough to conclude semipositivity in Zhang's sense [Zh5]. One may refer to [FG2, Lemma 3.11, Theorem 3.12] for an application of this argument in a dynamical context.

Remark 4.2.7. We point out here that a result analogous to [Th, Theorem 4.3.3] also holds for the Berkovich analytification of projective varieties X of arbitrary dimension defined over a complete discrete valuation field with residue characteristic zero; see [BFJ, Theorem B]. The authors in [BFJ] define the notion of a singular semipositive metric on an ample line bundle on X analogous to [Th, Definition 3.4.17] and show that it can be approximated by increasing nets of subharmonic "model" metrics.

We now introduce the notion of a continuous semipositive adelic metric on an ample line bundle $\mathscr{L}$ of X.

Definition 4.2.8. Let $\mathscr{L} \to X$ be an ample line bundle on X. A **continuous, semipositive, adelic metric** $(\mathscr{L}, \{||\cdot||_v\}_{v\in M_K})$ on $\mathscr{L}$ is a family of continuous semipositive metrics $||\cdot||_v$, one for each place $v \in M_K$, that is *coherent* in the sense of [Yu, Section 1.2], [CL2, Section 3.1.1]. That is, there is an integer $e \geq 1$ and an integral model $(\mathfrak{X}, \mathfrak{L})$ of $(X, \mathscr{L}^e)$ that induces the given metric $||\cdot||_v$ at all but finitely many places in the following sense. If $\mathfrak{U}$ is an open subset of $\mathfrak{X}$ over which $\mathfrak{L}$ admits a local frame $\varepsilon_{\mathfrak{U}}$, then for a section s of $\mathscr{L}$ over the generic fiber U of $\mathfrak{U}$ we may write $s^e = f\varepsilon_{\mathfrak{U}}$ where $f \in \mathscr{O}_X(U)$. We set $||s||_v = |f|_v^{1/e}$.

We denote a continuous, semipositive, adelic metric on an ample line bundle $\mathscr{L}$ by

$$\overline{\mathscr{L}} := (\mathscr{L}, \{||\cdot||_v\}_{v\in M_K}).$$

Example 4.2.9. We can equip $\mathscr{O}_{\mathbb{P}^1}(1)$ with a continuous, semipositive, adelic metric as

$$\overline{\mathscr{O}_{\mathbb{P}^1}(1)} = (\mathscr{O}_{\mathbb{P}^1}(1), \{||\cdot||_{W,v}\}_{v\in M_{\mathbb{Q}}});$$

see [CL2, §1.3.2]. Here, $\|\cdot\|_{W,v}$ is the Weil metric from Example 4.2.4.

A continuous, semipositive, adelic metric $\overline{\mathscr{L}}$ on an ample line bundle $\mathscr{L}$, induces a height function $h_{\overline{\mathscr{L}}}(\cdot)$, which associates a number $h_{\overline{\mathscr{L}}}(Y)$ to each subvariety Y of X; see [Zh5] or [CL2]. The height $h_{\overline{\mathscr{L}}}(X)$ of the curve X is also denoted as $\frac{(\hat{c}_1(\overline{\mathscr{L}}^2)|X)}{2(c_1(\mathscr{L})|X)}$ in [CL2]. In the case that Y is a point $x \in X(\overline{K})$, its height is given by

$$h_{\overline{\mathscr{L}}}(x) := \frac{1}{|\mathrm{Gal}(\overline{K}/K)\cdot x|} \sum_{y\in \mathrm{Gal}(\overline{K}/K)\cdot x} \sum_{v\in M_K} -n_v \log \|\phi(y)\|_v,$$

where ϕ is any global section of $\mathscr{L}$ which is non-vanishing along the Galois orbit of x.

Example 4.2.10. We have

$$h_{\overline{\mathscr{O}_{\mathbb{P}^1}(1)}}([w_0 : w_1]) = h([w_0 : w_1]),$$

where $h : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ denotes the logarithmic Weil height. Indeed, assume that $w_0 \neq 0$ and let $w = \frac{w_1}{w_0}$. Assume that $|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot w| = \ell$ and denote by $w^{(j)}$ for $j = 1, \ldots, \ell$ the ℓ distinct Galois conjugates of w. The section $\varepsilon_0 \in \mathscr{O}_{\mathbb{P}^1}(1)(U_0)$ does not vanish on the $w^{(i)}$'s and we have

$$\begin{aligned} h_{\overline{\mathscr{O}_{\mathbb{P}^1}(1)}}([w_0 : w_1]) &= \frac{1}{\ell} \sum_{j=1}^{\ell} \sum_{v \in M_{\mathbb{Q}}} -\log \max\{1, |w^{(j)}|_v\} \\ &= h([w_0 : w_1]). \end{aligned}$$

If on the other hand $w_0 = 0$, then to compute the height of $[0 : 1]$ we may choose the section $\varepsilon_1 \in \mathscr{O}_{\mathbb{P}^1}(1)(U_1)$. We get

$$h_{\overline{\mathscr{O}_{\mathbb{P}^1}(1)}}([0 : 1]) = \sum_{v \in M_{\mathbb{Q}}} -\log \max\{1, 0\} = 0.$$

For the purpose of proving Theorem 1.0.4, we refrain from using the definition of $h_{\overline{\mathscr{L}}}(X)$. To compute this number, we instead use Zhang's inequalities of successive minima [Zh2, Zh5]. Zhang defines the *successive minima* as

$$\begin{aligned} e_1(\overline{\mathscr{L}}) &= \sup_{s \in X(\overline{K})} \inf_{t \in X(\overline{K}) \setminus \{s\}} h_{\overline{\mathscr{L}}}(t) \\ e_2(\overline{\mathscr{L}}) &= \inf_{t \in X(\overline{K})} h_{\overline{\mathscr{L}}}(t). \end{aligned}$$

He proves the following.

Theorem 4.2.11. *[Zh5, Theorem 1.10] Let $\overline{\mathscr{L}}$ be a continuous, semipositive, adelic metrized ample line bundle. Then*

$$\frac{e_1(\overline{\mathscr{L}}) + e_2(\overline{\mathscr{L}})}{2} \leq h_{\overline{\mathscr{L}}}(X) \leq e_1(\overline{\mathscr{L}}).$$

Example 4.2.12. From Example 4.2.10, we have

$$h_{\overline{\mathscr{O}_{\mathbb{P}^1}(1)}}(z) \geq 0,$$

for all $z \in \mathbb{P}^1(\overline{\mathbb{Q}})$. Moreover, for any root of unity ξ we have

$$h_{\overline{\mathscr{O}_{\mathbb{P}^1}(1)}}([1:\xi]) = h([1:\xi]) = 0.$$

Therefore,

$$e_1(\overline{\mathscr{O}_{\mathbb{P}^1}(1)}) = e_2(\overline{\mathscr{O}_{\mathbb{P}^1}(1)}) = 0.$$

Hence, by Theorem 4.2.11 we get $h_{\overline{\mathscr{O}_{\mathbb{P}^1}(1)}}(\mathbb{P}^1) = 0$.

If the equality $h_{\overline{\mathscr{L}}}(X) = e_1(\overline{\mathscr{L}})$ is attained in Zhang's inequality, then we can find an infinite sequence $\{x_n\} \subset X(\overline{K})$ such that $h_{\overline{\mathscr{L}}}(x_n) \to h_{\overline{\mathscr{L}}}(X)$ as $n \to \infty$. Such a sequence is referred to as an $h_{\overline{\mathscr{L}}}$–small sequence. We are now ready to state the equidistribution theorem for small points.

Theorem 4.2.13. *[CL1], [Th], [Yu] Let X be a projective curve over K and let $\overline{\mathscr{L}}$ be an ample, continuous, semipositive, adelic metrized line bundle on X such that $h_{\overline{\mathscr{L}}}(X) = e_1(\overline{\mathscr{L}})$. Let $\{x_n\} \subset X(\overline{K})$ be an infinite sequence such that $h_{\overline{\mathscr{L}}}(x_n) \to h_{\overline{\mathscr{L}}}(X)$ as $n \to \infty$. For each $v \in M_K$, let δ_{x_n} be the of probability measure on X_v^{an} weighted equally on the points in the* $\mathrm{Gal}(\overline{K}/K)$*−orbit of x_n. Then δ_{x_n} converges weakly to the probability measure $d\mu_v := \frac{c_1(\overline{\mathscr{L}})_v}{(c_1(\overline{\mathscr{L}})|X)}$ on X_v^{an}. In other words,*

$$\lim_{n\to\infty} \frac{1}{|\mathrm{Gal}(\overline{K}/K)\cdot x_n|} \sum_{y\in \mathrm{Gal}(\overline{K}/K)\cdot x_n} \delta_{x_n} \to \mu_v.$$

Here, when v is archimedean the curvature $c_1(\overline{\mathscr{L}})_v$ is defined locally as the Laplacian of $-\log||\varepsilon_U||_v$ for a local frame ε_U on U. For the non-archimedean places, the curvature $c_1(\overline{\mathscr{L}})_v$ is defined by Chambert-Loir [CL1].

