Constraints on Geometry from Causal Holographic Information and Relative Entropy

by

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Abstract

In this thesis we find constraints to asymptotically anti de-Sitter space dual to holographic conformal field theory states using the holographic duality. A recent conjecture involving the causal holographic information surface propsed that for smooth asymptotically anti de-Sitter spacetimes that obey the null energy conditions, the area of the Ryu-Takayanagi surface will always be less than or equal to the area of the causal holographic information surface. This conjecture is explored in three dimensional spacetimes that are dual to translation invariant states on the boundary conformal field theory in two dimensions. A series expansion of the Ryu-Takayanagi surface and causal holographic information surface is derived, and is used to translate the constraint between the areas of the two surfaces into a constraint on the asymptotic structure of such geometries near the conformal boundary. The translated constraints are compared to the constraints given by the null energy condition - and it is found that the first two leading order constraints are the same. We then outline some preliminary results of an ongoing project whose goal is to understand the dual of relative entropy of holographic states defined on null cone regions on the conformal boundary. We derive the modular Hamiltonian for vacuum states defined on null cone regions in a conformal field theory using known results for modular Hamiltonians on null planes. We also derive the Ryu-Takayanagi surface associated with such null cone regions. Using these results, it is argued that, for null cones whose base is cut by a constant time cut, will not give new constraints beyond what is already known for ball shaped regions.

Lay Summary

A quantum field theory describes particles as excitations of an underlying quantum field. These theories usually exhibit some kind of symmetries. In particular, a conformal field theory is a special type of quantum field theory that has a large number of symmetries, which include conformal and Poincare symmetries. The anti de-Sitter conformal field theory correspondence tells us that special states of a conformal field theory, called holographic states, are related to a special class of spacetime geometries called asymptotically anti de-Sitter geometries. In this thesis we use this relation between holographic states and asymptotically anti de-Sitter spacetimes to translate constraints on the field theory to constraints on geometry.

Preface

The questions the author explores in this thesis were posed by the author's supervisor Dr. Mark Van Raamsdonk. The calculations done in chapter 2 were done independently and guided by suggestions from the author's supervisor. The materials in chapter 3 are a result of a collaborative effort with Dominik Neuenfeld under the guidance of the author's supervisor. The calculations in appendix A.4, A.6, and A.7 were originally done by Dominik. The calculations in section 3.3 and 3.4 were done jointly by both Dominik and the author and then checked for consistency.

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Dedication

To my parents, who always encourage and support me.

Chapter 1

Introduction

1.1 Historical Overview of the AdS/CFT Correspondence and Holography

The Anti-de Sitter/Conformal Field Theory (AdS_{d+1}/CFT_d) correspondence was first proposed by Juan Maldacena in 1997 in the context of string theory [1]. The conjecture states that a certain class of conformal field theories, which are sometimes called holographic, defined on a d-dimensional Minkowski background, are equivalent to theories of quantum gravity on a $d+1$-dimensional asymptotically Anti-de Sitter ($AAdS_{d+1}$) background [2]. Since its initial conception it has proved to be a powerful tool to do calculations in strongly coupled quantum field theories, [3], as well as being a promising approach to formulating a consistent theory of quantum gravity [4]. In particular one important development that has come out of studying the duality is the Ryu-Takayanagi (RT) conjecture. The conjecture states that the entanglement entropy of some sub-region of a CFT_d is proportional to the area of a co-dimension 2 extremal surface in the dual $AAdS_{d+1}$ geometry [5]. Since the area of an extremal surface is a geometric quantity which depends on the metric, the conjecture provides a direct relation between the quantum information quantity of entanglement entropy and the geometry of $AAdS_{d+1}$ spacetimes. Since there are certain constraints for the quantum information quantities on CFT_d's one can use the RT conjecture to translate these quantum information constraints to constraints on the dual spacetimes [4, 6, 7]. By understanding such constraints one can understand what types of geometries and energy conditions are physically allowed in any consistent theory of quantum gravity.

1.2 Basics of Entanglement Entropy

In the context of holography entanglement entropy usually refers to the entanglement entropy of some sub-region of spacetime over which the state of a quantum field theory is defined. Before understanding this notion of entanglement entropy, we start by introducing the density matrix formalism of quantum mechanics. In this formalism the central object that describes the state of a quantum system is called the density matrix which is a non-negative hermitian operator with unit trace. More explicitly, given a complete set of orthonormal quantum states, $\{|\psi_1\rangle, ..., |\psi_n\rangle\}$, for an n-dimensional Hilbert space, along with a set of "classical" probabilities that add to one, $\{p_1, ..., p_n\}$, the density matrix of the system can be written as:

$$\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle \langle\psi_i| \tag{1.2.1}$$

The operator defined above is a hermitian operator with non-negative eigenvalues $\{p_i\}_{i=1}^{n}$ and clearly the trace of the operator is equal to 1 since $\sum_{i=1}^{n} p_i = 1$. The expectation values of an operator $\mathcal{O}$ with respect to the density matrix ρ can be defined by using the trace:

$$\langle\mathcal{O}\rangle_\rho = Tr(\rho\mathcal{O}) \tag{1.2.2}$$

Density matrices in which only one eigenvalue is one and the rest are zero are referred to as pure states. In particular one can show that a density matrix defines a pure state iff $\rho = \rho^2$ or alternatively iff $Tr(\rho^2) = 1$. For a pure state defined in terms of a state vector, $\sigma = |\psi\rangle\langle\psi|$, the definition of the expectation value with respect to σ simplifies to the usual expression using the state vector $|\psi\rangle$. This can be seen by calculating the trace using a complete set of basis states $|\psi_i\rangle$:

$$\langle\mathcal{O}\rangle_\sigma = \sum_{i=1}^{n} \langle\psi_i| \sigma\mathcal{O} |\psi_i\rangle = \sum_{i=1}^{n} \langle\psi_i|\psi\rangle \langle\psi| \mathcal{O} |\psi_i\rangle = \sum_{i=1}^{n} \langle\psi|\psi_i\rangle \langle\psi_i| \mathcal{O} |\psi\rangle = \langle\psi| \mathcal{O} |\psi\rangle \tag{1.2.3}$$

We can also define density matrices for composite systems. Suppose we have two quantum systems A and B each with its own complete set of states $\{|e_1\rangle, |e_2\rangle, ..., |e_n\rangle\}$ and $\{|\tilde{e}_1\rangle, |\tilde{e}_2\rangle, ..., |\tilde{e}_m\rangle\}$ respectively. The composite system is in a mn-dimensional Hilbert space that is spanned by the following basis vectors $\{|e_i\rangle \bigotimes |\tilde{e}_j\rangle\}$, where $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. Using this basis we can write a general state vector $|\Psi\rangle$ in the composite system as:

$$|\Psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \psi_{ij} |e_i\rangle \bigotimes |\tilde{e}_j\rangle \tag{1.2.4}$$

As before, given a state vector, we can define a corresponding density matrix for the composite system:

$$\rho_{AB} = |\Psi\rangle\langle\Psi| = \sum_{i,l=1}^{n}\sum_{j,k=1}^{m} \psi_{ij}\psi_{kl}^* \left(|e_i\rangle \bigotimes |\tilde{e}_j\rangle\right) \left(\langle\tilde{e}_k| \bigotimes \langle e_l|\right) \tag{1.2.5}$$

Given a density matrix for a composite system, such as the one defined above, one can define the reduced density matrix for the subsystem A denoted ρ_A by taking a partial trace with respect to subsystem B. Explicitly we find:

$$\rho_A = Tr_B(\rho_{AB}) = \sum_{i,l=1}^{n} \sum_{p,j,k=1}^{m} \psi_{ij}\psi^*_{kl} |e_i\rangle \langle e_l| \delta_{jp}\delta_{kp} = \sum_{i,l=1}^{n} \sum_{j=1}^{m} \psi_{ij}\psi^*_{jl} |e_i\rangle \langle e_l| \tag{1.2.6}$$

The reduced density matrix can then be used to quantify the amount of entanglement between the sub-systems A and B. This is done by calculating the Von Neumann entropy of the reduced density matrix. The Von Neumann entropy for the density matrix ρ which will be denoted as $S(\rho)$ is given by the following equation:

$$S(\rho) = -Tr(\rho ln(\rho)) = -\sum_{i=1}^{n} p_i ln(p_i) \tag{1.2.7}$$

Where p_i are the eigenvalues of the density matrix ρ. Most of the time we will be interested in density matrices for composite systems in pure states. One special property of pure states is that the entanglement entropy of a sub-system is equal to the entanglement entropy of its complement. This can be seen by using an important theorem in quantum information called the Schmidt decomposition theorem. It states that if $|\Psi\rangle$ is a state vector for a composite system AB then there exists an orthonormal bases $\{|A_i\rangle\}_{i=1}^{n}$ and $\{|B_i\rangle\}_{i=1}^{m}$ for subsystems A and B respectively such that the state can be written as:

$$|\Psi\rangle = \sum_{i=1}^{min(n,m)} \sqrt{p_i} |A_i\rangle \bigotimes |B_i\rangle \tag{1.2.8}$$

Using this we can construct a density matrix which is given as:

$$\rho_{AB} = |\Psi\rangle \langle\Psi| = \sum_{i,j=1}^{min(n,m)} \sqrt{p_i p_j} |A_i\rangle \langle A_j| \bigotimes |B_i\rangle \langle B_j| \tag{1.2.9}$$

It is important to note that if we are to think of the operator above as a matrix it will be a nm-dimensional square matrix however the non-zero information will be contained in an $min(n,m)$ dimensional square sub-block. Now, we can compute the reduced density matrices for the subsystems by taking a partial trace:

$$\begin{aligned}\rho_A = Tr_B(\rho_{AB}) &= \sum_{k=1}^{m} \sum_{i,j=1}^{min(n,m)} \sqrt{p_i p_j} |A_i\rangle \langle A_j| \bigotimes \langle B_k|B_i\rangle \langle B_j|B_k\rangle = \sum_{i,j,k=1}^{min(n,m)} \sqrt{p_i p_j}\delta_{ki}\delta_{jk} |A_i\rangle \langle A_j| \\ &= \sum_{k=1}^{min(n,m)} p_k |A_k\rangle \langle A_k|\end{aligned} \tag{1.2.10}$$

$$\rho_B = Tr_A(\rho_{AB}) = \sum_{k=1}^{n} \sum_{i,j=1}^{min(n,m)} \sqrt{p_i p_j} \langle A_k|A_i\rangle \langle A_j|A_k\rangle \bigotimes |B_i\rangle \langle B_j| = \sum_{i,j,k=1}^{min(n,m)} \sqrt{p_i p_j}\delta_{ki}\delta_{jk} |B_i\rangle \langle B_j|$$
$$= \sum_{k=1}^{min(n,m)} p_k |B_k\rangle \langle B_k| \tag{1.2.11}$$

As one can see, the reduced density matrices will have the exact same non-zero eigenvalues which implies that the entanglement entropy of the sub-system A will be equal to the entanglement entropy of subsystem B. Also note that we did not make any assumptions on the sizes of the Hilbert spaces of the two subsystems. This means that for pure states, entanglement entropy does not scale with the volume of the Hilbert space of the subsystems. The formalism discussed above can be applied to any quantum system whose state can be summarized in terms a density matrix. Now we want to define entanglement entropy of a subregion of a CFT_d. To start one chooses some $d-1$ dimensional Cauchy slice of the background spacetime. On this slice we define a state using an Euclidean path integral. This defines a global state over the whole Cauchy slice which is often called the wave functional denoted, $|\Psi[\Phi(x)]\rangle$. The wave functional is defined by the fields, $\Phi(x)$, which depend on the coordinates on the slice. Using this wave functional one can define an assoiciated density matrix $\rho = |\Psi\rangle \langle\Psi|$ over the entire slice. After doing this, one can restrict themselves to some subregion on the slice- call it A, it will have a boundary, ∂A, which is sometimes called the entangling surface. This splits the slice into two regions A and its complement A^c. Naturally, one can now define the reduced density matrix, ρ_A, on A by integrating out the field configurations in the complement. The Von-Neumann entropy of the reduced density matrix can now be computed and this is defined to be the entanglement entropy of the subregion A of the CFT_d. Unsurprisingly when one calculates the entanglement entropy it diverges due to the fact that one is dealing with a continuous system. This is why the entanglement entropy is usually given in terms of some lattice spacing cutoff which regulates the UV divergence. Typically ground states of CFT_d's obey what is known as an area law of entanglement, which states that the entanglement entropy has a leading order UV divergence that scales like the area of the entangling surface ∂A for a fixed lattice spacing. For a more complete discussion of how to calculate entanglement entropy using the ideas discussed above one should refer to [8].

1.3 Holographic Entanglement Entropy and the Ryu-Takayanagi Conjecture

So far our discussions of Entanglement entropy had nothing to do with the AdS_{d+1}/CFT_d correspondence. The AdS_{d+1}/CFT_d correspondence comes into the picture when we consider a special sub-class of CFT_d states which are called holographic. For such CFT_d states it can be shown [5, 8], using the prescription outlined in the previous section, that the result for calculating the entanglement entropy of some subregion A will give the same leading order divergent term as the area of a co-dimension 2 surface in the bulk that is anchored to the entangling surface on the conformal boundary of AdS_{d+1}. This observation leads to the more general conjecture first formulated by Ryu and Takayanagi called the Ryu-Takayanagi conjecture. This is summarized by the Ryu-Takayanagi formula [5, 8]:

$$S_{EE} = \frac{Area(\gamma_A)}{4G_N} \tag{1.3.1}$$

Where γ_A is a co-dimension 2 extremal surface in the bulk which extremizes the area functional. This surface obeys the following boundary condition $\partial A = \partial \gamma_A$, which is to say that the extremal surface in the bulk ends on the conformal boundary on the entangling surface of the sub-region A. Sometimes there can be more than one extremal surface that satisfies this boundary condition in such a case one would choose the surface that minimizes the area. This formula will be used throughout this thesis and is sometimes referred to as the holographic entanglement entropy since it computes the entanglement entropy of holographic CFT_d states.

To make the definition more concrete and illustrate how such calculations will be done, we will do the calculation for the RT surface in pure AdS_{d+1} anchored to the boundary of a ball shaped region ∂B on a constant time slice $t = 0$, on the conformal boundary. The first thing that we must do is write down the line element for pure AdS_{d+1}. A convenient coordinate system to write it in is called Poincare coordinates given by the line element:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{z^2}\left[dz^2 - dt^2 + (dx^1)^2 + ... + (dx^{d-1})^2\right] \tag{1.3.2}$$

In these coordinates z is a space-like coordinate that goes into the bulk and the rest are boundary coordinates. The conformal boundary exists at $z = 0$ and the metric on the conformal boundary is given by $z^2 g_{\mu\nu}|_{z=0}$. In this case we have the flat Minkowski metric on the conformal boundary. We will change the space-like boundary coordinates to hyper-spherical coordinates $(\rho, \phi^1, ..., \phi^{d-2})$ due to the fact we want to consider ball shaped regions on the boundary. The line element becomes:

$$ds^2 = \frac{1}{z^2}\left(dz^2 - dt^2 + d\rho^2 + \rho^2 g^\Omega_{ij} d\phi^i d\phi^j\right) \tag{1.3.3}$$

Where g^Ω_{ij} is the metric on the unit $d-2$ sphere. Now we need to write the area functional for surfaces that are anchored to the entangling surface of the ball on the conformal boundary on the constant time slice, $t = 0$. To do this we apply the formalism discussed in appendix A.1 by setting two of the coordinates equal to functions of the other $d-1$ coordinates. In particular we define:

$$X^t = t = 0 \tag{1.3.4}$$

$$X^z = f(\rho) \tag{1.3.5}$$

$$X^\rho = \rho = \sigma^\rho \tag{1.3.6}$$

$$X^{\phi^i} = \phi^i = \sigma^i \tag{1.3.7}$$

The first embedding equation is simple; since we are considering a static slice of the boundary we know the surface will be on the same static slice in the bulk. The second embedding equation is some function of the radial boundary coordinate; of course, a more general anzatz would be to include ϕ^i. However, since the boundary entangling surface has no ϕ^i dependence we can eliminate such dependence from our anzatz. The coordinates on the surface will be the remaining coordinates of the background space, $\sigma^a = (\sigma^\rho = \rho, \sigma^i = \phi^i)$. Now we can write the induced metric, γ_{ab}, on the surface which is given by:

$$\gamma_{ab} = g_{\mu\nu}\frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b} = g_{zz}\partial_a X^z \partial_b X^z + g_{\rho\rho}\delta^\rho_a\delta^\rho_b + g_{ij}\delta^i_a\delta^j_b = \frac{1}{f^2(\rho)}\left[\delta^\rho_a\delta^\rho_b(1+\partial_\rho f(\rho)\partial_\rho f(\rho)) + \rho^2 g^\Omega_{ij}\delta^i_a\delta^j_b\right] \tag{1.3.8}$$

Using this we define the area functional:

$$A = \int \sqrt{\gamma} d^{d-1}\sigma \tag{1.3.9}$$

The integral goes over the boundary coordinates within the ball shaped region. To find the extremal surface we must extremize the Area functional defined above. In appendix A.1 we derived the equation that needs to be satisfied which is given by:

$$\frac{\delta A}{\delta X^B} = \frac{1}{2}\sqrt{\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu \partial_B g_{\mu\nu} - \partial_a\left(\sqrt{\gamma}\gamma^{ab}\partial_b X^\mu g_{\mu B}\right) \tag{1.3.10}$$

Where B is given by the two coordinates that we used to define the co-dimension 2 surface. In our case we have $B = t, z$. We see that $t = 0$ trivially satisfies the equation, so all we need to do is solve the equation when $B = z$. We use the fact that $\partial_z g_{\mu\nu} = -\frac{2}{z} g_{\mu\nu}$ this simplifies the first term. We can also simplify the second term by noting that X^z only depends on ρ. This implies we have only one non-zero term in the sum thus, we find that:

$$\frac{d-1}{f(\rho)}\sqrt{\gamma} + \partial_\rho\left(\sqrt{\gamma}\gamma^{\rho\rho}\partial_\rho f(\rho)\frac{1}{f^2(\rho)}\right) = 0 \tag{1.3.11}$$

Since the induced metric is diagonal it, is easy to see $\sqrt{\gamma} = \frac{\rho^{d-2}\sqrt{g^\Omega}\sqrt{1+(\partial_\rho f)^2}}{f^{d-1}(\rho)}$ and $\gamma^{\rho\rho} = \frac{f^2(\rho)}{1+(\partial_\rho f)^2}$. Plugging everything in, one can check that $f(\rho) = \sqrt{R^2 - \rho^2}$ solves the equation where R is the radius of the ball on the boundary. This reproduces the well known result that the Ryu-Takayanagi surface for pure AdS_{d+1} anchored to the entangling surface of a ball on a constant time slice is a $d-2$-dimensional hemisphere in the bulk.

One should keep in mind that the calculation we just did for the Ryu-Takayanagi surface was relatively simple for a number of reasons. The first is that the background geometry was a maximally symmetric space called pure AdS_{d+1}. This resulted in the metric having only explicit dependence on ρ and z. We also utilized the fact that the metric is diagonal. In more general $AAdS_{d+1}$ geometries these facts will no longer hold true, which will lead to more complicated equations. The second reason is that the entangling surface on which the extremal surface is anchored to has a very simple coordinate description for our choice of coordinates. In fact, this allowed us to justify the anzatz that f was only a function of ρ and not ϕ^i. For more general entangling surfaces on the boundary, we obviously cannot assume this making the equations more difficult to solve. If one plugs in the solution for the extremal surface back into the area functional one will find that the integral will diverge. Therefore one must introduce a cutoff in the bulk near the boundary this corresponds to the UV lattice cutoff we described in the previous section when one does the calculation on the CFT_d side. For holographic states of a CFT_d one will find that the leading order divergent term in the area for the Ryu-Takayanagi surface will coincide with the leading order divergent term in the entanglement entropy of the CFT_d. In particular for the extremal surface we calculated here its area divided by $4G_N$ would correspond to the entanglement entropy of a vacuum state of a dual CFT_d.

