Birational Models of Geometric Invariant Theory Quotients

by

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Abstract

In this thesis, we study the problem of finding birational models of projective G-varieties with tame stabilizers. This is done with linearizations, so that each birational model may be considered as a (modular) compactification of an orbit space (of properly stable points). The main portion of the thesis is a re-working of a result in Kirwan's paper "Partial Desingularisations of Quotients of Nonsingular Varieties and their Betti Numbers" [3], written in a purely algebro-geometric language. As such, the proofs are novel and require the Luna Slice Theorem as their primary tool.

Chapter 1 is devoted to preliminary material on Geometric Invariant Theory and the Luna Slice Theorem.

In Chapter 2, we present and prove a version of "Kirwan's procedure". This chapter concludes with an outline of some differences between the current thesis and Kirwan's original paper.

In Chapter 3, we combine the results from Chapter 2 and a result from a paper by Reichstein and Youssin to provide another type of birational model with tame stabilizers (again, with respect to an original linearization).

Preface

The topic of this thesis is based on the work of Kirwan (in [3]) and Reichstein (in [6]).

The content of Chapter 2 is based on [3]. However, the author does not directly use any results from [3], and provides a novel variation of the original ideas. Therefore, the content of Chapter 2 is ultimately independent of [3], and is original work by the author.

Chapter 3 is based on the work of Reichstein and Youssin (in [7]). The main technical results required for this chapter are directly borrowed from [7]. However, it is an original observation that one may combine the work of [3] (or Chapter 2 of this thesis), and [7] to provide a new result. This was an original unpublished idea by the author's supervisor, Zinovy Reichstein.

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Chapter 1

Introduction

1.1 The purpose of this thesis is to investigate birational equivariant models $\tilde{X}$ of smooth projective G-varieties X, such that there exists a linearization of $\tilde{X}$ with the property that all stabilizers of semi-stable points are finite (i.e. there are no strictly semi-stable points). If a linearization is provided for X, the linearization of $\tilde{X}$ may be arranged so that $\tilde{X} \to X$ is an isomorphism along the properly stable locus of X. This work is heavily based on Kirwan's paper [3], where the author investigates such models. In the second chapter, we provide an alternative exposition of this work, including proofs entirely written in the algebro-geometric language instead of the language of symplectic geometry. These new proofs make heavy use of the Luna Slice theorem and its various corollaries. In particular, the technical details involved in Kirwan's procedure may be interpreted and expressed in terms of Luna's stratification. In the case where X is generically free, we may find a birational model $\widetilde{X_{\text{ab}}} \to X$ of X so that $\widetilde{X_{\text{ab}}}$ has the property that all stabilizers of semi-stable points are finite, but in addition they are also abelian. If G acts freely on X^{ps} (given a linearization for X), then $\widetilde{X_{\text{ab}}} \to X$ is an isomorphism along the properly stable locus of X.

We assume that our base field κ is algebraically closed, and $\text{char}(\kappa) = 0$.

1.2 We will prove the following theorem:

Theorem (A). *Let X be a G-linearized smooth projective variety. If $X^{ps} \neq \emptyset$, then there exists a sequence of blow ups:*

$$\sigma : X_N := \tilde{X} \to X_{N-1} \to \cdots \to X_0 := X$$

such that X_N satisfies:

1. *$\tilde{X}^{ps} = \tilde{X}^{ss}$. That is, $\tilde{X}$ contains only unstable and properly stable points.*
2. *$\sigma : \tilde{X} \to X$ is an isomorphism along X^{ps}.*

The above theorem describes a sequence of blow ups performed on X so that $\tilde{X}^{ss}$ consists only of points with *finite* stabilizers. We will call such a variety $\tilde{X}$ with a birational morphism $\sigma : \tilde{X} \to X$ satisfying the conclusions of the above theorem, a **stable resolution** of X. In fact, by [6] this may indeed be alternatively achieved by performing a resolution of singularities on the strictly semistable locus $X^{s.ss}$ of X.

1.3 If the action of G on X is generically free, then [7] constructs a sequence of blow-ups peformed on X so that every reductive stabilizer subgroup of G is diagonalizable. We may combine this result with Theorem **(A)**:

Theorem (B). *Suppose that X is a generically free G-linearized smooth projective variety, and that $\sigma : \tilde{X} \to X$ is a smooth stable resolution of X. Then, there exists a sequence of blow-ups:*

$$\rho : \tilde{X}_M = \tilde{X}_{ab} \to \tilde{X}_{M-1} \cdots \to \tilde{X}$$

such that

1. *$\tilde{X}^{ps}_{ab} = \tilde{X}^{ss}_{ab}$. That is, $\tilde{X}_{ab}$ contains only unstable and properly points. Furthermore, any point $x \in \tilde{X}^{ps}_{ab}$ has a finite and abelian stabilizer subgroup.*

Furthermore, if we have that G acts on X^{ps} freely, then we may find a sequence of blow-ups such that $\tilde{X}_M$ satisfies:

2. *$\rho : \tilde{X}_{ab} := \tilde{X}_M \to \tilde{X}$ is an isomorphism along $\tilde{X}^{ps}$. In particular, $\rho \circ \sigma$ is an isomorphism along X^{ps}.*

In particular, if X is a smooth G-linearized projective variety such that $X^{ps} = X^{ss}$, we may take $\tilde{X} = X$ in the above statement.