4.3 A dynamical perspective

The Néron-Tate height $\hat{h}_E$ and its local counterparts $\hat{\lambda}_{E,v}$ can be defined by means of the dynamics of the projective line $\mathbb{P}^1$. Letting E be an elliptic curve defined over a number field K, the multiplication-by-2 endomorphism ϕ on E descends to a rational function of degree 4 on $\mathbb{P}^1$, via the standard quotient identifying a point P with its additive inverse:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "f_\phi"] & \mathbb{P}^1 \end{tikzcd} \tag{4.12}$$

Recall the elementary, but key, observation that a point is torsion on E if and only if its quotient in $\mathbb{P}^1$ is preperiodic for f_ϕ. The height $\hat{h}_E$ on $E(\overline{K})$ satisfies

$$\hat{h}_E(P) = \lim_{n\to\infty} \frac{1}{4^n} h(f_\phi^n(\pi P)),$$

where h is the standard logarithmic Weil height on $\mathbb{P}^1(\overline{K})$. Now let $E \to B$ be an elliptic surface defined over a number field K, and let $P : B \to E$ be a section, also defined over K. In this Section, we use this perspective to give a proof of subharmonicity of the local height functions $t \mapsto \hat{\lambda}_{E_t,v}(P_t)$ and the extensions $V_{P,t_0,v}$ of (4.10). We will present this fact as an immediate consequence of now-standard complex-dynamical convergence arguments, at least when the fiber E_t is smooth and the local height $\hat{\lambda}_{E_t,v}(P_t)$ is finite. Near singular fibers, we utilize the maximum principle and standard results on removable singularities for subharmonic functions. The same reasoning applies in both archimedean and non-archimedean settings.

In §4.3.3 we provide the background to justify the explicit description of the limiting distribution $\mu_{P,v}$ at the archimedean places v of K, as mentioned in Remark 1.0.6.

4.3.1 Canonical height and escape rates

As in §4.1.2, we let E be an elliptic curve defined over a product formula field $\mathscr{F}$ of characteristic 0 and given in Weierstrass form as

$$E \ : \ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

We denote its discriminant by Δ. We define a rational function $f = \phi/\psi$ on $\mathbb{P}^1$ by the formula

$$f(x(P)) = x([2]P),$$

for all $P \in E$. Here, $x(P)$ is given by the x–coordinate for a point $P \in E$. That is, for $P = (x_P, y_P) \in E \setminus \{O\}$ we have $x(P) = (x_P : 1)$. This function x plays the role of π in (4.12). In affine coordinates, for $P = (x,y) \in E$ we have

$$\begin{aligned}
&\phi(x) = x^4 - b_4x^2 - 2b_6x - b_8 \text{ and} \\
&\psi(x) = 4x^3 + b_2x^2 + 2b_4x + b_6 = (2y + a_1x + a_3)^2.
\end{aligned}$$

By a *lift* of f, we mean any homogeneous polynomial map F on $\mathbb{A}^2$, defined over $\mathscr{F}$, so that $\tau \circ \mathrm{F} = f \circ \tau$, where $\tau : \mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$ is the tautological projection. A *lift* of a point $x \in \mathbb{P}^1$ is a choice of $X \in \mathbb{A}^2 \setminus \{(0,0)\}$ so that $\tau(X) = x$. We let the *standard lift* of f to be the map $F : \mathbb{A}^2 \to \mathbb{A}^2$ defined by

$$F(z,w) = \left(w^4\phi(z/w), w^4\psi(z/w)\right), \tag{4.13}$$

where the coefficient of z^4 is equal to 1. Notice that F is given by two homogeneous polynomials of degree 4. We say that F has degree 4. Moreover, by the homogeneity of F any other lift of f is of the form cF for some $c \in \mathscr{F}^*$.

For each $v \in \mathscr{M}_{\mathscr{F}}$, we let

$$\|(z,w)\|_v = \max\{|z|_v, |w|_v\}.$$

For any lift F of f we define the Green function of F, $\mathscr{G}_{\mathrm{F},v} : \mathbb{C}_v^2 \setminus \{(0,0)\} \to \mathbb{R}$, by

$$\mathscr{G}_{\mathrm{F},v}(z,w) = \lim_{n\to\infty} \frac{\log ||\mathrm{F}^n(z,w)||_v}{4^n}.$$

It is not hard to prove that this limit actually exists; see [Si6, Proposition 5.8 (a)]. When F is the standard lift of f, we call $\mathscr{G}_{F,v}$ the v–adic *escape rate*. We refer the reader to [Si6, §5.9] for the definition of the v–adic escape rate for arbitrary rational maps as well as an account of its properties. We summarize some important ones in the following proposition.

Proposition 4.3.1. *For any lift F of f, and for each place v of $\mathscr{F}$, the following are satisfied.*

1. $\mathscr{G}_{\mathrm{F},v}(\mathrm{F}(z,w)) = 4\mathscr{G}_{\mathrm{F},v}(z,w)$ *for all* $(z,w) \in \mathbb{C}_v^2 \setminus \{(0,0)\}$.

2. $\mathscr{G}_{\mathrm{F},v}(z,w) = \log \|(z,w)\|_v + O(1)$ *for all* $(z,w) \in \mathbb{C}_v^2 \setminus \{(0,0)\}$, *where the implicit constant in the* $O(1)$ *term depends only on the coefficients of F and the place* $v \in M_{\mathscr{F}}$.

3. *For all* $(z,w) \in \mathbb{C}_v^2 \setminus \{(0,0)\}$ *and* $c, \alpha \in \mathbb{C}_v^*$ *the following homogeneity properties hold.*
$$\mathscr{G}_{c\mathrm{F},v}(z,w) = \mathscr{G}_{\mathrm{F},v}(z,w) + \frac{1}{3}\log|c|_v,$$
$$\mathscr{G}_{\mathrm{F},v}(\alpha z, \alpha w) = \mathscr{G}_{\mathrm{F},v}(z,w) + \log|\alpha|_v.$$
In particular, the function
$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{R} \\ (z:w) &\mapsto \mathscr{G}_{\mathrm{F},v}(z,w) - \log\|(z,w)\|_v \end{aligned}$$
is well-defined. That is, its value does not depend on the choice of homogeneous coordinates for $(z:w)$.

4. *The function* $\mathscr{G}_{\mathrm{F},v} : \mathbb{C}_v^2 \setminus \{(0,0)\} \to \mathbb{R}$ *is continuous.*

Proof. This proposition essentially follows from [Si6, Proposition 5.8]. More precisely, the statements in 1 and 2 follow from [Si6, Proposition 5.8 (b)] and the statement in 3 follows from [Si6, Proposition 5.8 (d)]. We provide a proof for the latter here to demonstrate the techniques. To this end, notice that if we set $\mathrm{F}_{(c)} = c\mathrm{F}$, an easy induction argument using the homogeneity of F yields
$$\mathrm{F}_{(c)}^n(z,w) = c^{1+4+\cdots+4^{n-1}}\mathrm{F}(z,w) = c^{\frac{4^n-1}{3}}\mathrm{F}(z,w).$$

Thus,

$$\begin{aligned}
\mathscr{G}_{c\mathrm{F},v}(z,w) &= \lim_{n\to\infty} \frac{\log \|\mathrm{F}^n_{(c)}(z,w)\|_v}{4^n} \\
&= \lim_{n\to\infty} \frac{\log \|\mathrm{F}^n(z,w)\|_v + \frac{4^n-1}{3}\log|c|_v}{4^n} \\
&= \mathscr{G}_{\mathrm{F},v}(z,w) + \frac{1}{3}\log|c|_v.
\end{aligned}$$

Further, since F^n is homogeneous of degree 4^n we have

$$\mathrm{F}^n(\alpha z, \alpha w) = \alpha^{4^n}\mathrm{F}^n(z,w).$$

Hence,

$$\begin{aligned}
\mathscr{G}_{\mathrm{F},v}(\alpha z,\alpha w) &= \lim_{n\to\infty} \frac{\log \|\mathrm{F}^n(\alpha z,\alpha w)\|_v}{4^n} \\
&= \lim_{n\to\infty} \frac{\log \|\mathrm{F}^n(z,w)\|_v + 4^n\log|\alpha|_v}{4^n} \\
&= \mathscr{G}_{\mathrm{F},v}(z,w) + \log|\alpha|_v.
\end{aligned}$$

In particular, the difference $\mathscr{G}_{F,v}(z,w) - \log\|(z,w)\|_v$ gives a well-defined function on $\mathbb{P}^1$. The fact that $\mathscr{G}_{\mathrm{F},v}$ is continuous on $\mathbb{C}_v^2 \setminus \{(0,0)\}$ as in part 4 follows from [Si6, Proposition 5.58 (e)] ; see also [HP, FS] for a proof in the archimedean case.

