Essays on Sorting with Financial Securities

by

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Abstract

This dissertation studies one-to-one matching between workers and assets in a market where financial securities are offered. The quality of an asset is publicly known, but a worker's productivity is private information. The asset side first posts contracts, under which the payment is contingent on the realized output. Then the workers direct their search based on the offers. Production exhibits complementarity so that the efficient allocation features positive assortative matching (PAM).

I consider a frictionless setting in the first chapter. First, I characterize the sufficient and necessary conditions for decentralizing PAM. For any distribution of types, these conditions ensure that the set of posted contracts not only induces the workers to sort assortatively but also precludes the asset owners from poaching. In comparison with the case of full information, the asset side's share of the matching surplus is always greater and increases with the asset quality at a faster rate in equilibrium. Second, I show that all asset owners will always be better off if the feasible contracts are replaced with steeper ones, which cost better workers more than weaker workers.

The second chapter focuses on the class of output sharing contracts. I study how it affects the matching efficiency and sorting pattern in the presence of search friction. The unique equilibrium features inefficient PAM. The matched pairs fully separate into a continuum of markets, where the queue length in each market still maximizes the expected surplus given the worker's equilibrium payoff. However, regardless of the distribution of types, all but the best workers pair up with better assets compared to the Second Best allocation. There is either an excessive entry of workers or an insufficient entry of assets. Sorting is inefficient because a reduction in the output share costs less to weaker workers than better workers. This handicaps their competition for better assets, driving up the output share of the best assets. These asset owners then induce an inefficiently long queue of workers to increase their matching probability.

Lay Summary

This dissertation studies matching between parties on the two sides of the markets where financial securities are commonly offered. An example is that firms hire CEOs from outside and offer equity shares in the remuneration packages. The agents can be ranked by their types, say firm size and candidate's ability. Since both sides compete for better partners from the given pools, the equilibrium matching pattern and the divisions of the matching surpluses vary with the distribution of types. My contributions are to provide qualitative results which hold for all distributions of types. The first chapter studies when the efficient allocation can always be decentralized in a frictionless environment, and how the forms of financial securities available affect the distribution of the surplus. The second chapter analyzes how the offering of output shares affect the matching pattern and entry decisions in the presence of search friction.

Preface

This dissertation is my original, unpublished and solo work. All errors are mine.

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List of Symbols

p	Worker's productivity
q	Asset quality
y	Output level
Ω_y	Support of output distribution
$F(y\|p,q)$	C.D.F. of conditional output distribution
$f(y\|p,q)$	P.D.F. of conditional output distribution
$\underline{V}$	Workers' outside option
$\underline{U}$	Assets' outside option
$\{p_l\}_{l=1}^{L}$	Support of worker's type in finite types setting
$\{q_k\}_{k=1}^{K}$	Support of Asset quality in finite types setting
$P(p_l)$	Measure of workers with productivity p_l in finite types setting
$Q(q_k)$	Measure of assets with quality q_k in finite types setting
r	Public belief about a worker's type
$\Delta(X)$	Set of probability distributions defined over X
$t(y)$	Contract
Ω_t	Set of feasible contracts
$\mathcal{T}$	Sigma algebra for Ω_t
$v(p,q,t)$	Worker's expected payoff

$u(q,p,t)$	Asset owner's expected payoff
S_t	Ordered set of securities
s	Contract term indexing S_t
μ	Tightness ratio
$\eta(\mu)$	Matching Probability for workers
$v(p)$	Worker's equilibrium payoff
$u(q)$	Asset owner's equilibrium payoff
$\psi(q)$	Set of contracts posted by owners of asset q
Ψ	Set of active markets
W	Measure of participating workers
$W_{pq}(p_l, q_k)$	Measure of workers assigned to the match (p_l, q_k)
$C_{pq}(p_l, q_k)$	Measure of assets assigned to the match (p_l, q_k)
T	Direct revelation mechanisms
$\Omega_T^{DRM}(q)$	Feasible set of T for owners of asset q
$\underline{l}$	Threshold type of workers
$\underline{k}$	Threshold type of assets
$r_q^{FB}(q)$	Distribution of workers who match with asset of quality q under PAM
$\{(t(.;\widetilde{s}_k), q_k)\}_{k\geq\underline{k}}$	Set of active markets supporting PAM
$F(p)$	Measure of workers of productivity below p in continuous types setting
f	Derivative of F
$G(q)$	Measure of assets with qualities below q in continuous types setting
g	Derivative of G
λ	Queue length
$\delta(\lambda)$	Matching probability for asset side

$K(q, s)$	Measure of asset owners in markets $(q', s') \leq (q, s)$
$L(p, q, s)$	Measure of workers with types $p' \leq p$ in markets $(q', s') \leq (q, s)$
$\Lambda(q, s; K, L)$	Belief of queue length in market (q, s)
$R(q, s; K, L)$	Belief of workers' types in market (q, s)
x_{SB}	Object of interest x in price competition
$\widetilde{x}$	Object of interest x in equilibrium

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Chapter 1

Introduction

This dissertation studies two-sided one-to-one matching in markets where the offering of financial securities or, more generally, contingent contracts is common. In many circumstances, the two sides can be ranked by some characteristics, or simply their types. Since both sides are competing for better partners from the given pools, the set of equilibria varies with the distribution of types. As the distribution of types is constantly changing and may not be observable to outsiders, qualitative results applicable to all distributions of types are of particular interest.

The seminal work of Becker (1973) shows that if the matching surplus exhibits supermodularity in types, positive assortative matching (PAM) maximizes the total surplus, and therefore prevails in a frictionless competitive market.[1] However, the agents on one side are often better informed about their own types, or the surplus from a match.[2] The forms of contingent payments offered then determine how much an informed party gains from a better partner, and more importantly, how such gain depends on his private

[1]Suppose a better agent generates a larger increase in the matching surplus when pairing up with a better partner, then the matching surplus is said to exhibit supermodularity. An allocation features PAM if the matches consist of the highest types on both sides, the second highest types, and so on.

[2]I use feminine pronouns for the uninformed side and masculine one for the informed side.

type. I study how the offering of contingent contracts affects matching and the divisions of surpluses in equilibrium.

I consider the following framework. There are continuums of asset owners and workers. A worker's productivity is privately known, whereas the quality of an asset is publicly observable. Each worker may operate an asset. The types on both sides determine the output distribution.[3] The asset side first posts contracts tied to the future outputs. The owners who have the same asset quality and post the same contracts will gather and form a (sub-)market. Observing the contracts posted, each worker may visit at most one market. The participants on both sides of a (sub-)market pair up randomly. Those who end up unmatched will stay idle. A worker forms his belief about his peers' choices, and hence his matching probability in each market. When an asset owner is deciding her contract offer, the contracts offered by her peers restrict the distribution of workers it may attract. Specifically, she believes that a deviating offer may only attract the workers who accept the lowest matching probability, or equivalently, the greatest percentage gain relative to their equilibrium payoff.

In the literature on assortative matching, the environment described is closest to the papers where the informed parties make up-front payments and select their partners based on the offered prices. As the informed side assumes the residual claim, the incentives for both sides is the same as in the full information case. Hence, price competition still decentralizes efficient allocations in a competitive market. The price competition has served

[3] Note that the distribution of the output does not depend on the contract chosen. Here the sole purpose of a contract is to determine the split of the matching surplus. This simplification allows me to concentrate on the potential distortions in the matching pattern.

as the benchmark case in the existing literature. However, the feasibility of the buyout arrangement may be undermined by the presence of wealth constraints and incentive provisions for other stakeholders. They motivate the use of contingent contracts.

The difference between this dissertation and competitive screening literature is also noteworthy. In the latter, only the principals compete for informed agents but not the other way around. Specifically, the equilibrium analysis can be conducted in a sequential game in which multiple principals simultaneously post their offers, then one single agent decides among them. The lack of competition among agents implies that the set of the separating equilibria and the associated distortion depend on neither the number of agents nor the distribution of their types.[4] This implication is the major convenience, but also the restriction, of the competitive screening models. Here both sides are competing for better partners from the exogenously given pools of assets and workers respectively. Finding "distribution-free" results is a non-trivial task because of the competition on both sides and their interaction.

Chapter 2 considers a benchmark environment, in which the participants in a market pair up frictionlessly.[5] It focuses on two questions: When PAM can always be decentralized in an equilibrium, and in this case, how the divisions of matching surpluses depend on the form of contingent payments available. Apart from theoretical interests, the conditions for decentralizing PAM provide guidance on how to restrict the set of contracts available to en-

[4] This is because the support of the type distribution alone pins down the set of incentive compatibility conditions.

[5] That is, the short side, which has fewer participants, will get matched for sure. Only the long side, with more participants, will be rationed.

sure matching efficiency and redistribute welfare. For empirical researchers, The comparative statics on the equilibrium payoffs produce testable implications on the presence of private information in these markets against competing theories.

Chapter 3 introduces search friction into the matching process and focuses on the class of output sharing contracts such as equity shares. I identify a novel source of inefficiency in such markets and analyze the form of distortion on the matching pattern and entry decisions. In particular, the equilibrium is unique and still features PAM.[6] I compare the equilibrium allocation with the Second Best one and obtain qualitative conclusions which are universal to all distributions of types.

Chapter 4 concludes.

[6]With search friction, PAM only requires that a better worker searches for weakly better assets. This is satisfied by infinitely many allocations, but only a subset of them is Second Best.

Chapter 2

Decentralizing Assortative Matching With Financial Securities

2.1 Introduction

In many matching markets, financial securities is the prevailing form of contracts between partners. It specifies how the payment between the partners depends on certain outcomes, such as the realized output, and imposes little restrictions on the actions taken by the partners. The following applications are some examples:

Market for top management: Firms are hiring top executives, whose contribution to the firm's profit depends on his ability and the firm size. However, the candidates know their own abilities better than the hiring firms. Nowadays, stock and stock warrant are major components of the remuneration packages. Frydman and Jenter (2010) look at the composition of CEO pay in S&P 500 firms: During the period 2000 to 2008, base salary makes up less than 20% of the remuneration, and over half of it are option grants and restricted stock grants.

Market for venture capitals: Venture capitalists can be ranked by their reputation and the prospect of the entrepreneurs' projects vary. Entrepreneurs initially know certain aspects of their projects better than the outsiders. Kaplan and Strömberg (2003) analyze 213 rounds of investments: In over 90% of financing rounds, venture capitalists obtain convertible preferred stock, sometimes along with other financial securities, from the entrepreneurs in return for their assistance and financing.

The defining feature is the contingent nature of the payment, which needs not involve any direct financial claims or explicit contracts between the partners. When forming business partnerships, contingent payments are often implemented via the capital structure of a joint venture. Even when a seller of an asset demands a fixed price, any postponed installment is still contingent on the outcome. This is because the buyer is protected by limited liability and may default if the business turns sour or the news reveals dim prospects. Clearly, a variety of forces can be at play in the above examples. The focus here is how the form of contingent payments available affects two-sided matching in the presence of information asymmetry. I will adopt the term contingent contract to underline this focus. [7]

Under a contingent contract, the expected payment made by a worker depends on his private type. More importantly, the asset owners become concerned about the types of their partners and take screening into account when deciding contract offer. They may attempt to poach better candidates. PAM, though efficient, needs not occur in decentralized markets using contingent contracts.

To have a direct comparison with price competition, the analysis centers on a class of contracts, which are ranked by the division of surplus. Specif-

[7] In particular, I sidestep the issue on control rights specified in the financial securities.

ically, these contracts can be indexed by a single contract term, for which workers of all types prefer a more generous term given the asset quality whereas the asset owners prefer a less generous term for a given worker.[8] Examples are equity contracts indexed by the percentage share and defaultable debts indexed by the principal amount.

I start by constructing a candidate equilibrium decentralizing PAM. By virtue of the order structure in the setup, the conditions for voluntary participation by both sides and the incentive compatibility for the workers pin down a unique set of contracts for generic distributions of types. This candidate set of contracts requires the workers to accept a less generous term for a better asset. More importantly, it defines indirect mappings from the distribution of types to the equilibrium payoffs, and the deviating payoffs for both sides. To obtain "distribution-free" results, the analysis then centers on the relation of these mappings with the output distribution and the feasible set of contracts.

I first study whether a single worker or asset owner may profit from a deviation under the candidate set of contracts. The analysis culminates with the necessary and sufficient condition for the decentralization of PAM. It can be decomposed into three conditions, which separately address the incentives of workers, the participating asset owners and the asset owners who shall take their outside option. These conditions are stated in terms of

[8]Namely, the set of feasible contracts is an ordered set of securities introduced in DeMarzo, Kremer and Skrzypacz (2005).

In the other extreme, Riordan and Sappington (1988) characterize the condition that perfect screening can be achieved costlessly using contingent payments. In this scenario, the equilibrium allocation and payoffs are the same as in the full information case. The discussed order structure for the contract space precludes this possibility.

the worker's induced preference over the contract term, his partner's type, and his matching probability.[9] They apply to arbitrary distributions of types. The conditions are the same when the uninformed side may post single contracts only or menus of contracts. For each of the conditions violated, I provide a procedure constructing distributions of types for which the corresponding group must profit from a deviation. This illustrates the forces against PAM.

I also provide a unifying sufficient condition on the worker's expected payoff: When switching to a better partner offering a less generous term, a better worker always sees a larger increase (or smaller reduction) in his expected payoff, measured by either amount or percentage. This increasing difference condition, termed as Global ID, is sufficient for decentralizing PAM by serving two purposes. The first one is to induce workers to sort assortatively. The second purpose is to preclude poaching offers from asset owners. I first show that under the set of candidate contracts inducing PAM, an asset owner never profits from poaching weaker workers. She may still attempt to poach better workers. To compete with better assets, the asset owner must offer a more favorable term to maintain its appeal. This generous offer will also interest workers of lower types. Since at most one worker will be hired, the workers must face rationing when seeking for this deviating contract. They have to trade off a better asset and a higher matching probability against a more favorable contract term. Global ID ensures that in the candidate equilibrium, the better workers always prefer

[9] The output distribution and the forms of feasible contracts jointly induce preferences for both workers and assets workers. The conditions on the worker's preference involve the asset owner's preference and the outside options for both sides indirectly.

the former while the weaker workers prefer the generous term.[10] As a result, the poaching offer will fail to attract better workers.

In some sense, Global ID is satisfied if the variation in the contingent payments aligns with the form of production complementarity. Based on this observation, I propose two notions of production complementarity, which manifest as a shift in the output distribution toward higher levels. For each of these conditions on the output distribution, I provide the corresponding sufficient condition for the set of feasible contracts. One condition applies to mixtures of cash and securities. The other applies to securities such as debt contracts, or stock options if the asset side makes the payment.

The second part is to study the effect of changes in the set of feasible contracts. A contract is steeper than another if it costs more to the better workers but less to the lower types.[11] The other contract is said to be flatter. Suppose that the entire set of feasible contracts is replaced by a steeper set and the equilibrium matching remains PAM. The use of steeper contracts handicaps the competition among workers. In particular, a low type worker is willing to accept a less favorable term in exchange for a better asset. The intensified competition for better assets will drive up the asset side's share of the surplus, despite the same allocation. While the first result is based on the competition among workers, the second result stems from the competition among asset owners. The asset side always prefers the flattest contracts

[10] Such monotonicity in the preference is stronger than necessary. The poaching offers can be deterred if they attract only the lowest type among all participating workers. The necessary and sufficient conditions for the asset side relaxes the above increasing difference condition by exploiting this observation.

[11] For example, DeMarzo, Kremer and Skrzypacz (2005) show that under the assumption of MLRP, the upfront payment, defaultable debt, equity share, call option are in ascending order by their steepness.

available, which are prone to attract better workers. Putting together, we obtain comparative statics on how the introduction (or exclusion) of steeper (or flatter) contracts affects the divisions of the matching surpluses under assortative matching.

Since the inclusion of steeper contracts in the feasible set has no impacts on the sets of equilibrium allocations and payoffs, all results can be extended to larger sets of contracts, which are not fully ranked. In particular, I consider examples that the workers have wealth constraints and that the asset owners may misappropriate the outputs.

This also leads to the main testable implication of the model here. Suppose a new regulation restricts the feasible set of contracts so that steeper contracts are offered, my result on comparative statics predicts that all asset owners are better off under PAM. The opposite shall occur following a slash of regulation. Such test, albeit demanding, is powerful. In pure moral hazard or risk sharing models, a restriction in the feasible set would only result in the offering of a suboptimal contract.

I then provide examples demonstrating that when offered flatter contracts, better workers may benefit less from a match with a better asset or an increase in matching probability. As a result, inefficiency may arise after the introduction of the flatter contracts. In this sense, these examples illustrate how restricting the feasible set of contracts may improve total surplus.

This chapter is organized as follows: Section 2.2 discusses the related literature and the contribution of the present work. I illustrate the main elements of the analysis casually using an example in Section 2.3.

Section 2.4 details the model setting and the equilibrium definition. Section 2.5 formally defines PAM, the First Best allocation in the Utilitarian framework. It always occurs under symmetric information or in price competition. Section 2.6 characterizes the conditions on the workers' preference for PAM decentralization. Joint sufficient conditions on the feasible contracts and the distribution of output are provided in Section 2.7. Section 2.8 discusses the effects of changes in the set of feasible contracts. Section 2.9 concludes. All proofs are relegated to the Appendix A.

2.2 Related Literature

Assortative matching in directed search

Though PAM in various environments, including non-transferable utility and random search, have been studied in the literature, the previous work with similar environments has exclusively considered the case that the informed side makes up-front payments. Mailath, Postlewaite and Samuelson (2016) (and the references therein) study the agents' decisions of privately observed pre-investment when they pair up with uninformed partners in a price competition afterward. Damiano and Li (2007) consider the rent extraction problem of a matchmaker who decides a menu of meeting places and admission fees for agents with private types. My setting is closely related to that in Eeckhout and Kircher (2010). The authors study price competition using a competitive search framework. They show that a stronger form of production complementarity is required to support (imperfect) PAM in the presence of search friction.

The use of contingent contracts not only changes the sorting incentives for the informed but also gives rise to the screening problems for the uninformed side. The latter has never been studied in this literature. For an asset owner, the pool of workers attracted by a deviating offer varies with the distribution of types indirectly through the set of contracts posted. Characterizing the conditions for deterring poaching offers is the main challenge in my analysis.

Competitive screening with bilateral matching

My setting is related to the competitive screening models with bilateral matching including Gale (1996) and Guerrieri, Shimer and Wright (2010). My equilibrium definition closely follows that the latter. They consider a competitive search setting with free entry of principals, who have both contract and matching probability as screening instruments. Facing the offers posted, the matching probabilities for the agents are jointly determined with their choices of contracts.[12] The authors characterize the equilibrium and study the form of distortion in various applications.

My point of departure is to consider two-sided one-to-one matching in which both sides compete for partners from given pools.[13] As a result, the distribution of types determines the set of feasible allocations, and hence the sets of efficient and equilibrium allocations. In general, the distortion in two-sided matching varies with the distribution of types. Instead of targeting at a specific form of distortion, the policy recommendation here is restricting the feasible set of contracts to ensure efficient matching for any distribution of types. To this end, I characterize the conditions for decentralizing PAM.

Security-bid auction

My formulation of the contract space is closely related to DeMarzo, Kre-

[12] This also implies that the introduction of bilateral matching changes the principals' deviating payoff in comparison with the textbook competitive screening models, ensuring the existence of an equilibrium.

[13] I also focus on a set of fully ordered contracts, so that any separation among workers' types must be induced by some variation in the asset quality and their matching probability. This allows me to concentrate on the distortions in the matching pattern.

mer and Skrzypacz (2005). They consider an auction of an asset, in which the buyers bid in the form of securities. The authors introduce the concept of ordered sets of securities, for which the expected payment can be ranked unambiguously for all types of buyers. They compare different ordered sets of securities in terms of their steepness. The auctioneer can improve her revenue by requiring the buyers to bid from a steeper ordered set of securities, provided that the equilibrium allocation remains efficient.[14] This is because steeper securities strengthen the linkage between the winner's type and the payment he makes, handicapping the competition among buyers.

I adopt their definitions of an ordered set of securities and security steepness. The comparative statics on the divisions of matching surpluses can be attributed to the insight in DeMarzo, Kremer and Skrzypacz (2005). My contribution is to establish the connection between the security-bid auction and assortative matching in this aspect. I further show that the intensified competition among workers for the same type of assets spills over to the competition for the better assets under PAM. Besides, the change in the feasible contracts will affect how the partners divide the gain from production complementarity, potentially changing the equilibrium allocation. This paper provides conditions ensuring that PAM always occurs in equilibrium.

Moral hazard and assortative matching

This paper is related to the literature on how incentive provision affects the matching pattern. This strand of literature also considers the use of contracts in two-sided matching markets. Serfes (2005) studies the equilibrium matching pattern in a principal-agent setup. Legros and Newman

[14] DeMarzo, Kremer and Skrzypacz (2005) assume that the payoff for the winner is strictly log-supermodular in his type and the ranking of his security-bid. This ensures that the buyer of the highest type always outbids the others.

(2007) discuss a related example. Kaya and Vereshchagina (2014) study how ownership structure and production technology shape the cost of incentive provision in team production. In these models, the parties' types are publicly observable. One side or both will take a private action after pairing up. Despite production complementarity, assortative matching needs not arise in equilibrium because of two channels. First, utility is not perfectly transferable. When adjusting the term of a contract to transfer utility between two sides, the conversion rate is not constant and dependent on types. Second, types affect productivity as well as the cost of incentive provision, so the matching surplus does not inherit supermodularity from the production technology.

I consider information asymmetry when forming matches. To focus on the potential distortion of the matching pattern, the choice of contract in my setting affects only the division, but not the size, of the matching surplus. This ensures PAM if the types are publicly observable. Here inefficiency can only arise because of both the use of contingent contracts and private information. Furthermore, it allows me to obtain general results on how the form of the contingent payments affects the divisions of surpluses in equilibrium.

2.3 An Illustrative Example

This section illustrates the main elements of the analysis. I also casually sketch out the equilibrium definition along the way. The formal setup will be laid out in the next section.

Consider an economy with two types of workers $p \in \{1, 2\}$. Each type has a unit measure. There are three types of assets $q \in \{1, 2, 3\}$ with $\frac{1}{2}q$ measure respectively. Both sides have the same outside option, say 1. $E(Y|p,q)$ denotes the expected output from a match (p,q). The matching surplus is supermodular (SPM) in types, so the First Best allocation is PAM. That is, better workers pair up with better assets. In this example, all high type workers are allocated the best assets. Half of the low type workers match with the best assets, and the other half match with median quality assets. The owners for the remaining half of the median quality assets and the lowest quality assets take their outside option.

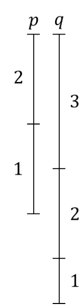

Figure 2.1: An example of PAM

For simplicity, I only consider two classes of contracts. The workers either pay a fixed price upfront or promise the asset owner a share of the future output.

It is well known that PAM prevails in a competitive market under full information. In such equilibrium, the owners of the median quality assets must be indifferent about their outside option, and the low type workers are indifferent about matches with the two types of assets. These indifference conditions uniquely pin down the division of matching surplus.

$$\begin{cases} 1 = U^{FB}(1) = U^{FB}(2), \\ E(Y|1,2) = V^{FB}(1) + U^{FB}(2), \\ E(Y|1,3) = V^{FB}(1) + U^{FB}(3), \\ E(Y|2,3) = V^{FB}(2) + U^{FB}(3), \end{cases}$$

where $V^{FB}(p)$ and $U^{FB}(q)$ denote the equilibrium payoffs for the workers and asset owners respectively. Observe that

$$V^{FB}(p) + U^{FB}(q) \geq E(Y|p,q),$$

no pairs will profit from switching their partners. Furthermore, the division of matching surplus is the same under the two classes of contracts. If upfront payment is feasible, the participating asset owners simply post the price $U^{FB}(q)$. Under full information, an equity contract may function as a posted price by making its term contingent on worker's type to implement the intended transfer $U^{FB}(q)$.

Now suppose workers have private types. If the asset side may post prices, the equilibrium allocation and payoffs remain the same. This is because the payment made by the worker does not depend on his private type, so the deviating payoff for both sides is always the same as in the full information case. The class of fixed prices, or cash payment, is in fact the only contracts for which the division of matching surplus is unaffected by information asymmetry.

Now we turn to the case that only equity contracts are available. Let $\widetilde{s}_q$ denote the equity shares for the assets of quality q. To induce PAM, the indifference conditions mentioned must continue to hold. They uniquely determine the contracts offered, and hence the equilibrium payoffs

$$\begin{cases} U(1) = U(2) = \widetilde{s}_2 E(Y|1,2) = 1, \\ V(1) = (1-\widetilde{s}_2)E(Y|1,2) = (1-\widetilde{s}_3)E(Y|1,3), \\ V(2) = (1-\widetilde{s}_3)E(Y|2,3), \\ U(3) = (1-\widetilde{s}_3)[\frac{1}{3}E(Y|1,3) + \frac{2}{3}E(Y|2,3)]. \end{cases} \tag{2.1}$$

In this example, all but the highest types have the same equilibrium payoff as in the the full information case. Yet the competition for the high quality assets has intensified because the equity shares cost the high type workers more than the low type. As a result, the high type workers end up paying more for the best assets,

$$\widetilde{s}_3 E(Y|2,3) > U^{FB}(3) = \widetilde{s}_3 E(Y|1,3).$$

This in turn leads to

$$V(2) + U(q) < E(Y|2,q), q = 1,2.$$

So far we have pin down the set of posted contracts and the equilibrium payoffs. The next step is to investigate the conditions ensuring that no agents may profit from a deviation. The market structure and the belief restriction provide the foundation for the deviating payoffs.

Facing the posted contracts $\widetilde{s}_2$ and $\widetilde{s}_3$, a high type worker may only switch to the median quality assets, earning $(1-\widetilde{s}_2)E(Y|2,2)$. Substituting $\widetilde{s}_2$ and $\widetilde{s}_3$, he will not profit from such deviation if

$$\frac{E(Y|2,3)}{E(Y|2,2)} > \frac{E(Y|1,3)}{E(Y|1,2)}. \tag{2.2}$$

(2.2) is stronger than SPM of the matching surplus. SPM merely ensures that better workers benefit more from an improvement in the asset quality under the same contract. Yet, they also suffer more from the reduction in their share. So PAM requires high type workers to have a greater percentage gain from a better asset.

Unlike workers, the deviating payoff for the asset side depends on the pool of workers attracted by a deviating offer. I adopt the belief restriction that such offer will only attract the workers, if any, who accept the lowest matching probability, or equivalently, see the greatest percentage gain relative to their equilibrium payoff. This restriction ensures that the pool of the workers attracted depends on the strength of complementarity in types. Take the lowest quality asset as an example. Suppose an owner posts a share s', a worker will be interested in this offer only if $(1-s')E(Y|p,1) > V(p)$. In this case, his percentage gain is simply $\frac{(1-s')E(Y|p,1)}{V(p)}$. Combining with the expression $V(p) = (1-\widetilde{s}_3)E(Y|1,3)$, any deviating offer will at most draw only the low-type workers if

$$\frac{E(Y|2,3)}{E(Y|1,3)} > \frac{E(Y|2,1)}{E(Y|1,1)}. \tag{2.3}$$

However, the asset owners never profit from a partner weaker than the one under PAM in this framework.

$$V(1) + U(q) \geq E(Y|1,q), q = 1,2,3.$$

One can check that the owners of the median and highest quality assets have no profitable deviations as well. Hence, PAM can be supported.[15]

Nevertheless, the conditions (2.2) and (2.3) only apply to the given distribution of types. The counterpart of the conditions (2.1) for general contracts defines indirect mappings from a generic distribution of types to the set of

[15] Furthermore, I will show that the asset owners can do no better by offering menus of contracts.

contracts posted, hence the equilibrium payoffs and the deviating payoffs. These mappings and their relation to the model primitives will be the central object of the analysis.

2.4 Model Setting

2.4.1 Two-sided matching

Production is carried out by a single worker using an asset. There are continuums of workers and asset owners. Each asset owner owns an asset. Assets can be ranked according to their publicly known qualities $q \in [\underline{q}, \overline{q}]$. All workers are ex-ante homogeneous but differ in their actual productivity $p \in [\underline{p}, \overline{p}]$, Every worker privately knows his productivity. All parties are risk neutral and have a quasi-linear preference.

Production takes place after a worker pairs up with an asset. The output Y is stochastic and contractible. Given the pair of types (p, q), the conditional distribution of output $Y|(p, q)$ has C.D.F. $F(y|p, q)$ with common support $\Omega_y \subseteq \mathbb{R}_+$. The outside options for the workers and assets are given by $\underline{V} > 0$ and $\underline{U} > 0$ respectively, so the matching surplus is $E(Y|p, q) - \underline{V} - \underline{U}$.[16]

Assumption (P)**.** $Y|(p, q)$ *has the following properties:*

1. *For any* $y \in \Omega_y$*,* $F(y|p, q)$ *is continuous and strictly decreasing in* p *and* q*.*

2. $E(Y|p, q)$ *is strictly supermodular(SPM) in* p *and* q*.*

3. $E(Y|\underline{p}, \underline{q}) = \underline{U} + \underline{V}$.

[16] Measurability and integrability are tacitly assumed whenever they are required.

Assumption (P) states that a higher worker's productivity or asset quality always improves the output distribution in a strict F.O.S.D. sense. The matching surplus is strictly increasing and supermodular in types p and q. Furthermore, the matching surplus is positive for any pair. Therefore, the total surplus is maximized under PAM.

For tractability, the type distribution for workers and assets assume to have finite supports $\{p_l\}_{l=1}^{L} \subseteq (\underline{p}, \overline{p}]$ and $\{q_k\}_{k=1}^{K} \subseteq (\underline{q}, \overline{q}]$ respectively, where $L \geq 2$ and $K \geq 1$.[17] Higher types of worker and asset refer to greater l and k respectively, i.e. $p_l > p_{l-1}$ and $q_k > q_{k-1}$.

The measure of workers with productivity p_l is denoted by $P(p_l)$. $r \in \Delta(\{p_l\}_{l=1}^{L})$ denotes the public belief about a worker's type p, where $\Delta(\{p_l\}_{l=1}^{L})$ is the set of probability distributions defined over $\{p_l\}_{l=1}^{L}$. Likewise, the measure of assets with quality q_k is $Q(q_k)$. Taking outside options by the asset owner and worker are referred as p_0 and q_0 respectively. $Q(q_0)$ and $P(p_0)$ are defined as $+\infty$.

In the analysis, types with subscripts, p_l and q_k, are used when considering a particular distribution of types. Otherwise, the discussion is referring to general types in the type space.

[17] It is more convenient to work with a finite distribution of types when discussing general contracts or menus of contracts.

Notice that there are no workers of type $\underline{p}$ and assets of type $\underline{q}$. Together with $E(Y|\underline{p}, \underline{q}) = \underline{U} + \underline{V}$, the matching surplus is always positive but can be arbitrarily small for some pairs. Section 2.6 characterizes the conditions for First Best decentralization. This property is exploited in establishing necessity of theses conditions. The conditions remain sufficient if $E(Y|\underline{p}, \underline{q})) > \underline{U} + \underline{V}$.

2.4.2 Contingent contract and division of surplus

Suppose a worker pairs up with an asset. The two parties may enter an agreement on the contingent payments, denoted by $t : \Omega_y \to \mathbb{R}$. When the production concludes, the worker receives the realized output y and makes payment $t(y)$ to the asset owner accordingly. The contingent payment scheme $t(y)$ will be referred as a contract. It inherently satisfies ex-post budget balance. The argument y will be omitted from $t(y)$ if no confusion will arise. Only bilateral contracts are considered.

All types of asset owners have access to the same set of feasible contracts, which is denoted by Ω_t. $\mathcal{T}$ denotes a sigma algebra for Ω_t. The set of feasible contracts captures various restrictions such as contract incompleteness and limited liability.

Assumption (C)**.** *For all $t \in \Omega_t$, $t(y)$ and $y - t(y)$ are not constant functions over Ω_y and weakly increasing in y.*

Assumption (C) rules out contracts specifying only a fixed payment to either side.[18] When the output level increases, one side must receive a higher payment under ex-post budget balance. Assumption (C) further requires that the asset owner's and worker's payoff are always increasing in the production outcome under any contract. This can be motivated by a threat of sabotage on both sides. It also implies that the payoffs for both sides are continuous in the output level. Such double monotonicity is common in financial agreements.

[18] This can easily be motivated by incentive provision for both sides. For example, the production may require both partners to make an arbitrarily small investment or effort, which are privately observed.

Give the pair of types, the contract determines the surplus division between the two parties. The worker's share of surplus is denoted by

$$v(p,q,t) \coloneqq E(Y - t(Y)|p,q),$$

and that for the asset owner is denoted by

$$u(q,p,t) \coloneqq E(t(Y)|p,q).$$

$v(p,q,t)$ and $u(q,p,t)$ will be referred as expected payoffs. Since a higher type always improves the output distribution in a strict F.O.S.D. sense and the contingent payment is monotonic in output, an agent, who has a higher type himself or has a better partner, always enjoy a higher expected payment.

Remark 1. *For any $t \in \Omega_t$, $u(q,p,t)$ and $v(p,q,t)$ are continuous and strictly increasing in p and q.*

Assumption (P) and (C) jointly introduce a uniform and monotonic preference for the partner's type. This leads to competition for a better partner on both sides. More importantly, an asset owner concerns about the type of workers attracted by her offer.

When entering the same contract, a better worker not only pays out more to his partner but also keeps a higher amount of residual payment. Consequentially, only the workers of the lowest type may take their outside options. However, the same conclusion does not automatically hold for the asset side as an owner of a better asset may end up with a weaker worker.

To facilitate the comparison with price competition and security-bid auction, I adopt the convention that the informed party is entitled to the full output and makes payment. This may differ from the default output division in the application considered. One must make adjustments to the forms of contracts accordingly, so that the payoffs at each output level are the same as in the setup here.

Ordered set of securities The analysis focuses on a special class of contracts, which has a complete order. The formulation here is built on DeMarzo, Kremer and Skrzypacz (2005).

Definition. *S_t is called an ordered set of securities if there exists a mapping $t(.;.) : \Omega_y \times [0,1] \to \mathbb{R}$ such that*

1. *$S_t = \{t(.;s) : s \in [0,1]\} \subseteq \Omega_t$, and*

2. *$t(.;s)$ is continuous in s with respect to supremum norm, and*

3. *$v(p,q,t(;s))$ is strictly decreasing in s, whereas $u(q,p,t(.;s))$ is strictly increasing in s for any $(p,q) \in [\underline{p},\overline{p}] \times [\underline{q},\overline{q}]$, and*

4. *For any $(p,q) \in [\underline{p},\overline{p}] \times [\underline{q},\overline{q}]$, $v(p,q,t(.;1)) \leq \underline{V}$ and $u(q,p,t(.;0)) \leq \underline{U}$.*

An ordered set of securities is a subset of feasible contracts indexed by s, which will be referred as the contract term. $v(p,q,t(.;s))$ and $u(q,p,t(.;s))$ are continuous in all arguments under the second condition. The third condition states that for any pair of types, a higher value of s represents a greater share of surplus for the asset owner. For a given partner, workers of all types unanimously prefer a lower term s, whereas the asset owners always prefer the opposite. Many standard securities in practice can be ranked in this manner. Examples of contract term s include the amount of cash payment, the equity share, the principal amount of debt and the strike price for options. These examples satisfy the third condition because $t(.;s^H) \geq t(.;s^L)$ whenever $s^H > s^L$.

Now consider a matched pair (p, q) who may only enter a contract in S_t. Under the second and third condition, $[v(p, q, t(.;1)), v(p, q, t(.;0))]$ is the range of the feasible payoff for the worker and that for the asset owner is $[u(q, p, t(.;0), u(q, p, t(.;1)]$. The fourth condition then implies that any value v' between $\underline{V}$ and $E(Y|p, q) - \underline{U}$ can be achieved by a contract in S_t. v' represents a split of matching surplus as the payoffs for both parties are above their outside options. An example when the fourth condition holds is that $t(y;0) = 0$ and $t(y;1) \geq y$.

In summary, an ordered set of securities shares two similarities with prices: a monotonic preference for all types and perfect transferability of the matching surplus under full information. On the other hand, it renders the expected payment from the worker dependent on his private type.

Throughout the analysis, an ordered set of securities is always feasible, $S_t \subseteq \Omega_t$. As we shall see, this ensures that the equilibrium allocation under full information is always the First Best.

2.4.3 Market structure

For a given distribution of types, there are continuums of (sub-)markets indexed by (t, q). An owner of asset quality q_k may decide between her outside option and one of the markets (t, q_k) while a worker may take his outside option or participate in any one of the markets. The market structure is interpreted as follows: Asset owners decide what contract they post.[19] Owners of the same asset quality q_k posting the same contract t gather into one meeting place, which forms the (sub-)market (t, q_k). When a worker decides to accept the offer t posted by the owners of asset quality q_k, he participates

[19]Section 2.4.5 will address the possibility of a menu of contracts.

in the market (t, q_k).

The participants on the two sides of a market will pair up randomly. Define the tightness ratio $\mu \in [0, \infty]$ as the ratio between the measure of assets and that of participating workers in the market. A worker gets matched with probability $\eta(\mu)$ while the matching probability for an asset owner is $\frac{\eta(\mu)}{\mu}$. The matching is frictionless, so $\eta(\mu) = \min\{\mu, 1\}$. The payoffs for those left unmatched are normalized to zero. Hence, the values of outside option are the cost of participation for the two sides.

Consider a market (t, q) with tightness ratio μ and the distribution of participating workers is given by $r \in \Delta(\{p_l\}_{l=1}^{L})$. By a slight abuse of notation, I denote the expected surplus for a matched asset owner by

$$u(q, r, t) \coloneqq \sum\nolimits_{l=1}^{L} u(q, p_l, t) r(p_l).$$

When participating in this market, the expected payoff for an asset owner is given by $\frac{\eta(\mu)}{\mu} u(q, r, t)$ and that for a worker of type p is $\eta(\mu) v(p, q, t)$. A worker is said to prefer the contract (t, q) to (t', q') if $v(p, q, t) \geq v(p, q', t')$. He prefers the market (t, q) to (t', q') if $\eta(\mu(t, q)) v(p, q, t) \geq \eta(\mu(t', q')) v(p, q', t')$.

The timing of the events is as follows: In the contract posting stage, the asset owners make their participation decisions simultaneously. At the beginning of acceptance stage, the workers observe the measure of asset owners across markets. They simultaneously make their participation decisions. Matches are then formed.

A market (t, q) is active if if it is chosen by some asset owners in equilibrium. Otherwise, it is inactive. An active market clears if it has a unity tightness ratio. The workers are said to be rationed in the market (t, q) if its tightness ratio is below unity. In the opposite case, the asset owners are said to be rationed.

2.4.4 Equilibrium definition

In this subsection, I first propose a formal definition of an equilibrium. I then explain the terminologies and motivate the belief restriction.

In the equilibrium definition, each market (t,q) is associated with a tightness ratio $\mu(t,q)$ and a distribution of participating workers $r(t,q)$. $r(p_l|t,q)$ is the proportion of workers of type p_l and the support is denoted as $\Omega_p(t,q)$. Everyone takes μ and r as given.[20] For the active markets, μ and r capture the participation decision of the workers and asset owners. For an inactive market $(\widetilde{t},\widetilde{q})$, $\mu(\widetilde{t},\widetilde{q})$ and $r(\widetilde{t},\widetilde{q})$ are interpreted as the public belief regarding the tightness ratio and the composition of participating workers in that market after an owner of asset quality $\widetilde{q}$ deviates to it. This notation eliminates the need to distinguish between deviations to active markets or inactive markets by an asset owner. The equilibrium payoffs for both sides are included in the definition to capture the optimality of the participation decisions.

Definition. *A competitive matching equilibrium consists of the asset owners' equilibrium payoff* $U : \{q_k\}_{k=1}^K \to \mathbb{R}_+$*, workers' equilibrium payoff* $V : \{p_l\}_{l=1}^L \to \mathbb{R}_+$*, asset owners' contract posting set* $\psi : \{q_k\}_{k=1}^K \to \Omega_t \cup \{p_0\}$*, the set of active markets* $\Psi \subseteq \Omega_t \times \{q_k\}_{k=1}^K$*, the measure of participating workers* $W : \mathcal{T} \times \mathcal{P}(\{q_k\}_{k=1}^K) \to [0,1]$*, the distribution of workers* $r : \Omega_t \times \{q_k\}_{k=1}^K \to \Delta(\{p_l\}_{l=1}^L)$ *and market tightness* $\mu : \Omega_t \times \{q_k\}_{k=1}^K \to [0,\infty]$ *such that*

1. *Asset Owners' Optimal Contract Posting:*

[20] The tightness ratio is often interpreted as the market clearing price of the concerned market. All parties take μ as given, therefore the environment here is "competitive".

For all $(t,q) \in \Omega_t \times \{q_k\}_{k=1}^K$, $U(q) \geq \frac{\eta(\mu(t,q))}{\mu(t,q)} u(q, r(t,q), t)$ *with equality if* $t \in \psi(q)$.

2. *Workers' Optimal Acceptance:*

   i)For all $(t,q) \in \Omega_t \times \{q_k\}_{k=1}^K$,

$$V(p) \geq \eta(\mu(t,q)) v(p,q,t) \tag{2.4}$$

   with equality if $p \in \Omega_p(t,q)$ *and* $\mu(t,q) < \infty$.

   ii)$\mu(t,q) = \infty$ *if* $V(p) > v(p,q,t)$ *for all* $p \in \{p_l\}_{l=1}^L$

3. *Active Markets:*

   $\Psi := \{(t,q) \in \Omega_t \times \{q_k\}_{k=1}^K : t \in \psi(q)\}$ *is the support of* W.

4. *Optimal Participation:*

   i)$U(q) \geq \underline{U}$ *and* $V(p) \geq \underline{V}$.

   ii)$\int_{\Omega_t \times \{q_k\}_{k=1}^K} r(p_l|t,q) dW \leq P(p_l)$ *with equality if* $V(p_l) > \underline{V}$.

   iii)$\int_{\Omega_t} \mu(t,q_k) dW \leq Q(q_k)$ *with equality if* $U(q_k) > \underline{U}$.

An allocation is denoted by $(W_{pq}, C_{pq}) \in \mathbb{R}_+^{K\times L} \times \mathbb{R}_+^{K\times L}$, where $W_{pq}(p_l, q_k)$ and $C_{pq}(p_l, q_k)$ denote the measure of the workers with type p_l and that for the assets of quality q_k assigned to the match (p_l, q_k) respectively. For a given equilibrium, the allocation is given by

$$\begin{aligned} W_{pq}(p_l, q_k) &= \int_{\Omega_t \times \{q_k\}} r(p_l|t, q_k) dW, \text{ and} \\ C_{pq}(p_l, q_k) &= \int_{\Omega_t \times \{q_k\}} \mu(t, q_k) dW. \end{aligned}$$

Active Markets $\psi(q_k)$ denotes the subset of contracts or outside option chosen by the owners of asset quality q_k. $t' \in \psi(q_k)$ if and only if a positive measure of asset owners participate in the market (t', q_k). In equilibrium, the asset owners participate in a market only when they anticipate a positive measure of workers on the opposite side. Therefore, the set of active markets, denoted by Ψ, is the union of $\psi(q_k) \times q_k$ across all asset qualities, excluding the outside option. For any subset of markets A, $W(A)$ denotes the total measure of participating workers on the equilibrium path. The support of W must be Ψ by the same reasoning.

For the active markets, r and μ have to be consistent with the participation decisions for both sides. The feasibility constraints require that for any p_l and q_k,

$$\begin{aligned} Q(q_k) &\geq \int_{\Omega_t} \mu(t, q_k) dW, \text{ and} \\ P(p_l) &\geq \int_{\Omega_t \times \{q_k\}_{k=1}^K} r(p_l | t, q) dW. \end{aligned}$$

The equality must hold for type q_k if these asset owners strictly prefer the active markets to their outside option in equilibrium, and likewise for the workers' side. These lead to the optimal participation condition. With this condition in place, one can recover the participation decisions for both sides from the market tightness $\mu|\Psi$, the workers' composition $r|\Psi$, and the measure of participating workers W, where $g|\Psi$ denotes the restriction of g to the set Ψ.

Since workers and asset owners may ensure themselves a payoff of $\underline{V}$ and $\underline{U}$ respectively, so $V(p) \geq \underline{V}$ and $U(q) \geq \underline{U}$ are called participation constraints for workers and asset owners.

A worker will never get matched if he unilaterally switches to an inactive market. So workers only consider deviations to active markets or outside option. This is captured by the inequality (2.4) for the active markets and condition ii) in Optimal Participation. Abusing the terminology, I will also call this set of conditions as incentive compatibility (IC) condition.

In a worker's perspective, the set of active markets Λ represents the competition between the asset owners. The competition from other workers is summarized by the matching probabilities $\eta(\mu(t, q))$ in these markets.

Belief restriction Since there are continuums of workers and assets, switching between active markets or taking outside option by a single party has negligible impacts. The same is true when a worker unilaterally switches to an inactive market. The focus here is the deviation to some inactive market by an asset owner, and her belief about the pool of workers who will, in response, participate in that market.

The workers' optimal acceptance condition impose restrictions on this off-equilibrium-path belief. In particular, the inequality (2.4) is required to hold for the inactive markets as well. The conditions implicitly require all parties to share the same belief off the equilibrium path.

Suppose an owner of asset quality $\widetilde{q}$ deviates to post a contract $\widetilde{t}$. The belief restriction states that if $V(p_l) > v(p_l, \widetilde{q}, \widetilde{t})$ for all types, then no workers will be attracted, and so $\mu(\widetilde{t}, \widetilde{q}) = \infty$. No restrictions are imposed on the $r(\widetilde{t}, \widetilde{q})$, which has no bearing in such case. Now suppose that $V(p) < v(p, \widetilde{q}, \widetilde{t})$ for a subset of types. Then $\mu(\widetilde{t}, \widetilde{q})$ is given by the greatest value of tightness ratio for which the inequality (2.4) holds for all types of workers. Put it differently, $\mu(\widetilde{t}, \widetilde{q})$ is the lowest matching probability some types of workers are willing to endure. Furthermore, the support of $r(\widetilde{t}, \widetilde{q})$ contains only those types. The remaining possibility is that $V(p) \geq v(p, \widetilde{q}, \widetilde{t})$ holds for all types of workers and with equality for some type. The inequality (2.4) requires that $\mu(\widetilde{t}, \widetilde{q}) \geq 1$. The restriction on $r(\widetilde{t}, \widetilde{q})$ in the preceding discussion still applies if $\mu(\widetilde{t}, \widetilde{q})$ is finite. All results in this study are robust to additional restrictions on $\mu(\widetilde{t}, \widetilde{q})$ in this case.

The belief restriction is interpreted as follows: Suppose an owner of asset quality $\widetilde{q}$ is pondering a deviating offer $(\widetilde{t}, \widetilde{q})$. Workers of type p_l is interested in it if $V(p_l) < v(p_l, \widetilde{q}, \widetilde{t})$. If multiple workers are interested in the deviating offer, the competition between them manifests as a reduction in their matching probability, dissipating any gain from the offer. The worker, who ends up matching with her, must be among those who are willing to endure the lowest matching probability.

In the view of a single asset owner, $\{V(p_l)\}_{l=1}^{L}$ reflects the competition from other asset owners of various qualities. This in turns affects the competition among the workers for the deviating offer, and hence the distribution of workers it attracts. These are captured by $\mu(\widetilde{t}, \widetilde{q})$ and $r(\widetilde{t}, \widetilde{q})$ respectively.

