Lollipop diagrams in defect $\mathcal{N} = 4$ super Yang-Mills theory

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B.Sc., Lanzhou University, 2012
M.Sc., Lanzhou University, 2015

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

The Faculty of Graduate and Postdoctoral Studies

(Physics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

August 2017

Abstract

In this thesis, we have studied the lollipop diagrams in defect $\mathcal{N} = 4$ super Yang-Mills field theory with nontrivial background, which is dual to the D3-D5 brane system with the probe D5 brane carrying k units of flux. Using the framework for performing loop computations for this system built by Buhl-Mortensen, Leeuw, Ipsen, Kristjansen and Wilhelm [2, 3], we prove that for arbitrary N and k, the contribution of the lollipop diagrams to the one-point function is zero. This improves their result, where they take the planar limit $N \gg 1$ and the probe brane limit $k/N \ll 1$.

Lay Summary

All particles are either bosons, such as photons, or fermions, such as electrons. Supersymmetry is a symmetry relating bosons and fermions. In a supersymmetric theory, there are some cancellations of quantum effects between bosons and fermions. In this thesis, we generalize such a proof that for a particular system, even the supersymmetry is partially broken, a special kind of cancellation still exists.

Preface

Chapter 1 contains chosen material done by others. Chapters 2, 3 and the appendix are based on my research under the supervision of Professor Gordon Semenoff.

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Acknowledgements

I would like to thank my supervisor Gordon Semenoff. He has helped me throughout my research and I have benefited tremendously from his courses. I am grateful to have worked with someone with such a strong physical insight. I would like to thank Mark Van Raamsdonk for being the second reader of this thesis. I would like to thank Shuailiang Ge and Krishan Saraswat for useful discussion. I would also like to thank all my friends at UBC. Finally, I would like to thank my family for their support.

Chapter 1

Introduction

1.1 AdS/dCFT

The AdS/CFT conjecture [1, 7] relates 4-dimensional $\mathcal{N}=4$ SU(N) supersymmetric Yang-Mills (SYM) theory, which is a conformal field theory, with Yang-Mills coupling g_{YM} to type IIB string theory with string length l_s and coupling g_s on $AdS_5 \times S^5$ with radius L and N units of $F_{(5)}$ flux on S^5. The strongest version states that these two theories are dynamically equivalent if the parameters satisfy

$$g_{\mathrm{YM}}^2 = 2\pi g_s, \qquad 2g_{\mathrm{YM}}^2 N = L^4/l_s^4. \tag{1.1}$$

Usually we define the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$. However it is difficult to do calculation on both sides to check the strongest version of the correspondence. So people usually consider the weaker form of the conjecture by taking some limits.

In the strong version, we take the limit $g_s \to 0$ and hold L/l_s fixed. The limit $g_s \to 0$ allows us to do perturbative calculation in string theory and the leading order is the classical string theory. This limit also implies $N \to \infty$, which is the planar limit of the SU(N) gauge field theory. To go back to the strongest version, notice that for fixed λ we have $1/N \sim g_s$. Then the $1/N$ expansion on the field theory side should correspond to the g_s expansion in string theory side, which is the genus expansion of the string worldsheet.

In the weak form we further take the limit $\lambda \to \infty$. In the string theory side $L/l_s \gg 1$, then we have an effective type IIB supergravity on weakly curved background. The field theory side becomes a strong coupled field theory. So it is usually called strong/weak duality.

The field theory living on the boundary can be derived from the low energy excitation of open string attached to a stack of N D3-branes. For the bosonic part, the massless excitations parallel to the brane correspond to gauge field A_μ and the excitations transversal to the brane correspond to the six scalar fields ϕ_i. Further, there are four Majorana fermions, which make the bosons and fermions have the same degrees of freedom. The $SO(4,2)$ symmetry of the AdS_5 corresponds to the 4-d conformal symmetry, which is

$SO(4,2)$, in field theory side. The $SO(6)$ symmetry of the S^5 corresponds to the rotational symmetry $SO(6)$ among the six real scalars and the $SU(4)$ symmetry among the four fermions since $SU(4) \sim SO(6)$.

The generalization of AdS/CFT with probe branes are widely studied [4–6]. The classical one is to add a probe D5 brane. If the coordinates of the 10-dimensional space are labeled by $x^0, x^1, \ldots, x^9$, the D3-branes occupy x^0, x^1, x^2, x^3 and the D5-brane with geometry $AdS_4 \times S^2$ occupies $x^0, x^1, x^2, x^4, x^5, x^6$. Then the field theory side is a defect conformal field theory with a co-dimension one defect. Because of the defect, the symmetry $SO(4,2)$ breaks to $SO(3,2)$, which is the conformal symmetry of the defect, and symmetry $SO(6)$ breaks to $SO(3) \times SO(3)$.

An interesting case is to let the D5 brane carry k units of Dirac monopole flux on the S^2. It can be realized as k of N D3 branes get dissolved in the D5 brane. Then on the field theory side, the defect connecting SU(N) gauge theory in the region $x^3 > 0$ and SU(N-k) gauge theory in the region $x^3 < 0$. Further, three of the six scalar fields carry nontrivial background.

The extra parameter k gives the system interesting new features [8]. A large enough k allows us to have both $\lambda \gg 1$, which ensure a valid supergravity description on string theory side, and $\lambda/k^2 \ll 1$, which gives us a small expansion parameter on field thoery side and allows us to do perturbative calculations. Then we can do calculations on both side and compare them directly. Further we need $k \ll N$ to ensure the probe brane limit, where the added D5 brane would not change the supergravity background.

In the recent papers [2, 3], they provide a systematic way to do perturbative calculations on the field theory side. They prove that the one-loop one point effective action is zero in the probe brane limit $k \ll N$. Noticing that half of the original supersymmetries remain when a defect is added to the field theory, it is natural to guess the remaining supersymmetries would ensure a zero one-loop one point effective action for arbitrary k and N. In this thesis, we prove that the one-loop one point effective action is zero for arbitrary k and N.

1.2 The action and the background solution

The field theory side is a defect CFT consists of a 4-dimensional bulk field and fields living on the codimension-one defect. The defect fields have no effect to the one-loop lollipop diagram that we will consider. Hence we start with the 4-dimensional $\mathcal{N}=4$ SYM with the action

$$\begin{aligned} S_{\mathcal{N}=4} = \frac{2}{g_{\mathrm{YM}}^2} \int d^4x \ \mathrm{tr}\Big[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \phi_i D^\mu \phi_i + \frac{i}{2} \bar{\Psi} \Gamma^\mu D_\mu \Psi \\ + \frac{1}{2} \bar{\Psi} \tilde{\Gamma}^i [\phi_i, \Psi] + \frac{1}{4} [\phi_i, \phi_j][\phi_i, \phi_j] \Big]. \end{aligned} \tag{1.2}$$

Here the indices are $\mu, \nu = 0, 1, 2, 3$ and $i, j = 1, 2, \ldots, 6$. The field Ψ is a 10-d Majorana-Weyl fermion and $\Gamma^\mu, \tilde{\Gamma}^i$ are the 10-d gamma matrices.

The background is the classical solution

$$\phi_i^{cl} = -\frac{1}{x_3} t_i^{(k)} \oplus 0^{(N-k)}, \quad \text{for} \quad i = 1, 2, 3, \tag{1.3}$$

where $t_i^{(k)}$ is the k-d irreducible representation of su(2) and $0^{(N-k)}$ is a $(N-k)$-d zero matrix. All the other fields have vanishing background. Following papers [2, 3], after expanding around the background $\phi_i \to \phi_i^{cl} + \phi_i$ and gauging it, the perturbative action is given by

$$S_{\mathcal{N}=4} + S_{\mathrm{gh}} = S_{\mathrm{kin}} + S_{\mathrm{m,b}} + S_{\mathrm{m,f}} + S_{\mathrm{cubic}} + S_{\mathrm{quartic}}, \tag{1.4}$$

where S_{gh} is the ghost action. The quartic terms are not relevant if we are only interested in the one-loop one-point effective action. The kinetic terms are

$$S_{\mathrm{kin}} = \frac{2}{g_{\mathrm{YM}}^2} \int d^4x \ \mathrm{tr}\Big[\frac{1}{2} A_\mu \partial_\nu \partial^\nu A^\mu + \frac{1}{2} \phi_i \partial_\nu \partial^\nu \phi_i + \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi + \bar{c} \partial_\mu \partial^\mu c \Big]. \tag{1.5}$$

The mass terms of bosonic fields are

$$\begin{aligned} S_{\mathrm{m,b}} = \frac{2}{g_{\mathrm{YM}}^2} \int d^4x \ \mathrm{tr}\Big[\frac{1}{2} [\phi_i^{\mathrm{cl}}, \phi_j^{\mathrm{cl}}][\phi_i, \phi_j] + \frac{1}{2} [\phi_i^{\mathrm{cl}}, \phi_j][\phi_i^{\mathrm{cl}}, \phi_j] + \frac{1}{2} [\phi_i^{\mathrm{cl}}, \phi_j][\phi_i, \phi_j^{cl}] \\ + \frac{1}{2} [\phi_i^{\mathrm{cl}}, \phi_i][\phi_j^{\mathrm{cl}}, \phi_j] + \frac{1}{2} [A_\mu, \phi_i^{\mathrm{cl}}][A^\mu, \phi_i^{\mathrm{cl}}] + 2i [A^\mu, \phi_i] \partial_\mu \phi_i^{\mathrm{cl}} \Big]. \end{aligned} \tag{1.6}$$

The mass terms of fermionic fields are

$$S_{\mathrm{m,f}} = \frac{2}{g_{\mathrm{YM}}^2} \int d^4x \ \mathrm{tr}\Big[\frac{1}{2} \bar{\psi} G^i [\phi_i^{\mathrm{cl}}, \psi] - \bar{c} [\phi_i^{\mathrm{cl}} [\phi_i^{\mathrm{cl}}, c]] \Big]. \tag{1.7}$$

The cubic vertices are

$$S_{\text{cubic}} = \frac{2}{g_{\text{YM}}^2}\int d^4x \,\text{tr}\Big[i[A^\mu, A^\nu]\partial_\mu A_\nu + [\phi_i^{\text{cl}}, \phi_j][\phi_i, \phi_j] + i[A^\mu, \phi_i]\partial_\mu \phi_i + [A_\mu, \phi_i^{\text{cl}}][A^\mu, \phi_i] + \frac{1}{2}\bar{\psi}\gamma^\mu[A_\mu, \psi] + \sum_{i=1}^{3}\frac{1}{2}\bar{\psi}G^i[\phi_i, \psi] + \sum_{i=4}^{6}\frac{1}{2}\bar{\psi}G^i[\phi_i, \gamma_5\psi] + i(\partial_\mu \bar{c})[A^\mu, c] - \bar{c}[\phi_i^{\text{cl}}, [\phi_i, c]]\Big]. \tag{1.8}$$

Here the 10-d Majorana-Weyl fermion Ψ is reduced to four 4-d Majorana fermions $\psi_1, \ldots, \psi_4$ and G^i are the resulting matrices whose properties can be found in [3].

