Critical Branching Random Walks, Branching Capacity and Branching Interlacements

by

Qingsan Zhu

B.Sc., Peking University, 2008
M.Sc., Peking University, 2011

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate and Postdoctoral Studies

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

August 2017

Abstract

This thesis concerns critical branching random walks. We focus on supercritical ($d \geq 5$) and critical ($d = 4$) dimensions.

In this thesis, we extend the potential theory for random walk to critical branching random walk. In supercritical dimensions, we introduce branching capacity for every finite subset of $\mathbb{Z}^d$ and construct its connections with critical branching random walk through the following three perspectives.

1. The visiting probability of a finite set by a critical branching random walk starting far away;

2. Branching recurrence and branching transience;

3. Local limit of branching random walk in torus conditioned on the total size.

Moreover, we establish the model which we call 'branching interlacements' as the local limit of branching random walk in torus conditioned on the total size.

In the critical dimension, we also construct some parallel results. On the one hand, we give the asymptotics of visiting a finite set and the convergence of the conditional hitting point. On the other hand, we establish the asymptotics of the range of a branching random walk conditioned on the total size.

Also in this thesis, we analyze a small game which we call the Majority-Markov game and give an optimal strategy.

Lay Summary

This thesis investigates a probabilistic model called branching random walk, which is a combination of two classical subjects in probability theory: random walk and branching process. A branching random walk is a random process consisting of a finite number of particles doing independent random walks, which at every time step, will give birth to a random number of new particles (particles are added) and then die (particles are removed), the new particles then begin independent random walks from the location of their parents. Of particular challenge to the analysis is a critical branching random walk, where the expected number of offsprings of each particle is one. The main contribution of this thesis is to develop new knowledge about critical branching random walks by building an analogy with classical results on random walk.

Preface

This dissertation is ultimately based on the original work of the author, under the supervisor Professor Omer Angel.

A version of Chapter 2 has been divided into several preprint papers, which are currently under review for publication and put on the arXiv ([26–29]). I am responsible for all of the proofs and writing of Chapter 2 and Chapter 4.

Chapter 3 is based on a joint work together with Professor Omer Angel and Dr. Balázs Ráth ([2]), being under review of publication.

Table of Contents

Appendix

List of Figures

Acknowledgements

I am greatly indebted to my supervisor Professor Omer Angel for his guidance throughout my PhD study. He always gives me interesting questions and useful suggestions.

I would like to thank Professor Edwin A. Perkins and Professor Gordon Slade for serving on my advisory committee, Dr. Balázs Ráth and the members of the Probability Group at the University of British Columbia for their help and discussions.

I also thank my colleagues in the Mathematics Department in UBC, especially to Li Xiaowei, Wang Li, Yang Wen and Ye Zichun for the happy time we study and play together.

Last but not least, I want to show special thanks to my parents for their support.

Dedication

To my parents.

Chapter 1

Introduction

This thesis studies some properties of critical branching random walks in dimension four and higher.

In the first part, we extend the theory of discrete capacity for random walk to critical branching random walk. We introduce branching capacity for any finite subset of $\mathbb{Z}^d$, for $d \geq 5$ and establish its connections with critical branching random walks. In 4-dimension, we give some parallel results.

In the second part, we introduce the model of branching interlacements. We show that this model turns out to be the local limit of the critical branching random walk in torus.

1.1 Critical branching random walks

As the name suggests, a branching random walk can be viewed as a system of particles performing random walks while branching (deterministically or randomly). We are mainly interested in the case when the branching is also random. For branching random walks with deterministic branching, one can refer to the lecture notes [18]. In our situation, there are two levels of random mechanism. One is for the branching and the other is for the random walk. To define a branching random walk, we need to fix two distributions μ and θ for the randomness.

1. μ is a probability measure on $\mathbb{N}$;

2. θ is a probability measure on $\mathbb{Z}^d$.

Definition 1.1.1. *A branching random walk starting from $x \in \mathbb{Z}^d$, with offspring distribution μ and jump distribution θ, can be described as follows.*

At time 0*, a particle is located at* x*. Suppose that, at each time* n*, a particle* v *is located at* $\mathcal{S}(v)$*. At time* $n+1$*,* v *dies and gives birth to a random number, distributed according to* μ*, of children. Each child then moves to a new location* $\mathcal{S}(v)+Y$*, with increment* Y *distributed according to* θ*. Different particles behave independently. Let* T *be the collection of all particles at all times. Then* $(\mathcal{S}(v))_{v\in T}$ *forms a branching random walk.*

In this thesis, we study critical branching random walk. By 'critical' we mean that

$$\mathbb{E}\mu = 1.$$

We always assume this and rule out the degenerate case, i.e. $\mu(1) = 1$ (unless otherwise specified). For the jump distribution we assume that it is centered in the following sense:

$$\mathbb{E}\theta = 0.$$

In addition, for technical reasons, we always assume the following moment conditions unless otherwise specified:

- μ has finite variance $\sigma^2 > 0$;
- θ is irreducible (i.e. not supported on a strict subgroup of $\mathbb{Z}^d$), and 'weak' L^d in the following sense: there exists $C > 0$, such that for any $r \geq 1$,
$$\theta(\{x \in \mathbb{Z}^d : |x| > r\}) < C \cdot r^{-d}. \tag{1.1.1}$$

Note that (1.1.1) holds if θ has finite d-th moment and if (1.1.1) holds, then θ has finite b-th moment, for any $0 < b < d$. For some results, we need stronger assumptions, which will be stated explicitly.

Remark 1.1.1. *Why the critical case? Branching random walk generalizes both branching process (no geometry) and random walk (no branching). The corresponding branching process for our branching random walk is the so-called Galton-Watson process. It is classical that for nondegenerate (i.e.*

$\mu(1) \neq 1$) Galton-Watson processes, the extinction probability is one, if $\mathbb{E}\mu \leq 1$, and strictly less than one, if $\mathbb{E}\mu > 1$. Moreover, the probability of survival to n-th generation decays (as $n \to \infty$) exponentially, if $\mathbb{E}\mu < 1$, and polynomially, if $\mathbb{E}\mu = 1$. Similarly, it turns out that the probability of visiting a distant point by a branching random walk starting from the origin has different asymptotics for three different regions: when $\mathbb{E}\mu > 1$, the probability is bigger than a positive constant (which is just the probability for survival of the corresponding Galton-Watson process); when $\mathbb{E}\mu = 1$, it decays to zero polynomially; when $\mathbb{E}\mu < 1$, it decays to zero exponentially.

Remark 1.1.2. *We have not striven for the greatest generality about the assumptions on μ and θ, and it is plausible that many results also hold under weaker assumptions, especially for θ.*

On the other hand, the dimension d plays an important role in the study of critical branching random walk: there are three regions:

1. Supercriticality: $d \geq 5$;
2. Criticality: $d = 4$;
3. Subcriticality: $d \leq 3$.

One can get an analogous feeling from random walk about the critical dimension. It is well-known that the critical dimension for random walk is $d = 2$. There are many results reflecting this fact. Here are a couple. The famous Pólya's Recurrence Theorem states that a simple random walk on a d-dimension is recurrent for $d = 1, 2$ and transient for $d > 2$. On the other hand, the range of a random walk with n-steps behaves sublinearly (when n goes to infinity), if $d = 1$; linearly with logarithm correction, if $d = 2$; linearly, if $d \geq 3$. One will see that both results (together with many others) have analogues in the setting of critical branching random walk (see Corollary 1.3.11, Proposition 2.3.3, and the rest of this chapter).

In this thesis, we mainly focus on supercritical and critical dimensions.

1.2 Range of critical branching random walk conditioned on total number of progeny

Le Gall and Lin ([13, 14]) have established the following result about the number of occupied sites by a critical branching random walk conditioned on the total number of offsprings being n, denoted by R_n: (under some regular conditions on μ and θ)

$$\frac{1}{n}R_n \xrightarrow{P} c_1 \quad \text{as } n \to \infty, \text{ when } d \geq 5; \tag{1.2.1}$$

$$\frac{\log n}{n}R_n \xrightarrow{L^2} c_2 \quad \text{as } n \to \infty, \text{ when } d = 4;$$

$$n^{-d/4}R_n \xrightarrow{d} c_3\lambda_d(supp(\mathcal{I})) \quad \text{as } n \to \infty, \text{ when } d \leq 3;$$

where $c_i, i = 1, 2, 3$ are some constants and $\lambda_d(supp(\mathcal{I}))$ stands for the Lebesgue measure of the support of the random measure on $\mathbb{R}^d$ known as Integrated Super-Brownian Excursion.

In the critical dimension, they assume that the offspring distribution μ is the critical geometric distribution (i.e. with parameter 1/2), while in other dimensions, they could handle very general offspring distributions.

In subcritical dimensions, they also established the asymptotics of the hitting probability of a distant point by critical branching random walk:

$$\lim_{x\to\infty} \|x\|^2 \cdot P(\mathcal{S}_x \text{ visits } 0) = \frac{2(4-d)}{d\sigma^2}, \tag{1.2.2}$$

where $\mathcal{S}_x$ is a critical branching random walk starting at x, $\|x\| = \sqrt{x \cdot Q^{-1}x}/\sqrt{d}$ with Q being the covariance matrix of θ, and σ^2 is the variance of μ.

They presented the following questions:

1. The asymptotic of $P(\mathcal{S}_x$ visits $0)$ in other dimensions ($d \geq 4$);

2. The range R_n in the critical dimension for general offspring distributions;

3. The range of branching random walk with a general initial configuration.

We answer the first two questions in this thesis. We show that:
When $d \geq 5$, we have

$$\lim_{x\to\infty} \|x\|^{d-2} \cdot P(\mathcal{S}_x \text{ visits } 0) = a_d c_1;$$

When $d = 4$, we have

$$\lim_{x\to\infty} \|x\|^2 \log \|x\| \cdot P(\mathcal{S}_x \text{ visits } 0) = 1/(2\sigma^2);$$

and

$$\frac{\log n}{n} R_n \xrightarrow{P} \frac{16\pi^2\sqrt{\det Q}}{\sigma^2} \quad \text{as } n \to \infty;$$

where c_1 is the same constant in (1.2.1) and a_d is some constant depending on θ.

To summarize, we have:

$$P(\mathcal{S}_x \text{ visits } 0) \approx \begin{cases} a_d c_1/\|x\|^{d-2}, & \text{when} \quad d \geq 5; \\ 1/(2\sigma^2\|x\|^2 \log \|x\|), & \text{when} \quad d = 4; \\ 2(4-d)/(d\sigma^2\|x\|^2), & \text{when} \quad d \leq 3. \end{cases}$$

Here are the heuristics for the exponents. We know that a typical random walk sample path connecting x and 0 is with length of order $\|x\|^2$. In order to reach a point with distance $\|x\|$ away (with not too small probability), the corresponding branching process should survive at least for generations with order $\|x\|^2$. This event has probability of $\|x\|^{-2}$. This incidence is enough to give order one probability for visiting x, when the dimension is low enough. On the other hand, it is not difficult to see that the expectation of the visiting times of 0 is the same as that for random walk, which is of order $\|x\|^{2-d}$ (for $d \geq 3$). Note that when $d = 4$ (the critical dimension), it is just $\|x\|^{-2}$! This maybe one of the most natural ways to remember the critical dimension. When the dimension is high ($d \geq 5$ is enough), the conditional visiting times, conditioned on visiting, is of order one, hence the probability of visiting should have the same order as the expectation, which is just $\|x\|^{2-d}$.

1.3 Potential theory for random walk and our parallel results

There are two essential theories on random walk, one is the discrete potential theory, the other is about the scaling limit of random walk, i.e. Brownian motion. It is well known that the scaling limit of critical branching random walk is the integrated super-Brownian excursion, or Brownian snake. For more details about this, we refer the reader to [12], [8] and the references therein.

In this thesis, we focus on the discrete potential theory and extend it to critical branching random walk. For the discrete potential theory for random walk, one can see e.g. [10, 11, 20]. Let us first review some results on regular (discrete) capacity. For any finite subset K of $\mathbb{Z}^d, d \geq 3$, the escape probability $\mathrm{ES}_K(x)$ is defined to be the probability that a random walk starting from $x \in \mathbb{Z}^d$ with symmetric jump distribution, denoted by $S_x = (S_x(n))_{n\in\mathbb{N}}$, never returns to K. The capacity of K, $\mathrm{Cap}(K)$ is given by:

$$\mathrm{Cap}(K) = \sum_{a\in K} \mathrm{ES}_K(a).$$

We have

$$\lim_{x\to\infty} \|x\|^{d-2} \cdot P(S_x \text{ visits } K) = a_d \mathrm{Cap}(K),$$

where $a_d = \frac{1}{2d^{(d-2)/2}\sqrt{\det Q}}\Gamma(\frac{d-2}{2})\pi^{-d/2}$. Moreover, let $\tau_K = \inf\{n \geq 1 : S_x(n) \in K\}$, then for any $a \in K$, we have

$$\lim_{x\to\infty} P(S_x(\tau_K) = a | S_x \text{ visits } K) = \mathrm{ES}_K(a)/\mathrm{Cap}(K).$$

$\mathrm{ES}_K(a)$ is usually called the equilibrium measure and the normalized measure $\mathrm{ES}_K(a)/\mathrm{Cap}(K)$ is called the harmonic measure of set K. In fact, not only the distribution of the first visiting point, but also that of the last visiting point, conditioned on visiting K, converge to the same measure:

$$\lim_{x\to\infty} P(S_x(\xi_K)=a|S_x \text{ visits } K) = \mathrm{ES}_K(a)/\mathrm{Cap}(K),$$

where $\xi_K = \sup\{n \geq 1 : S_x(n) \in K\}$.

The results above apply to any symmetric irreducible jump distribution with some finite moment conditions. Unfortunately we do not find any reference for the nonsymmetric walks. However the following result is well-known and can be proved similarly to the symmetric case (see the Preface of [11]). When the jump distribution is irreducible, nonsymmetric, with mean zero and, for simplicity, finite range, we have:

$$\lim_{x\to\infty} P(S_x(\tau_K)=a|S_x \text{ visits } K) = \mathrm{ES}^-_K(a)/\mathrm{Cap}(K),$$
$$\lim_{x\to\infty} P(S_x(\xi_K)=a|S_x \text{ visits } K) = \mathrm{ES}_K(a)/\mathrm{Cap}(K),$$

where ES^- is the escape probability for the reversed random walk.

In this thesis, we extend the theory of capacity for random walk to critical branching random walk. As we have mentioned, for critical branching random walk, the critical dimension is $d=4$ instead of $d=2$ (for random walk). This fact is also reflected in many of our results.

1.3.1 Supercritical dimensions ($d \geq 5$)

In $\mathbb{Z}^d, d \geq 5$, we introduce branching capacity for any finite subset. In order to define branching capacity, one needs to introduce analogues of the escape probability. For a finite set K of $\mathbb{Z}^d$, denoted by $K \subset\subset \mathbb{Z}^d$, one could consider the probability that the branching random walk starting at x, denoted by $\mathcal{S}_x$, avoids K. However, this turns out not to be the right generalization. Two different extensions of the escape probability need to be defined: one for the first and one for the last visiting point of K. We denote these by $\mathrm{Es}_K(x)$ and $\mathrm{Esc}_K(x)$. Both correspond to infinite versions of branching random walk. We defer the complete definitions to Chapter 2.

Formally one can define the branching capacity of K by

$$\mathrm{BCap}(K) = \sum_{z\in K} \mathrm{Es}_K(z) \ \left(\text{also } = \sum_{z\in K} \mathrm{Esc}_K(z) \right).$$

Then, we have:

Theorem 1.3.1. *For any nonempty finite subset K of $\mathbb{Z}^d$ and $a \in K$, we have*

$$\lim_{x\to\infty} \|x\|^{d-2} \cdot P(\mathcal{S}_x \text{ visits } K) = a_d BCap(K); \tag{1.3.1}$$

$$\lim_{x\to\infty} P(\mathcal{S}_x(\tau_K) = a|\mathcal{S}_x \text{ visits } K) = Es_K(a)/BCap(K),$$
$$\lim_{x\to\infty} P(\mathcal{S}_x(\xi_K) = a|\mathcal{S}_x \text{ visits } K) = Esc_K(a)/BCap(K),$$

where τ_K and ξ_K respectively are the first, and the last respectively, visiting time of K in a Depth-First search and $a_d = \frac{1}{2d^{(d-2)/2}\sqrt{\det Q}}\Gamma(\frac{d-2}{2})\pi^{-d/2}$ is the same constant as in the random walk case.

Let us make some comments here. First, if μ is the degenerate measure (that is $\mu(1) = 1$), then the branching random walk is just the regular random walk, and Es_K (Esc_K respectively) is just ES_K^- (ES_K respectively). In this case, Theorem 1.3.1 is the classical result for random walk.

Second, this result tells us that conditioned on visiting a fixed set, the 'first' (or the last) visiting point converges in distribution. It turns out that we can say more about this. In fact, we also show that (see Section 2.2.6) conditioned on visiting K, the set of entering points converges in distribution. Since the distribution of the intersection between K and the range of $\mathcal{S}_x$ can be determined by the entering points, hence we have

Theorem 1.3.2. *Conditioned on $\mathcal{S}_x$ visiting K, the intersection between K and the range of $\mathcal{S}_x$ converges in distribution, as $x \to \infty$.*

Third, this result gives the asymptotic behavior of the probability of visiting a fixed finite set by critical branching random walk starting from far away (for dimension $d \geq 5$), answering the first question in the end of

Section 1.2.1, for supercritical dimensions. We also establish the asymptotic behavior of the visiting probability for the case of critical dimension $d = 4$, (see Theorem 1.3.12). Note that we give the asymptotics of the probability of visiting any finite set while the original question is stated for one single point.

We mentioned in Section 1.2.3, that Le Gall and Lin establish the asymptotic of the range of a critical branching random walk conditioned on total size being n. In supercritical dimensions, the range divided by n converges in probability to a constant, c_1 in (1.2.1), which they interpret as some escape probability. This constant is just $\mathrm{BCap}(\{0\})$ in our notation.

We also construct the following bounds for the visiting probability by critical branching random walk when the distance $\rho(x, A)$ between x and A is not too small, compared with the diameter of A, $\mathrm{diam}(A)$.

Theorem 1.3.3. *For any finite $A \subseteq \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ with $\rho(x, A) \geq 0.1 diam(A)$, we have:*

$$P(\mathcal{S}_x \text{ visits } A) \asymp \frac{BCap(A)}{(\rho(x, A))^{d-2}}, \tag{1.3.2}$$

where $f(x, A) \asymp g(x, A)$ indicates that there exists positive constants c_1, c_2 independent of x, A such that $c_1 f(x, A) \leq g(x, A) \leq c_2 f(x, A)$.

One might compare this with the corresponding result for random walk:

$$P(S_x \text{ visits } A) \asymp \frac{\mathrm{Cap}(A)}{(\rho(x, A))^{d-2}}. \tag{1.3.3}$$

Similarly to random walk, computing escape probabilities can be very difficult. Hence it might not be practical to estimate the branching capacity by definition directly. However we can use (1.3.1) in reverse: by estimating the probability of visiting a set, we can give bounds for the branching capacity of that set. Through this, we find the order of the magnitude of the branching capacity of low dimensional balls:

Theorem 1.3.4. *Let $B^m(r)$ be the m-dimensional balls with radius r (as a subset of $\mathbb{Z}^d$), i.e. $\{z = (z_1, 0) \in \mathbb{Z}^m \times \mathbb{Z}^{d-m} = \mathbb{Z}^d : |z_1| \leq r\}$. For any*

$r > 2$, we have:

$$BCap(B^m(r)) \asymp \begin{cases} r^{d-4} & \text{if} \quad m \geq d-3; \\ r^{d-4}/\log r & \text{if} \quad m = d-4; \\ r^m & \text{if} \quad m \leq d-5. \end{cases} \tag{1.3.4}$$

One might compare this with the corresponding result about regular capacity:

$$\mathrm{Cap}(B^m(r)) \asymp \begin{cases} r^{d-2} & \text{if} \quad m \geq d-1; \\ r^{d-2}/\log r & \text{if} \quad m = d-2; \\ r^m & \text{if} \quad m \leq d-3. \end{cases} \tag{1.3.5}$$

Our definition of branching capacity depends on the offspring distribution μ and the jump distribution θ. From the previous result, one can see that branching capacities of a ball for different μ's and θ's are comparable. We believe this is generally true for any finite subset but can only show one part of it:

Theorem 1.3.5. *Suppose that μ_1, μ_2 are two nondegenerate critical offspring distributions with finite second moment and let $BCap_{\mu_1,\theta}$ and $BCap_{\mu_2,\theta}$ denote the corresponding branching capacities (with the same jump distribution θ). Then, there is a $C = C(\mu_1, \mu_2) > 0$ such that for all finite $A \subseteq \mathbb{Z}^d$,*

$$C^{-1} \cdot BCap_{\mu_1,\theta}(A) \leq BCap_{\mu_2,\theta}(A) \leq C \cdot BCap_{\mu_1,\theta}(A). \tag{1.3.6}$$

One might compare this with the corresponding result about regular capacity:

Suppose that θ_1 and θ_2 are two irreducible distributions on $\mathbb{Z}^d$ (for $d \geq 3$) with mean zero and finite range. Then, there is a $C = C(\theta_1, \theta_2) > 0$ such that,

$$C^{-1} \cdot \mathrm{Cap}_{\theta_1}(A) \leq \mathrm{Cap}_{\theta_2}(A) \leq C \cdot \mathrm{Cap}_{\theta_1}(A), \quad \text{for all finite } A \subseteq \mathbb{Z}^d.$$

We believe the following and can not prove at this time:

Conjecture 1.3.6. *Suppose that θ_1 and θ_2 are two irreducible distributions on $\mathbb{Z}^d$($d \geq 3$) with mean zero and finite range. Then, there is a $C = C(\mu, \theta_1, \theta_2) > 0$ such that,*

$$C^{-1} \cdot BCap_{\mu,\theta_1}(A) \leq BCap_{\mu,\theta_2}(A) \leq C \cdot BCap_{\mu,\theta_1}(A), \quad \textit{for all finite } A \subseteq \mathbb{Z}^d.$$

Furthermore, we construct an analogous version of Wiener's Test. Let us first review the classical Wiener's Test. A subset $K \subseteq \mathbb{Z}^d$ is called recurrent if

$$P(S_0(n) \in K \text{ for infinite } n \in \mathbb{N}) = 1;$$

and transient if

$$P(S_0(n) \in K \text{ for infinite } n \in \mathbb{N}) = 0.$$

For the recurrence and transience of a set, Wiener's Test says that:
Suppose $K \subseteq \mathbb{Z}^d, d \geq 3$ and let $K_n = \{a \in K : 2^n \leq |a| < 2^{n+1}\}$. Then,

$$K \text{ is recurrent } \Leftrightarrow \sum_{n=1}^{\infty} \frac{\mathrm{Cap}(K_n)}{2^{n(d-2)}} = \infty.$$

Inspired by this, we give the definition of branching recurrence and branching transience by using branching random walk conditioned on survival instead of random walk (see Chapter 2 for exact definitions). We have the following version of Wiener's Test:

Theorem 1.3.7. *Assume further that μ has finite third moment and θ is with finite range. Then for any $K \subseteq \mathbb{Z}^d, d \geq 5$, we have*

$$K \textit{ is branching recurrent } \Leftrightarrow \sum_{n=1}^{\infty} \frac{BCap(K_n)}{2^{n(d-4)}} = \infty.$$

Meanwhile, we give the asymptotics and bounds for the visiting probability of a finite set by critical branching random walk conditioned on survival starting from x (denoted by $\overline{\mathcal{S}}_x^{\infty}$):

Proposition 1.3.8. *For every finite $A \subseteq \mathbb{Z}^d$, we have:*

$$\lim_{x\to\infty} \|x\|^{d-4} \cdot P(\overline{\mathcal{S}}_x^\infty \text{ visits } A) = t_d \cdot a_d^2 \sigma^2 BCap(K), \tag{1.3.7}$$

where $t_d = t_d(\theta) = d^{d/2}\sqrt{\det Q}\int_{t\in\mathbb{R}^d} |t|^{2-d}|h' - t|^{2-d}dt$, and $h' \in \mathbb{R}^d$ is any vector with $|h'| = 1$. Recall σ^2 is the variance of μ.

Theorem 1.3.9. *For every finite $A \subseteq \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ with $\rho(x, A) \geq 0.1 diam(A)$, we have (assume further that θ has finite range):*

$$P(\overline{\mathcal{S}}_x^\infty \text{ visits } A) \asymp \frac{BCap(A)}{(\rho(x,A))^{d-4}}. \tag{1.3.8}$$

In particular, if we let M be the $(d-i)$-dimensional ($i = 1, 2, 3, 4$) linear subspace, i.e. $\{z = (z_1, z_2) \in \mathbb{Z}^{d-i} \times \mathbb{Z}^i : z_2 = 0\}$, then by Theorem 1.3.4, Theorem 1.3.7 and the monotonicity of branching capacity, we can see that M is branching recurrent. By projecting to $\mathbb{Z}^i$, we get that for $\mathbb{Z}^i$ ($i \leq 4$), the projection version of $\mathcal{S}_0^\infty$ will visit any vertex infinitely often, almost surely. Hence we can get the following result which appeared in [3]:

Corollary 1.3.10. *The critical branching random walk conditioned on survival in $\mathbb{Z}^d$ ($d \leq 4$) almost surely visits any vertex infinitely often, provided that the offspring distribution has finite third moment and that the step distribution is irreducible, centered, with finite range.*

In [3], this is proved when μ is the critical geometric distribution and θ is simple, i.e. uniform on the unit vectors. It is mentioned there that their method works for general critical offspring distribution with finite second moment, see Section 3.1 in [3]. It seems that their method requires the symmetry assumption for θ.

From Theorem 1.3.5 and Theorem 1.3.7, we can see that whether a set is branching recurrent or branching transient is independent of the choice of the offspring distribution, as long as that offspring distribution is nondegenerate, critical and with finite third moment:

Corollary 1.3.11. *Let θ be some fixed centered, irreducible distribution on $\mathbb{Z}^d$ with finite range. Then for any $K \subseteq \mathbb{Z}^d$, if there exists one nondegenerate* *critical offspring distribution μ with finite third moment, such that K is branching recurrent (corresponding to μ and θ), then for any such offspring distribution, K is branching recurrent.*

1.3.2 The critical dimension ($d = 4$)

In the critical dimension, we also establish the asymptotics of the visiting probability with a logarithmic correction:

Theorem 1.3.12. *Assume further that θ has finite exponential moments. Then, for every finite subset K of $\mathbb{Z}^4$, we have:*

$$\lim_{x\to\infty}(\|x\|^2\log\|x\|)\cdot P(\mathcal{S}_x \text{ visits } K)=\frac{1}{2\sigma^2}. \tag{1.3.9}$$

We also show the convergence for the first visiting point conditioned on visiting:

Theorem 1.3.13. *Assume further that θ has finite range. Then, for any finite nonempty subset K of $\mathbb{Z}^4$ and $a\in K$, we have*

$$\lim_{x\to\infty}P(\mathcal{S}_x(\tau_K)=a|\mathcal{S}_x \text{ visits } K)=\frac{\sigma^2}{4\pi^2\sqrt{\det Q}}\mathcal{E}_K(a),$$

where $\mathcal{E}_K(a)$ is defined later in (2.4.27).

Remark 1.3.1. *Recall that, in supercritical dimensions, we mention further that the conditional entering measure converges in distribution. However this is false in the critical dimension (and in subcritical dimensions). It turns out that, the conditional entering measure will blow up, as the starting point tends to infinity.*

Note that in the random walk case, the random walk in 2-d is recurrent and hence $P(S_x \text{ visits } K)=1$. However the harmonic measure does exist:

$$\lim_{x\to\infty}P(S_x(\tau_K)=a|S_x \text{ visits } K)=\frac{1}{\pi^2\sqrt{\det Q}}E_K(a),$$

where $E_K(x) = \lim_{n\to\infty} \log n \cdot P(\tau_n < \overline{\tau}_K)$ exists with τ_n being the hitting time of $B^c(n)$ by a random walk starting at x. As we will see, $\mathcal{E}_K(a)$ has a similar form.

Furthermore, recall that R_n, the range of the critical branching random walk conditioned on the total size being n has the following asymptotics:

$$\frac{\log n}{n} R_n \overset{n\to\infty}{\longrightarrow} 8\pi^2\sqrt{\det Q} \quad \text{in } L^2,$$

provided that μ is the geometric distribution with parameter $1/2$ and θ is symmetric and has exponential moments.

We establish the following:

Theorem 1.3.14.

$$\frac{\log n}{n} R_n \overset{n\to\infty}{\longrightarrow} \frac{16\pi^2\sqrt{\det Q}}{\sigma^2} \quad \textit{in probability},$$

assuming further that θ is symmetric and has finite exponential moments.

1.4 Branching interlacements

Sznitman introduced the model of random interlacements which consist of a countable collection of trajectories of doubly infinite random walks on the lattice $\mathbb{Z}^d$, for $d \geq 3$ ([21]). Since this seminal work, many aspects of the model of random interlacements have been studied by numerous authors. The basic results of the theory of random interlacements can be found in the lecture notes [5] and [22]. The interlacement $\mathcal{I}^u$ at level $u \geq 0$ is the trace left on $\mathbb{Z}^d$ by a cloud of paths constituting a Poisson point process on the space of doubly infinite transient trajectories modulo time-shift. Its law is characterized by:

$$P(\mathcal{I}^u \cap K = \emptyset) = \exp(-u \cdot Cap(K)), \text{ for every finite } K \subseteq \mathbb{Z}^d. \tag{1.4.1}$$

There are two main initial results about random interlacements. On the one hand, the random interlacement at level u turns out to be the local limit of the set of sites on the discrete torus $\mathbb{T}_N^d := (\mathbb{Z}/N\mathbb{Z})^d$ visited by the simple random walk up to $\lfloor uN^d \rfloor$ steps ([25]). On the other hand, as a percolation model, the complement of the interlacement, the so-called vacant set, exhibits a phase transition ([21] and [19]): there is a critical value $u^* \in (0, \infty)$ such that the vacant set percolates for $u < u^*$ and does not percolate for $u > u^*$.

Inspired by this, we introduce another kind of interlacements consisting of a countable collection of doubly infinite trajectories that encode infinite trees embedded in $\mathbb{Z}^d, d \geq 5$. We restrict ourself to a special case $\mu = Geo(1/2)$ and simply assume that θ is the uniform measure on the set of unit vectors in $\mathbb{Z}^d$. Similarly, a non-negative parameter u governs the amount of trajectories entering the picture. We show that:

Theorem 1.4.1. *For any $u > 0$, we can construct a random subset $\mathcal{I}^u$ of $\mathbb{Z}^d$, which is characterized by:*

$$P(\mathcal{I}^u \cap K = \emptyset) = \exp(-u \cdot BCap(K)), \text{ for every finite } K \subseteq \mathbb{Z}^d. \tag{1.4.2}$$

Furthermore, we prove: similar to the case of random interlacement, branching interlacement at level u, turns out to be the local limit of the law of the trace of branching random walk on torus with side-length N, conditioned to have $\lfloor uN^d \rfloor$ progeny. More precisely, let R_N be the occupied sites by a critical branching random walk, conditioned on the total size being $\lfloor uN^d \rfloor$, with uniform starting point in the torus with side-length N. Then for any fixed finite subsets $B \subseteq A \subseteq \mathbb{Z}^d$, we have:

Theorem 1.4.2.

$$\lim_{N \to \infty : N \equiv 1 \mod 2} P(R_N \cap A = B) = P(\mathcal{I}^u \cap A = B).$$

Note that when N is large enough, we can regard A, B as subsets of the torus with side-length N.

The reason we need to assume $N \equiv 1 \mod 2$ is due to the periodicity of simple random walk.

1.5 An optimal strategy for the Majority-Markov game

Let's begin with a little game. Three tokens begin on vertices -2, -1 and 1 of a path connecting vertices $-3, -2, ..., 3$ (see the figure below). At any time the player may pay one dollar and choose a token; that token will then move randomly, with equal probability to its left or right neighboring vertex. Different tokens move independently without interfering. There are holes at the endpoints. Hence once the token reaches the endpoint, it falls into that hole and cannot get out. If the player is curious about which hole finally contains more tokens, the negative side, or the positive side, which token should be chosen to move, with the goal of minimizing expected cost?

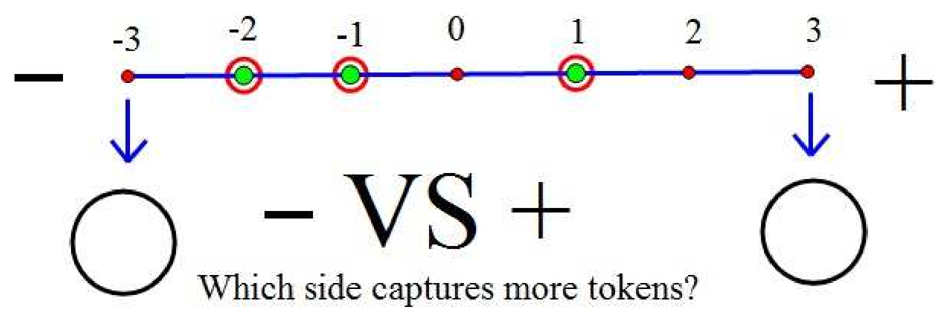

Figure 1.1: A simple example of Majority-Markov game

Note that though the player wants to know which side wins, he, as a matter of fact, has no influence on where the tokens go when they move, hence no influence at all, on the result about which side to win. We can think that the trajectories are pre-determined and the player do not know these trajectories. He needs to buy this information. At each step, the player can only decide, at cost of one dollar, which token's next position to be revealed to him, based on the current positions of the tokens. Only when the player sees two tokens in the same hole, he is sure which side has won. Then he stops paying and leaves. Therefore, his strategy has an effect on his wallet, but no on which side to win.

It turns out that the optimal strategy is always moving the middle one (if there are two at the same site, then choose either) and this is the unique optimal strategy.

In this thesis, we will analyze and solve a type of games which we call Majority-Markov games, as the following. There are an odd number of finite Markov chains. Each Markov chains contains two absorbing target states, one specified as positive kind, the other as negative kind. Since the targets are absorbing, finally, in each Markov chain, a target state will be reached, sometimes positive kind, sometimes negative kind. The player can decide which Markov chain to advance at every step. The goal of the player is to know which kind of target states reached is in the majority. Then, what is the best strategy to minimize the expected time?

The solution involves computing functions called grades, which is introduced in [6] for the states of the individual chains. In some sense, the 'middle' one produces an optimal strategy. See Theorem 4.1.1 and Chapter 4 for more details.

Remark 1.5.1. *Though the subjects of Chapter 2 and Chapter 3 are closely related, Chapter 2, 3 and 4 are written in an independent way. Each chapter is self-contained. The notations may differ in different chapters.*

Remark 1.5.2. *Note that for notational ease, we sometimes use the same notation for a random variable and its law. However, the reader can judge by the text.*

Chapter 2

Critical branching random walks

2.1 Preliminaries

We begin with some notations. For a set $K \subseteq \mathbb{Z}^d$, we write $|K|$ for its cardinality. We write $K \subset\subset \mathbb{Z}^d$ to express that K is a finite nonempty subset of $\mathbb{Z}^d$. For $x \in \mathbb{Z}^d$ (or $\mathbb{R}^d$), we denote by $|x|$ the Euclidean norm of x. We will mainly use the norm $\|\cdot\|$ corresponding the jump distribution θ, i.e. $\|x\| = \sqrt{x \cdot Q^{-1}x}/\sqrt{d}$, where Q is the covariance matrix of θ. For convenience, we set $|0| = \|0\| = 0.5$. We denote by $\mathrm{diam}(K) = \sup\{\|a-b\| : a, b \in K\}$, the diameter of K and by $\mathrm{Rad}(K) = \sup\{\|a\| : a \in K\}$, the radius of K with respect to 0. We write $\mathcal{C}(r)$ for the ball $\{z \in \mathbb{Z}^d : \|z\| \leq r\}$ and $\mathcal{B}(r)$ for the Euclidean ball $\{z \in \mathbb{Z}^d : |z| \leq r\}$. For any subsets A, B of $\mathbb{Z}^d$, we denote by $\rho(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}$ the distance between A and B. When $A = \{x\}$ consists of just one point, we just write $\rho(x, B)$ instead. For any path $\gamma : \{0, \dots, k\} \to \mathbb{Z}^d$, we let $|\gamma|$ stand for k, the length, i.e. the number of edges of γ , $\widehat{\gamma}$ for $\gamma(k)$, the endpoint of γ and $[\gamma]$ for $k+1$, the number of vertices of γ. Sometimes we just use a sequence of vertices to express a path. For example, we may write $(\gamma(0), \gamma(1), \dots, \gamma(k))$ for the path γ. For any $B \subseteq \mathbb{Z}^d$, we write $\gamma \subseteq B$ to express that all vertices of γ except the starting point and the endpoint, lie inside B, i.e. $\gamma(i) \in B$ for any $1 \leq i \leq k-1$. If the endpoint of a path $\gamma_1 : \{0, \dots, |\gamma_1|\} \to \mathbb{Z}^d$ coincides with the starting point of another path $\gamma_2 : \{0, \dots, |\gamma_2|\} \to \mathbb{Z}^d$, then we can define the composite of γ_1 and γ_2 by concatenating γ_1 and γ_2:

$$\gamma_1 \circ \gamma_2 : \{0, \dots, |\gamma_1| + |\gamma_2|\} \to \mathbb{Z}^d,$$

$$\gamma_1 \circ \gamma_2(i) = \begin{cases} \gamma_1(i), & \text{for } i \le |\gamma_1|; \\ \gamma_2(i - |\gamma_1|), & \text{for } i \ge |\gamma_1|. \end{cases}$$

We now state our convention regarding constants. Throughout the text (unless otherwise specified), we use C and c to denote positive constants depending only on dimension d, the critical distribution μ and the jump distribution θ, which may change from place to place. Dependence of constants on additional parameters will be made or stated explicit. For example, $C(\lambda)$ stands for a positive constant depending on d, μ, θ, λ. For functions $f(x)$ and $g(x)$, we write $f \sim g$ if $\lim_{x\to\infty}(f(x)/g(x)) = 1$. We write $f \preceq g$, respectively $f \succeq g$, if there exist constants C such that, $f \le Cg$, respectively $f \ge Cg$. We use $f \asymp g$ to express that $f \preceq g$ and $f \succeq g$. We write $f \ll g$ for that $\lim_{x\to\infty}(f(x)/g(x)) = 0$.

2.1.1 Finite and infinite trees

We are interested in rooted ordered trees (plane trees), in particular, Galton-Watson (GW) trees and its companions. Recall that $\mu = (\mu(i))_{i\in\mathbb{N}}$ is a critical distribution with finite variance $\sigma^2 > 0$. We exclude the trivial case that $\mu(1) = 1$. Throughout this chapter, μ will be fixed. Define another probability measure $\widetilde{\mu}$ on $\mathbb{N}$, call the **adjoint measure** of μ by setting $\widetilde{\mu}(i) = \sum_{j=i+1}^{\infty} \mu(j)$. Since μ has mean 1, $\widetilde{\mu}$ is indeed a probability measure. The mean of $\widetilde{\mu}$ is $\sigma^2/2$. A Galton-Watson process with distribution μ is a process starting with one initial particle, with each particle having independently a random number of children due to μ. The Galton-Watson tree is just the family tree of the Galton-Watson process, rooted at the initial particle. We simply write μ-GW tree for the Galton-Watson tree with offspring distribution μ. If we just change the law of the number of children for the root, using $\widetilde{\mu}$ instead of μ (for other particles still use μ), the new tree is called an **adjoint μ-GW tree**. The **infinite μ-GW tree** is constructed in the following way: start with a semi-infinite line of vertices, called the spine, and graft to the left of each vertex in the spine an independent adjoint μ-GW tree, called a bush. The infinite μ-GW tree is rooted at the first vertex of the spine. Here the left means that we assume every vertex in spine except the root is the youngest child (the latest in the Depth-First search order) of its parent. The **invariant μ-GW tree** is defined as the infinite μ-GW tree, except that for the root, we graft to the left of it, a μ-GW tree, in stead of an adjoint μ-GW tree. We also need to introduce the so-called **μ-GW tree conditioned on survival**. Start with a semi-infinite path, called the spine, rooted at the starting point. For each vertex in the spine, with probability $\mu(i+j+1)$ $(i, j \in \mathbb{N})$, it has totally $i+j+1$ children, with exactly i children elder than the child corresponding to the next vertex in the spine, and exactly j children younger. For any vertex not in the spine, it has a random number of children due to μ. The number of children for different vertices are independent. The random tree generated in this way is just the μ-GW tree conditioned on survival. Each tree is ordered using the classical order according to Depth-First search starting from the root. Note that the subtree generated by the vertices of the spine and all vertices on the left of the spine of the μ-GW tree conditioned on survival has the same distribution as the infinite μ-GW tree.

2.1.2 Tree-indexed random walk

Now we introduce the random walk in $\mathbb{Z}^d$ with jump distribution θ, indexed by a random plane tree T. First choose some $a \in \mathbb{Z}^d$ as the starting point. Conditionally on T we assign independently to each edge of T a random variable in $\mathbb{Z}^d$ according to θ. Then we can uniquely define a function $\mathcal{S}_T : T \to \mathbb{Z}^d$, such that, for every vertex $v \in T$ (we also use T for the set of all vertices of the tree T), $\mathcal{S}_T(v) - a$ is the sum of the variables of all edges belonging to the unique simple path from the root o to the vertex u (hence $\mathcal{S}_T(o) = a$). A plane tree T together with this random function $\mathcal{S}_T$ is called T-indexed random walk starting from a. When T is a μ-GW tree, an adjoint μ-GW tree, an infinite μ-GW tree, and a μ-GW tree conditioned on survival respectively, we simply call the tree-indexed random walk a **snake**, an **adjoint snake**, an **infinite snake** and **incipient infinite snake** (also called branching random walk conditioned on survival) respectively. We write $\mathcal{S}_x$, $\mathcal{S}'_x$, $\mathcal{S}^\infty_x$ and $\overline{\mathcal{S}}^\infty_x$ for a snake, an adjoint snake, and an infinite snake, respectively, starting from $x \in \mathbb{Z}^d$. Note that a snake is just the branching random walk with offspring distribution μ and jump distribution θ. We also need to introduce the **reversed infinite snake** starting from x, $\mathcal{S}_x^-$, which is constructed in the same way as $\mathcal{S}_x^\infty$ except that the variables assigned to the edges in the spine are now due to not θ but the reverse distribution θ^- of θ (i.e. $\theta^-(x) := \theta(-x)$ for $x \in \mathbb{Z}^d$) and similarly the **invariant snake** starting from x, $\mathcal{S}_x^I$, which is constructed by using the invariant μ-GW tree as the random tree T and using θ^- for all edges of the spine of T and θ for all other edges. For an infinite snake (or reversed infinite snake, invariant snake), the random walk indexed by its spine, called its backbone, is just a random walk with jump distribution θ (or θ^-). Note that all snakes here certainly depend on μ and θ. Since μ and θ are fixed throughout this chapter, we omit their dependence in the notation.

2.1.3 Random walk with killing

We will use the tools of random walk with killing. Suppose that when the random walk is currently at position $x \in \mathbb{Z}^d$, then it is killed, i.e. jumps to a 'cemetery' state ϖ, with probability $\mathbf{k}(x)$, where $\mathbf{k} : \mathbb{Z}^d \to [0,1]$ is a given function. In other words, the random walk with killing rate $\mathbf{k}(x)$ (and jump distribution θ) is a Markov chain $\{X_n : n \geq 0\}$ on $\mathbb{Z}^d \cup \{\varpi\}$ with transition probabilities $p(\cdot,\cdot)$ given by: for $x, y \in \mathbb{Z}^d$,

$$p(x,\varpi) = \mathbf{k}(x), \quad p(\varpi,\varpi) = 1, \quad p(x,y) = (1-\mathbf{k}(x))\theta(y-x).$$

For any path $\gamma : \{0,\dots,n\} \to \mathbb{Z}^d$ with length n, its probability weight $\mathbf{b}(\gamma)$ is defined to be the probability that the path consisting of the first n steps for the random walk with killing starting from $\gamma(0)$ is γ. Equivalently,

$$\mathbf{b}(\gamma) = \prod_{i=0}^{|\gamma|-1} (1-\mathbf{k}(\gamma(i)))\theta(\gamma(i+1)-\gamma(i)) = \mathbf{s}(\gamma) \prod_{i=0}^{|\gamma|-1} (1-\mathbf{k}(\gamma(i))), \quad (2.1.1)$$

where $\mathbf{s}(\gamma) = \prod_{i=0}^{|\gamma|-1} \theta(\gamma(i+1)-\gamma(i))$ is the probability weight of γ corresponding to the random walk with jump distribution θ. Note that $\mathbf{b}(\gamma)$

depends on the killing. We delete this dependence on the notation for simplicity.

Now we can define the corresponding Green function for $x, y \in \mathbb{Z}^d$:

$$G_{\mathbf{k}}(x,y) = \sum_{n=0}^{\infty} P(S_x^{\mathbf{k}}(n) = y) = \sum_{\gamma: x \to y} \mathbf{b}(\gamma).$$

where $S_x^{\mathbf{k}} = (S_x^{\mathbf{k}}(n))_{n \in \mathbb{N}}$ is the random walk (with jump distribution θ) starting from x, with killing function $\mathbf{k}$, and the last sum is over all paths from x to y. For $x \in \mathbb{Z}^d, A \subseteq \mathbb{Z}^d$, we write $G_{\mathbf{k}}(x, A)$ for $\sum_{y \in A} G_{\mathbf{k}}(x,y)$.

For any $B \subseteq \mathbb{Z}^d$ and $x, y \in \mathbb{Z}^d$, define the harmonic measure (when exactly one of $\{x, y\}$ is in B):

$$\mathcal{H}_{\mathbf{k}}^B(x,y) = \sum_{\gamma: x \to y, \gamma \subseteq B} \mathbf{b}(\gamma).$$

Note that when the killing function $\mathbf{k} \equiv 0$, the random walk with this killing is just random walk without killing and we write $\mathcal{H}^B(x,y)$ for this case.

We will repeatedly use the following First-Visit Lemma. The idea is to decompose a path according to the first or last visit of a set.

Lemma 2.1.1. *For any $B \subseteq \mathbb{Z}^d$ and $a \in B, b \notin B$, we have:*

$$\begin{aligned} G_{\mathbf{k}}(a,b) &= \sum_{z \in B^c} \mathcal{H}_{\mathbf{k}}^B(a,z) G_{\mathbf{k}}(z,b) = \sum_{z \in B} G_{\mathbf{k}}(a,z) \mathcal{H}_{\mathbf{k}}^{B^c}(z,b); \\ G_{\mathbf{k}}(b,a) &= \sum_{z \in B} \mathcal{H}_{\mathbf{k}}^{B^c}(b,z) G_{\mathbf{k}}(z,a) = \sum_{z \in B^c} G_{\mathbf{k}}(b,z) \mathcal{H}_{\mathbf{k}}^{B}(z,a). \end{aligned}$$

2.1.4 Some facts about random walk and the Green function

From now on, we assume $d \geq 4$. For $x \in \mathbb{Z}^d$, we write $S_x = (S_x(n))_{n \in \mathbb{N}}$ for the random walk with jump distribution θ starting from $S_x(0) = x$. The norm $\|\cdot\|$ corresponding to θ for every $x \in \mathbb{Z}^d$ is defined to be $\|x\| = \sqrt{x \cdot Q^{-1} x}/\sqrt{d}$, where Q is the covariance matrix of θ. Note that $\|x\| \asymp |x|$, especially, there exists $c > 1$, such that $\mathcal{C}(c^{-1}n) \subseteq \mathcal{B}(n) \subseteq \mathcal{C}(cn)$, for any $n \geq 1$. The Green function $g(x, y)$ is defined to be:

$$g(x, y) = \sum_{n=0}^{\infty} P(S_x(n) = y) = \sum_{\gamma : x \to y} \mathbf{s}(\gamma).$$

We write $g(x)$ for $g(0, x)$.

Our assumptions about the jump distribution θ guarantee the standard estimate for the Green function (see e.g. Theorem 2 in [23]):

$$g(x) \sim a_d \|x\|^{2-d}; \tag{2.1.2}$$

and (e.g. one can verify this using the error estimate of Local Central Limit Theorem in [23]) when $d \geq 5$,

$$\sum_{n=0}^{\infty} (n+1) \cdot P(S_0(n) = x) = \sum_{\gamma : 0 \to x} [\gamma] \cdot \mathbf{s}(\gamma) \asymp \|x\|^{4-d} \asymp |x|^{4-d}. \tag{2.1.3}$$

where $a_d = \frac{\mathbf{\Gamma}((d-2)/2))}{2d^{(d-2)/2}\pi^{d/2}\sqrt{\det Q}}$.

Also by LCLT, one can get the following lemma.

Lemma 2.1.2.

$$\lim_{n \to \infty} sup_{x \in \mathbb{Z}^d} \left(\sum_{\gamma : 0 \to x, |\gamma| \geq n|x|^2} \mathbf{s}(\gamma)/g(x) \right) = 0. \tag{2.1.4}$$

The following lemma is natural from the perspective of Brownian motion, the scaling limit of random walk.