The belief restriction here is often motivated by the "subgame perfection" in the competitive search literature. Suppose only ϵ-measure of the owners of asset quality $\widetilde{q}$ deviate to some inactive market $(\widetilde{t}, \widetilde{q})$. Observing the measure of the asset owners in every market, a worker has to anticipate his matching probability in each of the markets accordingly. When $\epsilon \to 0^+$, no types of workers can strictly gain from participating in the market $(\widetilde{t}, \widetilde{q})$ in the equilibrium of this "subgame". Otherwise, workers of all such types will turn up in this market but only ϵ-measure of them will get matched, resulting in an expected payoff below their outside option. It follows that the participating workers in the market $(\widetilde{t}, \widetilde{q})$, if any, are willing to accept the lowest matching probability. By continuity, the workers' payoff in the equilibrium of this "subgame" must converge to $V(p)$. This justifies the belief restriction discussed.[21]

Discussion Gale (1996) considers a continuum of markets indexed by all possible contracts and defines a notion of competitive equilibrium in the presence of adverse selection. In his definition of a "refined equilibrium", the restriction on the type of informed parties attracted by an off-equilibrium-path contract is the same as here.[22] The author suggests that this belief restriction is analogous to the "Universal Divinity" in Banks and Sobel (1988).

Eeckhout and Kircher (2010) define an equilibrium as a pair of measures of workers and assets across the markets. The set of active markets and the equilibrium payoffs are then derived from the pair of equilibrium measures. Eeckhout and Kircher (2010) adopt the same restriction on the market tightness for inactive markets. Since uninformed parties post prices in their setting, they leave out the off-equilibrium-path belief on the worker's type.

This study focuses on an equilibrium supporting positive assortative matching. As I will show, the corresponding incentive compatibility condition for workers and optimal participation conditions pin down a unique pair of measures of workers and assets across the markets for generic distributions of types. The remaining analysis is to verify that both sides have no profitable deviations and study the comparative statics of the equilibrium payoffs. To simplify the notation, I define an equilibrium using a set of equilibrium conditions directly involving the equilibrium payoffs and other

[21] The case that $V(p) \geq v(p, \widetilde{q}, \widetilde{t})$ for all types of workers and equality holds for some type, say p', is intricate. In such case, the limit $\mu(\widetilde{t}, \widetilde{q})$ may depend on the equilibrium strategies of both the workers and the asset owners. Nevertheless, the inequality (2.4) remains valid and equality holds for type p'.

[22] To be precise, Gale (1996) imposes the restriction only on r but not μ.

equilibrium objects of interest. This equilibrium definition closely follows that of Guerrieri, Shimer and Wright (2010), which share the same belief restriction.

I assume a finite distribution of types as a formal analysis for menus of general contracts with continuums of types invites substantial technical complications. Therefore, I define an equilibrium with respect to the support of a given finite distribution of types. The underlying arguments do not hinge on the assumption of a finite distribution. One may approximate any given distribution of types with a finite distribution close by. The results here apply all such finite distributions. On the other hand, the type space takes the form of an interval. This is because I have to vary the distribution of types in a continuous manner when establishing the necessity of the conditions for decentralizing PAM.

2.4.5 Menu of contracts

The baseline setting assumes that an asset owner may post only a single contract. Now consider a more general setting, in which an asset owner of q_k may post a menu of contracts specifying the asset quality $q \leq q_k$, the separation probability $\pi \leq 1$ and the associated payment scheme t. After the matching stage, the worker selects (π', q', t') from the menu. A lottery of the stated probability π' will be conducted publicly. If the lottery outcome is separation, then two parties will get their unmatched payoff. If the lottery outcome is continuation, the asset owner will impair the asset quality to q'. The pair will then perform production and split the output according to the contract t'. The key departure from the baseline setting is that the term of the contract, the asset quality and the matching probability for workers, after adjusted for the separation probability, can be made contingent on the worker's type, subject to additional incentive compatibility conditions.

It is without loss to focus on direct revelation mechanisms (DRM). The superscript and subscript denote the reported type and true type respectively. $T = \{q^l, \pi^l, t^l\}_{l=1}^L$ denotes a DRM, and satisfies the incentive compatibility conditions,

$$(1 - \pi^l)v(p_l, q^l, t^l) \geq (1 - \pi^{l'})v(p_l, q^{l'}, t^{l'})$$

for all l and l'. The set of DRM differs across asset qualities, and $\Omega_T^{DRM}(q)$ denotes that for owners of asset quality q.

The continuum of markets is now indexed by (T, q). An owner of asset quality q_k may take her outside option or participate in one of the markets $T \in \Omega_T^{DRM}(q_k)$. A worker may take his outside option or participate in any market. It is noteworthy that even for the same menu of contracts T, (T, q) and (T, q') are two distinct markets in this formulation. The workers' participation decisions in these two markets are allowed to differ. It is straightforward to modify the definition of equilibrium accordingly. The formal definition is relegated to the appendix.

Lemma 1 states that when the asset owners are allowed to post a menu of contracts, the set of equilibrium payoffs and allocations weakly expands.

Lemma 1. *For every competitive matching equilibrium, there exists an equilibrium using direct revelation mechanisms which supports the same allocation and equilibrium payoffs, and the asset owners post only degenerate direct revelation mechanisms.*

This result stems from two features of the current model. First, the pair of types determines the output distribution once a match is formed.

Any post-matching or post-contracting messaging is a zero-sum game. An asset owner may elicit a truthful report by posting a menu of contracts, but this confers the worker information rent without improving the matching surplus. Therefore, a menu of contracts is useful only if it affects the workers' participation decisions before the matching stage.

Second, an asset owner may pair up with at most one worker. Suppose an asset owner deviates to post a menu of contracts. As she cannot contract with a continuum of workers, the competition between workers drives down their matching probability to the level that no workers will gain from the deviating offer. The workers, who are willing to remain despite being rationed, are indifferent between the deviating offer and some other active markets. They may be of different types and select a different contract from the menu. When constructing an equilibrium using DRM, the asset owner is assumed to believe that only the lowest type among these workers will be attracted by her deviating offer.[23] If she profits from contracting with such type of workers, she must also profit from posting only the contract chosen by that type. In the presence of capacity constraint, this single contract will result in the same market tightness as the menu of contracts and attract a pool of workers of potentially higher types. Or put it differently, if an asset owner cannot profit from posting any contracts, deviation to a menu comprising these contracts neither improves her matching probability nor the worst admissible belief about her partner's type. Therefore, the set of

[23]First, this off-equilibrium-path belief is adopted in constructing the corresponding equilibrium using DRM but not required for the equilibrium using single contracts.

Second, the described belief is not the most pessimistic one allowed in the equilibrium definition. A contract chosen by workers of a higher type may provide the asset owner a lower payment.

equilibria in the baseline model is robust to the modification that the asset owners can post a menu of contracts.[24]

In the baseline setting, owners of the same asset quality may post several different contracts only if they are indifferent about these contracts in equilibrium. This is no longer true when the asset owners may post a menu of contracts. Even though an asset owner prefers some contracts in her posted menu over the rest, she refrains from posting only her favoured contracts if doing so will lead to a deterioration of her partner's type.[25] As a result, allowing menus of contracts potentially expands the set of equilibrium allocations and payoffs.

For notational simplicity, we restriction our attention to the baseline setting, where the asset owners post only a single contract, in the subsequent sections. All the results are robust to the introduction of the menus of contracts, including the necessary conditions for decentralizing PAM. We will revisit this issue at the end of Section 2.6.

2.5 First Best Allocation

I now formally define the First Best program. Since participation is costly, it is without loss to assume that the Utilitarian planner will pool all workers and assets assigned to the same match (p', q') into one meeting

[24]In fact, the argument underlying Lemma 1 applies to more general forms of menus beyond the scope of the subsequent analysis.

[25]This can be prevented by imposing some local sorting condition or stronger belief restriction. Though the choice of contract affects the matching surplus in Guerrieri, Shimer and Wright (2010), introducing menus of contracts has no effects in their setting because of a local sorting condition.

place. The total surplus for an allocation is then given by

$$TS(W_{pq}, C_{pq}) := \sum_{k=1}^{K} \sum_{l=1}^{L} E(Y|p_l, q_k) W_{pq}(p_l, q_k) \eta(\frac{C_{pq}(p_l, q_k)}{W_{pq}(p_l, q_k)}) - \sum_{k=1}^{K} \sum_{l=1}^{L} \left[\underline{V} W_{pq}(p_l, q_k) + \underline{U} C_{pq}(p_l, q_k) \right].$$

Definition. *Given the distribution of types* (P, Q)*, a First Best allocation* $(W_{pq}^{FB}, C_{pq}^{FB})$ *maximizes the total surplus* $TS(W_{pq}, C_{pq})$ *subject to the resources constraints:*

$$\begin{cases} \sum_{k=1}^{K} W_{pq}(p_l, q_k) \le P(p_l), \forall l \in [1, L] \\ \sum_{l=1}^{L} C_{pq}(p_l, q_k) \le Q(q_k), \forall k \in [1, K] \end{cases}$$

The set of First Best allocations is well-defined and always exists. $W_{pq}^{FB} = C_{pq}^{FB}$ because participation is costly. Hence, every active market clears in an equilibrium supporting a First Best allocation.

The First Best allocation is indeed unique under Assumption (P). Since the matching surplus is always positive, $\min\left\{\sum_{k=1}^{K} P(p_l), \sum_{l=1}^{L} Q(q_k)\right\}$ measure of agents on both sides participate in matching. The rest take their outside options. Only the highest types participate and pair up assortatively because the matching surplus is strictly increasing and SPM in types.

Definition. *Positive Assortative matching (PAM) is an allocation* (W_{pq}, C_{pq}) *satisfying* $W_{pq} = C_{pq}$ *and for any* $l' \ge 1$ *and* $k' \ge 1$*,*

$$\sum_{l \ge l'}^{L} \sum_{k \ge k'}^{K} W_{pq}^{FB}(p_l, q_k) = \min\{\sum_{l=l'}^{L} P(p_l), \sum_{k=k'}^{K} Q(q_k)\}. \tag{2.5}$$

Remark 2. *Positive Assortative matching is the unique First Best allocation.*

The lowest participating type on each side is referred as the threshold type. The threshold types are given by

$$\begin{aligned}
\underline{k} &= \max_{k' \geq 1} \left\{ \sum\nolimits_{k=k'}^{K} Q(q_k) \geq \min \left\{ \sum\nolimits_{l=1}^{L} P(p_l), \sum\nolimits_{k=1}^{K} Q(q_k) \right\} \right\}, \\
\underline{l} &= \max_{l' \geq 1} \left\{ \sum\nolimits_{l=l'}^{L} P(p_l) \geq \min \left\{ \sum\nolimits_{l=1}^{L} P(p_l), \sum\nolimits_{k=1}^{K} Q(q_k) \right\} \right\}.
\end{aligned}$$

Note that $\underline{k} = 1$ in the case $\sum_{k=1}^{K} Q(q_k) \leq \sum_{l=1}^{L} P(p_l)$ and $\underline{l} = 1$ if it is the other way around. For any $k \geq \underline{k}$, define $r_q^{FB}(q_k) = \{r_q^{FB}(p_l|q_k)\}_{l=1}^{L}$ where

$$r_q^{FB}(p_l|q_k) = \frac{W_{pq}^{FB}(p_l, q_k)}{\sum_{l'=1}^{L} W_{pq}^{FB}(p_{l'}, q_k)}. \tag{2.6}$$

$r_q^{FB}(q_k)$ is the distribution of workers that the asset owners of quality q_k match with under the First Best allocation.

2.5.1 First Best decentralization under full information

When the workers' types are public, the contract posted by an asset owner not only specifies the contingent payment scheme, but also the type of worker she commits to pair with. Owners of assets with the same quality may match with different type of workers, provided that they are indifferent between these contract offers. In this formulation, (t, p, q) indexes the continuums of markets. The market (t, p', q') only opens to the workers of type p' and the owners of asset quality q'. The equilibrium definition in Section 2.4.4 is adapted to this market structure to define an equilibrium under full information. The formal definition is provided in Appendix A.2.

Proposition 1. *Under full information, an allocation is supported by an equilibrium if and only if it is First Best.*

Proposition 1 does not require Assumption (P). It only relies on the availability of an ordered set of securities, $S_t \subseteq \Omega_t$, so that any fraction of matching surplus can be transferred between partners.

Let $V^{FB}(p_l) = \underline{V} + \Delta V^{FB}(p_l)$ and $U^{FB}(q_k) = \underline{U} + \Delta U^{FB}(q_k)$, where $\Delta V^{FB}(p_l) \geq 0$ and $\Delta U^{FB}(q_k) \geq 0$ are the shadow prices of the respective resource constraint in the First Best program. Under Assumption (P), $V^{FB}(p_l)$ and $U^{FB}(q_k)$ are monotonic in types, and strictly increasing for those above threshold types $p_{\underline{l}}$ and $q_{\underline{k}}$. The first order conditions for the First Best program are given by

$$V^{FB}(p_l) + U^{FB}(q_k) \geq E(Y|p_l, q_k) \tag{2.7}$$

with equality if $W_{pq}^{FB}(p_l, q_k) > 0$, and that the corresponding resource constraint must bind if $V^{FB}(p_l) > \underline{V}$ or $U^{FB}(q_k) > \underline{U}$.

A First Best allocation can be decentralized in the following manner. Asset owners of q_k, with a measure of $W_{pq}^{FB}(p_l, q_k) > 0$ will post the contract $t(.; s^{lk})$ for the workers of type p_l, where $t(.; s^{lk})$ provides the payoff $u(q_k, p_l, t(.; s^{lk})) = U^{FB}(q_k)$. If an asset owner posts a contract to attract workers of type $p_{l'}$, she has to offer $v(p_{l'}, q_k, t') \geq V^{FB}(p_{l'})$. This leaves her at most $u(q_k, p_{l'}, t') \leq U^{FB}(q_k)$ under the first order condition in (2.7). In this equilibrium, a worker is indifferent between any contracts available to him. Therefore, no one can gain from deviation.

Now consider an equilibrium under full information. Suppose $E(Y|p_l, q_k) > V(p_l) + U(q_k)$, then an asset owner of q_k will profit from offering the workers of p_l a contract with payoff slightly above $V(p_l)$. Her contract will be accepted and leaves her a payoff above $U(q_k)$. Since participation is costly, rationing necessarily results in $E(Y|p_l, q_k) > V(p_l) + U(q_k)$ for any pair of participants in that market. This immediately implies market clearing in every active market. If the resource constraint is not binding for some type of agents, say p_l, the decision that some of these workers are taking their outside option indicates that $V(p_l) = \underline{V}$. Hence, the equilibrium payoffs and allocation for every equilibrium conform with the first order conditions of the First Best program. It follows that the equilibrium allocation is a First Best.

Corollary 1. *Under full information, the First Best allocation can be decentralized using any set of contracts with transfers $U^{FB}(q_k)$ for q_k above the threshold type. The equilibrium divisions of the surpluses are invariant to the set of feasible contracts.*

Suppose that the asset owners may post a menu of contracts, which the contract term is contingent on the type of the worker. The markets are indexed by the pair of types and the menu of contracts. The meeting is bilateral. Proposition 1 can be extended to this setting using essentially the same argument.

2.5.2 First Best decentralization in price competition

I now return to the case of one-sided private information but omit Assumption (C) in this subsection. I will consider the class of cash payment, or fixed prices, which is represented by

$$t_c(y; s) = sE(Y|\overline{p}, \overline{q}).$$

Price competition refers to a setting where the class of cash payment is feasible.

Proposition 2 states that in price competition, the equilibrium allocation and payoffs are irrespective of whether the workers' types are private or public. Therefore, price competition always leads to the First Best outcome. This result is not surprising at all. With cash payment, the workers receive the residual claims, while the asset owners find the belief about the worker's type irrelevant. As a result, the incentives for both sides are the same as in the full information case.

Proposition 2. *When cash payment is feasible, an allocation is supported by an equilibrium if and only if it is First Best. Furthermore, the equilibrium payoffs in this equilibrium are the same as under full information.*

Like Proposition 1, Proposition 2 does not hinge on Assumption (P). A First Best allocation is decentralized in an equilibrium, in which owners of quality q_k post the cash payment $U^{FB}(q_k)$. In this equilibrium, a worker of type p_l receives $E(Y|p_l, q_k) - U^{FB}(q_k)$ from a match with an asset of quality q_k. From the first order condition in (2.7), he can earn no more than his equilibrium payoff $V^{FB}(p_l)$ by deviating to other active markets. As in the full information, this first order condition also rules out profitable deviations on the asset side. The converse is also true. Specifically, if $E(Y|p_l, q_k) > V(p_l) + U(q_k)$, then an asset owner of q_k will post a cash payment slightly above $V(p_l)$, earning a payoff above $u(q_k)$. We then conclude that in every equilibrium, the allocation is a First Best and in particular, all active markets clear.

Consider a generic distribution of types which satisfies $\sum_{k=k'}^{K} Q(q_k) \neq \sum_{l=l'}^{L} P(p_l)$ for all $k' \geq 1$ and $l' \geq 1$. Under Assumption (P), the first order conditions of the First Best program uniquely determine the equilibrium payoffs, and hence the set of prices posted in equilibrium. The asset owners of the threshold type, if participating, will post the cash payment

$$U^{FB}(q_{\underline{k}}) = \begin{cases} E(Y|p_{\underline{l}}, q_1) - \underline{V}, & \text{if } \sum_{k=1}^{K} Q(q_k) < \sum_{l=1}^{L} P(p_l); \text{ and} \\ \underline{U}, & \text{if } \sum_{k=1}^{K} Q(q_k) > \sum_{l=1}^{L} P(p_l). \end{cases}$$

Let $\bar{l}(k)$ denote $\max\{l \geq 1 : W_{pq}^{FB}(p_l, q_k) > 0, q_k \geq q_{\underline{k}}\}$, the highest type of workers pairing up with $q_k \geq q_{\underline{k}}$ in the First Best allocation. $\bar{l}(k)$

is increasing in l under PAM. Generically, the First Best allocation also involves matches between workers of type $p_{\bar{l}(k)}$ and the assets of q_{k+1}. From the first order condition in (2.7),

$$V^{FB}(p_{\bar{l}(k)}) = E(Y|p_{\bar{l}(k)}, q_k) - U^{FB}(q_k) = E(Y|p_{\bar{l}(k)}, q_{k+1}) - U^{FB}(q_{k+1}),$$

which allows us to construct the set of posted prices recursively,

$$U^{FB}(q_{k+1}) - U^{FB}(q_k) = E(Y|p_{\bar{l}(k)}, q_{k+1}) - E(Y|p_{\bar{l}(k)}, q_k). \tag{2.8}$$

2.6 Decentralization Of Positive Assortative Matching

This section studies whether the First Best allocation, defined in the equality (2.5), can be decentralized in an equilibrium. For exposition, I first analyze the case that the whole set of feasible contracts is fully ordered.[26]

Assumption (S)**.** *The feasible set of contracts is an ordered set of securities,* $\Omega_t = S_t$.

I will construct a candidate equilibrium in the process. In general, the expected payment between partners is not separable in the contract term and the expected output. So I characterize the existence conditions in terms of the worker's expected payoff, representing his trade-off between the asset quality q, the contract term s and his matching probability $\eta(\mu)$. These conditions apply to any distribution of types. They are Condition Sorting-p which renders deviations to other active markets unprofitable for the workers, Condition Screening-q and Condition Entry-q which deter deviations by

[26] Section 2.8 will relax this assumption and discuss what other contracts can be made feasible without affecting the results.

the asset owners above and below the threshold quality respectively. In the subsequent sections, I will explain how they address the potential deviations by workers and asset owners in the candidate equilibrium. The argument for necessity is relegated to the appendix. This section culminates with a unifying sufficient condition, Global ID, for the decentralization of the First Best allocation. I will evaluate the robustness of the result at the end of the section.

2.6.1 Sorting of workers

It is instructive to start with the problem of implementing the First Best allocation, in which the Utilitarian planner decides the contracts for each type of assets subject to their voluntary participation.[27] The implementation problem highlights the limitations due to the incentive compatibility constraints for the workers. Its solution then serves as the set of active markets in the candidate equilibrium supporting the First Best allocation.

All workers, regardless of their types, share the same preference over an ordered set of securities. The First Best allocation requires market clearing in every active market, and therefore owners of the same asset quality, if participating, must post the same contract. Otherwise, either the asset owners posting the contract with the highest term s will be left unmatched or the workers will be rationed in the market with the lowest term s. Hence, the solution to the implementation problem can be written as $\{(t(.;s_k), q_k)\}_{k \geq \underline{k}}$, where $t(.;s_k)$ denotes the contract for the owners of asset quality $q_k \geq q_{\underline{k}}$.

[27] In the market structure here, it is straightforward to implement such policy by restricting the set of markets available to the asset owners.

A worker has to decide among the active markets and his outside option,

$$V(p_l) = \max\{\underline{V}, \{v(p_l, q_k, t(.; s_k))\}_{k \geq \underline{k}}\}.$$

The implementation of the First Best allocation entails two sets of incentive compatibility (IC) conditions on the workers' side. The first set of conditions is given by $V(p_l) = \underline{V}$ if $\sum_{l=1}^{L} W_{pq}^{FB}(p_l, q_k) < P(p_l)$, so that these workers are indifferent about taking their outside options. The second set concerns the choice of contracts, requiring $V(p_l) = v(p_l, q_k, t(.; s_k))$ if (p_l, q_k) is in the support of W_{pq}^{FB}.

The distributions of workers in the active markets are consistent with the First Best allocation, which are given by r_q^{FB} defined in equality (2.6). Voluntary participation of the asset owners requires that $u(q_k, r_q^{FB}(q_k), t(.; \widetilde{s}_k)) \geq \underline{U}$ for $q_k \geq q_{\underline{k}}$. Furthermore $u(q_{\underline{k}}, r_q^{FB}(q_{\underline{k}}), t(.; s_{\underline{k}})) = \underline{U}$ if $\sum_{l=1}^{L} W_{pq}^{FB}(p_l, q_{\underline{k}}) < Q(q_{\underline{k}})$, so that the corresponding active market clears.

Now consider a generic distribution of types with $\sum_{k=k'}^{K} Q(q_k) \neq \sum_{l=l'}^{L} P(p_l)$ for all $k' \geq 1$ and $l' \geq 1$. In this case, the outlined conditions uniquely pin down the set of contracts offered . Let $\{(t(.; \widetilde{s}_k), q_k)\}_{k \geq \underline{k}}$ denote this set of contracts.

$\widetilde{s}_{\underline{k}}$ is chosen to ensure efficient participation of the threshold type on the long side, which requires the following indifference condition.[28]

$$\begin{cases} v(p_{\underline{l}}, q_{\underline{k}}, t(.; \widetilde{s}_{\underline{k}})) = \underline{V}, & \text{if } \sum_{k=1}^{K} Q(q_k) < \sum_{l=1}^{L} P(p_l); \text{ and} \\ u(q_{\underline{k}}, r_q^{FB}(q_{\underline{k}}), t(.; \widetilde{s}_{\underline{k}})) = \underline{U}, & \text{if } \sum_{k=1}^{K} Q(q_k) \geq \sum_{l=1}^{L} P(p_l). \end{cases} \tag{2.9}$$

This avoids excessive participation of the threshold type on the long side by leaving them indifferent between participation and their outside options.

[28] In equation (2.9), I assume that the workers receive the entire matching surplus when both sides are of equal measure. This is innocuous and not required for any of my results.

Recall that $p_{\bar{l}(k)}$ denotes the highest type of workers pairing up with the assets of $q_k \geq q_{\underline{k}}$ in the First Best allocation. For generic distributions of types, the First Best allocation also involves matches between workers of type $p_{\bar{l}(k)}$ and the assets of q_{k+1}. As a result, these workers must be indifferent between the market $(t(.;s_k), q_k)$ and $(t(.;s_{k+1}), q_{k+1})$, so that the local upward IC condition must hold with equality.

$$v(p_{\bar{l}(k)}, q_k, t(.;\widetilde{s}_k)) = v(p_{\bar{l}(k)}, q_{k+1}, t(.;\widetilde{s}_{k+1})). \tag{2.10}$$

For $k > \underline{k}$, $\widetilde{s}_k$ is defined recursively by the indifference condition in (2.10).[29]

Remark 3. *Give any generic distribution of types, the set of active markets for any equilibrium supporting the First Best allocation, if exists, is unique and given by* $\{(t(.;\widetilde{s}_k), q_k)\}_{k \geq \underline{k}}$*, where* $\{\widetilde{s}_k\}_{k \geq \underline{k}}$ *is determined by equations (2.9) and (2.10) recursively.*

$\{\widetilde{s}_k\}_{k \geq \underline{k}}$ is a strictly increasing sequence, representing the trade-off between the asset quality and the contract term facing the workers. More importantly, $\{(t(.;\widetilde{s}_k), q_k)\}_{k \geq \underline{k}}$ define indirect mappings from the distribution of types to the equilibrium payoffs, as well as the deviating payoffs for both sides. Decentralization of the First Best allocation requires the former to be always above the latter. The subsequent analysis studies how these mappings depend on the properties of the worker's and asset owner's expected payoff. For non-generic distributions of types, the First Best allocation can also be supported by a continuum of equilibria, including the

[29] Equation (2.10) should not be interpreted as binding local upward IC conditions. A subset of IC conditions is said to be binding only with respect to certain optimization problems, such as profit maximization or information rent minimization. The equality in (2.10) holds merely because the First Best allocation involves matches between workers of type $p_{\bar{l}(k)}$ and the asset of q_{k+1} as well as those of q_k.

candidate equilibrium with the active markets $\{(t(.;\widetilde{s}_k), q_k)\}_{k\geq\underline{k}}$. Hence, all results apply to any distribution of types.

Condition (Sorting-p)**.** *$v(p, q, t(.; s))$ satisfies Condition Sorting-p if for any $p^H > p^L$, $q^H > q^L$ and $s^H > s^L$ satisfying $v(p^L, q^L, t(.; s^L)) \geq \underline{V}$ and $u(q^L, p^H, t(.; s^L)) \geq \underline{U}$, the following holds:*

$$\begin{aligned} v(p^H, q^H, t(.; s^H)) &\geq (>) v(p^H, q^L, t(.; s^L)) \\ \textit{if } v(p^L, q^H, t(.; s^H)) &\geq (>) v(p^L, q^L, t(.; s^L)). \end{aligned} \tag{2.11}$$

And $v(p, q, t(.; s))$ satisfies Condition strict Sorting-p if (2.11) is replaced by

$$v(p^H, q^H, t(.; s^H)) > v(p^H, q^L, t(.; s^L)) \textit{ if } v(p^L, q^H, t(.; s^H)) \geq v(p^L, q^L, t(.; s^L)).$$

Condition Sorting-p is a single crossing property on the worker's preference over asset quality and contract term, provided that he will get matched. Consider two contracts $(t(.; s^H), q^H)$ and $(t(.; s^L), q^L)$, the line (2.11) states that a high type worker must strictly prefer the contract $(t(.; s^H), q^H)$ to $(t(.; s^L), q^L)$ if a worker of lower type does so. Its contrapositive requires that a low type worker must strictly prefer the contract $(t(.; s^L), q^L)$ to $(t(.; s^H), q^H)$ if a worker of higher type does so. Nevertheless, only the contracts that may be posted in the candidate equilibrium have to satisfy this property. This is achieved by the restrictions $v(p^L, q^L, t(.; s^L)) \geq \underline{V}$ and $u(q^L, p^H, t(.; s^L)) \geq \underline{U}$, which ensure that the outside options for the workers and the asset owners are no more attractive than the contracts.[30] Hence, Condition Sorting-p is weaker than the standard single crossing property, which applies to the entire domain. Condition strict Sorting-p is a stronger

[30]For the worker's type in $\Omega_p(t(.; \widetilde{s}_{\underline{k}}), q_{\underline{k}})$, $p_{\bar{l}(\underline{k})}$ plays the role of p^H and $u(q_{\underline{k}}, p_{\bar{l}(\underline{k})}, \widetilde{s}_{\underline{k}}) \geq \underline{U}$ by construction. Therefore, Condition Sorting-p imposes the restriction $u(q^L, p^H, t(.; s^L)) \geq \underline{U}$.

version of Sorting-p. It further requires that a high type worker strictly prefer the contract $(t(.; s^H), q^H)$ to $(t(.; s^L), q^L)$ when a worker of lower type is indifferent between the two.

It is the well-known that the single crossing property in Condition Sorting-p ensures the set of adjacent upward IC conditions in (2.10) are sufficient for the workers' incentive compatibility.

The construction of $\{(t(.; \widetilde{s}_k), q_k)\}_{k \geq \underline{k}}$ implies that the equilibrium payoffs are monotonic in types, and hence the participation constraints for the both sides are satisfied. Therefore, Condition Sorting-p allows the implementation of the First Best allocation.[31] As we will see, it is also necessary for the implementation for arbitrary distributions of types.[32]

Lemma 2. *Suppose Condition Sorting-p holds. Then the participation constraints for both sides and the workers' incentive compatibility condition are always satisfied given the set of active markets* $\{(t(.; \widetilde{s}_k), q_k)\}_{k \geq \underline{k}}$.

Recall that in the candidate equilibrium, the distributions of workers in the active markets are given by

$$r(t(.; \widetilde{s}_k), q_k) = r_q^{FB}(q_k), q_k \geq q_{\underline{k}}.$$

Hence, the equilibrium payoff for the asset owners is given by

$$U(q_k) = \begin{cases} u(q_k, r_q^{FB}(q_k), t(.; \widetilde{s}_k)), & \text{if } q_k \geq q_{\underline{k}}\text{; and} \\ \underline{U}, & \text{if } q_k < q_{\underline{k}}. \end{cases}$$

On the other side, a participating worker pays the asset owner $u(q_k, p_l, t(.; \widetilde{s}_k))$ in expectation, leaving him

$$V(p_l) = E(Y|p_l, q_k) - u(q_k, p_l, t(.; \widetilde{s}_k)).$$

[31] The argument for Lemma 2 does not hinge on the equality in (2.10) and applies to non-generic distribution of types.

[32] The necessity will be established in the proof of Proposition 3.

In price competition, he pays $U^{FB}(q_k)$ in equilibrium. Since $u(q_k, p_l, t(.; s))$ strictly increases with the worker's type, $V(p_l)$ must increase at a slower rate than in price competition, and the opposite is true for the asset side.

In both cases, the equilibrium payoff for the threshold type on the long side is the same as his outside option, hence

$$u(q_{\underline{k}}, p_{\bar{l}(\underline{k})}, t(.; \widetilde{s}_{\underline{k}})) \geq U^{FB}(q_{\underline{k}}).$$

The adjacent upward IC conditions (2.8) and (2.10) in these two cases can be rewritten as

$$u(q_{k+1}, p_{\bar{l}(k)}, t(.; \widetilde{s}_{k+1})) - U^{FB}(q_{k+1}) = u(q_k, p_{\bar{l}(k)}, t(.; \widetilde{s}_k)) - U^{FB}(q_k).$$

As the payment made by the worker increases with his type, the workers all pay more in the candidate equilibrium when they are on the long side. When the workers are on the short side, those of types above $p_{\bar{l}(\underline{k})}$ must pay more while their peers of the threshold type are better off.

Remark 4. *Comparing with the equilibria under full information or price competition, in the candidate equilibrium,*

1. *the equilibrium payoff for the asset owners is higher, and increases with their types at a faster rate, and*

2. *the equilibrium payoff for the workers increases with their types at a slower rate, and is lower for those matching with assets of quality strictly above the threshold type.*

The shift in the equilibrium divisions of the matching surpluses is related to the linkage principle in DeMarzo, Kremer and Skrzypacz (2005). To see the connection, consider a setting with a continuum of worker's types. The contract term $\widetilde{s}_{k+1}$, and hence the payment received by an asset owner of q_{k+1}, is determined by the local competition between the workers of $p_{\bar{l}(k)}$ and those of slightly higher types for the same assets q_{k+1}. This resembles an auction of an asset of q_{k+1}. Since the linkage between the worker's type and the expected payment he made is greater under a contingent contract than a posted price, DeMarzo, Kremer and Skrzypacz (2005) states that the competition between workers will intensify, bidding up the payment to the asset owner.

What is novel in assortative matching is the additional spillover effect to the competition for better assets. As the workers of $p_{l(k+1)}$ find the contract term for assets q_{k+1} less favorable, they are willing to pay more for an asset of q_{k+2}, further intensifying the competition for such assets. This spillover effect keeps growing when moving up to better assets. As a result, all asset owners are better off in comparison with price competition.[33] Section 2.8.1 will generalize this comparative statics for two different sets of feasible contracts.

2.6.2 Screening by asset owners

In this subsection, we assume that the IC conditions on the workers' side are all met and turn to the incentives for the asset side.[34] An asset owner chooses between her outside option and posting a contract. The First Best allocation can be decentralized if no asset owners may profit from posting a

[33] Note that the comparison with the full information case is due to the competition among the asset owners. The auctioneer will extract all the surplus if she knows the bidder's types.

[34] Even if Condition Sorting-p is not met, there are distributions of types for which the workers' IC conditions are satisfied in the candidate equilibrium. Condition Sorting-p plays no role in the analysis in this subsection.

deviating offer,

$$U(q_k) = \max\{\underline{U}, \{u(q_k, r(t(.;s), q_k), t(.;s), \mu(t(.;s), q_k))\}_{s\in[0,1]}\}.$$

Remark 5. *An owner of asset $q_k \geq q_{\underline{k}}$ will never profit from a match with a worker of type no higher than the lowest type in the support of $r_q^{FB}(q_k)$.*

Let p^L denote the lowest type of workers whom the asset owner may pair with in the First Best allocation. She does not gain from matching with such workers under full information because the decline in the expected output outweighs the savings in the payment for the worker,

$$U^{FB}(q_k) \geq E(Y|p^L, q_k) - V^{FB}(p^L) > E(Y|p_l, q_k) - V^{FB}(p_l).$$

Remark 4 establishes that the equilibrium payoff for workers increases at a slower rate in the candidate equilibrium. Hence, the amount of savings is even lower now,

$$U(q_k) \geq E(Y|p^L, q_k) - V(p^L) > E(Y|p_l, q_k) - V(p_l).$$

Yet the asset owner can receive at most $E(Y|p_l, q_k) - V(p_l)$ when pairing up with a worker of p_l. Otherwise, the worker will not accept the offered contract. Therefore, an asset owner will never attempt to poach weaker workers! On another hand, poaching a better worker is potentially profitable because they are willing to accept a lower payoff than in the full information case.

The worker's gain from a match $v(p, q_k, t(.;s))$ depends on the worker's type. As the quality of her asset is given, an asset owner may screen out better workers by varying both the contract term and the matching probability for the workers. To induce rationing, the asset owner must accept a lower term s compensating the workers for their risk of leaving unmatched.[35]

The distribution of workers attracted by a deviating offer depends on other options available to them, which are the active markets $\{(t(.;\widetilde{s}_k), q_k)\}_{k \geq \underline{k}}$ from the preceding section. I now construct the off-equilibrium-path belief in the candidate equilibrium. For any inactive market $(t(.;s'), q_k)$, no workers will participate if the contract $(t(.;s'), q_k)$ provides them no more than their equilibrium payoffs.[36] In this case, $r(t(.;s'), q_k)$ can be set arbitrarily, say the prior distribution for the worker's side. Suppose certain types of workers strictly gain from the contract $(t(.;s'), q_k)$, competition among workers pushes down the tightness ratio until no one gains from participating in that market,

$$\mu(t(.;s'), q_k) = \max\{\mu' \leq 1 : V(p') \geq \mu' v(p', q_k, t(.;s')), p' \in \{p_l\}_{l=1}^{L}\}.$$

Our equilibrium definition restricts the support of $r(t(.;s'), q_k)$ to the workers who are willing to endure the lowest matching probability. In the candidate equilibrium, $r(t(.;s'), q_k)$ is taken to be degenerate at the lowest type among these workers,

$$\min\{p' \in \{p_l\}_{l=1}^{L} : V(p') = \mu(t(.;s'), q_k) v(p', q_k, t(.;s'))\}$$

This is the most pessimistic belief allowed.

I will address the screening incentive for an owner of asset quality $q_k \geq q_{\underline{k}}$ and $q_k < q_{\underline{k}}$ in sequence. An owner of asset quality $q_k \geq q_{\underline{k}}$ will face competition from other owners of the same or lower asset quality while

[35] Section 2.6.4 provides further discussion on the screening instrument available to the asset owner.

[36] In the candidate equilibrium, I take $\mu(t(.;s'), q') = \infty$ when $V(p_l) = v(p_l, q', t(.;s'))$ for some types. All results remain valid if $\mu(t(.;s'), q')$ is finite in such case.

an owner of asset quality $q_k < q_{\underline{k}}$ will compete only with those of higher asset quality, in particular the threshold type $q_{\underline{k}}$. As we are looking for necessary and sufficient conditions for arbitrary distributions of types, the most substantial difference between the two cases lies in the restrictions imposed on the set of active markets in the candidate equilibrium. Note that for any asset quality $q \in (\underline{q}, \overline{q})$, it is below the threshold type in the First Best allocation for some distributions of types, while above the threshold for other distributions of types.

Let us consider the screening problem for the owner of asset quality $q_k \geq q_{\underline{k}}$. Suppose an asset owner posts a contract with $s > \widetilde{s}_k$, this contract is dominated by the contract $t(.; \widetilde{s}_k)$ posted by other owners of asset quality q_k. As the active market clears in the candidate equilibrium, the deviating contract will attract no workers. Now consider a contract with lower $s < \widetilde{s}_k$, competition among workers drives down their matching probability in the market $(t(.; s), q_k)$, raising the possibility of screening.

Condition (Strong Screening-q)**.** *$v(p, q, t(.; s))$ satisfies Strong Screening-q if for any $q \in (\underline{q}, \overline{q}]$, $p^H > p^L$ and $s^H > s^L$ satisfying $u(q, p^H, t(.; s^L)) > \max\{\underline{U}, u(q, p^L, t(.; s^H))\}$ and $v(p^L, q, t(.; s^H)) \geq \underline{V}$, then*

$$\frac{v(p^H, q, t(.; s^H))}{v(p^H, q, t(.; s^L))} > \frac{v(p^L, q, t(.; s^H))}{v(p^L, q, t(.; s^L))}. \tag{2.12}$$

Condition Strong Screening-q concerns the worker's preference over the contract term and his matching probability for the same type of assets. The inequality (2.12) is an increasing difference (ID) condition (in ratio) on $v(p, q, t(.; s))$, under which an increase in the term s reduces the value of a match for a low type worker proportionally more than a high type worker. This in turn implies that when facing a trade-off between matching probability and contract term, a high type worker prefers a higher matching probability whereas a low type worker prefers a more generous term. As a result, rationing can never improve the distribution of workers.

Fix a distribution of types, the contract $(t(.;\widetilde{s}_k), q_k)$ corresponds to $(t(.;s^H), q)$ in Condition Strong Screening-q. p^L is the lowest type of workers whom the asset owner may match with in the First Best allocation. An owner of asset quality q_k gains from posting a contract with $s^L < \widetilde{s}_k$ only if the deviating offer will be accepted by some workers of p^H, satisfying $u(q_k, p^H, t(.;s^L)) > U(q_k)$. Condition Strong Screening-q ensures the workers of p^L are willing to endure a lower matching probability in the market $(t(.;s^L), q_k)$, crowding out workers of p^H. To see this, the equilibrium tightness ratio in the market $(t(.;s^L), q_k)$ must satisfy

$$V(p^L) = v(p^L, q_k, t(.;\widetilde{s}_k)) \geq \mu(t(.;s^L), q_k)v(p^L, q_k, t(.;s^L)).$$

The IC condition for workers of p^H and the inequality (2.12) together yield

$$V(p^H) \geq v(p^H, q_k, t(.;\widetilde{s}_k)) > \mu(t(.;s^L), q_k)v(p^H, q_k, t(.;s^L)),$$

so no workers of p^H will accept the deviating offer.

Condition Strong Screening-q, though intuitive, is stronger than necessary. The reason is that it fails to capitalize on the presence of workers matching with assets of other qualities. Condition Screening-q is weaker than Condition Strong Screening-q in this regard. Condition Screening-q factors in the possibility that there can be some workers of type $p_l \leq p^L$, who needs not be participating in the market $(t(.;\widetilde{s}_k), q_k)$, are willing to endure a lower matching probability than those of p^H in the market $(t(.;s^L), q_k)$. Given the equilibrium payoffs, this happens if and only if

$$\frac{V(p^H)}{V(p_l)} \geq \frac{v(p^H, q_k, t(.;s^L))}{v(p_l, q_k, t(.;s^L))}.$$

Since the equilibrium payoffs are endogenous, the challenge is to characterize when this will be the case for arbitrary distributions of types and worker's expected payoff. PAM and the equalities conditions (2.9) and (2.10) restrict the type distributions for which $(t(.;\widetilde{s}_k), q_k)$ is an active market, and hence the active markets for assets of lower qualities. Condition Screening-q exploits this restriction. It is necessary and sufficient for preventing owners of asset quality $q_k \geq q_{\underline{k}}$ from deviating in the candidate equilibrium.

Condition (Screening-q)**.** *$v(p,q,t(.;s))$ is said to satisfy Screening-q if for any $q \in \Omega_q, p^H > p^L$ and $s^H > s^L$ satisfying $v(p^L,q,t(.;s^H)) \geq \underline{V}$ and $u(q,p^H,t(.;s^L)) > \max\{\underline{U}, u(q,p^L,t(.;s^H))\}$, then either*

$$\frac{v(p^H,q,t(.;s^H))}{v(p^H,q,t(.;s^L))} \geq \frac{v(p^L,q,t(.;s^H))}{v(p^L,q,t(.;s^L))}, \tag{2.13}$$

Or the followings hold:

1. *$v(p^L,q,t(.;s^H)) \in (\underline{V}, E(Y|p^L,\underline{q}) - \underline{U}]$, and*
2. *Let $q' \leq q$ and $s' \leq s^H$ satisfy $v(p^L,q',t(.;s')) = v(p^L,q,t(.;s^H))$. If $v(p',q',t(.;s')) = \underline{V}$ or $u(q',p',t(.;s')) \leq \underline{U}$ for some $p' < p^L$, then*

$$\frac{v(p^H,q,t(.;s^H))}{v(p^H,q,t(.;s^L))} \geq \frac{v(p',q',t(.;s'))}{v(p',q,t(.;s^L))}. \tag{2.14}$$

The inequality (2.13) merely replaces the strict inequality (2.12) in Condition Strong Screening-q with a weak one. In comparison with Condition Strong Screening-q, Condition Screening-q allows the situation that the workers of p^H are willing to endure a lower matching probability than workers of p^L for the contract term s^L in the candidate equilibrium. Such exception is permitted only under two additional conditions, which jointly guarantee that there are other workers of type below p^L prepared to accept a lower matching probability than those of type p^H.

The first condition ensures that some workers of types below p^L are participating in other active markets. Recall that the contract $(t(.;\widetilde{s}_k), q_k)$ plays the role of $(t(.;s^H), q)$ and p^L is the lowest type of workers participating in this market. For any asset quality q' below q_k, define the contract $t(.;\widehat{s}(q'))$ such that a worker of type p^L is indifferent between the contracts $(t(.;\widetilde{s}_k), q_k)$ and $(t(.;\widehat{s}(q')), q')$. Condition Screening-q requires $v(p^L, q_k, t(.;\widetilde{s}_k) > \underline{V}$ and $u(q', p^L, t(.;\widehat{s}(q'))) > \underline{U}$ for any $q' < q_k$. By construction of $\widetilde{s}_{\underline{k}}$, the very fact that some workers of p^L are participating in the active market $(t(.;\widetilde{s}_k), q_k)$ in the candidate equilibrium implies that the First Best allocation involves matches between workers of types below p^L and assets of quality below q_k.

The second condition ensures that workers of p^H will not participate in the market $(t(.;s^L), q_k)$. $p_{\underline{l}}$ is the lowest type among the participating workers. If workers are on the short side, there always exists $q' < q_k$ such that $V(p_{\underline{l}}) \leq v(p_{\underline{l}}, q', t(.;\widehat{s}(q')))$ and $u(q', p_{\underline{l}}, t(.;\widehat{s}(q'))) \leq \underline{U}$. Conversely, there exists $q' < q_k$ satisfying $V(p_{\underline{l}}) = v(p_{\underline{l}}, q', t(.;\widehat{s}(q'))) = \underline{V}$ if workers are on the long side. Given the continuity of the expected payoff, the existence of q' is an implication of the workers' incentive compatibility conditions and the construction of $\widetilde{s}_{\underline{k}}$. In general, q' is not in the support of the distribution of types and $q' \neq q_{\underline{k}}$ in particular.[37]

Under Condition Screening-q, the inequality (2.14) implies that in the candidate equilibrium,

$$V(p_{\underline{l}}) \leq v(p_{\underline{l}}, q', t(.;\widehat{s}(q'))) < v(p^H, q_k, t(.;s^L)),$$

and

$$\frac{V(p^H)}{V(p_{\underline{l}})} \geq \frac{v(p^H, q_k, t(.;s^H))}{v(p_{\underline{l}}, q', t(.;\widehat{s}(q')))} \geq \frac{v(p^H, q_k, t(.;s^L))}{v(p_{\underline{l}}, q_k, t(.;s^L))}.$$

[37] The support of type distribution Q is $\{q_k\}_{k=1}^K$, a finite subset of the types space $(\underline{q}, \overline{q}]$, and $q' \in (\underline{q}, \overline{q})$.

It follows that workers of $p_{\underline{l}}$ accept a lower matching probability in the market $(t(.; s^L), q_k)$ than their peers of p^H, and hence the asset owner believes that her deviating offer will attract no workers of p^H. Her deviating offer may be accepted other workers of types above p^L but she will not profit from such match.

Lemma 3. *Suppose participation constraints for both sides and the workers' incentive compatibility condition are met. Under Condition Screening-q, an owner of asset quality $q_k \geq q_{\underline{k}}$ cannot profit from posting a contract $t(.; s)$, where $s \neq \widetilde{s}_k$, in the candidate equilibrium.*

We now proceed to the participation decision of the asset owners of $q_k < q_{\underline{k}}$ in the candidate equilibrium. For them, only the deviating offers with $s^L < \widetilde{s}_{\underline{k}}$ are relevant. Otherwise, the offer will be dominated by the market $(t(.; \widetilde{s}_{\underline{k}}), q_{\underline{k}})$. A direct consequence of Condition Sorting-p is that a deviating offer from an owner of asset quality $q_k < q_{\underline{k}}$, if attracts any workers at all, will also interest the workers of the lowest type p_1. The construction of $\widetilde{s}_{\underline{k}}$ implies that workers of type p_1 pay the asset owners of $q_{\underline{k}}$ no more than the latter's outside option, so they will pay the deviating asset owner even less under the contract $t(.; s^L)$. Consequently, the asset owner cannot profit from such offer if it will attract only the workers of p_1.

Condition Entry-q builds on this observation. It is an increasing difference condition concerning the worker's preference over a more generous contract term and improvements in both asset quality and his matching probability. Condition Entry-q is necessary and sufficient to render any deviation by the owners of asset quality $q_k < q_{\underline{k}}$ unprofitable in the candidate equilibrium. It shall be stressed that the sufficiency is irrespective of whether Condition Sorting-p holds.

Condition (Entry-q). $v(p, q, t(.; s))$ *is said to satisfy Entry-q if for any* $p^H > p^L$, $q^H > q^L$ *and* $s^H > s^L$ *satisfying*

$$\begin{cases} v(p^H, q^L, t(.; s^L)) > v(p^H, q^H, t(.; s^H)), \text{ and} \\ v(p^L, q^H, t(.; s^H)) \geq \underline{V}, \text{ and} \\ u(q^L, p^H, t(.; s^L)) > \underline{U} \geq u(q^H, p^L, t(.; s^H)), \end{cases} \tag{2.15}$$

then

$$\frac{v(p^H, q^H, t(.; s^H))}{v(p^L, q^H, t(.; s^H))} \geq \frac{v(p^H, q^L, t(.; s^L))}{v(p^L, q^L, t(.; s^L))}. \tag{2.16}$$

Suppose that workers of some type p^H prefer the contract $(t(.; s^L), q^L)$ to $(t(.; s^H), q^H)$. Under these two contracts, a worker of type p^H pays more than the asset owner's outside option, whereas a worker of some type p^L does not. Then the inequality (2.16) in Condition Entry-q has two implications. First, workers of type p^L also prefer the contract $(t(.; s^L), q^L)$ to $(t(.; s^H), q^H)$.[38] Second, for any pair of matching probabilities in the market $(t(.; s^H), q^H)$ and $(t(.; s^L), q^L)$, workers of type p^L always prefer the latter market to the former if their peers of type p^H do so.