$\square$

Remark 4.3.2. Recall that $\mathbb{C}_v^2 \setminus \{(0,0)\}$ is a dense subset of the product of Berkovich affine lines $\mathbb{A}_v^{1,an} \times \mathbb{A}_v^{1,an} \setminus \{(0,0)\}$. We point out here that the function $\mathscr{G}_{\mathrm{F},v}$, which is continuous on $\mathbb{C}_v^2 \setminus \{(0,0)\}$, extends to a continuous function on $\mathbb{A}_v^{1,an} \times \mathbb{A}_v^{1,an} \setminus \{(0,0)\}$; see [BR2, Chapter 10].

Proposition 4.3.3. *For the standard lift F of f, and for each place v of $\mathscr{F}$, the local canonical height function satisfies*

$$\hat{\lambda}_{E,v}(P) = \frac{1}{2}\mathscr{G}_{F,v}(z,w) - \frac{1}{2}\log|w|_v - \frac{1}{12}\log|\Delta|_v,$$

where $x(P) = (z:w)$.

Proof. Define the function $G_v : E(\mathbb{C}_v) \setminus \{O\} \to \mathbb{R}$ as follows. For $P = (x_P, y_P) \in E(\mathbb{C}_v) \setminus \{O\}$, write $x(P) = (z : w)$ and let

$$G_v(P) = \frac{1}{2}\mathscr{G}_{F,v}(z,w) - \frac{1}{2}\log|w|_v - \frac{1}{12}\log|\Delta|_v.$$

Recall from part 3 of Proposition 4.3.1 that the number $G_v(P)$ does not depend on the choice of homogeneous coordinates for $x(P)$. We will show that G_v satisfies the three characterizing conditions for $\hat{\lambda}_{E,v}$ from Proposition 4.1.2 that uniquely define it. More specifically, we will show that the following hold.

(a) G_v is continuous on $E(\mathbb{C}_v) \setminus \{O\}$ and bounded on the complement of any v-adic neighborhood of O;

(b) The limit of $G_v(P) - \frac{1}{2}\log|x_P|_v$ exists as $P \xrightarrow{v} O$ in $E(\mathbb{C}_v)$.

(c) For all $P \in E$ with $[2]P \neq O$ we have

$$G_v([2]P) = 4G_v(P) - \frac{1}{2}\log|\psi(z/w)|_v + \frac{1}{4}\log|\Delta|_v.$$

By part 4 of Proposition 4.3.1, we know that $\mathscr{G}_{F,v}(z,w)$ is continuous for $(z,w) \in \mathbb{C}_v^2 \setminus \{(0,0)\}$. Moreover, $\log|w|_v$ is continuous when $w \neq 0$. Hence, the difference $\mathscr{G}_{F,v}(z,w) - \log|w|_v = \mathscr{G}_{F,v}(z/w,1)$ descends to a continuous function on $\mathbb{P}^1(\mathbb{C}_v) \setminus \{[1:0]\}$. Moreover, by part 2 of Proposition 4.3.1 it is easy to see that $\mathscr{G}_{F,v}(z/w,1)$ is bounded when w/z is bounded away from 0. Hence, (a) follows. For (b), notice that as $P \xrightarrow{v} O$ we have $|x_P|_v \to \infty$. Hence, we may assume $|x_P|_v \geq 1$ and get

$$G_v(P) - \frac{1}{2}\log|x_P|_v = \frac{1}{2}(\mathscr{G}_{F,v}(x_P,1) - \log\max\{|x_P|_v,1\}) - \frac{1}{12}\log|\Delta|_v.$$

This difference is bounded and continuous by parts 2 and 4 of Proposition 4.3.1. Hence, (b) follows. For (c), let $P \in E(\mathbb{C}_v) \setminus \{O\}$ be such that $[2]P \neq O$ and recall that

$$x([2]P) = \tau \circ F(z,w) = \left(w^4\phi(z/w) : w^4\psi(z/w)\right).$$

We have

$$G_v([2]P) = \frac{1}{2}\mathscr{G}_{F,v}(F(z,w)) - \frac{1}{2}\log|w^4\psi(z/w)|_v - \frac{1}{12}\log|\Delta|_v.$$

By part 1. of Proposition 4.3.1 this yields that

$$G_v([2]P) = 2\mathscr{G}_{F,v}(z,w) - \frac{1}{2}\log|w^4\psi(z/w)|_v + \frac{1}{4}\log|\Delta|_v.$$

The proof follows. □

4.3.2 Variation of canonical height: subharmonicity

Let K be a number field and $E \to B$ an elliptic surface defined over K with zero section $O: B \to E$. Let $k = \overline{K}(B)$; viewing E as an elliptic curve defined over k, we also fix a point $P \in E(k)$. Recall the function $V_{P,t_0,v}(t)$ defined in (4.10).

Theorem 4.3.4. *For every $t_0 \in B(\overline{K})$ and uniformizer u in k at t_0, the function*

$$V_{P,t_0,v}(t) := \hat{\lambda}_{E_t,v}(P_t) + \hat{\lambda}_{E,\mathrm{ord}_{t_0}}(P)\log|u(t)|_v$$

extends to a continuous and subharmonic function on a neighborhood of t_0 in the Berkovich analytification B_v^{an}.

The continuity was already established in Theorem 4.1.5, though it was not explicitly stated for the Berkovich space. We begin with a lemma.

Lemma 4.3.5. *Fix $\alpha \in k^*$ and $t_0 \in B(\overline{K})$. Let $u \in k$ be a uniformizer at t_0. For each place v of K, the function*

$$t \mapsto \log|\alpha_t|_v - (\mathrm{ord}_{t_0}\alpha)\log|u(t)|_v$$

is harmonic in a neighborhood of t_0 in the Berkovich analytification B_v^{an}.

Proof. This is Silverman's [Si4, Lemma II.1.1(c)] combined with a removable singularities lemma for harmonic functions. See also [BR2, Proposition 7.19] for the extension of a harmonic function to a disk in the Berkovich space B_v^{an}. □

Fix $P \in E(k)$. Let F and X be lifts of f and $x(P)$ to k^2, respectively. Iterating F, we set

$$(A_n, B_n) := F^n(X) \in k^2$$

and observe that from the definition of the escape rate, we have

$$\mathscr{G}_{F,\mathrm{ord}_{t_0}}(X) = -\lim_{n\to\infty} \frac{\min\{\mathrm{ord}_{t_0} A_n, \mathrm{ord}_{t_0} B_n\}}{4^n}. \tag{4.14}$$

We let F_t and X_t denote the specializations of F and X at a point $t \in B(\overline{K})$; they are well defined for all but finitely many t. Observe that if F is the standard lift for E then so is F_t for all t.

Proposition 4.3.6. *Fix $P \in E(k)$, $t_0 \in B(\overline{K})$, and $v \in M_K$. For any choice of lifts F of f and X of $x(P)$, the function*

$$G_P(t;v) := \mathscr{G}_{F_t,v}(X_t) + \mathscr{G}_{F,t_0}(X)\log|u(t)|_v$$

extends to a continuous and subharmonic function in a neighborhood of t_0 in B_v^{an}.

Proof. First observe that the conclusion does not depend on the choices of F and X. Indeed for any $c, \alpha \in k^*$, we have

$$\begin{aligned} \mathscr{G}_{c_tF_t,v}(\alpha_t X_t) + \mathscr{G}_{cF,t_0}(\alpha X)\log|u(t)|_v \quad = \quad & \mathscr{G}_{F_t,v}(X_t) + \mathscr{G}_{F,t_0}(X)\log|u(t)|_v \\ & + \frac{1}{3}\left(\log|c_t|_v - (\mathrm{ord}_{t_0} c)\log|u(t)|_v\right) \\ & + \log|\alpha_t|_v - (\mathrm{ord}_{t_0}\alpha)\log|u(t)|_v. \end{aligned}$$

So by Lemma 4.3.5 the function $G_P(t;v)$ is continuous and subharmonic for one choice if and only if it is continuous and subharmonic for all choices.

