A Three Dimensional Set of Limit Points Related to the abc Conjecture

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE AND POSTDOCTORAL STUDIES

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

December 2017

Abstract

We study the limit points $\mathfrak{Q}'$ of a three-dimensional set $\mathfrak{Q}$ which encodes the reciprocal quality of *abc* triples as the components of a vector of the form $(\log \operatorname{Rad} a, \log \operatorname{Rad} b, \log \operatorname{Rad} c)/\log c$. We establish that if the *abc* Conjecture holds, $\mathfrak{Q}'$ is contained in a heptahedron. Unconditionally, we establish the existence of a subset of $\mathfrak{Q}'$ with non-zero measure. We determine the implications of previous research on related problems involving limit points of *abc* triples in one-dimensional sets on $\mathfrak{Q}'$ and discuss possible avenues for future study.

Lay Summary

This thesis determines some of the mathematical properties of a three dimensional shape that represents the factorization of triples of positive whole numbers (a, b, c) where $a + b = c$. In particular, this work builds on the work of researchers who studied a related problem involving a one-dimensional shape, and proves some new results about the three dimensional shape.

Preface

This thesis is comprised almost entirely of original, unpublished work that I wrote. Though I wrote the text, chapters 4 through 6 contain original research done jointly with Greg Martin, for which we take equal credit. Both Julia Gordon and Greg Martin contributed to the editing of this work.

Two published results from other researchers are quoted in this thesis. They are the BFGS Theorem from the beginning of Section 3 and Lemma 6.13. These two mathematical results are the only unoriginal, previously published material in this thesis.

The subject of this thesis was suggested by Greg Martin. Though it has not, as of December 2017, been submitted for publication anywhere, Greg Martin and I have plans to prepare some of the material from Sections 4 to 6 for publication.

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Acknowledgements

I would like to thank both my advisors, Julia Gordon and Greg Martin, for their support and guidance throughout my studies. I would also like to thank my mother, Carol Simpson, for her support of my education, and for her love.

Chapter 1

Introduction

In the study of the *abc* conjecture, the quality $\mathfrak{q}$ of an element $(a, b) \in \mathbb{N}^2$ with $\gcd(a, b) = 1$ is defined to be

$$\mathfrak{q}(a, b) = \frac{\log(a+b)}{\log \operatorname{Rad} ab(a+b)},$$

where $\operatorname{Rad} n$ denotes the radical (the largest squarefree divisor) of the integer n. In fact, one formulation of the *abc* conjecture is expressed solely in terms of quality [1, 3].

The *abc* Conjecture (Limit Point Formulation)**.** The set $S = \{\mathfrak{q}(a, b) \colon a, b \in \mathbb{N}, \gcd(a, b) = 1\}$ has a maximum limit point of 1. In other words, $\sup S' = 1$.

This set of limit points has been studied by other researchers, and some interesting conclusions have been drawn. A paper by Browkin, Filaseta, Greaves and Schinzel [1] proves:

1. Unconditionally, $[1/3, 15/16] \subset S'$ [1, Theorem 1].
2. If the *abc* conjecture holds, $S' = [1/3, 1]$ [1, Theorem 4].

Similarly, in a paper by Filaseta and Konyagin [3] it was shown that

$$S' \cap [1, 3/2) \neq \emptyset.$$

In this work, we shall investigate a three-dimensional analog of S, which shall be called $\mathfrak{Q} \subset \mathbb{R}^3$. To define this three-dimensional analogue, we shall start with the inverse quality $\mathfrak{iq}(a, b) = \mathfrak{q}(a, b)^{-1}$. First, we define

$$\overrightarrow{\mathfrak{iq}}(a, b) = \frac{1}{\log(a+b)} \big(\log \operatorname{Rad} a, \log \operatorname{Rad} b, \log \operatorname{Rad}(a+b) \big).$$

Observe that this is a natural way of splitting $\mathfrak{iq}(a,b)$. Hence for $\overrightarrow{\mathfrak{iq}}(a,b) = (x,y,z)$, $x+y+z = \mathfrak{iq}(a,b)$. $\mathfrak{iq} = \overrightarrow{\mathfrak{iq}}(a,b) \cdot (1,1,1)$. This gives us our set:

$$\mathfrak{Q} = \{\overrightarrow{\mathfrak{iq}}(a,b) \colon a,b \in \mathbb{N}, \gcd(a,b) = 1\}.$$

We shall denote by $\mathfrak{H} \subset \mathbb{R}^3$ the heptahedron enclosed by the seven planes

$$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1, x + y + z = 1.$$

Proposition 1.1. *If the abc conjecture holds, $\mathfrak{Q}'$ is a subset of $\mathfrak{H}$.*

Proof. 1. First, we prove that if $(\alpha,\beta,\gamma) \in \mathfrak{Q}'$, then $\alpha+\beta+\gamma \geq 1$.

If $(\alpha,\beta,\gamma) \in \mathfrak{Q}'$, it follows that there exists a sequence $\{(a_n,b_n)\}_{n=1}^{\infty}$ with $\gcd(a_n,b_n) = 1$ for all n and $\lim_{n\to\infty} \overrightarrow{\mathfrak{iq}}(a_n,b_n) = (\alpha,\beta,\gamma)$. As $(x,y,z) \mapsto x+y+z$ is continuous, $\lim_{n\to\infty} \mathfrak{iq}(a_n,b_n) = \alpha+\beta+\gamma$. Thus if $\alpha+\beta+\gamma \neq 0$, then $(\alpha+\beta+\gamma)^{-1} \in S'$ and if $\alpha+\beta+\gamma = 0$, then $\infty \in S'$.

If the *abc* conjecture holds, $\sup S' = 1$, so it follows $\infty \notin S'$ and for $\alpha+\beta+\gamma \neq 0$, $(\alpha+\beta+\gamma)^{-1} \in S'$ implies $\alpha+\beta+\gamma \geq 1$. Thus it follows that if $(\alpha,\beta,\gamma) \in \mathfrak{Q}'$, and the *abc* conjecture holds, then $\alpha+\beta+\gamma \geq 1$.

2. Second, we observe that trivially from the definition $\mathfrak{Q} \subseteq [0,1]^3$.

   Thus $\mathfrak{Q} \subset [0,1]^3$, and thus $\mathfrak{Q}' \subset [0,1]^3$.

Hence, by the definition of $\mathfrak{H}$, we have that $\mathfrak{H}$ contains $\mathfrak{Q}'$. □

In other words, if the *abc* conjecture holds, $\mathfrak{H}$ contains all possible limit points of $\mathfrak{Q}$. In this work, we seek to explore the shape and structure of $\mathfrak{Q}'$.

Chapter 2

Summary of Results

Our first approach, which is detailed in §3, will attempt to apply the same techniques as found in [1] to prove the existence of points in S', so as to determine the existence of points in $\mathfrak{Q}'$.

For the following results, the notation $(x_1, y_1, z_1) \to (x_2, y_2, z_2)$ denotes a closed line segment with endpoints $(x_1, y_1, z_1), (x_2, y_2, z_2)$. We prove that:

Theorem 2.1. *Unconditionally, the following hold:*

- *If* $(x, y, z) \in \mathfrak{Q}'$, *then* $(y, x, z) \in \mathfrak{Q}'$.
- *For* $n = 1, 2, 4, 6$, *the line segments* $(\frac{1}{n}, 1, \frac{1}{n}) \to (0, 1, \frac{1}{n})$ *are subsets of* $\mathfrak{Q}'$.
- *For* $n = 1, 2, 4, 6$, *the line segments* $(1, \frac{1}{n}, \frac{1}{n}) \to (1, 0, \frac{1}{n})$ *are subsets of* $\mathfrak{Q}'$.
- *The line segments* $(\frac{3}{4}, \frac{1}{2}, \frac{1}{3}) \to (\frac{1}{4}, \frac{1}{2}, \frac{1}{3})$ *and* $(\frac{1}{2}, \frac{3}{4}, \frac{1}{3}) \to (\frac{1}{2}, \frac{1}{4}, \frac{1}{3})$ *are subsets of* $\mathfrak{Q}'$.
- *The line segments* $(\frac{2}{5}, \frac{7}{20}, \frac{7}{20}) \to (\frac{2}{5}, \frac{19}{60}, \frac{7}{20})$ *and* $(\frac{7}{20}, \frac{2}{5}, \frac{7}{20}) \to (\frac{19}{60}, \frac{2}{5}, \frac{7}{20})$ *are subsets of* $\mathfrak{Q}'$.
- *For* $n = 1, 2, 3$, *the line segments* $(\frac{1}{2} + \frac{1}{2n}, \frac{1}{2}, \frac{1}{2n}) \to (\frac{1}{2}, \frac{1}{2}, \frac{1}{2n})$ *are contained in* $\mathfrak{Q}'$.
- *For* $n = 1, 2, 3$, *the line segments* $(\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}, \frac{1}{2n}) \to (\frac{1}{2}, \frac{1}{2}, \frac{1}{2n})$ *are contained in* $\mathfrak{Q}'$.

Furthermore, if the abc conjecture holds, then the following holds:

- *For* $n \in \mathbb{N}$, $t \in [0, 1]$ *the point* $(\frac{t}{n}, \frac{1}{n}, 1)$ *lies in* $\mathfrak{Q}'$ *or the point* $(\frac{t}{n}, 1, \frac{1}{n})$ *lies in* $\mathfrak{Q}'$.
- *The points* $(0, 1, 0)$, $(1, 0, 0)$, *and* $(0, 0, 1)$ *are in* $\mathfrak{Q}'$.

Proof. First, we see the first statement is clear:

Observe that if $(x,y,z) \in \mathfrak{Q}'$ there exists a sequence of $\{(a_n,b_n)\}_{n=1}^{\infty} \subseteq \mathbb{N}^2$, such that

$$\lim_{n\to\infty} \overrightarrow{\mathfrak{iq}}(a_n,b_n) = (x,y,z).$$

Thus it follows that $\lim_{n\to\infty} \overrightarrow{\mathfrak{iq}}(b_n,a_n) = (y,x,z)$, and thus $(y,x,z) \in \mathfrak{Q}'$. This completes the proof of our first statement.

The remaining statements follow from various lemmas and the symmetry implied by the first statement. The unconditional statements are proven in Lemmata 3.1, 3.2, 3.2, and 3.5. When assuming the *abc* conjecture, we have the remaining statements by Lemma 3.4 and that $\mathfrak{Q}'$ is a closed set so one may take $1/n \to 0$. □

As is expected, all of the above statements, when projected back onto one dimension, give us the statements found in [1]. What is of interest, however, is that while the results of [1] almost completely characterize the set S', the same techniques when applied to the study of the three-dimensional object $\mathfrak{Q}'$ give only disjoint line segments.

It is also known as a consequence of [3] that there exists a point on the plane $x+y+z=1$ in $\mathfrak{Q}'$ or below it. However, the technique cannot be easily extended to three dimensions. See §3.3 for details.

After some examination of what it would take to modify the results of [1] to gain access to a volume, it was decided to approach the problem using other methods. The method of choice, in this case, being the circle method, with specific techniques taken from a paper [2] by Brüdern, Granville, Perelli, Vaughan, and Wooley. The application of this method to this problem is non-trivial, but a volume was obtained.

Theorem 2.2. *The volume enclosed by the following planes lies in* $\mathfrak{Q}'$*:*

$$\begin{array}{ccc} 0.7\alpha+\beta+\gamma = 2.1; & \alpha+0.7\beta+\gamma=2.1; & \alpha+\beta+0.7\gamma=2.1; \\ 13\alpha = 7; & 13\beta = 7; & 13\gamma = 7; \\ \alpha = 1; & \beta = 1; & \gamma = 1. \end{array}$$

The proof of this theorem comes at the end of §6, and depends on work done in §4 and §5.

Note that when projected onto one dimension, this theorem gives us a lower bound for S' substantially inferior to the one found in [1]. Specifically the above volume implies $[1/3, 3/7] \subseteq S'$. However, it does show that $\mathfrak{Q}'$ contains a volume, while the techniques used in [1] and [3] could not.

Chapter 3

The Study of the Limit Points of $\mathfrak{Q}$ with Polynomials and Binary Forms

In this chapter, we shall apply the techniques from [1] to the problem of finding the limit points of $\mathfrak{Q}'$. The main theorem which the authors use to obtain most of their results is as follows:

BFGS Theorem (Theorem 2 of [1])**.** *1. Let $1 \le Y \le X$, where X is sufficiently large, and let $f(x, y) \in \mathbb{Z}[x, y]$ be a binary form whose irreducible factors f_i are distinct and all have degrees not exceeding μ. Let D denote the largest fixed divisor of $f(x, y)$, and let $S = D/\operatorname{Rad}(D)$. Let $N(X, Y)$ denote the number of pairs (x, y) such that*

$$X \le x \le 2X; \quad Y \le y \le 2Y$$

for which $f(x, y)/S$ is squarefree.

Suppose for some $\epsilon > 0$, $X^{\mu} < (XY)^{3-\epsilon}$, $Y > X^{\epsilon}$.

Then

$$N(X, Y) = C_f XY \left\{1 + O\left(\frac{1}{\log X}\right)\right\} \tag{3.1}$$

where the constant $C_f > 0$ depends only on f, and the O constant depends only on ϵ.

2. Suppose $X = Y^{\alpha}$, where $\alpha > 1$ is fixed. It follows that 3.1 holds:

(a) for any α when $\mu \le 3$,

(b) if $\alpha < 3$ when $\mu = 4$,

(c) if $\alpha < 3/2$ when $\mu = 5$.

As described earlier, the BFGS Theorem is used in [1] to prove their main result, that unconditionally, $[1/3, 15/16] \subseteq S'$.

Ideally, one would hope that just as the BFGS Theorem is used to fill in most of the possible space of S', the BFGS Theorem would do the same for $\mathfrak{Q}'$; but as the following proofs shall demonstrate, this is not the case. In particular, we shall find that their results naturally produce parameterized, one-dimensional curves in three dimensional space.

3.1 Translation of the Results of [1] to Three Dimensions

First, we shall apply the methods of [1] to find points in $\mathfrak{Q}'$.

The following lemma uses the polynomial identities that gave [1] the result that $[\frac{1}{3}, \frac{6}{7}] \subseteq S'$, to give us the equivalent limit points of $\mathfrak{Q}'$:

Lemma 3.1. *For $n = 1, 2, 4, 6$, the line segments with endpoints*

$$\left(\frac{1}{n}, 1, \frac{1}{n}\right) \to \left(0, 1\frac{1}{n}\right),$$

lie in $\mathfrak{Q}'$.

Proof. Let $n \in \{1, 2, 4, 6\}$.

Observe that $x^n - y^n$ has irreducible factors of degree at most 2. Thus by the BFGS Theorem for any $\alpha > 1$, there exist squarefree integer values of $xy(x^n - y^n)$ for sufficiently large X, Y, with $X = Y^\alpha$, and $X \le x \le 2X$, $Y \le y \le 2Y$.

Thus there exists a function $f(Y, \alpha) = (a, b)$ that maps each real Y (sufficiently large) and $\alpha > 0$ to two numbers a, b such that

$$a = y^n; b = x^n - y^n; a + b = x^n; Y^\alpha \le x \le 2Y^\alpha; Y \le y \le 2Y; xy(x^n - y^n) \text{ is squarefree.}$$

Consequently, it follows for Y sufficiently large that

$$\begin{aligned}
\overrightarrow{\mathfrak{iq}}(f(Y,\alpha)) &= \frac{1}{n\log x}(\log \operatorname{Rad} y, \log \operatorname{Rad}(x^n - y^n), \log \operatorname{Rad} x)\\
&= \frac{1}{n\log x}(\log y, \log(x^n - y^n), \log x)\\
&= \left(\frac{\log y}{n\log x}, \frac{\log(x^n-y^n)}{n\log x}, \frac{1}{n}\right)\\
&= \left(\frac{\log Y + O(1)}{n\alpha\log Y + O(1)}, \frac{n\alpha \log Y + O(1)}{n\alpha\log Y + O(1)}, \frac{1}{n}\right)\\
&= \left(\frac{1+O(\log^{-1} Y)}{n\alpha + O(\log^{-1} Y)}, \frac{1+O(\log^{-1} Y)}{1+O(\log^{-1} Y)}, \frac{1}{n}\right).
\end{aligned}$$

Thus it follows that

$$\lim_{Y\to\infty} \overrightarrow{\mathfrak{iq}}(f(Y,\alpha)) = \left(\frac{1}{n\alpha}, 1, \frac{1}{n}\right).$$

In $\mathfrak{Q}'$ a line segment is obtained by considering all $\alpha > 1$. By varying α, it is clear that it has endpoints at $\left(\frac{1}{n}, 1, \frac{1}{n}\right)$ and $\left(0, 1, \frac{1}{n}\right)$.

Hence, for $n = 1, 2, 4, 6$ the technique produces four line segments:

$$(1,1,1) \to (0,1,1), \left(\frac{1}{2}, 1, \frac{1}{2}\right) \to \left(0, 1, \frac{1}{2}\right), \left(\frac{1}{4}, 1, \frac{1}{4}\right) \to \left(0, 1, \frac{1}{4}\right), \left(\frac{1}{6}, 1, \frac{1}{6}\right) \to \left(0, 1, \frac{1}{6}\right).$$

□

As can be seen, the techniques used in [1] cover most of S', but only covers a set of four disjoint line segments in the three dimensional case of $\mathfrak{Q}'$.

We now move on to the binomial form that was used in [1] to show that $[\frac{6}{7}, \frac{12}{13}] \subseteq S'$.

Lemma 3.2. *The line segment with endpoints* $(3/4, 1/2, 1/3)$ *and* $(1/4, 1/2, 1/3)$ *lies in* $\mathfrak{Q}'$.

Proof. As in [1], we start with the polynomial identity

$$y^3(2x+y) + (x+y)^3(x-y) = x^3(x+2y),$$

and observe by cubing the variables that

$$y^9(2x^3+y^3) + (x^3+y^3)^3(x^3-y^3) = x^9(x^3+2y^3).$$

Thus for $f(x,y) = xy(x^3+2y^3)(2x^3+y^3)(x^3+y^3)(x^3-y^3)$, we can apply the BFGS Theorem with $S = 2$, $\mu = 3$. Thus for all $\alpha > 1$, and Y sufficiently large, given $X = Y^\alpha$ there exist $x, y \in \mathbb{N}$ such that $X \le x \le 2X$, $Y \le y \le 2Y$, and $f(x,y)/2$ is squarefree.

Thus there exists a function $g(Y,\alpha) = (a,b)$ such that

$$a = y^9(2x^3+y^3); b = (x^3+y^3)^3(x^3-y^3); Y^\alpha \le x \le 2Y^\alpha, Y \le y \le 2Y, a, b \text{ are squarefree integers.}$$

Observe that

$$
\begin{aligned}
\overrightarrow{\mathfrak{iq}}(g(Y,\alpha)) &= \frac{1}{12\alpha\log Y + O(1)} \\
&\quad \cdot (\log \operatorname{Rad} y^9(2x^3+y^3), \log \operatorname{Rad}(x^3+y^3)^3(x^3-y^3), \log \operatorname{Rad} x^9(x^3+2y^3)) \\
&= \frac{1}{12\alpha\log Y + O(1)} \\
&\quad \cdot (\log y(2x^3+y^3)+O(1), \log(x^3+y^3)(x^3-y^3)+O(1), \log x(x^3+2y^3)+O(1)) \\
&= \frac{1}{12\alpha\log Y + O(1)}\big((3\alpha+1)\log Y + O(1), 6\alpha\log Y + O(1), 4\alpha\log Y + O(1)\big) \\
&= \left(\frac{(3\alpha+1)\log Y + O(1)}{12\alpha\log Y + O(1)} \frac{6\alpha\log Y + O(1)}{12\alpha\log Y + O(1)}, \frac{4\alpha\log Y + O(1)}{12\alpha\log Y + O(1)}\right) \\
&= \left(\frac{1+O(\log^{-1}Y)}{4+O(\log^{-1}Y)} + \frac{1+O(\log^{-1}Y)}{12\alpha+O(\log^{-1}Y)}, \frac{1+O(\log^{-1}Y)}{2+O(\log^{-1}Y)}, \frac{1+O(\log^{-1}Y)}{3+O(\log^{-1}Y)}\right).
\end{aligned}
$$

Thus it follows, for all $\alpha > 1$, that

$$
\lim_{Y\to\infty} \overrightarrow{\mathfrak{iq}}(g(Y,\alpha)) = \left(\frac{1}{4} + \frac{1}{2\alpha}, \frac{1}{2}, \frac{1}{3}\right).
$$

Hence we obtain the endpoints

$$
\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{3}\right)
$$

in $\mathfrak{Q}'$. □

Finally, we work on extending the application of the polynomial identity and binary form that was used in [1] to show that $[\frac{12}{13}, \frac{15}{16}] \subseteq S'$ to the three dimensional problem.

Lemma 3.3. *The line segment with endpoints* $(2/5, 7/20, 7/20)$ *and* $(2/5, 19/60, 7/20)$ *lies in* $\mathfrak{Q}'$.

Proof. As described in [1], one has that

$$
(x+y)^7(x-y)(x^2-xy+y^2) + y^7(2x+y)(3x^2+3xy+y^2) = x^7(x+2y)(x^2+3xy+3y^2).
$$

We shall note as an aside that [1] briefly discusses the origin of this identity, with further references for those interested.

Thus it follows that

$$
\begin{aligned}
(x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4) + y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4) \\
= x^{14}(x^2+2y^2)(x^4+3x^2y^2+3y^4).
\end{aligned}
$$

Observe that given

$$
\begin{aligned}
f_1(x,y) &= xy(x+y)(x-y)(x^2+y^2)(2x^2+y^2)(x^2+2y^2),\\
f_2(x,y) &= (x^4-x^2y^2+y^4)(3x^4+3x^2y^2+y^4)(x^4+3x^2y^2+3y^4),\\
f(x,y) &= f_1(x,y)f_2(x,y),
\end{aligned}
$$

as in [1], $f(x,y)$ is separable, $S=6$, $\mu=4$, and thus by the BFGS theorem, for all $\alpha\in(1,3)$ and Y sufficiently large with $X=Y^\alpha$, there exists $x,y\in\mathbb{N}$ such that $X\le x\le 2X$, $Y\le y\le 2Y$ where $f(x,y)/6$ is squarefree.

Given $k=\gcd(a,b)\in[1,6]$, there exists a function $g(Y,\alpha)=(a,b)/k$ such that for Y sufficiently large,

$$
a=(x^2+y^2)^7(x^2-y^2)(x^4-x^2y^2+y^4)/k;\quad b=y^{14}(2x^2+y^2)(3x^4+3x^2y^2+y^4)/k;
$$
$$
Y^\alpha\le x\le 2Y^\alpha;\quad Y\le y\le 2Y;\quad a,b \text{ are squarefree natural numbers.}
$$

Thus one has that

$$
\begin{aligned}
\overrightarrow{\mathfrak{iq}}(g(Y,\alpha)) &= \frac{1}{\log(a+b)}(\log\operatorname{Rad}a,\log\operatorname{Rad}b,\log\operatorname{Rad}a+b)\\
&= \frac{1}{20\alpha\log Y+O(1)}\\
&\Big(\log\big((x^2+y^2)(x^2-y^2)(x^4-x^2y^2+y^4)\big)+O(1),\\
&\log\big(y(2x^2+y^2)(3x^4+3x^2y^2+y^4)\big)+O(1),\\
&\log\big(x(x^2+2y^2)(x^4+3x^2y^2+3y^4)\big)+O(1)\Big)\\
&= \left(\frac{8\alpha\log Y+O(1)}{20\alpha\log Y+O(1)},\frac{(6\alpha+1)\log Y+O(1)}{20\alpha\log Y+O(1)},\frac{7\alpha\log Y+O(1)}{20\alpha\log Y+O(1)}\right),\\
&= \left(\frac{2+O(\log^{-1}Y)}{5+O(\log^{-1}Y)},\frac{(3+(2\alpha)^{-1})+O(\log^{-1}Y)}{10+O(\log^{-1}Y)},\frac{7+O(\log^{-1}Y)}{20+O(\log^{-1}Y)}\right).
\end{aligned}
$$

Thus one has, for $\alpha\in(1,3)$, that

$$
\lim_{Y\to\infty}\overrightarrow{\mathfrak{iq}}(g(Y,\alpha))=\left(\frac{2}{5},\frac{3}{10}+\frac{1}{20\alpha},\frac{7}{20}\right).
$$

Considering all possible values of α and noting that $\mathfrak{Q}'$ is a closed set, one has that this implies the existence of the line segment with endpoints

$$
\left(\frac{2}{5},\frac{7}{20},\frac{7}{20}\right),\left(\frac{2}{5},\frac{19}{60},\frac{7}{20}\right)
$$

in $\mathfrak{Q}'$. $\square$

The paper [1] demonstrated that if the *abc* conjecture holds, $S' = [1/3, 1]$. We shall now apply the same technique to determine what this implies for $\mathfrak{Q}'$. This technique does not rely on the BFGS Theorem, and is derived from the *treatment of theorem 4* in [1].

Lemma 3.4. *If the abc conjecture holds, then for each $n \in \mathbb{N}$, and $t \in [0,1]$ at least one of the following points lies in $\mathfrak{Q}'$:*

$$\left(\frac{t}{n}, \frac{1}{n}, 1\right), \left(\frac{t}{n}, 1, \frac{1}{n}\right)$$

Additionally, $(0,0,1), (0,1,0)$ *lie in* $\mathfrak{Q}'$.

Proof. Let $n \in \mathbb{N}$, and let

$$S_1(Y,\alpha) = \{(x,y) \in \mathbb{N}^2 \colon Y^\alpha < x \leq 2Y^\alpha, Y < y \leq 2Y, xy(x^n + y^n) \text{ is squarefree}\},$$
$$S_2(Y,\alpha) = \{(x,y) \in \mathbb{N}^2 \colon Y^\alpha < x \leq 2Y^\alpha, Y < y \leq 2Y, xy(x^n - y^n) \text{ is squarefree}\},$$

for $\alpha > 0$.

If one assumes the *abc* conjecture, it follows by the argument from the *treatment of theorem 4* found in [1] that for any Y sufficiently large and $\alpha > 1$,
$S_1(Y,\alpha) \neq \emptyset$ or $S_2(Y,\alpha) \neq \emptyset$.

Thus it is the case that either there are infinitely many Y such that $S_1(Y,\alpha) \neq \emptyset$ or infinitely many Y such that $S_2(Y,\alpha) \neq \emptyset$.

1. Fix α, and suppose there exist infinitely many Y such that $S_1(Y,\alpha) \neq \emptyset$.

   It follows there exists a function $F \colon (Y,\alpha) \mapsto (a,b)$, such that for Y sufficiently large, $\alpha > 1$, one has that

   $$a = y^n, b = x^n, \text{ for some } (x,y) \in S_1(Y,\alpha) \text{ if } S_1(Y,\alpha) \neq \emptyset,$$

   or, to deal with the possibility that $S_1(Y,\alpha)$ is sometimes empty,

   $$(a,b) = \max\{(m,n) \in \mathbb{N}^2 \colon (m,n) = F(Y',\alpha) \text{ for some } Y' \leq Y, S_1(Y,\alpha) \neq \emptyset\}.$$

   Observe that when $S_1(Y,\alpha) \neq \emptyset$,

   $$\begin{aligned}\overrightarrow{\mathfrak{iq}}(F(Y,\alpha)) &= \frac{1}{\log(x^n+y^n)}\big(\log \operatorname{Rad} y^n, \log \operatorname{Rad} x^n, \log \operatorname{Rad}(x^n+y^n)\big)\\ &= \frac{1}{n\alpha \log Y + O(1)}\big(\log Y + O(1), \alpha \log Y + O(1), n\alpha \log Y + O(1)\big)\\ &= \left(\frac{1+O(\log^{-1} Y)}{n\alpha + O(\log^{-1} Y)}, \frac{1+O(\log^{-1} Y)}{n + O(\log^{-1} Y)}, \frac{1+O(\log^{-1} Y)}{1+O(\log^{-1} Y)}\right),\end{aligned}$$

   and that when $S_1(Y,\alpha) = \emptyset$, the function F remains constant.

Thus it follows that

$$\lim_{Y\to\infty} \overrightarrow{\mathfrak{iq}}(F(Y,\alpha)) = \left(\frac{1}{n\alpha}, \frac{1}{n}, 1\right).$$

Thus the point $(\frac{1}{n\alpha}, \frac{1}{n}, 1)$ lies in $\mathfrak{Q}'$.

2. Fix α and suppose there exist infinitely many Y such that $S_2(Y,\alpha) \neq \emptyset$.

   It follows there exists a function $F(Y,\alpha) \to (a,b)$ such that for Y sufficiently large, $\alpha > 1$, one has that

$$a = y^n, b = x^n - y^n, \text{ for some } (x,y) \in S_2(Y,\alpha) \text{ if } S_2(Y,\alpha) \neq \emptyset,$$

   and behaves analogously to the function in the previous case when when $S_2(Y,\alpha) = \emptyset$. Observe that when $S_2(Y,\alpha) \neq \emptyset$,

$$\begin{aligned}
\overrightarrow{\mathfrak{iq}}(F(Y,\alpha)) &= \frac{1}{\log x^n}\big(\log \operatorname{Rad} y^n, \log \operatorname{Rad} x^n - y^n, \log \operatorname{Rad} x^n\big) \\
&= \frac{1}{n\alpha \log Y + O(1)}\big(\log Y + O(1), n\alpha \log Y + O(1), \alpha \log Y + O(1)\big) \\
&= \left(\frac{1 + O(\log^{-1} Y)}{n\alpha + O(\log^{-1} Y)}, \frac{1 + O(\log^{-1} Y)}{1 + O(\log^{-1} Y)}, \frac{1 + O(\log^{-1} Y)}{n + O(\log^{-1} Y)}\right),
\end{aligned}$$

   and that when $S_2(Y,\alpha) = \emptyset$, the function remains constant. Thus it follows that

$$\lim_{Y\to\infty} \overrightarrow{\mathfrak{iq}}(F(Y,\alpha)) = \left(\frac{1}{n\alpha}, 1, \frac{1}{n}\right).$$

   Thus the point $(\frac{1}{n\alpha}, \frac{1}{n}, 1)$ lies in $\mathfrak{Q}'$.

This completes the proof of the first statement. For the second, simply observe that

$$\lim_{n\to\infty} \overrightarrow{\mathfrak{iq}}(2^n, 3^n) = (0, 0, \alpha)$$

with $\alpha \leq 1$. But by the *abc* conjecture, $0 + 0 + \alpha \geq 1$, so $\alpha = 1$.

Likewise,

$$\lim_{n\to\infty} \overrightarrow{\mathfrak{iq}}(2^n, 3^n - 2^n) = (0, \alpha, 0)$$

with $\alpha \leq 1$ and thus by the *abc* conjecture, $\alpha = 1$. □

From this, it is clear that the results of [1] are not as capable of characterizing the set of limit points in the three-dimensional case as they were in the one-dimensional case. Their result that $[1/3, 15/16] \subseteq S'$ covers almost all of expected S', but in the three dimensional case amounts to a set of disjointed line segments, covering no volume at all.

The same situation occurs when assuming the *abc* conjecture. In [1], it is shown that the *abc* conjecture implies $[1/3, 1] = S'$. The three dimensional equivalent of that statement would be that the *abc* conjecture implies that $\mathfrak{Q}' = \mathfrak{H}$, but instead their technique only yields here that there exist points split across pairs of line segments in $\mathfrak{Q}'$ that get arbitrarily close to the face of $\mathfrak{H}$ nearest the origin.

The primary limitation of using the BFGS Theorem (and the *abc* variant) is that it naturally produces line segments. In the *abc* case, it was possible to produce an infinite number of such segments, giving us additional limit points that lie on the boundary of the heptahedron $\mathfrak{H}$, but still no volume.

However, it may be possible to extend this proof to either planes or volumes, but it will likely require either modification of the theorem (for example, adding a third variable like $Z \leq z \leq 2Z$, $Z = X^{\beta}$), or a method of adjusting the line segments by an ϵ to "fill in" volumes of the heptahedron. Alternative binary forms or alternative factorizations of binary forms may be necessary.

3.2 Applying the BFGS Theorem to Other Binary Forms

We shall now move on to look at some other polynomial identities and the limit points they produce when the BFGS Theorem is applied. This section will show that a great diversity of lines and points can be found, but that this gets one no closer to showing a set with a positive volume lies in $\mathfrak{Q}'$.

Lemma 3.5. *The line segments* $(1, \frac{1}{2}, \frac{1}{2}) \to (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ *and* $(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}) \to (\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$ *both lie in* $\mathfrak{Q}'$.

Proof. Consider, for $n = 1, 2, 3$ the following binary form

$$f(x, y) = xy(x^n - y^n)(2x^n - y^n).$$

By taking $f(x, y)$ and applying the BFGS Theorem, one observes that $\mu \leq 3$, and thus it follows that for any sufficiently large Y, $X = Y^{\alpha}, \alpha > 1$, there exist (x, y) such that $X < x \leq 2X, Y < y \leq 2Y$ and $f(x, y)$ is squarefree.

Thus there exists a function $F(Y, \alpha) = (a, b)$ such that

$$a = y^n(2x^n - y^n);\ b = (x^n - y^n)^2; a + b = x^{2n};\ Y^{\alpha} < x \leq 2Y^{\alpha};\ Y < y \leq 2Y;\ f(x, y) \text{ is squarefree.}$$

Thus one has that

$$\begin{aligned}
\overrightarrow{\mathfrak{iq}}(F(Y,\alpha)) &= \frac{1}{2n\log x}(\log \operatorname{Rad} y^n + \log \operatorname{Rad}(2x^n - y^n), \log \operatorname{Rad}(x^n - y^n)^2, \log \operatorname{Rad} x^{2n}) \\
&= \frac{1}{2n\log x}(\log y + \log(2x^n - y^n), \log(x^n - y^n), \log x) \\
&= \left(\frac{(1+n\alpha)\log Y + O(1)}{2n\alpha\log Y + O(1)}, \frac{n\alpha \log Y + O(1)}{2n\alpha\log Y + O(1)}, \frac{\alpha\log Y + O(1)}{2n\alpha \log Y + O(1)}\right) \\
&= \left(\frac{(1+n\alpha) + O(\log^{-1} Y)}{2n\alpha + O(\log^{-1} Y)}, \frac{1 + O(\log^{-1} Y)}{2 + O(\log^{-1} Y)}, \frac{1 + O(\log^{-1} Y)}{2n + O(\log^{-1} Y)}\right).
\end{aligned}$$

Thus one has for any $\alpha > 1$ that,

$$\lim_{Y\to\infty} F(Y,\alpha) = \left(\frac{1}{2n\alpha} + \frac{1}{2}, \frac{1}{2}, \frac{1}{2n}\right).$$

This gives the line segments:

$$\begin{gathered}
\left(1, \frac{1}{2}, \frac{1}{2}\right) \to \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \\
\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right) \to \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right) \\
\left(\frac{4}{6}, \frac{1}{2}, \frac{1}{6}\right) \to \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{6}\right)
\end{gathered}$$

□

If the above lemma could be proven to hold for all $n \in \mathbb{N}$, one would have a sequence of line segments that would eventually converge to the point $(1/2, 1/2, 0)$, which lies on the critical boundary. We shall note that it seems plausible in light of Lemma 3.4 that a proof that assumes the *abc* conjecture might be capable of proving the above statement for all $n \in \mathbb{N}$.

Additional polynomial and binary form identities are likely to result is similar sorts of patterns, but it is not clear at all that these patterns can generalize sufficiently well so as to form volumes.

3.3 Examining $\mathfrak{Q}' \cap ([0,1]^3 \setminus \mathfrak{H})$

From [3] one has the result that $S' \cap [1, 3/2) \neq \emptyset$.

The analogous property for $\mathfrak{Q}'$ would be to show that for every point (x,y,z) where $x+y+z = 1$, the line segment $(x,y,z) \to 2/3(x,y,z)$ is non-empty.

First, we construct the three dimensional analogue of the argument given in [3] that shows that $S \cap [1,\infty) \neq \emptyset$, for the purpose of illustrating the principle behind this technique.

Lemma 3.6. *Either there exist infinitely many points in the set* $\{(0, x, \frac{1}{n}) \colon n \in \mathbb{N}, x \in [0, 1 - \frac{1}{n})\} \cap \mathfrak{Q}'$, *or else* $(0,1,0) \in \mathfrak{Q}'$.

Proof. Fix some $k \in \mathbb{N}$. For any squarefree n, let

$$a_n = 1; \quad b_n = n^k - 1; \quad c_n = n^k.$$

Observe that for any squarefree n, by considering all possible values of $\mathrm{Rad}(n^k - 1)$, one has that

$$\begin{aligned}
\overrightarrow{\mathfrak{iq}}(a_n, b_n) &= \frac{1}{k \log n}(\log \mathrm{Rad}\, 1, \log \mathrm{Rad}(n^k - 1), \log \mathrm{Rad}\, n^k) \\
&= \frac{1}{k \log n}(0, \log \mathrm{Rad}(n^k - 1), \log n) \\
&= \left(0, \frac{\log \mathrm{Rad}(n^k - 1)}{k \log n}, \frac{1}{k}\right) \\
&\in \{(0, x, \frac{1}{k}) \in \mathbb{R}^3 \colon x \in [0, 1)\}.
\end{aligned}$$

By taking the subsequential limit of squarefree n, we observe

$$\lim_{\substack{n \to \infty \\ \mu_2(n) = 1}} \overrightarrow{\mathfrak{iq}}(a_n, b_n) \in \{(0, x, \frac{1}{k}) \colon x \in [0, 1]\}.$$

Thus it follows that for every $k \in \mathbb{N}$, there exists a point of the form $(0, x, \frac{1}{k}) \in \mathfrak{Q}'$ where $x \in [0, 1]$.