Lemma 4. *Suppose participation constraints for both sides and the workers' incentive compatibility condition are met. Under Condition Entry-q, an owner of asset quality* $q_k < q_{\underline{k}}$ *cannot profit from posting any contract in the candidate equilibrium.*

In the candidate equilibrium, p^L and $(t(.; s^H), q^H)$ correspond to p_1 and $(t(.; \widetilde{s}_{\underline{k}}), q_{\underline{k}})$ respectively. The contract $(t(.; s^L), q^L)$ is a deviating offer by an owner of asset quality $q^L < q_{\underline{k}}$. The construction of $\widetilde{s}_{\underline{k}}$ ensures that the

[38] Though the inequality (2.16) implies the statement (2.11), the former.is only required to hold for a smaller set of types and contracts. Therefore, Condition Entry-q is consistent with, but no stronger than, Condition Sorting-p.

preconditions in (2.15) are always met when the asset owners are on the long side. Under Condition Entry-q, the workers of p^1 are willing to endure a lower matching probability than any workers whose participation profits the deviating asset owner. Hence, the asset owners of q^L will take their outside options instead of posting the contract $t(.; s^L)$.

It is noteworthy that in both Condition Screening-q and Entry-q, workers of the threshold type $p_{\underline{l}}$, whose exact type depends on the distribution of types, play a key role in deterring deviations. Given the construction of the active markets $\{(t(.;\widetilde{s}_k), q_k)\}_{k \geq \underline{k}}$, a match with them always renders the concerned deviating offers unprofitable. This explains the sufficiency. The less obvious part is why the necessary and sufficient conditions pivot on them, but not their peers above the threshold type. This is because these workers choose the same contract in the candidate equilibrium for a larger subset of distributions of types. To see this, consider perturbing the distribution of types above the threshold types on both sides, the pairs of threshold types $(p_{\underline{l}}, q_{\underline{k}})$ and the contract term $\widetilde{s}_{\underline{k}}$ they choose remain unchanged while the other active markets $\{(t(.;\widetilde{s}_k), q_k)\}_{k > \underline{k}}$ and the types of their participants are potentially being affected. This feature stems from the fact that the set of active markets is determined recursively from bottom to top.

2.6.3 Conditions for Positive Assortative Matching

Proposition 3. *The First Best allocation can be supported by an equilibrium for any distribution of types if and only if Condition Sorting-p, Screening-q and Entry-q all hold.*

The preceding discussion covers the sufficiency of the conditions and the construction of the candidate equilibrium.[39] The proof for necessity is also constructive. For each of the conditions, I provide a procedure to construct a generic distribution of types, for which no equilibria can support the First Best allocation, if the condition is not met. The procedure identifies the corresponding profitable deviation in the process. This illustrates the incentives against assortative matching for an individual anticipating that the actions by the rest are consistent with the First Best allocation.

Given the order structure in the First Best allocation and the preference of the two sides, it is well known that some kinds of single crossing or increasing difference conditions are sufficient for supporting assortative matching. The novelty here is to characterize the exact conditions for all distributions of types, which involves two new complications.

The first complication stems from the fact that the expected payoff for workers is generally non-separable in the contract term and the matching surplus. As a result, the contract terms cannot be canceled out and will remain in the necessary and sufficient conditions. A prerequisite for such conditions is then characterizing the set of active markets and the types of the participants in the candidate equilibrium for all distributions of types because any restrictions inconsistent with this characterization are point-

[39] Condition Sorting-p, Screening-q and Entry-q should be viewed as three intersecting subsets in the space of the expected payoff functions $v(p, q, t(.; s))$. None of the subsets contains another. Each of them represents the exact subset of functions v for which the corresponding type of deviations is not profitable in the candidate equilibrium. The three conditions are separate in the sense that the arguments for sufficiency and necessity for each of them do not rely on the other two conditions. In this perspective, one may interpret that the intersection of the subsets corresponds to a grand necessary and sufficient condition, which Condition Sorting-p, Strong Screening-q and Entry-q are its decomposition.

less. This characterization must apply to a large class of expected payoff functions, covering those indeed satisfying the necessary and sufficient conditions.

In the setting here, the expected payoffs $v(p, q, t(.; s))$ and $u(q, p, t(.; s))$ are monotonic and continuous in all arguments and there exists pairs yielding arbitrarily small matching surplus, formally, $E(Y|\underline{p}, \underline{q}) = \underline{U} + \underline{V}$ in Assumption (P).[40] These two properties allow a closed-form characterization of the set of active markets and the types of associated participants, which is then incorporated as preconditions. The inclusion of preconditions allows me to decompose the grand necessary and sufficient condition into three separate conditions, underscoring the difference in the incentives in supporting PAM for various groups.

The second complication lies in the analysis of the screening problem for the uninformed side. The pool of workers attracted by a contract offer in the candidate equilibrium is determined by the workers' equilibrium payoff and their expected payoff function. This seems to suggest that finding out the exact condition involves characterizing the set of active markets and the workers' equilibrium payoff, which are dependent on the expected payoff

[40] The assumptions provide the following properties:

1. Given any (p, q^H, s^H) and $q^L < q^H$, one can find s^L such that $v(p, q^H, t(.; s^H)) = v(p, q^L, t(.; s^L))$.
2. Given any term $\widehat{s}$, one finds a term s' arbitrarily close to $\widehat{s}$ such that there exists either a pair of types (p', q') satisfying $v(p', q', t(.; s')) = \underline{V}$ and $u(q', p', t(.; s')) \geq \underline{U}$ or a pair of types (p'', q'') satisfying $v(p'', q'', t(.; s')) \geq \underline{V}$ and $u(q'', p'', t(.; s')) < \underline{U}$.

For each pair of types, I then find out the closure of the set of contract terms they may choose in the candidate equilibrium for some distribution of types using these two properties.

itself, for any distribution of types. I circumvent this task by introducing a hierarchy of preconditions. Given the distribution of types, the set of active markets must satisfy the equalities (2.9) and (2.10). Exploiting this system of equalities and PAM, the preconditions are constructed to categorize the candidate equilibrium into various scenarios. I then characterize the required property for the expected payoff for each type of scenario.

I now provide a unifying sufficient condition for Condition Sorting-p, Screening-q and Entry-q. It is an increasing difference condition stating that when switching to a better asset with a higher contract term s, a better worker will benefit more or suffer less, in term of either amount or percentage. Simplicity is its main advantage and allows me to provide sufficient conditions on the outcome distribution and contract space in Section 2.7.

Condition (Global ID). *$v(p, q, t(.;s))$ satisfies Global ID if for any $p^H > p^L$, $q^H \geq q^L$ and $s^H > s^L$, at least one of the following conditions hold:*

$$v(p^H, q^H, t(.;s^H)) - v(p^H, q^L, t(.;s^L)) \geq v(p^L, q^H, t(.;s^H)) - v(p^L, q^L, t(.;s^L)),$$

Or there exist $c^H \geq c^L \geq 0$ such that

$$\frac{v(p^H, q^H, t(.;s^H)) + c^L}{v(p^H, q^L, t(.;s^L)) + c^L} \geq \frac{v(p^L, q^H, t(.;s^H)) + c^H}{v(p^L, q^L, t(.;s^L)) + c^H}.$$

And $v(p, q, t(.;s))$ is said to satisfy strict Global ID if the weak inequalities in the two conditions are replaced with strict inequalities.

Condition Global ID consists of two inequalities, one in level and the other in ratio. It does not require the same inequality to hold for every pair of $\{p^H, p^L\}$ and $\{(q^H, s^H), (q^L, s^L)\}$. Instead, one of the inequalities must hold for any given pair of $\{p^H, p^L\}$ and $\{(q^H, s^H), (q^L, s^L)\}$. In this sense, Condition Global ID are not composed of two separate increasing difference conditions. Furthermore, the values of c^H and c^L may vary with the pair of $\{p^H, p^L\}$ and $\{(q^H, s^H), (q^L, s^L)\}$. The constants c^H and c^L can be thought to present in the level inequality but cancel out each other.

In the baseline setting, the workers and the asset owners receive nothing if they participate in matching but end up unmatched. This assumption is innocuous as it merely normalizes the parties' payoff with respect to the event they are left unmatched. Other reference points, the absolute output level and the worker's initial wealth for examples, are sometimes more convenient for the purpose of studying primitive conditions on the contract space and the output distribution. Taking the quantities c^H and c^L as the new workers' payoff when they left unmatched, Condition Global ID applies to the re-normalized worker's expected payoff. This property will be helpful in Section 2.7. The restriction $c^H \geq c^L$ reflects the requirement that the worker's unmatched payoff is weakly decreasing in his type. The restriction $c^L \geq 0$ ensures that the worker's expected payoff is always positive, otherwise the worker will simply take his outside option.

Proposition 4. *Condition Global ID implies all Condition Sorting-p, Screening-q and Entry-q. Condition strict Global ID implies all Condition Strict Sorting-p, Strong Screening-q and Entry-q.*

It is trivial that the inequalities in Condition Global ID imply Condition Sorting-p. Neither one of the inequalities is weaker than the other in supporting Condition Sorting-p. When a worker, be it high type or low type, finds the contract posted by the owner of higher quality asset more attractive, then the level inequality is the weaker of the two. The opposite is true if a worker prefers the contract posted by the owner of lower asset quality.

Recall from the previous discussion, Condition Strong Screening-q and Entry-q concern the circumstance that some workers of p^L are participating in the active market $(t(.;s^H), q^H)$ in the candidate equilibrium and both workers of types p^H and p^L strictly prefer the deviating offer $(t(.;s^L), q^L)$ to $(t(.;s^H), q^H)$.[41] In this case, the level inequality implies the strict ratio inequality with $c^H = c^L = 0$, and hence the two mentioned conditions. If the ratio inequality holds for some pairs of re-normalized expected payoff $v(p^H, q, t) + c^L$ and $v(p^L, q, t) + c^H$, the worker's marginal value of matching probabilities in the two markets also satisfies ratio inequality with $c^H = c^L = 0$. This is because a high type worker receives a lower payoff if he ends up unmatched.

2.6.4 Discussion

Menu of contracts revisited Proposition 3 applies to the setting in which the asset owners may post a menu of contracts. The sufficiency directly follows from Lemma 1 in section 2.4.5. The necessity merits some discussion. For each of the conditions violated, the proof of Proposition 3 details the construction of a generic distribution of types such that the corresponding First Best allocation cannot be supported by any equilibrium in which the asset owners may post only a single contract. Fix this distribution of types and let us conjecture an equilibrium which supports the First Best allocation using menus of contracts. In such equilibrium, the menu posted by an owner of asset quality $q_k \geq q_{\underline{k}}$ must include the contract $t(.;\widetilde{s}_k)$ and potentially some other contracts which are never chosen by workers. The

[41]Condition Strong Screening-q concerns the case that $q^H = q^L$.

reason is that all workers will pick the same contract from a menu because their preferences over the contracts are the same.[42] As the distribution of types is generic, the adjacent IC conditions imply that $\{(t(.;\widetilde{s}_k), q_k)\}_{k \geq \underline{k}}$ is the set of contracts chosen from the menus. This in turn implies that when an asset owner deviates to post a single contract, the most pessimistic belief about the pool of workers it attracts is the same as in an equilibrium which $\{(t(.;\widetilde{s}_k), q_k)\}_{k \geq \underline{k}}$ is the set of active markets. Therefore, the corresponding deviation identified in the proof of Proposition 3 remains profitable.

Destruction never improves screening This section shows that an asset owner never benefits from a lottery for separation, or a commitment to impair her asset under Condition Screening-q and Entry-q.[43]

Fix an equilibrium and consider two inactive markets $(1, q', t')$ and (π', q', t') with $\pi' < 1$. Suppose the market (π', q', t') is believed to attract some workers. As these workers are willing to endure the matching probability $(1 - \pi')\eta(\mu(\pi', q', t'))$, they must also accept the same matching probability in the market $(1, q', t')$, so

$$(1 - \pi')\eta(\mu(\pi', q', t')) = \mu(1, q', t').$$

It follows that any worker will anticipate the same payoff from these two inactive markets, so the sets of admissible beliefs regarding the distribution

[42] Asset owners of the same quality may still post different menus. Since the equilibrium supports the First Best allocation, workers face no variation in their matching probabilities in the active markets. They must pick the same contract in every menu for a given asset quality.

[43] According to Lemma 1, it is without loss to focus on the deviations to post a single contract. Therefore, the candidate equilibrium supporting the First Best allocation in Proposition 3 is robust to the introduction of menus of contracts specifying the separation probability $\pi \leq 1$, the asset quality $q \leq q_k$, and the contingent payment t.

of worker's type are the same for the two markets. An exogenous reduction in the matching probability for workers will not improve screening. Nevertheless, the separation lottery exposes the asset owner to a higher risk of ending up unmatched. We can exclude such lottery from our consideration, irrespective of Condition Screening-q and Entry-q.

We now turn to the option of asset impairment. Under Condition Screening-q and Entry-q, an asset owner cannot benefit from impairing her asset even when it is costless. The key reason is that the equilibrium payoff for an asset owner is increasing in her asset quality in the candidate equilibrium. Suppose, to the contrary, that for some distribution of types, an asset owner of $q_{\widehat{k}}$ gains from posting a contract $(t(.;s'), q')$ in the candidate equilibrium, where $q' \in [q_{\underline{k}}, q_{\widehat{k}})$. $\{q_k\}_{k \geq \underline{k}}^K$ denotes the set of asset qualities. Let $q' \in [q_{k'}, q_{k'+1})$, then we modify the distribution of types by adding the same measure of assets q' and workers of type $p_{\bar{l}(k')}$. This construction leads to the following properties for the resulting candidate equilibrium. First, the equilibrium payoffs for workers stay unchanged, so does the off-equilibrium-path belief for the inactive market $(t(.;s'), q')$ in the new candidate equilibrium. Second, the owners of the asset qualities $\{q_k\}_{k \geq \underline{k}}^K$ see no change in their equilibrium payoff. Third, the new candidate equilibrium retains the same set of active markets $\{(t(.;\widetilde{s}_k), q_k)\}_{k \geq \underline{k}}$ and includes a new active market $(t(.;\widetilde{s}'), q')$, where $\widetilde{s}'$ satisfies $v(p_{\bar{l}(k')}, q', t(.;\widetilde{s}')) = V(p_{\bar{l}(k')})$. Note that the owners of asset quality q' earn less than their peers of asset quality $q_{\widehat{k}}$. Hence, an owner of asset quality q' must profit from posting $t(.;s')$ in the new candidate equilibrium. This is impossible under Condition Screening-q. Condition Entry-q rules out the case $q' < q_{\underline{k}}$ with a similar argument.[44] Therefore, an asset

[44] This covers the case that the owner may reduce the quality of her asset to a level below $\underline{q}$ if Condition Entry-q is extended to hold for $q^L \leq \underline{q}$.

owner will never profit from a lower asset quality.

It should be highlighted that Proposition 3 is robust only to the commitments for deterministic impairment of the asset quality. The result does not apply to the setting, where asset owners may post joint lotteries over the asset quality and the contract term. For example, an asset owner may post a lottery over (q^H, s^L) and (q^L, s^H), where $q^H > q^L$ and $s^L \leq s^H$. In the context of the above argument, this joint lottery corresponds to side payments between owners of assets q^H and q^L. Condition Screening-q and Entry-q are silent on this example as they apply only to cases that a higher asset quality is accompanied with a higher term s. The underlying reason is that the analysis focuses on the workers' preference over the asset quality, the contract term and his matching probability, but not the joint lotteries over them.

Properties of other equilibria

Equilibrium rationing Suppose that in some active market $(t(.;s), q_k)$, workers are being rationed. A contract of a slightly higher term s^H will interest these workers. Among those are interested, Condition Strong Screening-q implies that the types of workers attracted to the contract $t(.;s^H)$ are no lower than those of the participants in the market $(t(.;s), q_k)$. An asset owner will profit from the contract $t(.;s^H)$ as it provides a greater division of surplus and attracts potentially better workers.

Lemma 5. *Under Condition Strong Screening-q, workers are never rationed in any equilibrium.*

In contrast, asset owners may be rationed in equilibrium. Define $I_p(t, q) := \{p \in \{p_l\}_{l=1}^L : V(p) = \eta(\mu(t, q))v(p, q, t)\}$. $I_p(t, q)$ denotes the types of workers, who obtain their equilibrium payoff if they participate in the market (t, q). It is non-empty for any active market. $\underline{I_p}(t, q)$ and $\overline{I_p}(t, q)$ denote the lowest and highest type in $I_p(t, q)$.

Suppose that there is an active market $(t(.; s'), q_k)$ with $\mu(t(.; s'), q_k) > 1$. Consider a contract $t(.; s^L)$, where $s^L < s'$. The incentive compatibility conditions and Condition Strong Screening-q jointly imply that workers of types greater than $\underline{I_p}(t(.; s'), q_k)$ will never accept such contract. Therefore, the asset owners have to trade off between a jump in matching probability and a less favorable distribution of partner's type. When $s^L < s'$ is sufficiently close to s', only workers of type $\underline{I_p}(t(.; s'), q_k)$ will be attracted. Such local deviation is the most profitable one among all contracts in S_t. It imposes an upper bound on the tightness ratio through the following condition,

$$U(q_k) = \frac{\eta(\mu(t(.; s'), q_k))}{\mu(t(.; s'), q_k)} u(q_k, r(t(.; s'), q_k), t(.; s')) \geq u(q_k, \underline{I_p}(t(.; s'), q_k), t(.; s')). \tag{2.17}$$

Note that the market $(t(.; s'), q_k)$ must clear if $I_p(t(.; s'), q_k)$ contains a single type only.

Assortative matching Lemma 5 has strong implications for the structure of an equilibrium. Since the workers are never rationed in equilibrium, the properties of $v(p, q, t(.; s))$ determine their choices of active markets. The first implication is that in any equilibrium, owners of the same asset quality, if participating, must post the same contract and the choice of contract term s increases with their asset quality. Second, workers' participation and equilibrium payoff are monotonic in their types. Now consider an equilibrium with active markets $(t(.; s^H), q^H)$ and $(t(.; s^L), q^L)$. Condition Sorting-p allows the possibility that a high type worker participates in $(t(.; s^L), q^L)$ while a low type worker participates in $(t(.; s^H), q^H)$. This occurs when these two types of workers are indifferent between the two active markets. Condition Strict Sorting-p rules out this case entirely. More importantly, Condition Strict Sorting-p implies that the asset owners' participation decision and the equilibrium payoff are monotonic as well. To see this, suppose that $(t(.;s^L), q^L)$ is an active market and p^H is the highest type of participating workers. Under Condition Strict Sorting-p, an owner of asset quality $q^H > q^L$ can find a contract $t(.;s^H)$, which gives workers of p^H a payoff just above what they may receive from the contract $(t(.;s^L), q^L)$ and attracts no workers of lower types. By posting this contract, her payoff will be strictly greater than that of the owners of q^L.

Lemma 6. *Under Condition Strict Sorting-p and Strong Screening-q, every equilibrium has the following properties:*

1. *Workers are not rationed.*
2. *Owners of the same asset quality, if participating, post the same contract.*
3. *Participation on both sides is monotonic in type.*
4. *The equilibrium payoffs are monotonic in types for both sides.*
5. *The types of workers matching with better assets must be no lower than those matching with assets of lower qualities.*

The characterization in Lemma 6 states that for any equilibrium, the set of active markets takes the form $\{(t(.;s_k), q_k)\}_{k \geq \widehat{k}}$, where $q_{\widehat{k}}$ is the lowest asset quality among the participating asset owners. The types of workers participating in an active market $\Omega_p(t(.;s_k), q_k)$ are increasing in the asset quality. Furthermore, only the highest type of workers in the market $(t(.;s_k), q_k)$ may also participate in the market $(t(.;s_{k+1}), q_{k+1})$.

Therefore, Lemma 6 implies that under Condition Strict Sorting-p and Strong Screening-q, rationing of the asset owners is the only form of inefficiency in the set of equilibrium allocations. Together with Condition Entry-q, a continuum of such equilibria always exists for generic finite distributions of types.[45] They can be constructed by perturbing the candidate equilibrium.

Under Condition Strict Sorting-p and Strong Screening-q, the equilibrium characterization in Lemma 6 allows us to construct a upper bound over $\frac{u(q_k,\overline{I_p}(t(.;s'),q_k),t(.;s'))}{u(q_k,\underline{I_p}(t(.;s'),q_k),t(.;s'))}$ for every active market. The inequality (2.17) then yields the following bound on the tightness ratio of an active market,

$$\mu(t(.;s'), q_k) \leq \max\left\{\frac{u(q_k, p_{l^H}, t(.;s'))}{u(q_k, p_{l^L}, t(.;s'))} : \sum_{l'=l^L}^{l^H} P(p_{l'}) \leq Q(q_k), L \geq l^H \geq l^L \geq 1\right\}.$$

Note that when the distribution of types converges to a continuous one, $p_{l^H} - p_{l^L} \to 0$ and hence $\mu(t(.;s'), q_k) \to 1$.

Belief restriction In the equilibrium definition, a deviating offer will only attract the types of workers who see the greatest proportional increase in their payoffs if they get matched. The belief restriction here is the same as in Guerrieri, Shimer and Wright (2010), who consider homogeneous principals.[46] With heterogeneity on both sides, this belief restriction is crucial for the decentralization of assortative matching. It establishes a linkage between

[45] Lemma 6 is silent on the existence of equilibria. The inclusion of Condition Entry-q ensures the existence of the candidate equilibrium.

[46] DeMarzo, Kremer and Skrzypacz (2005) also adopts a symmetric belief refinement when the informed party chooses the contract.

the complementarity in types and the pool of workers attracted by a deviating offer. Suppose that in the candidate equilibrium, an asset owner intends to poach workers from her peers of higher asset qualities. The workers accepting the deviating offer will suffer a reduction in the asset quality. The belief restriction implies that no workers of higher types will be attracted if they derive a sufficiently large gain from the complementarity in types. The exact requirement is captured by Condition Screening-q and Entry-q.[47]

2.7 Conditions On Contracts And Production Complementarity

This section discusses primitive conditions on the ordered set of securities and the conditional distribution of output, which give rise to Condition Global ID. The classes of output distributions we consider conform with Assumption (P). The type space and the values of outside options are assumed to satisfy $E(Y|\overline{p},\overline{q}) > \underline{V}+\underline{U} \geq E(Y|\underline{p},\underline{q})$.[48]

For notational simplicity, assume that the output is continuously distributed on an interval with a lower bound $\underline{y}$ and an upper bound $\overline{y} \in (\underline{y},\infty]$. Abusing the notation, $\Omega_y = [\underline{y},\infty)$ if $\overline{y}=\infty$. $F(y|p,q)$ and $f(y|p,q)$, respectively, denote the conditional distribution function and density function for

[47] Consider an alternative belief restriction. Suppose the asset owner believes that any workers who benefit from the deviating offer will accept it, and the distribution of her partner's type is proportional to the prior distribution. In this case, the significance of complementarity in types diminishes in the screening problem for the asset side. It is straightforward to construct a distribution of types for which assortative matching cannot be decentralized.

[48] If $\underline{V}+\underline{U} > E(Y|\underline{p},\underline{q})$, the type space $[\underline{p},\overline{p}]\times[\underline{q},\overline{q}]$ is truncated to $[\widetilde{p},\overline{p}]\times[\widetilde{q},\overline{q}]$, where $E(Y|\widetilde{p},\widetilde{q}) = \underline{V}+\underline{U}$. The choice of the pair $(\widetilde{p},\widetilde{q})$ is not unique.

$Y|(p,q)$. The results in this section can be extended to settings with discrete output distributions readily.

The joint conditions on the feasible contracts and the distribution of output are necessarily intertwined. I begin with conditions allowing decentralization of the First Best allocation if the assets are all homogeneous. For each of these conditions on the contingent contracts, I then proceed to the sufficient condition on the output distribution leading to Condition Global ID.[49]

Assumption (MLRP)**.** *$Y|(p,q)$ satisfies strict conditional Monotone Likelihood Ratio Property(MLRP). Given any asset quality $q \in [\underline{q}, \overline{q}]$, for all $y^H > y^L$ and $p^H > p^L$,*

$$\frac{f(y^H|p^H,q)}{f(y^L|p^H,q)} > \frac{f(y^H|p^L,q)}{f(y^L|p^L,q)}. \tag{2.18}$$

As is well known, a higher output level is a favorable signal for the worker's type under Assumption (MLRP), regardless of the asset quality.[50]

Remark 6. *When the assets are homogeneous, there exists an equilibrium supporting the First Best allocation if either of the following conditions hold:*

1. *$y - t(y;s)$ is SPM, or*
2. *$y - t(y;s) + c$ is non-negative and log-SPM for some constant $c \geq 0$.*

In the case of homogeneous assets, Remark 1 already ensures monotonic participation on the workers' side in any equilibrium. The First best allocation, which is still defined by the equality (2.5), can be decentralized if

[49] This approach rests on the assumption that uninformed parties of all types have access to the same set of feasible contracts.

[50] The strictness is not needed in Section 2.7. However, it will be required for Lemma 9 in Section 2.8.

a contract of a higher term s requires not only a greater expected transfer from the worker but also his payoff to be increasing with the output at a faster rate, either in level or percentage. Under Assumption (MLRP), a reduction of the term always benefits a low type more than a high type, preventing the asset owners from cream skimming. In the view of the uninformed side, screening and increasing transfer complement each other for such set of contracts. Formally, either of the conditions in Remark 6 ensures that Condition Global ID holds for $q^H = q^L$ and $c^H = c^L = c$.

Consider the condition $y - t(y;s)$ is SPM. When the contract term s increases, the worker has to pay the asset owner more on average but the contingent payment increases with the output at a slower rate. A consequence of these two requirements is that the worker must make a higher payment at the lowest output level under a contract of a higher term s. Presence of wealth constraint or limited liability may prevent the worker from making such a payment, and hence the SPM of his payoff as well.

Log-SPM of $y - t(y;s) + c$ is a promising alternative in this regard. It can be met even if all contracts specify the same payment at the lowest output level, and thus circumvents the wealth constraint. Log-SPM of $y - t(y;s) + c$ is a weaker condition than SPM of $y - t(y;s)$ if the contingent payment uniformly increases with the contract term s at all output levels. In general, neither one of them implies the other because how $t(y;s)$ changes with s is indefinite. As we shall see, a stronger notion of production complementarity is, nonetheless, needed for Condition Global ID under log-SPM of $y - t(y;s) + c$.

2.7.1 Applications to standard securities

The classes of standard securities introduced here satisfy Assumption (C). They also satisfy the definition of an ordered set of securities when indexed by appropriate contract terms. Note that the two conditions in Remark 6 concern the worker's payoff, which depends on both the initial division of output and the contract between the two parties. As in the baseline setting, we first consider the convention that the worker receives the entire output. The output level is normalized to be non-negative throughout this section, $\underline{y} \geq 0$.

I first discuss equity, debt and call option. These securities have the additional properties that i) $t(.;s^H) \geq t(.;s^L)$ if $s^H > s^L$, and ii) $y \geq t(y;s) \geq 0$. Thus, log-SPM of $y - t(y;s) + c$ is a weaker requirement than SPM.

Example (Equity)**.** *An equity contract is represented by $t(y) = \alpha y$. It can be indexed by the output share $\alpha = s$ so that*

$$y - t_E(y;s) = (1-s)y.$$

When the output level increases, the worker's payoff increases by the same percentage across all equity contracts. However, the increase in level is smaller if the asset owner is paid a greater share of output. Therefore, $y - t_E(y;s)$ is non-negative and log-SPM but not SPM. In fact, Global ID holds for equity contract whenever the expected output exhibits log-SPM.

Example (Debt)**.** *A debt or bond is represented by $t(y) = \min\{y, d\}$. It is indexed by the principal amount $d = s\widehat{y}$ such that*

$$y - t_D(y;s) = \max\{0, y - s\widehat{y}\}.$$

$\widehat{y}$ can be simply set as $\overline{y}$ if $\overline{y}$ is finite. Otherwise, $\widehat{y}$ has to be large enough so that $E(\min\{Y, \widehat{y}\}|\overline{p}, \overline{q}) \geq E(Y|\overline{p}, \overline{q}) - \underline{V}$. This ensures $v(p, q, t_D(y; 1)) \leq \underline{V}$.

Under the debt contract, the worker keeps the residual output after paying out the principal amount in full. For any output level above the principal amount, the worker's payoff increases with the output by the same amount. Nevertheless, the percentage increase will be greater if he pays a larger principal amount. Formally, $y - t_D(y; s)$ is non-negative and log-SPM but not SPM.

Example (Call option)**.** *A call option is represented by $t(y) = \max\{y - c, 0\}$. It is indexed using the strike price $c = (1 - s)\widehat{y}$, so that*

$$y - t_{CO}(y; s) = \min\{(1 - s)\widehat{y}, y\}.$$

Again, $\widehat{y}$ is chosen so that $E(\min\{Y, \widehat{y}\}|\overline{p}, \overline{q}) \geq E(Y|\overline{p}, \overline{q}) - \underline{U}$, ensuring $U(q, p, t_{CO}(y; 0)) \leq \underline{U}$.

When granting the asset owner a call option, the worker receives the output only up to the strike price of the option. His residual claim ceases to increase with the output when the latter exceeds the strike price. When s increases, the worker's residual claim starts flattening out at a lower output threshold, and so increases at a slower rate with the output level. Therefore, $y - t_{CO}(y; s)$ is neither log-SPM nor SPM. So the results in the subsequent section do not apply to the case that the uninformed parties are compensated using contracts within the class of call options.

In the baseline setting, the informed party receives the output and makes payment to the uninformed party. In a number of applications such as executive compensation, the flow of payment goes the other way around and the contingent payment is implemented using a different class of securities.

As an example, consider in our setting that the worker takes a debt of principal d from the asset owner in exchange for the use of the asset. The ex-post surplus division in this case coincides with the arrangement that the asset owner is entitled to the output and compensates the worker with a call option of strike price d. The previous discussion applies after re-indexing the class of call option as follows,

$$t_{CO}(y; 1-s) = \max\{0, y - s\widehat{y}\} = y - t_D(y; s).$$

Now suppose that the asset owner is entitled to the output and enjoys limited liability. If the asset owners compete for workers by offering fixed wage contract w, this compensation scheme is equivalent to the arrangement that the worker buys out the asset by granting its owner a call option with strike price w. From the preceding analysis, Condition Global ID is not met in general. This problem is more acute for small business and startups, who are likely to liquidate in case of a low revenue.

Example (Mixture of cash and standard securities)**.** *Another popular arrangement in practice is a mixture of cash and standard securities such as equity or bond. In this case, the contract can be indexed by the portfolio weight of cash payment. Fix a particular securities $\widehat{t}(y)$ satisfying Assumption (C), we can define an ordered set of securities as follows*

$$t_{MIX}(y; s|\widehat{t}) = s[E(Y|\overline{p}, \overline{q}) - \underline{V}] + (1-s)[\widehat{t}(y) - E(\widehat{t}(Y)|\overline{p}, \overline{q}) + \underline{U}].$$

$s[E(Y|\overline{p}, \overline{q}) - \underline{V}] - (1-s)[E(\widehat{t}(Y)|\overline{p}, \overline{q}) - \underline{U}]$ *is interpreted as the amount of up-front cash payment and the rest is the security portion of the offer. It is straightforward to verity that $t_{MIX}(y; s|\widehat{t})$ satisfies Assumption (C) and the conditions for an ordered set of securities. Since the security portion* *decreases with the term s, a contract of high term s renders the worker's payoff more sensitive to the output level. Hence, $y - t_{MIX}(y; s|\widehat{t})$ is SPM.*

So far, we have restricted attention to the circumstances, that the entire output produced by the match is contractible. The result can also be applied to another extreme, which the outcome is non-contractible and the parties exchange cash payment. Let $\widehat{h}(y)$ and $y - \widehat{h}(y)$ denote the non-contractible, possibly non-monetary, payoff for the asset owner and worker in the absence of transfers respectively. Mailath, Postlewaite and Samuelson (2013) term the pair $E(\widehat{h}(Y)|p,q) - \underline{U}$ and $E(Y - \widehat{h}(Y)|p,q) - \underline{V}$ as premuneration values. As the asset owners post prices, we define an ordered set of securities as

$$t_{PRE}(y; s|\widehat{h}) = \widehat{h}(y) - E(\widehat{h}(Y)|\overline{p}, \overline{q}) + \underline{U} + s[E(Y|\overline{p}, \overline{q}) - \underline{V} - \underline{U}].$$

If $\widehat{h}$ satisfies Assumption (C), then $t_{PRE}(y; s|\widehat{h})$ satisfies Assumption (C) and the conditions for an ordered set of securities. Lemma 7 applies because $y - t_{PRE}(y; s|\widehat{h})$ is SPM.

2.7.2 Conditions on production complementarity

Condition (Survival-SPM)**.** *For any $y \in (\underline{y}, \overline{y})$, $F(y|p,q)$ is strictly decreasing and weakly pairwise submodular in p and q, and $F(y|p,q)$ is strictly pairwise submodular in p and q for some subinterval of $(\underline{y}, \overline{y})$.*

Condition Survival-SPM is stronger than supermodularity of expected output. The interpretation of Condition Survival-SPM is that when there are two types of agents on both sides, with types $\{p^H, p^L\}$ and $\{q^H, q^L\}$, the distribution of total output under positive assortative matching F.O.S.D. that under negative assortative matching.

Lemma 7. *Condition Global ID holds if $y - t(y;s)$ is SPM and Condition Survival-SPM holds.*

The SPM of $y-t(y;s)$ implies that $v(p,q,t(.;s))$ is pairwise SPM in (p,s) while Condition Survival-SPM establishes that $v(p,q,t(.;s))$ is pairwise SPM in (p,q). Hence, the level inequality in Global ID always holds.[51]

To save on space, $\vee$ and $\wedge$ are used to denote maximum and minimum operator respectively.

Condition (Survival-logSPM)**.** *Given any $p^H \geq p^L$ and $q^H \geq q^L$, for any y and y' in Ω_y,*

$$\begin{aligned} & [1 - F(y \vee y'|p^H, q^H)][1 - F(y \wedge y'|p^L, q^L)] \\ \geq\ & [1 - F(y|p^L, q^H)][1 - F(y'|p^H, q^L)], \end{aligned} \tag{2.19}$$

with strict inequality if $y \neq y'$ and $(p^H, q^H) \neq (p^L, q^L)$.

Compared to strict log-SPM of $[1-F(y|p,q)]$, Condition Survival-logSPM is slightly weaker as it does not require strict pairwise log-SPM in (p,q). On the other hand, Condition Survival-logSPM is stronger than Condition Survival-SPM.[52]

Under Condition Survival-logSPM, conditioning on the event that the output is above some level $y > \underline{y}$, the public belief regarding the type of the workers will be higher in F.O.S.D. sense if he is operating with an asset of higher quality.[53] In this sense, pairing up with an asset of higher quality

[51] Lemma 7 does not require Assumption (MLRP).

[52] Note that by taking $y = \underline{y}$ and $y' = \underline{y}$, (2.19) implies that $F(y|p,q)$ is strictly decreasing in p and q. Take $y' = y$, $[1 - F(y|p,q)]$ is strictly increasing and log-SPM in (p,q). Thus, it must be strictly pairwise SPM in (p,q).

[53] Abusing the notation, let P denote the random variable for the worker's type. For any $y \in (\underline{y}, \overline{y})$ and $q^H > q^L$, the ratio $\frac{\Pr(Y \geq y, P \leq p'|q^H)}{\Pr(Y \geq y, P \leq p'|q^L)}$ is increasing in p'. Thus $\Pr(P \leq p'|Y \geq y, q^H) \leq \Pr(P \leq p'|Y \geq y, q^L)$.

makes a high output level an even more favorable signal for the worker's type.

Notice that the expected output can be written as

$$E(Y|p,q) = \underline{y} + \int_{\underline{y}}^{\overline{y}} 1 - F(y|p,q)dy.$$

Since log-SPM for a non-negative function is preserved under integration, the expected output must be log-SPM under Condition Survival-logSPM.

Lemma 8. *Condition Global ID holds if Condition Survival-logSPM holds and $y - t(y;s) + c$ is non-negative and log-SPM for some constant $c \geq 0$.*

Condition Survival-logSPM, together with Assumption (C), ensure that $v(p,q,t(.;s)) + c$ is pairwise log-SPM in (p,q). Assumption (MLRP) implies that $v(p,q,t(.;s)) + c$ is pairwise log-SPM in (p,s). These two properties yield the desired condition.[54]

Under Condition Survival-SPM and Survival-logSPM, production complementarity manifests as a shift in the entire output distribution toward the right. The worker's payoff is always increasing with the output level because of Assumption (C). So a worker benefits more from an improvement in the asset quality than his peers of lower types under the same contract. The conditions in Remark 6 further imply that the contracts posted in the candidate equilibrium amplifies such difference between workers' types. This is because the owners of higher asset quality demand a higher contract term, rendering the worker's payoff increasing with the output at a higher rate.

In general, the form of the production complementarity must align with the worker's compensation, improving the odds of the states in which he

[54]For the function $v(p,q,t(.;s)) + c$, pairwise log-SPM in (p,s) is weaker than SPM in (p,s), whereas pairwise log-SPM in (p,q) is stronger than SPM in (p,q). This explains why Lemma 8 requires a stronger condition on the output distribuition.

is generously rewarded. As a result, a worker enjoys a greater raise in expected payoff for a better partner than his peers of lower types for the same contract. The catch is that such differential among workers must be preserved after accounting for the difference in the contracts offered. This requires further alignment between the forms of feasible contracts and the form of production complementarity.

The well-known result that multiplication and integration preserve log-SPM for non-negative functions may be directly applied to establish a sufficient condition for log-SPM of $v(p, q, t(.; s))$, and hence Condition Global ID. It is interesting to compare Lemma 8 with such condition. For $[y - t(y; s) + c]f(y|p, q)$ to be log-SPM in all arguments, the sufficient condition on the conditional distribution is given by

$$f(y|p^L, q^H)f(y'|p^H, q^L) \leq f(y \vee y'|p^H, q^H)f(y \wedge y'|p^L, q^L). \tag{2.20}$$

When integrating the conditional density functions over their common support, both sides of the inequality will be unity. This turns out to impose strong restrictions on the conditional density functions satisfying the inequality (2.20).

Remark 7. *Suppose a conditional density function $f(y|p, q)$ satisfies Assumption (MLRP) and the inequality (2.20), then $f(y|p, q)$ is pairwise log-modular in (y, q) and (p, q).*

The set of conditions in Remark 7 is far more demanding than Condition Survival-logSPM. The monotonicity of the worker's payoff is instrumental for the weaker condition required in Lemma 8.

Examples of parametric distribution

Example (Bernoulli distribution)**.** *Suppose* $Y|(p,q)$ *is a Bernoulli distribution with support* $\{\underline{y}, \overline{y}\}$*. Assumption (P) and (MLRP) hold if and only if* $\Pr(Y = \overline{y}|p,q)$ *is strictly increasing and strictly SPM. This already implies Condition Survival-SPM. Condition Survival-logSPM holds if and only if strict SPM is strengthened to log-SPM.*

Example (Exponential distribution)**.** *Suppose* $Y|(p,q)$ *follows exponential distribution with mean* $\mu(p,q)$*, then Assumption (P) and (MLRP) hold if and only if* $\mu(p,q)$ *is strictly increasing and strict SPM. Both Condition Survival-SPM and Survival-logSPM are equivalent to the requirement that* $\mu^{-1}(p,q)$ *is submodular.*

Assumption (MLRP) and Condition Survival-SPM and Survival-logSPM are preserved under monotonic transformation of random variable Y. This observation leads to the following example.

Example (Transformed geometric distribution)**.** $\{\Delta_n\}_{n\in\mathbb{N}}$ *is a non-negative deterministic sequence.* $\{X_n\}_{n\in\mathbb{N}}$ *is a sequence of non-degenerate Bernoulli random variables with support* $\{0,1\}$ *and* $\{X_n\}_{n\in\mathbb{N}}|(p,q)$ *are independent. Define* $Y = \underline{y} + \sum_{n=1}^{M} \Delta_n$ *where* $M = \min\{n \geq 1 : X_n = 0\}$*. Suppose* $\Pr(X_n = 1|p,q) = G(p,q;n)$ *is strictly increasing in* (p,q)*, then*

1. *Assumption (MLRP) is met if* $\frac{G(p^H,q;n)}{1-G(p^H,q;n)} \frac{1-G(p^L,q;n)}{G(p^L,q;n)}$ *is weakly decreasing in* n *for any* q *and* $p^H > p^L$*.*

2. *Condition Survival-SPM is satisfied if* $G(p,q;n)$ *is strictly SPM in* (p,q) *for all* n*.*

3. *Condition Survival-logSPM is satisfied if* $G(p,q;n)$ *is log-SPM in* (p,q) *for all* n*.*

The condition for Assumption (MLRP) is always met if $G(p,q;n)$ can be written as $G_{pq}(p,q)G_n(n)$ where $G_n(n)$ is weakly decreasing.

The output distribution can be interpreted as follows: Once a match is formed, a base output $\underline{y}$ is produced immediately. A sequence of production stages has to take place in succession. Δ_n and $G(p,q;n)$ are the output and probability of success for the n-th stage. A failure is irrevocable and terminates the production. There may be a cap on the number of possible production stages, say N. This is accommodated by letting $\Delta_n = 0$ and $G(.,.;n) = G(.,.;N)$ for $n > N$. A continuous distribution counterpart for the example of Transformed geometric distribution can be obtained readily.

2.8 Feasible Contracts And Comparative Statics

This section studies how a change in the feasible set of contracts will affect the equilibrium allocation, and the divisions of the matching surplus. I will compare the contracts in term of their steepness, a partial order based on DeMarzo, Kremer and Skrzypacz (2005).

Definition. *Given $Y|(p,q)$, a contract t^s is steeper than another contract t^f if $E(t^s(Y)|p',q') = E(t^f(Y)|p',q')$ for some $(p',q') \in [\underline{p},\overline{p}] \times [\underline{q},\overline{q}]$, then for all $p^H > p' > p^L$,*

$$\begin{cases} E(t^s(Y)|p^H,q') > E(t^f(Y)|p^H,q') \\ E(t^s(Y)|p^L,q') < E(t^f(Y)|p^L,q') \end{cases}.$$

Furthermore, t^f is said to be flatter than t^s.

In essence, a contract is steeper if it costs more to workers of higher types but less to those of lower types, regardless of the asset quality. We say a single contract t is steeper(flatter) than a set of contracts Φ_t if t is steeper(flatter) than every contract from Φ_t. Likewise, a set of contracts is steeper(flatter) than another set of contracts if every member of the former is steeper(flatter) than every member of the latter.

Lemma 9 (DeMarzo, Kremer and Skrzypacz, 2005)**.** *A contract t^s is steeper than another contract t^f if there exists some $y^* \in (\underline{y}, \overline{y})$ such that $t^s(y) \geq t^f(y)$ if $y > y^*$ and $t^s(y) \leq t^f(y)$ if $y < y^*$, with strict inequality for some interval in $(\underline{y}, \overline{y})$.*

Under Assumption (MLRP), a higher output level is a favorable signal for the worker's type. Lemma 9 states that a contract t^s is steeper than a contract t^f if the former cuts the latter from below.[55] For examples, Lemma 9 can be applied to rank the classes of standard securities. Call option is the steepest, followed by equity. Equity is steeper than debt. Cash is the flattest under Assumption (C).

The analysis of the comparative statics consists of two parts. The first part considers the case that only an ordered set of securities is feasible and it is replaced by another ordered set. I study the changes in the equilibrium divisions of the surplus, provided that the equilibrium allocation remains First Best. This comparative statics is driven by the sorting of the workers. It applies whenever the conditions in Section 2.7 are met. The second part relaxes Assumption (S) and considers the introduction of new contracts into the feasible set. I study when the equilibrium allocation and payoffs will remain unchanged. Such invariance stems from the screening considerations by the asset side. This result also applies when contracts are made unavailable.

[55] An implication of Assumption (C) is that unless one contract always specifies a higher payment than the other, any two contracts must intersect at some output level.

The subsequent discussion assumes the baseline convention that the payment flows from the worker to the asset owner. If it is the asset owner is entitled to the output and the flow of payment goes the other way around, then a flatter contract costs more to high type workers, and all results will be flipped.

2.8.1 Steepness and division of surplus

Proposition 5. *With a steeper ordered set of securities, the equilibrium payoff for the asset owners will be higher in the candidate equilibrium, whereas the equilibrium payoff will be lower for the workers matching with assets of quality strictly above the threshold type.*

Fix a distribution of types where the workers are on the short side, and every type of workers match with two types of assets in the First Best allocation. When switching to a steeper ordered set of securities, consider the following the thought experiment: First starts with the contract terms $\{s_k\}_{k\geq 1}$ keeping the same equilibrium payoff for the asset side. As the new contracts are steeper, a worker of type p_1 will pay less if he deviates to match with the asset of quality q_2. To satisfy the IC condition for the workers of type p_1, the contract term s_2 must increase, driving up the equilibrium payoff for the owners of asset quality q_2. This in turns makes the deviation to the market with q_3 even more profitable for workers of type p_2, resulting in a greater increase in s_3.

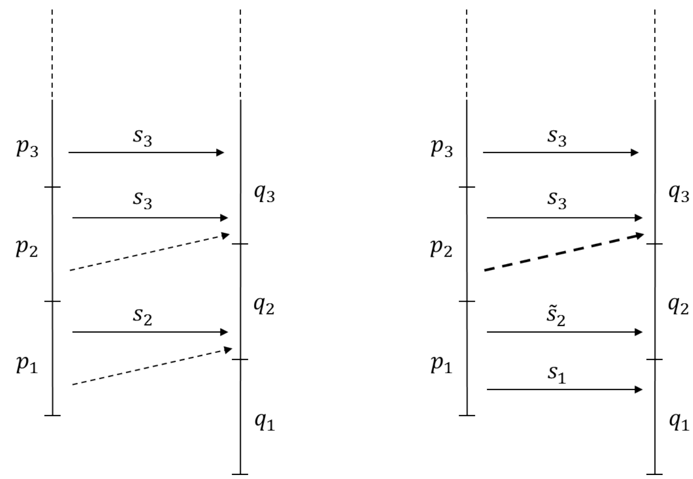

Figure 2.2: Spill-over effect of increased competition across assets

Inductively, all owners of asset quality above q_2 must post higher contract terms.

For any distribution of types, the above line of reasoning applies to the assets of quality above the threshold type and the workers they match with. For the workers matching with assets of the threshold type, the impact on their equilibrium payoff depends on the distribution of types. In particular, the workers of the threshold type are better off under a steeper ordered set of securities when the assets are on the long side. On another hand, all workers will be weakly worse off if they are on the long side.

Since the cash payment is the flattest ordered set of securities, Remark 4 in Section 2.6.1 is a special case of Proposition 5. In fact, Proposition 5 can be viewed as an extension of the Linkage principle in DeMarzo, Kremer and Skrzypacz (2005). The authors show that in the security-bid auction, a steeper ordered set of securities allows the auctioneer to extract more information rent from the bidders, provided that the equilibrium allocation is efficient and remains unchanged. Under assortative matching, the spillover effect across different types of assets further shifts the equilibrium division of matching surplus in the asset side's favor. Hence, I establish an analogous result in the context of assortative matching.