Lemma 2.1.3. *Let U, V be two connected bounded open subset of $\mathbb{R}^d$ such that $\overline{U} \subseteq V$. Then there exists a $C = C(U, V)$ such that if $A_n = nU \cap \mathbb{Z}^d, B_n = nV \cap \mathbb{Z}^d$ then when n is sufficiently large,*

$$\sum_{\gamma : x \to y, \gamma \subseteq B_n, |\gamma| \leq 2n^2} \mathbf{s}(\gamma) \geq C g(x, y), \text{ for any } x, y \in A_n \tag{2.1.5}$$

This Lemma may not be standard, hence we give a sketch of proof in Appendix.

Since our jump distribution θ may be unbounded, we need the following Overshoot Lemma:

Lemma 2.1.4. *For any $r, s > 1$, let $B = \mathcal{C}(r)$. Then for any $a \in B$, we have:*

$$\sum_{y \in (\mathcal{C}(r+s))^c} \mathcal{H}^B_{\mathbf{k}}(a, y) \preceq \frac{r^2}{s^d}, \qquad \sum_{y \in (\mathcal{C}(r+s))^c} \mathcal{H}^B_{\mathbf{k}}(y, a) \preceq \frac{r^2}{s^d}. \tag{2.1.6}$$

Proof. It suffices to show the case when $\mathbf{k} \equiv 0$. By considering where the last position is before leaving $\mathcal{C}(r)$, one can get:

$$\sum_{y \in (\mathcal{C}(r+s))^c} \mathcal{H}^B(a, y) \leq \sum_{z \in \mathcal{C}(r)} g(a, z) P(\text{the jump leaving } \mathcal{C}(r) \geq s)$$
$$\overset{(1.1.1)}{\leq} \Big(\sum_{z \in \mathcal{C}(r)} g(a, z)\Big) \cdot C/s^d \preceq \frac{r^2}{s^d}.$$

One can show the other inequality similarly. □

2.2 Branching capacity and visiting probabilities

In this and the following sections (Section 2.2 and Section 2.3), we focus on supercritical dimensions and always assume $d \geq 5$. For any $K \subset\subset \mathbb{Z}^d$, we are interested in the probability of visiting K by the critical branching random walk with offspring distribution μ and jump distribution θ, or equivalently, a snake. For any $x \in \mathbb{Z}^d$, write $\mathbf{p}(x)$, $\mathbf{r}(x)$, $\mathbf{q}(x)$ and $\mathbf{q}^-(x)$, respectively, for the probability that a snake, an adjoint snake, an infinite snake and a reversed infinite snake, respectively, starting from x visits K, i.e. $P((\mathcal{S}_T(T) \cap K) \neq \emptyset)$ where $T, \mathcal{S}_T$ are the corresponding random tree and random map. We write $\overline{\mathbf{p}}(x)$ and $\overline{\mathbf{r}}(x)$ respectively for the probability that a snake and an adjoint snake respectively, starting from x visits K strictly after time zero, i.e. $P((\mathcal{S}_T(T \setminus \{o\}) \cap K) \neq \emptyset)$. Note that when $x \notin K$, $\mathbf{p}(x) = \overline{\mathbf{p}}(x)$ and

$\mathbf{r}(x) = \bar{\mathbf{r}}(x)$. For simplicity, we delete the dependence on K in the notations. We fix K from now on until Section 2.2.6.

We first give some preliminary upper bounds for the visiting probabilities by computing the expectation of the number of visits. Here are the computations. When x is relatively far from K, say $\rho(x, K) \geq 2\mathrm{diam}(K)$. For the snake $\mathcal{S}_x$, the expectation of the number of offspring at n-th generation is one. Hence, the expectation of the number of visiting any $a \in K$ is just $g(x, a) \asymp \|x - a\|^{2-d} \asymp \|x\|^{2-d}$. For the adjoint snake $\mathcal{S}'_x$, the expectation of the number of offspring at n-th generation (for $n \geq 1$) is $\mathbb{E}\widetilde{\mu} = \sigma^2/2 \asymp 1$ (recall that μ is fixed). Hence the expectation of the total number of visiting a can also be bounded by $g(x, a)$ up to some constant multiplier. For the infinite snake $\mathcal{S}^\infty_x$, one can see that the expectation of the number of offspring at n-th generation is $1 + n \cdot \mathbb{E}\widetilde{\mu} \asymp n + 1$. Hence when $\rho(x, K) \geq 2\mathrm{diam}(K)$, the expectation of the total number of visiting a is bounded, up to some constant, by:

$$\sum_{n=0}^{\infty} (n+1) P(S_x(n) = a) \overset{(2.1.3)}{\asymp} \|x - a\|^{4-d} \asymp \|x\|^{4-d}.$$

Recall that $S_x = (S_x(n))_{n \in \mathbb{N}}$ is the random walk starting from x with jump distribution θ. Summing up over all $a \in K$, we get

$$\begin{aligned} \mathbf{p}(x) &\preceq |K|/\|x\|^{d-2}; \\ \mathbf{r}(x) &\preceq |K|/\|x\|^{d-2}; \\ \mathbf{q}(x) &\preceq |K|/\|x\|^{d-4}. \end{aligned} \tag{2.2.1}$$

For $\mathbf{q}^-(x)$, by considering the expectation of the number of visiting points, one can get (or use (2.2.22)):

$$\mathbf{q}^-(x) \preceq \sum_{y \in \mathbb{Z}^d} g^-(x, y) g(y, K) \preceq |K| \sum_{y \in \mathbb{Z}^d} \|x - y\|^{2-d} \|y\|^{2-d} \asymp |K|/\|x\|^{d-4},$$

where $g^-(x, y) = g(y, x)$ is the Green function for the reversed random walk.

From this, we see that when x tends to infinity, all four types of visiting probabilities tend to 0. Now we introduce the escape probabilities.

Definition 2.2.1. *K is a finite subset of $\mathbb{Z}^d$, for any $x \in \mathbb{Z}^d$, define $Es_K(x)$ to be the probability that a reversed infinite snake starting from x does not visit K except possibly for the image of the bush grafted to the root and $Esc_K(x)$ to be the probability that an invariant snake starting from x does not visit K except possibly for the image of the spine. Define the **Branching capacity** of K by:*

$$BCap(K) = \sum_{a \in K} Es_K(a) = \sum_{a \in K} Esc_K(a). \tag{2.2.2}$$

Remark 2.2.1. *In next chapter, we construct the model of branching interlacement. As a main step, we give the definition of branching capacity (only) when μ is the critical geometric distribution. In that case, the branching capacity here is equivalent to the branching capacity there, up to a constant factor 2. But here we do not need the so-called contour function which plays an important role there. Furthermore, we can construct the model of branching interlacement for general critical offspring distribution.*

The last equality can be seen from our main theorem of branching capacity, Theorem 1.3.1. We also introduce the escape probability for the infinite snake $\mathrm{Es}_K^+(x)$, which is defined to be the probability that an infinite snake starting from x does not visit K except possibly for the image of the bush grafted to the root. Note that $\mathrm{Es}_K^+(x) \geq 1 - \mathbf{q}(x) \to 1$, as $x \to \infty$.

Remark 2.2.2. *If we let μ be the degenerate measure, that is, $\mu(1) = 1$, then: the snake and the infinite snake are just the random walk with jump distribution θ; the reversed infinite snake and the invariant snake are the random walk with jump distribution θ^-. Therefore Es_K is just the escape probability for the 'reversed' walk and Esc_K is the escape probability for the 'original' walk. In that case, Theorem 1.3.1 is just the classical theorem for regular capacity. Note that when θ is symmetric, for random walk, $Es_K(a) = Esc_K(a)$. But this is generally not true for branching random walk even when θ is symmetric. If $K = \{a\}$ consists of only one point, then it is true by Theorem 1.3.1*

2.2.1 Monotonicity and subadditivity

We postpone the proof of Theorem 1.3.1 until Section 2.2.4. We now s-tate some basic properties about branching capacity. Like regular capacity, branching capacity is monotone and subadditive:

Proposition 2.2.2. *For any $K \subseteq K'$ finite subsets of $\mathbb{Z}^d$,*

$$BCap(K) \le BCap(K');$$

For any K_1, K_2 finite subsets of $\mathbb{Z}^d$,

$$BCap(K_1 \cap K_2) + BCap(K_1 \cup K_2) \le BCap(K_1) + BCap(K_2).$$

Proof. When $K \subseteq K'$, a snake visiting K must visit K'. So

$$P(\mathcal{S}_x \text{ visits } K) \le P(\mathcal{S}_x \text{ visits } K').$$

By (1.3.1), we get $\mathrm{BCap}(K) \le \mathrm{BCap}(K')$.

For the other inequality, we use a similar idea. First, we have:

$$\begin{aligned}
&P(\mathcal{S}_x \text{ visits } K_1) = P(\mathcal{S}_x \text{ visits } K_1 \text{ but not } K_2) + P(\mathcal{S}_x \text{ visits both } K_1 \& K_2);\\
&P(\mathcal{S}_x \text{ visits } K_2) = P(\mathcal{S}_x \text{ visits } K_2 \text{ but not } K_1) + P(\mathcal{S}_x \text{ visits both } K_1 \& K_2);\\
&P(\mathcal{S}_x \text{ visits } K_1 \cup K_2) = P(\mathcal{S}_x \text{ visits } K_1 \text{ but not } K_2) +\\
&\qquad\qquad P(\mathcal{S}_x \text{ visits } K_2 \text{ but not } K_1) + P(\mathcal{S}_x \text{ visits both } K_1 \& K_2).
\end{aligned}$$

Since $P(\mathcal{S}_x \text{ visits } K_1 \cap K_2) \le P(\mathcal{S}_x \text{ visits both } K_1 \& K_2)$, we have:

$$\begin{aligned}
P(\mathcal{S}_x \text{ visits } K_1 \cup K_2) + P(\mathcal{S}_x \text{ visits } K_1 \cap K_2) \le\\
P(\mathcal{S}_x \text{ visits } K_1) + P(\mathcal{S}_x \text{ visits } K_2).
\end{aligned}$$

This concludes the proposition by (1.3.1). □

2.2.2 A key observation

We begin with some straightforward computations. When a snake $\mathcal{S}_x = (T, \mathcal{S}_T)$ visits K, since T is an ordered tree, we have the unique first vertex, denoted by τ_K, in $\{v \in T : \mathcal{S}_T(v) \in K\}$ due to the default order. We say $\mathcal{S}_T(\tau_K)$ is the visiting point or $\mathcal{S}_x$ visits K at $\mathcal{S}_T(\tau_K)$. Assume $(v_0, v_1, \dots, v_k)$ is the unique simple path in T from the root o to τ_K. Define $\Gamma(\mathcal{S}_x) = (\mathcal{S}_T(v_0), \mathcal{S}_T(v_1), \dots, \mathcal{S}_T(v_k))$ and say $\mathcal{S}_x$ visits K via $\Gamma(\mathcal{S}_x)$. We now compute $P(\Gamma(\mathcal{S}_x) = \gamma)$, for any given $\gamma = (\gamma(0), \dots, \gamma(k)) \subseteq K^c$ starting from x, ending at K. Let $\widetilde{a}_i$ and $\widetilde{b}_i$ respectively, be the number of the older, and younger respectively, brothers of v_i, for $i = 1, \dots, k$. From the tree structure, one can see that, for any $l_1, \dots, l_k$, $m_1, \dots, m_k \in \mathbb{N}$,

$$
\begin{aligned}
&P(\mathcal{S}_x \text{ visits } K \text{ via } \gamma; \widetilde{a}_i = l_i, \widetilde{b}_i = m_i, \text{for } i = 1, \dots, k) \\
&\qquad = \mathbf{s}(\gamma) \prod_{i=1}^{k} \left(\mu(l_i + m_i + 1)(\widetilde{r}(\gamma(i-1)))^{l_i} \right),
\end{aligned} \tag{2.2.3}
$$

where $\widetilde{r}(z)$ is the probability that a snake starting from z does not visit K conditioned on the initial particle having only one child. Summing up, we get:

$$
\begin{aligned}
&P(\mathcal{S}_x \text{ visits } K \text{ via } \gamma) \\
=& \sum_{l_1,\dots,l_k; m_1,\dots,m_k \in \mathbb{N}} P(\mathcal{S}_x \text{ visits } K \text{ via } \gamma; \widetilde{a}_i = l_i, \widetilde{b}_i = m_i, \text{for } i = 1, \dots, k) \\
=& \sum_{l_1,\dots,l_k; m_1,\dots,m_k \in \mathbb{N}} \mathbf{s}(\gamma) \prod_{i=1}^{k} \left(\mu(l_i + m_i + 1)(\widetilde{r}(\gamma(i-1)))^{l_i} \right) \\
=& \mathbf{s}(\gamma) \prod_{i=1}^{k} \sum_{l_i, m_i \in \mathbb{N}} \left(\mu(l_i + m_i + 1)(\widetilde{r}(\gamma(i-1)))^{l_i} \right) \\
=& \mathbf{s}(\gamma) \prod_{i=1}^{k} \sum_{l_i \in \mathbb{N}} \left(\widetilde{\mu}(l_i)(\widetilde{r}(\gamma(i-1)))^{l_i} \right).
\end{aligned}
$$

Note that for any $z \notin K$,

$$\sum_{l \in \mathbb{N}} \widetilde{\mu}(l)(\widetilde{r}(z))^l$$

is just $1 - \mathbf{r}(z)$, the probability that an adjoint snake starting form z does not visit K. If we let the killing function be

$$\mathbf{k}(x) = P(\mathcal{S}'_x \text{ visits } K) = \mathbf{r}(x). \tag{2.2.4}$$

then we have (recall the definition of $\mathbf{b}(\gamma)$ from (2.1.1))

$$\begin{aligned}\mathbf{b}(\gamma) =& \mathbf{s}(\gamma) \prod_{i=1}^{k} (1 - \mathbf{k}(\gamma(i-1))) = \mathbf{s}(\gamma) \prod_{i=1}^{k} (1 - \mathbf{r}(\gamma(i-1))) \\ =& \mathbf{s}(\gamma) \prod_{i=1}^{k} \sum_{l_i \in \mathbb{N}} \left(\widetilde{\mu}(l_i)(\widetilde{r}(\gamma(i-1)))^{l_i} \right) = P(\mathcal{S}_x \text{ visits } K \text{ via } \gamma).\end{aligned}$$

This brings us to the key formula of this work:

Proposition 2.2.3.

$$\mathbf{b}(\gamma) = P(\mathcal{S}_x \textit{ visits } K \textit{ via } \gamma). \tag{2.2.5}$$

In words, the probability that a snake visits K via γ is just γ's probability weight according to the random walk with the killing function given by (2.2.4). Throughout this chapter, we will mainly use this killing function and write $G_K(\cdot, \cdot)$ for the corresponding Green function. By summing the last equality over γ, we get: for any $a \in K$,

$$P(\mathcal{S}_x \text{ visits } K \text{ at } a) = \sum_{\gamma : x \to a} \mathbf{b}(\gamma) = G_K(x, a); \tag{2.2.6}$$

and

$$\mathbf{p}(x) = P(\mathcal{S}_x \text{ visits } K) = \sum_{\gamma : x \to K} \mathbf{b}(\gamma) = G_K(x, K). \tag{2.2.7}$$

Note that since $\mathbf{r}(x) = 1$ for $x \in K$, when γ, except for the ending point, intersects K, $\mathbf{b}(\gamma) = 0$.

On the other hand, from the structure of the infinite snake, one can easily see that $\mathbf{q}(x)$ is just the probability that in this killing random walk, a particle starting at x will be killed at some finite time.

Now we turn to the last visiting point, which can be addressed similarly. When a snake $\mathcal{S}_x = (T, \mathcal{S}_T)$ visits K, denoted by ξ_K, the last vertex in $\{v \in T : \mathcal{S}_T(v) \in K\}$ due to the default order. Assume $(v_0, v_1, \dots, v_k)$ is the unique simple path in T from the root o to ξ_K. Define $\overline{\Gamma}(\mathcal{S}_x) = (\mathcal{S}_T(v_0), \mathcal{S}_T(v_1), \dots, \mathcal{S}_T(v_k))$ and say $\mathcal{S}_x$ leaves K at $\mathcal{S}_T(v_k)$, via $\overline{\Gamma}(\mathcal{S}_x)$. We would like to compute $P(\overline{\Gamma}(\mathcal{S}_x) = \gamma)$, for any $\gamma = (\gamma(0), \dots, \gamma(k))$ starting from x and ending at A (note that unlike the former case, the interior of γ now may intersect K). Let $\widetilde{a}_i$ ($\widetilde{b}_i$ respectively) be the number of the older (younger respectively) brothers of v_i, for $i = 1, \dots, k$. Similarly to the former case, one can see that, for any $l_1, \dots, l_k$, $m_1, \dots, m_k \in \mathbb{N}$,

$$
\begin{aligned}
&P(\mathcal{S}_x \text{ leaves } K \text{ via } \gamma; \widetilde{a}_i = l_i, \widetilde{b}_i = m_i, \text{for } i = 1, \dots, k) \\
&\qquad = \mathbf{s}(\gamma)(1 - \overline{\mathbf{p}}(\gamma(k))) \prod_{i=1}^{k} \left(\mu(l_i + m_i + 1)(\hat{r}(\gamma(i-1)))^{m_i}\right), \quad (2.2.8)
\end{aligned}
$$

where $\hat{r}(z)$ is the probability that a snake starting from z does not visit (except possibly for the root) K conditioned on the initial particle having only one child. Summing up, we get:

$$
P(\mathcal{S}_x \text{ leaves } K \text{ via } \gamma) = \mathbf{s}(\gamma)(1 - \overline{\mathbf{p}}(\widehat{\gamma})) \prod_{i=1}^{k} (1 - \overline{\mathbf{r}}(\gamma(i-1))). \qquad (2.2.9)
$$

If we let the killing function be $\mathbf{k}'(x) = \overline{\mathbf{r}}(x)$, then the last term is just $(1 - \overline{\mathbf{p}}(\widehat{\gamma}))\mathbf{b}_{\mathbf{k}'}(\gamma)$.

Remark 2.2.3. *We will always use the killing function in* (2.2.4)*, except in the proof of the third assertion in Theorem 1.3.1.*

Remark 2.2.4. *Now the reason for the introduction of the adjoint snake and the infinite snakes is clear: in order to understand* $\mathbf{p}(x)$*, the probability of visiting* K*, we need to study the random walk with killing where the killing function is just the probability of the adjoint snake visiting* K*.*

Remark 2.2.5. *The computations here are initiated in [29]. Note that in this subsection, we do not need the assumption $d \geq 5$. All results are valid for all dimensions.*

2.2.3 Convergence of the Green function

The goal of this subsection is to prove:

Lemma 2.2.4.

$$\lim_{x,y\to\infty} G_K(x,y)/g(x,y) = 1. \tag{2.2.10}$$

Proof. The part of '$\leq$' is trivial, since $G_K(x,y) \leq g(x,y)$. We need to consider the other part.

First, consider the case $\|x\|/2 \leq \|y\| \leq 2\|x\|^{1.1}$. Let

$$\Gamma_1 = \{\gamma : x \to y \,|\, |\gamma| \geq \|x\|^{0.1} \cdot \|x-y\|^2\};$$
$$\Gamma_2 = \{\gamma : x \to y \,|\, \gamma \text{ visits } \mathcal{C}(\|x\|^{0.9})\}.$$

By Lemma 2.1.2, one can see that $\sum_{\gamma\in\Gamma_1} \mathbf{s}(\gamma)/g(x,y)$ tends to 0. Similar to the First-Visit Lemma, by considering the first visiting place, we have (let $B = \mathcal{C}(\|x\|^{0.9})$):

$$\begin{aligned}\sum_{\gamma\in\Gamma_2} \mathbf{s}(\gamma) =& \sum_{a\in B} \mathcal{H}^{B^c}(x,a) g(a,y) \asymp \sum_{a\in B} \mathcal{H}^{B^c}(x,a)\|y\|^{2-d} \\ =& P(S_x \text{ visits } B) \cdot \|y\|^{2-d} \asymp (\|x\|^{0.9}/\|x\|)^{d-2}\|y\|^{2-d} \\ \preceq& \|x\|^{-0.1}\|x-y\|^{2-d} \asymp \|x\|^{-0.1} g(x,y).\end{aligned}$$

Note that the estimate of $P(S_x \text{ visits } \mathcal{C}(r)) \asymp (r/\|x\|)^{d-2}$ is standard, and for the second last inequality we use $\|y\| \geq (\|x\|+\|y\|)/3 \succeq \|x-y\|$. Hence, we get $\sum_{\gamma\in\Gamma_2} \mathbf{s}(\gamma)/g(x,y) \to 0$ and therefore,

$$\sum_{\gamma:x\to y,\ \gamma\notin\Gamma_1\cup\Gamma_2} \mathbf{s}(\gamma) \sim g(x,y). \tag{2.2.11}$$

For any $\gamma : x \to y,\ \gamma \notin \Gamma_1 \cup \Gamma_2$, using (2.2.1), one can see:

$$\begin{aligned}\mathbf{b}(\gamma)/\mathbf{s}(\gamma) = \prod_{i=0}^{|\gamma|-1}(1-\mathbf{k}(\gamma(i))) &\geq (1-c|K|/(\|x\|^{0.9})^{d-2})^{|\gamma|} \\ &\geq 1-c|K||\gamma|/(\|x\|^{0.9})^{d-2} \geq 1-c|K|\|x\|^{0.1}\|x-y\|^2/(\|x\|^{0.9})^{d-2} \\ &\geq 1-c|K|\|x\|^{0.1}\|x\|^{2.2}/\|x\|^{0.9\cdot 3} \geq 1-c|K|/\|x\|^{0.4} \to 1.\end{aligned}$$

Hence, we have:

$$\sum_{\gamma:x\to y,\ \gamma\notin\Gamma_1\cup\Gamma_2}\mathbf{b}(\gamma) \sim \sum_{\gamma:x\to y,\ \gamma\notin\Gamma_1\cup\Gamma_2}\mathbf{s}(\gamma).$$

Combining this and (2.2.11), we get: when $\|x\|/2 \leq \|y\| \leq 2\|x\|^{1.1}$, (2.2.10) is true.

When $\|y\| > 2\|x\|^{1.1}$, we know $g(x,y) \sim a_d\|y\|^{2-d}$. Hence, we need to show: $G_K(x,y) \sim a_d\|y\|^{2-d}$. Let $r = 2\|y\|^{1/1.1}$ and $B = \mathcal{C}(r)$. Then for any $a \in \mathcal{C}(2r) \setminus \mathcal{C}(r)$, $\|x\| < \|a\| < \|y\| \leq 2\|a\|^{1.1}$ (when $\|y\|$ is large). Hence $G_K(a,y) \sim g(a,y) \sim a_d\|y\|^{2-d}$. Applying the First-Visit Lemma, we have:

$$\begin{aligned}G_K(x,y) = \sum_{a\in B^c}\mathcal{H}_{\mathbf{k}}^B(x,a)G_K(a,y) &\geq \sum_{a\in\mathcal{C}(2r)\setminus B}\mathcal{H}_{\mathbf{k}}^B(x,a)G_K(a,y) \\ &\sim \sum_{a\in\mathcal{C}(2r)\setminus B}\mathcal{H}_{\mathbf{k}}^B(x,a)a_d\|y\|^{2-d} \\ &=(\sum_{a\in B^c}\mathcal{H}_{\mathbf{k}}^B(x,a) - \sum_{a\in(\mathcal{C}(2r))^c}\mathcal{H}_{\mathbf{k}}^B(x,a))a_d\|y\|^{2-d} \\ &\geq((1-\mathbf{r}(x))\mathrm{Es}_K^+(x) - C\frac{r^2}{r^d})a_d\|y\|^{2-d} \\ &\sim a_d\|y\|^{2-d}.\end{aligned}$$

In the second last inequality we use the Overshoot Lemma and

$$\sum_{a\in B^c}\mathcal{H}_{\mathbf{k}}^B(x,a) \geq \sum_{a\in B^c}\mathcal{H}_{\mathbf{k}}^B(x,a)(1-\mathbf{r}(a))\mathrm{Es}_K^+(a) = (1-\mathbf{r}(x))\mathrm{Es}_K^+(x) \to 1.$$

Now, we show (2.2.10) for the case $\|x\| \leq \|y\|$. The case of $\|x\| \geq \|y\|$ can be handled similarly. □

Remark 2.2.6. *As we have seen in the proof, since the jump distribution θ maybe unbounded, we need an extra step to control the long jump, via the Overshoot Lemma. This happens again and again later. It might be convenient, especially for a first-time reader, to restrict the attention to the jump distribution with finite range.*

2.2.4 Proof of Theorem 1.3.1

Now we are ready to prove Theorem 1.3.1. It is sufficient to show:

Lemma 2.2.5. *Under the same assumption of Theorem 1.3.1, we have:*

$$P(\mathcal{S}_x \textit{ visits } K \textit{ at } a) \sim a_d \|x\|^{2-d} Es_K(a);$$
$$P(\mathcal{S}_x \textit{ leaves } K \textit{ at } a) \sim a_d \|x\|^{2-d} Esc_K(a);$$

whenever the escape probability on the right hand side is nonzero.

Proof. Fix some $\alpha \in (0, 2/(d+2))$. Let $r = \|x\|^\alpha, s = \|x\|^{1-\alpha}$ and $B = \mathcal{C}(r), B_1 = \mathcal{C}(s) \setminus B$ and $B_2 = (\mathcal{C}(s))^c$. Note that our choice of α implies $r^2/s^d \ll \|x\|^{2-d}$. Then,

$$\begin{aligned} P(\mathcal{S}_x \text{ visits } K \text{ at } a) &\overset{(2.2.6)}{=} \sum_{\gamma: x \to a} \mathbf{b}(\gamma) = \sum_{b \in B^c} G_K(x,b)\mathcal{H}^B_{\mathbf{k}}(b,a) \\ &= \sum_{b \in B_1} G_K(x,b)\mathcal{H}^B_{\mathbf{k}}(b,a) + \sum_{b \in B_2} G_K(x,b)\mathcal{H}^B_{\mathbf{k}}(b,a). \end{aligned} \quad (2.2.12)$$

We argue that the first term has the desired asymptotics and the second is negligible:

$$\begin{aligned}
\sum_{b\in B_1} G_K(x,b)\mathcal{H}^B_{\mathbf{k}}(b,a) &\overset{(2.2.10)}{\sim} a_d\|x\|^{2-d}\sum_{b\in B_1}\mathcal{H}^B_{\mathbf{k}}(b,a)\\
&\sim a_d\|x\|^{2-d}(\mathrm{Es}_K(a)-\sum_{b\in B_2}\mathcal{H}^B_{\mathbf{k}}(b,a))\\
&\overset{(2.1.6)}{=} a_d\|x\|^{2-d}(\mathrm{Es}_K(a)-O(r^2/s^d))\sim a_d\|x\|^{2-d}\mathrm{Es}_K(a);\\
\sum_{b\in B_2} G_K(x,b)\mathcal{H}^B_{\mathbf{k}}(b,a) &\preceq \sum_{b\in B_2}\mathcal{H}^B_{\mathbf{k}}(b,a)\overset{(2.1.6)}{\preceq} r^2/s^d \ll \|x\|^{2-d}.
\end{aligned}$$

Note that the second line is due to $\mathrm{Es}_K(a)=\sum_{b\in B_1\cup B_2}\mathcal{H}^B_{\mathbf{k}}(b,a)\mathrm{Es}_K(b)$ and $\mathrm{Es}_K(x)\sim 1$.

Now we complete the proof of the first assertion. Very similar arguments can be used for the second assertion. Note that due to (2.2.9), we need to use the killing function $\mathbf{k}'(x)=\bar{\mathbf{r}}(x)$ and the analogous version of Lemma 2.2.4 for this killing. We leave the details to the reader. □

2.2.5 The asymptotics for $\mathbf{q}(x)$, $\mathbf{q}^-(x)$ and $\mathbf{r}(x)$

Thanks to Theorem 1.3.1, we also can find the exact asymptotics of the visiting probabilities by an adjoint snake, an infinite snake and a reversed infinite snake, i.e. $\mathbf{r}(x)$, $\mathbf{q}(x)$ and $\mathbf{q}^-(x)$:

Proposition 2.2.6.

$$\mathbf{r}(x)\sim\frac{a_d\sigma^2 BCap(K)}{2\|x\|^{d-2}}, \tag{2.2.13}$$

$$\mathbf{q}(x)\sim\frac{t_d\cdot a_d^2\sigma^2 BCap(K)}{2\|x\|^{d-4}}, \tag{2.2.14}$$

$$\mathbf{q}^-(x)\sim\frac{t_d\cdot a_d^2\sigma^2 BCap(K)}{2\|x\|^{d-4}}, \tag{2.2.15}$$

where σ^2 is the variance of μ, $t_d=t_d(\theta)=\int_{t\in\mathbb{R}^d}\|t\|^{2-d}\|h-t\|^{2-d}dt$, and $h\in\mathbb{R}^d$ is any vector satisfying $\|h\|=1$.

Remark 2.2.7. *In fact, $t_d = t_d(\theta)$ has the following form:*

$$t_d = \int_{t\in\mathbb{R}^d} \|t\|^{2-d}\|h-t\|^{2-d}dt = d^{d/2}\sqrt{\det Q}\int_{t\in\mathbb{R}^d} |t|^{2-d}|h'-t|^{2-d}dt,$$

where $h' \in \mathbb{R}^d$ is any vector with $|h'| = 1$ in $\mathbb{R}^d$.

Proof. Let $\widetilde{s}(x)$ be the probability that a snake starting from x visits K conditioned on the initial particle having exactly one child. Then it is straightforward to see that: when $x \notin K$,

$$1-\mathbf{p}(x) = \sum_{i\in\mathbb{N}} \mu(i)(1-\widetilde{s}(x))^i, \quad 1-\mathbf{r}(x) = \sum_{i\in\mathbb{N}} \tilde{\mu}(i)(1-\widetilde{s}(x))^i. \quad (2.2.16)$$

Note that

$$\sum_{i\in\mathbb{N}} \mu(i)(1-\widetilde{s}(x))^i \geq \sum_{i\in\mathbb{N}} \mu(i)(1-i\widetilde{s}(x)) = 1-(\mathbb{E}\mu)\widetilde{s}(x)$$

and

$$\sum_{i\in\mathbb{N}} \mu(i)(1-\widetilde{s}(x))^i \leq \mu(0) + \sum_{i=1}^{\infty} \mu(i)(1-\widetilde{s}(x)) = 1-(1-\mu(0))\widetilde{s}(x)$$

Hence we have

$$\mathbf{p}(x) \asymp \widetilde{s}(x), \quad (2.2.17)$$

and similarly one can get $\mathbf{r}(x) \asymp \widetilde{s}(x)$. Therefore,

$$\mathbf{r}(x) \asymp \mathbf{p}(x). \quad (2.2.18)$$

We will use the following easy lemma and omit its proof.

Lemma 2.2.7. *Let $(a_n)_{n\in\mathbb{N}}$ be a nonnegative sequence satisfying: $\sum_{n\in\mathbb{N}} a_n = 1$ and $\sum_{n\in\mathbb{N}} na_n < \infty$. Let $f(t) = \sum_{n\in\mathbb{N}} a_n t^n$. Then we have:*

$$\lim_{t\to 1^-} (1-f(t))/(1-t) = \sum_{n\in\mathbb{N}} na_n.$$

By this lemma and (2.2.16), we have

$$\mathbf{p}(x) \sim \sum_i i\mu(i)\widetilde{s}(x) = \widetilde{s}(x),$$

$$\mathbf{r}(x) \sim \sum_i i\tilde{\mu}(i)\widetilde{s}(x) = \frac{\sigma^2}{2}\widetilde{s}(x).$$

Hence,

$$\mathbf{r}(x) \sim \frac{\sigma^2}{2}\mathbf{p}(x) \sim \frac{\sigma^2 a_d \mathrm{BCap}(K)}{2\|x\|^{d-2}}.$$

Now we turn to the asymptotic of $\mathbf{q}(x)$. We point out two equalities for $\mathbf{q}(x)$:

$$\mathbf{q}(x) = \sum_{y\in\mathbb{Z}^d} G_K(x,y)\mathbf{r}(y); \tag{2.2.19}$$

$$\mathbf{q}(x) = \sum_{y\in\mathbb{Z}^d} g(x,y)\mathbf{r}(y)\mathrm{Es}_K^+(y). \tag{2.2.20}$$

The first can be easily derived by considering where the particle dies in the model of random walk with killing function $\mathbf{r}$. For the second one, we need to consider a bit different but equivalent model: a particle starting from x executes a random walk, but at each step, the particle has the probability $\mathbf{r}$ to get a flag (instead of to die) and its movements are unaffected by flags. Let τ and ξ be the first and last time getting flags (if there is no such times then denote $\tau = \xi = \infty$). Note that since $\mathbf{q}(z) < 1$ (when $|z|$ is large), the total number of flags gained is finite, almost surely. Hence $P(\tau < \infty) = P(\xi < \infty)$ and $\mathbf{q}(x)$ is just the probability that $\xi < \infty$. By considering where the particle gets its last flag, one can get (2.2.20).

We will use the following easy lemma and omit its proof:

Lemma 2.2.8.

$$\|x\|^{d-4}\sum_{z\in\mathbb{Z}^d}\frac{1}{\|z\|^{d-2}\|x-z\|^{d-2}} \sim \int_{t\in\mathbb{R}^d}\|t\|^{2-d}\|h-t\|^{2-d}dt. \tag{2.2.21}$$

For the asymptotics of $\mathbf{q}(x)$, one can use either (2.2.19) or (2.2.20) and the processes are similar to each other. Here we use (2.2.19). Let $B = \mathcal{C}(r)$ and r be very large. Divide the right hand side of (2.2.19) into three parts: $\sum_{y\in B}$, $\sum_{y\in x+B}$ and $\sum_{y\notin B\cup x+B}$. We will argue that the first two parts are negligible compared to $\|x\|^{4-d}$ and the third term has the desired asymptotics. For the first part, we have:

$$\|x\|^{d-4}\sum_{y\in B} G_K(x,y)\mathbf{r}(y) \preceq \|x\|^{d-4}\sum_{y\in B} g(x,y)\cdot 1$$
$$\preceq \|x\|^{d-4}\sum_{y\in B}\frac{1}{(\|x\|-r)^{d-2}} \preceq \|x\|^{d-4}r^d/(\|x\|-r)^{d-2} \to 0 \text{ (when } x\to\infty).$$

For the second part, we have:

$$\|x\|^{d-4}\sum_{y\in x+B} G_K(x,y)\mathbf{r}(y) \preceq \|x\|^{d-4}\sum_{y\in x+B} 1\cdot \mathbf{r}(y)$$
$$\overset{(2.2.1)}{\preceq} \|x\|^{d-4}\sum_{y\in x+B}|K|\|y\|^{2-d} \le \|x\|^{d-4}r^d|K|/(\|x\|-r)^{d-2}\to 0.$$

When r and $\|x\|$ are large and $y\notin B\cup(x+B)$, the ratio between $G_K(x,y)\mathbf{r}(y)$ and $a_d\|x-y\|^{2-d}a_d\sigma^2\mathrm{BCap}(K)\|y\|^{2-d}/2$ is very close to 1. On the other hand,

$$\begin{aligned}
&\|x\|^{d-4}\sum_{y\notin B\cup(x+B)} a_d\|x-y\|^{2-d}a_d\sigma^2\mathrm{BCap}(K)\|y\|^{2-d}/2\\
=&a_d^2\sigma^2\mathrm{BCap}(K)/2\cdot\|x\|^{d-4}\sum_{y\notin B\cup(x+B)}\|x-y\|^{2-d}\|y\|^{2-d}\\
=&a_d^2\sigma^2\mathrm{BCap}(K)/2\cdot(\|x\|^{d-4}\sum_{y\in\mathbb{Z}^d}\|x-y\|^{2-d}\|y\|^{2-d}-\\
&\|x\|^{d-4}\sum_{y\in(B\cup x+B)}\|x-y\|^{2-d}\|y\|^{2-d}).
\end{aligned}$$

By (2.2.21), the first term in the bracket tends to t_d. Similar to the estimate for the first two parts, one can verify that

$$\|x\|^{d-4} \sum_{y \in (B \cup x+B)} \|x-y\|^{2-d}\|y\|^{2-d} \preceq \|x\|^{d-4} r^d/(\|x\|-r)^{d-2} \to 0.$$

To sum up, we get

$$\|x\|^{d-4} \sum_{y \in \mathbb{Z}^d} G_K(x,y)\mathbf{r}(y) \sim a_d^2 \sigma^2 \mathrm{BCap}(K) \cdot t_d/2.$$

This completes the proof of (2.2.14).

(2.2.15) can be obtained in a very similar way and we leave the details to the reader. Note that one shall be a bit careful about whether to use the original walk and the reversed walk. For example, instead of (2.2.20), we have:

$$\mathbf{q}^-(x) = \sum_{y \in \mathbb{Z}^d} g(y,x)\mathbf{r}(y)\mathrm{Es}_K(y). \tag{2.2.22}$$

□

Remark 2.2.8. *The analogous result (Proposition 1.3.8) also holds for the incipient infinite snake and can be proved similarly:*

$$\lim_{x \to \infty} \|x\|^{d-4} \cdot P(\overline{\mathcal{S}}_x^\infty \textit{ visits } K) = t_d a_d^2 \sigma^2 BCap(K). \tag{2.2.23}$$

2.2.6 Convergence of the conditional entering measure

Theorem 1.3.1 implies that conditioned on visiting a finite set, the first visiting point and the last visiting point converge in distribution as the starting point tends to infinity. In fact, not only the first and last visiting points, but also the set of 'entering' points converge in distribution. Let us make this statement precise.

As before, we fix a $K \subset\subset \mathbb{Z}^d$. Let $\mathbb{M}_p(K)$ stand for the set of all finite point measures on K. The entering measure of a finite snake $\mathcal{S}_x = (T, \mathcal{S}_T)$

is defined by:

$$\Theta_x = \sum_{v \in T : \mathcal{S}_T(v) \in K, v \text{ has no ancestor lying in } K} \delta_{\mathcal{S}_T(v)}.$$

Note that Θ_x is a random element in $\mathbb{M}_p(K)$ and

$$P(\Theta_x \neq 0) = P(\mathcal{S}_x \text{ visits } K) \leq c_K |x|^{2-d},\ \mathbb{E}(\langle \Theta_x, 1 \rangle) \leq g(x, K) \leq c_K |x|^{2-d}.$$

We write $\overline{\Theta}_x$ for Θ_x conditioned on $\Theta_x \neq 0$. Now we can state our result:

Theorem 2.2.9.

$$\overline{\Theta}_x \xrightarrow{d} \mathbf{m}_K, \ \textit{as } |x| \to \infty, \tag{2.2.24}$$

where $\mathbf{m}_K$ *is defined later in* (2.2.30) *and* $\xrightarrow{d}$ *means convergence in distribution.*

Construction of the limiting measure.

There are two steps needed, to sample an element from $\mathbf{m}_K$. The first step is to sample the 'left-most' path ($\Gamma(\mathcal{S}_x)$) appeared in Section 2.2.2 and then run independent branching random walks from all vertices on that path.

We begin with the second step. We write $\widetilde{\Theta}_x$ for Θ_x conditioned on the initial particle having exactly one child. Inspired by (2.2.3), we introduce the position-dependent distribution μ_x on $\mathbb{N}$ and the random variable Λ_x on $\mathbb{M}_p(K)$:

$$\mu_x(m) = \sum_{l \geq 0} \mu(l + m + 1)(\widetilde{r}(x))^l / (1 - \mathbf{r}(x)), \quad \text{for } x \notin K, \tag{2.2.25}$$

$$\Lambda_x \overset{d}{=} \begin{cases} \Sigma_{i=1}^{Y} X_i, & \text{when } x \notin K; \\ \delta_x, & \text{when } x \in K; \end{cases} \tag{2.2.26}$$

where Y is an independent random variable with distribution μ_x and X_i are i.i.d. with distribution $\widetilde{\Theta}_x$. Write

$$N(x) = N_K(x) = \mathbb{E}\mu_x. \tag{2.2.27}$$

Note that

$$\lim_{x\to\infty} N(x) = \mathbb{E}\widetilde{\mu} = \frac{\sigma^2}{2}, \ \lim_{x\to\infty} \mu_x(m) = \widetilde{\mu}(m),$$
$$\mu_x(m) \le \widetilde{\mu}(m)/(1-\mathbf{r}(x)) \preceq \widetilde{\mu}(m),$$
$$\mathbb{E}\langle \Lambda_x, 1\rangle = (\mathbb{E}\mu_x)\mathbb{E}(\langle \Theta_x, 1\rangle) \le c_K|x|^{2-d}.$$

For any path γ, define $\mathcal{Z}(\gamma)$ and $\mathcal{Z}^-(\gamma)$ by:

$$\mathcal{Z}(\gamma) = \Sigma_{i=0}^{|\gamma|}\Lambda(i); \mathcal{Z}^-(\gamma) = \Sigma_{i=0}^{|\gamma|-1}\Lambda(i),$$

where $\Lambda(i) \stackrel{d}{=} \Lambda_{\gamma(i)}$ are independent random variables. Note that

$$\mathbb{E}\langle \mathcal{Z}(\gamma), 1\rangle \le c_K \Sigma_{i=0}^{|\gamma|}|\gamma(i)|^{2-d}.$$

Hence, for an infinite path $\gamma : \mathbb{N} \to \mathbb{Z}^d$, we can also define $\mathcal{Z}(\gamma)$:

$$\mathcal{Z}(\gamma) = \sum_{i=0}^{\infty} \Lambda(\gamma(i)) \in \mathbb{M}_p(K) \ a.s.,$$

as long as

$$\sum_{i=0}^{\infty} |\gamma(i)|^{2-d} < \infty. \tag{2.2.28}$$

Now we move to the first step and explain how to sample the left-most path. For any $x \in \mathbb{Z}^d$, let $h(x) = P(\mathcal{S}_x^-$ does not visit $K)$. Define P^∞ to be the transition probability of the Markov chain in $\{z \in \mathbb{Z}^d : \mathrm{Es}_K(z) > 0\}$ by:

$$P^\infty(x,y) = \frac{\theta(x-y)h(y)}{\sum_{z\in\mathbb{Z}^d}\theta(x-z)h(z)} = \frac{\theta(x-y)(1-\mathbf{k}(y))\mathrm{Es}_K(y)}{\mathrm{Es}_K(x)}. \tag{2.2.29}$$

For any x with $\mathrm{Es}_K(x) > 0$, define P_x^∞ to be the law of random walk starting from x with transition probability P^∞. Define P_K^∞ to be the law of random walk (with transition probability P^∞) starting at $a \in K$ with probability $\mathrm{Es}_K(a)/\mathrm{BCap}(K)$.

Now we can give the definition of $\mathbf{m}_K$:

$$\begin{aligned}\mathbf{m}_K = &\text{ the law of } Z: \\ &\quad \text{where first sample } \gamma \text{ by } P_K^\infty \text{ and then sample } Z \text{ by } \mathcal{Z}(\gamma). \quad (2.2.30)\end{aligned}$$

Note that under P_x^∞ (for those x with $\mathrm{Es}_K(x) > 0$),

$$\begin{aligned}\mathbb{E}_x^\infty \sum_{i=0}^{\infty} |\gamma(i)|^{2-d} &= \mathbb{E}_x^\infty \sum_{z\in\mathbb{Z}^d} \sum_{i\in\mathbb{N}} \mathbf{1}_{\gamma(i)=z} |z|^{2-d} \\ &= \sum_{z\in\mathbb{Z}^d} |z|^{2-d} \mathbb{E}_x^\infty \sum_{i\in\mathbb{N}} \mathbf{1}_{\gamma(i)=z} = \sum_{z\in\mathbb{Z}^d} |z|^{2-d} \frac{G_K(z,x)\mathrm{Es}_K(z)}{\mathrm{Es}_K(x)} \\ &\preceq \frac{1}{\mathrm{Es}_K(x)} \sum_{z\in\mathbb{Z}^d} |z|^{2-d}|z-x|^{2-d} \preceq \frac{|x|^{4-d}}{\mathrm{Es}_K(x)} < \infty.\end{aligned}$$

Therefore, under P_x^∞ (and hence P_K^∞), $\mathcal{Z}(\gamma)$ is well-defined a.s..

Convergence of the conditional entering measure

Since our sample space $\mathbb{M}_p(K)$ is discrete and countable, it is convenient to use the total variation distance. Recall that for two probability distributions ν_1, ν_2 on a discrete countable space Ω, the total variation distance is defined to be

$$d_{TV}(\nu_1, \nu_2) = \frac{1}{2} \sum_{\omega\in\Omega} |\nu_1(\omega) - \nu_2(\omega)| \in [0,1]$$

and $\nu_n \xrightarrow{d} \nu$ iff $d_{TV}(\nu_n, \nu) \to 0$.

Let us introduce some notations. Let Γ be a countable set of finite paths. For each $\gamma \in \Gamma$, assign to it, the weight $a(\gamma) \geq 0$ (assume that the total mass $\sum_{\gamma\in\Gamma} a(\gamma) \leq 1$) and a probability law $Z(\gamma)$ in $\mathbb{M}_p(K)$. We denote by $\bigsqcup_{\gamma\in\Gamma} a(\gamma) \cdot Z(\gamma)$ for the random element in $\mathbb{M}_p(K)$ as follows: pick a random path γ' among Γ with probability $P(\gamma' = \gamma) = a(\gamma)$ (with probability $1 - \sum_{\gamma\in\Gamma} a(\gamma)$ we do not get any path and in this case simply set $\bigsqcup_{\gamma\in\Gamma} a(\gamma) \cdot Z(\gamma) = 0$) and then use the law $Z(\gamma')$ to sample $\bigsqcup_{\gamma\in\Gamma} a(\gamma) \cdot Z(\gamma)$. One can easily verify the following proposition:

Proposition 2.2.10. *If* $\nu = \bigsqcup_{\gamma\in\Gamma} a(\gamma)\cdot Z(\gamma)$, $\nu_1 = \bigsqcup_{\gamma\in\Gamma} a_1(\gamma)\cdot Z(\gamma)$ *and* $\nu_2 = \bigsqcup_{\gamma\in\Gamma} a(\gamma)\cdot Z_1(\gamma)$, *then*

$$d_{TV}(\nu,\nu_1) \le \sum_{\gamma\in\Gamma} |a(\gamma)-a_1(\gamma)|, \quad d_{TV}(\nu,\nu_2) \le \sum_{\gamma\in\Gamma} a(\gamma) d_{TV}(Z(\gamma), Z_1(\gamma)). \tag{2.2.31}$$

For any $n > \mathrm{Rad}(K)$, write:

$$\mathbf{m}_K^n = \bigsqcup_{\gamma:(\mathcal{C}(n))^c\to K, \gamma\subseteq(\mathcal{C}(n)\setminus K)} \frac{\mathbf{b}(\gamma)\mathrm{Es}_K(\gamma(0))}{\mathrm{BCap}(K)}\cdot \mathcal{Z}(\gamma).$$

Note that $\mathbf{m}_K^n$ can be obtained equivalently as follows: first sample an infinite path γ' by P_K^∞ and cut γ' into two pieces at the hitting time of $(\mathcal{C}(n))^c$; let γ be the first part and then sample $\mathbf{m}_K^n$ by $\mathcal{Z}(\gamma)$. Hence, we have: $\mathbf{m}_K^n \xrightarrow{d} \mathbf{m}_K$ as $n\to\infty$.

Now we turn to $\overline{\Theta}_x$. Similar to the computations after (2.2.3), one can get, for $\gamma = (\gamma(0),\dots,\gamma(k)) \subseteq K^c$ with $\gamma(0)=x, \widehat{\gamma}=\gamma(k)\in K$ and $1\le j_1<j_2\le k$, (see the corresponding notations there)

$$P(\widetilde{b}_{j_1}=m|\Gamma(\mathcal{S}_x)=\gamma) = \frac{\sum_{l\in\mathbb{N}} \mu(l+m+1)(\widetilde{r}(\gamma(j_1-1)))^l}{1-\mathbf{r}(\gamma(j_1-1))};$$

$$P(\widetilde{b}_{j_i}=m_i, \text{for } i=1,2|\Gamma(\mathcal{S}_x)=\gamma) = \frac{\sum_{l\in\mathbb{N}} \mu(l+m_1+1)(\widetilde{r}(\gamma(j_1-1)))^l}{1-\mathbf{r}(\gamma(j_1-1))}\frac{\sum_{l\in\mathbb{N}} \mu(l+m_2+1)(\widetilde{r}(\gamma(j_2-1)))^l}{1-\mathbf{r}(\gamma(j_2-1))}.$$

From these (and the similar equations for more than two b_j's), one can see that conditioned on the event $\Gamma(\mathcal{S}_x)=\gamma$, $(\widetilde{b}_j)_{j=1,\dots,k}$ are independent and have the distribution of the form in (2.2.25). Hence, conditioned on $\Gamma(\mathcal{S}_x)=\gamma$, Θ_x has the law of $\mathcal{Z}(\gamma)$. Therefore, we have

Proposition 2.2.11.

$$\overline{\Theta}_x = \bigsqcup_{\gamma:x\to K} \frac{\mathbf{b}(\gamma)}{\mathbf{p}(x)}\cdot\mathcal{Z}(\gamma). \tag{2.2.32}$$

Remark 2.2.9. *Note that for this proposition, we do not need the assump-*

tion that $d \geq 5$.

