The Application of Shifted Frequency Analysis in Power System Transient Stability Studies

by

Andrea T.J. Martí

B.A.Sc., The University of British Columbia, 2015

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF APPLIED SCIENCE

in

The Faculty of Graduate and Postdoctoral Studies

(Electrical and Computer Engineering)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

February 2018

© Andrea T.J. Martí 2018
Abstract

Power system engineers use transient stability computer simulation programs to model the power grid’s behaviour when large disruptions cause the grid to deviate from its 60-Hz operating frequency. These programs must be able to capture these frequency dynamics around 60 Hz while being computationally efficient, as an extensive number of simulations are typically run for a given scenario. In the traditional power grid, the large mechanical inertia of the synchronous generators stabilizes the network during disturbances and maintains the system frequency close to the 60 Hz operating frequency. In the modern grid, however, the increase of renewable energy sources lowers the grid’s inertia and larger frequency deviations can occur. The phasor solution method employed in the traditional programs solves the network assuming a constant 60-Hz frequency. When deviations from 60 Hz are prominent, the Electromagnetic Transients Program (EMTP) is used as an alternative to the phasor solution to capture these fluctuations. The EMTP models the electrical network based on the differential equations of the network components, which allows the tracing of the network waveforms. However, this discretization requires small time-steps, which makes the solution method computationally expensive. The Shifted Frequency Analysis (SFA) method discussed in this work is an alternative to the traditional phasor solution and to the EMTP solution. In this work, a generalized SFA-based program is written and used for transient stability analysis. SFA is a discrete-time solution method, like the EMTP, but uses a frequency-shifting transformation to bring the solution domain down to 0 Hz. Because of this transformation, SFA can capture network dynamics around 60 Hz using large time-steps, making it suitable for transient stability analysis studies.
Lay Summary

Power system engineers use transient stability computer simulation programs to model the power grid’s behaviour when large disruptions cause the grid to deviate from its 60 Hz operating frequency. These programs must be able to capture the frequency dynamics around 60 Hz while being computationally efficient, as an extensive number of simulations are typically run for a given scenario. With the increase of renewable energy sources connected to the grid, the power system is more susceptible to network changes than the traditional power grid, whose high-inertial synchronous generators kept the grid close to 60 Hz. The traditional simulation programs use the constant 60 Hz frequency in their models and the alternative programs who can capture frequency dynamics can be computationally slow for the 60-Hz fluctuations.

This thesis proposes the solution method, Shifted Frequency Analysis (SFA), as a model to capture frequency dynamics around 60 Hz computationally efficiently for power system transient stability studies.
Preface

The research work in this thesis to the best of my knowledge is original and unpublished, except where references are made to previous works. The work is done by the author under the supervision of Prof. Juri Jatskevich.

# Table of Contents

Abstract ................................................................. ii

Lay Summary ............................................................ iii

Preface ................................................................. iv

Table of Contents ....................................................... v

List of Abbreviations .................................................. xvi

Nomenclature ............................................................ xvii

Acknowledgements ...................................................... xviii

Dedication ................................................................. xix

1 Introduction .......................................................... 1

1.1 Research Motivation ................................................ 1

1.2 The Role of AC Synchronization and Power Equilibrium For Power Grid Stability ........................................ 3

1.3 Transient Solution Methods Used in Transient Stability Studies .................................................. 6

1.3.1 The Traditional Phasor Solution Method ..................... 6

1.3.2 The EMTP Solution Method .................................. 9

1.3.3 Time-varying Phasors and the Introduction of the Shifted Frequency Analysis Solution Method .................. 11

1.4 Research Objectives ................................................ 17

1.5 Research Contributions ............................................. 18

1.6 Thesis Outline ....................................................... 18
# Table of Contents

2 Shifted Frequency Analysis: A Discrete-Time Solution Method and Solution Framework ................................................. 20

2.1 Shifted Frequency Analysis Solution Framework ................... 20
2.1.1 A Single-Frequency Signal ........................................ 21
2.1.2 The Time-Frequency Relationship ................................. 21
2.1.3 The SFA Solution Framework ..................................... 21

2.2 The SFA Solution Method .............................................. 23
2.2.1 The Choice of Time-Step in the SFA Solution Method ........ 26
2.2.2 A Summary of the SFA Solution Method ......................... 26

2.3 The SFA Equivalent Network Elements ............................... 27

2.4 The Discretization of the SFA Network Elements .................... 29
2.4.1 Equivalent SFA Network Elements Using the Trapezoidal Discretization Rule ................................................. 30
2.4.2 Equivalent SFA Network Elements Using the Backward Euler Discretization Rule ............................................... 32

2.5 A Comparison of the Traditional Phasor, the EMTP, and the SFA Network Elements ................................................ 34

2.6 An RL Circuit Energized with an AC Source Using the Traditional Phasor, the EMTP and SFA ................................. 36

3 The Numerical Discretization Methods Used in SFA .................. 39

3.1 The Evaluation of a Numerical Discretization Method Using the Frequency Response ................................................. 39

3.2 The System’s Frequency Response in the Physical Domain and in the SFA Domain .................................................... 40
3.2.1 The Frequency Response in the Physical Domain .............. 40
3.2.2 The Frequency Response in the SFA Domain .............. 42
3.2.3 The Continuous-Time Frequency Response in the SFA Domain .... 42

3.3 The Frequency Response of the Trapezoidal Discretization Rule in the SFA Domain .................................................. 43
3.3.1 Approach One: Applying the Single-Frequency Exponential ................................................................. 43
3.3.2 Approach Two: Applying the Z-Domain Transfer Function .... 44
3.3.3 The Trapezoidal Rule’s Accuracy in the SFA Domain ....... 46
3.3.4 The Numerical Accuracy of the Trapezoidal Rule: Magnitude and Phase Responses ............................................ 47
3.3.5 The Distortion of the Inductor Using the Trapezoidal Rule .... 50
### TABLE OF CONTENTS

3.4 The Frequency Response of the Backward Euler Discretization Rule in the SFA Domain ....................................................... 51
  3.4.1 Applying the Z-Domain Transfer Function ......................... 51
  3.4.2 The Backward Euler Rule’s Accuracy in the SFA Domain .... 52
  3.4.3 The Numerical Accuracy of the Backward Euler Rule: Magnitude and Phase Responses .......................... 53
  3.4.4 The Distortion of the Inductor Using the Backward Euler Rule 54
  3.4.5 A Comparison of The Trapezoidal and Backward Euler Rule Around 0 Hz (DC) ............................... 59
  3.4.6 A Summary of the Accuracy of the Two Discretization Rules 60
3.5 The Discretization Rule’s Stability in the SFA Domain .......... 61
3.6 The Response to a Step Input in the SFA Domain ............... 62
3.7 An RL Circuit Energized with a DC Source Using the Traditional Phasor, the EMTP and SFA .............................. 69

4 Transient Stability Analysis .................................................. 71
  4.1 A Physical Representation ................................................ 72
    4.1.1 The Equal Area Criteria and The Critical Clearing Time ... 73
    4.1.2 The Approximated Synchronous Machine Model ............. 74
  4.2 A Mathematical Representation .......................................... 74
    4.2.1 The Coupling of the Electromechanical and Electrical Network Equations in the Solution Methods .... 76
    4.2.2 The Discretization of the Electromechanical Equation ..... 77
  4.3 Implementation of SFA in Three Transient Stability Case Studies 78
    4.3.1 Fault Current Zero-Crossing Detection Algorithm in TSFA 82

5 Single-Generator Infinite-Bus Case Study ................................. 85
  5.1 Critical Clearing Times ................................................ 86
  5.2 A Three-Phase Fault at the Generator’s Terminals .......... 87
    5.2.1 A Comparison of Using the Actual Frequency with the Approximate Frequency .......................... 93
    5.2.2 A Comparison of the Discretization Methods in the TSFA Solution During the Three-Phase Fault. . . . . . . . . . 94

6 Three-Bus Network Case Study ............................................... 96
  6.1 Critical Clearing Times ................................................ 98
### TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.2 A Three-Phase Fault at the Terminals of Transformer Two</td>
<td>100</td>
</tr>
<tr>
<td>6.2.1 A Comparison of Using the Actual Velocity with the Approximate</td>
<td></td>
</tr>
<tr>
<td>6.2.2 A Comparison of the Discretization Rules in the TSFA Solution</td>
<td></td>
</tr>
<tr>
<td>6.2.3 The DC Offset Captured in the Fault Current in SFA</td>
<td>108</td>
</tr>
<tr>
<td>6.3 Generator One Disconnected From the Network</td>
<td>109</td>
</tr>
<tr>
<td>7 Thirty-Nine Bus Network Case Study</td>
<td>111</td>
</tr>
<tr>
<td>7.1 Critical Clearing Times</td>
<td>112</td>
</tr>
<tr>
<td>7.2 A Three-Phase Fault Applied to Bus 16</td>
<td>115</td>
</tr>
<tr>
<td>7.3 Generator Nine Disconnected from the Network</td>
<td>119</td>
</tr>
<tr>
<td>8 Conclusion and Future Research</td>
<td>120</td>
</tr>
<tr>
<td>8.1 Conclusion</td>
<td>120</td>
</tr>
<tr>
<td>8.2 Future Research</td>
<td>122</td>
</tr>
<tr>
<td>Bibliography</td>
<td>125</td>
</tr>
<tr>
<td>Appendix</td>
<td>131</td>
</tr>
</tbody>
</table>
List of Tables

Table 2.1  The branch equivalents for the traditional phasor, the EMTP, and the SFA methods using the trapezoidal discretization rule for the SFA and the EMTP. ........................................... 35
Table 2.2  The branch equivalents for the traditional phasor, the EMTP, and the SFA methods using the backward Euler discretization rule for the EMTP and SFA. ........................................... 35
Table 3.1  A comparison of the EMTP and SFA frequency response accuracy ratio’s when the trapezoidal rule is behaving as an integrator .... 47
Table 3.2  The discontinuities at the Nyquist frequency of different time-steps when using the trapezoidal rule. ................................. 47
Table 3.3  A comparison of the EMTP’s and SFA’s distortion of the inductor when using the trapezoidal rule. ................................. 51
Table 3.4  A comparison of the EMTP and SFA frequency response accuracy ratio for the backward Euler rule behaving as an integrator. .... 53
Table 3.5  A comparison of the EMTP’s and SFA’s distortion of the inductor when using the backward Euler rule. ................................. 59
Table 3.6  The response of a step input using the trapezoidal rule, comparing the SFA solution with the EMTP solution. ................................. 63
Table 3.7  The response of a step input using the backward Euler rule, comparing the SFA solution with the EMTP. ................................. 63
Table 5.1  Network parameters for the SGIB network (Figure 5.1). .... 85
Table 5.2  The CCTs for the three-phase fault on the generator’s terminals of the SGIB system in Figure 5.1 in 60-Hz cycles and milliseconds, and the critical clearing angle in degrees. ................................. 86
Table 5.3  A comparison of the error in frequency among TSAT (at $\Delta t = 8\text{ ms}$) and TSFA (at $\Delta t = 5\text{ ms}$ and $\Delta t = 8\text{ ms}$) with respect to the EMTP. 89
### LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 5.4</td>
<td>A comparison of the error in frequency between TSAT, TSAT*, TPhasor and TPhasor* with respect to the EMTP reference</td>
<td>93</td>
</tr>
<tr>
<td>Table 6.1</td>
<td>Network parameters for the three-bus network (Figure 6.1)</td>
<td>97</td>
</tr>
<tr>
<td>Table 6.2</td>
<td>The CCTs for three-phase faults at different locations of the three-bus system in Figure 6.1 in 60-Hz cycles and the critical clearing angle in degrees</td>
<td>99</td>
</tr>
<tr>
<td>Table 6.3</td>
<td>A comparison of the error in frequency between TSAT and TSFA with respect to the EMTP reference solution</td>
<td>100</td>
</tr>
<tr>
<td>Table 6.4</td>
<td>A comparison of the error in frequency between the different TSFA solutions with respect to the EMTP reference solution</td>
<td>105</td>
</tr>
<tr>
<td>Table 6.5</td>
<td>The four different conditions for the phasor solution in Figure 6.8</td>
<td>105</td>
</tr>
<tr>
<td>Table 6.6</td>
<td>A comparison of the error in frequency between the different TPhasor solutions with respect to the EMTP reference solution</td>
<td>106</td>
</tr>
<tr>
<td>Table 7.1</td>
<td>The CCTs for three-phase faults at different locations of the three-bus system in Figure 7.1 in 60-Hz cycles and the critical clearing angle in degrees</td>
<td>113</td>
</tr>
<tr>
<td>Table 7.2</td>
<td>A comparison of the error in frequency between TSAT and TSFA with respect to the EMTP reference solution</td>
<td>117</td>
</tr>
<tr>
<td>Table 7.3</td>
<td>A comparison of the error in frequency between the different TSFA solutions with respect to the EMTP reference solution</td>
<td>117</td>
</tr>
<tr>
<td>Table A1</td>
<td>Generator data for the IEEE 39-bus case study (Modified from [1])</td>
<td>131</td>
</tr>
<tr>
<td>Table A2</td>
<td>Transformer data for the IEEE 39-bus case study</td>
<td>131</td>
</tr>
<tr>
<td>Table A3</td>
<td>Power flow results for the IEEE 39-bus case study</td>
<td>132</td>
</tr>
<tr>
<td>Table A4</td>
<td>Transmission line data for the IEEE 39-bus case study</td>
<td>133</td>
</tr>
</tbody>
</table>
List of Figures

Figure 1.1 A classical equivalent representation of the power system shows that the electrical power delivered depends on the voltages, angles and network reactance. ................................................................. 4

Figure 1.2 AC power transmission limits for (a) steady-state conditions and (b) transient conditions. ................................................................. 5

Figure 2.1 The process to create a complex signal with frequency side-bands around 0Hz from a sinusoidal signal with frequency side-bands around $\omega_0$, such that a sinusoidal signal rotating around 60Hz can now rotate around 0Hz. ................................................................. 24

Figure 2.2 The SFA solution method: a real continuous-time signal is shifted to the SFA world, discretized and solved for in the discrete-time SFA domain, interpolated into a continuous-time signal in the SFA domain, and reverse transformed back to the physical domain. The imaginary portion is dropped and the real part of the signal is kept to create the physical continuous-time solution. ............... 25

Figure 2.3 Numerical discretization with the trapezoidal rule, where the area under the curve is $\text{Area} = \frac{v(t) + v(t - \Delta t)}{2} \Delta t$. ................................................................. 30

Figure 2.4 Numerical discretization with the backward Euler rule, where the area under the curve is $\text{Area} = v(t) \Delta t$. ................................................................. 32

Figure 2.5 Four representations of an RL circuit energized by an AC source: (a) the time-domain, (b) the phasor domain, (c) the EMTP, and (d) the SFA domain. ................................................................. 37

Figure 2.6 The node voltages and current through the inductor of the RL circuit in Figure 2.5 comparing the traditional phasor, the EMTP, and the SFA solution methods. ................................................................. 38
Figure 3.1 An LTI system’s response to an input signal that is the single-frequency exponential [2].

Figure 3.2 The frequency responses of an inductor and a capacitor whose values are L=1 and C=1 are the frequency responses of a differentiator and an integrator when current is the input and voltage is the output.

Figure 3.3 The accuracy of the trapezoidal rule acting as a differentiator (magnitude and phase response).

Figure 3.4 The accuracy of the trapezoidal rule acting as an integrator (magnitude and phase response).

Figure 3.5 The accuracy of the trapezoidal rule acting as a differentiator (around 0% error, which occurs when $|H_{\text{approx}}(\omega)/H_{\text{exact}}(\omega)| = 1$).

Figure 3.6 The accuracy of the backward Euler rule acting as a differentiator (magnitude and phase response).

Figure 3.7 The accuracy of the backward Euler rule acting as an integrator (magnitude and phase response).

Figure 3.8 The accuracy of the backward Euler rule acting as a differentiator (around 0% error, which occurs when $|H_{\text{approx}}(\omega)/H_{\text{exact}}(\omega)| = 1$).

Figure 3.9 A comparison of the accuracy of the trapezoidal and backward Euler rules acting as an integrator around 0 Hz.

Figure 3.10 A comparison of the magnitude of the accuracy of the trapezoidal and backward Euler frequency responses acting as a differentiator around 60 Hz using $\Delta t = 8$ ms and $\Delta t = 10$ ms.

Figure 3.11 A DC current source of I=1 A energizes an inductor of L=1 mH.

Figure 3.12 The voltage across an inductor in response to a 1A DC current in the EMTP: (a) the response given when using the trapezoidal rule and (b) the response given when using the backward Euler rule.

Figure 3.13 The voltage across an inductor in response to a 1A DC current using the trapezoidal rule in SFA: (a) the response given in the shifted domain and (b) the response given in the physical domain.

Figure 3.14 The voltage across an inductor in response to a 1A DC current using the backward Euler rule in SFA: (a) the response given in the shifted domain and (b) the response given in the physical domain.

Figure 3.15 An RL circuit energized with a DC voltage source.
LIST OF FIGURES

Figure 3.16 The RL circuit discretized with the EMTP (left) and the SFA (right). ............................................................... 69
Figure 3.17 A comparison of the trapezoidal and backward Euler rules in response to a DC source, using the EMTP as the reference solution. 70
Figure 4.1 The rotor angle of the generator is the relative angle between the stator winding and the rotor’s field winding, taking the stator as the reference frame and using the simplified model of the synchronous generator (Section 4.1.2). ............................ 72
Figure 4.2 The TSFA solution algorithm. .......................... 84
Figure 5.1 The single-line diagram of the Single-Generator Infinite-Bus (SGIB) network, where the electrical network is represented as an infinite bus. A three-phase balanced fault is applied to the terminals of the generator. ................................................................. 85
Figure 5.2 A comparison of the TSFA and TSAT solutions compared with the EMTP solution for the electromechanical variables in the system of Figure 5.1 during a three-phase fault at the terminals of the generator. ................................................................. 88
Figure 5.3 A comparison of the EMTP, TSFA using $\Delta t = 5$ ms and $\Delta t = 8$ ms and the phasor solution using $\Delta t = 5$ ms and $\Delta t = 8$ ms for the electromechanical variables during the three-phase fault in the system of Figure 5.1 ................................................................. 90
Figure 5.4 The voltage at the faulted bus (terminals of Generator 1) during the 3-phase fault. ............................................. 91
Figure 5.5 The fault current during a 3-phase fault at the terminals of Generator 1 (Top: using $\Delta t = 8$ ms for TSFA, Bottom: using $\Delta t = 1$ ms for TSFA). ................................................................. 92
Figure 5.6 A comparison of using the grid frequency, $\omega$, with the approximate frequency, $\omega_o$, in the SFA solution and in the phasor solution compared to the EMTP for the three-phase fault on the SGIB. .......... 93
Figure 5.7 A comparison of the trapezoidal and backward Euler discretization rules on the rotor angle, frequency, electrical power, and fault voltage for the three-phase fault condition. Top left: Generator’s rotor angle, Top right: Generator’s frequency, Bottom left: Generator’s electrical power, Bottom right: Fault voltage. ........... 95
LIST OF FIGURES

Figure 6.1 The one-line diagram of the three-bus network, where two generators feed a constant impedance load. ............................. 96

Figure 6.2 A comparison of the EMTP, TSFA, and TSAT solutions for the electromechanical variables of Generator 1 in the system of Figure 6.1 during a three-phase fault at the terminals of Transformer 2. 101

Figure 6.3 A comparison of the EMTP, TSFA, and TSAT solutions for the electromechanical variables of Generator 2 in the system of Figure 6.1 during a three-phase fault at the terminals of Transformer 2. 102

Figure 6.4 The voltage at the faulted bus during a 3-phase fault at the terminals of Transformer 2. ................................. 103

Figure 6.5 The fault current during a 3-phase fault at the terminals of Transformer 2 (using a $\Delta t = 8$ ms for the SFA solution). ......................... 103

Figure 6.6 The fault current during a 3-phase fault at the terminals of Transformer 2 (using a $\Delta t = 1$ ms for the SFA solution). ......................... 103

Figure 6.7 A comparison of using the grid frequency, $\omega$, with the approximate frequency, $\omega_0$, in the SFA solution in the electromechanical equation for the three-phase fault. ................................. 104

Figure 6.8 A comparison of using the grid frequency, $\omega$, with the approximate frequency, $\omega_0$, in the phasor solution for the three-phase fault. 106

Figure 6.9 A comparison of the trapezoidal and backward Euler discretization rules in SFA on the rotor angle, frequency, electrical power, and fault voltage during the three-phase fault at the terminals of Transformer 2. ................................. 107

Figure 6.10 A three-phase fault is cleared when the voltage crosses zero to capture the maximum DC offset. The SFA solution, using both the trapezoidal and backward Euler rules, and TPhasor* solution are compared to the EMTP. ................................. 108

Figure 6.11 The frequency of Generator 2 after disconnecting Generator 1 from the three-bus network in Figure 6.1 using the backward Euler rule (top) and both rules (bottom). ................................. 110

Figure 6.12 The insert of Figure 6.11. Left: without the trapezoidal rule, Right: with the trapezoidal rule. ................................. 110
LIST OF FIGURES

Figure 6.13 A comparison of the traditional phasor solution using the corrected electromechanical equation. .................................................. 110

Figure 7.1 The IEEE 39-bus test system. .................................................. 111

Figure 7.2 A comparison of the EMTP, TSFA, and TSAT solutions for the electromechanical variables of Generator 5 in the system of Figure 7.1 during a three-phase fault on Bus 16. .................................................. 116

Figure 7.3 A comparison of the SFA solution using the actual frequency $\omega$ instead of the approximated frequency $\omega_0$. ............................... 117

Figure 7.4 The voltage at the faulted bus during a 3-phase fault at Bus 16 in Figure 7.1. ................................................................. 118

Figure 7.5 The fault current during a 3-phase fault at Bus 16 in Figure 7.1. 118

Figure 7.6 The frequency of Generator 5 after losing Generator 9 in the 39-bus Network of Figure 7.1 using the backward Euler rule (top) and both the backward Euler and trapezoidal rules (bottom). Using the trapezoidal rule gives numerical oscillations. ............................... 119
# List of Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CCT</td>
<td>Critical Clearing Time</td>
</tr>
<tr>
<td>EMTP</td>
<td>Electromagnetic Transients Program</td>
</tr>
<tr>
<td>p.u.</td>
<td>Per Unit</td>
</tr>
<tr>
<td>SFA</td>
<td>Shifted Frequency Analysis</td>
</tr>
<tr>
<td>SGIB</td>
<td>Single-Generator Infinite-Bus</td>
</tr>
<tr>
<td>TPhasor</td>
<td>Transient Phasor</td>
</tr>
<tr>
<td>TSAT</td>
<td>Transient Security Assessment Tool</td>
</tr>
<tr>
<td>TSFA</td>
<td>Transient Shifted Frequency Analysis</td>
</tr>
</tbody>
</table>
Nomenclature

In this thesis, three types of representations for the electrical variables are used: instantaneous time-varying scalars, steady-state phasors, and time-varying phasors. To differentiate among the representations, the following nomenclature is used (unless used in another format in the literature):

- Instantaneous time-varying scalars are written in italics. E.g., \( v(t) \) and \( i(t) \).

- Phasors are written in italics with a bar, but independent of time. E.g., \( \bar{V} \) and \( \bar{I} \).

- Time-varying phasors are written in italics with a bar, as functions of time. E.g., \( \bar{V}(t) \) and \( \bar{I}(t) \).
Acknowledgements

I would like to express my deepest gratitude to my supervisor Prof. Juri Jatskevich. His gentle kindness and patience during my journey in the graduate program allowed me to not only grow in technical knowledge, but to mature as a person. His technical advice and guidance in this research allowed the ideas to strengthen and be dynamic. His ingrained teachings to take things one step at a time is a life-long lesson. I am grateful for his kindness, not only in this research, but in life.

I would also like to express my gratitude to Prof. KD Srivastava, whose wisdom guided me at the beginning of my journey, teaching me to reach out to all areas in the world and society beyond the scope of engineering. Our long conversations would leave me in dumbfounded awe and with the notion that even as a small entity, I have the commitment and desire to do as much good as possible in this world.

I am thankful to my friends in the power group for every conversation we have had over the years has opened my eyes to new beliefs and understandings about our role in the community, and our purpose in this world. I am grateful for your kindness and deep desires to respect knowledge and freedom. Thank you for teaching me that all ideas and feelings are integral.

To my Papa, Mama and Catherine. The values they have instilled in me through their own lives have taught me more than anything else. Their strengths to overcome any obstacles, optimisms in humanity and quests for the truth resonant throughout each of their lives, providing me with a little spark of courage to live my own life as such. Their unwavering patience, wisdom and love for me and their belief in me never falter and each time I fall, they pick me up with trust that I will go from crawling to walking a little farther each time. Thank you for imagining far beyond what is physically seen.
Dedication

To my Papa, Mama and Catherine. You are my reason for being.
Chapter 1

Introduction

1.1 Research Motivation

Frequency fluctuations around the grid frequency are prominent on the electric power grid when large disturbances perturb the network. Traditional transient stability simulation programs assume that these fluctuations are close to grid frequency, \( f_o \), which is 60 Hz in North America and the solution methods used in these programs use the constant 60-Hz frequency when modelling the electric network [3]. This assumption was acceptable when modelling the traditional power grid, where heavy, high-inertial synchronous generators convert resources, such as coal, oil, hydro and natural gas, into electricity. Physically, these generators store kinetic energy in their rotating components, which is proportional to their mass and their frequency of rotation

\[
E_k = \frac{1}{2} I \omega^2, \tag{1.1}
\]

where \( E_k \) is the amount of stored kinetic energy (J), \( I \) is the generator’s inertia constant (kg·m²) and \( \omega \) is the frequency of rotation (\( \text{rad/s} \)), which in this case is the grid frequency: \( \omega = \omega_o = 2\pi f_o \) [4]. The stored kinetic energy contributes to the generator’s ability to react and to adapt to external forces. The heavy hydroelectric and thermoelectric generators store large amounts of kinetic energy and have a large constant of inertia, which gives them the ability to remain relatively stable during disturbances. When there are no disturbances on the grid, the generators rotate in synchronicity with each other at \( \omega_o \). When a disturbance occurs on the grid, the generators will deviate from their steady-state frequency, or velocity, but are inertially reluctant to do so, as mathematically captured in Newton’s second law for rotational bodies

\[
T_{\text{net}} = I \alpha, \tag{1.2}
\]
1.1. Research Motivation

which, for power systems, is used in the following form \[4\]

\[
T_{\text{net}} = I \frac{d^2 \delta}{dt^2}, \tag{1.3a}
\]

\[
T_{\text{net}} = I \frac{d\omega}{dt}, \tag{1.3b}
\]

\[
P_{\text{net}} = \omega_o I \frac{d\omega}{dt}. \tag{1.3c}
\]

where in (1.3a), \(I\) is the machine’s inertia constant (kg\(\cdot\)m\(^2\)), \(\delta\) is the machine’s rotor angle (radians), \(T_{\text{net}}\) is the net difference between the mechanical input torque and the electrical output torque (Nm) and in (1.3b), \(\omega\) is the machine’s velocity (\(\text{rad}\)/s). In (1.3c), net torque is converted to net power using the grid frequency \(\omega_o\), as it is net power that is measured in the power network. These equations show that when there is a power imbalance, the rate of frequency change is determined by the machine’s inertia constant. Due to the large inertia in these synchronous generators, when such a disturbance changes the power balance, the frequency can deviate \(\approx \pm 0.1\)-0.5 Hz from the grid frequency\[5\]. Even so, larger frequency excursions from the grid frequency can occur, such as in the 2003 Italian blackout, when the grid frequency decreased 2.5 Hz, from 50 Hz to below 47.5 Hz, before the frequency controllers interfered \[6\].

On the modern grid, there is an increase of renewable energy plants, such as wind farms and solar photovoltaic systems contributing to power generation. The power electronic controllers, which couple these renewable energy plants to the grid, have no rotational motion, and therefore, no rotational inertia \[7\][8]. The lack of inertia from these plants lowers the grid’s overall inertia and now, when a disturbance perturbs the system, the frequency deviations will be greater than on the traditional high-inertial grid. For example, in the South Australian grid, where 48.4% of generation comes from wind and rooftop PV \[9\], during the 2016 South Australian blackout, there was a frequency drop of 3 Hz, from the 50 Hz grid frequency to below 47 Hz \[10\]. In addition to the grid’s decrease in inertia from the renewable sources replacing the high-inertial generators, certain power grid codes mandate that below a given frequency, the renewable energy source must be completely disconnected from the grid. For example, in the EDF Island Energy System, the renewable energy sources, such as wind plants and PV arrays, are required to disconnect when the frequency falls below a set value, in one case, 49.5 Hz \[11\]. The disconnection
The Role of AC Synchronization and Power Equilibrium For Power Grid Stability

of these renewable energy sources further decreases the grid’s inertia, increasing its susceptibility to frequency deviations and decreasing its ability to remain stable during large disturbances.

As the share of renewable energy sources is expected to provide 40% of the world’s total electricity by 2040 [12], the solution method in the simulation programs used to model the grid’s behaviour must capture the frequency deviations. However, the traditional transient stability tools that assume the constant 60-Hz frequency in their models [3][13] are not able to accurately model these fluctuations and the alternative solution programs that can capture these frequency deviations [14][15] are computationally slower than the constant frequency solution methods. There is a need to develop new solution methods for transient stability tools that can capture frequency deviations around the 60-Hz frequency while maintaining the high computational speed of the traditional tools.

This thesis proposes the Shifted Frequency Analysis (SFA) solution method for transient stability studies. SFA is able to model these grid frequency fluctuations without sacrificing computational time. This thesis is motivated to determine if SFA can be an alternative solution method to the traditional methods used in current transient stability tools.

1.2 The Role of AC Synchronization and Power Equilibrium For Power Grid Stability

This section provides a brief overview of two factors integral to the AC power grid’s stability: the synchronization of the generators and the network’s power equilibrium.

Synchronization

The AC power grid’s stability depends on the generators rotating in synchronization with each other. One concern with large frequency deviations is that they create a net power imbalance (1.3c). In balanced steady-state operating conditions, $P_{\text{out}} = P_{\text{in}}$ or $P_{\text{mech}} = P_{\text{elec}} + P_{\text{losses}}$ and the generators rotate and produce AC electricity at the grid frequency $f_0$. Sudden disturbances, such as large loads connecting, generators disconnecting, or faults occurring, change the net power balance. When the power is unbalanced, each of the generators releases or intakes more internal power.
The Role of AC Synchronization and Power Equilibrium For Power Grid Stability

kinetic energy \((1.1)\) and \((1.3c)\), \(\uparrow \downarrow T_{\text{net}} \rightarrow \uparrow \downarrow E_k\), moving their frequency away from \(f_0\) and “out of step”\([4]\) with each other. This asynchronicity impacts the grid’s stability. One primary concern of power system stability studies is the grid’s ability to maintain synchronicity after such sudden disturbances perturb the system \([4]\).