2.8.2 Steepness and contract offering

This subsection relaxes Assumption (S), which requires the feasible contracts to be fully ordered by a contract term, and analyzes the asset owners' choice of contract in a larger feasible set. The main conclusion is that all the results remain valid if the asset owners may post steeper contracts. Since a flatter contract costs the workers of higher types less than a steeper contract, the asset owners always prefer posting the former as it is less prone to attract the low type workers. Yet posting a flatter contract will has no effects on the pool of workers in certain situations, so that the asset owners are indifferent between the two contracts. Proposition 6 formalizes this observation. It states that when steeper contracts are made available, the set of equilibria weakly expands. Nevertheless, the set of equilibrium allocations and payoffs remain the same.[56]

Proposition 6. *Suppose $S_t \subseteq \Phi_t$, and $t^s \notin \Phi_t$ is a contract steeper than S_t.*

1. *For every equilibrium under the contract space Φ_t, there is a corresponding equilibrium under the contract space $\Phi_t \cup \{t^s\}$ with the same equilibrium payoffs $\{U, V\}$, the same active markets $\Psi \subseteq \Phi_t$ and the same distribution of participants in every active market.*

2. *For every equilibrium under the contract space $\Phi_t \cup \{t^s\}$, there must be an equilibrium under the same contract space, which supports the*

[56]Note that the cash payment satisfies Condition Global ID. Nevertheless, Proposition 2 is not a corollary to Proposition 3 and Lemma 6 as it holds without Assumption (P).

same equilibrium payoffs and allocation, and the asset owners only post contracts in Φ_t.

Note that Proposition 6 allows Φ_t to contain other contracts flatter than S_t. With an ordered set of securities available, the introduction of the new contract does not improve the transferability of surplus within the pair. From the preceding discussion, posting a steeper contract never benefits an asset owner. Therefore, any equilibrium is robust to the introduction of a steeper contract. It is obvious that the converse of the first statement is also true.

We proceed to the case that the ordered set of securities and a steeper contract t^s are both available. Notice that for any match (p, q), there is a contract $t(.; s(p, q))$ in S_t providing both parties the same payoff as the steeper contract t^s. As the former contract is flatter, workers of lower types all strictly prefer the contract t^s to $t(.; s(p, q))$. With these flatter contracts available, the asset owners are willing to post the steeper contract in an equilibrium only when the corresponding market (t^s, q) clears and attracts exactly a single type of workers.[57]

[57] To see this, suppose multiple types of workers participate in this active market and p^H is the highest type. an asset owner must profit from posting a contract $t(.; s')$ with a term s' slightly below $s(p^H, q)$. This is because the flatter contract will attract no workers of type below p^H and weakly improve her matching probability. Hence, p^H must be the only type of workers attracted to the market (t^s, q). This argument further implies that workers of higher types will be strictly worse off if they deviate to the market (t^s, q), and workers are not rationed in this market. Suppose the asset side is being rationed in the market (t^s, q), then an asset owner will deviate to a contract $t(.; s)$ where s is slightly above $s(p^H, q)$. Posting such contract will lead to a jump in the matching probability and attracts only workers of p^H.

This argument actually does not hinge on the assumption of finite distribution of types.

Now suppose that all the asset owners and workers in the market (t^s, q) switch to the market $(t(.; s(p', q)), q)$. As the contract $(t(.; s(p', q)), q)$ is flatter, it will not attract workers of lower types. It will not attract workers of higher types either, otherwise the asset owners would have deviated to post it after the first place. So all equilibrium conditions will still be satisfied. The equilibrium payoffs and the result allocation remain unchanged in the new equilibrium. This argument holds irrespective of the presence of other contracts in Φ_t.

An immediate corollary is that it is without loss to focus on the ordered set of securities if all other feasible contracts are steeper.

Corollary 2. *Suppose that A_t is steeper than S_t. an allocation and a pair of equilibrium payoffs can be supported by an equilibrium under the contract space S_t if and only if they can be supported by an equilibrium under the contract space $A_t \cup S_t$.*

DeMarzo, Kremer and Skrzypacz (2005) also consider informal auctions, in which the buyers may submit their bids from a larger set of securities and the seller selects the winning bid based on her belief. A worker signals his type by bidding with a flatter security, which is costlier to the low types. In equilibrium, all buyers bid with the flattest securities available. Here the competition among uninformed parties drives them to post the flattest securities available because of the screening incentive.

Corollary 2 allows us to generalize the results to larger sets of contracts.

Example (Linear compensation contracts)**.** *Consider $\Omega_t = \{t(y; \alpha, w) = \alpha y - w : \alpha \in [0, 1],\ w \geq 0\}$, which satisfies Assumption (C) and limited liability of the worker. The worker's payoff is given by $y - t(y; \alpha, w) = (1 - \alpha)y + w$, so Ω_t represents a class of linear compensation contracts. If* *a contract with $w = 0$ intersects with another one with $w > 0$, the former must cut the latter from below. The subclass of contracts with $w = 0$, which are effectively equity contracts, is flatter than any contracts with $w > 0$. Therefore, it is without loss to focus on the class of equity contracts $t_E(y; s)$.*

Example (Wealth constraint)**.** *Fix some $\pi \geq 0$, let Ω_t be the set of contracts satisfying Assumption (C) and $t(y) \leq y + \pi$. π is interpreted as the initial wealth level for the worker, so that the payment he made to the asset owner cannot exceed $y + \pi$ for any output level y. Take $\overline{y}$ as finite for simplicity. Under Assumption (C), any $t \in \Omega_t$ is absolutely continuous. Hence, $t(y)$ can be expressed as $t(\underline{y}) + \int_{\underline{y}}^{y} t'(z)dz$, where $t'(z) \in [0, 1]$. $t(\underline{y}) \leq \underline{y} + \pi$ is necessary and sufficient for $t(y) \leq y + \pi$ for any output level $y \in [\underline{y}, \overline{y}]$.*

Define the following ordered set of securities

$$t_{D+\pi}(y; s) = \min\{y + \pi, s(\overline{y} + \pi)\}.$$

For $s \leq \frac{\pi}{\overline{y}+\pi}$, the worker pays out cash up to his wealth level π. For $s > \frac{\pi}{\overline{y}+\pi}$, the worker tops up the cash payment π with a debt of principal amount $s\overline{y} + (1-s)\pi$. This ordered set of securities is flatter than any other contracts in Ω_t.[58] *Therefore, it is without loss to focus on the class $t_{D+\pi}(.; s)$. Since $y - t_{D+\pi}(y; s) + \pi = \max\{0, y + \pi - s(\overline{y} + \pi)\}$ is non-negative and log-SPM,*

[58]Fix some $s' \in [0, 1]$, let $\widehat{t}$ be a contract in Ω_t such that $\widehat{t}$ and $t_{D+\pi}(.; s')$ intersects at least once somewhere in $(\underline{y}, \overline{y})$. Note that $\widehat{t}(\underline{y}) \leq t_{D+\pi}(\underline{y}; s') = \underline{y} + \pi$. For $s' \leq \frac{\pi}{\overline{y}+\pi}$, $t_{D+\pi}(y; s') = s'(\overline{y} + \pi)$. Since $\widehat{t}(y)$ is increasing in y, $\widehat{t}(y)$ must cut $t_{D+\pi}(y; s')$ from below at some $y^* \in (\underline{y}, \overline{y})$. For $s' > \frac{\pi}{\overline{y}+\pi}$, $\widehat{t}(y)$ increases no faster than $t_{D+\pi}(y; s')$ for $y < s'(\overline{y} + \pi)$. Furthermore, $t_{D+\pi}(y; s')$ is constant in the region $(s''(\overline{y} + \pi), \overline{y}]$. Hence, $\widehat{t}(y)$ must cut $t_{D+\pi}(y; s')$ from below at some $y^* \in (s'(\overline{y} + \pi), \overline{y})$.

The above construction can be generalized to the constraint that $t'(y) \in [\underline{\alpha}(y), \overline{\alpha}(y)] \subseteq [0, 1]$ in the following manner: $t(\underline{y}; s) = \underline{y} + \pi$, $t'(y; s) = \underline{\alpha}(y)$ if $y < s(\overline{y} + \pi)$ and $t'(y; s) = \overline{\alpha}(y)$ if $y > s(\overline{y} + \pi)$. An example for such case is the threat of diversion.

Condition Global ID holds under Condition Survival-logSPM.

Example (A threat of misappropriation). *Let Ω_t be the set of contracts satisfying Assumption (C) and $t'(y) \geq \kappa$. This can be motivated as follows: The asset owner receives the output and pays the worker compensation. She may underreport the output level and misappropriate the unreported portion at a unit cost $1-\kappa$. Incentive provision for the asset owner leads to the constraint $t'(y) \geq \kappa$. Take $\overline{y}$ as finite for simplicity.*

Define the following ordered set of securities

$$t_\kappa(y; s) = \kappa y + s\overline{y} - (1-s)\kappa E(Y|\overline{p}, \overline{q}).$$

$t_\kappa(y; s)$ is essentially a cash payment topped with equity share κ. Any contract in Ω_t, if intersecting at all, must cut $t_\kappa(y; s)$ from below. Therefore, it suffices to consider only the class $t_\kappa(.; s)$. As $y - t_\kappa(y; s)$ is SPM, Condition Global ID holds under Condition Survival-SPM.

Testable Implications Proposition 5 and 6 together yield a testable implication regarding private types on one side.

Corollary 3. *Suppose an ordered set of securities is always available, then under assortative matching, exclusion of the flattest contracts increases the equilibrium payoff for the asset side.*

The preceding comparative statics predict an increase in the asset side's equilibrium payoff when the prevailing form of contracts offered, presumably the flattest available, switches to a steeper one, due to exogenous reasons such as financial regulation. It is hard to justify this prediction without information asymmetry when forming matches.

Under full information, a contingent payment is merely an instrument for transferring the matching surplus. Proposition 1 states that the equilibrium payoffs are invariant to changes in the feasible set of contracts. Now suppose at least one side is risk-averse, so the partners share the risk using the contingent contract. When contracts are excluded from the feasible set, the partners may either stay with the same contract or move to a suboptimal one. Some asset owners above threshold type must not gain from the exclusion of the contracts. The same argument applies in the case of pure moral hazard. Restriction on the incentive contracts will not benefit all asset owners.

Introduction Of Flatter Contracts Proposition 6 states that the introduction of steeper contracts have no effects on the set of equilibrium allocations and payoffs. When flatter contracts are introduced, the same cannot be said. If only particular contracts, say t^f, are introduced, the asset owners will gain from posting lotteries over t^f and S_t, which effectively form a flatter ordered set of securities. Therefore, I only consider the introduction of a flatter ordered set of securities. From the previous section, we know that assortative matching is still decentralized if the flatter ordered set of securities and the distribution of outputs satisfy the joint conditions in Lemma 7 and Lemma 8. The following examples illustrate that inefficiency may occur if the conditions are not all met. In other words, restricting the feasible set of contracts can improve welfare in these examples.

Introduction of a flatter ordered set of securities Consider the following example: The type space is given by $[\underline{p}, \overline{p}] = [\underline{q}, \overline{q}] = [0, 1]$. Production may result in three outcomes, $\Omega_y = \{0, \frac{1}{2}, 1\}$. The output distribution is given by

$$f_\epsilon(y|p,q) = \begin{cases} \frac{1}{8}p(1+2\epsilon q) & \text{, if } y = 1\text{; and} \\ \frac{1}{2}(1-\epsilon)pq + \frac{1}{4}q + \frac{1}{8} & \text{, if } y = \frac{1}{2}, \end{cases}$$

where $\epsilon \in (0,1)$.

The output distribution has the following properties:

1. $F_\epsilon(y|p,q)$ is continuous and strictly decreasing in p and q, and

2. $f_\epsilon(y|p,q)$ satisfies Assumption(MLRP), and

3. $F_\epsilon(y|p,q)$ satisfies Condition Survival-SPM, and

4. The expected output $E(Y|p,q) = \frac{1}{4}(p+\frac{1}{2})(q+\frac{1}{2})$ is log-modular, and hence SPM.

It is noteworthy that the survival function $1 - F_\epsilon(y|p,q)$ is pairwise log-SPM in (p,q) and (p,y) but not (q,y), so Condition Survival-logSPM is not met. The values of outside options can be chosen to satisfy $\underline{V} + \underline{U} = \frac{1}{16}$, and hence Assumption (P).

Suppose that only the class of equity contracts is feasible. For a given $\epsilon \in (0,1)$, the worker's expected payoff is denoted by

$$v_\epsilon(p,q,t_E(.;s)) = \frac{1}{4}(p+\frac{1}{2})(q+\frac{1}{2})(1-s).$$

It immediately follows that $v_\epsilon(p,q,t_E(.;s))$ satisfies Global ID, and hence the First Best allocation can always be decentralized.

Now suppose that the class of debt contracts is also made available. Since the class of debt contracts is flatter than that of equity, Corollary 2 states that it is without loss to assume the asset owners post only debt contracts.

$$v_\epsilon(p,q,t_D(.;s)) = \begin{cases} \frac{1}{2}(p+\frac{1}{2})[\frac{1}{2}(q+\frac{1}{2}) - s(q+\frac{1}{4})] - \frac{1}{4}(\epsilon pq + \frac{1}{4})s & \text{, if } s < \frac{1}{2}\text{; and} \\ \frac{1}{8}(1-s)p(1+2\epsilon q) & \text{, if } s \geq \frac{1}{2}. \end{cases}$$

The key property is that workers of all types share the same preference over (q,s,η) in the region $s \geq \frac{1}{2}$. From now on, we focus on the limiting case of $\epsilon \to 0^+$ and v_{0^+} denotes the worker's expected payoff at the limit. All the results hold when ϵ is sufficiently small.

For the workers side, $v_{0^+}(p,q,t_D(.;s))$ satisfies Condition Sorting-p. For the asset side, it is easy to verify that for any $\epsilon \in (0,1)$ and $s^H \geq \frac{1}{2}$,

$$\begin{aligned} &v_\epsilon(p^H,q^H,t_D(.;s^H))v_\epsilon(p^L,q^L,t_D(.;s^L)) \\ \geq &v_\epsilon(p^L,q^H,t_D(.;s^H))v_\epsilon(p^H,q^L,t_D(.;s^L)). \end{aligned} \tag{2.21}$$

For the case that $\frac{1}{2} \geq s^H > s^L$, the above inequality (2.21) holds at the limit $\epsilon \to 0^+$ if and only if

$$s^H[\frac{1}{2}(q^L+\frac{1}{2}) - s^L(q^L+\frac{1}{4})] \geq s^L[\frac{1}{2}(q^H+\frac{1}{2}) - s^H(q^H+\frac{1}{4})].$$

This is true whenever $q^H = q^L$, so Condition Screening-q is met.

However, the inequality (2.21) is violated for a large range of (p,q,s) where $q^H > q^L$. Condition Entry-q is not satisfied for certain values of outside options. Hence, there are distributions of types for which owners of asset quality below the threshold type profit from deviations in the candidate equilibrium. It is noteworthy that in this example, the First Best allocation can be decentralized if the asset side is homogeneous.

The concept of "steepness" does not concern about how the gain from production complementarity is shared within pair. In the current example, the worker's reward becomes more concentrated in the highest output level under the flatter class of contracts. While the highest output level is the most informative about the worker's productivity, the probability of its occurrence exhibits very weak complementarity between types. As a result, the workers benefit less from production complementarity to the extent that Condition Entry-q is no longer met, while Condition Sorting-p remains valid.

Introduction of a slightly flatter ordered set of securities Consider the following example: The type space is given by $[\underline{p}, \overline{p}] = [\underline{q}, \overline{q}] = [0, 1]$. $Y|(p, q)$ is a Bernoulli distribution with support $\{\underline{y}, \overline{y}\}$, where $1 > \overline{y} > \underline{y} \geq 0$. $\Pr(Y = \overline{y}|p, q) = pq$ is the probability of the good state, so Condition Survival-logSPM is met. The values of outside options satisfy $\underline{V} + \underline{U} = \underline{y}$ and $\underline{y}^2 \geq \underline{U}$, and so Assumption (P).

For the classes of equity contracts and debt contracts, the worker's expected payoffs are given by

$$v(p, q, t_E(.; s)) = (1 - s)[\underline{y} + (\overline{y} - \underline{y})pq]$$

$$v(p, q, t_D(.; s)) = \begin{cases} \max(\overline{y} - s, 0)pq & \text{, if } s \in [\underline{y}, 1]\text{; and} \\ [\underline{y} + (\overline{y} - \underline{y})pq] - s & \text{, if } s \in [0, \underline{y}). \end{cases}$$

For a given $\epsilon \in (0, 1)$, define a mixture of debt and equity contracts by

$$t^\epsilon(.; s) = (1 - \epsilon)sy + \epsilon \min(y, s),$$

so that

$$v(p, q, t^\epsilon(.; s)) = (1 - \epsilon)v(p, q, t_E(.; s)) + \epsilon v(p, q, t_D(.; s)).$$

ϵ is the weight on the debt contract while $1 - \epsilon$ is the weight on the equity contract.[59] Though $[y - t_E(y;s)]$ and $[y - t_D(y;s)]$ are log-SPM in (y,s), $y - t^\epsilon(y;s)$ is neither log-SPM nor SPM for any $\epsilon \in (0,1)$.

If only equity contracts are feasible, $v(p,q,t_E(.;s)$ satisfies Global ID, and hence the First Best allocation can always be decentralized. Now suppose mixtures of debt and equity with a fixed ϵ are introduced, it is without loss to consider only such mixtures because they are flatter than the equity contracts.

First, $v(p,q,t^\epsilon(.;s))$ satisfies Condition Sorting-p for any $\epsilon \in (0,1)$. The reason is that $v(p,q,t_E(.;s))$ and $v(p,q,t_D(.;s))$ both satisfy Condition Sorting-p and are linear in p for any given pair of (q,s), so do any linear combination of the two.

For a given asset quality, all workers share the same preference over (η,s) in the region $s \in [\overline{y},1]$. When ϵ is sufficiently small, a high type worker is willing to endure a lower matching probability for an incremental reduction in contract term than a low type worker if $s \in (\underline{y},\overline{y})$. The opposite happens if $s \in [0,\underline{y})$. This holds for all asset qualities. It can be shown that $v(p,q,t^\epsilon(.;s))$ does not satisfy Condition Screening-q for sufficiently small ϵ using a more involved argument.[60]

[59] When only equity contracts or debt contracts are available, the analysis is invariant to any order-preserving transformation of s in the definitions of $t_E(.;s)$ and $t_D(.;s)$. This is no longer true for a mixture of debt and equity contracts because the contract term s determines the pair of debt and equity contracts forming the mixture. The definition of $t^\epsilon(.;s)$ here implies that the principal amount is capped at $\overline{y}$ for debt contracts for $s > \overline{y}$.

[60] Suppose, to the contrary, that Condition Screening-q is met. Consider $\overline{y} > s^H > s^L > \underline{y}$. q, p^H and p^L are chosen so that $v(p^L,q,t^\epsilon(.;s^H)) \geq \underline{V}$ and $u(q,p^H,t^\epsilon(.;s^L)) > \underline{U}$. $u(q,p^H,t(.;s^L)) > u(q,p^L,t(.;s^H))\}$ if s^H and s^L are close enough. As discussed, the inequality (2.13) does not hold. It follows that the inequality (2.14) must hold for some $q' \leq q$ and $s' \leq s^H$ satisfying $v(p^L,q',t^\epsilon(.;s')) = v(p^L,q,t^\epsilon(.;s^H))$. The inequality (2.14) implies that $v(p',q,t^\epsilon(.;s^L)) > v(p',q',t^\epsilon(.;s'))$, while Condition Sorting-p requires that

For small values of ϵ, $v(p, q, t^\epsilon(.; s))$ indeed satisfies Condition Entry-q. Condition Entry-q has the pre-condition $U \geq u(q^H, p^L, t^\epsilon(.; s^H))$. At $\epsilon = 0$, $\underline{y}^2 \geq \underline{U} \geq s^H[\underline{y} + (\overline{y} - \underline{y})p^L q^H]$, and hence $\underline{y} > s^H$. For any pair of s^H and s^L in this range, $v(p, q, t_E(.; s))$ is strictly pairwise log-SPM in (p, s). Since $v(p, q, t_E(.; s))$ is also strictly pairwise log-SPM in (p, q), we must have

$$\frac{v(p^H, q^H, t^\epsilon(.; s^H))}{v(p^L, q^H, t^\epsilon(.; s^H))} > \frac{v(p^H, q^L, t^\epsilon(.; s^L))}{v(p^L, q^L, t^\epsilon(.; s^L))},$$

when $\epsilon \to 0^+$.

In summary, when a small component of the debt contract is introduced, there exist distributions of types for which owners of asset quality above the threshold type have profitable deviations in the candidate equilibrium. This is in stark contrast to the case with only equity contracts. Under a debt contract with term $s \in (\underline{y}, \overline{y})$, a worker will repay the principal in full only in the good state. A tiny reduction in the contract term reduces the expected payment from a worker more if his productivity is higher. Yet, the workers of lower types still see a greater decline in percentage. However, if the contract includes an overwhelming component of equity, it turns out that the reduction in contract term will result in a greater percentage gain for the workers of high types. So the asset owners will succeed in poaching the better workers.

$v(p', q', t^\epsilon(.; s')) \geq v(p', q, t^\epsilon(.; s^H))$. Since this holds for s^L arbitrarily close to s^H, it follows that $v(p', q', t^\epsilon(.; s')) = v(p', q, t^\epsilon(.; s^H))$. Suppose that the workers of p^L are indifferent between the contract $(t^\epsilon(.; s^H), q)$ and $(t^\epsilon(.; s''), q'')$, where $q'' \in (q, q')$. Condition Sorting-p then requires that all workers of types between p' and p^L are also indifferent between these two contracts. This is impossible because $v(p, q, t_E(.; s))$ satisfies Condition strict Sorting-p and $v(p, q, t^\epsilon(.; s))$ is a linear combination of $v(p, q, t_E(.; s))$ and $v(p, q, t_D(.; s))$.

2.9 Concluding Remarks

In this chapter, I study how the use of the contingent payment affects the matching efficiency and the divisions of surpluses in a market where the types on one side are privately known. I propose a stylized framework to address these questions. To uncouple the potential sources of inefficiencies, I focus on an equilibrium decentralizing PAM. I analyze the sorting decision for the informed side, and the uninformed side's choices of contracts in such equilibrium. I characterize the conditions under which PAM can be decentralized for any distribution of types. If these conditions are not all met, I detail how to construct some distributions of types and identify the profitable deviations by the corresponding group, illustrating the incentives against assortative matching. The convenience, which is also its limitation, of this approach is that it leaves out the interaction among these incentives. Studying their interactions and the resulting allocation is left for future research.

I then provide a unifying sufficient condition, Global ID, which is intuitive and easy to interpret. Its simplicity allows me to provide joint sufficient conditions on the contingent contracts and the form of production complementarity. When these primitive conditions are not all met for the application at hand, one shall directly check whether and which of the necessary and sufficient conditions on the expected payoff is violated. This points to the groups who potentially have incentives against assortative matching. I provide examples illustrating how restricting the feasible set of contracts can align the incentives for such group, and hence improve the matching efficiency in the decentralized market. In this light, this paper is a first step toward how a benevolent planner mitigates the potential inefficiency caused by screening in the matching markets. This is a promising direction for further studies.

Though the use of contingent contracts may leave the equilibrium allocation unchanged, it does affect the divisions of the matching surpluses between the two sides. Comparing with the full information case, the asset owners will enjoy higher payoffs at the expense of the workers above the threshold types. Furthermore, the equilibrium payoffs for the workers increase with their types at a slower rate, and the opposite holds for the asset owners. Recall that the equilibrium payoff for the workers and firm owners in the full information case represents their shadow value in a social planner's problem maximizing the total surplus. In this light, my result indicates there is a wedge between the social and private benefit of getting matched, and how the size of this wedge changes with the feasible set of contracts. This opens avenues for future research on how this wedge will interact with other channels such as pre-investment and search friction in richer models.

Chapter 3

Inefficient Sorting Under Output Sharing

3.1 Introduction

This paper studies sorting in a frictional market where the two sides can be ranked by some characteristics, or simply their types. One side competes for partners by offering financial securities or contracts specifying how the payment between the partners is contingent on certain outcomes, say the realized output. An example which has received much attention is the market for top executives. Firms are ordered by their size, while the candidates are ranked by their productivity or talents. Gabaix and Landier (2008); Terviö (2008) apply a frictionless assignment model to study how assortative matching accounts for the empirical distribution of the amount of CEOs pay among the largest publicly traded companies in the United States. Frydman and Jenter (2010) look at the composition of CEO pay in S&P 500 firms and document that base salary makes up less than 20% of the remuneration, and over half of it are option grants and restricted stock grants during the period 2000 to 2008. Other applications include the sorting between the entrepreneurs and venture capitals (Sørensen (2007)), or between the acquiring firms and target firms in M&A (Rhodes-Kropf and

Robinson (2008)).

In many circumstances, the parties on one side, say candidates for the CEO positions, are better informed about their own types. Little is known about how such information asymmetry may interact with search friction and contingent payment, and the overall effect on sorting.

I address this question for the class of output sharing contracts in a competitive search framework. There are double continuums of types of assets and workers. A worker's productivity is privately known, whereas the quality of an asset is publicly observable. Each worker may operate an asset. The types on both sides determine the output.[61] The asset owners first post sharing contracts specifying the payment contingent on the future outputs. Then the workers decide which type of asset and contract they search for. The meeting is bilateral and subject to search friction. Production exhibits complementarity, so the Second Best allocation always features positive assortative matching (PAM) despite search friction.[62] I identify a novel source of inefficiency in this environment and analyze the resulting distortion.

To better understand the source of inefficiency, let us first consider the benchmark result in Eeckhout and Kircher (2010). They study price competition in the described environment, in which the asset side post fixed prices. Hence the informed workers, once matched, will pay the asset owners upfront and assume the residual claim. The authors show that the equilibria supports Second Best allocations. Furthermore everyone receives her "social

[61] Here the choice of contract determines the split of the output, while leaving its size unaffected. This simplification allows me to concentrate on the potential distortions in the matching pattern.

[62] In the presence of search friction, PAM occurs if better workers always search for better assets.

value", the shadow price in the Utilitarian planner's problem, in equilibrium. However, wealth constraint of the workers and incentive provision for the asset owners may undermine the feasibility of the buyout arrangement, calling for the use of sharing contracts.

The Second Best allocations can no longer be decentralized using the sharing contracts. A low-type worker pays less than a high-type worker when conceding a larger output share to the asset owner. So the offering of sharing contracts handicaps the competition among workers for the *same* type of assets, increasing the expected payoff for the asset owners. This shift in the divisions of matching surpluses can be attributed to the linkage principle in auction theory (DeMarzo, Kremer, and Skrzypacz, 2005) because the allocation is held *unchanged*. Assortative matching gives rise to an additional spillover effect. Since the workers pay more for their partners in the Second Best allocation, they will find better assets more attractive, further intensifying the competition among workers for better assets. As a result, the set of incentive compatible contracts supporting a Second Best allocation must provide the asset side a larger slice of the matching surplus than in price competition.

Consequently, the private benefit for an asset to get matched is above the social benefit. The wedge is the largest at the top. Facing search friction, the owners of the best assets increase their matching probability by inducing an inefficiently long queue of workers. This leads to the unravelling of the Second Best allocation. Inefficiency here is caused by the interplay between three elements: sharing contracts, private types and search friction.

Two questions naturally arise from the preceding discussion. Unlike fixed prices, the asset owners are now concerned about the types of their partners, which affect their expected payment under sharing contracts. An asset owner will take screening into account when deciding contract offer, and may attempt to poach better workers.[63] The first question is whether PAM can still be supported by an equilibrium. In such equilibria, the pool of workers left to the lower quality assets must deteriorate amid an increase in the queue length for the best assets. This presents a countervailing force as an asset owner gains less from a match with a weaker worker, and may induce a shorter queue of workers instead. Hence, the distortions in the queue lengths and the sorting pattern are intertwined. The second question is what form of distortion arises in equilibrium.

To distinguish the channel of inefficiency here from those in the search and matching literature literature, I consider the setting that all workers have the same preference over the contract term and the matching probability the two given the asset quality. This property ensures that asset owners never use queue length as an instrument to screen out better workers. The stylized setting yields a unique equilibrium, which still features PAM. In this equilibrium, the matched pairs of types fully separate into a continuum of (sub-)markets. The term of the contract offered in every market is given by the Hosios (1990) condition, under which the equilibrium payoff of an agent is the reduction in the aggregate surplus if she is removed from the population.[64] The standard interpretation is that both sides fully internal-

[63] An asset owner believes that an off-equilibrium-path contract will only attract the types of workers accepting the lowest matching probability, given all other contract offers. Guerrieri, Shimer, and Wright (2010) motivates this belief restriction using "subgame perfection" with bilateral matching.

[64] The equilibrium allocation here is inefficient. Unlike the standard setting where both sides are homogeneous, the equilibrium payoff for an agent is no longer the same as her "social value", the maximum increase in the aggregate surplus a Utilitarian planner may achieve from assigning a new agent of the same type. I will provide a precise interpretation

ize their search externality on other participants in the same market. So the asset owners in every market induce a queue length maximizing the expected surplus for the pair of types, subject to free-entry of the workers at their equilibrium payoff. Nevertheless, the matching pattern and the workers' equilibrium payoff are endogenously determined. The asset owners do not account for the effects of their contract offers on the information rent for other types of workers and on the remaining pool of workers left to other types of assets. Therefore, the sorting inefficiency only arises in two-sided matching.

The equilibrium and the Second Best allocation vary with the entire distribution of types. The key contribution of this paper is the qualitative features of the distortion which are universal for *all* distributions. Such "distribution-free" result is of theoretical interest as the argument illuminates general economic forces which are always at play.

As one may have expected, the queue length for the best assets is always inefficiently high. The surprising and novel result is that all but the best assets will always pair up with weaker workers. Depending on the distribution of types, there is either an excessive entry of workers or an insufficient entry of assets. As a result, the best workers will suffer while the weakest workers gain from the offering of sharing contracts. The opposite is true for the asset side.

It is noted that here the comparative statics on the equilibrium payoffs differ from that in Proposition 5 in chapter 2. This is due to the adjustments in the equilibrium allocation following the change in the form of contingent payments.

of Hosios condition in the current setting with double continuums of types.

Section 3.2 discusses the related literature. Section 3.3 details the model setting and the equilibrium definition. Section 3.4 covers the benchmark results from Eeckhout and Kircher (2010). That is, the Second Best allocations feature PAM and can be decentralized when asset side post prices. Section 3.5 characterizes the equilibria when asset side offers output shares. Section 3.6 studies the forms of distortion in equilibrium. Section 3.7 concludes. All proofs are relegated to the Appendix B.

3.2 Related Literature

This paper is part of the literature on assortative matching. My setting is closely related to Eeckhout and Kircher (2010). The authors show that in price competition, the n-root-supermodularity condition is necessary and sufficient for any equilibrium to feature PAM, regardless of the distribution of types. In addition, the Second Best allocations are supported by the equilibria. The production and matching technology in my setting satisfy this condition. The offering of sharing contracts not only changes the sorting incentives for the workers but also renders poaching potentially profitable for the asset side. I address how such arrangement distorts sorting in equilibrium.

This paper also contributes to the literature on efficiency in search and matching models. Hosios (1990) considers a market where both sides are homogeneous. He provides the condition on the division of the matching surplus, under which the equilibrium queue length is constrained efficient. Albrecht, Navarro, and Vroman (2010) and Julien and Mangin (2016) show that the Hosios condition no longer ensures constrained efficiency when multiple types are pooled into a single market. This is because the participation of an agent also affects the distribution of the partners for the other side. In my setting, every active market features one pair of types and meets the Hosios condition. This distinguishes the channel of sorting inefficiency from the search externalities in the literature.

Guerrieri (2008) studies dynamic efficiency in a directed search model where a worker privately observes his match-specific productivity upon meeting. The worker then weighs the current match against his continuation value in the unemployed pool. As a result, the wage offers in the future periods determine the probability of workers' acceptance and their information rent in the present period. Under free entry, firms do not account for the effects of their offers on the markets in previous periods. The author shows that the convergence to the steady state is inefficiently slow in equilibrium. The source of sorting inefficiency here shares the similarity that the asset owners do not internalize the effects of their offers on the markets for other assets.

Guerrieri, Shimer, and Wright (2010) study competitive screening in a competitive search framework. They assume free entry of homogenous principals. These principals have both contract and matching probability as screening instruments, and the latter is endogenously determined. The authors characterize the equilibrium and study the form of distortion in various applications. I consider two-sided matching where both sides compete for partners from given pools.The key difference is that the distortion now depends on the distribution of types. I obtain "distribution-free" features of the form of distortion.

3.3 Model Setting

3.3.1 Production

There are continuums of workers and asset owners. Each asset owner owns a unit of asset. Assets can be ranked according to their publicly known qualities $q \in [0, 1]$. All workers are ex-ante homogeneous but differ in their actual productivity $p \in [0, 1]$. Every worker privately knows his productivity. The values of outside options for workers and asset owners are given by $\underline{V}$ and $\underline{U}$ respectively. $\emptyset$ denotes the choice of outside option. All parties are risk neutral and have a quasi-linear preference. I shall use feminine pronouns for asset owners and masculine one for the workers.

Production takes place after a worker pairs up with an asset. The matching surplus for the pair of types (p, q) is the output they produce, denoted by $y(p, q)$. $y : [0, 1]^2 \to \mathbb{R}_{++}$ is positive, strictly increasing and twice continuously differentiable (C^2) in (p, q).

Assumption (Y)**.** *The output $y(p, q)$ is strictly log-supermodular (log-SPM) in p and q.*

Assumption (Y) has two important implications in a frictionless setting. First, it represents a stronger form of production complementarity than strict supermodularity (SPM). Without search friction, the total surplus is maximized under perfect positive assortative matching. Second, log-SPM of the output $y(p, q)$ is also necessary and sufficient for decentralizing PAM in a frictionless world, when the asset side may only post output shares.

Example (O-ring production)**.** *Assumption (Y) is satisfied if the conditional distribution of output $Y|(p, q)$ is a Bernoulli distribution with support $\{\underline{y}, \overline{y}\}$, where $\overline{y} > \underline{y} > 0$ and $\Pr(Y = \overline{y}|p, q) = pq$: Production is composed* *of two tasks. The probability of success for the first task is p, and that of the second task is q. The production yields high output $\overline{y}$ if both tasks are successful. Otherwise, only base output $\underline{y}$ is produced.*

The types on both sides are continuously distributed with support $[0,1]^2$. $F(p)$ denotes the measure of workers of productivity below p and $G(q)$ is the measure of assets with qualities below q. F and G are C^2 and their derivatives are denoted by f and g respectively. f and g are positive and bounded over $[0,1]$.

Suppose a worker pairs up with an asset. The two parties may enter into a sharing contract $s \in [0,1]$ where s and $1-s$ are the shares of output for the asset owner and worker respectively.

3.3.2 Matching

There are continuums of (sub-)markets indexed by $(q,s) \in [0,1]^2$. An owner of asset quality q may participate in one of the markets (q,s) while a worker may participate in any one of the markets.

The timing of the events is as follows: In the contract posting stage, the asset owners make their participation decisions simultaneously. Observing the measure of asset owners in every market, the workers simultaneously make their participation decisions. Matches are then formed.

The participants on the two sides of a market will pair up randomly. Define the queue length $\lambda \in [0,\infty]$ as the ratio of the workers to the asset owners in the market. A worker gets matched with probability $\eta(\lambda)$ while the matching probability for a asset owner is $\delta(\lambda)$. Meeting is bilateral, so $\delta(\lambda) \leq min\{\lambda, 1\}$ and $\lambda\eta(\lambda) = \delta(\lambda)$. The payoffs for those who left unmatched are normalized to zero.

η is a strictly decreasing function. $\delta : [0, \infty] \to [0, 1]$ is C^2, strictly increasing and strictly concave. These properties jointly imply the following: For any positive $\lambda \in (0, \infty)$, $\frac{d \ln \delta}{d \ln \lambda} \in (0, 1)$.[65] and $1 > \eta(\lambda) > \delta'(\lambda)$. $\lim_{\lambda \to \infty} \delta'(\lambda) = 0$ because $\lim_{\lambda \to 0^+} \delta(\lambda) = 0$ and strictly concavity imply that $\delta(\lambda) > \lambda \delta'(\lambda)$ for all $\lambda > 0$.

Assumption (M). $\frac{d \ln \delta}{d \ln \lambda}$, *the elasticity for* $\delta(\lambda)$, *is decreasing.*

Following Eeckhout and Kircher (2010), I assume a decreasing elasticity for $\delta(\lambda)$.[66] Since $1 = \frac{d \ln \delta}{d \ln \lambda} - \frac{d \ln \eta}{d \ln \lambda}$, the elasticity for $\eta(\lambda)$ must be increasing.[67] The presence of search friction gives rise to an insurance motive against the risk of being unmatched. Assumption (M) states that an asset owner's marginal gain in her matching probability from an increase in the queue length is diminishing. Symmetrically, the workers see a diminishing marginal gain from a decrease in the queue length.

Example (Random matching). *Assumption (M) is satisfied for* $\delta(\lambda) = \frac{\lambda}{\lambda+1}$ *: All participants on both sides are pooled together to form pairs randomly. The pair may carry out production only when it consists of a worker and an asset owner.*

Example (Urn-ball matching). *Assumption (M) is satisfied for Urn-ball matching function,* $\delta(\lambda) = 1 - \exp(-\lambda)$ *: Every worker approaches one asset*

[65]In particular, $\frac{d \ln \delta}{d \ln \lambda} = 1 + \frac{d \ln \eta}{d \ln \lambda} < 1$ for $\lambda \in (0, \infty)$.

[66]Note that Assumption (M) is equivalent to a unit upper bound on the elasticity of substitution of the aggregate matching function. Suppose $M(L, K)$ is the number of matches in a market with L workers and K assets. Then $\frac{M_L(\lambda,1) M_K(\lambda,1)}{M_{LK}(\lambda,1) M(\lambda,1)} \leq 1$ for any $\lambda \geq 0$.

[67]Eeckhout and Kircher (2010) assume a strictly decreasing elasticity for $\delta(\lambda)$. Their results remain valid here because I strengthen the assumption on output $y(p, q)$ to strict log-SPM.

owner without coordination. An asset is utilized if its owner is approached by at least one worker.

Participation in matching is costly because the agent has to forgo her outside option. We assume that the total output from the matched pairs at the top can always cover the total opportunity costs of participation. Formally,

$$\max_{\lambda \geq 0}[\delta(\lambda)y(1,1) - \lambda \underline{V} - \underline{U}] > 0. \tag{3.1}$$

It ensures that for any distribution of types, it is always efficient to have the best agents on both sides searching for partners.

Example (Random matching)**.** *Given* $\delta(\lambda) = \frac{\lambda}{\lambda+1}$*, condition (3.1) is satisfied if and only if* $y(1,1) > (\sqrt{\underline{U}} + \sqrt{\underline{V}})^2$.

Example (Urn-ball matching)**.** *Given* $\delta(\lambda) = 1 - \exp(-\lambda)$*, condition (3.1) is satisfied if* $(1 - \sqrt{e})y(1,1) > \frac{1}{2}\underline{U} + \underline{V}$.

3.3.3 Equilibrium definition

$K(q,s)$ is the measure of asset owners participating in the markets $(q', s') \leq (q, s)$. $L(p,q,s)$ is the measure of workers with types $p' \leq p$ participating in the markets $(q', s') \leq (q, s)$. The marginal distributions are denoted with the corresponding variables as subscripts. (K, L) is feasible if $K_q \leq G$ and $L_p \leq F$. $G(q) - K_q(q)$ and $F(p) - L_p(p)$ are respectively the measures of assets of quality below q and workers with productivity below p assigned to the outside option. The support of K is denoted by Ψ. A market is active if it is in Ψ. Otherwise it is inactive. Since participation is costly, it is never optimal for workers to visit a market with no asset owners. Therefore, L_{qs} is required to be absolutely continuous w.r.t. K.[68]

[68] This requirement ensures the Radon-Nikodym derivative $\frac{dL_{qs}}{dK}$ is well-defined.

The equilibrium concept here follows the literature on large games (e.g. Mas-Colell, 1984). The payoff for every single agent depends on her own decision, and the decisions of all others only through K and L. K and L in turn are consistent with the optimal decisions of all individual agents.

Each market (q, s) is associated with a queue length $\Lambda(q, s; K, L)$ and a distribution of participating workers $R(q, s; K, L)$, where $R(.|q, s; K, L)$ is the C.D.F. for worker's type. The environment is competitive in the sense that everyone takes Λ and R as given. For the active markets, Λ is the Radon-Nikodym derivative, $\frac{dL_{qs}}{dK}$ and R is derived using Bayes' law.[69] By participating in an active market (q, s), a worker of type p receives an expected payoff

$$\eta(\Lambda(q, s; K, L))(1 - s)y(p, q), \tag{3.2}$$

while an asset owner receives an expected payoff

$$\delta(\Lambda(q, s; K, L))s \int y(p, q)dR(p|q, s; K, L). \tag{3.3}$$

I now extend the payoff functions to the inactive markets. I will elaborate on the belief restriction underlying the payoffs functions afterwards. A worker will never get matched if visiting an inactive market. A worker of type p can at most generate

$$V(p; K, L) = \sup\{\eta(\Lambda(q, s; K, L))(1 - s)y(p, q), (q, s) \in \Psi\} \cup \{\underline{V}\}.$$

$V(.; K, L)$ then determines the deviating payoff for the asset owners. For any inactive market,

$$\Lambda(q, s; K, L) = \inf\{\lambda \in [0, \infty] : V(p) \geq \eta(\lambda)(1-s)y(p, q), \forall p \in [0, 1]\}, \tag{3.4}$$

[69] Formally, $\int_\psi \Lambda(q, s)dK = \int_\psi dL_{qs}$, and $\int_\psi R(p'|q, s)dL_{qs} = \int_{\{p\leq p'\}\times\psi} dL$ for any measurable subset $\psi \subseteq \Psi$ and any $p' \in [0, 1]$.

and $R(q,s;K,L)$ is degenerate at

$$\inf\{p \in [0,1] : V(p;K,L) \le \eta(\Lambda(q,s;K,L))(1-s)y(p,q)\}. \tag{3.5}$$

The definition (3.4) and (3.5) represent two conditions. First, if $V(p;K,L) > \eta(0)(1-s)y(p,q)$ for all $p \in [0,1]$, then $\Lambda(q,s;K,L) = 0$ and $R(q,s;K,L)$ is degenerate at $p = 0$ for such inactive market. Second, for any market, be it active or inactive,

$$V(p;K,L) \ge \eta(\Lambda(q,s;K,L))(1-s)y(p,q)$$

for all types of workers, and equality holds if p is in the support of $R(q,s;K,L)$ and $\Lambda(q,s;K,L) > 0$.

Facing $\Lambda(q,s;K,L)$ and $R(q,s;K,L)$, an owner of asset quality q can receive

$$\begin{aligned} & U(q;K,L) \\ =& \sup\{\delta(\Lambda(q,s;K,L))s \int y(p,q)dR(p|q,s;K,L), (q,s) \in [0,1]^2\} \cup \{\underline{U}\}. \end{aligned}$$

Definition. *An equilibrium is a pair of distributions (K,L) satisfying:*

- *Asset owners' optimal contract posting: $(q,s) \in \Psi$ only if s maximizes the asset owner's expected payoff (3.3). $K_q'(q) \le g(q)$ with equality if $U(q;K,L) > \underline{U}$.*

- *Workers' optimal acceptance: (p,q,s) is in the support of L only if $(q,s) \in \Psi$ and maximizes the worker's expected payoff (3.2). $L_p'(p) \le f(p)$ with equality if $V(p;K,L) > \underline{V}$.*

Fix an equilibrium (K,L), $V(p;K,L)$ and $U(q;K,L)$ are the equilibrium payoff for workers and asset owners respectively. The argument K and L will be omitted from the equilibrium objects if no confusion arises.

Definition. *A pair of distributions (K, L) is incentive compatible if it satisfies workers' optimal acceptance condition in the definition of an equilibrium.*

Definition. *A pair of distributions (K, L) induces voluntary participation of the asset side if for any $(q, s) \in \Psi$, the asset owners' expected payoff in (3.3) is no less than $\underline{U}$.*

The notions of incentive compatibility and voluntary participation will be useful in the discussion of the Utilitarian planner's problem.

Belief restriction Since there are continuums of workers and assets, switching between active markets or taking outside option by a single party has negligible impacts. The same is true when a worker unilaterally switches to an inactive market. The focus here is the deviation to some inactive market by an asset owner. For an inactive market (q, s), $\Lambda(q, s)$ and $R(q, s)$ are interpreted as the public belief regarding the queue length and the composition of the workers attracted to that market after an owner of asset quality q deviates to it. An advantage of this notation is to eliminate the distinction between deviations to active markets or inactive markets by an asset owner.

Suppose an owner of asset quality q deviates to post a contract s. If $V(p) \geq \eta(0)(1 - s)y(p, q)$ for all types, then no workers will ever profit from accepting the deviating offer. The asset owner believes such offer will attract no workers and $R(q, s)$, which has no bearing in such case, is degenerate at $p = 0$. Now consider the case that $V(p) < \eta(0)(1 - s)y(p, q)$ for some types. Then $\Lambda(s, q)$ is uniquely determined by the lowest matching probability some workers are willing to endure. The asset owner believes that only the lowest type among these workers will be attracted.

The restriction on the "off-equilibrium-path" belief here is often motivated by the "subgame perfection" on the workers' side in the competitive search literature. Suppose only ϵ-measure of the owners of asset quality q deviate to some inactive market (s, q). Observing the measure of asset owners in every market, a worker has to anticipate his matching probability in each of the markets and adjust his participation decision accordingly. When $\epsilon \to 0^+$, no types of workers can strictly gain from participating in the market (s, q) in the equilibrium of this "subgame". Otherwise, workers of all such types will turn up in this market but only ϵ-measure of them will get matched, resulting in an expected payoff below their outside option. It follows that any workers attracted to the market (s, q), if any, are those willing to endure the lowest matching probability. By continuity, the workers' payoff in the equilibrium of this "subgame" must converge to $V(p)$. This justifies the belief restriction discussed.

In particular, the belief restriction here closely follows Guerrieri, Shimer, and Wright (2010). Eeckhout and Kircher (2010) adopt the same restriction on the queue length. Since asset owners post prices in their setting, they leave out the off-equilibrium-path belief on the worker's type.

3.3.4 Assortative matching

Definition. *A pair of distributions features positive assortative matching (PAM) if there exists a pair of threshold types* $(\underline{p}, \underline{q}) < (1, 1)$ *and an increasing function* $\kappa : [\underline{p}, 1] \to [\underline{q}, 1]$ *such that* $\kappa(\underline{p}) = \underline{q}$ *and* $L_{pq}(p, \kappa(p)) = F(p) - F(\underline{p})$.

$\kappa(p)$ denotes the quality of the asset assigned to a worker of type p. The above definition of PAM not only requires the participants to match assortatively, but also every worker above the threshold type to participate in matching. This is not restrictive because a better worker always gains more when entering the same market, so only the lowest types may take their outside options in equilibrium and in any efficient allocations.

This conclusion does not automatically extend to the asset side. Even when posting the same contract, an owner of a better asset may end up attracting weaker workers, gaining less from participation. In the subsequent sections, I will show that monotonic participation for both sides indeed occurs in any efficient allocations and equilibria. In this case, κ is bijective and strictly increasing. The inverse of κ is well-defined and denoted by $r : [\underline{q}, 1] \to [\underline{p}, 1]$. $r(q)$ is the type of worker assigned to the asset of quality q.