1.4 Boundary Stress Energy Tensor from Asymptotic Behaviour in Bulk

In this section we will do a brief review of the Einstein vacuum equations in $d+1$ dimensional spacetimes[1] for negative cosmological constant. We will give a formal definition of what it means for a space to be $AAdS_{d+1}$ and how exactly the asymptotics of the geometry of such spaces give us information about the boundary CFT_d stress energy tensor [9, 10]. The Einstein vacuum equations with non-zero cosmological constant, Λ, for $d+1$ dimensional spacetime is given by [11]:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{1.4.1}$$

In particular when $\Lambda < 0$, there exists a maximally symmetric spacetime that solves the equations known as pure AdS_{d+1}. This space can be written in Poincare coordinates and the line element will read:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{l^2}{z^2}\left[dz^2 - (dx^0)^2 + (dx^1)^2 + ... + (dx^{d-1})^2\right] = \frac{l^2}{z^2}\left[dz^2 + \eta_{ij}dx^i dx^j\right] \tag{1.4.2}$$

[1]Throughout this thesis we adopt the following convention for the spacetime signature (-,+,...,+) where the minus sign comes for time-like coordinates

Where the constant l is related to the cosmological constant through the following relation, $\Lambda = -\frac{d(d-1)}{2l^2}$, which can be obtained by plugging the metric into the vacuum equations. Throughout this thesis we will simply set $l = 1$ unless otherwise stated. A general $AAdS_{d+1}$ space can be written in the form:

$$ds^2 = \frac{1}{z^2}\left[dz^2 + g_{ij}(x,z)dx^i dx^j\right] = G_{\mu\nu}(z,x)dx^\mu dx^\nu \tag{1.4.3}$$

Where z is called the defining function. The defining function satisfies the following two conditions. The first is that $z \geq 0$ and only vanishes on the conformal boundary. The second is that $g^{\mu\nu}\partial_\mu z \partial_\nu z \neq 0$ on the boundary where $g_{\mu\nu} = z^2 G_{\mu\nu}$. It can be shown that there is always a preferred defining function in a small neighbourhood of the conformal boundary such that $g^{\mu\nu}\partial_\mu z \partial_\nu z = 1$. By choosing this preferred defining function, it was shown in that $g_{ij}(x,z)$ has the following asymptotic expansion near the boundary at $z = 0$ for pure Einstein gravity (i.e no matter fields) [9, 10]:

$$g_{ij}(x,z) = g^{(0)}_{ij}(x) + z^2 g^{(2)}_{ij} + ... + z^d g^{(d)}_{ij} + ... \tag{1.4.4}$$

Moreover it was shown that once one fixes $g^{(0)}_{ij}(x)$ the only non-zero higher order terms occur in even powers of z, which are can be determined order by order using the Einstein equations up to order z^d. Terms beyond order z^d are undetermined by the Einstein equations near the boundary. In this thesis we will mainly be interested in the case where the $AAdS_{d+1}$ space is conformally flat, $g^{(0)}_{ij}(x) = \eta_{ij}$. In this case it was shown that the asymptotic expansion of the $AAdS_{d+1}$ space is given by [10]:

$$ds^2 = \frac{1}{z^2}\left[z^2 + \eta_{ij}dx^i dx^j + z^d \Gamma^{(d)}_{ij}(x) + z^{d+1}\Gamma^{(d+1)}_{ij}(x) + ...\right] \tag{1.4.5}$$

From such an expansion it was shown using the AdS_{d+1}/CFT_d correspondence that the coefficient in the asymptotic expansion, $\Gamma^{(d)}_{ij}$, was fixed by the expectation value of the boundary CFT_d stress energy tensor through the following relation [10]:

$$\langle T_{ij}(x)\rangle = \frac{d}{16\pi G_N}\Gamma^{(d)}_{ij}(x) \tag{1.4.6}$$

The reader should keep in mind that the result above is only true if we are dealing with a conformally flat $AAdS_{d+1}$ spacetime. In general an extra term will be added that reflects conformal anomalies which occur due to lower order terms in the expansion [10]. Now we are ready to discuss the correspondence between states of a CFT_d on a Minkowski background and the corresponding $AAdS_{d+1}$ dual geometry. Start by noting that pure AdS_{d+1} occurs when all the terms $\Gamma^{(n\geq d)}_{ij}(x) = 0$. In particular the pure AdS_{d+1} geometry corresponds to a dual state on the CFT_d whose stress energy tensor expectation value is zero. This leads us to the correspondence that Pure AdS_{d+1} is dual to the vacuum state of a CFT_d and vice-versa. More generally when $\Gamma^{(n>d)}_{ij}(x) \neq 0$ then this will correspond to some state deformed away from the vacuum. Furthermore it is assumed that small perturbations away from the vacuum state of the CFT_d corresponds to small perturbations away from pure AdS_{d+1} dual geometry. Using this correspondence between the bulk $AAdS_{d+1}$ geometry and states of a CFT_d as a starting point one can begin to understand how constraints on states of a CFT_d translate to constraints in the bulk geometry.

Chapter 2

Constraints From Causal Holographic Information Surface

2.1 Defining the Causal Holographic Information Surface

Before understanding the causal holographic information (CHI) surface we need to understand how to construct a geometric quantity in the bulk called the causal wedge. The causal wedge construction in the bulk was motived by a need to understand how bulk geometry of $AAdS_{d+1}$ spacetimes emerged from the the dual CFT_d. In particular, it was argued that if one was given the density matrix on the CFT_d of some closed bounded region on the boundary, B, with boundary ∂B. Then the bulk geometry of the dual $AAdS_{d+1}$ spacetime could be reconstructed within a region known as the causal wedge of B [12]. We will give a quick summary of the construction of the wedge as given in [12]. We start with a Cauchy slice Σ of the spacetime the CFT_d resides on, then define a closed and bounded $d-1$ dimensional region on the slice and call it B. This region has an associated d dimensional future domain of dependence, denoted $D^+[B]$ and a past domain of dependence, denoted $D^-[B]$. More intuitively we can say that a point $p_- \in D^-[B]$ if all future oriented null geodesics originating from p_- intersect with B. Similarly we can say a point $p_+ \in D^+[B]$ if all past oriented null geodesics originating from p_+ intersect with B. Another way of saying this is that the set of points in $D^+[B]$ and $D^-[B]$ are determined by doing future and past time evolution of some prescribed data on the region B. Together the union of the past and future domain of dependence is known simply as the domain of dependence of the region B and is denoted as $D[B]$. Now that we have defined the domain of dependence of our region B we define the causal wedge of B to be the intersection of the future and past domains of influence of $D[B]$. This makes a wedge that extends into the bulk, the wedge at the conformal boundary coincides with $D[B]$. The boundary of the wedge is made of two null surfaces whose intersection defines a co-dimension 2 ($d-1$ -dimensional) space-like surface in the bulk which is called the causal holographic (CHI) surface which is denoted as Ξ_B. It is anchored to ∂B on the conformal boundary. Using this co-dimension 2 surface in the bulk one defines the CHI associated with the boundary region B as:

$$\chi_B = \frac{Area(\Xi_B)}{4G_N} \tag{2.1.1}$$

In this thesis we will be interested in the following property that Ξ is conjectured to obey for smooth spacetimes satisfying the null energy condition [13]:

$$\chi_B - S_B = \frac{1}{4G_N}(Area(\Xi_B) - Area(RT)) \geq 0 \tag{2.1.2}$$

Where $Area(RT)$ is the area of the Ryu-Takayanagi surface. We want to see what how this conjecture constraints the asymptotic geometry of highly symmetric and static $AAdS_3$ spacetimes.

2.2 Series Expansion for CHI Curve for $AAdS_3$ Spacetimes

We start with a general $AAdS_3$ metric given in Poincare coordinates (t, x, z) that exhibits translation invariance in the boundary coordinates x and y. The line element is given as:

$$ds^2 = \frac{l^2}{z^2}\left(dz^2 - g(z)dt^2 + f(z)dx^2\right) \tag{2.2.1}$$

Where $g(z)$ and $f(z)$ have the following asymptotic expansions near the boundary conformal boundary situated at $z = 0$:

$$f(z) = 1 + \frac{z^2 f_2}{2!} + \frac{z^3 f_3}{3!} + \frac{z^4 f_4}{4!} + ... \tag{2.2.2}$$

$$g(z) = 1 - \frac{z^2 f_2}{2!} + \frac{z^3 g_3}{3!} + \frac{z^4 g_4}{4!} + ... \tag{2.2.3}$$

From this point, throughout the rest of chapter 2 when we say $AAdS_3$ spacetime, we mean a spacetime that has the line element given by (2.2.1). The quadratic order coefficient in both expansions differ by a sign in order to have a traceless holographic stress energy tensor due to the fact that the spacetime is dual to a CFT_2 state. The goal now is to translate the constraint given by the inequality (2.1.2) into constraints on the coefficients in the asymptotic expansions given above. We will consider tilted intervals on the conformal boundary to be our region B from which we will construct the domain of dependence of the interval $D[B]$. We will in use $D[B]$ to define the CHI surface that extends into the bulk. To define the interval we start by looking at the induced line element on a constant z-slice:

$$ds^2_{z=z_0} = \frac{l^2}{z_0^2}\left(-g(z_0)dt^2 + f(z_0)dx^2\right) \tag{2.2.4}$$

Define an interval on the slice which is a straight line connecting the space-like separated points $P_1 = (-\delta t, -\delta x, z_0)$ and $P_2 = (\delta t, \delta x, z_0)$ where $\delta x - \delta t > 0$. Now we want to find the domain of dependence for this interval. To construct the domain of dependence we emit null geodesics from the end points of the interval. It is not difficult to see that the geodesics will intersect at two points. One will be a point to the past of the interval and another to the future which we will denote P_- and P_+ respectively. We can explicitly calculate these points by computing the null geodesics. We find that null geodesics emitted from the point P_1 are given by the equation:

$$t_{1\pm}(x) = \pm\sqrt{\frac{f(z_0)}{g(z_0)}}(x+\delta x) - \delta t \tag{2.2.5}$$

For null geodesics emitted at P_2:

$$t_{2\pm}(x) = \pm\sqrt{\frac{f(z_0)}{g(z_0)}}(x-\delta x) + \delta t \tag{2.2.6}$$

We get P_+ by setting $t_{1+} = t_{2-}$ and P_- by setting $t_{1-} = t_{2+}$ we find that $P_+ = \left(\sqrt{\frac{f(z_0)}{g(z_0)}}\delta x, \sqrt{\frac{g(z_0)}{f(z_0)}}\delta t, z_0\right)$ and $P_- = \left(-\sqrt{\frac{f(z_0)}{g(z_0)}}\delta x, -\sqrt{\frac{g(z_0)}{f(z_0)}}\delta t, z_0\right)$. The points P_1, P_+, P_2, and P_- are the vertices of a diamond shape which is the domain of dependence for the interval, sometimes it is called the causal diamond. We can get the domain of dependence on the conformal boundary by letting $z_0 \to 0$ then $\sqrt{\frac{f(z_0)}{g(z_0)}} \to 1$.

Now we will find a series expansion for the CHI surface associated with the space-like interval we defined. To do this we will emanate a family of null geodesics from the from the point $(t = \delta x, x = \delta t, z = 0)$ towards the past of the point, and another family of null geodesics from the point $(t = -\delta x, x = -\delta t, z = 0)$ towards the future of the point into the bulk geometry. The set of points where the geodesics intersect will form a curve and this will be the CHI surface. We will be interested in finding a series expansion for it. To begin, we will start by simplifying the problem of a tilted interval to a more simpler case where the interval is on a constant time slice. To do this we make use of a Lorentz transformation. Since we have a space-like interval on the conformal boundary that connects two points separated by $\Delta t = 2\delta t$ and $\Delta x = 2\delta x$. We want to choose a boost parameter v such that:

$$\delta t' = \gamma(\delta t - v\delta x) = 0 \tag{2.2.7}$$

$$\gamma = \frac{1}{\sqrt{1-v^2}} \tag{2.2.8}$$

Here we see that we require that $v = \frac{\delta t}{\delta x}$, in this new frame we have that:

$$L := \delta x' = \gamma(\delta x - v\delta t) = \frac{\delta x}{\gamma} \tag{2.2.9}$$

Note that L is the proper length of the interval which is a Lorentz invariant quantity. Since we transformed the interval we must also transform our original metric as well using the infinitesimal versions of the Lorentz transformations:

$$dt' = \gamma(dt - vdx) \Rightarrow dt = \gamma(dt' + vdx') \tag{2.2.10}$$

$$dx' = \gamma(dx - vdt) \Rightarrow dx = \gamma(dx' + vdt') \tag{2.2.11}$$

$$dz' = dz \tag{2.2.12}$$

Substituting into our original metric we obtain the transformed line element which will read:

$$ds'^2 = \frac{l^2}{z'^2}\left[dz'^2 + \gamma^2[-dt'^2(g(z') - v^2 f(z')) + dx'^2(f(z') - v^2 g(z')) + 2vdt'dx'(f(z') - g(z'))]\right] \tag{2.2.13}$$

Hence we have transformed our problem of finding the CHI curve associated with a tilted interval on the boundary with the bulk metric defined by equation (2.2.1) to an equivalent problem with a constant time interval on the boundary and a bulk metric defined by equation (2.2.13). The boost parameter, v, gives a way to go from one description to the other. Now we must compute null geodesics using the transformed metric. We will use the Lagrangian approach. Define the following Lagrangian:

$$\mathcal{L}' = \frac{l}{z'}\sqrt{\dot{z}'^2 + \gamma^2[-\dot{t}'^2(g(z') - v^2 f(z')) + \dot{x}'^2(f(z') - v^2 g(z')) + 2v\dot{t}'\dot{x}'(f(z') - g(z'))]} \tag{2.2.14}$$

Where we use the dot notation to represent the derivative of each coordinate with respect to some parameter along the geodesic which we will eventually take to be the coordinate time. We get geodesics by solving the Euler Lagrange equations. In particular since there is no explicit dependence on on either t' or x' we know that:

$$\frac{\gamma^2 l^2}{z'^2\mathcal{L}'}[\dot{x}'(f(z') - v^2 g(z')) + v\dot{t}'(f(z') - g(z'))] = c'_x \tag{2.2.15}$$

$$\frac{\gamma^2 l^2}{z'^2\mathcal{L}'}[v\dot{x}'(f(z') - g(z')) - \dot{t}'(g(z') - v^2 f(z'))] = c'_t \tag{2.2.16}$$

Where c'_x and c'_t are constants. Dividing one equation by the other and rearranging we obtain the following:

$$\frac{dx'}{dt'} = \frac{vf(z')(1+cv) - g(z')(c+v)}{vg(z')(c+v) - f(z')(1+cv)} \tag{2.2.17}$$

Where $c = -c'_x/c'_t$ is some constant (the negative sign is simply a convention we adopt). More physically, the parameter c will label the different geodesics, and different values of c will corrspond to geodesics being emitted along different directions from a given point in spacetime. Now we take the parameter along the geodesics in the Lagrangian as t' and substitute our above expression for $\frac{dx'}{dt'}$ into the Lagrangian (2.2.14). We then set the Lagrangian to zero (null condition) and solve for $\frac{dz'}{dt'}$. We obtain:

$$\frac{dz'}{dt'} = \gamma\sqrt{g(z')\left[v\frac{dx'}{dt'}+1\right]^2 - f(z')\left[v+\frac{dx'}{dt'}\right]^2} \tag{2.2.18}$$

Using the equations (2.2.17) and (2.2.18) we get the following two integral equations:

$$\int dt' = \int \frac{\sqrt{1-v^2}dz'}{\sqrt{g(z')\left[v\frac{dx'}{dt'}+1\right]^2 - f(z')\left[v+\frac{dx'}{dt'}\right]^2}} \tag{2.2.19}$$

$$\int dx' = \int \frac{vf(z')(1+cv)-g(z')(c+v)}{vg(z')(c+v)-f(z')(1+cv)} \frac{\sqrt{1-v^2}dz'}{\sqrt{g(z')\left[v\frac{dx'}{dt'}+1\right]^2 - f(z')\left[v+\frac{dx'}{dt'}\right]^2}} \tag{2.2.20}$$

Using the two integral equations above, we can describe how geodesics will evolve from a point of interest on the conformal boundary into the bulk geometry. In particular, for a past directed null geodesic starting at the future tip of the causal diamond on the boundary, $(t' = L, x' = 0, z = 0)$, we have that:

$$t_+(z, c_+) = L - \int_0^z \frac{\sqrt{1-v^2}dz'}{\sqrt{g(z')\left[v\frac{dx'}{dt'}+1\right]^2 - f(z')\left[v+\frac{dx'}{dt'}\right]^2}} \tag{2.2.21}$$

For future directed null geodesics starting at the past tip of the causal diamond on the boundary, $(t' = -L, x' = 0, z = 0)$, we get:

$$t_-(z, c_-) = -L + \int_0^z \frac{\sqrt{1-v^2}dz'}{\sqrt{g(z')\left[v\frac{dx'}{dt'}+1\right]^2 - f(z')\left[v+\frac{dx'}{dt'}\right]^2}} \tag{2.2.22}$$

We do a power series expansion of the right hand side of the equation at $z = 0$, and find power series of the form:

$$t_+(z, c_+) = L - \sum_{k=1}^{\infty} A_k(c_+)z^k \tag{2.2.23}$$

$$t_-(z, c_-) = -L + \sum_{k=1}^{\infty} A_k(c_-)z^k \tag{2.2.24}$$

We can also find similar integral expressions for the x coordinate along the past and future directed geodesics and expand in a power series in z with coefficients depending on $c_\pm$:

$$x_+(z, c_+) = \sum_{k=1}^{\infty} B_k(c_+)z^k \tag{2.2.25}$$

$$x_-(z, c_-) = \sum_{k=1}^{\infty} B_k(c_-) z^k \tag{2.2.26}$$

Now fix a particular c_+, this will correspond to a particular geodesic starting at the point $(t = L, x = 0, z = 0)$. This should intersect with some other geodesic characterized by c_- starting at the point $(t = -L, x = 0, z = 0)$. The intersection point between the two geodesics will correspond to one point on the CHI curve. To find this point we set $t_+ = t_-$ and $x_+ = x_-$. We obtain:

$$2L = \sum_{k=1}^{\infty} [A_k(c_+) + A_k(c_-)] z'^k_{int} \tag{2.2.27}$$

$$0 = \sum_{k=1}^{\infty} [B_k(c_+) - B_k(c_-)] z'^k_{int} \Rightarrow B_k(c_+) - B_k(c_-) = 0, \forall k \tag{2.2.28}$$

Where z_{int} is the z coordinate in the bulk where the geodesics intersect. Equation (2.2.28) will be satisfied if we set $c_+ = c_- = c$. By substituting this relation between c_+ and c_- into (2.2.27) we will find:

$$L = \sum_{k=1}^{\infty} A_k(c) {z'_{int}}^k \tag{2.2.29}$$

It states that the intersection point in the bulk of the two geodesics is controlled by the proper length of the interval on the conformal boundary. This implies that if the proper length of the interval on the boundary is sufficiently small, then the z-coordinate of the intersection point will also be small. To make this statement more precise we can invert the series given by equation (2.2.29) to write z'_{int}, as a power series in L with c and v dependent coefficients:

$$z'_{int}(c) = \sum_{k=1}^{\infty} \eta_k(c) L^k \tag{2.2.30}$$

From this point we must assume that $L << 1$ in order for the series to converge. Since we know what z_{int} is we can use equations (2.2.23)$-$(2.2.26) to obtain $t'_{int} = t_\pm(z'_{int}, c)$ and $x'_{int} = x_\pm(z'_{int}, c)$:

$$x'_{int}(c) = \sum_{k=1}^{\infty} B_k(c) z'_{int}(c)^k = \sum_{k=1}^{\infty} \zeta_k(c) L^k \tag{2.2.31}$$

$$t'_{int}(c) = 0 \tag{2.2.32}$$

The coefficients in the series expansions are well defined when $-1 \leq c \leq 1$. We now have the set of intersection points of geodesics that generate the null boundary of the causal wedge in the bulk. This gives us the CHI curve parameterized by $c \in [-1, 1]$. The curve is connected to the endpoints of the interval of interest $z'_{int}(\pm 1) = 0$ and $x'_{int}(\pm 1) = \pm L$ as expected. We can revert back to the original coordinates where the interval is tilted by applying the inverse Lorentz transformation to the CHI curve coordinates. We get:

$$z_{int}(c) = z'_{int}(c) = \sum_{k=1}^{\infty} \eta_k(c) L^k \tag{2.2.33}$$

$$x_{int}(c) = \gamma(x'_{int} + vt'_{int}) = \frac{x'_{int}(c)}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-v^2}} \sum_{k=1}^{\infty} \zeta_k(c) L^k \tag{2.2.34}$$

$$t_{int}(c) = \gamma(t'_{int} + vx'_{int}) = \frac{vx'_{int}(c)}{\sqrt{1-v^2}} = \frac{v}{\sqrt{1-v^2}} \sum_{k=1}^{\infty} \zeta_k(c) L^k \tag{2.2.35}$$

The three equations above give a complete perturbative expansion of the CHI curve associated with a tilted interval of proper length L on the boundary for $AAdS_3$ spaces whose metric can be written in the form given by equation (2.2.1). The coefficients in the expansions can be written in terms of the coefficients A_k and B_k defined in equations (2.2.23) – (2.2.26) through the procedure outlined above.