Stability

Stability is “the quality of the system or part of the system which enables it to develop restoring forces between these elements equal or greater than the disturbing forces so as to restore a state of equilibrium” \([16]\). In AC power systems, this state of equilibrium refers to the synchronous machines remaining in synchronism with each other \([4]\). There is a relationship between the maximum power transmitted and the generator’s rotor angle position. A classical equivalent representation of the power system \([4]\) is given in Figure 1.1.

In this representation, the generator delivers electrical power \(P_e\) (MW) to the load through the network’s equivalent reactance \(X_{\text{eq}}\) (\(\Omega\)). The voltages, \(E_{\text{gen}}\) and \(E_{\text{load}}\), are the voltages of the generator and load (V) and \(\delta_{\text{rel}}\) is the angle between the generator’s rotor and stator windings (radians or degrees). The voltage at the load is taken as the reference bus, with an angle of \(0^\circ\). The AC power transmission of Figure 1.1 is mathematically represented in \((1.4)\)

\[
P_e = \frac{E_{\text{gen}}E_{\text{load}}}{X_{\text{eq}}} \sin(\delta_{\text{rel}}).
\] \((1.4)\)
1.2. The Role of AC Synchronization and Power Equilibrium For Power Grid Stability

Maximum power is transferred when $\delta_{\text{rel}} = 90^\circ$, ($P_{\text{max}} = \frac{E_1 E_2}{X_{\text{eq}}}$) and this is called the system’s “steady-state stability limit” [4] because any extra power transmitted will cause the rotor to move out of synchronicity (Figure 1.2(a)).

After a disturbance, the grid is no longer in steady-state and its stability depends on the imbalance of power in the system. For example, in the case of a fault, the electrical power decreases and $P_{\text{mech}} > P_{\text{elec}}$. The excess mechanical energy is transferred to rotational kinetic energy and the rotor speeds up, increasing its relative angle away from its steady-state position ($\uparrow \delta_{\text{rel}}$). If the fault is cleared, but the rotor has accelerated past a critical point in which the generator has gained more kinetic energy than it can release, the system is unstable. The equal-area curve, shown in Figure (1.2(b)), depicts this energy balance. In Figure (1.2(b)), Area 1, from $\delta_1 \rightarrow \delta_2$, is the amount of kinetic energy that the generator has accumulated as it accelerates, where $\delta_2$ is the rotor angle at the time the fault is cleared. Area 2, from $\delta_2 \rightarrow \delta_3$ is the amount of kinetic energy that must be borrowed from the rotational masses so that the rotor can decelerate. When the two areas are equal, the system is stable. The transient stability limit occurs when the rotor angle is at a point where the two areas are equal. This angle is called the critical clearing angle, $\delta_c$. If the fault is cleared after the rotor angle has moved beyond $\delta_c$ then the system is unstable [4]. If the fault is cleared before the rotor angle has reached $\delta_c$, the system is stable.

![Figure 1.2: AC power transmission limits for (a) steady-state conditions and (b) transient conditions.](image)
1.3 Transient Solution Methods Used in Transient Stability Studies

Transient stability simulation tools are used to calculate the transient stability limits before the system becomes unstable. These programs depend on mathematical solution methods which model the power grid. The validity of the given solution method determines how close the model is to the physical network. This section presents an overview of two solution methods used to model the electrical network in power system dynamic studies: the phasor solution method and the Electromagnetic Transients Program (EMTP), and introduces the proposed time-domain solution method, SFA. Section 1.3.1 also provides a brief introduction to the electromechanical equation used to model the electromechanical dynamics, however Chapter 4 gives a more thorough discussion of the coupling between the electromechanical and electrical network dynamics.

1.3.1 The Traditional Phasor Solution Method

Background
The birth of calculating transient stability limits came in the early 20th century, when the introduction of automatic voltage regulators led to the increased development of transmission lines with high impedances. These lines allowed large amounts of power to be transmitted closer to the steady-state stability limit [4][16]. Consequently, the system became more susceptible to disturbances and instability frequently occurred [4]. The engineers saw a need to develop analytical methods that would determine power transfer limits such that the system would remain stable. These limits were called transient stability limits [16].

To determine these transient stability limits, the engineers modelled and analysed the power system using AC network analysers. The analysers would compute the power and rotor angle for off-line network contingencies at step-by-step time intervals [4]. From these results, time-angle curves called “swing-curves” would be drawn for the different contingencies. As faults were one of the most critical contingencies [4], these curves would determine the critical rotor angle, as explained in Section 1.2. The time that the critical angle occurred at is a critical parameter in transient stability studies as it determines the maximum time that the physical circuit breakers have to open and successfully clear the fault, called the critical clearing time.
1.3. Transient Solution Methods Used in Transient Stability Studies

To simplify the mathematical equations, an approximation was made to the main electromechanical equation that governs the machine’s dynamics (1.3a). Because the rotor angle and velocity determined in (1.3a)(1.3b) are used in the electrical network solution to calculate electrical power, which couples the electrical network and electromechanical dynamics; and it is electrical power, not torque, that is the output of the electrical network solution, the net torque in (1.3a) is converted to net power through frequency \( \omega \). However, using \( \omega \) makes the equations non-linear and for simplification, the constant \( \omega_0 \) is used in the conversion

\[
P_{\text{net}} = T_{\text{net}} \cdot \omega_0 \quad \text{and} \quad T_{\text{net}} = I \frac{d\omega}{dt} \rightarrow P_{\text{net}} = \omega_0 I \frac{d\omega}{dt}.
\]

(1.5)

To model the electrical network, the traditional phasor solution method was used.

**Phasor Solution Method**

In the phasor solution method, a sinusoidal time-domain signal is frequency transformed into a frequency-domain signal. The sinusoidal signal is represented with its Euler representation [17]

\[
v(t) = \hat{V} \cos(\omega_0 t + \delta_v) = \mathcal{R}(\hat{V}e^{j(\omega_0 t + \delta_v)}) = \mathcal{R}(\hat{V}e^{j\omega_0 t}e^{j\delta_v}) = \mathcal{R}(\hat{V}e^{j\omega_0 t}).
\]

\[
i(t) = \hat{I} \cos(\omega_0 t + \delta_i) = \mathcal{R}(\hat{I}e^{j(\omega_0 t + \delta_i)}) = \mathcal{R}(\hat{I}e^{j\omega_0 t}e^{j\delta_i}) = \mathcal{R}(\hat{I}e^{j\omega_0 t}).
\]

(1.6a) (1.6b)

where \( \hat{V} = \hat{V}e^{j\delta_v} \) and \( \hat{I} = \hat{I}e^{j\delta_i} \) are the “phasor-representations” of the sinusoidal signal. The phasor is independent of time and rotates at a constant frequency \( \omega_0 \) with a magnitude and an angle \((\hat{V} = \hat{V} \angle \delta_v, \hat{I} = \hat{I} \angle \delta_i)\).

Equations (1.6a) and (1.6b) are applied to the time-instantaneous voltages and currents, \( v(t) \) and \( i(t) \), in the differential equations that govern the network elements

\[
v_L(t) = L \frac{d}{dt} i_L(t) \quad \text{(1.7a)} \quad i_C(t) = C \frac{d}{dt} v_C(t) \quad \text{(1.7b)}
\]
1.3. Transient Solution Methods Used in Transient Stability Studies

to obtain their frequency-domain representations

\[ v_L(t) = L \frac{d}{dt} i_L(t) \]
\[ i_C(t) = C \frac{d}{dt} v_C(t) \]

\[ R \left( \tilde{V} e^{j\omega_0 t} \right) = L \frac{d}{dt} \left( R \left( \tilde{I} e^{j\omega_0 t} \right) \right) \]
\[ R \left( \tilde{V} e^{j\omega_0 t} \right) = j\omega_0 L \left( R \left( \tilde{I} e^{j\omega_0 t} \right) \right) \] \hspace{0.5cm} (1.8a)
\[ R \left( \tilde{I} e^{j\omega_0 t} \right) = C \frac{d}{dt} \left( R \left( \tilde{V} e^{j\omega_0 t} \right) \right) \]
\[ \tilde{I} = j\omega_0 CV. \] \hspace{0.5cm} (1.8b)

In (1.8a) and (1.8b), the operator \( \frac{d}{dt} \) becomes \( j\omega \) in the frequency domain because

\[ \frac{d}{dt} (e^{j\omega t}) = j\omega (e^{j\omega t}), \] \hspace{0.5cm} (1.9)

and the inductors and capacitors are represented as constant impedances and admittances rotating at \( \omega_0 \)

\[ Z_L = j\omega_0 L. \] \hspace{0.5cm} (1.10a)
\[ Y_C = j\omega_0 C. \] \hspace{0.5cm} (1.10b)

The transformation from time to frequency is the basis of the phasor solution method and as seen in (1.10a) and (1.10b), the constant frequency \( \omega_0 \) is used in this method.

**Constant Frequency Assumption**

Using the constant frequency \( \omega_0 \) for the electrical network model (1.8a)(1.8b) is an approximation to the actual grid frequency \( \omega \). If the frequency of the power network remains close to synchronous frequency, using the constant frequency in the phasor solution and in the electromechanical equation (1.5) helped to simplify the mathematical calculations in the early transient stability programs. The phasor solution method for modelling the electrical network and the approximate-frequency electromechanical equation are implemented in the current computer simulation programs used for transient stability analysis [3][13][18].
1.3. Transient Solution Methods Used in Transient Stability Studies

1.3.2 The EMTP Solution Method

In the 1960s, with the advent of digital computers, H. W. Dommel developed the Electromagnetic Transients Program (EMTP) [19]. The EMTP is a discrete, time-domain solution method, where the continuous-time differential equations (1.7a) and (1.7b) are replaced with difference equations through a numerical discretization method. For example, (1.11) derives the discretization of an inductor using the trapezoidal discretization rule and (1.12) derives the discretization of a capacitor using the trapezoidal rule [20].

\[
\begin{align*}
    v_L(t) &= L \frac{d}{dt} i_L(t) \\
    \int_{t-\Delta t}^{t} v_L(t) dt &= L[i_L(t) - i_L(t - \Delta t)] \\
    \frac{v_L(t) + v_L(t - \Delta t)}{2} \Delta t &= L[i_L(t) - i_L(t - \Delta t)] \\
    v_L(t) &= \frac{2L}{\Delta t} i_L(t) + [-v_L(t - \Delta t) - \frac{2L}{\Delta t} i_L(t - \Delta t)] \\
    v_L(t) &= \frac{2L}{\Delta t} i_L(t) + e_{hL}(t),
\end{align*}
\]

and

\[
\begin{align*}
    i_C(t) &= C \frac{d}{dt} v_C(t) \\
    \int_{t-\Delta t}^{t} i_C(t) dt &= C[v_C(t) - v_C(t - \Delta t)] \\
    \frac{i_C(t) + i_C(t - \Delta t)}{2} \Delta t &= C[v_C(t) - v_C(t - \Delta t)] \\
    v_C(t) &= \left(\frac{\Delta t}{2C}\right)i_C(t) + [v_C(t - \Delta t) + \frac{\Delta t}{2C} i_C(t - \Delta t)] \\
    v_C(t) &= \left(\frac{\Delta t}{2C}\right)i_C(t) + e_{hC}(t).
\end{align*}
\]

In (1.11) and (1.12), the terms \(e_{hL}(t)\) and \(e_{hC}(t)\), (where \(e_{hL}(t) = [-v_L(t - \Delta t) - \frac{2L}{\Delta t} i_L(t - \Delta t)]\) and \(e_{hC}(t) = [v_C(t - \Delta t) + \frac{\Delta t}{2C} i_C(t - \Delta t)]\)) are called history sources because they depend on the voltage and current of the inductor and capacitor at the previous time-step [19].

The EMTP solution method is highly accurate because the discretization of the differential equations retains the physical relationship between voltage and current.
in the inductors and capacitors, inherently tracking network fluctuations by tracing the voltage and current instantaneous waveforms. In addition, the equivalent voltage across the inductor and the current through the capacitor are dependent on the previous value of the voltage and current at the time-step before \( t - \Delta t \). This means that every discrete value of the solution, e.g., \( v(t) \) and \( i(t) \), remembers the value of all the previous solution points, also called the variable’s history. Keeping the variable’s history is important as it mathematically represents the physical nature of an inductor or capacitor, which store the kinetic energy, or memory, in their magnetic and electric fields.

The EMTP has been used as an alternative to the traditional phasor solution methods in transient stability studies [21][22]; however, due to the discretization of the continuous-time signal, its high level of accuracy comes with two limitations:

1. There exists a direct relationship when mapping a signal between time and frequency. A time-window of length \( T_c \), discretized into \( k = N \) samples at a time-interval of \( \Delta t \), corresponds to a frequency-window of length \( F_c \), also discretized into \( k = N \) samples. Frequency is inversely proportional to time: \( F = \frac{1}{T} \), so \( F_c = \frac{1}{N\Delta t} \). The frequency-window is centered around zero and includes positive frequencies up to half the of the frequency window, \( \frac{F_c}{2} \), and the complex conjugates of these frequencies. Half of the frequency window, \( \frac{F_c}{2} \), is the maximum frequency that can exist in the signal and it is called the Nyquist frequency [23].

2. It takes approximately 8-10 samples to sample one period of a sinusoidal waveform [20].

With the above conditions, if \( N = 10 \) samples and \( F = \frac{1}{T} \) and the trapezoidal rule is used to discretize the continuous-time signal, to ensure that there is less than 3% error in the discretization of the continuous-time signal, the maximum signal simulated is \( F_{\text{max}} = \frac{1}{5} f_{Ny} \) [20]. Therefore, the time-step used is

\[
\Delta t = \frac{1}{10 \times F_{\text{max}}}.
\] (1.13)

or

\[
\Delta t = \frac{1}{2 \times f_{Ny}}.
\] (1.14)
1.3. Transient Solution Methods Used in Transient Stability Studies

Based on (1.13), for transient stability studies around 60 Hz, the EMTP will require a time-step of approximately 1 ms to solve the electrical network. In addition, the EMTP time-step is limited by the travelling time of the waves in the transmission lines. If the transmission lines are modelled with the constant-parameter [19] or frequency-dependent [24] line models, the time-step may be smaller.

Although the EMTP solution can capture the detailed network frequency fluctuations due to the discretization of the differential equations, the small discretization time-step sizes required makes the computer simulation time longer than if the traditional phasor solution, which uses the constant 60-Hz frequency in its model of the electrical network solution, was used. This is a disadvantage in large systems; albeit, the phasor solution is a more approximate solution.

1.3.3 Time-varying Phasors and the Introduction of the Shifted Frequency Analysis Solution Method

In the past decade, there has been growing research interest in developing a new solution method that can capture the network fluctuations around the grid frequency, like the EMTP, whilst maintaining the computational speed of the traditional phasor method (for example, in [25][26][27][28][29][30][31][32][33]). The idea of a “time-varying phasor” germinated in the early 1990s when V. Venkatasubramanian noticed that the phasor method gave “dubious results” for voltage stability studies because it could not capture the large dynamics [25]. He began working on a phasor solution method which would be able to track the system’s dynamics during these transients. This phasor was called a “fast time-varying phasor” [34][35]. Around the same time, G. Verghese [26][36] developed a generalized averaging method based on time-varying Fourier coefficients for state-space averaging of power electronic converters, which he proposed could also be used as a type of time-varying phasor dynamic approach for the electric power network [36]. G. Verghese’s method and V. Venkatasubramanian’s method branched off into two directions for the research into dynamic phasors: the use of time-varying Fourier coefficients and the use of a phasor transformation.

G. Verghese and Dynamic Phasors

The generalized averaging method developed by G. Verghese [26][36] was based on Fourier analysis, in which a continuous-time signal is approximated into a sum of its
1.3. Transient Solution Methods Used in Transient Stability Studies

exponentials and a time-window of length \([t - T_c, t]\), “slides” across the continuous-time signal and changes the values of the coefficients \(X_n\) and \(\theta_n\)

\[
f_o = X_o + X_1 e^{j(\omega_1 t + \theta_1)} + X_1^* e^{-j(\omega_1 t + \theta_1)} + X_2 e^{j(\omega_2 t + \theta_2)} + X_2^* e^{-j(\omega_2 t + \theta_2)} + \ldots + X_n e^{j(\omega_n t + \theta_n)} + X_n^* e^{-j(\omega_n t + \theta_n)}
\]

or

\[
x(t - T_c + s) = \sum_{k=\infty}^{k=-\infty} (\langle x \rangle_k(t)e^{j\omega_s(t-T_c+s)}),
\]

and \(\omega_s = \frac{2\pi}{T_c}\), \(\epsilon(0, T_c]\) and \(\langle x \rangle_k(t)\) is the \(k^{th}\) time-varying Fourier coefficient [26]

\[
\langle x \rangle_k(t) = \frac{1}{T_c} \int_0^{T_c} x(t-T_c+s)e^{-j\omega_s(t-T_c+s)} ds.
\]

The purpose of decomposing the signal into Fourier coefficients was to create an equivalent state-space model where the coefficients \(\langle x \rangle_k(t)\) are the state variables. In the power system, these variables are the voltage and current sinusoids with time-varying amplitude and phase. Equivalent complex branch equations were derived for the inductor and capacitor, where the time-domain quantity in the original differential equation \((\frac{d}{dt})\) is replaced by its phasor representation \((j\omega_0 + \frac{d}{dt})\) [36]. The subsequent works of [27], [28], [30] further developed this method and it is commonly referred to as the dynamic phasor solution method. However, one concern with using this method is that because the signal is a continuous-time signal, the entire frequency spectrum is expressed in each new time-window. When approximating the signal, frequencies higher than the Nyquist frequency are then truncated, even though they still exist. This truncation may lead to frequency distortions if not carefully taken care of.

V. Venkatasubramanian and the Phasor Transformation

V. Venkatasubramanian [34], on the other hand, recognized that a phasor is simply a “mathematical transformation, where the only time-varying component, frequency, is eliminated” [34]. Therefore, a time-varying phasor is also a mathematical transformation, but one that does not eliminate the frequency component. V. Venkatasubramanian derived the time-varying phasor in the following manner [37].
1.3. Transient Solution Methods Used in Transient Stability Studies

1. He first defined time-varying phasors for voltage and current as

\[ \vec{E}(t) = E(t)e^{j\varphi_e(t)}, \quad (1.18a) \]
\[ \vec{I}(t) = I(t)e^{j\varphi_i(t)}. \quad (1.18b) \]

2. He then created a transformation operator, \( \mathcal{P} \) such that the instantaneous voltage and current sinusoids \( e(t), i(t) \)

\[ e(t) = \sqrt{2}E(t)\cos(\omega_0 t + \theta_e(t)), \quad (1.19a) \]
\[ i(t) = \sqrt{2}I(t)\cos(\omega_0 t + \theta_i(t)), \quad (1.19b) \]

can relate to the time-varying phasors

\[ \mathcal{P}(e(t)) := E(t)e^{j\varphi_e(t)}. \quad (1.20a) \]
\[ \mathcal{P}(i(t)) := I(t)e^{j\varphi_i(t)}. \quad (1.20b) \]

3. The network parameters were redefined as linear differential equations in the complex domain in terms of \( (1.18a), (1.18b) \). For example, the voltage and current relationship in an inductor \( (1.7a) \) is redefined using the transformation operator

\[ \vec{E}_L(t) = \mathcal{P}(e_L(t)) \]
\[ = \mathcal{P}(L\frac{d}{dt}(i_L(t))) \]
\[ = L\frac{d}{dt}(\vec{I}_L(t)) + j\omega_0 L\vec{I}_L(t), \quad (1.21) \]

where \( \vec{I}_L(t) \) is defined in \( (1.18b) \).

4. The transformation operator that allows the time-differential relationship to be defined in terms of phasors in the above equation \( (1.21) \) is \( \mathcal{P} = e^{j\omega_0 t} \). This transformation was derived from the Blondel transformation matrix used in power systems for balanced three-phase signals \[34\]. Two integral properties of \( \mathcal{P} \) are that: (1) \( \mathcal{P} \) is a linear transformation and (2) \( \mathcal{P}(\frac{d}{dt}e(t)) = \frac{d}{dt}(\vec{E}(t)) + j\omega \vec{E}(t) \). These properties are important because it demonstrates that using the single-frequency exponential can create a transformation from a time representation to a complex phasor representation. This was further expanded in \[31\].
1.3. Transient Solution Methods Used in Transient Stability Studies

The phasor representations in (1.18a) and (1.18b) can be rewritten in (1.22a), (1.22b) by decomposing the exponential into a sum of its real and imaginary components. The concept of real and imaginary components was again further expanded in [31].

\[
\vec{E}(t) = E(t) \angle \delta(t) = E_d(t) + jE_q(t), \quad (1.22a)
\]

\[
\vec{I}(t) = I(t) \angle \delta(t) = I_d(t) + jI_q(t). \quad (1.22b)
\]

These time-varying phasors were then used to decompose the three-phase power systems into its symmetrical components to model both balanced and unbalanced conditions. The hope was that the time-varying phasors would “unify transient analysis and steady-state analysis” [37].

H. W. Dommel, S. Henshel and modelling Dynamic Phasors using the EMTP

The work done by H. W. Dommel and S. Henshel in [31] further developed the concept of dynamic time-varying phasors. The objective was to unify the electromagnetic and electromechanical transients in the power system by applying the numerical discretization methods used in the EMTP to the new complex-valued differential equations created by time-varying phasors. Instead of applying three transformations to obtain one complex signal as in [37], the dynamic phasor method in [31] expressed each phase as one complex signal multiplied by \( e^{-j\omega_0 t} \), which generalized the dynamic phasor concept beyond three-phase networks. The general form of the dynamic phasor approach here is described next.

Similar to [34], a complex signal is first constructed from a real signal. Defining \( s(t) \) as the real signal, and \( S(\omega) \) as a phasor with real and imaginary components, the relationship between \( s(t) \) and \( S(\omega) \) is: \( s(t) = \mathcal{R}\left(S(\omega)\right) \). Treating the power system transients as bandpass signals around \( \omega_0 \) and applying Fourier analysis, the bandpass signal is decomposed into two low-pass signals, \( s_I(t) \) and \( s_Q(t) \), with two sinusoidal carrier signals surrounding \( \omega_0 \) and \( -\omega_0 \) (\( \cos(\omega_0 t) \) and \( \sin(\omega_0 t) \)).

\[
s(t) = s_I(t) \cos(\omega_0 t) - s_Q(t) \sin(\omega_0 t), \quad (1.23)
\]
1.3. Transient Solution Methods Used in Transient Stability Studies

where $s_I(t)$ and $s_Q(t)$ contain all the information about the dynamic transients surrounding $\omega_o$, which now surround 0 Hz. Equation (1.24) is called the “complex envelope of the real signal” or dynamic phasor

$$\mathcal{D}[S(t)] = s_I(t) + js_Q(t).$$  \hspace{1cm} (1.24)

Another way to represent the complex envelope is in (1.25), where $\mathcal{H}$ is the Hilbert transformation of the signal $s(t)$

$$S(t) = (s_I(t) + js_Q(t))e^{j\omega_o t} = s(t) + j\mathcal{H}[s(t)].$$  \hspace{1cm} (1.25)

In (1.25), the complex envelope is represented in terms of the real and imaginary part of the analytical signal. Replacing the arbitrary signal in (1.25) with the voltage and current sinusoids, the differential equations for the network elements (1.7a)-(1.7b) are now complex-valued

$$\mathcal{D}[V](t) = j\omega_o LD[I](t) + L\frac{d}{dt}\mathcal{D}[I](t),$$  \hspace{1cm} (1.26)

where $\mathcal{D}$ denotes the “dynamic phasor” notation.

Creating a low-pass signal surrounding 0 Hz from a band-pass signal surrounding $\omega_o$ is conceptually similar to defining a transformation that shifts a signal down from $\omega_o$ to 0 Hz. The concept of formulating a transformation that can shift the frequency of a signal was introduced in [32] and was named Shifted Frequency Analysis (SFA).

**J. R. Martí and Shifted Frequency Analysis**

Shifted Frequency Analysis (SFA) is a solution method developed by J. R. Martí [32] as a general framework to unify the concept of dynamic phasors with the discrete-time solution method of the EMTP. The SFA framework has its roots in discrete-time signal processing theory [32], where

1. The Fourier decomposition of a sinusoidal signal, $y(t) = A \cos(\omega_o t + \theta)$ has two frequency components: $\omega_o$ and $-\omega_o$.

2. The signal $e^{-j\omega t}$ is the only signal with a single-frequency [38] and using it as the input to a linear time-invariant system, allows all the properties of the original system to be retained in the output.
3. In Fourier analysis, the frequency-shifting property states that a time signal multiplied by a complex exponential at $\omega_o$ is equivalent to shifting the signal by the frequency $\omega_o$

$$x(t)e^{j\omega_o t} \rightarrow X(\omega - \omega_o)$$ (1.27)

By making use of these concepts, a frequency transformation $T = e^{-j\omega_o t}$ is defined [32], where $\omega_o$ is the signal’s fundamental frequency. The transformation $T$ “shifts” the original signal by $-\omega_o$ and the dynamics surrounding the original signal now surround the lower frequency-shifted signal. In the voltage and current electrical power signals, the fundamental frequency is the grid frequency (60 Hz). Using the above concepts 1 and 3 and applying $T$ to the 60 Hz voltage and current sinusoids, the 60-Hz frequency is shifted down by -60 Hz to 0 Hz (DC) and all the dynamics surrounding the original 60 Hz waveform are preserved in the shifted signal but now surrounding 0 Hz (DC). This is important because the SFA solution method, which is a discrete-time method, can now use large time-steps corresponding to frequency fluctuations around 0 Hz.

The SFA transformation is analogous to a change of variables from one domain (the physical domain) to another domain (the shifted domain)

$$v_{SFA}(t) = T v_{ph}(t), \quad (1.28a)$$
$$i_{SFA}(t) = T i_{ph}(t), \quad (1.28b)$$

where $v_{ph}(t) = V(t)e^{j(\omega_o t + \theta_v(t))}$ and $i_{ph}(t) = I(t)e^{j(\omega_o t + \theta_i(t))}$.

By applying $T$ to the time-differential equations (1.7a) and (1.7b), equivalent network branches for the inductor and capacitor are created. These branches preserve the physical nature of the elements through the differential term $\frac{d}{dt}$ but combined with a complex term $j\omega_o$. The SFA network elements are derived in Chapter 2.

The SFA solution method originated in [32], was formally described in [33][39] and was applied to synchronous and induction machine modelling in [40][41][42]. A hybrid multi-rate simulation framework that interfaces the EMTP solution with the SFA solution is currently being developed [43]. In this thesis, a generalized SFA-based solution algorithm is programmed and used for transient stability studies.
1.4 Research Objectives

The ability for the SFA solution to trace frequency fluctuations around the 60 Hz grid frequency using large discretization time-steps led to the following research question:

Can the SFA solution method be an alternative to the traditional phasor and the EMTP solution methods in transient stability studies due to its ability to trace the network dynamics using large time-steps?

The research in this thesis also reformulates how the electromechanical equation (1.3) used in transient stability studies is evaluated. Traditionally, in (1.3), net torque is converted to net power by using the approximate velocity $\omega_0$ (1.3b)(1.3c), which simplifies the equations. In the proposed solution algorithm, (1.3) is solved in discrete-time and the actual velocity $\omega$ is solved for at each time-step. This velocity is used in the conversion from net torque to net power. This observation led to the following research question:

Does using the actual velocity in the conversion between torque and power in the electromechanical equation lead to a difference in calculating transient stability limits, such as the critical clearing time?

To answer the above questions, the following objectives are set:

1. To compare the SFA solution method with the traditional phasor method and the EMTP in transient stability analysis case studies, using the EMTP as the reference solution method.

2. To determine the critical clearing times of different three-phase short-circuit fault contingencies in each of the case studies and to compare the results given by the three solution methods.

3. To determine if using the actual velocity instead of the approximate velocity in the electromechanical equation makes a difference in the solution accuracy.

My work on these research objectives led to the following contributions:
1.5 Research Contributions

The main contributions in this thesis are as follows:

1. A generalized SFA-based algorithm is written for a general power system network topology and the algorithm is used for transient stability analysis.

2. In this work, the SFA solution method is formulated in a generalized manner, such that SFA can be used with any arbitrary source type, not just sinusoidal sources. This is different than in the previous works, where SFA was only used with sinusoidal sources \[33\][39].

3. An interpolation procedure is introduced in the SFA algorithm, which allows the system to be solved with large time-steps when modelling the system in the shifted domain, while recapturing the physical solution with high resolution when transformed back into the physical 60 Hz domain.

4. The electromechanical equation is generalized to account the use of the actual velocity at every time-step of the simulation instead of using the approximate 60-Hz frequency in the equation. This is implemented in the SFA-based algorithm to compare the differences between using the known velocity and the approximate velocity.

5. A zero-crossing detection procedure for the fault current is implemented in the SFA algorithm for the transient stability studies. In this procedure, the fault current is detected to cross zero before clearing the fault in the shifted domain instead of detecting the zero-crossing of the fault current in the physical domain, which would have required shifting the solution back to the physical domain at every time-step. By detecting the zero-crossing in the shifted domain, the procedure saves computational time.

1.6 Thesis Outline

The thesis outline is as follows:

Chapter 2 describes the SFA solution framework and solution method. The equivalent network branches are derived in the SFA domain and discretized using two numerical discretization methods. An example of an RL circuit compares the SFA
solution method with the traditional phasor solution and the EMTP.

Chapter 3 presents a numerical error analysis of the two discretization rule used to discretize the SFA solution method in this work: the trapezoidal rule and the backward Euler rule, which demonstrates the accuracy of each discretization method in the SFA domain. In addition to numerical accuracy, the numerical method’s response to a discontinuity, such as a step input, is modelled and analysed. This analysis is useful in understanding how the solution method can handle network discontinuities, such as the sudden opening of a circuit breaker. In this chapter, an example of an RL circuit energized by a step input is modelled, which demonstrates that the SFA solution method can be used for any arbitrary source type.

Chapter 4 summarizes the main concepts of transient stability and derives the discretization of the electromechanical equation using the correct network frequency. The constraints and objectives for the case studies analysed in Chapter 5, Chapter 6 and Chapter 7 are also defined here.

Chapter 5 applies the SFA solution method to a Single-Generator Infinite-Bus system, Chapter 6 applies the SFA solution method to a three-bus system, and Chapter 7 applies the SFA solution method to the IEEE 39-bus system. For each case study, a three-phase fault is applied to the network and the generator’s electromechanical variables are compared using the three solution methods, with the EMTP solution as the reference. The SFA solution is solved using both the trapezoidal and backward Euler methods to compare the differences between the two discretization rules and the electromechanical equation is solved using both the approximate and exact network frequency. The critical clearing times and critical clearing angles are calculated using each solution method and compared with each other, again, using the EMTP solution as the reference. In the three-bus network of Chapter 6, a three-phase fault is applied such that the fault current has the maximum DC offset, to demonstrate that the SFA solution can capture the DC component of the fault current just like the EMTP. In the three-bus network and the 39-bus network, a generator is disconnected from the network and the solution is analysed to determine how the loss of a generator has a strong effect on the network frequency.