3.4 Second Best Allocation

Suppose that a Utilitarian planner, whose goal is to maximize the total output, have complete information and may dictate the participation decision for each type. Nevertheless, search friction remains present in the matching process. Her problem is given by

$$\max_{K,L} \int \eta(\Lambda(q,s))y(p,q)dL + [F(1) - L_p(1)]\underline{V} + [G(1) - K_q(1)]\underline{U}$$

subject to

$$K_q \leq G, L_p \leq F \text{ and } \Lambda(q,s) = \frac{dL_{qs}}{dK}.$$

The search friction introduces an insurance motive, which is conducive to negative assortative matching. This is because the most efficient way to increase the matching probability for high types is assigning them to a market flooded with low types from the other side. The more responsive the matching probabilities to a change in tightness ratio, the greater the insurance motive. The Utilitarian planner's solution always features PAM only if the production complementarity outweighs the insurance motive for all distributions of types.

Theorem (Eeckhout and Kircher, 2010). *Under Assumption (Y) and (M), Second Best allocations always feature PAM.*

The efficient allocation requires monotonic participation because the matching surplus is strictly increasing in types. The strict concavity of $\delta(\lambda)$ implies that it is efficient to pool the same pairs of types into one market. The contract term s can be omitted as it does not affect the size of the matching surplus. As a result, the Utilitarian planner's problem can be simplified as

$$\max_{\underline{p},\underline{q},r,\lambda} \int_{\underline{q}}^{1} \delta(\lambda(q))y(r(q),q)dG(q) + F(\underline{p})\underline{V} + G(\underline{q})\underline{U}$$

subject to

$$r(\underline{q}) = \underline{p}, r(1) = 1, \tag{3.6}$$

and for $q \geq \underline{q}$,

$$\int_{q}^{1} \lambda(q')dG(q') = F(1) - F(r(q)).$$

Abusing the terminology, a solution, denoted by $(r_{SB}, \lambda_{SB}, \underline{p}_{SB}, \underline{q}_{SB})$, is called a Second Best (SB) allocation.[70] The Utilitarian planner's problem can be reformulated as an optimal control problem with r as the state variable and λ as the control variable. The law of motion is given by

$$r'(q) = \frac{g(q)}{f(r(q))}\lambda(q). \tag{3.7}$$

[70] There is a continuum of (K, L) with the same matching pattern $(r, \lambda, \underline{p}, \underline{q})$ but different divisions of matching surpluses.

Introducing the co-state variable τ, the Hamiltonian is given by

$$H(q, r, \lambda, \tau) = g(q)[\delta(\lambda)y(r, q) - \tau(q)\frac{\lambda}{f(r)}].$$

Eeckhout and Kircher (2010) show that the first order conditions can be written as

$$v_{SB}(r_{SB}(q)) = \delta'(\lambda_{SB}(q))y(r_{SB}(q), q), \tag{3.8}$$

$$\left.\frac{\partial v_{SB}(p)}{\partial p}\right|_{p=r_{SB}(q)} = \eta(\lambda_{SB}(q)) \left.\frac{\partial y(p, q)}{\partial p}\right|_{(p,q)=(r_{SB}(q),q)}, \tag{3.9}$$

where $\tau(q) = f(r_{SB}(q))v_{SB}(r_{SB}(q))$. In particular, r_{SB} and λ_{SB} are continuously differentiable, C^1.

$v_{SB}(p)$ is the shadow value for a worker of type $p \geq \underline{p}_{SB}$. When comparing $v_{SB}(p)$ and $v'_{SB}(p)$, it is noteworthy that $\delta'(\lambda) < \eta(\lambda) < 1$ for $\lambda > 0$. The gap between $\delta'(\lambda)$ and $\eta(\lambda)$ reflects the benefit of a better asset. One can derive the shadow value for an asset of quality $q \geq \underline{q}_{SB}$ by a symmetric approach,

$$u_{SB}(q) = [\delta(\lambda_{SB}(q)) - \lambda_{SB}(q)\delta'(\lambda_{SB}(q))]y(r_{SB}(q), q),^{71} \tag{3.10}$$

and $u_{SB}(q)$ is strictly increasing. The boundary conditions at the bottom are given by

$$\underline{q}_{SB}[u_{SB}(\underline{q}_{SB}) - \underline{U}] = \underline{p}_{SB}[v_{SB}(\underline{p}_{SB}) - \underline{V}] = 0. \tag{3.11}$$

The above set of conditions, (3.6)-(3.11), defines a boundary value problem for $(\underline{p}_{SB}, \underline{q}_{SB}, r_{SB}, \lambda_{SB}, v_{SB}, u_{SB})$, which admits a unique solution under Assumption (Y) and (M). The assumption in (3.1) ensures participation at the top, $\underline{p}_{SB} < 1$ and $\underline{q}_{SB} < 1$. The shadow value of an agent below

[71] Note that $\delta - \lambda\delta' = \frac{d\eta(\lambda)}{d\lambda^{-1}}$.

the threshold type is simply the value of her outside option. I extend the definition of v_{SB} and u_{SB} to their entire type space, with $v_{SB}(p) = \underline{V}$ for $p < \underline{p}_{SB}$ and $u_{SB}(q) = \underline{U}$ for $q < \underline{q}_{SB}$.[72]

Remark 8. *The Second Best allocation is unique. For any p, q and λ,*

$$u_{SB}(q) + \lambda v_{SB}(p) \geq \delta(\lambda) y(p, q) \tag{3.12}$$

with equality if and only if $\lambda = \lambda_{SB}(q)$ and $p = r_{SB}(q)$.

The inequality in (3.12) is the counterpart of the well-known condition for stable matching. Without search friction, $\delta(\lambda) = min\{\lambda, 1\}$, the set of inequalities (3.12) collapses to $u_{SB}(q) + \lambda v_{SB}(p) \geq y(p, q)$.

When defining the Second Best allocation, it is assumed that the Utilitarian planner knows the workers' types. One may question if this is an appropriate benchmark when the workers' types are privately known. Suppose the Utilitarian planner observes only the types of the assets and may restrict the set of markets available. In essence, she may dictate the sharing contracts $s(q)$ for each type of assets, subject to their voluntary participation. The planner can induce the Second Best allocation by excluding all assets below $\underline{q}_{SB}$ from participation, and imposing $\widehat{s}(q)$ for $q \geq \underline{q}_{SB}$, where

$$\widehat{s}(q) = 1 - \frac{\delta'(\lambda_{SB}(\underline{q}_{SB}))}{\eta(\lambda_{SB}(q))} \exp\left(- \int_{\underline{q}_{SB}}^{q} \left. \frac{\partial \ln y(p, q')}{\partial q} \right|_{p = r_{SB}(q')} dq' \right).$$

At $q = \underline{q}_{SB}$, $\widehat{s}(\underline{q}_{SB}) = 1 - \left. \frac{d \ln \delta}{d \ln \lambda} \right|_{\lambda = \lambda_{SB}(\underline{q}_{SB})}$, so that the expected payoffs for the pair of threshold types are the same as their shadow values in (3.8) and (3.10). For $q > \underline{q}_{SB}$, $\widehat{s}(q)$ satisfies

$$\left. \frac{d}{dq} \ln(1 - \widehat{s}(q)) \eta(\lambda_{SB}(q)) y(p, q) \right|_{p = r_{SB}(q)} = 0.$$

[72] By an abuse of notation, v_{SB} and u_{SB} denote their respective restrictions over $[\underline{p}_{SB}, 1]$ and $[\underline{q}_{SB}, 1]$ when referring to the solution of the boundary value problem.

Under Assumption (Y), the workers of type $r_{SB}(q)$ strictly prefer the market $(\widehat{s}(q), q)$ to all other markets, receiving an expected payoff of $\widehat{V}(r_{SB}(q)) = \eta(\lambda_{SB}(q))(1-\widehat{s}(q))y(r_{SB}(q), q)$. By construction, $v_{SB}(p) \geq \widehat{V}(p) \geq v_{SB}(\underline{p}_{SB})$ for $p \geq \underline{p}_{SB}$ and equalities hold only at $p = \underline{p}_{SB}$.[73] This ensures incentive compatibility on the workers' side and voluntary participation for asset owners of $q \geq \underline{q}_{SB}$. Therefore, the Second Best allocation can be supported by $\widehat{s}(q)$.[74]

3.4.1 Price competition

Price competition refers to the benchmark setting that the asset owners may post prices, and the workers buy out the asset up front. When participating in a market with posted price w and queue length λ, a worker receives an expected payoff

$$\eta(\lambda)[y(p, q) - w],$$

while an asset owner receives an expected payoff of $\delta(\lambda)w$. An equilibrium in price competition can be defined analogously. It is essentially the equilibrium definition in Eeckhout and Kircher (2010).

Theorem (Eeckhout and Kircher, 2010)**.** *Under Assumption (Y) and (M), the Second Best allocation can be decentralized in price competition.*

For any $(\underline{p}_{SB}, \underline{q}_{SB}, r_{SB}, \lambda_{SB}, v_{SB}, u_{SB})$ satisfying (3.6)-(3.11), the authors construct an equilibrium which supports the Second Best allocation

[73] This is because $\frac{\partial}{\partial p} \ln v_{SB}(p) > \frac{\partial}{\partial p} \ln \widehat{V}(p) > 0$ for $p \geq \underline{p}_{SB}$. It also implies that the schedule $\widehat{s}(q) \in [s_{SB}(q), 1)$ is well-defined.

[74] One can recover the corresponding pair of distributions (K, L) from the Second Best allocation and the set of active markets $\{(q, \widehat{s}(q)) : q \in [\underline{q}_{SB}, 1]\}$ and check the conditions formally. The construction of (K, L) mirrors that in the proof of Proposition 7 in the Appendix.

$(r_{SB}, \lambda_{SB}, \underline{p}_{SB}, \underline{q}_{SB})$ and the equilibrium payoffs for the two sides are given by v_{SB} and u_{SB}. Let $w_{SB}(q)$ denote the price posted by the owners of asset quality $q \geq \underline{q}_{SB}$ in equilibrium. $w_{SB}(q)$ is determined by the FOC (3.8),

$$v_{SB}(r_{SB}(q)) = \delta'(\lambda_{SB}(q))y(r_{SB}(q), q) = \eta(\lambda_{SB}(q))[y(r_{SB}(q), q) - w_{SB}(q)],$$

so that

$$w_{SB}(q) = \left(1 - \left.\frac{d\ln\delta}{d\ln\lambda}\right|_{\lambda=\lambda_{SB}(q)}\right) y(r_{SB}(q), q). \tag{3.13}$$

This is known as the Hosios condition, for which a worker's share of the matching surplus is given by the elasticity of $\delta(\lambda)$ at the equilibrium queue length he is facing. Furthermore, Eeckhout and Kircher (2010) show that $w_{SB}(q)$ is increasing in q. The corresponding pair of distributions (K, L) can be again recovered from the Second Best allocation and $w_{SB}(q)$ readily.

The FOC (3.9) ensures that

$$v_{SB}(r_{SB}(q)) = \max_{q' \in [\underline{q}_{SB}, 1]} \{\eta(\lambda_{SB}(q'))[y(r_{SB}(q), q') - w_{SB}(q')]\}.$$

Together with the boundary condition(3.11), the incentive compatibility for workers is met.

Given the type of her potential partner $r_{SB}(q)$, an owner of asset quality q cannot profit from adjusting the price if and only if the Hosios condition holds. This is the standard result in settings with homogeneous workers. With heterogeneous workers,

$$\Lambda(q, w) = \inf_{\lambda \in [0,\infty]} \{v_{SB}(p) \geq \eta(\lambda)[y(p, q) - w], p \in [0, 1]\},$$

so a deviating offer may attract a longer queue of workers of other types.

Yet Assumption (Y) and (M) ensure such offer will not be profitable for the asset owners. The reason is that the price paid by the worker is independent of his type. Rearranging the inequality (3.12),

$$u_{SB}(q) \geq \max_{p,\lambda}[\delta(\lambda)y(p,q) - \lambda v_{SB}(p)] = \max_{w} \delta(\Lambda(q,w))w,$$

and equality holds for $q \geq \underline{q}_{SB}$.

3.5 Equilibrium Characterization

We now turn to the set of equilibria when only output sharing contracts are feasible. The workers' expected payoff can be separated into $y(p,q)$ and $\eta(\lambda)(1-s)$, and only the former depends on the private type. This has two important implications. First, the workers' preferences over q and $\eta(1-s)$ satisfy the strict single crossing property (SCP) as the matching surplus exhibits strict log-SPM. This reduces the multi-dimensional sorting into a familiar single dimensional one. Second, workers of all types share the same preference over their matching probability and the contract term for any given asset quality. The corresponding sets of indifference curves for workers are illustrated in the Figure 3.1

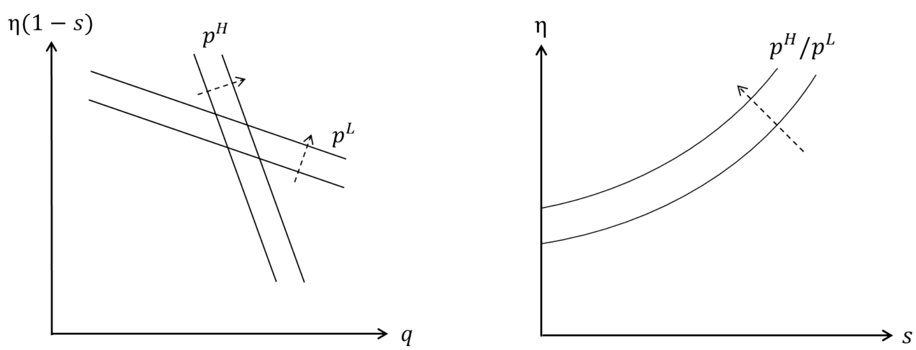

Figure 3.1: Properties of workers' preferences

The first property implies that if a worker prefers a market with better assets to another market with lower quality assets, then all better workers strictly prefer the one for better assets, vice versa. This holds regardless of the queue lengths and the contract terms in these markets. Therefore, the participants must match assortatively in any equilibrium.

This property also ensures monotonic participation on the asset side in any equilibrium. Suppose some type of workers participate in an active market (q^L, s^L), an owner of a better asset q^H can find a less generous contract s^H leaving these workers indifferent about accepting the two contracts. Hence, the queue length in the market (q^H, s^H) will be no lower than that in the active market (q^L, s^L). The strict SCP then implies that all weaker workers strictly prefer the latter market to the former. So posting the contract s^H provides an asset owner of q^H an expected payoff above the equilibrium payoff for her peers of q^L.

Let $(\underline{\widetilde{p}}, \underline{\widetilde{q}}, \widetilde{\kappa}, \widetilde{r})$ denote PAM in the equilibrium under consideration.

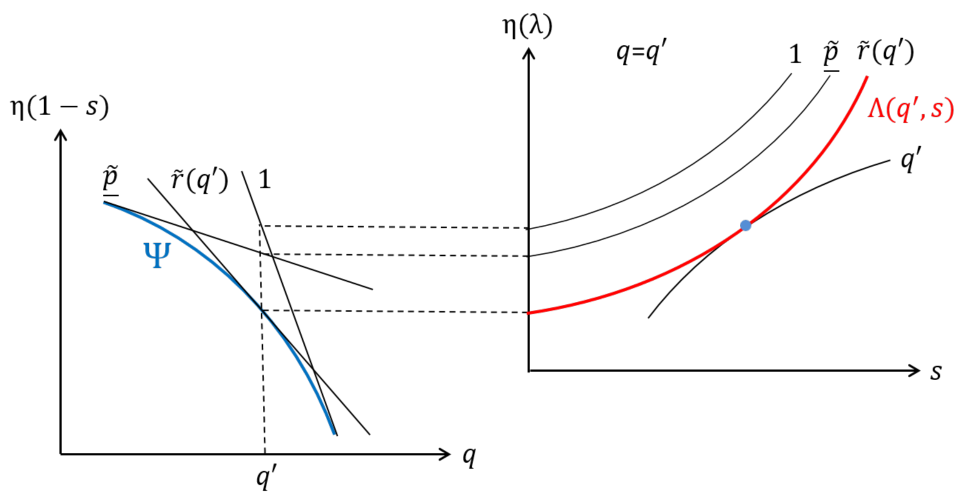

Figure 3.2: Characterization of active markets

The indifference curves over q and $\eta(1-s)$ of the participating workers, which yield their equilibrium payoff, are plotted in the left panel of Figure 3.2. The lower envelope of all these indifference curves must be $\eta(\Lambda(q,s))(1-s)$ for the set of active markets. Furthermore, the workers will have strict preference over the resulting set of active markets. That is, a worker of any type other than $\widetilde{r}(q')$ will be strictly worse off if he deviates to the active market (q', s'). Put it differently, the workers of $\widetilde{r}(q')$ accept a lower matching probability in the market (q', s') than anybody else.

The second property then implies that only the workers of $\widetilde{r}(q')$ will accept the lowest matching probability for any contract (q', s). This is because for a given asset, the share and the matching probability are perfect substitutes to the workers. The competition among workers of $\widetilde{r}(q')$ alone will result in an adjustment in their matching probability fully offsetting the variation in the posted share. A deviating asset owner of $q' > \underline{\widetilde{q}}$, if gets matched, always pair up with the same type of workers in equilibrium. She only trades off between her matching probability and her output share. This is illustrated in the right panel of Figure 3.2. Suppose we fix the pair of types $(\widetilde{r}(q'), q')$ and plot the indifference curves over $\eta(\lambda)$ and $(1-s)$. Taking the workers' equilibrium payoff as given, the tangent point of the indifferent curves of both sides is the optimal contract, and the associated queue length for the asset owner. This is exactly the Hosios condition. In equilibrium, owners of the same asset quality q, if participating, will post the same share

$$s = 1 - \left.\frac{d\ln\delta}{d\ln\lambda}\right|_{\lambda=\widetilde{\lambda}(q)},$$

where $\widetilde{\lambda}(q)$ is the resulting queue length. Hence, $\widetilde{r}$ and $\widetilde{\lambda}$ must satisfy the law of motion in (3.7).

The Hosios condition can be rearranged as

$$\delta'(\widetilde{\lambda}(q)) = \eta(\widetilde{\lambda}(q))(1-s'),$$

and thus,

$$V(\widetilde{r}(q)) = \delta'(\widetilde{\lambda}(q))y(\widetilde{r}(q), q).$$

This is the same condition in (3.8), which I will refer to as the Hosios condition as well. The violation of the Hosios condition is the reason why $\widehat{s}(q)$ in Section 3.4 cannot be supported by an equilibrium.

The incentive compatibility (IC) condition for the workers above the threshold type can also be rewritten as

$$V(\widetilde{r}(q)) = \max_{q' \in [\underline{\widetilde{q}}, 1]} \delta'(\widetilde{\lambda}(q'))y(\widetilde{r}(q), q'). \tag{3.14}$$

After accounting for the contract posting decisions, sorting of workers is induced by the variation in the queue length. The better the asset, the greater the queue length in the active market. Under Assumption (M), the output shares posted by the asset owners increase with their asset quality.

Apply the envelope theorem to (3.14), we obtain

$$\frac{\partial V(\widetilde{r}(q))}{\partial p} = \delta'(\widetilde{\lambda}(q))\frac{\partial y(\widetilde{r}(q), q)}{\partial p}. \tag{3.15}$$

Under the strict SCP over q and $\eta(1-s)$, the conditions (3.14) and (3.15) are in fact equivalent. Abusing the terminology, I will call the latter as the workers' IC condition.

The strict SCP also implies that the active market for asset quality $\underline{\widetilde{q}}$ is the most profitable deviation for the worker of type below $\underline{\widetilde{p}}$. Therefore, the workers of the threshold type $\underline{\widetilde{p}}$ must be indifferent about participation. This yields the boundary condition for the workers side, $\underline{\widetilde{p}}(V(\underline{\widetilde{p}}) - \underline{V}) = 0$.

The boundary condition for the asset side is more complicated as we have to find out the deviating payoff for the owners of asset quality below $\underline{\widetilde{q}}$. Suppose an owner of asset quality $q' < \underline{\widetilde{q}}$ post a deviating offer s'. We again look at the lower envelope of the workers' indifference curves over q

and $\eta(1-s)$ which yield their equilibrium payoff, including those below $\underline{\widetilde{p}}$, in Figure 3.3. The indifference curve for the workers attracted must be tangent to the lower envelope, which pins down $\eta(\Lambda(q', s'))(1-s')$.

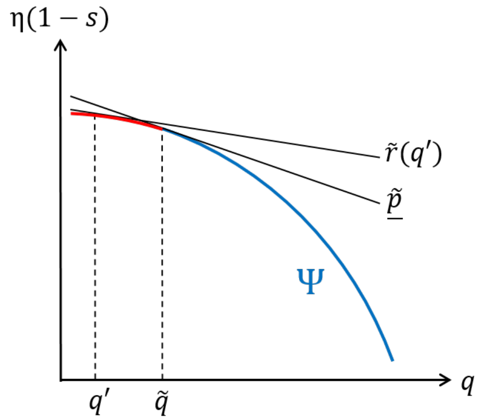

Figure 3.3: Deviations by assets below threshold type

Under the strict SCP, the workers of the threshold type $\underline{\widetilde{p}}$ is the highest type a deviating offer may attract. So the asset owner can never make more than her peers of the threshold type $\underline{\widetilde{q}}$. On the other hand, posting the same output share provides an asset owner slightly below $\underline{\widetilde{q}}$ a deviating payoff close to the equilibrium payoff of the threshold type $\underline{\widetilde{q}}$. The latter is given by $U(\underline{\widetilde{q}}) = [\delta(\widetilde{\lambda}(\underline{\widetilde{q}})) - \delta'(\widetilde{\lambda}(\underline{\widetilde{q}})\widetilde{\lambda}(\underline{\widetilde{q}})]y(\underline{\widetilde{p}}, \underline{\widetilde{q}})$ under the Hosios condition. So the boundary condition for the asset side, $\underline{\widetilde{q}}(U(\underline{\widetilde{q}}) - \underline{U}) = 0$, mirrors that for the workers. Notice that the boundary conditions at the bottom are the same as those for the Second Best allocation in (3.11).

The set of equilibrium conditions (3.6)-(3.8),(3.10), (3.11), and(3.15) defines a boundary value problem, for which the set of equilibria can be recovered from the solutions. In the appendix, I will analyze this boundary value problem. I establish the existence and the uniqueness of its solution, and hence the equilibrium. So Proposition 7 fully characterize the set of equilibria.

Proposition 7. *There exists a unique equilibrium. This equilibrium supports PAM and has the following properties:*

1. *Asset owners (workers) participate if and only if* $q \geq \underline{\widetilde{q}}$ $(p \geq \underline{\widetilde{p}})$, *and*

2. *Workers of* $p \geq \underline{\widetilde{p}}$ *have equilibrium payoffs* $\widetilde{v}(p)$, *and*

3. *The set of active markets* Ψ *is given by*

$$\left\{(q, s) : q \in [\underline{\widetilde{q}}, 1], s = 1 - \tfrac{d \ln \delta}{d \ln \lambda}\big|_{\lambda = \widetilde{\lambda}(q)} \right\},$$

   with $\Lambda(q, s) = \widetilde{\lambda}(q)$ *for* $(q, s) \in \Psi$, *and*

4. *For any* $s \in (0, 1)$, $R(q, s)$ *is degenerate at* $\widetilde{r}(q)$ *if* $q \geq \underline{\widetilde{q}}$ *and* $\Lambda(q, s) > 0$.

$\widetilde{r} : [\underline{\widetilde{q}}, 1] \to [\underline{\widetilde{p}}, 1]$, $\widetilde{\lambda} : [\underline{\widetilde{q}}, 1] \to \mathbb{R}_{++}$, *and* $\widetilde{v} : [\underline{\widetilde{p}}, 1] \to \mathbb{R}_{++}$ *are all continuously differentiable and strictly increasing. Together with the pair of threshold types* $(\underline{\widetilde{p}}, \underline{\widetilde{q}})$, *they satisfy the conditions*

$$\begin{cases} \widetilde{r}(\underline{\widetilde{q}}) = \underline{\widetilde{p}}, \widetilde{r}(1) = 1, \\ 0 = \underline{\widetilde{q}}[(\delta(\widetilde{\lambda}(\underline{\widetilde{q}})) - \delta'(\widetilde{\lambda}(\underline{\widetilde{q}}))\widetilde{\lambda}(\underline{\widetilde{q}}))y(\underline{\widetilde{p}}, \underline{\widetilde{q}}) - \underline{U}], \\ 0 = \underline{\widetilde{p}}(\widetilde{v}(\underline{\widetilde{p}}) - \underline{V}), \\ \widetilde{r}'(q) = \dfrac{g(q)}{f(r(q))}\widetilde{\lambda}(q), \\ \widetilde{v}(\widetilde{r}(q)) = \delta'(\widetilde{\lambda}(q))y(\widetilde{r}(q), q), \\ \dfrac{\partial \widetilde{v}(p)}{\partial p}\bigg|_{p=\widetilde{r}(q)} = \delta'(\widetilde{\lambda}(q)) \; \dfrac{\partial y(p, q)}{\partial p}\bigg|_{(p,q)=(\widetilde{r}(q),q)} . \end{cases} \tag{3.16}$$

3.6 Matching Efficiency

As explained in the introduction, the offering of output sharing contracts inevitably leads to an inefficient allocation. By comparing the set of conditions (3.16) in Proposition 7 with the set of conditions (3.6)-(3.11) for the Second Best allocation, I establish the form of distortion in equilibrium for any distribution of types.

Proposition 8. *In comparison with the Second Best allocation, the equilibrium allocation has the following features:*

1. *The queue length for the best assets is greater,* $\lambda_{SB}(1) < \widetilde{\lambda}(1)$*; and*

2. *All participating asset owners pair up with worse partners,* $r_{SB}(q) > \widetilde{r}(q)$ *for* $q \in (\underline{\widetilde{q}}, 1)$*; and*

3. *Higher participation on the workers' side,* $\underline{p}_{SB} \geq \underline{\widetilde{p}}$*, but lower participation on the asset side,* $\underline{\widetilde{q}} \geq \underline{q}_{SB}$*; and*

4. *The threshold type on one side remains unchanged only if that side features full participation, i.e., If* $\underline{p}_{SB} = \underline{\widetilde{p}}(\underline{\widetilde{q}} = \underline{q}_{SB})$*, then* $\underline{p}_{SB} = \underline{\widetilde{p}} = 0(\underline{\widetilde{q}} = \underline{q}_{SB} = 0)$*.*

Corollary 4. *In comparison with price competition,*

1. *the best workers are strictly worse off, whereas the lowest types of the participating workers of the threshold type,* $p \in (\underline{\widetilde{p}}, \underline{p}_{SB}]$ *strictly benefit; and*

2. *the owners of the highest quality assets are strictly better off, whereas those of the threshold asset quality* $q \in (\underline{q}_{SB}, \underline{\widetilde{q}}]$ *must be strictly worse off.*

To better understand the form of distortion, it is instructive to start with an equilibrium in price competition. Suppose we replace the posted prices with output shares $s_{SB}(q) = 1 - \frac{d\ln\delta}{d\ln\lambda}\big|_{\lambda=\lambda_{SB}(q)}$, keeping the same division of the matching surplus for every matched pair. This set of contracts is not incentive compatible for the workers. In comparison with a fixed price, a fixed share of output costs more to the better workers but less to the low types. The workers will have a higher deviating payoff from the active markets for better assets. In particular, the workers above the threshold type $\underline{p}_{SB}$ can always profit from searching for slightly better assets.

This must result in a longer queue of workers for the best assets.[75] In response, their owners will post a greater share to partially offset the increase in the queue length. These asset owners decide to retain a longer queue than in the Second Best allocation because their private value of matching probability increases with their share of the surplus.[76] On the other side, the best workers will suffer from the reductions in their share of surplus as well as their matching probability.

Under assortative matching, the pool of workers available to the lower quality assets must deteriorate. The asset owners in the intermediate range face two counteracting forces. First, a sharing contract costs less to weaker workers, intensifying the local competition among workers. Given the same type of workers, a less generous term is required to maintain sorting. With a greater share of the surplus, the asset owners gain by improving their

[75] $w_{SB}(q)$ is monotonic in q but $s_{SB}(q)$ needs not be. So we can only deduce a higher queue length for the best assets.

[76] In a setting with a large but finite number of agents, the average type of workers will decrease with the queue length. The owners of the best assets would still decide to retain a longer queue in equilibrium. Otherwise, a single asset owner will deviate to post a more generous term, drawing workers from other owners of the best assets.

matching probability. On the other hand, the asset owners are left with weaker workers amid the increase in the queue length for the better assets. As they see a lower gain from a match, they have incentive to induce a shorter queue of workers instead.

The relative strengths of the two forces depend on the distribution of types. Surprisingly, it turns out that all but the best assets must settle with weaker partners, regardless of the distribution of types. By continuity, this must happen to the assets of second highest quality. Suppose, to the contrary, that we move down from the top and find the owners of asset $\widehat{q} > \underline{\widetilde{q}}$ pairing with the same type $\widehat{p} = \widetilde{r}(\widehat{q}) = r_{SB}(\widehat{q})$ as in the Second Best allocation. Since the workers slightly better than $\widehat{p}$ now match with better assets, the queue length for the assets $\widehat{q}$ must be below than in the Second Best allocation, $\lambda_{SB}(\widehat{q}) \geq \widetilde{\lambda}(\widehat{q})$, under assortative matching. Figure 3.4 depicts the situation.

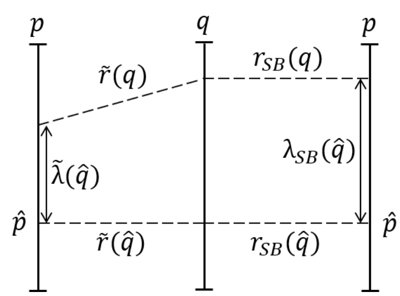

Figure 3.4: Law of motion in a thought experiment

Now consider the thought experiment of removing all workers above $\widehat{p}$ and assets above $\widehat{q}$. The Utilitarian planner still finds the original Second Best allocation optimal for this truncated distribution of types. Otherwise, she would have improved upon it. The same set of equilibrium conditions in Proposition 7 still applies to the truncated distribution of types. The underlying reason is that the set of contracts posted is to deter the agents below the threshold types from participating and the participating workers from deviating to match with slightly better assets. As $\widehat{p}$ and $\widehat{q}$ are now the highest types, we have argued that the owners of asset quality $\widehat{q}$ will induce an inefficiently long queue of workers, $\lambda_{SB}(\widehat{q}) < \widetilde{\lambda}(\widehat{q})$. This contradicts our previous claim!

To understand the distortion in the threshold types, let us return to our preceding discussion on the hypothetical set of contracts $s_{SB}(q)$. Suppose $\underline{p}_{SB} > 0$, the workers of the threshold type $\underline{p}_{SB}$ are indifferent about their outside option and entering the market $(s_{SB}(\underline{q}_{SB}), \underline{q}_{SB})$. Those with type below $\underline{p}_{SB}$ now pay less under the sharing contract $s_{SB}(\underline{q}_{SB})$. Some of them will be induced to participate. So the participation on the workers side must increase in equilibrium if $\underline{p}_{SB} > 0$.

The result on the matching pattern only states that the workers with type $\underline{p}_{SB}$ pair up with better assets in equilibrium, $\widetilde{\kappa}(\underline{p}_{SB}) \geq \kappa_{SB}(\underline{p}_{SB}) = \underline{q}_{SB}$. This raises the question on whether the workers with type $\underline{\widetilde{p}}$ may turn out matching with assets below $\underline{q}_{SB}$. The answer is negative because for the second best allocation, the Utilitarian planner would keep assigning agents into participation until the expected surplus for the last pair of types declines to zero. Suppose, to the contrary, that $\underline{p}_{SB} > \underline{\widetilde{p}}$ and $\underline{q}_{SB} > \underline{\widetilde{q}}$,

$$\begin{aligned} 0 &= (u_{SB}(\underline{q}_{SB}) - \underline{U}) + \lambda_{SB}(\underline{q}_{SB})(v_{SB}(\underline{p}_{SB}) - \underline{V}) \\ &= \max_{\lambda \geq 0}[\delta(\lambda) y(\underline{p}_{SB}, \underline{q}_{SB}) - \lambda \underline{V} - \underline{U}] > \max_{\lambda \geq 0}[\delta(\lambda) y(\underline{\widetilde{p}}, \underline{\widetilde{q}}) - \lambda \underline{V} - \underline{U}]! \end{aligned} \tag{3.17}$$

This is impossible as one side of the threshold types $(\underline{\widetilde{p}}, \underline{\widetilde{q}})$ will be better off taking outside option. Therefore, we conclude that the participation on the asset side can only be inefficiently low.[77]

[77] Since $\widetilde{\kappa}(p) > \kappa_{SB}(p)$ for $p \in (\underline{p}_{SB}, 1)$, the case $\underline{p}_{SB} = \underline{\widetilde{p}}$ and $\underline{q}_{SB} > \underline{\widetilde{q}}$ is impossible.

It remains to argue that some asset owner must be discouraged from participating if $\underline{q}_{SB} > 0$. Suppose, to the contrary, that $\underline{\widetilde{q}} = \underline{q}_{SB} > 0$, and hence $\widetilde{u}(\underline{\widetilde{q}}) = u_{SB}(\underline{\widetilde{q}}) = \underline{U}$. Again, the Utilitarian planner would assign some assets to their outside options only if the pair of threshold types yields zero expected surplus or all the workers are exhausted, $\underline{p}_{SB} = 0$. Recall that $\underline{p}_{SB} > \underline{\widetilde{p}}$ if $\underline{p}_{SB} > 0$. So the former case is impossible as the inequality (3.17) applies again. For the remaining possibility $\underline{p}_{SB} = \underline{\widetilde{p}} = 0$, the queue length for the pair of threshold types must also stay the same under the Hosios condition. Since the workers near the lowest type pay less under the sharing contract, the asset owners slightly above the threshold quality face a longer queue than in price competition. This is exactly opposite to the case in Figure 3.4. These asset owners must pair up with better workers than in the Second Best allocation, contracting our previous claim! With the distortion of the threshold types, $\widetilde{v}(\underline{p}_{SB}) > v_{SB}(\underline{p}_{SB})$ and $\widetilde{u}(\underline{\widetilde{q}}) < u_{SB}(\underline{\widetilde{q}})$ follow from the boundary conditions and the Hosios condition.

Notice that the above arguments only hinge on the property that the better workers always pay more under the sharing contracts.[78] Nonetheless, we are still able to draw conclusions on how the matching pattern and the participation margin are distorted.

Again, two counteracting forces affect the distortion of the queue length at the bottom. On one hand, the owners of assets slightly above $\underline{\widetilde{q}}$ now benefit less from a higher matching probability as they face weaker partners. On the other hand, their cost of increasing their matching probability may also fall. This happens when $\widetilde{v}(\underline{p}_{SB}) > v_{SB}(\underline{p}_{SB}) > \widetilde{v}(\underline{\widetilde{p}})$. Therefore, the queue length at the lower end can be distorted in either direction. This is

[78]Formally, the expected payment received by the asset owner strictly increases with the worker's type for any contracts and asset quality.

illustrated in the following example.

A symmetric example Suppose types on the two sides are both uniformly distributed over $[0, 1]$. The outside options for the two sides yield the same payoff $\underline{V} = \underline{U}$. Consider a market with O-ring production $y(p, q) = (\overline{y} - \underline{y})pq + \underline{y}$ and random matching technology $\delta(\lambda) = \frac{\lambda}{\lambda+1}$. The Second Best allocation inherits the symmetry between both sides in the setup. It is given by $r_{SB}(q) = q$, $\lambda_{SB}(q) = 1$, $v_{SB}(p) = \frac{1}{4}y(p, p)$ and $u_{SB}(q) = \frac{1}{4}y(q, q)$. $\underline{p}_{SB} = \underline{q}_{SB}$ satisfy $\frac{1}{4}[(\overline{y} - \underline{y})\underline{q}_{SB}^2 + \underline{y}] = \underline{U}$ if $\frac{1}{4}\underline{y} \leq \underline{U}$. Otherwise, $\underline{p}_{SB} = \underline{q}_{SB} = 0$.

We first consider the case $\frac{1}{4}\underline{y} \leq \underline{U}$. The boundary conditions at the bottom immediately imply that $\underline{\widetilde{pq}} > 0$ and

$$\left(\frac{1}{\widetilde{\lambda}(\underline{\widetilde{q}})+1}\right)^2 [(\overline{y} - \underline{y})\underline{\widetilde{pq}} + \underline{y}] = \underline{V} = \underline{U} = \left(\frac{\widetilde{\lambda}(\underline{\widetilde{q}})}{\widetilde{\lambda}(\underline{\widetilde{q}})+1}\right)^2 [(\overline{y} - \underline{y})\underline{\widetilde{pq}} + \underline{y}].$$

It follows that $\widetilde{\lambda}(\underline{\widetilde{q}}) = 1$ and $\underline{\widetilde{pq}} = \underline{p}_{SB}\underline{q}_{SB}$. Applying the characterization in Proposition 7 and 8, we conclude that $\underline{\widetilde{p}} < \underline{p}_{SB} = \underline{q}_{SB} < \underline{\widetilde{q}}$ and the queue length in almost every active market is inefficiently high, $\widetilde{\lambda}(q) > 1$.

For the case $\frac{1}{4}\underline{y} > \underline{U}$, Proposition 8 states that $0 = \underline{\widetilde{p}} = \underline{p}_{SB} = \underline{q}_{SB} < \underline{\widetilde{q}}$. The boundary condition for $\underline{\widetilde{q}} > 0$ implies that $\widetilde{\lambda}(\underline{\widetilde{q}}) < 1$. Hence, there is a threshold asset quality $\widehat{q}$ such that all active markets for $q > \widehat{q}$ feature an inefficiently high queue length and the opposite occurs to the active markets for $q < \widehat{q}$.

3.6.1 Discussion

The recipe for inefficiency here contains three ingredients: output sharing contracts, private types and search friction.

Eeckhout and Kircher (2010) show that Second Best allocations can always be decentralized in price competition. If the workers' types are contractible, a menu of output shares may function as a posted price as its term can be made contingent on types to implement the required transfer. So the Second Best allocation will be decentralized.[79]

Now consider an environment where the parties face no search friction. In the First Best allocation, the matching is perfectly assortative with a unit queue length for every matched pair. The First Best allocation always prevails if the asset side may post prices. When we replace the posted prices with output shares keeping the same division of the matching surplus, the workers will again deviate to better assets, resulting in a longer queue for the best assets. Without search friction, the owners of the best assets will increase their posted share until the queue length restores to unity. The asset owners still have a greater share of surplus upon matching, and hence a higher marginal value of matching probability. However, they cannot improve their matching probability by distorting the queue length. Their decisions in turn leave the same pool of workers to the asset owners of the second highest quality. Inductively, the equilibrium allocation remains First Best amid higher equilibrium payoff for all participating asset owners.

3.7 Concluding Remarks

This chapter studies how the output sharing arrangement affects sorting efficiency in a directed search framework where one side has private types. I

[79]Under condition 3.12 in Remark 8, an asset owner cannot gain from posting a menu of output shares specifying different expected payment for different types. This is because the meeting is bilateral in the setting here.

consider a stylized setting to disentangle the source of inefficiency from the well-known channels. I characterize the unique equilibrium which features inefficient PAM. I provide qualitative features of the distortion in equilibrium. These features applies to any distribution of types as the underlying forces are always present. In particular, the unique equilibrium features full separation of types and the Hosios condition is met in every active market. I then provide qualitative features of the distortion in the matching pattern and the participation thresholds.

For other forms of securities or contingent contract, the preference over the matching probability and the term of the contract differ across workers. In equilibrium, the matched pairs will not fully separate into a continuum of markets where the Hosios condition is satisfied. This is because in such case, the asset owners will distort the queue length to screen out better workers. [80] As a result, the channel of sorting inefficiency here will confound with the distortions associated with the screening by the asset owners as well as the search externalities such as thick market effect, congestion effect and compositional effect. An avenue for future research is to study such interactions and the resulting form of distortion. The results in this chapter will then serve as a useful benchmark.

[80] Under Hosios condition, an incremental distortion of the queue length leads to a second-order loss while an improvement in the worker's type yields a first-order gain.

Chapter 4

Conclusion

In two-sided one-to-one matching, the equilibrium matching pattern and the divisions of surpluses vary with the distribution of types. A central question is to identify "distribution-free" qualitative features and their relation to the model primitives such as the production technology and the market structure. These results yield testable implications and policy recommendation, which are robust to misspecification of the distribution of types. The arguments underlying the results illuminate general economic forces.

A recurrent focus in the literature has been the conditions for assortative matching, or lack thereof. This dissertation introduces private information and contingent payment, which are present in a number of circumstances. The contribution is twofold.

First, I provide economically meaningful conditions ensuring decentralization of efficient matching in a frictionless environment. The analysis extends our understanding of the potential forces against PAM from fixed prices to more general forms of contingent contracts. The policy implication here is to restrict the flattest contracts available so that Condition Global ID is met. Roughly speaking, assortative matching occurs whenever the variation in the contingent contracts aligns with the form of production complementarity. Such consideration has been absent from the literature of contract theory and in particular, security design.

Second, I provide two novel comparative statics. I show how the form of financial securities available affects the divisions of matching surpluses in a frictionless environment. As discussed, this comparative static not only has redistributive implications but also provides testable implications of information asymmetry in such markets. I then consider a frictional market in which the asset side posts output shares. I show that the inefficiency caused by the output sharing contracts manifests as monotonic changes in the matching pattern and the participation decision. These two comparative statics are qualitatively different from the existing results in the literature. The forces underlying them may open avenues for future research.

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Appendix A

Appendix for chapter 2

A.1 Proof of Lemma 1

I now formally define an equilibrium when DRM is feasible. It is straightforward to adapt the equilibrium definition for the setting that an owner of asset quality q_k may only post a single contract (q, π, t) with $q \leq q_k$.

Definition. *A competitive matching equilibrium using direct revelation mechanisms consists of the asset owners' equilibrium payoff* $U : \{q_k\}_{k=1}^K \to \mathbb{R}_+$, *workers' equilibrium payoff* $V : \{p_l\}_{l=1}^L \to \mathbb{R}_+$, *asset owners' contract posting set* $\psi^T : \{q_k\}_{k=1}^K \to \cup_{k=1}^K \Omega_T^{DRM}(q_k) \cup \{p_0\}$, *the set of active markets* $\Psi^T \subseteq \cup_{k=1}^K \Omega_T^{DRM}(q_k) \times \{q_k\}_{k=1}^K$, *the measure of participating workers* $W^T : \prod_{l=1}^L \{\mathcal{B}([\underline{q}, \overline{q}] \times [0,1]) \times \mathcal{T}\} \times \mathcal{P}(\{q_k\}_{k=1}^K) \to [0,1]$, *the distribution of workers* $r^T : \cup_{k=1}^K \Omega_T^{DRM}(q_k) \times \{q_k\}_{k=1}^K \to \Delta(\{p_l\}_{l=1}^L)$ *and market tightness* $\mu^T : \cup_{k=1}^K \Omega_T^{DRM}(q_k) \times \{q_k\}_{k=1}^K \to [0, \infty]$ *such that*

1. *Asset Owners' Optimal Contract Posting:*

   i)For all $q \in \{q_k\}_{k=1}^K$ *and* $T \in \Omega_T^{DRM}(q)$,

$$U(q) \geq \frac{\eta(\mu^T(T,q))}{\mu^T(T,q)} \sum_{l=1}^L r^T(p_l|T,q)(1-\pi^l)u(q^l, p_l, t^l)$$

   with equality if $T \in \psi^T(q)$.

   ii)$\mu^T(T,q) = 0$ *and* $T \notin \psi^T(q)$ *if* $T \notin \Omega_T^{DRM}(q)$.

2. *Workers' Optimal Acceptance:*

   i)For all $(T,q) \in \cup_{k=1}^{K} \Omega_T^{DRM}(q_k) \times \{q_k\}_{k=1}^{K}$,

$$V(p_l) \geq \eta(\mu^T(T,q))(1-\pi^l)v(p_l,q^l,t^l) \tag{A.1}$$

   with equality if $r^T(p_l|T,q) > 0$ *and* $\mu^T(T,q) < \infty$.

   ii)$\mu^T(T,q) = \infty$ *if* $V(p_l) > (1-\pi^l)v(p_l,q^l,t^l)$ *for all* $p \in \{p_l\}_{l=1}^{L}$.

3. *Active Markets:*

   $\Psi^T := \{(T,q) \in \cup_{k=1}^{K} \Omega_T^{DRM}(q_k) \times \{q_k\}_{k=1}^{K} : T \in \psi^T(q)\}$ *is the support of* W^T.

4. *Optimal Participation:*

   i)$U(q) \geq \underline{U}$ *and* $V(p_l) \geq \underline{V}$.

   ii)$\int_{\Omega_T^{DRM} \times \{q_k\}_{k=1}^{K}} r^T(p_l|T,q)dW^T \leq P(p_l)$ *with equality if* $V(p_l) > \underline{V}$.

   iii)$\int_{\Omega_T^{DRM}} \mu^T(T,q_k)dW^T \leq Q(q_k)$ *with equality if* $U(q_k) > \underline{U}$.

Notice that the inequality (A.1) in the workers' optimal acceptance decision holds for both active and inactive markets. Therefore, the worker's optimal acceptance condition captures the belief restriction in section 2.4.4.

Suppose that $\{U,V,\psi,\Psi,W,r,\mu\}$ *is a competitive matching equilibrium. Then there exists an equilibrium using direct revelation mechanisms* $\{U,V,\psi^T,\Psi^T,W^T,r^T,\mu^T\}$ *supporting the same equilibrium payoffs* $\{U,V\}$ *and same allocation,*

$$\begin{aligned}
\int_{\Omega_T^{DRM}(q_k)} r^T(p_l|T,q_k)dW^T &= \int_{\Omega_t} r(p_l|t,q_k)dW \\
\int_{\Omega_T^{DRM}(q_k)} r^T(p_l|T,q_k)\mu^T(t,q_k)dW^T &= \int_{\Omega_t} r(p_l|t,q_k)\mu(t,q_k)dW
\end{aligned}$$

for all p_l *and* q_k. *Furthermore, all active markets* Ψ^T *involve only degenerate direct revelation mechanisms.*

Proof. Fix an equilibrium $\{U, V, \psi, \Psi, W, r, \mu\}$, I now construct a corresponding equilibrium $\{U, V, \psi^T, \Psi^T, W^T, r^T, \mu^T\}$ involving only degenerate DRM.

The participation decisions are as follows: $\psi^T(q_k) = \{T = \{q', \pi', t'\}_{l=1}^L : (q', \pi', t') \in \psi(q_k)\}$ if $\{p_0\} \notin \psi(q_k)$. Otherwise, $\psi^T(q_k) = \{T = \{q', \pi', t'\}_{l=1}^L : (q', \pi', t') \in \psi(q_k)\} \cup \{p_0\}$. As required in the equilibrium definition, $\Psi^T = \{(T, q) \in \Omega_T^{DRM} \times \{q_k\}_{k=1}^K : T \in \psi^T(q)\}.W^T$ is defined as follows. For every set A,

$$W^T(A) = W(\{(q', \pi', t') : \{q', \pi', t'\}_{l=1}^L \in A \cap \Psi^T\}).$$

Consider $T' \in \psi^T(q_k)$. By construction, T' takes the form of $\{q', \pi', t'\}_{l=1}^L$, where $(q', \pi', t') \in \psi(q_k)$. Define $r^T(T', q_k) = r((q', \pi', t'), q_k)$ and $\mu^T(T', q_k) = \mu((q', \pi', t'), q_k)$. $\{U, V, \psi, \Psi, W, r, \mu\}$ and $\{U, V, \psi^T, \Psi^T, W^T, r^T, \mu^T\}$ support the same allocation by construction.