1.4 Note that although $\tilde{X}^{ps} \cong X^{ps}$, we do not necessarily have that $\tilde{X}^{ps}_{\mathrm{ab}} \cong \tilde{X}^{ps} \cong X^{ps}$. In fact, this will only be true if we can choose the blow up centers containing only unstable points. This is possible if G acts freely on X^{ps}.

However, the above theorem says that if $\tilde{X}$ has the property that $\tilde{X}$ contains *only* properly stable and unstable points, then the same is true for $\tilde{X}_{\mathrm{ab}}$. We will say that $\sigma : Y \dashrightarrow X$ is a **stable model** (of X) if: $\sigma : Y \to X$ is a birational morphism, and Y consists only properly stable and unstable points. In this case, we have that $y \in \sigma^{-1}(x)$ is properly stable if $x \in X$ is properly stable (see Proposition 3.4.1). In particular, a stable resolution of X is a stable model of X with restricts to an isomorphism along the properly stable loci. Thus, Theorem **(B)** claims that if $\tilde{X}$ is a stable model of X, then $\tilde{X}_{\mathrm{ab}}$ is one as well. If $\tilde{X}$ is a stable resolution of X, then $\tilde{X}_{\mathrm{ab}}$ is one as well, if G acts freely on X^{ps}.

1.5 Note that a stable resolution $\tilde{X} \to X$ is not guaranteed to be smooth if the centers of blow-up are singular. However, one may show that when choosing the centers of blow up in the above theorem, the centers choosen are possibly singular only at unstable points. By performing equivariant resolution of singularities, one may always find a smooth stable resolution $\tilde{X} \to X$. If this is done, then $\tilde{X}$ will be smooth. However, $\tilde{X}^{ss}$ will be smooth, provided that X^{ss} is smooth. Then, $\tilde{X} /\!\!/ G$ may be viewed as a partial desingularization of the GIT quotient $X /\!\!/ G$ in that the only singularities are finite quotient singularities. After applying the blow up sequence ρ, $\tilde{X}_{\mathrm{ab}} /\!\!/ G$ has at worse finite abelian quotient singularities.

Chapter 2

Preliminaries

Geometric Invariant Theory

We begin this chapter with a brief summary of GIT. We assume that the reader has had a previous encounter with the relevant definitions.

2.1 Suppose that X is a projective G-variety, and that there exists an ample line bundle $\mathcal{L}$ on X with a G-action on the total space compatible with the G-action on X. Then, there is a power $\mathcal{L}^{\otimes n}$ such that the linear system $X \hookrightarrow H^0(X, \mathcal{L}^{\otimes n})$ defines an equivariant embedding of G into $\mathbb{P}(H^0(X, \mathcal{L}^{\otimes n}))$. In this context, G-action on $\mathbb{P}(H^0(X, \mathcal{L}^{\otimes n}))$ is induced by a linear action on the vector space $H^0(X, \mathcal{L}^{\otimes n})$. We call a G-linearizable projective variety X along with such an equivariant embedding, a **linearized projective G-variety** X (or G-linearized projective variety X).

2.2 The linear equivariant embedding of X into the projective space $\mathbb{P}(H^0(X, \mathcal{L}^{\otimes n}))$ allows one to define various notions of stability for points of X (i.e. properly stable, unstable, semi-stable). This provides a decomposition of X into the disjoint union of three G-invariant sets $X^{ps}(\mathcal{L})$, $X^{s.ss}(\mathcal{L})$ and $X^{unst}(\mathcal{L})$ (properly stable, strictly semi-stable and unstable). Recall that $X^{unst}(\mathcal{L})$ is a closed subset of X. The significance of this decomposition is the following. One can show that a categorical quotient always exists for the open G-variety $X^{ps}(\mathcal{L}) \sqcup X^{s.ss}(\mathcal{L}) = X \setminus X^{unst}(\mathcal{L}) := X^{ss}$. This is of course the GIT quotient $X /\!\!/_{\mathcal{L}} G$ of X. On the locus of properly stable points X^{ps}, the restriction of π defines a geometric quotient (thus, an orbit space). That is, there is a categorical quotient $\pi : X^{ps}(\mathcal{L}) \to X^{ps}(\mathcal{L}) /\!\!/_{\mathcal{L}} G$ whose fibers parametrize precisely the orbits of G in $X^{ps}(\mathcal{L})$. One can show that the GIT quotient $X /\!\!/_{\mathcal{L}} G$ is a projective variety, if X is projective. Furthermore, $X^{ps} /\!\!/_{\mathcal{L}} G \subseteq X /\!\!/_{\mathcal{L}} G$ is an open subvariety, and hence is not necessarily projective. Hence, one may consider the GIT quotient $X /\!\!/_{\mathcal{L}} G$ as a compactification of the orbit space $X^{ps} /\!\!/_{\mathcal{L}} G$ where the orbits of G in $X^{s.ss}$ are the orbits which lie on the boundary