If there are not infinitely many points in the set $\{(0, x, \frac{1}{n}) \colon n \in \mathbb{N}, x \in [0, 1 - \frac{1}{n})\}$, then it follows by the above that there are infinitely many points in the set $\{(0, x, \frac{1}{n}) \colon n \in \mathbb{N}, x \in [1 - \frac{1}{n}, 1]\}$.

Thus it follows by the fact that $\mathfrak{Q}'$ is closed that $(0, 1, 0) \in \mathfrak{Q}'$. This completes the proof. □

Next, we move on to the translation of the proof that $S \cap [1, 2] \neq \emptyset$ from [3].

Lemma 3.7. *The set* $\{(0, x, y) \in \mathbb{R}^3 \colon \frac{1}{2} \leq x + y \leq 1\} \cap \mathfrak{Q}'$ *is non-empty.*

Proof. For each $n \in \mathbb{N}$, let t_n be the smallest integer such that

$$t_n > 2^n, \mathrm{Rad}\, t_n(t_n - 1) \leq 2t_n, \mathrm{Rad}\, t_n(t_n + 1) \geq 2(t_n + 1).$$

By [3] t_n exists for every n, and thus

$$2(t_n + 1) \leq \mathrm{Rad}(t_n(t_n^2 - 1)) \leq 2t_n(t_n + 1),$$

and therefore

$$\log(t_n) + O(1) \leq \log \mathrm{Rad}(t_n(t_n^2 - 1)) \leq 2 \log t_n + O(1).$$

Thus one has that

$$\overrightarrow{\mathfrak{iq}}(1, (t_n^2 - 1)) = \frac{1}{2 \log t_n}(0, \log \mathrm{Rad}(t_n^2 - 1), \log \mathrm{Rad}(t_n^2)),$$

and finally,

$$\lim_{n\to\infty} \overrightarrow{\mathfrak{iq}}(1,(t_n^2-1)) \in \{(0,x,y)\in\mathbb{R}^3 \colon \frac{1}{2} \le x+y \le 1\}.$$

□

Finally, we translate from [3] the proof that $S' \cap [1, 3/2] \neq \emptyset$.

Lemma 3.8. *The set* $\{(x,y,z)\in\mathbb{R}\colon 2/3 \le x+y+z \le 1\} \cap \mathfrak{Q}'$ *is non-empty..*

Proof. As described in [3], there are infinitely many $x \in \mathbb{N}$ such that $f(x) = x(x-1)(x-3)$ is squarefree. Let $t \in \mathbb{N}$, $\epsilon > 0$ be numbers such that $f(t)$ is squarefree, t sufficiently large, and ϵ sufficiently small so it follows that $\operatorname{Rad} t(t-1)(t-3) \ge t^{2+2\epsilon}$. As described in [3], for each t, ϵ there exists a minimal integer $m \in \mathbb{N}$ such that

$$(3^{m-1}t)^{2+2\epsilon} \le \operatorname{Rad} 3^m t(3^m t-1)(3^m t-3)(3^m t-9) \le (3^m t)^{3+2\epsilon}.$$

Thus for $a = (3^{m-1}t)^2(3^{m-1}t-3)$, $b = 3^m t - 1$, $a+b = (3^{m-1}t-1)^3$, one has that

$$\begin{aligned}\overrightarrow{\mathfrak{iq}}(a,b) &= \frac{1}{3\log(3^m t-1)}(\log\operatorname{Rad}(3^{m-1}t)^2(3^{m-1}t-3), \log\operatorname{Rad}(3^m t-1), \log\operatorname{Rad}(3^{m-1}t-1)^3)\\ &= \frac{1}{3\log(3^m t-1)}(\log\operatorname{Rad}(3^{m-1}t)(3^{m-1}t-3), \log\operatorname{Rad}(3^m t-1), \log\operatorname{Rad}(3^{m-1}t-1)).\end{aligned}$$

As there are infinitely many such t, considering a, b as functions of t one has that

$$\lim_{t\to\infty} \overrightarrow{\mathfrak{iq}}(a,b) \in \{(x,y,z)\in\mathbb{R}\colon 2/3 + 2/3\epsilon \le x+y+z \le 1 + 2/3\epsilon\}.$$

As this holds for $\epsilon > 0$ arbitrarily small, one has that there exists a point in

$$\{(x,y,z)\in\mathbb{R}\colon 2/3 \le x+y+z \le 1\} \cap \mathfrak{Q}' \neq \emptyset.$$

□

The structure of this proof is such that it gives no information in $\mathfrak{Q}'$ that cannot be gleaned from the existence of a limit point in S'.

So, a direct translation of the technique from S' to $\mathfrak{Q}'$ does not work, but perhaps a variant that chooses inequalities pairwise would. Given some of the flexibility that is afforded by the ϵ here, it may be useful to have two or more small variables to cover the volume contained in $x=0, y=0, z=0, x+y+z=1$.

3.4 Limitations of Current Polynomial Methods

The subject of this work is to find and state the volume of $\mathfrak{Q}'$. As can be seen earlier, the methods derived from [1] and [3] produce only line segments, and extending those segments to volumes does not seem to be possible without refinement of those techniques, or the determination of an infinite set of binomial forms compatible with the BFGS Theorem whose endpoints are dense in some surface contained in $\mathfrak{H}$.

Despite some investigation, it is not clear that such a set of binomial forms is known to exist. Some general sets of binomial forms used in [1] are not of this form in general, even if some chosen examples satisfy the requirements of the BFGS Theorem.

We shall now move on to discussing potential avenues for improvements to the BFGS Theorem that are plausible.

First, to discuss one such improvement, we need to define the following:

Definition 3.9 (Local Quality)**.** Let $\{a_n\}, \{b_n\} \subseteq \mathbb{N}$ be sequences of natural numbers.

The set of local qualities, $\mathfrak{Lq}$ of $\{(a_n, b_n)\}$ is defined to be

$$\mathfrak{Lq}(\{a_n, b_n\}) = \left\{ \frac{\log \operatorname{Rad} a_n}{\log b_n} : n \in \mathbb{N} \right\}',$$

that is to say, the set of limit points of real values of the form

$$\frac{\log \operatorname{Rad} a_n}{\log b_n}.$$

As a matter of notation, $\mathfrak{Lq}(\{a_n\}) = \mathfrak{Lq}(\{a_n, a_n\})$.

It is obvious that if $\{b_n\}$ is a subsequence of $\{a_n\}$, $\mathfrak{Lq}(\{b_n\}) \subseteq \mathfrak{Lq}(\{a_n\})$.

One improvements to the BFGS Theorem could involve extending it to demonstrate the existence of x_n, y_n such that $\limsup \mathfrak{Lq}(\{f(x_n, y_n)\})$ is equal to a fixed number β. This kind of extension would require some caution in its formulation as additional restrictions on the binary form and the value β would be needed to prevent contradicting the *abc* conjecture.

Some attempts to do this by way of a scaling sequence a_n ultimately failed due to the difficulty in finding a generalizable polynomial identity that could incorporate a_n and produce suitable *abc* triples.

Another possibility is to extend the BFGS Theorem to forms with three or more variables. Extending this result to 3 or 4 variable problems is difficult, and approaches that attempt to make f into a variable run into problems ensuring that the variables used to alter f remain of relevant scale as (x, y) become arbitrarily large. For example, one natural extension of $f(x, y)$ uses u, v, β, γ as variables (u, v squarefree):

$$f(x, y) = (xuyv((xu)^n - (yv)^n).$$

It does not seem possible to naturally extend the BFGS Theorem by relating u^β, v^γ to the size of x, y in any way: the structure of the proofs found in [1] are of no help, as they rely on starting with arbitrarily large x, y for each given f. Furthermore, this function would only produce a plane at best, despite the addition of two variables.

However, if for some $\alpha > 1$ one could show that for any $\beta \in [0,1]$ given that there exist infinitely many square-free (x, y) where $x \approx y^\alpha$ that there exists a subsequence where the ratio $\log \operatorname{Rad} f(x, y)$ over $\log f(x, y)$ approaches some constant $\beta \in [0, 1]$, this would then add one degree of freedom, giving for $a = x^n - y^n$, $b = y^n$, $c = x^n$:

$$\overrightarrow{\mathfrak{iq}}(a, b) = (\beta, \frac{1}{n\alpha}, \frac{1}{n}).$$

However, as discussed in the subsection below, there would be a significant constraint on the α (unless *abc* is false), for which this could be true, so any proof would need to take into account that $\alpha \leq (n + 1)/n$, meaning that the planes generated would be particularly useless for large n.

Overall, this suggests that different methods need to be explored, and that is why the primary focus for the future will be on the circle method.

3.5 Related Limit Points and Proof Limitations

In this section, we take a more general look at how the problem of finding points in $\mathfrak{Q}'$ relates to the problem of finding similar points in other sets.

We first observe a subtle implication of the *abc* conjecture, related to these local qualities.

Lemma 3.10. *Assume the abc conjecture holds.*

Let $\alpha > 1$, $\operatorname{Rad}(b_n)^\alpha \ll a_n$, and $\{a_n\}$ be is an increasing sequence, and $(a_n, b_n) = 1$ for all $n \in \mathbb{N}$. For any $x \in \mathfrak{Lq}(a_n, a_n + b_n)$, $y \in \mathfrak{Lq}(a_n + b_n)$, one has that

$$x + y \geq 1 - \alpha^{-1}.$$

Proof. Simply observe that

$$\begin{aligned}
&\liminf_{n\to\infty} \mathfrak{iq}(a_n, b_n) \\
&\qquad = \liminf_{n\to\infty} \frac{\log \operatorname{Rad} a_n}{\log(a_n + b_n)} + \frac{\log \operatorname{Rad} b_n}{\log(a_n + b_n)} + \frac{\log \operatorname{Rad}(a_n + b_n)}{\log(a_n + b_n)} \\
&\qquad = \inf \mathfrak{Lq}(a_n, a_n + b_n) + \inf \mathfrak{Lq}(b_n, a_n + b_n) + \inf \mathfrak{Lq}(a_n + b_n).
\end{aligned}$$

As $\operatorname{Rad}(b_n)^\alpha \ll a_n$, we have $0 \leq \inf \mathfrak{Lq}(b_n, a_n + b_n) \leq \alpha^{-1}$. By the *abc* conjecture, one has that $\mathfrak{iq}(a_n, b_n) \geq 1$. Thus $\inf \mathfrak{Lq}(a_n, a_n + b_n) + \inf \mathfrak{Lq}(a_n + b_n) \geq 1 - \inf \mathfrak{Lq}(b_n, a_n + b_n)$, and thus

$$\inf \mathfrak{Lq}(a_n, a_n + b_n) + \inf \mathfrak{Lq}(a_n + b_n) \geq 1 - \alpha^{-1}.$$

The lemma follows immediately from the definition of the infimums. □

Currently, every application of the BFGS Theorem involves choosing *abc* triples where $b \approx a^{\alpha}$ for some $\alpha > 1$, with the triple becoming smaller as α increases. This does have implications for the usefulness of proof techniques for finding limit points in $\mathfrak{Q}$. Since we believe that the *abc* conjecture is true, we should also believe that any proof technique that fails if the *abc* conjecture is true must fail in practice.

Any technique that produces sets of pairs (a, b) where $b^{\alpha} \ll a$, Lemma 3.10 ensures that for any limit point $(x, y, z) \in \mathfrak{Q}'$, $x + z > 1 - 1/\alpha$, as a consequence of the *abc* conjecture. Thus if one wants to find limit points where $x + z = 2/3$, one must ensure $\alpha \leq 3$. This is consistent with the lines generated so far by these techniques, and shows that naïve attempts to extend lines on the inwards is likely to fail if α can get arbitrarily large.

Chapter 4

Technical Results

This chapter deals with several technical lemmas needed to obtain results in later sections. As these results are technical, and difficult to give motivation for, readers interested in the bigger picture may wish to skip this chapter and reference it as needed.

4.1 Sets of *abc* and almost *abc* triples

The primary purpose of this section is to produce functions to compensate for the limitations of the circle method, which cannot distinguish between triples that are relatively prime and those that are not, and furthermore cannot restrict the components of the triple to being approximately the same size.

We shall define the notion of a pre-*abc* triple and a δ-triple and prove some of their properties.

Definition 4.1. A *pre-abc* triple is a triple (a, b, c) such that $a + b = c$, $a, b, c \in \mathbb{N}$.

A *proper set of pre-abc triples* is a set S of pre-*abc* triples where $(a, b, c) \in S$ implies for $d = (a, b)$ that $(a/d, b/d, c/d) \in S$.

In the context of the circle method, we will be counting squarefree numbers that are not necessarily relatively prime, that is to say, a set of pre-*abc* triples.

Definition 4.2. For a fixed $\delta > 0$, a δ-triple is a pre-*abc* triple $(a, b, c) \in \mathbb{N}^3$ satisfying the following condition:

There exists some $M \in \mathbb{R}$ such that $M < c \leq 2M$ and either $a \leq M^{1-\delta}$ or $b \leq M^{1-\delta}$.

This criterion will be used to ensure the triples counted are of appropriate size.

First, we show that for any function that counts pre-*abc* triples, there exists a related function that counts those triples where the gcd is large.

Lemma 4.3. *Let S be a set of pre-abc triples.*

Let $S(x) = \{(a,b,c) \in S \colon c \le x\}$, and let $S_\epsilon(x) = \{(a,b,c) \in S \colon \gcd(a,b) > x^\epsilon, c \le x\}$.

Let $F(x) = \#S(x)$, and let $F_\epsilon(x) = \#S_\epsilon(x)$.

1. *If S is proper either $F(x) \ll x$ or there exists an infinite $T \subset S$ where T is a set of abc triples.*

2. *For all $x > 0$, $F_\epsilon(x) \ll x^{2-\epsilon}$.*

Proof. 1. Suppose there does not exist an infinite $T \subset S$ where T is a set of *abc* triples. Hence there are finitely many *abc* triples in S. Let $N = \max\{c \colon (a,b,c) \in S, (a,b) = 1\}$. Hence, $F(N)$ is greater than or equal to the number of all *abc* triples in S. Since for all $x \in \mathbb{R}$, $S(x)$ is proper, all pre-*abc* triples that are not *abc* triples are multiples of some *abc* triple in $S(N)$. Hence $F(x) = \#S(x) \le F(N) \cdot x \ll x$. Hence $F(x) \ll x$.

2. Observe that for fixed d there exist at most x/d multiples of d less than or equal to x.

   Hence for $S(x,d) = \{(a,b,c) \in S \colon \gcd(a,b) = d, c \le x\}$ we have $|S(x,d)| \le x^2/d^2$. As $S_\epsilon(x) = \bigcup_{d > x^\epsilon} S(x,d)$,

$$F_\epsilon(x) = \#S_\epsilon(x) = \sum_{d > x^\epsilon} x^2/d^2 \le x^2 \int_{x^\epsilon - 1}^{\infty} y^{-2} dy \ll x^2 x^{-\epsilon}.$$

This completes the proof. □

Next, we consider functions that count the cardinality of subsets where one part of the triple is small.

Lemma 4.4. *Let S be a set of pre-abc triples.*

Let $\delta > 0$. Let $F(x) = \#\{(a,b,c) \in S \colon a,b,c \le x\}$. Let $F_\delta(x) = \#\{(a,b,c) \in S \colon a,b,c \le x, (a,b,c)$ is a δ-triple$\}$.

1. *If S is proper then the set of δ-triples in S is proper.*

2. *If $x \ge 2^{1/\delta+1}$, then $F_\delta(2x) - F_\delta(x) \le 8x^{2-\delta}$.*

Proof. To prove the first point, observe that if (a,b,c) is a δ triple, then for $d = (a,b), a' = a/d, b' = b/d, c' = c/d$, since there exists M such that (without loss of generality) $a \le M^{1-\delta}$ and $M < c$, it follows $a' \le M^{1-\delta}/d \le (M/d)^{1-\delta}$, and $M/d \le c'$, so (a',b',c') is a δ triple.

Now, we prove the second point.

Let $S^\star(x) \subseteq S$ be defined to be the set $\{(a,b,c) \in S | x < c \le 2x\}$. Observe that $F(2x) - F(x)$ counts exactly the triples (a,b,c) in S such that $x < c \le 2x$ and thus $F(2x) - F(x) = |S^\star(x)|$. Let $S_\delta(x)$ be the set of δ-triples in $S(x)$. It likewise follows that $|S^\star_\delta(x)| = F_\delta(2x) - F_\delta(x)$.

Let $x \geq 2^{1/\delta+1}$. It follows that δ triples in $S_\delta(x)$ cannot satisfy $a \leq M^{1-\delta}$ and $b \leq M^{1-\delta}$ for all possible $M \in [c/2, c)$ since $a+b \leq x$. Without loss of generality, assume $a < b$. Then it follows that for each triple $(a,b,c) \in S_\delta(x)$ there exists some M such that $M < c \leq 2M$ and $a < M^{1-\delta}$. Hence $a \leq M^{1-\delta} < c^{1-\delta} \leq (2x)^{1-\delta}$. Hence for each $(a,b,c) \in S_\delta(x)$, $a < (2x)^{1-\delta}$. Thus $1 \leq a < (2x)^{1-\delta}$ and thus, after multiplying by 2 to get the case where $b < a$, we have:

$$|S_\delta(x)| \leq 2(2x)^{1-\delta}(2x) = 2(2x)^{2-\delta} \leq 8x^{2-\delta}.$$

Hence $|S_\delta(x)| \ll x^{2-\delta}$. This completes the proof. □

As should be clear, once one subtracts from the count of a set of pre-*abc* triples the upper bounds for sets where the gcd is large and the sets where one of the terms of the triple is small, one is left with a lower bound on the number of triples that are approximately relatively prime and contain terms of approximately the same size.

4.2 The $G(q)$ function and related functions

In [2], part of their application of the circle method involves a function $G(q)$, which is also relevant to our proofs. This section concerns some results on the behaviour of this function that were not covered in the original paper.

First, we shall give a definition of the function G [2]:

Definition 4.5. The function $G\colon \mathbb{N} \to \mathbb{R}$ is a multiplicative function, where for a prime power p^ℓ:

$$G(p^\ell) = \begin{cases} -\frac{1}{(p^2-1)} & \text{if } 1 \leq \ell \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

As it is relevant to future results, we shall also note that μ_2 is the characteristic function of the squarefree integers.

The first result of this section is a generalization of Lemma 3.1 from [2] for square-free numbers. This proof uses a different method than [2] but gives the same bounds.

Lemma 4.6. *Let $n, m \in \mathbb{N}$, $m > n$. Let $R > 0$.*

If $m > n$,

$$\sum_{q \geq R} q^n |G(q)|^m \ll R^{n-m+1/2}.$$

If $m \leq n$,

$$\sum_{1 \leq q \leq R} q^n |G(q)|^m \ll R^{n-m+1/2}.$$

Proof. Observe that if $G(q) \neq 0$ then $q = ab^2$ where a, b are squarefree integers.

If $m > n$, then

$$\begin{aligned}
\sum_{q \geq R} q^n |G(q)|^m &\leq \sum_{a=1}^{\infty} \sum_{b \geq \sqrt{R/a}} \mu_2(a)\mu_2(b) a^n b^{2n} |G(a)|^m |G(b^2)|^m \\
&\leq \sum_{a=1}^{\infty} \mu_2(a) a^n |G(a)|^m \sum_{b \geq \sqrt{R/a}} \mu_2(b) b^{2n} |G(b^2)|^m \\
&\leq \sum_{a=1}^{\infty} a^{n-2m} \sum_{b \geq \sqrt{R/a}} b^{2n-2m} \\
&\ll \sum_{a=1}^{\infty} a^{n-2m} (R/a)^{n-m+1/2} \\
&\ll R^{n-m+1/2} \sum_{a=1}^{\infty} a^{-m-1/2} \ll R^{n-m+1/2}.
\end{aligned}$$

If $m \leq n$, then

$$\begin{aligned}
\sum_{1 \leq q \leq R} q^n |G(q)|^m &\leq \sum_{1 \leq a \leq R} \sum_{1 \leq b \leq \sqrt{R/a}} \mu_2(a)\mu_2(b) a^n b^{2n} |G(a)|^m |G(b^2)|^m \\
&\leq \sum_{1 \leq a \leq R} \mu_2(a) a^n |G(a)|^m \sum_{1 \leq b \leq \sqrt{R/a}} \mu_2(b) b^{2n} |G(b^2)|^m \\
&\leq \sum_{1 \leq a \leq R} a^{n-2m} \sum_{1 \leq b \leq \sqrt{R/a}} b^{2n-2m} \\
&\ll \sum_{1 \leq a \leq R} a^{n-2m} (R/a)^{n-m+1/2} \\
&\ll R^{n-m+1/2} \sum_{1 \leq a \leq R} a^{-m-1/2} \ll R^{n-m+1/2}.
\end{aligned}$$

This completes the proof. □

Due to its significance in later proofs, where it will be a component of the coefficient comprising the asymptotic formulas for functions counting a superset containing *abc*-triples, we shall resolve the infinite sum where $n = 1$, $m = 3$:

Lemma 4.7. *The sum* $\omega = \sum_{q=1}^{\infty} \varphi(q) G(q)^3$ *converges and* $\omega \approx 0.872985953173535$.

Furthermore, for any positive R,

$$\sum_{q \leq R} \varphi(q) G(q)^3 = \omega + O(R^{-3/2}).$$

Proof. By Lemma 4.6, with $n = 1, m = 3$, one has that the infinite sum converges and that for all $\epsilon > 0$,

$$\sum_{q=1}^{\infty} \varphi(q)|G(q)|^3 = \sum_{q \le R} \varphi(q)|G(q)|^3 + O(R^{-3/2+\epsilon}),$$

so ω exists.

Furthermore, we shall observe that

$$\begin{aligned}
\sum_{q=1}^{\infty} \varphi(q)G(q)^3 &= \prod_p \left(1 + \varphi(p)G(p)^3 + \varphi(p^2)G(p^2)^3\right) \\
&= \prod_p \left(1 - \frac{p-1}{(p^2-1)^3} - \frac{p(p-1)}{(p^2-1)^3}\right) \\
&= \prod_p \left(1 - \frac{(p+1)(p-1)}{(p^2-1)^3}\right) = \prod_p \left(1 - \frac{1}{(p^2-1)^2}\right)
\end{aligned}$$

and that since

$$\sum_p \left|\log\left(1 - \frac{1}{(p^2-1)^2}\right)\right| \le \sum_{n=2}^{\infty} \left|\log\left(1 - \frac{1}{(n^2-1)^2}\right)\right|$$

and

$$\left|\log\left(1 - \frac{1}{(n^2-1)^2}\right)\right| \ll \frac{1}{n^4}$$

it follows that $\sum_p |\log(1 - \frac{1}{(p^2-1)^2})| < \infty$ and thus $\prod_p (1 - \frac{1}{(p^2-1)^2}) > 0$. Thus the Euler product also converges, and must converge to the same quantity ω.

Now, given

$$\omega_R = \prod_{p \le R} \left(1 - \frac{1}{(p^2-1)^2}\right),$$

and the fact that for any positive R,

$$\begin{aligned}
\sum_{p>R} \left|\log\left(1 - \frac{1}{(p^2-1)}\right)\right| &\le \sum_{n>R} \left|\log\left(1 - \frac{1}{(n^2-1)^2}\right)\right| \\
&\le \sum_{n>R} 2n^{-4} \\
&\le \int_R^{\infty} 2x^{-4} dx \\
&\le 2R^{-3}/3,
\end{aligned}$$

it follows that

$$\begin{aligned}
\omega &= \sum_{q=1}^{\infty} \varphi(q) G(q)^3 \\
&= \prod_{p} \left(1 - \frac{1}{(p^2-1)^2}\right) \\
&= \omega_R \prod_{p>R} \left(1 - \frac{1}{(p^2-1)^2}\right) \\
&= \omega_R \exp\left(\sum_{p>R} \log\left(1 - \frac{1}{(p^2-1)^2}\right)\right) \\
&\in [\omega_R \exp(-2R^{-3}/3), \omega_R].
\end{aligned}$$

Furthermore, since the exponential has slope less than one over negative values, it follows that $\omega \in [\omega_R - 2R^{-3}/3, \omega_R]$.

By computation of ω_{100000}, one has that

$$\omega \in [0.8729859531735338, 0.8729859531735357].$$

□

In §6 some additional proofs related to functions derived from the G function are provided. They are not listed here as those variants of the G function depend on certain parameters derived from an independent variable only relevant to the problem described in that chapter.

Chapter 5

The Trivial Point Problem

In this chapter, we move on to using the Circle Method to prove the existence of a limit point at $(1,1,1) \in \mathfrak{Q}'$. This method involves calculating a function $F(N)$ which counts the number of triples $a+b=c$ (not necessarily relatively prime) such that a,b,c are squarefree and $c \leq N$. Then we show that the rate of growth of $F(N)$ exceeds the possible rate of growth expected if the number of triples $a+b=c$ where a,b,c are square-free and relatively prime is finite, which is what proves the existence of the limit point.

We are doing this to showcase the methods that will be used in later sections to tackle a more complex, general problem. We use a variant of the method described in [2] to show that the trivial point is a limit point in $\mathfrak{Q}'$. In this case, $F(N) = \int_0^1 f(x;N)f(x;N)f(-x;N)\,dx$, where f is the squarefree counting function described in [2]. That is to say,

$$f(x;N) = \sum_{1 \leq n \leq N} \mu_2(n)e(nx),$$

where μ_2 is the characteristic function of the squarefree integers.

Throughout this chapter, we shall use the same major $\mathfrak{M}$ and minor $\mathfrak{m}$ arcs as found in [2]. For reference, these depend on two parameters Q and N, with $1 \leq Q \leq N^{1/2}/2$:

$$\begin{aligned}
\mathfrak{M}(q,k) &= \{x \in [0,1) \colon |qx-k| \leq Q/N\}, \\
\mathfrak{M}(Q;N) &= \bigcup_{1 \leq q \leq Q} \bigcup_{\substack{k=0 \\ (q,k)=1}}^{q} \mathfrak{M}(q,k), \\
\mathfrak{m}(Q;N) &= \{x \in [0,1) \colon \forall q \in \mathbb{N}, k \in \mathbb{Z}, (k,q)=1, |qx-k| \leq Q/N \text{ implies } q > Q\}.
\end{aligned}$$

We now establish some bounds on the integral over the minor arcs.

Lemma 5.1. *For all $\epsilon > 0$,*

$$\int_{\mathfrak{m}} |f(x;N)^2 f(-x;N)|\, dx \leq \sup_{x\in\mathfrak{m}} |f(x;N)| \int_{\mathfrak{m}} |f(x;N)|^2\, dx \ll N^{2+\epsilon}Q^{-3/2} + N^{1+\epsilon}Q.$$

Proof. From [2, Theorem 2], for any $\epsilon > 0$, $\int_{\mathfrak{m}} |f(x;N)|^2\, dx \ll N^{1+\epsilon}Q^{-1/2} + N^{\epsilon}Q^2$.

From a paper by D.I. Tolev [5, Theorem 1] one has that $\sup_{x\in\mathfrak{m}} |f(x;N)| \ll N^{1+\epsilon}Q^{-1}$.

Thus by Hölder's inequality with $p = +\infty$ and $q = 1$ and the above observations, the desired statement naturally follows. □

In addition to the G function (first discussed in Definition 4.5) we shall use the following functions from [2]:

$$I(x) = \sum_{n=1}^{N} e(nx),$$

$$f^\star(x) = \begin{cases} \zeta(2)^{-1}G(q)I(x-a/q) & \text{if } x \in \mathfrak{M}(q,a) \subset \mathfrak{M}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Delta(x) = f(x) - f^\star(x).$$

For those unfamiliar with the circle method, $I(x)$ is the standard notation for the exponential sum in these problems, and $f^\star$ is the approximation of f used in [2].

From the definitions given above, we shall observe that:

$$\begin{aligned}\int_{\mathfrak{M}} f(x)^2 f(-x)\, dx &= \int_{\mathfrak{M}} (\Delta(x) + f^\star(x))^2(\Delta(-x) + f^\star(-x))\, dx \\ &= \int_{\mathfrak{M}} \Delta(x)^2\Delta(-x)\, dx \\ &\quad + \int_{\mathfrak{M}} 2\Delta(x)\Delta(-x)f^\star(x)\, dx + \int_{\mathfrak{M}} \Delta(x)^2 f^\star(-x)\, dx \\ &\quad + \int_{\mathfrak{M}} \Delta(-x)f^\star(x)^2\, dx + \int_{\mathfrak{M}} 2\Delta(x)f^\star(x)f^\star(-x)\, dx \\ &\quad + \int_{\mathfrak{M}} f^\star(x)^2 f^\star(-x)\, dx.\end{aligned}$$

Next, we place an upper bound on the growth of the difference between the integrals of the triple f and triple $f^\star$.

Lemma 5.2. *For every $\epsilon > 0$,*

$$\int_{\mathfrak{M}} f(x)^2 f(-x)\, dx - \int_{\mathfrak{M}} f^\star(x)^2 f^\star(-x)\, dx \ll \max\{N^{3/2+\epsilon}Q^{3/4}, N^{1+\epsilon}Q^2\}.$$

Proof. By Lemmata 3.2 and 4.2 in [2], we have that for all $\epsilon > 0$,

$$\int_{\mathfrak{M}} |\Delta(x)|^2 \, dx \ll N^{\epsilon} Q^2, \text{ and } \int_{\mathfrak{M}} |f^{\star}(x)\Delta(x)| \, dx \ll N^{1/2+\epsilon} Q^{3/4}$$

for any $Q \in \mathbb{N}$ where $Q \leq \frac{1}{2}\sqrt{N}$.

Now, observe that by definition, for any $x \in \mathfrak{M}(q, a)$ that

$$|f^{\star}(x)| \leq |\zeta(2)^{-1} G(q) I(x - a/q)| \leq \zeta(2)^{-1} |I(x - a/q)| \ll N.$$

Likewise,

$$|f(x)| \leq \sum_{n=1}^{N} 1 \ll N, \quad |\Delta(x)| \leq |f(x)| + |f^{\star}(x)| \ll N.$$

Thus we have:

$$\left| \int_{\mathfrak{M}} \Delta(x)^2 \Delta(-x) \, dx \right| \leq \sup_{x \in \mathfrak{M}} |\Delta(x)| \int_{\mathfrak{M}} |\Delta(x)^2| dx \ll N^{1+\epsilon} Q^2, \tag{5.1}$$

$$\left| \int_{\mathfrak{M}} \Delta(x)\Delta(-x) f^{\star}(x) \, dx \right| \leq \sup_{x \in \mathfrak{M}} |f^{\star}(x)| \int_{\mathfrak{M}} |\Delta(x)^2| dx \ll N^{1+\epsilon} Q^2, \tag{5.2}$$

$$\left| \int_{\mathfrak{M}} \Delta(-x) f^{\star}(x)^2 \, dx \right| \leq \sup_{x \in \mathfrak{M}} |f^{\star}(x)| \int_{\mathfrak{M}} |\Delta(x) f^{\star}(x)| dx \ll N^{3/2+\epsilon} Q^{3/4}, \tag{5.3}$$

$$\left| \int_{\mathfrak{M}} \Delta(x)^2 f^{\star}(-x) \, dx \right| = \left| \int_{\mathfrak{M}} \Delta(x)\Delta(-x) f^{\star}(x) \, dx \right| \ll N^{1+\epsilon} Q^2, \tag{5.4}$$

$$\left| \int_{\mathfrak{M}} \Delta(x) f^{\star}(x) f^{\star}(-x) \, dx \right| = \left| \int_{\mathfrak{M}} \Delta(-x) f^{\star}(x)^2 \, dx \right| \ll N^{3/2+\epsilon} Q^{3/4}. \tag{5.5}$$

This completes the proof. □

Now, it suffices to calculate $\int_{\mathfrak{M}} f^{\star}(x)^2 f^{\star}(-x) \, dx$.

Lemma 5.3. *The main term resolves as follows:*

$$\int_{\mathfrak{M}} f^{\star}(x)^2 f^{\star}(-x) \, dx = \frac{N(N-1)}{2} \zeta(2)^{-3} \left(\sum_{q=1}^{Q} \varphi(q) G(q)^3 \right) + O(N^2 Q^{-3/2}).$$

Proof. To start, recall that as in [2] for $x \in \mathfrak{M}(q, a)$, $f^{\star}(x) = \zeta(2)^{-1} G(q) I(x - a/q)$, where $G(n), I(x)$ are as defined earlier.

As $\mathfrak{M} = \bigcup_{q=1}^{Q} \bigcup_{\substack{0 \le a \le q \\ (a,q)=1}} \mathfrak{M}(a,q)$, it follows that:

$$\begin{aligned}
\int_{\mathfrak{M}} f^\star(x)^2 f^\star(-x)\,dx &= \sum_{q=1}^{Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q} \left(\frac{G(q)}{\zeta(2)}\right)^3 \int_{\mathfrak{M}(q,a)} I(x-a/q)^2 I(a/q-x)\,dx \\
&= \sum_{q=1}^{Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{G(q)}{\zeta(2)}\right)^3 \int_{-Q/qN}^{Q/qN} I(x)^2 I(-x)\,dx \\
&= \sum_{q=1}^{Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{G(q)}{\zeta(2)}\right)^3 \left(\int_{-1/2}^{1/2} I(x)^2 I(-x)\,dx + O\left(\int_{Q/qN}^{1/2} \|x\|^{-3}\,dx\right)\right) \\
&= \sum_{q=1}^{Q} \varphi(q) G(q)^3 \zeta(2)^{-3} \left(\int_{-1/2}^{1/2} I(x)^2 I(-x)\,dx + O\left(\int_{Q/qN}^{1/2} \|x\|^{-3}\,dx\right)\right).
\end{aligned}$$

Note that the transition from $a = 0$ to $a = 1$ is facilitated by the fact that for $q = 1$ both intervals are of half length, and since the exponential sum $I(x)$ is periodic modulo $\mathbb{Z}$, they can be stitched together into a single full-length interval. We now compute the integral for the main term:

$$\begin{aligned}
\int_{-1/2}^{1/2} I(x)^2 I(-x)\,dx &= \int_{-1/2}^{1/2} \left(\sum_{n=1}^{N} e(nx)\right)^2 \left(\sum_{m=1}^{N} e(-mx)\right) dx \\
&= \int_0^1 \sum_{m=1}^{N} e(-mx) \left(\sum_{n=1}^{N} e(nx)\right)^2 dx \\
&= \sum_{m=1}^{N} \int_0^1 \left(\sum_{n=1}^{N} e(nx)\right)^2 e(-mx)\,dx = \sum_{m=1}^{N} (m-1) = \frac{N(N-1)}{2}.
\end{aligned}$$

Now, for the integral in the error term:

$$\int_{Q/qN}^{1/2} \|x\|^{-3}\,dx \ll q^2 N^2 Q^{-2}.$$

By Lemma 4.6 one has that

$$\sum_{q=1}^{Q} \varphi(q) G(q)^3 \int_{Q/qN}^{1/2} \|x\|^{-3}\,dx \ll \sum_{q=1}^{Q} q|G(q)|^3 q^2 N^2 Q^{-2} \ll N^2 Q^{-2} \sum_{q=1}^{Q} q^3 |G(q)|^3 \ll N^2 Q^{-3/2+\epsilon}.$$

Hence, it follows that:

$$\int_{\mathfrak{M}} f^\star(x)^2 f^\star(-x)\,dx = \frac{N(N-1)}{2} \zeta(2)^{-3} \left(\sum_{q=1}^{Q} \varphi(q) G(q)^3\right) + O(N^2 Q^{-3/2+\epsilon}).$$

□

Putting it all together, the following theorem results:

Theorem 5.4. *The following holds for any $1 < Q \le \frac{1}{2}N^{1/2}$ and all $\epsilon > 0$:*

$$F(N) = \omega\zeta(2)^{-3}N(N+1)/2 + O(N^{2+\epsilon}Q^{-3/2} + N^{3/2+\epsilon}Q^{3/4} + N^{1+\epsilon}Q^2).$$

Proof. By Lemmata 5.3 and 4.7, for all $\epsilon > 0$:

$$\int_{\mathfrak{M}} f^\star(x)^2 f^\star(-x)\,dx = \omega\zeta(2)^{-3}N(N+1)/2 + O(N^2Q^{-3}) + O(N^2Q^{-3/2})$$

Hence, we conclude that for all $\epsilon > 0$, that

$$\int_{\mathfrak{M}} f^\star(x)^2 f^\star(-x)\,dx = \omega\zeta(2)^{-3}N(N+1)/2 + O(N^2Q^{-3/2}).$$

Combining this result with Lemmata 5.1 and 5.2 one has that:

$$\begin{aligned} F(N) &= \int_0^1 f(x)^2 f(-x)\,dx \\ &= \int_{\mathfrak{M}} f^\star(x)^2 f^\star(-x)\,dx + O(N^{3/2+\epsilon}Q^{3/4} + N^{1+\epsilon}Q + N^{2+\epsilon}Q^{-1/2}) \\ &= \omega\zeta(2)^{-3}N(N+1)/2 + O(N^2Q^{-3/2} + N^{3/2+\epsilon}Q^{3/4} + N^{1+\epsilon}Q + N^{2+\epsilon}Q^{-3/2}) + N^{1+\epsilon}Q^2. \end{aligned}$$

Noting that $N^2Q^{-3/2} \ll N^{2+\epsilon}Q^{-3/2}$ and $N^{1+\epsilon}Q \ll N^{1+\epsilon}Q^2$ for any $Q > 1$ completes the proof. □

Corollary 5.5. *For $Q = N^{2/9}$, and N sufficiently large,*

$$F(N) = \omega\zeta(2)^{-3}N^2/2 + O(N^{5/3+\epsilon}).$$

This suffices to show that the equation $a + b = c$ for $a, b, c \in \mathbb{N}$ has infinitely many square-free solutions. However, we need to show that $a + b = c$ for $a, b, c \in \mathbb{N}$ has infinitely many square-free solutions where (a, b) are relatively prime.