Let F be the standard lift of f. Suppose that $P = O$. Since $F(1,0) = (1,0)$, we compute that

$$G_O(t;v) = \mathscr{G}_{F_t,v}(1,0) + \mathscr{G}_{F,t_0}(1,0)\log|u(t)|_v \equiv 0.$$

Now suppose that $P \neq O$. Fix $t_0 \in B(\overline{K})$ and local uniformizer u at t_0. Choose a lift F of f so that the coefficients of F have no poles at t_0, with $F_{t_0} \neq (0,0)$. Choose lift X of $x(P)$ so that X_t is well defined for all t near t_0 and $X_{t_0} \neq (0,0)$. As above, we write

$$F^n(X) = (A_n, B_n),$$

and put

$$a_n = \min\{\mathrm{ord}_{t_0} A_n, \mathrm{ord}_{t_0} B_n\},$$

so that $a_n \geq 0$ for all $n \in \mathbb{N}$ and $a_0 = 0$. Set

$$F_n(t) = F_t^n(X_t)/u(t)^{a_n}.$$

For each place v, we set

$$h_{n,v}(t) = \frac{\log \|F_n(t)\|_v}{4^n}.$$

By construction, the limit of $h_{n,v}$ (for t near t_0 with $t \neq t_0$) is exactly the function G_P for these choices. In fact, for t in a small neighborhood of t_0, but with $t \neq t_0$, the function f_t on $\mathbb{P}^1$ is a well-defined rational function of degree 4; so the specialization of the homogeneous polynomial map F_t satisfies $F_t^{-1}\{(0,0)\} = \{(0,0)\}$. Furthermore, as the coefficients of F_t depend analytically on t, the functions $h_{n,v}$ converge locally uniformly to the function G_P away from $t = t_0$. This can be seen with a standard telescoping sum argument, used often in complex dynamics, as in [BH, Proposition 1.2]. In particular, G_P is continuous on a punctured neighborhood of t_0.

At the archimedean places v, and for each $n \in \mathbb{N}$, the function $h_{n,v}$ is clearly continuous and subharmonic in a neighborhood of t_0. At the non-archimedean places v, this definition extends to a Berkovich disk around t_0, setting

$$h_{n,v}(t) = \frac{1}{4^n} \max\{\log[A_n(T)/T^{a_n}]_t, \log[B_n(T)/T^{a_n}]_t\},$$

where $[\cdot]_t$ is the seminorm on $K[[T]]$ associated to the point t. Each of these functions $h_{n,v}$ is continuous and subharmonic for t in a Berkovich disk around t_0; compare [BR2] Example 8.7, Proposition 8.26(D), and equation (10.9).

Lemma 4.3.7. *For all v, and by shrinking the radius r if necessary, the functions $h_{n,v}$ are uniformly bounded from above on the (Berkovich) disk D_r.*

Proof. As observed above, the functions $h_{n,v}$ converge locally uniformly away from $t = t_0$ to the continuous function $G_P(t)$. Choose a small radius r, and let

$$M_v = \sup_n \max_{|t|_v = r} h_{n,v}(t),$$

which is finite by the convergence. Because the functions are subharmonic, the Maximum Principle implies that $h_{n,v}(t) \leq M_v$ throughout the disk of radius r, for all n. For the non-archimedean places, there is also a Maximum Principle on the Berkovich disk, where the role of the circle of radius r is played by the Type II point associated to the disk of radius r; see [BR2, Proposition 8.14]. □

We can now complete the proof of Proposition 4.3.6. As each $h_{n,v}$ is subharmonic, and the functions are uniformly bounded from above on the disk by Lemma 4.3.7, we know that the (upper-semicontinuous regularlization of the) limsup of these functions is subharmonic. See [BR2] Proposition 8.26(F) for a proof in the non-archimedean case. □

Proof of Theorem 4.3.4. Subharmonicity now follows combining Proposition 4.3.3, Lemma 4.3.5 and Proposition 4.3.6. The continuity at each archimedean place is the content of Theorem 4.1.5. The continuity at each non-archimedean place is a combination of the continuity on the punctured Berkovich disk (as in the proof of Proposition 4.3.6) and the continuity on Type I (classical) points given in Theorem 4.1.5. □

4.3.3 The measures on the base

Here we provide more details about the description of the measures appearing in the statement of Corollary 1.0.5, as discussed in Remark 1.0.6.

Fix an archimedean place v and a point $t_0 \in B(\overline{K})$. Choosing a uniformizer u at t_0, recall the definition of $V_{P,t_0,v}$ from (4.10). We define

$$\mu_{P,v} := dd^c V_{P,t_0,v}(t)$$

on a neighborhood of t_0 in B_v^{an}; note that this is indepedent of the choice of u. Note that $\mu_{P,v}$ can be expressed as

$$\mu_{P,v} = dd^c \hat{\lambda}_{E_t,v}(P_t),$$

for t outside of the finitely many points in the support of the divisor $D_E(P)$ or where the fiber E_t is singular. Note, further, that $\mu_{P,v}$ assigns no mass to any individual point t_0, because the potentials are bounded by Theorem 4.3.4. The details on the metric and the equidistribution theorem in §4.4 will show that these are exactly the measures that arise as the distribution of the points of small height in Corollary 1.0.5.

It is well known that the local height function on a smooth elliptic curve is a potential for the Haar measure. That is, for fixed t we have

$$dd^c \hat{\lambda}_{E_t,v}(\cdot) = \omega_t - \delta_o,$$

where ω_t is the normalized Haar measure on E_t and δ_o is a delta-mass supported at the origin of E_t; see, e.g., [La3, Theorem II.5.1]. We present an alternative proof of this fact related to dynamics as part of Proposition 4.3.8, as a consequence of Proposition 4.3.3.

Proposition 4.3.8. *Let $E \to B$ be an elliptic surface and $P : B \to E$ a section, both defined over a number field K. Let $\mathscr{S} \subset E$ be the union of the finitely many singular fibers in E. For each archimedean place v of K, there is a positive, closed (1,1) current T_v on $E \setminus \mathscr{S}$ with locally continuous potentials so that $T_v|_{E_t}$ is the Haar measure on each smooth fiber, and P^*T_v is equal to the measure $\mu_{P,v}$.*

Remark 4.3.9. As T_v has continuous potentials, the restriction $T_v|_{E_t}$ and the pull-back P^*T_v are well defined. That is, we have $T_v|_{E_t} = dd^c(u|_{E_t})$ where u is a locally defined potential of T_v, and $P^*T_v = dd^c(u \circ P)$ locally on B. The measure $\mu_{P,v}$ has no atoms, so it is determined by T_v along the image of P in $E \setminus \mathscr{S}$.

Proof of Proposition 4.3.8. Let us fix any small neighborhood U in the base curve $B(\mathbb{C})$ so that all fibers E_t are smooth for $t \in U$. Let f_t be the map on $\mathbb{P}^1$ defined in §4.3.1; by shrinking U if necessary, we can find lifts F_t of f_t that are holomorphic in $t \in U$. From [HP, FS] (or the proof of [BH, Proposition 1.2]), we know that the escape rate

$$\mathscr{G}_{F_t,v}(z,w) = \lim_{n\to\infty} \frac{\log ||F_t^n(z,w)||_v}{4^n}$$

is continuous and plurisubharmonic as a function of $(t,z,w) \in U \times (\mathbb{C}^2 \setminus \{(0,0)\})$. Therefore,

$$dd^c \mathscr{G}_{F_t,v}(z,w)$$

projects to a closed and positive (1,1)-current G_v on the complex surface $U \times \mathbb{P}^1$, with locally continuous potentials. This current G_v has the property that, restricted to each fiber $\mathbb{P}^1$, its total mass is 1; and the measure on the fiber is the measure of maximal entropy for the rational map f_t [Ly, HP].

The restriction $E|_U$ of the elliptic surface E over U maps with degree 2 to the complex surface $U \times \mathbb{P}^1$ by the projection π of (4.12). The current G_v can be pulled back to E as $\frac{1}{2} dd^c(g \circ \pi)$, where g is a locally-defined continuous potential for G_v. Covering the base of $E \setminus \mathscr{S}$ by sets of the form U, the local definitions glue to form the closed, positive $(1,1)$-current T_v on $E \setminus \mathscr{S}$.

If $P : B \to E$ is a section defined over the number field K, then P^*T_v has potential given locally by

$$\frac{1}{2}\, g \circ \pi \circ P = \frac{1}{2}\, \mathscr{G}_{F_t,v}(X_t),$$

for any lift X_t of $\pi(P_t) \in \mathbb{P}^1$. Proposition 4.3.3 yields that P^*T_v must coincide with $\mu_{P,v}$.

Finally, to conclude that $T_v|_{E_t}$ is equal to the normalized Haar measure ω_t, we may use the well-known dynamical fact that for each fixed t in the base, the measure ω_t projects by π to $\mathbb{P}^1$ to the unique measure of maximal entropy for the map f_t; see, e.g., [Mi, §7]. $\square$

4.4 The adelic metric and equidistribution

In this Section we give the proofs of Theorem 1.0.4 and Corollary 1.0.5. We first outline the proofs. Let $E \to B$ be an elliptic surface defined over a number field K with zero section $O : B \to E$, and let $P : B \to E$ be a section also defined over K so that $\hat{h}_E(P) \neq 0$. Recall from §4.1.3 that we introduced a $\mathbb{Q}$-divisor on B given by

$$D_E(P) = \sum_{\gamma \in B(\overline{K})} \hat{\lambda}_{E,\mathrm{ord}_\gamma}(P) \cdot (\gamma).$$

By enlarging K, we may assume that $\operatorname{supp} D_E(P)$ lies in $B(K)$. We will define an adelic metric on the ample line bundle $\mathscr{L}$ associated to the divisor $D_E(P)$, inducing a height function $h_{\mathscr{L}}$ such that

$$h_{\mathscr{L}}(t) = \hat{h}_{E_t}(P_t) \text{ for all but finitely many } t \in B(\overline{K}),$$

and

$$h_{\mathscr{L}}(t) \geq 0 \text{ for all } t \in B(\overline{K}).$$

Applying Silverman's results on the variation of the canonical height, Theorems 4.1.4 and 4.1.5, we will deduce that the metric is continuous and adelic. From Theorem 4.3.4, we will conclude that the metric is also semipositive in the sense of Zhang [Zh2]. We will use Zhang's inequalities [Zh5] to deduce that

$$h_{\mathscr{L}}(B) = 0.$$

Consequently, we will be able to apply the equidistribution results of Chambert-Loir, Thuillier, and Yuan [CL1, Th, Yu] to complete our proofs.