2.3 The boundary of the geometric quotient $X^{ps} /\!\!/ G$ in $X /\!\!/ G$ has the deficiency that its points only have "weak modular meaning". What this amounts to is that that the fibers above the points on the boundary are not in bijection with the orbits of G in $X^{s.ss}$. Instead, an orbit closure relation hold among the fibers: x and y in X^{ss} lie on the same fiber of the quotient map π if and only if $\overline{Gx} \cap \overline{Gy} \neq \emptyset$, where the closure is being taken in X^{ss}. Note that this relation is necessary for the map $X^{ss} \to X^{ss} /\!\!/ G$ to be continuous. However, one may prove that a fiber $\pi^{-1}(x)$ for $x \in X^{ss}$ contains a unique closed orbit. Therefore, one may say that the fibers of the GIT quotient map π are in bijection with *closed* orbits of X^{ss}. Recall that any orbit Gx in X has the property that $\overline{Gx}$ contains a closed orbit. We call a point semi-stable point x stable if Gv is closed in $H^0(X, \mathcal{L}^{\otimes n})$ for an affine representation v of x. In the sequel, X^s will denote the set of stable points of X.

Luna Slice Theorem

In this section, we introduce the Luna Slice theorem and its various corollaries. These are some of the main technical tools required to formulate and prove the results presented in this thesis.

2.4 The Luna slice theorem provides a local description of an orbit Gx of a point x in a G-variety X. In differential geometry, there is an analogous "slice theorem". It states that given a manifold X and a Lie group G acting on X diffeomorphically, then for any point $x \in M$, one may find a G-invariant open neighbourhood U containing Gx, such that U is equivariantly diffeomorphic to the homogeneous fiber space $G \times^{G_x} \mathcal{N}_x$. Here, $\mathcal{N}_x$ is a G_x-invariant direct sum complement to $T_x(Gx)$ in $T_x(X)$. In short, there is a tubular neighbourhood around Gx so that the action of G in this neighbourhood has a simple description as a homogeneous fiber space.

2.5 As one may expect, this theorem does not hold in the algebraic setting. Firstly, if G_x is not reductive, then a G_x-invariant complement of $T_x(Gx)$ in $T_x(X)$ may not exist. Furthermore, Zariski open sets are too large to expect to find a Zariski open set U which is equivariantly isomorphic to $G \times^{G_x} \mathcal{N}_x$. This is illustrated by the following example:

Example. *Suppose that* $X = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$*, and* $G = SL_2$*, whose action on* X *is induced by the standard representation on* $\mathbb{P}^1$*. Then, there is an* $x \in X$ *such that there is no open* U *such that* U *is equivariantly isomorphic to* $G \times^{G_x} \mathcal{N}_x$*.*

Indeed, consider $x = u^2v$. Note that any point in $U \cong G \times^{G_x} \mathcal{N}_x$ has a stabilizer which is conjugate to a subgroup of G_x (see Lemma 2.6.1). One may check that the stabilizer of x is trivial. However, one may also check that the action of G on X has a stabilizer in general position of order 3. This is because if κ is algebraically closed, any binary cubic with distinct roots may be transformed (under the action of SL_2, by 3-transitivity of the action of SL_2 on $\mathbb{P}^1$) to a multiple of the form $h = u^3 + v^3$. Clearly, for such a form, we have that $G_h = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$, where $\mu^3 = 1$. Thus, as any two non-empty Zariski open sets intersect, we see that such an open set U is not possible (i.e. an open set U such that every point has trivial stabilizer).

As we will see, this is resolved by the fact that $G \times^{G_x} \mathcal{N}_x$ is an étale neighbourhood of x containing Gx, as opposed to a Zariski neighbourhood. That is, there is an étale morphism from $G \times^{G_x} \mathcal{N}_x \to X$ with image containing Gx. For details, see paragraph 2.9.

Homogeneous Fiber Spaces

2.6 Suppose that G is a reductive group, and that H is a closed reductive subgroup of G. Then H acts on G by the formula $h.g = gh^{-1}$. Under this action, G is a principal H-bundle over the homogeneous space G/H.

If S is an affine H-variety, we can define a G variety by "twisting" S by G (as a H-variety as above). H acts naturally on $G \times S$ component-wise by the formula $h.(g, s) = (gh^{-1}, hs)$.