Chapter 8 summarizes the main findings of the thesis and presents ideas for future research.
Chapter 2

Shifted Frequency Analysis: A Discrete-Time Solution Method and Solution Framework

Shifted Frequency Analysis (SFA) is both a discrete-time solution method and a solution framework. In SFA, a continuous-time signal is transformed into an equivalent continuous-time complex-valued signal through a left-frequency shifting transformation. The transformation $T = e^{-j\omega_0 t}$ maps the signal from the physical domain to a shifted domain. This mapping preserves all the dynamics that surround the original signal.

Section 2.1 introduces the SFA framework. Section 2.2 describes the SFA solution method, Section 2.3 derives the equivalent SFA model for the electrical network components, Section 2.4 discretizes the equivalent SFA network elements using the trapezoidal and backward Euler rules, Section 2.5 compares the network branches of the SFA solution method, the EMTP solution method and the traditional phasor solution method, and Section 2.6 illustrates the similarities and differences among the three solution methods in an RL circuit energized by an AC source.

2.1 Shifted Frequency Analysis Solution Framework

The backbone of the SFA solution framework is built upon two concepts: (1) the use of a single-exponential frequency transformation, and (2) the time-frequency relationship in converting a continuous-time signal to a discrete-time signal.
2.1. Shifted Frequency Analysis Solution Framework

2.1.1 A Single-Frequency Signal

The SFA framework defines a single-frequency transformation as $T = e^{-j\omega_0 t}$ and applies it to a signal in the physical domain. Theoretically, this is equivalent to taking the sinusoidal time-domain signal of the form $y(t) = A(t)e^{j\omega_0 t + \theta(t)}$, which rotates around $\omega_0$, and left-frequency shift the signal by $-\omega_0$, such that the resulting signal rotates around 0 Hz. This is possible because the exponential signal $x(t) = Ae^{\pm j\omega t}$ has a single-frequency and when it is multiplied to another signal rotating at frequency $\omega_1$, the resulting signal will rotate at $\omega_1 \pm \omega$.

Once the physical domain signal have been frequency-shifted to the SFA domain the second property, the time-frequency relationship, is invoked to discretize the continuous-time signal into a discrete-time signal.

2.1.2 The Time-Frequency Relationship

As explained in Section 1.3.2, there is a relationship between time and frequency[23], such that any real continuous-time signal $x(t)$ can be segmented into time-windows of length $T_c$ and discretized into $k = N$ samples, where each sample has a time-width $\Delta t$. Each time-window corresponds to a frequency-window of length $F_c$, discretized into $k = N$ samples, with each sample having a frequency width $\Delta f$. The relationship is that $F_c = \frac{1}{\Delta t}$ and $\Delta t = \frac{1}{F_c}$, where $\Delta t$ is the discretization time-width, $F_c$ is the length of the frequency-window and $\frac{F_c}{2}$ is the maximum frequency that can exist in the signal, otherwise known as the Nyquist frequency. This relationship bears the following consequences:

1. The signal in each time-window can be expressed as a sum of its exponential components, with both the positive frequencies and their complex conjugate negative frequencies (Fourier decomposition).

2. Frequencies greater than the Nyquist frequency do not exist.

3. The value of $\Delta t$ is limited by (1.13).

2.1.3 The SFA Solution Framework

The SFA framework defines the single-frequency transformation as $T = e^{-j\omega_0 t}$ and applies it to a signal in the physical domain. After the signal has been left-frequency
2.1. Shifted Frequency Analysis Solution Framework

shifted into the SFA world, the continuous-time signal is discretized into a discrete-time signal, adhering to the time-frequency relationship and (1.13). Because the shifted signal is of the form \( Y(t) = A(t) \angle \theta(t) \), which is a time-varying complex number (phasor), the magnitude of the phasor is the envelope of the physical signal and it captures the signal’s dynamics. The transformation retains the rotational relationship between real and imaginary parts of the signal so that in the shifted domain, both real and imaginary components exist and comply with their perpendicular relationship (being 90° apart from each other).

In summary, the use of the single-frequency exponential and the discretization of the continuous-time signal provides the following groundwork for the SFA solution method:

1. As the system shifts from the 60 Hz domain to the 0 Hz domain, all the dynamics embodied in the original 60 Hz signal are preserved in the shifted 0 Hz signal.

2. The continuous-time differential equations are discretized into a discrete-time solution method and because the signal is shifted to 0 Hz, using (1.13), the time-steps theoretically can be infinitely large [32].

3. Because the SFA solution method is a discrete-time method and the time-frequency relationship of Sec.1.3.2 holds true, the time-step is fixed to the maximum frequency wished to be accurately simulated. Thus, frequencies outside the fixed frequency-window do not exist. This is one difference between SFA and the other dynamic phasor methods, such as [26][27][30].

4. Because the transformation is a frequency shifting rotational transformation, the perpendicularity between the real and imaginary components of a complex signal is preserved. Therefore, the voltage or current source used in the physical domain can be represented as a complex exponential, with both real and imaginary components existing in the shifted domain without need for the Hilbert Transform as used in [33] and [39].

Section 2.2 formulates the SFA solution method.
2.2 The SFA Solution Method

To model the electrical network with the SFA solution method, first the frequency-shifting transformation \( T = e^{-j\omega_0 t} \) is applied to the voltage and current sources.

In the power system, the voltage and current time-domain signals \( v(t) \) and \( i(t) \) rotate sinusoidally at the grid frequency, \( \omega_0 \). The frequency decomposition of a sinusoidal signal is two frequencies: \( \omega_0 \) and \( -\omega_0 \). When the exponential transformation is applied to \( v(t) \) and \( i(t) \), the frequency components become \( \omega + \omega_0 \) and \( \omega - \omega_0 \). Because the frequency of the transformation is chosen as \( \omega = -\omega_0 \), then the frequency components resulting are \( \omega = 0 \) and \( \omega = -2\omega_0 \). By disregarding the negative frequency component and multiplying the amplitude by two, the sinusoid can be represented as a single-frequency exponential rotating around 0 Hz. This is because the real part of the exponential is the sinusoidal signal \( \Re e^{j\omega t} = \cos(\omega t) \).

\[
\begin{align*}
\bar{V}(t) &= V(t)e^{j\theta_v(t)}, \\
\bar{I}(t) &= I(t)e^{j\theta_i(t)}.
\end{align*}
\]
Alternatively, (2.1) is equivalent to representing the voltage and current sinusoids using their Euler representation at the frequency $\omega_o$ and applying the frequency-shifting exponential with frequency $\omega = -\omega_o$ to create phasors

\[
\bar{V}(t) = V(t)e^{j(\omega_o t + \theta_v(t))}e^{-j\omega_o t} = V(t)e^{j\omega_o t}e^{-j\omega_o t}e^{j\theta_v(t)} = V(t)e^{j\theta_v(t)}. \\
\bar{I}(t) = I(t)e^{j(\omega_o t + \theta_i(t))}e^{-j\omega_o t} = I(t)e^{j\omega_o t}e^{-j\omega_o t}e^{j\theta_i(t)} = I(t)e^{j\theta_i(t)}.
\]

(2.2) (2.3)

The resulting signal shifts to the frequency $\omega = \omega - \omega_o$. When $\omega = \omega_o$, the frequency is 0 Hz ($\omega = \omega_o - \omega_o$) and all dynamics that surrounded the original signal around $\omega_o$ surround 0 Hz as is depicted in Figure 2.1.

Figure 2.1: The process to create a complex signal with frequency side-bands around 0 Hz from a sinusoidal signal with frequency side-bands around $\omega_o$, such that a sinusoidal signal rotating around 60 Hz can now rotate around 0 Hz.

The same discrete-time solution techniques employed in the EMTP for the electrical network are applied to the continuous-time signals in the complex SFA world. First, the differential equations governing the network elements, resistors, inductors and capacitors, are discretized into complex network equations using the trapezoidal or backward Euler numerical discretization methods. A discretization time-step which complies with the time-frequency relationship described in 2.1.2 is chosen. These complex network branches form the network admittance matrix $[Y]$ and through nodal analysis, $[Y][V] = [I]$, the node voltages are solved for as phasors with a time-varying magnitude and angle, where the magnitude is the envelope that traces
the dynamic behaviour. Like with the EMTP, the network branches depend on the solution at each previous time-step or the variable’s history. The history sources are included in the matrix \([I]\) when solving the nodal analysis, such that \([I_{\text{total}}]\) is a matrix of external current sources \([I]\) plus a matrix of history sources \([H]\). The discrete-time solution is then converted back to continuous-time using a linear interpolation scheme. This one-step linear interpolation allows the large time-steps to be used to solve the system in the SFA domain, while bringing back the detailed solution to the physical domain. The continuous-time complex signal in the SFA domain is now transformed back to the physical domain through the inverse transformation \(T^{-1} = e^{+j\omega_0 t}\). Finally, the imaginary portion of the complex-signal is dropped to obtain the real-valued physical solution. The SFA solution method is described in Figure 2.2.

![Diagram of the SFA solution method](image)

Figure 2.2: The SFA solution method: a real continuous-time signal is shifted to the SFA world, discretized and solved for in the discrete-time SFA domain, interpolated into a continuous-time signal in the SFA domain, and reverse transformed back to the physical domain. The imaginary portion is dropped and the real part of the signal is kept to create the physical continuous-time solution.
2.2. The SFA Solution Method

2.2.1 The Choice of Time-Step in the SFA Solution Method

The time-step used in the SFA discrete-time solution method can theoretically be infinitely large as formulated in (1.13): \( \Delta t = \frac{1}{10 \times 0} \) ms as the signal now rotates around 0 Hz. Physically, there will be frequency deviations in the dynamics of the transient. For example, if the frequency deviates 4 Hz from 60-64 Hz in the network, the time-step used can be \( \Delta t = \frac{1}{10 \times 4} = 25 \) ms.

With the un-shifted 60-Hz signal, the EMTP would require a time-step of \( \Delta t = \frac{1}{10 \times 64} = 1.56 \) ms. There is a time-step ratio between the EMTP and the SFA of 15 times. In reality, the EMTP time-step would be approximately 10 times less, or 100 \( \mu \)s and the time-step ratio between the EMTP and the SFA solution will be roughly 160 times. Note, however, that because the SFA solution is complex-valued, the total number of operations in the solution will decrease this ratio.

2.2.2 A Summary of the SFA Solution Method

A summary of the SFA solution method is provided below.

1. Apply \( T = e^{-j\omega t} \) to the continuous-time signals \( v_{ph}(t) \) and \( i_{ph}(t) \) in the sources and network elements.

2. Form a discrete-time signal from the shifted continuous-time signal using a numerical discretization method, such as the trapezoidal or backward Euler rules on the sources and network elements.

3. Form the admittance matrix \( [Y] \) and solve for node voltages using nodal analysis, where the current source vector \( [I_{total}] = [I_{external}] + [H] \), where \( [H] \) is the vector of history sources and \( [Y][V] = [I_{external} + H] \). The node voltages are phasors with a time-varying magnitude and angle.

4. Do a linear interpolation of the discrete-time SFA solution to convert back to a continuous-time SFA solution.

5. Apply the inverse transformation \( T^{-1} = e^{j\omega t} \) to the SFA solution and drop the imaginary portion of the complex signal.
2.3 The SFA Equivalent Network Elements

The shifting transformation $T$ is applied to the voltages and currents in the time-differential equations that define the network elements: resistors, inductors and capacitors. The SFA equivalent representation of these network elements are derived below. In the following derivations, the subscript “SFA” is used for the shifted domain and the subscript “ph” is used for the physical domain. Again, $v_{SFA} = Tv_{ph}$ or $v_{ph} = T^{-1}v_{SFA}$ where $T = e^{-j\omega t}$.

**Resistors**

The relationship between voltage and current in a resistor is

$$v_{R_{ph}}(t) = R_i_{R_{ph}}(t). \tag{2.4}$$

By applying $T$ to both sides of (2.4), one obtains

$$Tv_{R_{ph}}(t) = T(R_i_{R_{ph}}(t))$$

$$v_{R_{SFA}}(t) = R_i_{R_{SFA}}(t), \tag{2.5}$$

or in phasor notation:

$$\bar{V}_{R_{SFA}}(t) = R\bar{I}_{R_{SFA}}(t). \tag{2.6}$$

The equivalent relationship between voltage and current in the shifted domain is the same as in the physical domain, which matches the physical nature of a linear resistor, which dissipates energy.

**Inductors**

The relationship between voltage and current in an inductor is

$$v_{L_{ph}}(t) = L\frac{di_{L_{ph}}(t)}{dt}. \tag{2.7}$$

Applying the change of variables transformation $T$ to both sides of (2.7)

$$v_{ph} = T^{-1}v_{SFA}, \quad i_{ph} = T^{-1}i_{SFA}$$


2.3. The SFA Equivalent Network Elements

\[ T^{-1}v_{LSFA}(t) = L \frac{d}{dt}(T^{-1}i_{LSFA}(t)) \]

\[ e^{j\omega_0 t}v_{LSFA}(t) = L \frac{d}{dt}(e^{j\omega_0 t}i_{LSFA}(t)) \]

\[ e^{j\omega_0 t}v_{LSFA}(t) = L \frac{d}{dt}(i_{LSFA}(t))e^{j\omega_0 t} + j\omega_0 L i_{LSFA}(t)e^{j\omega_0 t} \]

\[ v_{LSFA}(t) = L \frac{d}{dt}(i_{LSFA}(t)) + j\omega_o Li_{LSFA}(t), \quad (2.8) \]

or in phasor notation

\[ \bar{V}_{LSFA}(t) = L \frac{d}{dt} \bar{I}_{LSFA}(t) + j\omega_o L \bar{I}_{LSFA}(t). \quad (2.9) \]

**Capacitors**

The relationship between voltage and current in a capacitor is

\[ i_{c_{ph}}(t) = C \frac{d}{dt}(v_{c_{ph}}(t)). \quad (2.10) \]

Applying the change of variables transformation \( T \) to both sides of (2.10)

\[ T^{-1}i_{c_{SFA}}(t) = C \frac{d}{dt}(T^{-1}v_{c_{SFA}}(t)) \]

\[ e^{j\omega_0 t}i_{c_{SFA}}(t) = C \frac{d}{dt}(e^{j\omega_0 t}v_{c_{SFA}}(t)) \]

\[ e^{j\omega_0 t}i_{c_{SFA}}(t) = C \frac{d}{dt}(v_{c_{SFA}}(t))e^{j\omega_0 t} + j\omega_o C v_{c_{SFA}}(t)e^{j\omega_0 t} \]

\[ i_{c_{SFA}}(t) = C \frac{d}{dt}(v_{c_{SFA}}(t)) + j\omega_o C v_{c_{SFA}}(t), \quad (2.11) \]

or in phasor notation:

\[ \bar{I}_{c_{SFA}}(t) = C \frac{d}{dt} \bar{V}_{c_{SFA}}(t) + j\omega_o C \bar{V}_{c_{SFA}}(t). \quad (2.12) \]

The equivalent network branch of either an inductor or a capacitor has both a differential part \( \frac{d}{dt} \) and a complex impedance \( j\omega_0 L \) or \( j\omega_0 C \). Therefore, the voltage across the inductor and the current through the capacitor are complex-valued or phasors.
2.4. The Discretization of the SFA Network Elements

Sources
The sinusoidal source is represented using Euler’s representation of a complex signal

\[ v_{ph}(t) = V(t)e^{j(\omega_o t + \theta_v(t))}, \quad (2.13) \]

and

\[ i_{ph}(t) = I(t)e^{j(\omega_o t + \theta_i(t))}. \quad (2.14) \]

Applying the transformation to both sides of (2.13) and (2.14) gives

\[ T v_{ph}(t) = TV(t)e^{j(\omega_o t + \theta_v(t))} \]

\[ T i_{ph}(t) = TI(t)e^{j(\omega_o t + \theta_i(t))} \]

\[ v_{SFA}(t) = e^{-j\omega_o t}V(t)e^{j\omega_o t}e^{j\theta_v(t)} \quad (2.15) \]

\[ i_{SFA}(t) = e^{-j\omega_o t}I(t)e^{j\omega_o t}e^{j\theta_i(t)} \quad (2.16) \]

Equations (2.15) and (2.16) are identical to (2.2) and (2.3).

2.4 The Discretization of the SFA Network Elements

A numerical discretization rule creates continuous-time differential equations that model the inductors and capacitors in (2.9) and (2.12) into discrete-time difference equations. The discretization rule chosen must be both numerically accurate in comparison with the continuous-time solution and must maintain numerical stability during discontinuities. J. R. Martí and J. Lin [38] validated that in the EMTP, both the trapezoidal and the backward Euler discretization rules maintain both numerical accuracy and stability when discretizing a first-order differential equation. Because the EMTP solution method is applied to the signals in the SFA domain, it is hypothesized that the trapezoidal and backward Euler rules hold the same numerical accuracy and stability in the SFA domain. This hypothesis is verified in Chapter 3 through the error analysis of the two rules in the SFA domain.

Sections 2.4.1 and 2.4.2 derive the discrete-time branch equivalents in the SFA domain using the trapezoidal and the backward Euler rules. For simplicity, the subscript “SFA” is dropped in the derivations in this section.
2.4. The Discretization of the SFA Network Elements

2.4.1 Equivalent SFA Network Elements Using the Trapezoidal Discretization Rule

The trapezoidal rule is applied to the SFA equivalent network elements. The trapezoidal rule calculates the area under the curve of a trapezoid as shown in Figure 2.3.

![Figure 2.3: Numerical discretization with the trapezoidal rule, where the area under the curve is Area = \( \frac{v(t) + v(t - \Delta t)}{2} (\Delta t) \).](image)

Resistors

In a resistor, there are no differentials in the relationship between voltage and current, and the continuous-time equation (2.4) remains the same in discrete-time.

\[
\bar{V}_R(t) = R\bar{I}_R(t). \tag{2.17}
\]

Inductors

The trapezoidal rule is applied to the SFA equivalent inductor given in (2.9)

\[
\frac{\bar{V}_L(t) + \bar{V}_L(t - \Delta t)}{2} \Delta t = L(\bar{I}_L(t) - \bar{I}_L(t - \Delta t)) + j\omega_0 L \left( \frac{\bar{I}_L(t) + \bar{I}_L(t - \Delta t)}{2} \Delta t \right). \tag{2.18}
\]
Rearranging for voltage on the left side and current on the right side

\[
\tilde{V}_L(t) + \tilde{V}_L(t - \Delta t) = \frac{2L}{\Delta t} \left( \tilde{I}_L(t) - \tilde{I}_L(t - \Delta t) \right) + j\omega_0 L (\tilde{I}_L(t) + \tilde{I}_L(t - \Delta t))
\]

\[
\tilde{V}_L(t) = -\tilde{V}_L(t - \Delta t) + \frac{2L}{\Delta t} \left( \tilde{I}_L(t) - \tilde{I}_L(t - \Delta t) \right) + j\omega_0 L (\tilde{I}_L(t) + \tilde{I}_L(t - \Delta t))
\]

\[
\tilde{V}_L(t) = \left( \frac{2L}{\Delta t} + j\omega_0 L \right) \tilde{I}_L(t) + [(\frac{-2L}{\Delta t} + j\omega_0 L)\tilde{I}_L(t - \Delta t) - \tilde{V}_L(t - \Delta t)]
\]

\[
\tilde{V}_L(t) = \left( \frac{2L}{\Delta t} + j\omega_0 L \right) \tilde{I}_L(t) - [(\frac{2L}{\Delta t} - j\omega_0 L)\tilde{I}_L(t - \Delta t) + \tilde{V}_L(t - \Delta t)].
\]

(2.19)

The equivalent inductor using the trapezoidal rule is an impedance \( \frac{2L}{\Delta t} + j\omega_0 L \) in series with a history voltage source, because it retains the voltage at the previous time-step \( t - \Delta t \) [19]. The history source is of the opposite polarity as the voltage across the inductor \( \tilde{V}_L \) and is a complex vector

\[
\bar{E}_h(t) = \left( \frac{2L}{\Delta t} - j\omega_0 L \right) \tilde{I}_L(t - \Delta t) + \tilde{V}_L(t - \Delta t).
\]

(2.20)

Capacitors

Similarly, the trapezoidal rule is applied to the SFA equivalent capacitor given in (2.12)

\[
\int_{t-\Delta t}^{t} \tilde{I}_C(t) \, dt = \int_{t-\Delta t}^{t} C \frac{d}{dt} \tilde{V}_C(t) + \int_{t-\Delta t}^{t} j\omega_0 C \tilde{V}_C(t) \, dt
\]

\[
\frac{\tilde{I}_C(t) + \tilde{I}_C(t - \Delta t)}{2} \Delta t = C(\tilde{V}_C(t) - \tilde{V}_C(t - \Delta t)) + j\omega_0 C \left( \frac{\tilde{V}_C(t) + \tilde{V}_C(t - \Delta t)}{2} \right) \Delta t.
\]

(2.21)

Rearranging (2.21), one obtains

\[
\tilde{I}_C(t) + \tilde{I}_C(t - \Delta t) = \frac{2C}{\Delta t} \left( \tilde{V}_C(t) - \tilde{V}_C(t - \Delta t) \right) + j\omega_0 C (\tilde{V}_C(t) + \tilde{V}_C(t - \Delta t))
\]

\[
\tilde{I}_C(t) = -\tilde{I}_C(t - \Delta t) + \frac{2C}{\Delta t} \left( \tilde{V}_C(t) - \tilde{V}_C(t - \Delta t) \right) + j\omega_0 C (\tilde{V}_C(t) + \tilde{V}_C(t - \Delta t))
\]

\[
\tilde{I}_C(t) = \left( \frac{2C}{\Delta t} + j\omega_0 C \right) \tilde{V}_C(t) + [(\frac{-2C}{\Delta t} + j\omega_0 C)\tilde{V}_C(t - \Delta t) - \tilde{I}_C(t - \Delta t)]
\]

\[
\tilde{I}_C(t) = \left( \frac{2C}{\Delta t} + j\omega_0 C \right) \tilde{V}_C(t) - [(\frac{2C}{\Delta t} - j\omega_0 C)\tilde{V}_C(t - \Delta t) + \tilde{I}_C(t - \Delta t)].
\]

(2.22)
2.4. The Discretization of the SFA Network Elements

The equivalent capacitance using the trapezoidal rule is an admittance \( \frac{2C}{\Delta t} + j\omega_0C \) in parallel with a history current source, which is a complex vector

\[
\bar{I}_h(t) = \left(\frac{2C}{\Delta t} - j\omega_0C\right)\bar{V}_C(t - \Delta t) + \bar{I}_C(t - \Delta t).
\] (2.23)

To parallel with the inductance, (2.22) can be rearranged in terms of the voltage across the capacitance

\[
\bar{V}_C(t) = \frac{1}{2C\Delta t + j\omega_0C}\bar{I}_C(t) + \left(\frac{2C}{\Delta t} - j\omega_0C\right)\bar{V}_C(t - \Delta t) + \bar{I}_C(t - \Delta t).
\] (2.24)

2.4.2 Equivalent SFA Network Elements Using the Backward Euler Discretization Rule

The backward Euler rule is applied to the SFA network elements. The backward Euler rule calculates the area of a rectangle under the curve, shown in Figure 2.4.

\[
\text{Area} = v(t)\Delta t.
\]

Figure 2.4: Numerical discretization with the backward Euler rule, where the area under the curve is \( \text{Area} = v(t)\Delta t \).

**Resistors**

Like with the trapezoidal rule, the continuous-time equation (2.4) remains the same in discrete-time

\[
\bar{V}_R(t) = R\bar{I}_R(t).
\] (2.25)
2.4. The Discretization of the SFA Network Elements

Inductors

The backward Euler rule is applied to the SFA inductor given in (2.9)

\[
\int_{t-\Delta t}^{t} \bar{V}_L(t) \, dt = \int_{t-\Delta t}^{t} L \frac{d}{dt} \bar{I}_L(t) \, dt + \int_{t-\Delta t}^{t} j\omega_0 L \bar{I}_L(t) \, dt
\]

\[
\bar{V}_L(t) = L(\bar{I}_L(t) - \bar{I}_L(t - \Delta t)) + j\omega_0 L \bar{I}_L(t) \Delta t
\]

\[
\bar{V}_L(t) = \frac{L}{\Delta t} \left( \bar{I}_L(t) - \bar{I}_L(t - \Delta t) \right) + j\omega_0 L \bar{I}_L(t)
\]

The equivalent inductor using the backward Euler method is an impedance \( \frac{L}{\Delta t} + j\omega_0 L \) in series with a history voltage source

\[
\bar{E}_h(t) = \frac{L}{\Delta t} (\bar{I}_L(t - \Delta t)).
\]

Capacitors

Similarly, the backward Euler rule is applied to the SFA capacitor given in (2.12)

\[
\int_{t-\Delta t}^{t} \bar{I}_C(t) \, dt = \int_{t-\Delta t}^{t} C \frac{d}{dt} \bar{V}_C(t) \, dt + \int_{t-\Delta t}^{t} (j\omega_0 C \bar{V}_C(t)) \, dt
\]

\[
\bar{I}_C(t) = C (\bar{V}_C(t) - \bar{V}_C(t - \Delta t)) + j\omega_0 C \bar{V}_C(t) \Delta t
\]

\[
\bar{I}_C(t) = \frac{C}{\Delta t} \left( \bar{V}_C(t) - \bar{V}_C(t - \Delta t) \right) + j\omega_0 C \bar{V}_C(t)
\]

The equivalent capacitance using the backward Euler rule is an admittance \( \frac{C}{\Delta t} + j\omega_0 C \) in parallel with a history current source

\[
\bar{I}_h(t) = \frac{C}{\Delta t} (\bar{V}_C(t - \Delta t)).
\]

To parallel with the inductance, (2.28) can be rearranged in terms of the voltage across the capacitance to obtain

\[
\bar{V}_C(t) = \frac{1}{\frac{C}{\Delta t} + j\omega_0 C} [\bar{I}_C(t) + \frac{C}{\Delta t} \bar{V}_C(t - \Delta t)].
\]
Comparing (2.19) and (2.22) with (2.26) and (2.28), the backward Euler discretization method is easier to implement than the trapezoidal discretization method.

2.5 A Comparison of the Traditional Phasor, the EMTP, and the SFA Network Elements

In the phasor domain, a change of operators transforms the time-domain differential equations into a frequency-domain complex number: \( \frac{d}{dt} \rightarrow s = j\omega_0 \).

In the EMTP domain, the continuous-time differential equations are discretized to discrete-time equations. The relationship between voltage and current in the differential equations are preserved in the discrete-time equivalent representation.

In the SFA domain, there is a change of variables from the physical time-domain differential equations to the complex time-domain differential equations through the transformation \( T \) as well as a discretization of the continuous-time equations. As in the EMTP, the relationship between voltage and current in the differential equations are preserved in the discrete-time equivalent representation, and as in the phasor representation, the equivalent branches are complex-valued.

Equation (2.31) gives the relationship between voltage and current in an inductor given by the traditional phasor, the EMTP and the SFA solution methods, using the backward Euler rule for the EMTP and SFA solutions.

\[
\begin{align*}
v_L(t) &= L \frac{d}{dt} i_L(t) \\
v_L(t) &= L \left( \frac{d}{dt} i_L(t) - (i_L(t - \Delta t)) \right) \\
\bar{V}_L(t) &= \left( \frac{L}{\Delta t} + j\omega_0 L \right) \bar{I}_L(t) - \frac{L}{\Delta t} (\bar{I}_L(t - \Delta t))
\end{align*}
\]

\( \bar{V}_L = j\omega_0 L \bar{I}_L \) Traditional Phasor
\( v_L(t) = \frac{L}{\Delta t} (i_L(t) - (i_L(t - \Delta t))) \) EMTP
\( \bar{V}_L(t) = \left( \frac{L}{\Delta t} + j\omega_0 L \right) \bar{I}_L(t) - \frac{L}{\Delta t} (\bar{I}_L(t - \Delta t)) \) SFA

(2.31)

Tables 2.1 and 2.2 compare the network elements in the traditional phasor domain, the EMTP domain, and the SFA domain using the trapezoidal discretization method (Table 2.1) and the backward Euler discretization method (Table 2.2). Both the traditional and SFA phasor equivalent branches are complex-valued whereas the EMTP’s equivalent branch is real-valued. It can be seen that the SFA solution is a hybrid between the traditional phasor and the EMTP solution.
### Table 2.1

The branch equivalents for the traditional phasor, the EMTP, and the SFA methods using the trapezoidal discretization rule for the SFA and the EMTP.

<table>
<thead>
<tr>
<th>Element</th>
<th>Traditional Phasor</th>
<th>EMTP</th>
<th>SFA</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Resistor</strong></td>
<td><img src="image" alt="Traditional Phasor Resistor" /></td>
<td><img src="image" alt="EMTP Resistor" /></td>
<td><img src="image" alt="SFA Resistor" /></td>
</tr>
<tr>
<td><strong>Inductor</strong></td>
<td><img src="image" alt="Traditional Phasor Inductor" /></td>
<td><img src="image" alt="EMTP Inductor" /></td>
<td><img src="image" alt="SFA Inductor" /></td>
</tr>
<tr>
<td><strong>Capacitor</strong></td>
<td><img src="image" alt="Traditional Phasor Capacitor" /></td>
<td><img src="image" alt="EMTP Capacitor" /></td>
<td><img src="image" alt="SFA Capacitor" /></td>
</tr>
</tbody>
</table>

### Table 2.2

The branch equivalents for the traditional phasor, the EMTP, and the SFA methods using the backward Euler discretization rule for the EMTP and SFA.

<table>
<thead>
<tr>
<th>Element</th>
<th>Traditional Phasor</th>
<th>EMTP</th>
<th>SFA</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Resistor</strong></td>
<td><img src="image" alt="Traditional Phasor Resistor" /></td>
<td><img src="image" alt="EMTP Resistor" /></td>
<td><img src="image" alt="SFA Resistor" /></td>
</tr>
<tr>
<td><strong>Inductor</strong></td>
<td><img src="image" alt="Traditional Phasor Inductor" /></td>
<td><img src="image" alt="EMTP Inductor" /></td>
<td><img src="image" alt="SFA Inductor" /></td>
</tr>
<tr>
<td><strong>Capacitor</strong></td>
<td><img src="image" alt="Traditional Phasor Capacitor" /></td>
<td><img src="image" alt="EMTP Capacitor" /></td>
<td><img src="image" alt="SFA Capacitor" /></td>
</tr>
</tbody>
</table>

---

2.5. A Comparison of the Traditional Phasor, the EMTP, and the SFA Network Elements
2.6 An RL Circuit Energized with an AC Source Using the Traditional Phasor, the EMTP and SFA

An RL circuit energized with an AC voltage source, Figure 2.5, is simulated to demonstrate the similarities and differences among the three solution methods. In Figure 2.5, the three solution methods’ equivalent circuits are depicted.

Nodal analysis is used for all three methods, \([Y][V] = [I_{\text{total}}]\), where

- \([Y]\) is the admittance matrix formed by the network’s branches.
- \([I_{\text{total}}] = [I] + [H]\) is a vector of all the external sources injected into the network plus the history source derived from the discretized L branch (in the EMTP and SFA solutions).
- \([V]\) is the vector of all the unknown bus voltages.