1.3 Fuzz sphere decomposition

In order to diagonalize the mass terms (1.6) and (1.7) that involve trace of matrix fields, it is convenient to introduce a method called fuzz sphere decomposition. Noticing that these traces contain commutator $[t_i^{(k)}, \phi]$, it is natural to look at eigenstates of the operator L_i, which acts on a $k \times k$ matrix ϕ as

$$L_i \phi = [t_i, \phi]. \tag{1.9}$$

L_i forms a representation of su(2) because

$$[L_i, L_j]\phi = [t_i, [t_j, \phi]] - [t_j, [t_i, \phi]] = [[t_i, t_j], \phi] = i\varepsilon_{ijk}L_k\phi. \tag{1.10}$$

This representation is highly reducible and can be reduced by explicitly constructing the basis Y_l^m of a $k \times k$ matrix, which is the eigenstate of the operators $L^2 = L_i L_i$ and L_3

$$\begin{aligned} L^2 Y_l^m &= l(l+1)Y_l^m, \\ L_3 Y_l^m &= m Y_l^m. \end{aligned} \tag{1.11}$$

Here the indices are $l = 0, 1, \ldots, k-1$ and $m = -l, \ldots, l$. Hence the total number of Y_l^m is k^2. Further they have the properties

$$L_i Y_l^m = Y_l^{m'}[t_i^{(2l+1)}]_{m',m}, \tag{1.12}$$

where $t_i^{(2l+1)}$ is the $(2l+1)$-d representation of su(2). By scaling the basis properly, the orthogonality relation can be normalized and hence

$$\mathrm{tr}[Y_l^{m\dagger} Y_{l'}^{m'}] = \delta_{ll'}\delta_{mm'}, \tag{1.13}$$

where

$$Y_l^{m\dagger} = (-1)^m Y_l^{-m}, \tag{1.14}$$

which is consistent with

$$L_3 Y_l^{m\dagger} = [t_3, Y_l^{m\dagger}] = -\left([t_3, Y_l^m]\right)^\dagger = -m Y_l^{m\dagger}. \tag{1.15}$$

For a general complex k-d matrix ϕ, it has k^2 complex degrees of freedom. It can be decomposed as

$$\phi = \phi_{l,m} Y_l^m, \tag{1.16}$$

with k^2 independent complex coefficients $\phi_{l,m}$. Because

$$\phi^\dagger = \phi_{l,m}^* Y_l^{m\dagger} = \phi_{l,m}^*(-1)^m Y_l^{-m}, \tag{1.17}$$

for a Hermitian matrix ϕ, $\phi^\dagger = \phi$ implies

$$\phi_{l,m}^*(-1)^m = \phi_{l,-m}. \tag{1.18}$$

From now on we consider ϕ as a Hermitian matrix. Notice that this decomposition is nondegenerate and hence by specifing the eigenvalues of L^2 and L_3 we can determine a unique degree of freedom in the matrix ϕ. For example, from

$$\begin{aligned} L^2 I = 0, \quad L_3 I = 0, \\ \mathrm{tr}\left(\frac{1}{\sqrt{k}} I \frac{1}{\sqrt{k}} I\right) = 1, \end{aligned} \tag{1.19}$$

we find $Y_{l=0}^{m=0} = \frac{1}{\sqrt{k}} I$.

Now we consider the case with the definition of L_i

$$L_i = [t_i^{(k)} \oplus 0^{(N-k)}, \quad], \tag{1.20}$$

where $t_i^{(k)}$ is the k-d representation of su(2) and $0^{(N-k)}$ is $(N-k)$-d zero matrix. Then

$$L_i \phi = \left[\begin{pmatrix} t_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} \right] = \begin{pmatrix} [t_i, A] & t_i B \\ -B^\dagger t_i & 0 \end{pmatrix}. \tag{1.21}$$

We decompose A, which is Hermitian, by basis Y_l^m,

$$A = \phi_{l,m} Y_l^m. \tag{1.22}$$

We decompose $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ by basis $E^n{}_a$,

$$B = \phi_{n,a} E^n{}_a, \tag{1.23}$$

where the indices are $n = 1, 2, \ldots, k$ and $a = k+1, k+2, \ldots, N$. $E^n{}_a$ is a N-d matrix defined as

$$(E^M{}_N)_{P,Q} = \delta_{M,P}\delta_{N,Q}, \tag{1.24}$$

and hence

$$E^M{}_N E^P{}_Q = \delta^P{}_N E^M{}_Q. \tag{1.25}$$

The orthogonality relation is

$$\mathrm{tr}\left(E^M{}_N E^P{}_Q\right) = \delta^P{}_N \delta^M{}_Q. \tag{1.26}$$

Equivalently, we have

$$\mathrm{tr}\left(E^{N\dagger}_{\ \ M} E^P{}_Q\right) = \delta^{NP}\delta_{MQ}, \tag{1.27}$$

that is

$$\mathrm{tr}\left(E^{n\dagger}_{\ \ a} E^{n'}{}_{a'}\right) = \delta^{nn'}\delta_{aa'}. \tag{1.28}$$

These matrices have the property

$$(L_i E^n{}_a)_{pq} = (t_i)_{pk}(E^n{}_a)_{kq} = (t_i)_{pn}\delta_{aq} = (t_i)_{n'n}\delta_{n'p}\delta_{aq} = (E^{n'}{}_a)_{pq}(t_i)_{n'n}. \tag{1.29}$$

Therefore, matrices $E^n{}_a$ with $n = 1, 2, \ldots, k$ form a k-d representation of su(2). Similarly for $E^a{}_n$, which is used to decompose $\begin{pmatrix} 0 & 0 \\ B^\dagger & 0 \end{pmatrix}$,

$$(L_i E^a{}_n)_{pq} = -(E^a{}_n)_{pk}(t_i)_{kq} = -\delta_{ap}(t_i)_{nq} = -(t_i)_{nn'}\delta_{ap}\delta_{n'q} = -(t_i)_{nn'}(E^a{}_{n'})_{pq}. \tag{1.30}$$

For a particular a, these matrices also form a k-d representation of su(2). The minus sign on the right hand side is consistent with t_i on the left,

$$\begin{aligned} [L_i, L_j]E^n{}_a &= E^{n'}{}_a [t_i, t_j]_{n'n} = E^{n'}{}_a i\varepsilon_{ijk}(t_k)_{n'n} = i\varepsilon_{ijk} L_k E^n{}_a, \\ [L_i, L_j]E^a{}_n &= [-t_j, -t_i]_{nn'} E^a{}_{n'} = -i\varepsilon_{ijk}(t_k)_{nn'} E^a{}_{n'} = i\varepsilon_{ijk} L_k E^a{}_n. \end{aligned} \tag{1.31}$$

In summary,

$$\begin{aligned} L_i E^n{}_a &= E^{n'}{}_a (t_i)_{n'n}, \\ L_i E^a{}_n &= -(t_i)_{nn'} E^a{}_{n'}. \end{aligned} \tag{1.32}$$

We can decompose $\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$ by basis $E^a{}_{a'}$. However since L_i annihilates it, it is not necessary. Further, these bases are orthogonal to each other,

$$\mathrm{tr}(Y_l^m E^a{}_n) = 0 = \mathrm{tr}(Y_l^m E^n{}_a). \tag{1.33}$$

Here and in the following, the Y_l^m should be undertstood as $(Y_l^m)^{(k)} \oplus 0^{(N-k)}$. A general Hermitian matrix ϕ can be decomposed as

$$\phi = \phi_{l,m} Y_l^m + \phi_{n,a} E^n{}_a + \phi_{a,n} E^a{}_n + \phi_{a,a'} E^a{}_{a'}. \tag{1.34}$$

The coefficients satisfy

$$\begin{aligned} \phi^*_{l,m}(-1)^m &= \phi_{l,-m}, \\ \phi^*_{a,n} &= \phi_{n,a}, \\ \phi^*_{a,a'} &= \phi_{a',a}. \end{aligned} \tag{1.35}$$

Consider a mass term

$$\begin{aligned} \mathrm{tr}\left[\phi L^2 \phi\right] &= \mathrm{tr}\left[\phi^\dagger L^2 \phi\right] \\ &= l(l+1)\phi^*_{l,m}\phi_{l,m} + \frac{k^2-1}{4}\phi^*_{n,a}\phi_{n,a} + \frac{k^2-1}{4}\phi^*_{a,n}\phi_{a,n} \\ &= l(l+1)\phi^\dagger_{l,m}\phi_{l,m} + 2\frac{k^2-1}{4}\phi^\dagger_{n,a}\phi_{n,a}. \end{aligned} \tag{1.36}$$

In the third line, to make it field theory we change $*$ to $\dagger$. We write the 2 in the second term explicitly because usually there is a 2 in the kinetic term.

1.4 Diagonalization of the mass terms

In the papers [2, 3], they diagonalized the mass terms (1.6) and (1.7) explicitly. By using the operator L_i, the mass term of the bosonic part can be written as

$$\begin{aligned} S_{\mathrm{m,b}} = \frac{2}{g^2_{\mathrm{YM}}} \int d^4x \frac{1}{x_3^2} \mathrm{tr}\Big[&-\frac{1}{2}\phi_i L^2 \phi_i - \frac{1}{2} A_3 L^2 A_3 + i\varepsilon_{ijk}\phi_i L_j \phi_k \\ &+ i\phi_i L_i A_3 - iA_3 L_i \phi_i - \frac{1}{2}\phi_\alpha L^2 \phi_\alpha - \frac{1}{2} A_i L^2 A_i \Big]. \end{aligned} \tag{1.37}$$

Here, repeated indices are summed over $i, j, k = 1, 2, 3$ and $\alpha = 4, 5, 6$ implicitly and A_i should be understood as the components A_0, A_1, A_2. The last two terms are decoupled from the rest of the mass terms, then they are diagonalized after using the fuzz sphere decomposition. Hence we call fields $\phi_4, \phi_5, \phi_6, A_0, A_1$ and A_2 the easy fields and call fields ϕ_1, ϕ_2, ϕ_3 and A_3, which are coupled together, the complicated fields.