2.3 Series Expansion for the Area of the CHI Curve

Now that we have a series expansion for the CHI curve we can find a series expansion for its length. To do this we will begin by defining a new parameter along the CHI curve, $\lambda = \sqrt{1-c^2}$, where $0 \leq \lambda \leq 1$. Then we can relate λ to c by using a piecewise definition, $c_\pm = \pm\sqrt{1-\lambda^2}$ where, $-1 \leq c_- \leq 0$ and $0 \leq c_+ \leq 1$. The main reason we choose to define this different parameter is because we will need to regulate the integrals involving the length of the CHI curve by introducing a cutoff near the conformal boundary. Doing this will simply amount to setting λ to a small parameter. This will make it easier to split the expressions into a finite part and a divergent part. Since we are going to use λ, we need to deal with the two halves of the CHI curve separately, in particular we define (t_L, x_L, z_L) to be points on the left half of the CHI curve and (t_R, x_R, z_R) to be the points on the right half of the CHI curve. We can define these points very easily using equations (2.2.33) – (2.2.35):

$$(t_L, x_L, z_L) = (t_{int}(c_-), x_{int}(c_-), z_{int}(c_-)) \tag{2.3.1}$$

$$(t_R, x_R, z_R) = (t_{int}(c_+), x_{int}(c_+), z_{int}(c_+)) \tag{2.3.2}$$

Now we can write the integral for the total length of the curve CHI curve as:

$$\begin{aligned} A_{CHI} = &\int_0^1 \frac{1}{z_L(\lambda)}\sqrt{\left(\frac{dz_L}{d\lambda}\right)^2 + f(z_L(\lambda))\left(\frac{dx_L}{d\lambda}\right)^2 - g(z_L(\lambda))\left(\frac{dt_L}{d\lambda}\right)^2}d\lambda \\ &+ \int_0^1 \frac{1}{z_R(\lambda)}\sqrt{\left(\frac{dz_R}{d\lambda}\right)^2 + f(z_R(\lambda))\left(\frac{dx_R}{d\lambda}\right)^2 - g(z_R(\lambda))\left(\frac{dt_R}{d\lambda}\right)^2}d\lambda \end{aligned} \tag{2.3.3}$$

Consider the first integral involving the length of the left half of the CHI curve and analyze the integrand near the boundary, $\lambda = 0$, as this is where any divergence in the integral will occur. To do this we need only to understand the asymptotic expansions of (t_L, x_L, z_L). They will have the form:

$$z_L = \lambda + \mathcal{O}(\lambda^2) \tag{2.3.4}$$

$$x_L = -L + \mathcal{O}(\lambda^2) \tag{2.3.5}$$

$$t_L = vx_L = -vL + \mathcal{O}(\lambda^2) \tag{2.3.6}$$

Where for equation (2.3.6) we used the fact that $t_{int}/x_{int} = v$. We can also give the asymptotic expansions for f and g in terms of λ:

$$f(z_L(\lambda)) = 1 + \mathcal{O}(\lambda^2) \tag{2.3.7}$$

$$g(z_L(\lambda)) = 1 + \mathcal{O}(\lambda^2) \tag{2.3.8}$$

This implies that near the boundary the integrand has the following asymptotic expansion:

$$\frac{1}{z_L(\lambda)}\sqrt{\left(\frac{dz_L}{d\lambda}\right)^2 + f(z_L(\lambda))\left(\frac{dx_L}{d\lambda}\right)^2 - g(z_L(\lambda))\left(\frac{dt_L}{d\lambda}\right)^2} = \frac{1}{\lambda} + \mathcal{O}(1) \tag{2.3.9}$$

This means the integral will diverge. The same will also hold true for the integrand involving the right side of the CHI curve. This is an expected result, what we can do now is rewrite it into a finite part and divergent part. This amounts to subtracting off the divergence and doing the integral from 0 to 1 and adding on the same divergence with cutoffs $\hat{\epsilon_L}$ and $\hat{\epsilon_R}$ going to zero this gives the following expression:

$$\begin{aligned} A_{CHI} = &\int_0^1 \left[\frac{d\lambda}{z_L(\lambda)} \sqrt{\left(\frac{dz_L}{d\lambda}\right)^2 + f(z_L(\lambda))\left(\frac{dx_L}{d\lambda}\right)^2 - g(z_L(\lambda))\left(\frac{dt_L}{d\lambda}\right)^2} - \frac{1}{\lambda} \right] d\lambda \\ &+ \int_0^1 \left[\frac{1}{z_R(\lambda)} \sqrt{\left(\frac{dz_R}{d\lambda}\right)^2 + f(z_R(\lambda))\left(\frac{dx_R}{d\lambda}\right)^2 - g(z_R(\lambda))\left(\frac{dt_R}{d\lambda}\right)^2} - \frac{1}{\lambda} \right] d\lambda \\ &- \lim_{\hat{\epsilon_R}\to 0} ln(\hat{\epsilon_R}) - \lim_{\hat{\epsilon_L}\to 0} ln(\hat{\epsilon_L}) \end{aligned} \tag{2.3.10}$$

Here the first integrals will converge and all the divergence is contained in the last two logarithmic terms as $\hat{\epsilon_L}$ and $\hat{\epsilon_R}$ go to zero. We can expand the finite part as a power series in L and explicitly do the integrals order by order. We find that:

$$\begin{aligned} A_{CHI} = &2ln(2) + \frac{1}{32}\frac{\pi(-g_3v^2 + f_3)}{1-v^2}L^3 + \frac{1}{90}\frac{3f_2^2v^2 - 2g_4v^2 - 3f_2^2 + 2f_4}{1-v^2}L^4 + \mathcal{O}(L^5) \\ &- \lim_{\hat{\epsilon_R}\to 0} ln(\hat{\epsilon_R}) - \lim_{\hat{\epsilon_L}\to 0} ln(\hat{\epsilon_L}) \end{aligned} \tag{2.3.11}$$

This gives us the area of the CHI curve as a power series in the proper length of the interval on the boundary with coefficients that depend on the asymptotic structure of the geometry and the tilt of the interval characterized by v.

2.4 Ryu-Takayanagi Curve for $AAdS_3$

Here we reproduce the expressions given in [4] that the Ryu-Takayanagi surface will obey. Start with the metric expressed in terms of a line element:

$$ds^2 = \frac{1}{z^2}(dz^2 + f(z)dx^2 - g(z)dt^2) \tag{2.4.1}$$

The Ryu-Takayanagi surface for this space will be a curve. We will parameterize the curve in terms the bulk coordinate z. The length functional will be:

$$L = \int_0^{z_0} \frac{dz}{z}\sqrt{1 + f(z)\left(\frac{dx}{dz}\right)^2 - g(z)\left(\frac{dt}{dz}\right)^2} \tag{2.4.2}$$

The functions $x(z)$ and $t(z)$ for the RT-curve will extremize the length functional. We can define the associated Lagrangian as:

$$\mathcal{L}(x, \dot{x}, t, \dot{t}, z) = \frac{1}{z}\sqrt{1 + f(z)\dot{x}^2 - g(z)\dot{t}^2} \tag{2.4.3}$$

Where we use Newton's dot notation for a derivative with respect to z. The associated Euler

Lagrange equations for $x(z)$ and $t(z)$ and given by:

$$\frac{d}{dz}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 \Rightarrow \frac{d}{dz}\left[\frac{1}{z}\frac{f(z)\dot{x}}{\sqrt{1+f(z)\dot{x}^2-g(z)\dot{t}^2}}\right] = 0 \tag{2.4.4}$$

$$\frac{d}{dz}\frac{\partial \mathcal{L}}{\partial \dot{t}} = 0 \Rightarrow \frac{d}{dz}\left[\frac{1}{z}\frac{g(z)\dot{t}}{\sqrt{1+f(z)\dot{x}^2-g(z)\dot{t}^2}}\right] = 0 \tag{2.4.5}$$

These tell us that along the curve parameterized by z the quantities in the square brackets are constants. In particular we can choose to evaluate the expression in the square brackets at the end point z_0 which is the maximal depth that the curve goes into the bulk. This implies that $[\dot{x}(z_0)]^{-1} = [\dot{t}(z_0)]^{-1} = 0$. Then it follows that the terms in the expressions in the square brackets evaluated at the point z_0 are:

$$\left[\frac{1}{z}\frac{f(z)\dot{x}}{\sqrt{1+f(z)\dot{x}^2-g(z)\dot{t}^2}}\right]_{z=z_0} = \frac{f_0}{z_0\sqrt{1-\beta_0^2}} \tag{2.4.6}$$

Where we defined $\beta_0 = \sqrt{\frac{g(z)}{f(z)}}\frac{dt}{dx}\Big|_{z=z_0}$, $f_0 = f(z_0)$, and $g_0 = g(z_0)$. Using the expression given in equation (2.4.4) we can write:

$$\frac{1}{z}\frac{f(z)\dot{x}}{\sqrt{1+f(z)\dot{x}^2-g(z)\dot{t}^2}} = \frac{f_0}{z_0\sqrt{1-\beta_0^2}} \tag{2.4.7}$$

Then we can write $\dot{t}$ in terms of $\dot{x}$ by noting that:

$$\frac{\frac{1}{z}\frac{g(z)\dot{t}}{\sqrt{1+f(z)\dot{x}^2-g(z)\dot{t}^2}}}{\frac{1}{z}\frac{f(z)\dot{x}}{\sqrt{1+f(z)\dot{x}^2-g(z)\dot{t}^2}}} = \frac{g(z)}{f(z)}\frac{\dot{t}}{\dot{x}} = \sqrt{\frac{g_0}{f_0}}\beta_0 \Rightarrow \dot{t} = \frac{f(z)}{g(z)}\sqrt{\frac{g_0}{f_0}}\beta_0\dot{x} \tag{2.4.8}$$

Plugging this into equation (2.4.7) and rearranging for $\dot{x}^2$ gives us:

$$\dot{x}^2 = \frac{z^2 f_0}{f^2 z_0^2}\frac{1}{\left[1-\frac{z^2 f_0}{z_0^2 f}\right] - \beta_0^2\left[1-\frac{z^2 g_0}{z_0^2 g}\right]} \tag{2.4.9}$$

Doing a similar calculation also shows that:

$$\dot{t}^2 = \beta_0^2\frac{z^2 g_0}{z_0^2 g^2}\frac{1}{\left[1-\frac{z^2 f_0}{z_0^2 f}\right] - \beta_0^2\left[1-\frac{z^2 g_0}{z_0^2 g}\right]} \tag{2.4.10}$$

This gives us equations that the Ryu-Takayanagi curve should obey.

2.5 Series Expansion for Area of the Ryu-Takayanagi Surface

Now we want to get a similar series expansion for the area of the RT curve. From the previous section we know that the RT curve satisfies the following differential equations:

$$\begin{aligned}\frac{dx}{dz} &= \frac{z\sqrt{f_0}}{z_0 f}\frac{1}{\sqrt{1-\frac{z^2 f_0}{z_0^2 f}-\beta_0^2\left[1-\frac{z^2 g_0}{z_0^2 g}\right]}}\\ \frac{dt}{dz} &= \beta_0\frac{z\sqrt{g_0}}{z_0 g}\frac{1}{\sqrt{1-\frac{z^2 f_0}{z_0^2 f}-\beta_0^2\left[1-\frac{z^2 g_0}{z_0^2 g}\right]}}\end{aligned} \tag{2.5.1}$$

Where z_0 is the maximal value of z reached by the RT curve and we defined $\beta_0 = \sqrt{\frac{g(z)}{f(z)}}\frac{dt}{dx}\Big|_{z=z_0}$, $f_0 = f(z_0)$, $g_0 = g(z_0)$, $g = g(z)$, and $f = f(z)$. This is the only information we need to find a series expansion of the length of the RT curve using the metric we are given. However, this expansion will have coefficients that depend on β_0 and z_0 which are not the parameters we used to express the CHI curve. This problem can be resolved by expressing the parameters β_0 and z_0 in terms of the parameters v and L used in the CHI expansion. To do this we start by integrating the expressions above to obtain:

$$\Delta x = \int_0^{z_0}\frac{z\sqrt{f_0}}{z_0 f}\frac{dz}{\sqrt{1-\frac{z^2 f_0}{z_0^2 f}-\beta_0^2\left[1-\frac{z^2 g_0}{z_0^2 g}\right]}} \tag{2.5.2}$$

$$\Delta t = \int_0^{z_0}\beta_0\frac{z\sqrt{g_0}}{z_0 g}\frac{dz}{\sqrt{1-\frac{z^2 f_0}{z_0^2 f}-\beta_0^2\left[1-\frac{z^2 g_0}{z_0^2 g}\right]}} \tag{2.5.3}$$

Note that these integrals only give half of the RT curve since we choose to parameterize in terms of the bulk coordinate z. However, due to the high degree of symmetry of the metric in the boundary coordinates, the total change in x and t are double the integrals we have above. We can now relate the proper length of the interval on the boundary to the integrals above $L = \frac{\delta x}{\gamma} = \sqrt{(\Delta x)^2 - (\Delta t)^2}$. We do a series expansion for the integrands at Δx and Δt at $z = 0$. We integrate the series term by term from $z = 0$ to $z = z_0$ and obtain a series expansion for δx and δt in terms of z_0. We can then plug in the series into the equation for $\frac{\delta x}{\gamma}$ and get:

$$L(z_0, \beta_0) = \frac{\delta x}{\gamma} = \sum_{k=1}^{\infty} b_k(\beta_0) z_0^k \tag{2.5.4}$$

Then we can do a series reversion and find z_0 as a power series in $\frac{\delta x}{\gamma} = L$:

$$z_0(L, \beta_0) = \sum_{k=1}^{\infty} c_k(\beta_0) L^k \tag{2.5.5}$$

Once again we find that the maximum depth the curve goes into the bulk is controlled by the proper length of the boundary interval. We can also do a series expansion of $v = \frac{\delta t}{\delta x} = \frac{\Delta t}{\Delta x}$ at $z = 0$ and integrate term by term to get v as a power series in z_0 with β_0 dependent coefficients:

$$v(z_0, \beta_0) = \beta_0 + \sum_{k=2}^{\infty} v_k(\beta_0) z_0^k \tag{2.5.6}$$

Since we know that z_0 is related to L by the series expansion given by equation (2.5.5), we can rewrite v as a power series in L by substituting $z_0(L, \beta_0)$ into the right had side of equation (2.5.6). Then we expand in powers of L, this will give a series expansion for v of the form:

$$v(L, \beta_0) = \beta_0 + \sum_{k=2}^{\infty} V_k(\beta_0) L^k \tag{2.5.7}$$

We can invert the series above and write β_0 as a power series in L with v dependent coefficients. To accomplish this we define the following series in L:

$$\beta_0(L, v) = v + \sum_{m=2}^{\infty} K_m(v) L^m \tag{2.5.8}$$

To find the coefficients $K_m(v)$, we substitute the series for $\beta_0(L, v)$ into the right hand side of equation (2.5.7) and expand as a power series in L and calculate up to whatever order in L we want. This will give us $\beta_0(L, v)$. Now we want to find $z_0(L, v)$. Going back to the series expansion given by equation (2.5.5) we note that the coefficients $c_k(\beta_0)$ are dependent on β_0. If we plug in our series expansion $\beta_0(L, v)$ into the coefficients $c_k(\beta_0(L, v))$ and expand, we will get a series expansion of $z_0(L, v)$. This will enable us to express the series expansion of the length of the RT curve in terms of the parameters L and v. The length of the curve parameterized in terms of z is given by:

$$A_{RT} = \int_0^{z_0} \frac{2}{z} \sqrt{1 + f(z) \left(\frac{dx}{dz}\right)^2 - g(z) \left(\frac{dt}{dz}\right)^2}\, dz \tag{2.5.9}$$

Just like for the CHI curve, we know that the length of the RT curve will diverge. We will split the area into a finite and divergent part:

$$\begin{aligned} A_{RT} &= \int_0^{z_0} \left[\frac{2}{z} \sqrt{1 + f(z) \left(\frac{dx}{dz}\right)^2 - g(z) \left(\frac{dt}{dz}\right)^2} - \frac{2}{z}\right] dz + \lim_{\epsilon \to 0} 2ln \left(\frac{z_0}{\epsilon}\right) \\ &= \int_0^1 \left[\frac{2}{u} \sqrt{\frac{1 - \beta_0^2}{1 - u^2 \frac{f(z_0)}{f(z_0 u)} - \beta_0^2 \left[1 - u^2 \frac{g(z_0)}{g(z_0 u)}\right]}} - \frac{2}{u}\right] du + \lim_{\epsilon \to 0} 2ln \left(\frac{z_0}{\epsilon}\right) \end{aligned} \tag{2.5.10}$$

The divergence is contained in the logarithmic terms expressed in terms of the cutoff ϵ. We can then substitute the series expansions of $z_0(L, v)$ and $\beta_0(L, v)$ and expand the finite part of the area as a power series in L. We find that:

$$\begin{aligned} A_{RT} =& 2ln(2) + \frac{1}{3}\frac{f_2(1+v^2)}{1-v^2}L^2 + \frac{1}{32}\frac{\pi(-g_3v^2+f_3)}{1-v^2}L^3 \\ &- \frac{1}{45}\frac{2f_2^2v^4 - g_4v^4 + f_4v^2 + g_4v^2 + 2f_2^2 - f_4}{(1-v^2)^2}L^4 + \mathcal{O}(L^5) + \lim_{\epsilon\to 0} 2ln\left(\frac{L}{\epsilon}\right) \end{aligned} \tag{2.5.11}$$

Which gives us the series expansion for the area of the RT curve in terms of the parameters defined for the CHI curve.