4.4.1 The metric and its properties

Let $m \in \mathbb{N}$ be such that

$$D = m \cdot D_E(P)$$

is an integral divisor. Let $\mathscr{L}_m$ be the associated line bundle on B. Note that $\deg(\mathscr{L}_m) = m\hat{h}_E(P) > 0$, so $\mathscr{L}_m$ is ample. By replacing m with a multiple, we may assume that $\mathscr{L}_m$ is very ample.

Fix a place v of K. Let U be an open subset of B_v^{an}. Each section $s \in \mathscr{L}_m(U)$ is identified with a meromorphic function f on U satisfying

$$(f) \geq -D.$$

We set

$$\|s(t)\|_v = \begin{cases} e^{-m\hat{\lambda}_{E_t,v}(P_t)}|f(t)|_v & \text{if } f(t) \text{ is finite and non-zero} \\ 0 & \text{if } \mathrm{ord}_t f > -m\hat{\lambda}_{E,\mathrm{ord}_t}(P) \\ e^{-mV_{P,t,v}(t)} & \text{otherwise,} \end{cases}$$

taking the locally-defined uniformizer $u = f^{1/\mathrm{ord}_t f}$ at t in the definition of $V_{P,t,v}$ from (4.10).

Theorem 4.4.1. *The metric $\|\cdot\| = \{\|\cdot\|_v\}_{v\in M_K}$ on $\mathscr{L}_m$ is continuous, semipositive, and adelic.*

Proof. The continuity and semipositivity follows from Theorem 4.3.4. Recall that although in [CL2], semipositivity of a continuously metrized line bundle on a curve is defined in terms of subharmonicity of potentials for the curvature form at each archimedean place, and as a uniform limit of "smooth semipositive" metrics at each non-archimedean place, in [Th, Theorem 4.3.3] it is established that subharmonicity of potentials is a sufficient notion at all places to yield semipositivity in the sense of Zhang [Zh2]. The adelic condition follows from Theorem 4.1.4. □

4.4.2 The associated height function

A height function on $B(\overline{K})$ is defined by setting

$$h_P(t) := \frac{1}{m}\frac{1}{|\mathrm{Gal}(\overline{K}/K)\cdot t|}\sum_{s\in\mathrm{Gal}(\overline{K}/K)\cdot t}\sum_{v\in M_K} -n_v \log\|\phi(s)\|_v,$$

where ϕ is any global section of $\mathscr{L}_m$ which is nonvanishing along the Galois orbit of t, and $\|\cdot\|_v$ is the metric of §4.4.1. Recall that $\mathrm{supp}\, D_E(P) \subset B(K)$; we may assume that our sections ϕ are defined over K, and the product formula guarantees our height is independent of the choice of ϕ.

Our next goal is to prove the following two important facts about this height function h_P.

Proposition 4.4.2. *The height function h_P satisfies*

$$h_P(t) = \hat{h}_{E_t}(P_t),$$

for all $t \in B(\overline{K})$ such that the fiber E_t is smooth.

Proposition 4.4.3. *The height function h_P satisfies*

$$h_P(t) \geq 0,$$

for all $t \in B(\overline{K})$.

Proof of Proposition 4.4.2. First fix $t \in B(\overline{K}) \setminus \operatorname{supp} D_E(P)$ with smooth fiber E_t. Choose a section ϕ defined over K that does not vanish along the Galois orbit of t, and let f be the associated meromorphic function on B. Then f takes finite and non-zero values along the Galois orbit of t. We have,

$$\begin{aligned}
h_P(t) &= \frac{1}{m} \frac{1}{|\operatorname{Gal}(\overline{K}/K) \cdot t|} \sum_{s \in \operatorname{Gal}(\overline{K}/K) \cdot t} \sum_{v \in M_K} n_v \left(m \hat{\lambda}_{E_s,v}(P_s) - \log |f(s)|_v\right) \\
&= \frac{1}{m} \frac{1}{|\operatorname{Gal}(\overline{K}/K) \cdot t|} \sum_{s \in \operatorname{Gal}(\overline{K}/K) \cdot t} \sum_{v \in M_K} m n_v \hat{\lambda}_{E_s,v}(P_s) \\
&= \hat{h}_{E_t}(P_t).
\end{aligned}$$

where the second equality follows from the product formula.

For $t_0 \in \operatorname{supp} D_E(P)$ such that E_{t_0} is smooth, it is necessarily the case that $P_{t_0} = O_{t_0}$, and therefore $\hat{h}_{E_{t_0}}(P_{t_0}) = 0$. To compute $h_P(t_0)$, observe that $t_0 \in B(K)$ so its Galois orbit is trivial; fixing a uniformizer $u \in K(B)$ at t_0, we have

$$h_P(t_0) = \sum_{v \in M_K} n_v V_{P,t_0,v}(t_0),$$

where $V_{P,t_0,v}$ is the function of (4.10) associated to the uniformizer u.

We can compute $h_P(t_0)$ using the dynamical interpretation of the local heights, described in Section 4.3.1. Fix a Weierstrass equation for E in a neighborhood of t_0 and write $P = (x_P, y_P)$. The assumption that $P_{t_0} = O_{t_0}$ is equivalent to $\operatorname{ord}_{t_0} x_P < 0$. After possibly shrinking U, write $x_P = (u)^{\operatorname{ord}_{t_0}(x_P)} A_0$ for the chosen uniformizer u and a function $A_0 \in K(B)$ that does not vanish in U. We choose a lift X of x_P on U defined as $X = (A_0, B_0)$, where $B_0 := (u)^{-\operatorname{ord}_{t_0}(x_P)}$. Notice that A_0 and B_0 are regular at t_0. Let F be the standard lift in these coordinates, defined in (4.13); it satisfies $F_{t_0}(1,0) = (1,0)$, and we have $\mathscr{G}_{F,\operatorname{ord}_{t_0}}(A_0, B_0) = 0$. Since $\operatorname{ord}_{t_0} \Delta_E = 0$, Proposition 4.3.3 implies that

$$V_{P,t_0,v}(t) = \frac{1}{2}\mathscr{G}_{F_t,v}(A_0(t),B_0(t)) - \frac{1}{12}\log|\Delta_E(t)|_v,$$

for all $t \in U$. Therefore,

$$\begin{aligned}
V_{P,t_0,v}(t_0) &= \frac{1}{2}\mathscr{G}_{F_{t_0},v}(A_0(t_0),0) - \frac{1}{12}\log|\Delta_E(t_0)|_v \\
&= \frac{1}{2}\lim_{n\to\infty}\frac{1}{4^n}\log\|F^n_{t_0}(A_0(t_0),0)\|_v - \frac{1}{12}\log|\Delta_E(t_0)|_v \\
&= \frac{1}{2}\lim_{n\to\infty}\frac{1}{4^n}\log|A_0(t_0)^{4^n}|_v - \frac{1}{12}\log|\Delta_E(t_0)|_v \\
&= \frac{1}{2}\log|A_0(t_0)|_v - \frac{1}{12}\log|\Delta_E(t_0)|_v.
\end{aligned}$$

The product formula now yields that $h_P(t_0) = 0$, as claimed. □

To prove Proposition 4.4.3, we first reduce to the case that the elliptic surface $E \to B$ has semi-stable reduction; that is all of its fibers are either smooth or have multiplicative reduction. The next lemma describes how the height associated with the divisor $D_E(P)$ behaves under base extensions of the elliptic surface $E \to B$. It is adapted from [Si4, Reduction Lemma II.2.1]. We include it here for completeness.