Definition 2.6.1. A homogeneous fiber space is the quotient $(G \times S) /\!\!/ H$, where H acts on each factor as described above. A homogeneous fiber space is denoted $G \times^H S$, and an element will be denoted by $[g, s]$. G acts on $G \times^H S$ on the left factor in the obvious way.

The affine GIT quotient $(G \times S) /\!\!/ H$ above is in fact a *geometric quotient*. Note that $G \times S$ is an affine free H-variety, so that $G \times S$ is a principal H-bundle over $(G \times S) /\!\!/ H$.

Lemma 2.6.1. *The stabilizer of a point $[g, x]$ in a homogeneous fiber space $G \times^H S$, under the action of G, is conjugate to a subgroup of the stabilizer $H_x \subset H$ of x.*

Proof. Since $g.[e, x] = [e, x]$, we have that $[e, x]$ and $[g, x]$ are in the same orbit. Hence, the stabilizers of $[g, x]$ and $[e, x]$ are conjugate. Therefore, to show this result, we may simply consider stabilizers for points of the form $[e, x]$. If $g.[e, x] = [g, x] = [e, x]$, we then have that $(gh^{-1}, hx) = (e, x)$ for some $h \in H$. From this, we see that $h \in H_x$ and $g = h$. Thus, $g \in H_x$. Conversely, it is clear that ghg^{-1} fixes $[g, x]$ for any $h \in H_x$. □

It is clear that S embeds into $G \times^H S$ by sending $x \in S$ to $[e, x]$. Furthermore, as $G \times S$ is a free H-variety, we have that for a fixed $\underline{x} = (g, x) \in G \times S$, $H \times \underline{x} \to G \times S$ defned by $(h, \underline{x}) \to (gh^{-1}, hx) \in G \times S$ defines an embedding of H into $G \times S$. For a point $\underline{x} = (e, x)$, this embedding induces an inclusion of tangent spaces $\mu_{\underline{x}} : T_e(H) \hookrightarrow T_{(e,x)}(G \times S) \cong T_e(G) \oplus T_x(S)$.

Lemma 2.6.2. *We have:*

1. $T_x(G \times^H S) = (T_e(G) \oplus T_x(S))/T_e(H)$.
2. $T_x(G \times^H S) \cong T_x(Gx) \oplus T_x(S)$ *as abstract vector spaces. If S is smooth at x, then $G \times^H S$ is smooth at $[e, x]$.*
3. *If x is an H-fixed point, then we have that the embedding $T_e(H) \xhookrightarrow{\mu_{\underline{x}}} T_{(e,x)}(G \times S)$ induces an identification $T_x(G \times^H S) \cong T_x(Gx) \oplus T_x(S)$.*

Proof. As $G \times S$ is a principal H-bundle over $(G \times S) /\!\!/ H$, we have an exact sequence of tangent spaces:

$$0 \to T_e(H) \xrightarrow{\mu_{\underline{x}}} T_{(e,x)}(G \times S) \to T_{[e,x]}((G \times S) /\!\!/ H) \to 0.$$

From this, 1. immediately follows.

So we have, $\dim_\kappa(T_x(Gx) \oplus T_x(S)) = \dim_\kappa(T_e(G)/T_e(H) \oplus T_x(S)) = \dim_\kappa(T_e(G)) - \dim_\kappa(T_e(H)) + \dim_\kappa(S) = \dim_\kappa((T_e(G) \oplus T_x(S))/T_e(H))$, and $T_x(G \times^H S) \cong T_x(Gx) \oplus T_x(S)$ as abstract vector spaces.

Note that if x is an H-fixed point, then the induced tangent map $\mu_{\underline{x}} : T_e(H) \hookrightarrow T_{(e,x)}(G \times S) \cong T_e(G) \oplus T_x(S)$ maps $T_e(H)$ into $T_e(H) \oplus \{0_{T_x(S)}\}$. Then, $(T_e(G) \oplus T_x(S))/T_e(H) = T_e(G)/T_e(H) \oplus T_x(S) \cong T_x(Gx) \oplus T_x(S)$. □

Remark 2.6.1. In fact, since $G \times S$ is a principal H-bundle over $G \times^H S$, $G \times S$ is smooth at (e, x) *if and only if* $G \times^H S$ is smooth at $[e, x]$. Therefore, $G \times^H S$ is smooth at $[e, x]$ *if and only if* S is smooth at $x \in S$.

Existence of Étale Slices

2.7 We discuss the existence of étale slices in the smooth setting. In this section, X is a smooth affine G-variety, where G is a reductive group.