Theorem 5.6. *There exist infinitely many abc triples where a, b, c are squarefree, and $(1, 1, 1) \in \mathfrak{Q}'$.*

Proof. Observe that by Corollary 5.5

$$\begin{aligned} F(M) &= \omega\zeta(2)^{-3}(M^2 + M)/2 + O(M^{5/3+\epsilon}) \\ F(2M) &= \omega\zeta(2)^{-3}(2M^2 + M) + O(M^{5/3+\epsilon}) \\ D_F(M) = F(2M) - F(M) &= \omega\zeta(2)^{-3}(3M^2 + M)/2 + O(M^{5/3+\epsilon}). \end{aligned}$$

Let $A_F(M, \epsilon)$ count the subset of squarefree pre-*abc* triples on the interval $(M, 2M]$ where $\gcd(a, b) > M^\epsilon$. Let $B_F(M, \delta)$ count the subset of squarefree pre-*abc* δ-triples on the interval

$(M, 2M]$. By Lemma 4.3, for any $\epsilon > 0$, $A_F(M, \epsilon) = O(M^{2-\epsilon})$. By Lemma 4.4, for any $\delta > 0$, $B_F(M, \delta) = O(M^{2-\delta})$.

Now, fix $\delta > 0$ and observe that $D(M, \delta) = D_F(M) - A_F(M, \delta/2) - B_F(M, \delta/2) = D_F(M) + O(M^{2-\delta/2})$, which is strictly greater than zero for sufficiently large M.

For M sufficiently large, $D(M, \delta) > 0$ and thus there exists a triple (a, b, c) such that $d = \gcd(a, b) \leq M^{\delta/2}$, $a, b \geq c^{\delta/2}$, and $M < c \leq 2M$.

Now, let $(a', b', c') = (a/d, b/d, c/d)$. It follows that $a' \geq aM^{-\delta/2}$, $b' \geq bM^{-\delta/2}$, $c' \geq cM^{-\delta/2} > M^{1-\delta/2}$. Furthermore, $a' \geq aM^{-\delta/2} \geq c^{1-\delta/2}M^{-\delta/2} \geq c^{1-\delta} \geq (c')^{1-\delta}$. By symmetric argument, $b' \geq (c')^{1-\delta}$.

Since M can be made arbitrarily large, and $c' > M^{1-\delta/2}$ it follows that there are infinitely many squarefree relatively prime triples (a, b, c) such that $a \geq c^{1-\delta}$, $b \geq c^{1-\delta}$ where $a + b = c$. Note that these triples are such that $\overrightarrow{\mathfrak{iq}}(a, b) \in [1 - \delta, 1]^2 \times \{1\} \subset \mathbb{R}^3$.

Since δ can be made arbitrarily small, this suffices to prove the existence of the limit point $(1, 1, 1)$ in Q. $\square$

This result is the end goal of this chapter. We have used the circle method to show that there exist infinitely many relatively prime positive squarefree integers a, b such that $a+b$ is also squarefree and of similar size.

Chapter 6

Determination of a Volume in $\mathfrak{Q}$

In this chapter, we shall use the same methods as in the Trivial Point Problem, but apply them to a more general problem, so as to allow the computation of a large volume of points in $\mathfrak{Q}$.

This chapter concerns the analysis of a counting function for a set of natural number triples $a + b = c$ which, in the limit as $N \to \infty$ have $\overrightarrow{\mathfrak{iq}}(a, b) = (\alpha, \beta, \gamma)$. Before giving a precise definition of our counting function, we will spend some time discussing the strategy of this proof.

We will employ the circle method to compute an asymptotic formula for a function $F(N)$ that depends on six parameters: α, β, γ and p_a, p_b, p_c. The parameters α, β, γ specify the limit point in $\mathfrak{Q}$ we seek to prove exists, and p_a, p_b, p_c are arbitrary distinct prime numbers. We have allowed for arbitrary distinct prime numbers instead of choosing specific primes (such as $2, 3, 5$) as it adds little complexity to the proof and greater generality.

The asymptotic formula of our counting function $F(N)$ will itself be insufficient to prove the existence of a limit point. This is because our function will count triples (a, b, c) that are not necessarily relatively prime to each other, and will count some numbers that are too small in a certain sense. However, these issues can be overcome, as the improper triples can be excluded by means of counting. Once this is done, constructing a limit point is simple.

6.1 Definitions

We now move on to the formal definitions.

Let $\alpha, \beta, \gamma \in (0, 1]$, and $p_a, p_b, p_c \in \mathbb{N}$ be distinct prime numbers. The choice of primes is arbitrary. Our proofs will show that the choice of primes affects the constant coefficients of the main term, and little else.

Now, we can move on to the definitions, with $F(N)$, we shall first give a small list of quantities that are functions of N and the parameters α, β, γ which will be used in the definition of $F(N)$ and in later proofs.

Definition 6.1 (Quantities that Depend on N)**.** For any given $N \in \mathbb{R}_{\geq 1}$, let

$$A = \lfloor (1-\alpha) \log_{p_a} N \rfloor, B = \lfloor (1-\beta) \log_{p_b} N \rfloor, C = \lfloor (1-\gamma) \log_{p_c} N \rfloor.$$

Let $h_a = N^{1-\alpha} p_a^{-A}$, $h_b = N^{1-\beta} p_b^{-B}$, and $h_c = N^{1-\gamma} p_c^{-C}$.

Note that h_a, h_b, h_c are technically functions of N, and that $h_a \colon \mathbb{R}_{>0} \to [1, p_a)$, $h_b \colon \mathbb{R}_{>0} \to [1, p_b)$, and $h_c \colon \mathbb{R}_{>0} \to [1, p_c)$. It should also be clear that they are periodic on a logarithmic scale.

We now have enough to give a precise definition of $F(N)$.

Definition 6.2 (The Counting Function $F(N)$)**.** The function $F(N)$ counts all triples of the form $(p_a^A a, p_b^B b, p_c^C c)$ where $a \leq N^\alpha$, $b \leq N^\beta$, $c \leq N^\gamma$ and a, b, c are squarefree. We shall also describe $F(N)$ as an integral of the function $f(x; M)$ as from [2]. The function $F(N)$ is thus represented by the following integral:

$$F(N) = \int_0^1 f(p_a^A x; N^\alpha) f(p_b^B x; N^\beta) f(-p_c^C x; N^\gamma) dx.$$

Note that $F(N)$ depends only on the choice of $(\alpha, \beta, \gamma) \in [0,1]^3$ and p_a, p_b, p_c. Once these are fixed, A, B, C are simply functions of N. Likewise, an application of the circle method requires a dissection of the unit interval $[0, 1)$ into major arcs. For reasons that will become apparent, the dissection used will be more complex than is usual, with three major arc components. For these major arcs, there will be three parameters $Q_a, Q_b, Q_c \in \mathbb{R}$ determining their size such that

$$1 \leq Q_a \leq \frac{1}{2} N^{\alpha/2}, 1 \leq Q_b \leq \frac{1}{2} N^{\beta/2}, 1 \leq Q_c \leq \frac{1}{2} N^{\gamma/2}.$$

Definition 6.3 (Major Arcs)**.** The major arcs are defined in terms of the following component sets:

$$\mathfrak{M}_\alpha(q, a) = \{x \in [0,1) \colon |q p_a^A x - a| \leq Q_a N^{-\alpha}\},$$
$$\mathfrak{M}_\beta(q, b) = \{x \in [0,1) \colon |q p_b^B x - b| \leq Q_b N^{-\beta}\},$$
$$\mathfrak{M}_\gamma(q, c) = \{x \in [0,1) \colon |q p_c^C x - c| \leq Q_c N^{-\gamma}\}.$$

These components become the local major arcs

$$\mathfrak{M}_\alpha = \bigcup_{q \leq Q_a} \bigcup_{\substack{a=0 \\ (q,a)=1}}^{p_a^A q} \mathfrak{M}_\alpha(q, a), \quad \mathfrak{M}_\beta = \bigcup_{q \leq Q_b} \bigcup_{\substack{b=0 \\ (q,b)=1}}^{p_b^B q} \mathfrak{M}_\beta(q, b), \quad \mathfrak{M}_\gamma = \bigcup_{q \leq Q_c} \bigcup_{\substack{c=0 \\ (q,c)=1}}^{p_c^C q} \mathfrak{M}_\gamma(q, c).$$

The entire major arc is thus defined to be the intersection of the three local major arcs, $\mathfrak{M} = \mathfrak{M}_\alpha \cap \mathfrak{M}_\beta \cap \mathfrak{M}_\gamma$. Thus one can say that the major arcs constitute the union of all intersections of $\mathfrak{M}_\alpha(q_a, a)$, $\mathfrak{M}_\beta(q_b, b)$ and $\mathfrak{M}_\gamma(q_c, c)$ where $q_a \le Q_a$, $q_b \le Q_b$, $q_c \le Q_c$ and $(q_a, a) = (q_b, b) = (q_c, c) = 1$.

Furthermore, an important subset of the major arcs, which shall be called the *harmonic major arcs* $\mathfrak{M}_H$, will have components

$$\mathfrak{M}_H(q,k) = \mathfrak{M}_\alpha\left(\frac{q}{(q,p_a^A)}, \frac{p_a^A}{(q,p_a^A)}k\right) \cap \mathfrak{M}_\beta\left(\frac{q}{(q,p_b^B)}, \frac{p_b^B}{(q,p_b^B)}k\right) \cap \mathfrak{M}_\gamma\left(\frac{q}{(q,p_c^C)}, \frac{p_c^C}{(q,p_c^C)}k\right).$$

And finally,

$$\mathfrak{M}_H = \bigcup_{\substack{1 \le q \\ q/(q,p_a^A) \le Q_a \\ q/(q,p_b^B) \le Q_b \\ q/(q,p_c^C) \le Q_c}} \bigcup_{\substack{k=0 \\ (q,k)=1}}^{q} \mathfrak{M}_H(q,k).$$

Note that $\mathfrak{M}_H$ is clearly a subset of $\mathfrak{M}$ by definition since it is a union of intersections of $\mathfrak{M}_\alpha(q_a, a)$, $\mathfrak{M}_\beta(q_b, b)$ and $\mathfrak{M}_\gamma(q_c, c)$ where by definition $q_a = q/(q, p_a^A) \le Q_a, q_b = q/(q, p_b^B) \le Q_b, q_c = q/(q, p_c^c) \le Q_c$ and $(q_a, a) = (q_b, b) = (q_c, c) = 1$.

The choice of the word *harmonic* to describe $\mathfrak{M}_H$ is due to the fact that, as will be proven later, the centres of the intervals are precisely the rational numbers such that one arc component of each type is centred on them, and this is considered a sort of "harmony." The harmonic major arcs are the arcs over which the asymptotic formula of $F(N)$ will actually be calculated.

As a shorthand, we will sometimes write

$$\tilde{Q} = \{q \in \mathbb{N} \colon q/(q,p_a^A) \le Q_a, q/(q,p_b^B) \le Q_b, q/(q,p_c^C) \le Q_c\}$$

to denote the set of natural numbers such that $\mathfrak{M}_H = \bigcup_{q \in \tilde{Q}} \bigcup_{\substack{k=0 \\ (q,k)}}^{q} \mathfrak{M}_H(q,k)$.

Unlike other arc dissections, which split $[0, 1)$ into the major arcs and minor arcs, we shall divide $[0, 1)$ into four different categories, the *major arcs* denoted $\mathfrak{M}$ as defined above, the *semi-major arcs* denoted $\partial\mathfrak{M}$, the *semi-minor arcs* $\partial\mathfrak{m}$, and the *minor arcs* $\mathfrak{m}$. While it is entirely possible to define these arcs as unions of intersections of the major arcs components, this somewhat obscures their structure. Instead we shall define them using the following characteristic functions:

Definition 6.4 (Characteristic Functions)**.** Define

$$\chi_\alpha(x) = \begin{cases} 1 & \text{if } x \in \mathfrak{M}_\alpha(q_a, a) \text{ where } (q_a, a) = 1,\ 1 \leq q_a \leq Q_a, \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi_\beta(x) = \begin{cases} 1 & \text{if } x \in \mathfrak{M}_\beta(q_b, b) \text{ where } (q_b, b) = 1,\ 1 \leq q_b \leq Q_b, \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi_\gamma(x) = \begin{cases} 1 & \text{if } x \in \mathfrak{M}_\gamma(q_c, c) \text{ where } (q_c, c) = 1,\ 1 \leq q_c \leq Q_c, \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi(x) = \chi_\alpha(x) + \chi_\beta(x) + \chi_\gamma(x).$$

From the definition of the major arcs, it follows that $x \in \mathfrak{M}$ if and only if $\chi(x) = 3$. We then define the major, semi-major, semi-minor, and minor arcs as follows:

Definition 6.5 (Other Arcs)**.**

$$\begin{aligned} \mathfrak{M} &= \{x[0,1) \colon \chi(x) = 3\}, \\ \partial\mathfrak{M} &= \{x \in [0,1) \colon \chi(x) = 2\}, \\ \partial\mathfrak{m} &= \{x \in [0,1) \colon \chi(x) = 1\}, \\ \mathfrak{m} &= \{x \in [0,1) \colon \chi(x) = 0\}. \end{aligned}$$

It should be obvious that these four sets are disjoint, and that $\mathfrak{M} \cup \partial\mathfrak{M} \cup \partial\mathfrak{m} \cup \mathfrak{m} = [0,1)$.

We shall now define the harmonic semi-major arcs, as well as a split of them into a low and high component. These serve a role analogous to the harmonic majors.

Definition 6.6 (Harmonic Semi-Major Arcs)**.** Let

$$\partial\tilde{Q} = \left\{ q \in \mathbb{N} \colon \text{ at least two of the following hold: } \frac{q}{(q, p_a^A)} \leq Q_a, \frac{q}{(q, p_b^B)} \leq Q_b, \frac{q}{(q, p_c^C)} \leq Q_c \right\}.$$

For any given $q \in \mathbb{N}, k \in \mathbb{Z}$ define the harmonic semi-major component

$$\partial\mathfrak{M}_H(q,k) = \partial\mathfrak{M} \cap \left(\mathfrak{M}_\alpha \left(\frac{q}{(q, p_a^A)}, \frac{p_a^A}{(q, p_a^A)} k \right) \cup \mathfrak{M}_\beta \left(q(q, p_b^B), \frac{p_b^B}{(q, p_b^B)} k \right) \cup \mathfrak{M}_\gamma \left(q(q, p_c^C), \frac{p_c^C}{(q, p_c^C)} k \right) \right).$$

The *harmonic semi-major arcs* $\partial\mathfrak{M}_H$ are the arcs such that

$$\partial\mathfrak{M}_H = \bigcup_{q \in \partial\tilde{Q}} \bigcup_{\substack{k=1 \\ (q,k)=1}}^{q} \partial\mathfrak{M}_H(q,k).$$

Finally, we shall define the *low harmonic semi-major arcs* to be

$$\partial\mathfrak{M}_{H0} = \bigcup_{q\in\tilde{Q}} \bigcup_{\substack{k=1\\(q,k)=1}}^{q} \partial\mathfrak{M}_H(q,k),$$

and also the *high harmonic semi-major arcs* to be

$$\partial\mathfrak{M}_{H1} = \partial\mathfrak{M}_H \setminus \partial\mathfrak{M}_{H0}.$$

In addition to the arcs defined above, it is necessary to define the *reduced major arcs* and *reduced minor arcs* which are not subsets of the arcs listed above, but serve as a bridge to results from [2]. In particular, the reduced major arcs are equivalent to the major arcs from [2] and the reduced minor arcs are equivalent to the minor arcs from [2].

Definition 6.7 (Reduced Major Arcs)**.** Let $M \in \mathbb{R}_{\geq 1}$, and $Q \in \mathbb{R}$ where $1 \leq Q \leq \frac{1}{2}N_0^{1/2}$.

The *reduced major arcs* at $Q;M$ are $\mathfrak{M}^\star(Q;M)$ and are defined to be a

$$\mathfrak{M}^\star(Q;M) = \bigcup_{q\leq Q} \bigcup_{\substack{k=1\\(q,k)=1}}^{q} \mathfrak{M}^\star(q,k;Q;M),$$

with

$$\mathfrak{M}^\star(q,k;Q;M) = \{x \in [0,1)\colon |qx-k| \leq QM^{-1}\}.$$

Likewise, the *reduced minor arcs* at $Q;M$ are $\mathfrak{m}^\star(Q;M) = [0,1) \setminus \mathfrak{M}^\star(Q;M)$, or in other words,

$$\mathfrak{m}^\star(Q;M) = \{x \in [0,1)\colon \forall q \in \mathbb{N}, a \in \mathbb{Z}, (q,a) = 1,\ \text{if } |qx-a| \leq QM^{-1} \text{ then } q > Q\}.$$

The reduced majors arcs and reduced minor arcs have definitions equivalent to the major arcs and minor arcs found in [2] but with different parameters, and on occasion some proofs will translate the complex major arcs of this paper into the reduced major arcs to apply results from [2].

We move on to defining some functions that will be used.

Definition 6.8. Define

$$G_3^\star(q) = G\left(\frac{q}{(q,p_a^A)}\right) G\left(\frac{q}{(q,p_b^B)}\right) G\left(\frac{q}{(q,p_c^C)}\right),$$
$$I_\alpha(x) = \sum_{n\leq N^\alpha} e(nx), \quad I_\beta(x) = \sum_{n\leq N^\beta} e(nx), \quad I_\gamma(x) = \sum_{n\leq N^\gamma} e(nx).$$

We shall then define the following functions, all analogous to the $f^\star$ function in [2] over different reduced major arcs.

$$f^\star_\alpha(x) = \begin{cases} \zeta(2)^{-1}G(q)I_\alpha(x-k/q) & \text{if } x \in \mathfrak{M}^\star(q,k;Q_a;N^\alpha) \subset \mathfrak{M}^\star(Q_a;N^\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

$$f^\star_\beta(x) = \begin{cases} \zeta(2)^{-1}G(q)I_\beta(x-k/q) & \text{if } x \in \mathfrak{M}^\star(q,k;Q_b;N^\beta) \subset \mathfrak{M}^\star(Q_b;N^\beta), \\ 0 & \text{otherwise.} \end{cases}$$

$$f^\star_\gamma(x) = \begin{cases} \zeta(2)^{-1}G(q)I_\gamma(x-k/q) & \text{if } x \in \mathfrak{M}^\star(q,k;Q_c;N^\gamma) \subset \mathfrak{M}^\star(Q_c;N^\gamma), \\ 0 & \text{otherwise.} \end{cases}$$

We shall also define the following functions, analogous to the $\Delta(x)$ functions in [2].

$$\begin{aligned} \Delta_\alpha(x) &= f(x;N^\alpha) - f^\star_\alpha(x), \\ \Delta_\beta(x) &= f(x;N^\beta) - f^\star_\beta(x), \\ \Delta_\gamma(x) &= f(x;N^\gamma) - f^\star_\gamma(x) \end{aligned}$$

Finally, unlike the trivial point problem discussed in the previous chapter, the central component of the main term is not an easily computed integral. The main term will be evaluated first in terms of this integral, and then this integral will be evaluated separately.

Definition 6.9 (Main Term Integral $U(N)$)**.** Define

$$U(N) = \int_0^1 I_\alpha(p_a^A x) I_\beta(p_b^B x) I_\gamma(p_c^C x) dx$$

We also have a related function, given below.

Definition 6.10. For every $\delta > 0$, define

$$U_\delta(N) = \#\{(a,b,c) \in \mathbb{N}\colon a \le N^{\alpha-\delta}, b \le N^{\beta-\delta}, c \le N^{\gamma-\delta}, p_a^A + p_b^B = p_c^C c\}.$$

We also have another two functions related to F.

Definition 6.11. We have for $\mathbb{N}_2$ being the set of squarefree natural numbers,

$$F(N,d) = \#\{(a,b,c) \in \mathbb{N}_2^3\colon p_a^A a + p_b^B b = p_c^C c, \gcd(a,b) = d, a \le N^\alpha, b \le N^\beta, c \le N^\gamma\}.$$

and the function

$$F_\epsilon(N) = \#\{(a,b,c) \in N_2^3\colon p_a^A a + p_b^B b = p_c^C c, \gcd(a,b) \ge N^\epsilon\}.$$

Note that $f_\alpha^\star, f_\beta^\star, f_\gamma^\star$ functions are analogous to the $f^\star$ function defined in [2], over different reduced major arcs. Likewise the $\Delta_\alpha, \Delta_\beta, \Delta_\gamma$ functions are analogous to the Δ function defined in [2], over different reduced major arcs with the corresponding $f^\star$ function.

We shall end the section with an definition of certain useful conditions in the calculation of the major and semi-major arcs and their relationship with their harmonic components. The use of these definitions is not readily apparent or obvious, but later in this work we shall show that when

$$Q_a \ll N^{\lambda_a \alpha}; \quad Q_b \ll N^{\lambda_b \beta}; \quad Q_c \ll N^{\lambda_c \gamma};$$

the weaker condition implies that the major arcs are equal to the harmonic major arcs, and the stronger condition implies that the semi-major arcs are equal to the harmonic semi-major arcs. It will be shown later that both of these conditions must be satisfied to find the asymptotic formula of the integral at the heart of the problem.

Definition 6.12. Let

$$D_H(\lambda_a, \lambda_b, \lambda_c) = \{(1-\lambda_a)\alpha + (1-\lambda_b)\beta, (1-\lambda_a)\alpha + (1-\lambda_c)\gamma, (1-\lambda_b)\beta + (1-\lambda_c)\gamma\},$$

be known as the *harmonic lambda factor*.

If $\max D_H(\lambda_a, \lambda_b, \lambda_c) > 1$, then $D_H(\lambda_a, \lambda_b, \lambda_c)$ is called *weakly harmonic*.

If $\min D_H(\lambda_a, \lambda_b, \lambda_c) > 1$, then $D_H(\lambda_a, \lambda_b, \lambda_c)$ is called *strongly harmonic*.

6.2 Evaluation of $U(N)$, and Bounds For U_δ and F_ϵ

The function $U(N)$ shall be evaluated by a simple argument involving the Geometry of Numbers.

We shall quote the following lemma proven in a paper by Kane [4], which deals with a similar problem, and is perfectly suited to solving this one:

Lemma 6.13 (Lemma 3 of [4]). *Let L be a lattice in a two-dimensional vector space V, and P a convex polygon in V. Let m be the minimum separation between points in L. Then*

$$\#(L \cap P) = \frac{\text{Volume}(P)}{\text{Covolume}(L)} + O\left(\frac{\text{Perimeter}(P)}{m} + 1\right).$$

It should be noted that this lemma, or its proof by Kane, does not in any way depend on which points in the boundary of P lie in P, as the number of points on the boundary is clearly contained in the error term.

Lemma 6.14. *Let $V_N = \{(x, y, z) \in \mathbb{R}^3 \colon p_a^A x + p_b^B y = p_c^C z\}$. Note that the dependence on N is via A, B, C.*

Let $\rho = \sqrt{1 + \frac{h_c^2}{h_a^2} N^{2\alpha-2\gamma} + \frac{h_c^2}{h_b^2} N^{2\beta-2\gamma}}$.

For any polygon $P \subset V_N$ *it follows that if* P' *is the projection of* P *onto the xy plane, then* $\text{Volume}(P) = \rho\,\text{Volume}(P')$.

Proof. Observe that for any $(x, y, z) \in V_N$, by rearrangement one has that z is a function of x, y and thus

$$z = \frac{p_a^A}{p_c^C} x + \frac{p_b^B}{p_c^C} y, \quad \frac{\partial z}{\partial x} = \frac{p_a^A}{p_c^C} = \frac{h_c}{h_a} N^{\alpha-\gamma}, \quad \frac{\partial z}{\partial y} = \frac{p_a^A}{p_c^C} = \frac{h_c}{h_b} N^{\beta-\gamma}.$$

Hence it follows that if $P \subset V_N$ is a polygon and P' is the projection of R onto the xy-plane,

$$\text{Volume}(P) = \iint_{P'} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_{P'} \rho\, dA = \rho\,\text{Volume}(P').$$

□

Lemma 6.15. *Let* $N > 0$, $P_N \subseteq \mathbb{R}^3$ *be the polygon produced by the intersection of the solid rectangular prism* $R_N = (0, N^\alpha] \times (0, N^\beta] \times (0, N^\gamma]$ *and the plane* $V_N = \{(x, y, z) \in \mathbb{R}^3 : p_a^A x + p_b^B y = p_c^C z\}$.

Let $\rho = \sqrt{1 + \frac{h_c^2}{h_a^2} N^{2\alpha-2\gamma} + \frac{h_c^2}{h_b^2} N^{2\beta-2\gamma}}$.

Then the polygon P_N *is convex and:*

1. *If* $h_a \le h_c$ *and* $h_b \le h_c$ *then* P_N *is a* triangle *and*

$$\text{Volume}(P_N) = \frac{1}{2} \frac{h_a h_b}{h_c^2} \rho N^{\alpha+\beta},$$

$$\text{Perimeter}(P_N) = \sqrt{\frac{h_a^2}{h_c^2} N^{2\alpha} + N^{2\gamma}} + \sqrt{\frac{h_a^2}{h_c^2} N^{2\alpha} + \frac{h_b^2}{h_c^2} N^{2\beta}} + \sqrt{\frac{h_b^2}{h_c^2} N^{2\beta} + N^{2\gamma}}.$$

2. *If* $h_a \le h_c$ *and* $h_b > h_c$ *then* P_N *is a* quadrilateral *and*

$$\text{Volume}(P_N) = \left(\frac{h_a}{h_c} - \frac{h_a}{2h_b}\right) \rho N^{\alpha+\beta},$$

$$\text{Perimeter}(P_N) = \sqrt{\frac{h_a^2}{h_c^2} N^{2\alpha} + N^{2\gamma}} + \sqrt{\frac{h_a^2}{h_b^2} N^{2\alpha} + N^{2\beta}}$$
$$+ \sqrt{\left(\frac{h_a}{h_c} - \frac{h_a}{h_b}\right)^2 N^{2\alpha} + \left(1 - \frac{h_c}{h_b}\right)^2 N^{2\gamma}} + \sqrt{N^{2\beta} + \frac{h_c^2}{h_b^2} N^{2\gamma}}.$$

3. If $h_a > h_c$ *and* $h_b \leq h_c$ *then* P_N *is a* quadrilateral *and*

$$\text{Volume}(P_N) = \left(\frac{h_b}{h_c} - \frac{h_b}{2h_a}\right)\rho N^{\alpha+\beta},$$

$$\text{Perimeter}(P_N) = \sqrt{N^{2\alpha} + \frac{h_c^2}{h_a^2}N^{2\gamma}} + \sqrt{\left(\frac{h_b}{h_c} - \frac{h_b}{h_a}\right)^2 N^{2\beta} + \left(1 - \frac{h_c}{h_a}\right)^2 N^{2\gamma}} + \sqrt{N^{2\alpha} + \frac{h_b^2}{h_a^2}N^{2\beta}} + \sqrt{\frac{h_b^2}{h_c^2}N^{2\beta} + N^{2\gamma}}.$$

4. If $h_a > h_c$ *and* $h_b > h_c$ *and* $\left(\frac{h_a}{h_c} - \frac{h_a}{h_b}\right) \geq 1$*, then* P_N *is a* parallelogram *and*

$$\text{Volume}(P_N) = \rho N^{\alpha+\beta},$$

$$\text{Perimeter}(P_N) = 2\sqrt{N^{2\alpha} + \frac{h_c^2}{h_a^2}N^{2\gamma}} + 2\sqrt{N^{2\beta} + \frac{h_c^2}{h_b^2}N^{2\gamma}}.$$

5. If $h_a > h_c$ *and* $h_b > h_c$ *and* $\left(\frac{h_a}{h_c} - \frac{h_a}{h_b}\right) < 1$ *then* P_N *is a* pentagon *and*

$$\text{Volume}(P_N) = \left(1 - \frac{h_b}{2h_a}\left(1 - \frac{h_a}{h_c} + \frac{h_a}{h_b}\right)^2\right)\rho N^{\alpha+\beta},$$

$$\text{Perimeter}(P_N) = \sqrt{N^{2\alpha} + {\frac{h_c^2}{h_a}}^2 N^{2\gamma}} + \sqrt{\left(\frac{h_b}{h_c} - \frac{h_b}{h_a}\right)^2 N^{2\beta} + \left(1 - \frac{h_c}{h_a}\right)^2 N^{2\gamma}} + \sqrt{\left(1 - \frac{h_a}{h_c} + \frac{h_a}{h_b}\right)^2 N^{2\alpha} + \left(1 - \frac{h_b}{h_c} + \frac{h_b}{h_a}\right)^2 N^{2\beta}} + \sqrt{\left(\frac{h_a}{h_c} - \frac{h_a}{h_b}\right)^2 N^{2\alpha} + \left(1 - \frac{h_c}{h_b}\right)^2 N^{2\gamma}} + \sqrt{N^{2\beta} + \frac{h_c^2}{h_b^2}N^{2\gamma}}.$$

Furthermore,

$$\frac{1}{2p_c^2}N^{\alpha+\beta} \leq \rho^{-1}\,\text{Volume}(P_N) \leq N^{\alpha+\beta}.$$

Proof. First, note given that $N > 0$, R_N is a three dimensional solid with one vertex at the origin and one vertex $(N^\alpha, N^\beta, N^\gamma)$ in the first octant. Since V_N is a plane containing the origin with all normal vectors pointing outside the first octant it follows that the intersection of R_N and V_N is non-empty. Thus P_N is a nonempty area lying in V_N, with $(0,0,0)$ on the boundary of P_N.

Since R_N is a convex volume, and V_N is a plane, and P_N a non-empty intersection of the two, it follows that P_N is a convex polygon whose edges lie on faces of R_N and whose vertices lie on the edges of R_N.

One may see that the five shapes listed in the lemma are exhaustive by the following argument. P_N contains the origin and two line segments from $(0,0,0)$ that extend along the $x=0$ and $y=0$ faces of the rectangle. These segments will either have endpoints on the top or sides of the rectangle.

If both endpoints are on the top, the resulting shape is a *triangle*. If one endpoint is on the top, the resulting shape is a *quadrilateral*. If neither endpoint is on the top, then one either gets a *parallelogram* or a *pentagon*, depending upon whether or not the plane intersects the top face of the rectangle at all.

The exact values given in the Lemma arise from the computation of the volume and perimeter of the polygons from the above description. For a full treatment of the geometric problem, see Appendix A. □

Lemma 6.16. *Let $\Lambda_N = \{(x,y,z) \in \mathbb{Z} \colon p_a^A x + p_b^B y = p_c^C z\}$ be a lattice lying the plane $V_N = \{(x,y,z) \in \mathbb{R} \colon p_a^A x + p_b^B y = p_c^C z\}$.*

Let $\rho = \sqrt{1 + \frac{h_c^2}{h_a^2} N^{2\alpha - 2\gamma} + \frac{h_c^2}{h_b^2} N^{2\beta-2\gamma}}$.

It follows that

$$\text{Covolume}(\Lambda_N) = \frac{\rho}{h_c} N^{1-\gamma},$$

and there exists some $\Psi_N \in \mathbb{N}$ such that $\Psi_N < p_c^C$, $(\Psi_N, p_c) = 1$ and the vectors

$$\vec{v}_{N1} = \left(1, \Psi_N, \frac{p_a^A}{p_c^C} + \frac{p_b^B}{p_c^C}\Psi_N\right), \vec{v}_{N2} = \left(0, p_c^C, p_b^B\right),$$

form a basis for Λ_N.

Proof. Before beginning with the proof, note that when dealing with equivalence classes modulo a natural number, a negative integer exponent $-n$ is understood to be the multiplicative inverse to the power of n in the appropriate group of residue classes. Let $\Psi_N \in \mathbb{N}$ be the number such that $\Psi_N \equiv -p_b^{-B} p_a^A \pmod{p_c^C}$ and $\Psi_N < p_c^C$.

For any $(a,b,c) \in \Lambda_N$ it holds that $p_a^A a + p_b^B b \equiv 0 \pmod{p_c^C}$, so it follows that since $(p_b, p_c) = 1$, and $b \equiv -p_b^{-B} p_a^A a \equiv \Psi_N a \pmod{p_c^C}$, we thus have for some $k \in \mathbb{Z}$ that $b = \Psi_N a + k p_c^C$. Consequently, since for any $(a,b,c) \in \Lambda_N$,

$$c = \frac{p_a^A}{p_c^C} a + \frac{p_b^B}{p_c^C} b = \frac{p_a^A}{p_c^C} a + \frac{p_b^B}{p_c^C} \Psi_N a + p_b^B k.$$

it follows that any $(a,b,c) \in \Lambda_N$ can be represented solely in terms of (a,k) and thus

$$v_1 = \left(1, \Psi_N, \frac{p_a^A}{p_c^C} + \frac{p_b^B}{p_c^C}\Psi_N\right),$$
$$v_2 = \left(0, p_c^C, p_b^B\right).$$

forms a basis for Λ_N.

The projection of v_1, v_2 onto the xy plane produces a lattice with basis $(1, \Psi_N)$, $(0, p_c^C)$ which since $1 \leq \Psi_N < p_c^C$ forms a parallelogram S of area p_c^C. By Lemma 6.14, it follows that the area of the parallelogram produced by v_1, v_2 is ρp_c^C. Thus, as the parallelogram produced by v_1, v_2 is the fundamental domain of Λ_N,

$$\text{Covolume}(\Lambda_N) = \frac{\rho}{h_c} N^{1-\gamma}.$$

□

Now, we have sufficient tools to determine the value of $U(N)$, as defined in Definition 6.9. The calculation of this value is critical, as it is in fact going to be the main term of the asymptotic formula of $F(N)$, our counting function.

Lemma 6.17. *For any real $N > 0$,*

$$U(N) = H(N) N^{\alpha+\beta+\gamma-1} + O(N^\alpha + N^\beta + N^\gamma),$$

$$H(N) = \begin{cases} \frac{h_a h_b}{2h_c} & \text{if } h_b \leq h_c \text{ and } h_a \leq h_c, \\ h_b\left(1 - \frac{h_c}{2h_a}\right) & \text{if } h_b \leq h_c \text{ and } h_a > h_c, \\ h_a\left(1 - \frac{h_c}{2h_b}\right) & \text{if } h_b > h_c \text{ and } h_a \leq h_c, \\ h_c & \text{if } h_b > h_c \text{ and } h_a > h_c \text{ and } \frac{h_a}{h_c} - \frac{h_a}{h_b} \geq 1, \\ h_c\left(1 - \frac{h_b}{2h_a}\left(1 - \frac{h_a}{h_c} + \frac{h_a}{h_b}\right)^2\right) & \text{if } h_b > h_c \text{ and } h_a > h_c \text{ and } \frac{h_a}{h_c} - \frac{h_a}{h_b} < 1 \end{cases}.$$

In particular,

$$\frac{1}{2p_c} N^{\alpha+\beta+\gamma-1} \leq U(N) + O(N^\alpha + N^\beta + N^\gamma) \leq N^{\alpha+\beta_\gamma-1}.$$

Proof. First, note by the definition of the integral, $U(N)$ counts all triples $(a, b, c) \in \mathbb{N}^3$ such that that $a \leq N^\alpha, b \leq N^\beta, c \leq N^\gamma$ and $p_a^A a + p_b^B b = p_c^C c$.

Thus for the lattice $\Lambda_N = \{(a, b, c) \in \mathbb{Z} \colon p_a^A a + p_b^B b = p_c^C\}$ lying in the plane $V_N = \{(x, y, z) \in \mathbb{R} \colon p_a^A x + p_b^B y = p_c^C z\}$, and the polygon $P_N = V_N \cap (0, N^\alpha] \times (0, N^\beta] \times (0, N^\gamma]$ we have that $U(N) = \#(P_N \cap \Lambda_N)$.

Thus by Lemma 6.13 (Kane),

$$U(N) = \frac{\text{Volume}(P_N)}{\text{Covolume}(\Lambda_N)} + O\left(\frac{\text{Perimeter}(P_N)}{m} + 1\right).$$

As m is the minimum separation between the points of Λ_N, and Λ_N is a lattice with integer coefficients, $1/m \leq 1$. By Lemma 6.15, for every possible shape P_N could take, $\text{Perimeter}(P_N) \ll N^\alpha + N^\beta + N^\gamma$. Thus it follows that

$$U(N) = \frac{\text{Volume}(P_N)}{\text{Covolume}(\Lambda_N)} + O(N^\alpha + N^\beta + N^\gamma).$$

To complete the proof of the formula for $U(N)$, for each possible shape of P_N given the parameters h_a, h_b, h_c, we divide the volume of P_N from Lemma 6.15 by the covolume of Λ_N from Lemma 6.16 to get the equation given by the lemma. Note that the constant ρ is cancelled by this division.