In the circuit of Figure 2.5, using the trapezoidal rule for the EMTP and SFA solutions, and where \(\omega_o = 2\pi 60\):

- \([V] = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}\), where the source value \(v_s(t) = 10 \cos(\omega_o t + 0^\circ)\) in Figures 2.5(a) and 2.5(c) and \(\bar{V}_s = 10 \angle 0^\circ\) in Figures 2.5(b) and 2.5(d).
- \([H_{\text{EMTP}}] = \begin{bmatrix} 0 \\ \frac{e_{L}(t)}{2L} \end{bmatrix}\), \([H_{\text{SFA}}] = \begin{bmatrix} 0 \\ \frac{E_{L}(t)}{2L} + j \omega_o L \end{bmatrix}\), \([H_{\text{phasor}}] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\).
- \([I] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\) and \([I_{\text{total}}] = [H] + [I]\).

The admittance matrix \([Y]\) is compared among the methods

- The admittance matrix for the traditional phasor method has complex terms:

\[
Y_{\text{phasor}} = \begin{bmatrix}
\frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\
-\frac{1}{R_2} & \frac{1}{R_2} + \frac{j \omega_o L}{2}\n\end{bmatrix}
\]

- The admittance matrix for the EMTP has no complex terms:

\[
Y_{\text{EMTP}} = \begin{bmatrix}
\frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\
-\frac{1}{R_2} & \frac{1}{R_2} + \frac{\Delta t}{2R_2}\n\end{bmatrix}
\]
2.6. An RL Circuit Energized with an AC Source Using the Traditional Phasor, the EMTP and SFA

- The admittance matrix for SFA has complex terms:

\[
Y_{\text{SFA}} = \begin{bmatrix}
\frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\
-\frac{1}{R_2} & \frac{1}{2\pi\tau} + \frac{1}{R_2} + j\omega L
\end{bmatrix}
\]

![RL circuit represented in time.](image)

![RL circuit represented in the phasor domain.](image)

![RL circuit represented in the EMTP.](image)

![RL circuit represented in the SFA domain.](image)

Figure 2.5: Four representations of an RL circuit energized by an AC source: (a) the time-domain, (b) the phasor domain, (c) the EMTP, and (d) the SFA domain.

It can be seen from Figure 2.6 that the phasor steady-state solution and the envelope in the SFA domain produce identical results and that when the SFA solution is interpolated back to continuous-time and transformed back into the physical domain, SFA and the EMTP produce the same results. This demonstrates the ability for the SFA solution to give both the dynamic envelope and the instantaneous-time solution.

Due to the time constant in the circuit, \( \tau = 1.98 \text{ ms} \), the time-step used in the EMTP is \( \Delta t = \frac{1}{10} \tau \) or \( \Delta t = 200 \mu s \) (0.2 ms). To solve for the SFA envelope, the time-step used is \( \Delta t = 50 \text{ ms} \). For interpolation back to the time-domain, the interpolation time-step used is \( \Delta t = 200 \mu s \) (0.2 ms).
Figure 2.6: The node voltages and current through the inductor of the RL circuit in Figure 2.5 comparing the traditional phasor, the EMTP, and the SFA solution methods.
Chapter 3

The Numerical Discretization Methods Used in SFA

3.1 The Evaluation of a Numerical Discretization Method Using the Frequency Response

A discrete-time signal discretized from a physical continuous-time signal is an approximation of the physical signal and therefore, the discrete-time solution is an approximation of the continuous-time solution. How close the approximated discrete-time solution is compared to the continuous-time solution depends on both the discretization time-step $\Delta t$ and the discretization method used [38]. The discretization time-step is important as it determines the maximum frequency that can be simulated without distortion introduced [38] as discussed in Section 1.3.2. The discretization method used is equally important as it determines how much distortion will be produced onto the discrete-time solution [2].

Different discretization methods use different approaches to approximate the continuous-time signal. For example, the trapezoidal discretization rule approximates the area of a trapezoid under the given signal (Figure 2.3), whereas the backward Euler rule approximates the area of a rectangle (Figure 2.4). Due to their different approaches, the two discretization rules contain different intrinsic errors and it is important to evaluate these errors when choosing when it is best to use one discretization rule over another. How large the error produced by its approximation to the physical solution is called the accuracy of the discretization rule.

Accuracy, however, is not the only aspect important in evaluating the behaviour of a discretization rule. The rule’s numerical stability is equally important [19], especially when evaluating how the numerical method will respond to discontinu-
3.2. The System’s Frequency Response in the Physical Domain and in the SFA Domain

In the EMTP, the trapezoidal and backward Euler discretization methods are proven to be both numerically stable and accurate for first-order systems \cite{19,38} and these two rules are used in this work for the discretization of the continuous-time SFA solution. The accuracy of a discretization rule can be analysed in the frequency domain by evaluating the frequency response of a discretized differentiator or an integrator and comparing that response with the frequency response of the continuous-time differentiator or integrator \cite{2}. This is discussed in detail in the next section, Section 3.2. Sections 3.3 and 3.4 analyse the accuracy of the trapezoidal and backward Euler rules in the SFA domain through the detailed frequency response of both rules. In Section 3.5, the numerical stability of the two rules is derived in the SFA domain and compared with the numerical stability of the rules in the physical domain, using the EMTP as the reference. The results from this section are illustrated in the response of an inductor to a step current source, shown in Section 3.6. This example illustrates the behaviour of the rules during discontinuities. Finally, the example of an RL circuit energized by a DC source in Section 3.7 demonstrates that the SFA solution is not limited to systems energized by sinusoidal sources.

3.2 The System’s Frequency Response in the Physical Domain and in the SFA Domain

3.2.1 The Frequency Response in the Physical Domain

When the input to a linear time-invariant system is the single-frequency exponential $e^{j\omega t}$ the output, $y(t) = H(\omega)e^{j\omega t}$ will retain the system’s inherent properties in its transfer function $H(\omega)$ as shown in Figure 3.1. The system’s transfer function is unique to each system and so by evaluating the transfer function at every frequency in a set range of frequencies, also known as the frequency response, how the system behaves and responds to the input is determined \cite{2}. If the continuous-time system’s frequency response matches the equivalent discrete-time system’s frequency response, the discrete-time solution is 100% accurate and the further the discrete-time solution’s response is from the continuous-time one, the more error has incurred due to the discretization rule. Therefore, the rule’s accuracy is determined by com-
3.2. The System’s Frequency Response in the Physical Domain and in the SFA Domain

Figure 3.1: An LTI system’s response to an input signal that is the single-frequency exponential [2].

Parling the frequency response of the continuous-time solution with the frequency response of the discrete-time solution, which has been discretized by that rule

\[ H_{\text{error}}(\omega) = \frac{H_{\text{DT}}(\omega)}{H_{\text{CT}}(\omega)}. \]  

(3.1)

One way to determine the frequency response of the rule is to evaluate the rule acting as an integrator or as a differentiator. This is possible to do by evaluating the response of an inductor or a capacitor:

\[ v_L(t) = L \frac{d}{dt} i_L(t) \]  

(3.2)

and

\[ v_C(t) = \frac{1}{C} \int i_C(t) dt. \]  

(3.3)

When \( L = 1 \) or \( C = 1 \) and current is the input to the system, then the frequency response of an inductor is the frequency response of a differentiator and the frequency response of a capacitance is the frequency response of an integrator [2]. In the discrete-time solution of, for example, (1.11)/(1.12), the relationship between the voltage and current input is the discrete-time transfer function and the frequency response is now the response of the discrete-time equivalent integrator or differentiator. Figure[3.2] shows the frequency response of an inductor acting as a differentiator and a capacitor acting as an integrator, where current is the input and voltage is the output. Here, current is replaced with \( e^{j\omega t} \) and voltage is replaced with \( H(\omega)e^{j\omega t} \), with \( H(\omega) \) as the transfer function.

The response of the differentiator and integrator are inversely related, and so it is sufficient to derive one or the other. In this thesis, the response of the inductor

41
3.2. The System’s Frequency Response in the Physical Domain and in the SFA Domain

act as a differentiator is derived for both the trapezoidal and backward Euler rules and the response of the integrator is the inverse of the differentiator. Alternatively, the response of the integrator could have been derived, with the response for the differentiator as its inverse. For each rule, the response of the rule acting as both a differentiator and an integrator are provided (Figures 3.3 3.4 3.6 and 3.7).

3.2.2 The Frequency Response in the SFA Domain

In the SFA domain, the discretization rule is applied to the already shifted signal and so to determine the rule’s accuracy, one takes the ratio of the SFA discrete-time solution’s frequency response to the SFA continuous-time solution’s frequency response

\[ H_{\text{error}}(\omega) = \frac{H_{\text{DT-SFA}}(\omega)}{H_{\text{CT-SFA}}(\omega)} \]  

(3.4)

3.2.3 The Continuous-Time Frequency Response in the SFA Domain

The SFA continuous-time (SFA-CT) frequency response is the frequency response of the continuous-time transformed inductor in the SFA domain (\( \hat{V}_L(t) = L \frac{d}{dt} \hat{I}_L(t) + j\omega_0 L \hat{I}_L(t) \))  

This is derived below in (3.5), where \( \hat{I}_L(t) \) is replaced with \( e^{j\omega t} \), \( \hat{V}_L(t) \) is replaced with \( H(\omega) e^{j\omega t} \), and the derivative operator is replaced with the frequency operator, \( \frac{d}{dt} \Rightarrow s = j\omega \).
3.3. The Frequency Response of the Trapezoidal Discretization Rule in the SFA Domain

\[ \tilde{V}_L(t) = L \frac{d}{dt} \tilde{I}_L(t) + j \omega_0 L \tilde{I}_L(t) \]

\[ H_{ct}(\omega) e^{j\omega t} = j \omega L e^{j\omega t} + j \omega_0 L e^{j\omega t} \]

\[ H_{ct}(\omega) = \frac{j \omega L e^{j\omega t} + j \omega_0 L e^{j\omega t}}{e^{j\omega t}} \]

By comparing the discrete-time frequency response in the SFA domain with the continuous-time frequency response in the SFA domain, the behaviour of the discretization rule applied in the SFA domain can be understood. Sections 3.3 and 3.4 derive the frequency response for the trapezoidal and backward Euler rules.

3.3 The Frequency Response of the Trapezoidal Discretization Rule in the SFA Domain

There are two approaches to obtain the frequency response of a discrete-time signal \[2\]. The first approach is the one used to obtain the SFA-CT frequency response, in which the system’s input was replaced with the single frequency exponential \( x(t) = e^{j\omega t} \), and the system’s output was replaced with \( y(t) = H(\omega) e^{j\omega t} \), where \( H(\omega) \) evaluated at every \( \omega \) is the frequency response. The second approach is to apply the \( z \)-operator to the discretized equation, where \( z \rightarrow t \) and \( z^{-1} \rightarrow t - \Delta t \). When \( z = e^{j\omega \Delta t} \), \( H(z) = H(\omega) \), which is the frequency response \[2\]. Both approaches are derived below and produce identical results, given in the subsequent equations (3.9) and (3.11).

3.3.1 Approach One: Applying the Single-Frequency Exponential

The equivalent inductor discretized using the trapezoidal rule was derived in (2.19) and is reiterated here

\[ \tilde{V}_L(t) = \frac{2L}{\Delta t} + j \omega_0 L \tilde{I}_L(t) - [\frac{2L}{\Delta t} - j \omega_0 L] \tilde{I}_L(t - \Delta t) + \tilde{V}_L(t - \Delta t)] \]

Replacing \( \tilde{I}_L(t) \) with \( e^{j\omega t} \) and \( \tilde{V}_L(t) \) with \( H(\omega) e^{j\omega t} \)

\[ H(\omega) e^{j\omega t} = \frac{2L}{\Delta t} + j \omega_0 L e^{j\omega t} + \left(-\frac{2L}{\Delta t} + j \omega_0 L\right) e^{j\omega (t-\Delta t)} - H(\omega) e^{j\omega (t-\Delta t)} \]
3.3. The Frequency Response of the Trapezoidal Discretization Rule in the SFA Domain

After cancelling out $e^{j\omega t}$ from both sides of the equation and rearranging terms, (3.7) gives the response $H(\omega)$ in terms of exponentials

$$H(\omega)e^{j\omega t} + H(\omega)e^{j\omega(t-\Delta t)} = \frac{2L}{\Delta t} + j\omega_0 L e^{j\omega t} + (-\frac{2L}{\Delta t} + j\omega_0 L) e^{j\omega(t-\Delta t)}$$

$$H(\omega)e^{j\omega t} + H(\omega)e^{-j\omega \Delta t} = \frac{2L}{\Delta t} e^{j\omega t} + j\omega_0 Le^{j\omega t} - \frac{2L}{\Delta t} e^{-j\omega \Delta t} + j\omega_0 Le^{-j\omega \Delta t}$$

$$H(\omega)(1 + e^{-j\omega \Delta t}) = \frac{2L}{\Delta t}(1 - e^{-j\omega \Delta t}) + j\omega_0 L(1 + e^{-j\omega \Delta t})$$

$$H(\omega) = \frac{2L}{\Delta t} + j\omega_0 L + (-\frac{2L}{\Delta t})e^{-j\omega \Delta t} + (j\omega_0 L)e^{-j\omega \Delta t}$$

$$H(\omega) = \frac{2L}{\Delta t} + j\omega_0 L + (-\frac{2L}{\Delta t} + j\omega_0 L)e^{-j\omega \Delta t}$$

To simplify the response, the exponentials are rewritten in terms of cosines and sines and $e^{-j\omega \Delta t}$ is rewritten as $e^{-j\omega \Delta t} = e^{-j\omega \Delta t}$

$$H(\omega) = \frac{e^{-j\omega \Delta t}}{e^{-j\omega \Delta t}} \times \frac{2L}{\Delta t} + j\omega_0 L) e^{j\omega \Delta t} + (-\frac{2L}{\Delta t} + j\omega_0 L) e^{-j\omega \Delta t}$$

$$\frac{2L}{\Delta t} + j\omega_0 L) e^{j\omega \Delta t} + (-\frac{2L}{\Delta t} + j\omega_0 L) e^{-j\omega \Delta t}$$

$$2 \cos(\omega \Delta t)$$

$$\frac{2L}{\Delta t} (e^{j\omega \Delta t} - e^{-j\omega \Delta t}) + j\omega_0 L(e^{j\omega \Delta t} + e^{-j\omega \Delta t})$$

$$2 \cos(\omega \Delta t)$$

$$\frac{2L}{\Delta t} (j 2 \sin(\omega \Delta t)) + j\omega_0 L(2 \cos(\omega \Delta t))$$

$$2 \cos(\omega \Delta t)$$

$$\frac{2L}{\Delta t} (j \sin(\omega \Delta t)) + j\omega_0 L(\cos(\omega \Delta t))$$

The frequency response of the trapezoidal rule in the SFA domain is given in (3.9), where $\omega$ is any frequency in a specified range, $L = 1$, and $\Delta t$ is the chosen time-step.

$$H_{\text{approx}}(\omega) = \frac{2L}{\Delta t} \tan(\omega \Delta t / 2) + j\omega_0 L$$

3.3.2 Approach Two: Applying the Z-Domain Transfer Function

Taking the discretized equation from (2.19), by rewriting the discretized equation in terms of $z$, where $t \rightarrow z$ and $t - \Delta t \rightarrow z^{-1}$, the $z$-domain transfer function is obtained. When $z = e^{j\omega \Delta t}$, one gets the frequency response of the discretized
3.3. The Frequency Response of the Trapezoidal Discretization Rule in the SFA Domain

inductor, which is equivalent to (3.9). This is derived as follows. For convenience, (2.19) is reiterated as

\[
\bar{V}_L(t) = \left(\frac{2L}{\Delta t} + j\omega_0L\right)\bar{I}_L(t) - \left[\left(\frac{2L}{\Delta t} - j\omega_0L\right)\bar{I}_L(t - \Delta t) + \bar{V}_L(t - \Delta t)\right].
\]

Replacing \(\bar{I}_L(t)\) with \(\bar{I}(z)\) and \(\bar{I}_L(t - \Delta t)\) with \(\bar{I}(z^{-1})\) (and similarly \(\bar{V}_L(t)\) with \(\bar{V}(z)\) and \(\bar{V}_L(t - \Delta t)\) with \(\bar{V}(z)z^{-1}\))

\[
\begin{align*}
\bar{V}(z) &= \left(\frac{2L}{\Delta t} + j\omega_0L\right)\bar{I}(z) + \left(-\frac{2L}{\Delta t} + j\omega_0L\right)\bar{I}(z)z^{-1} - \bar{V}(z)z^{-1} \\
\bar{V}(z) + \bar{V}(z)z^{-1} &= \frac{2L}{\Delta t}(\bar{I}(z) - \bar{I}(z)z^{-1}) + j\omega_0L(\bar{I}(z) + \bar{I}(z)z^{-1})) \\
\bar{V}(z)(1 + z^{-1}) &= \frac{2L}{\Delta t}\bar{I}(z)(1 - z^{-1}) + j\omega_0L\bar{I}(z)(1 + z^{-1}) \\
\frac{\bar{V}(z)}{\bar{I}(z)} &= \frac{2L}{\Delta t}\bar{I}(z)(1 - z^{-1}) + j\omega_0L\bar{I}(z)(1 + z^{-1}).
\end{align*}
\]

Defining \(\frac{\bar{V}(z)}{\bar{I}(z)} = H(z)\),

\[
H(z) = \frac{2L}{\Delta t}\left(1 - z^{-1}\right) + j\omega_0L
\]

\[
H(z) = \frac{2L}{\Delta t}\left(z - 1\right) + j\omega_0L.
\]

Replacing \(z\) with \(e^{j\omega\Delta t}\), and replacing the exponentials with sinusoids as in (3.8), and evaluating \(H(e^{j\omega\Delta t})\) at every frequency, results in

\[
H(e^{j\omega\Delta t}) = \frac{2L}{\Delta t}\left(e^{j\omega\Delta t} - 1\right) + j\omega_0L
\]

\[
= \frac{2L}{\Delta t}\left(e^{j\omega\frac{\Delta t}{2}} - e^{-j\omega\frac{\Delta t}{2}}\right) + j\omega_0L
\]

\[
= \frac{2L}{\Delta t}\left(e^{j\omega\frac{\Delta t}{2}} - e^{-j\omega\frac{\Delta t}{2}}\right) + j\omega_0L
\]

\[
= j\frac{2L}{\Delta t}\tan\left(\frac{\omega\Delta t}{2}\right) + j\omega_0L
\]

\[
H_{\text{approx}}(\omega) = j\frac{2L}{\Delta t}\tan\left(\frac{\omega\Delta t}{2}\right) + j\omega_0L.
\]

Equations (3.11) and (3.9) are identical.
3.3. The Frequency Response of the Trapezoidal Discretization Rule in the SFA Domain

3.3.3 The Trapezoidal Rule’s Accuracy in the SFA Domain

The trapezoidal rule’s accuracy when behaving as a differentiator is the ratio of its discrete-time frequency response in (3.11) to the continuous-time frequency response in (3.5) as shown in (3.12), where \( f_o \) is the fundamental frequency, \( \Delta t \) is the chosen time-step, and \( f \) is any frequency in a determined frequency range in the SFA domain (for example, from -60-Hz\( \rightarrow \)60-Hz):

\[
\text{Error}_{\text{diff.}} = \left| \frac{H_{\text{approx}}(\omega)}{H_{\text{exact}}(\omega)} \right| = \frac{j \frac{2L}{\Delta t} \tan(\omega \frac{\Delta t}{2}) + j \omega_o L}{jL(\omega + \omega_o)} = \frac{\frac{2}{\Delta t} \tan(\pi f \Delta t) + 2\pi f_o}{2\pi(f + f_o)} = \frac{1}{\pi \Delta t} \tan(\pi f \Delta t) + f_o \tag{3.12}
\]

The accuracy of the rule behaving as an integrator (for example, if voltage is the input, current is the output and the system is an inductor valued at 1 H) is the inverse of (3.12)

\[
\text{Error}_{\text{int.}} = \frac{1}{\frac{1}{\pi \Delta t} \tan(\pi f \Delta t) + f_o} = \frac{1}{\frac{\tan(\pi f \Delta t)}{\pi \Delta t(f + f_o)} + \frac{f_o}{(f + f_o)}} = \frac{1}{\frac{\tan(\pi f \Delta t)}{\pi \Delta t((f - f_o) + f_o)} + \frac{f_o}{(f - f_o) + f_o}} = \frac{1}{\frac{\tan(\pi f \Delta t)}{\pi f \Delta t} + \frac{f_o}{f}} \tag{3.13}
\]

The Accuracy of the Trapezoidal Rule in the Physical Domain, derived using the EMTP, compared to in the SFA Domain

Table 3.1 compares the EMTP and the SFA frequency response accuracy ratio’s both in per-unit (p.u.) of \( \Delta t \), where \( f_{\text{p.u.}} = f \Delta t \). In SFA, it is the term \( \frac{f_o}{f} \) which determines the Nyquist frequency, as this term shows that the SFA response depends on the chosen time-step \( (f_{\text{p.u.}} = f_o \Delta t) \). This differs from in the EMTP, in which the frequency response is independent of the chosen time-step and the Nyquist frequency is always 0.5 p.u. when using the trapezoidal rule [2].

46
3.3. The Frequency Response of the Trapezoidal Discretization Rule in the SFA Domain

### Table 3.1

<table>
<thead>
<tr>
<th></th>
<th>EMTP</th>
<th>SFA</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Error_{int.} = \frac{1}{\tan(\pi f_{p.u.})} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Error_{int.} = \frac{1}{\tan(\pi f_{p.u.})} + \frac{f_{o.p.u.}}{f_{p.u.}} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.3.4 The Numerical Accuracy of the Trapezoidal Rule: Magnitude and Phase Responses

In the SFA domain, the rotational transformation shifts the physical frequencies \( f \) by -60 Hz. To show the response in the physical domain in Figures (3.3) and (3.4), the frequencies \( f \) are re-shifted by +60 Hz, and the 0 Hz in SFA is re-shifted back to 60 Hz. Because the Nyquist frequency will be different for each time-step (\( f_{\text{max}} = \frac{1}{\Delta t} \) and \( f_{Ny} = \frac{1}{2} f_{\text{max}} \)) it is easier to examine the response in physical units (Hz). Figure (3.3) presents the magnitude and phase response of the trapezoidal rule acting as a differentiator, with chosen \( \Delta t \)'s ranging form 0.1 ms → 32 ms. Figure (3.4) presents the magnitude and phase response of the trapezoidal rule acting as an integrator.

In the trapezoidal rule, there is a discontinuity when the frequency approaches \( f = 60 \pm m f_{Ny} \) for \( m = (2m - 1 : m \in \mathbb{Z}) \). For example, at \( \Delta t = 10 \text{ ms} \), the Nyquist frequency is 50 Hz and there are discontinuities at 10 Hz (60-50) and 110 Hz (60+50). Because frequencies greater than the Nyquist frequency do not exist in the time-window, only positive frequencies lower than or equal to the Nyquist frequency have relevant discontinuities as provided in Table 3.2. When \( \Delta t < 8.33 \text{ ms} \), \( f_{Ny} > 60 \text{ Hz} \) and the discontinuities are less than 0 Hz.

### Table 3.2

<table>
<thead>
<tr>
<th>( \Delta t \text{(ms)} )</th>
<th>Nyquist Frequency(Hz)</th>
<th>Discontinuities at frequencies lower than the Nyquist frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.33</td>
<td>60</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>50</td>
<td>10</td>
</tr>
<tr>
<td>16.67</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>32</td>
<td>15.625</td>
<td>13.1</td>
</tr>
</tbody>
</table>

Regardless of the discontinuities, the trapezoidal rule is quite accurate in the SFA domain, as the error produced hovers around \( \left| \frac{H_{SFA-\text{DT}}(\omega)}{H_{SFA-\text{CT}}(\omega)} \right| = 1 \) as depicted in Figure...
3.3. The Frequency Response of the Trapezoidal Discretization Rule in the SFA Domain

\[ (3.5) \]

for all the time-steps and except for at the discontinuities, when the phase jumps from 0 to \( \pi \), there is no distortion in the phase response.

**Accuracy of the Trapezoidal Rule as a Differentiator**

(Magnitude Response)

![Magnitude Response Graph]

Accuracy of the Trapezoidal Rule as a Differentiator

(Phase Response)

![Phase Response Graph]

Figure 3.3: The accuracy of the trapezoidal rule acting as a differentiator (magnitude and phase response).
3.3. The Frequency Response of the Trapezoidal Discretization Rule in the SFA Domain

![Accuracy of the Trapezoidal Rule as an Integrator](image)

**Figure 3.4**: The accuracy of the trapezoidal rule acting as an integrator (magnitude and phase response).
3.3. The Frequency Response of the Trapezoidal Discretization Rule in the SFA Domain

Frequency Response (Magnitude) of SFA using the Trapezoidal Rule
(As a Differentiator) around 0% error

Figure 3.5: The accuracy of the trapezoidal rule acting as a differentiator (around 0% error, which occurs when \( \frac{H_{\text{approx}}(\omega)}{H_{\text{exact}}(\omega)} = 1 \)).

3.3.5 The Distortion of the Inductor Using the Trapezoidal Rule

Because the discretization rule creates an equivalent inductor that is a discretized version of the physical inductor (Tables 3.1 and 3.2), the equivalent inductance is a distorted version of the physical one. (Similarly, the capacitor is also distorted by the discretization rule). One way to define how much distortion is created by the discretization rule is to equate the discrete-time frequency response \( H_{\text{approx}}(\omega) \) (3.11) with the continuous-time frequency response \( H_{\text{ct}}(\omega) \) (3.5). By defining the continuous-time inductor in (3.5) as \( L_{\text{eq}} \) and the discrete-time inductance as the inductance \( L \) and equating \( H_{\text{approx}}(\omega) \) to \( Z(\omega) \), as the response in (3.11) is the response of the inductor acting as a differentiator,

\[
Z(\omega) = j \frac{2L}{\Delta t} \tan\left(\omega \frac{\Delta t}{2}\right) + j\omega_o L
\]

\[
j(\omega + \omega_o)L_{\text{eq}} = j \frac{2L}{\Delta t} \tan\left(\omega \frac{\Delta t}{2}\right) + j\omega_o L
\]

\[
L_{\text{eq}} = L \left( \frac{2}{\Delta t} \tan\left(\omega \frac{\Delta t}{2}\right) + \frac{\omega_o}{\omega + \omega_o} \right)
\]

\[
L_{\text{eq}} = L \left( \frac{\pi}{\Delta t} \tan\left(\pi f \Delta t\right) + \frac{f_o}{f + f_o} \right).
\]
3.4 The Frequency Response of the Backward Euler Discretization Rule in the SFA Domain

The distortion of the inductor evaluated in (3.14) also defines the boundary conditions of the frequency response, which correspond to the results obtained in Figures (3.3) and (3.4).

Boundary Conditions

1. When \( f = 0 \), \( L \to L_{ct} \)
2. When \( f = -f_o \), \( L \to \infty \)
3. When \( f \to f_{Ny} \), \( L \to \infty \). (\( f_{Ny} = \frac{1}{2\Delta t} \))

Table 3.3 compares the equivalent inductance in the shifted domain, obtained by the SFA solution, with the equivalent inductance in the EMTP solution.

<table>
<thead>
<tr>
<th></th>
<th>EMTP</th>
<th>SFA</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_{eq} )</td>
<td>( L - \frac{\tan(\pi f \Delta t)}{\pi f \Delta t} )</td>
<td>( L \left( \frac{1}{\Delta t} \tan(\pi f \Delta t) \right) \left( \frac{f_o}{f + f_o} \right) + \frac{f_o}{(f + f_o)} )</td>
</tr>
</tbody>
</table>

Table 3.3
A comparison of the EMTP’s and SFA’s distortion of the inductor when using the trapezoidal rule.

3.4 The Frequency Response of the Backward Euler Discretization Rule in the SFA Domain

The frequency response analysis performed with the trapezoidal rule is applied to the backward Euler rule. Equivalent results are obtained when using either the single-frequency exponential approach or the \( z \)-domain transfer function approach. For simplicity, only the derivation using the \( z \)-domain approach is provided.

3.4.1 Applying the Z-Domain Transfer Function

The equivalent inductor discretized using the backward Euler rule was derived in (2.26) and is reiterated here

\[
\bar{V}_L(t) = \left( \frac{L}{\Delta t} + j\omega_o L \right) \bar{I}_L(t) - \frac{L}{\Delta t} (\bar{I}_L(t - \Delta t)).
\]
3.4. The Frequency Response of the Backward Euler Discretization Rule in the SFA Domain

Replacing $\bar{I}_L(t)$ with $\bar{I}(z)$ and $\bar{I}_L(t - \Delta t)$ with $\bar{I}(z)^{-1}$ and $\bar{V}_L(t)$ with $\bar{V}(z)$,

$$\bar{V}(z) = (\frac{L}{\Delta t} + j\omega_o L)\bar{I}(z) - \frac{L}{\Delta t}\bar{I}(z)^{-1}$$  \hspace{1cm} (3.15)

Multiplying both sides by $I(z)$ and defining $\frac{\bar{V}(z)}{\bar{I}(z)} = H(z)$,

$$\frac{\bar{V}(z)}{\bar{I}(z)} = (\frac{L}{\Delta t} + j\omega_o L) - \frac{L}{\Delta t}z^{-1}$$

$$H(z) = (\frac{L}{\Delta t} + j\omega_o L) - \frac{L}{\Delta t}z^{-1}$$  \hspace{1cm} (3.16)

In terms of frequency, one can replace $z^{-1}$ with $e^{-j\omega \Delta t}$ and set $H(z)$ to be the response for all frequencies $\omega$,

$$H(\omega) = \frac{L}{\Delta t}(1 - e^{-j\omega \Delta t}) + j\omega_o L$$

$$\quad = \frac{L}{\Delta t}(e^{-j\omega \frac{\Delta t}{2}})(e^{j\omega \frac{\Delta t}{2}} - e^{-j\omega \frac{\Delta t}{2}}) + j\omega_o L$$  \hspace{1cm} (3.17)

$$\quad = \frac{L}{\Delta t}(e^{-j\omega \frac{\Delta t}{2}})(j2\sin(\omega \frac{\Delta t}{2})) + j\omega_o L,$$

with the final frequency response as

$$H_{\text{approx}}(\omega) = \frac{2L}{\Delta t}(e^{-j\omega \frac{\Delta t}{2}})(j\sin(\omega \frac{\Delta t}{2})) + j\omega_o L.$$  \hspace{1cm} (3.18)

3.4.2 The Backward Euler Rule’s Accuracy in the SFA Domain

The accuracy of the backward Euler rule acting as a differentiator given in (3.19) below is the ratio of the discrete-time frequency response in (3.18) to the continuous-time frequency response in (3.5), where in (3.19), $f_o$ is the fundamental frequency, $\Delta t$ is the chosen time-step for the response to be analysed at, and $f$ is any frequency in a determined frequency range in the SFA domain (for example from -60 Hz→60 Hz).
3.4. The Frequency Response of the Backward Euler Discretization Rule in the SFA Domain

\[ \text{Error}_{\text{diff.}} = \left| \frac{H_{\text{approx}}(\omega)}{H_{\text{exact}}(\omega)} \right| = \frac{j \frac{2L}{\Delta t}(e^{-j\omega \frac{\Delta t}{2}})(\sin(\omega \frac{\Delta t}{2})) + j\omega_o L}{jL(\omega + \omega_o)} \]
\[ = \frac{2}{\Delta t}(e^{-j\omega \frac{\Delta t}{2}})(\sin(\omega \frac{\Delta t}{2})) + \omega_o}{\omega + \omega_o} \]
\[ = \frac{1}{\pi \Delta t}(e^{-j\pi f \Delta t})(\sin(\pi f \Delta t)) + f_o}{f + f_0}. \] (3.19)

The accuracy of the rule behaving as an integrator is the inverse of (3.19)

\[ \text{Error}_{\text{int.}} = \frac{1}{\pi \Delta t}(e^{-j\pi f \Delta t})(\sin(\pi f \Delta t)) + f_o}{f + f_0}. \] (3.20)

The Accuracy of the Backward Euler Rule in the Physical Domain, derived using the EMTP, compared to in the SFA Domain

As with the trapezoidal discretization, the SFA’s frequency response accuracy ratio depends on the time-step. Table 3.4 compares the EMTP and the SFA frequency response accuracy ratio’s in per-unit (p.u.) of \( \Delta t \).