For infeasible menus $T' \notin \Omega_T^{DRM}(q_k)$, then $\mu^T(T', q) = 0$ and $r^T(p_1|T', q_k) = 1$. Now consider an inactive market $T \in \Omega_T^{DRM}(q_k) \backslash \psi^T(q_k)$. Define the market tightness by

$$\mu^T(T, q_k) = \sup\{\mu' \in [0, \infty] : V(p_l) \geq \eta(\mu')(1 - \pi^l)v(p_l, q^l, t^l), L \geq l \geq 1\}.$$

Notice that $\mu^T(T, q_k) = \infty$ if $V(p_l) \geq (1 - \pi^l)v(p_l, q^l, t^l)$ for all type p_l. Otherwise, $\mu^T(T, q_k) < 1$ and $r^T(T, q_k)$ is defined to be degenerate at

$$\min\{p \in \{p_l\}_{l=1}^L : V(p_l) = \mu^T(T, q_k)(1 - \pi^l)v(p_l, q^l, t^l)\}.$$

The construction of Ψ^T and the equilibrium conditions for $\{U, V, \psi, \Psi, W, r, \mu\}$ immediately imply that deviations to active markets are never profitable. It is sufficient to check if an asset owner of q_k cannot profit from posting $\widetilde{T} \in \Omega_T^{DRM}(q_k) \backslash \psi^T(q_k)$. Consider $\widetilde{T} = \{\widetilde{q}^l, \widetilde{\pi}^l, \widetilde{t}^l\}_{l=1}^L$ with $\mu^T(\widetilde{T}, q_k) < \infty$. By construction, $\mu^T(\widetilde{T}, q_k) < 1$ and $r^T(\widetilde{T}, q_k)$ must be degenerate at some $p_{\widehat{l}}$.

For any type p_l,

$$
\begin{aligned}
V(p_l) &\geq \mu^T(\widetilde{T}, q_k)(1-\widetilde{\pi}^l)v(p_l, \widetilde{q}^l, \widetilde{t}^l) \\
&\geq \mu^T(\widetilde{T}, q_k)(1-\widetilde{\pi}^{\widehat{l}})v(p_l, \widetilde{q}^{\widehat{l}}, \widetilde{t}^{\widehat{l}}).
\end{aligned}
$$

The first inequality is merely the definition of $\mu^T(\widetilde{T}, q_k)$ and the second inequality is the IC condition for DRM. As they hold with equalities for $p_l = p_{\widehat{l}}$, it follows that $\mu((\widetilde{\pi}^{\widehat{l}}, \widetilde{q}^{\widehat{l}}, \widetilde{t}^{\widehat{l}}), q_k) = \mu^T(\widetilde{T}, q_k) < 1$. Any $p_{l'}$ in the support of $r((\widetilde{\pi}^{\widehat{l}}, \widetilde{q}^{\widehat{l}}, \widetilde{t}^{\widehat{l}}), q_k)$ must satisfy

$$
V(p_{l'}) = \mu^T(\widetilde{T}, q_k)(1-\widetilde{\pi}^{\widehat{l}})v(p_{l'}, \widetilde{q}^{\widehat{l}}, \widetilde{t}^{\widehat{l}})
$$

The previous inequality immediately implies that $V(p_{l'}) = \mu^T(\widetilde{T}, q_k)(1 - \widetilde{\pi}^{l'})v(p_{l'}, \widetilde{q}^{l'}, \widetilde{t}^{l'})$ and hence, $p_{\widehat{l}} \leq p_{l'}$ by construction. From the optimal contract posting condition for $\{U, V, \psi, \Psi, W, r, \mu\}$,

$$
\begin{aligned}
U(q_k) &\geq (1-\widetilde{\pi}^{\widehat{l}})u(\widetilde{q}^{\widehat{l}}, r((\widetilde{\pi}^{\widehat{l}}, \widetilde{q}^{\widehat{l}}, \widetilde{t}^{\widehat{l}}), q_k), \widetilde{t}^{\widehat{l}}) \\
&\geq (1-\widetilde{\pi}^{\widehat{l}})u(\widetilde{q}^{\widehat{l}}, p_{\widehat{l}}, \widetilde{t}^{\widehat{l}}).
\end{aligned}
$$

The last inequality is due to Remark (1). Hence, the asset owner has no profitable deviations. □

A.2 Proof of Proposition 1

I first define an equilibrium in the full information case. Each market is associated with a tightness ratio $\mu^{FI}(t, p, q)$. The distribution of workers is trivial, and therefore omitted.

Definition. *A full information competitive matching equilibrium consists of the asset owners' equilibrium payoff* $U^{FI} : \{q_k\}_{k=1}^K \to \mathbb{R}_+$, *workers'* *equilibrium payoff* $V^{FI} : \{p_l\}_{l=1}^{L} \to \mathbb{R}_+$, *asset owners' contract posting set* $\psi^{FI} : \{q_k\}_{k=1}^{K} \to \Omega_t \times \{p_l\}_{l=1}^{L} \cup \{p_0\}$, *the set of active markets* $\Psi^{FI} \subseteq \Omega_t \times \{p_l\}_{l=1}^{L} \times \{q_k\}_{k=1}^{K}$, *the measure of participating workers* $W^{FI} : \mathcal{T} \times \mathcal{P}(\{p_l\}_{l=1}^{L} \times \{q_k\}_{k=1}^{K}) \to [0,1]$, *and market tightness* $\mu^{FI} : \Omega_t \times \{p_l\}_{l=1}^{L} \times \{q_k\}_{k=1}^{K} \to [0,\infty]$ *such that*

1. *Asset Owners' Optimal Contract Posting:*

   For all $(t,p,q) \in \Omega_t \times \{p_l\}_{l=1}^{L} \times \{q_k\}_{k=1}^{K}$, $U^{FI}(q) \geq \frac{\eta(\mu^{FI}(t,p,q))}{\mu^{FI}(t,p,q)} u(q,p,t)$ *with equality if* $t \in \psi^{FI}(q)$.

2. *Workers' Optimal Acceptance:*

   For all $(t,p,q) \in \Omega_t \times \{p_l\}_{l=1}^{L} \times \{q_k\}_{k=1}^{K}$,

   $$V^{FI}(p) \geq \eta(\mu^{FI}(t,p,q))v(p,q,t) \tag{A.2}$$

   with equality if $V^{FI}(p) \leq v(p,q,t)$. *Otherwise,* $\mu^{FI}(t,p,q) = \infty$.

3. *Active Markets:*

   $\Psi^{FI} := \{(t,p,q) \in \Omega_t \times \{p_l\}_{l=1}^{L} \times \{q_k\}_{k=1}^{K} : (t,p,q) \in \psi^{FI}(q)\}$ *is the support of* W^{FI}.

4. *Optimal Participation:*

   i)$U^{FI}(q) \geq \underline{U}$ *and* $V^{FI}(p) \geq \underline{V}$.

   ii)$W^{FI}(\Omega_t \times \{p_l\} \times \{q_k\}_{k=1}^{K}) \leq P(p_l)$ *with equality if* $V^{FI}(p_l) > \underline{V}$.

   iii)$\int_{\Omega_t \times \{p_l\}_{l=1}^{L}} \mu^{FI}(t,p,q_k) dW^{FI} \leq Q(q_k)$ *with equality if* $U^{FI}(q_k) > \underline{U}$.

I am going to show that the set of First Best allocations coincides with the set of equilibrium allocations under full information. The proof here does not require Assumption (P).

Every First Best allocation can be supported by an equilibrium under full information.

Lemma 10. *Suppose* $(W_{pq}^{FB}, C_{pq}^{FB})$ *is a First Best allocation and* $(\Delta V^{FB}, \Delta U^{FB})$ *is a set of associated Lagrange multipliers for the First Best program, where* $\Delta V^{FB}(p)$ *and* $\Delta U^{FB}(q)$ *denote the shadow price of the corresponding resource constraint respectively. Define* $V^{FB}(p) := \underline{V} + \Delta V^{FB}(p)$ *and* $U^{FB}(q) := \underline{U} + \Delta U^{FB}(q)$*. Then* $W_{pq}^{FB} = C_{pq}^{FB}$ *and*

1. $V^{FB}(p) + U^{FB}(q) \geq E(Y|p,q)$*. Equality holds if* $W_{pq}^{FB}(p,q) > 0$.
2. $V^{FB}(p) \geq \underline{V}$*.* $\sum_{k=1}^{K} W_{pq}^{FB}(p, q_k) = P(p)$ *if* $V^{FB}(p) > \underline{V}$.
3. $U^{FB}(q) \geq \underline{U}$*.* $\sum_{l=1}^{L} C_{pq}^{FB}(p_l, q) = Q(q)$ *if* $U^{FB}(q) > \underline{U}$.

Proof. For any (W'_{pq}, C'_{pq}), define $\min\{W'_{pq}, C'_{pq}\} = W''$ where $W''(p,q) := \min\{W'_{pq}(p,q), C'_{pq}(p,q)\}$ for all (p,q). If $W'_{pq} \neq C'_{pq}$, then $TS(W'_{pq}, C'_{pq}) < TS(\min\{W'_{pq}, C'_{pq}\}, \min\{W'_{pq}, C'_{pq}\})$. Therefore $W_{pq}^{FB} = C_{pq}^{FB}$. Substitute this into the First Best program, the Kuhn-Tucker conditions for the recasted program give rise to the remaining conditions. □

Fix a First Best allocation and a set of shadow prices. For all pairs (p_l, q_k) with $W_{pq}^{FB}(p_l, q_k) > 0$, Lemma 10 states that $V^{FB}(p_l) \in [\underline{V}, E(Y|p_l, q_k) - \underline{U}]$. Hence, there must exist some contract $t(.; s') \in S_t$ s.t. $U^{FB}(q_k) = u(q_k, p_l, t(.; s'))$ and $V^{FB}(p_l) = v(p_l, q_k, t(.; s'))$. Denote s' by $s^{FB}(p_l, q_k)$.

I now construct a candidate equilibrium with $V^{FI} = V^{FB}$ and $U^{FI} = U^{FB}$. The participation decisions are defined as follows: If $\sum_{l=1}^{L} W_{pq}^{FB}(p_l, q_{k'}) = Q(q_{k'})$,

$$\psi^{FI}(q_{k'}) = \{(t, p_l) : W_{pq}^{FB}(p_l, q_{k'}) > 0, t = t(.; s^{FB}(p_l, q_{k'}))\}.$$

If $\sum_{l=1}^{L} W_{pq}^{FB}(p_l, q_{k'}) \in (0, Q(q_{k'}))$,

$$\psi^{FI}(q_{k'}) = \{(t, p_l) : W_{pq}^{FB}(p_l, q_{k'}) > 0, t = t(.; s^{FB}(p_l, q_{k'}))\} \cup \{p_0\}.$$

$\psi^{FI}(q_{k'}) = \{p_0\}$ if $\sum_{l=1}^{L} W_{pq}^{FB}(p_l, q_{k'}) = 0$.

$\Psi^{FI} = \{(t, p_l, q_k) : W_{pq}^{FB}(p_l, q_k) > 0, t = t(.; s^{FB}(p_l, q_k))\}$. W^{FI} is degenerate with Ψ^{FI} as support and $W^{FI}(t(.; s^{FB}(p_l, q_k)), p_l, q_k) = W_{pq}^{FB}(p_l, q_k)$. $\mu^{FI}(t, p_l, q_k) = 1$ for all $(t, p_l, q_k) \in \Psi^{FI}$. Workers' optimal acceptance condition pins down μ^{FI} for inactive markets. For any $(t, p_l, q_k) \notin \Psi^{FI}$, $\mu^{FI}(t, p_l, q_k)$

$$\mu^{FI}(t, p_l, q_k) = \sup\{\mu' \in [0, \infty] : V^{FI}(p_l) \geq \eta(\mu')v(p_l, q_k, t)\}.$$

It remains to verify that ψ^{FI} is the set of optimal contracts for the asset owners. All other equilibrium conditions are met by construction. Suppose that for some $(\widehat{t}, p_{\widehat{l}}, q_{\widehat{k}}) \notin \Psi^{FI}$, $\frac{\eta(\mu^{FI}(\widehat{t}, p_{\widehat{l}}, q_{\widehat{k}}))}{\mu^{FI}(\widehat{t}, p_{\widehat{l}}, q_{\widehat{k}})} u(q_{\widehat{k}}, p_{\widehat{l}}, \widehat{t}) > U^{FI}(q_{\widehat{k}})$. It follows that $u(q_{\widehat{k}}, p_{\widehat{l}}, \widehat{t}) > U^{FI}(q_{\widehat{k}})$ and $\mu^{FI}(\widehat{t}, p_{\widehat{l}}, q_{\widehat{k}}) > 0$ so that $v(p_{\widehat{l}}, q_{\widehat{k}}, \widehat{t}) \geq V^{FI}(p_{\widehat{l}})$. These two inequalities jointly implies $V^{FB}(p_{\widehat{l}}) + U^{FB}(q_{\widehat{k}}) < E(Y|p_{\widehat{l}}, q_{\widehat{k}})$, contradicting the efficiency of the First Best allocation!

Every equilibrium allocation is a First Best.

Lemma 11. *An allocation $(\widehat{W}_{pq}, \widehat{C}_{pq})$ satisfying resources constraints is a First Best if $\widehat{W}_{pq} = \widehat{C}_{pq}$ and there exists a pair of functions $\widehat{U} : \{q_k\}_{k=1}^{K} \to \mathbb{R}_+$ and $\widehat{V} : \{p_l\}_{k=1}^{K} \to \mathbb{R}_+$ such that*

1. *$\widehat{V}(p) + \widehat{U}(q) \geq E(Y|p, q)$ for all $(p, q) \in \{p_l\}_{l=1}^{L} \times \{q_k\}_{k=1}^{K}$, and equality holds if $\widehat{W}_{pq}(p, q) > 0$.*

2. *$\widehat{V}(p) \geq \underline{V}$. $\sum_{k=1}^{K} \widehat{W}_{pq}(p', q_k) = P(p')$ if $\widehat{V}(p') > \underline{V}$.*

3. *$\widehat{U}(q) \geq \underline{U}$. $\sum_{l=1}^{L} \widehat{C}_{pq}(p_l, q') = Q(q')$ if $\widehat{U}(q') > \underline{U}$.*

Proof. For any (W'_{pq}, C'_{pq}), define $\min\{W'_{pq}, C'_{pq}\} = W''$ where $W''(p,q) := \min\{W'_{pq}(p,q), C'_{pq}(p,q)\}$ for all (p,q). Consider any allocation (W_{pq}, C_{pq}) satisfying the resource constraints,

$$\begin{aligned}
& TS(W_{pq}, C_{pq}) \leq TS(\min\{W_{pq}, C_{pq}\}, \min\{W_{pq}, C_{pq}\}) \\
= \quad & \sum_{k=1}^{K} \sum_{l=1}^{L} [E(Y|p_l, q_k) - \underline{U} - \underline{V}] \min\{W_{pq}(p_l, q_k), C_{pq}(p_l, q_k)\} \\
& + \underline{V} \sum_{l=1}^{L} P(p_l) + \underline{U} \sum_{k=1}^{K} Q(q_k) \\
\leq \quad & \sum_{l=1}^{L} \left\{ \left[\widehat{V}(p_l) - \underline{V} \right] \sum_{k=1}^{K} \min\{W_{pq}(p_l, q_k), C_{pq}(p_l, q_k)\} + P(p_l)\underline{V} \right\} \\
& + \sum_{k=1}^{K} \left\{ \left[\widehat{U}(q_k) - \underline{U} \right] \sum_{l=1}^{L} \min\{W_{pq}(p_l, q_k), C_{pq}(p_l, q_k)\} + Q(q_k)\underline{U} \right\} \\
\leq \quad & \sum_{l=1}^{L} P(p_l)\widehat{V}(p) + \sum_{k=1}^{K} Q(q_k)\widehat{U}(q) = TS(\widehat{W}_{pq}, \widehat{C}_{pq}).
\end{aligned}$$

The first inequality stems from the specification of matching function. The second inequality is obtained by substituting $\widehat{V}(p) + \widehat{U}(q) \geq E(Y|p,q)$. Since (W_{pq}, C_{pq}) satisfies the resource constraints, the second and third condition for $\widehat{V}$ and $\widehat{U}$ in Lemma 11 leads to the third inequality. The last equality is due to the equality conditions for the first condition in Lemma 11. Hence, $(\widehat{W}_{pq}, \widehat{C}_{pq})$ is a First Best allocation. $\square$

Now consider an equilibrium under full information. First, every active market clears. Suppose not, $\mu^{FI}(t', p_{l'}, q_{k'}) < 1$ for some $(t', p_{l'}, q_{k'}) \in \Psi^{FI}$. Then $v(p_{l'}, q_{k'}, t') > V^{FI}(p_{l'}) \geq \underline{V}$ and by ex-post budget balance, $E(Y|p_{l'}, q_{k'})) - v(p_{l'}, q_{k'}, t') = U^{FI}(q_{k'}) \geq \underline{U}$. There exist contracts $t(.;s'') \in S_t$ yielding $v(p_{l'}, q_{k'}, t(.;s''))$ arbitrarily close to but below $v(p_{l'}, q_{k'}, t')$, so that $\mu^{FI}(t(.;s''), p_{l'}, q_{k'}) < 1$ and $u(q_{k'}, p_{l'}, t(.;s'')) > U^{FI}(q_{k'})$. an asset owner will gain from offering such contract. A symmetric argument rules out $\mu^{FI}(t', p_{l'}, q_{k'}) > 1$. This argument remains valid when $V^{FI}(p_{l'}) = \underline{V}$ or $U^{FI}(q_{k'}) = \underline{U}$. Second, $V^{FI}(p_l) + U^{FI}(q_k) \geq E(Y|p_l, q_k)$ for all pairs of

(p_l, q_k), and equality holds if $(t', p_l, q_k) \in \Psi^{FI}$ for some $t' \in \Omega_t$. The equality is a direct consequence of ex-post budget balance and $\mu^{FI}(t, p_l, q_k) = 1$ for all $(t, p_l, q_k) \in \Psi^{FI}$. Now suppose $V^{FI}(p_{l'}) + U^{FI}(q_{k'}) < E(Y|p_{l'}, q_{k'})$, then an asset owner with quality $q_{k'}$ will deviate to a market $(t(.; s'), p_{l'}, q_{k'}) \in S_t$, in which $v(p_{l'}, q_{k'}, t(.; s'))$ is slightly above $V^{FI}(p_{l'})$!

Since $\mu^{FI}(t, p_l, q_k) = 1$ for all $(t, p_l, q_k) \in \Psi^{FI}$, the equilibrium allocation is given by $C^{FI}_{pq}(p_l, q_k) = W^{FI}_{pq}(p_l, q_k = W^{FI}(\Omega_t \times (p_l, q_k))$. Optimal participation conditions imply that C^{FI}_{pq} and W^{FI}_{pq} satisfy the resources constraints, with equality in the case $U^{FI}(q_k) > \underline{U}$ and $V^{FI}(p_l) > \underline{V}$ respectively. The equilibrium conditions also imply $U^{FI}(q_k) \geq \underline{U}$, $V^{FI}(p_l) \geq \underline{V}$ and $V^{FI}(p_l) + U^{FI}(q_k) = E(Y|p_l, q_k)$ whenever $W^{FI}_{pq}(p_l, q_k) > 0$. In summary, $(W^{FI}_{pq}, C^{FI}_{pq})$ satisfies the conditions in Lemma 11 with $(U^{FI}, V^{FI}) = (\widehat{U}, \widehat{V})$.

A.3 Proof of Proposition 2

I am going to prove that if $C_t \subseteq \Omega_t$, the set of First Best allocations coincides with the set of equilibrium allocations in price competition. The proof again does not require Assumption (P). The structure of this proof closely follows that for Proposition 1.

Every First Best allocation can be supported by an equilibrium under full information.

Proof. I now construct a candidate equilibrium with $V = V^{FB}$ and $U = U^{FB}$, where V^{FB} and U^{FB} are defined in the proof of Proposition 1. Let $\psi(q_k) = \{p_0\}$ if $\sum_{l=1}^{L} W^{FB}_{pq}(p_l, q_k) = 0$. For any $q_{\widehat{k}}$ satisfying $\sum_{l=1}^{L} W^{FB}_{pq}(p_l, q_{\widehat{k}}) > 0$, Lemma 10 implies that $U^{FB}(q_{\widehat{k}}) \in [\underline{U}, \max_k \{E(Y|p_l, q_{\widehat{k}}) - \underline{V}\}]$. Hence, these exists $s_{\widehat{k}} \in [0, 1]$ s.t. $t_c(.; s_{\widehat{k}}) = U^{FB}(q_{\widehat{k}})$. Let $\psi(q_{\widehat{k}}) = \{t_c(.; s_{\widehat{k}})\}$ if

$\sum_{l=1}^{L} W_{pq}^{FB}(p_l, q_{\widehat{k}}) = Q(q_{\widehat{k}})$ and $\psi(q_{\widehat{k}}) = \{t_c(.; s_{\widehat{k}})\} \cup \{p_0\}$ if $\sum_{l=1}^{L} W_{pq}^{FB}(p_l, q_{\widehat{k}}) < Q(q_{\widehat{k}})$. Hence, $\Psi = \{(t_c(.; s_k), q_k) : \sum_{l=1}^{L} W_{pq}^{FB}(p_l, q_k) > 0\}$. W is degenerate with Ψ as support and $W(t_c(.; s_k), q_k) = \sum_{l=1}^{L} W_{pq}^{FB}(p_l, q_k)$. For an active market $(t_c(.; s_k), q_k) \in \Psi$, $\mu(t_c(.; s_k), q_k) = 1$ and $r(t_c(.; s_k), q_k) = r_q^{FB}(q_k)$.

Workers' optimal acceptance condition pins down μ and r for the inactive markets. For any $(t', q') \notin \Psi$,

$$\mu(t', q') = \sup\{\mu' \in [0, \infty] : V^{FB}(p) \geq \eta(\mu')v(p, q', t')\}.$$

If $\mu(t', q') < \infty$, then $r(t', q')$ is degenerate at

$$\min\{p \in \{p_l\}_{l=1}^{L} : V^{FB}(p) = \mu(t', q')v(p, q', t')\}.$$

If $\mu(t', q') = \infty$, $r(t', q')$ has no bearings and is assumed to be the same as prior.

I first show that $\psi(q_k)$ is optimal for the owners of asset quality q_k. Suppose that for some $(\widetilde{t}, \widetilde{q}) \notin \Psi$, $\frac{\eta(\mu(\widetilde{t}, \widetilde{q}))}{\mu(\widetilde{t}, \widetilde{q})} u(\widetilde{q}, r(\widetilde{t}, \widetilde{q}), \widetilde{t}) > U^{FB}(\widetilde{q})$. By construction, $\mu(\widetilde{t}, \widetilde{q}) < \infty$ and $r(\widetilde{t}, \widetilde{q})$ is degenerate at some type $\widetilde{p}$. It follows that $u(\widetilde{q}, \widetilde{p}, \widetilde{t}) > U^{FB}(\widetilde{q})$ and $v(\widetilde{p}, \widetilde{q}, \widetilde{t}) \geq V^{FB}(\widetilde{p})$. These two inequalities jointly implies $V^{FB}(\widetilde{p}) + U^{FB}(\widetilde{q}) < E(Y|\widetilde{p}, \widetilde{q})$, contradicting Lemma 10! Second, the workers cannot gain by deviating to other active markets. For any $p \in \{p_l\}_{l=1}^{L}$ and $(t_c(.; s_k), q_k) \in \Psi, V(p) \geq E(Y|p, q_k) - U^{FB}(q_k) = v(p, q_k, t_c(.; s_k)\})$. The inequality is due to Lemma 10 and the last equality makes use of the property of cash payment. It suffices to check for these two types of deviations. All other equilibrium conditions are met by construction and Lemma 10. $\square$

Every equilibrium allocation is a First Best.

Proof. Fix an equilibrium. First, every active market clears. Suppose not, $\mu(t', q') > 1$ for some $(t', q') \in \Psi$ and let $r(p'|t', q') > 0$. Then $u(q', r(t', q'), t') >$

$U(q') \geq \underline{U}$. Then there exists contract $t_c(.;s'')$ yielding $v(p', q', t_c(.;s''))$ arbitrarily close to but above $v(p', q', t')$, so that $\mu(t_c(.;s''), q') < 1$, and $u(q', r(t_c(.;s''), q'), t_c(.;s'')) > U(q')$ for arbitrary $r(t_c(.;s''), q')$. An asset owner will gain from offering such contract. A symmetric argument rules out $\mu(t', q') < 1$. Second, $V(p) + U(q) \geq E(Y|p, q)$ for all pairs of (p, q), and equality holds if $(t', q) \in \Psi$ for some $t' \in \Omega_t$. The equality is a direct consequence of $\mu(t, q) = 1$ for all $(t, q) \in \Psi$. Now suppose $V(p') + U(q') < E(Y|p', q')$. There is some contract $t_c(.;s')$ in which $v(p', q', t_c(.;s'))$ is slightly above $V(p')$, so that $\mu(t_c(.;s'), q') < 1$, and $u(q', p'', t_c(.;s')) = t_c(.;s') > U(q')$ for any $p'' \in \{p_l\}_{l=1}^{L}$. An asset owner with quality q' will gain from offering such contract.

The allocation is given by

$$W'_{pq}(p_l, q_k) = C'_{pq}(p_l, q_k) = \int_{\Omega_t} r(p_l|t, q_k) dW(t, p_l, q_k).$$

Since $\mu(t, q) = 1$ in the support of W, optimal participation conditions imply that C'_{pq} and W'_{pq} satisfy the resources constraints, with equality in the case $U(q) > \underline{U}$ and $V(p) > \underline{V}$ respectively. In addition, $V(p) + U(q) = E(Y|p, q)$ whenever $W'_{pq}(p, q) > 0$. Applying Lemma 11 with $(U, V) = (\widehat{U}, \widehat{V})$, the equilibrium allocation (W'_{pq}, C'_{pq}) is a First Best. $\square$

A.4 Proof of Lemma 2-4 and Proposition 3

The sufficiency part in Proposition 3 directly follows from Lemma 2-4, which I will establish in sequence.

I first reiterate the candidate equilibrium for reference. The set of active markets is given by $\Psi = \{(t(.;\widetilde{s}_k), q_k)\}_{k \geq \underline{k}}$ where $\{\widetilde{s}_k\}_{k \geq \underline{k}}$ satisfies the equalities (2.9) and (2.10).

$$W(\{t,q\}) = \begin{cases} Q(q_k) & \text{if } \{t,q\} = \{(t(.;\widetilde{s}_k), q_k)\} \text{ and } q_k > q_{\underline{k}}; \\ \sum_{l=1}^{L} W_{pq}^{FB}(p_l, q_{\underline{k}}) & \text{if } \{t,q\} = \{(t(.;\widetilde{s}_{\underline{k}}), q_{\underline{k}})\}; \\ 0 & \text{Otherwise.} \end{cases}$$

$$\psi(q_k) = \begin{cases} \{(t(.;\widetilde{s}_k), q_k)\} & \text{if } q_k > q_{\underline{k}}; \\ \{(t(.;\widetilde{s}_{\underline{k}}), q_{\underline{k}})\} \cup \{p_0\} & \text{if } q_k = q_{\underline{k}} \text{ and } \sum_{l=1}^{L} W_{pq}^{FB}(p_l, q_{\underline{k}}) < Q(q_{\underline{k}}); \\ \{p_0\} & \text{if } q_k < q_{\underline{k}}. \end{cases}$$

The equilibrium payoffs for workers and asset owners are given by

$$V(p_l) = \begin{cases} \max\{v(p_l, q_k, t(.;\widetilde{s}_k))\}_{k \geq \underline{k}}, & \text{if } p_l \geq p_{\underline{l}}; \text{ and} \\ \underline{V}, & \text{if } p_l < p_{\underline{l}}. \end{cases}$$

$$U(q_k) = \begin{cases} u(q_k, r_q^{FB}(q_k), t(.;\widetilde{s}_k)), & \text{if } q_k \geq q_{\underline{k}}; \text{ and} \\ \underline{U}, & \text{if } q_k < q_{\underline{k}}. \end{cases}$$

For the active markets, $r(t(.;\widetilde{s}_k), q_k) = r_q^{FB}(q_k)$ and $\mu(t(.;\widetilde{s}_k), q_k) = 1$. For an inactive market $(t(.;s'), q_{k'}) \notin \Psi$,

$$\mu(t(.;s'), q_{k'}) = \sup\{\mu' \in [0,\infty] : V(p') \geq \eta(\mu')v(p', q_{k'}, t(.;s')), p' \in \{p_l\}_{l=1}^{L}\}.$$

If $\mu(t(.;s'), q_{k'}) < \infty$, $r(t(.;s'), q_{k'})$ is degenerate at

$$\min\{p' \in \{p_l\}_{l=1}^{L} : V(p') = \mu(t(.;s'), q_{k'})v(p', q_{k'}, t(.;s')))\}.$$

This is the most pessimistic belief allowed in our equilibrium definition. If $\mu(t(.;s'), q_{k'}) = \infty$, $r(t(.;s'), q_{k'})$ has no bearings and is assumed to be the same as prior.

A.4.1 Proof of Lemma 2

Lemma 12. *Suppose Condition Sorting-p holds. For any $k' \geq \underline{k}$, $\underline{V} \geq v(p_l, q_{k'}, t(.;\widetilde{s}_{k'}))$ if $p_l \leq p_{\underline{l}}$ and $v(p_l, q_k, t(.;\widetilde{s}_k)) \geq v(p_l, q_{k'}, t(.;\widetilde{s}_{k'}))$ if (p_l, q_k) is in the support of W_{pq}^{FB}.*

Proof. To save on space, I will denote $v(p_l, q_k, t(.;\widetilde{s}_k))$ by $g(l,k)$ for $k \geq \underline{k}$. $\underline{l}(k)$ denotes $\min\{l \geq 1 : W_{pq}^{FB}(p_l, q_k) > 0, q_k \geq q_{\underline{k}}\}$, the lowest type pairing up with $q_k \geq q_{\underline{k}}$. For generic distributions of types, $\underline{l}(k+1) = \bar{l}(k)$. Obviously, $v(p_l, q_k, t(.;\widetilde{s}_k)) \geq \underline{V}$ for $p_l \geq p_{\underline{l}}$ and $u(q_k, p_l, t(.;\widetilde{s}_k)) \geq \underline{U}$ for $p_l \geq p_{\bar{l}(\underline{k})}$.

For any $n \geq 1$, $g(\bar{l}(k+n-1), k+n-1) \geq g(\bar{l}(k+n-1), k+n)$ by (2.10). Condition Sorting-p implies that $g(\bar{l}(k), k+n-1) \geq g(\bar{l}(k), k+n)$. This argument holds for any $n \geq 1$. Inductively, $g(\bar{l}(k), k) \geq g(\bar{l}(k), k+n)$ for all $n \geq 1$. For any $n \in [1, k-\underline{k}+1]$, $g(\underline{l}(k-n), k-n) \geq g(\underline{l}(k-n), k-n-1)$. By a symmetric argument, $g(\underline{l}(k), k) \geq g(\underline{l}(k), k-n)$ for $n \in [1, k-\underline{k}]$.

Now consider p_l in the support of $r_q^{FB}(q_k)$ and $k^H > k \geq \underline{k}$. From Condition Sorting-p, $g(l,k) \geq g(l,k^H)$ because $g(\bar{l}(k), k) \geq g(\bar{l}(k), k^H)$. For the case $k > k^L \geq \underline{k}$, $g(\underline{l}(k), k) \geq g(\underline{l}(k), k^L)$, and so $g(l,k) \geq g(l,k^L)$. Putting together, we show that if $W_{pq}^{FB}(p_l, q_k) > 0$, then $g(l,k) \geq g(l,l')$ for any $k' \geq \underline{k}$.

For $p_l \leq p_{\underline{l}}$, $v(p_l, q_k, t(.;\widetilde{s}_k)) \leq v(p_{\underline{l}}, q_k, t(.;\widetilde{s}_k)) \leq v(p_{\underline{l}}, q_{\underline{k}}, t(.;\widetilde{s}_{\underline{k}}) = \underline{V}$. The first inequality is due to Remark 1 and the preceding argument establishes the second inequality. The last equality is from the construction of $\widetilde{s}_{\underline{k}}$. □

For workers of type $p_l \geq p_{\underline{l}}$, $V(p_l) = v(p_l, q_k, t(.;\widetilde{s}_k))$ for (p_l, q_k) in the support of W_{pq}^{FB}. For workers of type $p_l < p_{\underline{l}}$ cannot gain from participating in any active markets. For any $p_l \geq p_{\underline{l}}$, $V(p_l) \geq V(p_{\underline{l}}) \geq \underline{V}$. Hence, the participation constraints for workers hold. Therefore, workers have no profitable deviations in the candidate equilibrium.

For $q_k \geq q_{\underline{k}}$,

$$\begin{aligned} & U(q_{k+1}) - U(q_k) \geq u(q_{k+1}, p_{\bar{l}(k)}, t(.; \widetilde{s}_{k+1})) - u(q_k, p_{\bar{l}(k)}, t(.; \widetilde{s}_k)) \\ = \ & E(Y|p_{\bar{l}(k)}, q_{k+1}) - E(Y|p_{\bar{l}(k)}, q_k) > 0 \end{aligned}$$

The first inequality is due to Remark (1). The equality (2.10) gives rise to the equality on last line. The last inequality is due to Assumption (P). Our construction of $\widetilde{s}_{\underline{k}}$ ensures that $U(q_{\underline{k}}) \geq \underline{U}$. The participation constraints for the asset owners are met.

A.4.2 Proof of Lemma 3

Proof. For any $q_k \geq q_{\underline{k}}$, $V(p_l) \geq v(p_l, q_k, t(.; \widetilde{s}_k))$. These asset owners will not get matched if posting a contract with $s > \widetilde{s}_k$. I now show that owners of asset quality $q_k \geq q_{\underline{k}}$ have no incentive in posting a contract with $s < \widetilde{s}_k$. Suppose not, an asset owner of $q_{\widehat{k}} \geq q_{\underline{k}}$ profits from posting $s^{\widehat{L}} < \widetilde{s}_{\widehat{k}}$. By construction, $r(t(.; s^{\widehat{L}}), q_{\widehat{k}})$ is degenerate at some worker's type, say p^H. p^L denotes the lowest type in $\Omega_p(t(.; \widetilde{s}_k), q_k)$. It is trivial that $u(q_{\widehat{k}}, p^H, t(.; s^{\widehat{L}})) > U(q_{\widehat{k}}) \geq \max\{\underline{U}, u(q_{\widehat{k}}, p^L, t(.; \widetilde{s}_{\widehat{k}}))\}$ and $p^H > p^L$. Furthermore,

$$\begin{aligned} v(p^H, q_{\widehat{k}}, t(.; \widetilde{s}_{\widehat{k}})) &\leq V(p^H) = \mu(t(.; s^{\widehat{L}}), q_{\widehat{k}}) v(p^H, q_{\widehat{k}}, t(.; s^{\widehat{L}})), \\ v(p^L, q_{\widehat{k}}, t(.; \widetilde{s}_{\widehat{k}})) &= V(p^L) > \mu(t(.; s^{\widehat{L}}), q_{\widehat{k}}) v(p^L, q_{\widehat{k}}, t(.; s^{\widehat{L}})), \end{aligned}$$

which implies

$$\frac{v(p^H, q_{\widehat{k}}, t(.; \widetilde{s}_{\widehat{k}}))}{v(p^H, q_{\widehat{k}}, t(.; s^{\widehat{L}}))} < \frac{v(p^L, q_{\widehat{k}}, t(.; \widetilde{s}_{\widehat{k}}))}{v(p^L, q_{\widehat{k}}, t(.; s^{\widehat{L}}))}.$$

For all $q \leq q_{\widehat{k}}$, define $\widehat{s}(q)$ by $v(p^L, q, t(.; \widehat{s}(q))) = v(p^L, q_{\widehat{k}}, t(.; \widetilde{s}_{\widehat{k}}))$. Condition Screening-q implies that $V(p^L) = v(p^L, q_{\widehat{k}}, t(.; \widetilde{s}_{\widehat{k}})) > \underline{V}$ and

$u(q, p^L, \widehat{s}(q)) > \underline{U}$ for any $q \le q_{\widehat{k}}$. Our construction of $\{\widetilde{s}_k\}_{k \ge \underline{k}}$ implies that $\widehat{k} > \underline{k} \ge 1$ and $p^L > p_{\underline{l}}$. Note that

$$U(q_{\widehat{k}}) > u(q_{\widehat{k}}, p^L, t(.; s^{\widehat{L}}) > u(q_{\widehat{k}}, p_{\underline{l}}, t(.; s^{\widehat{L}}).$$

From the workers' IC condition, $\widetilde{s}_k \ge \widehat{s}(q_k)$ for $\underline{k} \le k \le \widehat{k} - 1$ because $v(p^L, q_k, t(.; \widehat{s}(q_k))) = v(p^L, q_{\widehat{k}}, t(.; \widetilde{s}_{\widehat{k}})) \ge v(p^L, q_k, t(.; \widetilde{s}_k)$.

Consider the case that $\sum_{k=1}^{K} Q(q_k) < \sum_{l=1}^{L} P(p_l)$, then $v(p_{\underline{l}}, q_1, t(.; \widehat{s}(q_1)) \ge v(p_{\underline{l}}, q_1, t(.; \widetilde{s}_1)) = V(p_{\underline{l}}) = \underline{V} \ge v(p_{\underline{l}}, q_{\widehat{k}}, t(.; \widehat{s}(q_{\widehat{k}})))$. The last inequality is due to Lemma 12. By continuity of $v(p, q, t(.; s))$ in q and s, there must exist $q' \in [q_1, q_{\widehat{k}}]$ such that $v(p_{\underline{l}}, q_1, t(.; \widetilde{s}_1)) = v(p_{\underline{l}}, q', t(.; \widehat{s}(q')) = \underline{V}$. Recall that $V(p^H) \ge v(p^H, q_{\widehat{k}}, t(.; \widetilde{s}_{\widehat{k}}))$. Putting together, Condition Screening-q implies that

$$\frac{V(p^H)}{V(p_{\underline{l}})} \ge \frac{v(p^H, q_{\widehat{k}}, t(.; \widetilde{s}_{\widehat{k}}))}{v(p_{\underline{l}}, q', t(.; \widehat{s}(q'))} \ge \frac{v(p^H, q_{\widehat{k}}, t(.; s^{\widehat{L}}))}{v(p_{\underline{l}}, q_{\widehat{k}}, t(.; s^{\widehat{L}}))}.$$

Therefore, $V(p_{\underline{l}}) \le \mu(t(.; s^{\widehat{L}}), q_{\widehat{k}}) v(p_{\underline{l}}, q_{\widehat{k}}, t(.; s^{\widehat{L}}))$. Our construction of off-equilibrium-path belief implies $r(t(.; s^{\widehat{L}}), q_{\widehat{k}})$ is degenerate at $p_{\underline{l}} < p^H$!

Consider the case that $\sum_{k=1}^{K} Q(q_k) \ge \sum_{l=1}^{L} P(p_l)$. Recall that $\widetilde{s}_{\underline{k}} \ge \widehat{s}(q_{\underline{k}})$, so $\underline{U} \ge u(q_{\underline{k}}, p_1, t(.; \widetilde{s}_{\underline{k}})) \ge u(q_{\underline{k}}, p_1, t(.; \widehat{s}(q_{\underline{k}})))$ and $v(p_1, q_{\underline{k}}, t(.; \widehat{s}(q_{\underline{k}})) \ge v(p_1, q_{\underline{k}}, t(.; \widetilde{s}_{\underline{k}})) = V(p_1)$. Combining with $V(p^H) \ge v(p^H, q_{\widehat{k}}, t(.; \widetilde{s}_{\widehat{k}}))$, Condition Screening-q implies

$$\frac{V(p^H)}{V(p_1)} \ge \frac{v(p^H, q_{\widehat{k}}, t(.; \widetilde{s}_{\widehat{k}}))}{v(p_1, q_{\underline{k}}, t(.; \widehat{s}(q_{\underline{k}}))} \ge \frac{v(p^H, q_{\widehat{k}}, t(.; s^{\widehat{L}}))}{v(p_1, q_{\widehat{k}}, t(.; s^{\widehat{L}}))}.$$

Therefore, $V(p_1) \le \mu(t(.; s^{\widehat{L}}), q_{\widehat{k}}) v(p_1, q_{\widehat{k}}, t(.; s^{\widehat{L}}))$, and hence $r(p_{\underline{l}} | t(.; s^{\widehat{L}}), q_{\widehat{k}}) = 1$! □

A.4.3 Proof of Lemma 4

Proof. Suppose to the contrary that an owner of asset quality $\widehat{q} < q_{\underline{k}}$ may gain from posting some contract $\widehat{s}$. It follows that $\widehat{s} < \widetilde{s}_{\underline{k}}$. Otherwise, no workers will accept such contract because $V(p_l) \geq v(p_l, q_{\underline{k}}, t(.;\widetilde{s}_{\underline{k}})) > v(p_l, \widehat{q}, t(.;\widehat{s}))$. By construction, $r(t(.;\widehat{s}), \widehat{q})$ is degenerate at some worker's type, say p^H. The deviation is profitable only if $u(\widehat{q}, p^H, t(.;\widehat{s})) > \underline{U}$. The construction of $\widetilde{s}_{\underline{k}}$ ensures that $\underline{U} \geq u(q_{\underline{k}}, p_1, t(.;\widetilde{s}_{\underline{k}}))$, and hence $p^H > p_1$. From the worker's IC condition, $v(p^H, \widehat{q}, t(.;\widehat{s})) > V(p^H) \geq v(p^H, q_{\underline{k}}, t(.;\widetilde{s}_{\underline{k}}))$. The inequality (2.16) in Condition Entry-q implies that

$$V(p_1) = v(p_1, q_{\underline{k}}, t(.;\widetilde{s}_{\underline{k}})) < v(p_1, \widehat{q}, t(.;\widehat{s})),$$

and

$$\frac{V(p^H)}{V(p_1)} \geq \frac{v(p^H, q_{\underline{k}}, t(.;\widetilde{s}_{\underline{k}}))}{v(p_1, q_{\underline{k}}, t(.;\widetilde{s}_{\underline{k}}))} \geq \frac{v(p^H, \widehat{q}, t(.;\widehat{s}))}{v(p_1, \widehat{q}, t(.;\widehat{s}))}.$$

The equilibrium condition requires $V(p_1) \geq \mu(t(.;\widehat{s}), \widehat{q}) v(p_1, \widehat{q}, t(.;\widehat{s}))$. It follows that $V(p^H) \geq \mu(t(.;\widehat{s}), \widehat{q}) v(p^H, \widehat{q}, t(.;\widehat{s}))$. Our construction of off-equilibrium-path belief implies $r(p^H | t(.;\widehat{s}), \widehat{q}) = 0$! □

A.4.4 Proof for necessity of conditions

If Condition Sorting-p fails, then there exists some distribution of types (P, Q) *such that the First Best allocation cannot be supported by an equilibrium.*

Proof. Suppose Condition Sorting-p fails, there exist $\widehat{p}^H > \widehat{p}^L$, $\widehat{q}^H > \widehat{q}^L$ and $\widehat{s}^H > \widehat{s}^L$ s.t. $v(\widehat{p}^L, \widehat{q}^L, t(.;\widehat{s}^L)) \geq \underline{V}$, $u(\widehat{q}^L, \widehat{p}^H, t(.;\widehat{s}^L)) \geq \underline{U}$ and

$$\begin{cases} v(\widehat{p}^H, \widehat{q}^H, t(.;\widehat{s}^H)) \leq v(\widehat{p}^H, \widehat{q}^L, t(.;\widehat{s}^L)), \\ v(\widehat{p}^L, \widehat{q}^H, t(.;\widehat{s}^H)) \geq v(\widehat{p}^L, \widehat{q}^L, t(.;\widehat{s}^L)), \end{cases} \tag{A.3}$$

where at least one of the inequalities in (A.3) is strict. Notice that equality cannot hold for both $v(\widehat{p}^L, \widehat{q}^L, t(.;\widehat{s}^L)) \geq \underline{V}$ and $u(\widehat{q}^L, \widehat{p}^H, t(.;\widehat{s}^L)) \geq \underline{U}$. By continuity, $\widehat{s}^L$ can be chosen such that $v(\widehat{p}^L, \widehat{q}^L, t(.;\widehat{s}^L)) > \underline{V}$, $u(\widehat{q}^L, \widehat{p}^L, t(.;\widehat{s}^L)) \neq \underline{U}$, and both inequalities in (A.3) are strict. Note that $u(\widehat{q}^H, \widehat{p}^H, t(.;\widehat{s}^H)) > \underline{U}$. Define $\widehat{s}(q)$ by $v(\widehat{p}^L, \widehat{q}^L, t(.;\widehat{s}^L)) = v(\widehat{p}^L, q, t(.;\widehat{s}(q)))$ for all $q \leq \widehat{q}^L$. $\widehat{s}(.)$ is continuous and increases with $\widehat{s}^L$. Furthermore, $\widehat{s}^L$ can be chosen so that $v(p, q, t(.;\widehat{s}(q))) \neq \underline{V}$ almost everywhere when $(p, q) \to (\underline{p}, \underline{q})$.

The first case is that $u(\widehat{q}^L, \widehat{p}^L, t(.;\widehat{s}^L)) < \underline{U}$. In this case, $\widehat{s}^L$ can be chosen such that $u(\widehat{q}^L, \widehat{p}^H, t(.;\widehat{s}^L)) > \underline{U}$. Consider the distribution of types with support $\{p_1, p_2\} = \{\widehat{p}^L, \widehat{p}^H\}$ and $\{q_1, q_2\} = \{\widehat{q}^L, \widehat{q}^H\}$. Moreover, $P(\widehat{p}^H) + P(\widehat{p}^L) < Q(\widehat{q}^H) + Q(\widehat{q}^L)$, $P(\widehat{p}^H) > Q(\widehat{q}^H)$ and $u(\widehat{q}^L, r_q^{FB}(\widehat{q}^L), t(.;\widehat{s}^L)) = \underline{U}$. In the First Best allocation, workers of type $\widehat{p}^H$ will match with assets of quality $\widehat{q}^H$ and $\widehat{q}^L$. In any equilibrium supporting the First Best, there must be two active markets $(t(.;\widehat{s}^L), \widehat{q}^L)$ and $(t(.;\widetilde{s}^H), \widehat{q}^H)$ s.t.

$$\begin{cases} v(\widehat{p}^H, \widehat{q}^H, t(.;\widetilde{s}^H)) = v(\widehat{p}^H, \widehat{q}^L, t(.;\widehat{s}^L)) \\ v(\widehat{p}^L, \widehat{q}^L, t(.;\widehat{s}^L)) \geq v(\widehat{p}^L, \widehat{q}^H, t(.;\widetilde{s}^H)) \end{cases} \tag{A.4}$$

Comparing with (A.3), the first equality requires $\widetilde{s}^H < \widehat{s}^H$ while the second equality requires $\widetilde{s}^H > \widehat{s}^H$!!!