Lemma 4.4.4. *Let $\mu : B' \to B$ be a finite map of smooth projective curves, let $E' \to B'$ be a minimal model for $E \times_B B'$, and let $P' : B' \to E'$ be the extension of the section P. For each $t_0 \in B(\overline{K})$ and $t'_0 \in \mu^{-1}(\{t_0\}) \subset B'(\mathbb{C}_v)$, there is a neighborhood U of t'_0 in $B'(\mathbb{C}_v)$ and a regular non-vanishing function f on U such that*

$$V_{P,t_0,v}(\mu(t')) - V_{P',t'_0,v}(t') = \log|f(t')|_v,$$

on $U \setminus \{t'_0\}$. In particular,

$$V_{P,t_0,v}(t_0) - V_{P',t'_0,v}(t'_0) = \log|f(t'_0)|_v.$$

Proof. Let u be a uniformizer at t_0, u' a uniformizer at t'_0 and $n = \mathrm{ord}_{t'_0}(\mu^* u)$. Since local heights are invariant under base extension we have

$$\hat{\lambda}_{E',\mathrm{ord}_{t'_0}}(P') = n\,\hat{\lambda}_{E,\mathrm{ord}_{t_0}}(P). \tag{4.15}$$

Notice that for all t' in a punctured neighborhood of t'_0 the fibers $E'_{t'}$ are smooth. Hence, the map $E' \to E$ gives an isomorphism between the fibers $E'_{t'} \to E_{\mu(t')}$. Under this isomorphism $P'_{t'} \in E'_{t'}$ is mapped to $P_{\mu(t')} \in E_{\mu(t')}$. Invoking now the uniqueness of the Néron local heights, we have

$$\hat{\lambda}_{E_{\mu(t')},v}(P_{\mu(t')}) = \hat{\lambda}_{E'_{t'},v}(P'_{t'}). \tag{4.16}$$

Combining (4.15) and (4.16) we get that for t' in a punctured neighborhood of t'_0,

$$V_{P,t_0,v}(\mu(t')) = V_{P',t'_0,v}(t') + \hat{\lambda}_{E,\mathrm{ord}_{t_0}}(P) \log \left| \frac{u(\mu(t'))}{u'^n(t')} \right|_v .$$

The definition of n yields that the function $f(t') = \left(\frac{u(\mu(t'))}{u'^n(t')}\right)^{\hat{\lambda}_{E,\mathrm{ord}_{t_0}}(P)}$ is regular and non-vanishing at t'_0. The first part of the lemma follows.

Finally, Theorem 4.1.5 allows us to conclude that

$$V_{P,t_0,v}(\mu(t'_0)) - V_{P',t'_0,v}(t'_0) = \log |f(t'_0)|_v$$

at the point t'_0, as claimed. □

The following lemma will allow us to prove Proposition 4.4.3 in the case that a fiber has multiplicative reduction. The proof is lengthy, but it is merely a collection of computations using the explicit formulas for the local height functions, as in [Si3, Theorem VI.3.4, VI.4.2].

Lemma 4.4.5. *Let $E \to B$ be an elliptic surface and let $P : B \to E$ be a non-zero section defined over K. Then there exists a finite extension L of the number field K so that, for each $t_0 \in B(\overline{K})$ such that E_{t_0} has multiplicative reduction, there exists an $x(t_0) \in L^*$ so that*

$$V_{P,t_0,v}(t_0) = \log |x(t_0)|_v,$$

at all places v of L.

Proof. We let

$$E : y^2 = x^3 + ax + b \tag{4.17}$$

be a minimal Weierstrass equation for E over an affine subset $W \subset B$ defined over K with $t_0 \in W$. Here $a, b \in K(B)$ are regular functions at t_0. Using this Weierstrass equation we write

$$P = (x_P, y_P),$$

where $x_P, y_P \in K(B)$. Since $E \to B$ has multiplicative reduction over $t_0 \in B(K)$, we have

$$N := \operatorname{ord}_{t_0} \Delta_E \geq 1 \text{ and } \min\{\operatorname{ord}_{t_0} a, \operatorname{ord}_{t_0} b\} = 0. \tag{4.18}$$

Let v be a place of K (archimedean or non-archimedean). We denote by j_E the $j-$invariant of $E \to W$, given by

$$j_E(t) = 1728 \frac{(4a(t))^3}{\Delta_E(t)}.$$

Notice that equation (4.18) yields that j_E has a pole at t_0. Hence, we can find a $v-$adic open neighborhood U of t_0 and an analytic map

$$\psi : U \to \{q \in \mathbb{C}_v \ : \ |q|_v < 1\},$$

such that the following holds: If j is the modular $j-$invariant [Si3, Chapter V], then we have

$$j_E(t) = j(\psi(t)) \text{ and } \operatorname{ord}_{t_0} \psi = N.$$

The function $\psi(t)$ is given as

$$\psi(t) = \frac{1}{j_E(t)} + \frac{744}{j_E^2(t)} + \frac{750420}{j_E^3(t)} + \ldots \in \mathbb{Z}[[(j_E(t))^{-1}]]. \tag{4.19}$$

In the following, we choose a uniformizer $u \in K(B)$ at t_0, and we identify ψ with its expression $\psi(t) \in \mathbb{C}_v[[u]]$ and write

$$\psi(t) = \beta u(t)^N + u(t)^{N+1} f(t), \text{ for } t \in U \setminus \{t_0\}. \tag{4.20}$$

Equation (4.19) yields that $\beta \in K \setminus \{0\}$ and $f(t) \in K[[u]]$. Following the proof of

[Si4, Section 6] and after possibly shrinking U we have isomorphisms

$$E_t(\mathbb{C}_v) \xrightarrow{\sim} \mathbb{C}_v^* \Big/ \psi(t)^{\mathbb{Z}} \xrightarrow{\sim} C_{\psi(t)} : y^2 = 4x^3 - g_2(\psi(t))x - g_3(\psi(t)), \tag{4.21}$$

for $t \in U \setminus \{t_0\}$. Under these isomorphisms, we have

$$P_t \mapsto \quad w(t) \quad \mapsto (\wp(w(t), \psi(t)), \wp'(w(t), \psi(t))).$$

Here g_2, g_3 are the modular invariants, given by their usual $q-$series

$$g_2(q) = \frac{1}{12}\left(1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}\right), \, g_3(q) = \frac{1}{216}\left(-1 + 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}\right),$$

and $\wp$ is the Weierstrass $\wp-$function given by

$$\begin{aligned} \wp(w,q) &= \frac{1}{12} + \sum_{n\in\mathbb{Z}}\frac{q^n w}{(1-q^n w)^2} - 2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \\ \wp'(w,q) &= \sum_{n\in\mathbb{Z}}\frac{q^n w(1+q^n w)}{(1-q^n w)^3}. \end{aligned} \tag{4.22}$$

In view of [Si4, Lemma II.6.2], after possibly replacing P by $-P$, we may assume that $w : U \to \mathbb{C}_v$ is an analytic map satisfying

$$0 \le \operatorname{ord}_{t_0} w \le \frac{1}{2}\operatorname{ord}_{t_0}\psi. \tag{4.23}$$

In the following we identify w with its series in $\mathbb{C}_v[[u]]$ and write

$$w(t) = \alpha u^m(t) + u^{m+1}(t)g(t), \tag{4.24}$$

where $\alpha \in \mathbb{C}_v$ and $g(t) \in \mathbb{C}_v[[u]]$.

We claim that $w(t) \in \overline{K}[[u]]$. To see this, notice that from [Si7, III] we have that for $t \in U$

$$(\wp(w(t), \psi(t)), \wp'(w(t), \psi(t)) = (\nu^{-2}(t)x_P(t), 2\nu^{-3}(t)y_P(t)),$$

where

$$\nu(t)^{12} = \frac{\Delta_E(t)}{\Delta(\psi(t))}.$$

In the equation above Δ denotes the modular discriminant given by

$$\Delta(q) = g_2(q)^3 - 27g_3(q)^3.$$

Since the functions ψ, Δ_E and Δ are defined over K, it follows that the function $Y(t) := 2\nu^{-3}(t)y_P(t)$ is also defined over K. Since $Y(t) = \wp'(w(t), \psi(t)) \in K[[u]]$ and $\psi(t) \in K[[u]]$ we get that $w(t) \in K[[u]]$.

Therefore, there are non-zero constants $\alpha, \beta, \gamma \in \overline{K} \setminus \{0\}$, non-negative integers $k, m \in \mathbb{N}$ and functions $f(t), g(t), h(t) \in \overline{K}[[u]]$ such that for all $t \in U$

$$\begin{aligned} \psi(t) = \beta u^N(t) + f(t)u^{N+1}(t),\, w(t) = \alpha u^m(t) + g(t)u^{m+1} \\ 1 - w(t) = \gamma u^k(t) + h(t)u^{k+1}(t). \end{aligned} \tag{4.25}$$

Next, we aim to express $x(t_0)$ (as in the statement of the lemma) in terms of $\alpha, \beta, \gamma \in \overline{K}$.