Table 3.4
A comparison of the EMTP and SFA frequency response accuracy ratio for the backward Euler rule behaving as an integrator.

<table>
<thead>
<tr>
<th>EMTP</th>
<th>SFA</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Error}<em>{\text{int.}} = \frac{\Delta t}{2L} + \frac{\Delta t}{2L} \tan(\pi f</em>{\text{p.u.}}) )</td>
<td>( \text{Error}<em>{\text{int.}} = \frac{1}{\pi \Delta t}(e^{-j\pi f</em>{\text{p.u.}}})(\sin(\pi f_{\text{p.u.}})) + f_{\text{op.u.}} )</td>
</tr>
</tbody>
</table>

3.4.3 The Numerical Accuracy of the Backward Euler Rule: Magnitude and Phase Responses

The numerical accuracy of the backward Euler rule is given in physical units (Hz) with the frequency response re-shifted to the physical domain through frequency shifting \( f \) by +60 Hz. Figures (3.6) and (3.7) show the magnitude and phase response of the backward Euler rule acting as a differentiator and an integrator, respectively. Again, this shift in frequencies back to the physical domain does not change the rule’s response at these frequencies; as it is just a frequency-shift. Un-
like the trapezoidal rule, there is a phase distortion in the accuracy response of the backward Euler rule, which comes from the fictitious damping resistance introduced in the distortion of the equivalent inductance (derived in Section 3.4.4). However, the error is constrained between $0^\circ \rightarrow 90^\circ$ as the system is minimum-phase. The accuracy of the magnitude response comparing of the backward Euler rule around $|H_{\text{approx}}(\omega)|/|H_{\text{exact}}(\omega)| = 1$ is shown in Figure (3.8).

### 3.4.4 The Distortion of the Inductor Using the Backward Euler Rule

Similar to the trapezoidal rule, the distortion created by the backward Euler rule on the inductor is derived by equating the discrete-time frequency response $H_{\text{approx}}(\omega)$ (3.18) with the continuous-time frequency response $H_{\text{ct}}(\omega)$ (3.5). Rewriting (3.18) in terms of cosines and sines, one obtains

$$Z(\omega) = j \frac{2L}{\Delta t} \left( \cos(\omega \frac{\Delta t}{2}) - j \sin(\omega \frac{\Delta t}{2}) \right) \sin(\omega \frac{\Delta t}{2}) + j 2\pi f_0 L$$

$$= j \frac{2L}{\Delta t} \left( \cos(\omega \frac{\Delta t}{2}) \sin(\omega \frac{\Delta t}{2}) - j^2 \frac{2L}{\Delta t} \sin(\omega \frac{\Delta t}{2}) \sin(\omega \frac{\Delta t}{2}) \right) + j 2\pi f_0 L$$

$$= \frac{2L}{\Delta t} \sin(\omega \frac{\Delta t}{2})^2 + j \frac{2L}{\Delta t} \left( \cos(\omega \frac{\Delta t}{2}) \sin(\omega \frac{\Delta t}{2}) \right).$$

Equation (3.21) has both a real and an imaginary part. The real part corresponds to a fictitious resistance, as seen in the EMTP [38][2]. This resistance allows energy to dissipate, which corresponds to a damping of the oscillations that are prominent with the trapezoidal rule. In the backward Euler rule, the equivalent resistance determines how much the oscillations will be dampened by and the equivalent inductance determines how much distortion has incurred. Deriving both the real and the imaginary parts are as follows. For the real and imaginary parts of the distorted inductor, the real part, $R_{eq}$ is (continued on page 57):
3.4. The Frequency Response of the Backward Euler Discretization Rule in the SFA Domain

Accuracy of the Backward Euler Rule as a Differentiator
(Magnitude Response)

Accuracy of the Backward Euler Rule as a Differentiator
(Phase Response)

Figure 3.6: The accuracy of the backward Euler rule acting as a differentiator (magnitude and phase response).
3.4. The Frequency Response of the Backward Euler Discretization Rule in the SFA Domain

Accuracy of the Backward Euler Rule as an Integrator
(Magnitude Response)

Accuracy of the Backward Euler Rule as an Integrator
(Phase Response)

Figure 3.7: The accuracy of the backward Euler rule acting as an integrator (magnitude and phase response).
3.4. The Frequency Response of the Backward Euler Discretization Rule in the SFA Domain

Accuracy of the Backward Euler Rule as a differentiator around 0% error

(Magnitude Response)

Figure 3.8: The accuracy of the backward Euler rule acting as a differentiator (around 0% error, which occurs when \[ \left| \frac{H_{\text{approx}}(\omega)}{H_{\text{exact}}(\omega)} \right| = 1 \].)

For the real and imaginary parts of the distorted inductor, the real part, \( R_{eq} \) is

\[
R_{eq} = \frac{2L}{\Delta t} \sin^2(\omega \frac{\Delta t}{2}). \tag{3.22}
\]

Applying the trigonometric law, \( \sin^2(\theta) = \frac{1}{2} (1 - \cos(2\theta)) \), (3.22) becomes

\[
R_{eq} = \frac{2L}{\Delta t} \left( \frac{1}{2} (1 - \cos(2(\omega \frac{\Delta t}{2}))) \right)
= \frac{L}{\Delta t} \left( 1 - \cos \left( 2(\omega \frac{\Delta t}{2}) \right) \right) \tag{3.23}
= \frac{L}{\Delta t} \left( 1 - \cos(\omega \Delta t) \right).
\]

Finally,

\[
R_{eq} = \frac{L}{\Delta t} \left( 1 - \cos(2\pi f \Delta t) \right). \tag{3.24}
\]
The imaginary portion is derived as follows

\[ j(\omega - \omega_o) L_{eq} = j\left[ \frac{2L}{\Delta t} \cos(\omega \frac{\Delta t}{2}) \sin(\omega \frac{\Delta t}{2}) + \omega_o L \right] \]

\[ L_{eq} = \frac{2L}{\omega \Delta t} \cos(\omega \frac{\Delta t}{2}) \sin(\omega \frac{\Delta t}{2}) + \omega_o L \]

\[ = \frac{2L}{\Delta t} \cos(\omega \frac{\Delta t}{2}) \sin(\omega \frac{\Delta t}{2}) + \frac{\omega_o}{\omega} \]

\[ = L \frac{2L}{\Delta t} \cos(\pi f \Delta t) \sin(\pi f \Delta t) + 2\pi f_o \]

\[ = L \frac{2L}{\pi \Delta t} \cos(\pi f \Delta t) \sin(\pi f \Delta t) + 2\pi f_o \]

\[ (3.25) \]

Finally,

\[ L_{eq} = L \frac{1}{\pi \Delta t} \cos(\pi f \Delta t) \sin(\pi f \Delta t) + \frac{f_o}{f + f_o} \]

\[ (3.26) \]

**Boundary Conditions**

The boundary conditions in SFA for the backward Euler rule are determined by the behaviour of the equivalent resistance and inductance. The damping resistance in SFA \((3.22)\) is a function of the frequency \((\cos(\omega \Delta t))\). This contrasts with the EMTP, where the damping resistance is independent of the frequency. The following deductions are made with regards to the equivalent resistance:

1. When \(f = f_{Ny}(\text{or any multiple of } f_{Ny}) \rightarrow R = \frac{L}{\Delta t} \).
2. When \(f = f_{max} \rightarrow R_{eq} = \frac{2L}{\Delta t} \), which is identical to the EMTP.
3. When \(f = 0 \rightarrow R = 0 \).

For the equivalent inductance \((3.25)\) the following boundary conditions are

1. When \(f \rightarrow -f_o, L \rightarrow \infty \).
2. When \(f = 0, \text{ or } L \rightarrow L_{ct} \).
3. When \(f \rightarrow f_{Ny}, L = \frac{f_o}{f + f_o} \).

The discontinuities observed using the trapezoidal rule do not occur with backward Euler, as observed in \((3.21)\), where at the Nyquist frequency \((f_{Ny})\), there exists a value \(Z_{eq} \rightarrow \frac{L}{\Delta t} + \frac{f_o}{f + f_o} \), which makes the response continuous for backward Euler.

Table \[3.5\] compares the behaviour of the equivalent inductance distorted by the backward Euler rule in the EMTP and the SFA.
3.4. The Frequency Response of the Backward Euler Discretization Rule in the SFA Domain

### Table 3.5
A comparison of the EMTP’s and SFA’s distortion of the inductor when using the backward Euler rule.

<table>
<thead>
<tr>
<th>EMTP</th>
<th>SFA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{eq} = \frac{2L}{\Delta t}$</td>
<td>$R_{eq} = \frac{L}{\Delta t} \left(1 - \cos(2\pi f \Delta t)\right)$</td>
</tr>
<tr>
<td>$L_{eq} = L \left(\tan\left(\frac{\omega \Delta t}{2}\right)\right)$</td>
<td>$L_{eq} = L \left(\frac{1}{\pi \Delta t} \cos(\pi f \Delta t) \sin(\pi f \Delta t) + \frac{f_o}{f + f_o}\right)$</td>
</tr>
</tbody>
</table>

3.4.5 A Comparison of The Trapezoidal and Backward Euler Rule Around 0 Hz (DC)

Around 0 Hz (DC), the trapezoidal rule is close to 100% (\(\left|\frac{H_{\text{approx}}(\omega)}{H_{\text{exact}}(\omega)}\right| = 1\)) accuracy when using a time-step smaller than 5 ms, whereas the backward Euler rule has high error around 0 Hz, as observed in Figure 3.9. This lends to one possibility in why the trapezoidal rule is more accurate in detecting the DC component of the fault current, as observed in the case study of Chapter 6. Recalling that the discretization response in the shifted domain retains its discretization properties when re-shifted back to the physical domain, the response at -60 Hz is the physical 0 Hz (Figure 3.9).

![Figure 3.9: A comparison of the accuracy of the trapezoidal and backward Euler rules acting as an integrator around 0 Hz.](image-url)

Accuracy of the Trapezoidal and Backward Euler Rules Around DC (As an Integrator)
3.4. The Frequency Response of the Backward Euler Discretization Rule in the SFA Domain

3.4.6 A Summary of the Accuracy of the Two Discretization Rules

Although the backward Euler rule does not have discontinuities close to the Nyquist frequency, in general, it is less accurate than the trapezoidal rule. For example, when using a $\Delta t = 10$ ms, the trapezoidal rule has 100% accuracy $\left( \left| \frac{H_{\text{approx}}(\omega)}{H_{\text{exact}}(\omega)} \right| = 1 \right)$ from 58 Hz to 62.1 Hz, which is a range of 4.1 Hz, whereas the backward Euler rule has 100% accuracy from 58.4 Hz to 61.6 Hz, which is a range of 3.2 Hz, approximately 1 Hz less than with the trapezoidal.

Both trapezoidal and backward Euler are suitable for the transient stability case studies studied in this thesis, where a time-step of $\Delta t = 8$ ms is used for the SFA solution. Figure (3.10) shows the error produced by the rules given a 20 Hz frequency deviation around the 60-Hz frequency using $\Delta t = 8.3$ ms and $\Delta t = 10$ ms.

Looking at Figure (3.10), one can see that trapezoidal introduces 0.55% less error than backward Euler using $\Delta t = 8.33$ ms and 0.78% less error using $\Delta t = 10$ ms at the 50 Hz frequency, which is a 10 Hz deviation from 60 Hz. At the 55 Hz frequency, which is a 5 Hz deviation, using $\Delta t = 10$ ms, trapezoidal has only an error of 0.05%, while backward Euler has an error of 0.16% in the magnitude response and a 0.8° error in the phase plot. Regardless, using backward Euler at $\Delta t = 8$ ms, as in the case studies, only introduces an error of 0.10%. In addition, due to the extra damping term in the backward Euler rule, backward Euler is a suitable choice for network discontinuities, such as switches opening to clear short-circuit faults, whereas trapezoidal introduces numerical oscillations.

To summarize, the trapezoidal rule reaches discontinuities when approaching the Nyquist frequency; however, it is more accurate than the backward Euler rule for frequencies about $\pm0.5$-1 Hz from 60 Hz and does not introduce a phase distortion. On the other hand, the backward Euler rule does not have discontinuities at the Nyquist frequency, but does have a phase distortion and is, on average, less accurate than trapezoidal. At frequencies around 0 Hz (DC), trapezoidal is highly accurate when using small time-steps, whereas backward Euler introduces large errors.
3.5. The Discretization Rule’s Stability in the SFA Domain

The second important property of a discretization rule is the rule’s stability, which can also be determined by observing the discrete-time transfer function \([2]\). Recalling that the solution to a first-order differential equation has both a transient part and a steady-state part \([38]\) \([2]\), if the transient part dies down to 0 as \(t \to \infty\), then the system is stable. The transient response of the discrete-time solution is of the form \(i_h[k] = C_1p_1^k + C_2p_2^k + ... + C_np_n^k\) for \(k = \frac{t}{\Delta t}\), which as a transfer function is \(H(z) = H\left(\frac{z+z_1}{z+p_1}\right)\left(\frac{z+z_2}{z+p_2}\right)\cdots\left(\frac{z+z_n}{z+p_n}\right)\), where \(z\) and \(p\) are the zeros and poles, respectively \([2]\). If the transfer function’s poles are on or inside the unit circle \(|p| \leq 1\), then the rule is stable, as in the case with the trapezoidal and backward Euler rules. In addition to stability, to ensure causality, the transfer function’s zeros must also be on or inside the unit circle, \(|z| \leq 1\) \([2]\). In the EMTP, both the trapezoidal and backward Euler discretization rules are derived as numerically stable \([19]\) \([38]\). The stability of the rules in the SFA domain are derived in this section.

Figure 3.10: A comparison of the magnitude of the accuracy of the trapezoidal and backward Euler frequency responses acting as a differentiator around 60 Hz using \(\Delta t = 8\,\text{ms}\) and \(\Delta t = 10\,\text{ms}\).
By discretizing the inductor using the trapezoidal rule and applying the z-transform on the discretized branch, as derived in (3.10), where \( H(z) = Z(\omega) \),

\[
Z(\omega) = \frac{2L}{\Delta t} \left( \frac{z - 1}{z + 1} \right) + j\omega_0 L. \tag{3.27}
\]

By discretizing the inductor using the backward Euler rule and applying the z-transform on the discretized branch, as derived in (3.16), where \( H(z) = Z(\omega) \)

\[
Z(\omega) = \frac{L}{\Delta t} \left[ 1 - e^{-j\omega \Delta t} \right] + j\omega_0 L
= \frac{L}{\Delta t} \left[ 1 - z^{-1} \right] + j\omega_0 L
= \frac{L}{\Delta t} \left[ \frac{z - 1}{z} \right] + j\omega_0 L. \tag{3.28}
\]

Equation (3.27) is the transfer function for the inductor discretized with trapezoidal, which shows that when the trapezoidal rule acts as a differentiator, there is a pole at -1 and a zero at 1 and (3.28) shows that when the backward Euler rule acts as a differentiator, there is a pole at 0 and a zero at 1. Therefore, the two discretization rules in the SFA domain are both stable and causality is maintained. By deriving these equations, certain properties of the rule, such as its behaviour during discontinuities can be explained. The following two sections demonstrate these behaviours.

3.6 The Response to a Step Input in the SFA Domain

The solution method’s response to a discontinuity can reveal an idea about the system’s behaviour to a discontinuity. This understanding is critical because discontinuities occur during switching action, which is prominent in power electronic converters. In the modern grid, where renewable energy sources are coupled to the grid through power electronic converters, if SFA was used in modelling the power electronics in future work, understanding the behaviour of the discretization rules is important. As described in [2], discontinuities occur whenever nature switches from one state to another instantaneously, such as the behaviour of the current through an inductor. One approach to model discontinuities in a system, therefore, is to energize an inductor of 1 mH with a DC current source of 1 A [2] as in Figure 3.11.
3.6. The Response to a Step Input in the SFA Domain

The relationship between voltage and current through an inductor in the SFA domain using the trapezoidal and backward Euler rules are given in (2.19) and (2.26), respectively. Tables 3.6 and 3.7 give four time-steps of the voltage across the inductor when the inductor is energized with a 1 A step input. These tables compare the response to a step input in the SFA domain with the response in the physical domain (using the EMTP solution).

**Table 3.6**
The response of a step input using the trapezoidal rule, comparing the SFA solution with the EMTP solution.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\bar{I}(t)$</th>
<th>$\bar{V}(t)$</th>
<th>i(t)</th>
<th>v(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>$e^{-j\omega_0 t}$</td>
<td>$(\frac{2L}{\Delta t} + j\omega_0 L)e^{-j\omega_0 t}$</td>
<td>1</td>
<td>$\frac{2L}{\Delta t}$</td>
</tr>
<tr>
<td>$2\Delta t$</td>
<td>$e^{-j\omega_0 t}$</td>
<td>$(-\frac{2L}{\Delta t} + j\omega_0 L)e^{-j\omega_0 t}$</td>
<td>1</td>
<td>$-\frac{2L}{\Delta t}$</td>
</tr>
<tr>
<td>$3\Delta t$</td>
<td>$e^{-j\omega_0 t}$</td>
<td>$(\frac{2L}{\Delta t} + j\omega_0 L)e^{-j\omega_0 t}$</td>
<td>1</td>
<td>$\frac{2L}{\Delta t}$</td>
</tr>
<tr>
<td>$4\Delta t$</td>
<td>$e^{-j\omega_0 t}$</td>
<td>$(-\frac{2L}{\Delta t} + j\omega_0 L)e^{-j\omega_0 t}$</td>
<td>1</td>
<td>$-\frac{2L}{\Delta t}$</td>
</tr>
</tbody>
</table>

**Table 3.7**
The response of a step input using the backward Euler rule, comparing the SFA solution with the EMTP.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\bar{I}(t)$</th>
<th>$\bar{V}(t)$</th>
<th>i(t)</th>
<th>v(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>$e^{-j\omega_0 t}$</td>
<td>$(\frac{L}{\Delta t} + j\omega_0 L)e^{-j\omega_0 t}$</td>
<td>1</td>
<td>$\frac{L}{\Delta t}$</td>
</tr>
<tr>
<td>$2\Delta t$</td>
<td>$e^{-j\omega_0 t}$</td>
<td>$(j\omega_0 L)e^{-j\omega_0 t}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$3\Delta t$</td>
<td>$e^{-j\omega_0 t}$</td>
<td>$(j\omega_0 L)e^{-j\omega_0 t}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$4\Delta t$</td>
<td>$e^{-j\omega_0 t}$</td>
<td>$(j\omega_0 L)e^{-j\omega_0 t}$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
## 3.6. The Response to a Step Input in the SFA Domain

The energization of a source into an inductor is shown in Figure 3.11. The EMTP and SFA responses are given in Figure 3.12 through Figure 3.14. The EMTP’s response using the trapezoidal rule is given in Figure 3.12(a) and the EMTP’s response using the backward Euler rule is given in Figure 3.12(b). Figure 3.13 shows the SFA response using the trapezoidal rule, with Figure 3.13(a) giving the response in the shifted domain and Figure 3.13(b) giving the response in the physical domain. Figure 3.14 shows the SFA response using the backward Euler rule, where Figure 3.14(a) shows the response in the shifted domain and Figure 3.14(b) shows the response in the physical domain.

![Figure 3.11: A DC current source of I=1 A energizes an inductor of L=1 mH.](image)

In Figures 3.13(a) and 3.13(b), the numerical oscillations that occur with the trapezoidal rule in SFA match the stability response of Table 3.6. In Table 3.6, we see that the trapezoidal response has sustained oscillations in both the SFA and the EMTP responses, where in the shifted domain, the sustained oscillations occur at \(\left(\frac{2L}{\Delta t} + j\omega_0 L \right)e^{-j\omega_0 t}\) and \(\left(-\frac{2L}{\Delta t} + j\omega_0 L \right)e^{-j\omega_0 t}\) whereas in the physical domain with the EMTP solution, the oscillations occur at \(\frac{2L}{\Delta t}\) and \(-\frac{2L}{\Delta t}\).

Using a time-step of \(\Delta t = 0.1\) ms, the oscillations occur at \(\pm20 + j0.377\) and using a time-step of \(\Delta t = 1\) ms, the oscillations occur at \(\pm 2 + j0.377\). Because both the real and imaginary parts have sustained oscillations, these are seen in both the real and imaginary components of the shifted signal (when the real part has a peak of \(\pm20\), the imaginary portion has a peak of \(\pm0.377\)). When transformed back into
the physical domain, these oscillations correspond to sustained beats between the real and imaginary part of the physical continuous-time domain solution peaking at \( \pm 20 \) or \( \pm 2 \) (Figure 3.13(b)). Contrastively, the EMTP trapezoidal response in Figure 3.12(a) has sustained oscillations at \( \frac{2L}{\Delta t} \).

Figure 3.12: The voltage across an inductor in response to a 1A DC current in the EMTP: (a) the response given when using the trapezoidal rule and (b) the response given when using the backward Euler rule.
3.6. The Response to a Step Input in the SFA Domain

Figures 3.14(a) and 3.14(b) above show the voltage across the inductor in response to the step current for two $\Delta t$'s: $\Delta t = 0.1\,\text{ms}$ and $\Delta t = 1$ using the backward Euler rule. Table 3.7 shows that the backward Euler response has sustained oscillations at $j\omega_0 Le^{-j\omega_0 t}$ after the first two time-steps in the SFA domain, whereas in the EMTP, the response is zero after the second time-step.

Using the larger time-step of $\Delta t = 1\,\text{ms}$, in the shifted domain, the response sustains a ripple in the real and imaginary part with a peak-peak value of $-0.07 \to 0.07$ (an amplitude of 0.14). The sustained ripples match the sustained imaginary component of the oscillations in Table 3.7. When $\Delta t = 1\,\text{ms}$ and $L=1\,\text{mH}$, evaluating $(j\omega_0 L)e^{-j\omega_0 t}$ gives the amplitude of 0.14. In the inverse transformed physical response, these ripples are sustained and instead of dropping to zero after the first time-step, the response is held at 0.07, due to the error in the backward Euler solution at this time-step. However, when the time-step is less than 0.5 ms, such as in the response when using $\Delta t = 0.1\,\text{ms}$, because the ripples are smaller (peak-peak value of $-0.007 \to 0.007$), in the transformed physical response, the response is sustained at approximately 0, just as in the EMTP (Figure 3.12(b)).
3.6. The Response to a Step Input in the SFA Domain

Figure 3.13: The voltage across an inductor in response to a 1A DC current using the trapezoidal rule in SFA: (a) the response given in the shifted domain and (b) the response given in the physical domain.
3.6. The Response to a Step Input in the SFA Domain

SFA Response to a Step Function using Backward Euler
(in the shifted domain)

(a) Backward Euler response in the SFA domain.

SFA Response to a Step Function using Backward Euler
(in the physical domain)

(b) Backward Euler response shifted back to the physical domain.

Figure 3.14: The voltage across an inductor in response to a 1A DC current using the backward Euler rule in SFA: (a) the response given in the shifted domain and (b) the response given in the physical domain.
3.7 An RL Circuit Energized with a DC Source Using the Traditional Phasor, the EMTP and SFA

In this scenario, a 10V DC source energizes an RL circuit, as shown in Figures 3.15 and 3.16. The response of the SFA and the EMTP solutions for the voltage across the inductor and current through the inductor are given in Figures 3.17(a) and 3.17(b). Because the time-constant of the circuit is 2 ms, the chosen time-step is 0.2 ms, using the rule that $\tau \approx \frac{1}{10} \Delta t$. Both trapezoidal and backward Euler rules are used for both the SFA and the EMTP solutions, with the EMTP being the reference solution for the SFA response. As analysed in Section 3.6, the response to a step function using the trapezoidal rule is accurate but contains sustained oscillations, whereas the response to a step function using the backward Euler rule, after it is transformed from the SFA domain back to the physical domain, is accurate only with small time-steps ($\Delta t < 0.5$ ms). This observation is verified in Figure 3.17 for when using the backward Euler rule, there is an error of 2.77%, whereas using the trapezoidal rule, there is no error. The error in the backward Euler drops to zero when the time-step is decreased, for example, if $\Delta t = 0.01$ ms.

![Figure 3.15: An RL circuit energized with a DC voltage source.](image1)

![Figure 3.16: The RL circuit discretized with the EMTP (left) and the SFA (right).](image2)
3.7. An RL Circuit Energized with a DC Source Using the Traditional Phasor, the EMTP and SFA

Voltage across an inductor when energized by a DC Source

Current through an inductor when energized by a DC Source

Figure 3.17: A comparison of the trapezoidal and backward Euler rules in response to a DC source, using the EMTP as the reference solution.
Chapter 4

Transient Stability Analysis

Transient stability, or rotor-angle stability, is concerned with the ability for the rotor angles of the generators to remain in synchronicity with each other within 3-5 seconds after a severe disturbance has occurred [44]. These studies analyze the behavior of the system after the disturbance disrupts the power flow balance in the electrical network. During such disturbances, the generator will accelerate or decelerate due to the imbalance of power between the input mechanical power and the output electrical power. As the machine’s velocity changes, the generator’s rotor field windings will deviate in position with respect to the generator’s stator windings.

If the stator is taken as the inertial reference frame, the relative position between the rotor and stator is \( \delta_{\text{rel}} \) [20]. \( \delta_{\text{rel}} \) is the angle observed during transient stability studies (as shown in Figure 4.1) to determine if the generator is “first-swing” unstable. First-swing instability means that \( \delta_{\text{rel}} \) will steadily increase without swinging back to its initial position and it is the assumption used when the synchronous machine is modeled using the equivalent voltage behind reactance model as described in Section 4.1.2 [4]. When there are multiple generators in the same synchronous area, the stability depends on the synchronicity among all the machines. The generators will accelerate and decelerate at different rates due to several external factors, such as their location to the fault and their internal parameters (such as reactances, constant of inertia, MVA capacity). Each generator’s relative angle \( \delta_{\text{rel}} \) will increase with respect to their own stator windings and with respect to each other. If the relative angle between two generators increases beyond the transient stability limit [4], the synchronicity between these two machines will be lost and the entire system will be considered unstable. The main objective of transient stability analysis studies is to ensure synchronism among all network components, in particular the rotating masses of the synchronous machines.

Due to the imbalance of power that creates the asynchronous behavior of the generators, it is relevant to understand the role of power equilibrium between the
mechanical power from the turbine into the generators and the electrical power delivered to the network from the generators, as briefly discussed in Section 1.2. Power equilibrium can be described using both the physical representation of the power imbalance and the mathematical model that describes the physics.

### 4.1 A Physical Representation

As described in Section 1.2, the ability for the power grid to maintain synchronicity depends on the power balance between the mechanical power into the generators from the turbine and the electrical power delivered to the network. During steady-state operation, the mechanical power provided by the turbines driving the synchronous generators is equal to the electrical power consumed by the network (plus the mechanical and electrical power losses): $P_{\text{elec}} = P_{\text{mech}} + P_{\text{losses}}$. However, sudden disturbances on the grid changes the power balance and $P_{\text{elec}} \neq P_{\text{mech}} + P_{\text{losses}}$.

The input mechanical power supplied by the prime mover (e.g., hydro or thermo), $P_{\text{turbine}}$ is converted into both electrical power that feeds the network, $P_{\text{elec}}$, and stored kinetic energy in the generators: $P_{\text{turbine}} \approx P_{\text{elec}} + \frac{dE_{\text{kinetic}}}{dt}$ [20], where in normal steady-state conditions, $P_{\text{turbine}} = P_{\text{elec}}$ and $\frac{dE_{\text{kinetic}}}{dt}$ remains as stored energy.

In the first few seconds after a disturbance, the mechanical output power $P_{\text{mech}}$ is slow to react to the changes of the electrical power due to the difference in time-constants (the mechanical time constants are larger than the electrical ones [45]) and
4.1. A Physical Representation

$P_{\text{turbine}}$ remains constant for about $\frac{3}{4}$ of a second before control action is initiated, such as automatic voltage control [4][46]. There are two conditions that can change the power balance: (1) If there is an increase in electrical power, for example, due to a large increase in load or a sudden loss of generation and (2) if there is a decrease in electrical power, for example, due to a loss of load or a fault.

In the first condition, when there is a large increase in load or loss of generation, there is a lack of $P_{\text{turbine}}$ to supply $P_{\text{elec}}$. During this time, the power is borrowed from the generator’s kinetic energy $P_{\text{kinetic}}$, which slows the machine down, or decreases its velocity away from the 60-Hz frequency. Conversely, when there is a loss of load due to, for example, a fault, there is an excess of $P_{\text{turbine}}$ compared with $P_{\text{elec}}$. The extra power is transferred to the kinetic energy of the rotor, causing the rotor to accelerate, which increases the electrical frequency.

4.1.1 The Equal Area Criteria and The Critical Clearing Time

The physical interpretation can be illustrated using the equal area criteria, which is a graphical representation of the kinetic energy balance in the network. If the network is represented as an equivalent single generator feeding an infinite bus [4], then the equal area criteria can be used to determine the maximum rotor angle position before instability occurs. As illustrated in the equal-area curve of Figure 1.2(b), when the fault occurs, the electrical power $P_e$ will decrease and the angle will begin to increase. When the fault is cleared, the electrical power will increase to its post-fault value and the angle will continue to increase due to the extra kinetic energy stored. The areas A1 and A2 represent the stored kinetic energy. The system is stable if A1 $<=$ A2.

The critical clearing angle of the machine is $\delta_c$ and the time at which the rotor angle reaches this position is called the critical clearing time (CCT). The CCT is a fundamental parameter necessary to compute when the disturbance is a fault as the CCT is the greatest time that the circuit breakers have to open to ensure that the system remains stable. When clearing a fault, the total clearing time needed is the total time it takes for the relay to detect the fault, for the relay to signal to the circuit breaker to open, for the circuit breaker to begin opening its contacts and create an arc, which will extinguish the fault, plus a margin of time delay [47]. The CCT is expressed in cycles on the basis of the grid’s frequency at 60 cycles per...
second [47]. For example, the breakers are designed as 5-cycle or 8-cycle breakers if the CCT is computed as 5 cycles or 8 cycles [47]. One reason for this is that the circuit breaker is set to open when the arc energy is as low as possible, which will occur after the fault current has crossed zero, any time between two 60-Hz cycles.