To diagonalize the mass terms of the complicated fields, define a field $C^T = (\phi_1, \phi_2, \phi_3, A_3)$ and Hermitian matrices S_i

$$\begin{aligned} (S_i)_{jk} &= -\frac{1}{2} i\varepsilon_{ijk}, \\ (S_i)_{j,4} = \frac{1}{2} i\delta_{ij}, &\quad (S_i)_{4,j} = -\frac{1}{2} i\delta_{ij}. \end{aligned} \tag{1.38}$$

Then the mass terms can be written as

$$S_{\mathrm{m,b}} = \frac{2}{g_{\mathrm{YM}}^2} \int d^4x \frac{1}{x_3^2} \mathrm{tr}\left[C^T(-\frac{1}{2}L^2 + 2S_i L_i)C \right]. \tag{1.39}$$

Notice the operator L_i, which acts on the color indices of the field C, has the su(2) algebra $[L_i, L_j] = i\epsilon_{ijk}L_k$. The operator S_i acting on the flavor indices also satisfies the su(2) Lie algebra

$$[S_i, S_j] = i\varepsilon_{ijk} S_k. \tag{1.40}$$

A general degree of freedom of C can be labeled by using the eigenvalues of L^2, L_3, S^2 and S_3. Because both L_i and S_i satisfy the su(2) algebra, we can introduce operator $J_i = L_i + S_i$, which satisfies the su(2) algebra. Therefore we can specify a degree of freedom by the eigenvalues of J^2, J_3, L^2 and S^2. Then the operator $-\frac{1}{2}L^2 + 2S_iL_i = -\frac{1}{2}L^2 + J^2 - L^2 - S^2$ are diagonalized and the mass coupled problem can be resolved. Similar method can be applied to the fermionic part.

1.5 The propagators

After resolving the mass coupled problem, the propagators can be found explicitly [2, 3] and we list the result here. The fuzz sphere decomposition of a field is

$$\phi = \phi_{lm} Y_l^m + \phi_{na} E^n{}_a + \phi_{an} E^a{}_n + \phi_{aa'} E^a{}_{a'}. \tag{1.41}$$

Consider the degrees of freedom coming from Y_l^m. For the bosonic easy fields ϕ_4, ϕ_5, ϕ_6, A_0, A_1 and A_2, propagators are the same as

$$\langle(\phi_4)_{lm}(\phi_4)^\dagger_{l'm'}\rangle = \delta_{ll'}\delta_{mm'}K^{l(l+1)}. \tag{1.42}$$

Here we make the spacetime coordinates implicit and the above formula should be understood as $\langle(\phi_4)_{lm}(x)(\phi_4)^\dagger_{l'm'}(y)\rangle = \delta_{ll'}\delta_{mm'}K^{l(l+1)}(x,y)$. $K^{l(l+1)}$ is a function whose property will be introduced later. For bosonic complicated fields, the propagators are

$$\begin{aligned}\langle(\phi_i)_{lm}(\phi_j)^\dagger_{l'm'}\rangle = \delta_{ij}\delta_{ll'}\delta_{mm'}&\left(\frac{l+1}{2l+1}K^{l(l-1)} + \frac{l}{2l+1}K^{(l+1)(l+2)}\right)\\ &-i\epsilon_{ijk}(t_k)_{mm'}\delta_{ll'}\frac{1}{2l+1}\left(K^{l(l-1)} - K^{(l+1)(l+2)}\right),\end{aligned} \tag{1.43}$$

$$\langle(A_3)_{lm}(A_3)^\dagger_{l'm'}\rangle = \delta_{ll'}\delta_{mm'}\left(\frac{l+1}{2l+1}K^{l(l-1)} + \frac{l}{2l+1}K^{(l+1)(l+2)}\right), \tag{1.44}$$

$$\langle(\phi_i)_{lm}(A_3)^\dagger_{l'm'}\rangle = i\delta_{ll'}(t_i)_{mm'}\frac{1}{2l+1}\left(K^{l(l-1)} - K^{(l+1)(l+2)}\right). \tag{1.45}$$

The propagators for fermions are

$$\begin{aligned}\langle(\psi_i)_{lm}\overline{(\psi_j)}_{l'm'}\rangle = \delta_{ij}\delta_{mm'}\delta_{ll'}&\left(\frac{l+1}{2l+1}K_F^{-l} + \frac{l}{2l+1}K_F^{l+1}\right)\\ &-\delta_{ll'}[G^k]_{ij}(t_k)_{mm'}\frac{1}{2l+1}\left(K_F^{-l} - K_F^{l+1}\right).\end{aligned} \tag{1.46}$$

Now consider the degrees of freedom come from $E^n{}_a$ and $E^a{}_n$. For the bosonic easy fields, the propagators are

$$\langle(\phi_4)_{na}(\phi_4)^\dagger_{n'a'}\rangle = \delta_{nn'}\delta_{aa'}K^{\frac{k^2-1}{4}}. \tag{1.47}$$

For bosonic complicated fields, the propagators are

$$\begin{aligned}\langle(\phi_i)_{na}(\phi_j)^\dagger_{n'a'}\rangle = \delta_{ij}\delta_{nn'}\delta_{aa'}&\left(\frac{k+1}{2k}K^{\frac{(k-2)^2-1}{4}} + \frac{k-1}{2k}K^{\frac{(k+2)^2-1}{4}}\right)\\ &-i\epsilon_{ijk}(t_k)_{nn'}\delta_{aa'}\frac{1}{k}\left(K^{\frac{(k-2)^2-1}{4}} - K^{\frac{(k+2)^2-1}{4}}\right),\end{aligned} \tag{1.48}$$

$$\langle(A_3)_{na}(A_3)^\dagger_{n'a'}\rangle = \delta_{nn'}\delta_{aa'}\left(\frac{k+1}{2k}K^{\frac{(k-2)^2-1}{4}} + \frac{k-1}{2k}K^{\frac{(k+2)^2-1}{4}}\right), \tag{1.49}$$

$$\langle(\phi_i)_{na}(A_3)^\dagger_{n'a'}\rangle = i\delta_{aa'}(t_i)_{nn'}\frac{1}{k}\left(K^{\frac{(k-2)^2-1}{4}} - K^{\frac{(k+2)^2-1}{4}}\right). \tag{1.50}$$

The propagators for fermions are

$$\begin{aligned}\langle(\psi_i)_{na}\overline{(\psi_j)}_{n'a'}\rangle = \delta_{aa'}\delta_{ij}\delta_{nn'}\frac{1}{k}\left(\frac{k+1}{2}K_F^{-\frac{k-1}{2}} + \frac{k-1}{2}K_F^{\frac{k+1}{2}}\right) \\ -\delta_{aa'}[G^k]_{ij}(t_k)_{nn'}\frac{1}{k}\left(K_F^{-\frac{k-1}{2}} - K_F^{\frac{k+1}{2}}\right).\end{aligned} \tag{1.51}$$

Notice that there is no correlation between the degrees of freedom from Y_l^m and $E^n_{\ a}$(or $E^a_{\ n}$). Hence they contribute to the one-loop one-point function separately. We don't need the propagators from $E^a_{\ a'}$ since they don't contribute to the one-loop one-point function.

1.6 Dimensional reduction regularization

In SYM, the dimensional regularization breaks the supersymmetry since it only change the number of degrees of freedom of gauge bosons, then makes the number of degrees of freedom of fermions and bosons unequal. In our case the 4-dimensional $\mathcal{N} = 4$ SYM, consider the on-shell degrees of freedom. There are one vector (matrix) gauge field A_μ with degrees of freedom of 2, six scalar fields with degrees of freedom 6 and four Majorana fermions with degrees of freedom of $4 \times 2 = 8$. Therefore we have equal numbers of fermions and bosons. However, after dimensional regularization to $D = 4 - 2\epsilon$, the degrees of freedom of the vector field is changed to $2 - 2\epsilon$ and other fields are the same. So the numbers of bosons and fermions are not equal since $8 - 2\epsilon \neq 8$.

Dimensional reduction regularization is a usual way to resolve this problem. The $D = 4$ $\mathcal{N} = 4$ SYM can be obtained from a dimensional reduction of $D = 10$ $\mathcal{N} = 1$ SYM. By dimensional reducing it to $D = 4 - 2\epsilon$, the number of scalar fields is $6 + 2\epsilon$. Hence the fermions and bosons have the same degrees of freedom $8 = 6 + 2\epsilon + 2 - 2\epsilon$.

In our case, the dimensional reduction should be adapted to the present of the defect. We keep the defect as codimension one with the dimension $3 - 2\epsilon$. The gauge field four components have been separated into easy fields, which are A_0, A_1, A_2 and complicated field, which is A_3, with the total number $n_A = n_{A,\text{easy}} + n_{A,\text{com}} = 3 + 1$. In the dimensional reduction, since the defect is codimension one, we set $n_{A,\text{easy}} = 3 - 2\epsilon$ and $n_{A,\text{com}} = 1$. Then $n_A = n_{A,\text{easy}} + n_{A,\text{com}} = 4 - 2\epsilon$. Perturbations of the scalars that have nonvanishing backgrounds are complicated fields and those have vanishing backgrounds are easy fields. If we keep the number of scalars that have nonvanishing backgrounds as 3, the separations are $n_{\phi,\text{com}} = 3$

and $n_{\phi,\text{easy}} = 3 + 2\epsilon$. It is a choice consistent with the classical equation of motion and the Nahm condition [3].