For a point $x \in X$, and a G_x-invariant affine subvariety S of X containing x, we may consider the homogeneous fiber space $G \times^{G_x} S$. There is a natural map $\psi_S : G \times^{G_x} S \to X$ defined by $\psi_S([g, s]) = gs$. If $\pi_1 : X \to X /\!\!/ G$ and $\pi_2 : G \times^{G_x} S \to G \times^{G_x} S /\!\!/ G$ are GIT quotient maps, then the composition $\pi_1 \circ \psi_S : G \times^{G_x} S \to X /\!\!/ G$ is G-invariant. Therefore, there is a factorization $\pi_1 \circ \psi_S = (\psi_S/G) \circ \pi_2$ for a unique map $\psi_S/G : G \times^{G_x} S /\!\!/ G \to X /\!\!/ G$. Thus, we have the following commutative diagram:

$$
\begin{tikzcd}
G \times^{G_x} S \arrow[r, "\psi_S"] \arrow[d, "\pi_2"] & X \arrow[d, "\pi_1"] \\
(G \times^{G_x} S) /\!\!/ G \arrow[r, "\psi_S/G"] & X /\!\!/ G
\end{tikzcd}
\tag{2.1}
$$

We say that a G_x-invariant affine subvariety S of X is an **étale slice** at x if in the above diagram, the following hold:

1. $x \in S$, and ψ_S is étale (in particular, the image of this map contains x).

2. ψ_S/G is étale.

3. The above diagram is cartestian. Thus, we have that the induced map $G \times^{G_x} S \to X \times_{X /\!\!/ G} ((G \times^{G_x} S) /\!\!/ G)$ is an isomorphism.

2.8 If an étale slice S exists, then the image of ψ_S is an open set U containing Gx. By Lemma 2.6.2, $T_x(U) = T_x(Gx) \oplus T_x(S)$. Thus, S is transverse to the orbit Gx at x. Conversely, we will see that if X is smooth at x, then there is an étale G_x-equivariant map $\phi : X \to T_x(X)$; under suitable hypotheses, if $T_x(Gx)$ has a G_x-invariant complement $\mathcal{N}_x$, one may take an appropriate open set of $\phi^{-1}(\mathcal{N}_x)$ to be an étale slice at x.

Lemma 2.8.1. *Suppose that X is an affine variety of dimension n, which is smooth at a point $x \in X$. Then, there is an étale (at x) morphism $\phi : X \to \mathbb{A}^n \cong T_x(X)$ such that $\phi(x) = 0 \in \mathbb{A}^n$.*

Proof. Since $x \in X$ is smooth, we may find *regular* functions $f_1, \ldots, f_n$ in $m_x \subset \mathcal{O}_X(X)$, such that their images $df_1, \ldots, df_n$ in $m_x/m_x^2 = (T_x(X))^\vee$ form a basis. Therefore, the map $(f_1, \ldots, f_n) : X \to \mathbb{A}^n$ is étale at x. Indeed, the induced map $T_0(\mathbb{A}^n)^\vee \to T_x(X)^\vee$ is given by the map $dx_i \mapsto df_i$ and is an isomorphism.

□

Lemma 2.8.2 (Luna). *Suppose that X is an affine G-variety of dimension n, which is smooth at a point $x \in X$. Suppose that G_x is* linearly reductive*. Then, there exists a morphism $\phi : X \to \mathbb{A}^n \cong T_x(X)$ such that:*

1. *ϕ is G_x-equivariant.*
2. *ϕ is étale at x.*
3. *$\phi(x) = 0$.*

Proof. One only needs to see that the map ϕ in Lemma 2.8.1 can be made G_x-equivariant. This will depend on choosing appropriate elements $f_1, \ldots, f_n$ in m_x which generate m_x/m_x^2. Both m_x and m_x/m_x^2 are G_x-modules, where G_x is linearly reductive. Furthermore, we clearly have that the natural map $\tau : m_x \to m_x/m_x^2$ is G_x-equivariant and surjective. By linear reductivity, we may define a G_x-equivariant section $s : m_x/m_x^2 \to m_x$. Indeed, the $\ker(\tau)$ has a G_x-invariant complement V in m_x, which is equivariantly isomorphic to m_x/m_x^2. Therefore, we may find $f_1, \ldots, f_n$ in V so that their images in m_x/m_x^2 form a basis (as a G_x-module).

□

2.9 We note that a G_x-invariant subvariety $S \subset X$ being an étale slice at x, is much stronger than only satisfying condition 1. (i.e. ψ_S is étale). For example, ψ_S/G being étale implies that the image of ψ_S is not only open (recall that étale maps have open image), but a saturated open set. Indeed, this follows from ψ_S/G being an open map, and that the square (1) commutes. Some authors may call such a subvariety S satisfying only condition 1. as a **weak étale slice**, and a subvariety S satisfying all three conditions an étale slice or a **strong étale slice**. This is a significant distinction, as one needs a strong étale slice to assert (for example), the existence of a Luna stratification (see Theorem 2.10.1). Furthermore, the example presented in paragraph 2.5 illustrates a situation where one only has a weak étale slice at x, and not an étale slice. The reason for this is that the orbit of x in that example is not closed.