4.1.2 The Approximated Synchronous Machine Model

As the main motivation for this work is to model the SFA solution for the electrical network compared with the traditional phasor and the EMTP, the simplified model for the synchronous machine is used in this thesis. The simplified generator model is an approximation of the full-order synchronous machine model and is reasonably accurate for the first few cycles of the transient disturbance [4][19]. The generator is modelled as a voltage source behind a transient reactance, where the rotor angle $\delta_{\text{rel}}$ is the same electrical phase angle of the generator (Figure 4.1). This representation assumes that the rotor flux linkages are constant, the damper winding currents have died down, and saliency is neglected [19]. Because the time constants of the sub-transient and armature currents are in the order of $0.03 \text{s} \rightarrow 0.04 \text{s}$ and $0.1 \text{s} \rightarrow 0.3 \text{s}$ [48], whereas the electrical transient phenomenon ranges from $0.5 \text{s} \rightarrow 10 \text{s}$ [48], it can be assumed that only the transient reactance $x_d'$ remains.

4.2 A Mathematical Representation

The core mathematical equation that models the machine’s dynamics is formulated in Newton’s second law of motion for rotational bodies

$$T_{\text{net}} = I \alpha$$

$$T_{\text{net}} = I \frac{d^2 \delta_{\text{rel}}}{dt^2},$$

where $T_{\text{net}}$ is the net torque between the mechanical and the electrical network, $I$ is the machine’s constant of inertia (kg·m²), and $\alpha$ is the rotational acceleration ($\text{rad/s}^2$). This equation, commonly referred to as the “Swing Equation”, is a second-order non-linear equation, which can be expressed more intuitively, as two first-order equations.
4.2. A Mathematical Representation

As a second-order equation,

$$T_m(t) - T_e(t) = I \frac{d^2 \delta_{rel}(t)}{dt^2} + K \frac{d \delta_{rel}(t)}{dt}$$  \hspace{1cm} (4.2)

As two first-order equations,

$$T_m(t) - T_e(t) = I \frac{d \omega_{rel}(t)}{dt} + K \omega_{rel}(t),$$ \hspace{1cm} (4.3)

$$\omega_{rel}(t) = \frac{d \delta_{rel}(t)}{dt}$$ \hspace{1cm} (4.4)

In (4.2), (4.3), and (4.4), $\delta_{rel}$ is the rotor angle (rad), $\omega_{rel}$ is the relative velocity between the rotor velocity and the rotating field created by the stator currents (rad/s), $T_m(t)$ is the torque produced by the prime mover moving the field winding (N·m), $T_e(t)$ is the torque produced by the rotating field due to the stator currents (N·m), $I$ is the moment of inertia (kg·m²), and $K$ is the self-damping coefficient of the machine (N·m rad/s), which ranges from 0.5-2.0 per-unit of the turbine rating for hydro turbines [21]. The damping coefficient allows the rotor’s oscillations to decrease in amplitude and to die out [48]. In stability studies, the inertia constant of the machine $H$ is commonly used, where $H = \frac{I \omega_o^2}{2S_{base}}$ (s), and $S_{base}$ is the MVA rating of the generator. Equation (4.4) is used for both the mechanical and the electrical velocity of the generator, where the relationship between the mechanical and electrical velocity is given as,

$$w_{mech} = \frac{2w_{elec}}{p},$$ \hspace{1cm} (4.5)

where $p$ is the number of poles in the machine. If the machine is two-poled, the mechanical and electrical frequency are equal $w_{mech} = w_{elec}$.

Equations (4.2) and (4.3) are non-linear because net power is known in the power network and the frequency, $\omega$ is used to convert net torque into net power. In the traditional transient stability programs, the net torque is converted to net power through $\omega_o$ under the assumption that the actual frequency $\omega$ does not deviate far from $\omega_o$. This assumption is approximately accurate only under small frequency deviations where $\omega \approx \omega_o$. When the frequency deviations are larger, $\omega$ will differ from $\omega_o$. For a more precise mathematical model, the correct $\omega$ should be used in (4.2) and (4.3) to convert net torque into net power, as done in this thesis.
4.2.1 The Coupling of the Electromechanical and Electrical Network Equations in the Solution Methods

To mathematically couple the electromechanical and the electrical networks, the phasor solution, the EMTP, and the SFA solution methods take different approaches.

Coupling in the Traditional Phasor Method

The traditional simulation programs (e.g., [3][18]) that model the electrical network with the 60-Hz frequency phasor solution method use step-by-step time-domain methods to simultaneously solve the electromechanical and electrical network equations.

\[
\frac{d}{dt} = f(x, V),
\]
\[
I(x, V) = Y_N V,
\]

where \( x \) is the vector of states (rotor angle, frequency), \( V \) is the vector of voltages, \( I \) is the current vector, and \( Y_N \) is the admittance matrix [3].

The differential electromechanical equation (4.2) is solved using continuous-time numerical integration techniques, such as the Runge-Kutte or the Trapezoidal technique [3] and the algebraic network equations (4.6) use the traditional phasor method, described in Section 1.3.1. In actuality, the phasor method is solved in the frequency domain through the phasor representation [21].

Coupling in the EMTP and SFA Methods

The EMTP solution is solved completely in the time-domain. The electromechanical equation is discretized using numerical discretization methods, such as the trapezoidal or the backward Euler, given in (4.9) and (4.10) or (4.11) and (4.12), and the electrical network equations are solved through the discrete-time EMTP equivalent discretization, as described in Section 1.3.2. Likewise, in the SFA solution, the electromechanical equation is discretized like the EMTP ((4.9), (4.10), (4.11), and (4.12)), and the electrical network equations are solved in the SFA domain as a discrete-time solution, as described in Chapter 2. The coupling of the differential and algebraic equations is completely in the discrete-time domain when using the EMTP and the SFA solutions due to the discretization of the continuous-time equations, as opposed to the phasor solution, where the equations are coupled between the continuous-time domain and the phasor domain.
4.2. The Discretization of the Electromechanical Equation

In this work, the actual velocity $\omega$ is used to convert between torque and power. Since the electromechanical equation is discretized using numerical discretization in the solution method, $\omega$ is known at every time-step. If the trapezoidal rule is used in this discretization, (4.3) and (4.4) are formulated as (4.9) and (4.10). If the backward Euler discretization rule is used in this discretization, (4.3) and (4.4) are formulated as (4.11) and (4.12). Net torque is converted into net power using

$$T_{\text{net}}(t) = \frac{P_{\text{net}}(t)}{w_{\text{rel}}(t - \Delta t)}.$$  \hspace{1cm} (4.7)

where

$$P_{\text{net}}(t) = P_m(t) - P_e(t).$$  \hspace{1cm} (4.8)

The following equations discretize the electromechanical equation using the trapezoidal and the backward Euler discretization rules.

**Applying the Trapezoidal Discretization Rule**

$$\omega_{\text{rel}}(t) = \frac{1 - K \frac{\Delta t}{I}}{1 + K \frac{\Delta t}{I}} (\omega_{\text{rel}}(t - \Delta t)) + \frac{\frac{\Delta t}{I}}{1 + K \frac{\Delta t}{I}} \left( \frac{P_m(t) - P_e(t)}{\omega_{\text{rel}}(t - \Delta t)} \right)$$  \hspace{1cm} (4.9)

and

$$\delta_{\text{rel}}(t) = \frac{\Delta t}{2} (\omega_{\text{rel}}(t) + \omega_{\text{rel}}(t - \Delta t)) + \omega_{\text{rel}}(t - \Delta t).$$  \hspace{1cm} (4.10)

**Applying the Backward Euler Discretization Rule**

$$\omega_{\text{rel}}(t) = \frac{1}{1 + \frac{K \Delta t}{I}} \left( \frac{\Delta t}{I} \left( \frac{P_m(t) - P_e(t)}{\omega_{\text{rel}}(t - \Delta t)} \right) \right) + \frac{1}{1 + \frac{K \Delta t}{I}} (\omega_{\text{rel}}(t - \Delta t))$$  \hspace{1cm} (4.11)

and

$$\delta_{\text{rel}}(t) = \delta_{\text{rel}}(t - \Delta t) + \omega_{\text{rel}}(t) \Delta t.$$  \hspace{1cm} (4.12)

The backward Euler discretization rule is used for the electromechanical equations in this work.
4.3 Implementation of SFA in Three Transient Stability Case Studies

The SFA solution method is implemented and tested in three power system transient stability case studies: a Single-Generator Infinite-Bus (SGIB) network [4] (Chapter 5), a three-bus network [45] (Chapter 6), and the IEEE 39-bus network [1] (Chapter 7). In these case studies, SFA’s capability to computationally and efficiently capture network dynamics is demonstrated during three-phase balanced short-circuit faults. In the three-bus network, two generators of different inertia constants feeding a constant impedance load are modelled with the different methods and with the IEEE 39-bus network, the feasibility of the SFA solution method is demonstrated in a large-scale network. The three-bus and the 39-bus networks also demonstrate the behaviour of the solution methods when a generator is disconnected from the network. For all case studies, the EMTP is used as the reference solution.

In the three case studies, some of the observations considered includes:

1. The similarities and differences in CCTs, in both 60-Hz cycles and milliseconds, among the three methods when different three-phase balanced short-circuit faults occur on various locations in the network.

2. The similarities and differences among the three solution methods after a three-phase fault has been cleared from the network.

3. The behaviour of the solution methods when one generator is suddenly disconnected in the three-bus network and the 39-bus network.

4. The differences between the trapezoidal and backward Euler discretization methods in the SFA solution.

5. The difference between using the approximate vs actual velocity in the electromechanical equation for the traditional phasor and the SFA solution methods.

The commercial Transient Security Assessment Tool (TSAT) software [3] is used for the phasor solution, the commercial MicroTran software [14] is used for the EMTP, and a self-written program TSFA is used for the SFA solution method. A
phasor-solution solver, TPhasor, was also written, which allows the comparison of the correct network frequency with the approximate network frequency in the electromechanical equation for the traditional phasor solution. Both the TSFA and TPhasor solutions were written in the Python computer language and are generalized solution algorithms.

For the discretization method of the electromechanical equations, the solution from TSAT uses the trapezoidal numerical integration method, MicroTran uses the trapezoidal method with critical damping adjustment (CDA), and unless otherwise stated, TSFA uses the backward Euler method for both the electrical and the electromechanical network equations. Backward Euler is used for the TSFA solution instead of trapezoidal because the backward Euler rule dampens the numerical oscillations as explained in Chapter [3]. However, as seen in Section 3.4.2, backward Euler introduces high error around 0 Hz, whereas the trapezoidal behaves more accurately at 0 Hz, which is demonstrated when observing the DC component of the fault current.

The time-steps chosen for the solution methods were the maximum values that the solutions would give without too much loss of accuracy in the electromechanical variables. For the EMTP $\Delta t = 100\,\mu s$, for TSAT $\Delta t = 8\,ms$, and for TSFA $\Delta t = 8\,ms$. The time-step for the reconstructed SFA solution (in the continuous-time, physical domain) is $100\,\mu s$ to match the EMTP solution. In all the cases, the EMTP is used as the benchmark solution.

**Conditions and Necessary Assumptions**

The following conditions were set for the case studies:

1. The power system is modelled as a three-phase balanced symmetrical network and only three-phase faults are applied on the system to maintain the balanced condition for all three phases. Future work will include unbalanced conditions, such as single line-ground faults.

2. The controllers, such as the automatic voltage controller and the turbine’s governor action, are not considered, and therefore $P_m$ is taken to be constant. This assumption is used because the first-swing of the rotor angle occurs within the first few seconds after the disturbance [4] and the behaviour of the generators
4.3. Implementation of SFA in Three Transient Stability Case Studies

given by the solution methods is observed for first-swing stability.

3. The simplified generator model is used which consists of a voltage source $E_{gen}$ and a direct-axis transient reactance $x'_d$ as described in Section 4.1.2 [4]. To facilitate the studies, the generators are assumed to be two-poled (3600 rpm in steady-state) and saliency is ignored. In TSAT, the classical model “CGEN” is used for the generators [3]. In MicroTran, because the time constants of the sub-transient and damper winding currents are small, it can be assumed that the it can be assumed that the damper winding currents have died down as the sub-transient reactance time constants are smaller than the the transient reactance time constants (i.e. $T''_d \approx 0.035 \text{s}$, whereas the time constant of the transient reactance, $T'_d \approx 1.5 \text{s}$ [48] and so only the transient reactances ($x'_d$) of the synchronous generator are used for the simplified generator model. To model the simplified generator in MicroTran, the machine’s reactances can all be set to equal $x'_d$ ($x_d = x_q = x'_d = x'_q = x''_d = x''_q$).

4. The loads are modelled as constant impedances, where $P_{load} \propto V^2$ and $Q_{load} \propto V^2$.

5. The EMTP solution in MicroTran updates the generator voltage with the frequency changes in the network; that is, $E_{gen} = \frac{\omega(t)}{\omega_0} E_{field}$, where $E_{field}$ is the voltage induced by the field winding current and is the generator’s voltage $E_{gen}$ and $\omega$ is the known grid frequency at that time-step. This is a type of internal control mechanism between the generator’s voltage and the network frequency, which is different from an external control mechanism, such as the automatic voltage regulator or the governor droop control action. This internal control mechanism is implemented in the TSFA solver.

6. The CCT is observed based on first-swing stability [4, 49]; that is, if the first swing of the rotor angle (or difference in rotor angles between the machines and chosen reference machine) is unstable after the fault has been cleared then the system is assumed unstable. The angle stability margin index used to determine the CCT in TSAT [3] is implemented in TSFA. This margin is calculated by finding the maximum generator angle separation between any two generators and by calculating an index $\eta$, which must be between -100 and 100 [3]. It is important to note that first-swing stability has relatively good accuracy only under the assumption that the synchronous machine is
modelled using the constant voltage behind reactance model, as described in Section 4.1.2, and if the critical clearing times are larger than 6 cycles \cite{48}. However, the first-swing approximation may not always be reliable, especially during fast clearing times (under 6 cycles)\cite{48}. When the clearing time is very fast, the transient stability limit may exceed the steady state stability limit, which may not show instability until after the first-swing\cite{48}. This, however, is not the case in the following case studies, where all clearing times are greater than 6 cycles.

7. As the circuit breaker must wait until after the fault current has crossed zero to clear the fault \cite{47}, both the TSFA and EMTP methods both use a zero-crossing fault detection algorithm. The fault current zero-crossing detection method implemented in TSFA is described in Section 4.3.1.

8. The power flow solution for the network’s initial conditions (initial Q and $\delta$ of the PV buses and V and $\delta$ of the PQ buses) is solved using the commercial product, Powerflow & Short-circuit Analysis Tool (PSAT) from Powertech laboratories\cite{50}. The initial generator voltages $E_{\text{gen}}$ and the initial value of the rotor angles $\delta_o$ are solved for using Kirchoff’s Voltage Law

$$E_{\text{gen}} = \bar{V}_{tn} e^{j\theta_n} + j x_d n \bar{I}_{tn},$$

(4.13)

where

$$\bar{I}_{tn} = \frac{\bar{S}^*}{\bar{V}_{tn} e^{-j\theta_n}}$$

$$\bar{I}_{tn} = \frac{P_n - jQ_n}{\bar{V}_{tn} e^{-j\theta_n}},$$

where $\bar{I}_t$ is the current out of the generator’s terminals, $\bar{V}_t$ is the voltage out of the generator’s terminals, and $n$ is the number of generators in the system.

9. The fault voltage is plotted as the peak, line-neutral per-unit voltage. Therefore there is a factor of $\sqrt{2}/\sqrt{3}$ when converting from the real values to the per-unit values.

10. The TSFA and TPhasor solutions refer to the solutions using the velocity $\omega$ in the electromechanical equation and TSFA* and TPhasor* use the approximated velocity $\omega_o$ in the electromechanical equation.
4.3. Implementation of SFA in Three Transient Stability Case Studies

4.3.1 Fault Current Zero-Crossing Detection Algorithm in TSFA

When a short-circuit fault occurs on the network, the protective devices, such as the circuit breaker, must open to isolate the fault. At the time of the fault, the relay detects the fault and sends a signal for the circuit breaker to open its contacts. The contacts are electrodes and as they separate, the medium that the circuit breaker is immersed in, such as oil, vacuum, or SF6, becomes ionized and an arc of energy is formed between the contacts [47]. As long as the arc is sustained in between the contacts, the current cannot be interrupted. The fault current will only be interrupted when the energy between the contacts has dissipated and the arc is as thin as possible [20]. This occurs when the current is as low as possible (i.e., when the current is zero). In AC systems, the fault current is sinusoidal, changing from positive to negative every half cycle, and therefore it crosses zero twice in one cycle. The circuit breakers will succeed in clearing the fault when the fault current has crossed zero.

A fault current zero-crossing detection algorithm is implemented in the TSFA algorithm to detect the zero-crossing in the shifted domain. Because the shifted domain only gives the envelope and phase angle of the current, however, detecting when the current’s amplitude crosses zero will not be feasible. Therefore, this algorithm detects when the phase of the fault current crosses $90^\circ$ after the set fault clearing time. The derivation is shown as follows:

$$i(t) = |I(t)| \cos(\omega(t) + \theta(t))$$

$$= \frac{|I(t)|}{2} \left( e^{j(\omega(t)+\theta(t))} + e^{-j(\omega(t)+\theta(t))} \right),$$

(4.14)

where $\omega(t)$ is the grid frequency, $|I(t)|$ is the magnitude of the fault current, and $\theta(t)$ is the phase of the fault current.

Current $i(t)$ is zero when $\cos(\omega(t) + \theta(t)) = 0^\circ$. This occurs when $\omega(t) + \theta(t) = 90^\circ$. Alternatively, when $\omega(t) + \theta(t) = 90^\circ$, $e^{j(\omega(t)+\theta(t))} + e^{-j(\omega(t)+\theta(t))} = 0$, which happens when $\omega(t)$ and $-\omega(t)$ are $180^\circ$ apart from each other.

$$(\omega(t) + \theta(t)) - (-\omega(t) + \theta(t)) = 180^\circ \quad \text{or} \quad (\omega(t) + \theta(t)) = 90^\circ. \quad (4.15)$$

Therefore, the fault clearing time in the SFA domain occurs the first moment that
ω(t) + θ(t) = 90° after the set clearing time. The algorithm ensures that the angle is phase angle is constrained to 360°. Without this fault current phase zero-crossing detection algorithm in the shifted domain, at every time-step the fault current would need to be re-transformed into the physical domain, heavily increasing the computational speed.

Considerations Made in the TSFA Algorithm
The following considerations are made in the SFA solution method for the TSFA algorithm:

1. For the change in network topology, such as the fault occurring and clearing, the history voltage is interpolated between the time-step before the topological change and the present time-step.

2. The initial conditions for the SFA solution at time $t = 0^-$ are set by solving the steady-state phasor solution for the solution at $t = 0$ instead of assuming zero initial conditions.

3. As described above, the fault current must cross zero before the fault is actually cleared.

The SFA solution algorithm is given in Figure 4.2.
4.3. Implementation of SFA in Three Transient Stability Case Studies

Figure 4.2: The TSFA solution algorithm.
Chapter 5

Single-Generator Infinite-Bus Case Study

The Single-Generator Infinite-Bus network is shown in Figure 5.1 [4], where a 25 MVA, 60-Hz generator delivers 20 MW through two parallel transmission lines to a large electrical network, represented as an equivalent infinite bus. The network parameters are provided in Table 5.1, with a system base of 25 MVA, 25 kV.

Figure 5.1: The single-line diagram of the Single-Generator Infinite-Bus (SGIB) network, where the electrical network is represented as an infinite bus. A three-phase balanced fault is applied to the terminals of the generator.

<table>
<thead>
<tr>
<th>Table 5.1</th>
<th>Network parameters for the SGIB network (Figure 5.1).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generator 1</td>
<td>Power Rating (MVA) 25</td>
</tr>
<tr>
<td>Generator’s Initial Conditions</td>
<td>Real Power (MW) 20</td>
</tr>
<tr>
<td>Transmission Lines</td>
<td>Reactance (Ω/km) 5</td>
</tr>
<tr>
<td>Line 1-2</td>
<td>Line 1-3</td>
</tr>
</tbody>
</table>

1 per-unit of generator base  
2 per-unit of torque base  
3 per-unit of system base
5.1 Critical Clearing Times

The CCTs given by the traditional phasor solution (TSAT), the traditional phasor solution with the exact velocity (TPhasor), the EMTP, TSFA, and TSFA* (which uses the approximate velocity $\omega_o$) are provided in Table 5.2 for the three-phase fault at the generator’s terminals. The CCT is in both 60-Hz cycles and in milliseconds and the critical clearing angle $\delta_c$ is in degrees. To be conservative, the 60-Hz cycle shown is rounded down to the lowest closest cycle (for example, in the EMTP, the rotor angle becomes unstable at 13.6 cycles, which is rounded down to 13 cycles).

<table>
<thead>
<tr>
<th>Three Phase Fault on Gen.1</th>
<th>EMTP ($\Delta t = 100,\mu s$)</th>
<th>TSAT ($\Delta t = 8,\text{ms}$)</th>
<th>TPhasor ($\Delta t = 8,\text{ms}$)</th>
<th>TSFA ($\Delta t = 8,\text{ms}$)</th>
<th>TSFA* ($\Delta t = 8,\text{ms}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_c$(cycles)</td>
<td>13°</td>
<td>14</td>
<td>14</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>$t_c$(ms)</td>
<td>216</td>
<td>233</td>
<td>234</td>
<td>248</td>
<td>248</td>
</tr>
<tr>
<td>$\delta_c$</td>
<td>80°</td>
<td>104.8°</td>
<td>106.3°</td>
<td>97.5°</td>
<td>98.1°</td>
</tr>
</tbody>
</table>

* rounded down from 13.8 to 13 cycles

From Table 5.2, the following observations are made:

1. There is a half-cycle difference in the CCT between all the solutions (TSAT, TPhasor, TSFA and TSFA*) and the reference EMTP. This is due to the smaller time-step used in the EMTP solution compared with the other solutions. If the EMTP time-step is raised to $\Delta = 1\,\text{ms}$, then the CCT for the EMTP solution matches the other solutions at 14 cycles.

2. Using the actual $\omega$ instead of $\omega_o$ in the electromechanical equations does not change the CCT for the SFA solution method; however, the traditional phasor solution changes by 1 ms.
5.2 A Three-Phase Fault at the Generator’s Terminals

A three-phase balanced fault is applied to the terminals of the generator (Bus 1 in Figure 5.1) at \( t = 1 \) s and is cleared after 8 cycles. The generator’s electromechanical variables: rotor-angle, frequency and electrical power as well as the fault voltage and fault current are simulated using the phasor, the EMTP and the SFA solution methods, with the EMTP as the reference solution.

The fault is set to clear at 8 cycles, but due to both the differences in the solution methods’ time-steps and the zero-crossing fault current detection employed in the EMTP and TSFA solutions, the actual clearing of the fault is different among the methods, varying between 8-9 cycles. This difference in clearing times means that extra kinetic energy has accumulated in the network [20], producing both a phase displacement and a magnitude difference among the methods.

When the fault happens at \( t = 1 \) s, the generator’s rotor angle and frequency begin to increase until the fault is cleared. When the fault is cleared, the frequency begins to decrease, but due to extra kinetic energy stored in the generator, the angle continues to increase for another 4.4 cycles (73.3 ms). Because the fault is cleared before its CCT of 14 cycles, the system is considered “first-swing stable”. The infinite bus acts like a constant voltage source with zero impedance and infinite inertia [4], fixing the system’s frequency to 60 Hz around the infinite bus. Figures 5.2 and 5.3 present the generator’s electromechanical variables: rotor angle, frequency and electrical power, where in Figure 5.3, the TSFA solution uses both \( \Delta t = 5 \) ms and \( \Delta t = 8 \) ms to compare the differences in time-steps with respect to the EMTP reference. In Figures 5.2(c) and 5.3(c), the electrical power oscillations in the EMTP solution are due to the EMTP calculating the instantaneous power, whereas the phasor and SFA solution calculate the average power, \( P = \mathcal{R}(VI^*) \).

The fault voltage and fault current are shown in Figures 5.4 and 5.5, where the difference in clearing times are prominent. In the EMTP, the fault is cleared at 8.46 cycles (140.6 ms) after zero-crossing detection and in the SFA solution, the fault is cleared at 8.63 cycles (143.9 ms) after zero-crossing detection. In the TSAT solution, the fault is set to clear at exactly 8 cycles (133 ms), while in the TPhasor* solution, used to simulate the fault current, due to the fixed discrete time-step, the fault current clears in 8.15 cycles (136 ms).
5.2. A Three-Phase Fault at the Generator’s Terminals

Figure 5.2: A comparison of the TSFA and TSAT solutions compared with the EMTP solution for the electromechanical variables in the system of Figure 5.1 during a three-phase fault at the terminals of the generator.
In Figure 5.2(b), there is a difference in the peak-value of the generator’s frequency among the methods. The frequency error between TSAT and the EMTP is 0.081%, which is smaller than the error between TSFA and the EMTP on the first swing, 0.114%. However, at the second-swing and onwards, the TSFA and the EMTP solution are consistently closer to each other than the TSAT and the EMTP solution (0.106% between TSFA and the EMTP and 0.146% between TSAT and the EMTP on the second-swing).

By using a smaller time-step of $\Delta t = 5$ ms for TSFA, as shown in Figure 5.3, the error between the TSFA and the EMTP solution is less than when using $\Delta t = 8$ ms in Figure 5.2. This is due to two factors. First, when the smaller time-step is used, the fault is cleared at 8.39 s (139 ms) (Figure 5.3), which is a difference of 1 ms between the TSFA and the EMTP, whereas when $\Delta t = 8$ ms, the fault was cleared at 8.63 cycles (143 ms) in the TSFA solution (Figure 5.2(b)). This is a 4 ms between the TSFA and the EMTP. When the fault is cleared in the TSFA solution at a closer time to when the fault is cleared in the EMTP solution and the time-step used is smaller, the frequency error between TSFA and the EMTP on the first-swing of the frequency is smaller (0.016% in Figure 5.3(b)). There is no difference when using a smaller $\Delta t$ in TSAT. The TSAT and TSFA solutions are close to each other in this case study because the infinite bus does not allow large frequency deviations as it fixes the network frequency close to 60 Hz.

A comparison of the error in frequency between the three solution methods is provided in Table 5.3.

<table>
<thead>
<tr>
<th></th>
<th>TSAT ($\Delta t = 5$ ms or $\Delta t = 8$ ms)</th>
<th>TSFA ($\Delta t = 8$ ms)</th>
<th>TSFA ($\Delta t = 5$ ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency error</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(first peak) (%)</td>
<td>0.083</td>
<td>0.116</td>
<td>0.016</td>
</tr>
<tr>
<td>frequency error</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(second peak) (%)</td>
<td>0.146</td>
<td>0.106</td>
<td>0.016</td>
</tr>
<tr>
<td>clearing time</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(first peak) (cycles)</td>
<td>8.15</td>
<td>8.63</td>
<td>8.39</td>
</tr>
</tbody>
</table>

Table 5.3

A comparison of the error in frequency among TSAT (at $\Delta t = 8$ ms) and TSFA (at $\Delta t = 5$ ms and $\Delta t = 8$ ms) with respect to the EMTP.
5.2. A Three-Phase Fault at the Generator’s Terminals

Figure 5.3: A comparison of the EMTP, TSFA using $\Delta t = 5 \text{ ms}$ and $\Delta t = 8 \text{ ms}$ and the phasor solution using $\Delta t = 5 \text{ ms}$ and $\Delta t = 8 \text{ ms}$ for the electromechanical variables during the three-phase fault in the system of Figure 5.1.
5.2. A Three-Phase Fault at the Generator’s Terminals

The fault voltage presented in Figure 5.4 and Figure 5.4(a) shows that the continuous-time TSFA solution matches the EMTP solution and the TSFA envelope is almost identical to the TSAT solution, except for the first time-step post-fault due to the differences in fault clearing times.

Voltage at the faulted bus (Terminals of Generator 1)

Due to different fault clearing times

Figure 5.4: The voltage at the faulted bus (terminals of Generator 1) during the 3-phase fault.

Voltage at the faulted Bus (Pre-Fault and Post-Fault)

Figure 5.4 (a) A close-up of the voltage at the faulted bus shows that TSAT and the envelope of TSFA match and the reconstructed TSFA and the EMTP match. (Left: pre-fault, Right: post-fault).
5.2. A Three-Phase Fault at the Generator’s Terminals

The fault current is simulated using both $\Delta t = 8$ ms and $\Delta t = 1$ ms in the TSFA solution (Figure 5.5), resulting in a 2.05% error difference when compared with the EMTP (Figure 5.5(a)). When $\Delta t = 8$ ms, the fault is cleared at 8.63 cycles (143.9 ms) and when $\Delta t = 1$ ms, the fault is cleared 1 ms earlier at 8.57 cycles (142.9 ms).

![Fault Current](image)

Figure 5.5: The fault current during a 3-phase fault at the terminals of Generator 1 (Top: using $\Delta t = 8$ ms for TSFA, Bottom: using $\Delta t = 1$ ms for TSFA).

(a) The fault current during a 3-phase fault at the terminals of Generator 1: (Left: TSFA at $\Delta t = 8$ ms and Right: TSFA at $\Delta t = 1$ ms).
5.2. A Three-Phase Fault at the Generator’s Terminals

5.2.1 A Comparison of Using the Actual Frequency with the Approximate Frequency

The SFA solution using the approximate frequency TSFA* is compared with TSFA to observe the differences between using the actual velocity \( \omega \) as opposed to the approximate velocity \( \omega_o \) in the electromechanical equation. As the EMTP solution uses the approximate velocity, the TSFA* solution is closer to the EMTP reference. The phasor solution is also simulated using the corrected equation in TPhasor and compared with using the approximate velocity in TPhasor*. The results are shown in Table 5.4, and Figure 5.6.

<table>
<thead>
<tr>
<th></th>
<th>TSFA</th>
<th>TSFA*</th>
<th>TPhasor</th>
<th>TPhasor*</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency error</td>
<td>0.114</td>
<td>0.0978</td>
<td>0.114</td>
<td>0.147</td>
</tr>
<tr>
<td>(first peak) (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>frequency difference (Hz)</td>
<td>0.07</td>
<td>0.06</td>
<td>0.07</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Table 5.4
A comparison of the error in frequency between TSAT, TSAT*, TPhasor and TPhasor* with respect to the EMTP reference

Changing the velocity \( \omega \) in the electromechanical equation comparing SFA and the phasor solution with the EMTP

![Graph showing frequency comparison](image)

Figure 5.6: A comparison of using the grid frequency, \( \omega \), with the approximate frequency, \( \omega_o \), in the SFA solution and in the phasor solution compared to the EMTP for the three-phase fault on the SGIB.
5.2. A Three-Phase Fault at the Generator’s Terminals

5.2.2 A Comparison of the Discretization Methods in the TSFA Solution During the Three-Phase Fault.

The trapezoidal and backward Euler discretization methods were compared for the same three-phase fault condition in Figure 5.1. Figure 5.7 shows that the error difference between TSFA-Trap and the EMTP reference is 0.26%, while the error difference between TSFA-BE and the EMTP reference is 0.11% on the first swing of the frequency, making the backward Euler rule more accurate in this scenario.