Using the isomorphisms in 4.21, the uniqueness of the local canonical heights and the explicit formulas for the local canonical heights [Si3, Theorem VI.3.4, VI.4.2], we get

$$\begin{aligned} \hat{\lambda}_{E_t,v}(P_t) &= \hat{\lambda}(w(t), \psi(t)) \\ &= -\frac{1}{2}B_2\left(\frac{\log|w(t)|_v}{\log|\psi(t)|_v}\right)\log|\psi(t)|_v - \log|1 - w(t)|_v \\ &\quad - \sum_{n\geq 1}\log|(1-\psi(t)^n w(t))(1-\psi(t)^n w(t)^{-1})|_v, \end{aligned} \tag{4.26}$$

where $B_2(s) = s^2 - s + 1/6$ is the second Bernoulli polynomial.

Since $\operatorname{ord}_{t_0}\psi = N \geq 1$ and using (4.23), we get

$$\lim_{t \xrightarrow{v} t_0} \sum_{n\geq 1}\log|(1-\psi(t)^n w(t))(1-\psi(t)^n w(t)^{-1})|_v = 0. \tag{4.27}$$

In what follows, for $F(t) \in \mathbb{C}_v[[u]]$ we write

$$F(t) := o_v(1), \text{ if } \lim_{t \xrightarrow{v} t_0} F(t) = 0.$$

In view of [Si2, Lemma I.5.1], we have

$$\begin{aligned} B_2\left(\frac{\log|w(t)|_v}{\log|\psi(t)|_v}\right)\log|\psi(t)|_v &= \frac{\log^2|w(t)|_v}{\log|\psi(t)|_v} - \log|w(t)|_v + \frac{1}{6}\log|\psi(t)|_v \\ &= \frac{m^2}{N}\log|u(t)|_v + \frac{m}{N^2}\log\left(\frac{|\alpha|_v^{2N}}{|\beta|_v^m}\right) \\ &- \log|\alpha|_v - m\log|u(t)|_v + \frac{\log|\beta|_v}{6} \\ &+ \frac{N}{6}\log|u(t)|_v + o_v(1). \end{aligned} \tag{4.28}$$

Using equations (4.27) and (4.28), equation (4.26) yields

$$\begin{aligned} &\hat{\lambda}_{E_t,v}(P_t) + \frac{1}{2}\left(\frac{m^2}{N} - m + \frac{N}{6} + 2k\right)\log|u(t)|_v = \\ &-\frac{1}{2}\left(\frac{m}{N^2}\log\left(\frac{|\alpha|_v^{2N}}{|\beta|_v^m}\right) - \log|\alpha|_v + \frac{\log|\beta|_v}{6}\right) \\ &-\log|\gamma|_v + o_v(1). \end{aligned} \tag{4.29}$$

Finally, notice that [Si3, Theorem VI.4.2] implies

$$\hat{\lambda}_{E,\mathrm{ord}_{t_0}}(P) = \mathrm{ord}_{t_0}(1-w) + \frac{1}{2}B_2\left(\frac{\mathrm{ord}_{t_0} w}{\mathrm{ord}_{t_0}\psi}\right)\mathrm{ord}_{t_0}\psi = \frac{1}{2}\left(\frac{m^2}{N} - m + \frac{N}{6} + 2k\right).$$

Therefore,

$$\begin{aligned} V_{P,t_0,v}(t_0) = \lim_{t \xrightarrow{v} t_0} V_{P,t_0,v}(t) &= -\frac{1}{2}\left(\frac{m}{N^2}\log\left(\frac{|\alpha|_v^{2N}}{|\beta|_v^m}\right) - \log|\alpha|_v + \frac{\log|\beta|_v}{6}\right) - \log|\gamma|_v \\ &= \log|x(t_0)|_v, \end{aligned}$$

where $x(t_0) = \frac{\beta^{m^2/2N^2-1/2}}{\gamma\alpha^{m/N-1/2}}$ belongs to a finite extension of K, denoted by L. $\square$

Proof of Proposition 4.4.3. By [Si4, Lemma II.2.2], there is a finite map of smooth projective curves $B' \to B$ such that if $E' \to B'$ is a minimal model for $E \times_B B'$, then E' has semi-stable reduction over the singular fibers of $E \to B$. Moreover, we may choose B' so that everything is defined over K. Thus, by Lemma 4.4.4 and using the product formula, we may assume that the singular fibers of our elliptic surface $E \to B$ have multiplicative reduction.

For all $t \in B(\overline{K})$ for which E_t is smooth, we know from Proposition 4.4.2 that $h_P(t) = \hat{h}_{E_t}(P_t)$. The canonical height is always non-negative, so we may conclude that $h_P(t) \geq 0$ for all such t.

Assume now that $t_0 \in B(\overline{K})$ has a fiber with multiplicative reduction. Enlarging the number field K if necessary we may assume that $t_0 \in B(K)$ and that its corresponding $x(t_0)$ defined in the statement of Lemma 4.4.5 is in K^*. Then, on using the product formula, Lemma 4.4.5 implies that $h_P(t_0) = 0$. This completes the proof. □

4.4.3 Proofs of Theorem 1.0.4 and Corollary 1.0.5

Proof of Theorem 1.0.4. Let $\mathscr{L}_P$ be the line bundle on B induced from the divisor $D_E(P)$. From Theorem 4.4.1, we know that its m-th tensor power can be equipped with a continuous, adelic, semipositive metric, so that the corresponding height function is (a multiple of) the canonical height $\hat{h}_{E_t}(P_t)$ on the smooth fibers. Thus, by pulling back the metric to $\mathscr{L}_P$, we obtain a continuous, semipositive, adelic metric on $\mathscr{L}_P$ inducing the desired height function.

It remains to show that this height h_P satisfies $h_P(B) = 0$. This is a consequence of Propositions 4.4.2 and 4.4.3 and Zhang's inequalities on successive minima recalled in 4.2.11. Since $\hat{h}_E(P) \neq 0$, we know that there are infinitely many $t \in B(\overline{K})$ for which P_t is a torsion point in E_t, or equivalently

$$\hat{h}_{E_t}(P_t) = 0.$$

For a complex-dynamical proof of this fact, we refer to [De1, Proposition 1.5, Proposition 2.3]. Therefore, from Proposition 4.4.2, we may deduce that the first essential minimum of h_P on B, that is $e_1(\mathscr{L}_P)$, is equal to 0. On the other hand, from Proposition 4.4.3, we know that $h_P(t) \geq 0$ for all $t \in B(\overline{K})$. Therefore, we also have $e_2(\mathscr{L}_P) = 0$. Now from Zhang's inequalities 4.2.11 we may conclude that $h_P(B) = 0$. $\square$

Proof of Corollary 1.0.5. When combined with the equidistribution theorem 4.2.13, we immediately obtain the corollary from Theorem 1.0.4. The measures $\mu_{P,v}$ are the curvature currents associated to the metrics $\|\cdot\|_v$ at each place v; for non-archimedean these are defined by Chambert-Loir [CL1]. From the definition of the metric in §4.4.1, we see that they are given locally by

$$\mu_{P,v} = dd^c V_{P,t_0,v}(t)$$

in a v-adic neighborhood of any point $t_0 \in B(\overline{K})$, and for any choice of uniformizer u at t_0. $\square$

4.5 Proof of Theorem 1.0.8.

4.5.1 Reduction to the case of a fiber product of elliptic surfaces

We first show that, to prove the theorem, it suffices to prove the result for sections of the fiber product $A = E_1 \times_B \cdots \times_B E_m$ of $m \geq 2$ elliptic surfaces $E_i \to B$ over the same base, and to assume that the line bundle $\mathscr{L}$ is generated by the divisor

$$\{O_{E_1}\} \times E_2 \times \cdots E_m + E_1 \times \{O_{E_2}\} \times \cdots \times E_m + \cdots + E_1 \times E_2 \times \cdots \times \{O_{E_m}\}.$$

Let B be a quasiprojective smooth algebraic curve defined over $\overline{\mathbb{Q}}$. Suppose that $A \to B$ is family of abelian varieties defined over $\overline{\mathbb{Q}}$, which is isogenous to a fibered product of $m \geq 2$ elliptic curves. That is, there is a branched cover $B' \to B$ and $m \geq 2$ elliptic surfaces $E_i \to B'$ that give rise to an isogeny

$$E_1 \times_{B'} \cdots \times_{B'} E_m \to A$$

over B'. Now let $\mathscr{L}$ be a line bundle on A which restricts to an ample and symmetric line bundle on each fiber A_t for $t \in B$. Then the line bundle $\mathscr{L}$ pulls back to a line bundle $\mathscr{L}'$ on $E_1 \times_{B'} \cdots \times_{B'} E_m$, and it again restricts to an ample and symmetric line bundle on each fiber over $t \in B'$.

Now suppose that we have a section $P : B \to A$. The section P pulls back to a section $P' : B' \to A$, and this in turn pulls back to a (possibly multi-valued) section of $E_1 \times_{B'} \cdots \times_{B'} E_m$. If multi-valued, we can perform a base change again, passing to a branched cover $B'' \to B'$, so that the induced section $P'' : B'' \to E_1 \times_{B''} \cdots \times_{B''} E_m$ is well defined. By definition, the assumption that P is non-special on A means that it is non-special as a section of $E_1 \times_{B''} \cdots \times_{B''} E_m$.