Set $n = n(x) = \|x\|^{\frac{d-1}{d}}$. We need to show:

$$\lim_{x\to\infty} d_{TV}(\overline{\Theta}_x, \mathbf{m}_K^n) = 0. \tag{2.2.33}$$

Let $B = \mathcal{C}(n)$ and $B_1 = \mathcal{C}(2n)$. For any $\gamma : x \to K$, we decompose γ into two pieces $\gamma = \gamma_1 \circ \gamma_2$ according to the last visiting time of B^c. We can rewrite $\overline{\Theta}_x$ as follows:

$$\overline{\Theta}_x = \bigsqcup_{\gamma:x\to K} \frac{\mathbf{b}(\gamma)}{\mathbf{p}(x)} \cdot \mathcal{Z}(\gamma) = \bigsqcup_{\gamma:x\to K} \frac{\mathbf{b}(\gamma_1)\mathbf{b}(\gamma_2)}{\mathbf{p}(x)} \cdot (\mathcal{Z}^-(\gamma_1) + \mathcal{Z}(\gamma_2)) =$$
$$\bigsqcup_{\gamma_2:B^c\to K,\gamma_2\subseteq B} \frac{\mathbf{b}(\gamma_2)g_K(x,\gamma_2(0))}{\mathbf{p}(x)} \cdot (\mathcal{Z}(\gamma_2) + \bigsqcup_{\gamma_1:x\to\gamma_2(0)} \frac{\mathbf{b}(\gamma_1)}{g_K(x,\gamma_2(0))} \cdot \mathcal{Z}^-(\gamma_1))$$

We point out that

$$\sum_{\gamma_2:B_1^c\to K,\gamma_2\subseteq B} \frac{\mathbf{b}(\gamma_2)g_K(x,\gamma_2(0))}{\mathbf{p}(x)} \ll 1. \tag{2.2.34}$$

This can be seen from: (by the Overshoot Lemma and (2.1.2))

$$\sum_{\gamma_2:\mathcal{C}(\|x\|/2)^c\to K,\gamma_2\subseteq B} \mathbf{b}(\gamma_2)g_K(x,\gamma_2(0)) \preceq n^2/\|x\|^d \ll \mathbf{p}(x);$$
$$\sum_{\gamma_2:\mathcal{C}(\|x\|/2)\setminus B_1\to K,\gamma_2\subseteq B} \mathbf{b}(\gamma_2)g_K(x,\gamma_2(0)) \preceq n^2/n^d \cdot \|x\|^{2-d} \ll \mathbf{p}(x).$$

Furthermore, when $y \in B_1$, by (2.2.10), (2.1.2) and (1.3.1), we have:

$$\frac{g_K(x,y)}{\mathbf{p}(x)} \sim \frac{1}{\mathrm{BCap}(K)} \sim \frac{\mathrm{Es}_K(y)}{\mathrm{BCap}(K)}. \tag{2.2.35}$$

Hence (by Proposition 2.2.10, (2.2.34) and (2.2.35)), we have:

$$d_{TV}(\overline{\Theta}_x, \bigsqcup_{\gamma:B_1\setminus B\to K,\gamma\subseteq B} \frac{\mathbf{b}(\gamma)\mathrm{Es}_K(\gamma(0))}{\mathrm{BCap}(K)}\cdot \left(\mathcal{Z}(\gamma)+\bigsqcup_{\gamma_1:x\to\gamma(0)}\frac{\mathbf{b}(\gamma_1)}{g_K(x,\gamma(0))}\cdot\mathcal{Z}^-(\gamma_1)\right))\to 0.$$

Similarly, we have:

$$d_{TV}\left(\mathbf{m}_K^n, \bigsqcup_{\gamma:B_1\setminus B\to K,\gamma\subseteq B}\frac{\mathbf{b}(\gamma)\mathrm{Es}_K(\gamma(0))}{\mathrm{BCap}(K)}\cdot\mathcal{Z}(\gamma)\right)\to 0.$$

On the other hand, for any $\gamma : B_1 \setminus B \to K, \gamma \subseteq B$, we have (let $y = \gamma(0)$):

$$\begin{aligned}
&d_{TV}\left(\mathcal{Z}(\gamma)+\bigsqcup_{\gamma_1:x\to\gamma(0)}\frac{\mathbf{b}(\gamma_1)}{g_K(x,\gamma(0))}\cdot\mathcal{Z}^-(\gamma_1),\mathcal{Z}(\gamma)\right)\\
&\quad\le P(\bigsqcup_{\gamma_1:x\to y}\frac{\mathbf{b}(\gamma_1)}{g_K(x,y)}\cdot\mathcal{Z}^-(\gamma_1)\neq 0)\le E\langle\bigsqcup_{\gamma_1:x\to y}\frac{\mathbf{b}(\gamma_1)}{g_K(x,y)}\cdot\mathcal{Z}^-(\gamma_1),1\rangle\\
&\quad\preceq\sum_{\gamma_1:x\to y}\frac{\mathbf{b}(\gamma_1)}{g_K(x,y)}\sum_{i=0}^{|\gamma_1|}|\gamma_1(i)|^{2-d}=\sum_{\gamma_1:x\to y}\frac{\mathbf{b}(\gamma_1)}{g_K(x,y)}\sum_{z\in\mathbb{Z}^d}|z|^{2-d}\sum_{i=0}^{|\gamma_1|}\mathbf{1}_{\boldsymbol{\gamma_1(i)=z}}.
\end{aligned}$$

We need to show the above term tends to 0. Note that $g_K(x,y)\asymp|x|^{2-d}$ and $\mathbf{b}(\gamma)\le\mathbf{s}(\gamma)$, it suffices to show:(when $x\to\infty$, uniformly for any $y\in B_1\setminus B$)

$$|x|^{d-2}\sum_{z\in\mathbb{Z}^d}|z|^{2-d}\sum_{\gamma_1:x\to y}\mathbf{s}(\gamma_1)\sum_{i=0}^{|\gamma_1|}\mathbf{1}_{\boldsymbol{\gamma_1(i)=z}}\to 0. \tag{2.2.36}$$

Note that

$$\sum_{\gamma_1:x\to y}\mathbf{s}(\gamma_1)\sum_{i=0}^{|\gamma_1|}\mathbf{1}_{\boldsymbol{\gamma_1(i)=z}}=\sum_{\gamma_3:x\to z}\mathbf{s}(\gamma_3)\sum_{\gamma_4:z\to y}\mathbf{s}(\gamma_4)=g(x,z)g(z,y).$$

Hence, the left hand side of (2.2.36) can be bounded by:

$$
\begin{aligned}
&|x|^{d-2}\sum_{z\in\mathbb{Z}^d}|z|^{2-d}g(x,z)g(z,y) \preceq |x|^{2-d}\sum_{z\in\mathbb{Z}^d}|z|^{2-d}|x-z|^{2-d}|y-z|^{2-d}\\
&= |x|^{d-2}\Big(\sum_{z:|z-x|\le|x|/2} + \sum_{z:|z-x|>|x|/2}\Big)|z|^{2-d}|x-z|^{2-d}|y-z|^{2-d}\\
&\preceq |x|^{d-2}\Big(\sum_{z:|z-x|\le|x|/2}|x|^{2-d}|z-x|^{2-d}|x|^{2-d}+\\
&\qquad\qquad \sum_{z:|z-x|>|x|/2}|z|^{2-d}|x|^{2-d}|y-z|^{2-d}\Big)\\
&\preceq |x|^{d-2}(|x|^{6-2d}+|y|^{4-d}|x|^{2-d}) \asymp |y|^{4-d} \preceq n^{4-d} \to 0.
\end{aligned}
$$

Now the proof is complete.

2.2.7 Branching capacity of balls.

In this subsection, we compute the branching capacity of balls. As mentioned before, we carry out this by estimating the visiting probability of balls and then use (1.3.1) in reverse. Let us set up the notations. For $x \in \mathbb{Z}^d$ and $A \subset\subset \mathbb{Z}^d$, we write $\mathbf{p}_A(x)$, $\mathbf{r}_A(x)$, $\mathbf{q}_A(x)$ and $\mathbf{q}_A^-(x)$ respectively, for the probability that a snake, an adjoint snake, an infinite snake and a reversed snake respectively, starting from x visits A.

Theorem 2.2.12. *Let $A = \{z = (z_1, 0) \in \mathbb{Z}^m \times \mathbb{Z}^{d-m} = \mathbb{Z}^d : \|z\| \le r\}$ be the m-dimensional ball ($1 \le m \le d$) with radius $r \ge 1$ and $x \in \mathbb{Z}^d \setminus A$. When $s = \rho(x, A) \ge 2$, we have*

$$
\mathbf{p}_A(x) \asymp \begin{cases}
r^{d-4}/s^{d-2}, & \text{if } m \ge d-3 \text{ and } s \ge r;\\
1/s^2, & \text{if } m \ge d-3 \text{ and } s \le r;\\
r^{d-4}/(s^{d-2}\log r), & \text{if } m = d-4 \text{ and } s \ge r;\\
1/(s^2\log s), & \text{if } m = d-4 \text{ and } s \le r;\\
r^m/s^{d-2}, & \text{if } m \le d-5 \text{ and } s \ge r;\\
1/s^{d-m-2}, & \text{if } m \le d-5 \text{ and } s \le r.
\end{cases}
$$

Proof. Let us first mention the organization of the proof. All lower bounds will be proved by the second moment method. So we first estimate the first and the second moments. For upper bounds, due to Markov property (from Proposition 2.2.3), the case for 'big s' (i.e. $s \geq r$) can be reduced to the case for 'small s' (i.e. $s \leq r$). For small s, visiting a large m-dimensional ball in $\mathbb{Z}^d$ behaves like visiting a point in $\mathbb{Z}^{d-m}$. Hence we can use the results on the latter case.

Upper bounds for $\mathbf{m \leq d-5}$ and lower bounds for all cases. Let N be the number of times the branching random walk visits A. Then $\mathbb{E}N = \sum_{z \in A} g(x,z) = g(x,A)$. For the first moment, we point out:

$$g(x,A) \asymp \begin{cases} r^m/s^{d-2}, & \text{for } s \geq r; \\ 1/s^{d-m-2}, & \text{for } s \leq r, m \leq d-3. \end{cases} \tag{2.2.37}$$

The computations are straightforward. When $s \geq r$, for any $a \in A$, $\rho(x,a) \asymp s$. Hence $g(x,A) \asymp |A| \cdot 1/s^{d-2} \asymp r^m/s^{d-2}$. When $s \leq r, m \leq d-3$, the part of $\succeq$ is easy. Let $b \in A$ satisfying $\rho(x,b) = \rho(x,A)$ and let $B = b + \mathcal{C}(s)$. Then for any $a \in B \cap A$, $\rho(x,a) \asymp s$ and $|B \cap A| \asymp s^m$. Hence $g(x,A) \succeq s^{2-d} \cdot s^m = 1/s^{d-m-2}$. For the other part, it needs a bit more work. Assume $x = (\bar{x}_1, \bar{x}_2) \in \mathbb{Z}^m \times \mathbb{Z}^{d-m}$ and let $x_1 = (\bar{x}_1, 0), x_2 = (0, \bar{x}_2) \in \mathbb{Z}^d$. Since $\rho(x,A) = s$, either $\rho(x,x_1) \geq s/2$ or $\rho(x_1,A) \geq s/2$. When $s/2 \leq \rho(x,x_1) = \|x_2\|$, note that $|x_2| \succeq \|x_2\| \succeq s$. We have:

$$\begin{aligned} g(x,A) \leq & \sum_{z \in \mathbb{Z}^m \times 0 \subseteq \mathbb{Z}^d} g(x,z) \asymp \sum_{z_1 \in \mathbb{Z}^m} (\sqrt{|z_1|^2 + |x_2|^2})^{2-d} \\ = & \sum_{z_1 \in \mathbb{Z}^m, |z_1| \leq s} (\sqrt{|z_1|^2 + |x_2|^2})^{2-d} + \sum_{z_1 \in \mathbb{Z}^m, |z_1| \geq s} (\sqrt{|z_1|^2 + |x_2|^2})^{2-d} \\ \leq & \sum_{z_1 \in \mathbb{Z}^m, |z_1| \leq s} |x_2|^{2-d} + \sum_{z_1 \in \mathbb{Z}^m, |z_1| \geq s} (|z_1|)^{2-d} \\ \preceq & s^m \cdot s^{2-d} + \sum_{n \geq s} \frac{n^{m-1}}{n^{d-2}} = s^{m+2-d} + \sum_{n \geq s} \frac{1}{n^{d-m-1}} \\ \asymp & 1/s^{d-m-2}. \end{aligned}$$

When $\rho(x_1, A) \geq s/2$, note that $|x_1| \asymp \|x_1\| \succeq s$. We have:

$$\begin{aligned} g(x,A) &\asymp \sum_{z\in\mathbb{Z}^m\times 0\subseteq\mathbb{Z}^d, \|z\|\leq r} \rho(x,z)^{2-d} \leq \sum_{z\in\mathbb{Z}^m\times 0\subseteq\mathbb{Z}^d, \|z-x_1\|\geq s/2} \rho(x,z)^{2-d} \\ &= \sum_{z\in\mathbb{Z}^m\times 0\subseteq\mathbb{Z}^d, \|z\|\geq s/2} \|z\|^{2-d} \asymp \sum_{z\in\mathbb{Z}^m\times 0\subseteq\mathbb{Z}^d, \|z\|\geq s/2} |z|^{2-d} \\ &\leq \sum_{n\geq Cs} \frac{n^{m-1}}{n^{d-2}} = \sum_{n\geq Cs} \frac{1}{n^{d-m-1}} \asymp 1/s^{d-m-2}. \end{aligned}$$

Now we finish the proof of (2.2.37). Note that (2.2.37) is also true even for $x \in A$ i.e. $g(x,A) \asymp 1$(recall that since we set $\|0\| = 1/2$, when $x \in A$, $\rho(x,A) = 1/2$ by our convention).

Using $P(N > 0) \leq \mathbb{E}N$, one can get the desired upper bounds for $m \leq d-5$.

For the lower bounds, we need to estimate the second moment and the following is a standard result for branching random walk (for example, see Remark 2 in Page 13 of [14]).

Lemma 2.2.13. *There exists a constant C, such that:*

$$\mathbb{E}N^2 \leq C \sum_{z\in\mathbb{Z}^d} g(x,z) g^2(z,A).$$

We need to estimate the above sum. First consider the case when A is a m-dimensional ball and $m \leq d-3, s \geq r$. Let $B_0 = \{z \in \mathbb{Z}^d : \rho(z,A) \leq r/6\}$ and $B_n = \mathcal{C}(2^n s/3)$, for $n \in \mathbb{N}^+$. Note that there exists some $c > 0$, such that $\mathcal{B}(c^{-1}r) \subseteq B_0 \subseteq \mathcal{B}(cr)$ and $\mathcal{B}(c^{-1}2^{n-1}s) \subseteq B_n \subseteq \mathcal{B}(c2^{n-1}s)$, for any $n \geq 1$. We will divide the sum into three parts and estimate separately: $\sum_{z\in B_0}$, $\sum_{z\in B_1\setminus B_0}$, and $\sum_{n\geq 2}\sum_{z\in B_n\setminus B_{n-1}}$. When $z = (z_1, z_2) \in B_0$, where $z_1 \in \mathbb{Z}^m$

and $z_2 \in \mathbb{Z}^{d-m}$, $\|x-z\| \asymp s$ and $\rho(z,A) \succeq |z_2|$. Hence

$$\begin{aligned}
\sum_{z\in B_0} g(x,z)g^2(z,A) &\asymp \sum_{z\in B_0} \frac{1}{\|x-z\|^{d-2}} \frac{1}{\rho(z,A)^{2d-2m-4}} \\
&\preceq \frac{1}{s^{d-2}} \sum_{z\in\mathcal{B}(cr)} \frac{1}{|z_2|^{2d-2m-4}} \preceq \frac{1}{s^{d-2}} \sum_{|z_1|\le cr, |z_2|\le cr} \frac{1}{|z_2|^{2d-2m-4}} \\
&\asymp \frac{r^m}{s^{d-2}} \sum_{|z_2|\le cr} \frac{1}{|z_2|^{2d-2m-4}} \asymp \frac{r^m}{s^{d-2}} \sum_{n\le cr} \frac{n^{d-m-1}}{n^{2d-2m-4}} \\
&= \frac{r^m}{s^{d-2}} \sum_{n\le cr} \frac{1}{n^{d-m-3}} \\
&\asymp \begin{cases} r^{m+1}/s^{d-2}, & \text{if } m=d-3; \\ r^m \log r/s^{d-2}, & \text{if } m=d-4; \\ r^m/s^{d-2}, & \text{if } m\le d-5. \end{cases}
\end{aligned}$$

When $z \in B_1 \setminus B_0$, $\|x-z\| \asymp s$, $\rho(z,A) \asymp |z|$ and $g(z,A) \asymp r^m/|z|^{d-2}$. Hence:

$$\begin{aligned}
\sum_{z\in B_1\setminus B_0} g(x,z)g^2(z,A) &\asymp \sum_{z\in B_1\setminus B_0} \frac{1}{\|x-z\|^{d-2}} \left(\frac{r^m}{|z|^{d-2}}\right)^2 \\
&\preceq \sum_{z\in\mathcal{B}(cs)\setminus\mathcal{B}(c^{-1}r)} \frac{r^{2m}}{s^{d-2}|z|^{2d-4}} \preceq \frac{r^{2m}}{s^{d-2}} \sum_{c^{-1}r\le n\le cs} \frac{n^{d-1}}{n^{2d-4}} \asymp \frac{r^{2m}}{s^{d-2}} \frac{1}{r^{d-4}}.
\end{aligned}$$

Note that this term is not bigger than the first term and hence negligible. The remaining part can be estimated similarly and is also negligible:

$$\begin{aligned}
\sum_{n\ge2} \sum_{z\in B_n\setminus B_{n-1}} g(x,z)g^2(z,A) &\asymp \sum_{n\ge2} \sum_{z\in B_n\setminus B_{n-1}} \frac{1}{|x-z|^{d-2}} \left(\frac{r^m}{|z|^{d-2}}\right)^2 \\
&\preceq \sum_{n\ge2} \sum_{z\in\mathcal{B}(c2^{n-1}s)\setminus\mathcal{B}(c^{-1}2^{n-1}s)} \frac{1}{|x-z|^{d-2}} \frac{r^{2m}}{(2^n s)^{2d-4}} \overset{(*)}{\preceq} \sum_{n\ge2} \frac{(2^n s)^2 r^{2m}}{(2^n s)^{2d-4}} \\
&= \sum_{n\ge2} \frac{r^{2m}}{s^{2d-6}} \frac{1}{(2^n)^{2d-6}} \asymp \frac{r^{2m}}{s^{2d-6}} \le \frac{r^{2m}}{s^{d-2}} \frac{1}{r^{d-4}}.
\end{aligned}$$

$(*)$ is due to the fact that $\sum_{z\in\mathcal{B}(n)} |x-z|^{2-d} \le \sum_{z\in\mathcal{B}(n)} |z|^{2-d} \asymp n^2$.

To summarize, we get:

$$\sum_{z\in\mathbb{Z}^d} g(x,z)g^2(z,A) \preceq \begin{cases} r^{m+1}/s^{d-2,} & \text{if } m = d-3; \\ r^m \log r/s^{d-2}, & \text{if } m = d-4; \\ r^m/s^{d-2}, & \text{if } m \le d-5. \end{cases}$$

For $r \ge s$, since we are considering lower bound on $\mathbf{p}_A(x)$, by monotonicity, we can assume $m \le d-3, r \in [s/2, s]$. Then, we can just let $r \asymp s$ in the last formula and get:

$$\sum_{z\in\mathbb{Z}^d} g(x,z)g^2(z,A) \preceq \begin{cases} s^{m+1}/s^{d-2} = 1, & \text{if } m = d-3; \\ s^m \log s/s^{d-2} = \log s/s^2, & \text{if } m = d-4; \\ s^m/s^{d-2} = 1/s^{d-m-2}, & \text{if } m \le d-5. \end{cases}$$

Using $\mathbf{p}_A(x) = P(N>0) \ge (\mathbb{E}N)^2/\mathbb{E}N^2$, one can get the required lower bounds for all cases.

***From small* s *to big* s.**We have proved the upper bound for $m \le d-5$ and now consider the case $m \ge d-4$. Assume that we have the desired upper bounds for small s. We want the upper bound for big s. Let $B = \{z \in \mathbb{Z}^d : \rho(z,A) \le r/2\}$ and $C = \{z \in \mathbb{Z}^d : \rho(z,A) \le r/4\}$. Then by the assumption, we know that for any $z \in B\backslash C$, $\mathbf{p}(z) \preceq \alpha(r)$, where $\alpha(r) = 1/r^2$ or $1/(r^2 \log r)$ depending on m. Let

$$\Gamma_1 = \{\gamma : x \to A | \gamma \subseteq A^c, \gamma \text{ visits } B \setminus C\},$$
$$\Gamma_2 = \{\gamma : x \to A | \gamma \subseteq A^c, \gamma \text{ avoids } B \setminus C\}.$$

We decompose $\mathbf{p}_A(x)$ into two pieces:

$$\mathbf{p}_A(x) = \sum_{\gamma:x\to A} \mathbf{b}(\gamma) = \sum_{\gamma\in\Gamma_1} \mathbf{b}(\gamma) + \sum_{\gamma\in\Gamma_2} \mathbf{b}(\gamma).$$

For the first term, by considering the first visiting point of $B \setminus C$, one can see:

$$\begin{aligned}\sum_{\gamma\in\Gamma_1}\mathbf{b}(\gamma) &\le \sum_{z\in B\setminus C}\ \sum_{\gamma:x\to z,\gamma\subseteq(B\setminus C)^c}\mathbf{b}(\gamma)\mathbf{p}_A(z) \le \sum_{z\in B\setminus C}\ \sum_{\gamma:x\to z,\gamma\subseteq(B\setminus C)^c}\mathbf{s}(\gamma)\alpha(r)\\ &\le \alpha(r)P(S_x \text{ visits } (B\setminus C)) \le \alpha(r)P(S_x \text{ visits } B)\\ &\asymp \alpha(r)(r/s)^{d-2} = \begin{cases} r^{d-4}/s^{d-2} & \text{if } m\ge d-3;\\ r^{d-4}/(s^{d-2}\log r) & \text{if } m=d-4.\end{cases}\end{aligned}$$

Recall that S_x is the random walk starting from x and we use the standard estimate of $P(S_x \text{ visits } B)\asymp(r/s)^{d-2}$. For the other term, by considering the first jump from B^c to C, one can see:

$$\begin{aligned}\sum_{\gamma\in\Gamma_2}\mathbf{b}(\gamma) &\le \sum_{z\in B^c}G_A(x,z)\sum_{y\in C}\theta(y-z)\mathbf{p}_A(y)\\ &= \sum_{z\in B_1^c}G_A(x,z)\sum_{y\in C}\theta(y-z)\mathbf{p}_A(y) + \sum_{z\in B_1\setminus B}G_A(x,z)\sum_{y\in C}\theta(y-z)\mathbf{p}_A(y),\end{aligned}$$

where $B_1=\{z:\rho(z,A)\le r/2+s/4\}$. Both terms are not more than the desired order:

$$\begin{aligned}&\sum_{z\in B_1^c}G_A(x,z)\sum_{y\in C}\theta(y-z)\mathbf{p}_A(y) \preceq \sum_{z\in B_1^c}1\cdot\sum_{y\in C}\theta(y-z)\alpha(r)\\ &\qquad\le \sum_{y\in C}\alpha(r)\sum_{z\in B_1^c}\theta(y-z) \preceq \sum_{y\in C}\alpha(r)s^{-d} \preceq \alpha(r)r^d/s^d \le \alpha(r)(r/s)^{d-2};\\ &\sum_{z\in B_1\setminus B}G_A(x,z)\sum_{y\in C}\theta(y-z)\mathbf{p}_A(y) \preceq \sum_{z\in B_1\setminus B}s^{2-d}\sum_{y\in C}\theta(y-z)\alpha(r)\\ &\le s^{2-d}\sum_{y\in C}\alpha(r)\sum_{z\in B^c}\theta(y-z) \preceq s^{2-d}\sum_{y\in C}\alpha(r)r^{-d} \preceq s^{2-d}\alpha(r) \le \alpha(r)(r/s)^{d-2}.\end{aligned}$$

***For small* $\mathbf{s}$ *and* $\mathbf{m\ge d-3}$.** The upper bound in this case relies on the corresponding bound for one dimensional branching random walk. Let H be a half space, say $H=\{z=(z_1,\dots,z_d)\in\mathbb{Z}^d : z_1\ge n\}$. The probability of visiting H is equivalent to the probability of 1-dimensional branching random walk visiting a half line. The asymptotic behavior of visiting a single point in 1-d is known. Recall (1.2.2),

$$\lim_{a\to\infty} \|x\|^2 P(\text{ Branching random walk from 0 visits } x) = 2(4-d)/d\sigma^2.$$

However, our situation is a bit different. For our purpose, we give a weaker result here under weaker assumptions:

Proposition 2.2.14. *Let $\mathcal{S}_x$ be 1-dimensional branching random walk starting from $x \in \mathbb{Z}$, given that the offspring distribution μ is critical and nondegenerate, and the jump distribution θ has zero mean and finite second moment, and satisfies $\sum_{i:i\le -k}\theta(i) \le Ck^{-4}$ for any $k\in\mathbb{N}^+$ and some C (independent of k). Then for some large constant $c = c(\theta,\mu) > 0$ (independent of x), we have: for any $x\in\mathbb{N}^+$,*

$$P(\mathcal{S}_x \text{ visits } \mathbb{Z}^-) \le c/|x|^2, \tag{2.2.38}$$

where $\mathbb{Z}^- = \{0,-1,-2,\dots\}$.

We postpone the proof of this proposition. Return to d dimension. Since we can find at most d half spaces $H_1, H_2, \dots, H_d$ satisfying: $\rho(x, H_i) \asymp s$ for any $i = 1,\dots,d$; and that any path from x to A must hit at least one of H_i. Then we have:

$$\mathbf{p}_A(x) \le \sum_{i=1}^{d} P(\mathcal{S}_x \text{ visits } H_i) \preceq d\cdot |s|^{-2} \asymp |s|^{-2}.$$

For small* $\mathbf{s}$ *and* $\mathbf{m} = \mathbf{d}-\mathbf{4}$*. Intuitively when the radius r is large, visiting a $m = d-4$ dimensional ball in $\mathbb{Z}^d$, is similar to visiting a point in $\mathbb{Z}^4$. This is indeed the case. In Section 2.4.1 we give the desired upper bound for the latter case and the method there also works here with slight modifications. We point out the major differences and leave the details to the reader. On the one hand, one should use $\tilde{g}(\gamma) := \sum_{i=0}^{|\gamma|-1} g(\gamma(i), A)$ instead of $g(\gamma)$ there. On the other hand, in proving an analogy of Lemma 10.1.2(a) in [11], one might use the stopping times:

$$\tilde{\xi}^i = \min\{k : \rho(S_k, A) \ge 2^i\} \wedge (\tilde{\xi}^{i-1} + (2^i)^2).$$

instead of ξ^i there. $\square$

Proof of Proposition 2.2.14. Write $p(x)$ for the left hand side in (2.2.38). In order to obtain upper bounds of $p(x)$, we use some of the ideas of [9] (Section 7.1), the techniques from nonlinear difference equations. We will exploit the fact that $p(x)$ satisfies a parabolic nonlinear difference equation and use the comparison principle.

Let $p_n(x) = P(\mathcal{S}_x \text{ visits } \mathbb{Z}^- \text{ within the first } n \text{ generations})$. Then $p_n(x)$ is increasing for n and converges to $p(x)$ when $n \to \infty$. On the other hand, one can easily verify that $p_n(x)$ satisfies the recursive equations:

$$p_0(x) = \mathbf{1}_{\mathbb{Z}^-}(x); \quad p_n(x) = 1 \text{ for } x \in \mathbb{Z}^-; \tag{2.2.39}$$

$$p_{n+1}(x) = f(\mathbb{A}p_n(x)), \text{ for } x \in \mathbb{N}^+; \tag{2.2.40}$$

where $f(t) = 1 - \sum_{k\geq 0} \mu(k)(1-t)^k$ and $\mathbb{A}$ is the Markov operator for the random walk, that is, for any bounded function $w : \mathbb{Z} \to \mathbb{R}$, $\mathbb{A}w(x) = \sum_{y\in\mathbb{Z}} \theta(y)w(x+y)$. One can see that $f : [0,1] \to [0, 1-\mu(0)]$ is in $C^1[0,1] \cap C^\infty(0,1]$ with the first 2 derivatives as follows:

$$f'(t) = \sum_{k\geq 1} k\mu(k)(1-t)^{k-1} > 0 \text{ for } t \in [0,1); \quad f'(0) = 1, f'(1) = \mu(1) \geq 0;$$
$$f''(t) = -\sum_{k\geq 2} k(k-1)\mu(k)(1-t)^{k-2} < 0 \text{ for } t \in (0,1).$$

From these, it is easy to obtain:

$$\inf_{t\in(0,1]} \frac{t - f(t)}{t^2} > 0.$$

Hence we can find some $a \in (0, 1/2)$, such that

$$f(t) \leq t - at^2, \text{ for any } t \in [0,1] \text{ and } t(1+at) \leq 1 \text{ for any } t \in [0, 1-\mu(0)]. \tag{2.2.41}$$

To extract information from (2.2.40), we will use the following standard comparison principle.

Lemma 2.2.15. *Let $u_n(x)$ and $v_n(x)$ be $\mathbb{Z} \to [0,1]$, satisfying*

$$u_n(x) = v_n(x) = 1, \quad \text{for any } x \in \mathbb{Z}^- \text{ and } n \in \mathbb{N};$$
$$u_{n+1}(x) = f(\mathbb{A}u_n(x)), \quad v_{n+1}(x) \geq f(\mathbb{A}v_n(x)) \quad \text{for any } x \in \mathbb{N}^+.$$

If $v_0(x) \geq u_0(x)$ for all x, then

$$v_n(x) \geq u_n(x) \quad \text{for all } n \in \mathbb{N}^+ \text{ and } x \in \mathbb{Z}.$$

Proof. Note that for $n > 0$ and $x \in \mathbb{N}^+$:

$$\begin{aligned} v_n(x) - u_n(x) &\geq f(\mathbb{A}v_{n-1}(x)) - f(\mathbb{A}u_{n-1}(x)) \\ &\geq \min_{t\in[0,1]}\{f'(t)\}(\mathbb{A}v_{n-1}(x) - \mathbb{A}u_{n-1}(x)) \\ &= \min_{t\in[0,1]}\{f'(t)\}\mathbb{A}(v_{n-1} - u_{n-1})(x). \end{aligned}$$

Since $f'(t) \geq 0$, one can use induction to finish the proof. □

Now let $u_n(x) = p_n(x)$ and $v_n(x) = v(x) = 1 \wedge (c/x^2)$ when $x \in \mathbb{N}^+$ for some large c (to be determined later). If we can show

$$v(x) \geq f(\mathbb{A}v(x)) \quad \text{for any } x \in \mathbb{N}^+, \tag{2.2.42}$$

then by the lemma above we conclude the proof of Proposition 2.2.14.

Let us write down our strategy for choosing c. First we fix some $\epsilon \in (0, 1/2)$, such that $(1-\mu(0))/(1-\epsilon)^2 < 1$. Choose c satisfying:

$$ac^2 \geq C/\epsilon^4 + 3(E|\theta|^2)c/(1-\epsilon)^4. \tag{2.2.43}$$

We argue that (2.2.42) holds for our choice of c. When $c/x^2 \geq 1 - \mu(0)$, (2.2.42) is obvious since $f(t) \leq 1 - \mu(0)$.

Now assume $c/x^2 < 1 - \mu(0)$. Since $f(t)$ is increasing, we need to find an upper bound of $\mathbb{A}v(x)$. We achieve this by decomposing $\mathbb{A}v(x)$ into two pieces and estimating each one separately:

$$\mathbb{A}v(x) = \sum_{y\in\mathbb{Z}} \theta(y)v(x+y) = \sum_{y\le -\epsilon x} \theta(y)v(x+y) + \sum_{y>-\epsilon x} \theta(y)v(x+y).$$

We can use our assumption of θ to bound the first term:

$$\sum_{y\le -\epsilon x} \theta(y)v(x+y) \le \sum_{y\le -\epsilon x} \theta(y) \le C/(\epsilon x)^4;$$

Using Taylor expansion, the second term can be bounded by:

$$\begin{aligned}
\sum_{y>-\epsilon x} \theta(y)v(x+y) &\le \sum_{y>-\epsilon x} \theta(y)(v(x) + yv'(x) + \frac{y^2}{2}v''((1-\epsilon)x)) \\
&= v(x)\sum_{y>-\epsilon x} \theta(y) + v'(x)\sum_{y>-\epsilon x} \theta(y)y + v''((1-\epsilon)x)\sum_{y>-\epsilon x} \theta(y)\frac{y^2}{2} \\
&\le v(x)\cdot 1 + v'(x)(-\sum_{y\le -\epsilon x} \theta(y)y) + v''((1-\epsilon)x)E|\theta|^2/2 \\
&\le v(x) + 0 + \frac{E|\theta|^2}{2}\cdot\frac{6c}{(1-\epsilon)^4x^4}.
\end{aligned}$$

To summarize, we get (let $K = (C/\epsilon^4 + 3E|\theta|^2c/(1-\epsilon)^4)$):

$$\mathbb{A}v(x) \le v(x) + (C/\epsilon^4 + 3E|\theta|^2c/(1-\epsilon)^4)x^{-4} = v(x) + Kx^{-4}.$$

Note that by (2.2.41) and (2.2.43), we have $v(x)+Kx^{-4} \le v(x)+a(v(x))^2 \le 1$. Hence:

$$\begin{aligned}
f(\mathbb{A}v(x)) &\le \mathbb{A}v(x)(1 - a\mathbb{A}v(x)) \le (v(x) + Kx^{-4})(1 - av(x)) \\
&\le v(x) + Kx^{-4} - a(v(x))^2 = v(x) + Kx^{-4} - ac^2x^{-4} \le v(x).
\end{aligned}$$

This completes the proof of (2.2.42) and hence the proof of Proposition 2.2.14. $\square$

For the future use, we give the following upper bound for the visiting probability of a ball by an infinite snake.

Lemma 2.2.16. *Let $A = \mathcal{C}(r)$ ($r \geq 1$) and $x \in \mathbb{Z}^d$ such that $s = \rho(x, A) \geq r$. Then we have:*

$$\mathbf{q}_A(x) \vee \mathbf{q}_A^-(x) \preceq (r/s)^{d-4}. \tag{2.2.44}$$

Proof. Consider a bigger ball $B = \mathcal{C}(1.5r)$. Then

$$\mathbf{q}_A(x) \leq P(\text{ backbone visits } B) + P(\text{ backbone avoids } B,\ \mathcal{S}_x^\infty \text{ visits } A).$$

Since the backbone is just a random walk, the first term is comparable to $(r/s)^{d-2}$, which is less that $(r/s)^{d-4}$. On the other hand, when the backbone does not visit B, by considering where the particle is killed, we have:

$$P(\text{backbone avoids } B, \mathcal{S}_x^\infty \text{ visits } A) \leq \sum_{z \in B^c} G_A(x,z)\mathbf{r}_A(z) \preceq \sum_{z \in B^c} g(x,z)\mathbf{p}_A(z)$$
$$\asymp \sum_{z \in B^c} \frac{1}{|x-z|^{d-2}} \frac{r^{d-4}}{|\rho(z,A)|^{d-2}} \asymp \sum_{z \in B^c} \frac{1}{|x-z|^{d-2}} \frac{r^{d-4}}{|z|^{d-2}} \asymp \frac{r^{d-4}}{|x|^{d-4}} \asymp (\frac{r}{s})^{d-4}.$$

This completes the proof of $\mathbf{q}_A(x) \preceq (r/s)^{d-4}$ and similarly one can show $\mathbf{q}_A^-(x) \preceq (r/s)^{d-4}$. □

2.2.8 Proof of Theorem 1.3.5

We use an equation approach similar to the proof of Proposition 2.2.14. Write $f_i(t) = 1 - \sum_{k \geq 0} \mu_i(k)(1-t)^k$, $i = 1, 2$. We need the following little lemma and postpone its proof.

Lemma 2.2.17. *There is a $C = C(\mu_1, \mu_2) > 1$ such that, for all $t \in [0, 1]$,*

$$f_1((Ct) \wedge 1) \leq (Cf_2(t)) \wedge 1. \tag{2.2.45}$$

For any $A \subset\subset \mathbb{Z}^d$ fixed, as in the proof of Proposition 2.2.14, denote $u_{i,n}(x)$ ($i = 1, 2$) recursively by:

$$u_{i,0}(x) = \mathbf{1}_A(x),\ u_{0,n}(a) = 1\ \forall a \in A; \quad u_{i,n+1}(x) = f_i(\mathbb{A}u_{i,n}(x))\ \forall a \notin A.$$

With the help of last lemma, one can see that $u_{1,n}(x) \leq Cu_{2,n}(x)$, for any

n, x. On the other hand, we know that $u_{i,n}(x) \to \mathbf{p}_{i,A}(x)$. Hence we have $\mathbf{p}_{1,A}(x) \le C\mathbf{p}_{2,A}(x)$. Then by Theorem 1.3.1, one can get Theorem 1.3.5.

Proof of Lemma 2.2.17. Since $\lim_{t\to 0} f_2(t)/t = 1$, when C is large enough, we have $Cf_2(C^{-1}) \ge 1 - \mu_1(0) = f_1(1)$. It suffices to show for $t \in [0, C^{-1}]$,

$$g(t) \doteq Cf_2(t) - f_1(Ct) \ge 0. \tag{2.2.46}$$

Note that $f_i(0) = 0, f_i'(0) = 1, f_i''(0) = -\mathrm{Var}(\mu_i), f_i''(t) \le 0$ and $|f_i''(t)|$ is non-increasing. Hence we can find some $C = C(\mu_1, \mu_2) > 1$ such that, $C|f_1''(1/2)| \ge 2|f_2''(0)|$ (and $Cf_2(C^{-1}) \ge 1 - \mu_1(0)$). Then we have

$$g''(t) = C(f_2''(t) - Cf_1''(Ct)) \ge \begin{cases} C|f_2''(0)|, & t \in [0, 1/(2C)]; \\ -C|f_2''(0)|, & t \in [1/(2C), 1/C]. \end{cases}$$

Together with $g(0) = g'(0) = 0$, one can get (2.2.46). $\square$

2.2.9 Bounds for the Green function

The speed of convergence in (2.2.10) depends on K, which maybe not convenient in some cases. For example, by that lemma, we know $G_K(x,y) \ge C_K g(x,y)$ (when $|x|, |y|$ are large), but the constant depends on K. The purpose of this section is to build up this type of bounds with constants independent of K.

Thanks to lemma 2.1.3, we have:

Lemma 2.2.18. *Let U, V be two connected bounded open subset of $\mathbb{R}^d$ such that $\overline{U} \subseteq V$. Then there exists a $C = C(U,V)$ such that if $A_n = nU \cap \mathbb{Z}^d, B_n = nV \cap \mathbb{Z}^d$ then when n is sufficiently large and $K \subseteq B_n^c$, we have*

$$G_K(x,y) \ge Cg(x,y) \text{ for any } x, y \in A_n. \tag{2.2.47}$$

Proof. Without loss of generality, we can assume $\rho(K, B_n) \succeq n$ (by shrinking V a bit). Hence for any $z \in B_n$, $\rho(z, K) \succeq n$. By Proposition 2.2.14, one can see that $\mathbf{p}_K(x) \preceq \rho(x,K)^{-2}$. Hence $\mathbf{k}(z) = \mathbf{r}_K(z) \asymp \mathbf{p}_K(z) \preceq n^{-2}$. Then we have, for any $\gamma : x \to y, \gamma \subseteq B_n, |\gamma| \le 2n^2$, $\mathbf{b}(\gamma)/\mathbf{s}(\gamma) \ge (1-c/n^2)^{2n^2} \succeq 1$ (provided that n is sufficiently large). Then we have:

$$G_K(x,y) \ge \sum_{\gamma:x\to y,\gamma\subseteq B_n,|\gamma|\le 2n^2} \mathbf{b}(\gamma) \succeq \sum_{\gamma:x\to y,\gamma\subseteq B_n,|\gamma|\le 2n^2} \mathbf{s}(\gamma) \overset{(2.1.5)}{\succeq} g(x,y).$$

□

Before giving a better form, we turn to the escape probability and prove:

Lemma 2.2.19. *For any $\lambda > 0$, there exists a positive $C = C(\lambda)$, such that, for any $A \subset\subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ satisfying $\|x\| \ge (1+\lambda)Rad(A)$, we have:*

$$Es_A(x) > C. \tag{2.2.48}$$

Proof. By lemma 2.2.16, we can find a positive constant $c_1 > 1$, such that, for any $z \in \mathbb{Z}^d$ with $\|z\| \ge c_1\mathrm{Rad}(A)$, we have $\mathrm{Es}_A(z) \ge 1/2$. Write $r = \mathrm{Rad}(A)$, $B = \mathcal{C}(2c_1 r)$ and $D = \mathcal{C}(4c_1 r)\setminus\mathcal{C}(3c_1 r)$. Without loss of generality, assume $1+\lambda < c_1/2$ and $\|x\| < c_1 r$. For any $y \in D$, by Lemma 2.2.18 (let $U = \{x \in \mathbb{R}^d : \|x\| \in (1+\lambda, 4c_1)\}$), we have (when r is large): $G_A(y,x) \asymp g(y,x) \asymp r^{2-d}$. Applying the First-Visit Lemma, we get:

$$G_A(y,x) = \sum_{z\in B^c} G_A(y,z)\mathcal{H}^B_{\mathbf{k}}(z,x).$$

Hence,

$$\sum_{y\in D} G_A(y,x) = \sum_{y\in D}\sum_{z\in B^c} G_A(y,z)\mathcal{H}^B_{\mathbf{k}}(z,x).$$

Note that the left hand side is $\asymp r^d \cdot r^{2-d} = r^2$ and the right hand side is not larger than:

$$\sum_{y\in D}\sum_{z\in B^c} g(y,z)\mathcal{H}^B_{\mathbf{k}}(z,x) = \sum_{z\in B^c} \mathcal{H}^B_{\mathbf{k}}(z,x) \sum_{y\in D} g(y,z) \preceq \sum_{z\in B^c} \mathcal{H}^B_{\mathbf{k}}(z,x)\cdot r^2.$$

This implies $\sum_{z\in B^c}\mathcal{H}^B_{\mathbf{k}}(z,x)\succeq 1$. Therefore we have:

$$\mathrm{Es}_A(x)=\sum_{z\in B^c}\mathcal{H}^B_{\mathbf{k}}(z,x)\mathrm{Es}_A(z)\geq 1/2\cdot\sum_{z\in B^c}\mathcal{H}^B_A(x,z)\succeq 1,$$

which completes the proof. ☐

Remark 2.2.10. *In fact we prove* (2.2.48) *only when Rad*(A) *is large. We ignore the case when Rad*(A) *is not large since this can be done by a s-tandard argument as follows. If Rad*(A) *is not sufficiently large, there are only finite possibilities of* A*. For each of those* A*, we have already known the asymptotics of* $Es_A(x)$ *(*$\lim_{x\to\infty} Es_A(x)=1$*). On the other hand, it is obvious that for any* $\|x\|>$ *Rad*(A)*,* $Es_A(x)>0$*. Hence we can find some* $C(A)>0$ *satisfying* (2.2.48)*. Since there are finite many* $C(A)$*'s, we can simply choose* C *to be the smallest one of those* $C(A)$ *(together with the one for sufficiently large* A*). We will also omit this type of standard arguments later. In fact, we have done this in the proof of Theorem 2.2.12.*

Now we are ready to prove the following bound of Green function:

Lemma 2.2.20. *For any* $\lambda>0$*, there exists* $C=C(\lambda)>0$*, such that: for any* $A\subset\subset\mathbb{Z}^d$ *and* $x,y\in\mathbb{Z}^d$ *with* $\|x\|,\|y\|>(1+\lambda)$*Rad*(A)*, we have:*

$$G_A(x,y)\geq Cg(x,y). \tag{2.2.49}$$

Proof. Without loss of generality, assume $\|x\|\leq\|y\|$. By Lemma 2.2.18 one can assume $\|y\|>10\|x\|$ and note that under this assumption $g(x,y)\asymp\|y\|^{2-d}$. Let $B=\mathcal{C}(\|y\|/2)$ and $C=\mathcal{C}(3\|y\|/4)$. For any $z\in C\setminus B$, also by Lemma 2.2.18, we have $G_A(y,z)\asymp\|y\|^{2-d}$. Applying the First-Visit Lemma, we have:

$$\begin{aligned}G_A(x,y)&=\sum_{z\in B^c}\mathcal{H}^B_{\mathbf{k}}(x,z)G_A(z,y)\succeq\sum_{z\in C\setminus B}\mathcal{H}^B_{\mathbf{k}}(x,z)\|y\|^{2-d}\\&=\|y\|^{2-d}(\sum_{z\in B^c}\mathcal{H}^B_{\mathbf{k}}(x,z)-\sum_{z\in C^c}\mathcal{H}^B_A(x,z))\\&\geq\|y\|^{2-d}(\mathrm{Es}_A(x)-c\|y\|^2/\|y\|^d),\end{aligned}$$

where for the last step we use the Overshoot Lemma. Therefore, when $\mathrm{Rad}(A)$ is large enough, by Lemma 2.2.19, $G_A(x,y) \succeq \|y\|^{2-d} \asymp g(x,y)$. $\square$

2.2.10 Proof of Theorem 1.3.3.

Proof. By cutting A into small pieces, it is enough to show (1.3.2) under the assumption of $\|x\| \geq 3\mathrm{Rad}(A)$. Also, as before, we can assume $r = \mathrm{Rad}(A)$ is sufficiently large. Let $B = \mathcal{C}(2r)$.

Upper bound. By (2.2.7) and the First-Visit Lemma, we have

$$\mathbf{p}_A(x) = \sum_{y \in A} G_A(x,y) = \sum_{y \in A} \sum_{z \in B^c} G_A(x,z) \mathcal{H}_{\mathbf{k}}^B(z,y).$$

We will decompose it into two parts and estimate them separately.

Let $D = \{z \in \mathbb{Z}^d : \rho(z,x) \leq 0.1\rho(x,A)\}$. Note that when $z \in B^c \setminus D$, $\rho(x,z) \succeq \rho(x,A)$ and $\mathrm{Es}_A(z) \asymp 1$ by Lemma 2.2.19. Hence,

$$\begin{aligned}
\sum_{y \in A} \sum_{z \in B^c \setminus D} G_A(x,z) \mathcal{H}_{\mathbf{k}}^B(z,y) &\preceq \sum_{y \in A} \sum_{z \in B^c \setminus D} \rho(x,A)^{2-d} \mathcal{H}_{\mathbf{k}}^B(z,y) \\
&\asymp \rho(x,A)^{2-d} \sum_{y \in A} \sum_{z \in B^c \setminus D} \mathcal{H}_{\mathbf{k}}^B(z,y) \mathrm{Es}_A(z) \\
&\leq \rho(x,A)^{2-d} \sum_{y \in A} \mathrm{Es}_A(y) = \rho(x,A)^{2-d} \mathrm{BCap}(A).
\end{aligned}$$

When $z \in D$, $\rho(z,B) \asymp \rho(x,A)$. By considering the position where the first jump falls into, we have:

$$\mathcal{H}_{\mathbf{k}}^B(z,y) \leq \sum_{w \in B} \theta(w-z) G_A(w,y).$$

Hence,

$$\begin{aligned}\sum_{y\in A}\sum_{z\in D} G_A(x,z)\mathcal{H}^B_{\mathbf{k}}(z,y) &\preceq \sum_{z\in D} g(x,z)\sum_{w\in B}\theta(w-z)\sum_{y\in A} G_A(w,y)\\ &= \sum_{z\in D} g(x,z)\sum_{w\in B}\theta(w-z)\mathbf{p}_A(w) \le \sum_{z\in D} g(x,z)\sum_{w\in B}\theta(w-z)\\ &\overset{(1.1.1)}{\preceq} \sum_{z\in D} g(x,z)\rho(z,B)^{-d} \asymp \sum_{z\in D} g(x,z)\rho(x,A)^{-d} \preceq \rho(x,A)^{2-d}.\end{aligned}$$

This completes the proof of the upper bound.

Lower bound. First choose some $a>1$, such that for any $s\ge 1$,

$$|\mathcal{C}(s)|\cdot\theta\{(\mathcal{C}((a-1)s))^c\} \le \frac{\mathrm{BCap}(\{0\})}{2}, \tag{2.2.50}$$

Note that our assumption of θ guarantees that $\theta\{(\mathcal{C}((a-1)s))^c\} \preceq ((a-1)s)^{-d}$.