2.6 Constraints on $AAdS_3$ Spacetimes from CHI Inequality

Now that we have a series expansion for both the CHI and RT curve associated with the interval on the conformal boundary, we will see what kind of constraint we can get by using the conjecture that $A_{CHI} - A_{RT} \geq 0$. Before doing this, we must recall that the cutoff was described differently for the RT and CHI curve. We must first relate the cutoff $\hat{\epsilon}$ for the CHI curve to the cutoff ϵ for the RT curve. In particular we must satisfy:

$$\epsilon = z_L(\lambda = \hat{\epsilon_L}) = \Omega_L(L,v)\hat{\epsilon}_L + \mathcal{O}(\hat{\epsilon}_L^2) \tag{2.6.1}$$

$$\epsilon = z_R(\lambda = \hat{\epsilon_R}) = \Omega_R(L,v)\hat{\epsilon}_R + \mathcal{O}(\hat{\epsilon}_R^2) \tag{2.6.2}$$

We expand the right hand side of the two equations as power series in $\hat{\epsilon_L}$ and $\hat{\epsilon_R}$ with v and L dependent coefficients. We will only need to retain terms up to first order since we are taking a limit to zero in the end. Then:

$$\hat{\epsilon_L} = \frac{\epsilon}{\Omega_L(L,v)} \tag{2.6.3}$$

$$\hat{\epsilon_R} = \frac{\epsilon}{\Omega_R(L,v)} \tag{2.6.4}$$

We will then substitute these expressions into our cutoff for the CHI curve and and expand the cutoff term in a series in L when we do this we obtain the following expansion with a cutoff that is identical to the RT curve cutoff:

$$\begin{aligned} A_{CHI} =& 2ln(2) + \frac{1}{3}\frac{f_2(1+v^2)}{1-v^2}L^2 + \frac{1}{32}\frac{\pi(-g_3v^2+f_3)}{1-v^2}L^3 \\ &- \frac{1}{45}\frac{2f_2^2v^4 - g_4v^4 + f_4v^2 + g_4v^2 + 2f_2^2 - f_4}{(1-v^2)^2}L^4 + \mathcal{O}(L^5) + \lim_{\epsilon\to 0} 2ln\left(\frac{L}{\epsilon}\right) \end{aligned} \tag{2.6.5}$$

Now we can compute the difference in area between the CHI and RT which will be finite. We get:

$$
\begin{aligned}
A_{CHI} - A_{RT} =& \frac{1}{192}\frac{\pi f_2 v^2(g_3 - f_3)}{(1-v^2)^2}L^5 \\
&+ \frac{1}{2764800(1-v^2)^2}[(1440f_3^2 + 8640g_3f_3 + (6075\pi^2 - 44640)g_3^2)v^4 \\
&+ ((2700\pi^2 - 41280)(f_3^2 + g_3^2) + (-17550\pi^2 + 151680)g_3f_3 + 16384f_2(g_4 - f_4))v^2 \\
&+ (6075\pi^2 - 44640)f_3^2 + 8640f_3g_3 + 1440g_3^2]L^6 + \mathcal{O}(L^7)
\end{aligned}
\tag{2.6.6}
$$

As we can see, the area of RT curve and CHI curve area are identical up to fourth order in L. In particular when $v = 0$, the leading order term is of order L^6 given by:

$$
\frac{1}{61440}(135\pi^2 - 992)f_3^2 + \frac{1}{320}g_3f_3 + \frac{1}{1920}g_3^2 \tag{2.6.7}
$$

We claim that the leading order term for $v = 0$ is always non-negative for any $AAdS_3$ geometry given. To see this fix g_3 to any arbitrary value then the sixith order coefficient is a quadratic in f_3. We want to start by understanding the zeros of the quadratic by solving for f_3 in terms of g_3 we find:

$$
f_3 = \frac{4g_3(-24 \pm i\sqrt{270\pi^2 - 2560})}{135\pi^2 - 992} \tag{2.6.8}
$$

From this we see that f_3 is complex and can only be real if $g_3 = 0$ which, in turn, implies that $f_3 = 0$. Which makes the entire 6th order term equal to zero. This means that if $g_3 \neq 0$ the parabola in f_3 will not cross the f_3 axis. If $g_3 = 0$, then there is a doubly degenerate zero at the origin and the parabola will not cross the f_3 axis. Hence to check positivity it suffices to choose any value for f_3 and g_3 and see that it is greater than zero. In particular let $g_3 = 0$, and let f_3 be arbitrary. Then the sixth order term is positive because $135\pi^2 - 992 > 0$. This proves that for $v = 0$ the leading order term is always non-negative. This makes sense because we know that for a constant time slice the RT curve is a minimal length curve that is anchored to the boundary interval. Now we consider the case in which v is non-zero. The leading order term is is 5th order in L. We require that the term be non-negative for $v \in (0, 1)$. This gives the constraint:

$$
f_2(g_3 - f_3) \geq 0 \tag{2.6.9}
$$

We should assume that $f_2 > 0$ because f_2 is related to the expectation value of the stress energy tensor of the CFT_2. In particular, f_2 is proportional to the energy density which should be non-negative. Using this fact we have that:

$$
f_3 \leq g_3 \tag{2.6.10}
$$

Now suppose that $g_3 = f_3$, then the leading order term will be 6th order in L. Requiring non-negativity implies that:

$$\begin{aligned}
&Ax^2 - 2Bx + A \geq 0 \\
&A = \left(\pi^2 - \frac{256}{45}\right) f_3^2 \geq 0 \\
&B = A + \frac{8192}{6075} f_2(f_4 - g_4) \\
&x = v^2 \in (0,1)
\end{aligned} \tag{2.6.11}$$

Start with the case when $f_3 = 0$ then we are dealing with a linear equation in x. Clearly in order for the expression to be non-negative we require that:

$$B \leq 0 \Rightarrow f_2(f_4 - g_4) \leq 0 \Rightarrow f_4 \leq g_4 \tag{2.6.12}$$

Now, we deal with the case that $f_3 \neq 0$ start by rewriting the inequality as follows:

$$x^2 - \frac{2B}{A}x + 1 \geq 0 \tag{2.6.13}$$

Using results from appendix A we show that the quadratic satisfies the inequality in the interval $x \in (0,1)$ when:

$$f_2(f_4 - g_4) \leq 0 \Rightarrow f_4 \leq g_4 \tag{2.6.14}$$

We can compare these leading order constraints to the constraints derived in [4] which we will quickly review. It was shown that for $AAdS_3$ spacetimes we are considering the non-vanishing components of the stress energy tensor are given as:

$$T_{zz} = -\frac{1}{2z}\frac{g'}{g} - \frac{1}{2z}\frac{f'}{f} + \frac{1}{4}\frac{f'}{f}\frac{g'}{g} \tag{2.6.15}$$

$$T_{tt} = \frac{g}{4z}\left(2\frac{f'}{f} + z\frac{f'^2}{f^2} - 2z\frac{f''}{f}\right) \tag{2.6.16}$$

$$T_{xx} = -\frac{f}{4z}\left(2\frac{g'}{g} + z\frac{g'^2}{g^2} - 2z\frac{g''}{g}\right) \tag{2.6.17}$$

We can do an asymptotic expansion of these expressions to obtain:

$$T_{tt} = -\frac{1}{4}f_3 z - \frac{1}{6}f_4 z^2 + \frac{1}{6}f_2 f_3 z^3 + \mathcal{O}(z^4) \tag{2.6.18}$$

$$T_{xx} = \frac{1}{4}g_3 z + \frac{1}{6}g_4 z^2 + \frac{1}{6}f_2 g_3 z^3 + \mathcal{O}(z^4) \tag{2.6.19}$$

We will apply the null energy condition (NEC) which states $T_{\mu\nu}u^\mu u^\nu \geq 0$. We take u to be a null vector in the boundary directions, $u = u^x \partial_x + u^t \partial_t$ this gives $T_{tt}\frac{f(z)}{g(z)} + T_{xx} \geq 0$. We can calculate in terms of the asymptotic expansion we get:

$$T_{tt}\frac{f(z)}{g(z)} + T_{xx} = -\frac{1}{4}(f_3 - g_3)z - \frac{1}{6}(f_4 - g_4)z^2 + \mathcal{O}(z^3) \geq 0 \tag{2.6.20}$$

The leading order term in z gives us $f_3 \leq g_3$, which is exactly the same constraint we got in (2.6.10). When we assume $f_3 = g_3$ then the leading order term is of order z^2 and the NEC gives $f_4 \leq g_4$, which we got in (2.6.14). In conclusion, we find that the conjecture $A_{CHI} - A_{RT} \geq 0$ at leading orders does not give any tighter constraints to the asymptotic structure of $AAdS_3$ spacetimes than what we obtain using the NEC in the boundary field theory directions. In fact, we found that to the first two leading orders the constraints are identical.

A natural question to ask in light of these leading order results is whether this is also true for higher order terms. That is, do constraints from the series expansion of $A_{CHI} - A_{RT} \geq 0$ in L give the same constraints as the NEC for null vectors in the boundary directions at higher than the first two leading orders? A good way to start answering this question is to simply calculate and compare a few more terms in the series expansions. If we find that the higher order terms match, then this might indicate a more deeper relation between the NEC in $AAdS_3$ spacetimes and the constraint that $A_{CHI} - A_{RT} \geq 0$. If the higher order constraints between the two conditions do not match, then one could ask why the leading order terms match and whether the results we obtained are a result of choosing $AAdS_3$ metrics that are translation invariant in the boundary coordinates. In either case, there are still some open questions that one could try to answer that would give a better insight of what causal holographic information can tell us about bulk spacetimes.

Chapter 3

Constraints from Relative Entropy

3.1 Basic Properties of Relative Entropy

Relative entropy is yet another quantum information quantity that can be defined for states on some subregion of a CFT_d. This means that we can translate constraints on relative entropy to constraints in bulk geometry. Such constraints have been extensively studied in [6, 7, 14] for ball shaped regions. We will review some of these results and use them as a basis for understanding the dual of relative entropy for holographic states defined on null cone sub-regions. Start by defining relative entropy. Suppose we are given two states of a quantum system in terms of the density matrices ρ and σ. We can define a quantity called relative entropy in terms of these two states:

$$S(\rho||\sigma) = Tr(\rho ln\rho) - Tr(\rho ln\sigma) \tag{3.1.1}$$

Here, σ is often called the reference state. Relative entropy is always greater than or equal to zero and will be equal to zero iff $\rho = \sigma$. This property is often referred to as the positivity of relative entropy. Furthermore, for reduced density matrices ρ_A and σ_A obtained by a partial trace operation from ρ and σ, one can show:

$$S(\rho_A||\sigma_A) \leq S(\rho||\sigma) \tag{3.1.2}$$

This is called the monotonicity of relative entropy. If we consider the case where the density matrices ρ and σ describe states in a subregion B of a CFT. We can view ρ_A and σ_A as the same states defined in a subregion A such that $A \subset B$.

Now that we have introduced the notion of relative entropy we will move on and reformulate it in terms of a quantity called the modular Hamiltonian. The modular Hamiltonian for a state σ is defined by the formula $H_\sigma = -ln\sigma$. Using this definition we can recast the equation of relative entropy as follows:

$$
\begin{aligned}
S(\rho||\sigma) &= Tr(\rho ln\rho) - Tr(\rho ln\sigma) + Tr(\sigma ln\sigma) - Tr(\sigma ln\sigma) \\
&= [Tr(\rho H_\sigma) - Tr(\sigma H_\sigma)] - [-Tr(\rho ln\rho) + Tr(\sigma ln\sigma)] \\
&= \left[\langle H_\sigma\rangle_\rho - \langle H_\sigma\rangle_\sigma\right] - [S(\rho) - S(\sigma)] \\
&= \Delta\langle H_\sigma\rangle - \Delta S
\end{aligned} \tag{3.1.3}
$$

Where $\Delta\langle H_\sigma\rangle$ is the difference in the expectation value of the modular Hamiltonian H_σ with respect to the states ρ and σ and ΔS is the difference between the Von Neumann entropies of the states ρ and σ. Before explaining how this formula will be used in the setting of holography, we want to prove the so called first law of entanglement entropy for states close to the reference state σ. Let $\rho = \sigma + \epsilon X$ where $0 < \epsilon << 1$ and X is a traceless hermitian matrix. We then substitute this into the equation for relative entropy given at the beginning of the section and do a series expansion in ϵ. We find that:

$$
\begin{aligned}
&S(\sigma+\epsilon X||\sigma) = Tr\left[(\sigma+\epsilon X)\left(ln\sigma + \epsilon X\sigma^{-1} - \frac{1}{2}\epsilon^2 X^2\sigma^{-2} + \mathcal{O}(\epsilon^3)\right)\right] - Tr\left[(\sigma+\epsilon X)ln\sigma\right] \\
&= \epsilon Tr(X) + \frac{1}{2}\epsilon^2 Tr(X\sigma^{-1}X) + \mathcal{O}(\epsilon^3) = \frac{1}{2}\epsilon^2 Tr(X\sigma^{-1}X) + \mathcal{O}(\epsilon^3)
\end{aligned} \tag{3.1.4}
$$

Where we used the cyclic property of trace and the fact that X is traceless. The leading order non-zero term is of order ϵ^2 and is called quantum Fisher information. Furthermore, since the first order term in ϵ vanishes, this tells us that the first order variation of relative entropy for states near the reference state σ vanish. We can use this result to see that the first order variation of the modular hamiltonian equals to the first order variation in the Von Neumann entropy:

$$
\delta S = \delta\langle H_\sigma\rangle \tag{3.1.5}
$$

The equation above is often called the first law of entanglement entropy. It was shown in [14] that when one considers the bulk dual of the first law for ball shaped regions on the boundary CFT_d, one finds the linearized Einstein equations in the bulk.

3.2 Relative Entropy for Ball Shaped Regions in Terms of Bulk Quantities

Here we will review relative entropy in the context of AdS_{d+1}/CFT_d. We start with the formula for relative entropy in terms of the modular Hamiltonian.

$$
S(\rho_B||\sigma_B) = \Delta\langle H_B\rangle - \Delta S \tag{3.2.1}
$$

Here we let $\sigma = \sigma_B$ be the vacuum state of a CFT_d on a ball shaped region B for a constant time slice. Let $\rho = \rho_B$ be some other excited state on B. In this case, the modular Hamiltonian for the vacuum state σ_B takes a simple form given by the following integral expression [6, 7, 14]:

$$H_B = 2\pi \int_{ball} \frac{R^2 - |\vec{x} - \vec{x}_c|^2}{2R} T_{tt}(t_c, \vec{x}) d^{d-1}x \tag{3.2.2}$$

Where R is the radius of the $d-1$ ball centred at the point $(t_c, \vec{x}_c)$. $T_{tt}(t_c, \vec{x})$ is the time-time component of the stress energy tensor on the constant time slice $t = t_c$. This can be generalized to a more covariant version given by the equation below [6, 7, 14]:

$$H_{\zeta_B} = \int_{B' \in D[B]} \zeta_B^\mu T_{\mu\nu} \hat{\epsilon}^\nu \tag{3.2.3}$$

Where the integral is over a $d-1$ space-like surface B' that has the same domain of dependence as the ball of radius R centred at $(t_c, \vec{x}_c)$. The vector field ζ_B is a conformal killing vector field defined as [6, 7, 14]:

$$\zeta_B^\mu = \frac{\pi}{R}\left[R^2 - (t-t_c)^2 - |\vec{x} - \vec{x}_c|^2\right]\partial_t - \frac{2\pi}{R}(t - t_c)\sum_{i=1}^{d-1}(x - x_c)^i \partial_i \tag{3.2.4}$$

The vector field defines what is known as the modular flow associated with the domain of dependence $D[B]$. The stress energy tensor $T_{\mu\nu}$ is now on B' and ϵ^ν is a $d-1$ form defined using the d dimensional background Minkowski metric $\eta_{\mu\nu}$:

$$\hat{\epsilon}_\nu = \frac{\sqrt{\eta}}{(d-1)!}\epsilon_{\nu a_1 a_2 \ldots a_{d-1}} dx^{a_1} \wedge \ldots \wedge dx^{a_{d-1}} \tag{3.2.5}$$

It has a property such that by contracting the form with a normal vector n^ν to the surface B', we get the volume form for the $d-1$ dimensional surface B'. One can check that this covariant version reproduces the older result for the ball on the constant time slice $t = t_c$ given by equation (3.2.2). Using the covariant formula along with the fact that $\langle T_{\mu\nu}\rangle = \frac{d}{16\pi G_N}\Gamma^{(d)}_{\mu\nu}$, we can write the quantity $\Delta \langle H_{\zeta_B}\rangle$ as follows [2]:

$$\Delta \langle H_{\zeta_B}\rangle = \int_{B'} \zeta_B^\mu \Delta \langle T_{\mu\nu}\rangle \hat{\epsilon}^\nu = \frac{d}{16\pi G_N}\int_{B'} \zeta_B^\mu \Gamma^{(d)}_{\mu\nu}\hat{\epsilon}^\nu \tag{3.2.6}$$

This gives the change in the modular Hamiltonian in terms of metric quantities of the dual $AAdS_{d+1}$ spacetimes. We can address the term ΔS using the RT formula which will tell us that ΔS is the difference in the areas of the RT surfaces in the different backgrounds:

$$\Delta S = \frac{\Delta A}{4 G_N} \tag{3.2.7}$$

[2] We will assume from this point onwards that σ is the vacuum state, we know that its dual geometry is pure AdS_{d+1}. We also know that the excited state ρ will be dual to some other $AAdS_{d+1}$ spacetime which will have the following asymptotic expansion in Poincare coordinates $ds^2 = \frac{1}{z^2}\left[dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu + z^d \Gamma^{(d)}_{\mu\nu} dx^\mu dx^\nu + \mathcal{O}(z^{d+1})\right]$.

Combining the two results together gives us relative entropy expressed in terms of the bulk quantities:

$$S(\rho_B||\sigma_B) = \frac{d}{16\pi G_N}\int_{B'} \zeta_B^{\mu}\Gamma_{\mu\nu}^{(d)}\epsilon^{\nu} - \frac{\Delta A}{4G_N} \tag{3.2.8}$$

The quantity above is called holographic relative entropy. We defined it for states on $d-1$ dimensional sub-regions within the domain of dependence of a ball. We see that to calculate this quantity we need two things; the first is the modular Hamiltonian of the sub-region on the boundary, and the second is the area of the RT surfaces. For ball shaped regions it has been shown that holographic relative entropy can be interpreted as a quasi-local bulk energy [6]. At first order the vanishing of the variation of this quantity leads to the linearized Einstein equations. At second order, it was shown that quantum Fisher information was dual to a canonical energy defined in the bulk [6].

One should keep in mind that the results discussed above apply to ball shaped regions. This is because for more arbitrary shaped sub-regions, the modular Hamiltonian is unknown and is assumed to take on a non-local form. Furthermore, calculating RT surfaces anchored to arbitrary entangling surfaces can be a difficult problem as we already discussed in the introduction. However, recent results by Casini and collaborators showed that if we restrict ourselves to the future horizon of some cut on a null plane, then one we can write the modular Hamiltonian as a simple integral. We will review this recent result and apply a conformal transformation to get the corresponding modular Hamiltonian on a past light-cone whose base can be defined by an arbitrary cut.