Consider the case $v(\widehat{p}^L, \widehat{q}^L, t(.;\widehat{s}^L)) \in [E(Y|\widehat{p}^L, \underline{q}) - \underline{U}, E(Y|\widehat{p}^L, \widehat{q}^L) - \underline{U})$. It follows that $v(\widehat{p}^L, \widehat{q}^L, t(.;\widehat{s}^L)) = E(Y|\widehat{p}^L, q') - \underline{U}$ for some $q' \in (\underline{q}, \widehat{q}^L)$. In this case, consider the distribution of types with support $\{p_1, p_2\} = \{\widehat{p}^L, \widehat{p}^H\}$ and $\{q_1, q_2, q_3\} = \{q', \widehat{q}^L, \widehat{q}^H\}$. Moreover,

$$Q(\widehat{q}^H) + Q(\widehat{q}^L) + Q(q') > P(\widehat{p}^H) + P(\widehat{p}^L) > Q(\widehat{q}^H) + Q(\widehat{q}^L)$$

and $P(\widehat{p}^H) > Q(\widehat{q}^H)$. In any equilibrium supporting the First Best, there must be three active markets $(t(.;\widehat{s}(q')), q'), (t(.;\widehat{s}^L), \widehat{q}^L)$ and $(t(.;\widetilde{s}^H), \widehat{q}^H)$, where $u(q', \widehat{p}^L, t(.; \widehat{s}(q'))) = \underline{U}$ and $v(\widehat{p}^L, \widehat{q}^L, t(.; \widehat{s}^L)) = v(\widehat{p}^L, q', t(.; \widehat{s}(q')))$. (A.4) must be satisfied and the previous argument applies.

The last case is $v(\widehat{p}^L, \widehat{q}^L, t(.; \widehat{s}^L)) \in (\underline{V}, E(Y|\widehat{p}^L, \underline{q}) - \underline{U}]$. Recall that $E(Y|\underline{p}, \underline{q})) = \underline{V} + \underline{U}$, our choice of $\widehat{s}^L$ ensures that when (p, q) is sufficiently close to $(\underline{p}, \underline{q})$, $v(p, q, t(.; \widehat{s}(q))) \notin [\underline{V}, E(Y|p, q) - \underline{U}]$. Either there exists $(p', q') < (\widehat{p}^L, \widehat{q}^L)$ satisfying $v(p', q', t(.; \widehat{s}(q')) = \underline{V}$ and $u(q', p', t(.; \widehat{s}(q')) > \underline{U}$ or there exists $(p'', q'') < (\widehat{p}^L, \widehat{q}^L)$ such that $v(p', q', t(.; \widehat{s}(q')) > \underline{V}$ and $u(q', p', t(.; \widehat{s}(q')) < \underline{U}$.

For the first possibility, consider the distribution of types with support $\{p_1, p_2, p_3\} = \{p', \widehat{p}^L, \widehat{p}^H\}$ and $\{q_1, q_2, q_3\} = \{q', \widehat{q}^L, \widehat{q}^H\}$.

$$\begin{aligned} & P(\widehat{p}^H) + P(\widehat{p}^L) + P(p') > Q(\widehat{q}^H) + Q(\widehat{q}^L) + Q(q') \\ > \quad & P(\widehat{p}^H) + P(\widehat{p}^L) > Q(\widehat{q}^H) + Q(\widehat{q}^L), \end{aligned}$$

and $P(\widehat{p}^H) > Q(\widehat{q}^H)$. In any equilibrium supporting the First Best, there must be three active markets $(t(.; \widehat{s}(q')), q'), (t(.; \widehat{s}^L), \widehat{q}^L)$ and $(t(.; \widetilde{s}^H), \widehat{q}^H)$. (A.4) must be satisfied and the previous argument applies again.

For the remaining possibility, consider the distribution of types with support $\{p_1, p_2, p_3\} = \{p'', \widehat{p}^L, \widehat{p}^H\}$ and $\{q_1, q_2, q_3\} = \{q'', \widehat{q}^L, \widehat{q}^H\}$ satisfying $Q(\widehat{q}^H) + Q(\widehat{q}^L) + Q(q'') > P(\widehat{p}^H) + P(\widehat{p}^L) + P(p'')$, $P(\widehat{p}^H) + P(\widehat{p}^L) > Q(\widehat{q}^H) + Q(\widehat{q}^L)$, $P(\widehat{p}^H) > Q(\widehat{q}^H)$ and $u(q'', r_q^{FB}(q''), t(.; \widehat{s}(q'')) = \underline{U}$. In any equilibrium supporting the First Best, there must be three active markets $(t(.; \widehat{s}(q'')), q''), (t(.; \widehat{s}^L), \widehat{q}^L)$ and $(t(.; \widetilde{s}^H), \widehat{q}^H)$. The previous argument applies again. □

Suppose Entry-q fails, then there exists some distribution of types (P, Q) such that the First Best allocation cannot be supported by an equilibrium.

Proof. Suppose that Condition Entry-q fails for some $\widehat{q}^H > \widehat{q}^L$, $\widehat{p}^H > \widehat{p}^L$

and $\widehat{s}^H > \widehat{s}^L$ satisfying

$$\begin{aligned} v(\widehat{p}^H, \widehat{q}^L, t(.;\widehat{s}^L)) &> v(\widehat{p}^H, \widehat{q}^H, t(.;\widehat{s}^H)) > v(\widehat{p}^L, \widehat{q}^H, t(.;\widehat{s}^H)) \geq \underline{V}, \\ u(\widehat{q}^L, \widehat{p}^H, t(.;\widehat{s}^L)) &> \underline{U} \geq u(\widehat{q}^H, \widehat{p}^L, t(.;\widehat{s}^H)), \text{ and} \\ \frac{v(\widehat{p}^H, \widehat{q}^H, t(.;\widehat{s}^H))}{v(\widehat{p}^L, \widehat{q}^H, t(.;\widehat{s}^H))} &< \frac{v(\widehat{p}^H, \widehat{q}^L, t(.;\widehat{s}^L))}{v(\widehat{p}^L, \widehat{q}^L, t(.;\widehat{s}^L))}. \end{aligned}$$

Since $u(\widehat{q}^H, \widehat{p}^L, t(.;\widehat{s}^H)+v(\widehat{p}^L, \widehat{q}^H, t(.;\widehat{s}^H)) > \underline{U}+\underline{V}$, $\widehat{s}^H$ can be chosen so that $u(\widehat{q}^H, \widehat{p}^L, t(.;\widehat{s}^H)) < \underline{U}$. Consider the distribution of types with support $\{p_1, p_2\} = \{\widehat{p}^L, \widehat{p}^H\}$ and $\{q_1, q_2\} = \{\widehat{q}^L, \widehat{q}^H\}$. Moreover, $P(\widehat{p}^H) + P(\widehat{p}^L) < Q(\widehat{q}^H)$ and $u(\widehat{q}^L, r_q^{FB}(\widehat{q}^H), t(.;\widehat{s}^H)) = \underline{U}$. In the First Best allocation, all workers will match with assets of quality $\widehat{q}^H$ and the asset owners of $\widehat{q}^L$ will take their outside options. In any equilibrium supporting the First Best, there is a single market $(t(.;\widehat{s}^H), \widehat{q}^H)$. Consider the deviation that an asset owner of $\widehat{q}^L$ posts the contract $t(.;\widehat{s}^L)$. Since

$$\frac{V(\widehat{p}^H)}{V(\widehat{p}^L)} = \frac{v(\widehat{p}^H, \widehat{q}^H, t(.;\widehat{s}^H))}{v(\widehat{p}^L, \widehat{q}^H, t(.;\widehat{s}^H))} < \frac{v(\widehat{p}^H, \widehat{q}^L, t(.;\widehat{s}^L))}{v(\widehat{p}^L, \widehat{q}^L, t(.;\widehat{s}^L))},$$

It follows that $V(\widehat{p}^L) > \mu(t(.;\widehat{s}^L), \widehat{q}^L) v(\widehat{p}^L, \widehat{q}^L, t(.;\widehat{s}^L))$ and $r(\widehat{p}^H | t(.;\widehat{s}^L), \widehat{q}^L) = 1$. Such contracts yield the asset owner $u(\widehat{q}^L, \widehat{p}^H, t(.;\widehat{s}^L)) > \underline{U} = U(\widehat{q}^L)$! $\square$

Suppose Screening-q fails. There exists some distribution of types (P, Q) such that the First Best allocation cannot be supported by an equilibrium.

Proof. Suppose that Condition Screening-q fails for some $\widehat{q}$, $\widehat{p}^H > \widehat{p}^L$ and $\widehat{s}^H > \widehat{s}^L$ s.t. $v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^H)) \geq \underline{V}$, $u(\widehat{q}, \widehat{p}^H, t(.;\widehat{s}^L)) > \max\{\underline{U}, u(\widehat{q}, \widehat{p}^L, t(.;\widehat{s}^H))\}$ and

$$\frac{v(\widehat{p}^H, \widehat{q}, t(.;\widehat{s}^H))}{v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^H))} < \frac{v(\widehat{p}^H, \widehat{q}, t(.;\widehat{s}^L))}{v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^L))}.$$

Define $\widehat{s}(q)$ by $v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^H)) = v(\widehat{p}^L, q, t(.;\widehat{s}(q)))$ for $q \leq \widehat{q}$. $\widehat{s}^H$ can be chosen such that $v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^H)) > \underline{V}$, $u(\widehat{q}, \widehat{p}^L, t(.;\widehat{s}^H)) \neq \underline{U}$, and $v(p, q, \widehat{s}(q)) \neq \underline{V}$ almost everywhere when $(p, q) \to (\underline{p}, \underline{q})$.

The first case is $u(\widehat{q}, \widehat{p}^L, t(.;\widehat{s}^H)) < \underline{U}$. Consider the distribution of types with support $\{p_1, p_2\} = \{\widehat{p}^L, \widehat{p}^H\}$ and $\{q_1\} = \{\widehat{q}\}$. $P(\widehat{p}^H) + P(\widehat{p}^L) < Q(\widehat{q})$ and $u(\widehat{q}, r_q^{FB}(\widehat{q}), t(.;\widehat{s}^H)) = \underline{U}$. In the First Best allocation, workers of both types match with assets of quality $\widehat{q}$. In any equilibrium supporting the First Best, there is only one active market $(t(.;\widehat{s}^H), \widehat{q})$ with $\mu(t(.;\widehat{s}^H), \widehat{q}) = 1$. Consider the deviation that an owner of asset quality $\widehat{q}$ posts the contract $t(.;\widehat{s}^L)$. Since

$$\frac{V(\widehat{p}^H)}{V(\widehat{p}^L)} = \frac{v(\widehat{p}^H, \widehat{q}, t(.;\widehat{s}^H))}{v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^H))} < \frac{v(\widehat{p}^H, \widehat{q}, t(.;\widehat{s}^L))}{v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^L))}, \tag{A.5}$$

It follows that $V(\widehat{p}^L) > \mu(t(.;\widehat{s}^L), \widehat{q}) v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^L))$ and $r(\widehat{p}^H | t(.;\widehat{s}^L), \widehat{q}) = 1$. By posting the contract $t(.;\widehat{s}^L)$, the asset owner can earn a payoff $u(\widehat{q}, \widehat{p}^H, \widehat{s}^L) > \underline{U} = U(\widehat{q})$.

For the case $v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^H)) \in [E(Y|\widehat{p}^L, \underline{q}) - \underline{U}, E(Y|\widehat{p}^L, \widehat{q}) - \underline{U})$, there exists some $q' \in (\underline{q}, \widehat{q}^L)$ such that $u(q', \widehat{p}^L, t(.;\widehat{s}(q'))) = \underline{U}$ and $v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^H)) = v(\widehat{p}^L, q', t(.;\widehat{s}(q')))$. Consider the distribution of types with support $\{p_1, p_2\} = \{\widehat{p}^L, \widehat{p}^H\}$ and $\{q_1, q_2\} = \{q', \widehat{q}\}$. Moreover,

$$Q(\widehat{q}) + Q(q') > P(\widehat{p}^H) + P(\widehat{p}^L) > Q(\widehat{q}) > P(\widehat{p}^H).$$

In the First Best allocation, workers of both types match with assets of quality $\widehat{q}$ and only low type workers match with assets of quality q'. In any equilibrium supporting the First Best, there must be two active markets $(t(.;\widehat{s}(q')), q')$ and $(t(.;\widehat{s}^H), \widehat{q})$. (A.5) holds in equilibrium, so that $r(\widehat{p}^H | t(.;\widehat{s}^L), \widehat{q}) = 1$. Posting $t(.;\widehat{s}^L)$ is a profitable deviation for the owners of asset quality $\widehat{q}$.

The remaining case is $v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^H)) \in (\underline{V}, E(Y|\widehat{p}^L, \underline{q}) - \underline{U}]$. Suppose that for some $\widehat{p}_1 < \widehat{p}^L$ and $q' \leq \widehat{q}$, $v(\widehat{p}_1, q', t(.;\widehat{s}(q'))) = \underline{V}$ but

$$\frac{v(\widehat{p}^H, \widehat{q}, t(.;\widehat{s}^H))}{v(\widehat{p}^H, \widehat{q}, t(.;\widehat{s}^L))} < \frac{v(\widehat{p}_1, q', t(.;\widehat{s}(q')))}{v(\widehat{p}_1, \widehat{q}, t(.;\widehat{s}^L))}. \tag{A.6}$$

I will restrict the attention to the case $q' < \widehat{q}$. This also covers the case $q' = \widehat{q}$ because of continuity. Consider the distribution of types with support $\{p_1, p_2, p_3\} = \{\widehat{p}_1, \widehat{p}^L, \widehat{p}^H\}$ and $\{q_1, q_2\} = \{q', \widehat{q}\}$. Moreover,

$$P(\widehat{p}^H) + P(\widehat{p}^L) + P(\widehat{p}_1) > Q(\widehat{q}) + Q(q') > P(\widehat{p}^H) + P(\widehat{p}^L) > Q(\widehat{q}) > P(\widehat{p}^H).$$

In any equilibrium supporting the First Best, there must be two active markets $(t(.;\widehat{s}(q')), q')$ and $(t(.;\widehat{s}^H), \widehat{q})$. By construction, $v(\widehat{p}^L, \widehat{q}, t(.;\widehat{s}^H)) = v(\widehat{p}^L, q', t(.;\widehat{s}(q')))$. By construction,

$$\frac{V(\widehat{p}^H)}{V(\widehat{p}_1)} = \frac{v(\widehat{p}^H, \widehat{q}, t(.;\widehat{s}^H))}{v(\widehat{p}_1, q', t(.;\widehat{s}(q')))} < \frac{v(\widehat{p}^H, \widehat{q}, t(.;\widehat{s}^L))}{v(\widehat{p}_1, \widehat{q}, t(.;\widehat{s}^L))}.$$

Together with (A.5), we obtain $r(\widehat{p}^H | t(.;\widehat{s}^L), \widehat{q}) = 1$. Posting $t(.;\widehat{s}^L)$ is a profitable deviation for the owners of asset quality $\widehat{q}$.

A similar construction applies to the case $u(q', \widehat{p}_1, t(.;\widehat{s}(q'))) \leq \underline{U}$ and (A.6) holds. The only change is that $Q(\widehat{q}) + Q(q') > P(\widehat{p}^H) + P(\widehat{p}^L) + P(\widehat{p}_1)$, $P(\widehat{p}^H) + P(\widehat{p}^L) > Q(\widehat{q}) > P(\widehat{p}^H)$ and $u(q', r_q^{FB}(q'), t(.;\widehat{s}(q'))) = \underline{U}$. □

A.5 Proof of Proposition 4

It is trivial that Condition Global ID gives rise to Condition Sorting-p. The proof here is to establish the following

Condition (ID-q)**.** *For any $p^H > p^L$, $q^H \geq q^L$ and $s^H > s^L$ satisfying $v(p^L, q^H, t(.;s^H)) \geq \underline{V}$, $u(q^L, p^H, t(.;s^L)) > \underline{U}$ and $v(p^H, q^L, t(.;s^L)) > v(p^H, q^H, t(.;s^H))$. Then*

$$\frac{v(p^H, q^H, t(.;s^H))}{v(p^L, q^H, t(.;s^H))} \geq \frac{v(p^H, q^L, t(.;s^L))}{v(p^L, q^L, t(.;s^L))}. \tag{A.7}$$

Note that when $q^H = q^L = \widehat{q}$, $v(p^H, \widehat{q}, t(.;s^L)) > v(p^H, \widehat{q}, t(.;s^H))$ is always true. Condition ID-q implies both Condition Screening-q and Entry-

q. Condition strong Screening-q holds if (A.7) always holds with strict inequality.

Proof. Fix the pairs $\{p^H, p^L\}$ and $\{(q^H, s^H), (q^L, s^L)\}$, for any $c^H \geq c^L \geq 0$, denote

$$\begin{aligned}
\Delta_{product}(v; c^H, c^L) &= [v(p^H, q^H, t(.; s^H)) + c^L][v(p^L, q^L, t(.; s^L)) + c^H] \\
&\quad -[v(p^H, q^L, t(.; s^L)) + c^L][v(p^L, q^H, t(.; s^H) + c^H] \\
\Delta_{sum}(v) &= v(p^L, q^L, t(.; s^L)) + v(p^H, q^H, t(.; s^H)) \\
&\quad -v(p^H, q^L, t(.; s^L)) - v(p^L, q^H, t(.; s^H).
\end{aligned}$$

Notice that

$$\begin{aligned}
&\Delta_{product}(v; c^H, c^L) - \Delta_{product}(v; 0, 0) \\
=&c^H \Delta_{sum}(v) - (c^H - c^L)[v(p^L, q^L, t(.; s^L)) - v(p^L, q^H, t(.; s^H)].
\end{aligned}$$

Recall the pre-condition $v(p^L, q^H, t(.; s^H)) \geq \underline{V} > 0$ and $v(p^H, q^L, t(.; s^L)) > v(p^H, q^H, t(.; s^H))$ in Condition ID-q. Global ID immediately implies that $v(p^L, q^L, t(.; s^L)) > v(p^L, q^H, t(.; s^H))$. The first case is that $\Delta_{sum}(v) \geq 0$, then

$$\begin{aligned}
&\Delta_{product}(v; 0, 0) \\
\geq&\Delta_{product}(v; c^H, c^L) + (c^H - c^L)[v(p^L, q^L, t(.; s^L)) - v(p^L, q^H, t(.; s^H)] \\
&-[v(p^L, q^H, t(.; s^H)) + c^H]\Delta_{sum}(v) \\
=&\Delta_{product}(v; -v(p^L, q^H, t(.; s^H)), -v(p^L, q^H, t(.; s^H))) \\
=&[v(p^H, q^H, t(.; s^H)) - v(p^L, q^H, t(.; s^H))][v(p^L, q^L, t(.; s^L)) - v(p^L, q^H, t(.; s^H))] \\
>&0
\end{aligned}$$

For the case that $\Delta_{sum}(v) < 0$, $\Delta_{product}(v; 0, 0) \geq 0$ if $\Delta_{product}(v; c^H, c^L) \geq 0$ and $\Delta_{product}(v; 0, 0) = 0$ if and only if $c^H = c^L = 0$.

Putting together, if $\Delta_{product}(v; c^H, c^L) \geq 0$, then $\Delta_{product}(v; 0, 0) \geq 0$ regardless of the sign for $\Delta_{sum}(v)$. Hence, Condition Global ID implies that $\Delta_{product}(v; 0, 0) \geq 0$. Furthermore, $\Delta_{product}(v; 0, 0) > 0$ if $\Delta_{sum}(v) \geq 0$ or $\Delta_{product}(v; c^H, c^L) \geq 0$ for some $c^H > 0$. □

A.6 Proof of Lemma 5

Proof. Suppose not, $(t(.; s'), q')$ is an active market with $\mu(t(.; s'), q') < 1$. Let p^H be the highest type in $I_p(t(.; s'), q')$. Pick a sufficiently small $\epsilon > 0$ such that $v(p^H, q', t(.; s'+\epsilon) > \mu(t(.; s'), q')v(p^H, q', t(.; s'))$. $\mu(t(.; s'+\epsilon), q') < 1$ and satisfies

$$V(p^H) = \mu(t(.; s'), q')v(p^H, q', t(.; s')) \geq \mu(t(.; s' + \epsilon), q')v(p^H, q', t(.; s' + \epsilon)$$

For all $p' < p^H$, Condition Strong Screening-q and the worker's IC condition,

$$V(p') \geq \mu(t(.; s'), q')v(p', q', t(.; s')) > \mu(t(.; s' + \epsilon), q')v(p', q', t(.; s' + \epsilon.)$$

Hence, $r(p'|t(.; s' + \epsilon), q') = 0$ only if $p' < p^H$. Posting this contract gives the asset owner a payoff $u(q', r(t(.; s'+\epsilon), q'), t(.; s'+\epsilon)) \geq u(q', p^H, t(.; s'+\epsilon)) > u(q', p^H, t(.; s')) \geq U(\widetilde{q})$! □

A.7 Proof of Lemma 6

Proof. Fix an equilibrium $\{U, V, \psi, \Psi, W, r, \mu\}$. From Lemma 5, $\mu(t(.; s'), q_k) \geq 1$ if $(t(.; s'), q_k) \in \Psi$. A worker of p_l obtains $v(p_l, q_k, t(.; s'))$ from participating in this active market. It follows that for any $s'' > s'$, $V(p_l) \geq v(p_l, q_k, t(.; s')) > v(p_l, q_k, t(.; s''))$ for all p_l, so that $\mu(t(.; s''), q_k) = 0$. $(t(.; s'), q_k)$ must be only active market for asset quality q_k. This shows that owners of the same quality , if participating, post the same contract.

For any equilibrium, the set of active markets can be written as $\Psi = \{(t(.;s_k), q_k)\}_{k \geq \widehat{k}}$. The equilibrium payoff is $V(p_l) = \max\{\underline{V}, v(p_l, q_k, t(.;s)) : (t(.;s), q_k) \in \Psi\}$, which is increasing in p. It also follows that worker's participation is monotonic.

Now consider an active market $(t(.;s'), q_k)$ and p_l is the highest type in $\Omega_p((t(.;s'), q_k)$. Then $V(p_l) = v(p_l, q_k, t(.;s'))$ and $V(p^L) \geq v(p^L, q_k, t(.;s'))$ for any $p^L < p_k$. For any $q^H > q_k$, there exists $s'' > s'$ satisfying $v(p_l, q^H, t(.;s''))$ $= v(p_l, q_k, t(.;s'))$. Condition Strict Sorting-p implies that $v(p^L, q^H, t(.;s'')) < v(p^L, q_k, t(.;s')) \leq V(p^L)$ for all $p^L < p_k$. For sufficiently small $\epsilon > 0$, $V(p_l) < v(p_l, q^H, t(.;s'' - \epsilon)$ and $V(p^L) > v(p^L, q^H, t(.;s'' - \epsilon))$. Hence, $\mu(t(.;s'' - \epsilon), q^H) < 1$ and $p^L \notin \Omega_p((t(.;s'' - \epsilon), q^H)$ for any $p^L < p_l$. Note that

$$\begin{aligned} u(q^H, p_k, t(.;s'')) &= [E(Y|p_l, q^H) - E(Y|p_l, q_k)] + u(q_k, p_l, t(.;s')) \\ &> u(q_k, p_l, t(.;s')) \geq U(q_k) \geq \underline{U} \end{aligned}$$

For sufficiently small $\epsilon > 0$, posting $t(.;s'' - \epsilon)$ provides the asset owner of q^H a payoff $u(q^H, p_k, t(.;s'' - \epsilon)) > U(q_k)$. This establishes that $U(q^H) > U(q_k) \geq \underline{U}$, so the asset owner's participation is monotonic and their equilibrium payoff is increasing in q.

Consider two active markets $(t(.;s^H), q^H)$ and $(t(.;s^L), q^L)$ where $q^H > q^L$. $\mu(t(.;s^H), q^H) \geq 1$ and $\mu(t(.;s^L), q^L) \geq 1$ requires $s^H > s^L$. Now suppose that there exists $p^H > p^L$ such that $p^L \in \Omega_p(t(.;s^H), q^H)$ and $p^H \in \Omega_p(t(.;s^L), q^L)$. IC condition for workers requires both $v(p^H, q^L, t(.;s^L)) \geq v(p^H, q^H, t(.;s^H))$ and $v(p^L, q^L, t(.;s^L)) \leq v(p^L, q^H, t(.;s^H))$. This contradicts Condition Strict Sorting-p. Therefore, if $p \in \Omega_p(t(.;s^H), q^H)$ and $p' \in \Omega_p(t(.;s^L), q^L)$, then $p \geq p'$. □

A.8 Proof of Remark 6

First, consider the case that $[y - t(y;s)]$ is SPM, which is the same as $[t(y;s^L) - t(y;s^H)]$ is weakly increasing in y. Since $Y|(p^H,q)$ F.O.S.D. $Y|(p^L,q)$,

$$\begin{aligned}
& v(p^H,q,t(.;s^H)) + v(p^L,q,t(.;s^L)) - v(p^H,q,t(.;s^L)) - v(p^L,q,t(.;s^H)) \\
= & \int_{\underline{y}}^{\overline{y}} [t(y;s^L) - t(y;s^H)] d[F(y|p^H,q) - F(y|p^L,q)] \geq 0.
\end{aligned}$$

Second, suppose that $[y - t(y;s) + c]$ is non-negative and log-SPM. Together with Assumption (MLRP), $[y-t(y;s)+c]f(y|p,q)$ is also non-negative and log-SPM in (y,s,p). It is well known that these two properties are jointly preserved under integration w.r.t. y, so that

$$\begin{aligned}
& [v(p^H,q,t(.;s^H)) + c][v(p^L,q,t(.;s^L)) + c] \\
\geq & [v(p^H,q,t(.;s^L)) + c][v(p^L,q,t(.;s^H)) + c].
\end{aligned}$$

When either of these conditions hold, the inequalities in Condition Global ID hold for $q^H = q^L$. This is a special case of the proof for Lemma 4 and hence,

$$\frac{v(p^H,q,t(.;s^H))}{v(p^L,q,t(.;s^H))} \geq \frac{v(p^H,q,t(.;s^L))}{v(p^L,q,t(.;s^L))} > 1.$$

This ensures the First Best allocation is supported by the candidate equilibrium with a single active market $(t(.;\widetilde{s}_1), q_1)$ in the case of homogeneous assets.

A.9 Proof of Lemma 7

Proof. Under Assumption (C) and Condition Survival-SPM,

$$v(p^H, q^H, t(.; s)) + v(p^L, q^L, t(.; s)) - v(p^H, q^L, t(.; s)) - v(p^L, q^H, t(.; s))$$
$$= -\int_{\underline{y}}^{\overline{y}} [F(y|p^H, q^H) + F(yp^L, q^L) - F(y|p^H, q^L) - F(y|p^L, q^H)]d[y - t(y; s)]$$
$$\geq 0.$$

Recall from the proof of Remark 6, SPM of $[y - t(y; s)]$ implies

$$v(p^H, q, t(.; s^H)) + v(p^L, q, t(.; s^L)) \geq v(p^H, q, t(.; s^L)) + v(p^L, q, t(.; s^H)).$$

Combining the two inequalities together yields Global ID. □

A.10 Proof of Lemma 8

Proof. Under Assumption (C), $[y - t(y; s) + c]$ is absolutely continuous, weakly increasing and strictly increasing for some interval of $[\underline{q}, \overline{q}]$.[81] By integration by parts,

$$v(p, q, t(.; s)) + c = [\underline{y} - t(\underline{y}; s) + c] + \int_{\underline{y}}^{\overline{y}} [1 - F(y|p, q)]d[y - t(y; s) + c].$$

The log-SPM of the survival function in Condition Survival-SPM is preserved under integration. $v(p, q, t(.; s)) + c$ then inherits log-SPM from $\int_{\underline{y}}^{\overline{y}}[1 - F(y|p, q)]d[y - t(y; s) + c]$, so that

$$[v(p^H, q^H, t(.; s)) + c][v(p^L, q^L, t(.; s)) + c]$$
$$\geq [v(p^H, q^L, t(.; s)) + c][v(p^L, q^H, t(.; s)) + c].$$

[81]For any interval (y^L, y^H), $y^H - y^L \geq [y^H - t(y^H; s)] - [y^L - t(y^L; s)] \geq 0$. Hence, $y - t(y; s) + c$ is absolutely continuous in y.

Recall from the proof of Remark 6, log-SPM of $[y - t(y; s) + c]$ and Assumption (MLRP) jointly imply

$$\begin{aligned}&[v(p^H, q, t(.; s^H)) + c][v(p^L, q, t(.; s^L)) + c]\\ \geq&[v(p^H, q, t(.; s^L)) + c][v(p^L, q, t(.; s^H)) + c].\end{aligned}$$

The two inequalities together yield Global ID. □

A.11 Proof of Remark 7

Proof. To prove this remark, I will make use of a result based on Theorem 2.1 in Karlin and Rinott (1980). Fix the pairs of $\{p^H, p^L\}$ and $\{q^H, q^L\}$. To saves on space, denote $f(y|p^H, q^H)$, $f(y|p^L, q^L)$, $f(y|p^L, q^H)$ and $f(y|p^H, q^L)$ by $f_{HH}(y)$, $f_{LL}(y)$, $f_{LH}(y)$ and $f_{HL}(y)$ respectively. For any $y, y' \in \Omega_y$, we have

$$f_{HH}(y \vee y')f_{LL}(y \wedge y') \geq f_{LH}(y)f_{HL}(y'). \tag{A.8}$$

Since $\int f(y|p, q)dy = 1$, we have the identity

$$\begin{aligned}0 =& \int f_{HH}(y)dy \int f_{LL}(z)dz - \int f_{LH}(y)dy \int f_{HL}(z)dz\\ =& \int_{\{y^H > y^L\}} \Big\{f_{HH}(y^H)f_{LL}(y^L) + f_{HH}(y^L)f_{LL}(y^H) - f_{LH}(y^H)f_{HL}(y^L)\\ &- f_{LH}(y^L)f_{HL}(y^H)\Big\}d(y^H, y^L).\end{aligned}$$

Fix a pair of (y^H, y^L). Denote $[f_{HH}(y^H)f_{LL}(y^L) - f_{LH}(y^H)f_{HL}(y^L)]$ by A, $[f_{HH}(y^H)f_{LL}(y^L) - f_{LH}(y^L)f_{HL}(y^H)]$ by B and

$$C = f_{HH}(y^H)f_{LL}(y^L)f_{HH}(y^L)f_{LL}(y^H) - f_{LH}(y^H)f_{HL}(y^L)f_{LH}(y^L)f_{HL}(y^H).$$

Applying the inequality (A.8), we obtain that A, B and C are all non-negative. Since $f_{HH}(y^H)f_{LL}(y^L)$ is positive throughout the support, the integrand can be expressed as

$$\frac{1}{f_{HH}(y^H)f_{LL}(y^L)}[AB + C)] \geq 0.$$

It follows that $AB = C = 0$ for any pair of (y^H, y^L).[82] The equality $C = 0$ and the inequality (A.8) imply that for $y \in \{y^H, y^L\}$,

$$f_{HH}(y)f_{LL}(y) = f_{LH}(y)f_{HL}(y).$$

Hence, $f(y|p,q)$ is pairwise log-modular in (p,q). Assumption (MLRP), together with the above equality imply that $A > 0$, and hence $B = 0$. The equality $B = 0$ again implies

$$f_{HH}(y^H)f_{HL}(y^L) = f_{HH}(y^L)f_{HL}(y^H).$$

Hence, $f(y|p,q)$ is pairwise log-modular in (y,q). □

A.12 Proof of Proposition 6

$\{U, V, \psi, \Psi, W, r, \mu\}$ is an equilibrium under the contract space Φ_t if and only if there exists a corresponding equilibrium $\{U, V, \psi, \Psi, W', r', \mu'\}$ under the contract space $\Phi_t \cup \{t^s\}$ with the same equilibrium payoff $\{U, V\}$, the same active markets $\Psi \subseteq \Phi_t$ and the same distribution of participants in every active market, $\{\mu|\Psi, r|\Psi, W|\Psi\} = \{\mu'|\Psi, r'|\Psi, W'|\Psi\}$.

[82] Alternatively, Condition Global ID can be obtained by replacing the inequality (A.8) with

$$\begin{aligned} & [y \vee y' - t(y \vee y'; s^H) + c][y \wedge y' - t(y \wedge y'; s^L) + c]f(y \vee y'|p^H, q^H)f(y \wedge y'|p^L, q^L) \\ \geq \quad & [y - t(y; s^H) + c][y' - t(y'; s^L) + c]f(y|p^L, q^H)f(y'|p^H, q^L). \end{aligned}$$

The same equality condition still applies when $s^H \to s^L$.

Proof. The if part is trivial because $\Psi \subseteq \Phi_t$. For only if part, I now construct a corresponding equilibrium.

For the markets $(t^s, q) \in \{t^s\} \times \{q_k\}_{k=1}^K$,

$$\mu(t^s, q) = \sup\{\mu' \in [0, \infty] : V(p) \geq \eta(\mu')v(p, q, t^s)\}.$$

If $\mu(t^s, q) < \infty$, then $r(t^s, q)$ is degenerate at the type

$$\min\{p \in \{p_l\}_{l=1}^L : V(p) = \mu(t^s, q)v(p, q, t^s)\}.$$

It is sufficient to show that an asset owner cannot gain from a posting the contract t^s. Suppose not, an owner of asset quality $q_{k'}$ profits from posting t^s. By construction, $r(t^s, q)$ is degenerate at some $p_{l'}$, where $v(p_{l'}, q_{k'}, t^s) \geq V(p_{l'})$ and $u(q_{k'}, p_{l'}, t^s) > U(q_{k'})$. s' can be chosen s.t. $v(p_{l'}, q_{k'}, t(.; s'))$ is slightly above $v(p_{l'}, q_{k'}, t^s)$ and hence $\mu(t(.; s'), q_{k'}) < \min\{\mu(t^s, q_{k'}), 1\}$. Since t^s is steeper than $t(.; s')$, $r(p_l|t(.; s'), q_{k'}) = 0$ if $p_l < p_{l'}$.[83] This contradicts with the equilibrium condition that the asset owner cannot gain from posting $t(.; s')$ as

$$U(q_{k'}) < \frac{\eta(\mu(t^s, q_{k'}))}{\mu(t^s, q_{k'})} u(q_{k'}, p_{l'}, t^s) < u(q_{k'}, r(t(.; s'), q_{k'}), t(.; s')).$$

□

For every equilibrium $\{U, V, \psi, \Psi, W, r, \mu\}$ under the contract space $\Phi_t \cup \{t^s\}$, there also exists an equilibrium $\{U, V, \psi', \Psi', W', r', \mu'\}$ under the same contract space, *for which the asset owners only contracts in Φ_t and supporting the same allocation, $\int_{\Omega_t \times \{q_k\}} r(p_l|t, q_k)dW = \int_{\Omega_t \times \{q_k\}} r'(p_l|t, q_k)dW'$ and $\int_{\Omega_t \times \{q_k\}} \mu(t, q_k)dW = \int_{\Omega_t \times \{q_k\}} \mu'(t, q_k)dW'$ for all p_l and q_k.*

[83] This result does not depend on the assumption of finite distribution of types. s' can be chosen such that $r(t(.; s'), q')$ contains types in some local neighborhood of p', which satisfies $u(q', p, t(.; s')) > U(q')$

Proof. I will first establish some properties of an equilibrium $\{U, V, \psi, \Psi, W, r, \mu\}$ with $t^s \in \psi(q_k)$ for some q_k.

The first property is that if $t^s \in \psi(q_{k'})$, then $\Omega_p(t^s, q_{k'})$ is a singleton. Suppose to the contrary that $\Omega_p(t^s, q_{k'})$ contains more than one type. Let $p_{l'}$ be the highest type in $\Omega_p(t^s, q_{k'})$ and so $u(q_{k'}, p_{l'}, t^s) > u(q_{k'}, r(t^s, q_{k'}), t^s)$. There exists $\widetilde{s}$ such that $E(t(Y; \widetilde{s})|p_{l'}, q_{k'}) = E(t^s(Y)|p_{l'}, q_{k'})$. It follows that for all $p < p_{l'}$, $v(p, q_{k'}, t^s) > v(p, q_{k'}, t(.; \widetilde{s}))$ because t^s is steeper than S_t. For sufficiently small $\epsilon > 0$,

$$V(p_{l'}) \leq \eta(\mu(t^s, q_{k'}))v(p_{l'}, q_{k'}, t^s) < \eta(\mu(t^s, q_{k'}))v(p_{l'}, q_{k'}, t(.; \widetilde{s} - \epsilon)),$$

but $V(p_l) > \eta(\mu(t^s, q_{k'}))v(p_l, q_{k'}, t(.; \widetilde{s} - \epsilon))$ for all $p_l < p_{l'}$ in the support of P.[84] The last inequality makes use of the incentive compatibility condition for workers with $p_l < p_{l'}$. Hence, $\mu(t(.; \widetilde{s} - \epsilon), q_{k'}) < \min\{\mu(t^s, q_{k'}), 1\}$ and $r(p_l|t(.; \widetilde{s} - \epsilon), q_{k'}) = 0$ if $p_l < p_{l'}$. Hence,

$$U(q_{k'}) < \frac{\eta(\mu(t^s, q_{k'}))}{\mu(t^s, q_{k'})}u(q_{k'}, r(t^s, q_{k'}), t^s) < u(q_{k'}, r(t(.; \widetilde{s} - \epsilon), q_{k'}), t(.; \widetilde{s} - \epsilon)).$$

and an asset owner will gain from posting a contract $t(.; \widetilde{s} - \epsilon)$.

The second property is that $\mu(t^s, q_{k'}) = 1$. Now let $\Omega_p(t^s, q_{k'}) = \{p_{l'}\}$.The case for $\mu(t^s, q_{k'}) > 1$ follows a similar argument. For sufficiently small $\epsilon > 0$, $V(p_{l'}) < v(p_{l'}, q_{k'}, t(.; \widetilde{s} - \epsilon))$ but $V(p_l) > v(p_l, q_{k'}, t(.; \widetilde{s} - \epsilon))$ for all $p_l < p_{l'}$. Hence, $\mu(t(.; \widetilde{s} - \epsilon), q_{k'}) < 1$ and $r(p|t(.; \widetilde{s} - \epsilon), q_{k'}) > 0$ only if $p \geq p_{l'}$. an asset owner will gain from posting $t(.; \widetilde{s} - \epsilon)$ for sufficiently small ϵ. Now consider $\mu(t^s, q_{k'}) < 1$. If a worker with type p is indifferent between participating in $(t^s, q_{k'})$ and a market $(t', q_{k'})$ with some tightness ratio μ', then define $\overline{\mu}(p|t', q_{k'}) = \mu'$. Otherwise, $\overline{\mu}(p|t', q_{k'}) = \infty$. Note that $\overline{\mu}(p_{l'}|t(.; \widetilde{s}), q_{k'}) = \mu(t^s, q_{k'})$. For sufficiently small $\epsilon > 0$, $\overline{\mu}(p_l|t(.; \widetilde{s}), q_{k'}) >$

[84] This result does not depend on the assumption of finite distribution of types.

$\overline{\mu}(p_{l'}|t(.;\widetilde{s}+\epsilon), q_{k'}) > \mu(t^s, q_{k'})$ whenever $p_l < p_{l'}$. Hence, $\mu(t(.;\widetilde{s}+\epsilon), q_{k'}) < 1$ and $r(p_l|t(.;\widetilde{s}+\epsilon), q_{k'}) > 0$ only if $p_l \geq p_{l'}$. an asset owner will gain from posting $t(.;\widetilde{s}+\epsilon)$ for sufficiently small ϵ.

It follows that $U(q_{k'}) = u(q_{k'}, p_{l'}, t(.;\widetilde{s}))$ and $V(p_{l'}) = v(p_{l'}, q_{k'}, t(.;\widetilde{s}))$. $V(p_l) > v(p_l, q_{k'}, t(.;\widetilde{s}))$ for all $p_l \neq p_{l'}$. This holds for $p_l < p_{l'}$ because $V(p_l) \geq v(p_l, q_{k'}, t^s)$ and t^s is steeper than $t(.;\widetilde{s})$. The part for $p_l > p_{l'}$ comes from the fact that an owner of asset quality $q_{k'}$ cannot profit from the deviation of posting $t(.;\widetilde{s})$. Note that if $\psi(q_{k'})$ also contains $t(.;\widetilde{s})$, it must be that $\Omega_p(t(.;\widetilde{s}), q_{k'}) = \{p_{l'}\}$ and $\mu(t(.;\widetilde{s}), q_{k'}) = 1$. Otherwise, the asset owners will not be indifferent between the two markets. Putting all together, all equilibrium conditions will still be met if those asset owners of $q_{k'}$ posting t^s and the workers participating in $(t^s, q_{k'})$ all switch to the market $(t(.;\widetilde{s}), q_{k'})$. In the new equilibrium, $\Omega_p(t(.;\widetilde{s}), q_{k'}) = \{p_{l'}\}$ and $\mu(t(.;\widetilde{s}), q_{k'}) = 1$. It is trivial that the resulting allocation is unchanged. $\square$

A.13 Proof of Proposition 5

Consider two ordered set of securities S_t^s and S_t^f, where S_t^s is steeper than S_t^f. Fix a distribution of types, the contracts $t^s(.;s)$ and $t^f(.;s)$ are indexed in a manner that same set of contract terms $\{\widetilde{s}_k\}_{k \geq \underline{k}}$ are posted in the two candidate equilibriums under the contract space S_t^s and S_t^f.

Proof. First, notice that $v(p_{\bar{l}(\underline{k})}, q_{\underline{k}}, t^s(.;\widetilde{s}_{\underline{k}})) \leq v(p_{\bar{l}(\underline{k})}, q_{\underline{k}}, t^f(.;\widetilde{s}_{\underline{k}}))$. If workers are on the long side, then $v(p_{\underline{l}}, q_1, t^s(.;\widetilde{s}_1)) = v(p_{\underline{l}}, q_1, t^f(.;\widetilde{s}_1)) = \underline{V}$. The above inequality holds because $t^f(.;\widetilde{s}_1)$ is flatter. If workers are on the short side, then $u(q_{\underline{k}}, r_q^{FB}(q_{\underline{k}}), t^s(.;\widetilde{s}_{\underline{k}})) = u(q_{\underline{k}}, r_q^{FB}(q_{\underline{k}}), t^f(.;\widetilde{s}_{\underline{k}})) = \underline{U}$. Suppose, to the contrary, that $v(p_{\bar{l}(\underline{k})}, q_{\underline{k}}, t^s(.;\widetilde{s}_{\underline{k}})) > v(p_{\bar{l}(\underline{k})}, q_{\underline{k}}, t^f(.;\widetilde{s}_{\underline{k}})) =$

$v(p_{\bar{l}(\underline{k})}, q_{\underline{k}}, t^s(.; s'))$. Then $v(p_l, q_{\underline{k}}, t^f(.; \widetilde{s}_{\underline{k}})) \geq v(p_l, q_{\underline{k}}, t^s(.; s'))$ whenever $\bar{l}(\underline{k}) \geq l \geq 1$. By ex-post budget balance,

$u(q_{\underline{k}}, r_q^{FB}(q_{\underline{k}}), t^s(.; \widetilde{s}_{\underline{k}})) > u(q_{\underline{k}}, r_q^{FB}(q_{\underline{k}}), t^s(.; s')) \geq u(q_{\underline{k}}, r_q^{FB}(q_{\underline{k}}), t^f(.; \widetilde{s}_{\underline{k}}))$!!!

The indifference condition in (2.10) implies that

$$\begin{aligned} v(p_{\bar{l}(\underline{k})}, q_{\underline{k}+1}, t^s(.; \widetilde{s}_{\underline{k}+1})) &= v(p_{\bar{l}(\underline{k})}, q_{\underline{k}}, t^s(.; \widetilde{s}_{\underline{k}})) \\ &\leq v(p_{\bar{l}(\underline{k})}, q_{\underline{k}}, t^f(.; \widetilde{s}_{\underline{k}})) = v(p_{\bar{l}(\underline{k})}, q_{\underline{k}+1}, t^f(.; \widetilde{s}_{\underline{k}+1})). \end{aligned}$$

Hence, there exists $v(p_{\bar{l}(\underline{k})}, q_{\underline{k}+1}, t^s(.; \widetilde{s}_{\underline{k}+1})) = v(p_{\bar{l}(\underline{k})}, q_{\underline{k}+1}, t^f(.; s''))$ for some $s'' \geq \widetilde{s}_{\underline{k}+1}$. Now consider any higher type $l \geq \bar{l}(\underline{k})$, in particular $\bar{l}(\underline{k}+1)$,

$$v(p_l, q_{\underline{k}+1}, t^s(.; \widetilde{s}_{\underline{k}+1})) \leq v(p_l, q_{\underline{k}+1}, t^f(.; s'')) \leq v(p_l, q_{\underline{k}+1}, t^f(.; \widetilde{s}_{\underline{k}+1})).$$

Hence, $u(q_{\underline{k}+1}, r_q^{FB}(q_{\underline{k}+1}), t^s(.; \widetilde{s}_{\underline{k}+1})) \geq u(q_{\underline{k}+1}, r_q^{FB}(q_{\underline{k}+1}), t^f(.; \widetilde{s}_{\underline{k}+1}))$.

By induction, it follows that under the contract space S_t^s, $U(q_k)$ is higher whereas $V(p_l)$ is lower for $p_l \geq p_{\bar{l}(\underline{k})}$. $\square$

Appendix B

Appendix for chapter 3

B.1 Proof of Remark 8

Eeckhout and Kircher (2010) show that the boundary value problem for the Second Best allocation admits a solution.[85] Fix $(\underline{p}_{SB}, \underline{q}_{SB}, r_{SB}, \lambda_{SB}, v_{SB}, u_{SB})$, I will first show that it satisfies the inequality (3.12), and use the inequality to establish uniqueness of the Second Best allocation. Define

$$\widehat{U}(p,q) = \max_{\lambda \geq 0}\{\delta(\lambda)y(p,q) - \lambda v_{SB}(p)\}.$$

Since $\delta(\lambda)$ is strictly concave, the unique maximizer, denoted by $\widehat{\lambda}(p,q)$, is determined by the FOC,

$$\delta'(\widehat{\lambda}(p,q))y(p,q) = v_{SB}(p).$$

We first consider $p \geq \underline{p}_{SB}$. From the condition (3.8),

$$v_{SB}(p) = \delta'(\lambda_{SB}(\kappa_{SB}(p)))y(p, \kappa_{SB}(p)).$$

Therefore, $\widehat{\lambda}(p,q) > (<) \lambda_{SB}(\kappa_{SB}(p))$ if $q > (<) \kappa_{SB}(p)$. By envelope theo-

[85] Though Eeckhout and Kircher (2010) assume the values of outside options to be zero, their proof is readily extended to cover the case with positive outside options.

rem,

$$
\begin{aligned}
&\frac{\partial}{\partial p}\widehat{U}(p,q) \\
&= \delta(\widehat{\lambda}(p,q))\frac{\partial}{\partial p}y(p,q) - \widehat{\lambda}(p,q)\frac{\partial}{\partial p}v_{SB}(p) \\
&= \delta(\widehat{\lambda}(p,q))\frac{\partial}{\partial p}y(p,q) - \widehat{\lambda}(p,q)\eta(\lambda_{SB}(\kappa_{SB}(p)))\frac{\partial}{\partial p}y(p,\kappa_{SB}(p)) \\
&= \widehat{\lambda}(p,q)v_{SB}(p)\left[\frac{\eta(\widehat{\lambda}(p,q))}{\delta'(\widehat{\lambda}(p,q))}\frac{\partial \ln y(p,q)}{\partial p} - \frac{\eta(\lambda_{SB}(\kappa_{SB}(p)))}{\delta'(\lambda_{SB}(\kappa_{SB}(p)))}\frac{\partial \ln y(p,\kappa_{SB}(p))}{\partial p}\right]
\end{aligned}
$$

The second inequality obtained by substituting the condition (3.9). Under Assumption (Y) and (M), $\frac{\partial}{\partial p}\widehat{U}(p,q) > (<)0$ if $q > (<)\kappa_{SB}(p)$. For $p < \underline{p}_{SB}$, $v_{SB}(p) = \underline{V} = v_{SB}(\underline{p}_{SB})$, so $\widehat{U}(p,q) < \max_{\lambda\geq 0}\{\delta(\lambda)y(\underline{p}_{SB},q) - \lambda v_{SB}(\underline{p}_{SB})\} = \widehat{U}(\underline{p}_{SB},q)$.