In the electrical power and fault voltage, the trapezoidal rule has oscillations post-fault. To show the oscillations in the fault voltage, the time-step used for both trapezoidal and backward Euler in the fault voltage is $\Delta t = 20$ ms. The oscillations are numerical as they occur peak-peak $\Delta t$ apart, which matches the behaviour of the trapezoidal method during discontinuities (Chapter 3), in which sustained numerical oscillations occur during a step discontinuity, which, like the fault clearing, is a topological change. If the time-step for the trapezoidal rule was smaller, the oscillations would be smaller.

Therefore, if the trapezoidal rule was used in TSFA, two approaches to eliminate these oscillations could be to either use a multi-rate technique [51] to change to a smaller time-step during the fault and back to the larger time-step after the fault, or to use a multi-rule technique [51], using backward Euler rule during the fault and one time-step after the fault, and then switch back to using the trapezoidal rule for the post-fault condition (similar to the critical damping adjustment used in the EMTP [38]).
5.2. A Three-Phase Fault at the Generator’s Terminals

Comparison of the Trapezoidal and Backward Euler Rules During a Three-Phase Fault

Figure 5.7: A comparison of the trapezoidal and backward Euler discretization rules on the rotor angle, frequency, electrical power, and fault voltage for the three-phase fault condition. Top left: Generator’s rotor angle, Top right: Generator’s frequency, Bottom left: Generator’s electrical power, Bottom right: Fault voltage.
Chapter 6

Three-Bus Network Case Study

The three-bus network is shown in Figure 6.1. The three-bus network topology is a traditional case study studied in transient stability analysis literature \[45\]. The network consists of two generators, a 25 kV, 300 MVA hydro generator and a 13.8 kV, 50 MVA hydro generator, delivering a total of 315 MW over a balanced 3-phase 250 kV system to a constant impedance load of 310 MW. The 300 MVA generator has a kinetic energy of 6 MJ per MVA (rated inertia of \( H = 6 \) s) at rated velocity and the 50 MVA generator has a kinetic energy of 2 MJ per MVA (rated inertia of \( H = 2 \) s) at rated velocity. The direct-axis transient reactance of both generators is \( x_d' = 0.3 \) p.u. on the generator base; and the voltage behind the reactance of Generator 1 and Generator 2 are 1.18 and 1.48 p.u. on the generator bases. The transmission line is modelled using the \( \pi \) model. The system is a single-line equivalent of a balanced, symmetrical, three-phase network and the system’s parameters are given in Table 6.1 below.

The simulations were performed under two conditions: a three-phase fault on the terminals of Transformer 2 and when Generator 1 is disconnected from the network.

Figure 6.1: The one-line diagram of the three-bus network, where two generators feed a constant impedance load.
### Table 6.1
**Network parameters for the three-bus network** (Figure 6.1)

<table>
<thead>
<tr>
<th>Generators</th>
<th>Power Rating (MVA)</th>
<th>Voltage Rating (kV)</th>
<th>Transient Reactance ($x_d'$, p.u.)</th>
<th>Inertia Constant (H) (sec)</th>
<th>Moment of Inertia (I) (kg·m²)</th>
<th>Damping Coefficient (K, p.u.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generator 1</td>
<td>300</td>
<td>25</td>
<td>0.3</td>
<td>6</td>
<td>25330.3</td>
<td>1.5</td>
</tr>
<tr>
<td>Generator 2</td>
<td>50</td>
<td>13.8</td>
<td>0.3</td>
<td>2</td>
<td>1407.23</td>
<td>1.0</td>
</tr>
</tbody>
</table>

**Generator Initial Conditions**

<table>
<thead>
<tr>
<th>Generators</th>
<th>Real Power (MW)</th>
<th>Reactive Power (MVAR)</th>
<th>Terminal Voltage ($V_t$, p.u. $\angle$°)</th>
<th>$E_o \angle \delta_o$ (kV°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generator 1</td>
<td>270.1</td>
<td>106.58</td>
<td>1$\angle$0°</td>
<td>23.25$\angle$13.72°</td>
</tr>
<tr>
<td>Generator 2</td>
<td>45</td>
<td>17.40</td>
<td>0.95$\angle$7.35°</td>
<td>12.36$\angle$7.66°</td>
</tr>
</tbody>
</table>

**Transformers**

<table>
<thead>
<tr>
<th>Transformers</th>
<th>Rated Power (MVA)</th>
<th>Voltage Turns Ratio (kV:kV)</th>
<th>Reactance $X_t$, p.u.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trans. 1</td>
<td>600</td>
<td>25:250</td>
<td>0.15</td>
</tr>
<tr>
<td>Trans. 2</td>
<td>300</td>
<td>250:13.8</td>
<td>0.12</td>
</tr>
</tbody>
</table>

**Loads**

<table>
<thead>
<tr>
<th>Loads</th>
<th>Real Power (MW)</th>
<th>Reactive Power (MVAR)</th>
<th>Resistance $R$, Ω</th>
<th>Reactance $X$, Ω</th>
</tr>
</thead>
<tbody>
<tr>
<td>Load 1</td>
<td>310</td>
<td>150</td>
<td>164.45</td>
<td>339.86</td>
</tr>
</tbody>
</table>

**Transmission Lines**

<table>
<thead>
<tr>
<th>Line</th>
<th>Length (km)</th>
<th>R ($\Omega$/km)</th>
<th>X ($\Omega$/km)</th>
<th>$B^8$ (S/km)</th>
<th>$R$, p.u.</th>
<th>X, p.u.</th>
<th>$B^8$, p.u.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line 1-2</td>
<td>180</td>
<td>0.04 ($7.2\Omega$)</td>
<td>0.4 ($72\Omega$)</td>
<td>4.3 $\mu$S, ($774\mu$S)</td>
<td>0.0115</td>
<td>0.115</td>
<td>0.4836</td>
</tr>
<tr>
<td>Line 1-3</td>
<td>150</td>
<td>0.0267 ($41\Omega$)</td>
<td>0.267 ($40\Omega$)</td>
<td>4.3 $\mu$S, ($645\mu$S)</td>
<td>0.0064</td>
<td>0.064</td>
<td>0.4032</td>
</tr>
<tr>
<td>Line 2-3</td>
<td>80</td>
<td>0.04 ($3.2\Omega$)</td>
<td>0.4 ($32\Omega$)</td>
<td>4.3 $\mu$S, ($344\mu$S)</td>
<td>0.00512</td>
<td>0.0512</td>
<td>0.215</td>
</tr>
</tbody>
</table>

1 per-unit of generator base; 2 per-unit of torque base; 3 per-unit of system base; 4 per-unit of transformer base; 5 $V_{load} = 225.79$ kV; 6 kV, line-neutral peak value [19]; 7 in total Ω or S;

---

7 not $\frac{4}{7}$
6.1 Critical Clearing Times

Three-phase balanced faults were applied at different locations in the network to observe the different CCTs given by the EMTP, TSAT, the TPhasor method, which uses the correct velocity in the electromechanical equation, TSFA and TSFA*, which uses the approximated velocity. The comparisons are presented in Table 6.2. The results show that the only time the TSFA solution does not match the EMTP solution is when the fault is at the terminals of Generator 1. In this scenario, the TSFA solution is 8 ms smaller than the EMTP solution due to two reasons. First, if the EMTP does not check for zero-crossing, then the EMTP solution gives a CCT=284 ms (17 cycles). Second, when the TSFA solution has a reduced time-step to $\Delta t = 1$ ms, the TSFA solution also gives a CCT=284 ms. When these two conditions happen, the EMTP and the TSFA solutions will match. Therefore, the discrepancy in the CCT in this case is not due to the SFA solution method because with the other fault cases, the TSFA and the EMTP solutions are either identical or only 1 ms apart. In the case with the phasor solution, TSAT, there is approximately a 20 ms difference between TSAT and the EMTP, which corresponds to 1 cycle. This signifies that the SFA and the EMTP solutions are closer than the TSAT and EMTP solution.

Observing the differences between the CCTs given in the TSAT/TPhasor methods and the TSFA/TSFA* methods, there is approximately a consistent 10 ms difference between the TPhasor and TSAT solutions. In the case when the fault is placed at the terminals of Transformer 2, in the middle of Line 1-2, and in the middle of Line 1-3, the CCT given by the TSFA* solution is TSFA* is 1 cycle lower than when using TSFA. These differences indicate that using the correct frequency in the electromechanical equation can make a slight difference in the CCT, even in the traditional phasor solution method. The critical clearing angles is the rotor angle of Generator 2 and they are all within 10-15° apart from each other. For the EMTP solution, the CCT is calculated with the zero-crossing of the fault current being detected, but for the TSFA solution, the CCT is calculated without the zero-crossing detection because if the zero-crossing detection was implemented to calculate the clearing times, there would be approximately a 3-cycle difference between the stable and unstable clearing time. This is due to the large time-step used in the solution method. A more accurate method to capture the clearing time in TSFA using the
### 6.1. Critical Clearing Times

Zero-crossing detection would be to use multi-rate to change the time-step to a smaller one at the time of the fault clearing such that there would not be this large difference. This improvement is something to be done in future research.

#### Table 6.2

The CCTs for three-phase faults at different locations of the three-bus system in Figure 6.1 in 60-Hz cycles and the critical clearing angle in degrees.

<table>
<thead>
<tr>
<th>Contingency</th>
<th>EMTP $(\Delta t = 100,\mu s)$</th>
<th>TSAT $(\Delta t = 8,\text{ms})$</th>
<th>TPhasor $(\Delta t = 8,\text{ms})$</th>
<th>TSFA $(\Delta t = 8,\text{ms})$</th>
<th>TSFA* $(\Delta t = 8,\text{ms})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fault on the terminals of Generator 1 (Location A)</td>
<td>17 (288 ms) 99.5°</td>
<td>16 (267 ms) 90.7°</td>
<td>16 (272 ms) 103.8°</td>
<td>17 (280 ms) 101.0°</td>
<td>17 (280 ms) 104.5°</td>
</tr>
<tr>
<td>Fault on the terminals of Generator 2 (Location G)</td>
<td>15 (247 ms) 87.5°</td>
<td>13 (225 ms) 83.2°</td>
<td>14 (240 ms) 100.6°</td>
<td>15 (248 ms) 96.7°</td>
<td>15 (248 ms) 98.9°</td>
</tr>
<tr>
<td>Fault at the load (Bus 3)</td>
<td>17 (296 ms) 101.4°</td>
<td>16 (277 ms) 97.6°</td>
<td>17 (288 ms) 112.0°</td>
<td>17 (296 ms) 107.8°</td>
<td>17 (296 ms) 110.3°</td>
</tr>
<tr>
<td>Fault on the terminals of Transformer 1 (Location B)</td>
<td>16 (272 ms) 91.9°</td>
<td>15 (254 ms) 87.0°</td>
<td>15 (264 ms) 103.4°</td>
<td>16 (272 ms) 100.1°</td>
<td>16 (272 ms) 102.7°</td>
</tr>
<tr>
<td>Fault on the terminals of Transformer 2 (Location F)</td>
<td>15 (255 ms) 91.5°</td>
<td>14 (233 ms) 85.4°</td>
<td>14 (240 ms) 100.4°</td>
<td>15 (256 ms) 100.1°</td>
<td>14 (248 ms) 98.9°</td>
</tr>
<tr>
<td>Fault in the middle of Line 1-2 (Location C)</td>
<td>18 (313 ms) 104.2°</td>
<td>17 (287 ms) 96.1°</td>
<td>17 (296 ms) 108.0°</td>
<td>18 (312 ms) 110.0°</td>
<td>18 (304 ms) 107.0°</td>
</tr>
<tr>
<td>Fault in the middle of Line 1-3 (Location D)</td>
<td>18 (304 ms) 104.0°</td>
<td>16 (274 ms) 92.2°</td>
<td>17 (288 ms) 110.1°</td>
<td>18 (304 ms) 110.0°</td>
<td>17 (296 ms) 107.8°</td>
</tr>
<tr>
<td>Fault in the middle of Line 2-3 (Location E)</td>
<td>16 (280 ms) 97.8°</td>
<td>15 (260 ms) 92.4°</td>
<td>15 (272 ms) 106.1°</td>
<td>16 (280 ms) 105.0°</td>
<td>16 (280 ms) 107.9°</td>
</tr>
</tbody>
</table>
6.2 A Three-Phase Fault at the Terminals of Transformer Two

A three-phase fault is applied to the terminals of Transformer 2 at \( t = 1 \) s and is cleared in 8 cycles by opening circuit breaker B (Figure 6.1). Figures 6.2 and 6.3 present the electromechanical variables: rotor angle, frequency and electrical power. Figures 6.4, 6.5 and 6.6 present the fault voltage and fault currents, where in Figure 6.6 the fault current is simulated using a \( \Delta t = 1 \) ms. The TPhasor* solution is used instead of TSAT for the fault current in Figure 6.5 and Figure 6.6 because TSAT does not explicitly give the fault current.

The fault occurs at \( t = 1 \) s, and both generators begin to accelerate. As Generator 2 has a lower inertia, its frequency deviations are larger than Generator 1’s deviations. Generator 2 deviates 2 Hz from the 60-Hz frequency while Generator 1 only deviates 0.7 Hz from the 60-Hz frequency. Once the fault is cleared, the frequency begins to decrease, but due to the extra kinetic energy stored in the generators, the angles continue to increase for another 2.7 cycles (45.4 ms). As the fault is cleared before the CCT of 14 cycles, the rotor angle is considered “first-swing stable.”

Both the TSFA and EMTP solutions use zero-crossing detection for the fault current and so the clearing times differ among the solution methods. The EMTP solution clears the fault after 7.9 cycles (132 ms) and the TSFA solution clears the fault after 8.63 cycles (144 ms). The TSAT solution clears the fault at exactly 8 cycles (133 ms). Frequency errors are observed for the first swing in Generator 1 and 2 when the results are compared among the benchmark, EMTP, with the TSFA and TSAT solutions. For Generator 1, the frequency error was 0.116% between TSAT and EMTP, and 0.033% between TSFA and EMTP (Figure 6.2). For Generator 2, the frequency error was 0.355% between TSAT and EMTP and 0.161% between TSFA and EMTP (Figure 6.3). This is shown in Table 6.3.

<table>
<thead>
<tr>
<th></th>
<th>TSAT (( \Delta t =8 ) ms)</th>
<th>TSFA (( \Delta t =8 ) ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen.1 frequency error (%)</td>
<td>0.116</td>
<td>0.033</td>
</tr>
<tr>
<td>Gen.2 frequency error (%)</td>
<td>0.355</td>
<td>0.161</td>
</tr>
</tbody>
</table>

Table 6.3
A comparison of the error in frequency between TSAT and TSFA with respect to the EMTP reference solution.
6.2. A Three-Phase Fault at the Terminals of Transformer Two

Figure 6.2: A comparison of the EMTP, TSFA, and TSAT solutions for the electromechanical variables of Generator 1 in the system of Figure 6.1 during a three-phase fault at the terminals of Transformer 2.

Figure 6.2 (a) Gen.2-Gen.1 angle.

Figure 6.2 (b) Gen.1's frequency.

Figure 6.2 (c) Gen.1's power.
6.2. A Three-Phase Fault at the Terminals of Transformer Two

Figure 6.3: A comparison of the EMTP, TSFA, and TSAT solutions for the electromechanical variables of Generator 2 in the system of Figure 6.1 during a three-phase fault at the terminals of Transformer 2.

Fig.6.3 (a) Gen.2’s frequency.

Fig.6.3 (b) Gen.2’s electrical power.
6.2. A Three-Phase Fault at the Terminals of Transformer Two

![Voltage at the faulted bus (Terminals of Transformer 2)](image)

Figure 6.4: The voltage at the faulted bus during a 3-phase fault at the terminals of Transformer 2.

![Fault Current](image)

Figure 6.5: The fault current during a 3-phase fault at the terminals of Transformer 2 (using a $\Delta t = 8$ ms for the SFA solution).

![Fault Current](image)

Figure 6.6: The fault current during a 3-phase fault at the terminals of Transformer 2 (using a $\Delta t = 1$ ms for the SFA solution).
As in the SGIB case study, the fault voltage using the TSFA solution interpolated back into the physical domain is identical to the EMTP solution and the TSFA envelope is the same as the phasor solution. For the fault current, using a smaller time-step in TSFA makes a difference as shown in Figures 6.5 and 6.6. In Figure 6.6 when the $\Delta t$ for the TSFA solution is reduced to 1 ms from 8 ms, the difference in fault current between the EMTP and the TSFA solutions at the second swing post-fault reduces by 1.31%.

6.2.1 A Comparison of Using the Actual Velocity with the Approximate Velocity

The SFA solution using the approximate velocity TSFA* is compared with TSFA to observe the differences between using the actual velocity $\omega$ as opposed to the approximate velocity $\omega_o$ in the electromechanical equation. This comparison is shown in Figure 6.7. Between the TSFA and TSFA* solution there is only a 0.01 Hz difference in the frequency of Generator 1, but a 0.04 Hz difference in the frequency of Generator 2. This difference indicates that using the correct network frequency has an effect on the simulation results. These results are tabulated in Table 6.4.

![Comparison of the SFA Solution changing $\omega$ with respect to the EMTP](image)

Figure 6.7: A comparison of using the grid frequency, $\omega$, with the approximate frequency, $\omega_o$, in the SFA solution in the electromechanical equation for the three-phase fault.
6.2. A Three-Phase Fault at the Terminals of Transformer Two

Table 6.4
A comparison of the error in frequency between the different TSFA solutions with respect to the EMTP reference solution.

<table>
<thead>
<tr>
<th></th>
<th>TSFA</th>
<th>TSFA*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen.1 frequency error (%)</td>
<td>0.033</td>
<td>0.049</td>
</tr>
<tr>
<td>Gen.2 frequency error (%)</td>
<td>0.259</td>
<td>0.323</td>
</tr>
</tbody>
</table>

The phasor solution method using the approximate frequency, TPhasor*, is also compared with the phasor solution using the correct frequency, TPhasor. In the case of the phasor solution, to compare with the EMTP benchmark, the internal generator voltage (Assumption 5 of Section 4.3) was also implemented in the phasor solution. The four conditions for the phasor solution are tabulated in Table 6.5 and shown in Figure 6.8. For these comparisons, TPhasor* was used instead of TSAT to ensure consistency among the solution methods.

Table 6.5
The four different conditions for the phasor solution in Figure 6.8.

<table>
<thead>
<tr>
<th></th>
<th>Omega (Constant)</th>
<th>Omega (Updated)</th>
<th>Generator Voltage (Constant)</th>
<th>Generator Voltage (Updated)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TPhasor*</td>
<td>✓</td>
<td>✗</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>TPhasor</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>TPhasor**</td>
<td>✓</td>
<td>✗</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>TPhasor***</td>
<td>✗</td>
<td>✓</td>
<td>✗</td>
<td>✓</td>
</tr>
</tbody>
</table>

Similar to the comparison between TSFA and TSFA*, between TPhasor and TPhasor*, there is a 0.01 Hz difference in the frequency of Generator 1 but a 0.04 Hz difference in the frequency of Generator 2. On the other hand, the TPhasor** and TPhasor*** are closer to the EMTP on the second swing than the TPhasor or TPhasor* solutions, because correcting the generator voltage allows the phasor solution to settle. In the case when the generator voltage is internally updated using the grid frequency $\omega$, there is a 0.57 Hz difference when using $\omega_0$ in TPhasor** and $\omega$ in TPhasor***, which indicates that using the correct grid frequency does make a slight difference in the simulation results. Although there is only a slight difference when using the corrected frequency, this does not mean that the correction will not make a difference in a situation where the frequency of the generator deviates more than 2 Hz from the 60-Hz frequency. These results are tabulated in Table 6.6.
6.2. A Three-Phase Fault at the Terminals of Transformer Two

Comparison of the Phasor Solutions changing $\omega$ with respect to the EMTP

Figure 6.8: A comparison of using the grid frequency, $\omega$, with the approximate frequency, $\omega_0$, in the phasor solution for the three-phase fault.

Table 6.6
A comparison of the error in frequency between the different TPhasor solutions with respect to the EMTP reference solution.

<table>
<thead>
<tr>
<th></th>
<th>TPhasor</th>
<th>TPhasor*</th>
<th>TPhasor**</th>
<th>TPhasor***</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen.1 frequency error (%)</td>
<td>3.002</td>
<td>2.986</td>
<td>0.112</td>
<td>0.096</td>
</tr>
<tr>
<td>Gen.2 frequency error (%)</td>
<td>0.419</td>
<td>0.484</td>
<td>0.387</td>
<td>0.323</td>
</tr>
</tbody>
</table>

6.2.2 A Comparison of the Discretization Rules in the TSFA Solution During the Three-Phase Fault.

There is negligible difference between using the trapezoidal and backward Euler discretization rules in the rotor angle difference between the two generators and in the generators’ frequencies; however, in the electrical power and post-fault fault voltage, the solution using the trapezoidal rule has sustained numerical oscillations. These oscillations are different from the EMTP oscillations during the fault. In the EMTP, the oscillations are due to the instantaneous power captured instead of the average power. When using a smaller time-step for the trapezoidal rule, such as in the electrical power of Generator 2 in Figure 6.9, the numerical oscillations become smoother and the trapezoidal solution is closer to the EMTP solution than the backward Euler solution (by 0.1 Hz on the first-swing of Generator 1).
6.2. A Three-Phase Fault at the Terminals of Transformer Two

Comparison of the Trapezoidal and Backward Euler Rules During a Three-Phase Fault

Figure 6.9: A comparison of the trapezoidal and backward Euler discretization rules in SFA on the rotor angle, frequency, electrical power, and fault voltage during the three-phase fault at the terminals of Transformer 2.
6.2.3 The DC Offset Captured in the Fault Current in SFA

The solution of a first-order differential equation consists of a transient solution and a steady-state solution. Because the SFA and the EMTP solutions discretize the continuous-time differential equations, they can capture the DC offset in the fault current that happens in the network due to the transient part of the differential equation. The amount of DC offset in the fault current depends on when the fault occurs. If the fault occurs when the fault voltage crosses zero, the DC offset is at its maximum value. This offset is not captured in the traditional phasor solution. To capture the maximum DC offset, the time-step used in SFA must be around 1.67 ms due to the time-frequency relationship (Section 2.1.2). The 0 Hz (DC) frequency from the physical domain is a -60-Hz frequency in the shifted domain and the time-step needed to capture it is \( \Delta t = \frac{1}{10 \times 60} = 1.67 \text{ ms} \). To model the DC offset, a three-phase fault was applied to the middle of Line 1-2 when the voltage crossed zero. Both the trapezoidal and the backward Euler rules were used in SFA and compared with the reference EMTP solution (Figure 6.10). The DC component is accurately captured when using the trapezoidal rule, but not when using the backward Euler rule. The error from using the backward Euler rule can be explained in the error analysis of Chapter 3, where it is observed that at \( \Delta t = 1 \text{ ms} \), the backward Euler has a large magnitude error \( \frac{H_{DT}}{H_{CT}} \approx 5 \) and a large phase distortion \( \approx 80^\circ \) at frequencies between 0-0.5 Hz (Figure 3.5), whereas at \( \Delta t = 1 \text{ ms} \), the solution using trapezoidal has a smaller magnitude error \( \frac{H_{DT}}{H_{CT}} \approx 0.4 \rightarrow 2.5 \) and no phase distortion (Figure 3.2).

![Fault Current with DC Component](image)

Figure 6.10: A three-phase fault is cleared when the voltage crosses zero to capture the maximum DC offset. The SFA solution, using both the trapezoidal and backward Euler rules, and TPhasor* solution are compared to the EMTP.

108
6.3 Generator One Disconnected From the Network

The second scenario simulated is the situation when Generator 1 is suddenly disconnected from the system at \( t = 1 \) s. In this scenario, because the load remains constant, Generator 2 is not capable of supplying all of the load by itself and because governor control action is not considered, the energy that supplies the load comes from the kinetic energy of Generator 2. Therefore, Generator 2’s frequency will decrease and the network frequency will change drastically from 60 Hz. As shown in Figures 6.11 and 6.12 the system is very sensitive to using the actual value of \( \omega \) instead of the constant \( \omega_0 \). The TSFA solution with the corrected frequency drops to 54.93 Hz whereas the EMTP solution settles at 55.4 Hz. In this case, the TSFA solution is actually more accurate than the EMTP solution because the EMTP does not use the actual velocity \( \omega \) when converting between torque and power in the electromechanical equations; hence, TSFA solution has a 0.47 Hz difference compared to the EMTP, whereas the TSFA* solution has a 0.2 Hz difference.

When the trapezoidal rule used, there are strong numerical oscillations are given by the trapezoidal rule. It is interesting to note that the trapezoidal in SFA has oscillations whilst the EMTP solution, which uses the trapezoidal rule with CDA, does not have these oscillations.

TPhasor* was used instead of TSAT because TSAT failed after 2 ms of simulation time. Because the TPhasor* solution does not use the internal voltage control mechanism, the frequency continues to decrease without stabilizing. Figure 6.13 shows the difference in the phasor solution with the different frequency changing combinations of Table 6.5 above. As predicted, when the generator’s voltage is updated with the corrected frequency, the TPhasor solution stabilizes; however, the solution stabilizes to a higher frequency than with the EMTP and SFA.
6.3. Generator One Disconnected From the Network

**Figure 6.11:** The frequency of Generator 2 after disconnecting Generator 1 from the three-bus network in Figure 6.1 using the backward Euler rule (top) and both rules (bottom).

**Figure 6.12:** The insert of Figure 6.11. Left: without the trapezoidal rule, Right: with the trapezoidal rule.

**Figure 6.13:** A comparison of the traditional phasor solution using the corrected electromechanical equation.
Chapter 7

Thirty-Nine Bus Network Case Study

The IEEE 39-bus test system [1] was studied to demonstrate the behaviour of the SFA solution method in a large network. This system is shown in Figure 7.1. The data for this system is taken from [1], with the modifications provided in Tables A1, A2, A3, and A4 of the Appendix.

Figure 7.1: The IEEE 39-bus test system.
In this system, ten generators of varying MVA ratings feed eighteen loads. Generator 5 represents a lower-inertial machine as it is rated at 50 MVA, with an inertia constant $H = 1\text{ s}$. Generator 1 is the reference generator, rated at 1100 MVA, with an inertia constant of $H = 5\text{ s}$. To model the power imbalance, two scenarios are applied to the network. In the first scenario, a three-phase balanced fault occurs on Bus 16. In the second scenario, Generator 9 is disconnected from the grid. In addition, the CCTs of three-phase faults occurring on different locations along the 39-bus network are given in Table 7.1.

### 7.1 Critical Clearing Times

The CCTs for 3-phase balanced faults at different locations along the 39-bus network are given in Table 7.1. The CCTs are in terms of 60-Hz cycles, (rounded to the lowest cycle) and in milliseconds. In addition, the critical clearing angles of the generators that are most affected by the fault are recorded. The results show that the generators with the lowest moment of inertia, Generators 5, 2, 7, and 8, (Table A1) are also the ones that are the most affected when the fault does not occur on the terminals of one of the generators.

The TSFA and the EMTP solution methods give identical CCTs (in 60-Hz cycles) for almost all the cases that were simulated. On average, the difference in milliseconds in CCTs between the TSFA and the EMTP range by approximately 10 ms (Table 7.1). The difference in milliseconds between the two solutions could be due to the difference in time-steps that were used for each solution method. In addition, the EMTP solution checks for the zero-crossing of the fault current to calculate the CCT, whereas the TSFA does not (due to its very large time-steps), an additional procedure which could account for the small differences. Likewise, the critical clearing angle between the two solution methods are very similar, sometimes with only 1° difference. Within the SFA solution, the CCTs show that there are differences between the TSFA and TSFA* solutions observed on Bus 14 and Bus 16 (both buses are near the middle of the network), which indicates that using the corrected velocity in the electromechanical equation can make a difference. Interestingly, when the fault occurs on Bus 7, all the solution methods (TSAT, TSFA, TSFA* and the EMTP) give the identical CCT of 20 cycles.
7.1. Critical Clearing Times

The CCTs for three-phase faults at different locations of the three-bus system in Figure 7.1 in 60-Hz cycles and the critical clearing angle in degrees.