3.3 Modular Hamiltonian on the Cone

In this section, we want to consider the modular Hamiltonian of regions on a CFT_d bounded by an entangling surface that lies on a light cone. Our starting point will be a result derived by Casini, Teste, and Gonzalo [15]. To start we consider the CFT_d on a flat Minkowski background with coordinates $x^{\mu} = (x^0, x^1, ..., x^{d-1})$. In these coordinates the Minkowski metric is diagonal, $diag(-1, 1, ..., 1)$. Now we change coordinates to what we will call null plane coordinates defined as $x^- = x^0 - x^1$ and $x^+ = x^0 + x^1$ leaving the other transverse coordinates the same. The line element becomes:

$$ds^2 = -dx^+dx^- + \sum_{i=2}^{d-1}(dx^i)^2 \tag{3.3.1}$$

In particular, if we set $x^- = 0$, then this defines a null plane hyper-surface. Now we consider some cut along the null plane defined by setting the null coordinate x^+ equal to some function, $\gamma(x^{\perp})$, of the transverse coordinates $x^{\perp} = (x^2, .., x^{d-1})$. Then the modular Hamiltonian on the future horizon to the cut, γ, is given by the following integral expression [15]:

$$H_\gamma = 2\pi \int dx^2...dx^{d-1} \int_{\gamma(x^\perp)}^{\infty} (x^+ - \gamma(x^\perp))T_{++}(x^- = 0)dx^+ \tag{3.3.2}$$

We know from the results in appendix A.3 that there exists a special conformal transformation (SCT) that maps this null sheet to a null cone. This means that by doing a conformal transformation of the integral expression above, we can obtain the corresponding modular Hamiltonian on the light cone. To start, we want to see how the integration measure $dx^+dx^2...dx^{d-1}$ on the plane transforms after applying the SCT. The first step will be to do a transformation of the transverse coordinates $(x^+, x^2, .., x^{d-1}) \to (x^+, y^2, ..., y^{d-1})$ where the coordinates on y are obtained after applying the SCT map given in appendix A.3:

$$\begin{aligned} y^i &= \frac{x^i}{\Omega(x)} \\ \Omega(x) &= \frac{-(x^+ + 2R)(x^- - 2R) + (x^\perp)^2}{4R^2} \Rightarrow \Omega(x^- = 0) = 1 + \frac{x^+}{2R} + \left(\frac{x^\perp}{2R}\right)^2 \\ (x^\perp)^2 &= \sum_{i=2}^{d-1} (x^i)^2 \end{aligned} \tag{3.3.3}$$

Using the map above one can calculate the elements of the Jacobian matrix associated with changing coordinates. The result is:

$$\frac{\partial y^i}{\partial x^k} = \Omega^{-1}\left[\delta^i_k - \frac{x^i x_k}{2R^2\Omega}\right] \tag{3.3.4}$$

Where $i, k \in \{2, 3, ..., d-1\}$. The determinant of the matrix can be found by finding the eigenvalues of the matrix which is outlined in appendix A.4, the result is:

$$J_\perp = det\left(\frac{\partial y^i}{\partial x^k}\right) = \Omega^{2-d}\left(1 - \frac{(x^\perp)^2}{2R^2\Omega}\right) \tag{3.3.5}$$

If we restrict ourselves to the null plane $x^- = 0$ then:

$$J_\perp\big|_{x^-=0} = \frac{2 + \frac{x^+}{R} - \Omega}{\Omega^{d-1}}\bigg|_{x^-=0} \tag{3.3.6}$$

This allows us to make the following statement:

$$dx^+dy^2dy^3...dy^{d-1} = J_\perp dx^+dx^2dx^3...dx^{d-1} \Rightarrow dx^+dx^2...dx^{d-1}\big|_{x^-=0} = \frac{\Omega^{d-1}}{2 + \frac{x^+}{R} - \Omega}\bigg|_{x^-=0} dx^+dy^2...dy^{d-1} \tag{3.3.7}$$

Now we use the fact that on the null plane $x^- = 0$ and this implies $x^+ = 2x^1$, thus:

$$dx^+ dx^2 ... dx^{d-1}\big|_{x^-=0} = \frac{2\Omega^{d-1}}{1 + \frac{2x^1}{R} - \Omega}\bigg|_{x^-=0} dx^1 dy^2 ... dy^{d-1} = \frac{2\Omega^{d-1}}{2 + \frac{2x^1}{R} - \Omega}\frac{\partial x^1}{\partial y^1}\bigg|_{x^-=0} dy^1 dy^2 ... dy^{d-1} \tag{3.3.8}$$

Since we know that points on $x^- = 0$ get mapped to points on $y^0 + |\vec{y}| = R$ we will calculate the partial derivative under this restriction and find:

$$\frac{\partial x^1}{\partial y^1}\bigg|_{y^0+|\vec{y}|=R} = \frac{1}{\omega}\frac{R - |\vec{y}| - y^1}{|\vec{y}|}\bigg|_{y^0+|\vec{y}|=R} \tag{3.3.9}$$

Combining everything and using that $\Omega = 1/\omega$ we find that:

$$dx^+ dx^2 ... dx^{d-1}\bigg|_{x^-=0} = \frac{2R}{|\vec{y}|\omega^{d-1}} dy^1 ... dy^{d-1}\bigg|_{y^0+|\vec{y}|=R} \tag{3.3.10}$$

We can change to hyper-spherical coordinates with ρ as the radial coordinate and $\phi^1, ..., \phi^{d-2}$ the angular coordinates. This gives:

$$dx^+ dx^2 ... dx^{d-1}\bigg|_{x^-=0} = \frac{2R\sqrt{g^\Omega}}{\omega^{d-1}} \rho^{d-3} d\rho d\phi^1 ... d\phi^{d-2}\bigg|_{y^0+\rho=R} \tag{3.3.11}$$

Where g^Ω is the determinant of the metric on a unit $d-2$ sphere. Finally of define radial null coordinates by letting $\rho^\pm = y^0 \pm \rho$ it follows that:

$$dx^+ dx^2 ... dx^{d-1}\bigg|_{x^-=0} = -\frac{R\sqrt{g^\Omega}}{\omega^{d-1}}\left(\frac{R-\rho^-}{2}\right)^{d-3} d\rho^- d\phi^1 ... d\phi^{d-2}\bigg|_{\rho^+=R} \tag{3.3.12}$$

This tells us how the area measure on the plane changes when to change coordinates using the SCT that maps a half plane to a ball.

We want to understand exactly how the cut $x^+ = \gamma(x^\perp)$ on the null plane is mapped to the null cone. We need to understand this due to the fact that the integral involving x^+ is not over all space, but rather only to the future of the cut and also because $\gamma(x^\perp)$ shows up in the integrand. To do this we use the result from appendix A.5 that, ρ^- on the cone is related to $x+$ on the plane via:

$$\rho^- = \frac{x^+}{1 + \frac{x^+}{2R} + \left(\frac{x^\perp}{2R}\right)^2} - R \tag{3.3.13}$$

Now we will write the cut on the plane in the following form:

$$x^+(x^\perp) = 2R\left(\frac{1}{g(x^\perp)} - 1\right)\left(1 + \left(\frac{x^\perp}{2R}\right)^2\right) \tag{3.3.14}$$

Where $g(x^{\perp})$ is some arbitrary function whose properties we can understand by substituting this expression for x^{+} into the equation (3.3.13). We find that:

$$\rho^{-} = R(1 - 2g(x^{\perp})) \tag{3.3.15}$$

This tells us that if $g(x^{\perp}) = 0 \Rightarrow x^{+} = \infty$. This we are sitting on the tip of the cone. If $g(x^{\perp}) = 1 \Rightarrow x^{+} = 0$, then we are on the boundary of the ball on the zero time slice. This tells us any cut that we make on the cone that is between the tip and ball on the zero time slice will be specified by some function that obeys the following inequality $0 \leq g(x^{\perp}) \leq 1$. For example, if the function is a constant, then this will correspond to a constant cut of the light cone. One may actually be concerned because the function we are specifying is not a function of the angular coordinates. However, one can check that any function of the transverse coordinates on the plane will give some function of the angular coordinates on the cone. This means that in principle we could pick some cut on the cone that we want $g(\phi)$ and then use the relations in the appendix to express all the ϕ dependence in terms of $x^{\perp}$ and vice-versa. Using the integration limit given by equation (3.3.14) we can write:

$$H_{\gamma} = 2\pi \int \int_{2R\left(\frac{1}{g(x^{\perp})}-1\right)\left(1+\left(\frac{x^{\perp}}{2R}\right)^2\right)}^{\infty} \left[x^{+} - 2R\left(\frac{1}{g(x^{\perp})} - 1\right)\left(1 + \left(\frac{x^{\perp}}{2R}\right)^2\right)\right] T_{++} dx^{+} dx^{2} ... dx^{d-1} \tag{3.3.16}$$

When we change coordinates to the light cone we can write the terms in the large square bracket as:

$$x^{+} - 2R\left(\frac{1}{g(x^{\perp})} - 1\right)\left(1 + \left(\frac{x^{\perp}}{2R}\right)^2\right) = \frac{\rho^{-} - R + 2Rg(\phi)}{\omega g(\phi)} \tag{3.3.17}$$

Where we use the fact that:

$$1 + \left(\frac{x^{\perp}}{2R}\right)^2 = \frac{1}{\omega}\left(1 - \frac{R + \rho^{-}}{2R}\right) \tag{3.3.18}$$

Using the result from appendix A.6, for doing a conformal transformation of the stress energy tensor in a CFT_d allows us to write:

$$T_{++} = \frac{\omega^{d}}{R^{2}}\left(\frac{R - \rho^{-}}{2}\right)^{2} \tilde{T}_{--} \tag{3.3.19}$$

The upper integration limit becomes R and the lower integration limit becomes $R(1 - 2g(\phi))$. Combining everything gives us the result for the modular Hamiltonian on a cone:

$$H_{cone} = 2\pi \int \int_{R(1-2g(\phi))}^{R} \sqrt{g^{\Omega}} \left(\frac{R - \rho^{-}}{2}\right)^{d-1} \left[\frac{R(1 - 2g(\phi)) - \rho^{-}}{Rg(\phi)}\right] \tilde{T}_{--} d\rho^{-} d\phi^{1} ... d\phi^{d-2} \tag{3.3.20}$$

We define $\rho_0^-(\phi) = R(1-2g(\phi))$ and rewrite the result as:

$$H_{cone} = 4\pi \int \int_{\rho_0^-(\phi)}^{R} \sqrt{g^\Omega} \left(\frac{R-\rho^-}{2}\right)^{d-1} \left[\frac{\rho_0^-(\phi)-\rho^-}{R-\rho_0^-(\phi)}\right] \tilde{T}_{--} d\rho^- d\phi^1 ... d\phi^{d-2} \tag{3.3.21}$$

Where $-R \leq \rho_0^-(\phi) \leq R$. We can do one more change in the integration variable to make the integral start at zero. Introducing the new integration parameter $u = \frac{R-\rho^-}{2}$ and defining $\gamma(\phi) = \frac{R-\rho_0^-}{2}$ will give[3]:

$$H_{cone} = 2\pi \int \int_0^{\gamma(\phi)} \sqrt{g^\Omega} u^{d-1} \left(\frac{u}{\gamma(\phi)} - 1\right) \bar{T}_{uu} du d\phi^1 ... d\phi^{d-2} \tag{3.3.22}$$

Where $0 < \gamma(\phi) < R$.

Now that we have an expression for the modular Hamiltonian on a light cone cone with a cut base we know we can calculate the modular Hamiltonian term in the relative entropy formula. We still need a description of the corresponding RT surface anchored to this cone on the boundary. This will be address this in the following section in the case of a pure AdS_{d+1} background.

3.4 Ryu-Takayanagi Surface Anchored to Light Cone on CFT_d Boundary

In this section we want to derive the Ryu-Takayanagi surface for pure AdS_{d+1} anchored to some region on the boundary light cone. To do this start by writing the pure AdS_{d+1} metric in Poincare coordinates and rewrite the boundary coordinates in hyper-spherical coordinates, the line element reads:

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dz^2 + d\rho^2 + \rho^2 g_{ij}^\Omega d\phi^i d\phi^j\right) \tag{3.4.1}$$

where ρ is the radial distance on the boundary, $i, j \in \{1, 2, ..., d-2\}$, and g_{ij}^Ω are components of the metric on the unit $d-2$ sphere with angular coordinates $(\phi^1, ..., \phi^{d-2})$. Then we define another change of coordinates by defining:

$$\begin{aligned} z &= rsin(\theta) \\ \rho &= rcos(\theta) \\ \theta &\in [0, \pi/2] \\ r &\in (0, \infty) \end{aligned} \tag{3.4.2}$$

[3]At the time of writing this thesis we determined that the result in the paper [15] for the light cone modular Hamiltonian is not correct. We verified this with the authors and used a conformal transformation outlined in this section to get the correct result.

The line element will now read:

$$ds^2 = \frac{1}{r^2 sin^2(\theta)}\left(-dt^2 + dr^2 + r^2 d\theta^2 + r^2 cos^2(\theta) g^{\Omega}_{ij} d\phi^i d\phi^j\right) \tag{3.4.3}$$

In these coordinates, r defines a radial coordinate for the bulk geometry which will simplify to radial coordinates to ρ on the boundary situated at $\theta = 0$. We do one final transformation by defining $r^{\pm}$ as follows:

$$r^{\pm} = t \pm r \tag{3.4.4}$$

This gives us the final form of the line element we will need expressed in the coordinates $(r^+, r^-, \theta, \phi^1, .., \phi^{d-2})$:

$$ds^2 = \frac{1}{sin^2(\theta)}\left[\frac{-4dr^+dr^-}{(r^+ - r^-)^2} + d\theta^2 + cos^2(\theta) g^{\Omega}_{ij} d\phi^i d\phi^j\right] \tag{3.4.5}$$

We will refer to these coordinates as bulk radial null coordinates. In these coordinates we will define the following co-dimension 2 surface through the following two embedding equations:

$$r^+ = R \tag{3.4.6}$$

$$r^- = f(\theta, \phi^i) \tag{3.4.7}$$

Where R is a constant and $f(\theta, \phi^i)$ is some function that will be fixed by solving some PDEs which we will write down shortly. More intuitively equation (3.4.6) specifies a past bulk null cone whose tip is at $(t = R, z = 0, x^{bdry} = 0)$. Equation (3.4.7) will specify a cut at the base of the bulk cone. The induced metric on this co-dimension 2 hyper-surface is given by the following $d-1$ dimensional metric:

$$G_{ab} = \frac{\delta^{\theta}_a \delta^{\theta}_b + cos^2(\theta) g^{\Omega}_{ij} \delta^i_a \delta^j_b}{sin^2(\theta)} \tag{3.4.8}$$

Where the indices $a, b \in (\theta, \phi^1, .., \phi^{d-2})$. The hyper-surface will be extremal if it satisfies the following two PDE equations which we derived in appendix A.1:

$$\partial_a \left[\frac{\sqrt{G} G^{ab} \partial_b r^{\mp}}{(r^+ - r^-)^2 sin^2(\theta)}\right] = 0 \tag{3.4.9}$$

Clearly $r^+ = R$ will satisfy the PDE above. This leaves us with the following equation for the function $f(\theta, \phi^i)$:

$$\partial_a \left[\frac{\sqrt{G} G^{ab} \partial_b f(\theta, \phi^i)}{(r^+_0 - f(\theta, \phi^i))^2 sin^2(\theta)}\right] = \partial_a \left[\frac{\sqrt{G} G^{ab}}{sin^2(\theta)} \partial_b \left(\frac{1}{\tilde{f}(\theta, \phi^i)}\right)\right] = 0 \tag{3.4.10}$$

Where we defined $\tilde{f}(\theta,\phi) = R - f(\theta,\phi^i)$. It is not difficult to see that:

$$\sqrt{G} = \frac{cot^{d-2}(\theta)}{sin(\theta)}\sqrt{g^{\Omega}} \tag{3.4.11}$$

$$G^{ab} = sin^2(\theta)\left(\delta^{\theta a}\delta^{\theta b} + \frac{g^{\Omega}_{ij}\delta^{ai}\delta^{bj}}{cos^2(\theta)}\right) \tag{3.4.12}$$

Substituting these expressions into equation (3.4.10) and expanding the sum we find that:

$$tan^{d-2}(\theta)sin(\theta)cos^2(\theta)\partial_\theta\left[\frac{cot^{d-2}(\theta)}{sin(\theta)}\partial_\theta\left(\frac{1}{\tilde{f}}\right)\right] + \frac{1}{\sqrt{g^{\Omega}}}\partial_i\left[\sqrt{g^{\Omega}}(g^{\Omega})^{ij}\partial_j\left(\frac{1}{\tilde{f}}\right)\right] = 0 \tag{3.4.13}$$

Now we will apply separation of variables between the boundary angular coordinates ϕ^i and the bulk angle θ. Define $\frac{1}{\tilde{f}} = h(\theta)\Phi(\phi^i)$. Substituting this into the equation above gives:

$$tan^{d-2}(\theta)sin(\theta)cos^2(\theta)\frac{1}{h(\theta)}\frac{d}{d\theta}\left[\frac{cot^{d-2}(\theta)}{sin(\theta)}\frac{dh}{d\theta}\right] + \frac{1}{\Phi(\phi^i)}\frac{1}{\sqrt{g^{\Omega}}}\frac{\partial}{\partial\phi^i}\left[\sqrt{g^{\Omega}}(g^{\Omega})^{ij}\frac{\partial\Phi}{\partial\phi^j}\right] = 0 \tag{3.4.14}$$

Now define the constant of separation to be α. Then we know:

$$\frac{1}{\sqrt{g^{\Omega}}}\partial_i\left[\sqrt{g^{\Omega}}(g^{\Omega})^{ij}\partial_j\Phi\right] = -\alpha\Phi(\phi^i) \tag{3.4.15}$$

The PDE given by equation (3.35) is the Laplace-Beltrami operator acting on the function Φ on the unit $d-2$ sphere. The solutions to the PDE are well known and are called hyper-spherical harmonics, $\Phi_n(\phi^i)$ with eigenvalues $-\alpha = n(3-d-n)$, where $n \in \{0,1,2,...\}$ [4]. This means that the ODE involving θ will be:

$$-sin(\theta)cos^2(\theta)\frac{d^2h}{d\theta^2} + cos(\theta)\left(cos^2(\theta)+d-2\right)\frac{dh}{d\theta} - n(3-d-n)sin(\theta)h(\theta) = 0 \tag{3.4.16}$$

The general solution is given by hypergeometric functions:

$$\begin{aligned} h_n(\theta) &= C_1 cos^n(\theta)\,{}_2F_1\left(\frac{n}{2},\frac{n-1}{2};\frac{2n+d-1}{2},cos^2(\theta)\right) \\ &+ C_2 cos^{3-d-n}(\theta)\,{}_2F_1\left(\frac{2-d-n}{2},\frac{3-d-n}{2};\frac{5-d}{2}-n,cos^2(\theta)\right) \end{aligned} \tag{3.4.17}$$

[4] Note that we decide to be sloppy in labeling the hyper-spherical harmonic functions. It should be noted that in high dimensions there is more than one function that can give the same eigenvalues these degenerate functions are orthogonal and have their own labels but we choose to suppress these labels and only write the label that tells us the eigenvalues. For a more complete treatment one can look at [16].