Write $\rho=\rho(x,A)$ and let $C=\mathcal{C}(a\rho)$. Note that $\rho\ge 2r$ and $\rho(x,B)\ge r$. Hence $r\le\rho/2$, $B\subseteq\mathcal{C}(\rho)$ and for any $w\in B, z\in C^c$,

$$\rho(w,z)\ge(a-1)\rho. \tag{2.2.51}$$

Then

$$\begin{aligned}\mathbf{p}_A(x) &= \sum_{y\in A} G_A(x,y) = \sum_{y\in A}\sum_{z\in B^c} G_A(x,z)\mathcal{H}^B_{\mathbf{k}}(z,y)\\ &\ge \sum_{y\in A}\sum_{z\in C\setminus B} G_A(x,z)\mathcal{H}^B_{\mathbf{k}}(z,y) \succeq \sum_{y\in A}\sum_{z\in C\setminus B}(2a\rho)^{2-d}\mathcal{H}^B_{\mathbf{k}}(z,y),\end{aligned}$$

We use the last Lemma in the last step. It is sufficient to show:

$$\sum_{y\in A}\sum_{z\in C\setminus B}\mathcal{H}^B_{\mathbf{k}}(z,y) \succeq \mathrm{BCap}(A). \tag{2.2.52}$$

Note that:

$$\begin{aligned}&\sum_{y\in A}\sum_{z\in C\setminus B}\mathcal{H}^B_{\mathbf{k}}(z,y)\geq\sum_{y\in A}\sum_{z\in C\setminus B}\mathcal{H}^B_{\mathbf{k}}(z,y)\mathrm{Es}_A(z)\\&\quad=\sum_{y\in A}(\mathrm{Es}_A(y)-\sum_{z\in C^c}\mathcal{H}^B_{\mathbf{k}}(z,y)\mathrm{Es}_A(z))\geq\mathrm{BCap}(A)-\sum_{y\in A}\sum_{z\in C^c}\mathcal{H}^B_{\mathbf{k}}(z,y).\end{aligned}$$

As in the proof for the upper bound, we have:

$$\begin{aligned}&\sum_{y\in A}\sum_{z\in C^c}\mathcal{H}^B_{\mathbf{k}}(z,y)\leq\sum_{y\in A}\sum_{z\in C^c}\sum_{w\in B}\theta(w-z)G_A(w,y)\\&\quad=\sum_{y\in A}\sum_{w\in B}G_A(w,y)\sum_{z\in C^c}\theta(w-z)\overset{(2.2.51)}{\leq}\sum_{w\in B}\sum_{y\in A}G_A(w,y)\theta\{(\mathcal{C}((a-1)\rho))^c\}\\&\quad=\theta\{(\mathcal{C}((a-1)\rho))^c\}\sum_{w\in B}p_A(w)\leq\theta\{(\mathcal{C}((a-1)\rho))^c\}|B|\overset{(2.2.50)}{\leq}\frac{\mathrm{BCap}(A)}{2}.\end{aligned}$$

Now (2.2.52) follows and this completes the proof of the lower bound. □

2.3 Branching capacity and branching recurrence

We now give the definitions of branching recurrence and branching transience. Recall that we always assume $d\geq 5$ in this section. In addition, we assume further that θ has finite range throughout this section.

Definition 2.3.1. *Let A be a subset of $\mathbb{Z}^d$. We call A a **branching recurrent (B-recurrent)** set if*

$$P(\mathcal{S}_0^\infty \text{ visits } A \text{ infinitely often})=1, \tag{2.3.1}$$

*and a **branching transient (B-transient)** set if*

$$P(\mathcal{S}_0^\infty \text{ visits } A \text{ infinitely often})=0. \tag{2.3.2}$$

In fact, it is equivalent to use the incipient infinite snake in the definition of branching recurrence and branching transience.

Proposition 2.3.2.

$$P(\mathcal{S}_0^\infty \textit{ visits } A \textit{ infinitely often}) = 1 \Leftrightarrow P(\overline{\mathcal{S}}_0^\infty \textit{ visits } A \textit{ infinitely often}) = 1.$$

Proof. The necessity is trivial. For the sufficiency, we use the following coupling between $\overline{\mathcal{S}}_0^\infty$ and $\mathcal{S}_0^\infty$. First sample $\overline{\mathcal{S}}_0^\infty$. Then we can construct $\mathcal{S}_0^\infty$ as follows: for the backbone of $\mathcal{S}_0^\infty$, just use the backbone of $\overline{\mathcal{S}}_0^\infty$; for each vertex in the backbone, we graft to it an adjoint snake, independently, using either the left adjoint snake or the right one, corresponding to the same vertex in $\overline{\mathcal{S}}_0^\infty$, with equal probability. When $\overline{\mathcal{S}}_0^\infty$ visits A infinitely often, there are infinite adjoint snakes on $\overline{\mathcal{S}}_0^\infty$ visiting A. For each vertex on the backbone, either the left adjoint snake or the right one is chosen, independently with equal probability. Therefore, by the strong law of large numbers, an infinite number of adjoint snakes that visits A will be chosen, on the process of producing $\mathcal{S}_0^\infty$, almost surely. It means that $\mathcal{S}_0^\infty$ visits A infinitely often almost surely. $\square$

Proposition 2.3.3. *Every set $A \subseteq \mathbb{Z}^d$ is either B-recurrent or B-transient.*

Proof. Let $f(x) = P(\mathcal{S}_x^\infty$ visits A infinitely often). It is easy to see that f is a bounded harmonic function. But every bounded harmonic function in $\mathbb{Z}^d$ is constant. Hence $f \equiv t$ for some $t \in [0,1]$. Let V be the event $\mathcal{S}_0^\infty$ visits A infinitely often. Since $f \equiv t$, we have $P(V|\mathcal{F}_n) = t$ for any n, where $\mathcal{F}_n$ is the σ-field generated by all 'information' (the tree structure and the random variables corresponding to the edges) after n-th vertex of the spine. Then V is a tail event. By the Kolmogorov 0-1 Law, t is either 0 or 1. $\square$

If A is finite, since $\mathbf{q}_A(x) < 1$ for large x, $f(x) < 1$ and must be 0. Hence we have:

Proposition 2.3.4. *Every finite subset of $\mathbb{Z}^d$ is B-transient.*

For some technical reasons, we assume further that θ has finite range in this section.

2.3.1 Inequalities for convolved sums

We need the following two inequalities in the proof of our version of Wiener's Test.

Lemma 2.3.5. *For any $n \in \mathbb{N}^+$, let $B = \mathcal{C}(n)$. When $A \subseteq B$ and $x \in \mathbb{Z}^d$, we have:*

$$\sum_{z \in B} G_A(x,z)\mathbf{q}_A(z) \preceq (diam(B))^2 \mathbf{q}_A(x); \tag{2.3.3}$$

$$\sum_{z \in B} G_A(x,z)\mathbf{p}_A(z) \preceq (diam(B))^2 \mathbf{p}_A(x). \tag{2.3.4}$$

We prove (2.3.3) here and postpone the proof of (2.3.4) until Section 2.3.3.

Proof of (2.3.3). For (2.3.3), we do not need to assume that B is a ball and $A \subseteq B$. In fact, we will prove (2.3.3) for any finite subsets A, B of $\mathbb{Z}^d$ and $x \in \mathbb{Z}^d$.

We are working at the random walk with killing function $\mathbf{r}_A$. Consider the following equivalent model: a particle starting from x executes a random walk $S = (S(k))_{k \in \mathbb{N}}$, but at each step, the particle has the probability $\mathbf{r}_A$ to get a flag (instead of to die) and its movements are unaffected by flags. Let τ and ξ be the first and last time getting flags (if there is no such time, define both to be infinity). Note that since $\mathbf{q}_A(z) < 1$ (when $|z|$ is large), the total number of flags gained is finite, almost surely. Hence $P(\tau < \infty) = P(\xi < \infty)$. Under this model, one can see that

$$G_A(x,z)\mathbf{q}_A(z) = P(\tau < \infty, S(k) = z \text{ for some } k \leq \tau).$$

Hence it is not more than $\mathbb{E}(\sum_{i=0}^{\tau} \mathbf{1}_{\{S(i)=z\}}; \tau < \infty)$ and the L.H.S. of (2.3.3) is not more than

$$\mathbb{E}(\sum_{i=0}^{\tau} \mathbf{1}_{\{S_x(i) \in B\}}; \tau < \infty) \leq \mathbb{E}(\sum_{i=0}^{\xi} \mathbf{1}_{\{S_x(i) \in B\}}; \xi < \infty).$$

By considering the place where the particle gets its last flag, one can see:

$$\mathbb{E}(\sum_{i=0}^{\xi} 1_{\{S_x(i)\in B\}}; \xi < \infty) = \sum_{w\in\mathbb{Z}^d} \left(\sum_{\gamma:x\to w} \mathbf{s}(\gamma)(\sum_{i=0}^{|\gamma|} \mathbf{1}_{\gamma(i)\in B}) \right) \cdot \mathbf{r}_A(w)\mathrm{Es}_A^+(w).$$

We point out a result about random walk and prove it later:

$$\sum_{\gamma:x\to w} \mathbf{s}(\gamma)(\sum_{i=0}^{|\gamma|} \mathbf{1}_{\gamma(i)\in B}) \preceq (\mathrm{diam}(B))^2 \sum_{\gamma:x\to w} \mathbf{s}(\gamma). \tag{2.3.5}$$

Hence we get:

$$\begin{aligned}\sum_{z\in B} G_A(x,z)\mathbf{q}_A(z) \preceq& (\mathrm{diam}(B))^2 \sum_{w\in\mathbb{Z}^d}\sum_{\gamma:x\to w} \mathbf{s}(\gamma)\mathbf{r}_A(w)\mathrm{Es}_A^+(w)\\ =&(\mathrm{diam}(B))^2 \sum_{w\in\mathbb{Z}^d} g(x,w)\mathbf{r}_A(w)\mathrm{Es}_A^+(w)\\ \overset{(2.2.20)}{=}&(\mathrm{diam}(B))^2\mathbf{q}_A(x).\end{aligned}$$

Now we just need to prove (2.3.5). First we assume $x, w \in B$, then

$$\begin{aligned}&\sum_{\gamma:x\to w} \mathbf{s}(\gamma)(\sum_{i=0}^{|\gamma|} \mathbf{1}_{\gamma(i)\in B}) \le \sum_{\gamma:x\to w} \mathbf{s}(\gamma)[\gamma] \overset{(2.1.3)}{\asymp} |x-w|^{4-d}\\ &\le (\mathrm{diam}(B))^2|x-w|^{2-d} \asymp (\mathrm{diam}(B))^2 g(x,w) = (\mathrm{diam}(B))^2 \sum_{\gamma:x\to w} \mathbf{s}(\gamma).\end{aligned}$$

For general x, w, one just need to decompose γ into pieces according to the first and last visiting time of B. For example, when $x, w \notin B$, we have:

$$\sum_{\gamma:x\to w} \mathbf{s}(\gamma)(\sum_{i=0}^{|\gamma|} \mathbf{1}_{\gamma(i)\in B}) = \sum_{y,z\in B} \mathcal{H}^{B^c}(x,y) \left(\sum_{\gamma':y\to z} \mathbf{s}(\gamma') \sum_{i=0}^{|\gamma'|} \mathbf{1}_{\gamma'(i)\in B} \right) \mathcal{H}^{B^c}(z,w)$$

$$\begin{aligned}
&\preceq \sum_{y,z\in B} \mathcal{H}^{B^c}(x,y) \left((\mathrm{diam}(B))^2 \sum_{\gamma':y\to z} \mathbf{s}(\gamma') \right) \mathcal{H}^{B^c}(z,w) \\
&= (\mathrm{diam}(B))^2 \sum_{y,z\in B} \mathcal{H}^{B^c}(x,y) \sum_{\gamma':y\to z} \mathbf{s}(\gamma') \mathcal{H}^{B^c}(z,w) \\
&= (\mathrm{diam}(B))^2 \sum_{\gamma:x\to w, \gamma \text{ visits } B} \mathbf{s}(\gamma) \le (\mathrm{diam}(B))^2 \sum_{\gamma:x\to w} \mathbf{s}(\gamma).
\end{aligned}$$

When just one of x and w is in B, the proof is similar but easier. □

2.3.2 Restriction lemmas

Recall that we have: (see (2.2.7))

$$\mathbf{p}_A(x) = \sum_{\gamma:x\to A} \mathbf{b}(\gamma).$$

Our goals of this section are to show:

Proposition 2.3.6. *For any $n \in \mathbb{N}^+$ sufficiently large and $A \subseteq \mathcal{C}(n), x \in \mathcal{C}(n)$, we have:*

$$\mathbf{p}_A(x) \asymp \sum_{\gamma:x\to A, \gamma\subseteq \mathcal{C}(1.1n)} \mathbf{b}(\gamma). \tag{2.3.6}$$

Proposition 2.3.7. *For any $n \in \mathbb{N}^+$ sufficiently large and $A \subseteq \mathcal{C}(n), x \in \mathcal{C}(n)$, we have:*

$$\mathbf{q}_A(x) \asymp \sum_{\gamma:x\to A, \gamma\subseteq \mathcal{C}(4n)} [\gamma] \cdot \mathbf{b}(\gamma). \tag{2.3.7}$$

We first introduce some notations. Since θ has finite range, we can define the outer boundary $\partial_o B$ for any $B \subseteq \mathbb{Z}^d$ by

$$\partial_o B = \{z \in \mathbb{Z}^d \setminus B : \exists y \in B, \theta(z-y) \vee \theta(y-z) > 0\}.$$

Note that for any $y \in \partial_o B$, $\rho(y, B)$ is bounded above by a constant depending on θ. For $A \subseteq B \subseteq \mathbb{Z}^d$ and $x, y \in B \cup \partial_o B$, write

$$G_A^B(x,y) = \sum_{\gamma: x \to y, \gamma \subseteq B} \mathbf{b}(\gamma).$$

Lemma 2.3.8. *For any $\lambda_1, \lambda_2, \lambda_3 > 0$, there exists $C = C(\lambda_1, \lambda_2, \lambda_3) > 0$ satisfying the following. When n is sufficiently large, let $B_0 = \mathcal{C}(n)$,$B_1 = \mathcal{C}((1+\lambda_1)n)$, $B_2 = \mathcal{C}((1+\lambda_1+\lambda_2)n)$ and $B = \mathcal{C}((1+\lambda_1+\lambda_2+\lambda_3)n)$. Then for any $x, y \in B_2 \setminus B_1$ and $A \subseteq B_0$, we have:*

$$G_A^B(x,y) \geq C G_A(x,y). \tag{2.3.8}$$

Proof. Let $B' = \mathcal{C}((1+\lambda_1/2)n)$. Note that for any $y \in B \setminus B'$, by (1.3.2) and (2.2.18), we have $\mathbf{r}_A(y) \asymp \mathbf{p}_A(y) \preceq n^{-2}$. Hence, we have: for any $\gamma \subseteq B \setminus B'$ with $|\gamma| \leq 2n^2$, $\mathbf{b}(\gamma)/\mathbf{s}(\gamma) \geq (1 - c/n^2)^{2n^2} \succeq 1$.

Therefore, by Lemma 2.1.3, one can see that:

$$\begin{aligned} G_A^B(x,y) = \sum_{\gamma: x \to y, \gamma \subseteq B} \mathbf{b}(\gamma) &\geq \sum_{\gamma: x \to y, \gamma \subseteq B \setminus B', |\gamma| \leq 2n^2} \mathbf{b}(\gamma) \\ &\succeq \sum_{\gamma: x \to y, \gamma \subseteq B \setminus B', |\gamma| \leq 2n^2} \mathbf{s}(\gamma) \asymp g(x,y) \geq G_A(x,y). \end{aligned}$$

□

Lemma 2.3.9. *For any $\lambda > 0, \iota > 0$, there exists $C = C(\lambda, \iota) > 0$ satisfying the following. When n is sufficiently large, let $B_0 = \mathcal{C}(n)$, $B_1 = \mathcal{C}((1+\lambda)n)$, $B = \mathcal{C}((1+\lambda+\iota)n)$. Then for any $x, y \in B_1$ and $A \subseteq B_0$, we have:*

$$G_A^B(x,y) \geq C G_A(x,y). \tag{2.3.9}$$

Proof. By last lemma, one can get, for any $z, w \in \partial_o B_1$,

$$G_A^B(z,w) \succeq G_A(z,w).$$

For any $x, y \in B_1$, we have:

$$G_A^B(x,y) = G_A^{B_1}(x,y) + \sum_{\gamma:x\to y,\gamma \text{ visits } B_1^c, \gamma\subseteq B} \mathbf{b}(\gamma).$$

By considering the first and last visits in B_1^c, we have:

$$\begin{aligned}\sum_{\gamma:x\to y,\gamma \text{ visits } B_1^c,\gamma\subseteq B} \mathbf{b}(\gamma) &= \sum_{z,w\in\partial_o B_1} \mathcal{H}_A^{B_1}(x,z) G_A^B(z,w) \mathcal{H}_A^{B_1}(w,y) \\ \overset{(2.3.8)}{\succeq} & \sum_{z,w\in\partial_o B_1} \mathcal{H}_A^{B_1}(x,z) G_A(z,w) \mathcal{H}_A^{B_1}(w,y) = \sum_{\gamma:x\to y,\gamma \text{ visits } B_1^c} \mathbf{b}(\gamma).\end{aligned}$$

Hence, we have:

$$\begin{aligned} G_A^B(x,y) =& G_A^{B_1}(x,y) + \sum_{\gamma:x\to y,\gamma \text{ visits } B_1^c,\gamma\subseteq B} \mathbf{b}(\gamma) \\ \succeq & G_A^{B_1}(x,y) + \sum_{\gamma:x\to y,\gamma \text{ visits } B_1^c} \mathbf{b}(\gamma) = G_A(x,y).\end{aligned}$$

□

Now we can show Proposition 2.3.6:

Proof of Proposition 2.3.6. Let $B = \mathcal{C}(1.1n)$. We have:

$$\mathbf{p}_A(x) = \sum_{\gamma:x\to A} \mathbf{b}(\gamma) = \sum_{z\in A} G_A(x,z) \overset{(2.3.9)}{\asymp} \sum_{z\in A} G_A^B(x,z) = \sum_{\gamma:x\to A,\gamma\subseteq B} \mathbf{b}(\gamma).$$

□

Now we turn to $\mathbf{q}_A(x)$. The starting point is:

Lemma 2.3.10. *For any $a \in \mathbb{Z}^d$, $A \subset\subset \mathbb{Z}^d, B \subseteq \mathbb{Z}^d$, we have:*

$$\sum_{z\in B} G_A(x,z)\mathbf{p}_A(z) = \sum_{\gamma:x\to A} \mathbf{b}(\gamma) \sum_{i=0}^{|\gamma|} \mathbf{1}_{\gamma(i)\in B}. \tag{2.3.10}$$

Proof.

$$\begin{aligned}\sum_{z\in B} G_A(x,z)\mathbf{p}_A(z) &= \sum_{z\in B}\sum_{\gamma_1:x\to z}\mathbf{b}(\gamma_1)\sum_{\gamma_2:z\to A}\mathbf{b}(\gamma_2)\\ &= \sum_{z\in B}\sum_{\gamma_1:x\to z}\sum_{\gamma_2:z\to A}\mathbf{b}(\gamma_1)\mathbf{b}(\gamma_2)\\ &= \sum_{z\in B}\sum_{\gamma_1:x\to z}\sum_{\gamma_2:z\to A}\mathbf{b}(\gamma_1\circ\gamma_2)\\ &= \sum_{\gamma:x\to A}\mathbf{b}(\gamma)\cdot\sum_{i=0}^{|\gamma|}\mathbf{1}_{\gamma(i)\in B}.\end{aligned}$$

The last equality is due to the fact that for any $\gamma : x \to A$, there are exactly $\sum_{i=0}^{|\gamma|}\mathbf{1}_{\gamma(i)\in B}$ ways to rewrite γ as the composite of two paths γ_1 and γ_2 such that the common point of γ_1 and γ_2 is in B. $\square$

Corollary 2.3.11.

$$\mathbf{q}_A(x) \asymp \sum_{\gamma:x\to A}[\gamma]\cdot\mathbf{b}(\gamma). \tag{2.3.11}$$

Proof. By (2.2.19) and (2.2.18), we have:

$$\mathbf{q}_A(x) \asymp \sum_{z\in\mathbb{Z}^d} G_A(x,z)\mathbf{p}_A(z). \tag{2.3.12}$$

By last lemma, we have $\sum_{z\in\mathbb{Z}^d} G_A(x,z)\mathbf{p}_A(z) = \sum_{\gamma:x\to A}[\gamma]\cdot\mathbf{b}(\gamma)$. $\square$

Lemma 2.3.12. *For any n sufficiently large, $A\subseteq\mathcal{C}(n), x\in\mathbb{Z}^d$ with $\|x\|\geq 1.1n$, we have:*

$$\mathbf{q}_A(x) \asymp \sum_{\gamma:x\to A,\gamma\subseteq\mathcal{C}(3\|x\|)}[\gamma]\cdot\mathbf{b}(\gamma). \tag{2.3.13}$$

Proof. The part of '$\succeq$' is trivial by the last corollary. It suffices to show the other part. As in the last corollary, we have:

$$\mathbf{q}_A(x) \asymp \sum_{z\in\mathbb{Z}^d} G_A(x,z)\mathbf{p}_A(z).$$

First by (2.1.2),(1.3.2) and (2.2.49), one can see that:

$$\begin{aligned}\sum_{z\in\mathcal{C}(2\|x\|)\setminus\mathcal{C}(1.5\|x\|)} G_A(x,z)\mathbf{p}_A(z) &\asymp \sum_{z\in\mathcal{C}(2\|x\|)\setminus\mathcal{C}(1.5\|x\|)} g(x,z)\frac{\mathrm{BCap}(A)}{(\rho(z,A))^{d-2}} \\ &\asymp \sum_{z\in\mathcal{C}(2\|x\|)\setminus\mathcal{C}(1.5\|x\|)} \frac{1}{|x|^{d-2}}\frac{\mathrm{BCap}(A)}{|x|^{d-2}} \\ &\asymp |x|^d\frac{1}{|x|^{d-2}}\frac{\mathrm{BCap}(A)}{|x|^{d-2}} = \frac{\mathrm{BCap}(A)}{|x|^{d-4}}.\end{aligned}$$

Similarly, we can get:

$$\begin{aligned}\sum_{z\in\mathcal{C}(2\|x\|)^c} G_A(x,z)\mathbf{p}_A(z) &\asymp \sum_{z\in\mathcal{C}(2\|x\|)^c} g(x,z)\frac{\mathrm{BCap}(A)}{(\rho(z,A))^{d-2}} \\ &\asymp \sum_{z\in\mathcal{C}(2\|x\|)^c} \frac{1}{|z|^{d-2}}\frac{\mathrm{BCap}(A)}{|z|^{d-2}} \\ &\asymp \mathrm{BCap}(A)\sum_{z\in\mathcal{C}(2\|x\|)^c}\frac{1}{|z|^{2d-4}} \asymp \frac{\mathrm{BCap}(A)}{|x|^{d-4}}.\end{aligned}$$

Hence we have:

$$\mathbf{q}_A(x) \asymp \sum_{z\in\mathbb{Z}^d} G_A(x,z)\mathbf{p}_A(z) \asymp \sum_{z\in\mathcal{C}(2\|x\|)} G_A(x,z)\mathbf{p}_A(z).$$

By Lemma 2.3.9, we have (let $B=\mathcal{C}(3\|x\|)$):

$$\begin{aligned}\sum_{z\in\mathcal{C}(2\|x\|)} G_A(x,z)\mathbf{p}_A(z) &= \sum_{z\in\mathcal{C}(2\|x\|)} G_A(x,z)\sum_{y\in A} G_A(z,y) \\ &\asymp \sum_{z\in\mathcal{C}(2\|x\|)} G_A^B(x,z)\sum_{y\in A} G_A^B(z,y) \\ &= \sum_{z\in\mathcal{C}(2\|x\|)}\sum_{\gamma_1:x\to z,\gamma_1\subseteq B}\mathbf{b}(\gamma_1)\sum_{\gamma_2:z\to A,\gamma_2\subseteq B}\mathbf{b}(\gamma_2) \\ &= \sum_{z\in\mathcal{C}(2\|x\|)}\sum_{\gamma_1:x\to z,\gamma_1\subseteq B}\sum_{\gamma_2:z\to A,\gamma_2\subseteq B}\mathbf{b}(\gamma_1\circ\gamma_2) \\ &\le \sum_{\gamma:x\to A,\gamma\subseteq B}[\gamma]\mathbf{b}(\gamma).\end{aligned}$$

This completes the proof. □

Proof of Proposition 2.3.7. Let $B = \mathcal{C}(1.1n)$ and $B' = \mathcal{C}(4n)$. We have:

$$\mathbf{q}_A(x) \asymp \sum_{\gamma:x\to A} [\gamma]\mathbf{b}(\gamma) = \sum_{\gamma:x\to A,\gamma\subseteq B} [\gamma]\mathbf{b}(\gamma) + \sum_{\gamma:x\to A,\gamma \text{ visits } B^c} [\gamma]\mathbf{b}(\gamma).$$

By considering the first visit of B^c, the second term is equal to:

$$\begin{aligned}
&\sum_{y\in\partial_o B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B}\sum_{\gamma_2:y\to A}(|\gamma_1| + [\gamma_2])(\mathbf{b}(\gamma_1)\mathbf{b}(\gamma_2))\\
&= \sum_{y\in\partial_o B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B}|\gamma_1|\mathbf{b}(\gamma_1)\sum_{\gamma_2:y\to A}\mathbf{b}(\gamma_2) +\\
&\qquad\qquad \sum_{y\in\partial_o B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B}\mathbf{b}(\gamma_1)\sum_{\gamma_2:y\to A}[\gamma_2]\mathbf{b}(\gamma_2)\\
&\overset{(2.2.7)(2.3.11)}{\asymp} \sum_{y\in\partial_o B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B}|\gamma_1|\mathbf{b}(\gamma_1)\mathbf{p}_A(y) + \sum_{y\in\partial_o B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B}\mathbf{b}(\gamma_1)\mathbf{q}_A(y)\\
&\overset{(2.3.13),(2.3.6)}{\asymp} \sum_{y\in\partial_o B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B}|\gamma_1|\mathbf{b}(\gamma_1)\sum_{\gamma_2:y\to A,\gamma_2\subseteq B'}\mathbf{b}(\gamma_2) +\\
&\qquad\qquad \sum_{y\in\partial_o B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B}\mathbf{b}(\gamma_1)\sum_{\gamma_2:y\to A,\gamma_2\subseteq B'}[\gamma_2]\mathbf{b}(\gamma_2)\\
&= \sum_{y\in\partial_o B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B}\sum_{\gamma_2:y\to A,\gamma_2\subseteq B'}(|\gamma_1| + [\gamma_2])(\mathbf{b}(\gamma_1)\mathbf{b}(\gamma_2))\\
&= \sum_{\gamma:x\to A,\gamma \text{ visits } B^c,\gamma\subseteq B'}[\gamma]\mathbf{b}(\gamma).
\end{aligned}$$

Hence, we get

$$\mathbf{q}_A(x) \asymp \sum_{\gamma:x\to A,\gamma\subseteq B}[\gamma]\mathbf{b}(\gamma) + \sum_{\gamma:x\to A,\gamma \text{ visits } B^c,\gamma\subseteq B'}[\gamma]\mathbf{b}(\gamma) = \sum_{\gamma:x\to A,\gamma\subseteq B'}[\gamma]\mathbf{b}(\gamma).$$

This completes the proof. □

2.3.3 Visiting probability by an infinite snake

In this subsection we establish the following bounds analogous to (1.3.2):

Theorem 2.3.13. *For any $A \subset\subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ with $\|x\| \geq 2Rad(A)$, we have:*

$$\mathbf{q}_A(x) \asymp \frac{BCap(A)}{(\rho(x,A))^{d-4}}. \tag{2.3.14}$$

Remark 2.3.1. *By cutting A into small pieces, one can replace $\|x\| \geq 2Rad(A)$ by $\rho(x,A) \geq \epsilon diam(A)$, for any $\epsilon > 0$.*

Remark 2.3.2. *The analogous result for $\overline{\mathcal{S}}_x^\infty$ (Theorem 1.3.9) can be proved in a similar way.*

Proof. It suffices to show the case when $\mathrm{Rad}(A)$ is sufficiently large since we know the asymptotical behavior when x is far away (see (2.2.14)). The part for $\succeq$ is straightforward and similar to the first part of the proof of Lemma 2.3.12:

$$\begin{aligned}\mathbf{q}_A(x) &\overset{(2.3.12)}{\asymp} \sum_{z\in\mathbb{Z}^d} G_A(x,z)\mathbf{p}_A(z) \geq \sum_{2\|x\|\leq\|z\|\leq 4\|x\|} G_A(x,z)\mathbf{p}_A(z)\\ &\overset{(1.3.2)(2.2.49)}{\asymp} \sum_{2\|x\|\leq\|z\|\leq 4\|x\|} \frac{1}{|x-z|^{d-2}}\frac{\mathrm{BCap}(A)}{(\rho(z,A))^{d-2}} \asymp \sum_{2\|x\|\leq\|z\|\leq 4\|x\|}\frac{1}{|z|^{d-2}}\frac{\mathrm{BCap}(A)}{|z|^{d-2}}\\ &\asymp |x|^d\frac{1}{|x|^{d-2}}\frac{\mathrm{BCap}(A)}{|x|^{d-2}} = \frac{\mathrm{BCap}(A)}{|x|^{d-4}} \asymp \frac{\mathrm{BCap}(A)}{(\rho(x,A))^{d-4}}.\end{aligned}$$

The other part can be implied by (1.3.2) and the following lemma (let $n = \|x\|$). □

Lemma 2.3.14. *For any $n \in \mathbb{N}^+$ sufficiently large, $A \subset \mathcal{C}(n), y \in \mathcal{C}(n)$, we have:*

$$\mathbf{q}_A(y) \preceq n^2\mathbf{p}_A(y). \tag{2.3.15}$$

Proof. Let $B = \mathcal{C}(4n)$. By (2.3.7) and (2.2.7), it suffices to prove:

$$\sum_{\gamma:y\to A,\gamma\subseteq B}[\gamma]\mathbf{b}(\gamma) \preceq n^2 \sum_{\gamma:y\to A,\gamma\subseteq B}\mathbf{b}(\gamma). \tag{2.3.16}$$

By (2.3.3) and (2.3.7), one can get:

$$\sum_{z\in B} G_A(y,z)\mathbf{q}_A(z) \preceq n^2 \sum_{\gamma:y\to A,\gamma\subseteq B}[\gamma]\mathbf{b}(\gamma). \tag{2.3.17}$$

For the left hand side, we have:

$$\begin{aligned}
&\sum_{z\in B} G_A(y,z)\mathbf{q}_A(z) \overset{(2.3.11)}{\asymp} \sum_{z\in B}\sum_{\gamma_1:y\to z}\mathbf{b}(\gamma_1)\sum_{\gamma_2:z\to A}[\gamma_2]\mathbf{b}(\gamma_2)\\
\geq &\sum_{z\in B}\sum_{\gamma_1:y\to z,\gamma_1\subseteq B}\sum_{\gamma_2:z\to A,\gamma_2\subseteq B}[\gamma_2]\mathbf{b}(\gamma_1\circ\gamma_2)\\
= &\sum_{\gamma:y\to A,\gamma\subseteq B}(1+2+...+[\gamma])\mathbf{b}(\gamma) \asymp \sum_{\gamma:y\to A,\gamma\subseteq B}[\gamma]^2\mathbf{b}(\gamma).
\end{aligned}$$

Hence, we have:

$$\sum_{\gamma:y\to A,\gamma\subseteq B}[\gamma]^2\mathbf{b}(\gamma) \preceq n^2 \sum_{\gamma:y\to A,\gamma\subseteq B}[\gamma]\mathbf{b}(\gamma). \tag{2.3.18}$$

By Cauchy-Schwarz inequality:

$$\begin{aligned}
\left(\sum_{\gamma:y\to A,\gamma\subseteq B}[\gamma]\mathbf{b}(\gamma)\right)^2 &\leq \left(\sum_{\gamma:y\to A,\gamma\subseteq B}[\gamma]^2\mathbf{b}(\gamma)\right)\cdot\left(\sum_{\gamma:y\to A,\gamma\subseteq B}\mathbf{b}(\gamma)\right)\\
&\preceq n^2\sum_{\gamma:y\to A,\gamma\subseteq B}[\gamma]\mathbf{b}(\gamma)\cdot\left(\sum_{\gamma:y\to A,\gamma\subseteq B}\mathbf{b}(\gamma)\right).
\end{aligned}$$

Then (2.3.16) follows and we complete the proof. □

Proof of (2.3.4). When $x \in B$, by the last lemma (recall that $\mathbf{q}_A(x) \asymp \sum_{z\in\mathbb{Z}^d} G_A(x,z)\mathbf{p}_A(x)$), we have the desired bound. Now we assume $x \notin B$.

By considering the first visit of B, we have

$$\begin{aligned}
\sum_{z\in B} G_A(x,z)\mathbf{p}_A(z) &\overset{(2.3.10)}{=} \sum_{\gamma:x\to A}\Big(\sum_{i=0}^{|\gamma|}\mathbf{1}_{\gamma(i)\in B}\Big)\mathbf{b}(\gamma)\\
&= \sum_{y\in B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B^c}\sum_{\gamma_2:y\to A}\mathbf{b}(\gamma_1\circ\gamma_2)\Big(\sum_{i=0}^{|\gamma_2|}\mathbf{1}_{\gamma_2(i)\in B}\Big)\\
&= \sum_{y\in B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B^c}\mathbf{b}(\gamma_1)\sum_{\gamma_2:y\to A}\Big(\sum_{i=0}^{|\gamma_2|}\mathbf{1}_{\gamma_2(i)\in B}\Big)\mathbf{b}(\gamma_2)\\
&\overset{(2.3.10)}{=} \sum_{y\in B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B^c}\mathbf{b}(\gamma_1)\sum_{z\in B}G_A(y,z)\mathbf{p}_A(z)\\
&\overset{(*)}{\preceq} \sum_{y\in B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B^c}\mathbf{b}(\gamma_1)(\mathrm{diam}(B))^2\mathbf{p}_A(y)\\
&= (\mathrm{diam}(B))^2\sum_{y\in B}\sum_{\gamma_1:x\to y,\gamma_1\subseteq B^c}\mathbf{b}(\gamma_1)\mathbf{p}_A(y)\\
&= (\mathrm{diam}(B))^2\mathbf{p}_A(x).
\end{aligned}$$

$(*)$ is because we have proved that (2.3.4) is true for $x\in B$ and for the last line, we use the First-Visit Lemma and (2.2.7). □

2.3.4 Upper bounds for the probabilities of visiting two sets

In this subsection we aim to prove the following inequalities which we will use in the proof of Wiener's Test.

Lemma 2.3.15. *For any disjoint nonempty subsets $A, B \subset\subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we have:*

$$P(\mathcal{S}_x \textit{ visits both } A\&B) \preceq \sum_{z\in\mathbb{Z}^d} G_{A\cup B}(x,z)\mathbf{p}_A(z)\mathbf{p}_B(z); \tag{2.3.19}$$

$$P(\mathcal{S}_x^\infty \textit{ visits both } A\&B) \preceq \sum_{z\in\mathbb{Z}^d} G_{A\cup B}(x,z)\left(\mathbf{p}_A(z)\mathbf{q}_B(z)+\mathbf{q}_A(z)\mathbf{p}_B(z)+P(\mathcal{S}'_z \textit{ visits both } A\&B)\right). \tag{2.3.20}$$

Proof. (2.3.20) is a bit easier and we prove it first. When an infinite snake $\mathcal{S}_x^\infty=(T,\mathcal{S}_T)$ visits both A and B, let u be the first vertex in the spine such that the image of the bush graft to u under $\mathcal{S}_T$ intersects $A\cup B$. Assume $(v_0,\dots,v_k)$ is the unique simple path in the spine from o to u. Define $\Gamma_{(A,B)}(\mathcal{S}_x^\infty)=(\mathcal{S}_T(v_0),\dots,\mathcal{S}_T(v_k))$. For any path $\gamma=(\gamma(0),\dots,\gamma(k))$ starting from x with length $|\gamma|=k$, we would like to estimate $P(\Gamma_{(A,B)}(\mathcal{S}_x^\infty)=\gamma)$. If we can show that:

$$P(\Gamma_{(A,B)}(\mathcal{S}_x^\infty)=\gamma)\preceq \mathbf{b}(\gamma)\Big(\mathbf{p}_A(\widehat{\gamma})\mathbf{q}_B(\widehat{\gamma})+\mathbf{q}_A(\widehat{\gamma})\mathbf{p}_B(\widehat{\gamma})+P(\mathcal{S}'_{\widehat{\gamma}} \text{ visits both } A\&B)\Big), \tag{2.3.21}$$

then by summation, one can get (2.3.20).

Now we argue that (2.3.21) is correct. Let $\mathbf{t}$ be the bush grafted to u. There are three possibilities: $\mathcal{S}_T(\mathbf{t})$ visits A but not B, visits B but not A or visits both A and B. For the first one, to guarantee $\Gamma_{(A,B)}(\mathcal{S}_x^\infty)=\gamma$, we need three conditions to be true. The first is that $\mathcal{S}_T$ maps $(v_0,\dots,v_k)$ to γ and that the image of each bush grafted to v_i does not intersect $A\cup B$, for $i=0,\dots,k-1$. The probability of this condition being true is $\mathbf{b}(\gamma)$. The second condition is that $\mathcal{S}_T(\mathbf{t})$ intersects A but not B. The probability of this condition being true is at most $\mathbf{r}_A(\widehat{\gamma})\asymp\mathbf{p}_A(\widehat{\gamma})$. The last condition is that the image of the bushes after u intersects B. The probability of this condition being true is at most $\mathbf{q}_B(\widehat{\gamma})$. Note that for fixed γ, the three conditions are independent. Hence we have:

$$P(\Gamma_{(A,B)}(\mathcal{S}_x^\infty)=\gamma,\mathcal{S}_T(\mathbf{t}) \text{ visits } A \text{ not } B)\le \mathbf{b}(\gamma)\mathbf{p}_A(\widehat{\gamma})\mathbf{q}_B(\widehat{\gamma}).$$

Similarly, one can get the other two inequalities. This completes the proof of (2.3.20).

For (2.3.19), we use a similar idea. When a snake $\mathcal{S}_x = (T, \mathcal{S}_T)$ visits both A and B, then $V_A := \{v \in T : \mathcal{S}_T(v) \in A\}$ and $V_B := \{v \in T : \mathcal{S}_T(v) \in B\}$ are nonempty. We call a vertex $v \in T$ good, if v is the last common ancestor for some $u_1 \in V_A$ and $u_2 \in V_B$ (any vertex is regarded as an ancestor of itself). Since for any $u_1 \in V_A$ and $u_2 \in V_B$, they have the unique last common ancestor. Hence there exists at least one good vertex and we choose the first good one (due to the default order, Depth-First order), say u. Assume $\gamma = (v_0, \dots, v_k)$ is the unique simple path in T from the root o to u. Define $\Gamma_{(A,B)}(\mathcal{S}_x) = (\mathcal{S}_x(v_0), \dots, \mathcal{S}_x(v_k))$. As before, we would like to estimate $P(\Gamma_{(A,B)}(\mathcal{S}_x) = \gamma)$, for a fixed path $\gamma = (\gamma(0), \dots, \gamma(k))$ starting from x, with length $|\gamma| = k$. We argue that:

$$P(\Gamma_{(A,B)}(\mathcal{S}_x) = \gamma) \preceq \mathbf{b}(\gamma)\mathbf{p}_A(\widehat{\gamma})\mathbf{p}_B(\widehat{\gamma}). \tag{2.3.22}$$

Since u is the first good vertex, one can see that all vertices in $V_A \cup V_B$ are descendants of u or u itself. In particular,

$$\text{any vertex before } u \text{ is not in } V_A \cup V_B. \tag{2.3.23}$$

Here, 'before' is due to the Depth-First search order. This is the first necessary condition for the event $\Gamma_{(A,B)}(\mathcal{S}_x) = \gamma$ being true. Similar to the computations in Section 2.2.2, the probability for (2.3.23) being true is $\mathbf{b}(\gamma)$. Note that this condition just depends on $(T \setminus T_u, \mathcal{S}_T|_{T \setminus T_u})$, where T_u is the subtrees generated by u and its descendants, and $T \setminus T_u$ is the tree generated by u and those vertices outside T_u.

On the other hand, since u is the last common ancestor for some $u_1 \in V_A$ and $u_2 \in V_B$, when $u \notin V_A \cup V_B$, u must have two different children u^1 and u^2, such that $\mathcal{S}_T(T_{u^1}) \cap A \neq \emptyset$ and $\mathcal{S}_T(T_{u^2}) \cap B \neq \emptyset$. This is the second necessary condition for the event $\Gamma_{(A,B)}(\mathcal{S}_x) = \gamma$ being true. Note that for fixed γ, this condition is independent of (2.3.23), and its probability is at most

$$\sum_{n=2}^{\infty} \mu(n)n(n-1)\mathbf{p}_A(\widehat{\gamma})\mathbf{p}_B(\widehat{\gamma}) = \sigma^2 \mathbf{p}_A(\widehat{\gamma})\mathbf{p}_B(\widehat{\gamma}).$$

When $u \in V_A \cup V_B$, say $\in A$ (it implies $\widehat{\gamma} \in A$), then similarly, u must have a descendant mapped into B. The probability for this condition is: $\mathbf{p}_B(\widehat{\gamma}) = \mathbf{p}_A(\widehat{\gamma})\mathbf{p}_B(\widehat{\gamma})$. Combining the two conditions one can get (2.3.22). By summation, one can get (2.3.19). This completes the proof of (2.3.19). □

We require the assumption of the finite third moment of μ only for the following lemma.

Lemma 2.3.16. *When μ has finite third moment, we have:*

$$P(\mathcal{S}_x \textit{ visits both } A\&B) \asymp P(\mathcal{S}'_x \textit{ visits both } A\&B). \tag{2.3.24}$$

Proof. In fact, we will show:

$$P(\mathcal{S}_x \text{ visits both } A\&B) \asymp \mathbf{p}_A(x)\mathbf{p}_B(x) + P(\overline{\mathcal{S}}_x \text{ visits both } A\&B); \tag{2.3.25}$$

$$P(\mathcal{S}'_x \text{ visits both } A\&B) \asymp \mathbf{p}_A(x)\mathbf{p}_B(x) + P(\overline{\mathcal{S}}_x \text{ visits both } A\&B); \tag{2.3.26}$$

where $\overline{\mathcal{S}}_x$ is the finite snake from x conditioned on the initial particle having only one child.

For the upper bound of the first assertion, consider whether $\mathcal{S}_x$ visits A via the same child of the initial particle as it visits B via. If it does, this probability is at most

$$\sum_{i=1}^{\infty} \mu(i) \cdot iP(\overline{\mathcal{S}}_x \text{ visits both } A\&B) = \mathbb{E}(\mu)P(\overline{\mathcal{S}}_x \text{ visits both } A\&B).$$

If it does not, this probability is at most

$$\sum_{i=2}^{\infty} \mu(i) \cdot i(i-1)P(\overline{\mathcal{S}}_x \text{ visits } A)P(\overline{\mathcal{S}}_x \text{ visits } B) \asymp \mathbf{p}_A(x)\mathbf{p}_B(x).$$

Note that we use the fact that $\sum_{i=2}^{\infty} \mu(i) \cdot i(i-1)$ is bounded by the second moment of μ and $P(\overline{\mathcal{S}}_x \text{ visits } A) \asymp \mathbf{p}_A(x)$, which can be proved similar to

(2.2.18). Combining the last two inequalities, we get the upper bound of (2.3.25).

For the lower bound, it is easy to see that

$$\begin{aligned} P(\mathcal{S}_x \text{ visits both } A\&B) &\geq \left(\sum_{i\geq 2}\mu(i)\right) P(\overline{\mathcal{S}}_x \text{ visits } A)P(\overline{\mathcal{S}}_x \text{ visits } B) \\ &\asymp \mathbf{p}_A(x)\mathbf{p}_B(x); \\ P(\mathcal{S}_x \text{ visits both } A\&B) &\geq \left(\sum_{i\geq 1}\mu(i)\right) P(\overline{\mathcal{S}}_x \text{ visits both } A\&B). \end{aligned}$$

Combining these two, we can the lower bound of (2.3.25).

Similarly one can get (2.3.26). Note that for the upper bound, we require that $\widetilde{\mu}$ has finite second moment which is equivalent to the assumption that μ has finite third moment. $\square$

2.3.5 Proof of Wiener's Test

We first divide $\{x \in \mathbb{R}^d : 1 \leq |x| < 2\}$ into a finite number of small pieces with diameter less than 1/32: $B_1, \ldots, B_N$. Let $K_n^k = K_n \cap (2^n B_k)$ for any $n \in \mathbb{N}^+, 1 \leq k \leq N$. For any nonempty set K_n^k, we have $\text{diam}(K_n^k) \leq 2^n/32$ and $\rho(0, K_n^k) \in [2^n, 2^{n+1})$. Let V_n^k be the event that $\mathcal{S}_0^\infty$ visits K_n^k. Applying Theorem 2.3.13, we can get:

$$P(V_n^k) \asymp \frac{\text{BCap}(K_n^k)}{2^{n(d-4)}}. \tag{2.3.27}$$

Since each K_n^k is finite (any finite set is B-transient), we have

$$P(\mathcal{S}_0^\infty \text{ visits } K \text{ } i.o.) = P(V_n^k \text{ } i.o.).$$

When $\sum_{n=0}^\infty \text{BCap}(K_n)/2^{n(d-4)} < \infty$, by monotonicity, for any $1 \leq k \leq N$,

$$\sum_{n=1}^\infty \frac{\text{BCap}(K_n^k)}{2^{n(d-4)}} < \sum_{n=0}^\infty \frac{\text{BCap}(K_n)}{2^{n(d-4)}} < \infty.$$

Hence,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{N} P(V_n^k) \asymp \sum_{n=1}^{\infty}\sum_{k=1}^{N} \frac{\mathrm{BCap}(K_n^k)}{2^{n(d-4)}} = \sum_{k=1}^{N}\left(\sum_{n=1}^{\infty} \frac{\mathrm{BCap}(K_n^k)}{2^{n(d-4)}}\right) < \infty.$$

Then by Borel-Cantelli Lemma, almost surely, only finite V_n^k occurs and hence K is B-transient.

When $\sum_{n=0}^{\infty} \mathrm{BCap}(K_n)/2^{n(d-4)} = \infty$, by subadditivity of branching capacity (see Section 2.2.1), we have:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{N} \frac{\mathrm{BCap}(K_n^k)}{2^{n(d-4)}} \geq \sum_{n=1}^{\infty} \frac{\mathrm{BCap}(K_n)}{2^{n(d-4)}} = \infty.$$

Hence for some $1 \leq k \leq N$, $\sum_{n=1}^{\infty} \mathrm{BCap}(K_n^k)/2^{n(d-4)} = \infty$. Suppose

$$\sum_{n=1}^{\infty} \mathrm{BCap}(K_n^1)/2^{n(d-4)} = \infty.$$

We need the following Lemma whose proof we postpone.

Lemma 2.3.17. *There exists some $C > 0$, such that, for any $n < m$, we have:*

$$P(V_n^1 \cap V_m^1) \leq CP(V_n^1)P(V_m^1).$$

Let $I_n = \sum_{i=1}^{n} \mathbf{1}_{V_i^1}$ and $F = \mathbf{1}_{\{I_n \geq \mathbb{E}(I_n)/2\}}$. By the lemma above, we have:

$$\mathbb{E}(I_n^2) \leq C(\mathbb{E}(I_n))^2.$$

Note that

$$\mathbb{E}(FI_n) = \mathbb{E}I_n - \mathbb{E}(I_n \mathbf{1}_{\{I_n < \mathbb{E}(I_n)/2\}}) \geq \mathbb{E}(I_n)/2.$$

Hence,

$$\begin{aligned} P(I_n \geq \mathbb{E}(I_n)/2) &= \mathbb{E}(F) = \mathbb{E}(F^2) \\ &\geq (\mathbb{E}(FI_n))^2/\mathbb{E}(I_n^2) \geq (\mathbb{E}(I_n)/2)^2/C(\mathbb{E}(I_n))^2 = 1/(4C). \end{aligned}$$

Since $\mathbb{E}I_n \to \infty$, let $n \to \infty$, we get

$$P(I_n = \infty) \geq 1/(4C).$$

By Proposition 2.3.3, we get that K is B-recurrent.