In characteristic 0, the only condition required for the existence of an étale slice at x (which may be taken to be an appropriate open subset of $\phi^{-1}(\mathcal{N}_x)$) is that the orbit Gx is closed in X.

Theorem 2.9.1 (Matsushima). *Suppose that X is an affine G-variety. Suppose that $x \in X$ has a closed orbit Gx. Then, G_x is a reductive subgroup of G.*

Theorem 2.9.2 (Luna's Slice Theorem). *Suppose that X is an affine G-variety over a field κ which is algebraically closed and of characteristic 0. Suppose that $x \in X$ is so that Gx is closed. Then, there exists an étale slice at x of X. Furthermore, the image of the map ψ_S is a saturated open set of X.*

Proof sketch of Luna's Slice Theorem. One may consider $\phi^{-1}(\mathcal{N}_x)$ for some G_x-invariant complement to $T_x(Gx) \subset T_x(X)$ so that the natural map $\phi : G*_{\mathrm{Stab}\, G(x)}\phi^{-1}(\mathcal{N}_x) \to X$ is étale whose image contains x (note that $T_x(X) = T_x(Gx) \oplus \mathcal{N}_x \cong T_x(G \times^{G_x} \phi^{-1}(\mathcal{N}_x))$, by Lemma 2.6.2.). Therefore, we may find a *weak* étale slice at x. To find a strong étale slice, we may restrict ϕ to an appropriate open subset of $G *_{\mathrm{Stab}\, G(x)} \phi^{-1}(\mathcal{N}_x)$. This is provided by the lemma below (Luna's fundamental lemma). □

Lemma 2.9.1 (Luna's fundamental lemma). *Suppose that $\phi : Y \to X$ is a G-equivariant étale (at y) morphism of affine G-varieties, where Y is normal at a $y \in Y$. Suppose that $\phi(Gy)$ is closed in X, and that $\phi|_{Gy}$ is injective. Then there exists an open $V \subset Y$ containing y, such that*

1. *ϕ/G is étale at $\pi_Y(y)$.*
2. *The following square is cartesian:*

$$\begin{array}{ccc} V & \xrightarrow{\phi} & \phi(V) \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_X} \\ V /\!\!/ G & \xrightarrow{\phi/G} & \phi(V) /\!\!/ G \end{array} \tag{2.2}$$

3. *V is saturated. That is, $V = \pi_Y^{-1}(V')$ for some V' open in $Y /\!\!/ G$.*

4. *The image $\phi(V) = U$ is a saturated open set of X.*

Luna Stratification

2.10 Recall that the affine GIT quotient map (i.e. the categorical quotient) $\pi : X \to Z := X /\!\!/ G$ maps saturated open sets of X to open sets of $X /\!\!/ G$. Recall that for any $z \in Z$, there exists a unique closed orbit contained in the fiber $\pi^{-1}(z)$. Let us write x_z for a choice of an element in $\pi^{-1}(z)$ with closed orbit. By Luna's slice theorem, we have a map ψ_S, obtained from an étale slice, whose image is a saturated open set U_{x_z}. Thus, $\pi(U_{x_z})$ is an open set in Z, containing z. Therefore, we may cover Z with open sets of the form $\pi(U_{x_z})$. By quasi-compactness of Z, we may also find a finite cover $\{\pi(U_{x_{z_i}}) := V_i\}$ covering Z. Note further that $\pi(U_{x_z})$ is a constructible set (i.e. a finite union of locally closed subsets of Z). Thus, we have the theorem:

Theorem 2.10.1 (Luna Stratification). *There is finite collection of locally closed subsets V_i of $Z := X /\!\!/ G$ such that $\cup V_i = Z$, with the following properties:*

1. *Corresponding to the collection $V_1, \dots V_m$, there is a finite list of reductive subgroups $R_1, \dots, R_m$ so that any stable point in $\pi^{-1}(V_i)$ has a stabilizer conjugate to R_i.*

2. *If $x \in \pi^{-1}(V_i)$, then G_x is conjugate to a subgroup of R_i*

Now, Luna's stratification provides a stratification of the quotient with *finitely* many strata of the form $Z^{\langle H \rangle} := \{z \in Z \mid \mathrm{Stab}(x_z) \in \langle H \rangle\}$. Furthermore, let us define $X^{\langle H \rangle} := \{x \in X \mid G.x \text{ is closed and } \mathrm{Stab}(x) \in \langle H \rangle\}$. Thus, semi-stable points in X with closed orbit are stratified as above (i.e.

stratification induced by the stratification of the quotient), with strata labelled by reductive subgroups H of G. Let us call the strata $X^{\langle H \rangle}$, *isotropy strata* of X. For an affine G-variety as above, let us denote the set of V_i by $\mathcal{L}(X)$, and the set of corresponding subgroups R_i by $\mathcal{R}(X)$. Note that for a fixed $\mathcal{L}(X)$, the subgroups R_i in $\mathcal{R}(X)$ are well-defined up to conjugacy. In many cases, the strata $Z^{\langle H \rangle}$ are *intrinsic* in that they are fixed by any automorphisms of Z. For details, see [4].