In dimension $D = 4 - 2\epsilon$, the regularized bosonic propagators have the property [3]

$$K^{\nu}(x,x) = \frac{g_{\text{YM}}^2}{2}\frac{1}{16\pi^2 x_3^2}\left(m^2\left[-\frac{1}{\epsilon}-\log(4\pi)+\gamma_E-2\log(x_3)+2\Psi(\nu+\frac{1}{2})-1\right]-1\right), \tag{1.52}$$

where γ_E is the Euler-Mascheroni constant and Ψ is the digamma function. For propagators of fermions, we have

$$\begin{aligned}\text{tr}K_F^m(x,x) =& \text{sign}(m)\frac{g_{\text{YM}}^2}{2}\frac{1}{4\pi^2 x_3^3}\Big[|m|^3 + |m|^2 - 3|m| - 1 \\ &+ |m|(|m|^2-1)\Big(-\frac{1}{\epsilon} - \log(4\pi) + \gamma_E - 2\log(x_3) + 2\Psi(|m|) - 2\Big)\Big].\end{aligned} \tag{1.53}$$

Chapter 2

Lollipop diagram

2.1 A useful formula of the fuzz sphere Harmonics

Any complex k-d matrix can be decomposed by the basis Y_l^m, where $l = 0, 1, \ldots, k-1$, as a complex vector space. Now consider the matrix

$$\sum_m Y_l^{m\dagger} Y_{l'}^{m'} Y_l^m, \tag{2.1}$$

where the indices l are not summed over. By using (1.12), which is

$$[t_i, Y_l^m] = Y_l^{m'} (t_i)_{m'm}, \tag{2.2}$$

where $(t_i)_{m'm}$ should be understood as $(t_i^{(2l+1)})_{m'm}$, we have

$$[t_i, Y_l^{m\dagger}] = -[t_i, Y_l^m]^\dagger = -Y_l^{m'\dagger} (t_i)^*_{m'm} = -(t_i)_{mm'} Y_l^{m'\dagger}. \tag{2.3}$$

Therefore

$$\begin{aligned}
&[t_i, \sum_m Y_l^{m\dagger} Y_{l'}^{m'} Y_l^m] \\
&= \sum_{mm''} -(t_i)_{mm''} Y_l^{m''\dagger} Y_{l'}^{m'} Y_l^m + \sum_{mm''} (t_i)_{m''m} Y_l^{m\dagger} Y_{l'}^{m'} Y_l^{m''} \\
&\quad + \sum_m Y_l^{m\dagger} [t_i, Y_{l'}^{m'}] Y_l^m \\
&= \sum_m Y_l^{m\dagger} [t_i, Y_{l'}^{m'}] Y_l^m.
\end{aligned} \tag{2.4}$$

It implies that $\sum_m Y_l^{m\dagger} Y_{l'}^{m'} Y_l^m$ has the same transformation property as $Y_{l'}^{m'}$. From the nondegenerate of the eigenvalues l, m of Y_l^m, we have

$$\sum_m Y_l^{m\dagger} Y_{l'}^{m'} Y_l^m = C(l, l', m') Y_{l'}^{m'}, \tag{2.5}$$

where $C(l,l',m')$ is a constant, which may depend on l, l' and m'. By applying a unitary transformation $U = e^{-i\lambda_i t_i}$,

$$\begin{aligned} C(l,l',m')Y_{l'}^{\tilde{m}'} &= U\left(\sum_m Y_l^{m\dagger} Y_{l'}^{m'} Y_l^m\right)U^\dagger \\ &= \sum_m UY_l^{m\dagger}U^\dagger UY_{l'}^{m'}U^\dagger UY_l^m U^\dagger \\ &= \sum_{\tilde{m}} Y_l^{\tilde{m}\dagger} Y_{l'}^{\tilde{m}'} Y_l^{\tilde{m}} = C(l,l',\tilde{m}')Y_{l'}^{\tilde{m}'}, \end{aligned} \tag{2.6}$$

which implies that $C(l,l',m')$ can not depend on m'. Then equation (2.5) becomes

$$\sum_m Y_l^{m\dagger} Y_{l'}^{m'} Y_l^m = C(l,l')Y_{l'}^{m'}. \tag{2.7}$$

In our calculation, we are interested in

$$\sum_m [Y_l^{m\dagger}, [t_i, Y_l^m]] = C_{li} t_i. \tag{2.8}$$

To calculate C_{li}, we first define a constant

$$l^2 + (l-1)^2 + ... + (-l+1)^2 + (-l)^2 = c_l. \tag{2.9}$$

We have

$$\mathrm{tr}(t_i^{(2l+1)} t_i^{(2l+1)}) = \mathrm{tr}(t_3^{(2l+1)} t_3^{(2l+1)}) = c_l. \tag{2.10}$$

Here indices i are not summed over. It implies that the trace doesn't depend on indices i. From

$$\mathrm{tr}(t_i^{\frac{N-1}{2}} t_i^{\frac{N-1}{2}}) = c_{\frac{N-1}{2}}, \tag{2.11}$$

and

$$\begin{aligned} \sum_m \mathrm{tr}(t_i[Y_l^{m\dagger}, [t_i, Y_l^m]]) &= \sum_m \mathrm{tr}([t_i, Y_l^{m\dagger}][t_i, Y_l^m]) \\ &= \sum_m (-1)(t_i)_{mm'}(t_i)_{m''m}\mathrm{tr}(Y_l^{m'\dagger} Y_l^{m''}) = (-1)\mathrm{tr}(t_i^{2l+1} t_i^{2l+1}) = -c_l, \end{aligned} \tag{2.12}$$

we have

$$\sum_m [Y_l^{m\dagger}, [t_i, Y_l^m]] = -\frac{c_l}{c_{\frac{N-1}{2}}} t_i = C_l t_i. \tag{2.13}$$

Notice that the constant C_l doesn't depend on i, which is consistent with (2.7).

2.2 One-loop effective action for one-point function

By contracting two fields in the cubic interaction terms, we will get the effective action that will contribute to the one-point function. Because there is no propagator that correlates the degrees of freedom from the Y_l^m and the $E^n{}_a$ ($E^a{}_n$) parts, the contributions from these two parts can be calculated separately.

2.2.1 Contribution from the Y_l^m part

Based on the expectation values calculated in Appendix (A), we summarize the contribution from the Y_l^m part in this subsection. Consider the contribution from $\mathrm{tr}(i[A^\mu, A^\nu]\partial_\mu A_\nu)$. We have

$$\mathrm{tr}(i[\hat{A}^\mu, \hat{A}^\nu]\partial_\mu A_\nu) = 0. \tag{2.14}$$

Here and in the following we use hats to indicate the two fields contracted together by a propagator. Because $\langle A^\mu A^\nu \rangle \neq 0$ only when $\mu = \nu$, antisymmetry of $\mu\nu$ makes the above contribution zero. Further, we have

$$\mathrm{tr}(i[A^\mu, \hat{A}^\nu]\partial_\mu \hat{A}_\nu) = \mathrm{tr}(iA^\mu[\hat{A}^\nu, \partial_\mu \hat{A}_\nu]) = \mathrm{tr}(iA^\mu \frac{1}{2}\partial_\mu[\hat{A}^\nu, \hat{A}_\nu]) = 0, \tag{2.15}$$

and similarly

$$\mathrm{tr}(i[\hat{A}^\mu, A^\nu]\partial_\mu \hat{A}_\nu) = 0. \tag{2.16}$$

Then the total contribution from $\mathrm{tr}(i[A^\mu, A^\nu]\partial_\mu A_\nu)$ is zero.

Consider the contribution from term $\mathrm{tr}([\phi_i^{\mathrm{cl}}, \phi_j][\phi_i, \phi_j])$. We have

$$\mathrm{tr}([\phi_i^{\mathrm{cl}}, \hat{\phi}_j][\phi_i, \hat{\phi}_j])_{\mathrm{easy}} = -\frac{1}{y_3}K^{l(l+1)}C_l\mathrm{tr}(\phi_i t_i)n_{\phi,\mathrm{easy}}, \tag{2.17}$$

and

$$\begin{aligned}\mathrm{tr}([\phi_i^{\mathrm{cl}}, \hat{\phi}_j][\phi_i, \hat{\phi}_j])_{\mathrm{com}} &= -\frac{1}{y_3}\mathrm{tr}(\phi_i[\hat{\phi}_j[t_i, \hat{\phi}_j]])_{\mathrm{com}} \\ &= -\frac{1}{y_3}\left(\frac{l+1}{2l+1}K^{l(l-1)} + \frac{l}{2l+1}K^{(l+1)(l+2)}\right)C_l\mathrm{tr}(\phi_i t_i)n_{\phi,\mathrm{com}}.\end{aligned} \tag{2.18}$$

Further, we have

$$\begin{aligned}\mathrm{tr}([\phi_i^{\mathrm{cl}}, \phi_j][\hat{\phi}_i, \hat{\phi}_j]) &= \frac{1}{y_3}\mathrm{tr}([\hat{\phi}_i, \hat{\phi}_j][t_i, \phi_j])_{\mathrm{com}} \\ &= -\frac{1}{y_3}(-1)\frac{n_{\phi,\mathrm{com}}-1}{2l+1}(K^{l(l-1)} - K^{(l+1)(l+2)})C_l\mathrm{tr}(t_i\phi_i).\end{aligned} \tag{2.19}$$

Here, the contribution only comes from the complicated part. Furthermore,

$$\begin{aligned}\mathrm{tr}([\phi_i^{\mathrm{cl}},\hat{\phi}_j][\hat{\phi}_i,\phi_j]) &= -\frac{1}{y_3}\mathrm{tr}(\hat{\phi}_i[\phi_j[t_i,\hat{\phi}_j]]) \\ &= -\frac{1}{y_3}\Big\{-\left(\frac{l+1}{2l+1}K^{l(l-1)}+\frac{l}{2l+1}K^{(l+1)(l+2)}\right)C_l\mathrm{tr}(\phi_i t_i) \\ &\quad -(n_{\phi,\mathrm{com}}-1)\frac{1}{2l+1}(K^{l(l-1)}-K^{(l+1)(l+2)})C_l\mathrm{tr}(\phi_i t_i)\Big\}.\end{aligned} \tag{2.20}$$

Consider the contribution from $\mathrm{tr}([A_\mu,\phi_i^{\mathrm{cl}}][A^\mu,\phi_i])$. We have

$$\mathrm{tr}([\phi_i^{\mathrm{cl}},\hat{A}_\mu][\phi_i,\hat{A}_\mu])_{\mathrm{easy}} = -\frac{1}{y_3}K^{l(l+1)}C_l\mathrm{tr}(\phi_i t_i)n_{A,\mathrm{easy}}, \tag{2.21}$$

and

$$\begin{aligned}&\mathrm{tr}([\phi_i^{\mathrm{cl}},\hat{A}_\mu][\phi_i,\hat{A}_\mu])_{\mathrm{com}} \\ &= -\frac{1}{y_3}\left(\frac{l+1}{2l+1}K^{l(l-1)}+\frac{l}{2l+1}K^{(l+1)(l+2)}\right)C_l\mathrm{tr}(\phi_i t_i)n_{A,\mathrm{com}}.\end{aligned} \tag{2.22}$$

Further, we have

$$\mathrm{tr}([\phi_i^{\mathrm{cl}},A_\mu][\hat{\phi}_i,\hat{A}_\mu]) = 0, \tag{2.23}$$

$$\mathrm{tr}([\phi_i^{\mathrm{cl}},\hat{A}_\mu][\hat{\phi}_i,A_\mu]) = \mathrm{tr}(\phi_i^{\mathrm{cl}}[\hat{A}_\mu[\hat{\phi}_i,A_\mu]]) = 0. \tag{2.24}$$

The last equality is due to the Jacobi identity.