Finally, we observe that the conclusion of Theorem 1.0.8 does not depend on the choice of line bundle. (We thank Joe Silverman for his help with this argument.) Recall that, on any abelian variety A defined over $\overline{\mathbb{Q}}$, the notion of a "small sequence" of points is independent of the choice of ample and symmetric line bundle. That is, if we take two ample and symmetric divisors D_1 and D_2, then we know that there exists an integer $m_1 > 0$ so that $m_1 D_1 - D_2$ is ample; similarly there exists $m_2 > 0$ so that $m_2 D_2 - D_1$ is ample. It follows from properties of the Weil height machine that the heights h_{D_1} and h_{D_2} will then satisfy

$$\frac{1}{m_1} h_{D_2} + C_1 \leq h_{D_1} \leq m_2 h_{D_2} + C_2,$$

for real constants C_1, C_2. Upon passing to the canonical height, we conclude that

$$\frac{1}{m_1} \hat{h}_{D_2} \leq \hat{h}_{D_1} \leq m_2 \hat{h}_{D_2} \tag{4.30}$$

on the abelian variety. In particular, $\hat{h}_{D_1}(a_i) \to 0$ for some sequence in $A(\overline{\mathbb{Q}})$ if and only if $\hat{h}_{D_2}(a_i) \to 0$. Now suppose we have a family of abelian varieties $A \to B$. Two line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ associated to relatively ample and symmetric divisors induce a canonical heights $\hat{h}_{\mathscr{L}_{1,t}}$ and $\hat{h}_{\mathscr{L}_{2,t}}$ on each fiber A_t. But recalling that amplitude persists on Zariski open sets [La4, Theorem 1.2.17], there exist positive integers m_1 and m_2 so that the line bundles $\mathscr{L}_1^{m_1} \otimes \mathscr{L}_2^{-1}$ and $\mathscr{L}_2^{m_2} \otimes \mathscr{L}_1^{-1}$ are relatively ample on a Zariski open subset of the base B. Passing to the canonical heights once again, we find that the relation (4.30) holds uniformly over B (after possibly excluding finitely many points). Therefore, for any section $P : B \to A$, there exists a positive constant $c(\mathscr{L}_1, P)$ of Theorem 1.0.8 for the height $\hat{h}_{\mathscr{L}_1}$ if and only if it there exists such a constant $c(\mathscr{L}_2, P)$ for $\hat{h}_{\mathscr{L}_2}$.

4.5.2 Proof for a fiber product of elliptic curves

Fix integer $m \geq 2$, and let $E_i \to B$ for $i = 1, \ldots, m$ be elliptic surfaces over the same base curve B, defined over $\overline{\mathbb{Q}}$. Let $A = E_1 \times_B \cdots \times_B E_m$, and let $\mathscr{L}$ be the line bundle on $E_1 \times_B \cdots \times_B E_m$ associated to the divisor

$$D = \{O_{E_1}\} \times E_2 \times \cdots E_m + E_1 \times \{O_{E_2}\} \times \cdots \times E_m + \cdots + E_1 \times E_2 \times \cdots \times \{O_{E_m}\}.$$

For all but finitely many $t \in B(\overline{\mathbb{Q}})$, the canonical height $\hat{h}_{\mathscr{L}_t}$ on the fiber A_t is easily seen to be the sum of canonical heights (see, e.g., [HS] for properties of the height functions), so that

$$\hat{h}_{\mathscr{L}_t} = \sum_{i=1}^{m} \hat{h}_{E_{i,t}}.$$

Now assume that $P = (P_1, \ldots, P_m)$ is a section of $A \to B$. Define

$$\hat{h}_i(t) := \hat{h}_{E_{i,t}}(P_i(t)),$$

for $i = 1, \ldots, m$ and for all $t \in B(\overline{\mathbb{Q}})$ where all $E_{i,t}$ are smooth elliptic curves. Suppose there exists an infinite sequence $\{t_n\} \subset B(\overline{\mathbb{Q}})$ for which

$$\hat{h}_i(t_n) \to 0 \text{ for all } i = 1, \ldots, m. \tag{4.31}$$

as $n \to \infty$. We will prove that for every pair (i, j), there exists an infinite sequence $\{s_n\} \subset B(\overline{\mathbb{Q}})$ so that

$$\hat{h}_i(s_n) = \hat{h}_j(s_n) = 0,$$

for all n. In this way, we reduce our problem to the main results of [MZ2, MZ3], which imply that the pair (P_i, P_j) must be a special section of $E_i \times_B E_j$. Finally, we observe that our definition of a special section $P = (P_1, P_2, \ldots, P_m)$ is equivalent to the statement that every pair (P_i, P_j) is special. Therefore, for any non-special section P, we can conclude that there exists a constant $c = c(P) > 0$ so that the set

$$\{t \in B(\overline{\mathbb{Q}}) : \hat{h}_{\mathscr{L}_t}(P_t) < c\}$$

is finite.

Fix a pair (i,j). First assume that neither E_i nor E_j is isotrivial. If P_i or P_j is torsion, then the section (P_i,P_j) is special. Otherwise, we have $\hat{h}_{E_i}(P_i) \neq 0$ and $\hat{h}_{E_j}(P_j) \neq 0$, and we may apply Theorem 1.0.4 to deduce that the height functions h_i and h_j are "good" on B. More precisely, we let M_i and M_j be the adelically metrized line bundles on the base curve B associated to the height functions $\hat{h}_i$ and $\hat{h}_j$, from Theorem 1.0.4. They are both equipped with continuous, semipositive, adelic metrics. By assumption, we have

$$\hat{h}_i(t_n) \to 0 \quad \text{and} \quad \hat{h}_j(t_n) \to 0, \tag{4.32}$$

as $n \to \infty$. Note that $M_i \otimes M_j$ also comes equipped with continuous, semipositive, adelic metrics induced by the metrics of M_i and M_j. Its associated height is $h_{M_i \otimes M_j} = \hat{h}_i + \hat{h}_j$, which is a non-negative function on B by Proposition 4.4.3. Moreover, equation 4.32 yields

$$h_{M_i \otimes M_j}(t_n) = \hat{h}_i(t_n) + \hat{h}_j(t_n) \to 0.$$

Thus, we have $e_1(M_i \otimes M_j) = e_2(M_i \otimes M_j) = 0$ and by Zhang's inequalities 4.2.11, we get

$$h_{M_i \otimes M_j}(B) = 0.$$

Recall also from Theorem 1.0.4 that $h_{M_i}(B) = h_{M_j}(B) = 0$. Therefore, we may apply the observation of Chambert-Loir [CL2, Proposition 3.4.2], which builds upon on Zhang's inequalities [Zh5], to the metrized line bundle $M_i^{\deg(M_j)} \otimes M_j^{-\deg(M_i)}$. We conclude that there exist integers n_i and n_j so that $M_i^{n_i}$ and $M_j^{n_j}$ are isomorphic as line bundles on B and their metrics are scalar multiples of one another. In particular, there is a constant $c \in \mathbb{R}$ such that their associated heights satisfy $n_i\hat{h}_i(t) - n_j\hat{h}_j(t) = c$ for all $t \in B(\overline{\mathbb{Q}})$ and in fact by (4.32) we get that $c = 0$. It follows that the height functions $\hat{h}_i$ and $\hat{h}_j$ are the same, up to scale, and in particular they have the same zero sets. In other words, $P_i(t)$ is a torsion point on $E_{i,t}$ if and only if $P_j(t)$ is a torsion point on $E_{j,t}$ (for all but finitely many t in B), and there are infinitely many such parameters $t \in B(\overline{\mathbb{Q}})$. Hence, by Masser-Zannier's theorems [MZ2, MZ3], we get that (P_i,P_j) is a special section of $E_i \times_B E_j$.

Now suppose that E_i is isotrivial. The existence of the small sequence t_n in

(4.32) implies that either $\hat{h}_{E_i}(P_i) \neq 0$ or P_i is torsion on E_i, and furthermore, if P_i is torsion, then it follows that (P_i, P_j) is a special section of $E_i \times_B E_j$. Similarly if E_j is isotrivial. In other words, the existence of the sequence t_n in (4.32) allows us to conclude that either (P_i, P_j) is a special pair, or we have that both $\hat{h}_{E_i}(P_i) \neq 0$ and $\hat{h}_{E_j}(P_j) \neq 0$. Therefore, we may proceed as above in the nonisotrivial case, applying Theorem 1.0.4 to deduce that the heights $\hat{h}_i$ and $\hat{h}_j$ coincide, up to scale, and in particular there are infinitely many parameters $s \in B(\overline{\mathbb{Q}})$ where

$$\hat{h}_i(s) = \hat{h}_j(s) = 0.$$

This concludes the proof of Theorem 1.0.8. $\square$

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