2.3.6 Proof of Lemma 2.3.17

Write $A = K_n^1, B = K_m^1$ and $M = 2^m$. Without loss of generality, assume $A, B \neq \emptyset$. We know

$$\mathrm{diam}(A) \leq 2^n/32, \mathrm{diam}(B) \leq 2^m/32.$$

Fix any $a \in A$ and $b \in B$. Let $\widehat{A} = a + \mathcal{C}(2^n/8)$ and $\widehat{B} = b + \mathcal{C}(2^m/8)$. Then we have

$$\rho(A, \widehat{A}^c) \asymp \rho(0, \widehat{A}) \asymp 2^n; \rho(B, \widehat{B}^c) \asymp \rho(0, \widehat{B}) \asymp 2^m; \rho(a, b) \asymp \rho(\widehat{A}, \widehat{B}) \asymp 2^m. \tag{2.3.28}$$

We need to show:

$$P(\mathcal{S}_0^\infty \text{ visits both } A \,\&\, B) \preceq \mathbf{q}_A(0)\mathbf{q}_B(0). \tag{2.3.29}$$

In the proof, we will repeatedly use (1.3.2), Theorem 2.3.13, (2.3.12) and Lemma 2.3.5 without mention. Since (see (2.3.20) and (2.3.24))

$$P(\mathcal{S}_0^\infty \text{ visits both } A \,\&\, B) \preceq \sum_{z \in \mathbb{Z}^d} G_{A\cup B}(0, z) \cdot \left(\mathbf{p}_A(z)\mathbf{q}_B(z) + \mathbf{p}_B(z)\mathbf{q}_A(z) + P(\mathcal{S}_z \text{ visits both } A \,\&\, B)\right),$$

it suffices to show:

$$\sum_{z\in\mathbb{Z}^d} G_{A\cup B}(0,z)\mathbf{p}_A(z)\mathbf{q}_B(z) \preceq \mathbf{q}_A(0)\mathbf{q}_B(0); \tag{2.3.30}$$

$$\sum_{z\in\mathbb{Z}^d} G_{A\cup B}(0,z)\mathbf{p}_B(z)\mathbf{q}_A(z) \preceq \mathbf{q}_A(0)\mathbf{q}_B(0); \tag{2.3.31}$$

$$\sum_{z\in\mathbb{Z}^d} G_{A\cup B}(0,z)P(\mathcal{S}_z \text{ visits both } A\,\&\,B) \preceq \mathbf{q}_A(0)\mathbf{q}_B(0). \tag{2.3.32}$$

Note that by monotonicity, $G_{A\cup B}(x,y) \le \min\{G_A(x,y), G_B(x,y)\}$. For (2.3.30), we have:

$$\begin{aligned}
&\sum_{z\in\mathbb{Z}^d} G_{A\cup B}(0,z)\mathbf{p}_A(z)\mathbf{q}_B(z)\\
&\quad=\sum_{z\in\widehat{B}} G_{A\cup B}(0,z)\mathbf{p}_A(z)\mathbf{q}_B(z) + \sum_{z\in\widehat{B}^c} G_{A\cup B}(0,z)\mathbf{p}_A(z)\mathbf{q}_B(z)\\
&\quad\preceq\sum_{z\in\widehat{B}} G_{B}(0,z)\mathbf{p}_A(b)\mathbf{q}_B(z) + \sum_{z\in\widehat{B}^c} G_{A}(0,z)\mathbf{p}_A(z)\mathbf{q}_B(0)\\
&\quad\preceq(\mathrm{diam}(\widehat{B}))^2\mathbf{q}_B(0)\mathbf{p}_A(b) + \mathbf{q}_A(0)\mathbf{q}_B(0) \preceq \mathbf{q}_A(0)\mathbf{q}_B(0).
\end{aligned}$$

Similarly one can show (2.3.31).

We just need to show (2.3.32). We first argue that:

$$P(\mathcal{S}_z \text{ visits both } A\,\&\,B) \preceq \begin{cases} \mathbf{p}_A(b)\mathbf{q}_B(z) + \mathbf{p}_B(0)\mathbf{q}_A(z); & \text{when } z\in\mathcal{C}(4M);\\ \mathbf{p}_A(z)\mathbf{q}_B(a); & \text{when } z\notin\mathcal{C}(4M). \end{cases} \tag{2.3.33}$$

By (2.3.19), we need to estimate:

$$\sum_{w\in\mathbb{Z}^d} G_{A\cup B}(z,w)\mathbf{p}_A(w)\mathbf{p}_B(w).$$

When $z \in \mathcal{C}(4M)$, we have

$$\begin{aligned}
&\sum_{w\in\mathbb{Z}^d} G_{A\cup B}(z,w)\mathbf{p}_A(w)\mathbf{p}_B(w) \\
&\qquad= \sum_{w\in\widehat{B}} G_{A\cup B}(z,w)\mathbf{p}_A(w)\mathbf{p}_B(w) + \sum_{w\in\widehat{B}^c} G_{A\cup B}(z,w)\mathbf{p}_A(w)\mathbf{p}_B(w) \\
&\qquad\preceq \sum_{w\in\widehat{B}} G_A(z,w)\mathbf{p}_A(b)\mathbf{p}_B(w) + \sum_{w\in\widehat{B}^c} G_B(z,w)\mathbf{p}_A(w)\mathbf{p}_B(0) \\
&\qquad\preceq \mathbf{p}_A(b)\mathbf{q}_B(z) + \mathbf{p}_B(0)\mathbf{q}_A(z).
\end{aligned}$$

When $z \notin \mathcal{C}(4M)$, let $\widehat{C} = \mathcal{C}(3M)$. We divide the sum into three parts:

$$\sum_{w\in\widehat{B}}, \sum_{w\in\widehat{C}\setminus\widehat{B}}, \sum_{w\in\widehat{C}^c}.$$

$$\begin{aligned}
\sum_{w\in\widehat{B}} G_{A\cup B}(z,w)\mathbf{p}_A(w)\mathbf{p}_B(w) &\preceq \sum_{w\in\widehat{B}} G_B(z,w)\mathbf{p}_A(b)\mathbf{p}_B(w) \\
&\preceq (\mathrm{diam}\widehat{B})^2\mathbf{p}_B(z)\mathbf{p}_A(b) \asymp (\rho(a,b))^2\frac{\mathrm{BCap}(A)\mathrm{BCap}(B)}{(\rho(z,B))^{d-2}(\rho(a,b)^{d-2})} \\
&\asymp \mathbf{p}_A(z)\mathbf{q}_B(a); \\
\sum_{w\in\widehat{C}\setminus\widehat{B}} G_{A\cup B}(z,w)\mathbf{p}_A(w)\mathbf{p}_B(w) &\preceq \sum_{w\in\widehat{C}\setminus\widehat{B}} G_A(z,w)\mathbf{p}_A(w)\mathbf{p}_B(a) \\
&\preceq (\mathrm{diam}\widehat{C})^2\mathbf{p}_A(z)\mathbf{p}_B(a) \asymp \mathbf{p}_A(z)\mathbf{q}_B(a); \\
\sum_{w\in\widehat{C}^c} G_{A\cup B}(z,w)\mathbf{p}_A(w)\mathbf{p}_B(w) &\preceq \sum_{w\in\widehat{C}^c} g(z,w)\frac{\mathrm{BCap}(A)\mathrm{BCap}(B)}{|w-a|^{d-2}|w-b|^{d-2}} \\
&\asymp \mathrm{BCap}(A)\mathrm{BCap}(B)\sum_{w\in\widehat{C}^c}\frac{1}{|w-z|^{d-2}|w-a|^{2d-4}} \\
&\overset{(*)}{\preceq} \frac{\mathrm{BCap}(A)\mathrm{BCap}(B)}{|z-a|^{d-2}|b-a|^{d-4}} \asymp \mathbf{p}_A(z)\mathbf{q}_B(a).
\end{aligned}$$

Combining all three above, we get (2.3.33). Note that for $(*)$, we use:

$$
\begin{aligned}
&\sum_{w\in\widehat{C}^c}\frac{1}{|w-z|^{d-2}|w-a|^{2d-4}}\\
&\leq \sum_{\|w-z\|\leq\|z\|/8}\frac{1}{|w-z|^{d-2}|w-a|^{2d-4}}+\\
&\qquad\qquad\sum_{\|w-z\|\geq\|z\|/8,w\in\widehat{C}^c}\frac{1}{|w-z|^{d-2}|w-a|^{2d-4}}\\
&\preceq \sum_{\|w-z\|\leq\|z\|/8}\frac{1}{|w-z|^{d-2}|z-a|^{2d-4}}+\sum_{\|w-z\|\geq\|z\|/8,w\in\widehat{C}^c}\frac{1}{|z|^{d-2}|w-a|^{2d-4}}\\
&\preceq \frac{|z|^2}{|z-a|^{2d-4}}+\frac{1}{|z|^{d-2}}\sum_{w\in\widehat{C}^c}\frac{1}{|w-a|^{2d-4}}\asymp\frac{1}{|z|^{2d-6}}+\frac{1}{|z|^{d-2}}\sum_{n\geq 3M}\frac{n^{d-1}}{n^{2d-4}}\\
&\asymp \frac{1}{|z|^{2d-6}}+\frac{1}{|z|^{d-2}}\frac{1}{M^{d-4}}\asymp\frac{1}{|z|^{d-2}}\frac{1}{M^{d-4}}\asymp\frac{1}{|z-a|^{d-2}|b-a|^{d-4}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{z\in\mathbb{Z}^d}G_{A\cup B}(0,z)P(\mathcal{S}_z\text{ visits both }A\,\&\,B)\\
&\preceq \sum_{z\in\mathcal{C}(4M)}G_{A\cup B}(0,z)(\mathbf{p}_A(b)\mathbf{q}_B(z)+\mathbf{p}_B(0)\mathbf{q}_A(z))+\\
&\qquad\qquad\sum_{z\in\mathcal{C}(4M)^c}G_A(0,z)\mathbf{p}_A(z)\mathbf{q}_B(a)\\
&\preceq M^2\mathbf{p}_A(b)\mathbf{q}_B(0)+M^2\mathbf{p}_B(0)\mathbf{q}_A(0)+\mathbf{q}_A(0)\mathbf{q}_B(a)\asymp\mathbf{q}_B(0)\mathbf{q}_A(0).
\end{aligned}
$$

This is just (2.3.32) and we finish the proof.

2.4 The critical dimension: d=4

In this section, we focus on the critical dimension $d=4$. Note that now (2.1.2) is just:

$$g(x)\sim a_4\|x\|^{-2}; \tag{2.4.1}$$

where $a_4=1/(8\pi^2\sqrt{\det Q})$ with Q being the covariant matrix of θ.

For some technical reasons, we assume further that, in this section, θ has finite exponential moments, i.e. for some $\lambda > 0$,

$$\sum_{z \in \mathbb{Z}^4} \theta(z) \cdot \exp(\lambda |z|) < \infty.$$

2.4.1 An upper bound

In this subsection, we construct a weaker result which will be used in the proof of Theorem 1.3.12:

Theorem 2.4.1.

$$\mathbf{p}_{\{0\}}(x) \preceq (|x|^2 \log |x|)^{-1}. \tag{2.4.2}$$

Remark 2.4.1. *On the other hand, the reversed inequality can be obtained by the second moment method. This process is similar to and easier than the proof for the lower bound in Theorem 2.2.12.*

The idea of proof is as follows. From simple calculation one can see that the expectation of the times of visiting x is $g(x) \asymp |x|^{-2}$. If conditioned on visiting, the expectation of the visiting times is of order $\log |x|$, then we can get (2.4.2). In fact, we will show that this is true with high probability.

Let N_0 be the number of times of visiting 0. We need to estimate $\mathbb{E}(N_0 | \mathcal{S}_x \text{ visits } K \text{ via } \gamma)$. For any finite path γ, define

$$\mathcal{N}_0(\gamma) = \sum_{i=0}^{|\gamma|} N(\gamma(i)) g(\gamma(i), 0), \tag{2.4.3}$$

where $N(x) = N_K(x) = \mathbb{E}\mu_x$ (see (2.2.25), (2.2.27) and set $K = \{0\}$).

By Proposition 2.2.11, we have:

$$\mathbb{E}(N_0 | \mathcal{S}_x \text{ visits } 0 \text{ via } \gamma) = \mathcal{N}_0(\gamma).$$

For $N(x)$, we have:

$$\begin{aligned}N(x)=&\mathbb{E}\mu_x=\frac{\sum_{l\geq 0,m\geq 0} m\mu(l+m+1)(\widetilde{r}(x))^l}{\sum_{l\geq 0,m\geq 0}\mu(l+m+1)(\widetilde{r}(x))^l}\\ \geq&\frac{\sum_{l=0,m\geq 0} m\mu(m+1)}{\sum_{l\geq 0,m\geq 0}\mu(l+m+1)}\\ =&\frac{\sum_{m\geq 0} m\mu(m+1)}{\sum_{l\geq 0,m\geq 0}\mu(l+m+1)}=\mu(0).\end{aligned}$$

Write $g(\gamma)=\sum_{i=0}^{|\gamma|} g(\gamma(i))$. Then we have:

$$\mathcal{N}_0(\gamma)\geq \mu(0)g(\gamma)\asymp g(\gamma). \tag{2.4.4}$$

We need the following lemma and postpone its proof:

Lemma 2.4.2. *There exists a $c>0$, such that for any x, we have:*

$$\sum_{\gamma:x\to 0,g(\gamma)\leq c\log|x|}\mathbf{b}(\gamma)\preceq |x|^{-2.1}. \tag{2.4.5}$$

Now we start the proof of Theorem 2.4.1. First we have

$$\mathbb{E}N_0=g(x,0)\asymp |x|^{-2}.$$

Hence,

$$\begin{aligned}|x|^{-2}\asymp\mathbb{E}N_0=&\sum_{\gamma:x\to 0}\mathbf{b}(\gamma)\mathcal{N}_0(\gamma)\overset{(2.4.4)}{\succeq}\sum_{\gamma:x\to 0}\mathbf{b}(\gamma)g(\gamma)\\ \geq&\sum_{\gamma:x\to 0,g(\gamma)\geq c\log|x|}\mathbf{b}(\gamma)g(\gamma)\succeq\sum_{\gamma:x\to 0,g(\gamma)\geq c\log|x|}\mathbf{b}(\gamma)\log|x|.\end{aligned}$$

Therefore we have:

$$\sum_{\gamma:x\to 0,g(\gamma)\geq c\log|x|}\mathbf{b}(\gamma)\preceq 1/(|x|^2\log|x|).$$

Then we have:

$$\mathbf{p}_{\{0\}}(x) = \sum_{\gamma:x\to 0} \mathbf{b}(\gamma) = \sum_{\gamma:x\to 0, g(\gamma)\geq c\log|x|} \mathbf{b}(\gamma) + \sum_{\gamma:x\to 0, g(\gamma)<c\log|x|} \mathbf{b}(\gamma)$$
$$\preceq 1/|x|^{2.1} + 1/(|x|^2\log|x|) \preceq 1/(|x|^2\log|x|).$$

We still need to show (2.4.5). Note that $\mathbf{b}(\gamma) \leq \mathbf{s}(\gamma)$. Hence (2.4.5) can be obtained by

Proposition 2.4.3. *There exist c_1, c_2 such that for $x \in \mathbb{Z}^4$ with $|x|$ sufficiently large,*

$$P(\tau_x < \infty, \sum_{i=0}^{\tau_x} g(S_i) \leq c_1 \log|x|) \leq c_2|x|^{-2.1},$$

where $(S_i)_{i\in\mathbb{N}}$ is a random walk starting from 0 with distribution θ^- and τ_x is the hitting time for x.

This proposition is an adjusted version of Lemma 10.1.2 (a) in [11]. It is assumed there that θ has finite support which is stronger than our assumptions, though its conclusion is also stronger than ours. The argument is similar to the one there with small adjustments. We mention the main difference here and leave the details to the reader. It suffices to prove:

$$P(\sum_{i=0}^{\tau_n} g(S_i) \leq c_1 \log n) \leq c_2 n^{-2.1}, \tag{2.4.6}$$

where $\tau_n = \min\{k \geq 0 : |S_k| \geq n\}$.

Let $N = \lfloor n^{0.9} \rfloor$. Let A be the event that $|X_i| \leq N$, for $i = 1, 2, \ldots, 2n^2 \wedge \tau_n$ (where $X_i = S_i - S_{i-1}$). Note that $P(A^c) \preceq n^{-2.1}$. When A happens, the range of the random walk is bounded by N for the first $2n^2$ steps. Since only first $2n^2$ steps are bounded, we need to change the stopping times there a bit. Let $\xi^0 = 0$, $\xi^i = \min\{k : |S_k| \geq 2^i N\} \wedge (\xi^{i-1} + (2^i N)^2)$, for $i = 1, 2, \ldots, L$, where $L = \max\{k : 2^k N \leq n\} \asymp \log n$. Now (2.4.6) can be obtained by following the argument of the proof of Lemma 10.1.2 (a) in [11]
.

2.4.2 The visiting probability

The main goal of this subsection is to prove Theorem 1.3.12. In this subsection and the next, we fix a finite nonempty subset $K \subset\subset \mathbb{Z}^4$ and therefore the corresponding constants may also depend on K.

The first step is to construct the following estimate of the Green function:

Lemma 2.4.4. *For any $\alpha \in (0, 1/2)$, we have:*

$$\lim_{x,y\to\infty:\|x\|/(\log\|x\|)^\alpha \le \|y\| \le \|x\|\cdot(\log\|x\|)^\alpha} \frac{G_K(x,y)}{g(x,y)} = 1. \tag{2.4.7}$$

Remark 2.4.2. *In supercritical dimensions, we have $G_K(x,y) \sim g(x,y)$ (see Lemma 2.2.4). In the critical dimension, this holds only when x, y are not too far away from each other, compared with their norms. We will give a more precise asymptotic behavior of G_K in next subsection.*

Proof of Lemma 2.4.4. We use the same idea in the proof for a similar form in supercritical dimension (see Section 2.2.3). Since $\alpha < 1/2$, we can pick up some $\beta, \epsilon > 0$, such that $\epsilon + 2\alpha + 2\beta < 1$. Without loss of generality, we assume $\|y\| \ge \|x\|$. Let $r = \|x\|/\log^\beta\|x\|$ and

$$\Gamma_1 = \{\gamma : x \to y |\ |\gamma| \ge (\log\|x\|)^\epsilon \|x-y\|^2\};$$
$$\Gamma_2 = \{\gamma : x \to y | \gamma \text{ visits } \mathcal{C}(r)\}.$$

We just need to check: (when $\|x\| \to \infty$)

$$\sum_{\gamma\in\Gamma_1} \mathbf{s}(\gamma)/g(x,y) \to 0; \quad \sum_{\gamma\in\Gamma_2} \mathbf{s}(\gamma)/g(x,y) \to 0;$$
$$\mathbf{b}(\gamma)/\mathbf{s}(\gamma) \to 1, \text{ for any } \gamma : x \to y, \notin \Gamma_1 \cup \Gamma_2.$$

The first one follows from Lemma 2.1.2. The second one can be obtained by: (let $B=\mathcal{C}(r)$):

$$
\begin{aligned}
\sum_{\gamma\in\Gamma_2}\mathbf{s}(\gamma)=&\sum_{a\in B}\mathcal{H}^{B^c}(x,a)g(a,y)\asymp\sum_{a\in B}\mathcal{H}^{B^c}(x,a)\|y\|^{-2}\\
=&P(S_x \text{ visits } B)\cdot\|y\|^{-2}\asymp(r/\|x\|)^2\|y\|^{-2}\\
\preceq&(\log\|x\|)^{-2\beta}\|x-y\|^{-2}\ll g(x,y).
\end{aligned}
$$

Note that the estimate of $P(S_x \text{ visits } \mathcal{C}(r))\asymp(r/\|x\|)^2$ is standard, and for the second last inequality we use $\|y\|\geq(\|x\|+\|y\|)/2\succeq\|x-y\|$.

For the third one, note that in the critical dimension $d=4$, by (2.4.2) and (2.2.18), the killing function $\mathbf{k}(z)=\mathbf{r}_K(z)\asymp\mathbf{p}_K(z)\preceq 1/(\|z\|^2\log\|z\|)$. Hence, we have:
for any $\gamma:x\to y,\notin\Gamma_1\cup\Gamma_2$,

$$
\begin{aligned}
\mathbf{b}(\gamma)/\mathbf{s}(\gamma)=&\prod_{i=0}^{|\gamma|-1}(1-\mathbf{k}(\gamma(i)))\geq(1-c/(r^2\log r))^{|\gamma|}\geq 1-c|\gamma|/(r^2\log r)\\
\geq&1-c(\log\|x\|)^{\epsilon}\|y\|^2/((\|x\|/\log^{\beta}\|x\|)^2(\log\|x\|))\\
\geq&1-c(\log\|x\|)^{\epsilon}(\|x\|\log^{\alpha}\|x\|)^2/((\|x\|/\log^{\beta}\|x\|)^2(\log\|x\|))\\
\geq&1-c(\log\|x\|)^{\epsilon+2\alpha+2\beta}/\log\|x\|\to 1.
\end{aligned}
$$

□

Let N be the number of times of visiting K. We need to estimate $\mathbb{E}(N|\mathcal{S}_x \text{ visits } K \text{ via } \gamma)$. For any finite path γ, define

$$
\mathcal{N}(\gamma)=\sum_{i=0}^{|\gamma|}N(\gamma(i))g(\gamma(i),K);\quad \mathcal{N}^-(\gamma)=\sum_{i=0}^{|\gamma|-1}N(\gamma(i))g(\gamma(i),K). \tag{2.4.8}
$$

By Proposition 2.2.11, we have:

$$
\mathbb{E}(N|\mathcal{S}_x \text{ visits } K \text{ via } \gamma))=\mathcal{N}(\gamma).
$$

Hence, we have:

$$\sum_{\gamma:x\to K} \mathbf{b}(\gamma)\mathcal{N}(\gamma) = g(x,K) \sim a_4|K|\|x\|^{-2}. \tag{2.4.9}$$

The main step is to control the sum of the escape probabilities:

Proposition 2.4.5.

$$\sum_{\gamma:\mathcal{C}(2n)\setminus\mathcal{C}(n)\to K,\gamma\subseteq\mathcal{C}(n)} \mathbf{b}(\gamma) \sim \frac{4\pi^2\sqrt{\det Q}}{\sigma^2}\frac{1}{\log n}. \tag{2.4.10}$$

In order to prove this proposition, we need two lemmas about random walks. They are adjusted versions of Lemma 17 and Lemma 18 in [14]. As before, write $(S_j)_{j\in\mathbb{N}}$ for the random walk (starting from 0). Let τ_n be the first visiting time of $\mathcal{C}(n)^c$ by the random walk and $h(x):\mathbb{Z}^4\to\mathbb{R}^+$ is a fixed positive function satisfying $h(x)\sim a_4\|x\|^{-2}$.

Lemma 2.4.6. *For $p=1,2$, there exists a constant $C(p)$ (also depending on h) such that, for every $n\geq 2$,*

$$\mathbb{E}(\sum_{j=0}^{\tau_n} h(S_j))^p \leq C(p)(\log n)^p. \tag{2.4.11}$$

.

Lemma 2.4.7. *For every $\alpha,p>0$, there exists a constant $C_\alpha(p)$ (also depending on h) such that, for every $n\geq 2$, we have*

$$P(|\sum_{k=0}^{\tau_n} h(S_j) - 4a_4\log n| \geq \alpha\log n) \leq C_\alpha(p)(\log n)^{-p}. \tag{2.4.12}$$

In fact, we apply both lemmas for the reversed random walk (that is, with jump distribution θ^-) other than the original random walk. Let $h(x) = 2N(x)g(x,K)/(\sigma^2|K|)$. Recall that $N(x) = E\mu_x \sim \sigma^2/2$ and

$\mathcal{N}(\gamma) = \sum \sigma^2 |K| h(\gamma(i))/2)$. Hence, for any $a \in K$ we have:

$$\sum_{\gamma:\mathcal{C}(n)^c \to a, \gamma \subseteq \mathcal{C}(n)} \mathbf{s}(\gamma)(\mathcal{N}(\gamma))^2 \preceq (\log n)^2,$$
$$\sum_{\gamma:\mathcal{C}(n)^c \to a, \gamma \subseteq \mathcal{C}(n), |\mathcal{N}(\gamma) - 2a_4|K|\sigma^2 \log n| \geq \alpha \log n} \mathbf{s}(\gamma) \leq C_\alpha(p)(\log n)^{-p}.$$

By monotonicity and summation, we get:

$$\sum_{\gamma:\mathcal{C}(n)^c \to K, \gamma \subseteq \mathcal{C}(n) \setminus K} \mathbf{s}(\gamma)(\mathcal{N}(\gamma))^2 \preceq (\log n)^2, \tag{2.4.13}$$
$$\sum_{\gamma:\mathcal{C}(n)^c \to K, \gamma \subseteq \mathcal{C}(n) \setminus K, |\mathcal{N}(\gamma) - 2a_4|K|\sigma^2 \log n| \geq \alpha \log n} \mathbf{s}(\gamma) \preceq C_\alpha(p)(\log n)^{-p}. \tag{2.4.14}$$

Let us make some comments about the proofs. Lemma 18 in [14] states that

$$P(|\sum_{k=0}^{n} g(S_j) - 2a_4 \log n| \geq \alpha \log n) \leq C_\alpha (\log n)^{-3/2}.$$

where g is the Green function. Their argument is to derive an analogous result for Brownian motion and then to transfer this result to the random walk via the strong invariance principle. This argument also works here with small adjustments. Note that it is assumed there that the jump distribution θ is symmetric (besides having exponential tail). However if one checks the proof there, one can see that the assumption of symmetry is not needed and $g(x)$ can be replaced by any $h(x)$ satisfying $h(x) \sim a_4 \|x\|^{-2}$. Moreover, the exponent $3/2$ can be replaced by any positive constant p with minor modifications. Combing this with the fact that for any fixed $\epsilon > 0$, $P(\tau_n \notin [n^{2-\epsilon}, n^{2+\epsilon}]) = o((\log n)^{-p})$, one can get Lemma 2.4.7.

For Lemma 2.4.6, we give a direct proof here:

Proof of Lemma 2.4.6. **For p=1**,

$$\begin{aligned}\mathbb{E}(\sum_{j=0}^{\tau_n} h(S_j)) \preceq \sum_{z\in\mathcal{C}(n)} h(z)\mathbb{E}(\sum_{j=0}^{\tau_n}\mathbf{1}_{S_j=z}) \preceq \sum_{z\in\mathcal{C}(n)} |z|^{-2}\mathbb{E}(\sum_{j=0}^{\infty}\mathbf{1}_{S_j=z})\\ = \sum_{z\in\mathcal{C}(n)} |z|^{-2}g(0,z) \asymp \sum_{z\in\mathcal{C}(n)} |z|^{-4} \asymp \log n.\end{aligned}$$

For p=2,

$$\begin{aligned}&\mathbb{E}(\sum_{j=0}^{\tau_n} h(S_j))^2 \preceq \mathbb{E}(\sum_{z\in\mathcal{C}(n)} h(z)\sum_{j=0}^{\tau_n}\mathbf{1}_{S_j=z})^2 \preceq \mathbb{E}(\sum_{z\in\mathcal{C}(n)} |z|^{-2}\sum_{j=0}^{\infty}\mathbf{1}_{S_j=z})^2\\ &= \sum_{z,w\in\mathcal{C}(n)} |z|^{-2}|w|^{-2}\mathbb{E}(\sum_{j=0}^{\infty}\mathbf{1}_{S_j=z}\sum_{i=0}^{\infty}\mathbf{1}_{S_i=w}).\end{aligned}$$

Write $A_x = \sum_{j=0}^{\infty}\mathbf{1}_{S_j=x}$ and $A = A_z + A_w$. We point out that

$$\mathbb{E}(A_zA_w) \preceq (|z|^{-2}+|w|^{-2})|z-w|^{-2}. \tag{2.4.15}$$

If so, note that

$$\begin{aligned}&\sum_{z,w\in\mathcal{C}(n)} |z|^{-2}|w|^{-2}(|z|^{-2}+|w|^{-2})|z-w|^{-2}\\ \preceq &\sum_{z,w\in\mathcal{C}(n):|z|\le|w|} |z|^{-4}|w|^{-2}|z-w|^{-2}\\ \le &\sum_{w\in\mathcal{C}(n)}(\sum_{z:|z|\le|w|,|z-w|\ge|w|/2} + \sum_{z:|z|\le|w|,|z-w|\le|w|/2})|z|^{-4}|w|^{-2}|z-w|^{-2}\\ \preceq &\sum_{w\in\mathcal{C}(n)}(\sum_{z:|z|\le|w|,|z-w|\ge|w|/2}|z|^{-4}|w|^{-4} + \sum_{z:|z|\le|w|,|z-w|\le|w|/2}|w|^{-6}|z-w|^{-2})\\ \preceq &\sum_{w\in\mathcal{C}(n)}((\log|w|)|w|^{-4}+|w|^{-4}) \asymp (\log n)^2,\end{aligned}$$

and then one can get $\mathbb{E}(\sum_{j=0}^{\tau_n} h(S_j))^2 \preceq (\log n)^2$. We now only need to show (2.4.15). Without loss of generality, assume $z \neq w$ (the case $z = w$ can be addressed similarly with small adjustments). Note that

$$\mathbb{E}(A_z A_w) \le \mathbb{E}(A^2; A_z > 0, A_w > 0) \asymp \sum_{k\ge 2} kP(A \ge k, A_z > 0, A_w > 0).$$

By Markov property, one can see that:

$$\begin{aligned} &P(A \ge k, A_z > 0, A_w > 0) \le \\ P(A_z > 0)&((k-1)P_z(A_w > 0)c^{k-2}) + P(A_w > 0)((k-1)P_w(A_z > 0)c^{k-2}), \end{aligned}$$

where we write P_x for the law of random walk starting from x and

$$c = \sup_{x\ne y\in\mathbb{Z}^4} P_x(A_x + A_y > 1) < 1.$$

Hence, we have:

$$\begin{aligned} &\sum_{k\ge 2} kP(A \ge k, A_z > 0, A_w > 0) \\ \le&(\sum_{k\ge 2} k(k-1)c^{k-2})(P(A_z > 0)P_z(A_w > 0) + P(A_w > 0)P_w(A_z > 0)) \\ \preceq& P(A_z > 0)P_z(A_w > 0) + P(A_w > 0)P_w(A_z > 0) \\ \asymp&(|z|^{-2} + |w|^{-2})|z - w|^{-2}. \end{aligned}$$

□

Proof of Proposition 2.4.5. We first show the following weaker result:

Lemma 2.4.8.

$$\sum_{\gamma:\mathcal{C}(2n)\backslash\mathcal{C}(n)\to K, \gamma\subseteq\mathcal{C}(n)} \mathbf{b}(\gamma) \preceq (\log n)^{-1}, \quad \text{as } n \to \infty. \tag{2.4.16}$$

Proof. By (2.4.2) and (2.2.7), we have:

$$\sum_{\gamma:x\to K} \mathbf{b}(\gamma) \preceq (\|x\|^2 \log \|x\|)^{-1}. \tag{2.4.17}$$

Pick some $x \in \mathbb{Z}^d$ such that $n = \lfloor \|x\| (\log \|x\|)^{-1/4} \rfloor$. Let $B = \mathcal{C}(n)$ and $B_1 = \mathcal{C}(2n)$. By the First-Visit Lemma, we have:

$$\sum_{\gamma : x \to K} \mathbf{b}(\gamma) = \sum_{a \in K} \sum_{z \in B^c} G_K(x,z) \mathcal{H}^B_{\mathbf{k}}(z,a) \geq \sum_{a \in K} \sum_{z \in B_1 \setminus B} G_K(x,z) \mathcal{H}^B_{\mathbf{k}}(z,a)$$
$$\overset{(2.4.7)}{\succeq} \sum_{a \in K} \sum_{z \in B_1 \setminus B} g(x,z) \mathcal{H}^B_{\mathbf{k}}(z,a) \succeq \|x\|^{-2} \sum_{\gamma : \mathcal{C}(2n) \setminus \mathcal{C}(n) \to K, \gamma \subseteq \mathcal{C}(n)} \mathbf{b}(\gamma).$$

Combining this with (2.4.17) gives (2.4.16). □

We need to transfer (2.4.9) to the following form:

Lemma 2.4.9.

$$\lim_{n \to \infty} \sum_{\gamma : \mathcal{C}(2n) \setminus \mathcal{C}(n) \to K, \gamma \subseteq \mathcal{C}(n)} \mathcal{N}(\gamma) \mathbf{b}(\gamma) = |K|. \tag{2.4.18}$$

Proof of (2.4.18). Pick some $x \in \mathbb{Z}^d$ such that $n = \lfloor \|x\| (\log \|x\|)^{-1/4} \rfloor$. Let $B = \mathcal{C}(n)$ and $B_1 = \mathcal{C}(2n)$. By decomposing γ at the last step in B, one can get:

$$\sum_{\gamma : x \to K} \mathbf{b}(\gamma) \mathcal{N}(\gamma) = \sum_{z \in B^c} \sum_{\gamma_1 : x \to z} \sum_{\gamma_2 : z \to K, \gamma_2 \subseteq B} \mathbf{b}(\gamma_1) \mathbf{b}(\gamma_2) (\mathcal{N}^-(\gamma_1) + \mathcal{N}(\gamma_2))$$
$$= \sum_{z \in B^c} \sum_{\gamma_1 : x \to z} \sum_{\gamma_2 : z \to K, \gamma_2 \subseteq B} \mathbf{b}(\gamma_1) \mathbf{b}(\gamma_2) \mathcal{N}^-(\gamma_1) +$$
$$\sum_{z \in B^c} \sum_{\gamma_1 : x \to z} \sum_{\gamma_2 : z \to K, \gamma_2 \subseteq B} \mathbf{b}(\gamma_1) \mathbf{b}(\gamma_2) \mathcal{N}(\gamma_2)$$
$$= \sum_{z \in B^c} \sum_{\gamma_2 : z \to K, \gamma_2 \subseteq B} \mathbf{b}(\gamma_2) \sum_{\gamma_1 : x \to z} \mathbf{b}(\gamma_1) \mathcal{N}^-(\gamma_1) +$$
$$\sum_{z \in B^c} \sum_{\gamma_2 : z \to K, \gamma_2 \subseteq B} \mathbf{b}(\gamma_2) \mathcal{N}(\gamma_2) \sum_{\gamma_1 : x \to z} \mathbf{b}(\gamma_1).$$

We argue that the first term is negligible:

$$\sum_{z \in B^c} \sum_{\gamma_2 : z \to K, \gamma_2 \subseteq B} \mathbf{b}(\gamma_2) \sum_{\gamma_1 : x \to z} \mathbf{b}(\gamma_1) \mathcal{N}^-(\gamma_1) \ll \|x\|^{-2}. \tag{2.4.19}$$

Note that

$$\sum_{\gamma:x\to z}\mathbf{b}(\gamma)\mathcal{N}^-(\gamma)\le\sum_{w\in\mathbb{Z}^d}N(w)g(w,K)\sum_{\gamma:x\to z}\mathbf{b}(\gamma)\sum_{i=0}^{|\gamma|}\mathbf{1}_{\gamma(i)=w}$$
$$\preceq\sum_{w\in\mathbb{Z}^d}|w|^{-2}\sum_{\gamma:x\to w}\mathbf{b}(\gamma)\sum_{\gamma:w\to z}\mathbf{b}(\gamma)\le\sum_{w\in\mathbb{Z}^d}|w|^{-2}g(x,w)g(w,z).$$

In order to estimate the term above, we need the following easy lemma whose proof we postpone

Lemma 2.4.10. *For any $a,b,c\in\mathbb{Z}^4$, we have:*

$$\sum_{z\in\mathbb{Z}^4}|z-a|^{-2}|z-b|^{-2}|z-c|^{-2}\preceq\frac{1\vee\log(M/m)}{M^2},\tag{2.4.20}$$

where $M=\max\{|a-b|,|b-c|,|c-a|\}$ and $m=\min\{|a-b|,|b-c|,|c-a|\}$.

By this lemma, when $z\in B_1\setminus B$, $\sum_{\gamma:x\to z}\mathbf{b}(\gamma)\mathcal{N}^-(\gamma)\preceq\frac{\log(\|x\|/n)}{\|x\|^2}$. Together with (2.4.16), we have

$$\sum_{z\in B_1\setminus B}\sum_{\gamma_2:z\to K,\gamma_2\subseteq B}\mathbf{b}(\gamma_2)\sum_{\gamma_1:x\to z}\mathbf{b}(\gamma_1)\mathcal{N}^-(\gamma_1)\ll\|x\|^{-2}.$$

Also by Lemma 2.4.10 when $z\in B_1^c$, $\sum_{\gamma:x\to z}\mathbf{b}(\gamma)\mathcal{N}^-(\gamma)\preceq\frac{\log(\|x\|)}{\|x\|^2}$. On the other hand, by the Overshoot Lemma, we have $\sum_{z\in B_1^c}\sum_{\gamma_2:z\to K,\gamma_2\subseteq B}\mathbf{b}(\gamma_2)\preceq n^{-4}$. Hence,

$$\sum_{z\in B_1^c}\sum_{\gamma_2:z\to K,\gamma_2\subseteq B}\mathbf{b}(\gamma_2)\sum_{\gamma_1:x\to z}\mathbf{b}(\gamma_1)\mathcal{N}^-(\gamma_1)\ll\|x\|^{-2}.$$

This completes the proof of (2.4.19). Combining (2.4.19) with (2.4.9) gives:

$$\sum_{z\in B^c}\sum_{\gamma_2:z\to K,\gamma_2\subseteq B}\mathbf{b}(\gamma_2)\mathcal{N}(\gamma_2)\sum_{\gamma_1:x\to z}\mathbf{b}(\gamma_1)\sim a_4|K|\|x\|^{-2}.$$

Now we aim to show

$$\sum_{z\in B_1^c}\sum_{\gamma_2:z\to K,\gamma_2\subseteq B}\mathbf{b}(\gamma_2)\mathcal{N}(\gamma_2)\sum_{\gamma_1:x\to z}\mathbf{b}(\gamma_1)\ll a_4|K|\|x\|^{-2}. \tag{2.4.21}$$

If so, then we have:

$$\sum_{z\in B_1\setminus B}\sum_{\gamma_2:z\to K,\gamma_2\subseteq B}\mathbf{b}(\gamma_2)\mathcal{N}(\gamma_2)\sum_{\gamma_1:x\to z}\mathbf{b}(\gamma_1)\sim a_4|K|\|x\|^{-2}, \tag{2.4.22}$$

and combining this with Lemma 2.4.4 gives (2.4.18). Since $\sum_{\gamma_1:x\to z}\mathbf{b}(\gamma_1)=G_K(x,z)\preceq 1$ and $\mathbf{b}(\gamma)\le\mathbf{s}(\gamma)$. It suffices to show:

$$\sum_{\gamma:B_1^c\to K,\gamma\subseteq B}\mathbf{s}(\gamma)\mathcal{N}(\gamma)\ll\|x\|^{-2}. \tag{2.4.23}$$

By Cauchy-Schwarz inequality, one can get:

$$\sum_{\gamma:B_1^c\to K,\gamma\subseteq B}\mathbf{s}(\gamma)\mathcal{N}(\gamma)\le(\sum_{\gamma:B_1^c\to K,\gamma\subseteq B}\mathbf{s}(\gamma))^{1/2}(\sum_{\gamma:B_1^c\to K,\gamma\subseteq B}\mathbf{s}(\gamma)(\mathcal{N}(\gamma))^2)^{1/2}.$$

By the Overshoot Lemma, the first term in the right hand side decays faster than any polynomial of n. On the other hand, due to (2.4.13), the second term in the right hand side is less than $\log n$ by a constant multiplier. Combining both gives (2.4.23) and finishes the proof of (2.4.18). $\square$

Now we are ready to prove Proposition 2.4.5. Fix any small $\epsilon>0$. Let $n=\|x\|/(\log\|x\|)^{1/4}$,

$$\Gamma=\{\gamma:\mathcal{C}(2n)\setminus\mathcal{C}(n)\to K,\gamma\subseteq\mathcal{C}(n)\setminus K\},$$
$$\Gamma_1=\{\gamma\in\Gamma:|\mathcal{N}(\gamma)-2a_4\sigma^2|K|\log n|>\epsilon\log n\}\quad\text{and}\quad\Gamma_2=\Gamma\setminus\Gamma_1.$$

By (2.4.14), we have: (when $\|x\|$ and hence n are large)

$$\sum_{\gamma\in\Gamma_1}\mathbf{s}(\gamma)\preceq(\log n)^{-4}. \tag{2.4.24}$$

Hence, we have (when n is large):

$$\sum_{\gamma\in\Gamma_1}\mathbf{b}(\gamma)\mathcal{N}(\gamma)\le\sum_{\gamma\in\Gamma_1}\mathbf{s}(\gamma)\mathcal{N}(\gamma)\le(\sum_{\gamma\in\Gamma_1}\mathbf{s}(\gamma)\cdot\sum_{\gamma\in\Gamma_1}\mathbf{s}(\gamma)(\mathcal{N}(\gamma))^2)^{1/2}$$
$$\overset{(2.4.24),(2.4.13)}{\preceq}((\log n)^{-4}(\log n)^2)^{1/2}=(\log n)^{-1}\ll|K|.$$

Combing this with (2.4.18) gives:

$$\sum_{\gamma\in\Gamma_2}\mathbf{b}(\gamma)\mathcal{N}(\gamma)\sim|K|.$$

Hence, we have (when n is large):

$$\frac{(1-\epsilon)|K|}{(2a_4\sigma^2|K|+\epsilon)\log n}\le\sum_{\gamma\in\Gamma_2}\mathbf{b}(\gamma)\le\frac{(1+\epsilon)|K|}{(2a_4\sigma^2|K|-\epsilon)\log n}.$$

On the other hand, $\sum_{\gamma\in\Gamma_1}\mathbf{b}(\gamma)\ll(\log n)^{-1}$. Let $\epsilon\to 0^+$, one can get Proposition 2.4.5. □

Proof of Lemma 2.4.10. Without loss of generality, assume $m=|a-b|$. Let $B_a=\{z:|z-a|\le 3m/4\}$, $B_b=\{z:|z-b|\le 3m/4\}$ and $B_c=\{z:|z-c|\le M/4\}$. Write $t=(a+b)/2$ and $B=\{z:|z-t|\le 2M\}$. Then we can estimate separately:

$$\sum_{z\in B_a}|z-a|^{-2}|z-b|^{-2}|z-c|^{-2}\asymp\sum_{z\in B_a}\frac{1}{|z-a|^2m^2M^2}$$
$$\preceq\frac{1}{m^2M^2}\sum_{z\in B_a}\frac{1}{|z-a|^2}\asymp\frac{m^2}{m^2M^2}\le\frac{1}{M^2};$$
$$\sum_{z\in B_b}|z-a|^{-2}|z-b|^{-2}|z-c|^{-2}\preceq\frac{1}{M^2}\text{ (similarly);}$$

$$\sum_{z\in B_c}|z-a|^{-2}|z-b|^{-2}|z-c|^{-2}\asymp\sum_{z\in B_a}\frac{1}{|z-c|^2M^2M^2}$$
$$\preceq\frac{1}{M^4}\sum_{z\in B_c}\frac{1}{|z-c|^2}\asymp\frac{M^2}{M^4}\leq\frac{1}{M^2};$$

$$\sum_{z\in B\setminus(B_a\cup B_b\cup B_c)}|z-a|^{-2}|z-b|^{-2}|z-c|^{-2}\asymp\sum_{z\in B\setminus(B_a\cup B_b\cup B_c)}\frac{1}{|z-t|^2|z-t|^2M^2}$$
$$\preceq\frac{1}{M^2}\sum_{z:m/4\leq|z-t|\leq 2M}\frac{1}{|z-t|^4}\asymp\frac{1}{M^2}\sum_{:m/4\leq n\leq 2M}\frac{n^3}{n^4}\preceq\frac{1\vee\log(M/m)}{M^2};$$
$$\sum_{z\in B^c}|z-a|^{-2}|z-b|^{-2}|z-c|^{-2}\asymp\sum_{z\in B^c}\frac{1}{|z-t|^6}\preceq\sum_{n\geq 2M}\frac{n^3}{n^6}\preceq\frac{1}{M^2}.$$

This completes the proof. □

Now we are ready to prove Theorem 1.3.12.

Proof of Theorem 1.3.12. Let $n=\|x\|/(\log\|x\|)^{1/4}$, $B=\mathcal{C}(n)$, $B_1=\mathcal{C}(2n)\setminus B$ and $B_2=\mathcal{C}(2n)^c$. As before, by (2.2.7) and the First-Visit Lemma, we have:

$$P(\mathcal{S}_x\text{ visits }K)=\sum_{\gamma:x\to K}\mathbf{b}(\gamma)=\sum_{b\in B^c}G_K(x,b)\sum_{a\in K}\mathcal{H}^B_{\mathbf{k}}(b,a)$$
$$=\sum_{b\in B_1}G_K(x,b)\sum_{a\in K}\mathcal{H}^B_{\mathbf{k}}(b,a)+\sum_{b\in B_2}G_K(x,b)\sum_{a\in K}\mathcal{H}^B_{\mathbf{k}}(b,a).$$

We argue that the first term has the desired asymptotics and the second is negligible:

$$\sum_{b\in B_1}G_K(x,b)\sum_{a\in K}\mathcal{H}^B_{\mathbf{k}}(b,a)\overset{(2.4.7)}{\sim}a_4\|x\|^{-2}\sum_{b\in B_1}\sum_{a\in K}\mathcal{H}^B_{\mathbf{k}}(b,a)$$
$$\overset{(2.4.10)}{\sim}a_4\|x\|^{-2}\frac{4\pi^2\sqrt{\det Q}}{\sigma^2\log n}\sim\frac{1}{2\sigma^2\|x\|^2\log\|x\|};$$

$$\sum_{b\in B_2} G_K(x,b)\sum_{a\in K}\mathcal{H}^B_{\mathbf{k}}(b,a)\preceq\sum_{a\in K}\sum_{b\in B_2}\mathcal{H}^B_{\mathbf{k}}(b,a)$$
$$\overset{(2.1.6)}{\preceq}|K|n^2/n^5\ll 1/\|x\|^2\log\|x\|.$$

□

2.4.3 Convergence of the first visiting point

We aim to show Theorem 1.3.13. For simplicity, we assume in this subsection that θ has finite range. Then, for any subset $B\subset\subset\mathbb{Z}^4$, we can denote its outer boundary and inner boundary by:

$$\partial_o B\doteq\{y\notin B:\exists x\in B,\ \text{such that }\theta(x-y)\vee\theta(y-x)>0\};$$
$$\partial_i B\doteq\{y\in B:\exists x\notin B,\ \text{such that }\theta(x-y)\vee\theta(y-x)>0\}.$$

The first step is to construct the following asymptotical behavior of the Green function:

Lemma 2.4.11.

$$\lim_{x,y\to\infty:\|x\|\geq\|y\|}\frac{G_K(x,y)}{(\log\|y\|/\log\|x\|)g(x,y)}=1;\tag{2.4.25}$$

Remark 2.4.3. *It is a bit unsatisfactory that we need to require $\|x\|\geq\|y\|$ in the limit. When θ is symmetric, this requirement can be removed since $G_K(x,y)/(1-\mathbf{k}(x))=G_K(y,x)/(1-\mathbf{k}(y))$.*

Proof. By Lemma 2.4.4, we can assume $\|x\|\geq\|y\|(\log\|y\|)^{1/4}$. Let $n=\|y\|(\log\|y\|)^{1/8}$ and $B=\mathcal{C}(n)$. As before, we have:

$$\mathbf{p}_K(x)=G_K(x,K)=\sum_{z\in\partial_i B}\mathcal{H}^{B^c}_{\mathbf{k}}(x,z)G_K(z,K)=\sum_{z\in\partial_i B}\mathcal{H}^{B^c}_{\mathbf{k}}(x,z)\mathbf{p}_K(z).$$

By Theorem 1.3.12, we get:

$$\sum_{z\in\partial_i B}\mathcal{H}^{B^c}_{\mathbf{k}}(x,z)\sim\frac{n^2\log n}{\|x\|^2\log\|x\|}.\tag{2.4.26}$$

By Lemma 2.4.4, we have $G_K(z,y) \sim g(z,y) \sim a_4 n^{-2}$ for any $z \in \partial_i B$. Therefore,

$$G_K(x,y) = \sum_{z\in\partial_i B} \mathcal{H}_{\mathbf{k}}^{B^c}(x,z) G_K(z,y) \sim \sum_{z\in\partial_i B} \mathcal{H}_{\mathbf{k}}^{B^c}(x,z) a_4 n^{-2}$$
$$\sim a_4 \log n/(\|x\|^2 \log\|x\|) \sim a_4 \log\|y\|/(\|x\|^2 \log\|x\|).$$

This finishes the proof. □

Now we give the following asymptotics of the escape probability by a reversed snake.