Likewise, to prove the upper and lower bounds, take the bounds on the area of P_N given by Lemma 6.15, and divide by the covolume of Λ_N from 6.15. □

We shall now move on to estimating an upper bound for a related function, $U_\delta(N)$, also using similar methods.

Lemma 6.18. *For all $\delta > 0$,*

$$U_\delta(N) \ll N^{\alpha+\beta+\gamma-1-\delta} + N^\alpha + N^\beta + N^\gamma,$$

where the implicit constant depends only on p_a, p_b, p_c.

Proof. By Definition 6.10, for any $\delta > 0$, $U_\delta(N)$ counts all triples $(a, b, c) \in \mathbb{N}$ such that $a \leq N^{\alpha-\delta}$, $b \leq N^{\beta-\delta}$, $c \leq N^{\gamma-\delta}$ and $p_a^A a + p_b^B = p_c^C c$.

Let

$$\begin{aligned} R_a &= \{(x, y) \in \mathbb{R}^2 \colon 0 < x \leq N^{\alpha-\delta}, 0 < y \leq N^\beta\}, \\ R_b &= \{(x, y) \in \mathbb{R}^2 \colon 0 < x \leq N^\alpha, 0 < y \leq N^{\beta-\delta}\}, \\ T_c &= \left\{(x, y) \in \mathbb{R}^2 \colon 0 < x, 0 < y, \frac{p_a^A}{p_c^C} x + \frac{p_b^B}{p_c^C} y \leq N^{\gamma-\delta}\right\}. \end{aligned}$$

It is clear from the definitions given above that the projection of every $(a, b, c) \in \mathbb{N}^3$ onto the x, y-plane is contained in the union of R_a, R_b, T_c. Thus it follows by Lemmata 6.14, 6.16 and 6.13 that

$$\begin{aligned} U_\delta(N) \leq\ & h_c N^{\gamma-1}(\text{Volume}(R_a) + \text{Volume}(R_b) + \text{Volume}(T_c)) \\ & + O(\text{Perimeter}(R_a) + \text{Perimeter}(R_b) + \text{Perimeter}(T_c) + 1). \end{aligned}$$

It is clear that R_a, R_b are rectangles and that $\text{Volume}(R_a) = N^{\alpha+\beta-\delta}$, and $\text{Volume}(R_b) = N^{\alpha+\beta-\delta}$. Likewise, T_c is a triangle, with vertices clearly at

$$\Big(0, 0\Big), \Big(0, \frac{h_b}{h_c} N^{\beta-\delta}\Big), \text{ and } \Big(\frac{h_a}{h_c} N^{\alpha-\delta}, 0\Big),$$

so $\text{Volume}(T_c) = \frac{h_a h_b}{2h_c^2} N^{\alpha+\beta-2\delta}$. It is geometrically clear that the perimeters have similar magnitude in terms of N as the perimeters of the polygon P_N, adjusted at most by constant factors that depend on p_a, p_b, p_c. Thus it follows that

$$U_\delta(N) \leq \frac{h_a h_b}{2h_c} N^{\alpha+\beta+\gamma-1-2\delta} + 2h_c N^{\alpha+\beta+\gamma-1-\delta} + O(N^\alpha + N^\beta + N^\gamma).$$

Which implies that

$$U_\delta(N) \ll N^{\alpha+\beta+\gamma-1-\delta} + N^\alpha + N^\beta + N^\gamma.$$

□

Now, we shall follow up with some bounds on some additional functions related to F from Definition 6.11.

Lemma 6.19. *For all $d \in \mathbb{N}$ that*

$$F(N,d) \ll N^{\alpha+\beta+\gamma-1} d^{-2},$$

where the implicit coefficient depends only on the choice of primes p_a, p_b, p_c.

Proof. For $\mathbb{N}_2$ being the set of squarefree natural numbers, let

$$S(N,d) = \{(a,b,c) \in \mathbb{N}_2^3 \colon p_a^A a + p_b^B b = p_c^C c, \gcd(a,b) = d, a \le N^\alpha, b \le N^\beta, c \le N^\gamma\},$$
$$T(N,d) = \{(a,b,c) \in \mathbb{N}^3 \colon p_a^A a + p_b^B b = p_c^C c, \gcd(a,b) = 1, a \le N^\alpha/d, b \le N^\beta/d, c \le N^\gamma/d\},$$
$$V(N,d) = \{(a,b,c) \in \mathbb{N}^3 \colon p_a^A a + p_b^B b = p_c^{C-1} c, \gcd(a,b) = 1, a \le N^\alpha/d, b \le N^\beta/d, c \le p_c N^\gamma/d\}.$$

By definition, $F(N,d) = \#S(N,d)$.

First, we consider the case where d is divisible by a square. Observe that if $(a,b,c) \in S(N,d)$ then d divides a, and since d is not squarefree, a is not squarefree, which contradicts the definition of $S(N,d)$. So it follows $S(N,d) = \emptyset$ and thus $|F(N,d)| = 0$.

Now, we move onto the case where d is squarefree, and p_c does not divide d. Observe that if $(a,b,c) \in S(N,d)$, since d divides a,b there exists $a', b' \in \mathbb{N}$ such that $a/d = a', b/d = b'$ and $(a',b') = 1$. Thus $p_a^A a' d + p_b^B b' d = p_c^C c$. Since $(d, p_c) = 1$, one has that d divides c and thus there exists $c' \in \mathbb{N}$ such that $c/d = c'$. Thus by dividing all by d one has that

$$p_a^A a' + p_b^B b' = p_c^C c'; a' \le N^\alpha/d; b' \le N^\beta/d; c' \le N^\gamma/d.$$

Thus $(a',b',c') \in T(N,d)$. If one takes two triples $(a_1,b_1,c_1), (a_2,b_2,c_2) \in S(N,d)$, observe that $(a_1/d, b_1/d, c_1/d) = (a_2/d, b_2/d, c_2/d)$ if and only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$. Thus for each $(a,b,c) \in S(N,d)$ there exists a unique $(a/d, b/d, c/d) \in T(N,d)$ and thus

$$F(N,d) = \#S(N,d) \le \#T(N,d).$$

Observe that for any $(a,b,c) \in T(N,d)$, one has that

$$b \equiv p_b^{-B} p_a^A a \pmod{p_c^C}.$$

Thus there exists $\Psi_N \in \mathbb{N}$, $\Psi_N \equiv p_b^{-B} p_a^A a \pmod{p_c^C}$ with $\Psi_N < p_c^C$. Thus for any $(a,b,c) \in T(N,d)$, there exists some $k \in \mathbb{Z}_{\geq 0}$ where

$$b = \Psi_N a + p_c^C k.$$

Thus, it follows that

$$c = \frac{p_a^A}{p_c^C} a + \frac{p_b^B}{p_c^C} \Psi_N a + p_b^B k.$$

Solving for k and using the inequality $c \leq N^\gamma/d$,

$$\begin{aligned} k &\leq p_b^{-B} N^\gamma/d - \frac{p_a^A + p_b^B \Psi_N}{p_b^B p_c^C} a \\ &\leq h_b N^{\beta+\gamma-1}/d. \end{aligned}$$

Let $K(a)$ denote the number of possible values k may take for a given a. From the above inequality, it is clear that

$$K(a) \leq h_b N^{\beta+\gamma-1}/d.$$

By definition, $\#T(N,d) = \sum_{a=1}^{N^\alpha/d} K(a)$, and thus

$$\#T(N,d) = \sum_{a=1}^{N^\alpha/d} K(a) \leq \sum_{a=1}^{N^\alpha/d} h_b N^{\beta+\gamma-1}/d \leq h_b N^{\alpha+\beta+\gamma-1} d^{-2} \leq p_b N^{\alpha+\beta+\gamma-1} d^{-2}.$$

Thus it follows that $F(N,d) \ll N^{\alpha+\beta+\gamma-1}$ when $(p_c, d) = 1$.

Finally, we consider the case where d is squarefree, and p_c divides d. As in the previous case, obverse that for $(a,b,c) \in S(N,d)$, since d divides a,b there exists $a', b' \in \mathbb{N}$ such that $a/d = a', b/d = b'$ and $(a',b') = 1$. Thus $p_a^A a' d + p_b^B b' d = p_c^C c$. As d is squarefree, $(d, p_c^C) = p_c$ and thus it follows that there exists $c' \in \mathbb{N}$ such that $p_c c = c' d$. Thus, dividing by d one has that

$$p_a^A a' + p_b^B b' = p_c^{C-1} c'; a' \leq N^\alpha/d; b' \leq N^\beta/d; c' \leq p_c N^\gamma/d.$$

Thus $(a',b',c') \in V(N,d)$. It is clear that for every $(a,b,c) \in S(N,d)$ there exists a unique triple

$$(a/d, b/d, p_c c/d) \in V(N,d).$$

Thus

$$F(N,d) = \#S(N,d) \leq \#V(N,d).$$

For any $(a,b,c) \in V(N,d)$ one has that

$$b \equiv p_b^{-B} p_a^A a \pmod{p_c^{C-1}}.$$

Thus there exists $\Psi_N \in \mathbb{N}$, where $\Psi_N \equiv p_b^{-B} p_a^A a \pmod{p_c^{C-1}}$ and $\Psi_N < p_c^{C-1}$. Thus for any $(a,b,c) \in T(N,d)$, there exists some $k \in \mathbb{Z}_{\geq 0}$ such that

$$b = \Psi_N a + p_c^{C-1} k.$$

Thus it follows that

$$c = \frac{p_a^A}{p_c^{C-1}} a + \frac{p_b^B}{p_c^{C-1}} \Psi_N a + p_b^B k.$$

Solving for k with the inequality $c \leq N^\gamma/d$,

$$\begin{aligned} k &\leq p_b^{-B} N^\gamma/d - \frac{p_a^A + p_b^B \Psi_N}{p_b^B p_c^{C-1}} a \\ &\leq h_b N^{\beta+\gamma-1}/d. \end{aligned}$$

Let $K'(a)$ denote the number of possible values k may take for a given a. From the above inequality, it is clear that

$$K'(a) \leq h_b N^{\beta+\gamma-1}/d.$$

By definition, $\#V(N,d) = \sum_{a=1}^{N^\alpha/d} K'(a)$, and thus

$$\#V(N,d) = \sum_{a=1}^{N^\alpha/d} K'(a) \leq \sum_{a=1}^{N^\alpha/d} h_b N^{\beta+\gamma-1}/d \leq p_b N^{\alpha+\beta+\gamma-1} d^{-2}.$$

Thus it follows that $F(N,d) \ll N^{\alpha+\beta+\gamma-1}$ when d is squarefree and p_c divides d. □

Lemma 6.20. *For all $\epsilon > 0$,*

$$F_\epsilon(N) \ll N^{\alpha+\beta+\gamma-1-\epsilon},$$

with the implicit constant depending only on p_a, p_b, p_c.

Proof. Observe by Lemma 6.19 that for all $d \in \mathbb{N}$,

$$F(N,d) \ll N^{\alpha+\beta+\gamma-1} d^{-2},$$

with the implicit constant depending only on p_a, p_b, p_c.

From the definition of $F_\epsilon(N)$ it follows that

$$F_\epsilon(N) \leq \sum_{d \geq N^\epsilon} |F(N,d)| \ll \sum_{d \geq N^\epsilon} N^{\alpha+\beta+\gamma-1} d^{-2} \ll N^{\alpha+\beta+\gamma-1} \int_{N^\epsilon}^{\infty} x^{-2}\, dx \ll N^{\alpha+\beta+\gamma-1-\epsilon}.$$

This completes the proof. □

6.3 Major Arc Structure

This section deals with the structure of the major arcs and the harmonic arcs contained within them. We shall also derive the component of the integral of $F(N)$ that provides what will become the asymptotic formula.

First, we shall prove a fairly simple statement about the structure of the harmonic major arcs. Though the structure of many of these objects is simple, there are a large number of them, and several parameters are required to describe them. So for the sake of clarity, some simple statements such as this are given full proofs.

Lemma 6.21. *Let $a, b, c \in \mathbb{Z}$, and $q_a, q_b, q_c \in \mathbb{N}$ where $q_a \le Q_a$, $q_b \le Q_b$, $q_c \le Q_b$, and $(q_a, a) = 1$, $(q_b, b) = 1$, $(q_c, c) = 1$. If $a/(p_a^A q_a) = b/(p_b^B q_b) = c/(p_c^C q_c)$ then there exists $q \in \mathbb{N}$ $k \in \mathbb{Z}$ such that*

$$\mathfrak{M}_\alpha(q_a, a) \cap \mathfrak{M}_\beta(q_b, b) \cap \mathfrak{M}_\gamma(q_c, c) = \mathfrak{M}_H(q, k).$$

Furthermore, $\mathfrak{M}_H(q, k) \subseteq \mathfrak{M}_H$ provided $0 \le c \le p_c^C q_c$.

Proof. If $a/(p_a^A q_a) = b/(p_b^B q_b) = c/(p_c^C q_c)$, it follows that since these three rational numbers are equal, there exist $q \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $(q, k) = 1$ and $q/k = a/(p_a^A q_a) = b/(p_b^B q_b) = c/(p_c^C q_c)$. Thus it follows that

$$\frac{p_a^A k}{q} = \frac{a}{q_a}; \quad \frac{p_b^B k}{q} = \frac{b}{q_b}; \quad \frac{p_c^C k}{q} = \frac{c}{q_c}.$$

Since k/q, a/q_a, b/q_b, c/q_c are all in lowest form, it follows from the above equalities that

$$a = \frac{p_a^A k}{(p_a^A, q)}; q_a = \frac{q}{(p_a^A, q)}; \quad b = \frac{p_b^B k}{(p_b^B, q)}; q_b = \frac{q}{(p_b^B, q)}; \quad c = \frac{p_c^C k}{(p_c^C, q)}; q_c = \frac{q}{(p_c^C, q)}.$$

Thus it follows that, by the definition of $\mathfrak{M}_H(q, k)$, that

$$\begin{aligned}\mathfrak{M}_\alpha(q_a, a) \cap & \mathfrak{M}_\beta(q_b, b) \cap \mathfrak{M}_\gamma(q_c, c) \\ &= \mathfrak{M}_\alpha\left(\frac{q}{(q, p_a^A)}, \frac{p_a^A}{(q, p_a^A)} k\right) \cap \mathfrak{M}_\beta\left(\frac{q}{(q, p_b^B)}, \frac{p_b^B}{(q, p_b^B)} k\right) \cap \mathfrak{M}_\gamma\left(\frac{q}{(q, p_c^C)}, \frac{p_c^C}{(q, p_c^C)} k\right) \\ &= \mathfrak{M}_H(q, k).\end{aligned}$$

As $(q, k) = 1$, to show that $\mathfrak{M}_H(q, k) \subseteq \mathfrak{M}_H$, we must first show that $q/(q, p_a^A) \le Q_a$, $q/(q, p_b^B) \le Q_b$, and $q/(q, p_c^C) \le Q_c$, which holds since by assumption $q_a \le Q_a$, $q_b \le Q_b$, $q_c \le Q_c$. We must second show that $k \le q$ which holds since $0 \le p_c^C q_c = q/k \le 1$. This completes the proof. □

Next, we shall demonstrate that an integral over the harmonic arcs can be decomposed into several smaller integrals over the harmonic arc components, in a manner that is analogous to the standard kind of decomposition of the major arcs found in the trivial point problem.

Lemma 6.22. *Suppose* $1 \le Q_a \le \frac{1}{2}N^{\alpha/2}$, $1 \le Q_b \le \frac{1}{2}N^{\beta/2}$, $1 \le Q_c \le \frac{1}{2}N^{\gamma/2}$. *Let* $m(q) = \min\{h_a Q_a(q,p_a^A), h_b Q_b(q,p_b^B), h_c Q_c(q,p_c^C)\}/qN$. *Let* $g : \mathbb{R} \to \mathbb{C}$ *be a function periodic modulo* $\mathbb{Z}$.

Then for any $q \in \mathbb{N}$,

$$\sum_{\substack{k=0\\(q,k)=1}}^{q} \int_{\mathfrak{M}_H(q,k)} g(x+k/q)\,dx = \varphi(q)\int_{-m(q)}^{m(q)} g(x)\,dx.$$

Proof. First observe that by definition,

$$\begin{aligned}\mathfrak{M}_H(q,k) &= \mathfrak{M}_\alpha\left(\frac{q}{(q,p_a^A)}, \frac{p_a^A}{(q,p_a^A)}k\right) \cap \mathfrak{M}_\beta\left(\frac{q}{(q,p_b^B)}, \frac{p_b^B}{(q,p_b^B)}k\right) \cap \mathfrak{M}_\gamma\left(\frac{q}{(q,p_c^C)}, \frac{p_c^C}{(q,p_c^C)}k\right)\\
&= \{x \in [0,1) \colon |qp_a^A x - p_a^A k| \le (q,p_a^A)Q_a N^{-\alpha}\}\\
&\quad \cap \{x \in [0,1) \colon |qp_b^B x - p_b^B k| \le (q,p_b^B)Q_b N^{-\beta}\}\\
&\quad \cap \{x \in [0,1) \colon |qp_c^C x - p_c^C k| \le (q,p_c^C)Q_c N^{-\gamma}\}\\
&= \{x \in [0,1) \colon |qx - k| \le (q,p_a^A)h_a Q_a N^{-1}\}\\
&\quad \cap \{x \in [0,1) \colon |qx - k| \le (q,p_b^B)h_b Q_b N^{-1}\}\\
&\quad \cap \{x \in [0,1) \colon |qx - k| \le (q,p_c^C)h_c Q_c N^{-1}\}.\end{aligned}$$

Thus it follows that by the definition of $m(q)$ given above that

$$\mathfrak{M}_H(q,k) = \{x \in [0,1) \colon |qx-k| \le qm(q)\} = \{x \in [0,1) \colon |x - k/q| \le m(q)\}.$$

Also note that by the bounds on Q_a, Q_b, Q_c, that

$$\begin{aligned}qm(q) &\le \min\{h_a N^{\alpha/2}(q,p_a^A), h_b N^{\beta/2}(q,p_b^B), h_c N^{\gamma/2}(q,p_c^C)\}/(2N)\\
&\le \min\{h_a N^{\alpha/2}p_a^A, h_b N^{\beta/2}p_b^B, h_c N^{\gamma/2}p_c^C\}/(2N)\\
&\le \min\{N^{-\alpha/2}, N^{-\beta/2}, N^{-\gamma/2}\}/2 \le 1/2. \qquad (6.1)\end{aligned}$$

1. Suppose $q > 1$.

   For any $k \in \mathbb{Z}_{\ge 0}$ where $k \le q$ and $(q,k) = 1$ observe that $1 \le k < q$ since $(0,q) > 1$ and $(q,q) > 1$, and thus

   $$|q^{-1}| \le |k/q|, |q^{-1}| \le |1 - k/q|.$$

   As $qm(q) \le 1/2$ and thus $m(q) \le q^{-1}/2 < q^{-1}$, we have $|k/q| > m(q)$ and $|1-k/q| > m(q)$, so the set $\{x \in [0,1) \colon |x - k/q| \le m(q)\}$ is the closed interval $[k/q - m(q), k/q + m(q)]$. Thus it follows that

$$\begin{aligned}\sum_{\substack{k=0\\(q,k)=1}}^{q} \int_{\mathfrak{M}_H(q,k)} g(x)\,dx &= \sum_{\substack{k=0\\(q,k)=1}}^{q} \int_{k/q-m(q)}^{k/q+m(q)} g(x+k/q)\,dx \\ &= \sum_{\substack{k=0\\(q,k)=1}}^{q} \int_{-m(q)}^{m(q)} g(x)\,dx \\ &= \varphi(q)\int_{-m(q)}^{m(q)} g(x)dx.\end{aligned}$$

Thus for $q > 1$ the lemma holds.

2. Suppose $q = 1$.

First note from (6.1) $m(1) \leq 1/2$, $0 + m(1) \leq 1/2 < 1$ and $1 - m(1) \geq 1/2 > 0$, so it follows that

$$\mathfrak{M}_H(1,0) = \{x \in [0,1) \colon |x| \leq m(1)\} = [0, m(1)],$$

and

$$\mathfrak{M}_H(1,1) = \{x \in [0,1) \colon |x-1| \leq m(1)\} = [1-m(1), 1).$$

Thus it follows that since g is periodic modulo $\mathbb{Z}$ that

$$\begin{aligned}\sum_{\substack{k=0\\(q,k)=1}}^{q} \int_{\mathfrak{M}_H(q,k)} g(x-k/q)\,dx &= \int_0^{m(1)} g(x)dx + \int_{1-m(1)}^{1} g(x-1)\,dx \\ &= \int_0^{m(1)} g(x)dx + \int_{-m(1)}^{0} g(x)\,dx \\ &= \int_{-m(1)}^{m(1)} g(x)\,dx = \varphi(1)\int_{-m(1)}^{m(1)} g(x)\,dx = \varphi(q)\int_{-m(q)}^{m(q)} g(x)\,dx.\end{aligned}$$

Thus for $q = 1$ the lemma holds.

□

Ultimately, this shows that the harmonic component of the major arcs $\mathfrak{M}_H$ can be decomposed into a simple integral that does not depend on k. With the next two lemmas, we shall show that by restricting the values of Q_a, Q_b, Q_c to certain functions of N, we will have that $\mathfrak{M} = \mathfrak{M}_H$ for N sufficiently large.

Lemma 6.23. *Let $\lambda_a, \lambda_b, \lambda_c \in (0, 1/2]$, and N sufficiently large.*

1. *Suppose* $1 \leq Q_a \ll N^{\lambda_a \alpha}$ *and* $1 \leq Q_b \ll N^{\lambda_b \beta}$.

   Let $q_a, q_b \in \mathbb{N}$, $a, b \in \mathbb{Z}$ *where* $a/p_a^A q_a \neq b/p_b^B q_b$, *and* $q_a \leq Q_a$, *and* $q_b \leq Q_b$. *If* $\lambda_a \alpha + \lambda_b \beta < \alpha + \beta - 1$, *then* $\mathfrak{M}_\alpha(q_a, a) \cap \mathfrak{M}_\beta(q_b, b) = \emptyset$.

2. *Suppose* $1 \leq Q_a \ll N^{\lambda_a \alpha}$ *and* $1 \leq Q_c \ll N^{\lambda_c \gamma}$.

   Let $q_a, q_c \in \mathbb{N}$, $a, c \in \mathbb{Z}$ *where* $a/p_a^A q_a \neq c/p_c^C q_c$, *and* $q_a \leq Q_a$, *and* $q_c \leq Q_c$. *If* $\lambda_a \alpha + \lambda_c \gamma < \alpha + \gamma - 1$, *then* $\mathfrak{M}_\alpha(q_a, a) \cap \mathfrak{M}_\gamma(q_c, c) = \emptyset$.

3. *Suppose* $1 \leq Q_b \ll N^{\lambda_b \beta}$ *and* $1 \leq Q_c \ll N^{\lambda_c \gamma}$.

   Let $q_b, q_c \in \mathbb{N}$, $b, c \in \mathbb{Z}$ *where* $b/p_b^B q_b \neq c/p_c^C q_c$, *and* $q_b \leq Q_b$, *and* $q_c \leq Q_c$. *If* $\lambda_b \beta + \lambda_c \gamma < \beta + \gamma - 1$, *then* $\mathfrak{M}_\beta(q_b, b) \cap \mathfrak{M}_\gamma(q_c, c) = \emptyset$.

Proof. It suffices to prove the first case, as the other cases follow by substituting terms. As $a/p_a^A q_a \neq b/p_b^B q_b$, $|a/p_a^A q_a - b/p_b^B q_b| > 0$. Since the smallest possible nonzero difference between two integers is 1, it follows that

$$\left| \frac{a}{p_a^A q_a} - b p_b^B q_b \right| = \frac{1}{p_a^A q_a p_b^B q_b} \left| p_b^B q_b a - p_a^A q_a b \right| \geq \frac{1}{p_a^A p_b^B q_a q_b}.$$

Thus $|a/p_a^A q_a - b/p_b^B q_b| \geq h_a h_b N^{\alpha+\beta-2}/q_a q_b$. Consequently,

$$\mathfrak{M}_\alpha(q_a, a) \subseteq [a/p_a^A q_a - h_a Q_a/q_a N, a/p_a^A q_a + h_a Q_a/q_a N],$$
$$\mathfrak{M}_\beta(q_b, b) \subseteq [b/p_b^B q_b - h_b Q_b/q_b N, b/p_b^B q_b + h_b Q_b/q_b N],$$

so it suffices to show $|a/p_a^A q_a - b/p_b^B q_b| > h_a Q_a/q_a N + h_b Q_b/q_b N$ to prove that $\mathfrak{M}_\alpha(q_a, a) \cap \mathfrak{M}_\beta(q_b, b) = \emptyset$. Therefore, it suffices to show that

$$\mu = \frac{h_a h_b N^{\alpha+\beta-2} q_a^{-1} q_b^{-1}}{h_a Q_a/q_a N + h_b Q_b/q_b N} > 1.$$

Observe that if $Q_a \ll N^{\lambda_a \alpha}$ and $Q_b \ll N^{\lambda_b \beta}$, then since $Q_a, Q_b \geq 1$,

$$\mu = \frac{h_a h_b N^{\alpha+\beta-1}}{h_a Q_a q_b + h_b q_a Q_b} \geq \frac{h_a h_b N^{\alpha+\beta-1}}{(h_a + h_b) Q_a Q_b} \gg N^{(1-\lambda_a)\alpha + (1-\lambda_b)\beta - 1}.$$

Thus by our hypothesis, it follows that for N sufficiently large that $\mu > 1$ and thus $\mathfrak{M}_\alpha(q_a, a) \cap \mathfrak{M}_\beta(q_b, b) = \emptyset$. □

Now, we use the above lemma to prove that $\mathfrak{M} = \mathfrak{M}_H$ for N sufficiently large when a special condition is satisfied. Recall Definition 6.12 of $D_H(\lambda_a, \lambda_b, \lambda_c)$, the harmonic lambda factor, and the related definition for weakly and strongly harmonic lambdas.

Lemma 6.24. *Suppose $Q_a \ll N^{\lambda_a \alpha}$, $Q_b \ll N^{\lambda_b \beta}$, and $Q_c \ll N^{\lambda_c \gamma}$. If $D_H(\lambda_a, \lambda_b, \lambda_c)$ is weakly harmonic, then for N sufficiently large, $\mathfrak{M} = \mathfrak{M}_H$.*

Proof. First, note that since $\mathfrak{M}_H \subseteq \mathfrak{M}$ by definition, it suffices to prove that $\mathfrak{M} \subseteq \mathfrak{M}_H$.

Since $\max D_H(\lambda_a, \lambda_b, \lambda_c) > 1$ (the weakly harmonic condition) then by Lemma 6.23 it follows that for any $q_a, q_b, q_c \in \mathbb{N}$, $a, b, c \in \mathbb{Z}$ where $a/p_a^A q_a \neq b/p_b^B q_b$ or $a/p_a^A q_a \neq c/p_c^C q_c$ or $b/p_b^B q_b \neq c/p_c^C q_c$, and $q_a \leq Q_a$, $q_b \leq Q_b$, and $q_c \leq Q_c$, that

$$\mathfrak{M}_\alpha(q_a, a) \cap \mathfrak{M}_\beta(q_b, b) \cap \mathfrak{M}_\gamma(q_c, c) = \emptyset \subset \mathfrak{M}_H.$$

The only other components of $\mathfrak{M}$ remaining are those where $a/p_a^A q_a = b/p_b^b q_b = c/p_c^C q_c \in [0, 1]$, where $q_a \leq Q_a$, $q_b \leq Q_b$, and $q_c \leq Q_c$, a/q_a, b/q_b, c/q_c in lowest form, which are by Lemma 6.21 all in $\mathfrak{M}_H$. Thus it follows that $\mathfrak{M}_H = \mathfrak{M}$. □

The following lemma is a useful tool for translating between different kinds of integral, and will be used in later sections to compute certain error terms.

Lemma 6.25. *Let $g \colon \mathbb{R} \to \mathbb{C}$ be a function periodic modulo $\mathbb{Z}$ (that is, $g(x) = g(x + n)$ for any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$). Then,*

$$\int_{\mathfrak{M}_\alpha} g(p_a^A x) dx = \int_{\mathfrak{M}^\star(Q_a, N^\alpha)} g(x) dx,$$
$$\int_{\mathfrak{M}_\beta} g(p_b^B x) dx = \int_{\mathfrak{M}^\star(Q_b, N^\beta)} g(x) dx,$$
$$\int_{\mathfrak{M}_\gamma} g(-p_c^C x) dx = \int_{\mathfrak{M}^\star(Q_c, N^\beta} g(-x) dx.$$

Proof. It suffices to prove only the first equation. Proofs for the other equations can easily be constructed by substituting terms.

Observe that by Definitions 6.3 and 6.7 of $\mathfrak{M}_\alpha(q, a)$ and $\mathfrak{M}^\star(q, a; Q_a; N^\alpha)$ respectively that $x \in \mathfrak{M}_\alpha(q, a)$ if and only if $p_a^A x \in \mathfrak{M}^\star(q, a; Q_a; N^\alpha)$. Thus by the definition of $\mathfrak{M}_\alpha$ and $\mathfrak{M}^\star(Q_a; N^\alpha)$, it follows that $x \in \mathfrak{M}_\alpha$ if and only if $p_a^A x \in (\mathfrak{M}^\star(Q_a, N^\alpha) + k)$ for some integer k where $0 \leq k < p_a^A$.

Consequently note that $p_a^A \cdot \mathfrak{M}_\alpha$, the set produced by pointwise multiplication of the elements of $\mathfrak{M}_\alpha$ is precisely the set

$$S = \bigcup_{n=0}^{p_a^A - 1} (\mathfrak{M}^\star(Q_a, N^\alpha) + n),$$

where $\mathfrak{M}_\alpha + n$ is the set produced by pointwise addition of n. Thus it follows that by the substitution $u = p_a^A x$,

$$\begin{aligned}\int_{\mathfrak{M}_\alpha} g(p_a^A x)\,dx &= p_a^{-A} \int_{p_a^A \mathfrak{M}^\star(Q_a,N^\alpha)} g(u)\,du \\ &= p_a^{-A} \sum_{n=0}^{p_a^A-1} \int_{\mathfrak{M}^\star(Q_a,N^\alpha)+n} g(u)\,du \\ &= p_a^{-A} \sum_{n=0}^{p_a^A-1} \int_{\mathfrak{M}^\star(Q_a,N^\alpha)} g(u-n)\,du \\ &= p_a^{-A} \sum_{n=0}^{p_a^A-1} \int_{\mathfrak{M}^\star(Q_a,N^\alpha)} g(u)\,du = \int_{\mathfrak{M}^\star(Q_a,N^\alpha)} g(x)\,dx.\end{aligned}$$

This completes the proof. □

We now move on to the triple product of the $G(q)$ function, $G_3^\star(q)$. Note that by Definition 6.8 that this function is also technically a function of N, and depends on the parameters α, β, γ and p_a, p_b, p_c.

We will make use of the following definition in the next lemma:

Definition 6.26. Let $P \subseteq \{p_a, p_b, p_c\}$ be the set such that $p_a \in P$ if and only if $\alpha < 1$, and $p_b \in P$ if and only if $\beta < 1$, and $p_c \in P$ if and only if $\gamma < 1$.

Lemma 6.27. *The function $G_3^\star(q)$ is a multiplicative function, and for any prime p, and integer $\ell \in \mathbb{Z}_{\geq 0}$, for N sufficiently large,*

$$G_3^\star(p^\ell) = \begin{cases} -1/(p^2-1)^3 & \text{if } p \notin P \text{ and } 1 \leq \ell \leq 2, \\ 1/(p^2-1)^2 & \text{if } p \in P \text{ and } 1 \leq \ell \leq 2, \\ 0 & \text{if } \ell > 2. \end{cases}$$

Furthermore, when N is sufficiently large, $G_3^\star(q) \ll |G(q)|^3$, with the sufficiently large condition and implicit coefficients depending only on p_a, p_b, p_c.

Proof. Let $N^{\max\{1-\alpha,1-\beta,1-\gamma\}/2} \geq \sup P$. This is our sufficiently large condition. Note the choice of $\sup P$ allows for the condition to be trivially true when $P = \emptyset$.

Let p be any prime. When $\ell > 2$, observe that

$$G_3^\star(p^\ell) = G(p^\ell/(p^\ell, p_a^A))G(p^\ell/(p^\ell, p_b^B))G(p^\ell/(p^\ell, p_c^C)),$$

and as p is relatively prime to at least two of p_a, p_b, p_c and since $G(p^\ell) = 0$ it follows that $G_3^\star(p^\ell) = 0$. Thus the rest of the proof need only concern the cases where $0 \leq \ell \leq 2$.

Now let $p \in P$, $\ell \in \{1,2\}$. Observe that $G_3^\star(p^\ell) = G(p^\ell/(p^\ell, p_a^A))G(p^\ell/(p^\ell, p_b^B))G(p^\ell/(p^\ell, p_c^C))$. Also note that p is equal to one of p_a, p_b, p_c and relatively prime to the other two. Thus if N is large enough that $p_a \in P$ implies $p_a^2 \mid p_a^A$, $p_b \in P$ implies $p_b^2 \mid p_b^B$, and $p_c \in P$ implies $p_c^2 \mid p_c^C$, it follows that

$$G_3^\star(p^\ell) = G(p^\ell)^2 G(1) = 1/(p^2-1)^2.$$

Now suppose $p \notin P$, $\ell \in \{1,2\}$. If $p = p_a$ then $A = 0$ and so $(p^\ell, p_a^A) = 1$ regardless of p. Likewise, $(p^\ell, p_b^B) = 1$ and $(p^\ell, p_c^C) = 1$ regardless of p. Thus

$$G_3^\star(p^\ell) = G(p^\ell)^3 = -1/(p^2-1)^3.$$

To show $G_3^\star(q) \ll |G(q)|^3$ it suffices to observe that as $N^{\max\{1-\alpha,1-\beta,1-\gamma\}/2} \geq \sup P$.

$$\begin{aligned} G_3^\star(q) &= G\left(\frac{q}{(q,p_a^A)}\right) G\left(\frac{q}{(q,p_b^B)}\right) G\left(\frac{q}{(q,p_c^C)}\right) \\ &\leq (p_a^2-1)(p_b^2-1)(p_c^2-1)|G(q)|^3. \end{aligned}$$

□

We now show that just as the infinite sum $\sum \varphi(q)G(q)^3 = \omega$, that for N sufficiently large, the infinite sum $\sum \varphi(q)G_3^\star(q)$ stabilizes to a value close to ω.

Lemma 6.28. *Let P be as in Definition 6.26, and ω the number calculated in Lemma 4.7. For N sufficiently large,*

$$\sum_{q=1}^{R} \varphi(q)G_3^\star(q) = \left(\prod_{p\in P} \frac{p^2-1}{p^2-2}\right) \omega + O(R^{-3/2}).$$

Proof. Recall from Lemma 4.7 that the series

$$\omega = \sum_{q=1}^{\infty} \varphi(q)G(q)^3$$

converges and can be rewritten as an Euler product,

$$\omega = \prod_p (1 + \phi(p)G(p)^3 + \varphi(p^2)G(p^2)^3).$$

For N sufficiently large, Lemma 6.27 implies

$$\sum_{q=1}^{\infty} \phi(q)G_3^\star(q) \ll \sum_{q=1}^{\infty} \phi(q)G(q)^3 \ll 1,$$

and thus one can take the Euler product of $G_3^\star$:

$$\begin{aligned}\sum_{q=1}^{\infty}\varphi(q)G_3^\star(1) &= \prod_p (1+\phi(p)G_3^\star(p)+\varphi(p^2)G_3^\star(p^2))\\
&= \left(\prod_{p\in P}\left(1+\varphi(p)G_3^\star(p)+\varphi(p^2)G_3^\star(p^2)\right)\right)\left(\prod_{p\notin P}\left(1+\varphi(p)G_3^\star(p)+1+\varphi(p^2)G_3^\star(p^2)\right)\right)\\
&= \left(\prod_{p\in P}\left(1+\frac{p-1}{(p^2-1)^2}+\frac{p(p-1)}{(p^2-1)^2}\right)\right)\left(\prod_{p\notin P}\left(1+\varphi(p)G(p)^3+\varphi(p^2)G(p^2)^3\right)\right)\\
&= \left(\prod_{p\in P}\left(1+\frac{1}{p^2-1}\right)(1+\varphi(p)G(p)^3+\varphi(p^2)G(p^2)^3)^{-1}\right)\omega\\
&= \omega\prod_{p\in P}\left(1+\frac{1}{p^2-1}\right)\left(1-\frac{1}{(p^2-1)^2}\right)^{-1} = \omega\prod_{p\in P}\frac{p^2-1}{p^2-2}.\end{aligned}$$

Observing that by Lemmata 6.27 and 4.6, for N sufficiently large,

$$\sum_{q>Q}\varphi(q)G_3^\star(q) \ll \sum_{q\geq Q}|\varphi(q)G(q)^3| \ll Q^{-3/2},$$

so we have

$$\omega\prod_{p\in P}\left(\frac{p^2-1}{p^2-2}\right) = \sum_{q=1}^{\infty}\varphi(q)G_3^\star(q) = \sum_{q=1}^{Q}\varphi(q)G_3^\star(q)+O(Q^{-3/2}).$$

The proof follows immediately on rearrangement. □

Finally, we use the previous lemmas in this section to calculate what will eventually become the asymptotic formula of $F(N)$.