We throw out the second term due to the fact it is ill defined for certain values of n and d and also generally vanishes on the boundary which are not the type of solutions we are looking for. This means:

$$h_n(\theta) = C_1 cos^n(\theta)_2F_1\left(\frac{n}{2}, \frac{n-1}{2}; \frac{2n+d-1}{2}, cos^2(\theta)\right) \tag{3.4.18}$$

This solution satisfies the following conditions:

$$h_0(\theta) = 1 \tag{3.4.19}$$

$$\lim_{\theta\to 0} h_{n>0}(\theta) = \frac{\Gamma(\frac{2n+d-1}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{n+d-1}{2})\Gamma(\frac{n+d}{2})} \neq 0 \tag{3.4.20}$$

$$\lim_{\theta\to \pi/2} h_{n>0} = 0 \tag{3.4.21}$$

In summary, we find that the Ryu-Takayanagi surface can be written defined by the embedding equations:

$$r^+ = R \tag{3.4.22}$$

$$r^- = f(\theta, \phi^i) = R - \frac{1}{C_0 + \sum_{n=1}^{\infty} C_n h_n(\theta)\Phi_n(\phi^i)} \tag{3.4.23}$$

Where we define $C_n = \frac{c_n}{h_n(0)}$ as an arbitrary constant normalized by the value of $h_n(\theta = 0)$. By doing this we can see that on the boundary we have that:

$$r^+ = R = \rho^+ \tag{3.4.24}$$

$$r^-(\theta = 0, \phi^i) = R - \frac{1}{C_0 + \sum_{n=1}^{\infty} c_n \Phi_n(\phi^i)} = \rho^-(\phi^i) \tag{3.4.25}$$

Where we used the fact:

$$\rho^{\pm}(\theta, \phi^i) = R - \frac{1 \mp cos(\theta)}{2(C_0 + \sum_{n=1}^{\infty} C_n h_n(\theta)\Phi_n(\phi^i))} \tag{3.4.26}$$

We see that, on the extremal surface ends on some boundary light cone whose base is cut by a function written as a series in hyper-spherical harmonics. To get an intuitive sense of the equations lets go back to hyper-spherical coordinates in the bulk the equations tell us the extremal surface is described by the set of points satisfying:

$$r = \frac{1}{2C_0 + 2\sum_{n=1}^{\infty} C_n h_n(\theta)\Phi_n(\phi^i)} \tag{3.4.27}$$

$$t = R - \frac{1}{2C_0 + 2\sum_{n=1}^{\infty} C_n h_n(\theta)\Phi_n(\phi^i)} \tag{3.4.28}$$

If we set the higher order terms to zero then $C_0 \to \infty$ places us at the tip of the cone where $r = 0$ and $t = R$. If we set $C_0 = 1/2R$ then we have a constant time slice cut at $t = 0$ of the cone which is a hyper-sphere in the bulk of radius R. If we restrict or cut to be between the coordinate time $0 \leq t < R$, then we have to require that $\frac{1}{2R} \leq C_0 + \sum_{n=1}^{\infty} h_n(\theta)\Phi_n(\phi^i) < \infty$. Hence we see that the higher order terms can be thought of as perturbing away from the constant time cut to a more general cut which can be expressed in terms of hyper-spherical harmonics.

3.5 Relative Entropy as Quasi-Local Bulk Energy

Using the formula for holographic relative entropy discussed in the previous section, one could try to directly calculate the quantities. Generally this will be quite difficult due to the fact that it is hard to solve the equations describing the RT surface in an arbitrary $AAdS_{d+1}$ spacetime. To work around this issue we will review the formalism discussed in [6]. This allows us to write relative entropy in a form that is more naturally suited to handle arbitrary perturbations in the bulk. To start we define a $d+1$ form related to the Lagrangian density $\mathcal{L}$ of our system:

$$L(g) = \mathcal{L}\hat{\epsilon} \tag{3.5.1}$$

Where g is a shorthand for the fields in the Lagrangian. In our case since we are interested in pure gravity the Lagrangian density for our system will be the usual Einstein-Hilbert density given by:

$$\mathcal{L} = \frac{1}{16\pi G_N} R - \Lambda \tag{3.5.2}$$

The $d+1$ form, $\hat{\epsilon}$, is defined in terms of the determinant of $d+1$ dimensional background metric, g_{ab}:

$$\hat{\epsilon} = \frac{\sqrt{-g}}{(d+1)!}\epsilon_{a_1 a_2 \ldots a_{d+1}} dx^{a_1} \wedge dx^{a_2} \wedge \ldots \wedge dx^{a_{d+1}} \tag{3.5.3}$$

We define $(d+1-n)$- dimensional forms $\hat{\epsilon}_{c_1 c_2 \ldots c_n}$, where $n < d+1$ as follows:

$$\hat{\epsilon}_{c_1 c_2 \ldots c_n} = \frac{\sqrt{-g}}{(d+1-n)!}\epsilon_{c_1 \ldots c_n a_{n+1} \ldots a_{d+1}} dx^{n+1} \wedge \ldots \wedge dx^{d+1} \tag{3.5.4}$$

Where $\sqrt{-g}$ is still the determinant of the d+1 dimensional metric. It is useful for describing volume forms on co-dimension n surfaces embedded in the $d+1$ dimensional background.

As a quick example to see how these forms operate we will look at the case where g_{ab} is the metric in pure AdS_{d+1} in Poincare coordinates. In this case the indices we sum over will take values, $a_1, a_2, .., a_{d+1} \in \{t, z, x^1, .., x^{d-1}\}$. It is easy to see that the form ϵ will be given as:

$$\hat{\epsilon} = \frac{1}{z^{d+1}} \epsilon_{tx^1..x^{d-1}z} dx^t \wedge dx^1 \wedge .... \wedge dx^{d-1} \wedge dx^z = \frac{1}{z^{d+1}} dt dx^1 ... dx^{d-1} dz \tag{3.5.5}$$

Which is what we would expect for the pure AdS_{d+1} metric in Poincare coordinates. Now suppose that we want to embed a co-dimension 1 surface. For simplicity let it be a constant z slice. It is clear that the normalized unit normal vector $n = \frac{1}{\sqrt{g_{zz}}} \partial_z$. If we contract this with the $d+1$ form $\hat{\epsilon}$ with the unit normal we will get:

$$\hat{\epsilon} \cdot n^z \partial_z = \hat{\epsilon_z} n^z = \frac{1}{\sqrt{g_{zz}} z^{d+1}} \epsilon_{tx^1..x^{d-1}} dx^t \wedge dx^1 \wedge .... \wedge dx^{d-1} = \frac{1}{z^d} dt dx^1 ... dx^{d-1} \tag{3.5.6}$$

Which gives the correct form on the slice. For co-dimension 2 surfaces, we would have to find a unit binormal to the surface to define the $d-1$ volume form.

Having defined the forms we can continue and take a variation of the $d+1$ form given by equation (3.5.1). One will get the following results:

$$\delta L(g) = (-E^g) \delta g \hat{\epsilon} + d\Theta(g, \delta g) \tag{3.5.7}$$

The first term is the equations of motion associated with the Lagrangian density $\mathcal{L}$. For us they will be the Einstein vacuum equations with the cosmological constant. The second term is boundary term which is defined in terms of a d-form, $\Theta(g, \delta g)$ and $d\Theta$ is the exterior derivative of the d-form. This will be used to define another d-form called the symplectic d-form which will be defined as:

$$\omega(\delta_1 g, \delta_2 g) = \delta_1 \Theta(g, \delta_2 g) - \delta_2 \Theta(g, \delta_1 g) \tag{3.5.8}$$

Note that it is defined in terms of metric perturbations $\delta_1 g$ and $\delta_2 g$. Now we want to use this in the context of holography. We consider the subregion given by the light-cone regions, $\hat{A}$, for which the modular Hamiltonian H_{cone} is known for vacuum states. This subregion, $\hat{A}$, has an associated co-dimension 2 extremal surface that extends into the bulk $\tilde{A}$ which we found in section 3.4 for pure AdS_{d+1}. One then defines a d dimensional surface in the bulk that is bounded by the extremal surface, $\tilde{A}$, in the bulk and $\hat{A}$ on the boundary which we denote as Σ. In our case, for pure AdS_{d+1} bulk geometry, we know that this surface, Σ, will be the bulk light cone $r^+ = R$, whose base is cut by a function described by $r^- = f(\theta, \phi^i)$. On Σ, we define a vector field ξ_c whose purpose will be to generate diffeomorphisms. Finally, we can define δH_{ξ_c} (not to be confused with the modular Hamiltonian) in terms of the d-form ω and the vector field ξ_c that exists on Σ:

$$\delta H_{\xi_c} = \int_\Sigma \omega(\delta g, \mathcal{L}_{\xi_c} g) = \int_\Sigma [\delta \Theta(g, \mathcal{L}_{\xi_c} g) - \mathcal{L}_{\xi_c} \Theta(g, \delta g)] \tag{3.5.9}$$

We can expand the Lie derivative of the d-form Θ along the vector field ξ_c using the identity $\mathcal{L}_{\xi_c}\Theta = \xi_c \cdot d\Theta + d(\xi_c \cdot \Theta)$. Where $\xi_c \cdot \Theta$ is notation that tells us to contract Θ with the vector field ξ_c. Using the identity we get:

$$\delta H_{\xi_c} = \int_\Sigma [\delta\Theta - \xi_c \cdot d\Theta - d(\xi_c \cdot \Theta)] = \int_\Sigma [\delta\Theta - \xi_c \cdot d\Theta] - \int_{\partial\Sigma} \xi_c \cdot \Theta \tag{3.5.10}$$

Where $\partial\Sigma = \tilde{A} \cup \hat{A}$. We define the Noether current d-form associated with the diffeomorphism generated by ξ_c to be, $J_{\xi_c} = \Theta - \xi_c \cdot L$. Assuming that the equations of motion are satisfied we know from equation (3.5.7) that $d\Theta(\delta g) = \delta L(g)$. We can show that the exterior derivative of the Noether current form vanishes through the following calculations:

$$\begin{aligned} dJ_{\xi_c} &= d\Theta(\mathcal{L}_{\xi_c} g) - d(\xi_c \cdot L(\mathcal{L}_{\xi_c} g)) = \delta L(\mathcal{L}_{\xi_c} g) - d(\xi_c \cdot L(\mathcal{L}_{\xi_c} g)) \\ &= \mathcal{L}_{\xi_c} L(g) + \xi_c \cdot dL(g) - \mathcal{L}_{\xi_c} L(g) = \xi_c \cdot dL(g) = 0 \end{aligned} \tag{3.5.11}$$

Where for the last equality we used that fact that L is a $d+1$-form and $dL = 0$. This means that when the equation of motion is satisfied then we can find a $d-1$ form, Q_{ξ_c}, such that $J_\xi = dQ_{\xi_c}$. Now consider the variation of Noether current form as follows:

$$\delta J_{\xi_c} = \delta\Theta - \xi_c \cdot \delta L = \delta\Theta - \xi_c \cdot d\Theta \tag{3.5.12}$$

This allows us to write:

$$\delta H_{\xi_c} = \int_\Sigma \delta J_{\xi_c} - \int_{\partial\Sigma} \xi_c \cdot \Theta \tag{3.5.13}$$

Now one needs to find a d-form K on the boundary, $\partial\Sigma$, that has the following property:

$$\delta(\xi_c \cdot K)|_{\partial\Sigma} = \xi_c \cdot \Theta|_{\partial\Sigma} \tag{3.5.14}$$

It turns out that such a K exists if the following condition holds true:

$$\int_{\partial\Sigma} \xi_c \cdot \omega(\delta_1 g, \delta_2 g) = 0, \forall \delta_1 g, \delta_2 g \tag{3.5.15}$$

Using this K along with the fact that $J_{\xi_c} = dQ_{\xi_c}$, we can write:

$$H_{\xi_c} = \int_\Sigma J_{\xi_c} - \int_{\partial\Sigma} \xi_c \cdot K = \int_{\partial\Sigma} [Q_{\xi_c} - \xi_c \cdot K] \tag{3.5.16}$$

Here H_{ξ_c} is referred to as the quasi-local energy associated with the vector field ξ_c on Σ. The important fact to note here is that H_{ξ_c} can be written completely in terms of a oriented integral over $\partial\Sigma = \tilde{A} \cup \hat{A}$. Due to this fact it has been shown in case for ball shaped subregions[5] that one can choose a particular ξ_B such that the integral reproduces the holographic relative entropy given

[5] The arguments made to this point also apply to ball shaped regions simply replace the cone region on boundary with ball regions along with the extremal surface anchored to the ball region

by equation (3.2.8). This tells us that for ball shaped regions, the relative entropy has a bulk dual in the form of the quasi-local energy we defined. What we want to do is use similar arguments but adapt them to regions on light-cones to reproduce the formulas we have for holographic relative entropy for light-cone subregions and their associated extremal surfaces. What we have outlined here is a starting point to doing this. In the following sections we will give a sketch as to how some of the arguments will be used in the case of null cone subregions on the boundary.

3.6 Writing Modular Hamiltonian in Covariant Form

In this section we want to gain insight as to the form of ξ_c on the conformal boundary. We will do this by assuming that modular Hamiltonians on cone shaped regions on the boundary take the following form:

$$H_{cone} = \int_{cone} \zeta_c^{\mu} T_{\mu\nu} \hat{\epsilon}^{\nu} \tag{3.6.1}$$

Where ζ_c will be a vector field such that when it is restricted to the surface of the cone we are integrating over will, it reproduce the modular Hamiltonian given by equation (3.3.21). We define $\hat{\epsilon}$ is a d-form and $\hat{\epsilon}_\nu$ is a $d-1$ form such that when it is contracted with a unit normal vector n^ν it gives the $d-1$ dimensional volume form on the perpendicular subspace:

$$\hat{\epsilon}_\nu = \frac{\sqrt{-g}}{(d-1)!} \epsilon_{\nu a_2 a_3 \ldots a_d} dx^{a_2} \wedge \ldots \wedge dx^{a_d} \tag{3.6.2}$$

We will work in boundary radial null coordinates $(\rho^+, \rho^-, \phi^1, ..., \phi^{d-2})$ and adopt the convention $\epsilon_{+-\phi^1..\phi^{d-2}} = 1$. For the past light cone volume element we are interested in a normal vector to the surface $\rho^+ = R$. We calculate it as follows:

$$n^\mu = g^{\mu\nu} \partial_\nu (-\rho^+ + R) = -g^{\mu +} \Rightarrow n = n^\mu \partial_\mu = -g^{+-} \partial_- \tag{3.6.3}$$

We contract this normal vector with the d-form $\hat{\epsilon}$ to get the $d-1$ form on the cone:

$$\hat{\epsilon} \cdot n = \left[\frac{-g^{+-}\sqrt{-g}}{d!} \epsilon_{a_1 \ldots a_d} dx^{a_1} \wedge \ldots \wedge dx^{a_d} \right] \cdot \partial_- = - \left[\frac{\sqrt{g^{\Omega}}}{d!} \left(\frac{R - \rho^-}{2} \right)^{d-2} \epsilon_{a_1 \ldots a_d} dx^{a_1} \wedge \ldots \wedge dx^{a_d} \right] \cdot \partial_- \tag{3.6.4}$$

We use the fact that $dx^+ \cdot \partial_- = -1$ and we find the following $d-1$ form:

$$\hat{\epsilon}_+ = \frac{\sqrt{g^{\Omega}}}{(d-1)!} \left(\frac{R - \rho^-}{2} \right)^{d-2} \epsilon_{+a_2 \ldots a_d} dx^{a_2} \wedge \ldots \wedge dx^{a_d} = \sqrt{g^{\Omega}} \left(\frac{R - \rho^-}{2} \right)^{d-2} d\rho^- d\phi^1 \ldots d\phi^{d-2} \tag{3.6.5}$$

Which gives us the correct volume form for the past light-cone. Finally, we use the fact that

$\hat{\epsilon}^- = g^{\nu -}\hat{\epsilon_\nu} = g^{+-}\hat{\epsilon}_+$, this gives:

$$\hat{\epsilon}^- = -2\sqrt{g^\Omega}\left(\frac{R-\rho^-}{2}\right)^{d-2} d\rho^- d\phi^1 ... d\phi^{d-2} \tag{3.6.6}$$

We also require that on the cone $\rho^+ = R$, the vector field obeys $\zeta_c|_{\rho^+=R} = \zeta_c^- \partial_-$. Which gives:

$$H_{cone} = \int_{cone} \zeta_c^- T_{--}\hat{\epsilon}^- = -\int_{cone} 2\zeta_c^- T_{--}\left(\frac{R-\rho^-}{2}\right)^{d-2} \sqrt{g^\Omega} d\rho^- d\phi^1 ... d\phi^{d-2} \tag{3.6.7}$$

Comparing this expression to equation (3.3.21), we find that we require the vector field:

$$\zeta_c^- = -4\pi(R-\rho^-)\left[\frac{R-\rho^-}{R-\rho_0^-} - 1\right] = 4\pi\frac{(R-\rho^-)(-\rho_0^- + \rho^-)}{R-\rho_0^-} \tag{3.6.8}$$

By choosing this vector field, we will reproduce the modular Hamiltonian given by equation (3.3.21). Hence we have found the following boundary condition that the bulk vector field ξ_c must satisfy on the conformal boundary:

$$\xi_c|_{\hat{A}} = \frac{4\pi(R-\rho^-)(\rho^- - \rho_0^-)}{R-\rho_0^-}\partial_- \tag{3.6.9}$$

3.7 Extending Boundary Vector Field into Bulk

Now we want to extend this vector field on the boundary cone to a bulk vector field on the surface Σ. An obvious and simple way of doing this in pure AdS_{d+1} is to simply replace the boundary radial null coordinates ρ^- and ρ^+ with bulk radial null coordinates r^+ and r^-. In these coordinates Σ is the bulk cone $r^+ = R$ with a base that is cut by some function of θ and ϕ^i. Explicitly we have that:

$$\xi_c|_\Sigma = \frac{4\pi(R-r^-)(r^- - f(\theta,\phi^i))}{R-f(\theta,\phi^i)}\partial_- \tag{3.7.1}$$

Where $f(\theta = 0, \phi^i) = \rho_0^-(\phi^i)$. One can check that this trivial extension of the boundary vector field will satisfy the following conditions on the RT surface, $\tilde{A}$, given by $r^+ = R$ and $r^- = f(\theta,\phi^i)$:

$$\xi_c|_{\tilde{A}} = \frac{4\pi(R-r^-)(r^- - f(\theta,\phi^i))}{R-f(\theta,\phi^i)}\partial_-\bigg|_{\tilde{A}} = 0 \tag{3.7.2}$$

$$\nabla^a\xi_c^b - \nabla^b\xi_c^a|_{\tilde{A}} = 4\pi n^{ab} \tag{3.7.3}$$

Where we define n^{ab} as the unit binormal tensor to the RT surface in pure AdS_{d+1}, which we derive in appendix A.7. Note that the boundary conditions we have defined by equations (3.6.9), (3.6.11), and (3.6.12) are analogous to the boundary conditions given in [6] for the bulk vector field ξ_B, corresponding to ball shaped regions. Due to this, we expect that many of the same arguments used in [6] for ball shaped regions will also apply to our light-cone regions. For example, we can consider how the quasi-local energy integral gives us the area term in the holographic relative entropy formula. We use the result from [6], which states $Q_{\xi_c} = \frac{-1}{16\pi G_N}\nabla^a \xi_c^b \epsilon_{\hat{ab}}$ where $\epsilon_{\hat{ab}}$ is a $d-1$ form defined in our discussion of forms in section 3.5. We can use the boundary conditions to get:

$$\begin{aligned}\int_{\tilde{A}} [Q_{\xi_c} - \xi_c \cdot K] = \int_{\tilde{A}} Q_{\xi_c} = -\frac{1}{16\pi G_N}\int_{\tilde{A}} \frac{1}{2}\epsilon_{\hat{ab}}\left[\nabla^a \xi_c^b - \nabla^b \xi_c^a\right] = -\frac{1}{8G_N}\int_{\tilde{A}} \epsilon_{\hat{ab}} n^{ab} \\ = -\frac{1}{4G_N}\int_{\tilde{A}} \epsilon_{+-} = -\frac{Area(\tilde{A})}{4G_N}\end{aligned} \tag{3.7.4}$$

Which gives the area term in the holographic relative entropy formula in the pure AdS_{d+1}[6].