Lemma 2.4.12. *For any $x \in \mathbb{Z}^4$, we have:*

$$\mathcal{E}_K(x) \doteq \lim_{n\to\infty} \log n \cdot \sum_{z\in\partial_o\mathcal{C}(n)} \mathcal{H}_{\mathbf{k}}^{\mathcal{C}(n)}(z,x) \text{ exists.} \tag{2.4.27}$$

Remark 2.4.4. *Note that $\mathcal{H}_{\mathbf{k}}^{\mathcal{C}(n)}(z,x) = \mathcal{H}_{\mathbf{k}}^{\mathcal{C}(n)\setminus K}(z,x)$ and $\sum_{z\in\partial_o\mathcal{C}(n)} \mathcal{H}_{\mathbf{k}}^{\mathcal{C}(n)}(z,x)$ is the probability that a reversed snake starting from x does not return to K, except for the bush grafted to the root, until the backbone reaches outside of $\mathcal{C}(n)$. For the random walk in critical dimension ($d=2$), we also have (e.g. see Section 2.3 in [10]):*

$$E_K(x) \doteq \lim_{n\to\infty} \log n \cdot \sum_{z\in\partial_o\mathcal{C}(n)} \mathcal{H}^{\mathcal{C}(n)\setminus K}(z,x) \text{ exists,} \quad \text{for any } x\in\mathbb{Z}^2, K \subset\subset \mathbb{Z}^2;$$

and

$$\lim_{x\to\infty} P(S_x(\tau_K) = a | S_x \text{ visits } K) = \frac{1}{\pi^2\sqrt{\det Q}} E_K(a).$$

Proof. We first need to show:

$$\lim_{n\to\infty, y\to\infty:\ \|y\|\le n} \frac{\log n}{\log\|y\|} \sum_{z\in\partial_o\mathcal{C}(n)} \mathcal{H}_{\mathbf{k}}^{\mathcal{C}(n)}(z,y) = 1. \tag{2.4.28}$$

Choose some $x \in \mathbb{Z}^4$ such that $\|x\| \ge n\log n$. By the First-Visit Lemma, we have:

$$G_K(x,y) = \sum_{z\in\partial_o\mathcal{C}(n)} G_K(x,z)\mathcal{H}_{\mathbf{k}}^{\mathcal{C}(n)}(z,y). \tag{2.4.29}$$

Due to last lemma, $G_K(x,y) \sim a_4\|x\|^{-2}\cdot\log\|y\|/\log\|x\|$, $G_K(x,z) \sim a_4\|x\|^{-2}\cdot\log n/\log\|x\|$. Together with (2.4.29), one can get (2.4.28).

Now we are ready to show (2.4.27). Without loss of generality, assume $\|x\| > \mathrm{Rad}(K)$. Write

$$a(n) = \log n \cdot \sum_{z\in\partial_o\mathcal{C}(n)} \mathcal{H}_{\mathbf{k}}^{\mathcal{C}(n)}(z,x).$$

Note that, for any (large) $m > n$,

$$\sum_{w\in\partial_o\mathcal{C}(m)} \mathcal{H}_{\mathbf{k}}^{\mathcal{C}(m)}(w,x) = \sum_{z\in\partial_o\mathcal{C}(n)} \mathcal{H}_{\mathbf{k}}^{\mathcal{C}(n)}(z,x) \sum_{w\in\partial_o\mathcal{C}(m)} \mathcal{H}_{\mathbf{k}}^{\mathcal{C}(m)}(w,z).$$

By (2.4.28), we have $\sum_{w\in\partial_o\mathcal{C}(m)} \mathcal{H}_{\mathbf{k}}^{\mathcal{C}(m)}(w,z) \sim \log n/\log m$. This implies $a(n)/a(m) \sim 1$ and hence the convergence of $a(n)$. □

Proof of Theorem 1.3.13. Let $n = \|x\|/\log\|x\|$ and $B = \mathcal{C}(n)$. Then,

$$\begin{aligned} P(\mathcal{S}_x(\tau_K) = a|\mathcal{S}_x \text{ visits } K) &= \frac{\sum_{\gamma:x\to a}\mathbf{b}(\gamma)}{\mathbf{p}_K(x)} \sim \frac{\sum_{z\in\partial_o B} G_K(x,z)\mathcal{H}_{\mathbf{k}}^B(z,a)}{1/2\sigma^2\|x\|^2\log\|x\|} \\ &\sim \frac{a_4\|x\|^{-2}\sum_{z\in\partial_o B}\mathcal{H}_{\mathbf{k}}^B(z,a)}{1/2\sigma^2\|x\|^2\log\|x\|} \sim \frac{a_4\|x\|^{-2}\mathcal{E}_K(a)\log^{-1} n}{1/2\sigma^2\|x\|^2\log\|x\|} \\ &\sim 2\sigma^2 a_4\mathcal{E}_K(a) = \frac{\sigma^2\mathcal{E}_K(a)}{4\pi^2\sqrt{\det Q}}. \end{aligned}$$

□

2.4.4 The range of branching random walk conditioned on the total size

The main goal of this subsection is to construct the asymptotics of the range of the branching random walk conditioned on the total size, i.e. Theorem 1.3.14. Our proof of this theorem is based on some ideas from [14]. Especially, we need to use the invariant shift on the invariant snake, $\mathcal{S}^I$.

For the invariant snake $\mathcal{S}^I$, recall that its backbone is just a random walk with jump distribution θ^-. We write τ_n for the hitting time (vertex) of $(\mathcal{C}(n))^c$ by the backbone. Thanks to Proposition 2.4.5, we can obtain the following:

Proposition 2.4.13.

$$P(\mathcal{S}_0^I(v) \neq 0,\ \forall v \leq \tau_n \textit{ not on the spine}) \sim \frac{4\pi^2\sqrt{\det Q}}{\sigma^2}\frac{1}{\log n};$$
$$P(\mathcal{S}_0^I(v_i) \neq 0,\ i = 1, 2, \ldots, n) \sim \frac{16\pi^2\sqrt{\det Q}}{\sigma^2}\frac{1}{\log n};$$

where $v_1 < v_2 < v_3 < \ldots$ are all vertices of $\mathcal{S}_0^I$ that are not on the spine.

Proof. By Proposition 2.4.5 (set $K = \{0\}$) and the Overshoot Lemma, we have:

$$\sum_{\gamma:(\mathcal{C}(n))^c \to 0, \gamma \subseteq \mathcal{C}(n)} \mathbf{b}(\gamma) \sim \frac{4\pi^2\sqrt{\det Q}}{\sigma^2}\frac{1}{\log n}.$$

Hence, the first assertion can be obtained if we can show

$$P(\mathcal{S}_0^I(v) \neq 0,\ \forall v \leq \tau_n \text{ not on the spine}) \sim \sum_{\gamma:(\mathcal{C}(n))^c \to 0, \gamma \subseteq \mathcal{C}(n)} \mathbf{b}(\gamma).$$

Let $p_0 = P(\mathcal{S}_0$ does not visit 0 except at the root) and the new killing function $k'(x)$ be the probability that $\mathcal{S}'_x$ visits to 0 (except possibly for the starting point). Note that $k'(x) = \mathbf{k}_K(x)$ when $x \neq 0$. We write $\mathbf{b}_{k'}(\gamma)$ for the probability weight of γ with this killing function. Then, we have

$$P(\mathcal{S}_0^I(v) \neq 0,\ \forall v \leq \tau_n \text{ not on the spine}) \sim p_0 \sum_{\gamma:(\mathcal{C}(n))^c \to 0,\ \gamma \subseteq \mathcal{C}(n)} \mathbf{b}_{k'}(\gamma)$$
$$= p_0\Big(\sum_{\gamma:(\mathcal{C}(n))^c \to 0,\ \gamma \subseteq \mathcal{C}(n)\setminus\{0\}} \mathbf{b}_{k'}(\gamma)\Big)\Big(\sum_{\gamma:0 \to 0,\ \gamma \subseteq \mathcal{C}(n)} \mathbf{b}_{k'}(\gamma)\Big).$$

Note that $\lim_{n\to\infty}\sum_{\gamma:0\to 0,\ \gamma\subseteq\mathcal{C}(n)}\mathbf{b}_{k'}(\gamma)=\sum_{\gamma:0\to 0}\mathbf{b}_{k'}(\gamma)$ and

$$\sum_{\gamma:(\mathcal{C}(n))^c\to 0,\ \gamma\subseteq\mathcal{C}(n)}\mathbf{b}(\gamma)=\sum_{\gamma:(\mathcal{C}(n))^c\to 0,\ \gamma\subseteq\mathcal{C}(n)\setminus\{0\}}\mathbf{b}_{k'}(\gamma).$$

Hence, for the first assertion, it is sufficient to show:

$$p_0\sum_{\gamma:0\to 0}\mathbf{b}_{k'}(\gamma)=1. \tag{2.4.30}$$

Note that this is just (2.2.9) (note that we set $x=0,K=\{0\}$). We finish the proof of the first assertion. The second assertion is an easy consequence of the first one, noting that, for any $\epsilon\in(0,1/4)$ fixed, $P(v_n\le\tau_{\lfloor n^{1/4-\epsilon}\rfloor})$ and ,$P(v_n\ge\tau_{\lfloor n^{1/4+\epsilon}\rfloor})$ are $o((\log n)^{-1})$. $\square$

Now we can construct the following result about the range of S^I:

Theorem 2.4.14. *Set $R_n^I:=\#\{\mathcal{S}_0^I(o),\mathcal{S}_0^I(v_1),\dots,\mathcal{S}_0^I(v_n)\}$ for every integer $n\ge 0$. We have:*

$$\frac{\log n}{n}R_n^I\xrightarrow{L^2}\frac{16\pi^2\sqrt{\det Q}}{\sigma^2}\quad \text{as } n\to\infty,$$

where $v_1,v_2,\dots$ are the same as in Proposition 2.4.13. Hence, we have:

$$\frac{\log n}{n}R_n^I\xrightarrow{P}\frac{16\pi^2\sqrt{\det Q}}{\sigma^2}\quad \text{as } n\to\infty.$$

Remark 2.4.5. *Since the typical number of vertices in the spine that come before v_n is of order $\sqrt{n}$, which is much less than $n/\log n$, one can get,*

$$\frac{\log n}{n}\#\{\mathcal{S}_0^I(\bar v_0),\mathcal{S}_0^I(\bar v_1),\dots,\mathcal{S}_0^I(\bar v_n)\}\xrightarrow{P}\frac{16\pi^2\sqrt{\det Q}}{\sigma^2}\quad \text{as } n\to\infty,$$

where $\bar v_0,\bar v_1,\dots$ are all vertices due to the default order in the corresponding plane tree T in $\mathcal{S}_0^I$.

Proof of Theorem 2.4.14. As mentioned before, we need to use the invariant shift ς on spacial trees, which appeared in [14]. For any spacial tree $(T,\mathcal{S}_T)$, set $\varsigma(T,\mathcal{S}_T)=(T',\mathcal{S}'_{T'})$. Roughly speaking, one can get T' by 'rerooting' T

at the first vertex that is not in the spine and then removing the vertices that are strictly before the parent of the new root. For $\mathcal{S}'_{T'}$, just set:

$$\mathcal{S}'_{T'}(v) = \mathcal{S}_T(v) - \mathcal{S}_T(o'), \quad \text{for any } v \in \varsigma(T),$$

where o' is the new root. The key result is that ς is invariant under the law of the invariant snake from the origin. For more details about this shift transformation, see Section 2 in [14].

Now we start our proof. For simplicity, write $\hat{v}_0 = 0 (\in \mathbb{Z}^4)$ and $\hat{v}_i = \mathcal{S}_0^I(v_i)$. First observe that:

$$\mathbb{E}(R_n^I) = \mathbb{E}(\sum_{i=0}^n \mathbf{1}_{\{\hat{v}_j \neq \hat{v}_i, \forall j \in [i+1,n]\}}) = \sum_{i=0}^n P(\hat{v}_j \neq \hat{v}_i, \forall j \in [i+1, n]).$$

From the invariant shift mentioned in the beginning, we have

$$P(\hat{v}_j \neq \hat{v}_i, \forall j \in [i+1, n]) = P(\hat{v}_j \neq \hat{v}_0, \forall j \in [1, n-i]).$$

Therefore by Proposition 2.4.13, we get

$$\mathbb{E}(R_n^I) = \sum_{i=0}^n P(\hat{v}_j \neq \hat{v}_0, \forall j \in [1, n-i]) \sim \frac{16\pi^2\sqrt{\det Q}}{\sigma^2} \frac{n}{\log n}. \tag{2.4.31}$$

Now we turn to the second moment. Similarly, we have

$$\begin{aligned}
\mathbb{E}((R_n^I)^2) &= \mathbb{E}(\sum_{i=0}^n \sum_{j=0}^n \mathbf{1}_{\{\hat{v}_k \neq \hat{v}_i, \forall k \in [i+1,n]; \hat{v}_l \neq \hat{v}_j, \forall l \in [j+1,n]\}}) \\
&= 2 \sum_{0 \le i < j \le n} P(\hat{v}_k \neq \hat{v}_i, \forall k \in [i+1, n]; \hat{v}_l \neq \hat{v}_j, \forall l \in [j+1, n]) + E(R_n) \\
&= 2 \sum_{0 \le i < j \le n} P(\hat{v}_k \neq 0, \forall k \in [1, n-i]; \hat{v}_l \neq \hat{v}_{j-i}, \forall l \in [j-i+1, n-i]) \\
&\quad + \mathbb{E}(R_n),
\end{aligned}$$

where the last equality again follows from the invariant shift. For any fixed

$\alpha \in (0, 1/4)$ define

$$\sigma_n := \sup\{k \geq 0 : v_k \leq u_{\lfloor n^{\frac{1}{2}-\alpha} \rfloor}\},$$

where $u_0 \leq u_1 \leq \ldots$ are the all vertices on the spine. By standard arguments, one can show

$$P(\sigma_n \notin [n^{1-3\alpha}, n^{1-\alpha}]) = o(\log^{-2} n).$$

Therefore we have

$$\limsup_{n\to\infty} (\frac{\log n}{n})^2 \mathbb{E}((R_n^I)^2) = \limsup_{n\to\infty} 2(\frac{\log n}{n})^2 \sum_{0\leq i<j\leq n} P(\hat{v}_k \neq 0,$$
$$\forall k \in [1, n-i]; \hat{v}_l \neq \hat{v}_{j-i}, \forall l \in [j-i+1, n-i]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}]).$$

Obviously, in order to study the limsup in the right-hand side, we can restrict the sum to indices i and j such that $j - i > n^{1-\alpha}$. However, when i and j are fixed and satisfied with $j - i > n^{1-\alpha}$,

$$\begin{aligned}
&P(\hat{v}_k \neq 0, \forall k \in [1, n-i]; \hat{v}_l \neq \hat{v}_{j-i}, \forall l \in [j-i+1, n-i]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}]) \\
&\leq P(\hat{v}_k \neq 0, \forall k \in [1, \sigma_n]; \hat{v}_l \neq \hat{v}_{j-i}, \forall l \in [j-i+1, n-i]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}]) \\
&= P(\hat{v}_k \neq 0, \forall k \in [1, \sigma_n]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}]) P(\hat{v}_l \neq \hat{v}_{j-i}, \forall l \in [j-i+1, n-i]) \\
&= P(\hat{v}_k \neq 0, \forall k \in [1, \sigma_n]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}]) P(\hat{v}_l \neq 0, \forall l \in [1, n-j]).
\end{aligned}$$

Note that for the second last line, we use the fact that after conditioning on $\sigma_n = m(< n^{1-\alpha})$, the event on the second probability is independent to the event on the first one, and for the last line, we use the invariant shift. Now,

$$P(\hat{v}_k \neq 0, \forall k \in [1, \sigma_n]; \sigma_n \in [n^{1-3\alpha}, n^{1-\alpha}]) \leq P(\hat{v}_k \neq 0, \forall k \in [1, n^{1-3\alpha}]),$$

and then we have

$$\begin{aligned}\limsup_{n\to\infty}(\frac{\log n}{n})^2\mathbb{E}((R_n^I)^2) &\leq \limsup_{n\to\infty} 2(\frac{\log n}{n})^2\cdot \\ &\sum_{0\leq i<j\leq n, j-i>n^{1-\alpha}} P(\hat{v}_k \neq 0, \forall k \in [1, n^{1-3\alpha}])P(\hat{v}_l \neq 0, \forall l \in [1, n-j]) \\ &= \frac{1}{1-3\alpha}(\frac{16\pi^2\sqrt{\det Q}}{\sigma^2})^2.\end{aligned}$$

Let $\alpha \to 0^+$, we get

$$\limsup_{n\to\infty}(\frac{\log n}{n})^2\mathbb{E}((R_n^I)^2) \leq (\frac{16\pi^2\sqrt{\det Q}}{\sigma^2})^2.$$

Combining this with (2.4.31), we finish the proof of Theorem 2.4.14. □

Noting that $\mathcal{S}_0^-$ is different to $\mathcal{S}_0^I$ only at the subtree grafted to the root, one can also obtain the range of the infinite snake $\mathcal{S}^-$:

Corollary 2.4.15. *Set $R_n^- := \#\{\mathcal{S}_0^-(v_0), \mathcal{S}_0^-(v_1), \dots, \mathcal{S}_0^-(v_n)\}$. Then,*

$$\frac{\log n}{n}R_n^- \xrightarrow{P} \frac{16\pi^2\sqrt{\det Q}}{\sigma^2} \quad \text{as } n\to\infty,$$

where $v_0, v_1, \dots$ are all vertices of the corresponding plane tree due to the default order in the reversed snake.

Now we are ready to prove our main result about the range of branching random walk conditioned on the total size. This result will follow from Corollary 2.4.15 by an absolute continuity argument, which is similar to the one in the proof of Theorem 7 in [14]. The idea is as follows. We write Ξ for the law of the μ-GW tree. For every $a \in (0,1)$, the law under $\Xi^n := \Xi(\cdot|\#T = n)$ of the subtree obtained by keeping only the first $\lfloor an \rfloor$ vertices of T is absolutely continuous with respect to the law under $\Xi^\infty(\cdot) := \Xi(\cdot|\#T = \infty)$ of the same subtree, with a density that is bounded independently of n. Then a similar property holds for spatial trees, and hence we can use the convergence in Corollary 2.4.15, for a tree distributed according to Ξ^∞, to get a similar convergence for a tree distributed according to Ξ^n.

Proof of Theorem 1.3.14. Let $\mathcal{G}$ be the smallest subgroup of $\mathbb{Z}$ that contains the support of μ. In fact, the cardinality of the vertex set of a μ-GW tree belongs to $1+\mathcal{G}$. For simplicity, we assume in the proof that $\mathcal{G} = \mathbb{Z}$. Minor modifications are needed for the general case. On the other hand, for any sufficiently large integer $n \in 1+\mathcal{G}$, we can define the conditional probability $\mathcal{S}^n$ to be $\mathcal{S}_0$ conditioned on the total number of vertices being n (this event is with strictly positive probability).

For a finite plane tree T, write $v_0(T), v_1(T), \dots, v_{\#T-1}(T)$ for the vertices of T by the default order. The Lukasiewisz path of T is then the finite sequence $(X_l(T), 0 \le l \le \#T)$, which can be defined inductively by

$$X_0(T) = 0, X_{l+1} - X_l = k_{v_l(T)}(T) - 1, \quad \text{for every } 0 \le l < \#T,$$

where $k_u(T)$(for $u \in T$) is the number of children of u. The tree T is determined by its Lukasiewisz path. A key result says that under Ξ, the Lukasiewisz path is distributed as a random walk on $\mathbb{Z}$ with jump distribution ν determined by $\nu(j) = \mu(j+1)$ for any $j \ge -1$, which starts from 0 and is stopped at the hitting time of -1 (in particular, the law or $\#T$ coincides with the law of that hitting time). For notational convenience, we let $(Y_k)_{k\ge 0}$ be a random walk on $\mathbb{Z}$ with jump distribution ν, which starts form i under $P_{(i)}$, and set

$$\tau := inf\{k \ge 0 : Y_k \le -1\}.$$

We can also do this for infinite trees. When T ia an infinite tree with only one infinite ray, now the Depth-First search sequence $o = v_0 < v_1 < v_2 < \dots < v_n < \dots$ only examines part of the vertex set of T. We could also define the Lukasiewisz path of T to be the infinite sequence $(X_i(T), i \in \mathbb{N})$:

$$X_0(T) = 0, X_{l+1} - X_l = k_{v_l(T)}(T) - 1, \quad \text{for every } l \in \mathbb{N}.$$

Now, only the 'left half' of T (precisely, the subtree generated by $v_0, v_1, \ldots$), not the whole tree T, is determined by its Lukasiewisz path. It is not difficult to verify that when T is a μ-GW tree conditioned on survival, its

Lukasiewisz path is distributed as the random walk on the last paragraph conditioned on $\tau = \infty$, i.e, a Markov chain on $\mathbb{N}$ with transition probability $p(i,j) = \frac{j+1}{i+1}\nu(j-i)$. Recall that the infinite μ-GW tree is just the 'left half' of the μ-GW tree conditioned on survival.

Next, take n large enough such that $\Xi(\#T = n) > 0$. Fix $a \in (0,1)$, and consider a tree (finite or infinite) T with $\#T \geq n$. Then, the collection of vertices $v_0(T), \ldots, v_{\lfloor an \rfloor}(T)$ forms a subtree of T (because in the Depth-First search order the parent of a vertex comes before this vertex), and we denote this tree by $\rho_{\lfloor an \rfloor}(T)$. It is elementary to see that $\rho_{\lfloor an \rfloor}(T)$ is determined by the sequence $(X_l(T), 0 \leq l \leq \lfloor an \rfloor)$. Let f be a bounded function on $\mathbb{Z}^{\lfloor an \rfloor}$. One can verify that

$$\Xi^n(f((X_k)_{0\leq k\leq \lfloor an \rfloor})) = \frac{1}{P_{(0)}(\tau = n+1)}\Xi^\infty(f((X_k)_{0\leq k\leq \lfloor an \rfloor})\frac{\psi_n(X_{\lfloor an \rfloor})}{X_{\lfloor an \rfloor}+1}), \tag{2.4.32}$$

where for every $j \in N$, $\psi_n(j) = P_{(j)}(\tau = n+1-\lfloor an \rfloor)$.

We now let $n \to \infty$. Using Kemperman's formula and a standard local limit theorem, one can get,

$$\lim_{n\to\infty}\left(\sup_{j\in A_n}|\frac{\psi_n(j)}{P_{(0)}(\tau = n+1)(j+1)} - \Gamma_a(\frac{j}{\sigma\sqrt{n}})|\right) = 0, \tag{2.4.33}$$

where $\Gamma_a(x) = \exp(-\frac{x^2}{2(1-a)})/(1-a)^{\frac{3}{2}}$ and $A_n := \{i \in \mathbb{N} : P_{(i)}(\tau = n + 1 - \lfloor an \rfloor) > 0\}$. By combining (2.4.32) and (2.4.33), we get that, for any uniformly bounded sequence of functions $(f_n)_{n\geq 1}$ on $\mathbb{Z}^{\lfloor an \rfloor + 1}$, we have

$$\lim_{n\to\infty}|\Xi^n(f_n((X_k)_{0\leq k\leq \lfloor an \rfloor})) - \Xi^\infty(f_n((X_k)_{0\leq k\leq \lfloor an \rfloor})\Gamma_a(\frac{X_{\lfloor an \rfloor}}{\sigma\sqrt{n}}))| = 0.$$

Clearly, the above still holds after we add the spatial random mechanism. Therefore, when $\epsilon > 0$ is fixed, we have

$$\lim_{n\to\infty}|\Xi^n_\theta(\mathbf{1}_{\{|R_{\lfloor an \rfloor} - tan/\log n| > \epsilon n/\log n\}}) - \Xi^\infty_\theta(\mathbf{1}_{\{|R_{\lfloor an \rfloor} - tan/\log n| > \epsilon n/\log n\}}\Gamma_a(\frac{X_{\lfloor an \rfloor}}{\sigma\sqrt{n}}))| = 0,$$

where $t = \frac{16\pi^2\sqrt{\det Q}}{\sigma^2}$, Ξ_θ^n, Ξ_θ^∞ are the laws of the corresponding tree-indexed random walks, and $R_{\lfloor an \rfloor}$ is the range of the subtree $\rho_{\lfloor an \rfloor}(T)$. Note that the function Γ_a is bounded and under Ξ_θ^∞, $R_{\lfloor an \rfloor}$ is just the range of $\mathcal{S}_0^\infty$ for the first $\lfloor an \rfloor$ vertices. Hence, by Corollary 2.4.15 (note that $\mathcal{S}_0^- = \mathcal{S}_0^\infty$ since we assume that θ is symmetry), we obtain that

$$\lim_{n\to\infty} \Xi_\theta^n(\mathbf{1}_{\{|R_{\lfloor an \rfloor} - tan/\log n| > \epsilon n/\log n\}}) = 0.$$

Note that $R_n \geq R_{\lfloor an \rfloor}$ (under Ξ_θ^n) and a can be chosen arbitrarily close to 1, this finishes the proof of the lower bound.

We also need to show the upper bound. Note that $\rho_{\lfloor an \rfloor}(T)$ is the subtree lying on the 'left' side, generated by the first $\lfloor an \rfloor$ vertices of T. Similarly, one can consider the subtree lying on the 'right' side. Strictly speaking, to get the subtree lying on the right side, denoted by $\rho^-_{\lfloor an \rfloor}(T)$, we first reverse the order of children for each vertex in T, and then $\rho_{\lfloor an \rfloor}$ of the same tree T with the new order is just $\rho^-_{\lfloor an \rfloor}(T)$. Write $R^-_{\lfloor an \rfloor}$ for the range of $\rho^-_{\lfloor an \rfloor}(T)$ corresponding to $\mathcal{S}^n$. By symmetry, we also have

$$\lim_{n\to\infty} \Xi_\theta^n(\mathbf{1}_{\{|R^-_{\lfloor an \rfloor} - tan/\log n| > \epsilon n/\log n\}}) = 0.$$

Now fix some $a \in (0,1)$. Note that $\rho_{\lfloor an \rfloor}(T)$ and $\rho^-_{\lfloor (1-a)n \rfloor}(T)$ cover the whole tree T except for a number of vertices. This number is not more than $|\rho_{\lfloor an \rfloor}(T) \cap \rho^-_{\lfloor (1-a)n \rfloor}(T)| + 2$. Note that on each generation, there is at most one vertex that is in both $\rho_{\lfloor an \rfloor}(T)$ and $\rho^-_{\lfloor (1-a)n \rfloor}(T)$. Hence $|\rho_{\lfloor an \rfloor}(T) \cap \rho^-_{\lfloor (1-a)n \rfloor}(T)|$ is not more than the number of generations, which is typically of order $\sqrt{n}$ (under Ξ^n). Hence, $R_n - (R_{\lfloor an \rfloor}(\mathcal{S}^n) + R^-_{\lfloor (1-a)n \rfloor}(\mathcal{S}^n))$ is less than $n^{0.6}$ with high probability (tending to 1). This finishes the proof of the upper bound. $\square$

Chapter 3

Branching interlacements

3.1 Preliminaries

3.1.1 Plane trees, contour function and branching random walk

We are interested in (finite or infinite) rooted ordered trees, called plane trees. A rooted tree $\mathbf{t}$ is a tree with a distinguished vertex o called the root. $\mathbf{t}$ can be regarded as a family tree with ancestor o. A plane tree is a rooted tree in which an ordering for the children of each vertex is specified. The size $|\mathbf{t}|$ is the number of edges of $\mathbf{t}$. We denote by $\mathbf{A}$ the set of all finite plane trees and by $\mathbf{A}_n$ the set of all plane trees with $n \in \mathbb{N}$ edges.

Let $\mathbf{t}$ be a plane tree and $k \in \mathbb{N}$, we write $[\mathbf{t}]_k$ for the subtree obtained by keeping only the first k generations of $\mathbf{t}$. Let T be a GW (Galton-Watson) tree with geometric offspring distribution of parameter $1/2$ (throughout this chapter our GW tree will always be with this offspring distribution). It is classical that the distribution of T conditioned on having n edges is the uniform probability measure on $\mathbf{A}_n$. The following result is also standard (e.g. see [1]):

Proposition 3.1.1. *Let T_n be uniform on $\mathbf{A}_n$. Then there exists a random infinite plane tree T_∞ such that for every $k \in \mathbb{N}$ we have*

$$[T_n]_k \xrightarrow{d} [T_\infty]_k, \text{ as } n \to \infty. \tag{3.1.1}$$

Moreover, this random infinite plane tree T_∞, called the critical Galton-Watson tree conditioned to survive, can be constructed in the following way: begin with a semi-infinite line of vertices called the spine and graft to the left and to the right of each vertex in the spine an independent GW tree. It is rooted at the first vertex in the spine.

A nice way to code plane trees is the so-called contour function. Assume $\mathbf{t}$ is a plane tree with k edges. Let v_0 be the root of $\mathbf{t}$. Define v_i to be the first unexplored child of v_{i-1} if v_{i-1} has such children, or the parent of v_i if not, for $i = 1, \dots, 2k$. Let $C(i)$ be the tree distance between the root and v_i. Then $(C(i))_{i\in\{0,\dots,2k\}}$ is the contour function of $\mathbf{t}$.

For $k \in \mathbb{N}$, a Dyck path of length $2k$ is a sequence $(s_0, s_1, ..., s_{2k})$ of integers such that $s_0 = s_{2k} = 0$, $s_i \geq 0$ and $|s_i - s_{i-1}| = 1$, for every $i = 1, ..., 2k$. If $\mathbf{t}$ is a plane tree of size k, then its contour function $(C(0), C(1), \dots, C(2k))$ is a Dyck path of length $2k$. Moreover, we have (e.g. see the lecture notes [15])

Proposition 3.1.2. *The mapping* $\mathbf{t} \to (C(0), C(1), \dots, C(2k))$ *is a bijection from* $\mathbf{A}_k$ *onto the set of all Dyck paths of length* $2k$. *Therefore, the contour function of a GW tree conditioned on having* k *edges is uniform on all Dyck paths of length* $2k$.

There is a similar result for the unconditioned GW tree. Assume $S = (S_n)_{n\in\mathbb{N}}$ is simple random walk on $\mathbb{Z}$ (starting from 0). Let $\tau = \inf\{n \in \mathbb{N} : S_n = -1\} < \infty$ a.s. Then, the distribution of the contour function of a unconditioned GW tree is the same as $(S_i)_{0\leq i\leq \tau-1}$.

Now we introduce the simple random walk in $\mathbb{Z}^d$ indexed by a random plane tree T. Conditionally on T we assign independently to each edge of T a variable uniform on all unit vectors in $\mathbb{Z}^d$. Then for every vertex v in T, we assign to v the sum of the variables of all edges belonging to the unique simple path from the root o to the vertex v. This gives a random function $\mathcal{S}_T : T \to \mathbb{Z}^d$ from the vertices of T to the vertices of $\mathbb{Z}^d$ (note that $\mathcal{S}_T(o) = 0$). A plane tree T together with this random function $\mathcal{S}_T$ is called a spatial tree. When T is an unconditioned GW tree, a GW tree conditioned on having n edges or a GW tree conditioned to survive, the spatial tree is called finite branching random walk, branching random walk conditioned to have n progeny or branching random walk conditioned to survive. When $T = T_\infty$, we can talk about recurrence and transience. If $|\mathcal{S}_{T_\infty}^{-1}(0)| < \infty$ a.s., we say that the branching random walk conditioned to survive is transient. If $|\mathcal{S}_{T_\infty}^{-1}(0)| = \infty$ a.s., we say that it is recurrent. About recurrence and transience, we have (see [3] or see Corollary 1.3.10 and Proposition 2.3.4) :

Proposition 3.1.3. *Branching random walk on $\mathbb{Z}^d$ conditioned to survive is transient if and only if $d > 4$.*

3.1.2 Some results on simple random walk

Let us now collect some facts about random walks for later use. We use C, c to denote positive constants, depending only on dimension d, which may change from line to line. If a constant depends on some other variable, this will be made explicit. We use $a \vee b$ and $a \wedge b$ for $\max\{a, b\}$ and $\min\{a, b\}$ respectively. We will write $f \preceq g$ ($f \succeq g$ resp.), if there exists a positive constant C (depending on dimension only), such that $f \leq Cg$ ($f \geq Cg$ resp.) and write $f \asymp g$ if $f \preceq g$ and $f \succeq g$. For $x \in \mathbb{Z}^d$, we write P_x (just in this subsection) for the law of simple random walk $(Z_n)_{n\geq 0}$ on $\mathbb{Z}^d$ starting at $Z_0 = x$. Define:

$$p_n(x) = P_0[Z_n = x]; \quad \bar{p}_n(x) := 2(d/2\pi n)^{d/2} \exp(-d|x|_2^2/2n), \tag{3.1.2}$$

where we write $|\cdot|_2$ for the Euclidean norm (and reserve $|\cdot|$ for the ∞-norm). Then we have the so-called Local Central Limit Theorem (LCLT) (e.g. see Chapter 1.2 in [10]):

Proposition 3.1.4. *For $x \in \mathbb{Z}^d$, we have*

$$p_n(x) \preceq n^{-d/2}. \tag{3.1.3}$$

If $\delta < 2/3$ and $|z| \leq n^\delta$ such that z and n have the same parity, then we have

$$p_n(z) = \bar{p}_n(z)(1 + O(n^{3\delta-2})). \tag{3.1.4}$$

The next proposition follows from an application of the Azuma-Hoeffding inequality (e.g. Proposition 2.1.2 [11]).

Proposition 3.1.5. *There exist positive C and c, such that for all n and $s > 0$,*

$$P_0[\max_{0 \le j \le n} |Z_j| \ge s\sqrt{n}] \le C \exp(-cs^2). \tag{3.1.5}$$

The Green function of simple random walk on $\mathbb{Z}^d$ is defined by

$$G(x,y) = \sum_{n=0}^{\infty} P_x[Z_n = y] = \sum_{n=0}^{\infty} p_n(y-x), \quad x, y \in \mathbb{Z}^d. \tag{3.1.6}$$

Using LCLT, one can get the standard estimate for the Green function (for $d \ge 3$)

$$G(x,y) = \sum_{n=0}^{\infty} p_n(y-x) \asymp (|x-y| \vee 1)^{2-d}. \tag{3.1.7}$$

Using the same method, one can also get (for $d \ge 5$):

$$\sum_{n=0}^{\infty} n \cdot p_n(y-x) \asymp (|x-y| \vee 1)^{4-d}. \tag{3.1.8}$$

We are particularly interested in one-dimensional simple random walk. The following is a special case of Kemperman's formula (Lemma 2.12 in [15]).

Proposition 3.1.6. *Let τ be the hitting time of -1. We have, for any $k \in \mathbb{N}$ and $n \in \mathbb{N}^+$,*

$$P_k[\tau = n] = \frac{k+1}{n} P_k[S_n = -1], \tag{3.1.9}$$

where P_k is the probability measure under which the simple random walk S starts from k.

We will also use the so-called heat kernel bound (Lemma 2.1 [7]):

Proposition 3.1.7. *There exists positive C and c, such that for all $n \in \mathbb{N}^+$ and $k \in \mathbb{N}$, (P_0 has the same meaning as in last proposition)*

$$P_0[S_n = k] \le Cn^{-1/2} \exp(-ck^2/n). \tag{3.1.10}$$

3.2 Basic model and some first properties

In this section we give the definition of branching interlacements at level u as the range of a countable collection of doubly-infinite trajectories in $\mathbb{Z}^d$. As we mentioned before, the model of branching interlacements is an analogous model to random interlacements. Many definitions here are similar or even the same as in [21]. The collection of doubly-infinite trajectories will arise from a certain Poisson point process, called the branching interlacements point process. The main task is to construct the intensity measure of this Poisson point process.

3.2.1 Notations

We denote with $|\cdot|_2$ and $|\cdot|$ the Euclidean and ∞-norm on $\mathbb{Z}^d$. We write $B_x(r)$ and $S_x(r)$ for the closed $|\cdot|$-ball and $|\cdot|$-sphere with center x in $\mathbb{Z}^d$ and radius $r \geq 0$. We say that x, y in $\mathbb{Z}^d$ are neighbors (denoted by $x \sim y$), respectively *-neighbors, if $|x-y|_2 = 1$, respectively $|x-y| = 1$. The notion of nearest neighbor or *-nearest neighbor paths in $\mathbb{Z}^d$ is defined accordingly. For a subset K of $\mathbb{Z}^d$, we define

$$\partial_o K := \{x \in \mathbb{Z}^d \backslash K : \exists y \in K \text{ such that } x \sim y\} \tag{3.2.1}$$

its external boundary and

$$\partial_i K := \{x \in K : \exists y \in \mathbb{Z}^d \backslash K \text{ such that } x \sim y\} \tag{3.2.2}$$

its internal boundary. We consider W and W_+ the space of 2-sided and 1-sided nearest neighbor transient trajectories on $\mathbb{Z}^d$:

$$W = \{w : \mathbb{Z} \to \mathbb{Z}^d; \lim_{|n| \to \infty} |w(n)| = \infty \text{ and } w(n+1) \sim w(n), \forall n \in \mathbb{Z}\}, \tag{3.2.3}$$

$$W_+ = \{w : \mathbb{N} \to \mathbb{Z}^d; \lim_{n \to \infty} |w(n)| = \infty \text{ and } w(n+1) \sim w(n), \forall n \in \mathbb{N}\}. \tag{3.2.4}$$

If $w = (w(n))_{n\in\mathbb{Z}} \in W$, we define $w_+ \in W_+$ to be the part of w which is indexed by nonnegative coordinates, i.e., $w_+ = (w(n))_{n\in\mathbb{N}}$. We denote by $\mathcal{W}$, the product σ-algebra on W generated by coordinates, and by $\mathcal{W}_+$ the product σ-algebra on W_+. For $w \in W$ or W_+ and $x \in \mathbb{Z}^d$, we denote the space translation by $w + x$, i.e. $(w + x)(n) = w(n) + x$. We define the shift operators $\theta_k : W \to W$, $k \in \mathbb{Z}$ and $\theta_k : W_+ \to W_+$, $k \in \mathbb{N}$ by

$$(\theta_k(w))(n) = w(n + k). \tag{3.2.5}$$

Next we will define the space $(W^*, \mathcal{W}^*)$, which will play an important role in our construction of branching interlacement. Define the set of paths modulo time-shift by $W^* = W/ \simeq$, where $\simeq$ is the equivalence relation

$$w_1 \simeq w_2, \text{ if } \theta_k(w_1) = w_2 \text{ for some } k \in \mathbb{Z}. \tag{3.2.6}$$

Denote the canonical projection by $\pi : W \to W^*$ which sends each element in W to its equivalence class in W^*. We endow W^* with the shift invariant σ-field:

$$\mathcal{W}^* = \{A \subseteq W^* : \pi^{-1}(A) \in \mathcal{W}\}. \tag{3.2.7}$$

For any finite subset K of $\mathbb{Z}^d$ (we will write $K \subset\subset \mathbb{Z}^d$ for this), define:

$$W_K =\{w \in W : w(n) \in K, \text{ for some } n \in \mathbb{Z}\} \quad \text{and} \quad W_K^* = \pi(W_K), \tag{3.2.8}$$

$$W_{K+} = \{w \in W_+ : w(n) \in K, \text{ for some } n \in \mathbb{N}\}. \tag{3.2.9}$$

It follows from (3.2.3) that, for any trajectory $w \in W_K$ or W_{K+}, the set $\{n : w(n) \in K\}$ is finite. Hence, we can define the 'entrance time':

$$H_K(w) = \inf\{n : w(n) \in K\}. \tag{3.2.10}$$

Thus $H_K(w) < \infty$ if $w \in W_K$ or W_{K+}. We can partition W_K according to the time of the first entrance:

$$W_K = \bigcup_{n \in \mathbb{Z}} W_K^n, \quad \text{where} \quad W_K^n = \{w \in W_K : H_K(w) = n\}. \tag{3.2.11}$$

We define $t_K : W_K \to W_K^0$, respectively $t_K^* : W_K^* \to W_K^0$ with $t_K(w) = w_0$, respectively $t_K^*(w^*) = w_0$, where w_0 is the unique element w_0 in W_K^0 with $w_0 \simeq w$, respectively $\pi(w_0) = w^*$. Also we can define $t_{K+}^* : W_K^* \to W_+$ with $t_{K+}^*(w) = (t_K^*(w))_+$.

3.2.2 Simple random walk as a contour function and snakes

Our goal is to construct a σ-finite measure on $(W^*, \mathcal{W}^*)$. Before doing this, we need to introduce the finite measure Q_K for every $K \subset\subset \mathbb{Z}^d$ on $(W, \mathcal{W})$. The first step is to build a random matching on $E(\mathbb{Z}) = \{e_i = (i-1, i); i \in \mathbb{Z}\}$, the set of all edges of the lattice $\mathbb{Z}$. Let $S = (S_i)_{i \in \mathbb{Z}}$ be 1-dimensional two-sided simple random walk. Then S almost surely determines a matching of $E(\mathbb{Z})$, or more precisely, a bijection f_S between the set of upsteps $M(S) = \{e_i : S_i - S_{i-1} = +1\}$ and the set of downsteps $N(S) = \{e_i : S_i - S_{i-1} = -1\}$:

$$\begin{aligned} &f_S(e_k) = e_l \qquad \text{if and only if} \\ &k < l, S_k - S_{k-1} = 1, S_l - S_{l-1} = -1, \quad \& \; S_k = S_{l-1} = \min\nolimits_{k \le n \le l-1} S_n. \end{aligned} \tag{3.2.12}$$

Remark 3.2.1. *If we glue edges through the matching, the resulting quotient of the graph $\mathbb{Z}$ (rooted at 0) becomes an infinite plane tree. Precisely, for any $x, y \in \mathbb{Z}$, let $d(x,y) = S_x + S_y - 2\min\{S_t : t \in [x,y]\}$. If we identify x and y when $d(x,y) = 0$, then under this equivalence, the quotient space with the metric d is an infinite plane tree. One can check that using the description after Proposition 3.1.1 , this tree is just T_∞ and S is just its contour function if we let the spine go downwards and the finite trees attached to the spine grow upwards as usual. Note that for a finite tree, since we place the root at the bottom, its contour function is always non-negative. But here we let the spine go downwards hence the contour function can be negative.*

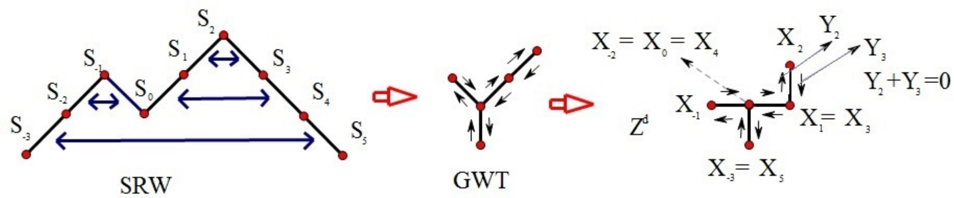

Figure 3.1: Construction of 2-sided infinite snake

Now we combine the contour function and the simple random walk indexed by a random tree. At the moment we have a mapping from $\mathbb{Z}$ to T_∞ along the contour. A transient mapping from T_∞ to $\mathbb{Z}^d$ can be written as a trajectory $X \in W$ with the property that if $n, n' \in \mathbb{Z}$ correspond to the same vertex of T_∞ then $X_n = X_{n'}$. We shall denote the increments of such a mapping by $Y_n = X_n - X_{n-1}$. Since in the contour exploration of the tree, each edge is crossed once in each direction, the corresponding increment variables should be opposite to each other, and otherwise they are independent.

Definition 3.2.1. *Let $S = (S_n)_{n\in\mathbb{Z}}$ be two-sided simple random walk and f_S be the corresponding matching. For each upstep e_n of S, where $S_n = S_{n-1}+1$ let Y_n be a uniform unit vector in $\mathbb{Z}^d$, all independent. For any downstep $e_m = (m-1, m)$ let $Y_m = -Y_n$, where n is such that $f_S(e_n) = e_m$ (see figure.) The starting point $X_0 = 0$ and relation $Y_n = X_n - X_{n-1}$ determine X_n for all n. The trajectory $(X)_{n\in\mathbb{Z}}$ is called the* ***2-sided infinite snake****. The half process $(X)_{n\in\mathbb{N}}$ is called the (1-sided)* ***infinite snake****. In this section, we write P_0 for the law of X. If the starting point is x (i.e. $X_0 = x$), we use P_x.*

Claim 3.2.2. *Since S is invariant under time-reversal and Y is symmetric, we can see that X is also invariant under time-reversal, i.e. $(X_n)_{n\in\mathbb{Z}}$ and $(X_{-n})_{n\in\mathbb{Z}}$ have the same distribution. Similarly, one can see that the law of the increment sequence Y is also invariant under time shift.*

Remark 3.2.2. *X is not a branching random walk, but the contour function of the branching random walk conditioned to survive. We primarily focus* *on the range of X, which has the same distribution as the range of the corresponding branching random walk. The reason for the introduction of snakes is that the contour function provides us a nice way to code branching random walk. As we mentioned in last section, this branching random walk is transient if and only if $d \geq 5$. Hence for $d \geq 5$, P_x is indeed a probability measure supported on W. This is why we assume $d \geq 5$ throughout this chapter.*

The **finite snake** is defined similarly, as follows. For the simple random walk S, set $\tau = \inf\{n \geq 0 : S_n = -1\} < \infty$ a.s. . The finite path $\{S_0, S_1, \dots, S_{\tau-1}\}$ is called an excursion of the simple random walk. We can also define the matching on the finite edge set $\{e_i = (i-1, i) : 1 \leq i \leq \tau - 1\}$, in the same way as before. Then we may also define $(Y_i)_{1 \leq i \leq \tau-1}$ and $(X_i)_{0 \leq i \leq \tau-1}$ as before. This finite process X is called the finite snake and is the contour function of the unconditioned branching random walk.

Note: the process X is just the restriction of the infinite snake X to the random time interval $[0, \tau - 1]$. It is possible that $\tau = 1$, in which case the branching random walk dies immediately and its image is the single point X_0.

If we condition on $\tau = 2L + 1$, then $(X_i)_{0 \leq i \leq 2L}$ is called the **snake conditioned to have length** $2L$ and is the contour function of the branching random walk conditioned to have L progeny.

3.2.3 Construction of the branching interlacement intensity measure

Once we have the definition of P_x (see Definition 3.2.1), we can define a measure Q_K on $(W, \mathcal{W})$ for any $K \subset\subset \mathbb{Z}^d$. In fact, Q_K will be supported on W_K^0 (see (3.2.11)). For any $K \subset\subset \mathbb{Z}^d$ and $A \in \mathcal{W}$, define:

$$Q_K(A) = \sum_{x \in K} P_x[A \cap W_K^0]. \tag{3.2.13}$$

Note that since P_x is a probability measure, $Q_K(A) \leq |K|$, so Q_K is a finite measure.

For different $K \subseteq K' \subset\subset \mathbb{Z}^d$, Q_K and $Q_{K'}$ are consistent in the following sense:

Proposition 3.2.3. *For any $A \in \mathcal{W}^*$, and $K \subseteq K' \subset\subset \mathbb{Z}^d$, we have:*

$$Q_K(\pi^{-1}(A) \cap W_K) = Q_{K'}(\pi^{-1}(A) \cap W_K). \tag{3.2.14}$$

With the help of this proposition, we can define a measure on $(W^*, \mathcal{W}^*)$:

Theorem 3.2.4. *There exists a unique σ-finite measure ν on $(W^*, \mathcal{W}^*)$ which satisfies: for all $K \subset\subset \mathbb{Z}^d$*

$$1\{W_K^*\} \cdot \nu = \pi \circ Q_K. \tag{3.2.15}$$

Proof of Proposition 3.2.3. Write $B = t_K(\pi^{-1}(A) \cap W_K)$. Since Q_K ($Q_{K'}$ resp.) is supported on W_K^0 ($W_{K'}^0$ resp.), we have:

$$Q_K(\pi^{-1}(A) \cap W_K) = Q_K(B) \quad \text{and} \quad Q_{K'}(\pi^{-1}(A) \cap W_K) = Q_{K'}(t_{K'}(B)) \tag{3.2.16}$$

So, we need to prove:

$$Q_K(B) = Q_{K'}(t_{K'}(B)). \tag{3.2.17}$$

We partition W_K^0 according to the hitting time and hitting point of K and K'. For any $x \in K, y \in K'$ and $n \in \mathbb{Z}^- = \{0, -1, -2, ...\}$, define:

$$A_{x,n,y} = \{w \in W : w(0) = x, H_K(w) = 0, w(n) = y, H_{K'}(w) = n\}. \tag{3.2.18}$$

On $A_{x,n,y}$, $t_{K'}$ is injective, $t_{K'}(w)(\bullet) = w(\bullet + n)$ and:

$$t_{K'}(A_{x,n,y}) = \{w \in W : w(0) = y, H_K(w) = -n, w(-n) = x, H_{K'}(w) = 0\}.$$

Let $B_{x,n,y} = B \cap A_{x,n,y}$. Then B has a countable partition:

$$B = \bigcup_{x \in K, y \in K', n \in \mathbb{Z}^-} B_{x,n,y}. \tag{3.2.19}$$

In order to show (3.2.17), it is enough to prove:

$$Q_K(B_{x,n,y}) = Q_{K'}(t_{K'}(B_{x,n,y})). \tag{3.2.20}$$

By definition of Q_K (see (3.2.13)), the left hand side is:

$$\begin{aligned} Q_K(B_{x,n,y}) &= P_x(B_{x,n,y}) = P_x[X_n = y, H_K(X) = 0, H_{K'}(X) = n, X_0 = x] \\ &\overset{(*)}{=} P_y[X_{-n} = x, H_K(X) = -n, H_{K'}(X) = 0, X_0 = y] = Q_{K'}(t_{K'}(B_{x,n,y})) \end{aligned}$$

Since $\{X_n = y, H_K(X) = 0, H_{K'}(X) = n, X_0 = x\}$ is the translation of $\{X_{-n} = x, H_K(X) = -n, H_{K'}(X) = 0, X_0 = y\}$ by n, $(*)$ is due to the translation invariance of Y (see Claim 3.2.2). □

Proof of Theorem 3.2.4. Uniqueness is obvious by (3.2.15).