Lemma 6.29. *If* $Q_0=\min\{Q_a,Q_b,Q_c\}$*, then*

$$\int_{\mathfrak{M}_H} f_\alpha^\star(p_a^A x)f_\beta^\star(p_b^B x)f_\gamma^\star(-p_c^C x)\,dx = \zeta(2)^{-3}\left(\sum_{q\leq Q_0}\varphi(q)G_3^\star(q)\right)U(N)+O(N^{\alpha+\beta+\gamma-1}Q_0^{-3/2}).$$

Proof. Let $\tilde{Q}=\{q\in\mathbb{N}\colon q\leq\min\{(q,p_a^A)Q_a,(q,p_b^B)Q_b,(q,p_c^C)Q_c\}\}$, so that

$$\mathfrak{M}_H = \bigcup_{q\in\tilde{Q}}\bigcup_{\substack{k=0\\(q,k)=1}}^{q}\mathfrak{M}_H(q,k).$$

Next, we break the integral of $\mathfrak{M}_H$ into two components,

$$\int_{\mathfrak{M}_H} f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x) f_\gamma^\star(-p_c^C x)\,dx$$
$$= \sum_{q \le Q_0} \sum_{\substack{k=0\\(q,k)=1}}^{q} \int_{\mathfrak{M}_H(q,k)} f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x) f_\gamma^\star(-p_c^C x)\,dx$$
$$+ \sum_{\substack{q>Q_0\\ q\in\tilde{Q}}} \sum_{\substack{k=0\\(q,k)=1}}^{q} \int_{\mathfrak{M}_H(q,k)} f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x) f_\gamma^\star(-p_c^C x)\,dx$$
$$= \sum_{q \le Q_0} \sum_{\substack{k=0\\(q,k)=1}}^{q} \int_{\mathfrak{M}_H(q,k)} f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x) f_\gamma^\star(-p_c^C x)\,dx + E_H,$$

where E_H is the second double sum in the centre expression.

Define

$$m(q) = \min\left\{h_a Q_a(q,p_a^A), h_b Q_b(q,p_b^B), h_c Q_c(q,p_c^C)\right\}/(qN).$$

Observe that by Lemma 6.27 if $G_3^\star(q) \neq 0$, $m(q) \le \max\{p_a^2, p_b^2, p_c^2\} h_0 Q_0/qN$, so $m(q) = O(Q_0/qN)$. Also note that regardless of $G_3^\star(q)$,

$$m(q)^{-1} = qN \max\left\{\frac{1}{h_a Q_a(q,p_a^A)}, \frac{1}{h_b Q_b(q,p_b^B)}, \frac{1}{h_c Q_c(q,p_c^C)}\right\} \le qN/Q_0,$$

so it follows that $m(q)^{-1} \ll qN/Q_0$.

For fixed $q \in \tilde{Q}$, we apply Lemma 6.22 with

$$g(x-k/q) = I_\alpha(p_a^A x - p_a^A k/q) I_\beta(p_b^B x - p_b^B k/q) I_\gamma(p_c^C x - p_c^C k/q)$$

to show that:

$$\sum_{\substack{k=0\\(q,k)=1}}^{q} \int_{\mathfrak{M}_H(q,k)} f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x) f_\gamma^\star(-p_c^C x) dx$$
$$= \zeta(2)^{-3}\varphi(q) G\left(\frac{q}{(q,p_a^A)}\right) G\left(\frac{q}{(q,p_b^B)}\right) G\left(\frac{q}{(q,p_c^C)}\right) \int_{-m(q)}^{m(q)} I_\alpha(p_a^A x) I_\beta(p_b^B x) I_\gamma(-p_c^C x) dx$$
$$= \zeta(2)^{-3}\varphi(q) G_3^\star(q) \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} I_\alpha(p_a^A x) I_\beta(p_b^B x) I_\gamma(-p_c^C x) dx + O\left(\int_{m(q)}^{\frac{1}{2}} \|p_a^{-A} p_b^{-B} p_c^{-C} x^{-3}\| dx\right)\right)$$
$$= \zeta(2)^{-3}\varphi(q) G_3^\star(q) \left(\int_0^1 I_\alpha(p_a^A x) I_\beta(p_b^B x) I_\gamma(-p_c^C x) dx + O\left(N^{\alpha+\beta+\gamma-3}\int_{m(q)}^{\frac{1}{2}} \|x\|^{-3} dx\right)\right)$$
$$= \zeta(2)^{-3}\varphi(q) G_3^\star(q) \left(U(N) + O(N^{\alpha+\beta+\gamma-3} m(q)^{-2})\right)$$
$$= \zeta(2)^{-3}\varphi(q) G_3^\star(q) \left(U(N) + O(q^2 N^{\alpha+\beta+\gamma-1} Q_0^{-2})\right).$$

Thus by Lemmata 6.27 and 4.6 one has that for all $\epsilon > 0$:

$$\begin{aligned}\sum_{q\le Q_0}\varphi(q)G_3^\star(q)(q^2N^{\alpha+\beta+\gamma-1}Q_0^{-2}) &\ll \sum_{q\le Q_0} q^3|G(q)|^3N^{\alpha+\beta+\gamma-1}Q_0^{-2}\\ &\ll N^{\alpha+\beta+\gamma-1}Q_0^{-2}\sum_{q\le Q_0}q^3|G(q)|^3\\ &\ll N^{\alpha+\beta+\gamma-1+\epsilon}Q_0^{-3/2}.\end{aligned}$$

Thus it follows that

$$\int_{\mathfrak{M}_H} f_\alpha^\star(p_a^Ax)f_\beta^\star(p_b^Bx)f_\gamma^\star(-p_c^Cx)dx = \sum_{q\le Q_0}\left(\zeta(2)^{-3}\phi(q)G_3^\star(q)U(N)\right) + O(N^{\alpha+\beta+\gamma-1}Q_0^{-3/2}) + E_H,$$

Thus to prove the lemma, it now suffices to show that $E_H \ll N^{\alpha+\beta+\gamma-1}Q_0^{-3/2}$. First, observe that for any $q\in\tilde{Q}$, if $G_3^\star(q)\neq 0$ it follows by Lemma 6.27 that $(q,p_a^A)\le p_a^2, (q,p_b^B)\le p_b^2, (q,p_c^C)\le p_c^2$, so $q\le p_0^2Q_0$ and $q\ll Q_0$. Consequently it follows by Lemma 6.27 that

$$\begin{aligned}E_H &\le \zeta(2)^{-3}\sum_{Q_0<q\le p_0^2Q_0}\varphi(q)G_3^\star(q)\int_{-m(q)}^{m(q)} I_\alpha(x)I_\beta(x)I_\gamma(x)dx\\ &\ll \sum_{Q_0<q\le p_0^2Q_0} qG_3^\star(q)N^{\alpha+\beta+\gamma}m(q)\\ &\ll \sum_{Q_0<q\le p_0^2Q_0}|G(q)|^3N^{\alpha+\beta+\gamma-1}Q_0 \ll N^{\alpha+\beta+\gamma-1}Q_0\sum_{Q_0<q\le p_0^2Q_0}|G(q)|^3.\end{aligned}$$

Now observe that by Lemma 4.6, one has that for all $\epsilon > 0$

$$\sum_{Q_0<q\le p_0^2Q_0}|G(q)|^3 \ll Q_0^{-5/2+\epsilon}.$$

Combining the two relations gives the result that $E_H \ll N^{\alpha+\beta+\gamma-1}Q_0^{-3/2}$. □

The formula given above can be simplified further, leading to the following, final lemma of this section.

Lemma 6.30. *Let $Q_0 = \min\{Q_a, Q_b, Q_c\}$ and P be as Definition 6.26. Then*

$$\int_{\mathfrak{M}_H} f_\alpha^\star(p_a^Ax)f_\beta^\star(p_b^Bx)f_\gamma^\star(-p_c^Cx)dx = \omega\zeta(2)^{-3}\left(\prod_{p\in P}\frac{p^2-1}{p^2-2}\right)U(N) + O(N^{\alpha+\beta+\gamma-1}Q_0^{-3/2}).$$

Proof. This follows from Lemmata 6.28 and 6.29, and observing that the hypotheses for both are satisfied once N is sufficiently large, and removing the smaller error term. □

The above lemma contains the core of the integral, the component that will be the main term in the future. The magnitude of the main term is $N^{\alpha+\beta+\gamma-1}$, which can be seen by combining it with Lemma 6.17. Observe that the error on this term is always strictly smaller than the main term as long as there exists some $\delta > 0$ such that $Q_0 \gg N^{\delta}$. Practically speaking, this will always hold in this work.

6.4 Bounds of Functions on Component Arcs

We now move on to a discussion of some technical lemmas which will be used to determine the sizes of various error terms.

First, we estimate the delta functions over their component major arcs.

Lemma 6.31. *For all $\epsilon > 0$,*

$$\int_{\mathfrak{M}_\alpha} |\Delta_\alpha(p_a^A x)|^3 dx \ll N^{\alpha+\epsilon} Q_a^2,$$
$$\int_{\mathfrak{M}_\beta} |\Delta_\beta(p_b^B x)|^3 dx \ll N^{\beta+\epsilon} Q_b^2,$$
$$\int_{\mathfrak{M}_\gamma} |\Delta_\gamma(-p_c^C x)|^3 dx \ll N^{\gamma+\epsilon} Q_c^2.$$

Proof. The proof for the first equation is easily generalized to the other two equations, so it suffices to prove only the first one. By Lemma 6.25,

$$\int_{\mathfrak{M}_\alpha} |\Delta_\alpha(p_a^A x)|^3 dx = \int_{\mathfrak{M}^\star(Q_a, N^\alpha)} |\Delta_\alpha(x)|^3 dx.$$

First, note that as $f_\alpha, f_\alpha^\star$ are exponential sums with $N^\alpha + O(1)$ terms,

$$|\Delta_\alpha(x)| = |f_\alpha(x) - f_\alpha^\star(x)| \leq |f_\alpha(x)| + |f_\alpha^\star(x)| \ll N^\alpha,$$

so $\sup |\Delta_\alpha(x)| \ll N^\alpha$. By application of Lemma 3.2 from [2], after taking the sup-norm one has that for any $\epsilon > 0$,

$$\int_{\mathfrak{M}^\star(Q_a, N^\alpha)} |\Delta_\alpha(x)|^3 dx \leq \sup_{x \in \mathfrak{M}^\star(Q_a, N^\alpha)} |\Delta_\alpha(x)| \int_{\mathfrak{M}^\star(Q_a, N^\alpha)} |\Delta_\alpha(x)|^2 \, dx \ll N^{\alpha+\epsilon} Q_a^2.$$

This completes the proof. □

The above bound for the triple Δ component may not be ideal, and, as will be seen later, this bound will be one of the most significant contributors to the size of the error terms.

Next, we estimate the $f^\star$ functions over their component arcs.

Lemma 6.32. *For all* $\epsilon > 0$,

$$\int_{\mathfrak{M}_\alpha} |f_\alpha^\star(p_a^A x)|^3 dx \ll N^{2\alpha},$$
$$\int_{\mathfrak{M}_\beta} |f_\beta^\star(p_b^B x)|^3 dx \ll N^{2\beta},$$
$$\int_{\mathfrak{M}_\gamma} |f_\gamma^\star(-p_c^C x)|^3 dx \ll N^{2\gamma}.$$

Proof. The proof of the three equations is symmetric, so it suffices to prove only the first one. By Lemma 6.25,

$$\int_{\mathfrak{M}_\alpha} |f_\alpha^\star(p_a^A x)|^3 dx = \int_{\mathfrak{M}^\star(Q_a,N^\alpha)} |f_\alpha^\star(x)|^3 dx.$$

Note that $f_\alpha^\star(x)$ is an exponential sum with $N^\alpha + O(1)$ terms so $\sup |f_\alpha^\star(x)| \ll N^\alpha$. By application of Lemma 4.1 from [2], after taking the sup-norm one has that for any $\epsilon > 0$,

$$\int_{\mathfrak{M}^\star(Q_a,N^\alpha)} |f_\alpha^\star(x)|^3 dx \leq \sup_{x \in \mathfrak{M}^\star(Q_a,N^\alpha)} |f_\alpha^\star(x)| \int_{\mathfrak{M}^\star(Q_a;N^\alpha)} |f_\alpha^\star(x)|^2 dx \ll N^{2\alpha}(1 + Q_a^{-1/2+\epsilon}).$$

This completes the proof. □

The $f^\star$ bounds are the best possible bounds, as we know the exponential sums get large on portions of the arc.

Next, we provide a function for converting from our minor arc components to minor arcs as defined in [2].

Lemma 6.33. *Let* $g\colon \mathbb{R} \to \mathbb{C}$ *be a function periodic modulo* $\mathbb{Z}$. *Then*

$$\int_{\mathfrak{m}_\alpha} g(p_a^A x)\, dx = \int_{\mathfrak{m}^\star(Q_a;N^\alpha)} g(x)\, dx,$$
$$\int_{\mathfrak{m}_\beta} g(p_b^B x)\, dx = \int_{\mathfrak{m}^\star(Q_b;N^\beta)} g(x)\, dx,$$
$$\int_{\mathfrak{m}_\gamma} g(-p_c^C x)\, dx = \int_{\mathfrak{m}^\star(Q_c;N^\gamma)} g(-x)\, dx.$$

Proof. Since the proof of each equation is symmetric, it suffices to prove only the first. Observe that by Lemma 6.25 that

$$\int_{\mathfrak{M}_\alpha} g(p_a^A x)\, dx = \int_{\mathfrak{M}^\star(Q_a,N^\alpha)} g(x)\, dx.$$

Also note that by substitution of $u = p_a^A x$ that,

$$\int_0^1 g(p_a^A x)\, dx = p_a^{-A} \int_0^{p_a^A} g(u)\, du = \int_0^1 g(x)\, dx.$$

Now, observe that

$$\int_{\mathfrak{m}_\alpha} g(p_a^A x)\,dx = \int_0^1 g(p_a^A x)dx - \int_{\mathfrak{M}_\alpha} g(p_a^A x)\,dx$$
$$= \int_0^1 g(x)dx - \int_{\mathfrak{M}^\star(Q_a;N^\alpha)} g(x)\,dx = \int_{\mathfrak{m}^\star(Q_a;N^\alpha)} g(x)\,dx.$$

This completes the proof. □

We now provide a bound on certain products of functions related to $G(q)$. This is much like the earlier proof for $G_3^\star(q)$, but only an upper bound is needed as these only appear in certain error terms.

Lemma 6.34. *Uniformly as a function of N,*

$$G\left(\frac{q}{(q,p_a^A)}\right)G\left(\frac{q}{(q,p_b^B)}\right) \ll |G(q)|^2,$$
$$G\left(\frac{q}{(q,p_a^A)}\right)G\left(\frac{q}{(q,p_c^C)}\right) \ll |G(q)|^2,$$
$$G\left(\frac{q}{(q,p_b^B)}\right)G\left(\frac{q}{(q,p_c^C)}\right) \ll |G(q)|^2.$$

Proof. It suffices to prove only the first equation as the proofs for the other two are symmetric. For any q, let $q' \in \mathbb{N}$, $k_a, k_b \in \mathbb{Z}_{\geq 0}$ where

$$q = p_a^{k_a} p_b^{k_b} q'; \quad \gcd(p_a, q') = \gcd(p_b, q') = 1.$$

If $k_a > 2$ then $G(q/(q,p_b^B)) = 0$. If $k_b > 2$ then $G(q/(q,p_a^A)) = 0$.

Thus if $G(q/(q,p_a^A))G(q/(q,p_b^B)) \neq 0$, then $k_a \leq 2$, $k_b \leq 2$. Thus $p_a^2 p_b^2 q' \geq q$, so

$$|G(q/(q,p_a^A))G(q/(q,p_b^B))| \leq |G(q')|^2 \leq |G(q)|^2(|G(p_a^2)G(p_b^2)|)^{-1}.$$

Since $|G(p_a^2)G(p_b^2)|$ is a constant, it follows that $G(q/(q,p_a^A))G(q/(q,p_b^B)) \ll |G(q)|^2$. □

Just as a bound for products of two values of G will be needed for a component of the error term, an estimate for an integral of a pair of I functions over the unit interval will be needed as well.

Lemma 6.35. *For all $\epsilon > 0$,*

$$\int_0^1 |I_\alpha(p_a^A x) I_\beta(p_b^B x)|^2\,dx \ll N^{2\alpha+2\beta-1} + N^{\alpha+\beta},$$
$$\int_0^1 |I_\alpha(p_a^A x) I_\gamma(p_c^C x)|^2\,dx \ll N^{2\alpha+2\gamma-1} + N^{\alpha+\beta},$$
$$\int_0^1 |I_\beta(p_b^B x) I_\gamma(p_c^C x)|^2\,dx \ll N^{2\beta+2\gamma-1} + N^{\alpha+\beta}.$$

Proof. It suffices to prove only the first equation as the arguments for the second and third are symmetric. Observe that $\int_0^1 |I_\alpha(p_a^A x)I_\beta(p_b^B x)|^2\,dx$ is clearly a counting function which counts the elements of the set

$$S = \{(x,y,z,w) \in \mathbb{N} \colon p_a^A x + p_b^B y = p_a^A z + p_b^B w, x, z \le N^\alpha, y, w \le N^\beta\}.$$

Observe that for any $(x,y,z,w) \in S$, one has that $p_a^A x \equiv p_a^A z \pmod{p_b^B}$, and since p_a, p_b are relatively prime, it follows that $x \equiv z \pmod{p_b^B}$. Thus $z = x + p_b^B k$ for some $k \in \mathbb{Z}$. Similarly, $y \equiv w \pmod{p_a^A}$, so there exists $\ell \in \mathbb{Z}$ such that $w = y + p_a^A \ell$.

Rewriting in terms of k, ℓ instead of z, w, the equation becomes

$$p_a^A x + p_b^B y = p_a^A x + p_a^A p_b^B k + p_b^B y + p_a^A p_b^B \ell,$$

and thus after simplifying it follows that $\ell = -k$. So one can represent $(x,y,z,w) \in S$ uniquely with just (x,y) and k. Observe that as $1 \le z \le N^\alpha$, $1 \le x + p_b^B k \le N^\alpha$, so

$$\frac{1-x}{p_b^B} \le k \le h_b N^{\alpha+\beta-1} - \frac{x}{p_b^B}$$

If $\alpha + \beta - 1 > 0$, there are $h_b N^{\alpha+\beta-1} + O(1)$ possible values for k to take. For x, y, there are at most $N^{\alpha+\beta}$ possible values to take, so therefore $\#S \ll N^{2\alpha+2\beta-1}$. If $\alpha + \beta \le 1$, there is only one value of k, $k = 0$ that is valid. Hence $\#S = N^{\alpha+\beta}$. The desired result follows immediately. □

We now move on an observation about the structure of the semi-major arcs that will allow for substantial reduction of the error term.

Lemma 6.36. *Suppose there exist $\lambda_a, \lambda_b, \lambda_c \in [0, 1/2]$ such that*

$$Q_a \ll N^{\lambda_a \alpha}; \quad Q_b \ll N^{\lambda_b \beta}; \quad Q_c \ll N^{\lambda_c \gamma}.$$

If $D_H(\lambda_a, \lambda_b, \lambda_c)$ is strongly harmonic, $\partial\mathfrak{M}_H = \partial\mathfrak{M}$.

Proof. First observe by Definition 6.6 that $\partial\mathfrak{M}_H$ is trivially a subset of $\partial\mathfrak{M}$, since $\partial\mathfrak{M}_H$ is a union of subsets of $\partial\mathfrak{M}$, so to prove equality it suffices to show that $\partial\mathfrak{M} \subseteq \partial\mathfrak{M}_H$.

If $\min D_H(\lambda_a, \lambda_b, \lambda_c) > 1$ then every inequality in Lemma 6.23 is satisfied. Thus for all $q_a, q_b, q_c \in \mathbb{N}$, and $a, b, c \in \mathbb{Z}$, where $(q_a, a) = (q_b, b) = (q_c, c) = 1$ and $a \le p_a^A q_a$, and $b \le p_b^B q_b$, and $c \le p_c^C q_c$, one has that

$$\mathfrak{M}_\alpha(q_a, a) \cap \mathfrak{M}_\beta(q_b, b) \neq \emptyset \text{ iff } \frac{a}{p_a^A q_a} = \frac{b}{p_b^B q_b}$$
$$\mathfrak{M}_\alpha(q_a, a) \cap \mathfrak{M}_\gamma(q_c, c) \neq \emptyset \text{ iff } \frac{a}{p_a^B q_a} = \frac{c}{p_c^C q_c}$$
$$\mathfrak{M}_\beta(q_b, b) \cap \mathfrak{M}_\gamma(q_c, c) \neq \emptyset \text{ iff } \frac{b}{p_b^B q_b} = \frac{c}{p_c^C q_C}.$$

Since by definition, $\partial\mathfrak{M}$ is a subset of the union of the sets of one of the three the forms above, and those sets are non-empty if and only if they lie in the harmonic semi-majors, it follows that $\partial\mathfrak{M} \subseteq \partial\mathfrak{M}_H$, thus completing the proof. □

This suffices as a set of tools to resolve most of the components of the error term that remain. We shall now move on to actually calculating the various components of the error term.

6.5 Triple Δ Error Terms

In this section, we compute the error terms over components of the major, semi-major, and semi-minor arcs that have the form $\int |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(p_c^C x)|\, dx$.

All the proofs in this section are straightforward applications of Hölder's inequality to decompose the integral into components resolvable by lemmata in the previous sections, followed by a final lemma which summarizes the overall result.

First, we compute them over the major arcs:

Lemma 6.37. *For all* $\epsilon > 0$*,*

$$\int_{\mathfrak{M}} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|dx \ll N^{(\alpha+\beta+\gamma)/3+\epsilon} Q_a^{2/3} Q_b^{2/3} Q_c^{2/3}.$$

Proof. Recall that $\mathfrak{M} \subseteq \mathfrak{M}_\alpha$. Thus by Hölder's inequality

$$\begin{aligned}
&\int_{\mathfrak{M}} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|dx \\
&\quad\le \left(\int_{\mathfrak{M}_\alpha} |\Delta_\alpha(p_a^A x)|^3 dx \int_{\mathfrak{M}_\beta} |\Delta_\beta(p_b^B x)|^3 dx \int_{\mathfrak{M}_\gamma} |\Delta_\gamma(-p_c^C x)|^3 dx \right)^{1/3}.
\end{aligned}$$

To complete the proof, simply apply Lemma 6.31 to the above equation. □

Second, we compute them over the semi-major arcs:

Lemma 6.38. *For all* $\epsilon > 0$ *one has that*

$$\begin{aligned}
&\int_{\partial\mathfrak{M}} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|dx \\
&\quad\ll N^{(\alpha+\beta+\gamma)/3+\epsilon} Q_a^{2/3} Q_b^{2/3} (N^{\gamma/3} Q_c^{-1/2} + Q_c^{1/3}) \\
&\qquad + N^{(\alpha+\beta+\gamma)/3+\epsilon} Q_a^{2/3} Q_c^{2/3} (N^{\beta/3} Q_b^{-1/2} + Q_b^{1/3}) \\
&\qquad + N^{(\alpha+\beta+\gamma)/3+\epsilon} Q_b^{2/3} Q_c^{2/3} (N^{\alpha/3} Q_a^{-1/2} + Q_a^{1/3}).
\end{aligned}$$

Proof. First observe that by the definition of the semimajor arcs,

$$\int_{\partial\mathfrak{M}} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(p_c^c x)|dx$$
$$= \int_{\mathfrak{M}_\alpha\cap\mathfrak{M}_\beta\cap\mathfrak{m}_\gamma} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|dx$$
$$+ \int_{\mathfrak{M}_\alpha\cap\mathfrak{m}_\beta\cap\mathfrak{M}_\gamma} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|dx$$
$$+ \int_{\mathfrak{m}_\alpha\cap\mathfrak{M}_\beta\cap\mathfrak{M}_\gamma} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|dx.$$

Thus by Hölder's inequality, and Lemmata 6.25, 6.33, 5.1, and 6.31 that:

$$\int_{\mathfrak{M}_\alpha\cap\mathfrak{M}_\beta\cap\mathfrak{m}_\gamma} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|dx$$
$$\leq \left(\int_{\mathfrak{M}_\alpha} |\Delta_\alpha(p_a^A x)|^3 dx \int_{\mathfrak{M}_\beta} |\Delta_\beta(p_b^B x)|^3 dx \int_{\mathfrak{m}_\gamma} |\Delta_\gamma(p_c^C x)|^3 dx\right)^{1/3}$$
$$\ll \left(N^{\alpha+\beta+\epsilon} Q_a^2 Q_b^2 \int_{\mathfrak{m}^\star(Q_c;N^\gamma)} |f(x;N^\gamma)|^3 dx\right)^{1/3}$$
$$\ll N^{(\alpha+\beta)/3+\epsilon} Q_a^{2/3} Q_b^{2/3} (N^{2\gamma} Q_c^{-3/2} + Q_c)^{1/3}$$
$$\ll N^{(\alpha+\beta+\gamma)/3+\epsilon} Q_a^{2/3} Q_b^{2/3} (N^{\gamma/3} Q_c^{-1/2} + Q_c^{1/3}).$$

To complete the proof, apply the same lemmas to the other two components of the integral. □

Second, we compute them over the semi-minor arcs:

Lemma 6.39. *For all $\epsilon > 0$,*

$$\int_{\partial\mathfrak{m}} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx$$
$$\ll N^{(\alpha+\beta+\gamma)/3+\epsilon} Q_a^{2/3} (N^{\beta/3} Q_b^{-1/2} + Q_b^{1/3})(N^{\gamma/3} Q_c^{-1/2} + Q_c^{1/3})$$
$$+ N^{(\alpha+\beta+\gamma)/3+\epsilon} Q_b^{2/3} (N^{\alpha/3} Q_a^{-1/2} + Q_a^{1/3})(N^{\gamma/3} Q_c^{-1/2} + Q_c^{1/3})$$
$$+ N^{(\alpha+\beta+\gamma)/3+\epsilon} Q_c^{2/3} (N^{\alpha/3} Q_a^{-1/2} + Q_a^{1/3})(N^{\beta/3} Q_b^{-1/2} + Q_b^{1/3}).$$

Proof. First observe that by the definition of the semi-minor arcs,

$$\begin{aligned}\int_{\partial\mathfrak{m}} &|\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx \\ &= \int_{\mathfrak{M}_\alpha\cap\mathfrak{m}_\beta\cap\mathfrak{m}_\gamma} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx \\ &\quad + \int_{\mathfrak{M}_\beta\cap\mathfrak{m}_\alpha\cap\mathfrak{m}_\gamma} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx \\ &\quad + \int_{\mathfrak{M}_\gamma\cap\mathfrak{m}_\alpha\cap\mathfrak{m}_\beta} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx.\end{aligned}$$

Now observe that by Hölder's inequality, followed by Lemmata 6.25, 6.33, 5.1, and 6.31 that:

$$\begin{aligned}\int_{\mathfrak{M}_\alpha\cap\mathfrak{m}_\beta\cap\mathfrak{m}_\gamma} &|\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx \\ &\le \left(\int_{\mathfrak{M}_\alpha} |\Delta_\alpha(p_a^A x)|^3\, dx \int_{\mathfrak{m}_\beta} |\Delta_\beta(p_b^B x)|^3\, dx \int_{\mathfrak{m}_\gamma} |\Delta_\gamma(-p_c^C x)|^3\, dx\right)^{1/3} \\ &\ll \left(N^{\alpha+\epsilon} Q_a^2 \int_{\mathfrak{m}^\star(Q_b;N^\beta)} |f(x;N^\beta)|^3\, dx \int_{\mathfrak{m}^\star(Q_b;N^\gamma)} |f(x;N^\gamma)|^3\, dx\right)^{1/3} \\ &\ll \left(N^{\alpha+\epsilon} Q_a^2 (N^{2\beta+\epsilon} Q_b^{-3/2} + N^{\beta+\epsilon} Q_b)(N^{2\gamma+\epsilon} Q_c^{-3/2} + N^{\gamma+\epsilon} Q_c)\right)^{1/3} \\ &\ll N^{(\alpha+\beta+\gamma)/3+\epsilon} Q_a^{2/3} (N^{\beta/3} Q_b^{-1/2} + Q_b^{1/3})(N^{\gamma/3} Q_c^{-1/2} + Q_c^{1/3}).\end{aligned}$$

To complete the proof, apply the above lemmas to the other two components of the arc. □

Finally, we compute them over the minor arcs:

Lemma 6.40. *For all* $\epsilon > 0$*,*

$$\begin{aligned}\int_{\mathfrak{m}} &|\Delta_\alpha(p_a^A x; N^\alpha)\Delta_\beta(p_b^B x; N^\beta)\Delta_\gamma(-p_c^C x; N^\gamma)|\, dx \\ &\ll N^{(\alpha+\beta+\gamma)/3+\epsilon} (N^{\alpha/3} Q_a^{-1/2} + Q_a^{1/3})(N^{\beta/3} Q_b^{-1/2} + Q_b^{1/3})(N^{\gamma/3} Q_c^{-1/2} + Q_c^{1/3}).\end{aligned}$$

Proof. First note that as the minor arcs is exactly the subset of $[0, 1)$ that is minor for all three components,

$$\int_{\mathfrak{m}} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx = \int_{\mathfrak{m}} |f(p_a^A x; N^\alpha) f(p_b^B x; N^\beta) f(p_c^C x; N^\gamma)\, dx.$$

Observe that by Hölder's Inequality, Lemma 6.33, and Lemma 5.1, that for all $\epsilon > 0$,

$$\begin{aligned}
&\int_{\mathfrak{m}} |f(p_a^A x; N^\alpha) f_\beta(p_b^B x; N^\beta) f_\gamma(-p_c^C x; N^\gamma)|\, dx \\
&\quad \le \left(\int_{\mathfrak{m}} |f(p_a^A x; N^\alpha)|^3\, dx \int_{\mathfrak{m}} |f(p_b^B x; N^\beta)|^3\, dx \int_{\mathfrak{m}} |f(p_c^C x; N^\gamma)|^3 \right)^{1/3} dx \\
&\quad \le \left(\int_{\mathfrak{m}^\star(Q_a, N^\alpha)} |f(x; N^\alpha)|^3\, dx \int_{\mathfrak{m}^\star(Q_b, N^\beta)} |f(x; N^\beta)|^3\, dx \int_{\mathfrak{m}^\star(Q_c, N^\gamma)} |f(x; N^\gamma)|^3\, dx \right)^{1/3} \\
&\quad \ll (N^{2\alpha+\epsilon} Q_a^{-3/2} + N^{\alpha+\epsilon} Q_a)^{1/3} (N^{2\beta+\epsilon} Q_b^{-3/2} + N^{\beta+\epsilon} Q_b)^{1/3} (N^{2\gamma+\epsilon} Q_c^{-3/2} + N^{\gamma+\epsilon} Q_c)^{1/3} \\
&\quad \ll N^{(\alpha+\beta+\gamma)/3+\epsilon} (N^{\alpha/3} Q_a^{-1/2} + Q_a^{1/3})(N^{\beta/3} Q_b^{-1/3} + Q_b^{1/3})(N^{\gamma/3} Q_c^{-1/2} + Q_b^{1/3}).
\end{aligned}$$

□

By combining all these results, we get following bound for the integral over the entire interval:

Lemma 6.41. *For all $\epsilon > 0$,*

$$\begin{aligned}
&\int_0^1 |\Delta_\alpha(p_a^A x) \Delta_\beta(p_b^B x) \Delta_\gamma(-p_c^C x)|\, dx \\
&\qquad \ll N^{(\alpha+\beta+\gamma)/3+\epsilon} (N^{\alpha/3} Q_a^{-1/2} + Q_a^{2/3})(N^{\beta/3} Q_b^{-1/2} + Q_b^{2/3})(N^{\gamma/3} Q_c^{-1/2} + Q_c^{2/3}).
\end{aligned}$$

If we choose $\lambda_a, \lambda_b, \lambda_c$ so that

$$Q_a = N^{\lambda_a \alpha}; \quad Q_b = N^{\lambda_b \beta}; \quad Q_c = N^{\lambda_c \gamma};$$

then this error term is optimal when $\lambda_a = \lambda_b = \lambda_c = 2/7$. This optimal bound exceeds the main term $N^{\alpha+\beta+\gamma-1}$ when $\alpha + \beta + \gamma \le 21/10$.

Proof. The above equation follows immediately by combining the results of Lemmata 6.37, 6.38, 6.39, 6.40. Consequently, if $Q_a = N^{\lambda_a \alpha}$, $Q_b = N^{\lambda_b \beta}$, $Q_c = N^{\lambda_c \gamma}$, then it follows by that $\lambda_a = \lambda_b = \lambda_c = 2/7$ is the point where the error term is minimal. Above this threshold, the error increases with $\lambda_a, \lambda_b, \lambda_c$.

Thus, at best,

$$\int_0^1 |\Delta_\alpha(p_a^A x) \Delta_\beta(p_b^B x) \Delta_\gamma(-p_c^C x)|\, dx \ll N^{11(\alpha+\beta+\gamma)/21}.$$

To observe the limit on the error, combine the above equation with the observation that

$$\frac{11}{21}(\alpha + \beta + \gamma) < \alpha + \beta + \gamma - 1 \text{ if and only if } \alpha + \beta + \gamma > \frac{21}{10}.$$

□

While the error given above can be made smaller than the main term for $\alpha+\beta+\gamma > 2.1$, it is clear that the minor arc component alone has a far superior ideal bound. If one sets $Q_a = N^{2\alpha/5}$, $Q_b = N^{2\beta/5}$ and $Q_c = N^{2\gamma/5}$, by Lemma 6.40 the bound on the minor arcs for the triple-delta component is

$$N^{7(\alpha+\beta+\gamma)/15+\epsilon},$$

which is strictly smaller than the main term when $\alpha+\beta+\gamma > 1.875$.

The primary cause of the larger size in the earlier terms comes from the relatively large bound on $\int |\Delta(x)|^3\,dx$. If a superior technique were found to optimize the bound on this component alone, it is plausible that the minor arcs could become the limiting factor.

6.6 Single $f^\star$, Double Δ Error Terms

We now move on to error terms of the form $\int |f^\star \Delta\Delta|\,dx$.

As in the previous section, we shall find bounds for the errors over the major, semi-major, and semi-minor arcs. Over the minor arcs, errors of this form are zero, as the $f^\star$ functions are truncated to be zero over the minor arcs.

Unlike the previous section, some of these proofs involve slightly more complex machinery. The technique used in these proofs is similar to the one used to prove Lemma 4.2 in [2]. As these will use common notation, we shall provide it here, rather than repeat it over multiple lemmas.

For any $R \in \mathbb{R}$ where $1 \le R \le Q_a$, let

$$\sigma_b = \frac{\log Q_b}{\log Q_a}, \quad \sigma_c = \frac{\log Q_c}{\log Q_a},$$
$$R_a = R, \quad R_b = R^{\sigma_b}, \quad R_c = R^{\sigma_c},$$
$$\mathfrak{N}_\alpha(q,a;R) = \{x \in [0,1) \colon |qp_a^A x - a| \le R_a/N^\alpha\},$$
$$\mathfrak{N}_\beta(q,b;R) = \{x \in [0,1) \colon |qp_b^B x - b| \le R_b/N^\beta\},$$
$$\mathfrak{N}_\gamma(q,c;R) = \{x \in [0,1) \colon |qp_c^C x - c| \le R_c/N^\gamma\},$$
$$\mathfrak{N}_\alpha(R) = \bigcup_{q \le R_a} \bigcup_{\substack{a=1 \\ (q,a)=1}}^{q} \mathfrak{N}_\alpha(q,a;R),$$
$$\mathfrak{N}_\beta(R) = \bigcup_{q \le R_b} \bigcup_{\substack{b=1 \\ (q,b)=1}}^{q} \mathfrak{N}_\beta(q,b;R),$$
$$\mathfrak{N}_\gamma(R) = \bigcup_{q \le R_c} \bigcup_{\substack{c=1 \\ (q,c)=1}}^{q} \mathfrak{N}_\gamma(q,c;R),$$
$$\mathfrak{P}_\alpha(R) = \mathfrak{N}_\alpha(2R) \setminus \mathfrak{N}_\alpha(R).$$

Observe that $\mathfrak{N}_\alpha, \mathfrak{N}_\beta, \mathfrak{N}_\gamma$ are just rescalings of the major arc components by some power of R, so any lemmas that apply to major arcs can apply to their rescalings.