Chapter 3

Kirwan's Procedure and Stable Resolutions

In this chapter, we assume that X is a G-linearized projective variety, such that X^{ss} is smooth.

3.1 Under the assumption that X contains at least one properly stable point, Kirwan defines a sequence of blow-ups $\sigma_i : X_i \to X_{i-1}$ starting with a linearization of $X = X_0$, which produces a G-linearized variety $\tilde{X} = X_N$ such that every semi-stable point is properly stable [3]. Furthermore, the composition σ of the blow-ups σ_i is an isomorphism along the properly stable locus X^{ps} of X. Such a birational model $\tilde{X}$ is known as a *stable resolution* of X.

Thus, the stable resolution X has the property that $\tilde{X} /\!\!/ G$ is projective with an inclusion $X^{ps} /\!\!/ G \hookrightarrow \tilde{X} /\!\!/ G$. Hence, $\tilde{X} /\!\!/ G$ may be considered as a compactification of $X^{ps} /\!\!/ G$ with the property that the fibers of the boundary of $X^{ps} /\!\!/ G$ in $\tilde{X} /\!\!/ G$ are in bijection with orbits of G in $\tilde{X}^{ss} \setminus X^{ps}$.

3.2 The construction of a stable resolution is based on an analysis of stability of points in a G-linearized Y, in terms of X, where Y and X are related by an equivariant morphism $f : Y \to X$. Particularly, Kirwan in [3] works with the scenario when f is a blow-up map. In the case where f is a blowing up along a smooth center, a complete characterization of the properly stable, semi-stable and unstable points of Y can be provided in terms of the corresponding stability loci of X (see [6]).

3.3 Suppose that $\sigma : \hat{X} \to X$ is a blowing-up of X along a closed G-invariant center $\mathcal{C}$. Then $\hat{X}$ is naturally G-linearized via the line bundle

$L_d := \sigma^*(\mathcal{L}^d) \otimes \mathcal{O}(-E)$ for sufficiently large d. In the sequel, we will assume that all blow-ups of linearized varieties X are linearized in this way. Furthermore, we let X_{aff} denote the affine cone of the projective variety $X \subseteq \mathbb{P}(H^0(X, \mathcal{L}))$. That is, X_{aff} is a G-invariant subvariety of the vector space $H^0(X, \mathcal{L})$. For $x \in X$, we define $x_{\text{aff}} \in X_{\text{aff}}$ to be a lift of x in the affine cone X_{aff} of X.

3.4 In birationally modifying X (e.g. by a sequence of blow-ups) to arrive at a stable resolution $\tilde{X}$, one only has to modify the strictly semi-stable locus of X. That is, for any equivariant map $Y \to X$, there is a linearization of Y ensuring that the properly stable and unstable loci of Y do not get any smaller than those of X. In particular, the fibers in Y above such points do not introduce new strictly semi-stable points.

Proposition 3.4.1. *Suppose that X is a G-linearized projective variety. Suppose that $f : Y \to X$ is a G-equivariant map. Then for a suitable linearization of Y,*

1. *If $x \in X$ is unstable, then so is $y \in f^{-1}(x)$.*
2. *If If $x \in X$ is properly stable, then so is $y \in f^{-1}(x)$.*

Proof. See Mumford's Geometric Invariant Theory [5] □

Thus, to define a stable resolution of X by a sequence of blow-ups, one chooses the respective centers according to this observation by choosing centers intersecting strictly semi-stable points of X. In fact, we have the following lemma:

Lemma 3.4.1. *If $X^{ps} \neq \emptyset$, the following are equivalent:*

1. *$X^{ss} = X^{ps}$.*
2. *For every stable point x, $\mathrm{Stab}_G(x_{\text{aff}})$ is finite.*

Proof. ($1 \implies 2$): This is obvious.
($2 \implies 1$): Suppose that $y \in X^{ss}$. $\overline{Gy_{\text{aff}}}$ contains a closed orbit Gx_{aff}. If $Gx_{\text{aff}} \neq \overline{Gy_{\text{aff}}}$, then $Gx_{\text{aff}} \subset \overline{Gy_{\text{aff}}} \backslash Gy_{\text{aff}}$. Then, Gx_{aff} is strictly smaller than Gy_{aff} in dimension. However, as $\mathrm{Stab}_G(x_{\text{aff}})$ is finite, Gx_{aff} is of maximal dimension. Therefore, $Gx_{\text{aff}} = \overline{Gy_{\text{aff}}} = Gy_{\text{aff}}$ and y_{aff} is properly stable.