For the existence of ν, fix a sequence $K_1 \subseteq K_2 \subseteq \dots$ converging to $\mathbb{Z}^d$, define: $\nu(A) = \lim_{n\to\infty} Q_{K_n}((\pi)^{-1}(A \cap W^*_{K_n}))$ (This sequence is increasing and hence the limit exists). We just need to check that ν does not depend on the choice of the sequence. The following is enough: if $K \subseteq K' \subset\subset \mathbb{Z}^d$ and $A \in \mathcal{W}^*, A \subseteq W^*_K \subseteq W^*_{K'}$, then

$$Q_{K'}(\pi^{-1}(A)) = Q_K(\pi^{-1}(A)). \tag{3.2.21}$$

Note that $A \subseteq W^*_K$, so $\pi^{-1}(A) \cap W_K = \pi^{-1}(A)$. The equality above is what Proposition 3.2.3 tells us. □

One can easily check, by definition, the following proposition, which we state here for future use:

Proposition 3.2.5. *1. ν is invariant under the time inversion: $w^* \to \check{w}^*$, where $\check{w}^* = \pi(\check{w})$, with $\pi(w) = w^*$ and $\check{w}(n) = w(-n)$, for $n \in \mathbb{Z}$;*

2. ν is invariant under spatial translations: $w^* \to w^* + x, x \in \mathbb{Z}^d$, where $w^* + x = \pi(w + x)$, with $\pi(w) = w^*$.

Given $K \subset\subset \mathbb{Z}^d$, we define the escape probability, similarly to the analogous notion for simple random walks.

$$\overline{e}_K(x) := P_x[H_K(X) = 0] = 1_{\{x \in K\}} \cdot P_x[\cup_{n<0}\{X_n\} \cap K = \emptyset] \tag{3.2.22}$$
$$= 1_{\{x \in K\}} \cdot P_x[\cup_{n>0}\{X_n\} \cap K = \emptyset]. \tag{3.2.23}$$

The last equality is due to the fact that the law of X is invariant under time-reversal by Claim 3.2.2. Note that $\overline{e}_K$ is supported on $\partial_i K$. We write $P_{(x,K)}$ for the restriction to $(W_+, \mathcal{W}_+)$ of $P_x(\cdot | H_K(X) = 0)$, and write $P_{\overline{e}_K}$ for the normalized measure:

$$P_{\overline{e}_K} = \frac{1}{\sum_{x \in Supp(\overline{e}_K)} \overline{e}_K(x)} \sum_{x \in Supp(\overline{e}_K)} \overline{e}_K(x) P_{(x,K)}. \tag{3.2.24}$$

It is straightforward to check that (see the end of Section 3.2.1 for the definition of t^*_{K+}):

$$\sum_{x \in Supp(\overline{e}_K)} \overline{e}_K(x) P_{(x,K)} = t^*_{K+} \circ (1\{W^*_K\}\nu). \tag{3.2.25}$$

Remark 3.2.3. *In fact, $P_{(x,K)}$ is the law of the positive part of a infinite 2-sided snake starting from x, conditioned on its negative part avoiding K. The positive part and negative part are only related at the spine. Hence, compared to P_x (restricted to $(W_+, \mathcal{W}_+)$), $P_{(x,K)}$ just changes the law of the spatial spine, not the law of the spatial trees grafted through the spine. Moreover, under $P_{(x,K)}$, the spatial spine is a biased Random walk on $\mathbb{Z}^d$. The transition probability of this biased random walk can be expressed as follows: for $x \sim y$, the transition probability $p(x,y) = P_y[(X)_{n \le 0} \cap K = \emptyset] / \sum_{z \sim x} P_z[(X)_{n \le 0} \cap K = \emptyset]$.*

We now define the branching capacity by:

$$\mathrm{BCap}(K) := \nu(W^*_K). \tag{3.2.26}$$

Analogously to the standard capacity, branching capacity is the total mass of escape probability:

$$\begin{aligned}\mathrm{BCap}(K) &= \nu(W_K^*) = Q_K(W_K) = Q_K(W_K^0)\\ &= \sum_{x\in K} P_x[H_K(X)=0] \overset{(3.2.22)}{=} \sum_{x\in K} \overline{e}_K(x).\end{aligned}$$

Remark 3.2.4. *The definition of branching capacity in this chapter is a bit different to the one in previous chapters. In fact, one can verify that* $\overline{e}_K(x) = Es_K(x)/2$. *Therefore, K's branching capacity here equals half of the branching capacity before.*

Also, branching capacity is monotone and subaddictive:

Proposition 3.2.6. *For any* $K \subset\subset K' \subset\subset \mathbb{Z}^d$,

$$BCap(K) \le BCap(K'); \tag{3.2.27}$$

For any $K_1, K_2 \subset\subset \mathbb{Z}^d$,

$$BCap(K_1 \cup K_2) \le BCap(K_1) + BCap(K_2). \tag{3.2.28}$$

Proof. For monotonicity, assume $K \subset\subset K' \subset\subset \mathbb{Z}^d$. Any trajectory hitting K must hit K'. Hence $W_K^* \subseteq W_{K'}^*$. Therefore $\mathrm{BCap}(K) = \nu(W_K^*) \le \nu(W_{K'}^*) = \mathrm{BCap}(K')$.

Similarly, any trajectory hitting $K_1 \cup K_2$ must hit either K_1 or K_2. Hence $W_{K_1\cup K_2}^* \subseteq W_{K_1}^* \cup W_{K_2}^*$. Therefore $\mathrm{BCap}(K_1 \cup K_2) = \nu(W_{K_1\cup K_2}^*) = \nu(W_{K_1}^* \cup W_{K_2}^*) \le \nu(W_{K_1}^*) + \nu(W_{K_2}^*) = \mathrm{BCap}(K_1) + \mathrm{BCap}(K_2)$. □

3.2.4 Branching interlacement point process

We further need to introduce the space of locally finite point measures on W^*:

$$\Omega := \Big\{\omega = \sum_{n\ge 0} \delta_{w_n^*},\ \text{where } w_n^* \in W^*,\ n \ge 0 \\ \text{and } \omega(W_K^*) < \infty \text{ for any } K \subset\subset \mathbb{Z}^d\Big\} \tag{3.2.29}$$

We endow W^* with the σ-algebra $\mathcal{A}$ generated by the evaluation maps of form

$$\omega \mapsto \omega(D) = \sum_{n\geq 0} 1[w_n^* \in D], \quad \text{if } \omega = \sum_{n\geq 0} \delta_{w_n^*}, \quad D \in \mathcal{W}^*. \tag{3.2.30}$$

For any $u \in \mathbb{R}^+$, the probability space of the branching interlacement Poisson point process (PPP) at level u is $(\Omega, \mathcal{A}, \mathbb{P}^u)$, where

$$\omega = \sum_{n\geq 0} \delta_{w_n^*} \quad \text{is a PPP with intesity measure } u\cdot\nu \ \text{ on } \ W^* \ \text{ under } \mathbb{P}^u, \tag{3.2.31}$$

where ν is defined in Theorem 3.2.4.

Up to now, we have constructed the branching interlacement point process. In addition, we would like to introduce some relative PPP on W_+. Consider the space of countable point measures on W_+:

$$M := \left\{\mu = \sum_{i\in I} \delta_{w_i}, \text{ where } w_i \in W_+, \ I \text{ is a finite or infinite subset of } \mathbb{N}\right\} \tag{3.2.32}$$

endowed with the canonical σ-fields $\mathcal{M}$, i.e. generated by the evaluation maps.

For $K \subset\subset \mathbb{Z}^d$ define μ_K and Θ_K in the following way: if $\omega = \sum_{n\geq 0} \delta_{w_n^*} \in \Omega$, then $\mu_K(\omega) = \sum_{n\geq 0} \delta_{t^*_{K+}(w_n^*)} 1\{w_n^* \in W_K^*\}$; if $\mu = \sum_{i\in I} \delta_{w_i}$, then $\Theta(\mu) = \sum_{i\in I} 1\{H_K(w_i) < \infty\}\delta_{\theta_{H_K}(w)}$. In words: in $\mu_K(\omega)$ (or $\Theta(\mu)$) we only collect the trajectories from ω (or μ) which hit the set K, and keep the part of each trajectory which comes after hitting K, and reparameterize the time of the trajectories in a way such that the hitting time of K is 0. We record here the straightforward identities valid for $K \subseteq K' \subset\subset \mathbb{Z}^d$:

$$\Theta_K \circ \mu_{K'} = \mu_K; \tag{3.2.33}$$

$$\Theta_K \circ \Theta_{K'} = \Theta_K. \tag{3.2.34}$$

We have built the Poisson point measure $\mathbb{P}^u$ on $(\Omega, \mathcal{A})$ with intensity $u\cdot\nu$

(see (3.2.31)). Since $P_{\bar{e}_K}$ is a measure on $(W_+, \mathcal{W}_+)$, we can also realize on $(M, \mathcal{M})$ a Poisson point measure with intensity $u \cdot P_{\bar{e}_K}$. We denote the law of this Poisson point measure by $\mathbb{P}^u_K$. Given $\omega = \sum_{i\geq 0} \delta_{w_i^*}$, we write:

$$\check{\omega} = \sum_{i\geq 0} \delta_{\check{w}_i^*} \in \Omega; \qquad \vartheta_x(\omega) = \sum_{i\geq 0} \delta_{w_i^* - x} \in \Omega. \tag{3.2.35}$$

The following follows from (3.2.25), (3.2.33) and Proposition 3.2.5:

Proposition 3.2.7. *For any $K \subseteq K' \subset\subset \mathbb{Z}^d$, $u \in [0, \infty)$ and $x \in \mathbb{Z}^d$, we have:*

1. *$\mathbb{P}^u_K$ is the law of μ_K under $\mathbb{P}^u$;*
2. *$\Theta_K \circ \mathbb{P}^u_{K'} = \mathbb{P}^u_K$;*
3. *$\mathbb{P}^u$ is invariant under $\omega \to \check{\omega}$;*
4. *$\mathbb{P}^u$ is invariant under ϑ_x.*

We can now define the branching interlacement at level u:

Definition 3.2.8. *Branching interlacement at level u is defined to be the random subset of $\mathbb{Z}^d$ given by*

$$\mathcal{I} = \mathcal{I}(\omega) := \bigcup_{n\geq 0} \mathrm{Range}(w_n^*), \quad \textit{where} \quad \omega = \sum_{n\geq 0} \delta_{w_n^*} \ \textit{has law}\ \mathbb{P}^u, \tag{3.2.36}$$

where for $w^* \in W^*$, $\mathrm{Range}(w^*) = w(\mathbb{Z})$, for any $w \in W$ with $\pi(w) = w^*$. The vacant set of branching interlacement at level u is defined by

$$\mathcal{V} = \mathcal{V}(\omega) := \mathbb{Z}^d \setminus \mathcal{I}(\omega). \tag{3.2.37}$$

Sometimes we use $\mathcal{I}^u$ and $\mathcal{V}^u$ instead of $\mathcal{I}$ and $\mathcal{V}$ to emphasize the dependence of u. Note that in view of (3.2.33), we have:

$$\mathcal{I}(\omega) \cap K = \bigcup_{w \in \mathrm{Supp}\, \mu_{K'}(\omega)} w(\mathbb{N}) \cap K, \tag{3.2.38}$$

for any $K \subset\subset K' \subset\subset \mathbb{Z}^d$.

Proposition 3.2.9. *For any $u \geq 0$ and $K \subset\subset \mathbb{Z}^d$, we have:*

$$\mathcal{I}(\omega) \cap K \neq \emptyset \Leftrightarrow \mu_K(\omega) \neq 0; \tag{3.2.39}$$

$$\mathbb{P}^u[K \subseteq \mathcal{V}(\omega)] = \exp(-uBCap(K)). \tag{3.2.40}$$

Proof. (3.2.39) follows immediately from (3.2.38).

$$\begin{aligned}\mathbb{P}^u[K \subseteq \mathcal{V}(\omega)] &= \mathbb{P}^u[K \cap \mathcal{I}(\omega) = \emptyset] = \mathbb{P}^u[\mu_K(\omega) = 0] \\ &= \exp(-u \cdot \nu(W_K^*)) = \exp(-u\mathrm{BCap}(K)).\end{aligned}$$

□

Remark 3.2.5. *Analogously to the case of random interlacements in [21, (2.17)]), using the inclusion-exclusion principle, one can see that* (3.2.40) *uniquely determines the law of $\mathcal{V}$ and $\mathcal{I}$.*

In view of Proposition 3.2.7, there is an equivalent way to construct a set with the same law as $\mathcal{I} \cap K$.

Proposition 3.2.10. *For any $K \subset\subset \mathbb{Z}^d$, let N_K be a Poisson random variable with parameter $u \cdot BCap(K)$, and $(X^j)_{j\geq 1}$ i.i.d. with the law $P_{\overline{e}_K}$ and independent from N_K. Then $K \cap \left(\cup_{j=1}^{N_K} X^j(\mathbb{N})\right)$ has the same distribution as $\mathcal{I}^u \cap K$.*

3.3 Branching random walk on the torus and branching interlacements

In this section we consider branching random walk on the discrete torus $\mathbb{T}_N := (\mathbb{Z}/N\mathbb{Z})^d$ of side-length N (for any $d \geq 5$ fixed). For some technical reason due to the periodicity of simple random walk on the torus, we assume N is an odd number, see Remark 3.3.4. We prove that for any fixed $u > 0$, the local limit (as $N \to \infty$) of the set of vertices in $\mathbb{T}_N$ visited by the branching random walk with a uniformly distributed starting point, conditioned to have $\lfloor uN^d \rfloor$ progeny is given by branching interlacement at level $2u$.

We write $\varphi : \mathbb{Z}^d \to \mathbb{T}_N$ for the canonical projection map induced by mod N. Recall that (Definition 3.2.1) P_x is the law of infinite snake. We write $P_x^{L,N}$ (respectively P_x^L) for the law of snake conditioned to have length $2L$ on the torus $\mathbb{T}_N$ (respectively on $\mathbb{Z}^d$) with starting point x. If the starting point is uniform on $\mathbb{T}_N$, we will use $P^{L,N}$. P^L will be reserved for the law of 1-dimensional simple random walk excursion conditioned to have length $2L$ (i.e. $\tau = 2L + 1$).

Theorem 3.3.1. *For any $K \subset\subseteq \mathbb{Z}^d$ and $u > 0$, if N is odd and (X_n) is a snake on $\mathbb{T}_N$, conditioned to have length $2L$ with uniform starting distribution, where $L = \lfloor uN^d \rfloor$, then*

$$\lim_{N\to\infty} P^{L,N}[\{X_0, X_1, ..., X_{2L}\} \cap \varphi(K) = \emptyset] = e^{-2u\cdot BCap(K)}. \qquad (3.3.1)$$

Remark 3.3.1. *By* (3.2.40)*, the right hand side is $\mathbb{P}^{2u}[\mathcal{I} \cap K = \emptyset]$.*

Remark 3.3.2. *Note that the statement here is a bit different to the statement (Theorem 1.4.1 and Theorem 1.4.2) in Section 1.4 . The reason is that the branching capacity in this chapter differs from the one in the previous chapters by a multiplicative constant,* 1/2*. See Remark 3.2.4.*

Through the inclusion-exclusion principle, this theorem implies the local convergence of the configuration:

Corollary 3.3.2. *Under the same assumptions on Theorem 3.3.1, for any $A \subseteq K$, we have:*

$$\lim_{N\to\infty} P^{L,N}[\{X_0, X_1, ..., X_{2L}\} \cap \varphi(K) = A] = \mathbb{P}^{2u}[\mathcal{I} \cap K = A]. \qquad (3.3.2)$$

The idea of the proof of Theorem 3.3.1 is to use the 'law of rare events', i.e., to decompose the event into the intersection of weakly dependent rare events. Hence the proof consists of two main ingredients. One is to estimate the hitting probability of $\varphi(K)$ by a small snake, see Section 3.3.1; the other is to cut a large tree into small subtrees, see Section 3.3.2 .

3.3.1 Hitting probability of a set by a small snake

Theorem 3.3.1 gives an asymptotic formula for the probability that the snake visits a subset on $\mathbb{T}_N$ with length proportional to the volume of the torus, N^d. The main result of this section gives an asymptotic formula for the probability of the event that a set is hit by a much shorter snake.

Proposition 3.3.3. *For $\alpha_1 < \alpha_2 \in (0,d)$ fixed, $L = L(N)$ is any integer-valued function of N satisfying $L(N) \in [N^{\alpha_1}, N^{\alpha_2}]$, then*

$$\lim_{N\to\infty} \frac{N^d}{2L} P^{L,N}(\{X_0, X_1, ..., X_{2L}\} \cap \varphi(K) \neq \emptyset) = BCap(K). \tag{3.3.3}$$

In order to prove this proposition, we need the following lemma.

Lemma 3.3.4. *If $S = (S_i)$ is one-dimensional simple random walk excursion conditioned to have length $2L$, then for any $L \in \mathbb{N}^+$, $i \in [[0, L]]$ and $x \in [[0, i]]$, we have:*

$$P^L[S_i = x] \preceq (x+1)^2(i+1)^{-\frac{3}{2}}; \tag{3.3.4}$$

$$P^L[S_i \leq x] \preceq (x+1)^3(i+1)^{-\frac{3}{2}}; \tag{3.3.5}$$

$$P^L[S_i = x] \preceq (i+1)^{-\frac{1}{2}}; \tag{3.3.6}$$

For any $\epsilon \in (0, 1/2)$ and $n \in \mathbb{N}$, there exists $C(\epsilon, n) > 0$, such that, for any $L \in \mathbb{N}^+$ and $i \in [[0, L]]$, we have:

$$P^L[S_i \geq i^{\frac{1}{2}+\epsilon}] \leq C(\epsilon, n) i^{-n}. \tag{3.3.7}$$

In words, the L.H.S. decays much faster than any polynomial of i.

Proof. In view of (3.1.9), we have (when i and x have the same parity):

$$\begin{aligned}
P^L[S_i = x] &\overset{(*)}{=} \frac{(P_x[\tau = i+1]\cdot 2^{i+1})(P_x[\tau = 2L-i+1]\cdot 2^{2L-i+1})}{P_0[\tau = 2L+1]\cdot 2^{2L+1}} \\
&= \frac{2\frac{x+1}{i+1}P_x[S_{i+1} = -1]\frac{x+1}{2L-i+1}P_x[S_{2L-i+1} = -1]}{\frac{P_0[S_{2L+1}=-1]}{2L+1}} \\
&\asymp \frac{(x+1)^2}{i+1}\frac{P_0[S_{i+1} = -1-x]P_0[S_{2L-i+1} = -1-x]}{P_0[S_{2L+1} = -1]} \\
&\overset{(3.1.4),(3.1.10)}{\preceq} \frac{(x+1)^2}{i+1}\frac{P_0[S_{i+1} = -1-x]\frac{1}{\sqrt{2L-i+1}}}{\frac{1}{\sqrt{2L+1}}} \\
&\preceq \frac{(x+1)^2}{i+1}P_0[S_{i+1} = -1-x] \\
&\overset{(3.1.10)}{\preceq} \frac{(x+1)^2}{i+1}(i+1)^{-1/2}\exp(-c\frac{(x+1)^2}{i+1}) \\
&= \frac{(x+1)^2}{(i+1)^{3/2}}\exp(-c\frac{(x+1)^2}{i+1}),
\end{aligned}$$

where in $(*)$ we used the time-reversibility of the random walk. Since $\exp(-c(x+1)^2/(i+1)) \le 1$, we have (3.3.4). By summing (3.3.4), we get (3.3.5). Because $\big((x+1)^2/(i+1)\big)\exp\big(-c\cdot(x+1)^2/(i+1)\big) = t\exp(-ct)$ is less than a constant (which only depends on c) we have (3.3.6). For (3.3.7), we have:

$$\begin{aligned}
P^L[S_i \ge i^{\frac{1}{2}+\epsilon}] &\le \sum_{x\ge i^{\frac{1}{2}+\epsilon}} \frac{1}{\sqrt{i+1}}\left(\frac{x+1}{\sqrt{i+1}}\right)^2\exp(-c\left(\frac{x+1}{\sqrt{i+1}}\right)^2) \\
&\preceq \int_{\frac{i^{\frac{1}{2}+\epsilon}}{\sqrt{i+1}}}^{\infty} t^2\exp(-ct^2)dt \le \int_{i^\epsilon/2}^{\infty} t^2\exp(-ct^2)dt \\
&= \int_{i^{2\epsilon}/4}^{\infty}\frac{u}{2}\exp(-cu)du = \frac{1}{2c}(\frac{i^{2\epsilon}}{4}+\frac{1}{c})\exp(-c\frac{i^{2\epsilon}}{4}).
\end{aligned}$$

The last term decays faster than any polynomial of i. □

Proof of Proposition 3.3.3. We start with recalling some combinatorial properties of Dyck paths (see the discussion before Proposition 3.1.2 for the definition of Dyck paths). Fix $k \ge 1$ and $j \in \{0, 1, ..., 2k\}$, for any $i \in$

$\{0, 1, ...2k-1\}$ and Dyck path $(s_0, s_1, ...s_{2k})$ with length $2k$, , define

$$s_j^{(i)} = s_i + s_{i\oplus j} - 2 \min_{i\wedge(i\oplus j)\leq n\leq i\vee(i\oplus j)} s_n \tag{3.3.8}$$

where $i \oplus j = i + j$, if $i + j \leq 2k$ and $i \oplus j = i + j - 2k$, if $i + j \geq 2k$. It is elementary to see that $(s_0^{(i)}, s_1^{(i)}, ...s_{2k}^{(i)})$ is still a Dyck path with length $2k$ and that the mapping $\Phi_i : (s_0, s_1, ...s_{2k}) \to (s_0^{(i)}, s_1^{(i)}, ...s_{2k}^{(i)})$ is a bijection from the set of all Dyck paths with length $2k$ onto itself (e.g. see page 14 of [15]). Recall that, under P^L (or $P^{L,N}$), $(S_0, S_1, ...S_{2k})$ is uniformly distributed on the set of all Dyck paths with length $2L$. Hence $(S_0^{(i)}, S_1^{(i)}, ...S_{2k}^{(i)})$ is distributed identically as $(S_0, S_1, ...S_{2k})$. From this and the fact that the starting measure is uniform on the torus, one can see that, under $P^{L,N}$, $(X_0, X_1, ..., X_{2L})$ and $(X_i, X_{i+1}, ..., X_{2L}, X_1, ..., X_{i-1}, X_i)$ have the same law.

On the other hand, the 'time reversal' map $s = (s_0, s_1, ...s_{2k}) \to \check{s} = (s_{2k}, s_{2k-1}, ..., s_0)$ is also a bijection on the set of all Dyck paths with length $2k$. Hence, under $P^{L,N}$, $(X_0, X_1, ..., X_{2L})$ and $(X_{2L}, X_{2L-1}, ..., X_0)$ have the same law (here we also use the fact that the increment variables Y_i have symmetric distribution).

Write $K' = \varphi(K)$, we have (when $N > \mathrm{diam}(K) := \max\{|a - b| : a, b \in K\}$):

$$\begin{aligned}
&P^{L,N}(\{X_0, X_1, ..., X_{2L}\} \cap K' \neq \emptyset) \times \frac{N^d}{2L}\\
&= \sum_{x\in\partial_i K'} \sum_{k=1}^{2L-1} P^{L,N}[X_0, ...X_{k-1} \notin K'; X_k = x] \cdot \frac{N^d}{2L} + \sum_{x\in K'} P^{L,N}(X_0 = x) \cdot \frac{N^d}{2L}\\
&\overset{(*)}{=} \sum_{x\in\partial_i K'} \sum_{k=1}^{2L-1} P^{L,N}[X_0 = x; X_1, ...X_k \notin K'] \cdot \frac{N^d}{2L} + \sum_{x\in K'} \frac{1}{2L}\\
&= \sum_{x\in\partial_i K'} \frac{1}{2L} \sum_{k=1}^{2L-1} P_x^{L,N}[X_1, ...X_k \notin K'] + \frac{|K|}{2L},
\end{aligned}$$

$(*)$ is due to:

$$\begin{aligned} P^{L,N}[X_0, ...X_{k-1} \notin K'; X_k = x] &= P^{L,N}[X_{2L-k}, ...X_{2L-1} \notin K'; X_{2L} = x] \\ &= P^{L,N}[X_k, ...X_1 \notin K'; X_0 = x]. \end{aligned}$$

Hence in order to prove Proposition 3.3.3 it suffices to show that for $x \in \partial_i K$

$$\lim_{N\to\infty} \frac{1}{2L} \sum_{k=1}^{2L-1} P^{L,N}_{\varphi(x)}[X_1, ...X_k \notin K'] = \overline{e}_K(x), \tag{3.3.9}$$

where $\overline{e}_K(x)$ is the escape probability (see (3.2.22)).

For the above, it is enough to prove:

$$\lim_{N\to\infty} \max_{L'<k<2L-L'} |P^{L,N}_{\varphi(x)}[X_1, ...X_k \notin K'] - P_x[X_n \notin K \text{ for any } n > 0]| = 0. \tag{3.3.10}$$

for some $L' = L'(N)$, a function of N satisfying $L'(N) \to \infty$ and $\frac{L'(N)}{L(N)} \to 0$ as $N \to \infty$ (e.g. we can fix $L' = \lfloor L^{0.2} \rfloor$ which satisfies also the condition in Lemma 3.3.7).

The proof of Proposition 3.3.3 is now reduced to the following lemmas:

Lemma 3.3.5. *For any $x \in \partial_i K$,*

$$\lim_{N\to\infty} \max_{L'<k<2L-L'} |P^{L,N}_{\varphi(x)}[X_1, ...X_k \notin K'] - P^L_x[X_1, ...X_k \notin K]| = 0. \tag{3.3.11}$$

Lemma 3.3.6. *For any $x \in \partial_i K$,*

$$\lim_{N\to\infty} \max_{L'<k<2L-L'} |P^L_x[X_1, ...X_k \notin K] - P^L_x[X_1, ...X_{L'} \notin K]| = 0. \tag{3.3.12}$$

Lemma 3.3.7. *For any $x \in \partial_i K$, if $L' = o(\sqrt{L})$, then:*

$$\lim_{N\to\infty} |P^L_x[X_1, ...X_{L'} \notin K] - P_x[X_1, ...X_{L'} \notin K]| = 0. \tag{3.3.13}$$

Note that $P_x[X_1, ...X_{L'} \notin K]$ converges to the escape probability $\overline{e}_K(x) = P_x(\cup_{n>0}\{X_n\} \notin K)$, so (3.3.10) indeed follows from Lemmas 3.3.5, 3.3.6, 3.3.7.

Without loss of generality, we assume $x = 0 \in \partial_i K$ and $N > 2\mathrm{diam}(K)$.

Proof of Lemma 3.3.5. Let $\varphi^{-1}(K') = \bigcup_{i=0}^{\infty} K_i$, such that, $x \in K_0 = K$ and K_i is a translated copy of K_0. Recall from the statement of Proposition 3.3.3 that $\alpha_2 < d$ and choose $\lambda \in (\frac{1}{4}, \frac{d}{4\alpha_2})$ and let $b = \lfloor \frac{L^\lambda}{N} \rfloor + 1$.

$$\begin{aligned}
&P_0^L[X_1, ...X_k \notin K] - P_{\varphi(0)}^{L,N}[X_1, ...X_k \notin K'] \\
=&P_0^L[X_1, ...X_k \notin K] - P_0^L[X_1, ...X_k \notin \varphi^{-1}(K)] \\
=&P_0^L[X_1, ...X_k \notin K] - P_0^L[X_1, ...X_k \notin K, X_1, ...X_k \notin K_i \,\text{for}\, i \geq 1] \\
=&P_0^L[X_1, ...X_k \notin K, \{X_1, ...X_k\} \cap (\cup_{i\geq 1} K_i) \neq \emptyset] \\
\leq&P_0^L[\{X_1, ...X_k\} \cap (\cup_{i\geq 1} K_i) \neq \emptyset] \\
\leq&P_0^L[\sup_{0\leq i\leq 2L} |X_i| > bN] + P_0^L[\sup_{0\leq i\leq 2L} |X_i| \leq bN, \{X_1, ...X_k\} \cap (\cup_{i\geq 1} K_i) \neq \emptyset]
\end{aligned} \tag{3.3.14}$$

The first term above goes to 0, due to the following (since $bN \geq L^\lambda, \lambda > 1/4$):

Proposition 3.3.8. *For any $c > \frac{1}{4}$,*

$$\lim_{L\to\infty} P_0^L[\sup_{0\leq i\leq 2L} |X_i| > L^c] \to 0. \tag{3.3.15}$$

This Proposition is an easy corollary in the theory of convergence of discrete snakes (see e.g.[15], or more generally [8]). In fact, $\sup_{0\leq i\leq 2L} |X_i|/(2L)^{1/4}$ converges in distribution as $L \to \infty$ to an a.s. finite random variable.

For the estimate of the second term in (3.3.14), we will use the following (a special case of Theorem 1.13 in [7]):

Proposition 3.3.9. *There exists a constant C, such that for all $n \in \mathbb{N}$, if T_n is GW tree conditioned to have n progeny and $w_k(T_n)$ is the number of vertices in the k-th generation of T_n, then we have*

$$\mathbb{E}(w_k(T_n)) \leq C \cdot k. \tag{3.3.16}$$

With the help of this, for any $y \in \mathbb{Z}^d$, let $M = |\{i \in [0, 2L] : X_i = y\}|$. We have:

$$\begin{aligned} P_0^L[X_i = y \text{ for some } 0 \le i \le 2L] &= P_0^L[M > 0] \le \mathbb{E}(M) \\ &\overset{(3.3.16)}{\preceq} \sum_{k=0}^{\infty} k \cdot p_k(0, y) \overset{(3.1.8)}{\asymp} |y|^{4-d}. \end{aligned} \tag{3.3.17}$$

Now we estimate the second term in (3.3.14) for any $1 \le k \le 2L$:

$$\begin{aligned} &P_0^L[\sup_{0 \le i \le 2L} |X_i| \le bN, \{X_1, ...X_k\} \cap (\cup_{i \ge 1} K_i) \neq \emptyset] \\ \le &\sum_{i: K_0 \neq K_i \subseteq B_0((b+1)N)} P_0^L[\{X_1, ...X_k\} \cap K_i \neq \emptyset] \\ = &\sum_{i=1}^{b+1} \sum_{j: K_j \cap S_0(iN) \neq \emptyset} P_0^L[\{X_1, ...X_k\} \cap K_j \neq \emptyset] \\ \overset{(3.3.17)}{\preceq} &\sum_{i=1}^{b+1} \sum_{j: K_j \cap S_0(iN) \neq \emptyset} \frac{|K|}{(iN)^{d-4}} \preceq \sum_{i=1}^{b+1} i^{d-1} \cdot \frac{|K|}{(iN)^{d-4}} \\ \preceq &|K| \frac{b^4}{N^{d-4}} \preceq |K| \frac{L^{4\lambda}/N^4 + 1}{N^{d-4}} \to 0, \end{aligned}$$

where the last convergence follows from $\lambda < \frac{d}{4\alpha_2}$, $\alpha_2 < d$ and $L \le N^{\alpha_2}$. $\square$

Proof of Lemma 3.3.6. Recall that we have assumed $x = 0 \in \partial_i K$. Let $L' < k < 2L - L'$.

$$\begin{aligned} &P_0^L[X_1, ...X_{L'} \notin K] - P_0^L[X_1, ...X_k \notin K] \\ \le &P_0^L[\exists i \in (L', k], X_i \in K] \\ \le &P_0^L[\exists i \in (L', 2L - L'), X_i \in K] \\ \le &P_0^L[\exists i \in (L', L], X_i \in K] + P_0^L[\exists i \in [L, 2L - L'), X_i \in K] \\ = &2P_0^L[\exists i \in (L', L], X_i \in K], \end{aligned}$$

where in the last line we used the reversal property described in the beginning of the proof of Proposition 3.3.3.

Let us estimate $P_0^L[X_i = y]$ for any $y \in \mathbb{Z}^d$:

$$P_0^L[X_i = y] = \sum_{l=0}^{\infty} P^L[S_i = l] \cdot P_0^L[X_i = y | S_i = l] \tag{3.3.18}$$

Recall from Section 3.2.2 that under P^L, $S_i = S_i(U)$ is the contour function of the random tree conditioned to have size L. $P_0^L[X_i = y | S_i = l]$ is the probability of $Z_l = y$, where $Z = (Z_n)_{n \in \mathbb{N}}$ is the simple random walk from 0 in $\mathbb{Z}^d$. Recall that we have ((3.1.3) and Lemma 3.3.4) :

$$P^L[S_i = l] \preceq (l+1)^2 \cdot i^{-\frac{3}{2}},\ P^L[S_i = l] \preceq i^{-\frac{1}{2}},\ P(Z_l = y) \preceq l^{-\frac{d}{2}}.$$

Therefore:

$$\begin{aligned}
P_0^L[X_i = y] &\overset{(3.3.18)}{\leq} P_0^L[S_i = 0] + \sum_{0 < l \leq \sqrt{i}} P^L[S_i = l] \cdot P_0^L[X_i = y | S_i = l] + \\
&\qquad \sum_{l > \sqrt{i}} P^L[S_i = l] \cdot P_0^L[X_i = y | S_i = l] \\
&\preceq i^{-\frac{3}{2}} + \sum_{0 < l \leq \sqrt{i}} (l+1)^2 i^{-\frac{3}{2}} \cdot l^{-\frac{d}{2}} + \sum_{l > \sqrt{i}} i^{-\frac{1}{2}} \cdot l^{-\frac{d}{2}} \\
&\preceq i^{-\frac{3}{2}} + i^{-\frac{3}{2}} \sum_{0 < l \leq \sqrt{i}} (l+1)^{2-\frac{d}{2}} + i^{-\frac{1}{2}} \sum_{l > \sqrt{i}} l^{-\frac{d}{2}}.
\end{aligned}$$

Note that every term is decreasing in d. Hence we can assume $d = 5$:

$$\begin{aligned}
P_0^L[X_i = y] &\preceq i^{-\frac{3}{2}} + i^{-\frac{3}{2}} \sum_{0 < l \leq \sqrt{i}} (l+1)^{2-\frac{5}{2}} + i^{-\frac{1}{2}} \sum_{l > \sqrt{i}} l^{-\frac{5}{2}} \\
&\preceq i^{-\frac{3}{2}} + i^{-\frac{3}{2}} (\sqrt{i})^{\frac{1}{2}} + i^{-\frac{1}{2}} (\sqrt{i})^{-\frac{3}{2}} \preceq i^{-\frac{5}{4}}.
\end{aligned}$$

So $P_0^L[X_i = y]$ is summable in i and:

$$\begin{aligned}
P_0^L[\exists i \in (L', L], X_i \in K] &\leq 2|K| \sum_{L' < i \leq L} P_0^L[X_i = y] \\
&\preceq 2|K| \sum_{i > L'} i^{-\frac{5}{4}} \overset{L' \to \infty}{\longrightarrow} 0.
\end{aligned}$$

□

Let us do some preparations before proving Lemma 3.3.7. When considering a 1-sided snake conditioned on survival, it is convenient to introduce the discrete Bessel Processes. We use the setting of 1-sided snake conditioned on survival:

$$U'_i,\ i \in \mathbb{N}^+, \quad i.i.d. \quad with\ P(U'_i = 1) = P(U'_i = -1) = \frac{1}{2};$$
$$S'_n(U') = U'_1 + ... + U'_n.$$

$S'_n(U')$ is the 1-sided simple random walk. Let

$$M_n = M_n(U') = \max_{k \le n} S'_k, \qquad R_n = R_n(U') = 2M_n - S'_n,$$

then the process $(R_n)_{n\in\mathbb{N}}$ is called the discrete Bessel Process (DBP). We also can define a partial matching on the set $E(\mathbb{N}) = \{e_i = (i-1, i);\ i \in \mathbb{N}^+\}$ of all edges of the lattice $\mathbb{N}$. Any edge is either in the set of upsteps $M(U') = \{e_i : R_i - R_{i-1} = 1\}$ or the set of downsteps $N(U') = \{e_i : R_i - R_{i-1} = -1\}$. For any edge $e_l \in N(U')$, we can find a unique edge $f'_U(e_l) = e_k \in M(U')$, such that:

$$k < l,\ R_{k-1} = R_l,\ R_k = R_{l-1} = \min_{k \le i \le l-1} R_i = R_l + 1. \tag{3.3.19}$$

Note that there are some upsteps with no downstep matched to them. Similarly to the construction given in Section 3.2.2, under gluing through this matching, we get a (random) tree such that $(R_n)_{n\in\mathbb{N}}$ is the contour function. It is elementary to see that the tree has the same distribution as the tree corresponding to the one-sided infinite snake.

Another equivalent definition of the law of DBP is as follows (e.g. see [17]). $(R_n)_{n\in\mathbb{N}}$ is the $\mathbb{N}$-valued Markov process starting at zero and having the transition function specified by the relation:

$$P[R_{n+1} - R_n = \Delta | R_n] = \frac{R_n + 1 + \Delta}{2(R_n + 1)}, \quad \Delta = \pm 1. \tag{3.3.20}$$

Denote by

$$\Gamma_n = \{(s_1, s_2, ...s_n) : s_i \in \mathbb{N}, s_1 = 1, |s_{i+1} - s_i| = 1\}$$

the set of all sample paths of $(R_i)_{1\leq i\leq n}$. In view of (3.3.20), for any $s = (s_1, ...s_n) \in \Gamma_n$, one obtains

$$a(s) := P[(R_1, ...R_n) = s] = \frac{s_n + 1}{2^n}; \tag{3.3.21}$$

On the other hand, similarly to the computation in the proof of Lemma 3.3.4, we obtain that for any $1 \leq n \leq 2L$ we have

$$a_L(s) := P^L[(S_1, ...S_n) = s] = \frac{\frac{s_n+1}{2L+1-n}\begin{pmatrix} 2L-n+1 \\ L-\frac{n+s_n}{2}\end{pmatrix}}{\frac{1}{2L+1}\begin{pmatrix} 2L+1 \\ L\end{pmatrix}}. \tag{3.3.22}$$

Let us estimate $a_L(s)/a(s)$:

$$\begin{aligned}\frac{a_L(s)}{a(s)} &= \frac{(s_n+1)\cdot(L+1)...(L-\frac{n-s_n}{2}+2)\cdot L...(L-\frac{n+s_n}{2}+1)}{(2L)...(2L-n+1)} \Big/ \frac{(s_n+1)}{2^n} \\ &= \frac{(L+1)L...(L-\frac{n-s_n}{2}+2)\cdot L(L-1)...(L-\frac{n+s_n}{2}+1)}{L(L-\frac{1}{2})(L-\frac{2}{2})...(L-\frac{n-1}{2})} \\ &\in \left((\frac{L-n}{L})^n, \frac{L+1}{L}\cdot(\frac{L}{L-n})^n \right).\end{aligned}$$

Hence, if $n = o(\sqrt{L})$, then $a_L(s)/a(s) \to 1$ uniformly for all $s \in \Gamma_n$.

Proof of Lemma 3.3.7. Using our new description of the law P_x of the one-sided snake in terms of the DBP, the definitions (3.3.21) and (3.3.22) as well as the fact

$$P_x[X_1, ...X_{L'} \notin K | (R_1, ...R_{L'}) = s] = P_x^L[X_1, ...X_{L'} \notin K | (S_1, ...S_{L'}) = s],$$

we obtain

$$P_x^L[X_1, ...X_{L'} \notin K] = \sum_{s \in \Gamma_{L'}} a_L(s) \cdot P_x^L[X_1, ...X_{L'} \notin K | (S_1, ...S_{L'}) = s];$$
$$P_x[X_1, ...X_{L'} \notin K] = \sum_{s \in \Gamma_{L'}} a(s) \cdot P_x^L[X_1, ...X_{L'} \notin K | (S_1, ...S_{L'}) = s].$$

Since $P_x[X_1, ...X_{L'} \notin K] \in (0,1)$ if $x \in \partial_i K$, we obtain

$$P_x^L[X_1, ...X_{L'} \notin K]/P_x[X_1, ...X_{L'} \notin K] \to 1$$

as a consequence of $L'/\sqrt{L} \to 0$. The proof of Lemma 3.3.7 is complete. $\square$

$\square$

3.3.2 Cutting trees

Our goal for this subsection is to construct the following 'cutting tree' lemma:

Lemma 3.3.10. *Assume $d \geq 5, u > 0$ fixed. Let T be a uniform tree in $\mathbf{A}_L$ where $L = \lfloor uN^d \rfloor$. Then, there are some $\epsilon, \eta > 0, a_1, a_2 \in (4, d)$ (depending on u, d only) such that for any sufficiently large $N \in \mathbb{N}$, with probability at least $1 - N^{-\epsilon}$ we can find a number of rooted subtrees $T_1, \ldots T_{n'}$ (T_i is rooted at v_i, the unique vertex in T_i closest to o, the root of T) satisfying the following:*

1. *For every $i \in \{1, ..., n'\}$, $N^{a_1} \leq |T_i| \leq N^{a_2}$ and the distance between v_i and o is bigger than $N^{2+\eta}$;*

2. *Let $\hat{T}$ be the graph generated by the all edges not in any T_i. Then $\hat{T}$ is a tree and $|\hat{T}| \leq N^{d-\epsilon}$;*

3. *Let ι_i ($i = 1, \ldots, n'$) be the unique path starting from v_i towards the root of T, with length $\lfloor N^{2+\eta} \rfloor + 1$. Then for any $i \in \{1, \ldots, n'\}$, all T_j except T_i, are in the same component of $T \setminus \iota_i$;*

4. *Conditioned on $\{n'; |T_1|, \dots, |T_{n'}|; \hat{T}\}$ (and even the places of v_i in $\hat{T}$), the trees T_i are independent and uniform on all plane trees with their sizes.*

As before, we will use contour function to represent a tree. For simple random walk $(S_n)_{n\in\mathbb{N}}$, conditioned on $\tau\,(=\inf\{n : S_n = -1\}) = 2L+1$, $(S_n)_{n\in[[0,2L]]}$ is the contour function of a random tree T which is uniformly distributed over $\mathbf{A}_L$. If for some subinterval $I = [[a,b]] \subseteq [[0,2L]]$ $(a, b \in \mathbb{N})$ we have:

$$S_a = S_b = \min_{a\le n\le b} S_n, \tag{3.3.23}$$

then $(S_n)_{n\in I}$ is the contour function of a subtree of T (rooted at the vertex corresponding to a and b). We denote by ξ the size of the unconditioned GW tree. It is standard that

$$\begin{aligned} P[\xi = j] &= P[\inf\{n : S_n = -1\} = 2j+1] \\ &\overset{(3.1.9)}{=} \frac{1}{2j+1}\cdot\frac{1}{2^{2j+1}}\binom{2j+1}{j} \overset{(3.1.4)}{\asymp} (j+1)^{-\frac{3}{2}}. \end{aligned} \tag{3.3.24}$$

First we introduce some lemmas which will be used in the proof of Lemma 3.3.10.

Lemma 3.3.11. *For any $\beta, \epsilon \in (0, 1/2)$, $\epsilon < \beta/2$, there exist positive constants C_1 and C_2 (depending on β, ϵ) satisfying the following. For $\xi_1, ..., \xi_m$, i.i.d. with the distribution of ξ, let*

$$\tilde{\xi}_i = \xi\cdot 1_{\{\xi\le m^{2-\beta}\}}, \quad \sigma_m = \xi_1 + ... + \xi_m, \quad \tilde{\sigma}_m = \tilde{\xi}_1 + ... + \tilde{\xi}_m.$$

Then, for any integer $M \in [\frac{1}{10}m^{2-\epsilon}, 10m^{2.5}]$, we have:

$$P[\tilde{\sigma}_m > C_1 m^{2-\beta/2} | \sigma_m = M] \le \exp(-C_2 m^{\beta/2}). \tag{3.3.25}$$

We need the so-called Bernstein Inequality (see, e.g. the part of 'Existing Inequalities' in [4]) to prove the lemma above:

Proposition 3.3.12. *Let $X_1, X_2, ...X_n$ be independent zero-mean random variables. Suppose that $|X_i| \le \bar{M}$ almost surely, for all i. Then, for all* *positive t:*

$$P[\sum_{i=1}^{n} X_i > t] \le \exp(-\frac{\frac{1}{2}t^2}{\sum E(X_j^2) + \frac{1}{3}\bar{M}t}) \tag{3.3.26}$$

Proof of Lemma 3.3.11. Since $E\tilde{\xi} \asymp m^{1-\beta/2}$ and $E\tilde{\xi}^2 \asymp m^{3-3\beta/2}$, using Bernstein inequality (let $t = m^{2-\beta/2}$), we get: for some positive constants C_1, C_2,

$$P[\tilde{\sigma}_m > C_1 m^{2-\beta/2}] \le \exp(-C_2 m^{\beta/2}). \tag{3.3.27}$$

On the other hand, when $M \in [\frac{1}{10}m^{2-\epsilon}, 10m^{2.5}]$, we have:

$$\begin{aligned}
P[\sigma_m = M] &= P_{m-1}[\tau = 2M+m] \overset{(3.1.9)}{=} \frac{m}{2M+m} P_{m-1}[S_{2M+m} = -1] \\
&= \frac{m}{2M+m} P_0[S_{2M+m} = m] \overset{(3.1.4)}{\asymp} \frac{m}{(2M+m)^{\frac{3}{2}}} \exp(-\frac{m^2}{2(2M+m)}) \\
&\ge \exp(-Cm^{\epsilon}).
\end{aligned}$$

Combining (3.3.27) and the inequality above, we get: when $\epsilon < \beta/2$,

$$P[\tilde{\sigma}_m > C_1 m^{2-\beta/2} | \sigma_m = M] \le \exp(-C_3 m^{\beta/2}). \tag{3.3.28}$$

□

Another lemma we need is:

Lemma 3.3.13. *For any positive η, δ, there exists a positive constant $C(\eta,\delta)$, such that, for any $L \ge 2N^{4+2\eta+\delta}$, we have:*

$$P^L\left[S_i < N^{2+\eta},\ \textit{for some } i \in [N^{4+2\eta+\delta}, 2L - N^{4+2\eta+\delta}]\right] \le C(\eta,\delta)/N^{\frac{3}{8}\delta}. \tag{3.3.29}$$

Recall that P^L is the law of SRW conditioned on $\tau = 2L+1$.

If we let

$$\begin{aligned}
b_1 &= \max\{n \in [[0,L]], S_n = \lfloor N^{2+\eta} \rfloor\} + 1, \\
b_2 &= \min\{n \in [[L,2L]], S_n = \lfloor N^{2+\eta} \rfloor\} - 1,
\end{aligned}$$

then by Lemma 3.3.13, with probability at least $1-C(\eta,\delta)N^{-3\delta/8}$, the length of $[[b_1,b_2]]$ is bigger than $2(L-N^{4+2\eta+\delta})$. Since S on $[[b_1,b_2]]$ satisfies tree condition ((3.3.23)), it means that S on $[[b_1,b_2]]$ is a subtree. Note that the distance between this subtree and the root is $\lfloor N^{2+\eta}\rfloor+1$. Hence we can interpret Lemma 3.3.13 in the language of random tree.

Corollary 3.3.14. *For any $\eta,\delta>0$, there exists a positive constant $C(\eta,\delta)$, satisfying the following: if T is uniform on $\mathbf{A}_L$ with $L\geq 2N^{4+2\eta+\delta}$, then with high probability, at least $1-C(\eta,\delta)/N^{\frac{3}{8}\delta}$, we can find a rooted subtree T' which is rooted at the vertex closet to the original root, such that, the distance between this subtree and the original root is equal to $\lfloor N^{2+\eta}\rfloor+1$ and the number of edges we discard ($|T\setminus\widetilde{T}|$) is at most $N^{4+2\eta+\delta}$. Moreover, conditioned on the size of T', it is uniform on all plane trees of that size.*

The last conclusion is simply from the fact that conditioned on the length, each Dyck path of that length has the same probability weight. Note that if $L\gg N^{4+2\eta+\delta}$, then the ratio of edges discarded is less than $N^{4+2\eta+\delta}/L$, which would be very small.