Consider the contribution from $\mathrm{tr}(i[A^\mu,\phi_i]\partial_\mu\phi_i)$. We have

$$\begin{aligned}\mathrm{tr}(i[\hat{A}^\mu,\hat{\phi}_i]\partial_\mu\phi_i) &= \mathrm{tr}(i[\hat{A}^3,\hat{\phi}_i]\partial_3\phi_i)n_{A,\mathrm{com}} \\ &= -\frac{1}{2l+1}(K^{l(l-1)}-K^{(l+1)(l+2)})C_l\mathrm{tr}(t_i\partial_3\phi_i)n_{A,\mathrm{com}}.\end{aligned} \tag{2.25}$$

Further, we have

$$\begin{aligned}\mathrm{tr}(i[\hat{A}^\mu,\phi_i]\partial_\mu\hat{\phi}_i) &= \mathrm{tr}(i[\hat{A}^3,\phi_i]\partial_3\hat{\phi}_i)n_{A,\mathrm{com}} \\ &= \frac{1}{2}\frac{1}{2l+1}\partial_3(K^{l(l-1)}-K^{(l+1)(l+2)})C_l\mathrm{tr}(t_i\phi_i)n_{A,\mathrm{com}}.\end{aligned} \tag{2.26}$$

Furthermore,

$$\mathrm{tr}(i[A^\mu,\hat{\phi}_i]\partial_\mu\hat{\phi}_i) = \mathrm{tr}(iA^\mu[\hat{\phi}_i,\partial_\mu\hat{\phi}_i]) = \mathrm{tr}(iA^\mu\frac{1}{2}\partial_\mu[\hat{\phi}_i,\hat{\phi}_i]) = 0. \tag{2.27}$$

Consider the contribution from $\mathrm{tr}(-\bar{c}[\phi_i^{\mathrm{cl}},[\phi_i,c]])$. We have

$$\mathrm{tr}(-\hat{\bar{c}}[\phi_i^{\mathrm{cl}},[\phi_i,\hat{c}]]) = \mathrm{tr}([\phi_i^{\mathrm{cl}},\hat{\bar{c}}][\phi_i,\hat{c}]) = -\frac{1}{y_3}K^{l(l+1)}C_l\mathrm{tr}(\phi_i t_i)(-n_c). \tag{2.28}$$

It is similar to $\mathrm{tr}([\phi_i^{\mathrm{cl}}, \hat{\phi}_j][\phi_i, \hat{\phi}_j])_{\mathrm{easy}}$ except the minus sign for the fermion loop.

Consider the contribution

$$\mathrm{tr}(i\partial_\mu \hat{\bar{c}}[A^\mu, \hat{c}]) = 0, \tag{2.29}$$

which is similar to $\mathrm{tr}(i[A^\mu, \hat{\phi}_i]\partial_\mu \hat{\phi}_i) = 0$.

Consider the contribution from vertices

$$\frac{1}{2}\sum_{i=1}^{3} \mathrm{tr}(\bar{\psi} G^i[\phi_i, \psi]) + \frac{1}{2}\sum_{i=4}^{3+n_{\phi,\mathrm{easy}}} \mathrm{tr}(\bar{\psi} G^i[\phi_i, \gamma_5\psi]) + \frac{1}{2}\mathrm{tr}(\bar{\psi}\gamma^\mu[A_\mu, \psi]). \tag{2.30}$$

Only the first term has contribution

$$\frac{1}{2}\mathrm{tr}(\hat{\bar{\psi}} G^i[\phi_i, \hat{\psi}]) = -\frac{1}{2} n_\psi \frac{1}{2l+1}(K_F^{-l} - K_F^{l+1}) C_l \mathrm{tr}(\phi_i t_i). \tag{2.31}$$

Finally the one-loop effective action from the Y_l^m part that will contribute to the one-point function is

$$\begin{aligned}
V_{\mathrm{eff}}^{lm}(y) = &-\frac{1}{y_3} K^{l(l+1)} C_l \mathrm{tr}(\phi_i t_i) n_{\mathrm{easy}} \\
&- \frac{1}{y_3}\left(\frac{l+1}{2l+1}K^{l(l-1)} + \frac{l}{2l+1}K^{(l+1)(l+2)}\right) C_l \mathrm{tr}(\phi_i t_i)(n_{\phi,\mathrm{com}} + n_{A,\mathrm{com}} - 1) \\
&- \frac{1}{y_3}(-\frac{3}{2})\frac{1}{2l+1}(K^{l(l-1)} - K^{(l+1)(l+2)}) C_l \mathrm{tr}(t_i \phi_i)(n_{\phi,\mathrm{com}} - 1) \\
&- \frac{1}{2l+1}(K^{l(l-1)} - K^{(l+1)(l+2)}) C_l \mathrm{tr}(t_i \partial_3 \phi_i) n_{A,\mathrm{com}} \\
&+ \frac{1}{2}\frac{1}{2l+1}\partial_3(K^{l(l-1)} - K^{(l+1)(l+2)}) C_l \mathrm{tr}(t_i \phi_i) n_{A,\mathrm{com}} \\
&- \frac{1}{2} n_\psi \frac{1}{2l+1}(\mathrm{tr} K_F^{-l} - \mathrm{tr} K_F^{l+1}) C_l \mathrm{tr}(\phi_i t_i),
\end{aligned} \tag{2.32}$$

where $n_{\mathrm{easy}} = n_{\phi,\mathrm{easy}} + n_{A,\mathrm{easy}} - n_c$. The function K here should be understood as the regularized $K(y, y)$ (1.52)(1.53). Further, the fourth term can be integrated by parts, since we consider the spacetime integral of the effective action.

Then the effective action becomes

$$
\begin{aligned}
V_{\text{eff}}^{lm} = -\frac{1}{y_3}C_l \text{tr}(\phi_i t_i)\frac{1}{l}\Big\{ & \Big[-\frac{1}{\epsilon} - \log(4\pi) + \gamma_E - 2\log(x_3) + 2\Psi(l) - 1\Big] \\
& \times\Big[(l^3+l^2)(n_{\text{easy}} + n_{\phi,\text{com}} + n_{A,\text{com}} - 2n_\psi - 1) \\
& \qquad + 3l(n_{\phi,\text{com}} - 2n_{A,\text{com}} - 1)\Big] \\
& + 2l^2(n_{\text{easy}} + n_{\phi,\text{com}} + n_{A,\text{com}} - 2n_\psi - 1) \\
& + l(n_{\text{easy}} + 6n_{\phi,\text{com}} - 3n_{A,\text{com}} - 2n_\psi - 12) \\
& + 6(n_{\phi,\text{com}} - 2n_{A,\text{com}} - 1)\Big\}.
\end{aligned}
\tag{2.33}
$$

In the dimensional reduction regularization, we have $n_{\text{easy}} = n_{\phi,\text{easy}} + n_{A,\text{easy}} - n_c = 3 + 2\epsilon + 3 - 2\epsilon - 1 = 5$, $n_{\phi,\text{com}} = 3$, $n_{A,\text{com}} = 1$ and $n_\psi = 4$. Notice that all these don't depend on ϵ. After substituting these, we have the one-loop effective action, which will contribute to one-point function,

$$
V_{\text{eff}}^{lm} = 0. \tag{2.34}
$$

2.2.2 Contribution from the ${E^n}_a$ (${E^a}_n$) part

From Appendix (B), we find the contribution from the ${E^n}_a$ (${E^a}_n$) part is the same as the contribution from the Y_l^m part if we make the correspondence

$$
k = 2l + 1, \qquad -2(N-k) = C_l. \tag{2.35}
$$

Therefore, from the result of the Y_l^m part we have

$$
\begin{aligned}
V_{\text{eff}}^{na}(y) = & \frac{2(N-k)}{y_3} K^{\frac{k^2-1}{4}} \text{tr}(\phi_i t_i) n_{\text{easy}} \\
& + \frac{N-k}{y_3}\Big(\frac{k+1}{k}K^{\frac{(k-2)^2-1}{4}} + \frac{k-1}{k}K^{\frac{(k+2)^2-1}{4}}\Big)\text{tr}(\phi_i t_i)(n_{\phi,\text{com}} + n_{A,\text{com}} - 1) \\
& - 3\frac{N-k}{y_3}\frac{1}{k}\Big(K^{\frac{(k-2)^2-1}{4}} - K^{\frac{(k+2)^2-1}{4}}\Big)\text{tr}(\phi_i t_i)(n_{\phi,\text{com}} - 1) \\
& + \frac{2(N-k)}{k}\Big(K^{\frac{(k-2)^2-1}{4}} - K^{\frac{(k+2)^2-1}{4}}\Big)\text{tr}(t_i \partial_3 \phi_i) n_{A,\text{com}} \\
& - \frac{(N-k)}{k}\partial_3\Big(K^{\frac{(k-2)^2-1}{4}} - K^{\frac{(k+2)^2-1}{4}}\Big)\text{tr}(t_i \phi_i) n_{A,\text{com}} \\
& + \frac{N-k}{k}\Big(\text{tr}K_F^{-\frac{k-1}{2}} - \text{tr}K_F^{\frac{k+1}{2}}\Big)\text{tr}(t_i \phi_i) n_\psi,
\end{aligned}
\tag{2.36}
$$

and finally the contribution is

$$V_{\text{eff}}^{na} = 0. \tag{2.37}$$

2.2.3 one-loop one-point function

The total contribution to the one-loop effective action is

$$V_{\text{eff}}^{lm} = V_{\text{eff}}^{lm} + V_{\text{eff}}^{na} = 0. \tag{2.38}$$

By connecting the field in the one-loop effective action to an external field, we get the one-loop one-point function. This diagram is called lollipop diagram. From equation (2.38), we find for any field the one-loop one-point function (lollipop diagram) is zero

$$\langle \phi \rangle_{1-\text{loop}} = 0. \tag{2.39}$$

Chapter 3

Conclusion

In this thesis, we have studied the lollipop diagram in a defect field theory with nontrivial background. This field theory is originated from the AdS/dCFT correspondence, where the gravity side is a D3-D5 brane system with the probe D5 branes carrying k units of flux. Due to the existence of the defect and the nontrivial background, half of the supersymmetries are broken. However, there may still be enough supersymmetries to make the contribution of lollipop diagrams to the one-point function zero. In [3], they prove that the contribuiton is indeed zero in the planar limit $N \to \infty$ and the probe limit $\frac{k}{N} \ll 1$. Further, based on their explicit result for $N, k < 9$, they argued that the contribution should be zero for arbitrary N and k. In this thesis, we proved the contribution of lollipop diagrams is zero for arbitrary N and k.