First, the major arcs:

Lemma 6.42. *For all $\epsilon > 0$,*

$$\int_{\mathfrak{M}} |f_\alpha^\star(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)| \, dx \ll N^{(2\alpha+\beta+\gamma)/3+\epsilon}(1 + Q_a^{-1/2}Q_b^{2/3}Q_c^{2/3})$$
$$\int_{\mathfrak{M}} |\Delta_\alpha(p_a^A x) f_\beta^\star(p_b^B x)\Delta_\gamma(-p_c^C x)| \, dx \ll N^{(\alpha+2\beta+\gamma)/3+\epsilon}(1 + Q_a^{2/3}Q_b^{-1/2}Q_c^{2/3})$$
$$\int_{\mathfrak{M}} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x) f_\gamma^\star(-p_c^C x)| \, dx \ll N^{(\alpha+\beta+2\gamma)/3+\epsilon}(1 + Q_a^{2/3}Q_b^{2/3}Q_c^{-1/2}).$$

Proof. It suffices to prove the first equation, as the others follow by similar logic. Observe that $\mathfrak{N}_\alpha(R), \mathfrak{N}_\beta(R), \mathfrak{N}_\gamma(R)$ are the major arc components $\mathfrak{M}_\alpha, \mathfrak{M}_\beta, \mathfrak{M}_\gamma$ with Q_a, Q_b, Q_c rescaled to R_a, R_b, R_c respectively. Also observe that the sets

$$\{\mathfrak{P}_\alpha(R) \colon 1 \le R \le Q_a/2\}; \quad \{\mathfrak{P}_\beta(R) \colon 1 \le R \le Q_a/2\}; \quad \{\mathfrak{P}_\gamma(R) \colon 1 \le R \le Q_a/2\};$$

each cover $\mathfrak{M}_\alpha$, $\mathfrak{M}_\beta$, and $\mathfrak{M}_\gamma$ respectively. It is also clear that it takes $O(\log Q_a)$ distinct R to cover each. Combining this dyadic dissection with Hölder's inequality,

$$\int_{\mathfrak{M}} |f_\alpha^\star(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)| \, dx \ll \log(2Q_a) \max_{1 \le R \le Q_a/2} (U_1(R)U_2(R)U_3(R))^{1/3},$$

where

$$U_1(R) = \int_{\mathfrak{P}_\alpha(R)} |f_\alpha^\star(p_a^A x)|^3\, dx; U_2(R) = \int_{\mathfrak{N}_\beta(2R)} |\Delta_\beta(p_b^B x)|^3\, dx; U_3(R) = \int_{\mathfrak{N}_\gamma(2R)} |\Delta_\gamma(p_c^C x)|^3\, dx.$$

By Lemma 6.31, for all $\epsilon > 0$,

$$\int_{\mathfrak{N}_\beta(R)} |\Delta_\beta(p_b^B x)|^3\, dx \ll N^{\beta+\epsilon} R_b^2; \quad \int_{\mathfrak{N}_\gamma(R)} |\Delta(p_c^C)|^3\, dx \ll N^{\gamma+\epsilon} R_c^2.$$

By Lemma 6.25,

$$\int_{\mathfrak{P}_\alpha(R)} |f_\alpha^\star(p_a^A x)|^3\, dx = \int_{\mathfrak{M}^\star(2R;N^\alpha)\setminus\mathfrak{M}^\star(R;N^\alpha)} |f_\alpha^\star(x)|^3\, dx.$$

We expand the above expression in a manner similar to that found in Lemma 4.2 of [2]. Observe that $\mathfrak{P}_\alpha(R)$ is the set of major arcs $\mathfrak{N}_\alpha(q,k;2R)$ on the interval for $q \in (R, 2R]$, but is a union of annular regions of the form $\mathfrak{N}_\alpha(q,k;2R) \setminus \mathfrak{N}_\alpha(q,k;R)$, $q \in [1, R]$, so

$$\begin{aligned}
U_1(R) &= \int_{\mathfrak{P}_\alpha(R)} |f_\alpha^\star(p_a^A x)|^3\, dx \\
&= \sum_{R<q\le 2R} \phi(q)|G(q)|^3 \int_{-2R/qN^\alpha}^{2R/qN^\alpha} |I_\alpha(x)|^3\, dx \\
&\quad + \sum_{q\le R} \phi(q)|G(q)|^3 \int_{R/qN^\alpha}^{2R/qN^\alpha} |I_\alpha(x)|^3\, dx,
\end{aligned}$$

and follow with the application of Lemma 4.6 to give us that for all $\epsilon > 0$,

$$\begin{aligned}
U_1(R) &\ll \sum_{R<q\le 2R} \phi(q)|G(q)|^3 \int_{-1/2}^{1/2} |I_\alpha(x)|^3\, dx + \sum_{1\le q\le R} \phi(q)|G(q)|^3 \int_{R/qN^\alpha}^{1/2} \|x\|^3\, dx \\
&\ll N^{2\alpha}\left(\sum_{R<q\le 2R} q|G(q)|^3\right) + N^{2\alpha}R^{-2} \sum_{1\le q\le R} q^3|G(q)|^3 \\
&\ll N^{2\alpha}R^{-3/2+\epsilon} + N^{2\alpha}R^{-3/2+\epsilon}.
\end{aligned}$$

By combining these results, it follows that

$$\max_{1\le R\le Q_a/2} (U_1(R)U_2(R)U_3(R))^{1/3} \ll N^{(2\alpha+\beta+\gamma)/3+\epsilon}(1 + Q_a^{-1/2}Q_b^{2/3}Q_c^{2/3}),$$

and thus

$$\int_{\mathfrak{M}} |f_\alpha^\star(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)| dx \ll N^{(2\alpha+\beta+\gamma)/3+\epsilon}(1 + Q_a^{-1/2}Q_b^{2/3}Q_c^{2/3}).$$

□

Now, the semimajor arcs.

Lemma 6.43. *For all $\epsilon > 0$,*

$$\begin{aligned}\int_{\partial\mathfrak{M}} &|f^\star_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \\ &\ll N^{(2\alpha+\beta+\gamma)/3+\epsilon}\big((1+Q_a^{-1/2}Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2}+Q_c^{1/3}) \\ &\quad + (1+Q_a^{-1/2}Q_c^{2/3})(N^{\beta/3}Q_b^{-1/2}+Q_b^{1/3})\big)\\ \int_{\partial\mathfrak{M}} &|\Delta_\alpha(p_a^A x)f^\star_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \\ &\ll N^{(\alpha+2\beta+\gamma)/3+\epsilon}\big((1+Q_a^{2/3}Q_b^{-1/2})(N^{\gamma/3}Q_c^{-1/2}+Q_c^{1/3}) \\ &\quad + (1+Q_b^{-1/2}Q_c^{2/3})(N^{\alpha/3}Q_a^{-1/2}+Q_a^{1/3})\big)\\ \int_{\partial\mathfrak{M}} &|\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)f^\star_\gamma(-p_c^C x)|\,dx \\ &\ll N^{(\alpha+\beta+2\gamma)/3+\epsilon}\big((1+Q_a^{2/3}Q_c^{-1/2})(N^{\beta/3}Q_b^{-1/2}+Q_b^{1/3}) \\ &\quad + (1+Q_b^{2/3}Q_c^{-1/2})(N^{\alpha/3}Q_a^{-1/2}+Q_a^{1/3})\big).\end{aligned}$$

Proof. It suffices to prove the first equation as the others can be proven by similar logic. Observe that $\mathfrak{N}_\alpha(R), \mathfrak{N}_\beta(R), \mathfrak{N}_\gamma(R)$ are the major arc components $\mathfrak{M}_\alpha, \mathfrak{M}_\beta, \mathfrak{M}_\gamma$ with Q_a, Q_b, Q_c rescaled to R_a, R_b, R_c respectively. Also observe that the sets

$$\{\mathfrak{P}_\alpha(R)\colon 1 \le R \le Q_a/2\}; \quad \{\mathfrak{P}_\beta(R)\colon 1 \le R \le Q_a/2\}; \quad \{\mathfrak{P}_\gamma(R)\colon 1 \le R \le Q_a/2\};$$

each cover $\mathfrak{M}_\alpha$, $\mathfrak{M}_\beta$, and $\mathfrak{M}_\gamma$ respectively. It is also clear that it takes $O(\log Q_a)$ distinct R to cover each.

Now, note that if $|f^\star_\alpha(p_a^A x)| \neq 0$ then $x \in \mathfrak{M}_\alpha$, and that implies for $x \in \partial\mathfrak{M}$, that $x \in \mathfrak{M}_\alpha \cap \mathfrak{M}_\beta \cap \mathfrak{m}_\gamma$ or $x \in \mathfrak{M}_\alpha \cap \mathfrak{m}_\beta \cap \mathfrak{M}_\gamma$. Thus by Hölder's inequality,

$$\begin{aligned}\int_{\partial\mathfrak{M}} &|f^\star_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)\,dx \\ &\le \left(\int_{\mathfrak{M}_\alpha\cap\mathfrak{M}_\beta} |f^\star_\alpha(p_a^A x)\Delta_\beta(p_b^B x)|^{3/2}\,dx\right)^{2/3}\left(\int_{\mathfrak{m}_\gamma}|\Delta_\gamma(p_c^C x)|^3 dx\right)^{1/3} \\ &\quad + \left(\int_{\mathfrak{M}_\alpha\cap\mathfrak{M}_\gamma} |f^\star_\alpha(p_a^A x)\Delta_\gamma(p_c^C x)|^{3/2}\,dx\right)^{2/3}\left(\int_{\mathfrak{m}_\beta}|\Delta_\gamma(p_b^B x)|^3 dx\right)^{1/3}.\end{aligned}$$

Observe that $\Delta = f$ on the minor arcs. Combining this with the dyadic dissection above and applying Cauchy's inequality,

$$\begin{aligned}
&\int_{\partial\mathfrak{M}} |f_\alpha^\star(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \\
&\qquad \ll \log(2Q_a) \max_{1\le R\le Q_a/2} (U_1(R)U_2(R))^{1/3} \left(\int_{\mathfrak{m}_\gamma} |f(p_c^C x; N^\gamma)|^3\,dx \right)^{1/3} \\
&\qquad\qquad + \log(2Q_a) \max_{1\le R\le Q_a/2} (U_1((R)U_3(R))^{1/3} \left(\int_{\mathfrak{m}_\beta} |f(p_b^B x; N^\beta)|^3\,dx \right)^{1/3},
\end{aligned}$$

where

$$U_1(R) = \int_{\mathfrak{P}_\alpha(R)} |f_\alpha^\star(p_a^A x)|^3\,dx; U_2(R) = \int_{\mathfrak{N}_\beta(2R)} |\Delta_\beta(p_b^B x)|^3\,dx; U_3(R) = \int_{\mathfrak{N}_\gamma(2R)} |\Delta_\gamma(p_c^C x)|^3\,dx.$$

By Lemma 6.31, for all $\epsilon > 0$,

$$\int_{\mathfrak{N}_\beta(R)} |\Delta_\beta(p_b^B x)|^3\,dx \ll N^{\beta+\epsilon}R_b^2; \quad \int_{\mathfrak{N}_\gamma(R)} |\Delta(p_c^C)|^3\,dx \ll N^{\gamma+\epsilon}R_c^2.$$

By Lemmata 6.33 and 5.1,

$$\int_{\mathfrak{m}_\beta} |f(p_b^B x; N^\beta)|^3\,dx \ll N^\beta Q_b^{-3/2} + Q_b; \quad \int_{\mathfrak{m}_\gamma} |f(p_c^C x; N^\gamma)|^3\,dx \ll N^\gamma Q_c^{-3/2} + Q_c.$$

By Lemma 6.25,

$$\int_{\mathfrak{P}_\alpha(R)} |f_\alpha^\star(p_a^A x)|^3 dx = \int_{\mathfrak{M}^\star(2R;N^\alpha)\setminus\mathfrak{M}^\star(R;N^\alpha)} |f_\alpha^\star(x)|^3\,dx.$$

By the same technique described in the previous lemma, for all $\epsilon > 0$,

$$\begin{aligned}
U_1(R) &\ll \sum_{R<q\le 2R} \phi(q)|G(q)|^3 \int_{-1/2}^{1/2} |I_\alpha(x)|^3\,dx + \sum_{1\le q\le R} \phi(q)|G(q)|^3 \int_{R/qN^\alpha}^{1/2} \|x\|^3\,dx \\
&\ll N^{2\alpha} \left(\sum_{R<q\le 2R} q|G(q)|^3 \right) + N^{2\alpha}R^{-2} \sum_{1\le q\le R} q^3|G(q)|^3 \\
&\ll N^{2\alpha}R^{-3/2+\epsilon} + N^{2\alpha}R^{-3/2+\epsilon}.
\end{aligned}$$

By combining these results, we have

$$\begin{aligned}
\max_{1\le R\le Q_a/2} (U_1(R)U_2(R))^{1/3} &\ll N^{(2\alpha+\beta)/3+\epsilon}(1 + Q_a^{-1/2}Q_b^{2/3}), \\
\max_{1\le R\le Q_a/2} (U_1(R)U_3(R))^{1/3} &\ll N^{(2\alpha+\gamma)/3+\epsilon}(1 + Q_a^{-1/2}Q_c^{2/3}).
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_{\mathfrak{M}} |f_\alpha^\star(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \\
&\qquad \ll N^{(2\alpha+\beta)/3+\epsilon}(1 + Q_a^{-1/2}Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{1/3}) \\
&\qquad\qquad + N^{(2\alpha+\gamma)/3+\epsilon}(1 + Q_a^{-1/2}Q_c^{2/3})(N^{\beta/3}Q_b^{-1/2} + Q_b^{1/3}).
\end{aligned}$$

∎

We now move on to the error term of this form over the semi-minor arcs. Unlike the earlier proofs, there is nothing to gain by breaking the arcs into dyadic intervals, so this is a straightforward application of Hölder's inequality.

Lemma 6.44. *For all $\epsilon > 0$,*

$$\int_{\partial\mathfrak{m}} |f_\alpha^\star(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \ll N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3}Q_b^{-1/2} + Q_b^{1/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{1/3}),$$
$$\int_{\partial\mathfrak{m}} |\Delta_\alpha(p_a^A x)f_\beta^\star(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \ll N^{(\alpha+2\beta+\gamma)/3+\epsilon}(N^{\alpha}Q_a^{-1/2} + Q_a^{1/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{1/3}),$$
$$\int_{\partial\mathfrak{m}} |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)f_\gamma^\star(-p_c^C x)|\,dx \ll N^{(\alpha+\beta+2\gamma)/3+\epsilon}(N^{\alpha/3}Q_a^{-1/2} + Q_a^{1/3})(N^{\beta/3}Q_b^{-1/2} + Q_b^{1/3}).$$

Proof. It suffices to prove the first equation as the others can be proven by symmetric logic.

First note that $f^\star(p_a^A x) \neq 0$ implies that $x \in \mathfrak{M}_\alpha$. By the definition of the semi-minor arcs, this implies $x \in \mathfrak{m}_\beta \cap \mathfrak{m}_\gamma$. Thus observe that by Hölder's inequality,

$$\int_{\partial\mathfrak{M}} |f^\star(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx$$
$$\leq \left(\int_{\mathfrak{M}_\alpha} |f^\star(p_a^A x)|^3\,dx \int_{\mathfrak{m}_\beta} |\Delta_\beta(p_b^B x)|^3\,dx \int_{\mathfrak{m}_\gamma} |\Delta_\gamma(p_c^C x)|^3\,dx\right)^{1/3}.$$

By Lemmata 6.33 and 5.1, for all $\epsilon > 0$

$$\int_{\mathfrak{m}_\beta} |\Delta_\beta(p_b^B x)|^3\,dx \ll N^{2\beta}Q_b^{-3/2} + N^\beta Q_b; \quad \int_{\mathfrak{m}_\gamma} |\Delta_\gamma(p_c^C x)|^3\,dx \ll N^{2\gamma}Q_c^{-3/2} + N^\gamma Q_c.$$

By Lemma 6.32,

$$\int_{\mathfrak{M}_\alpha} |f_\alpha^\star(p_a^A x)|^3\,dx \ll N^{2\alpha}.$$

Thus it follows that

$$\int_{\partial\mathfrak{M}} |f^\star(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx$$
$$\ll N^{2\alpha/3+\epsilon}(N^{2\beta/3}Q_b^{-1/2} + N^{\beta/3}Q_b^{1/3})(N^{2\gamma/3}Q_c^{-1/2} + N^{\gamma/3}Q_c^{1/3}),$$

completing the proof. ∎

This leads to the final result of the section:

Lemma 6.45. *For all $\epsilon > 0$,*

$$\int_0^1 |f^\star_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \ll N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3}Q_b^{-1/2} + Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{2/3}),$$
$$\int_0^1 |\Delta_\alpha(p_a^A x)f^\star_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \ll N^{(\alpha+2\beta+\gamma)/3+\epsilon}(N^{\alpha/2}Q_a^{-1/2} + Q_a^{2/3})(N^{\gamma/2}Q_c^{-1/2} + Q_c^{2/3}),$$
$$\int_0^1 |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)f^\star_\gamma(-p_c^C x)|\,dx \ll N^{(\alpha+\beta+2\gamma)/3+\epsilon}(N^{\alpha/2}Q_a^{-1/2} + Q_a^{2/3})(N^{\beta/2}Q_b^{-1/2} + Q_b^{2/3}).$$

If there exists some $\lambda_a, \lambda_b, \lambda_c$ such that

$$Q_a = N^{\lambda_a \alpha}; \quad Q_b = N^{\lambda_b \beta}; \quad Q_c = N^{\lambda_c \gamma},$$

then the error terms for all three equations are optimal when $\lambda_a = \lambda_b = \lambda_c = 2/7$. This optimal bound is strictly smaller than the main term when the following inequalities are satisfied:

$$0.7\alpha + \beta + \gamma > 2.1; \quad \alpha + 0.7\beta + \gamma > 2.1; \quad \alpha + \beta + 0.7\gamma > 2.1.$$

Proof. The above equations follow immediately by combining Lemmata 6.42, 6.43, and 6.44 with the observation that on the minor arcs, the integrals are zero since all $f^\star$ functions are zero there. When $Q_a \ll N^{\lambda_a \alpha}$, $Q_b \ll N^{\lambda_b \beta}$ $\lambda_a = \lambda_b = \lambda_c = 2/7$, the error term is of the form

$$\int_0^1 |f^\star_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \ll N^{2\alpha/3+11(\beta+\gamma)/21+\epsilon},$$
$$\int_0^1 |\Delta_\alpha(p_a^A x)f^\star_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \ll N^{2\beta/3+11(\alpha+\gamma)/21+\epsilon},$$
$$\int_0^1 |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)f^\star_\gamma(-p_c^C x)|\,dx \ll N^{2\gamma/3+11(\alpha+\beta)/21+\epsilon}.$$

Observe that as

$$2\alpha/3 + 19(\beta+\gamma)/21 < \alpha+\beta+\gamma-1 \text{ iff } \alpha/3 + 10(\beta+\gamma)/21 > 1,$$
$$2\beta/3 + 19(\alpha+\gamma)/21 < \alpha+\beta+\gamma-1 \text{ iff } \beta/3 + 10(\alpha+\gamma)/21 > 1,$$
$$2\gamma/3 + 19(\alpha+\beta)/21 < \alpha+\beta+\gamma-1 \text{ iff } \gamma/3 + 10(\alpha+\beta)/21 > 1.$$

the region stated in the proof follows immediately. □

Observe that the bound is primarily determined by the contribution from the semi-minor arcs.

6.7 Harmonic Arc Error Terms

Before moving to the actual double $f^\star$, single Δ error terms, we shall first investigate the behaviour of these integrals on the harmonic majors, and on an extension of the concept of the harmonic majors to the semi-major arcs.

First, we prove a result about the double $f^\star$, single Δ error terms on the major arcs. This uses a technique similar to the one used in the previous section, but with some changes, so it is fully explained within the proof of the lemma.

Lemma 6.46. *Let $Q_0 = \min\{Q_a, Q_b, Q_c\}$. For all $\epsilon > 0$,*

$$\int_{\mathfrak{M}_H} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx \ll N^{\alpha+\beta-1/2+\epsilon},$$
$$\int_{\mathfrak{M}_H} |f^\star_\alpha(p_a^A x)\Delta_\beta(p_b^B x) f^\star_\gamma(-p_c^C x)|\, dx \ll N^{\alpha+\gamma-1/2+\epsilon},$$
$$\int_{\mathfrak{M}_H} |\Delta_\alpha(p_a^A x) f^\star_\beta(p_b^B x) f^\star_\gamma(-p_c^C x)|\, dx \ll N^{\beta+\gamma-1/2+\epsilon},$$

provided $\alpha+\beta > 1$, $\alpha+\gamma > 1$, and $\beta+\gamma > 1$ for each equation respectively.

Proof. Let $X \subseteq [0,1)$ be the set such that $f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)\Delta_\gamma(-p_c^C x) \neq 0$.

For any R such that $1 \le R \le Q_0$:

$$\sigma_a = \frac{\log Q_a}{\log Q_0}, \quad \sigma_b = \frac{\log Q_b}{\log Q_0}, \quad \sigma_c = \frac{\log Q_c}{\log Q_0},$$
$$R_a = R^{\sigma_a}, \quad R_b = R^{\sigma_b}, \quad R_c = R^{\sigma_c},$$
$$m(q;R) = \min\left\{\frac{h_a R_a (q, p_a^A)}{qN}, \frac{h_b R_b (q, p_b^B)}{qN}, \frac{h_c R_c (q, p_c^C)}{qN}\right\},$$
$$\tilde{Q}(R) = \{q \in \mathbb{N}\colon q \le R_a (q, p_a^A), q \le R_b (q, p_b^B), q \le R_c (q, p_c^C)\},$$
$$\mathfrak{N}_\alpha(q,a;R) = \{x \in [0,1)\colon |q p_a^A x - a| \le R_a/N^\alpha\},$$
$$\mathfrak{N}_\beta(q,b;R) = \{x \in [0,1)\colon |q p_b^B x - b| \le R_b/N^\beta\},$$
$$\mathfrak{N}_\gamma(q,c;R) = \{x \in [0,1)\colon |q p_c^C x - c| \le R_c/N^\gamma\},$$
$$\mathfrak{N}(q,k;R) = \mathfrak{N}_\alpha\Big(\frac{q}{(q,p_a^A)}, \frac{p_a^A}{(q,p_a^A)}k; R\Big) \cap \mathfrak{N}_\beta\Big(\frac{q}{(q,p_b^B)}, \frac{p_b^B}{(q,p_b^B)}k; R\Big) \cap \mathfrak{N}_\gamma\Big(\frac{q}{(q,p_c^C)}, \frac{p_c^C}{(q,p_c^C)}k; R\Big),$$
$$\mathfrak{N}(R) = \bigcup_{q \in \tilde{Q}(R)} \bigcup_{\substack{k=1 \\ (q,k)=1}}^{q} \mathfrak{N}(q,k;R),$$
$$\mathfrak{P}(R) = \mathfrak{N}(2R) \setminus \mathfrak{N}(R).$$

The above definitions are directly analogous to the definition of the harmonic majors, with Q_a, Q_b, Q_c rescaled. Observe that by definition, $\mathfrak{M}_H$ can be covered by the set of dyadic intervals

$$\{\mathfrak{P}(R)\colon R \in [1, Q_1/2]\},$$

and since the number of dyadic intervals needed to cover $\mathfrak{M}_H$ is proportional to $\log Q_0$ it thus follows that

$$\int_{\mathfrak{M}_H} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx \ll \log(2Q_0) \max_{1\le R\le Q_0} \left(U_1(R)U_2(R)\right)^{1/2},$$

where

$$U_1(R) = \int_{\mathfrak{P}(R)} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x)|^2\, dx; \quad U_2(R) = \int_{\mathfrak{N}(2R)\cap X} |\Delta_\gamma(p_c^C x)|^2\, dx.$$

Observe that if $\mathfrak{N}(q,k;2R) \subseteq \mathfrak{N}(2R)\cap X$, this implies that $q \le \min\{p_a^2 R_a, p_b^2 R_b, p_c^2 R_c\}$, which is less than $q \le \max\{p_a^2, p_b^2, p_c^2\}R$. For $K = \max\{p_a^2, p_b^2, p_c^2\}$,

$$\mathfrak{N}(2R)\cap X \subseteq \bigcup_{q=1}^{KR} \bigcup_{\substack{c=1\\(q,c)=1}}^{q} \{x\in[0,1)\colon |qp_c^C x - c| \le KR/N^\gamma\}.$$

From Lemma 3.2 of [2] we have a bound on the L^2 norm of Δ type functions over the appropriate major arc. As this set is of the form of $\mathfrak{M}_\gamma$ but with a different maximum, it follows by Lemma 6.25 that for all $\epsilon > 0$, that

$$U_2(R) = \int_{\mathfrak{M}^\star(2R;N^\gamma)} |\Delta_\gamma(p_c^C x)|^2\, dx \ll R^{2+\epsilon}.$$

To estimate $U_1(R)$, first observe that for $q \in \tilde{Q}(2R)\setminus\tilde{Q}(R)$ that if $G(q/(q,p_a^A))G(q/(q,p_b^B)) \neq 0$ then $Q_a < q \le 2p_a^2 Q_a$, $Q_b < q \le 2p_b^2 Q_b$ and $Q_c < q \le 2p_c^2 Q_c$.

By Lemma 6.34, $G(q/(q,p_a^A)G(q/(q,p_b^B)) \ll G(q)^2$.

Let $p_1 = \max\{p_a, p_b, p_c\}$, Thus for $U_1(R)$, observe that by the definition of $\mathfrak{P}(\vec{R})$, Lemma 6.35, and the fact that $\alpha+\beta>1$ that

$$\begin{aligned}
&\int_{\mathfrak{P}(R)} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x)|^2\,dx \\
&\quad = \sum_{q\in\tilde{Q}(2R)\setminus\tilde{Q}(R)} \sum_{\substack{k=1\\(q,k)=1)}}^{q} \int_{\mathfrak{N}(q,k;2R)} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x)|^2\,dx \\
&\qquad + \sum_{q\in\tilde{Q}(R)} \sum_{\substack{k=1\\(q,k)=1}}^{q} \int_{\mathfrak{N}(q,k;2R)\setminus\mathfrak{N}(q,k,R)} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B)|^2\,dx \\
&\quad \le \sum_{q\in tildeQ(2R)\setminus\tilde{Q}(R)} \phi(q)|G(q/(q,p_a^A))G(q/(q,p_b^B)|^2 \int_0^1 |I_\alpha(p_a^A x) I_\beta(p_b^B)|^2\,dx \\
&\qquad + \sum_{q\in\tilde{Q}(R)} \phi(q)|G(q/(q,p_a^A))G(q/(q,p_b^B))|^2 \int_{m(q;R)}^{1/2} |I_\alpha(p_a^A x) I_\beta(p_b^B)|^2\,dx \\
&\quad \ll \sum_{R<q\le 2p_0^2 R} q|G(q)|^4 (N^{2\alpha+2\beta-1} + N^{\alpha+\beta}) + \sum_{q\le R} q|G(q)|^4 \int_{m(q;R)}^{1/2} |I_\alpha(p_a^A x) I_\beta(p_b^B x)|^2\,dx \\
&\quad \ll \sum_{R<q\le 2p_0^2 R} q|G(q)|^4 N^{2\alpha} + \sum_{q\le R} q|G(q)|^4 p_a^{-2A} p_b^{-2B} \int_{m(q;R)}^{1/2} \|x\|^{-4}\,dx \\
&\quad \ll \sum_{R<q\le 2p_0^2 R} q|G(q)|^4 N^{2\alpha} + \sum_{q\le R} q|G(q)|^4 N^{2\alpha+2\beta-4} m(q;R)^{-3} \\
&\quad \ll N^{2\alpha+2\beta-1} \sum_{R<q\le 2p_0^2 R} q|G(q)|^4 + N^{2\alpha+2\beta-1} R^{-3} \sum_{q\le R_0} q^4 |G(q)|^4.
\end{aligned}$$

By Lemma 4.6 it follows that

$$\sum_{R_0<q\le 2p_1 R_0} q|G(q)|^4 \ll R^{-5/2+\epsilon}; \quad \sum_{q\le R} q^4|G(q)|^4 \ll R^{1/2+\epsilon},$$

and thus

$$U_1(R) \ll N^{2\alpha+2\beta-1} R^{-5/2}.$$

Hence, it follows that

$$\begin{aligned}
&\int_{\mathfrak{M}_H} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x) \Delta_\gamma(-p_c^C x)|\,dx \\
&\quad \ll \log(2Q_0) \max_{1\le R\le Q_0/2} \left(N^{2\alpha+2\beta-1+\epsilon} R^{-5/2} R^2\right)^{1/2} \\
&\quad \ll \log(2Q_0) \max_{1\le R\le Q_0/2} \left(N^{2\alpha+2\beta-1+\epsilon} R^{-1/2}\right)^{1/2} \\
&\quad \ll N^{\alpha+\beta-1/2+\epsilon}.
\end{aligned}$$

This completes the proof. □

Note that the condition that $\alpha+\beta>1$, $\alpha+\gamma>1$, and $\beta+\gamma>1$ are there simply to ensure that the bound from Lemma 6.35 is of the $N^{\alpha+\beta-1/2}$ form and not the full $N^{\alpha+\beta-1/2}+N^{\alpha/2+\beta/2}$ form. This restriction is not particularly cumbersome, and the main term cannot actually be resolved completely unless all three of the above are satisfied, since when one is not satisfied the error term of $U(N)$ from Lemma 6.17 becomes too large. Removing the condition produces an equally usable lemma, but with a more cumbersome to write error term. This observation will apply to the last lemma of this section as well.

We now move on to the semi-major arcs. Recall from Definition 6.6 the harmonic semimajor arcs, and their low and high components. We shall now bound the error terms on these arcs for integrals with two $f^\star$ functions and one Δ function.

We first compute the low harmonic semi-major arcs. These are annular regions near the major arcs.

Lemma 6.47. *For all $\epsilon>0$, it follows that*

$$\int_{\partial\mathfrak{M}_{H0}} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \ll N^{\alpha+\beta+\gamma/2-1/2+\epsilon}Q_c^{-7/4}Q_0^{1/4}+N^{\alpha+\beta-1/2+\epsilon}Q_c^{-1/2}Q_0^{1/4},$$
$$\int_{\partial\mathfrak{M}_{H0}} |f^\star_\alpha(p_a^A x) \Delta_\beta(p_b^B x)f^\star_\gamma(-p_c^C x)|\,dx \ll N^{\alpha+\beta/2+\gamma-1/2+\epsilon}Q_b^{-7/4}Q_0^{1/4}+N^{\alpha+\gamma-1/2+\epsilon}Q_b^{-1/2}Q_0^{1/4},$$
$$\int_{\partial\mathfrak{M}_{H0}} |\Delta_\alpha(p_a^A x) f^\star_\beta(p_b^B x)f^\star_\gamma(-p_c^C x)|\,dx \ll N^{\alpha/2+\beta+\gamma-1/2+\epsilon}Q_a^{-7/4}Q_0^{1/4}+N^{\beta+\gamma-1/2+\epsilon}Q_a^{-1/2}Q_0^{1/4}.$$

Proof. Observe that if $p_c^C x\in\mathfrak{M}_\gamma$, this implies one of

$$p_a^A x\notin\mathfrak{M}_\alpha, p_b^B x\notin\mathfrak{M}_\beta,$$

and thus $|f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)|=0$. Thus it follows that

$$\int_{\partial\mathfrak{M}_{H0}} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx = \int_{\partial\mathfrak{M}_{H}0\cap\mathfrak{m}_\gamma} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx.$$

Thus it follows that

$$\int_{\partial\mathfrak{M}_{H0}\cap\mathfrak{m}_\gamma} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx$$
$$=\sum_{q\in\partial\tilde{Q}}\sum_{\substack{k=1\\(q,k)=1}}^{q}\int_{\partial\mathfrak{M}_H(q,k)\cap\mathfrak{m}_\gamma} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx.$$

Thus it follows by Cauchy's inequality that

$$\int_{\partial\mathfrak{M}_{H0}\cap\mathfrak{m}_\gamma} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)\Delta_\gamma(p_c^C)|\, dx$$

$$\leq \left(\int_{\partial\mathfrak{M}_{H0}} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)|^2\, dx \int_{\mathfrak{m}_\gamma} |\Delta_\gamma(p_c^C x)|^2\, dx\right)^{1/2}$$

$$\leq \left(\int_{\partial\mathfrak{M}_{H0}} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)|^2\, dx \int_{\mathfrak{m}_\gamma} |f(p_c^C x; N^\gamma)|^2\, dx\right)^{1/2}.$$

By Lemma 6.33, followed by Theorem 2 of [2] one has that:

$$\int_{\mathfrak{m}_\gamma} |f(-p_c^C x; N^\gamma)|^2\, dx = \int_{\mathfrak{m}^\star(Q_c, N^\gamma)} |f_\gamma(x)|^2\, dx \ll N^\gamma Q_c^{-1/2} + Q_c^2.$$

Let

$$m_c(q) = \frac{h_c Q_c(q, p_b^B)}{qN}.$$

Observe that $\partial\mathfrak{M}_{H0} \cap \mathfrak{m}_\gamma$ is a union of annular regions around each major arc, and when one fixes q, each annular region is bounded within $[K - 1/2, K - m_c(q)] \cup [m_c(q) + K, 1/2 + K]$ where K is the centre of the annulus. Thus by Lemma 6.34,

$$\int_{\partial\mathfrak{M}_{H0}\cap\mathfrak{m}_\gamma} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)|^2\, dx$$

$$= \sum_{q\in\tilde{Q}} \sum_{\substack{k=1\\(q,k)=1}}^{q} \int_{\partial\mathfrak{M}_{H(q,k)}\cap\mathfrak{m}_\gamma} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)|^2\, dx$$

$$\ll \sum_{q\in\tilde{Q}} \phi(q) |G(q/(q,p_a^A)) G(q/(q,p_b^B))|^2 \int_{m_c(q)}^{1/2} |I_\alpha(p_a^A x) I_\beta(p_b^B x)|^2\, dx$$

$$\ll \sum_{q\in\tilde{Q}} q |G(q/(q,p_a^A)) G(q/(q,p_b^B))|^2 \int_{m_c(q)}^{1/2} p_a^{-2A} p_b^{-2B} \|x\|^{-4}\, dx$$

$$\ll \sum_{q\in\tilde{Q}} q |G(q/(q,p_a^A)) G(q/(q,p_b^B))|^2 N^{2\alpha+2\beta-4} m_c(q)^{-3}$$

$$\ll \sum_{q\leq p_0^2 Q_0} q|G(q)|^4 N^{2\alpha+2\beta-4} q^3 N^3 Q_c^{-3}$$

$$\ll N^{2\alpha+2\beta-1} Q_c^{-3} \sum_{q\in\partial\tilde{Q}} q^4 |G(q)|^4.$$

Thus by Lemma 4.6,

$$\int_{\partial\mathfrak{M}_{H0}} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x)|^2\, dx \ll N^{2\alpha+2\beta-1+\epsilon} Q_c^{-3} Q_0^{1/2},$$

and therefore,

$$\int_{\partial\mathfrak{M}_{H0}} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \ll N^{\alpha+\beta+\gamma/2-1/2+\epsilon} Q_c^{-7/4} Q_0^{1/4} + N^{\alpha+\beta-1/2+\epsilon} Q_c^{-1/2} Q_0^{1/4}.$$

This completes the proof for this case. As the others are symmetric, this suffices. □

Second, we compute the high harmonic semi-major arcs. These are the remains of the harmonic arcs after q becomes large.

Lemma 6.48. *Let $Q_0 = \min\{Q_a, Q_b, Q_c\}$. For all $\epsilon > 0$,*

$$\int_{\partial\mathfrak{M}_{H1}} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx \ll N^{\alpha+\beta-1/2+\epsilon}(N^{\gamma/2} Q_c^{-3/2} + Q_c^{-1/4}),$$
$$\int_{\partial\mathfrak{M}_{H1}} |f_\alpha^\star(p_a^A x) \Delta_\beta(p_b^B x) f_\gamma^\star(-p_c^C x)|\,dx \ll N^{\alpha+\gamma-1/2+\epsilon}(N^{\beta/2} Q_b^{-3/2} + Q_b^{-1/4}),$$
$$\int_{\partial\mathfrak{M}_{H1}} |\Delta_\alpha(p_a^A x) f_\beta^\star(p_b^B x) f_\gamma^\star(-p_C^C x)|\,dx \ll N^{\beta+\gamma-1/2+\epsilon}(N^{\alpha/2} Q_a^{-3/2} + Q_a^{-1/4}),$$

when $\alpha+\beta > 1$, $\alpha+\gamma > 1$, and $\beta+\gamma > 1$ respectively.

Proof. We shall prove this for the first equation only, as the other two equations can be proven by symmetric logic. First, by Cauchy's inequality, and the fact that $f_\alpha^\star$ and $f_\beta^\star$ are non-zero only (in terms of the semimajor arcs) on the intersection of $\mathfrak{M}_\alpha \cap \mathfrak{M}_\beta \cap \mathfrak{m}_\gamma$,

$$\int_{\partial\mathfrak{M}_{H1}} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x)\Delta_\gamma(-p_c^C x)|\,dx$$
$$\leq \left(\int_{\partial\mathfrak{M}_{H1}} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x)|^2\,dx \int_{\mathfrak{m}_\gamma} |\Delta_\gamma(-p_c^C x)|^2\,dx\right)^{1/2}.$$

First, the minor arc integral is resolved by observing that by Lemma 6.33 followed by Theorem 2 of [2],

$$\int_{\mathfrak{m}_\gamma} |\Delta_\gamma(p_c^C x)|^2\,dx = \int_{\mathfrak{m}^\star(Q_c;N^\gamma)} |\Delta_\gamma(x)|^2\,dx \ll N^\gamma Q_c^{-1/2} + Q_c^2.$$

Now, we resolve the remaining integral. Let

$$\partial\tilde{Q}_H = \{q \in \mathbb{N}\colon \text{ exactly two of the inequalities } q \leq Q_a(q,p_a^A), q \leq Q_b(q,p_b^B), q \leq Q_c(q,p_c^C) \text{ hold}\}.$$

Observe that by Lemmata 6.34, 6.35, and 4.6, it follows that

$$\begin{aligned}
&\int_{\partial\mathfrak{M}_{H1}} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x)|^2 \, dx \\
&\qquad \le \sum_{q \in \tilde{Q}_H} \sum_{\substack{k=0 \\ (q,k)=1}}^{q} \int_{\partial\mathfrak{M}(q,k)} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x)|^2 \, dx \\
&\qquad \le \sum_{q \in \tilde{Q}_H} \varphi(q) |G(q/(q,p_a^A)) G(q/(q,p_b^B))|^2 \int_{-1/2}^{1/2} |I_\alpha(p_a^A x) I_\beta(p_b^B x)|^2 \, dx \\
&\qquad \ll \sum_{q \in \tilde{Q}_H} q |G(q)|^4 (N^{2\alpha+2\beta-1} + N^{\alpha+\beta}) \\
&\qquad \ll N^{2\alpha+2\beta-1} \sum_{q \ge Q_c} q |G(q)|^4 \\
&\qquad \ll N^{2\alpha+2\beta-1} Q_c^{-5/2}.
\end{aligned}$$

The proof is completed by combining the two components with the earlier inequality. □

This completes this section, and prepares all the tools needed to compute the double $f^\star$, single Δ integral error terms.