Putting together, for $q \geq \underline{q}_{SB}$ and any $p \in [0,1]$

$$
\begin{aligned}
u_{SB}(q) &= \delta(\lambda_{SB}(q))y(r_{SB}(q),q) - \lambda_{SB}(q)v_{SB}(r_{SB}(q)) \\
&= \widehat{U}(r_{SB}(q),q) = \max_{\lambda\geq 0}\{\delta(\lambda)y(p,q) - \lambda v_{SB}(p)\}
\end{aligned}
$$

For $q < \underline{q}_{SB}$, the boundary condition requires that

$$
u_{SB}(q) = \underline{U} = \max_{\lambda\geq 0}\{\delta(\lambda)y(p,\underline{q}_{SB}) - \lambda v_{SB}(p)\} > \max_{\lambda\geq 0}\{\delta(\lambda)y(p,q) - \lambda v_{SB}(p)\}.
$$

This establishes the inequality (3.12).

Let $supp(L)$ denote the support of measure L. The total surplus for (K,L) is given by

$$
\begin{aligned}
&\int_{supp(L)} \eta(\frac{dL_{qs}}{dK})y(p,q)dL + [F(1) - L_p(1)]\underline{V} + [G(1) - K_q(1)]\underline{U} \\
\leq\ & \int_{supp(L)} \frac{dK}{dL_{qs}}u_{SB}(q) + v_{SB}(p)dL + [F(1) - L_p(1)]\underline{V} + [G(1) - K_q(1)]\underline{U} \\
\leq\ & \int u_{SB}(q)dG(q) + \int v_{SB}(p)dF(p) \\
=\ & \int_{\underline{q}}^{1} \delta(\lambda_{SB}(q))y(r_{SB}(q),q)dG(q) + F(\underline{p})\underline{V} + G(\underline{q})\underline{U}
\end{aligned}
$$

The first inequality is due to the inequality (3.12) and the second inequality stems from the boundary conditions for the Second Best allocation. The above inequality holds with equality if and only if (K, L) features PAM with $(\underline{p}_{SB}, \underline{q}_{SB}, \kappa_{SB})$ and $\frac{dL_{qs}}{dK} = \lambda_{SB}(q)$ almost everywhere in the support of L.

B.2 Proof of Proposition 7

In this proof, I first show the properties listed in Proposition 7. I then proceed to show that the boundary value problem in system (3.16) admits a unique solution. Note that $(\widetilde{r}, \widetilde{\lambda}, \widetilde{v})$ in any solution must be continuously differentiable and strictly increasing. In particular, $\frac{\partial \ln \widetilde{v}(p)}{\partial p}\Big|_{p=\widetilde{r}(q)} = \frac{\partial \ln y(p,q)}{\partial p}\Big|_{(p,q=(\widetilde{r}(q),q)}$, so the gain from matching with a better asset must be offset by a reduction in $\delta'(\widetilde{\lambda}(q))$. Hence, $\widetilde{\lambda}$ is strictly increasing.

Characterization of the equilibria

Any equilibrium satisfies the properties in Proposition 7.

"Only if"

First fix an equilibrium (K, L). The assumption in (3.1) ensures that the set of active markets is non-empty. Suppose not, consider the inactive market $(1, s')$ where s' and λ' satisfy $\underline{V} = \delta'(\lambda')y(1,1) = \eta(\lambda')(1-s')y(1,1)$. Since $\eta(\lambda')(1-s')y(p,1) < \underline{V}$ if $p < 1$, only the best workers will be attracted to this inactive market, and the resulting queue length is λ'. The deviating payoff for an owner of asset quality q is $[\delta(\lambda') - \delta'(\lambda')\lambda']y(1,1) > \underline{U}$! Since participation is costly, every active market must have a positive finite queue length, and $s \in (0,1)$. Furthermore, the participation on the workers side must be monotonic.

Step 1: All equilibria feature PAM

Suppose not, then there must exist $\{(q^H, s_1), (q^L, s_0)\} \in \Psi$, where $q^H > q^L$, and $p^H > p^L$ where p^L and p^H are in the support of $R(q^H, s_1)$ and $R(q^L, s_0)$ respectively. The workers' acceptance decisions are optimal only if

$$\begin{aligned} \eta(\Lambda(q^L, s_0))(1 - s_0)y(p^H, q^L) &\geq \eta(\Lambda(q^H, s_1))(1 - s_1)y(p^H, q^H), \text{ and} \\ \eta(\Lambda(q^H, s_1))(1 - s_1)y(p^L, q^H) &\geq \eta(\Lambda(q^L, s_0))(1 - s_0)y(p^L, q^L), \end{aligned}$$

This implies $y(p^H, q^L)y(p^L, q^H) \geq y(p^H, q^H)y(p^L, q^L)$, which contradicts strict log-SPM of $y(p, q)$!

Since the equilibrium allocation features PAM, the threshold types $(\underline{p}, \underline{q})$ and $\kappa(p)$ are well-defined. κ is strictly increasing because the distribution of types is atomless and every active market must have a positive finite queue length.

Step 2: $U(q)$ is strictly increasing for $q \geq \underline{q}$

Suppose $(q^L, s_0) \in \Psi$ and $q^L = \kappa(p^L)$. Fix $\widehat{q} > q^L$. Consider the inactive market $(\widehat{q}, \widehat{s})$, where $\widehat{s}$ is given by

$$(1 - \widehat{s})y(p^L, \widehat{q}) = (1 - s_0)y(p^L, q^L).$$

$\Lambda(\widehat{q}, \widehat{s}) \geq \Lambda(q^L, s_0)$ because $V(p^L) \geq \eta(\Lambda(\widehat{q}, \widehat{s}))(1 - \widehat{s})y(p^L, \widehat{q})$. For all $p < p^L$,

$$V(p) \geq \eta(\Lambda(q^L, s_0))(1 - s_0)y(p, q^L) > \eta(\Lambda(\widehat{q}, \widehat{s}))(1 - \widehat{s})y(p, \widehat{q}).$$

The strict inequality follows from log-SPM of $y(p, q)$. It follows that the support of $R(\widehat{q}, \widehat{s})$ contains no type below p^L. An owner of asset $\widehat{q}$ can ensure herself a payoff of

$$\delta(\Lambda(\widehat{q}, \widehat{s}))\widehat{s} \int y(p, \widehat{q})dR(\widehat{q}, \widehat{s}) > U(q^L).$$

Therefore, the participation of the asset side is monotonic, and the function $r(q)$ is well-defined. By definition,

$$V(p) = \eta(\Lambda(q,s))(1-s)y(p,q) \ \textit{if} \ q = \kappa(p) \text{ and } (q,s) \in \Psi.$$

Step 3: For any $p \in [0,1]$ *and* $(q,s) \in \Psi$,

$$V(p) > \eta(\Lambda(q,s))(1-s)y(p,q) \ \textit{if} \ q \neq \kappa(p).$$

Consider two active markets (q^H, s_1) and (q^L, s_0), where $q^H > q^L$. Suppose a worker of type $r(q^H) > 0$ is indifferent between these two markets. His acceptance decision is optimal only if

$$\begin{aligned} \eta(\Lambda(q^L,s_0))(1-s_0)y(r(q^H),q^L) &= \eta(\Lambda(q^H,s_1))(1-s_1)y(r(q^H),q^H) \\ &\geq \eta(\Lambda(q,s'))(1-s')y(r(q^H),q) \end{aligned}$$

for all $(q,s') \in \Psi$ where $q \in (q^L, q^H)$. Strict log-SPM of $y(p,q)$ implies that for any $p < r(q^H)$,

$$\frac{\eta(\Lambda(q^L,s_0))(1-s_0)}{\eta(\Lambda(q^H,s_1))(1-s_1)} = \frac{y(r(q^H),q^H)}{y(r(q^H),q^L)} > \frac{y(p,q^H)}{y(p,q^L)}$$

and

$$\frac{\eta(\Lambda(q^L,s_0))(1-s_0)}{\eta(\Lambda(q',s'))(1-s')} \geq \frac{y(r(q^H),q')}{y(r(q^H),q^L)} > \frac{y(p,q')}{y(p,q^L)},$$

for all $(q',s') \in \Psi$ where $q' \in (q^L, q^H)$. PAM then implies that $\Lambda(q',s') = 0$ if $(q',s') \in \Psi$ and $q' \in (q^L, q^H)$. Hence, $U(q') = \underline{U} \leq U(q^L)$, contradicting our previous claim!

Suppose $1 > q^L = k(p^L)$ and $(q^L, s_0) \in \Psi$. A symmetric argument rules out the case that a worker of type p^L is indifferent between (q^L, s_0) and another active market (q^H, s_1) where $q^H > q^L$.

Step 4:Characterize active marketsΨ.

Lemma 13. *Suppose* $(q,s') \in \Psi$ *and for any* $p \in [0,1]$,

$$V(p) \geq \eta(\Lambda(q,s'))(1-s')y(p,q),$$

with equality if and only if $q = \kappa(p)$. *Then for any* $s \in [0,1)$, $R(q,s)$ *is degenerate at* $r(q)$ *if* $\Lambda(q,s) > 0$. *Furthermore, an owner of asset quality* q *has no profitable deviations if and only if* $\Lambda(q,s')$ *satisfies*

$$\delta'(\Lambda(q,s')) = \eta(\Lambda(q,s'))(1-s'). \tag{B.1}$$

Proof. For $s \in [0,1)$ and $p \neq r(q)$,

$$\frac{V(p)}{V(r(q))} > \frac{y(p,q)}{y(r(q),q)} = \frac{\eta(\Lambda(q,s))(1-s)y(p,q)}{\eta(\Lambda(q,s))(1-s)y(r(q),q)}.$$

Suppose $\Lambda(q,s) > 0$, then $V(p) = \eta(\Lambda(q,s))(1-s)y(p,q)$ if and only if $p = r(q)$, and hence $R(q,s)$ is degenerate at $r(q)$. In this case, $\Lambda(q,s)$ is determined by

$$V(r(q)) = \eta(\Lambda(q,s'))(1-s')y(r(q),q) = \eta(\Lambda(q,s))(1-s)y(r(q),q).$$

An asset owner has no profitable deviations if and only if

$$U(q) \geq \delta(\Lambda(q,s))sy(r(q),q),$$

with equality at $s = s'$. This can further simplified as

$$\Lambda(q,s') \in \arg\max_{\lambda\in[0,\infty]} \delta(\lambda) - \lambda\eta(\Lambda(q,s'))(1-s').$$

Since $\delta(\lambda)$ is strictly concave and $\Lambda(q,s') \in (0,\infty)$, the above holds if and only if the equality (B.1) holds. □

Step 5: Establish the boundary value problem (3.16).

There exists a function $\widetilde{\lambda} : [0,1] \to (0,\infty)$ *such that*

$$\Psi = \{(q,s) : q \in [\underline{q},1], s = 1 - \frac{d\ln\delta}{d\ln\lambda}\bigg|_{\lambda=\widetilde{\lambda}(q)}\},$$

and $\Lambda(q,s) = \widetilde{\lambda}(q)$ *for* $(q,s) \in \Psi$. $(\widetilde{\lambda}, V, r)$ *is continuously differentiable and satisfies the differential equation system in (3.16) along with* $\underline{q}$.

Recall that the participation is monotonic on both sides, so there is at least one active market (q', s') for any $q' \in [\underline{q}, 1]$. Consider the workers of $p \geq \underline{p}$. Substitute the equality (B.1), the expression of their equilibrium payoff can be expressed as

$$V(r(q')) = \delta'(\Lambda(q', s'))y(r(q'), q'),$$

and incentive compatibility requires

$$V(r(q')) = \max_{(q,s)\in\Psi} \{\delta'(\Lambda(q,s))y(r(q'), q)\}.$$

The envelope theorem implies that V is continuously differentiable (C^1). Since $\delta'(\lambda)$ and $y(p,q)$ are C^1, there must exist a C^1 function $\lambda : [0,1] \to (0,\infty)$ such that

$$\left.\frac{\partial V(p)}{\partial p}\right|_{p=r(q)} = \delta'(\lambda(q)) \left.\frac{\partial y(p,q)}{\partial p}\right|_{(p,q)=(r(q),q)}.$$

This also establishes that for each asset quality $q \geq \underline{q}$, there is exactly one active market (q', s') with $s' = 1 - \frac{d\ln\delta}{d\ln\lambda}\big|_{\lambda=\lambda}$ Furthermore, $\Lambda(q', s') = \lambda(q')$.

Given full participation for $p \in [\underline{p}, 1]$ and $q \in [\underline{q}, 1]$, λ and r must satisfy the law of motion (3.7).

It remains to show the boundary conditions for the threshold types. Suppose $\underline{p} > 0$. Since $y(p,q)$ is strictly increasing in p, the workers $p < \underline{p}$ all take their outside option only if $V(\underline{p}) = \underline{V}$. Now suppose $\underline{q} > 0$, $(\underline{q}, \underline{s}) \in \Psi$ and $U(\underline{q}) > \underline{U}$. If $\underline{p} = 0$, an owner of q' slightly below $\underline{q}$ can secure a payoff $\delta(\lambda(\underline{q}))s'y(0,q') > \underline{U}$ by posting a share s' satisfying $\eta(\lambda(\underline{q}))(1-s')y(0,q') = V(0)$. We turn to the case $\underline{p} > 0$ so that $V(\underline{p}) = \eta(\lambda(\underline{q}))(1-\underline{s})y(\underline{p}, \underline{q}) = \underline{V}$.

By continuity, for q' slightly below $\underline{q}$, there must exist $s' < \underline{s}$ and $p' < \underline{p}$ satisfying both

$$\delta(\lambda(\underline{q}))s'y(p', q') > \underline{U}$$
$$\eta(\lambda(\underline{q}))(1-s')y(p', q') = \underline{V}$$

It follows that $\Lambda(q', s') \geq \lambda(\underline{q})$, and $\eta(\Lambda(q', s'))(1-s')y(p, q') < \underline{V}$ for any $p < p'$. Hence, the expected payoff for an asset owner to participate in (q', s') must be above $\underline{U}$, rendering the deviation profitable. Therefore, $U(\underline{q}) = \underline{U}$ if $\underline{q} > 0$.

The preceding analysis verifies all properties in Proposition 7.

"If"

Fix a solution $(\underline{\widetilde{p}}, \underline{\widetilde{q}}, \widetilde{r}, \widetilde{\lambda}, \widetilde{v})$ to the boundary value problem in system (3.16). One can recover a unique candidate equilibrium $(\widetilde{K}, \widetilde{L})$ satisfying the properties in listed Proposition 7. Let $\widetilde{\kappa}$ denote the inverse of $\widetilde{r}$. Define $\widetilde{s}(q) = 1 - \frac{d \ln \delta}{d \ln \lambda}\big|_{\lambda = \widetilde{\lambda}(q)}$. Since $\widetilde{\lambda}(q)$ is continuous and strictly increasing, $\widetilde{s}(q)$ is also continuous and increasing in q under Assumption (M). $\widetilde{K}(q', s') = 0$ if $q' \leq \underline{\widetilde{q}}$ or $s' \leq \widetilde{s}(\underline{\widetilde{q}})$. Otherwise, $\widetilde{K}(q', s') = G(\sup\{q \leq q' : \widetilde{s}(q) \leq s'\}) - G(\underline{\widetilde{q}})$. $\widetilde{L}(p', q', s') = F(\sup\{p \leq p' : \widetilde{\kappa}(p) \leq q', \widetilde{s}(\widetilde{\kappa}(p)) \leq s'\}) - F(\underline{\widetilde{p}})$ if $p > \underline{\widetilde{p}}$, $q' > \underline{\widetilde{q}}$ and $s' > \widetilde{s}(\underline{\widetilde{q}})$. Otherwise, $\widetilde{L}(p', q', s') = 0$.

I first verify that the workers' acceptance decision is optimal. $V(p) = \widetilde{v}(p) > \underline{V}$ for $p > \underline{\widetilde{p}}$. Combining the conditions (3.8) and (3.14),

$$\frac{\partial \ln \widetilde{v}(p)}{\partial p} = \frac{\partial \ln y(p, q)}{\partial p}\bigg|_{q = \widetilde{\kappa}(p)}, p \geq \underline{\widetilde{p}}.$$

Consider any $p_0 \geq \underline{\widetilde{p}}$ and $p_1 \neq p_0$,

$$\begin{aligned}
& \ln V(p_1) - \ln \delta'(\widetilde{\kappa}(p_0))y(p_1, \widetilde{\kappa}(p_0)) \\
= \;& [\ln V(p_1) - \ln \widetilde{v}(p_0)] - [\ln y(p_1, \widetilde{\kappa}(p_0)) - \ln y(p_0, \widetilde{\kappa}(p_0))] \\
= \;& \int_{p_0}^{p_1} \frac{\partial \ln V(p)}{\partial p} - \left.\frac{\partial \ln y(p,q)}{\partial p}\right|_{q=\widetilde{\kappa}(p_0)} dp \\
\geq \;& \int_{p_0}^{\max\{p_1, \underline{\widetilde{p}}\}} \left.\frac{\partial \ln y(p,q)}{\partial p}\right|_{q=\widetilde{\kappa}(p)} - \left.\frac{\partial \ln y(p,q)}{\partial p}\right|_{q=\widetilde{\kappa}(p_0)} dp > 0.
\end{aligned}$$

The last strict inequality is due to the strict log-SPM of $y(p,q)$ and $\widetilde{\kappa}$ is strictly increasing. Recall that $\eta(\Lambda(q,s))(1-s) = \delta'(\widetilde{\lambda}(q))$ holds for any active market (q,s). Therefore, a worker of $p = \widetilde{r}(q)$ receives his highest payoff only at $(q,s) \in \Psi$, and the outside option is optimal for the workers of $p < \underline{\widetilde{p}}$.

We now turn to the asset side. For $q \geq \underline{\widetilde{q}}$,

$$U(q) = [\delta(\widetilde{\lambda}(q)) - \widetilde{\lambda}(q)\delta'(\widetilde{\lambda}(q))]y(\widetilde{r}(q), q) \geq \underline{U}.$$

Together with Lemma 13, the contract posting decision is optimal for owners of $q \geq \underline{\widetilde{q}}$.

Now suppose $\underline{\widetilde{q}} > 0$. Consider an inactive market (q^L, s') where $q^L < \underline{\widetilde{q}}$. For any $p^H > \underline{\widetilde{p}}$, $R(p^H|q^L, s') = 0$ because

$$\frac{V(p^H)}{V(\underline{\widetilde{p}})} > \frac{\delta'(\widetilde{\lambda}(\underline{\widetilde{q}}))y(p^H, \underline{\widetilde{q}})}{\delta'(\widetilde{\lambda}(\underline{\widetilde{q}}))y(\underline{\widetilde{p}}, \underline{\widetilde{q}})} > \frac{(1-s')y(p^H, q^L)}{(1-s')y(\underline{\widetilde{p}}, q^L)}.$$

Hence, $R(q^L, s')$ is degenerate at some $p^L \leq \underline{\widetilde{p}}$. The case $\Lambda(q^L, s') = 0$ is trivial. For the case $\Lambda(q^L, s') > 0$, $\Lambda(q^L, s')$ satisfies $\eta(\Lambda(q^L, s'))(1 - s')y(p^L, q^L) = V(p^L)$. If deviating to the market (q^L, s'), an asset owner will receive

$$\begin{aligned}\delta(\Lambda(q^L,s'))s'y(p^L,q^L) &= \delta(\Lambda(q^L,s'))y(p^L,q^L)-\Lambda(q^L,s')V(p^L)\\ &\leq \max_{\lambda}[\delta(\lambda)y(p^L,q^L)-\lambda V(p^L)]\\ &= \max_{\lambda}[\delta(\lambda)y(p^L,q^L)-\lambda V(\underline{\widetilde{p}})] < U(\underline{\widetilde{q}}) = \underline{U}.\end{aligned}$$

The second equality holds because of the boundary condition $\underline{\widetilde{p}}(V(\underline{\widetilde{p}})-\underline{V}) = 0$. So it is never optimal for an owner of $q < \underline{\widetilde{q}}$ to participate.

Analysis of the boundary value problem

By differentiating the Hosios condition w.r.t. q and subtracting it with the expression $\left.\frac{\partial \ln \widetilde{v}(p)}{\partial p}\right|_{p=\widetilde{r}(q)}$, we obtain

$$\frac{\partial \ln \delta'(\widetilde{\lambda}(q))}{\partial q} = -\frac{\partial \ln y(\widetilde{r}(q),q)}{\partial q}$$

There exists a unique pair of $\underline{\lambda}$ and $\overline{\lambda}$ satisfying $[\delta(\underline{\lambda})-\delta'(\underline{\lambda})\underline{\lambda}]y(1,1)=\underline{U}$ and $\delta'(\overline{\lambda})y(1,1)=\underline{V}$ respectively. Note that $\overline{\lambda}>\underline{\lambda}$. Consider the following initial value problem (IPV-$\lambda(1)$):

$$\begin{aligned} r'(q) &= \frac{g(q)}{f(r(q))}\lambda(q),\\ \frac{\partial \ln \delta'(\lambda(q))}{\partial q} &= -\frac{\partial \ln y(r(q),q)}{\partial q},\end{aligned}$$

where the initial values are given by $r(1)=1$ and $\lambda(1)=\lambda^1\in[\underline{\lambda},\overline{\lambda}]$. Since the differential equation system is locally Lipschitz, Picard's existence theorem ensures that IPV-$\lambda(1)$ (in the downward direction) admits a unique solution $\{r(q;\lambda^1),\lambda(q;\lambda^1)\}$ over the interval $[\underline{q}(\lambda^1),1]$, where $\underline{q}(\lambda^1)$ is the first level of q where either of the following cases occurs:

$$0 = r(q;\lambda^1)[\delta'(\lambda(q;\lambda^1))y(r(q;\lambda^1),q)-\underline{V}]=0,\ \text{or} \tag{B.2}$$

$$0 = q[(\delta(\lambda(q;\lambda^1))-\delta'(\lambda(q;\lambda^1))\lambda(q;\lambda^1))y(r(q;\lambda^1),q)-\underline{U}]. \tag{B.3}$$

Furthermore, $\underline{q}(\lambda^1)$ and $\underline{p}(\lambda^1) := r(\underline{q}(\lambda^1); \lambda^1)$ are continuous in λ^1. For $q > \underline{q}(\lambda^1)$, $\lambda(q; \lambda^1)$ and $r(q; \lambda^1)$ are strictly increasing.

For $p \in [\underline{p}(\lambda^1), 1], \kappa(p; \lambda^1)$ denote the inverse of $r(q; \lambda^1)$. Also define

$$\begin{aligned} v(p; \lambda^1) &= \delta'(\lambda(\kappa(p; \lambda^1); \lambda^1))y(p, \kappa(p; \lambda^1)), p \in [\underline{p}(\lambda^1), 1], \text{ and} \\ u(q; \lambda^1) &= [\delta(\lambda(q; \lambda^1)) - \delta'(\lambda(q; \lambda^1))\lambda(q; \lambda^1)]y(r(q; \lambda^1), q \in [\underline{q}(\lambda^1), 1]. \end{aligned}$$

Notice that for $q > \underline{q}(\lambda^1)$, $r(q; \lambda^1)[v(r(q; \lambda^1); \lambda^1) - \underline{V}]$ and $q[u(q; \lambda^1) - \underline{U}]$ are positive and strictly increasing in q.[86]

Existence of a solution

The boundary value problem has a solution if there exists some λ^1 such that the solution to the IPV-$\lambda(1)$ with $\lambda(1) = \lambda^1$ satisfies both condition (B.2) and (B.3) at $q = \underline{q}(\lambda^1)$. By construction, condition (B.2) holds at $q = \underline{q}(\overline{\lambda})$ and condition (B.3) holds at $q = \underline{q}(\underline{\lambda})$. Consider

$$\widehat{\lambda} = \inf\{\lambda' \geq \underline{\lambda} : [v(\underline{p}(\lambda^1); \lambda^1) - \underline{V}]\underline{p}(\lambda^1) = 0, \forall \lambda^1 \geq \lambda'\}.$$

By continuity, $[v(\underline{p}(\widehat{\lambda}); \widehat{\lambda}) - \underline{V}]\underline{p}(\widehat{\lambda}) = 0$. If $\widehat{\lambda} = \underline{\lambda}$, then we have argued that condition (B.3) also holds at $q = \underline{q}(\widehat{\lambda})$. Suppose $\widehat{\lambda} > \underline{\lambda}$, the construction of $\widehat{\lambda}$ ensures that there is a convergent sequence $\{\lambda_n^1\}$ with limit $\widehat{\lambda}$ such that $\lambda_n^1 < \widehat{\lambda}$ and only condition (B.3) holds at $q = \underline{q}(\lambda_n^1)$ for $\lambda^1 = \lambda_n^1$. By continuity, $\underline{q}(\widehat{\lambda})[u(\underline{q}(\widehat{\lambda}); \widehat{\lambda}) - \underline{U}] = 0$ must hold as well. Therefore the solution to the IPV-$\lambda(1)$ with $\lambda(1) = \widehat{\lambda}$ solves the boundary value problem.

Uniqueness of the solution

Suppose $\lambda^H > \lambda^L$. *For* $p \in [\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$, $\kappa(p; \lambda^H) > \kappa(p; \lambda^L)$, $v(p; \lambda^H) < v(p; \lambda^L)$ *and* $\lambda(\kappa(p; \lambda^H); \lambda^H) > \lambda(\kappa(p; \lambda^L); \lambda^L)$.

Proof. Since $r'(1; \lambda^H) > r'(1; \lambda^L)$, $\kappa(p; \lambda^H) > \kappa(p; \lambda^L)$, $\lambda(\kappa(p; \lambda^H); \lambda^H) >$

[86] $v(r(q; \lambda^1); \lambda^1)$ is stricly increasing in q because $\frac{\partial \ln \delta'(\lambda(q;\lambda^1))}{\partial q} + \frac{\partial \ln y(r(q;\lambda^1),q)}{\partial q} = 0$.

$\lambda(\kappa(p;\lambda^L);\lambda^L)$ and $v(p;\lambda^H) < v(p;\lambda^L)$ must hold in some neighborhood of $p = 1$.

Consider the case that $\kappa(.;\lambda^H)$ and $\kappa(.;\lambda^L)$ intersects somewhere in $[\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$. $p_\kappa = \max\{p < 1 : \kappa(p;\lambda^H) = \kappa(p;\lambda^L)\}$ is then well-defined and by construction, $\kappa(p;\lambda^H) > \kappa(p;\lambda^L)$ for all $p \in (p_\kappa, 1)$. It follows that $v(.;\lambda^H)$ and $v(.;\lambda^L)$ must intersect somewhere in $[p_\kappa, 1)$. Otherwise, for all $p \in (p_\kappa, 1]$,

$$\begin{aligned} \delta'(\lambda(\kappa(p;\lambda^H);\lambda^H))y(p,\kappa(p;\lambda^H)) &= v(p;\lambda^H) \\ < v(p;\lambda^L) &= \delta'(\lambda(\kappa(p;\lambda^L);\lambda^L))y(p,\kappa(p;\lambda^L)), \end{aligned}$$

and hence $\lambda(\kappa(p;\lambda^H);\lambda^H) > \lambda(\kappa(p;\lambda^L);\lambda^L)$. This contradicts the law of motion,

$$\begin{aligned} 0 > &\int_{p_\kappa}^1 \frac{1}{\lambda(\kappa(p;\lambda^H);\lambda^H)} - \frac{1}{\lambda(\kappa(p;\lambda^L);\lambda^L)} dF \\ = &[G(1) - G(\kappa(p_\kappa;\lambda^H))] - [G(1) - G(\kappa(p_\kappa;\lambda^L))] = 0! \end{aligned}$$

Consider the case that $v(.;\lambda^H)$ and $v(.;\lambda^L)$ intersects somewhere in $[\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$. Define $p_v = \max\{p < 1 : v(p;\lambda^H) = v(p;\lambda^L)\}$. Since $v(p;\lambda^H) < v(p;\lambda^L)$ for $p > p_v$,

$$\begin{aligned} \left.\frac{\partial \ln y(p,q)}{\partial p}\right|_{(p,q)=(p_v,\kappa(p_v;\lambda^H))} &= \frac{\partial \ln v(p_v;\lambda^H)}{\partial p} \\ &\le \frac{\partial \ln v(p_v;\lambda^L)}{\partial p} = \left.\frac{\partial \ln y(p,q)}{\partial p}\right|_{(p,q)=(p_v,\kappa(p_v;\lambda^L))}. \end{aligned}$$

Assumption (Y) implies that $\kappa(p_v;\lambda^H) \le \kappa(p_v;\lambda^L)$. So $\kappa(.;\lambda^H)$ and $\kappa(.;\lambda^L)$ intersects somewhere in $[p_v, 1]$.

Putting together, it must be that $\kappa(p;\lambda^H) > \kappa(p;\lambda^L)$ and $v(p;\lambda^H) < v(p;\lambda^L)$ throughout $[\max\{\underline{p}(\lambda^H), \underline{p}(\lambda^L)\}, 1)$. Otherwise, p_v and p_κ are well-defined satisfying $p_v > p_\kappa$ and $p_\kappa \ge p_v$. $\lambda(\kappa(p;\lambda^H);\lambda^H) > \lambda(\kappa(p;\lambda^L);\lambda^L)$ then follows from the definition of $v(p;\lambda^H)$. □

There is a unique initial value of $\lambda(1)$ *for which the solution to the IPV-*$\lambda(1)$ *satisfying both condition (B.2) and (B.3) at* $q = \underline{q}(\lambda^1)$.

Suppose not, the solutions to the IPV-$\lambda(1)$ with $\lambda(1) = \lambda^H$ and $\lambda(1) = \lambda^L$ satisfy both condition (B.2) and (B.3) at $q = \underline{q}(\lambda^1)$, where $\lambda^H > \lambda^L$.

Consider the case $\underline{p}(\lambda^L) > \underline{p}(\lambda^H)$. Since $\underline{p}(\lambda^L) > 0$, $\underline{V} = v(\underline{p}(\lambda^L); \lambda^L) > v(\underline{p}(\lambda^L); \lambda^H)$. Condition (B.2) cannot be met at $\underline{p}(\lambda^H)$!

Now consider the case $\underline{p}(\lambda^L) \leq \underline{p}(\lambda^H)$, then $\underline{q}(\lambda^H) = \kappa(\underline{p}(\lambda^H); \lambda^H) > \kappa(\underline{p}(\lambda^H); \lambda^L) \geq \underline{q}(\lambda^L)$ and $\lambda(\underline{q}(\lambda^H); \lambda^H) > \lambda(\kappa(\underline{p}(\lambda^H); \lambda^L); \lambda^L)$. This is impossible because $\underline{q}(\lambda^H) > 0$ implies

$$\underline{U} = u(\underline{q}(\lambda^H); \lambda^H) > u(\kappa(\underline{p}(\lambda^H); \lambda^L); \lambda^L)$$

Condition (B.3) cannot be met at $\underline{q}(\lambda^L)$!

B.3 Proof of Proposition 8

Denote the equilibrium payoff for the asset owners by

$$\widetilde{u}(q) = (\delta(\widetilde{\lambda}(q)) - \delta'(\widetilde{\lambda}(q))\widetilde{\lambda}(q))y(\widetilde{r}(q), q), q \geq \underline{\widetilde{q}}.$$

The equilibrium allocation and the Second Best allocation (or the equilibrium allocation in price competition) can be respectively recovered from $(\underline{\widetilde{p}}, \underline{\widetilde{q}}, \widetilde{r}, \widetilde{\lambda}, \widetilde{v}, \widetilde{u})$ and $(r_{SB}, \lambda_{SB}, \underline{p}_{SB}, \underline{q}_{SB}, v_{SB}, u_{SB})$, which both satisfy the following conditions

Boundary conditions:	$r(\underline{q}) = \underline{p}, r(1) = 1, \underline{q}[u(\underline{q}) - \underline{U}] = \underline{p}(v(p) - \underline{V}) = 0,$
Law of motion:	$r'(q) = \frac{g(q)}{f(r(q))}\lambda(q),$
Hosios condition:	$v(r(q)) = \delta'(\lambda(q))y(r(q), q).$

The only difference is in the worker's IC conditions, which are given by

$$\begin{aligned}\left.\frac{\partial \ln \widetilde{v}(p)}{\partial p}\right|_{p=\widetilde{r}(q)} &= \left.\frac{\partial \ln y(p,q)}{\partial p}\right|_{(p,q)=(\widetilde{r}(q),q)},\\ \left.\frac{\partial \ln v_{SB}(p)}{\partial p}\right|_{p=r_{SB}(q)} &= \frac{\eta(\lambda_{SB}(q))}{\delta'(\lambda_{SB}(q))}\left.\frac{\partial \ln y(p,q)}{\partial p}\right|_{(p,q)=(r_{SB}(q),q)}.\end{aligned}$$

The listed set of conditions defines two boundary value problems, and we are going to compare their solutions.

Step 1: For any $\widehat{p} > \max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}$ *and* $\widehat{q} > 0$ *satisfying* $\widehat{q} = \widetilde{\kappa}(\widehat{p}) = \kappa_{SB}(\widehat{p})$, *then* $\lambda_{SB}(\widehat{q}) < \widetilde{\lambda}(\widehat{q})$.

Suppose, to the contrary that, $\lambda_{SB}(\widehat{q}) \geq \widetilde{\lambda}(\widehat{q})$. Then $v_{SB}(\widehat{p}) \leq \widetilde{v}(\widehat{p})$ and $\frac{\partial \ln \widetilde{v}(\widehat{p})}{\partial p} < \frac{\partial \ln v_{SB}(\widehat{p})}{\partial p}$. There must exist some $\epsilon > 0$ such that for all $p \in (\widehat{p}-\epsilon, \widehat{p})$, $v_{SB}(p) < \widetilde{v}(p)$ and $\kappa_{SB}(p) > \widetilde{\kappa}(p)$.[87]

Consider the case that $\widetilde{\kappa}$ and κ_{SB} intersect in $[\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, \widehat{p})$. p_κ denotes the first intersection point of $\widetilde{\kappa}$ and κ_{SB} in $[\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, \widehat{p})$, so that $\kappa_{SB}(p) > \widetilde{\kappa}(p)$ for all $p \in (p_\kappa, \widehat{p})$. Then v_{SB} and $\widetilde{v}$ must intersect somewhere in between p_κ and $\widehat{p}$. Otherwise, for all $p \in (p_\kappa, \widehat{p})$, the Hosios condition implies $\delta'(\widetilde{\lambda}(\widetilde{\kappa}(p)))y(p, \widetilde{\kappa}(p)) > \delta'(\lambda_{SB}(\kappa_{SB}(p)))y(p_v, \kappa_{SB}(p))$, and hence $\widetilde{\lambda}(\widetilde{\kappa}(p)) < \lambda_{SB}(\kappa_{SB}(p))$. This contradicts the law of motion,

$$\begin{aligned}0 &< \int_{p_\kappa}^{\widehat{p}} \frac{1}{\widetilde{\lambda}(\widetilde{\kappa}(p))} - \frac{1}{\lambda_{SB}(\kappa_{SB}(p))} dF\\ &= [G(\widehat{q}) - G(\widetilde{\kappa}(p_\kappa))] - [G(\widehat{q}) - G(\kappa_{SB}(p_\kappa))] = 0!\end{aligned}$$

Consider the case that v_{SB} and $\widetilde{v}$ intersect in $[\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, \widehat{p})$. Let p_v be the first intersection point of v_{SB} and $\widetilde{v}$ in $[\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, \widehat{p})$, so that $v_{SB}(p) < \widetilde{v}(p)$ for all $p \in (p_v, \widehat{p})$. Then $\widetilde{\kappa}$ and κ_{SB} must intersect at some point between p_v and $\widehat{p}$. Suppose not, the continuity of $\widetilde{\kappa}$ and κ_{SB} imply

[87] For the case $\lambda_{SB}(\widehat{q}) = \widetilde{\lambda}(\widehat{q})$, one can show $\frac{\partial \ln \delta'(\lambda_{SB}(\widehat{q}))}{\partial q} > \frac{\partial \ln \delta'(\widetilde{\lambda}(\widehat{q}))}{\partial q}$ by differentiating Hosios condition $v(r(q)) = \delta'(\lambda(q))y(r(q), q)$ w.r.t. q and combining it with $\left.\frac{\partial \ln v(p)}{\partial p}\right|_{p=r(q)}$.

that $\kappa_{SB}(p) > \widetilde{\kappa}(p)$ for $p \in (p_v, \widehat{p})$. The Hosios condition again requires $\widetilde{\lambda}(\widetilde{\kappa}(p)) < \lambda_{SB}(\kappa_{SB}(p))$ for all $p \in (p_\kappa, \widehat{p})$. Under Assumption (Y) and (M), for all $p \in (p_\kappa, \widehat{p})$,

$$\begin{aligned}\frac{\partial \ln v_{SB}(p)}{\partial p} &= \frac{\eta(\lambda_{SB}(\kappa_{SB}(p)))}{\delta'(\lambda_{SB}(\kappa_{SB}(p)))}\frac{\partial \ln y(p, \kappa_{SB}(p))}{\partial p} \\ &> \frac{\eta(\widetilde{\lambda}(\widetilde{\kappa}(p)))}{\delta'(\widetilde{\lambda}(\widetilde{\kappa}(p)))}\frac{\partial \ln y(p, \widetilde{\kappa}(p))}{\partial p} > \frac{\partial \ln \widetilde{v}(p)}{\partial p}.\end{aligned}$$

Hence, v_{SB} and $\widetilde{v}$ cannot intersect at p_v.

It follows that $v_{SB}(p) < \widetilde{v}(p)$ and $\kappa_{SB}(p) > \widetilde{\kappa}(p)$ for $p \in [\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, \widehat{p})$. Otherwise, p_v and p_κ will co-exist, satisfying $\widehat{p} > p_\kappa > p_v$ and $\widehat{p} > p_v > p_\kappa$! Again $\widetilde{\lambda}(\widetilde{\kappa}(p)) < \lambda_{SB}(\kappa_{SB}(p))$ throughout $[\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, \widehat{p})$ because of the Hosios condition.

These conclusions cannot be consistent with the boundary conditions. Suppose $\underline{\widetilde{p}} > \underline{p}_{SB}$, then the boundary condition for $\underline{\widetilde{p}} > 0$ requires $\underline{V} = \widetilde{v}(\underline{\widetilde{p}}) > v_{SB}(\underline{\widetilde{p}})$! Suppose $\underline{\widetilde{p}} \leq \underline{p}_{SB}$, then $\underline{q}_{SB} = \kappa_{SB}(\underline{p}_{SB}) > \widetilde{\kappa}(\underline{p}_{SB}) \geq \underline{\widetilde{q}}$ and $\lambda_{SB}(\underline{q}_{SB}) > \widetilde{\lambda}(\widetilde{\kappa}(\underline{p}_{SB}))$. The boundary condition for $\underline{q}_{SB} > 0$ then requires $\underline{U} = u_{SB}(\underline{q}_{SB}) > \widetilde{u}(\widetilde{\kappa}(\underline{p}_{SB})) \geq \widetilde{u}(\underline{\widetilde{q}})$!

Corollary: $\lambda_{SB}(1) < \widetilde{\lambda}(1)$, $v_{SB}(1) > \widetilde{v}(1)$ *and* $u_{SB}(1) < \widetilde{u}(1)$.

Step 2: $\widetilde{\kappa}(p) > \kappa_{SB}(p)$ *for any* $p \in (\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, 1)$.

Since $\lambda_{SB}(1) < \widetilde{\lambda}(1)$, the law of motion implies $r'_{SB}(1) < \widetilde{r}'(1)$, and hence $\widetilde{\kappa}(p) > \kappa_{SB}(p)$ for sufficient large p. Suppose $\widetilde{\kappa}(.)$ and $\kappa_{SB}(.)$ intersects somewhere in $(\max\{\underline{p}_{SB}, \underline{\widetilde{p}}\}, 1)$. Consider the first intersection point $\widehat{p} = \max\{p \in (0,1) : \widetilde{\kappa}(p) = \kappa_{SB}(p)\}$. Let $\widehat{q} = \widetilde{\kappa}(\widehat{p}) = \kappa_{SB}(\widehat{p})$. The previous claim states that $\lambda_{SB}(\widehat{q}) < \widetilde{\lambda}(\widehat{q})$. However, $r_{SB}(q) > \widetilde{r}(q)$ for $q > \widehat{q}$ by construction. From the law of $r'_{SB}(\widehat{q}) \geq \widetilde{r}'(\widehat{q})$ only if $\lambda_{SB}(\widehat{q}) \geq \widetilde{\lambda}(\widehat{q})$!

Step 3: $\underline{p}_{SB} \geq \underline{\widetilde{p}}$ *and* $\underline{\widetilde{q}} \geq \underline{q}_{SB}$, *and one of the inequalities must be strict.*

First, suppose, to the contrary that, $\underline{\widetilde{p}} > \underline{p}_{SB} \geq 0$. Then $\underline{\widetilde{q}} = \widetilde{\kappa}(\underline{\widetilde{p}}) \geq$

$\kappa_{SB}(\underset{\sim}{p}) > \kappa_{SB}(\underline{p}_{SB}) = \underline{q}_{SB}$. The boundary conditions must be violated if $\underset{\sim}{p} > \underline{p}_{SB}$ and $\underset{\sim}{q} > \underline{q}_{SB}$.

$$\begin{aligned} \underline{U} &= \widetilde{u}(\underset{\sim}{q}) + \widetilde{\lambda}(\underset{\sim}{q})[\widetilde{v}(\underset{\sim}{p}) - \underline{V}] \\ &= \max_{\lambda \geq 0}[\delta(\lambda) y(\underset{\sim}{p}, \underset{\sim}{q}) - \lambda \underline{V}] > \max_{\lambda \geq 0}[\delta(\lambda) y(\underline{p}_{SB}, \underline{q}_{SB}) - \lambda v_{SB}(\underline{p}_{SB})] \\ &= u_{SB}(\underline{q}_{SB})! \end{aligned} \tag{B.4}$$

The first equality is due to the boundary conditions for $\underset{\sim}{q} > 0$ and $\underset{\sim}{p} > 0$ while the second equality and the last equality come from their FOCs and the Hosios condition. Interchanging the role of $(\underline{p}_{SB}, \underline{q}_{SB})$ and $(\underset{\sim}{p}, \underset{\sim}{q})$ in the inequality (B.4), the case $\underline{p}_{SB} > \underset{\sim}{p}$ and $\underline{q}_{SB} > \underset{\sim}{q}$ is also ruled out. A continuity argument rules out the case $\underline{p}_{SB} = \underset{\sim}{p}$ and $\underline{q}_{SB} > \underset{\sim}{q}$. For any p' slightly above $\underset{\sim}{p}$, $\widetilde{\kappa}(p') > \kappa_{SB}(p') > \underline{q}_{SB}$ and $\widetilde{\kappa}(\underset{\sim}{p}) = \underset{\sim}{q} < \underline{q}_{SB}$. $\widetilde{\kappa}$ must be discontinuous at $\underset{\sim}{p}$! Therefore, we establish that $\underline{p}_{SB} \geq \underset{\sim}{p}$ and $\underset{\sim}{q} \geq \underline{q}_{SB}$.

Step 4: $\underline{p}_{SB} = \underset{\sim}{p}$ only if $\underline{p}_{SB} = \underset{\sim}{p} = 0$ and $\widetilde{v}(0) > v_{SB}(0)$. $\underset{\sim}{q} = \underline{q}_{SB}$ only if $\underset{\sim}{q} = \underline{q}_{SB} = 0$ and $u_{SB}(0) > \widetilde{u}(0)$.

First, consider the case $\underline{p}_{SB} = \underset{\sim}{p}$ and $\underset{\sim}{q} = \underline{q}_{SB}$. Differentiating the Hosios condition $\ln v(r(q)) = \ln \delta'(\lambda(q)) + \ln y(r(q), q)$ and subtracting it with the respective expressions of $\left.\frac{\partial \ln v_{SB}(p)}{\partial p}\right|_{p=r_{SB}(q)}$ and $\left.\frac{\partial \ln \widetilde{v}(p)}{\partial p}\right|_{p=\widetilde{r}(q)}$, we obtain $\frac{\partial \ln \delta'(\lambda_{SB}(\underset{\sim}{q}))}{\partial q} > \frac{\partial \ln \delta'(\widetilde{\lambda}(\underset{\sim}{q}))}{\partial q}$. It follows that $\widetilde{\lambda}(\underset{\sim}{q}) < \lambda_{SB}(\underset{\sim}{q})$. Otherwise, $\widetilde{\lambda}(q') > \lambda_{SB}(q')$ for q' slightly above $\underset{\sim}{q}$. The law of motion in turn implies that $\widetilde{r}(q') > r_{SB}(q')$, contradicting the previous claim in Step 2! We can immediately rule out the cases with $\underline{p}_{SB} = \underset{\sim}{p} > 0$ or $\underset{\sim}{q} = \underline{q}_{SB} > 0$. This is because the boundary conditions and the Hosios condition in such case require $\widetilde{\lambda}(\underset{\sim}{q}) = \lambda_{SB}(\underset{\sim}{q})$! The remaining possibility is that $\underline{p}_{SB} = \underset{\sim}{p} = \underset{\sim}{q} = \underline{q}_{SB} = 0$. $\widetilde{v}(0) > v_{SB}(0)$ and $u_{SB}(0) > \widetilde{u}(0)$ because $\widetilde{\lambda}(0) < \lambda_{SB}(0)$.

Consider the case $\underline{p}_{SB} = \underset{\sim}{p}$ and $\underset{\sim}{q} > \underline{q}_{SB}$. From the boundary conditions and Hosios condition,

$$\max_{\lambda\geq 0}[\delta(\lambda)y(\underline{p}_{SB},\underline{q}_{SB})-\lambda v_{SB}(\underline{p}_{SB})]=u_{SB}(\underline{q}_{SB})$$
$$\geq \underline{U}=\widetilde{u}(\underline{\widetilde{q}})=\max_{\lambda\geq 0}[\delta(\lambda)y(\underline{p}_{SB},\underline{\widetilde{q}})-\lambda\widetilde{v}(\underline{p}_{SB})].$$

This immediately implies that $\widetilde{v}(\underline{p}_{SB}) > v_{SB}(\underline{p}_{SB}) \geq \underline{V}$, and hence $\underline{p}_{SB} = \underline{\widetilde{p}} = 0$. The case $\underline{p}_{SB} > \underline{\widetilde{p}}$ and $\underline{\widetilde{q}} = \underline{q}_{SB}$ follows from a symmetric argument.

Corollary: $\widetilde{v}(\underline{p}_{SB}) > v_{SB}(\underline{p}_{SB})$ and $u_{SB}(\underline{\widetilde{q}}) > \widetilde{u}(\underline{\widetilde{q}})$

The previous claim establishes the cases of $\underline{p}_{SB} = \underline{\widetilde{p}}$ or $\underline{\widetilde{q}} = \underline{q}_{SB}$. Suppose $\underline{p}_{SB} > \underline{\widetilde{p}}$, the boundary condition immediately implies $\widetilde{v}(\underline{p}_{SB}) > v_{SB}(\underline{p}_{SB}) = \underline{V}$. Similarly, $u_{SB}(\underline{\widetilde{q}}) > \widetilde{u}(\underline{\widetilde{q}}) = \underline{U}$ if $\underline{\widetilde{q}} > \underline{q}_{SB}$.

Notice that the above arguments only require the workers' IC condition to satisfy $\left.\frac{\partial \widetilde{v}(p)}{\partial p}\right|_{p=\widetilde{r}(q)} < \eta(\widetilde{\lambda}(q)) \left.\frac{\partial y(p,q)}{\partial p}\right|_{(p,q)=(\widetilde{r}(q),q)}$ for $\widetilde{v}(p) \geq \underline{V}$.