In the usual supersymmetric QFT, the zero contribution is due to the cancellation between the boson loops and fermion loops with the same mass. In our calculation, we calculated the contributions from the degrees of freedom from the Y_l^m part and the $E^n{}_a$ $(E^a{}_n)$ part separately. We found that for the Y_l^m part, the cancellation is between the boson loops and fermion loops with the same l and for the $E^n{}_a$ $(E^a{}_n)$ part, the cancellation is between the boson loops and fermion loops with same a. This is partly because the commutators with t_i mix Y_l^m of different m and mix $E^n{}_a$ $(E^a{}_n)$ of different n.

In our calculation, we used a modified version of dimensional reduction regularization from [3]. An explicit regularization is needed in this method. It would be interesting to find a proof based on the remaining supersymmetries and prove these results without any explicit regularization.

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Appendix A

Useful expectation values contributed from the Y_l^m part

In this appendix, we calculated some useful expectation values from the contribution of the degrees of freedom of the Y_l^m part. They will be useful to calculate one-loop effective action for one-point function.

• Contribution to $\mathrm{tr}([t_i,\hat{\phi}_4][\phi_i,\hat{\phi}_4])$

The hats means contraction of the two ϕ_4

$$\begin{aligned}
&\mathrm{tr}([t_i,\hat{\phi}_4][\phi_i,\hat{\phi}_4]) \\
&\quad = \langle(\phi_4)^\dagger_{l'm'}(\phi_4)_{lm}\rangle \mathrm{tr}(\phi_i[Y_{l'}^{m'\dagger}[t_i,Y_l^m]]) \\
&\quad = \sum_l K^{l(l+1)} \sum_m \mathrm{tr}(\phi_i[Y_l^{m\dagger}[t_i,Y_l^m]]) \\
&\quad = K^{l(l+1)} C_l \mathrm{tr}(\phi_i t_i).
\end{aligned} \tag{A.1}$$

In the last line, there is a sum over l implicitly.

• Contribution to $\mathrm{tr}(A^\mu[\hat{\phi}_4,\partial_\mu\hat{\phi}_4])$

Because

$$\lim_{x\to y}\langle\phi_4(x)\partial_\mu\phi_4(y)\rangle = \frac{1}{2}\partial_\mu \lim_{x\to y}\langle\phi_4(x)\phi_4(y)\rangle = \frac{1}{2}\partial_\mu\langle\phi_4^2\rangle, \tag{A.2}$$

and the antisymmetry of the commutator, we have

$$\mathrm{tr}(A^\mu[\hat{\phi}_4,\partial_\mu\hat{\phi}_4]) = 0. \tag{A.3}$$

• Contribution to $\mathrm{tr}(\phi_i[\hat{\phi}_1[t_i,\hat{\phi}_1]])$

We have

$$\mathrm{tr}(\phi_i[\hat{\phi}_1[t_i,\hat{\phi}_1]]) = \left(\frac{l+1}{2l+1}K^{l(l-1)} + \frac{l}{2l+1}K^{(l+1)(l+2)}\right) C_l \mathrm{tr}(\phi_i t_i). \tag{A.4}$$

• Contribution to $\mathrm{tr}(\hat{\phi}_i[\hat{\phi}_j[t_i,\phi_j]])$

Because of the t_i and the sum over i, the i, j here label the complicated fields

$$\begin{aligned}
&\mathrm{tr}(\hat{\phi}_i[\hat{\phi}_j[t_i,\phi_j]]) = \mathrm{tr}([\hat{\phi}_i,\hat{\phi}_j][t_i,\phi_j]) \\
&= \langle(\phi_i)^\dagger_{l'm'}(\phi_j)_{lm}\rangle_A \mathrm{tr}([Y_{l'}^{m'\dagger},Y_l^m][t_i,\phi_j]) \\
&= i\epsilon_{ijk}(t_k)_{mm'}\frac{1}{2l+1}(K^{l(l-1)}-K^{(l+1)(l+2)})\mathrm{tr}([Y_l^{m'\dagger},Y_l^m][t_i,\phi_j]) \\
&= i\epsilon_{ijk}\frac{1}{2l+1}(K^{l(l-1)}-K^{(l+1)(l+2)})\mathrm{tr}(C_l t_k[t_i,\phi_j]) \\
&= i\epsilon_{ijk}\frac{1}{2l+1}(K^{l(l-1)}-K^{(l+1)(l+2)})C_l i\epsilon_{kip}\mathrm{tr}(t_p\phi_j) \\
&= -\frac{n_{\phi,\mathrm{com}}-1}{2l+1}(K^{l(l-1)}-K^{(l+1)(l+2)})C_l\mathrm{tr}(t_i\phi_i).
\end{aligned} \tag{A.5}$$

In the fourth line we use

$$(t_k)_{mm'}[Y_l^{m'\dagger},Y_l^m] = [Y_l^{m'\dagger}[t_k,Y_l^{m'}]] = C_l t_k. \tag{A.6}$$

In the sixth line we use

$$\epsilon_{kij}\epsilon_{kip} = 2\delta_{jp} = (n_{\phi,\mathrm{com}}-1)\delta_{jp}, \tag{A.7}$$

since in our regularization method we have $n_{\phi,\mathrm{com}} = 3$.

- Contribution to $\mathrm{tr}(\hat{\phi}_i[\phi_j[t_i,\hat{\phi}_j]])$

$$\begin{aligned}
&\mathrm{tr}(\hat{\phi}_i[\phi_j[t_i,\hat{\phi}_j]]) = -\mathrm{tr}(\phi_j[\hat{\phi}_i[t_i,\hat{\phi}_j]]) \\
&\quad = -\langle(\phi_i)^\dagger_{l'm'}(\phi_j)_{lm}\rangle\mathrm{tr}(\phi_j[Y_{l'}^{m'\dagger}[t_i,Y_l^m]]).
\end{aligned} \tag{A.8}$$

For the complicated fields

$$\langle(\phi_i)^\dagger_{l'm'}(\phi_j)_{lm}\rangle = \delta_{ij}\delta_{ll'}\delta_{mm'}S_{\mathrm{com}} + i\epsilon_{ijk}(t_k)_{mm'}\delta_{ll'}A_{\mathrm{com}}, \tag{A.9}$$

where the symmetric and antisymmetric part is

$$\begin{aligned}
S_{\mathrm{com}} &= \frac{l+1}{2l+1}K^{l(l-1)} + \frac{l}{2l+1}K^{(l+1)(l+2)}, \\
A_{\mathrm{com}} &= \frac{1}{2l+1}\left(K^{l(l-1)}-K^{(l+1)(l+2)}\right).
\end{aligned} \tag{A.10}$$

For the symmetric part,

$$\langle(\phi_i)^\dagger_{l'm'}(\phi_j)_{lm}\rangle_S\mathrm{tr}(\phi_j[Y_{l'}^{m'\dagger}[t_i,Y_l^m]]) = S_{\mathrm{com}}\ C_l\mathrm{tr}(\phi_i t_i). \tag{A.11}$$

For the antisymmetric part,

$$\begin{aligned}
&\langle(\phi_i)^\dagger_{l'm'}(\phi_j)_{lm}\rangle_A \mathrm{tr}(\phi_j[Y_{l'}^{m'\dagger}[t_i, Y_l^m]]) \\
&= i\epsilon_{ijk}(t_k)_{mm'}\mathrm{tr}(\phi_j[Y_l^{m'\dagger}[t_i, Y_l^m]])A_{\mathrm{com}} \\
&= i\epsilon_{ijk}\mathrm{tr}(\phi_j[Y_l^{m'\dagger}[t_i[t_k, Y_l^{m'}]]])A_{\mathrm{com}} \\
&= i\epsilon_{ijk}\mathrm{tr}(\phi_j[Y_l^{m'\dagger}[t_q, Y_l^{m'}]])i\epsilon_{ikq}A_{\mathrm{com}} \\
&= (n_{\phi,\mathrm{com}} - 1)C_l\mathrm{tr}(\phi_i t_i)A_{\mathrm{com}},
\end{aligned} \tag{A.12}$$

where in the fourth line we use the antisymmetry of i, k. The total contribution is

$$\begin{aligned}
\mathrm{tr}(\hat{\phi}_i[\phi_j[t_i, \hat{\phi}_j]]) =& - S_{\mathrm{com}}\ C_l\mathrm{tr}(\phi_i t_i) - (n_{\phi,\mathrm{com}} - 1)C_l\mathrm{tr}(\phi_i t_i)A_{\mathrm{com}} \\
=& - \left(\frac{l+1}{2l+1}K^{l(l-1)} + \frac{l}{2l+1}K^{(l+1)(l+2)}\right) C_l\mathrm{tr}(\phi_i t_i) \\
& - (n_{\phi,\mathrm{com}} - 1)\frac{1}{2l+1}\left(K^{l(l-1)} - K^{(l+1)(l+2)}\right) C_l\mathrm{tr}(\phi_i t_i).
\end{aligned} \tag{A.13}$$

- Contribution to $\mathrm{tr}(\hat{\phi}_i[\hat{A}_3[t_i, A_3]])$

$$\begin{aligned}
\mathrm{tr}(\hat{\phi}_i[\hat{A}_3[t_i, A_3]]) &= \mathrm{tr}([\hat{\phi}_i, \hat{A}_3][t_i, A_3]) \\
&= \langle(\phi_i)_{lm}(A_3)^\dagger_{l'm'}\rangle \mathrm{tr}([Y_l^m, Y_{l'}^{m'\dagger}][t_i, A_3]).
\end{aligned} \tag{A.14}$$

Because

$$\langle(\phi_i)_{lm}(A_3)^\dagger_{l'm'}\rangle \sim \delta_{ll'}(t_i)_{mm'}, \tag{A.15}$$

we have

$$\mathrm{tr}(\hat{\phi}_i[\hat{A}_3[t_i, A_3]]) \sim \mathrm{tr}(t_i[t_i, A_3]) = 0. \tag{A.16}$$