6.8 Double $f^\star$, Single Δ Error Terms

We shall now combine the above lemmas about behaviour on the harmonics with our understanding of the relation between the harmonic major and semimajor arcs and the entire major and semimajor arcs to give use the error terms derived from the integrals with two $f^\star$ and one Δ component.

First, the major arcs:

Lemma 6.49. *Fix $\lambda_a, \lambda_b, \lambda_c \in [0, 1/2]$ so*

$$Q_a \ll N^{\lambda_a \alpha}; \quad Q_b \ll N^{\lambda_b \beta}; \quad Q_c \ll N^{\lambda_c \gamma}.$$

If $D_H(\lambda_a, \lambda_b, \lambda_c)$ is weakly harmonic, then for all $\epsilon > 0$,

$$\begin{aligned}
\int_{\mathfrak{M}} |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x) \Delta_\gamma(-p_c^C x)| \, dx &\ll N^{\alpha+\beta-1/2+\epsilon}, \\
\int_{\mathfrak{M}} |f_\alpha^\star(p_a^A x) \Delta_\beta(p_b^B x) f_\gamma^\star(-p_c^C x)| \, dx &\ll N^{\alpha+\gamma-1/2+\epsilon}, \\
\int_{\mathfrak{M}} |\Delta_\alpha(p_a^A x) f_\beta^\star(p_b^B x) f_\gamma^\star(-p_c^C x)| \, dx &\ll N^{\beta+\gamma-1/2+\epsilon}.
\end{aligned}$$

Proof. This follows immediately from the application of Lemma 6.24 to Lemma 6.46. □

Second, the semimajor arcs:

Lemma 6.50. *Fix* $\lambda_a, \lambda_b, \lambda_c \in [0, 1/2]$ *so*

$$Q_a \ll N^{\lambda_a \alpha}; \quad Q_b \ll N^{\lambda_b \beta}; \quad Q_c \ll N^{\lambda_c \gamma}.$$

If $D_H(\lambda_a, \lambda_b, \lambda_c)$ *is strongly harmonic, then for all* $\epsilon > 0$,

$$\int_{\partial\mathfrak{M}} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x) \Delta_\gamma(-p_c^C x)|\, dx \ll N^{\alpha+\beta-1/2+\epsilon}(N^{\gamma/2} Q_c^{-3/2} + Q_c^{-1/4}),$$
$$\int_{\partial\mathfrak{M}} |f^\star_\alpha(p_a^A x) \Delta_\beta(p_b^B x) f^\star_\gamma(-p_c^C x)|\, dx \ll N^{\alpha+\gamma-1/2+\epsilon}(N^{\beta/2} Q_b^{-3/2} + Q_b^{-1/4}),$$
$$\int_{\partial\mathfrak{M}} |\Delta_\alpha(p_a^A x) f^\star_\beta(p_b^B x) f^\star_\gamma(-p_c^C x)|\, dx \ll N^{\beta+\gamma-1/2+\epsilon}(N^{\alpha/2} Q_a^{-3/2} + Q_a^{-1/4}).$$

Proof. We shall give an explicit proof for the first equation only. By Lemmata 6.47 and 6.48,

$$\begin{aligned}
&\int_{\partial\mathfrak{M}_H} |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x) \Delta_\gamma(-p_c^C x)|\, dx \\
&\qquad \ll N^{\alpha+\beta-1/2+\epsilon}\big((N^{\gamma/2} Q_c^{-7/4} + Q_c^{-1/2}) Q_0^{1/4} + (N^{\gamma/2} Q_c^{-3/2} + Q_c^{-1/4})\big) \\
&\qquad \ll N^{\alpha+\beta-1/2+\epsilon}(N^{\gamma/2} Q_c^{-3/2} + Q_c^{-1/4}).
\end{aligned}$$

By the strongly harmonic condition and Lemma 6.36, it follows by that $\partial\mathfrak{M}_H = \partial\mathfrak{M}$, completing the proof. □

As the semiminor and minor arcs have at least two minor components, any error term with two $f^\star$ components must be zero as $f^\star$ is by definition zero on the minor arcs, so we can now move on to the error over the whole integral:

Lemma 6.51. *Let*

$$Q_0 = \min\{Q_a, Q_b, Q_c\}.$$

Let $\lambda_a, \lambda_b, \lambda_c \in [0, 1/2]$ *such that*

$$Q_a \ll N^{\lambda_a \alpha}; \quad Q_b \ll N^{\lambda_b \beta}; \quad Q_c \ll N^{\lambda_c \gamma}.$$

If $D_H(\lambda_a, \lambda_b, \lambda_c)$ *is strongly harmonic, it follows that for all* $\epsilon > 0$,

$$\int_0^1 |f^\star_\alpha(p_a^A x) f^\star_\beta(p_b^B x) \Delta_\gamma(-p_c^C x)|\, dx \ll N^{\alpha+\beta-1/2+\epsilon}(1 + N^{\gamma/2} Q_c^{-3/2}),$$
$$\int_0^1 |f^\star_\alpha(p_a^A x) \Delta_\beta(p_b^B x) f^\star_\gamma(-p_c^C x)|\, dx \ll N^{\alpha+\gamma-1/2+\epsilon}(1 + N^{\beta/2} Q_b^{-3/2}),$$
$$\int_0^1 |\Delta_\alpha(p_a^A x) f^\star_\beta(p_b^B x) f^\star_\gamma(-p_c^C x)|\, dx \ll N^{\beta+\gamma-1/2+\epsilon}(1 + N^{\alpha/2} Q_a^{-3/2}).$$

Proof. This follows immediately from Lemmata 6.49 and 6.50 and the observation that on the semiminor and minor arcs, the integrals with two $f^\star$ components must be zero as each x in those arcs is in at most one major arc component. □

With this, we can conclude our work on the $1f^\star 2\Delta$ type integrals, and move on to the computation of the whole integral.

6.9 Computation of the Integral

In this section, and the next, we shall unite all the errors from the various cross products of $f^\star$ and Δ into a single function of Q_a, Q_b, Q_c.

Definition 6.52. Let us define the error term

$$E(Q_a, Q_b, Q_c; N) = \int_0^1 f(p_a^A x; N^\alpha) f(p_b^B x; N^\beta) f(-p_c^C x; N^\gamma)\, dx - \int_0^1 f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x) f_\gamma^\star(-p_c^C x)\, dx,$$

recalling that the definition of the $f^\star$ functions depend on Q_a, Q_b, Q_c.

From our earlier results, we construct a bound on E.

Lemma 6.53. *Suppose there exist $\lambda_a, \lambda_b, \lambda_c \in [0, 1/2]$ such that $Q_a \ll N^{\lambda_a \alpha}$, $Q_b \ll N^{\lambda_b \beta}$, $Q_c \ll N^{\lambda_c \gamma}$. If $D_H(\lambda_a, \lambda_b, \lambda_c)$ is strongly harmonic, then for all $\epsilon > 0$,*

$$\begin{aligned} E(Q_a, Q_b, Q_c; N) \ll\ & N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3}Q_b^{-1/2} + Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{2/3}) \\ &+ N^{(\alpha+2\beta+\gamma)/3+\epsilon}(N^{\alpha/3}Q_a^{-1/2} + Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{2/3}) \\ &+ N^{(\alpha+\beta+2\gamma)/3+\epsilon}(N^{\alpha/3}Q_a^{-1/2} + Q_a^{2/3})(N^{\beta/3}Q_b^{-1/2} + Q_b^{-2/3}) \\ &+ N^{\alpha+\beta-1/2+\epsilon}(1 + N^{\gamma/2}Q_c^{-3/2}) \\ &+ N^{\alpha+\gamma-1/2+\epsilon}(1 + N^{\beta/2}Q_b^{-3/2}) \\ &+ N^{\beta+\gamma-1/2+\epsilon}(1 + N^{\alpha/2}Q_a^{-3/2}). \end{aligned}$$

Proof. First, we observe that by definition,

$$
\begin{aligned}
&E(Q_a, Q_b, Q_c; N)\\
&= \int_0^1 f(p_a^A x; N^\alpha) f(p_b^B x; N^\beta) f(-p_c^C x; N^\gamma) dx - \int_0^1 f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x) f_\gamma^\star(-p_c^C x) dx\\
&= \int_0^1 (\Delta_\alpha(p_a^A x) - f_\alpha^\star(p_a^A x))(\Delta_\beta(p_a^A x) - f_\beta^\star(p_b^B x))(\Delta_\gamma(-p_c^C x) - f_\gamma^\star(-p_c^C x))\, dx\\
&\quad - \int_0^1 f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x) f_\gamma^\star(-p_c^C x)\, dx\\
&\le \int_0^1 |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx + \int_0^1 |f_\alpha^\star(p_a^A x)\Delta_\beta(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx\\
&\quad + \int_0^1 |\Delta_\alpha(p_a^A x) f_\beta^\star(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx + \int_0^1 |\Delta_\alpha(p_a^A x)\Delta_\beta(p_b^B x) f_\gamma^\star(-p_c^C x)|\, dx\\
&\quad + \int_0^1 |f_\alpha^\star(p_a^A x) f_\beta^\star(p_b^B x)\Delta_\gamma(-p_c^C x)|\, dx + \int_0^1 |f_\alpha^\star(p_a^A x)\Delta_\beta(p_b^B x) f_\gamma^\star(-p_c^C x)|\, dx\\
&\quad + \int_0^1 |\Delta_\alpha(p_a^A x) f_\beta^\star(p_b^B x) f_\gamma^\star(-p_c^C x)|\, dx.
\end{aligned}
$$

Thus by application of Lemmata 6.41, 6.45, and 6.51, one has that provided the conditions for Lemma 6.51 are satisfied, that for all $\epsilon > 0$,

$$
\begin{aligned}
E(Q_a, Q_b, Q_c; N) \ll\ & N^{(\alpha+\beta+\gamma)/3+\epsilon}(N^{\alpha/3}Q_a^{-1/2} + Q_a^{2/3})(N^{\beta/3}Q_b^{-1/2} + Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{2/3})\\
&+ N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3}Q_b^{-1/2} + Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{2/3})\\
&+ N^{(\alpha+2\beta+\gamma)/3+\epsilon}(N^{\alpha/3}Q_a^{-1/2} + Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{2/3})\\
&+ N^{(\alpha+\beta+2\gamma)/3+\epsilon}(N^{\alpha/3}Q_a^{-1/2} + Q_a^{2/3})(N^{\beta/3}Q_b^{-1/2} + Q_b^{-2/3})\\
&+ N^{\alpha+\beta-1/2+\epsilon}(1 + N^{\gamma/2}Q_c^{-3/2})\\
&+ N^{\alpha+\gamma-1/2+\epsilon}(1 + N^{\beta/2}Q_b^{-3/2})\\
&+ N^{\beta+\gamma-1/2+\epsilon}(1 + N^{\alpha/2}Q_a^{-3/2}).
\end{aligned}
$$

This gives us the general bound for this lemma, after noting that the first term is strictly smaller than the next three for any $\lambda \in [0, 1/2]$, and removing the strictly smaller terms. □

From this, we determine the geometric description of regions where it is possible to ensure that $E(Q_a, Q_b, Q_c)$

Lemma 6.54. *Let $\lambda_a, \lambda_b, \lambda_c \in (0, 1/2]$. Let $Q_a = N^{\lambda_a}, Q_b = N^{\lambda_b}, Q_c = N^{\lambda_c}$. Let*

$$
\begin{aligned}
\eta_a = \min\{1 + 3\lambda_a/2, 2 - 2\lambda_a\};\quad \eta_b = \min\{1 + 3\lambda_b/2, 2 - 2\lambda_b\};\quad \eta_c = \min\{1 + 3\lambda_c/2, 2 - 2\lambda_c\},\\
\nu_a = \min\{1 + 3\lambda_a, 2\};\quad \nu_b = \min\{1 + 3\lambda_b, 2\};\quad \nu_c = \min\{1 + 3\lambda_c, 2\}.
\end{aligned}
$$

For each $(\alpha,\beta,\gamma)\in\mathfrak{R}(\lambda_a,\lambda_b,\lambda_c)$ *there exists some* $\delta>0$ *such that*

$$E(Q_a,Q_b,Q_c;N)\ll N^{\alpha+\beta+\gamma-1-\delta},$$

where $\mathfrak{R}(\lambda_a,\lambda_b,\lambda_c)$ *is the intersection of the following volumes:*

$$\begin{aligned}
V_1&=\{(\alpha,\beta,\gamma)\in[0,1]^3\colon (1-\lambda_a)\alpha+(1-\lambda_b)\beta>1\},\\
V_2&=\{(\alpha,\beta,\gamma)\in[0,1]^3\colon (1-\lambda_a)\alpha+(1-\lambda_c)\gamma>1\},\\
V_3&=\{(\alpha,\beta,\gamma)\in[0,1]^3\colon (1-\lambda_b)\beta+(1-\lambda_c)\gamma>1\},\\
V_4&=\{(\alpha,\beta,\gamma)\in[0,1]^3\colon \alpha+\eta_b\beta+\eta_c\gamma>3\},\\
V_5&=\{(\alpha,\beta,\gamma)\in[0,1]^3\colon \eta_a\alpha+\beta+\eta_c\gamma>3\},\\
V_6&=\{(\alpha,\beta,\gamma)\in[0,1]^3\colon \eta_a\alpha+\eta_b\beta+\gamma>3\},\\
V_7&=\{(\alpha,\beta,\gamma)\in[0,1]^3\colon \nu_c\gamma>1\},\\
V_8&=\{(\alpha,\beta,\gamma)\in[0,1]^3\colon \nu_b\beta>1\},\\
V_9&=\{(\alpha,\beta,\gamma)\in[0,1]^3\colon \nu_a\alpha>1\}.
\end{aligned}$$

Proof. Observe that in order to satisfy the conditions for Lemma 6.53, we must have that

$$(1-\lambda_a)\alpha+(1-\lambda_a)\beta>1,\alpha(1-\lambda_a)+\gamma(1-\lambda_c)>1,\beta(1-\lambda_b)+\gamma(1-\lambda_c)>1,$$

which can be represented geometrically as a set of volumes V_1,V_2,V_3 with corresponding planar boundaries π_1,π_2,π_3:

$$\begin{aligned}
V_1&=\{(\alpha,\beta,\gamma)\colon (1-\lambda_a)\alpha+(1-\lambda_b)\beta>1\},\\
V_2&=\{(\alpha,\beta,\gamma)\colon (1-\lambda_a)\alpha+(1-\lambda_c)\gamma>1\},\\
V_3&=\{(\alpha,\beta,\gamma)\colon (1-\lambda_b)\beta+(1-\lambda_c)\gamma>1\},\\
\pi_1&=\{(\alpha,\beta,\gamma)\colon (1-\lambda_a)\alpha+(1-\lambda_b)\beta=1\},\\
\pi_2&=\{(\alpha,\beta,\gamma)\colon (1-\lambda_a)\alpha+(1-\lambda_c)\gamma=1\},\\
\pi_3&=\{(\alpha,\beta,\gamma)\colon (1-\lambda_b)\beta+(1-\lambda_c)\gamma=1\}.
\end{aligned}$$

Let $V=V_1\cap V_2\cap V_3$. By Lemma 6.53, for all $(\alpha,\beta,\gamma)\in V$ and all $\epsilon>0$,

$$\begin{aligned}
E(N^{\lambda_a},N^{\lambda_b},N^{\lambda_c};N)\ll{}& N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3}Q_b^{-1/2}+Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2}+Q_c^{2/3})\\
&+N^{(\alpha+2\beta+\gamma)/3+\epsilon}(N^{\alpha/3}Q_a^{-1/2}+Q_a^{2/3})(N^{\gamma/3}Q_c^{-1/2}+Q_c^{2/3})\\
&+N^{(\alpha+\beta+2\gamma)/3+\epsilon}(N^{\alpha/3}Q_a^{-1/2}+Q_a^{2/3})(N^{\beta/3}Q_b^{-1/2}+Q_b^{2/3})\\
&+N^{\alpha+\beta-1/2+\epsilon}(1+N^{\gamma/2}Q_c^{-3/2})\\
&+N^{\alpha+\gamma-1/2+\epsilon}(1+N^{\beta/2}Q_b^{-3/2})\\
&+N^{\beta+\gamma-1/2+\epsilon}(1+N^{\alpha/2}Q_a^{-3/2}).
\end{aligned}$$

It is clear from the above that there are two distinct forms of terms, with Q_a, Q_b, Q_c and α, β, γ permuted for a total of six terms.

First, we evaluate the geometric space on which $N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3}Q_b^{-1/2}+Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2}+Q_c^{2/3})$ is strictly less than $N^{\alpha+\beta+\gamma-1}$, by splitting the evaluation into four cases:

1. If $\lambda_b \leq 2/7$ and $\lambda_c \leq 2/7$,

$$\begin{aligned} N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3}Q_b^{-1/2} + Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{2/3}) \\ \ll N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3-\lambda_b\beta/2+\gamma/3-\lambda_c\gamma/2}) \\ \ll N^{2(\alpha+\beta+\gamma)/3-\lambda_b\beta/2-\lambda_c\gamma/2+\epsilon}, \end{aligned}$$

   which implies that the condition that the error be smaller than the main term is

$$\begin{aligned} 2(\alpha+\beta+\gamma)/3 - \lambda_b\beta/2 - \lambda_c\gamma/2 < \alpha+\beta+\gamma-1 \\ 1 - \lambda_b\beta/2 - \lambda_c\gamma/2 < (\alpha+\beta+\gamma)/3 \\ 3 < \alpha + (1+3\lambda_b/2)\beta + (1+3\lambda_c/2)\gamma, \end{aligned}$$

   and thus the volume V_4 where the error term is smaller than the main term is

$$V_4 = \{(\alpha,\beta,\gamma)\colon \alpha + (1+3\lambda_b/2)\beta + (1+3\lambda_c/2)\gamma > 3\},$$

   with corresponding boundary plane

$$\pi_4 = \{(\alpha,\beta,\gamma)\colon \alpha + (1+3\lambda_b/3)\beta + (1+3\lambda_c/2)\gamma = 3\}.$$

2. If $\lambda_b \leq 2/7$ and $\lambda_c > 2/7$,

$$\begin{aligned} N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3}Q_b^{-1/2} + Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{2/3}) \\ \ll N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3-\lambda_b\beta/2+2\lambda_c\gamma/3}) \\ \ll N^{(2\alpha+2\beta+\gamma)/3-\lambda_b\beta/2+2\lambda_c\gamma/3+\epsilon}, \end{aligned}$$

   which implies that the condition that the error be smaller than the main term is

$$\begin{aligned} (2\alpha+2\beta+\gamma)/3 - \lambda_b\beta/2 + 2\lambda_c\gamma/3 < \alpha+\beta+\gamma-1 \\ 1 - \lambda_b\beta/2 + 2\lambda_c\gamma/3 < (\alpha+\beta+2\gamma)/3 \\ 3 < \alpha + (1+3\lambda_b/2)\beta + (2-2\lambda_c)\gamma, \end{aligned}$$

   and thus the volume V_4 where the error term is smaller than the main term is

$$V_4 = \{(\alpha,\beta,\gamma)\colon \alpha + (1+3\lambda_b/2)\beta + (2-2\lambda_c)\gamma > 3\},$$

   with corresponding boundary plane

$$\pi_4 = \{(\alpha,\beta,\gamma)\colon \alpha + (1+3\lambda_b/3)\beta + (2-2\lambda_c)\gamma = 3\}.$$

3. If $\lambda_b > 2/7$ and $\lambda_c \leq 2/7$,

$$\begin{aligned} N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3}Q_b^{-1/2} + Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{2/3}) \\ \ll N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{2\lambda_b\beta/3+\gamma/3-\lambda_c\gamma/2}) \\ \ll N^{(2\alpha+\beta+2\gamma)/3+2\lambda_b\beta/3-\lambda_c\gamma/2+\epsilon}, \end{aligned}$$

which implies that the condition that the error be smaller than the main term is

$$\begin{aligned} (2\alpha+\beta+2\gamma)/3 + 2\lambda_b\beta/2 - \lambda_c\gamma/2 < \alpha+\beta+\gamma-1 \\ 1 + 2\lambda_b\beta/3 - \lambda_c\gamma/2 < (\alpha+2\beta+\gamma)/3 \\ 3 < \alpha + (2-2\lambda_b)\beta + (1+3\lambda_c/2)\gamma, \end{aligned}$$

and thus the volume V_4 where the error term is smaller than the main term is

$$V_4 = \{(\alpha,\beta,\gamma) \colon \alpha + (2-2\lambda_b)\beta + (1+3\lambda_c/2)\gamma > 3\},$$

with corresponding boundary plane

$$\pi_4 = \{(\alpha,\beta,\gamma) \colon \alpha + (2-2\lambda_b)\beta + (1+3\lambda_c/2)\gamma = 3\}.$$

4. If $\lambda_b > 2/7$ and $\lambda_c > 2/7$:

$$\begin{aligned} N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{\beta/3}Q_b^{-1/2} + Q_b^{2/3})(N^{\gamma/3}Q_c^{-1/2} + Q_c^{2/3}) \\ \ll N^{(2\alpha+\beta+\gamma)/3+\epsilon}(N^{2\lambda_b\beta/3+2\lambda_c\gamma/3}) \\ \ll N^{(2\alpha+\beta+\gamma)/3+2\lambda_b\beta/3+2\lambda_c\gamma/3+\epsilon}, \end{aligned}$$

which implies that the condition that the error be smaller than the main term is

$$\begin{aligned} (2\alpha+\beta+\gamma)/3 + 2\lambda_b\beta/3 + 2\lambda_c\gamma/3 < \alpha+\beta+\gamma-1 \\ 1 + 2\lambda_b\beta/3 + 2\lambda_c\gamma/3 < (\alpha+2\beta+2\gamma)/3 \\ 3 < \alpha + (2-2\lambda_b)\beta + (2-2\lambda_c)\gamma, \end{aligned}$$

and thus the volume V_4 where the error term is smaller than the main term is

$$V_4 = \{(\alpha,\beta,\gamma) \colon \alpha + (2-2\lambda_b)\beta + (2-2\lambda_c)\gamma > 3\},$$

with corresponding boundary plane

$$\pi_4 = \{(\alpha,\beta,\gamma) \colon \alpha + (2-2\lambda_b)\beta + (2-2\lambda_c)\gamma = 3\}.$$

By our choice of η_a, η_b, η_c, it is clear that this can be simplified so that

$$V_4 = \{(\alpha, \beta, \gamma) \colon \alpha + \eta_b\beta + \eta_c\gamma > 3\},$$

with boundary

$$\pi_4 = \{(\alpha, \beta, \gamma) \colon \alpha + \eta_b\beta + \eta_c\gamma = 3\}.$$

As there are two error terms that are merely permutations of the previous, it is clear that they are strictly smaller than $N^{\alpha+\beta+\gamma-1}$ on the corresponding volumes

$$V_5 = \{(\alpha, \beta, \gamma) \colon \eta_a\alpha + \beta + \eta_c\gamma > 3\},$$
$$V_6 = \{(\alpha, \beta, \gamma) \colon \eta_a\alpha + \eta_b\beta + \gamma > 3\}.$$

with corresponding boundaries

$$\pi_5 = \{(\alpha, \beta, \gamma) \colon \eta_a\alpha + \beta + \eta_c\gamma = 3\},$$
$$\pi_6 = \{(\alpha, \beta, \gamma) \colon \eta_a\alpha + \eta_b\beta + \gamma = 3\}.$$

We shall now consider the geometric space where $N^{\alpha+\beta-1/2+\epsilon}(1+N^{\gamma/2}Q_c^{-3/2})$ is strictly smaller than $N^{\alpha+\beta+\gamma-1}$. We split this problem into two cases:

1. If $\lambda_c \geq 1/3$,
$$N^{\alpha+\beta-1/2+\epsilon}(1 + N^{\gamma/2}Q_c^{-3/2}) = N^{\alpha+\beta-1/2+\epsilon},$$
so it follows the condition that the error be smaller than the main term is
$$\alpha + \beta - 1/2 < \alpha + \beta + \gamma - 1,$$
$$1/2 < \gamma,$$
and thus the volume V_7 where the error is smaller than the main term is
$$V_7 = \{(\alpha, \beta, \gamma) \colon 2\gamma > 1\},$$
with boundary
$$\pi_7 = \{(\alpha, \beta, \gamma) \colon 2\gamma = 1\}.$$

2. If $\lambda_c \leq 1/3$,
$$N^{\alpha+\beta-1/2+\epsilon}(1 + N^{\gamma/2}Q_c^{-3/2} = N^{\alpha+\beta+\gamma/2-3\lambda_c\gamma/2-1/2+\epsilon},$$
so it follows the condition that the error be smaller than the main term is
$$\alpha + \beta + \gamma/2 - 3\lambda_c\gamma/2 - 1/2 < \alpha + \beta + \gamma - 1,$$
$$1/2 < \gamma/2 + 3\lambda_c\gamma/2,$$
$$1 < (1 + 3\lambda_c)\gamma,$$

and thus the volume V_7 where the error is smaller than the main term is

$$V_7 = \{(\alpha, \beta, \gamma) \colon (1 + 3\lambda_c)\gamma > 1\},$$

with boundary

$$\pi_7 = \{(\alpha, \beta, \gamma) \colon (1 + 3\lambda_c)\gamma = 1\}.$$

By our choice of ν_a, ν_b, ν_c, it is clear that this simplifies to

$$V_7 = \{(\alpha, \beta, \gamma) \colon \nu_c\gamma > 1\},$$

with boundary

$$\pi_7 = \{(\alpha, \beta, \gamma) \colon \nu_c\gamma = 1\}.$$

The two remaining error terms are permutations of this one, so it follows the corresponding volumes on which those error terms are smaller than $N^{\alpha+\beta+\gamma-1}$ are

$$V_8 = \{(\alpha, \beta, \gamma) \colon \nu_b\beta > 1\},$$
$$V_9 = \{(\alpha, \beta, \gamma) \colon \nu_a\alpha > 1\},$$

with boundaries

$$\pi_8 = \{(\alpha, \beta, \gamma) \colon \nu_b\beta = 1\},$$
$$\pi_9 = \{(\alpha, \beta, \gamma) \colon \nu_a\alpha = 1\}.$$

Combining all these together gives the geometric object described by the lemma. □

Let us define the following volume:

Definition 6.55. Let $\mathfrak{R}$ be the following union of volumes:

$$\mathfrak{R} = \bigcup_{\lambda_a \in (0,1/2]} \bigcup_{\lambda_b \in (0,1/2]} \bigcup_{\lambda_c \in (0,1/2]} \mathfrak{R}(\lambda_a, \lambda_b, \lambda_c),$$

where $\mathfrak{R}(\lambda_a, \lambda_b, \lambda_c)$ is as defined in Lemma 6.54.

Observe that by Lemma 6.54, it is clear that $\mathfrak{R}$ is precisely the set of parameters (α, β, γ) such that there exist $\lambda_a, \lambda_b, \lambda_c$ where

$$E(N^{\lambda_a}, N^{\lambda_b}, N^{\lambda_c}; N) \ll N^{\alpha+\beta+\gamma-1-\delta}$$

for some $\delta > 0$.

We shall now give a geometric description of $\mathfrak{R}$:

Lemma 6.56. *The volume*

$$\mathfrak{R} = \bigcup_{2/7 \leq \lambda \leq 1/3} \big(\mathfrak{R}(\lambda, 2/7, 2/7) \cup \mathfrak{R}(2/7, \lambda, 2/7) \cup \mathfrak{R}(2/7, 2/7, \lambda)\big).$$

which contains the non-empty open volume bounded by

$$\begin{aligned} 0.7\alpha + \beta + \gamma > 2.1; &\qquad \alpha + 0.7\beta + \gamma > 2.1; &\qquad \alpha + \beta + 0.7\gamma > 2.1; \\ 13\alpha > 7; &\qquad 13\beta > 7; &\qquad 13\gamma > 7; \\ \alpha = 1; &\qquad \beta = 1; &\qquad \gamma = 1, \end{aligned}$$

and is contained in the volume

$$\begin{aligned} 0.7\alpha + \beta + \gamma > 2.1; &\qquad \alpha + 0.7\beta + \gamma > 2.1; &\qquad \alpha + \beta + 0.7\gamma > 2.1; \\ \alpha > 1/2; &\qquad \beta > 1/2; &\qquad \gamma > 1/2; \\ \alpha = 1; &\qquad \beta = 1; &\qquad \gamma = 1. \end{aligned}$$

Proof. First, we describe the geometric shape of $\mathfrak{R}(\lambda_a, \lambda_b, \lambda_c)$ for some fixed $\lambda_a, \lambda_b, \lambda_c \in (0, 1/2]$. Let $V_1, \dots, V_9$ be the volumes described by Lemma 6.54. Let η_a, η_b, η_c and ν_a, ν_b, ν_c be as defined in Lemma 6.54. Let $\pi_1, \dots, \pi_9$ be the planes bounding the corresponding volume.

1. The rectangular prism $X = V_7 \cap V_8 \cap V_9$.

   By the definition of the three volumes, it is clear they form a rectangular prism:

   $$X = (\frac{1}{\nu_a}, 1] \times (\frac{1}{\nu_b}, 1] \times (\frac{1}{\nu_c}, 1].$$

   Furthermore, it is clear that $X = (1/2, 1]^3$ if and only if $\lambda_a, \lambda_b, \lambda_c \geq 1/3$, with $X \subseteq (1/2, 1]^3$ otherwise.

2. The triangular pyramid $Y = V_4 \cap V_5 \cap V_6$.

   Observe that the boundary $\pi_4 \cap \pi_5$ forms the line

   $$\ell_1 = \left(\frac{1}{\eta_a - 1}, \frac{1}{\eta_b - 1}, \frac{\eta_a \eta_b - 1}{\eta_c(\eta_a - 1)(\eta_b - 1)}\right) t + (0, 0, 3),$$

   and likewise $\pi_4 \cap \pi_6$ forms the line

   $$\ell_2 = \left(\frac{1}{\eta_a - 1}, \frac{\eta_a \eta_c - 1}{\eta_b(\eta_a - 1)(\eta_c - 1)}, \frac{1}{\eta_c - 1}\right) t + (0, 3, 0),$$

   and finally $\pi_5 \cap \pi_6$ forms the line

   $$\ell_3 = \left(\frac{\eta_b \eta_c - 1}{\eta_a(\eta_b - 1)(\eta_c - 1)}, \frac{1}{\eta_b - 1}, \frac{1}{\eta_c - 1}\right) t + (3, 0, 0).$$

These three lines intersect at the point

$$P = 3(\eta_a - 1)(\eta_b - 1)(\eta_c - 1) \cdot \Bigg(\frac{1}{\eta_a(\eta_b - 1)(\eta_c - 1) + (\eta_a - 1)(\eta_b\eta_c - 1)}, \frac{1}{\eta_b(\eta_a - 1)(\eta_c - 1) + (\eta_b - 1)(\eta_a\eta_c - 1)}, \frac{1}{\eta_c(\eta_a - 1)(\eta_b - 1) + (\eta_c - 1)(\eta_a\eta_b - 1)} \Bigg).$$

Furthermore, it is clear that since $\eta_a, \eta_b, \eta_c \leq 10/7$ (with equality if and only if $\lambda = 2/7$),

$$V_4 \subseteq W_4 = \{(\alpha, \beta, \gamma) \colon \alpha + 10\beta/7 + 10\gamma/7 > 3\},$$
$$V_5 \subseteq W_5 = \{(\alpha, \beta, \gamma) \colon 10\alpha/7 + \beta + 10\gamma/7 > 3\},$$
$$V_6 \subseteq W_6 = \{(\alpha, \beta, \gamma) \colon 10\alpha/7 + 10\beta/7 + \gamma > 3\},$$

and that by the relation between η and λ as one moves away from maximum values one has that if

$$|\lambda_a - 2/7| \geq |\lambda'_a - 2/7|, \quad |\lambda_b - 2/7| \geq |\lambda'_b - 2/7|, \quad |\lambda_c - 2/7| \geq |\lambda'_c - 2/7|,$$

then for $j = 4, 5, 6$,

$$V_j(\lambda'_a, \lambda'_b, \lambda'_c) \subseteq V_j(\lambda_a, \lambda_b, \lambda_c).$$

Let $Z = V_1 \cap V_2 \cap V_3$. When $2/7 \leq \lambda_a, \lambda_b, \lambda_c \leq 1/3$, one has that $X \cap Y \subseteq Z$ since in this range, $V_1 \subseteq V_6$, $V_2 \subseteq V_5$ and $V_3 \subseteq V_4$.

We also know from manner in which X and Y shrink as the λ parameters move from $2/7$ and $1/3$ that it is necessarily the case that

$$\mathfrak{R} = \bigcup_{2/7 \leq \lambda_a \leq 1/3} \bigcup_{2/7 \leq \lambda_b \leq 1/3} \bigcup_{2/7 \leq \lambda_c \leq 1/3} \mathfrak{R}(\lambda_a, \lambda_b, \lambda_c),$$

and that furthermore,

$$\mathfrak{R} = \bigcup_{2/7 \leq \lambda \leq 1/3} \big(\mathfrak{R}(\lambda, 2/7, 2/7) \cup \mathfrak{R}(2/7, \lambda, 2/7) \cup \mathfrak{R}(2/7, 2/7, \lambda) \big).$$

The non-empty open volume given by the proof is the volume of $\mathfrak{R}(2/7, 2/7, 2/7)$, which is clearly the majority of $\mathfrak{R}$. The upper bound on the volume is given by the union of the individual maximums for X and Y in the range $2/7$ to $1/3$. □

Now, with a bound for the error term, when α, β, γ lies in $\mathfrak{R}$, it is possible to construct an asymptotic formula for the function $F(N)$.

Theorem 6.57. *Let P be as in Definition 6.26. If $(\alpha, \beta, \gamma) \in \mathfrak{R}$, then there exists some $\delta > 0$ such that*

$$F(N) = \omega\zeta(2)^{-3}\left(\prod_{p\in P}\frac{p^2-1}{p^2-2}\right)U(N) + O(N^{\alpha+\beta+\gamma-1-\delta}),$$

and specifically,

$$F(N) \leq \omega\zeta(2)^{-3}\left(\prod_{p\in P}\frac{p^2-1}{p^2-2}\right)N^{\alpha+\beta+\gamma-1} + O(N^{\alpha+\beta+\gamma-1-\delta}),$$

$$F(N) \geq \frac{\omega}{2p_c}\zeta(2)^{-3}\left(\prod_{p\in P}\frac{p^2-1}{p^2-2}\right)N^{\alpha+\beta+\gamma-1} + O(N^{\alpha+\beta+\gamma-1-\delta}).$$

Proof. By Lemma 6.24, for any $(\alpha, \beta, \gamma) \in \mathfrak{R}$, there exist $\lambda_a, \lambda_b, \lambda_c \in (0, 1/2]$ such that

$$\min D_H(\lambda_a, \lambda_b, \lambda_c) > 1 \text{ and } \mathfrak{M} = \mathfrak{M}_H.$$

Let $Q_a = N^{\lambda_a\alpha}$, $Q_b = N^{\lambda_b\beta}$, $Q_c = N^{\lambda_c\gamma}$. By definition,

$$F(N) = \int_0^1 f^\star_\alpha(p_a^A x)f^\star_\beta(p_b^B x)f^\star_\gamma(-p_c^C x)dx + E(Q_a, Q_b, Q_c; N).$$

Since for any $x \in [0, 1)$, $f^\star_\alpha(p_a^A x) \neq 0$ if and only if $x \in \mathfrak{M}_\alpha$, $f^\star_\beta(p_b^B x) \neq 0$ if and only if $x \in \mathfrak{M}_\beta$, and $f^\star_\gamma(-p_c^C x) \neq 0$ if and only if $x \in \mathfrak{M}_\gamma$, it follows that

$$F(N) = \int_{\mathfrak{M}} f^\star_\alpha(p_a^A x)f^\star_\beta(p_b^B x)f^\star_\gamma(-p_c^C x)dx + E(Q_a, Q_b, Q_c; N).$$

As $\mathfrak{M} = \mathfrak{M}_H$ and $(\alpha, \beta, \gamma) \in \mathfrak{R}$, by Lemmata 6.30 and 6.56, it follows that there exists some $\delta > 0$ such that

$$F(N) = \omega\zeta(2)^{-3}\left(\prod_{p\in P}\frac{p^2-1}{p^2-2}\right)U(N) + O(N^{\alpha+\beta+\gamma-1-\delta}.$$

To complete this proof, simply factor in the bounds on $U(N)$ as determined in Lemma 6.17.