So far, we have argued that in the light-cone case relative entropy is dual to a bulk quasi-local energy on Σ, and many of the same arguments used to show this in the ball case also apply to cut light-cone regions. These statements were made by only knowing how the vector field ξ_c behaves on $\partial\Sigma$. Now we want to consider extending this vector field away from Σ. As a first step we can consider this bulk vector field, ξ_c, defined on Σ in pure AdS_{d+1} and compare it to the bulk ξ_B given in [14]. The vector field, ξ_B, plays the role of ξ_c for ball shaped regions on the boundary and is given as:

$$\xi_B = \frac{\pi}{R}\left[R^2 - z^2 - t^2 - |\vec{x}|^2\right]\partial_t - \frac{2\pi}{R} t\left(\sum_{i=1}^{d-1} x^i\partial_i + z\partial_z\right) \tag{3.7.5}$$

Where the ball shaped region on the boundary is centred at $t_c = 0, \vec{x}_c = 0$, and simplifies to the conformal Killing vector field given by equation (3.2.4) on the boundary $z = 0$. We can rewrite this in the coordinate basis given by the coordinates (t, r, θ, ϕ^i) by noting that $r\partial_r = z\partial_z + \sum_{i=1}^{d-1} x^i\partial_i$. This allows us to write:

$$\xi_B = \frac{\pi}{R}\left[R^2 - t^2 - r^2\right]\partial_t - \frac{2\pi}{R} tr\partial_r \tag{3.7.6}$$

Using this we can easily write down the non-zero components of the vector fields in bulk radial null coordinates $(r^+, r^-, \theta, \phi^i)$:

$$\xi_B^+ = \frac{\partial r^+}{\partial t}\xi_B^t + \frac{\partial r^+}{\partial r}\xi_B^r = \frac{\pi}{R}\left[R^2 - (r^+)^2\right] \tag{3.7.7}$$

$$\xi_B^- = \frac{\partial r^-}{\partial t}\xi_B^t + \frac{\partial r^-}{\partial r}\xi_B^r = \frac{\pi}{R}\left[R^2 - (r^-)^2\right] \tag{3.7.8}$$

Now we want to compare this vector field for the ball shaped region to the vector field we defined

[6]This argument is exactly the same as for ball shaped sub-regions on the boundary we also expect that our derivation of the term that will reproduces the modular Hamiltonian term from the quasi-local energy will be identical to the derivation given in [6] for ball shaped regions.

in equation (3.7.1). This time we we take the cut of the cone defined by the function $f(\theta, \phi^i) = -R$. We do this because a cut $r^- = f(\theta, \phi^i) = -R$ corresponds to a constant time slice cut of the cone at $t = 0$. By doing this, the null boundary of the causal wedge for the ball and Σ will coincide in Pure AdS_{d+1} between the coordinate times $t \in [0, R]$. We find that:

$$\xi_B|_\Sigma = \frac{\pi}{R}\left[R^2 - (r^-)^2\right]\partial_- \tag{3.7.9}$$

$$\xi_c|_\Sigma = \frac{2\pi}{R}\left[R^2 - (r^-)^2\right]\partial_- \tag{3.7.10}$$

The vector fields are identical up to a factor of two. We could then extend the vector field ξ_c away from the surface Σ by simply defining its extension away from the surface to be the same as in ξ_B by a factor of two. By doing this we know that, up to a constant, ξ_c will have all the same properties as ξ_B and the arguments used for ball shaped regions in [6] should also apply to constant cut cones. This suggests that for constant time slices of the cone we do not expect to get any new constraints from relative entropy inequalities. For more arbitrary cut cones, we know the vector field ξ_c will not coincide with the vector field ξ_B on Σ. In this case it is not as obvious how to extend the vector field away from Σ. We hope that new constraints will arise by understanding ξ_c away from Σ for arbitrary cuts and using it in the formalism described in section 3.5.

Chapter 4

Conclusion

In this thesis we have tried to obtain constraints on $AAdS$ spacetimes using information theoretic quantities for holographic CFT states that are dual to such geometries. In chapter 2, we translated the constraint that $A_{CHI} - A_{RT} \geq 0$ to statements about the asymptotic structure of $AAdS_3$ spacetimes that have translation invariance in the boundary coordinates. This was done by finding series expansions in the proper length of the boundary intervals for both the area of the CHI and RT curves. These series expansions were used to construct the series expansion for the quantity $A_{CHI} - A_{RT}$. The constraints on the asymptotic geometry from this conjecture was obtained by requiring that the leading order term be positive. We found that the first two leading order constraints provided no new information about the possible asymptotic structure of $AAdS_3$ spacetimes. However, when the results were compared to the constraints obtained from the series expansion in small z of the null energy condition $T_{\mu\nu}u^\mu u^\nu \geq 0$ for null vectors parallel to the boundary; we found that the first two leading order terms gave the exact same constraints as what we got by simply considering $A_{CHI} - A_{RT} \geq 0$. We proposed that this observation may be a result of some interesting connection between the constraint $A_{CHI} - A_{RT} \geq 0$ and the null energy condition in the bulk. In chapter 3 we reviewed the progress of a ongoing research project whose goal is to understand the bulk dual of relative entropy constraints for holographic states defined on cut null cone regions on the boundary CFT_d. We derived the modular Hamiltonian cone regions whose base is cut. Our strategy was to start with the result for the null plane given in [15], and using a conformal transformation to get the result on the cone. We then derived the RT surface in the bulk for pure AdS_{d+1} spacetime anchored to the cut cone region on the boundary. Using these results we argued that for sub-regions on the boundary that are on null cones, one could still use the machinery developed in [6] for ball shaped regions. We gave a rough sketch as to how one can begin to prove that relative entropy between states on cone subregions is dual to quasi-local energy in the bulk. We argued that for cone regions with constant time cuts, the constraints would be identical to the constraints from ordinary ball shaped regions. For future work we stated that we need to understand the quasi-local energy for more generally cut cone sub-regions, and what it has to say about the constraints that relative entropy imposes. The work in chapter 3 can be thought of as a starting point for understanding relative entropy duals for regions on the boundary that are deformed away from the ball. By carefully studying these quantities we hope to sharpen our understanding of the role that relative entropy plays in the reconstruction and dynamics of $AAdS_{d+1}$ spacetimes.

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Appendix A

Supplementary Material

A.1 Co-Dimension 2 Extremal Surface in $d+1$ Dimensional Spacetime

In this section, we will go over how one can define a Co-dimension 2 surface in a $d+1$ dimensional spacetime as well as the equations the surface must obey to be extremal. Start with a metric for the $d+1$ dimensional spacetime $g_{\mu\nu}(X)$. The metric is a function of the coordinates of the spacetime X^{μ}. To define a co-dimension two surface in the spacetime we write two of the coordinates which label with capital letter indices X^{B_1} and X^{B_2}. The remaining coordinates of the $d-1$ coordinates which we label by lower case latin letters X^a will serve as the coordinates on the co-dimension 2 surface we collectively label these coordinates as $\sigma^a = X^a$. Now one can define the $d-1$ dimensional induced metric, γ_{ab},on the surface as follows:

$$\gamma_{ab}(\sigma) = g_{\mu\nu}(X(\sigma))\frac{\partial X^{\mu}}{\partial \sigma^a}\frac{\partial X^{\nu}}{\partial \sigma^b} \tag{A.1.1}$$

Using the induced metric one can define an area functional for the surface in terms of the determinant of the induced metric γ:

$$A = \int \sqrt{\gamma} d^{d-1}\sigma \tag{A.1.2}$$

Now we can consider fixing the background metric $g_{\mu\nu}$ and doing a variation to the surface. We want to know when the variation of the area functional vanishes. This amounts to having $X^{B_1} \to X^{B_1} + \delta X^{B_1}$ and $X^{B_2} \to X^{B_2} + \delta X^{B_2}$ and calculating the difference to first order in δX:

$$\begin{aligned}
\delta A &= \int \sqrt{det\left[g_{\mu\nu}(X+\delta X)\frac{\partial(X^{\mu}+\delta X^{\mu})}{\partial \sigma^a}\frac{\partial(X^{\nu}+\delta X^{\nu})}{\partial \sigma^b}\right]} - \sqrt{det\left[g_{\mu\nu}(X)\frac{\partial X^{\mu}}{\partial \sigma^a}\frac{\partial X^{\nu}}{\partial \sigma^b}\right]} \\
&= \int \sqrt{det\left[\gamma_{ab} + g_{\mu\nu}\left(\partial_a \delta X^{\mu}\partial_b X^{\nu} + \partial_a X^{\mu}\partial_b \delta X^{\nu}\right) + \partial_a X^{\mu}\partial_b X^{\nu}\partial_{\rho} g_{\mu\nu}\delta X^{\rho} + ...\right]} - \sqrt{\gamma}
\end{aligned} \tag{A.1.3}$$

Note that we have the functional in the form $\delta A = \int \left(\sqrt{det(\gamma_{ab}+\delta\gamma_{ab}+...)}-\sqrt{\gamma}\right) d^{d-1}\sigma$, this is easily expanded to first order using the formula $\delta A = \int \frac{1}{2}\sqrt{\gamma}\gamma^{ab}\delta\gamma_{ab}d^{d-1}\sigma$ where $\delta\gamma_{ab} = g_{\mu\nu}\left(\partial_a\delta X^\mu\partial_b X^\nu + \partial_a X^\mu \partial_b \delta X^\nu\right) + \partial_a X^\mu \partial_b X^\nu \partial_\rho g_{\mu\nu}\delta X^\rho$. Plugging into our formula we find:

$$\delta A = \int \frac{1}{2}\sqrt{\gamma}\gamma^{ab}\left(2\partial_a\delta X^\rho\partial_b X^\mu g_{\mu\rho} + \partial_a X^\mu\partial_b X^\nu\partial_\rho g_{\mu\nu}\delta X^\rho\right) d^{d-1}\sigma \tag{A.1.4}$$

After integrating the first term by parts and using the fact the variation should vanish at the boundary we are left with the result:

$$\delta A = \int \left[\frac{1}{2}\sqrt{\gamma}\gamma^{ab}\partial_a X^\mu\partial_b X^\nu\partial_\rho g_{\mu\nu} - \partial_a\left(\sqrt{\gamma}\gamma^{ab}\partial_b X^\mu g_{\mu\rho}\right)\right]\delta X^\rho \tag{A.1.5}$$

This gives us the condition for the co-dimension 2 surface to be extremal:

$$\frac{\delta A}{\delta X^B} = \frac{1}{2}\sqrt{\gamma}\gamma^{ab}\partial_a X^\mu\partial_b X^\nu\partial_B g_{\mu\nu} - \partial_a\left(\sqrt{\gamma}\gamma^{ab}\partial_b X^\mu g_{\mu B}\right) = 0 \tag{A.1.6}$$

Where $B = B_1, B_2$.

A.2 Quadratic Analysis

We want to understand for what values of C the following quadratic will be greater than zero on the interval $x \in (0,1)$:

$$x^2 + Cx + 1 \geq 0 \tag{A.2.1}$$

Start by noting that if $C \geq 0$ then the inequality holds trivially on our interval. The only possible way it could be less than zero is for some set of values $C \leq 0$. Start by calculating the roots which will be given by:

$$x = \frac{-C \pm \sqrt{C^2-4}}{2} = \frac{-C \pm \sqrt{(C+2)(C-2)}}{2} \tag{A.2.2}$$

We assume that $C \leq 0$ then $C-2 \leq 0$ in order for the root to be real we require that $C+2 \leq 0$. If $C = -2$, then $x = 1$. Now consider $C = -2-\epsilon$, $\epsilon > 0$. It follows that the real roots are:

$$x_\pm = 1 + \frac{\epsilon}{2} \pm \frac{\sqrt{\epsilon(\epsilon+4)}}{2} \tag{A.2.3}$$

Considering the minus root we have that:

$$x_- = 1 + \frac{\epsilon}{2} - \frac{\sqrt{\epsilon^2+4\epsilon}}{2} \leq 1 + \frac{\epsilon}{2} - \frac{\sqrt{\epsilon^2}}{2} = 1 \Rightarrow x \leq 1 \tag{A.2.4}$$

Hence if $C < -2$ there will always be a root in the interval $x \in (0,1)$. Now consider the case when $-2 \leq C \leq 0$. These two conditions imply that $C^2 - 4 \leq 0$. These automatically tell us that there are no real roots and since the quadratic has a y-intercept of 1, then the quadratic is positive. Hence we find that the inequality $(A.1)$ is satisfied in the interval $(0,1)$ if $C \geq -2$.

A.3 Mapping Half Space to a Ball

In this section we will go over the special conformal transformation (SCT) that will map the half space on a constant time slice to a ball shaped region on a Minkowski background with signature $(-1, 1, ..., 1)$. We start by introducing coordinates to the half space given as $x^\mu = (x^0, x^1, ..., x^{d-1})$ then define the following change of coordinates:

$$y^\mu(x) = \frac{x^\mu - (x \cdot x)c^\mu}{1 - 2(c \cdot x) + (c \cdot c)(x \cdot x)} + 2R^2 c^\mu \tag{A.3.1}$$

Where $c^\mu = (0, -1/(2R), 0, ..., 0)$ and $x \cdot x = \eta_{\mu\nu} x^\mu x^\nu$. It is straight forward to check that these change of coordinates will change the flat Minkowski metric by a local scale factor, which implies that this is a conformal change of coordinates. In particular one can show:

$$\begin{aligned} \eta_{\mu\nu} \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta} &= \frac{1}{\Omega^2(x)} \eta_{\alpha\beta} \\ \Omega(x) &= 1 - 2(c \cdot x) + (c \cdot c)(x \cdot x) \end{aligned} \tag{A.3.2}$$

This implies that:

$$ds^2 = \eta_{\mu\nu} dy^\mu dy^\nu = \frac{1}{\Omega^2(x)} \eta_{\alpha\beta} dx^\alpha dx^\beta \tag{A.3.3}$$

Now we will show that the half space described by the points $\{x^\mu : x^1 > 0, x^0 = 0\}$ are mapped to a ball. To do this we start by calculating $y \cdot y = \eta_{\mu\nu} y^\mu y^\nu$ in terms of the coordinates x. We find that:

$$y \cdot y = \frac{\Omega(x) R^2 - 2x^1 R}{\Omega(x)} \Rightarrow x^1 = \frac{\Omega(x)}{2R} \left(R^2 - y \cdot y\right) = \frac{\Omega(x)}{2R} \left(R^2 + (y^0)^2 - |\vec{y}|^2\right) \tag{A.3.4}$$

By using the SCT one can verify that the set of points on the constant time slice $x^0 = 0$ are mapped to points on the constant time slice $y^0 = 0$. On this time slice one can verify that $\Omega(x^0 = 0) \geq 0$. Using this information one can see that points in the region $\{x^\mu : x^1 > 0, x^0 = 0\}$ are mapped to points in the region $\{y^\mu : |\vec{y}| \leq R, y^0 = 0\}$. This proves the statement that the half space is mapped to a ball shaped region on a constant time slice. We can also calculate the inverse of the SCT transformation given by equation $(B.1)$. This will amount to finding $x^\mu(y)$ with the property that $x^\mu(y^\mu(\bar{x})) = \bar{x}^\mu$. To do this we split the SCT given by $(B.1)$ into two parts given by:

$$y^\mu(x) = y'^\mu(x) + 2R^2 c^\mu = y'^\mu(x) + \frac{c^\mu}{2c^2} \tag{A.3.5}$$

Where we defined:

$$y'^\mu(x) = \frac{x^\mu - (x \cdot x)c^\mu}{1 - 2(c \cdot x) + c^2(x \cdot x)} \tag{A.3.6}$$

We make the claim that the inverse of $y'^{\mu}(x)$ is given by the following:

$$x^{\mu}(y'^{\mu}) = \frac{y'^{\mu} + (y' \cdot y')c^{\mu}}{1 + 2(c \cdot y') + c^2(y' \cdot y')} \tag{A.3.7}$$

To verify this claim start by noting that:

$$y'(\bar{x}) \cdot y'(\bar{x}) = \frac{\bar{x} \cdot \bar{x}}{\Omega(\bar{x})} \tag{A.3.8}$$

Using equation $(B.6)$ we find that:

$$y'^{\mu}(\bar{x}) + y'(\bar{x}) \cdot y'(\bar{x})c^{\mu} = \frac{\bar{x}^{\mu}}{\Omega(\bar{x})} \tag{A.3.9}$$

Using this, calculate $x^{\mu}(y'^{\mu}(\bar{x}))$ and find:

$$x^{\mu}(y'^{\mu}(\bar{x})) = \frac{\bar{x}}{\omega(y'^{\mu}(\bar{x}))\Omega(\bar{x})} \tag{A.3.10}$$

Where we defined $\omega(y) as:$

$$\omega(y) = 1 + 2(c \cdot y) + c^2(y \cdot y) = \frac{1}{4} - \frac{y^1}{2R} + \frac{y \cdot y}{4R^2} \tag{A.3.11}$$

One can check that $\omega(y'^{\mu}(\bar{x}))\Omega(\bar{x}) = 1$ this proves our claim. Now we can use the result $(B.7)$ and substitute for the argument $y'^{\mu} = y^{\mu} - \frac{c^{\mu}}{2c^2}$ this will give us the inverse of the SCT given by (B.1) we find that:

$$x^{\mu}(y) = \frac{y^{\mu} - \frac{c^{\mu}}{2c^2} + (y^{\nu} - \frac{c^{\nu}}{2c^2})(y_{\nu} - \frac{c_{\nu}}{2c^2})c^{\mu}}{1 + 2c_{\nu}(y^{\nu} - \frac{c^{\nu}}{2c^2}) + c^2(y^{\nu} - \frac{c^{\nu}}{2c^2})(y_{\nu} - \frac{c_{\nu}}{2c^2})} = \frac{y^{\mu} + 2(y \cdot y)c^{\mu}}{\frac{1}{4} + c \cdot y + c^2(y \cdot y)} - \frac{c^{\mu}}{c^2} \tag{A.3.12}$$

We can also give an interpretation of $\omega(y)$ by the following argument. Using $(B.12)$ we can see that:

$$x^1(y) = \frac{R^2 - y \cdot y}{2R\omega(y)} \tag{A.3.13}$$

Comparing this with (B.4) tells us that:

$$\Omega = \frac{1}{\omega(y)} \tag{A.3.14}$$

Hence ω will be the scale local scale factor in particular we can see by rearranging (B.2) that:

$$\eta_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial y^{\mu}}\frac{\partial x^{\beta}}{\partial y^{\nu}} = \frac{1}{\omega^2(y)}\eta_{\mu\nu} \tag{A.3.15}$$

Now we want to show that the co-dimension 1 null surface given by setting $x^- := x^0 - x^1 = 0$ gets mapped to the past null cone of the point $(y^0 = R, 0, ..., 0)$. To see this we begin by calculating

$|\vec{y}|^2$ in terms of y we find that:

$$|\vec{y}|^2 = \frac{(x^0 - R\Omega)^2 + 2R\Omega x^-}{\Omega^2} \tag{A.3.16}$$

Now we set $x^- = 0$ and use the fact $y^0 = x^0/\Omega$ this gives us:

$$|\vec{y}|^2 = (y^0 - R)^2 \Rightarrow |\vec{y}| = R - y^0 \tag{A.3.17}$$

Which defines the surface of a past null cone with its tip at $(y^0 = R, 0, ..., 0)$.