<table>
<thead>
<tr>
<th>Contingency</th>
<th>EMTP $(\Delta t = 100\mu s)$</th>
<th>TSAT $(\Delta t = 8\text{ms})$</th>
<th>TSFA $(\Delta t = 8\text{ms})$</th>
<th>TSFA* $(\Delta t = 8\text{ms})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fault on Bus 16</td>
<td>17 (285 ms)</td>
<td>14 (238 ms)</td>
<td>18 (304 ms)</td>
<td>17 (296 ms)</td>
</tr>
<tr>
<td>(Gen 5’s angle)</td>
<td>171°</td>
<td>60.2°</td>
<td>107°</td>
<td>111°</td>
</tr>
<tr>
<td>Fault on Bus 31</td>
<td>15 (260 ms)</td>
<td>14 (244 ms)</td>
<td>14 (240 ms)</td>
<td>14 (240 ms)</td>
</tr>
<tr>
<td>(Gen 2’s angle)</td>
<td>104°</td>
<td>64°</td>
<td>101°</td>
<td>111°</td>
</tr>
<tr>
<td>Fault on Bus 32</td>
<td>14 (240 ms)</td>
<td>12 (201 ms)</td>
<td>14 (240 ms)</td>
<td>14 (240 ms)</td>
</tr>
<tr>
<td>(Gen 3’s angle)</td>
<td>108°</td>
<td>86°</td>
<td>106°</td>
<td>108°</td>
</tr>
<tr>
<td>Fault on Bus 33</td>
<td>15 (258 ms)</td>
<td>12 (210 ms)</td>
<td>15 (248 ms)</td>
<td>15 (248 ms)</td>
</tr>
<tr>
<td>(Gen 4’s angle)</td>
<td>165°</td>
<td>91°</td>
<td>106°</td>
<td>107°</td>
</tr>
<tr>
<td>Fault on Bus 34</td>
<td>11 (195 ms)</td>
<td>9 (148 ms)</td>
<td>11 (184 ms)</td>
<td>11 (184 ms)</td>
</tr>
<tr>
<td>(Gen 5’s angle)</td>
<td>126°</td>
<td>112°</td>
<td>147°</td>
<td>150°</td>
</tr>
<tr>
<td>Fault on Bus 35</td>
<td>22 (370 ms)</td>
<td>18 (304 ms)</td>
<td>22 (384 ms)</td>
<td>22 (384 ms)</td>
</tr>
<tr>
<td>(Gen 6’s angle)</td>
<td>110°</td>
<td>99°</td>
<td>118.4°</td>
<td>119°</td>
</tr>
<tr>
<td>Fault on Bus 36</td>
<td>10 (175 ms)</td>
<td>9 (152 ms)</td>
<td>10 (176 ms)</td>
<td>10 (176 ms)</td>
</tr>
<tr>
<td>(Gen 7’s angle)</td>
<td>96°</td>
<td>165°</td>
<td>104°</td>
<td>105°</td>
</tr>
<tr>
<td>Fault on Bus 37</td>
<td>12 (205 ms)</td>
<td>10 (179 ms)</td>
<td>12 (208 ms)</td>
<td>12 (208 ms)</td>
</tr>
<tr>
<td>(Gen 8’s angle)</td>
<td>99°</td>
<td>93°</td>
<td>113°</td>
<td>113°</td>
</tr>
<tr>
<td>Fault on Bus 38</td>
<td>12 (205 ms)</td>
<td>10 (170 ms)</td>
<td>12 (208 ms)</td>
<td>12 (208 ms)</td>
</tr>
<tr>
<td>(Gen 9’s angle)</td>
<td>85°</td>
<td>74°</td>
<td>91°</td>
<td>91.2°</td>
</tr>
</tbody>
</table>

Continued on next page
### Table 7.1 – Critical Clearing Times

<table>
<thead>
<tr>
<th>Contingency</th>
<th>EMTP ((\Delta t = 100 \mu s))</th>
<th>TSAT ((\Delta t = 8\text{ms}))</th>
<th>TSFA ((\Delta t = 8\text{ms}))</th>
<th>TSFA* ((\Delta t = 8\text{ms}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fault on Bus 30</td>
<td>40</td>
<td>30</td>
<td>39</td>
<td>39</td>
</tr>
<tr>
<td>(Gen 10’s angle)</td>
<td>(680 ms)</td>
<td>(502 ms)</td>
<td>(664 ms)</td>
<td>(664 ms)</td>
</tr>
<tr>
<td></td>
<td>169°</td>
<td>152°</td>
<td>168°</td>
<td>171°</td>
</tr>
<tr>
<td>Fault on Bus 4</td>
<td>21</td>
<td>19</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>(Gen 3’s angle)</td>
<td>(365 ms)</td>
<td>(330 ms)</td>
<td>(360 ms)</td>
<td>(360 ms)</td>
</tr>
<tr>
<td></td>
<td>150°</td>
<td>132°</td>
<td>149°</td>
<td>150°</td>
</tr>
<tr>
<td>Fault on Bus 7</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>(Gen 2’s angle)</td>
<td>(348 ms)</td>
<td>(334 ms)</td>
<td>(336 ms)</td>
<td>(336 ms)</td>
</tr>
<tr>
<td></td>
<td>140°</td>
<td>118°</td>
<td>146°</td>
<td>146°</td>
</tr>
<tr>
<td>Fault on Bus 8</td>
<td>20</td>
<td>19</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>(Gen 2’s angle)</td>
<td>(340 ms)</td>
<td>(320 ms)</td>
<td>(328 ms)</td>
<td>(328 ms)</td>
</tr>
<tr>
<td></td>
<td>138°</td>
<td>126°</td>
<td>144°</td>
<td>146°</td>
</tr>
<tr>
<td>Fault on Bus 10</td>
<td>15</td>
<td>13</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>(Gen 3’s angle)</td>
<td>(257 ms)</td>
<td>(225 ms)</td>
<td>(256 ms)</td>
<td>(256 ms)</td>
</tr>
<tr>
<td></td>
<td>113°</td>
<td>83°</td>
<td>119°</td>
<td>119°</td>
</tr>
<tr>
<td>Fault on Bus 14</td>
<td>22</td>
<td>19</td>
<td>22</td>
<td>22</td>
</tr>
<tr>
<td>(Gen 3’s angle)</td>
<td>(370 ms)</td>
<td>(317 ms)</td>
<td>(368 ms)</td>
<td>(360 ms)</td>
</tr>
<tr>
<td></td>
<td>153°</td>
<td>127°</td>
<td>155°</td>
<td>152°</td>
</tr>
<tr>
<td>Fault on Bus 19</td>
<td>11</td>
<td>10</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>(Gen 5’s angle)</td>
<td>(194 ms)</td>
<td>(175 ms)</td>
<td>(192 ms)</td>
<td>(192 ms)</td>
</tr>
<tr>
<td></td>
<td>152°</td>
<td>138°</td>
<td>159°</td>
<td>159°</td>
</tr>
<tr>
<td>Fault on Bus 21</td>
<td>18</td>
<td>15</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td>(Gen 7’s angle)</td>
<td>(300 ms)</td>
<td>(260 ms)</td>
<td>(312 ms)</td>
<td>(312 ms)</td>
</tr>
<tr>
<td></td>
<td>155°</td>
<td>135°</td>
<td>167°</td>
<td>167°</td>
</tr>
<tr>
<td>Fault on Bus 25</td>
<td>12</td>
<td>11</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>(Gen 9’s angle)</td>
<td>(212 ms)</td>
<td>(191 ms)</td>
<td>(216 ms)</td>
<td>(216 ms)</td>
</tr>
<tr>
<td></td>
<td>113°</td>
<td>122°</td>
<td>73°</td>
<td>73°</td>
</tr>
</tbody>
</table>
7.2 A Three-Phase Fault Applied to Bus 16

In the first scenario, a balanced three-phase fault is applied to Bus 16 at \( t = 1 \) s and cleared after 8 cycles. Due to the difference in time-steps and the zero-crossing detection in the TSFA and the EMTP solutions, the fault is cleared at different times. In TSFA, the fault is cleared at 8.15 cycles (135 ms), in TSAT, the fault is cleared at 8 cycles (133 ms), and in the EMTP, the fault is cleared at 8.39 cycles (139 ms). The electromechanical variables of Generator 5 are analysed as Generator 5 deviates the most from 60 Hz (±3 Hz from 60 Hz) due to its low moment of inertia (Figure 7.2). Among the three solution methods (TSFA, TSAT and the EMTP), there is negligible error on the first frequency swing, but on the second frequency swing (Figure 7.2(b)), there is a 0.16% between the TSFA and the EMTP and a 0.25% error between the traditional phasor and the EMTP (Table 7.2).

In Figure 7.3 the TSFA solution is compared with the TSFA* solution to observe if changing the velocity in the electromechanical equation is noticeable. Because the EMTP uses \( \omega_0 \) in the electromechanical equation, the TSFA* is closer to the EMTP solution on the second swing (Table 7.3), which agrees with the 8ms (half-cycle) difference in CCTs between the TSFA and TSFA* when the fault was on Bus 16 (Table 7.1).
7.2. A Three-Phase Fault Applied to Bus 16

Rotor Angle Difference between Gen.1 and Gen.5 after a 3-Phase Fault on Bus 16

![Graph showing rotor angle difference](image1)

Frequency of Generator 5 after 3-Phase Fault on Bus 16

![Graph showing frequency](image2)

Electrical Power of Gen.5 after 3-Phase Fault on Bus 16

![Graph showing electrical power](image3)

Figure 7.2: A comparison of the EMTP, TSFA, and TSAT solutions for the electromechanical variables of Generator 5 in the system of Figure 7.1 during a three-phase fault on Bus 16.
7.2. A Three-Phase Fault Applied to Bus 16

Figure 7.3: A comparison of the SFA solution using the actual frequency $\omega$ instead of the approximated frequency $\omega_0$.

### Table 7.2
A comparison of the error in frequency between TSAT and TSFA with respect to the EMTP reference solution.

<table>
<thead>
<tr>
<th></th>
<th>TSAT</th>
<th>TSFA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen.5 frequency error (first swing, Figure (b)) (%)</td>
<td>0.61</td>
<td>0.035</td>
</tr>
<tr>
<td>Gen.5 frequency error (second top swing) (%)</td>
<td>0.252</td>
<td>0.16</td>
</tr>
</tbody>
</table>

### Table 7.3
A comparison of the error in frequency between the different TSFA solutions with respect to the EMTP reference solution.

<table>
<thead>
<tr>
<th></th>
<th>TSFA</th>
<th>TSFA*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen.5 frequency error (first swing) (%)</td>
<td>0.078</td>
<td>0.094</td>
</tr>
<tr>
<td>Gen.5 frequency error (second swing) (%)</td>
<td>0.158</td>
<td>0.016</td>
</tr>
</tbody>
</table>

Figures 7.4 and 7.5 (below) show the fault voltage and fault current during the three-phase fault at Bus 16. The TSFA solution, interpolated back into physical time, matches the EMTP references solution almost identically both in the fault voltage, pre-fault and post-fault, and in the fault current. In the fault voltage, there is a difference of $\approx 0.04$ per-unit between the TSFA solution and the EMTP pre-fault and a difference of $\approx 0.03$ per-unit post-fault. In the fault current, there is a difference of $\approx 4$ per-unit between the TSFA and the EMTP.
7.2. A Three-Phase Fault Applied to Bus 16

**Figure 7.4:** The voltage at the faulted bus during a 3-phase fault at Bus 16 in Figure 7.1.

(a) A close-up of the voltage at the faulted bus shows that the reconstructed TSFA and the EMTP match. (Left: pre-fault, Right: post-fault).

**Fault Current**

**Figure 7.5:** The fault current during a 3-phase fault at Bus 16 in Figure 7.1.
7.3 Generator Nine Disconnected from the Network

When Generator 9 is disconnected from the network, the frequency of the other generators in the network decreases. Using both trapezoidal and backward Euler in the TSFA solution, Figure 7.6 shows the frequency of Generator 5. There are strong oscillations with the trapezoidal rule. Between the TSFA and the EMTP solutions, there is a 0.1 Hz difference in frequency and when comparing the difference between using the corrected velocity ($\omega$) with using the approximate one ($\omega_o$), there is negligible difference. One possible reason for the change in frequency not making a difference is because the frequency drop in this network is small (only 2 Hz). These results differ from the three-bus network (Chapter 6), where there was a frequency drop of 5 Hz in the network and using the corrected network frequency had an effect on the solution.

![Frequency of Gen.5 after Gen.9 is Disconnected](image)

Figure 7.6: The frequency of Generator 5 after losing Generator 9 in the 39-bus Network of Figure 7.1 using the backward Euler rule (top) and both the backward Euler and trapezoidal rules (bottom). Using the trapezoidal rule gives numerical oscillations.
Chapter 8

Conclusion and Future Research

8.1 Conclusion

The research in this thesis demonstrates the application of Shifted Frequency Analysis for power system transient stability studies. The results show that through the use of a shifting transformation, SFA is able to trace the frequency dynamics around the 60-Hz grid frequency using large discretization time-steps. This ability allows the SFA solution method to be used as an alternative to the traditional solution methods in transient stability studies.

The research objectives and contributions defined in Chapter 1 are addressed and implemented in the case studies of Chapters 5, 6, and 7. A generalized SFA solution algorithm TSFA was written for a general power system network and applied in the transient stability case studies of a single-generator infinite-bus network, a three-bus network, and a 39-bus network. The TSFA algorithm implemented a number of novel algorithmic schemes. A zero-crossing detection method for the fault current was developed, which allows the zero-crossing of the fault current to be detected in the SFA domain, which saves computational time. An interpolation scheme was introduced, which allows the use of large time-steps in the SFA solution while keeping the high resolution of the reconstructed physical signal.

In addition to applying the SFA solution method to the electrical network, the transient stability electromechanical equation was generalized to allow the use of the known machine velocity at every time-step of the simulation instead of using the approximate 60-Hz frequency in the equation. This was implemented in the TSFA algorithm and the two methods: using the correct velocity and using the approximate 60-Hz velocity for the electromechanical equation, were compared in the TSFA algorithm. There were noticeable differences between the two SFA solutions, in particular when there were high frequency deviations away from the 60-Hz frequency,
such as in the case when a generator was disconnected from the network.

The TSFA solution was compared with the traditional phasor solution and the EMTP reference solution for the transient stability cases. These cases demonstrate that the TSFA solution matches the EMTP solution when calculating the critical clearing times of a number of three-phase balanced faults. The TSFA also matches the EMTP solution when capturing the frequency dynamics of the electromechanical variables of the generators, rotor angle, frequency, electrical power, and the electrical variables, fault voltage and fault current, including capturing the DC offset of the fault current during a three-phase fault. In addition, TSFA matches the EMTP solution when capturing the large frequency deviation from 60-Hz frequency when a generator is disconnected from the network, all while using large discretization time-steps. These findings validate SFA’s ability to capture the frequency deviations around the 60-Hz frequency, preserving the high fidelity of the EMTP solution, while matching the computational time-efficiency of the fixed 60-Hz frequency phasor solution. This is important for the modern power grid, where an increasing number of renewable energy sources create prominent frequency fluctuations around the 60-Hz. Being able to track these deviations in a computationally time-efficient way is important.

In addition to studying transient stability, the work done in this thesis re-formulates the SFA solution method in a generalized manner as discussed in Chapters 2 and 3. The basis of SFA solution framework is the single-exponential, frequency-shifting transformation, which shifts the system from a 60-Hz domain to a 0-Hz domain [32]. Because it is a rotational transformation, the shifting of frequencies preserves the dynamics from one domain to another and SFA is able to capture dynamics at a lower frequency. Because SFA is a discrete-time solution method that adheres to the time-frequency relationship, shifting the system down to deviate around 0 Hz allows large discretization time-steps to be used. In addition, because the reverse transformation is another rotational transformation, the dynamics captured in the lower frequency domain are maintained when transferred back to the physical domain. In Chapter 3, a detailed analysis of the accuracy and behaviour during discontinuities of the trapezoidal and backward Euler discretization methods used in this work to discretize the continuous-time SFA signals are discussed. The findings from this analysis indicate that the trapezoidal method is a more accurate solution method,
especially around DC, while the backward Euler method is more suited to network discontinuities as its internal damping dampens the numerical oscillations that appear when using the trapezoidal method.

One observation from this research is that the SFA solution is not restricted to networks energized using sinusoidal sources, but can solve networks with arbitrary sources, such as DC sources, as demonstrated in Section 3.7. This is advantageous for future work.

The beauty of the SFA solution method comes from its ability to derive a complex world from a frequency-shifting transformation of the physical world. There are an extensive number of future applications that can use SFA to model dynamic behaviour.

8.2 Future Research

The research presented in this thesis is one application of research using SFA. Because of the advantages above, SFA can be used in modelling the behaviour of power systems whenever there are system dynamics around the 60-Hz frequency. This section describes a few possible research ideas for SFA in both expanding the current work done in this thesis and in new topics.

1. This thesis only covers three-phase balanced networks with three-phase balanced faults. The TSFA algorithm can be expanded for unbalanced networks, such as unbalanced fault conditions and unbalanced loads. By expanding the program for unbalanced conditions, SFA can be used in other dynamic stability studies. For example, one prominent issue that occurs on both the transmission and the distribution network is the fault-induced delayed voltage recovery (FIDVR) event. FIDVR events occur when a group of single-phase induction motors, such as air-conditioning motors stall, due to a single line-to-ground fault on the feeder, causing the bus voltage to drop and stay depressed for a long time. As in transient stability, the current stability simulators do not capture the dynamics induced by the single line-to-ground fault accurately and the EMTP method is too computationally expensive to be used on the large network with many dynamic elements, such as the power electronic converters and many small induction motors [52]. Current approaches for this
phenomenon include creating a hybrid simulator that interfaces traditional phasors with the EMTP [52]. However, it may be possible to use SFA to model the induction motors and the electrical network to capture these dynamics without needing to use a hybrid solution.

2. The TSFA program can be extended by coupling the electrical and electromechanical network equation using the MATE [22] concept. Currently, the time-step used in TSFA is limited to around 10 ms because that is the highest time-step needed by the electromechanical equations. If, however, the two networks were decoupled, the electrical network could be solved with as large time-steps as necessary, as long as it complies with the time-frequency criteria, while the electromechanical network would be solved at a smaller time-step. The coupling would increase the solution’s computational speed.

3. Using multi-rate techniques,[22],[51], a more accurate zero-crossing detection mechanism can be done, which would allow the SFA solution to use the zero-crossing detection when calculating the critical clearing times. For example, the time-step would be smaller two time-steps before the fault is scheduled to clear and then once the fault is cleared, will return to the large time-steps.

4. Research is currently being done in developing a hybrid simulation platform that couples the EMTP solution with the SFA solution [43]. By employing the Multi-Area Thévenin Equivalent Multi-Rate (MATE-MR) framework [51], this hybrid solution can be an accurate and efficient platform for systems with multiple time-scales. Because the SFA and the EMTP solution methods use the same nodal analysis method and discretization of the continuous-time differential equations, the simulator would be able to capture the network dynamics without approximations[43].

5. The modelling of non-linear elements, such as non-linear inductors, can be done in the SFA domain. In addition, the work here does not include the controllers and assumes that the mechanical power is constant and so this work can be expanded to include controller action.

6. There has been work done in developing a discrete-time state-space model using the EMTP solution to follow the network’s trajectory and eigenvalues [53]. A derivation of this work can be done by deriving a SFA-state-space
8.2. Future Research

model. The SFA solution already solves for the network’s trajectory through the envelope’s dynamics, which identifies the network’s eigenvalues directly. There is potential research to be done in interfacing the SFA with state-space equations.

7. Finally, the SFA framework is not limited to power system studies. The technique of shifting frequencies down to more easily model dynamic behaviour can be applied in the field of neuroscience. In the brain, the neural oscillation activity in the different regions play an important role in how information is processed [54]. When information flows from one area to another, sometimes the frequencies are crossed. This is called cross-frequency coupling (CFC) [54] and is the interaction between oscillations at the different frequency bands and has been found to be prominent in regions of the brain associated with learning and memory [55]. One form of CFC is called phase-amplitude coupling, where high-frequency oscillations cross with the phases of low frequency oscillations [55]. Modelling and measuring PAC is challenging, as the noise affects the high-frequencies of the brain activity in the measuring devices of the EEG and MEGs. One method recently proposed [55] is a dynamic phase-amplitude coupling, which uses Fourier decomposition to extract the magnitude and phase of the signal to see dynamic changes, such as amplitude peaks. The idea of shifting the frequencies of interest down to a lower frequency, as in SFA, can be applied in a similar manner to develop a similar algorithm proposed in [55], in a discrete-time shifted frequency manner. Measuring CFC is important in understanding both normal and abnormal brain behaviour [54].
Bibliography


systems. In *Power and Energy Society General Meeting (PESGM)*, 2016, July 2016. [Pages 11, 12, and 22]


[37] V. Venkatasubramanian. Dynamic analysis of the general large power system using time-varying phasors. [Pages 12 and 14]


system dynamics. In 2014 Power Systems Computation Conference (PSCC),
pages 1–8, Wroclaw, Poland, Aug. 2014. [Pages 16, 18, and 22.]

[40] P. Zhang, J. R. Martí, and H. W. Dommel. Induction machine modeling
based on Shifted Frequency Analysis. IEEE Transactions on Power Systems,

based on Shifted Frequency Analysis. IEEE Transactions on Power Systems,

[42] Y. Huang, F. Therrien, J. Jatskevich, and L. Dong. State-space voltage-behind-
reactance modeling of induction machines based on Shifted Frequency Analysis.
[Page 16]

Multirate Simulation of Power Systems. Internal Proposal: A proposal for
the PhD Dissertation, The University of British Columbia, Vancouver, Canada,
August 2017. [Pages 16 and 123.]

Definition and classification of power system stability. IEEE/CIGRE Joint
Task Force on Stability Terms and Definitions. IEEE Transactions on Power

Control. John Wiley & Sons, Inc., 2008. [Pages 72, 78, and 96]


John Wiley & Sons, Inc, 1948. [Pages 74, 75, 80, and 81]


130
Appendix

Tables A1, A2, A3, and A4 are the modified data for the IEEE 39-bus case study [1].

### Table A1
**Generator data for the IEEE 39-bus case study (Modified from [1]).**

<table>
<thead>
<tr>
<th>Generators</th>
<th>Power Rating (MVA)</th>
<th>Voltage Rating (kV)</th>
<th>( x'd ) (per-unit(^1))</th>
<th>Inertia Constant (H) (sec)</th>
<th>Moment of Inertia (I) (kg·m(^2))</th>
<th>Damping Coefficient (K) (per-unit(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generator 1</td>
<td>1100</td>
<td>100</td>
<td>0.3</td>
<td>5</td>
<td>77398.1</td>
<td>1.5</td>
</tr>
<tr>
<td>Generator 2</td>
<td>350</td>
<td>22</td>
<td>0.3</td>
<td>3</td>
<td>14776.0</td>
<td>1.5</td>
</tr>
<tr>
<td>Generator 3</td>
<td>750</td>
<td>22</td>
<td>0.3</td>
<td>3</td>
<td>31662.8</td>
<td>1.5</td>
</tr>
<tr>
<td>Generator 4</td>
<td>650</td>
<td>22</td>
<td>0.3</td>
<td>4</td>
<td>36588.2</td>
<td>1.5</td>
</tr>
<tr>
<td>Generator 5</td>
<td>50</td>
<td>11</td>
<td>0.3</td>
<td>1</td>
<td>703.6</td>
<td>1.5</td>
</tr>
<tr>
<td>Generator 6</td>
<td>600</td>
<td>22</td>
<td>0.3</td>
<td>5.5</td>
<td>46438.8</td>
<td>1.5</td>
</tr>
<tr>
<td>Generator 7</td>
<td>700</td>
<td>22</td>
<td>0.3</td>
<td>2</td>
<td>19701.3</td>
<td>1.5</td>
</tr>
<tr>
<td>Generator 8</td>
<td>600</td>
<td>22</td>
<td>0.3</td>
<td>2.5</td>
<td>21108.5</td>
<td>1.5</td>
</tr>
<tr>
<td>Generator 9</td>
<td>900</td>
<td>22</td>
<td>0.3</td>
<td>3.5</td>
<td>44328.0</td>
<td>1.5</td>
</tr>
<tr>
<td>Generator 10</td>
<td>350</td>
<td>22</td>
<td>0.3</td>
<td>6</td>
<td>29552.0</td>
<td>1.5</td>
</tr>
</tbody>
</table>

\(^1\) per-unit of each generator’s system base  
\(^2\) per-unit of torque base

### Table A2
**Transformer data for the IEEE 39-bus case study.**

<table>
<thead>
<tr>
<th>From Bus</th>
<th>To Bus</th>
<th>R (per-unit(^1))</th>
<th>X (per-unit(^1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>31</td>
<td>0</td>
<td>0.025</td>
</tr>
<tr>
<td>10</td>
<td>32</td>
<td>0</td>
<td>0.020</td>
</tr>
<tr>
<td>12</td>
<td>11</td>
<td>0.0016</td>
<td>0.0435</td>
</tr>
<tr>
<td>12</td>
<td>13</td>
<td>0.0016</td>
<td>0.0435</td>
</tr>
<tr>
<td>19</td>
<td>33</td>
<td>0.0007</td>
<td>0.0142</td>
</tr>
<tr>
<td>19</td>
<td>20</td>
<td>0.0007</td>
<td>0.0138</td>
</tr>
<tr>
<td>20</td>
<td>34</td>
<td>0.0009</td>
<td>0.0180</td>
</tr>
<tr>
<td>23</td>
<td>36</td>
<td>0.0005</td>
<td>0.0272</td>
</tr>
<tr>
<td>29</td>
<td>38</td>
<td>0.0008</td>
<td>0.0156</td>
</tr>
<tr>
<td>30</td>
<td>2</td>
<td>0</td>
<td>0.0181</td>
</tr>
<tr>
<td>35</td>
<td>22</td>
<td>0</td>
<td>0.0143</td>
</tr>
<tr>
<td>37</td>
<td>25</td>
<td>0.0006</td>
<td>0.0232</td>
</tr>
</tbody>
</table>

\(^1\) Transformers are rated 100kV:22kV and reactances are per-unit on the 100kV, 100 MVA base
Table A3
Power flow results for the IEEE 39-bus case study.

<table>
<thead>
<tr>
<th>Bus</th>
<th>V (per-unit)</th>
<th>Angle (Deg)</th>
<th>$P_{Gen}$ (MW)</th>
<th>$Q_{Gen}$ (MVAR)</th>
<th>$P_{Load}$ (MW)</th>
<th>$Q_{Load}$ (MVAR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0364</td>
<td>1.1635</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.0185</td>
<td>3.2222</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.9908</td>
<td>-0.2058</td>
<td>322.00</td>
<td>2.40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.9557</td>
<td>-2.1849</td>
<td>500.00</td>
<td>184.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.9555</td>
<td>-1.7048</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.9567</td>
<td>-1.1545</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.9488</td>
<td>-3.2679</td>
<td>233.80</td>
<td>84.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.9494</td>
<td>-3.6643</td>
<td>522.00</td>
<td>176.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.0088</td>
<td>-1.4814</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.9628</td>
<td>2.0779</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.9593</td>
<td>0.9846</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.9398</td>
<td>1.1603</td>
<td>7.50</td>
<td>88.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>0.9612</td>
<td>1.4533</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>0.9617</td>
<td>0.0417</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.9695</td>
<td>0.7541</td>
<td>320.00</td>
<td>153.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>0.9890</td>
<td>2.7813</td>
<td>329.00</td>
<td>32.30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>0.9926</td>
<td>1.4812</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>0.9904</td>
<td>0.3742</td>
<td>158.00</td>
<td>30.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>0.9951</td>
<td>10.0384</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1.0021</td>
<td>10.0973</td>
<td>30.00</td>
<td>4.90</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>0.9961</td>
<td>5.0020</td>
<td>274.00</td>
<td>115.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>1.0220</td>
<td>9.3535</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>1.0201</td>
<td>9.6455</td>
<td>247.50</td>
<td>84.60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>0.9972</td>
<td>2.8785</td>
<td>308.60</td>
<td>-92.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>1.0276</td>
<td>4.7982</td>
<td>224.00</td>
<td>57.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>1.0174</td>
<td>3.4692</td>
<td>139.00</td>
<td>17.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>0.9997</td>
<td>1.3200</td>
<td>281.00</td>
<td>75.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>1.0189</td>
<td>7.1934</td>
<td>206.00</td>
<td>27.60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>1.0205</td>
<td>10.1108</td>
<td>283.50</td>
<td>26.90</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>1.0475</td>
<td>5.6530</td>
<td>250.00</td>
<td>173.09</td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>0.9820</td>
<td>3.2836</td>
<td>300.00</td>
<td>115.14</td>
<td>9.20</td>
<td>4.60</td>
</tr>
<tr>
<td>32</td>
<td>0.9831</td>
<td>9.9720</td>
<td>650.00</td>
<td>144.57</td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>0.9972</td>
<td>15.2223</td>
<td>632.00</td>
<td>11.98</td>
<td></td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>1.0123</td>
<td>10.4757</td>
<td>40.00</td>
<td>55.64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>1.0493</td>
<td>13.2379</td>
<td>508.00</td>
<td>217.81</td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>1.0635</td>
<td>18.9686</td>
<td>650.00</td>
<td>210.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>1.0278</td>
<td>11.8574</td>
<td>560.00</td>
<td>21.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>38</td>
<td>1.0265</td>
<td>17.1898</td>
<td>830.00</td>
<td>48.31</td>
<td></td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>1.0300</td>
<td>0.0000</td>
<td>23.25</td>
<td>-41.01</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Table A4
Transmission line data for the IEEE 39-bus case study.

<table>
<thead>
<tr>
<th>From Bus</th>
<th>To Bus</th>
<th>R (per-unit(^1))</th>
<th>X (per-unit(^1))</th>
<th>B(total(^2)) (per-unit(^1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>1</td>
<td>0.0010</td>
<td>0.0250</td>
<td>0.7500</td>
</tr>
<tr>
<td>39</td>
<td>9</td>
<td>0.0010</td>
<td>0.0250</td>
<td>1.2000</td>
</tr>
<tr>
<td>28</td>
<td>29</td>
<td>0.0014</td>
<td>0.0151</td>
<td>0.2490</td>
</tr>
<tr>
<td>27</td>
<td>17</td>
<td>0.0013</td>
<td>0.0173</td>
<td>0.3216</td>
</tr>
<tr>
<td>26</td>
<td>27</td>
<td>0.0014</td>
<td>0.0147</td>
<td>0.2396</td>
</tr>
<tr>
<td>26</td>
<td>28</td>
<td>0.0043</td>
<td>0.0474</td>
<td>0.7802</td>
</tr>
<tr>
<td>26</td>
<td>29</td>
<td>0.0057</td>
<td>0.0625</td>
<td>1.0290</td>
</tr>
<tr>
<td>25</td>
<td>26</td>
<td>0.0032</td>
<td>0.0323</td>
<td>0.5130</td>
</tr>
<tr>
<td>24</td>
<td>23</td>
<td>0.0022</td>
<td>0.0350</td>
<td>0.3610</td>
</tr>
<tr>
<td>22</td>
<td>23</td>
<td>0.0006</td>
<td>0.0096</td>
<td>0.1846</td>
</tr>
<tr>
<td>22</td>
<td>21</td>
<td>0.0008</td>
<td>0.0140</td>
<td>0.2565</td>
</tr>
<tr>
<td>21</td>
<td>16</td>
<td>0.0008</td>
<td>0.0135</td>
<td>0.2548</td>
</tr>
<tr>
<td>17</td>
<td>18</td>
<td>0.0007</td>
<td>0.0082</td>
<td>0.1319</td>
</tr>
<tr>
<td>16</td>
<td>24</td>
<td>0.0003</td>
<td>0.0059</td>
<td>0.0680</td>
</tr>
<tr>
<td>16</td>
<td>17</td>
<td>0.0007</td>
<td>0.0089</td>
<td>0.1342</td>
</tr>
<tr>
<td>16</td>
<td>19</td>
<td>0.0016</td>
<td>0.0195</td>
<td>0.3040</td>
</tr>
<tr>
<td>15</td>
<td>16</td>
<td>0.0009</td>
<td>0.0094</td>
<td>0.1710</td>
</tr>
<tr>
<td>14</td>
<td>15</td>
<td>0.0018</td>
<td>0.0217</td>
<td>0.3660</td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>0.0004</td>
<td>0.0043</td>
<td>0.0729</td>
</tr>
<tr>
<td>13</td>
<td>14</td>
<td>0.0009</td>
<td>0.0101</td>
<td>0.1723</td>
</tr>
<tr>
<td>11</td>
<td>10</td>
<td>0.0004</td>
<td>0.0043</td>
<td>0.0729</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>0.0023</td>
<td>0.0363</td>
<td>0.3804</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>0.0004</td>
<td>0.0046</td>
<td>0.0780</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>0.0006</td>
<td>0.0092</td>
<td>0.1130</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>0.0007</td>
<td>0.0082</td>
<td>0.1389</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>0.0002</td>
<td>0.0026</td>
<td>0.0434</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>0.0008</td>
<td>0.0128</td>
<td>0.1342</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>0.0008</td>
<td>0.0112</td>
<td>0.1476</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>0.0008</td>
<td>0.0129</td>
<td>0.1382</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.0013</td>
<td>0.0151</td>
<td>0.2572</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.0011</td>
<td>0.0133</td>
<td>0.2138</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.0013</td>
<td>0.0213</td>
<td>0.2214</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>0.0070</td>
<td>0.0086</td>
<td>0.1460</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.0035</td>
<td>0.0411</td>
<td>0.6987</td>
</tr>
</tbody>
</table>

\(^1\) per-unit on the 100kV, 100 MVA base \(^2\) total susceptance