Due to the Jacobi identity, we have

$$\mathrm{tr}(\hat{\phi}_i[A_3[t_i, \hat{A}_3]]) = \mathrm{tr}(\hat{\phi}_i[\hat{A}_3[t_i, A_3]]) = 0. \tag{A.17}$$

- Contribution to $\mathrm{tr}(i[\hat{A}_3, \hat{\phi}_i]\partial_3\phi_i)$

$$\begin{aligned}
&\mathrm{tr}(i[\hat{A}_3, \hat{\phi}_i]\partial_3\phi_i) \\
&= i\langle(\phi_i)_{lm}(A_3)^\dagger_{l'm'}\rangle \mathrm{tr}([Y_{l'}^{m'\dagger}, Y_l^m]\partial_3\phi_i) \\
&= -\frac{1}{2l+1}\left(K^{l(l-1)} - K^{(l+1)(l+2)}\right)\mathrm{tr}([Y_l^{m\dagger}[t_i, Y_l^m]\partial_3\phi_i) \\
&= -\frac{1}{2l+1}\left(K^{l(l-1)} - K^{(l+1)(l+2)}\right)C_l\mathrm{tr}(t_i\partial_3\phi_i).
\end{aligned} \tag{A.18}$$

- Contribution to $\mathrm{tr}(i[\hat{A}_3, \phi_i]\partial_3\hat{\phi}_i)$

$$\begin{aligned}
\mathrm{tr}(i[\hat{A}_3, \phi_i]\partial_3\hat{\phi}_i) &= i\mathrm{tr}([\partial_3\hat{\phi}_i, \hat{A}_3]\phi_i) \\
&= i\langle\partial_3(\phi_i)_{lm}(A_3)^\dagger_{l'm'}\rangle\mathrm{tr}([Y_l^m, Y_{l'}^{m'\dagger}]\phi_i) \\
&= \frac{i}{2}\partial_3(\langle(\phi_i)_{lm}(A_3)^\dagger_{l'm'}\rangle)\mathrm{tr}([Y_l^m, Y_{l'}^{m'\dagger}]\phi_i) \\
&= \frac{i}{2}\frac{1}{2l+1}\partial_3(K^{l(l-1)} - K^{(l+1)(l+2)})i(t_i)_{mm'}\mathrm{tr}([Y_l^m, Y_{l'}^{m'\dagger}]\phi_i) \\
&= -\frac{1}{2}\frac{1}{2l+1}\partial_3(K^{l(l-1)} - K^{(l+1)(l+2)})(-1)C_l\mathrm{tr}(\phi_i t_i) \\
&= \frac{1}{2}\frac{1}{2l+1}\partial_3(K^{l(l-1)} - K^{(l+1)(l+2)})C_l\mathrm{tr}(\phi_i t_i).
\end{aligned} \tag{A.19}$$

- Contribution to $\frac{1}{2}\mathrm{tr}(\hat{\bar{\psi}}G^i[\phi_i, \hat{\psi}])$

$$\begin{aligned}
\frac{1}{2}\mathrm{tr}(\hat{\bar{\psi}}_j(G^i)_{jk}[\phi_i, \hat{\psi}_k]) &= \frac{1}{2}(G^i)_{jk}\langle(\bar{\psi}_j)_{l'm'}(\psi_k)_{lm}\rangle\mathrm{tr}(Y_{l'}^{m'}[\phi_i, Y_l^m]) \\
&= \frac{1}{2}(G^i)_{jk}\langle(\bar{\psi}_j)_{l'-m'}(\psi_k)_{lm}\rangle\mathrm{tr}(\phi_i[Y_l^m, Y_{l'}^{-m'}]).
\end{aligned} \tag{A.20}$$

From the propagators of fermions (1.46), we have

$$\langle(\psi_i)_{lm}(\bar{\psi}_j)_{l'-m'}\rangle(-1)^{m'} = \delta_{ij}\delta_{mm'}\delta_{ll'}S_\mathrm{f} - \delta_{ll'}(G^k)_{ij}(t_k)_{mm'}A_\mathrm{f}. \tag{A.21}$$

The symmetric part is

$$\frac{1}{2}(G^i)_{jk}(-1)\delta_{jk}\mathrm{tr}(\phi_i[Y_l^m, Y_l^{m\dagger}])S_\mathrm{f} = 0, \tag{A.22}$$

since $\mathrm{tr}(G^i) = 0$. The antisymmetric part is

$$\begin{aligned}
&\frac{1}{2}(G^i)_{jk}(G^h)_{kj}(t_h)_{mm'}A\mathrm{tr}(\phi_i[Y_l^m, Y_l^{m'\dagger}]) \\
&= \frac{1}{4}\mathrm{tr}(\{G^i, G^h\})A(-1)\mathrm{tr}(\phi_i[Y_l^{m'\dagger}[t_h, Y_l^{m'}]]) \\
&= \frac{1}{4}\mathrm{tr}(2I_{n_\psi\times n_\psi})A(-1)C_l\mathrm{tr}(\phi_i t_i) \\
&= -\frac{1}{2}n_\psi\frac{1}{2l+1}(K_F^{-l} - K_F^{l+1})C_l\mathrm{tr}(\phi_i t_i),
\end{aligned} \tag{A.23}$$

where $n_\psi = 4$. The total contribution is

$$\frac{1}{2}\mathrm{tr}(\hat{\bar{\psi}}G^i[\phi_i, \hat{\psi}]) = -\frac{1}{2}n_\psi\frac{1}{2l+1}(K_F^{-l} - K_F^{l+1})C_l\mathrm{tr}(\phi_i t_i). \tag{A.24}$$

Appendix B

Useful expectation values contributed from the $E^n{}_a$ ($E^a{}_n$) part

In this appendix, we calculated some useful expectation values from the contribution of the degrees of freedom of the $E^n{}_a$ and $E^a{}_n$ part. They will be useful to calculate one-loop effective action for one-point function.

- Contribution to $\mathrm{tr}([t_i, \hat{\phi}_4][\phi_i, \hat{\phi}_4])$

$$\begin{aligned}
&\mathrm{tr}([t_i, \hat{\phi}_4][\phi_i, \hat{\phi}_4]) \\
&\quad = \langle(\phi_4)^\dagger_{a'n'}(\phi_4)_{an}\rangle \mathrm{tr}(\phi_i[E^{a'}{}_{n'}{}^\dagger[t_i, E^a{}_n]]) + \langle(\phi_4)^\dagger_{n'a'}(\phi_4)_{na}\rangle \mathrm{tr}(\phi_i[E^{n'}{}_{a'}{}^\dagger[t_i, E^n{}_a]]) \\
&\quad = \langle(\phi_4)^\dagger_{a'n'}(\phi_4)_{an}\rangle \mathrm{tr}(\phi_i[E^{n'}{}_{a'}, -E^a{}_n t_i]) + \langle(\phi_4)^\dagger_{n'a'}(\phi_4)_{na}\rangle \mathrm{tr}(\phi_i[E^{a'}{}_{n'}, t_i E^n{}_a]) \\
&\quad = \langle(\phi_4)^\dagger_{a'n'}(\phi_4)_{an}\rangle \mathrm{tr}(\phi_i(-\delta^a_{a'} E^{n'}{}_n t_i + E^a{}_n t_i E^{n'}{}_{a'})) \\
&\qquad + \langle(\phi_4)^\dagger_{n'a'}(\phi_4)_{na}\rangle \mathrm{tr}(\phi_i(E^{a'}{}_{n'} t_i E^n{}_a - t_i \delta^{a'}_a E^n{}_{n'})) \\
&\quad = K^{\frac{k^2-1}{4}} \mathrm{tr}(\phi_i(-2(N-k)t_i)) \\
&\quad = -2(N-k)K^{\frac{k^2-1}{4}} \mathrm{tr}(\phi_i t_i).
\end{aligned} \tag{B.1}$$

where we use

$$\begin{aligned}
\sum_{n,a} E^a{}_n t_i E^n{}_a &= (E^a{}_n)_{pq}(t_i)_{qh}(E^n{}_a)_{hl} \\
&= \delta_{ap}\delta_{nq}(t_i)_{qh}\delta_{nh}\delta_{al} = \sum_n (N-k)(t_i)_{nn} = 0.
\end{aligned} \tag{B.2}$$

- Contribution to $\mathrm{tr}(A^\mu[\hat{\phi}_4, \partial_\mu \hat{\phi}_4])$

Similar to our discussion for the Y^m_l part, the contribution is zero.

- Contribution to $\mathrm{tr}(\phi_i[\hat{\phi}_1[t_i, \hat{\phi}_1]])$

$$\mathrm{tr}(\phi_i[\hat{\phi}_1[t_i, \hat{\phi}_1]]) = -2(N-k)\Big(\frac{k+1}{2k}K^{\frac{(k-2)^2-1}{4}} + \frac{k-1}{2k}K^{\frac{(k+2)^2-1}{4}}\Big)\mathrm{tr}(\phi_i t_i). \tag{B.3}$$

- Contribution to $\mathrm{tr}(\hat{\phi}_i[\hat{\phi}_j[t_i,\phi_j]])$
  Here i,j label the complicated fields

$$\begin{aligned}
&\mathrm{tr}(\hat{\phi}_i[\hat{\phi}_j[t_i,\phi_j]]) = \mathrm{tr}([\hat{\phi}_i,\hat{\phi}_j][t_i,\phi_j]) \\
&= \langle(\phi_i)_{n'a'}(\phi_j)_{an}\rangle_A \mathrm{tr}([E^{n'}_{\ a'},E^a_{\ n}][t_i,\phi_j]) + \langle(\phi_i)_{a'n'}(\phi_j)_{na}\rangle_A \mathrm{tr}([E^{a'}_{\ n'},E^n_{\ a}][t_i,\phi_j]) \\
&= -i\epsilon_{ijk}(t_k)_{n'n}A\mathrm{tr}([E^{n'}_{\ a},E^a_{\ n}][t_i,\phi_j]) + i\epsilon_{ijk}(t_k)_{nn'}A\mathrm{tr}([E^a_{\ n'},E^n_{\ a}][t_i,\phi_j]) \\
&= -2i\epsilon_{ijk}A\mathrm{tr}(t_k[t_i,\phi_j])(N-k) \\
&= -2i\epsilon_{ijk}Ai\epsilon_{kih}\mathrm{tr}(t_h\phi_j)(N-k) \\
&= 2(N-k)(n_{\phi,\mathrm{com}}-1)\frac{1}{k}(K^{\frac{(k-2)^2-1}{4}} - K^{\frac{(k+2)^2-1}{4}})\mathrm{tr}(t_i\phi_i),
\end{aligned} \tag{B.4}$$

where we use

$$(t_i)_{n'n}[E^{n'}_{\ a},E^a_{\ n}] = (t_i)_{n'n}E^{n'}_{\ n}\delta^a_a - (t_i)_{n'n}\delta^{n'}_n E^a_{\ a} = (N-k)t_i. \tag{B.5}$$