Before proving Lemma 3.3.13 we introduce some notation. For any $n\in\mathbb{N}^+$ and $i\in[[0,n]]$, write $A(n,i)=\binom{n}{i}-\binom{n}{i+1}$. Using the reflection principle, one can see: for any $x,n\in\mathbb{N}$ with the same parity, $t\in\mathbb{N}$, $x+2t\leq n$,

$$\begin{aligned}
&|\{s:[[0,n]]\to\mathbb{Z}:s(0)=0,s(n)=x,\min_{0\leq i\leq n}S_i=-t;\forall i,|s(i)-s(i-1)|=1\}|\\
=&|\{s:[[0,n]]\to\mathbb{Z}:s(0)=0,s(n)=x,\min_{0\leq i\leq n}S_i\leq -t;\forall i,|s(i)-s(i-1)|=1\}|\\
-&|\{s:[[0,n]]\to\mathbb{Z}:\\
&\qquad s(0)=0,s(n)=x,\min_{0\leq i\leq n}S_i\leq -t-1;\forall i,|s(i)-s(i-1)|=1\}|\\
=&\binom{n}{\frac{n+x}{2}+t}-\binom{n}{\frac{n+x}{2}+t+1}=A(n,\frac{n+x}{2}+t).
\end{aligned}$$

Lemma 3.3.15 (Comparison between Combinations)**.** *For any $\epsilon\in(0,\frac{1}{2})$, $A>0$, there exists $C=C(\epsilon,A)>1$, satisfying the following:*
For any $n,k,k'\in\mathbb{N}^+$, $n/2\leq k<k'<n$, let $i=k-\frac{n-1}{2},i'=k'-\frac{n-1}{2}$. If

$i' < \epsilon n, i'(i'-i) < An$, *then:*

$$\frac{A(n,k)/i}{A(n,k')/i'} \in (1, C(\epsilon, A)). \tag{3.3.30}$$

Remark 3.3.3. *In fact, the case* $\epsilon = \frac{1}{4}, A = 1$ *is enough for our purpose and we only use this case.*

Proof. It is straightforward to get:

$$A(n,k) = \binom{n}{k} - \binom{n}{k+1} = \frac{n!(2k-n+1)}{(k+1)!(n-k)!}; \tag{3.3.31}$$

$$\frac{A(n,k)/i}{A(n,k+1)/(i+1)} = \frac{k+2}{n-k} = 1 + \frac{2i+1}{(n+1)/2-i} < 1 + \frac{4i}{(1/2-\epsilon)n}; \tag{3.3.32}$$

Hence,

$$\begin{aligned} \ln\left(\frac{A(n,k)/i}{A(n,k')/i'}\right) &< \sum_{i\le \bar{i} < i'} \ln(1+\frac{4\bar{i}}{(1/2-\epsilon)n}) \\ &\le \sum_{i\le \bar{i} < i'} \frac{4\bar{i}}{(1/2-\epsilon)n} \le \frac{4(i'-i)i'}{(1/2-\epsilon)n} \le \frac{4A}{1/2-\epsilon}. \end{aligned}$$

The upper bound follows. The lower bound is immediate from (3.3.32). □

Proof of Lemma 3.3.13. By symmetry, it suffices to show:

$$P^L\left[S_i < N^{2+\eta}, \text{ for some } i \in [N^{4+2\eta+\delta}, L]\right] \le C(\eta,\delta)/N^{\frac{3}{8}\delta}. \tag{3.3.33}$$

Let $j = \lfloor N^{4+2\eta+\delta} \rfloor$. By lemma 3.3.4,

$$P^L[S_j \le N^{2+\eta+\frac{3}{8}\delta}] \preceq N^{-\frac{3}{8}\delta}; \tag{3.3.34}$$

$$P^L[S_L \le N^{2+\eta+\frac{3}{8}\delta}] \preceq N^{-\frac{3}{8}\delta}; \tag{3.3.35}$$

$$P^L[S_j \ge \sqrt{L}\cdot N^{\frac{\delta}{10}}] \le C(\eta,\delta) N^{-\frac{3}{8}\delta}; \tag{3.3.36}$$

$$P^L[S_L \ge \sqrt{L}\cdot N^{\frac{\delta}{10}}] \le C(\eta,\delta) N^{-\frac{3}{8}\delta}. \tag{3.3.37}$$

Hence, we have:

$$P^L[S_j, S_L \in [N^{2+\eta+\frac{3}{8}\delta}, \sqrt{L}\cdot N^{\frac{\delta}{10}}] \geq 1 - C(\eta,\delta)/N^{\frac{3}{8}\delta}. \tag{3.3.38}$$

For $a_1, a_2 \in [[N^{2+\eta+\frac{3}{8}\delta}, \sqrt{L}\cdot N^{\frac{\delta}{10}}]]$, such that $2|(a_2-a_1)-(L-j)$ and $m \in [[0, \frac{1}{2}N^{2+\eta+\frac{3}{8}\delta}]]$, write

$$S(a_1,a_2,m) :=$$
$$|\{s : [[j,L]] \to \mathbb{N} : s(j) = a_1, s(L) = a_2, \min_{j\leq i\leq L} S_i = m; \forall i, |s(i)-s(i-1)| = 1\}|$$

We know $S(a_1,a_2,m) = A(L-j, \frac{L-j+a_2-a_1}{2} + a_1 - m)$ (see the discussion before Lemma 3.3.15). We would like to use Lemma 3.3.15 to compare $S(a_1,a_2,m_1)$ and $S(a_1,a_2,m_2)$. For any $m_1, m_2 \in [[0, \frac{1}{2}N^{2+\eta+\frac{3}{8}\delta}]]$, one can check that, if we let

$$i = (L-j) - \frac{(\frac{L-j+a_2-a_1}{2} + a_1 - m_1) - 1}{2},$$
$$i' = (L-j) - \frac{(\frac{L-j+a_2-a_1}{2} + a_1 - m_2) - 1}{2},$$

then,

$$i' \leq a_2 \leq \sqrt{L}N^{\frac{\delta}{10}} < \frac{L}{4}, \quad i' - i \leq \frac{1}{2}N^{2+\eta+\frac{3}{8}\delta};$$
$$i'(i'-i) \leq \sqrt{L}N^{\frac{\delta}{10}}\frac{1}{2}N^{2+\eta+\frac{3}{8}\delta} \leq L.$$

Also we have $i \asymp i'$ (since $i \geq a_1 - \frac{1}{2}N^{2+\eta+\frac{3}{8}\delta} \geq \frac{1}{2}N^{2+\eta+\frac{3}{8}\delta} \geq i'-i$). Hence, by Lemma (3.3.15), for any $m_1 \in [[N^{2+\eta}, \frac{1}{2}N^{2+\eta+\frac{3}{8}\delta}]]$ and $m_2 \in [[0, N^{2+\eta}]]$,

$$S(a_1,a_2,m_1) \geq C(\frac{1}{4},1)S(a_1,a_2,m_2). \tag{3.3.39}$$

Note that the left hand side may be zero (when $\frac{L-j+a_2-a_1}{2} + a_1 - m_1 > \frac{L-j}{2}$), but in that case the right hand side is also zero (since $m_1 > m_2$). Hence,

$$\frac{P^L(m \le N^{2+\eta}|S_j = a_1, S_L = a_2)}{P^L(m \le N^{2+\eta+\frac{3}{8}\delta}/2|S_j = a_1, S_L = a_2)} \le C\frac{N^{2+\eta}}{N^{2+\eta+\frac{3}{8}\delta}/2} \le \frac{2C}{N^{\frac{3}{8}\delta}}. \quad (3.3.40)$$

Combining this and (3.3.38) completes the proof. □

Proof of Lemma 3.3.10. Let us first explain the rough idea of the proof. We first divide the domain $[0, 2L]$ into subintervals. In each subinterval, since S does not necessarily satisfy the tree condition (3.3.23) generally, S restricted to that interval does not correspond a tree. But S restricted to that interval can still be regarded as a series of trees which are attached to the vertices of a segment, called the 'spine', which consists of those edges without matching. Then, we pick up those subtrees with large size. Assume those subtrees kept are $\widetilde{T}_1, \dots, \widetilde{T}_K$. For each $\widetilde{T}_i$, we can apply Corollary 3.3.14 to get its subtree T_i. This simple method can satisfy all requirements we need.

Let $k = \lfloor u \cdot N^\alpha \rfloor$ and $l = 2\lfloor N^\lambda \rfloor$, where $\lambda = d - \alpha$ and α is a parameter we will choose later. We will write down the constraints for α and other parameters later. Let $I_i = [[(i-1)l, il]]$, for any $i \in [[1, k]]$. Write $m_i = \min_{n \in I_i} S_n$ and $\Delta_i = S_{(i-1)l} + S_{il} - 2m_i$. In fact, Δ_i is the tree distance between the endpoint vertices corresponding to $(i-1)l$ and il. Note that $\Delta_1 = S_l$, thus by Lemma 3.3.4, we have:

$$P^L[\Delta_1 < N^{\lambda/2-\gamma}] \le CN^{-3\gamma};$$
$$P^L[\Delta_1 > N^{\lambda/2+\epsilon}] \le CN^{-3\gamma}.$$

Note that the constants here (and through out this proof) may depend on parameters α, γ, ϵ (and of course u, d), but definitely not on N. In fact we will choose $\alpha, \beta, \gamma, \delta$ (β, γ, δ will appear later) in the end and they will be chosen to be small. After that ϵ and η will be chosen to be even smaller numbers depending on $\alpha, \beta, \gamma, \delta$.

Using the root-changing method, (see the beginning of the proof of Proposition 3.3.3) we have the same inequalities for not only Δ_1, but every Δ_i, since Δ_i means the tree distance between the endpoints which is invariant under root-changing. The number of intervals is $k \asymp N^{\alpha}$. Hence, if

$$\alpha < 3\gamma, \tag{3.3.41}$$

then (by Lemma 3.3.4) we obtain that with a high probability, at least $1 - C/N^{3\gamma-\alpha}$, all Δ_i are in $[[N^{\lambda/2-\gamma}, N^{\lambda/2+\epsilon}]]$.

As mentioned earlier, the part of S in I_i can be regarded as a segment called 'spine', consisting of those edges which the contour walk crosses once inside I_i and once outside I, together with a set of subtrees of S (we call them bushes) attached to the vertices in the spine. The number of vertices, also the number of bushes (some maybe one-point trees) in the spine is $m = \Delta_i + 1$, and the total edges of these bushes is $M = \frac{l-\Delta_i}{2}$. Moreover, it is elementary to see that the joint law of the sizes of these bushes is $(\xi_1, \ldots, \xi_m)$ conditioned on $\sum \xi_j = M$. We know that with high probability $\Delta_i \in [[N^{\lambda/2-\gamma}, N^{\lambda/2+\epsilon}]]$. Since ϵ will be chosen very small ($N^{\lambda/2+\epsilon} \ll N^{\lambda}$), $M \approx \frac{l}{2} = \lfloor N^{\lambda} \rfloor$. When

$$\gamma/\lambda < 0.1 \tag{3.3.42}$$

and ϵ is very small, one can check that M and m are in the required relation for Lemma 3.3.11 to hold. Hence we have:

$$P[\tilde{\sigma}_m > C_1 m^{2-\beta/2} | \sigma_m = M] \leq \exp(-C_2 m^{\beta/2}). \tag{3.3.43}$$

It means that if we discard those bushes with edges less then $m^{2-\beta}$, with high probability, the total number of edges we lose is less than $C_1 m^{2-\beta/2}$. We do so and pick up those bushes with size bigger than $m^{2-\beta}$. The ratio of edges lost compared to total edges is less than:

$$\frac{m^{2-\beta/2}}{2l} \preceq \frac{N^{(\lambda/2+\epsilon)(2-\beta/2)}}{N^{\lambda}} = N^{-(\frac{\lambda\beta}{4}-(2-\frac{\beta}{2})\epsilon)}. \tag{3.3.44}$$

When ϵ is very small the exponent is negative, which is what we want (for Condition 2). The size of each bush we pick is less than $l/2 \leq N^{\lambda}$ and bigger than:

$$m^{2-\beta} \geq N^{(\frac{\lambda}{2}-\gamma)(2-\beta)}. \tag{3.3.45}$$

Note that the exponent λ is obviously less than d. The total number of bushes picked is less than:

$$k \cdot (l/(N^{(\frac{\lambda}{2}-\gamma)(2-\beta)})) \preceq N^{\alpha+2\gamma+\frac{\beta\lambda}{2}-\beta\gamma}. \tag{3.3.46}$$

Assume now that all bushes (for all k intervals) picked are $\widetilde{T}_1, \dots, \widetilde{T}_{n'}$ and the vertices where they are grafted in the spine are $\bar{v}_1, \dots, \bar{v}_{n'}$. Note that for each subinterval we have a spine and all spines together form the contour walk of a connected subtrees of T, which we will call the skeleton. All bushes (whether we picked or not) are grafted to the skeleton. Hence, the set $T \setminus (\cup \widetilde{T}_i)$ consists of the skeleton and the bushes we do not pick up and is connected. Since the root o is in the spine of the first interval, o is in the skeleton and in $T \setminus (\cup \widetilde{T}_i)$. Moreover, conditioned on their sizes and $T \setminus (\cup \widetilde{T}_i)$, the trees $\widetilde{T}_i$ are independent and uniform. This can be induced simply by the fact that conditioned on the length, each bush is independent of the spine and any other bush, and the fact that each Dyck path with that length has the same probability weight.

In view of Lemma 3.3.14, for each $\widetilde{T}_i$, with high probability, we can find its subtree T_i which is far from the root of $\bar{v}_i$. More precisely, for each $\widetilde{T}_i$, assume $|\widetilde{T}_i| = L_i$ and $(S_n)_{n \in [[0,2|L_i|]]}$ is the contour function. We know that with high probability, the event in (3.3.29) is true. Set

$$\begin{aligned} b_1 &= \max\{n \in [[0, L_i]], S_n = \lfloor N^{2+\eta} \rfloor\} + 1, \\ b_2 &= \min\{n \in [[L_i, 2L]], S_n = \lfloor N^{2+\eta} \rfloor\} - 1, \end{aligned}$$

and let T_i is the subtree corresponding to $[[b_1, b_2]]$. Then the distance from $\widetilde{T}_i$ and the root of $\widetilde{T}_i$ is $\lfloor N^{2+\eta} \rfloor + 1$. We use T_i to replace $\widetilde{T}_i$. If we can replace all $\widetilde{T}_i$ successfully, then Conditions 2 and 3 can be satisfied and $T_1, \dots, T_{n'}$ satisfy all conditions. When

$$(\frac{\lambda}{2} - \gamma)(2 - \beta) > 4 + 2\eta + \delta, \tag{3.3.47}$$

the probability of failure for one subtree has order $N^{-3\delta/8}$. Since there are at most $N^{\alpha+2\gamma+\frac{\beta\lambda}{2}-\beta\gamma}$ subtrees, if

$$\alpha+2\gamma+\frac{\beta\lambda}{2}-\beta\gamma<\frac{3}{8}\delta, \tag{3.3.48}$$

then the probability that we can replace all T_i' successfully is bigger than something like $1-N^{-\epsilon'}$.

The constraints (3.3.41),(3.3.42),(3.3.47) and (3.3.48) are not tight, e.g. $\alpha=0.001d$, $\gamma=0.002d$, $\delta=0.05d$ and $\beta=0.02$ (let ϵ and η be very small) satisfy all constraints. Then we conclude the lemma. $\square$

3.3.3 Proof of the main theorem

Let $\mathcal{S}:T\to\mathbb{T}_n$ be the random function corresponding to X_n. Then T is uniform on $\mathbf{A}_L$ and $\{X_0,\dots,X_{2L}\}=\mathcal{S}(T)$. Due to Lemma 3.3.10, with high probability $(1-C/N^{\epsilon})$, we can find subtrees $T_1,\dots,T_{n'}$ as in the lemma.

We denote with A this event. We write $P[\cdot|(n';L_1,\dots,L_{n'};\mathbf{t})]$ (respectively $p(n';L_1,\dots,L_{n'};\mathbf{t})$) for the conditional probability conditioned (respectively the probability) that A is true, the number of subtrees of T_i is n', the size of T_i is L_i $(i=1,\dots,n')$ and the subtree $\hat{T}$ with n' vertices indicating the places of v_i is $\mathbf{t}$ (we also assume that $\hat{T}$ is a rooted tree together with n' ordered vertices in it). Note that under $P[\cdot|(n';L_1,\dots,L_{n'};\mathbf{t})]$, the trees $T_1,\dots,T_{n'}$ are independent and uniform on all plane trees with the given size.

$$\begin{aligned}&P^{L,N}[\{X_0,X_1,...,X_{2L}\}\cap\varphi(K)=\emptyset]\\=&P^{L,N}[\mathcal{S}(T)\cap\varphi(K)=\emptyset]\\\le&P^{L,N}[A^c]+\sum p(n';L_1,\dots,L_{n'};\mathbf{t})P[\mathcal{S}(T)\cap\varphi(K)=\emptyset|(n';L_1,\dots,L_{n'};\mathbf{t})],\end{aligned}$$

where the sum runs over all possible values of $\Upsilon=(n';L_1,\dots,L_{n'};\mathbf{t})$ such that $p(\Upsilon)>0$ (depending on N).

Since $P^{L,N}[A^c]\to 0$, it suffices to prove

$$\lim_{N\to\infty}\max_{\Upsilon}|P[\mathcal{S}(T)\cap\varphi(K)=\emptyset|\Upsilon]-\exp\left(-2u\mathrm{BCap}(K)\right)|=0. \tag{3.3.49}$$

The above one can be reduced to (3.3.50)-(3.3.52):

$$\lim_{N\to\infty}\max_{\Upsilon}|P[\mathcal{S}(T)\cap\varphi(K)=\emptyset|\Upsilon]- P\left[\left(\cup_{i=1}^{n'}\mathcal{S}(T_i)\right)\cap\varphi(K)=\emptyset|\Upsilon\right]|=0; \quad (3.3.50)$$

$$\lim_{N\to\infty}\max_{\Upsilon}|P\left[\left(\cup_{i=1}^{n'}\mathcal{S}(T_i)\right)\cap\varphi(K)=\emptyset|\Upsilon\right]- \prod_{i=1}^{n'}P\left[(\mathcal{S}(T_i))\cap\varphi(K)=\emptyset|\Upsilon\right]|=0; \quad (3.3.51)$$

$$\lim_{N\to\infty}\max_{\Upsilon}\left|\prod_{i=1}^{n'}P\left[(\mathcal{S}(T_i))\cap\varphi(K)=\emptyset|\Upsilon\right]-\exp\left(-2u\mathrm{BCap}(K)\right)\right|=0. \quad (3.3.52)$$

The proof of (3.3.50) is easy.

$$\begin{aligned}&|P[\mathcal{S}(T)\cap\varphi(K)=\emptyset|\Upsilon]-P\left[\left(\cup_{i=1}^{n'}\mathcal{S}(T_i)\right)\cap\varphi(K)=\emptyset|\Upsilon\right]|\\ \le&|P\left[\mathcal{S}(T\setminus\left(\cup_{i=1}^{n'}\mathcal{S}(T_i)\right))\cap\varphi(K)\neq\emptyset|\Upsilon\right]|\le N^{d-\epsilon}\frac{1}{N^d}\to 0.\end{aligned}$$

The last inequality is due to Condition 2 in Lemma 3.3.10, the union bound and the fact that $S(v)$ is uniformly distributed on $\mathbb{T}_N$ for all $v\in T$.

For (3.3.52), by Condition 1 and 4 in Lemma 3.3.10, we know that $|T_i|\in[N^{a_1},N^{a_2}]$ and that conditioned on the size, T_i is uniform on $\mathbf{A}_{|T_i|}$. Hence we can apply Proposition 3.3.3. Then together with Condition 2, one can get (3.3.52).

Now we turn to (3.3.51). We need the following lemma.

Lemma 3.3.16. *There exist positive c and C (depending on those variables in Lemma 3.3.10 but not N), such that, for any $N\in\mathbb{N}^+$ and $\Upsilon=$*

$(n'; L_1, \dots, L_{n'}; \mathbf{t})$ with $p(\Upsilon) > 0$,$k \in [[1, n'-1]]$, then

$$|P\left[\left(\cup_{i=k}^{n'} \mathcal{S}(T_i)\right) \cap \varphi(K) = \emptyset | \Upsilon\right] - P\left[(\mathcal{S}(T_k)) \cap \varphi(K) = \emptyset | \Upsilon\right] \times P\left[\left(\cup_{i=k+1}^{n'} \mathcal{S}(T_i)\right) \cap \varphi(K) = \emptyset | \Upsilon\right] | \le C \exp(-cN^{\eta}), \quad (3.3.53)$$

where η is from Lemma 3.3.10.

With this Lemma one can use induction to show

$$|P\left[\left(\cup_{i=1}^{n'} \mathcal{S}(T_i)\right) \cap \varphi(K) = \emptyset | \Upsilon\right] - \prod_{i=1}^{n'} P[\mathcal{S}(T_i) \cap \varphi(K) = \emptyset | \Upsilon]| \le (n'-1) C \exp(-cN^{\eta}). \quad (3.3.54)$$

Since n' is bounded by a polynomial of N, the right hand side tends to 0, which implies (3.3.51).

Proof of Lemma 3.3.16. Let o_1 and o_2 be the ends of ι_k (say $o_1 \in T_k$). For any $x, y \in \mathbb{T}_N$, define

$$f(x) = P\left[(\mathcal{S}(T_k)) \cap \varphi(K) = \emptyset | \mathcal{S}(o_1) = x, \Upsilon\right], \quad (3.3.55)$$

$$h(y) = P\left[\left(\cup_{i=k+1}^{n'} \mathcal{S}(T_i)\right) \cap \varphi(K) = \emptyset | \mathcal{S}(o_2) = y, \Upsilon\right]. \quad (3.3.56)$$

By Condition 3, this path separates T_k and $\cup_{i=k+1}^{n'} T_i$, so we have

$$P\left[\left(\cup_{i=k}^{n'} \mathcal{S}(T_i)\right) \cap \varphi(K) = \emptyset | \mathcal{S}(o_1) = x, \mathcal{S}(o_2) = y, \Upsilon\right] = f(x) \times h(y). \quad (3.3.57)$$

Therefore,

$$\begin{aligned} & P\left[\left(\cup_{i=k}^{n'} \mathcal{S}(T_i)\right) \cap \varphi(K) = \emptyset | \Upsilon\right] \\ = & \sum_{x,y \in \mathbb{T}_N} f(x) h(y) P[\mathcal{S}(o_1) = x, \mathcal{S}(o_2) = y | \Upsilon] \\ = & N^{-d} \cdot \sum_{x,y \in \mathbb{T}_N} f(x) h(y) P_x^{SRW}[Z_{\lfloor N^{2+\eta} \rfloor + 1} = y], \end{aligned}$$

where P_x^{SRW} means the law of $Z = (Z_n)_{n\in\mathbb{N}}$ which is a simple random walk starting from x. Note that

$$P\left[(\mathcal{S}(T_k)) \cap \varphi(K) = \emptyset | \Upsilon\right] = N^{-d} \sum_{x\in\mathbb{T}_N} f(x); \tag{3.3.58}$$

$$P\left[\left(\cup_{i=k+1}^{n'} \mathcal{S}(T_i)\right) \cap \varphi(K) = \emptyset | \Upsilon\right] = N^{-d} \sum_{y\in\mathbb{T}_N} h(y). \tag{3.3.59}$$

Hence the left hand side of (3.3.53) is:

$$\begin{aligned}|N^{-d} \sum_{x,y\in\mathbb{T}_N} & f(x)h(y)(P_x^{SRW}[Z_{\lfloor N^{2+\eta}\rfloor+1} = y] - N^{-d})| \\ & \leq \max_{x\in\mathbb{T}_N} \sum_{y\in\mathbb{T}_N} |P_x^{SRW}[Z_{\lfloor N^{2+\eta}\rfloor+1} = y] - N^{-d}|.\end{aligned}$$

Now (3.3.53) can be implied by the following result in the theory of mixing time (e.g. see Chapter 5 in [16]).

Proposition 3.3.17. *Let $\kappa > 2$. There exist positive numbers c and C such that for any odd $N \in \mathbb{N}^+$, we have:*

$$\max_{x,y\in\mathbb{T}_N} |P_x^{SRW}[Z_{\lfloor N^{\kappa}\rfloor+1} = y] - N^{-d}| \leq C\exp(-cN^{\kappa-2}). \tag{3.3.60}$$

□

Remark 3.3.4. *The requirement of oddness is due to the periodicity of Simple Random Walk. If the random walk is lazy, then Proposition 3.3.10 is correct without assuming oddness. Hence if the branching random walk is lazy, we still have Theorem 3.3.1 without assuming oddness.*

Chapter 4

An optimal strategy for the Majority-Markov game

4.1 Definitions, settings and main result

Our theorem (Theorem 4.1.1) will be stated in terms of Markov game. We adopt some terminologies from [6]. Furthermore, our theorem is based on the key object 'grade', and its properties from [6].

4.1.1 Markov systems

A Markov system with one target (respectively with two targets) $S =< V, P, C, t >$ (resp. $S =< V, P, C, t^+, t^- >$) consists of a Markov chain (V, P), a cost function $C : V \to \mathbb{R}^+$, and a target $t \in V$ (resp. two targets t^+ and t^-). We assume that the targets are absorbing. We further assume that the state space V is finite and that every target is accessible from any non-target state. The cost of a 'trip' $v(0), v(1), ...v(k)$ on S is the sum $\sum_{i=0}^{k-1} C_{v(i)}$ of the costs of the visited states except the last. If $C \equiv 1$, then C could be regarded as the time or the number of steps.

4.1.2 Games

Let $S(1), S(2), ...S(n)$ be Markov systems with either one or two targets. For each $S(i)$, we fix a starting state, $u(i)$, and place a token i at that state. A 'game' consists of Markov systems with tokens on their starting states and a stopping rule $\Lambda \subseteq V(1) \times V(2) \times ... \times V(n)$: the set of configurations when the game ends.

A single player plays against a 'bank': chooses one (say token i at the state $v \in V(i)$) of the n tokens to move (according to its transition probability $P(i)$) and pays the cost ($C_v(i)$), then chooses and pays again... When all tokens form a configuration in the stopping rule Λ, the game ends and the player leaves. We assume that if all tokens are at targets, the game ends (this means that Λ contains those configurations in which all coordinates are targets). As targets are absorbing, we could assume that tokens at targets are not allowed to choose.

By setting different stopping rules, we have different games. A trivial stopping rule is that all tokens are at the targets. For a non-trivial example, [6] considers the simple multitoken game $Sim(S(1), S(2), ..., S(n); 1)$, whose stopping rule is (at least) one of the tokens at the targets. Similarly, we define $Sim(S(1), S(2), ..., S(n); k)$ to be the game whose stopping rule is at least k of the tokens at the targets. In this chapter, we address the 'Majority-Markov' game $Maj(S(1), ..., S(2k+1))$ with $n = 2k+1$ Markov systems with two targets, whose stopping rule is '$k+1$ tokens at positive targets or $k+1$ tokens at negative targets'.

4.1.3 Strategies and costs

A strategy tells us how to choose the token to move. Mathematically, by a strategy σ, we mean a function $\sigma : V(1) \times ... \times V(n) \backslash \Lambda \to \{1, 2, ...n\}$ satisfying $\sigma(u_1, u_2, ...u_n) \neq i$ if u_i is a target. When tokens are at the state $u = (u_1, u_2, ...u_n)$, under strategy σ, $\sigma(u_1, u_2, ...u_n)$ is chosen. Note that the inequality means that we cannot choose tokens at target.

The cost $E[G, \sigma]$ (or simply $E[\sigma]$) is the expected cost (for the player) of playing G under strategy σ. The cost $E[G]$ of a game G is the minimum expected cost of playing G, under all possible strategies. The optimal s-trategies are those strategies that reach E[G]. If we want to emphasize the starting state $u = (u_1, ...u_n)$, we use $E_u[G, \sigma]$, $E_u[\sigma]$, or even $E(u)$ (when the game G and the strategy σ are explicit).

4.1.4 Grades and positive-(negative-)grades

For a Markov system with one target $S =< V, P, C, t >$, a state $u \neq t$ (where the token is), and a positive real number g, consider a modified game where the player can leave at target t without money as usual, or leave at any other state by paying g dollars. This can be defined using our terminology by adding a Markov system T_g. Define the terminator T_g as the Markov system $< \{s, t\}, P, g, t >$ with starting state s, where $p_{s,t} = 1$. The terminator always hits its target in exactly one step, at cost g. The modified game is now the simple Markov game $Sim(S, T_g; 1)$. We can imagine that when g is small enough, the optimal strategy is to leave by paying g and when g is large enough, the optimal is to choose the token at the system S until it hits the target. The grade $\gamma_u(S)$ of the system S for state u is defined to be the unique value of g at which an optimal player is indifferent between the two possible first moves in the game $Sim(S, T_g; 1)$. Naturally, we set $\gamma_t(S) = 0$. It is possible to compute γ in a polynomial time (see [6] for this and more properties about grades).

For a Markov system with two targets $S =< V, P, C, t^+, t^- >$, we define the positive-grade $\gamma_u^+(S)$ for state $u \in V\backslash\{t^-\}$ to be the grade for u in $S^+ =< V, P, C, t^+ >$ and the negative-grade $\gamma_u^-(S)$ for $u \in V\backslash\{t^+\}$ to be the grade for u in $S^- =< V, P, C, t^- >$. For convenience, we set $\gamma_{t^+}^-(S) = \gamma_{t^-}^+(S) = \infty$. Note that either positive-grade or negative-grade is a nonnegative number.

4.1.5 Main result

Theorem 4.1.1. *A strategy for the Majority-Markov game* $Maj(S(1), ..., S(2k+1))$ *is optimal if and only if it always plays in a system in which neither the positive-grade (of the position of the token) is larger than the median of all* $2k+1$ *positive-grades, nor the negative-grade is larger than the median of all* $2k+1$ *negative-grades.*

4.2 Some known results about Markov games

It turns out that every Markov game (at least in our sense, i.e. when the stopping rule contains all configurations in which all coordinates are targets) has an optimal strategy. We refer the reader to [24] for more details.

From a given state $u = (u_1, ...u_n)$ of a Markov game (with n Markov systems), an action $\alpha \in \{1, ..., n\}$ (means choosing token α) gives an immediate cost $C_u(\alpha)$ (more precisely, $C_{u_\alpha}(\alpha)$) and a probability distribution $\{p_{u,\bullet}\}$ for the next state. Therefore a strategy σ with action α at the state u satisfies:

$$E_u[\sigma] = C_u(\alpha) + \sum_v p_{u,v}(\alpha) E_v[\sigma].$$

If among all possible actions at state u, α is the minimizer of the right-hand side of this expression, then σ is said to be consistent at u.

Proposition 4.2.1. *A strategy is optimal if and only if it is consistent at every state ($\notin \Lambda$).*

For the expected cost function, we have:

Proposition 4.2.2. *For the game G consisting of Markov systems $S(1), ...,$ $S(n)$ with stopping rule Λ, if a function $E : V(1) \times ... \times V(n) \to \mathbb{R}^+$ satisfies: $E|_\Lambda \equiv 0$ and*

$$E_u = \min_\alpha \{C_u(\alpha) + \sum_v p_{u,v}(\alpha) E_u\} \quad \forall u \in V(1) \times ... \times V(n) \setminus \Lambda, \tag{4.2.1}$$

where min *is under all possible actions α at state u, then, E is the (unique) cost function for G and a strategy is optimal if and only if, it, at every state $u \in V(1) \times ... \times V(n) \setminus \Lambda$, takes the action which reaches the* min*.*

The game $Sim(S(1), ..., S(n); 1)$ is analyzed in [6] where the optimal strategy is established. Their argument also works for $Sim(S(1), ..., S(n); k)$ with minor modifications.

Proposition 4.2.3. *A strategy for the game $Sim(S(1), S(2), ..., S(n); k)$ is optimal if and only if it always plays in a system whose current grade is not larger than the k-th smallest grade.*

4.3 Proof of the main theorem

We have $n = 2k + 1$ Markov systems: $S(i) =< V(i), P(i), C(i), t^+, t^- >$,$i = 1, ..., n$. For notational ease, we identify all positive (negative) states. Now recall and define some games by giving their stopping rules.

Games	Stopping rules
G^M	$k + 1$ tokens at positive targets or $k + 1$ at negative
G^0	all n tokens are at targets
G^+	$k + 1$ tokens at positive targets or all n tokens at targets
G^-	$k + 1$ tokens at negative targets or all n tokens at targets
GS^+	$k+1$ tokens at positive targets (redefine $p_{t^-,t^+} = 1, C_{t^-} = C_0$ for all Markov systems)
GS^-	$k + 1$ tokens at negative targets (redefine $p_{t^+,t^-} = 1, C_{t^+} = C_0$ for all Markov systems.)

Here, C_0 is a large real number (such that it is larger than all positive-grades and negative-grades for non-target states).

Let $E^M, E^0, E^\pm, ES^\pm$ be the expected cost functions for the corresponding games. The following lemma is natural in light of Proposition 4.2.3.

Lemma 4.3.1. *A strategy for G^+ is optimal if and only if it always plays in a system whose current positive-grade is not larger than the median of all positive-grades. For the game G^-, we have the similar conclusion.*

Proof. For simplicity, consider the case of $k = 1$. The general case is similar.

First consider GS^+ here. Note that GS^+ is a simple multitoken game. With Proposition 4.2.3 in mind, we need to consider the grade corresponding to the target t^+. We point out that since C_0 is larger than any other positive-grade, changing transition probability from t^- does not change the positive-grades of other states. Hence for other states, the new grade is the same as the old positive-grade (in G^+).

Under the adjustment $p_{t^-,t^+} = 1$, t^+ is accessible from t^-, so we can apply Proposition 4.2.3 for GS^+. The optimal strategies are those moving the token whose grade is not larger than the median. As C_0 is the positive-grade of t^-, which is larger than all other positive-grades. Hence for any optimal strategy, we can avoid playing tokens at t^-, unless all tokens are at $t^\pm$. So, if we forbid choosing tokens at t^- unless all tokens are at targets, the cost of GS^+ remains the same. Under this assumption, at state $u \notin B = \{u = (u_1, u_2, u_3) : u_i = t^\pm, i = 1, 2, 3\}$, G^+ and GS^+ has the same possible actions.

Consider the expected cost functions E^+ and ES^+ for games G^+ and GS^+. One can easily find out their values on the boundary set (denoted by $\partial\Lambda$):

$\partial\Lambda$	E^+	ES^+
(t^+, t^+, u_3)	0	0
(t^+, u_2, t^+)	0	0
(u_1, t^+, t^+)	0	0
(t^-, t^-, t^+)	0	C_0
(t^-, t^+, t^-)	0	C_0
(t^+, t^-, t^-)	0	C_0
(t^-, t^-, t^-)	0	$2C_0$

In order to remove the difference between E^+ and ES^+, we introduce $EL^+ : V(1) \times V(2) \times V(3) \to [0, +\infty)$ by:

$$EL^+(u_1, u_2, u_3) = (2p^-_{u_1}p^-_{u_2}p^-_{u_3} + (p^+_{u_1}p^-_{u_2}p^-_{u_3} + p^-_{u_1}p^+_{u_2}p^-_{u_3} + p^-_{u_1}p^-_{u_2}p^+_{u_3})) \times C_0, \tag{4.3.1}$$

where $p^+_{u_i}$ denote the probability of the event that a token starting from u_i visits t^+ (before visiting t^-), similarly for $p^-_{u_i}$.

Since $p^+_{t^+} = p^-_{t^-} = 1, p^+_{t^-} = p^-_{t^+} = 0$, we can check that $EL^+(u) = ES^+(u)$, for any $u \in \partial\Lambda$. On the other hand using the equalities for hitting probability: $p^\pm_{u_i} = \Sigma_{v_i} p_{u_i,v_i}(i) p^\pm_{v_i}$, we can see that EL^+ satisfies the linear part of (4.2.1):

$$EL^+(u) = \Sigma_v p_{u,v}(i) EL^+(v) \quad \text{for } i : u_i \neq t^\pm. \tag{4.3.2}$$

Because ES^+ satisfies (4.2.1) and EL^+ satisfies the linear part of (4.2.1), $ES^+ - EL^+$ also satisfies (4.2.1). On the other hand, $ES^+ - EL^+$ and E^+ are equal to 0 on $\partial\Lambda$. By the uniqueness of Proposition 4.2.2, $ES^+ - EL^+ = E^+$. In particular, the actions which reaches the min in (4.2.1) for E^+ and ES^+ are the same. Hence, for G^+ and GS^+, we has the same optimal strategies. This completes the proof of the first assertion. By symmetry, one can get the other assertion. □

Remark 4.3.1. *For general k, one should use:*

$$\begin{aligned} EL^+(u) &= C_0 \cdot \mathbb{E}_u(max\{0, \textit{the number of tokens hitting } t^- - k\}) \\ &= C_0 \cdot \sum_{\tau_1 \in \{+,-\}, \ldots \tau_n \in \{+,-\}} max\{0, (\sum_i 1_{\tau_i = -}) - k\} p_{u_1}^{\tau_1} \ldots p_{u_n}^{\tau_n}. \end{aligned}$$

Now we can build a connection between G^M, G^0 and $G^\pm$.

Claim 4.3.2.

$$E^M = E^+ + E^- - E^0. \tag{4.3.3}$$

For any strategy σ for game G^M, we could use it to play any of G^+, G^- and G^0: use strategy σ to play until the stopping rule for G^M happens. If at that time the game does not end, the stopping rule switches to the trivial stopping rule, i.e., all tokens are at targets. Hence the subsequent cost after G^M ends is independent of the strategy. Note first that G^+, G^- and G^0 will not end before G^M ends, since $\Lambda^+, \Lambda^-, \Lambda^0 \subseteq \Lambda^M$. And if G^M ends before all tokens reach targets, exactly one of G^+ and G^- ends at this time, and the other will end at the same time as G^0 ends, i.e when all tokens reach targets; if G^M ends when all tokens are at the targets, then all four games end at this time. Consider game pairs (G^M, G^0) and (G^+, G^-). We get that (under any strategy) when one of the games in the left pair ends, one in the right pair also ends and when the other game in the left pair ends, the other in the right also ends and vice versa. Note that at each step we pay the same amount of money for each pair. Hence, one can get:

$$E^M[\sigma] + E^0[\sigma] = E^+[\sigma] + E^-[\sigma] \quad \leq E^+ + E^-.$$

Therefore,

$$E^M[\sigma] \leq E^+ + E^- - E^0[\sigma] = E^+ + E^- - E^0.$$

The last equality is due to the fact that the cost of the trivial game E^0 is independent of strategies.

Furthermore, the equality holds if and only if $E^+[\sigma] = E^+$ and $E^-[\sigma] = E^-$. This means that σ is optimal for both G^+ and G^-. Hence, σ is an optimal strategy for G^M if and only if it is an optimal strategy for both G^+ and G^-. Therefore, by Lemma 4.3.1 we finish the proof of the main result.

Bibliography

[1] David Aldous. The continuum random tree. II. An overview. In *Stochastic analysis (Durham, 1990)*, volume 167 of *London Math. Soc. Lecture Note Ser.*, pages 23–70. Cambridge Univ. Press, Cambridge, 1991.

[2] Omer Angel, Balázs Ráth, and Qingsan Zhu. Local limit of branching random walk on the torus and branching interlacements. In preparation.

[3] Itai Benjamini and Nicolas Curien. Recurrence of the $\mathbb{Z}^d$-valued infinite snake via unimodularity. *Electron. Commun. Probab.*, 17:no. 1, 10, 2012.

[4] George Bennett. Probability inequalities for the sum of independent random variables. *Journal of the American Statistical Association*, 57(297):33–45, 1962.

[5] Alexander Drewitz, Balázs Ráth, and Artëm Sapozhnikov. *Lecture Notes on Random Interlacements.*

[6] Ioana Dumitriu, Prasad Tetali, and Peter Winkler. On playing golf with two balls. *SIAM J. Discrete Math.*, 16(4):604–615, 2003.

[7] Svante Janson. Random cutting and records in deterministic and random trees. *Random Structures Algorithms*, 29(2):139–179, 2006.

[8] Svante Janson and Jean-François Marckert. Convergence of discrete snakes. *J. Theoret. Probab.*, 18(3):615–647, 2005.

[9] Steven P. Lalley and Xinghua Zheng. Occupation statistics of critical

branching random walks in two or higher dimensions. *Ann. Probab.*, 39(1):327–368, 2011.

[10] Gregory F. Lawler. *Intersections of random walks*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2013. Reprint of the 1996 edition.

[11] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[12] Jean-François Le Gall. *Spatial branching processes, random snakes and partial differential equations*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1999.

[13] Jean-François Le Gall and Shen Lin. The range of tree-indexed random walk in low dimensions. *Ann. Probab.*, 43(5):2701–2728, 2015.

[14] Jean-François Le Gall and Shen Lin. The range of tree-indexed random walk. *J. Inst. Math. Jussieu*, 15(2):271–317, 2016.

[15] Jean-François Le Gall and Grégory Miermont. Scaling limits of random trees and planar maps. In *Probability and statistical physics in two and more dimensions*, volume 15 of *Clay Math. Proc.*, pages 155–211. Amer. Math. Soc., Providence, RI, 2012.

[16] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.

[17] Andrei S. Mishchenko. Discrete bessel process and its properties. *Theory Probab. Appl.*, 50:700–709, 2005.

[18] Zhan Shi. *Branching random walks*, volume 2151 of *Lecture Notes in Mathematics*. Springer, Cham, 2015. Lecture notes from the 42nd Probability Summer School held in Saint Flour, 2012, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[19] Vladas Sidoravicius and Alain-Sol Sznitman. Percolation for the vacant set of random interlacements. *Comm. Pure Appl. Math.*, 62(6):831–858, 2009.

[20] Frank Spitzer. *Principles of random walk*. Springer-Verlag, New York-Heidelberg, second edition, 1976. Graduate Texts in Mathematics, Vol. 34.

[21] Alain-Sol Sznitman. Vacant set of random interlacements and percolation. *Ann. of Math. (2)*, 171(3):2039–2087, 2010.

[22] Augusto Q. Teixeira. From random walk trajectories to random interlacements. *Ensaios Matemáticos*, 23:1–78, 2012.

[23] Kôhei Uchiyama. Green's functions for random walks on $\mathbf{Z}^N$. *Proc. London Math. Soc. (3)*, 77(1):215–240, 1998.

[24] Douglas J. White. *Markov decision processes*. John Wiley & Sons, Ltd., Chichester, 1993.

[25] David Windisch. Random walk on a discrete torus and random interlacements. *Electron. Commun. Probab.*, 13:140–150, 2008.

[26] Qingsan Zhu. On the critical branching random walk i: Branching capacity and visiting probability. Preprint, arXiv:1611.10324.

[27] Qingsan Zhu. On the critical branching random walk ii: Branching capacity and branching recurrence. Preprint, arXiv:1612.00161.

[28] Qingsan Zhu. On the critical branching random walk iii: the critical dimension. Preprint, arXiv:1701.08917.

[29] Qingsan Zhu. An upper bound for the probability of visiting a distant point by critical branching random walk in $\mathbf{Z}^4$. Preprint, arXiv:1503.00305.

Appendix A

Sketch of Proof of Lemma 2.1.3

Proof. Without loss of generality, one can assume θ is aperiodic. The first step is to show:

- There is a $\delta \in (0, 0.1)$, such that, for any $\epsilon > 0$ small enough, and $m \in \mathbb{N}^+$ large enough (depending on ϵ), we can find $c_1 = c_1(\epsilon)$, such that, for any $n \in [\epsilon m^2, 2\epsilon m^2]$, $z, w \in \mathcal{C}(3\delta m)$, we have:

$$p_n^m(z,w) \doteq \sum_{\gamma: z \to w, \gamma \subseteq \mathcal{C}(m), |\gamma| = n} \mathbf{s}(\gamma) \geq c_1 \cdot m^{-d}. \tag{A.0.1}$$

Indeed, the Markov property implies that:

$$p_n^m(z,w) \geq P(S_z(n) = w) - \max\{P(S_y(k) = w) : k \leq n, y \in (\mathcal{C}(m))^c\},$$

and the LCLT establishes (A.0.1). Using this estimate, one can see that:

- For any $\epsilon > 0$ small enough, and $m \in \mathbb{N}^+$ large enough, we can find $c_2 = c_2(\epsilon)$, such that, for any $z, w \in \mathcal{C}(3\delta m)$, we have (we write $\mathcal{C}_x(r)$ for the ball centered at x with radius r):

$$\sum_{\gamma: z \to w, |\gamma| \leq 2\epsilon m^2, \gamma \subseteq \mathcal{C}(m)} \mathbf{s}(\gamma) \geq c_2 m^{2-d}; \tag{A.0.2}$$

$$\sum_{\gamma: z \to \mathcal{C}_w(\delta m/10), |\gamma| \leq 2\epsilon m^2, \gamma \subseteq \mathcal{C}(m)} \mathbf{s}(\gamma) \geq c_2 m^2.$$

Note that in the first assertion, the left hand side is increasing for m when z, w are fixed. Due to this fact, one can get that

- For any $\epsilon > 0$ small enough, and $m \in \mathbb{N}^+$ large enough, we can find $c_2 = c_2(\epsilon)$, such that, for any $z, w \in \mathcal{C}(3\delta m)$, we have:

$$\sum_{\gamma: z \to w, |\gamma| \le 2\epsilon m^2, \gamma \subseteq \mathcal{C}(m)} \mathbf{s}(\gamma) \ge c_3 \|z - w\|^{2-d}; \tag{A.0.3}$$

By considering the first visit of $\mathcal{C}_w(\delta m/10)$, one can get:

$$\begin{aligned} & \sum_{\gamma: z \to \mathcal{C}_w(\delta m/10), |\gamma| \le 2\epsilon m^2, \gamma \subseteq \mathcal{C}(m) \setminus \mathcal{C}_w(\delta m/10)} \mathbf{s}(\gamma) \\ \ge & \sum_{\gamma: z \to \mathcal{C}_w(\delta m/10), |\gamma| \le 2\epsilon m^2, \gamma \subseteq \mathcal{C}(m)} \mathbf{s}(\gamma) / \max\{g(x, \mathcal{C}(\delta m/10)) : x \in \mathcal{C}(\delta m/10)\} \\ \succeq & m^2/m^2 \asymp 1. \end{aligned}$$

Hence we have:

- For any $\epsilon > 0$ small enough, and $m \in \mathbb{N}^+$ large enough, we can find $c_4 = c_4(\epsilon)$, such that, for any $z, w \in \mathcal{C}(3\delta m)$, we have:

$$\sum_{\gamma: z \to \mathcal{C}_w(\delta m/10), |\gamma| \le 2\epsilon m^2, \gamma \subseteq \mathcal{C}(m) \setminus \mathcal{C}_w(\delta m/10)} \mathbf{s}(\gamma) \ge c_4. \tag{A.0.4}$$

Now we are ready to show the lemma. Without loss of generality, assume $\rho(U, V^c) = 1$. First, choose a finite number of balls with radius δ and centers at U: $B_1, B_2, \ldots, B_k$ covering $\overline{U}$. Choose ϵ small enough for (A.0.2),(A.0.3), (A.0.4) and $\epsilon < 1/k$. Now we argue that when n is sufficiently large, (2.1.5) holds.

Write $B_i' = nB_i \cap \mathbb{Z}^d$ and $\overline{B}_i' = n\overline{B}_i \cap \mathbb{Z}^d$ for $i = 1, \ldots, k$, where $\overline{B}_i$ is the ball with radius 1 and the same center of B_i. When $\|x - y\| \le 2\delta n$, by (A.0.3) we have (2.1.5). Otherwise, x, y are not on the same B_i'. However, we can find at most $k+1$ points $x_0 = x, x_1, \ldots, x_l = y$,$(l \le k)$ such that x_j and x_{j+1} are in the same B_i', say B_j'. Note that when z, w are on the same

B_i', by (A.0.4), for any $z' \in \mathcal{C}_z(\delta n/10)$,

$$\sum_{\gamma: z' \to \mathcal{C}_w(\delta n/10), |\gamma| \le 2\epsilon n^2, \gamma \subseteq \overline{B}_i' \setminus \mathcal{C}_w(\delta n/10)} \mathbf{s}(\gamma) \ge c_4.$$

Hence, by connecting paths, one can get:

$$\begin{aligned} &\sum_{\gamma: x \to y, \gamma \subseteq B_n, |\gamma| \le 2n^2} \mathbf{s}(\gamma) \ge \sum_{\gamma_0: x_0 \to \mathcal{C}_{x_1}(\delta n/10), |\gamma_0| \le 2\epsilon n^2, \gamma_0 \subseteq \overline{B}_0' \setminus \mathcal{C}_{x_1}(\delta n/10)} \mathbf{s}(\gamma_0) \\ &\cdot \sum_1 \mathbf{s}(\gamma_1) \cdot \sum_2 \mathbf{s}(\gamma_2) \cdot \cdots \sum_{l-2} \mathbf{s}(\gamma_{l-2}) \cdot \sum_{\gamma_{l-1}: \widehat{\gamma}_{l-2} \to y, |\gamma_{l-1}| \le 2\epsilon n^2, \gamma_{l-1} \subseteq \overline{B}_{l-1}'} \mathbf{s}(\gamma_{l-1}) \\ &\ge (c_4)^{l-1} \cdot c_2(n^{2-d}) \ge (c_4)^k c_2 n^{2-d} \asymp g(x, y), \end{aligned}$$

where $\sum_j = \sum_{\gamma_j: \widehat{\gamma}_{j-1} \to \mathcal{C}_{x_{j+1}}(\delta n/10), |\gamma_j| \le 2\epsilon n^2, \gamma_j \subseteq \overline{B}_j' \setminus \mathcal{C}_{x_{j+1}}(\delta n/10)}$ for $j = 1, \ldots, l-2$.

$\square$