Note that the error terms of $U(N)$ can be accounted for by making δ small enough that

$$\alpha + \beta + \gamma - 1 - \delta > \max\{\alpha, \beta, \gamma\},$$

which is possible since for all $(\alpha, \beta, \gamma) \in \mathfrak{R}$, $\alpha + \beta + \gamma - 1 > \max\{\alpha, \beta, \gamma\}$. □

At the end of all this, we have an asymptotic formula which is valid for all parameters $(\alpha, \beta, \gamma) \in \mathfrak{R}$. However, we are not yet done. Our initial problem dealt with the limit points in the set $\mathfrak{Q}$. As discussed at the beginning of §6, this is not enough to prove immediately that $\mathfrak{R} \subseteq \mathfrak{Q}'$ as $F(N)$ does not distinguish between relatively prime triples and places no restriction on the relative sizes of the first two numbers, so it does not count only *abc*-triples. In the next section, we prove Theorem 2.2.

6.10 Proof of Theorem 2.2

Here, we show that the closure of $\mathfrak{R}$ (a volume whose bounds given in Lemma 6.56) lies in $\mathfrak{Q}'$. As $\mathfrak{Q}'$ is a closed set, it suffices to show that $\mathfrak{R}$ lies in $\mathfrak{Q}'$.

For any $\delta > 0$ let

$$\begin{aligned} S_\delta(N) = \{(a,b,c) \in \mathbb{N}^3 \colon & p_a^A a + p_b^B b = p_c^C c, \gcd(a,b) \geq N^\delta, \\ & N^{\alpha-\delta} < a \leq N^\alpha, N^{\beta-\delta} < b \leq N^\beta, N^{\gamma-\delta} < c < N^\gamma \\ & a,b,c \text{ are squarefree}\}. \end{aligned}$$

Let $F_\delta^\star(N) = \#(S_\delta(N))$.

Recall $U_\delta(N)$ from Definition 6.10 and $F_\epsilon(N)$ from Definition 6.11. For any $\delta > 0$ one has that

$$F_\delta^\star(N) \geq F(N) - U_\delta(N) - F_\delta(N).$$

If $(\alpha, \beta, \gamma) \in \mathfrak{R}_{2/7}$, by Theorem 6.57, one has that there exists some $\epsilon > 0$ such that

$$F(N) \geq \frac{\omega}{2p_c}\zeta(2)^{-3}\left(\prod_{p\in P}\frac{p^2-1}{p^2-2}\right)N^{\alpha+\beta+\gamma-1} + O(N^{\alpha+\beta+\gamma-1-\epsilon}).$$

where P is the set described in Definition 6.26, and N is sufficiently large.

It follows from Lemmata 6.18 and 6.20 that for all $\delta > 0$

$$F_\delta^\star(N) \geq \frac{\omega}{2p_c}\zeta(2)^{-3}\left(\prod_{p\in P}\frac{p^2-1}{p^2-2}\right)N^{\alpha+\beta+\gamma-1} + O(N^{\alpha+\beta+\gamma-1-\min\{\epsilon,\delta\}}).$$

for N sufficiently large. Thus for every $\delta > 0$ there exists some $M_\delta \in \mathbb{R}$ such that for all $N \geq M_\delta$, $S_\delta(N)$ is non-empty.

In the rest of the proof we take A, B, C to depend in the usual way on N, but with $N = \max_{1\leq m\leq n}\{M_{1/m}\} + n$. That is to say, A, B, C depend on n. From M_δ above, there exists a sequence of triples T where

$$T = \{(p_a^A a_n, p_b^B b_n, p_c^C c_n)\}_{n=1}^\infty, (a_n, b_n, c_n) \in S_{1/n}(\max_{1\leq m\leq n}\{M_{1/n}\} + n).$$

First, observe that for $(p_a^A a_n, p_b^B b_b, p_c^C c_n) \in T$ there is some $d_n = \gcd(a_n, b_n)$ where $d_n < (\max_{1\leq m\leq n}\{M_{1/m}\} + n)^{1/n}$. Let $a_n' = a_n/d_n, b_n' = b_n/d_n$, and $c_n' = \gcd(p_c, d_n)c_n/d_n$. As d_n divides a_n, d_n is squarefree, so $\gcd(p_c, d_n) = \gcd(p_c^C, d_n)$, and thus it follows that

$$p_a^A a_n' + p_b^B b_n' \in \{p_c^{C-1}c_n', p_c^C c_n'\}.$$

Thus for $N = \max_{1 \le m \le n}\{M_{1/m}\} + n$

$$N^{\alpha-2/n} \le a'_n \le N^{\alpha-1/n}; N^{\beta-2/n} \le b'_n \le N^{\beta-1/n}; N^{\gamma-2/n} \le c'_n \le p_c N^{\gamma-1/n}.$$

Hence, it follows that for A, B, C as functions of N that

$$\begin{aligned}
&\frac{\log \mathrm{Rad}(p_a^A a'_n)}{\log(p_a^A a'_n + p_b^B b'_n)} \\
&\quad \in \left[\frac{\log(a'_n)}{C\log(p_c) + \log(c'_n)}, \frac{\log(p_a) + \log(a'_n)}{(C-1)\log(p_c) + \log(c'_n)}\right] \\
&\quad \subseteq \left[\frac{(\alpha - 2/n)\log(N)}{(1-\gamma+\gamma-1/n)\log(N) + \log(p_c)}, \frac{(\alpha-1/n)\log(N) + \log(p_a)}{(1-\gamma+\gamma-2/n)\log(N) - \log(p_c)}\right] \\
&\quad = \left[\frac{\alpha - 2/n}{1 - 1/n + \log(p_c)/\log(N)}, \frac{\alpha - 1/n + \log(p_a)/\log(N)}{1 - 2/n - \log(p_c)/\log(N)}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\log \mathrm{Rad}(p_b^B b'_n)}{\log(p_a^A b'_n + p_b^B b'_n)} \\
&\quad \in \left[\frac{\log(b'_n)}{C\log(p_c) + \log(c'_n)}, \frac{\log(p_b) + \log(b'_n)}{(C-1)\log(p_c) + \log(c'_n)}\right] \\
&\quad \subseteq \left[\frac{(\beta - 2/n)\log(N)}{(1-\gamma+\gamma-1/n)\log(N) + \log(p_c)}, \frac{(\beta-1/n)\log(N) + \log(p_b)}{(1-\gamma+\gamma-2/n)\log(N) - \log(p_c)}\right] \\
&\quad = \left[\frac{\beta - 2/n}{1 - 1/n + \log(p_c)/\log(N)}, \frac{\beta - 1/n + \log(p_b)/\log(N)}{1 - 2/n - \log(p_c)/\log(N)}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\log \mathrm{Rad}(p_a^A a'_n + p_b^B b'n)}{\log(p_a^A a'_n + p_b^B b'_n)} \\
&\quad \in \left[\frac{\log(c'_n)}{C\log(p_c) + \log(c'_n)}, \frac{\log(p_c) + \log(c'_n)}{(C-1)\log(p_c) + \log(c'_n)}\right] \\
&\quad \subseteq \left[\frac{(\gamma - 2/n)\log(N)}{(1-\gamma+\gamma-1/n)\log(N) + \log(p_c)}, \frac{(\gamma-1/n)\log(N) + \log(p_c)}{(1-\gamma+\gamma-2/n)\log(N) - \log(p_c)}\right] \\
&\quad = \left[\frac{\gamma - 2/n}{1 - 1/n + \log(p_c)/\log(N)}, \frac{\gamma - 1/n + \log(p_c)/\log(N)}{1 - 2/n - \log(p_c)/\log(N)}\right].
\end{aligned}$$

As $N \ge n$, it follows that,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log \mathrm{Rad}(p_a^A a'_n)}{\log(p_a^A a'_n + p_b^B b'_n)} &= \alpha, \\
\lim_{n\to\infty} \frac{\log \mathrm{Rad}(p_b^B b'_n)}{\log(p_a^A a'_n + p_b^B b'_n)} &= \beta, \\
\lim_{n\to\infty} \frac{\log \mathrm{Rad}(p_a^A a'_n + p_b^B b'_n)}{\log(p_a^A a'_n + p_b^B b'_n)} &= \gamma.
\end{aligned}$$

Observe that since $a'_n \geq N^{\alpha - 2/n}$, it follows $a' \geq n^{\alpha-2/n}$. This implies that there are infinitely many unique triples (a'_n, b'_n, c'_n) derived from the triples in T. Recall from the beginning that for $x, y \in \mathbb{N}$,

$$\overrightarrow{\mathfrak{iq}}(x,y) = (\log x + y)^{-1}(\log \operatorname{Rad} x, \log \operatorname{Rad} y, \log \operatorname{Rad} x + y),$$

So we thus have that $\lim_{n\to\infty} \overrightarrow{\mathfrak{iq}}(p_a^A a'_n, p_b^B b'_n) = (\alpha, \beta, \gamma)$. This gives us $(\alpha, \beta, \gamma) \in \mathfrak{Q}'$. $\square$

Chapter 7

Conclusion

7.1 Binary Forms, Polynomial Identities, and Limit Points in Three Dimensions

Our work on polynomial and binary forms in terms of a three-dimensional vector of the inverse quality reveals that the techniques used in the one-dimensional case are something akin to an optical illusion. Results in terms of limit points in the one-dimensional case give closed lines that cover almost all (or, when the *abc* conjecture is assumed, all) of the possible space. However, when one determines what these results mean in three dimensions, one obtains a set of disjoint line segments and various points lying in an empty volume.

This is not to say that the one-dimensional results and techniques are not impressive. They still give us the points with minimum inverse quality: $(2/5, 19/60, 7/20), (19/60, 2/5, 7/20) \in \mathfrak{Q}'$ unconditionally. However, any attempt to translate or apply them to obtain a volume seems to fail. In fact, it seems that in order to obtain a volume with those techniques, it would be necessary to find an infinite set of polynomials (or binary forms) whose identities could be manipulated in the proper manner. This is not necessarily impossible, but given the restrictions that apply to say, the BFGS Theorem, it is certainly no easy task.

In many ways we are left with more questions than answers. Given the extent to which, without assuming the *abc* conjecture, it is possible to obtain points close to the boundary $x + y + z = 1$, it seems possible that the BFGS Theorem, or an improvement of it, combined with a sufficiently large family of the right binary forms, could produce not only a volume but a large volume. However, it is not clear at all where one would start to look for such forms; many of the families discussed in [1] seem interesting at first, but the best application of them to the problem at hand is not at all obvious.

7.2 Circle Method, Possible Optimizations and Generalizations

The end result of our work in §6 was to prove Theorem 2.2. The volume described in this theorem, $\mathfrak{R} \subseteq \mathfrak{Q}'$, has the following geometric properties:

- The point on the boundary of $\mathfrak{R}$ nearest the plane $\alpha + \beta + \gamma = 1$ is $(7/9, 7/9, 7/9)$. This corresponds to the inverse quality of 21/9. Polynomial methods get closer but this is the closest the circle method comes.
- The volume of $\mathfrak{R}$ is positive.

The volume $\mathfrak{R}$ falls far short of the ideal volume of $\mathfrak{H}$. This was expected, given the difficulty of obtaining one-dimensional limit points with inverse quality near one. However, there are several possible avenues for improving the result. As the volume $\mathfrak{R}$ is directly related to the size of the various error terms computed in §6, improving $\mathfrak{R}$ is simply a matter of finding better bounds on the various errors.

The first source of errors come from the $\Delta\Delta\Delta$ type integrals (see §6.5). In the current computation, these error bounds are strictly smaller than the error bounds for $f^{\star}\Delta\Delta$ (see §6.6) type integrals, so optimization of these should not be the highest priority. However, if these at some point need to be optimized, the most likely means of doing so is by finding a smaller bound on the L^3-norm for the Δ function from [2] over the major arcs, and a smaller bound on L^3-norm for the f function over the minor arcs. Such improved bounds may be difficult to find for the L^3-norm directly, so it may be better to first compute the L^4-norms for the Δ function over the major arcs, and the L^4-norms for the f function over the minor arcs.

Similarly, one may look at the $f^{\star}\Delta\Delta$ type integrals. These integrals have the most substantial error term and give rise to the pyramidal tip farthest from the trivial corner. Improving these errors will have an immediate effect on the volume of $\mathfrak{R}$. Like the $\Delta\Delta\Delta$ type integrals, optimizing the L^3-norms of the Δ function on the major arcs and the f function on the minor arcs will shrink all the errors in this section. The L^3-norm of $f^{\star}$ on the major arcs cannot be optimized further by the simple observation that the truncated exponential sum has an exact value, and that value is the growth rate used: it cannot be improved.

There are also possible improvements in technique in the $f^{\star}\Delta\Delta$ section. Lemma 6.43 is particularly bad, and it is possible a carefully constructed dyadic dissection would improve it. In general, in this section, it is likely that the dyadic dissections used could be improved upon. In particular, finding the right parameters to allow the dissection of the minor arcs along with the major arcs would yield a substantial improvement. It should be noted that a naïve attempt at this will fail, as the bound of f along the minor arcs increases once the R parameter is sufficiently small; the proper setup must account for this.

One final possible method to improve the $f^\star\Delta\Delta$ error terms is to somehow factor the notion that the minor arc integrals are not over whole minor arcs but a smaller subset of distorted major arc images lying in the minor arcs. No technique seems to work correctly here, but if it could be done, it may substantially improve the error terms.

The $f^\star f^\star\Delta$ type integrals (see §6.7 and §6.8 are fairly distinct. If one is content to remain in the cube $[1/2, 1]^3$, it may suffice to improve the other errors and accept the limitations of the technique. However, if the L^3-norms are sufficiently improved, and techniques for scaling the R parameters with regard to minor arcs are refined, it may be possible to replace the atypical split used in $f^\star f^\star\Delta$ with the three L^3-norm Hölder inequality used in the previous two sections. If this is done, and the bounds are good enough, it may be possible to escape from this cube.

Finally, regardless of how much these error terms improve, there is a hard limit on the volume of limit points that is independent of $\mathfrak{R}$: the function $U(N)$ described in §6. This function is essential to the computation of the limit points, as it gives the growth rate for the main term. Recall that its formula is $U(N) = N^{\alpha+\beta+\gamma-1} + O(N^\alpha + N^\beta + N^\gamma)$. Thus the main term is greater than the error term as long as $\alpha + \beta > 1$, $\alpha + \gamma > 1$, and $\beta + \gamma > 1$. Thus these three half-planes put an ultimate bound on the error terms.

Outside of improving the volume, one may also consider generalizations of the function $F(N)$ described in §6. One easy generalization, which was not done solely for the sake of simplifying calculations, is to switch from three arbitrary distinct primes p_a, p_b, p_c to instead three arbitrary, relatively prime squarefree natural numbers s_a, s_b, s_c. There is no reason for this not to work, as at no point is the primality of p_a, p_b, p_c invoked except to simplify some calculations. Going further, and replacing p_a, p_b, p_c with arbitrary, relatively prime natural numbers n_a, n_b, n_c may also work.

Finally, we shall note that the asymptotic formula for $F(N)$ may be of interest in and of itself, as it counts the multiples of large powers of primes.

7.3 Relationship with Kane's Paper

The work we have done is similar in several respects to a paper by Kane [4]. His results are of a sufficiently similar nature that a discussion the relation between his result and our result will aid in the understanding of both. We seek to show that both our result and Kane's are, despite being similar in form, sufficiently different that the task of showing whether or not they are equivalent problems is non-trivial.

Kane is concerned with the bounds of a function, $S_{\alpha,\beta,\gamma}(N)$:

Definition 7.1. For $\alpha, \beta, \gamma \in [0,1]$, let $S_{\alpha,\beta,\gamma}\colon \mathbb{N} \to \mathbb{N}$ be the function

$$S_{\alpha,\beta,\gamma}(N) = \#\{(a,b,c) \in \mathbb{Z}^3 \colon a+b+c=0,$$
$$\operatorname{Rad} a \le |a|^\alpha, \operatorname{Rad} b \le |b|^\beta, \operatorname{Rad} c \le |c|^\gamma, \max\{|a|,|b|,|c|\} \le N\}.$$

By comparison, our counting functions, described in §5 and §6 count subsets of $\mathbb{N}^3$ where $c = a+b$, thus ensuring that $c > \max\{a,b\}$. Kane's function counts a set closed under any permutation, while our functions count sets that only allow the permutation of a and b. Thus our counting functions are sensitive to asymmetries in the distribution of radical sizes between the sum and the summands, while Kane's is not. While there is no reason to expect there to be asymmetries of this sort, the absence of such an asymmetry would be desirable to prove, which is why we constructed our functions in this manner.

Returning to Kane's result, his paper was concerned with finding the largest set of $(\alpha,\beta,\gamma) \in [0,1]^3$ such that the following inequality held: that for all $\epsilon > 0$, and N sufficiently large,

$$N^{\alpha+\beta+\gamma-1-\epsilon} < S_{\alpha,\beta,\gamma}(N) < N^{\alpha+\beta+\gamma-1+\epsilon}.$$

Kane's exact results are slightly more precise than these bounds, but the set of $(\alpha,\beta,\gamma) \in [0,1]^3$ that Kane can prove have the desired bounds for any N sufficiently large is the region

$$\mathfrak{K} = \{(\alpha,\beta,\gamma) \in [0,1]^3 \colon \alpha+\beta+\gamma \ge 2\},$$

with the lower bound being close for $\alpha+\beta+\gamma > 1$.

It should be noted that our methods cannot improve on Kane's bounds for $S_{\alpha,\beta,\gamma}$. The function that we have that would be best suited to this would be $U(N)$ from §6, which is as described only suitable for finding a lower bound to $S_{\alpha,\beta,\gamma}$, and has error terms that dominate the main term before $\alpha+\beta+\gamma > 1$.

Likewise, it should be noted that from the definition of the bounds above, obtaining limit points from the bounds of the counting function of $S_{\alpha,\beta,\gamma}$ is non-trivial: since $S_{\alpha,\beta,\gamma}$ counts $a,b,c \in \mathbb{Z}$, where $\log \operatorname{Rad} a \le \alpha \log|a|$, $\log \operatorname{Rad} b \le \beta \log|b|$, and $\log \operatorname{Rad} c \le \gamma \log|c|$, the ranges of the sizes of the radicals are too broad to simply take elements of this form, take N to infinity, and construct a sequence that gives a limit point.

However, despite this obstacle, it should be noted that under reasonable assumptions about the distribution of squarefree numbers, it would seem that this problem could be overcome. However, such assumptions do not constitute proof, as it is unclear how the squarefree numbers are distributed in the set $\Lambda \cap P$.

Given the difficulties, above, the question arises: can one extract a limit point, not from the inequality, but from Kane's proof of the lower bound? To do so, we shall consider the techniques used by Kane to find his lower bound.

To prove his lower bound for a given α, β, γ and N, Kane constructs a lattice

$$\Lambda = \{(a, b, c) \colon a + b + c = 0, 2^x q^w \mid a, 3^y \mid b, 5^z \mid c\},$$

where $x, y, w, z, q \in \mathbb{N}$ are chosen such that

$$\begin{aligned} N^{1-\alpha} &\le 2^x q^w / \operatorname{Rad}(2^x q^w) \le 2qN^{1-\alpha}, \\ N^{1-\beta} &\le 3^y / \operatorname{Rad}(3^y) \le 3N^{1-\beta}, \\ N^{1-\gamma} &\le 5^z / \operatorname{Rad}(5^z) \le 5N^{1-\gamma}, \end{aligned}$$

and that $5 < q \ll \log N$ is prime and for any $A, B, C \in \mathbb{Z}$ where $2^x q^w \mid A$, $3^y \mid B$ and $5^z \mid C$, $\max(|A|, |B|, |C|) \ge \Omega(N^{1/2-\alpha-\beta-\gamma} / \log(N))$.

By means of a geometry of numbers argument involving the volume on the polygon P where $|A| \le N, |B| \le N, |C| \le N$, and Lemma 6.13, he proves that $S_{\alpha,\beta,\gamma} = \Omega(N^{\alpha+\beta+\gamma-1} \log(N)^{-2})$.

However, this system cannot be used to find limit points in $\mathfrak{Q}'$. The issue lies in the fact that for any $(a, b, c) \in \Lambda \cap P$, the vector inverse quality is of the form (assuming without loss of generality that $|c| > |a|, |b|$),

$$\overrightarrow{\mathfrak{iq}}(a, b) = \left(\frac{\log \operatorname{Rad} a}{\log c}, \frac{\log \operatorname{Rad} b}{\log c}, \frac{\log \operatorname{Rad} c}{\log c} \right).$$

It is clear from the nature of the problem that there is no non-constant lower bound for $\log \operatorname{Rad} a, \log \operatorname{Rad} b$ and only the upper bound of $\log \operatorname{Rad} a \ll \alpha \log N, \log \operatorname{Rad} b \ll \beta \log N$. This wide range effectively prevents the finding of a limit point, at least, not without substantial work. There are also some difficulties in reconciling the sizes of a, b, and c, but those could be overcome with similar techniques as found in the geometry of numbers section of §6.

Thus we can conclude that there is no simple logical bridge between Kane's result and ours. However, it may be the case that there is a bridge of sorts. Kane uses a slightly different formulation of a lattice than we do in order to obtain a superior range of lower bounds for his problem. In our treatment of the function $U(N)$, we use a weaker but simpler construction of a lattice to count possible solutions before filtering them to be squarefree. While our choice of lattice does not currently affect our set of limit points, as other error terms dominate long before the errors of $U(N)$ are reached, should improvement to those bounds result in the errors of $U(N)$ becoming limiting, reformulating the problem so that $U(N)$ can use Kane's more refined construction of a lattice might be a good idea, and would constitute a bridge between results, albeit a very non-trivial one.

As for the reverse, going from our results to Kane's result, since our result covers less three dimensional space than Kane's, there is little motivation to explore it in detail, but the counting function F described in §6 can also be applied to the problem in Kane's paper. We shall sketch a quick, informal argument here, as the full details would amount to a nearly complete repetition of the proof of our theorem. Let $F_{\alpha,\beta,\gamma}$ be this function with parameters α, β, γ. By the same argument as in the proof for Theorem 2.2, one obtains a sequence a_n, b_n such that $a_n = p_a^A x$, $b_n = p_b^B y$, $a_n + b_n = p_c^C z$, where for each a_n, b_n, $p_a^A \approx N^{1-\alpha}$, $p_b^B \approx N^{1-\beta}$, $p_c^C \approx N^{1-\gamma}$, and $x, y, z \in \mathbb{N}$ are squarefree and $x \approx N^\alpha$, $y \approx N^\beta$, and $z \approx N^\gamma$. This would then give a lower bound on $S_{\alpha,\beta,\gamma}$.

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Appendix A

Intersection of a Plane and Rectangular Prism

In this appendix, we write the technical details involved in computing the volume and perimeter of the polygon P_N described in Lemma 6.15

Let $N > 0$, $P_N \subseteq \mathbb{R}^3$ be the polygon produced by the intersection of the solid rectangular prism $R_N = (0, N^\alpha] \times (0, N^\beta] \times (0, N^\gamma]$ and the plane $V_N = \{(x, y, z) \in \mathbb{R}^3 \colon p_a^A x + p_b^B y = p_c^C z\}$.

Let $\rho = \sqrt{1 + \frac{h_c^2}{h_a^2} N^{2\alpha - 2\gamma} + \frac{h_c^2}{h_b^2} N^{2\beta - 2\gamma}}$.

By the definition of P_N, one may observe that its boundary is the closed loop contained in the intersection of the planes defining the faces of R_N and the planar vector space V_N, which is the following set of lines:

$$\begin{aligned}
\ell_{x1}(t) = (1, 0, \frac{p_a^A}{p_c^C})t, \ell_{x2}(t) &= (1, 0, \frac{p_a^A}{p_c^C})t + (0, 1, \frac{p_b^B}{p_c^C})N^\beta, \\
\ell_{y1}(t) = (0, 1, \frac{p_b^B}{p_c^C})t, \ell_{y2}(t) &= (0, 1, \frac{p_b^B}{p_c^C})t + (1, 0, \frac{p_a^A}{p_c^C})N^\alpha, \\
\ell_{z1}(t) = (1, -\frac{p_a^A}{p_b^B}, 0)t, \ell_{z2}(t) &= (1, -\frac{p_a^A}{p_b^B}, 0)t + (0, \frac{p_c^C}{p_b^B}, 1)N^\gamma.
\end{aligned}$$

Note that lines with the same letter in the subscript are distinct parallel lines as $N > 0$.

Thus the vertices of P_N must be a subset of the intersections of the lines given above. Excluding repeated vertices and those that unconditionally lie outside the first octant, the following points become the possible vertices of P_N in addition to the origin:

$$\pi_{X1} = \ell_{x1} \cap \ell_{y2} = (N^\alpha, 0, \frac{h_c}{h_a} N^\gamma),$$
$$\pi_{X2} = \ell_{x1} \cap \ell_{z2} = (\frac{h_a}{h_c} N^\alpha, 0, N^\gamma),$$
$$\pi_{Y1} = \ell_{x2} \cap \ell_{y1} = (0, N^\beta, \frac{h_c}{h_b} N^\gamma),$$
$$\pi_{Y2} = \ell_{y1} \cap \ell_{z2} = (0, \frac{h_b}{h_c} N^\beta, N^\gamma),$$
$$\pi_{Z1} = \ell_{x2} \cap \ell_{z2} = ((\frac{h_a}{h_c} - \frac{h_a}{h_b}) N^\alpha, N^\beta, N^\gamma),$$
$$\pi_{Z2} = \ell_{y2} \cap \ell_{z2} = (N^\alpha, (\frac{h_b}{h_c} - \frac{h_b}{h_a}) N^\beta, N^\gamma),$$
$$\pi_1 = \ell_{x2} \cap \ell_{y2} = (N^\alpha, N^\beta, (\frac{h_c}{h_a} + \frac{h_c}{h_b}) N^\gamma).$$

To complete the proof, start with the fact that $(0,0,0)$ is a vertex of P_N and ℓ_{x1} a line intersecting an edge of the boundary of P_N. By traversal of this line in the direction $(1,0,0)$ (since P_N lies in the first octant) until the first intersection of two lines on the boundary is reached, rotating the direction of traversal counterclockwise to match the new line, and repeating the process until arriving back at $(0,0,0)$ it is possible to trace the polygon P_N.

Triangle. Suppose $h_a \leq h_c$ and $h_b \leq h_c$.

Observe that traversal starts at $(0,0,0)$, goes along ℓ_{x1} until the intersection π_{X2} which occurs before π_{X1} since $h_a/h_c \leq 1$. Now traversal occurs on ℓ_{z2} until the intersection π_{Y2} which occurs before π_{Z1} since $h_b/h_c \leq 1$. Finally, traversal moves down along ℓ_{y1} and returns to $(0,0,0)$.

$$\pi_0 \to \pi_{X2} \to \pi_{Y2} \to \pi_0.$$

Thus P_N is a triangle and

$$\text{Perimeter}(P_N) = d(\pi_0, \pi_{X2}) + d(\pi_{X2}, \pi_{Y2}) + d(\pi_{Y2}, \pi_0).$$

The projection of P_N onto the xy plane S_N has vertices

$$(0,0), (\frac{h_a}{h_c} N^\alpha, 0), (0, \frac{h_b}{h_c} N^\beta)$$

and thus by the formula for the area of a triangle,

$$\text{Volume}(S_N) = \frac{1}{2} \frac{h_a h_b}{h_c^2} N^{\alpha+\beta}.$$

Quadrilateral I. Suppose $h_a \leq h_c$ and $h_b > h_c$.

Observe that traversal starts at $(0,0,0)$ along ℓ_{x1} until π_{X2}, same as in the previous case. Now traversal occurs along ℓ_{z2} until the intersection at π_{Z1} which occurs before π_{Y2} since $h_b/h_c > 1$. Now traversal moves along ℓ_{x2} towards the y-axis until the intersection at π_{Y1}. As $h_b/h_c > 1$, there are no intersections between π_{Y1} and the origin and so the traversal completes the circuit.

$$\pi_0 \to \pi_{X2} \to \pi_{Z1} \to \pi_{Y1} \to \pi_0.$$

Thus P_N is a quadrilateral and

$$\text{Perimeter}(P_N) = d(\pi_0, \pi_{X2}) + d(\pi_{X2}, \pi_{Z1}) + d(\pi_{Z1}, \pi_{Y1}) + d(\pi_{Y1}, \pi_0).$$

The projection of P_N onto the xy plane S_N has vertices

$$(0,0), (\frac{h_a}{h_c}N^\alpha, 0), \beta), ((\frac{h_a}{h_c} - \frac{h_a}{h_b})N^\alpha, N^\beta)(0, N^\beta)$$

and thus by representing the area of S_N as the subtraction of the area of a triangle from a rectangle,

$$\text{Volume}(S_N) = \frac{h_a}{h_c}N^{\alpha+\beta} - \frac{1}{2}\frac{h_a}{h_b}N^{\alpha+\beta}.$$

Quadrilateral II. Suppose $h_a > h_c$ and $h_b \leq h_c$.

Observe that traversal starts at $(0,0,0)$ along ℓ_{x1} until reaching π_{X1}, which occurs before π_{X2} since $h_a/h_c > 1$. Traversal continues along ℓ_{y2} until the intersection at π_{Z2} which occurs before π_1 since $h_b \leq h_c$ and lies in the first octant because $h_a > h_c$. From π_{Z2} traversal continues along ℓ_{z2} until π_{Y2} which occurs before π_{Z1} since $h_b/h_c \leq 1$. From there traversal descends back to $(0,0,0)$.

$$\pi_0 \to \pi_{X1} \to \pi_{Z2} \to \pi_{Y2} \to \pi_0.$$

Thus P_N is a quadrilateral and

$$\text{Perimeter}(P_N) = d(\pi_0, \pi_{X1}) + d(\pi_{X1}, \pi_{Z2}) + d(\pi_{Z2}, \pi_{Y2}) + d(\pi_{Y2}, \pi_0).$$

The projection of P_N onto the xy plane S_N has vertices

$$(0,0), (N^\alpha, 0), (N^\alpha, (\frac{h_b}{h_c} - \frac{h_b}{h_a})N^\beta), (0, \frac{h_b}{h_c}N^\beta)$$

and thus by representing the area of S_N as the subtraction of the area of a triangle from a rectangle,

$$\text{Volume}(S_N) = \frac{h_b}{h_c}N^{\alpha+\beta} - \frac{1}{2}\frac{h_b}{h_a}N^{\alpha+\beta}.$$

Parallelogram. Suppose $h_a > h_c$, $h_b > h_c$ and $\frac{h_a}{h_c} - \frac{h_a}{h_b} \geq 1$.

First note that

$$h_b/h_c - h_b/h_a = h_b(1/h_c - 1/h_a) = h_b(h_a/h_c - 1)/h_a \geq h_b(h_a/h_b)/h_a = 1.$$

Observe that traversal starts at $(0, 0, 0)$ along ℓ_{x1} until π_{X1} as before. Traversal continues along ℓ_{y2} until π_1 since $h_b/h_c - h_b/h_a \geq 1$ so π_1 occurs before π_{Z2}. From π_1 traversal continues along ℓ_{x2} until π_{Y1} as by assumption the x coordinate of π_{Z1} is larger than N^α. From π_{Y1} the traversal returns to the origin along ℓ_{y1}.

$$\pi_0 \to \pi_{X1} \to \pi_1 \to \pi_{Y1} \to \pi_0.$$

Thus P_N is a parallelogram and

$$\text{Perimeter}(P_N) = d(\pi_0, \pi_{X1}) + d(\pi_{X1}, \pi_1) + d(\pi_1, \pi_{Y1}) + d(\pi_{Y1}, \pi_0).$$

The projection of P_N onto the xy plane S_N has vertices

$$(0, 0), (N^\alpha, 0), (N^\alpha, N^\beta), (0, N^\beta)$$

and thus by the formula for the area of a rectangle,

$$\text{Volume}(S_N) = N^{\alpha+\beta}.$$

Pentagon. Suppose $h_a > h_c$, $h_b > h_c$ and $\frac{h_a}{h_c} - \frac{h_a}{h_b} < 1$.

First note that

$$h_b/h_c - h_b/h_a = h_b(1/h_c - 1/h_a) = h_b(h_a/h_c - 1)/h_a < h_b(h_a/h_b)/h_a = 1.$$

Observe that traversal starts at $(0, 0, 0)$ along ℓ_{x1} until π_{X1} as before. Traversal then goes along ℓ_{y2} until π_{Z2} since $h_b/h_c - h_b/h_a < 1$ so π_{Z2} occurs before π_1. From π_{Z2} traversal continues along ℓ_{z2} to π_{Z1} which occurs before π_{X2} since $h_b/h_c > 1$. From π_{Z1} along ℓ_{x2} traversal goes to the next intersection at π_{Y1}, then descends to along ℓ_{y1} to the origin.

$$\pi_0 \to \pi_{X1} \to \pi_{Z2} \to \pi_{Z1} \to \pi_{Y1} \to \pi_0.$$

Thus P_N is a pentagon and

$$\text{Perimeter}(P_N) = d(\pi_0, \pi_{X1}) + d(\pi_{X1}, \pi_{Z2}) + d(\pi_{Z2}, \pi_{Z1}) + d(\pi_{Z1}, \pi_{Y1}) + d(\pi_{Y1}, \pi_0).$$

Thus the projection of P_N onto the xy plane S_N has vertices

$$(0,0),(N^\alpha,0),(N^\alpha,(\frac{h_b}{h_c}-\frac{h_b}{h_a})N^\beta),((\frac{h_a}{h_c}-h_ah_b)N^\alpha,N^\beta),(0,N^\beta)$$

and thus by subtracting the upper right triangle from the rectangle, one has that

$$\begin{aligned}\text{Volume}(S_N) &= N^{\alpha+\beta}-\frac{1}{2}(1-\frac{h_a}{h_c}+\frac{h_a}{h_b})(1-\frac{h_b}{h_c}+\frac{h_b}{h_a})N^{\alpha+\beta},\\ &=(1-\frac{h_b}{2h_a}(1-\frac{h_a}{h_c}+\frac{h_a}{h_b})^2)N^{\alpha+\beta}.\end{aligned}$$

It is clear from the inequalities that this covers every possibility.

The determination of the area of P_N as opposed to S_N is done by application of Lemma 6.14.

The lower bounds on the coefficients of the area are determined by observing that quadrilateral I contains the triangle $\pi_0 \to \pi_{X2} \to \pi_{Y1} \to \pi_0$, quadrilateral II contains the triangle $\pi_0 \to \pi_{X1} \to \pi_{Y2} \to \pi_0$, the parallelogram and pentagon contain the triangle $\pi_0 \to \pi_{X1} \to \pi_{Y1} \to \pi_0$, so it follows that the area of the polygon is minimal implies that the polygon is a triangle. Then it is a simple observation that as $1 \le h_a < p_a$, $1 \le h_b < p_b$, $1 \le h_c < p_c$,

$$h_ah_b/(2h_c^2)\rho N^{\alpha+\beta} \ge 1/(2p_c^2)\rho N^{\alpha+\beta}.$$

The upper bound is determined by observing that all five polygons lie in the parallelogram $\pi_0 \to \pi_{X1} \to \pi_1 \to \pi_{Y1} \to \pi_0$, so the maximum area of P_N is thus $\rho N^\alpha + \beta$.

We now move on to the perimeter. We calculate the distance of each line segment.

- $\pi_0 \to \pi_{X1}$ has length $\sqrt{N^{2\alpha}+\frac{h_c^2}{h_a^2}N^{2\gamma}}$.
- $\pi_0 \to \pi_{X2}$ has length $\sqrt{\frac{h_a^2}{h_c^2}N^{2\alpha}+N^{2\gamma}}$.
- $\pi_0 \to \pi_{Y1}$ has length $\sqrt{N^{2\beta}+\frac{h_c^2}{h_b^2}N^{2\gamma}}$.
- $\pi_0 \to \pi_{Y2}$ has length $\sqrt{\frac{h_b^2}{h_c^2}N^{2\beta}+N^{2\gamma}}$.
- $\pi_{X1} \to \pi_{Z2}$ has length $\sqrt{\left(\frac{h_b}{h_c}-\frac{h_b}{h_a}\right)^2N^{2\beta}+\left(1-\frac{h_c}{h_a}\right)^2N^{2\gamma}}$.
- $\pi_{X1} \to \pi_1$ has length $\sqrt{N^{2\beta}+\frac{h_c^2}{h_b^2}N^{2\gamma}}$.
- $\pi_{X2} \to \pi_{Z1}$ has length $\sqrt{\frac{h_a^2}{h_b^2}N^{2\alpha}+N^{2\beta}}$.
- $\pi_{X2} \to \pi_{Y2}$ has length $\sqrt{\frac{h_a^2}{h_c^2}N^{2\alpha}+\frac{h_b^2}{h_c^2}N^{2\beta}}$.

- $\pi_{Y1} \to \pi_{Z1}$ has length $\sqrt{\left(\frac{h_a}{h_c} - \frac{h_a}{h_b}\right)^2 N^{2\alpha} + \left(1 - \frac{h_c}{h_b}\right)^2 N^{2\gamma}}$.
- $\pi_{Y1} \to \pi_1$ has length $\sqrt{N^{2\alpha} + \frac{h_c^2}{h_a^2} N^{2\gamma}}$.
- $\pi_{Y2} \to \pi_{Z2}$ has length $\sqrt{N^{2\alpha} + \frac{h_b^2}{h_a^2} N^{2\beta}}$.
- $\pi_{Z1} \to \pi_{Z2}$ has length $\sqrt{\left(1 - \frac{h_a}{h_c} + \frac{h_a}{h_b}\right)^2 N^{2\alpha} + \left(1 - \frac{h_b}{h_c} + \frac{h_b}{h_a}\right)^2 N^{2\beta}}$.

The perimeters given in the lemma are simply sums of these.