A.4 Calculating Jacobian for SCT

Here we derive the equation for the elements of the Jacobian matrix as well as its determinant. For the transverse coordinates we know that:

$$\begin{aligned} y^i &= \frac{x^i}{\Omega(x)} \\ \Omega(x) &= \frac{-(x^+ + 2R)(x^- - 2R) + (x^\perp)^2}{4R^2} \\ (x^\perp)^2 &= \sum_{i,j=2}^{d-1} \delta_{ij} x^i x^j \end{aligned} \tag{A.4.1}$$

Now we compute the elements of the Jacobian associated with the mapping above:

$$\frac{\partial y^i}{\partial x^k} = \frac{\frac{\partial x^i}{\partial x^k}\Omega - x^i \frac{\partial \Omega}{\partial x^k}}{\Omega^2} \tag{A.4.2}$$

We can calculate $\frac{\partial \Omega}{\partial x^k}$ as follows:

$$\frac{\partial \Omega}{\partial x^k} = \frac{\partial}{\partial x^k}\left[\frac{\sum_{i,j=2}^{d-1} \delta_{ij} x^i x^j}{4R^2}\right] = \frac{x_k}{2R^2} \tag{A.4.3}$$

Combining everything gives the result:

$$[J_\perp]^i_k = \frac{\partial y^i}{\partial x^k} = \Omega^{-1}\left[\delta^i_k - \frac{x_k x^i}{2R^2\Omega}\right] \tag{A.4.4}$$

We compute the determinant of the matrix by finding its eigenvalues. To find the eigenvalues we need eigenvectors. We can write the eigenvectors in a basis where the first eigenvector is given as $v_2 = \sum_{k=2}^{d-1} x^k \partial_k$. If we apply this vector to the Jacobian we will find:

$$\sum_{k=2}^{d-1} [J_\perp]^i_k v_2^k = \Omega^{-1}\left[1 - \frac{(x^\perp)^2}{2R^2\Omega}\right] x^i = \Omega^{-1}\left[1 - \frac{(x^\perp)^2}{2R^2\Omega}\right] v_2^i \tag{A.4.5}$$

It has an eigenvalue of $\Omega^{-1}\left[1 - \frac{(x^\perp)^2}{2R^2\Omega}\right]$. We can choose the other $d-3$ eigenvectors to be orthogonal to v_2, this implies that $\sum_{k=2}^{d-1} x_k v_b^k = 0, b \in \{3, 4, ..., d-1\}$. Hence we see that:

$$\sum_{k=2}^{d-1} [J_\perp]^i_k v_b^k = \Omega^{-1} v_b^k \tag{A.4.6}$$

Which states that we have $d-3$ eigenvalues of Ω^{-1}. The determinant of the matrix is the product of eigenvalues this gives the result:

$$det([J_\perp]^i_k) = J_\perp = \Omega^{2-d} \left[1 - \frac{(x^\perp)^2}{2R^2\Omega}\right] \tag{A.4.7}$$

A.5 Coordinates on Null Plane to Coordinates on Null Cone

In appendix A.3 we defined a change of coordinates which was a SCT that maps a null sheet to a null cone. The mapping was done between cartesian coordinates on the plane $(x^0, x^1, x^2, ..., x^{d-1})$ and cartesian coordinates on the cone $(y^0, y^1, y^2, ..., y^{d-1})$. We had a complete understanding of the maps that go from one coordinate to the other. Here we want to write the coordinates on the plane in terms of cartesian null coordinates $(x^+, x^-, x^2, ..., x^{d-1})$ and use the SCT to go to radial null coordinates on the cone $(\rho^+, \rho^-, \phi^1, ..., \phi^{d-2})$. Recall that from equation (A.3.12):

$$x^\mu(y) = \frac{y^\mu + 2(y\cdot y)c^\mu}{\frac{1}{4} + c\cdot y + c^2(y\cdot y)} - \frac{c^\mu}{c^2} = \frac{y^\mu + 2(y\cdot y)c^\mu}{\omega} - \frac{c^\mu}{c^2} \tag{A.5.1}$$

Now we compute $x^\pm = x^0 \pm x^1$ which is given by:

$$x^+ = \frac{y^0}{\omega} + \frac{R^2 - y\cdot y}{2R\omega} = \frac{(R+y^0)^2 - |\vec{y}|^2}{2R\omega} = \frac{(R+\rho^-)(R+\rho^+)}{2R\omega} \tag{A.5.2}$$

$$x^- = \frac{y^0}{\omega} - \frac{R^2 - y\cdot y}{2R\omega} = -\frac{(R-y^0)^2 - |\vec{y}|^2}{2R\omega} = -\frac{(R-\rho^+)(R-\rho^-)}{2R\omega} \tag{A.5.3}$$

$$x^i = \frac{y^i}{\omega} \tag{A.5.4}$$

where $\rho^\pm = y^0 \pm |\vec{y}|$ and $\omega = \frac{1}{4} - \frac{y^1}{2R} + \frac{y\cdot y}{4R^2}$. Notice from these coordinates that it is clear that if $\rho^+ = R$ then $x^- = 0$, applying these restrictions, one can easily relate the null coordinate on the plane which is x^+ and the null coordinate on the cone which is ρ^-:

$$x^+ = \frac{R + \rho^-}{\omega|_{\rho^+ = R}} \tag{A.5.5}$$

We can explicitly calculate $\omega|_{\rho^+=R}$ as follows:

$$\begin{aligned}\omega &= \frac{1}{4} - \frac{y^1}{2R} + \frac{y \cdot y}{4R^2} = \frac{R^2 - 2Ry^1 - (y^0)^2 + |\vec{y}|^2}{4R^2} = -\frac{\rho^+\rho^- + R(2y^1 - R)}{4R^2} \\ \Rightarrow \omega|_{\rho^+=R} &= -\frac{\rho^- + 2y^1 - R}{4R}\end{aligned} \tag{A.5.6}$$

We can also write ρ^- in terms of x^+ by rearranging $C.5$ and using $\Omega = 1/\omega$:

$$\rho^- = \frac{x^+}{\Omega|_{x^-=0}} - R \tag{A.5.7}$$

Where we explicitly can compute $\Omega|_{x^-=0}$ as follows:

$$\begin{aligned}\Omega &= 1 + \frac{x^1}{R} + \frac{x \cdot x}{4R^2} = \frac{-(x^0)^2 + (x^1 + 2R)^2 + (x^\perp)^2}{4R^2} = \frac{-(x^- - 2R)(x^+ + 2R) + (x^\perp)^2}{4R^2} \\ \Rightarrow \Omega|_{x^-=0} &= 1 + \frac{x^+}{2R} + \left(\frac{x^\perp}{2R}\right)\end{aligned} \tag{A.5.8}$$

This gives us the equation used in equation (3.12) :

$$\rho^- = \frac{x^+}{1 + \frac{x^+}{2R} + \left(\frac{x^\perp}{2R}\right)^2} - R \tag{A.5.9}$$

A.6 Conformal Transformation of the Stress Energy Tensor of a CFT_d

Here we go over the calculations for applying a conformal transformation to the stress energy tensor component when we apply the SCT defined in appendix A.3. The conformal transformation of the stress energy tensor associated with the SCT can be implemented through a standard change of coordinates to the tensor along with Weyl rescaling to make the background metric flat again. We start by calculating the general elements of the Jacobian matrix associated with the SCT:

$$\begin{aligned}\frac{\partial y^\mu}{\partial x^\nu} &= \frac{\partial}{\partial x^\nu}\left[\frac{x^\mu - (x \cdot x)c^\mu}{\Omega}\right] = \frac{\delta^\mu_\nu - 2x_\nu c^\mu}{\Omega} - \frac{\frac{\partial \Omega}{\partial x^\nu}(x^\mu - (x \cdot x)c^\mu)}{\Omega^2} \\ &= \frac{1}{\Omega}\left[\delta^\mu_\nu - 2x_\nu c^\mu - \frac{\partial \Omega}{\partial x^\nu}\left(y^\mu - \frac{c^\mu}{2c^2}\right)\right] = \frac{1}{\Omega}\left[\delta^\mu_\nu - 2x_\nu c^\mu - \left(-2c_\nu + 2c^2 x_\nu\right)\left(y^\mu - \frac{c^\mu}{2c^2}\right)\right] \\ &= \frac{1}{\Omega}\left[\delta^\mu_\nu - x_\nu c^\mu + 2c_\nu y^\mu - 2c^2 x_\nu y^\mu - \frac{c_\nu c^\mu}{c^2}\right] \\ &= \frac{1}{\Omega}\left[\delta^\mu_\nu - \delta^\mu_1 \delta^1_\nu - \frac{y^\mu}{R}\delta^1_\nu + \frac{x_\nu}{2R}\left(\delta^\mu_1 - \frac{y^\mu}{R}\right)\right]\end{aligned} \tag{A.6.1}$$

By similar calculations one can show:

$$\frac{\partial x^\mu}{\partial y^\nu} = \Omega \left[\delta^\mu_\nu - x^\mu c_\nu + 2c^\mu y_\nu - 2c^2 x^\mu y_\nu - \frac{c_\nu c^\mu}{c^2} \right] \tag{A.6.2}$$

We want to use these to calculate how the stress energy tensor will change from the change in coordinates:

$$T_{++} = \frac{\partial y^\mu}{\partial x^+} \frac{\partial y^\nu}{\partial x^+} T_{\mu\nu} \tag{A.6.3}$$

We can use the formula (A.6.1) to calculate the partial derivative:

$$\frac{\partial y^\mu}{\partial x^+} = \frac{1}{2} \left(\frac{\partial y^\mu}{\partial x^0} + \frac{\partial y^\mu}{\partial x^1} \right) = \frac{1}{2\Omega} \left[\delta^\mu_0 - \frac{y^\mu}{R} + \frac{x_+}{2R} \left(\delta^\mu_1 - \frac{y^\mu}{R} \right) \right] \tag{A.6.4}$$

From this point forward throughout this section we will be restricted to the null plane. We know that $x_+ = -x^- = 0$. This means that:

$$T_{++}\big|_{x^-=0} = \frac{1}{4\Omega^2} \left(\delta^\mu_0 - \frac{y^\mu}{R} \right) \left(\delta^\nu_0 - \frac{y^\nu}{R} \right) T_{\mu\nu}\big|_{\rho^+ = y^0 + |\vec{y}| = R} \tag{A.6.5}$$

Now we make the following claim:

$$\delta^\mu_0 - \frac{y^\mu}{R} = \frac{2|\vec{y}|}{R} \frac{\partial y^\mu}{\partial \rho^-} \tag{A.6.6}$$

To see this is true, we start by calculating the following quantity:

$$\begin{aligned} \frac{|\vec{y}|}{R} \frac{\partial \rho^-}{\partial y^\mu} + 2\delta^0_\mu \left(1 - \frac{|\vec{y}|}{R} \right) &= \frac{|\vec{y}|}{R} \delta^0_\mu - \frac{y_\mu}{R} (1 - \delta^0_\mu) + 2\delta^0_\mu \left(\frac{y^0}{R} \right) = \delta^0_\mu \left(\frac{y^0 + |\vec{y}|}{R} \right) - \frac{y_\mu}{R} \\ &= \delta^0_\mu - \frac{y_\mu}{R} \end{aligned} \tag{A.6.7}$$

Where we used the fact that $y^0 + |\vec{y}| = R$. Now we can compute the following quantity:

$$\left(\delta^0_\nu - \frac{y_\nu}{R} \right) \left(\delta^\nu_0 - \frac{y^\nu}{R} \right) = \frac{R^2 + y \cdot y}{R^2} = \frac{2|\vec{y}|}{R} \tag{A.6.8}$$

At the same time we also can compute:

$$\begin{aligned} &\left[\frac{|\vec{y}|}{R} \frac{\partial \rho^-}{\partial y^\mu} + 2\delta^0_\mu \left(1 - \frac{|\vec{y}|}{R} \right) \right] \left[\frac{2|\vec{y}|}{R} \frac{\partial y^\mu}{\partial \rho^-} \right] = \frac{2|\vec{y}|^2}{R^2} + \frac{4|\vec{y}|}{R} \left(1 - \frac{|\vec{y}|}{R} \right) \frac{\partial y^0}{\partial \rho^-} \\ &= \frac{2|\vec{y}|}{R} \end{aligned} \tag{A.6.9}$$

Where we used the fact that $y^0 = \frac{1}{2}(\rho^+ + \rho^-)$. This means that we know the left hand sides of (A.6.8) and (A.6.9) are equal. Furthermore, using equation (A.6.7), we can deduce that that the claim given in equation (A.6.6) is true. Using the result gives:

$$T_{++}\big|_{x^-=0} = \frac{1}{R^2\Omega^2}\left(\frac{R-\rho^-}{2}\right)^2 \tilde{T}_{--}\big|_{\rho^+=R} \tag{A.6.10}$$

This takes care of the coordinate transformation. In order to complete the conformal transformation, we need to do a Weyl rescaling of the stress energy tensor to figure out what power of the conformal factor we need we apply the following argument. Suppose we have a conformal change of coordinates such as in the SCT. Then we know that when we go from coordinates $(x^0, x^1, ..., x^{d-1}) \to (y^0, y^1, ..., y^{d-1})$ the new metric will be rescaled $\eta_{\mu\nu} \to \Omega^2\eta_{\mu\nu}$. Now consider the term in the action where the stress energy tensor couples to the metric:

$$\int d^dy\sqrt{det(\Omega^2\eta_{\mu\nu})}\frac{\eta_{\mu\nu}}{\Omega^2}\tilde{T}_{\mu\nu} = \int d^dx\Omega^{d-2}\eta^{\mu\nu}\tilde{T}_{\mu\nu} = \int d^dx\eta^{\mu\nu}T_{\mu\nu} \tag{A.6.11}$$

In order to cancel the conformal factor we need to rescale the stress energy tensor by Ω^{2-d}. Using this the conformally transformed stress energy tensor is now:

$$T_{++}\big|_{x^-=0} = \frac{1}{R^2\Omega^d}\left(\frac{R-\rho^-}{2}\right)^2 \tilde{T}_{--}\big|_{\rho^+=R} = \frac{\omega^d}{R^2}\left(\frac{R-\rho^-}{2}\right)^2 \tilde{T}_{--}\big|_{\rho^+=R} \tag{A.6.12}$$

Which gives the result in equation used in chapter 3 equation (3.3.19).

A.7 Unit Binormal to RT surface Anchored to Cone Regions

In this section we want to derive the unit binormal on the extremal surface we derived in the previous section. To calculate the unit binormal to our extremal surface start by calculating the $d-1$ tangent vectors to the surface which will be labeled with the index a as $\mathcal{T}_a = \mathcal{T}_a^\mu\partial_\mu$, $a \in \{1, 2, ..., d-1\}$. We can also write in component form $\mathcal{T}_a = (\mathcal{T}_a^+, \mathcal{T}_a^-, \mathcal{T}_a^\theta, \mathcal{T}_a^i)$ in the coordinate basis. These vectors will have components that satisfy the following equations:

$$\mathcal{T}_a^\mu\partial_\mu(r^+ - R) = \mathcal{T}_a^+ = 0 \tag{A.7.1}$$

$$\mathcal{T}_a^\mu\partial_\mu(r^- - \Lambda(\theta, \phi^i)) = \mathcal{T}_a^- - \mathcal{T}_a^\theta\partial_\theta\Lambda - \mathcal{T}_a^i\partial_i\Lambda = 0 \tag{A.7.2}$$

The first equation tells us that the tangent vectors will have no components in the direction ∂_+. Now we will introduce a slightly abusive labeling of the tangent vectors. We let labelling indices to be coordinate indices $a \in \{\theta, \phi^j\}$, then we can define the following vectors:

$$\mathcal{T}_a = (\partial_a\Lambda)\partial_- + \delta_a^\theta\partial_\theta + \delta_a^i\partial_i \tag{A.7.3}$$

We can clearly see that these satisfy the second equation. We can check for orthogonality:

$$g_{\mu\nu}\mathcal{T}_a^\mu\mathcal{T}_b^\nu = g_{\theta\theta}\mathcal{T}_a^\theta\mathcal{T}_b^\theta + \sum_{i=1}^{d-2} g_{ii}\mathcal{T}_a^i\mathcal{T}_b^i = g_{\theta\theta}\delta_a^\theta\delta_b^\theta + \sum_{i=1}^{d-2} g_{ii}\delta_a^i\delta_b^i = g_{aa}\delta_{ab} \tag{A.7.4}$$

Where we used the fact that the metric on the unit $d-2$ sphere is diagonal and $g_{--} = 0$, normalization can be trivially done by dividing by the metric components. These vectors form an orthogonal basis $d-1$ basis on the Ryu-Takayanagi surface. Now we need to find two more vectors n_1 and n_2 that are orthogonal to each other and the tangent vectors we defined. Start with the most general form for the normal vectors with no constraints on the components $n_1 = n_1^\mu\partial_\mu$ and $n_2 = n_2^\mu\partial_\mu$. Now we write the condition that the vectors should be orthogonal to the tangent vectors starting with n_1:

$$g_{\mu\nu}n_1^\mu\mathcal{T}_a^\nu = g_{+-}n_1^+\partial_a\Lambda + g_{\theta\theta}n_1^\theta\delta_a^\theta + \sum_{i=1}^{d-2} g_{ii}n_1^i\delta_a^i = 0 \tag{A.7.5}$$

This equation constrains the components $d-1$ components $n_1^a = -\frac{g_{+-}n_1^+\partial_a\Lambda}{g_{aa}}$. The exact same argument holds true for n_2 hence we have that the following two vectors with be normal to all tangent vectors:

$$n_1 = n_1^+\partial_+ + n_1^-\partial_- + n_1^a\partial_a \tag{A.7.6}$$

$$n_2 = n_2^+\partial_+ + n_2^-\partial_- + n_2^a\partial_a \tag{A.7.7}$$

To simplify calculations we let $n_1^+ = n_2^+ = g^{+-}$. This implies that $n_1^a = n_2^a$ and the orthogonality condition between the two vectors will be:

$$g_{\mu\nu}n_1^\mu n_2^\nu = (n_1^- + n_2^-) + \sum_{a=\theta,i} \frac{(\partial_a\Lambda)^2}{g_{aa}} = 0 \tag{A.7.8}$$

We also want $g_{\mu\nu}n_1^\mu n_1^\nu = 1$ this means that:

$$n_1^- = \frac{1 - \sum_{a=\theta,i}\frac{(\partial_a\Lambda)^2}{g_{aa}}}{2} \tag{A.7.9}$$

Finally, we can use the orthogonality condition to get the component n_2^-:

$$n_2^- = -\frac{1 + \sum_{a=\theta,i}\frac{(\partial_a\Lambda)^2}{g_{aa}}}{2} \tag{A.7.10}$$

We can verify that $g_{\mu\nu}n_2^\mu n_2^\nu = -1$. In summary we found the normalized normal vectors to the

Ryu-Takayanagi surface to be:

$$n_1 = g^{+-}\partial_+ + \frac{1-Z}{2}\partial_- - \sum_{a=\theta,i} \frac{\partial_a \Lambda}{g_{aa}}\partial_a \tag{A.7.11}$$

$$n_2 = g^{+-}\partial_+ - \frac{1+Z}{2}\partial_- - \sum_{a=\theta,i} \frac{\partial_a \Lambda}{g_{aa}}\partial_a \tag{A.7.12}$$

$$Z = \sum_{a=\theta,i} \frac{(\partial_a \Lambda)^2}{g_{aa}} \tag{A.7.13}$$

Now we can define the unit binormal components using the normal vector components:

$$n^{ab} = n_2^a n_1^b - n_2^b n_1^a \tag{A.7.14}$$

One can check that the only non-zero components of the binormal are given by:

$$\begin{aligned} n^{+-} &= \frac{1}{g_{+-}} \\ n^{\theta-} &= -\frac{\partial_\theta \Lambda}{g_{\theta\theta}} \\ n^{i-} &= -\frac{\partial_i \Lambda}{g_{ii}} \end{aligned} \tag{A.7.15}$$