- Contribution to $\mathrm{tr}(\hat{\phi}_i[\phi_j[t_i,\hat{\phi}_j]])$

$$\mathrm{tr}(\hat{\phi}_i[\phi_j[t_i,\hat{\phi}_j]]) = \mathrm{tr}([\hat{\phi}_i,\phi_j][t_i,\hat{\phi}_j]) = -\mathrm{tr}(\phi_j[\hat{\phi}_i[t_i,\hat{\phi}_j]]). \tag{B.6}$$

The symmetric part is

$$\mathrm{tr}(\phi_i t_i)(N-k)\left(\frac{k+1}{2k}K^{\frac{(k-2)^2-1}{4}} + \frac{k-1}{2k}K^{\frac{(k+2)^2-1}{4}}\right). \tag{B.7}$$

To calculate the antisymmetric part, first we find

$$\begin{aligned}
(t_k)_{n'n}[E^{n'}_{\ a}[t_i,E^a_{\ n}]] &= (t_k)_{n'n}[E^{n'}_{\ a},-E^a_{\ n}t_i] \\
&= (t_k)_{n'n}(-E^{n'}_{\ a}E^a_{\ n}t_i + E^a_{\ n}t_iE^{n'}_{\ a}) \\
&= (t_k)_{n'n}(-\delta^a_aE^{n'}_{\ n}t_i + (N-k)(t_i)_{nn'}P_{N-k}) \\
&= -(N-k)t_kt_i + (N-k)\mathrm{tr}(t_kt_i)P_{N-k},
\end{aligned} \tag{B.8}$$

where

$$\begin{aligned}
(E^a_{\ n}t_iE^{n'}_{\ a})_{pl} &= (E^a_{\ n})_{pq}(t_i)_{qh}(E^{n'}_{\ a})_{hl} \\
&= \delta_{ap}\delta_{nq}(t_i)_{qh}\delta_{n'h}\delta_{al} \\
&= (N-k)(t_i)_{nn'}(P_{N-k})_{pl}.
\end{aligned} \tag{B.9}$$

Here P_{N-k} is the projection operator to the $(N-k)\times(N-k)$ subspace. Similarly

$$(t_k)_{nn'}[E^a_{\ n'}[t_i,E^n_{\ a}]] = -(N-k)t_it_k + (N-k)\mathrm{tr}(t_kt_i)P_{N-k}. \tag{B.10}$$

Then the antisymmetric part is

$$
\begin{aligned}
&-\langle(\phi_i)_{n'a'}(\phi_j)_{an}\rangle_A \mathrm{tr}(\phi_j[E^{n'}{}_{a'}[t_i,E^a{}_n]]) - \langle(\phi_i)_{a'n'}(\phi_j)_{na}\rangle_A \mathrm{tr}(\phi_j[E^{a'}{}_{n'}[t_i,E^n{}_a]])\\
&= i\epsilon_{ijk}(t_k)_{n'n}\mathrm{tr}(\phi_j[E^{n'}{}_a[t_i,E^a{}_n]])A - i\epsilon_{ijk}(t_k)_{nn'}\mathrm{tr}(\phi_j[E^a{}_{n'}[t_i,E^n{}_a]])A\\
&= i\epsilon_{ijk}\mathrm{tr}\Big(\phi_j[-(N-k)t_kt_i + (N-k)\mathrm{tr}(t_kt_i)P_{N-k}]\\
&\quad -\phi_j[-(N-k)t_it_k + (N-k)\mathrm{tr}(t_kt_i)P_{N-k}]\Big)A\\
&= -i\epsilon_{ijk}(N-k)\mathrm{tr}(\phi_j[t_k,t_i])A\\
&= (N-k)(n_{\phi,\mathrm{com}}-1)\mathrm{tr}(\phi_it_i)A\\
&= \frac{N-k}{k}(n_{\phi,\mathrm{com}}-1)(K^{\frac{(k-2)^2-1}{4}} - K^{\frac{(k+2)^2-1}{4}})\mathrm{tr}(\phi_it_i)
\end{aligned}
\tag{B.11}
$$

- Contribution to $\mathrm{tr}(\hat{\phi}_i[\hat{A}_3[t_i,A_3]])$

$$
\begin{aligned}
\mathrm{tr}(\hat{\phi}_i[\hat{A}_3[t_i,A_3]]) &= \mathrm{tr}([\hat{\phi}_i,\hat{A}_3][t_i,A_3])\\
&= \langle(\phi_i)_{n'a'}(A_3)_{an}\rangle \mathrm{tr}([E^{n'}{}_{a'},E^a{}_n][t_i,A_3]) + \langle(\phi_i)_{a'n'}(A_3)_{na}\rangle \mathrm{tr}([E^{a'}{}_{n'},E^n{}_a][t_i,A_3])\\
&= \frac{1}{k}\left(K^{\frac{(k-2)^2-1}{4}} - K^{\frac{(k+2)^2-1}{4}}\right)\\
&\quad \times\Big(i(t_i)_{n'n}\mathrm{tr}([E^{n'}{}_a,E^a{}_n][t_i,A_3]) - i(t_i)_{nn'}\mathrm{tr}([E^a{}_{n'},E^n{}_a][t_i,A_3])\Big)\\
&= 2i\frac{1}{k}\left(K^{\frac{(k-2)^2-1}{4}} - K^{\frac{(k+2)^2-1}{4}}\right)\mathrm{tr}(t_i[t_i,A_3])\\
&= 0.
\end{aligned}
\tag{B.12}
$$

- Contribution to $\mathrm{tr}(i[\hat{A}^3,\hat{\phi}_i]\partial_3\phi_i)$

$$
\begin{aligned}
&\mathrm{tr}(i[\hat{A}^3,\hat{\phi}_i]\partial_3\phi_i)\\
&\quad = i\langle(A^3)_{n'a'}(\phi_i)_{an}\rangle \mathrm{tr}([E^{n'}{}_{a'},E^a{}_n]\partial_3\phi_i) + i\langle(A^3)_{a'n'}(\phi_i)_{na}\rangle \mathrm{tr}([E^{a'}{}_{n'},E^n{}_a]\partial_3\phi_i)\\
&\quad = i(-i)(t_i)_{n'n}\mathrm{tr}([E^{n'}{}_a,E^a{}_n]\partial_3\phi_i)Aii(t_i)_{nn'}\mathrm{tr}([E^a{}_{n'},E^n{}_a]\partial_3\phi_i)A\\
&\quad = (N-k)\mathrm{tr}(t_i\partial_3\phi_i)A + (N-k)\mathrm{tr}(t_i\partial_3\phi_i)A\\
&\quad = 2\frac{N-k}{k}(K^{\frac{(k-2)^2-1}{4}} - K^{\frac{(k+2)^2-1}{4}})\mathrm{tr}(t_i\partial_3\phi_i).
\end{aligned}
\tag{B.13}
$$

- Contribution to $\mathrm{tr}(i[\hat{A}^3, \phi_i]\partial_3\hat{\phi}_i)$

$$\begin{aligned}\mathrm{tr}(i[\hat{A}^3, \phi_i]\partial_3\hat{\phi}_i) = i\mathrm{tr}([\partial_3\hat{\phi}_i, \hat{A}^3]\phi_i]) &= -\frac{i}{2}\mathrm{tr}(\partial_3[\hat{A}^3, \hat{\phi}_i]\phi_i) \\ &= -\frac{N-k}{k}\partial_3(K^{\frac{(k-2)^2-1}{4}} - K^{\frac{(k+2)^2-1}{4}})\mathrm{tr}(t_i\phi_i).\end{aligned} \tag{B.14}$$

- Contribution to $\frac{1}{2}\mathrm{tr}(\hat{\bar{\psi}}G^i[\phi_i, \hat{\psi}])$

$$\begin{aligned}&\frac{1}{2}\mathrm{tr}(\hat{\bar{\psi}}_j(G^i)_{jk}[\phi_i, \hat{\psi}_k]) \\ &\quad= \frac{1}{2}(G^i)_{jk}\langle(\bar{\psi}_j)_{n'a'}(\psi_k)_{an}\rangle\mathrm{tr}(E^{n'}{}_{a'}[\phi_i, E^a{}_n]) \\ &\qquad+ \frac{1}{2}(G^i)_{jk}\langle(\bar{\psi}_j)_{a'n'}(\psi_k)_{na}\rangle\mathrm{tr}(E^{a'}{}_{n'}[\phi_i, E^n{}_a]) \\ &\quad= \frac{1}{2}(G^i)_{jk}\langle(\bar{\psi}_j)_{n'a'}(\psi_k)_{an}\rangle\mathrm{tr}([E^{n'}{}_{a'}, E^a{}_n]\phi_i)(-1) \\ &\qquad+ \frac{1}{2}(G^i)_{jk}\langle(\bar{\psi}_j)_{a'n'}(\psi_k)_{na}\rangle\mathrm{tr}([E^n{}_a, E^{a'}{}_{n'}]\phi_i).\end{aligned} \tag{B.15}$$

The symmetric part is zero because $\mathrm{tr}(G^i) = 0$. The antisymmetric part is

$$\begin{aligned}&\frac{1}{2}(G^i)_{jk}(G^l)_{kj}(t_l)_{n'n}A\mathrm{tr}([E^{n'}{}_a, E^a{}_n]\phi_i) \\ &+\frac{1}{2}(G^i)_{jk}(G^l)_{kj}(t_l)_{nn'}A\mathrm{tr}([E^n{}_a, E^a{}_{n'}]\phi_i) \\ &\quad= (G^i)_{jk}(G^l)_{kj}A\mathrm{tr}(t_l\phi_i)(N-k) \\ &\quad= \frac{1}{2}\mathrm{tr}(\{G^i, G^l\})A\mathrm{tr}(t_l\phi_i)(N-k) \\ &\quad= \mathrm{tr}(I_{n_{\psi\times\psi}})A\mathrm{tr}(t_i\phi_i)(N-k) \\ &\quad= n_\psi\frac{N-k}{k}(K_F^{-\frac{k-1}{2}} - K_F^{\frac{k+1}{2}})\mathrm{tr}(t_i\phi_i).\end{aligned} \tag{B.16}$$