□

Hence, it suffices to successively blow up X at stable (but not properly stable) points x in a way so that $\mathrm{Stab}_G(x_{\mathrm{aff}})$ strictly decreases in size with each blow up. This is the approach of Kirwan's resolution.

3.5 The following two paragraphs (3.5 and 3.6) will provide an overview of the general argument presented in this section. The required proofs will follow in paragraphs 3.7 and 3.8.

Note that *2.* in Lemma 3.4.1 is equivalent to $\mathrm{Stab}_G(x_{\mathrm{aff}})^0 = \{1_G\}$, or $\dim(\mathrm{Stab}_G(x_{\mathrm{aff}})^0) = 0$ for any stable point x. In fact, we have the following lemma.

Lemma 3.5.1. *Suppose that x is a semi-stable point. Then, any connected subgroup $R \subseteq G_x$ is contained in* $\mathrm{Stab}_G(x_{\mathrm{aff}})^0$*. In particular,* $G_x^0 = \mathrm{Stab}_G(x_{\mathrm{aff}})^0$.

Proof. If R is a connected subgroup of G_x, then R acts on the line generated by x_{aff}. Suppose that $\phi : R \to \mathbb{G}_{\mathrm{m}}$ is the corresponding linear representation of R. Since x is semi-stable, the image of ϕ cannot be dense. Otherwise, $0 \in \overline{Gx_{\mathrm{aff}}}$, and hence x would be unstable. Thus, by the connectedness of R, the image of ϕ is precisely the identity subgroup $\{1\} \subset \mathbb{G}_{\mathrm{m}}$. Therefore, R acts trivially on the line $\mathbb{G}_{\mathrm{m}}.x_{\mathrm{aff}}$. Thus, $R \subseteq \mathrm{Stab}_G(x_{\mathrm{aff}})^0$.

□

Thus, *1.* in Lemma 3.4.1 occurs when no connected positive dimensional subgroup R of G fixes a stable point x.

Recall that for a stable point x, we have that Gx_{aff} is closed X_{aff}. Further, $\mathrm{Stab}_G(x_{\mathrm{aff}})$ is a reductive subgroup of G, and x_{aff} will be contained in $(X_{\mathrm{aff}}^{ss})^{\langle H\rangle}$ for some reductive subgroup H in G. Luna's stratification tells us that there are finitely many possible connected components H^0 of reductive stabilizers, such that $(X_{\mathrm{aff}}^{ss})^{\langle H\rangle}$ is not empty.

Suppose that R is a connected reductive subgroup of G. Let $\langle R\rangle$ denote the conjugacy class formed by the subgroup R. We define:

$Z_R(X_{\mathrm{aff}}^{ss}) := \bigcup_{H^0 \in \langle R\rangle} (X_{\mathrm{aff}}^{ss})^{H^0}$

This is the union of all $V_i \in \mathcal{L}(X_{\mathrm{aff}}^{ss})$ such that the corresponding stabilizer R_i has an identity component conjugate to a fixed connected reductive subgroup R of G.

Corollary 3.5.1. *$x \in X^{ss}$ is a stable point if and only if $x_{\text{aff}} \in Z_R(X_{\text{aff}}^{ss})$ for some connected reductive group R.*

Therefore, X is a linearized projective G-variety with no strictly semi-stable points if and only if $\mathbb{P}(Z_R(X_{\text{aff}}^{ss}))$ is empty for any connected reductive subgroup R of positive dimension. Furthermore, X contains *only* properly stable and unstable points if and only if we also have that $\mathbb{P}(Z_R(X_{\text{aff}}^{ss}))$ is non-empty for $R = \{1_G\}$. In this case, we have that $\mathbb{P}(Z_R(X_{\text{aff}}^{ss})) = X^{ps}$.

3.6 In essence, Kirwan's procedure works by blowing up X at $\overline{\mathbb{P}(Z_R(X_{\text{aff}}^{ss}))}$, for connected reductive subgroups R of positive dimension. This is done so that in the resulting blow-up space $X^{'}$ the projectivized isotropy strata intersecting the exceptional divisor are labelled by reductive subgroups $H^{'}$ whose identity components have dimension strictly smaller than the dimension of R. As there are only finitely many isotropy strata, one may perform finitely many blow-ups (starting with X) and arrive at a $\tilde{X}$ with no strictly semi-stable points. The blow-ups are defined iteratively: one begins with reductive subgroups of maximal dimension ($:= d_X$), labelling the isotropy strata (there are finitely many). After finitely many blow-ups, one arrives at a blow up space $X^{'}$ such that the maximal dimension among reductive subgroups labelling the isotropy strata of $(X_{\text{aff}}^{'})^{ss}$ is strictly less than d_X. This is repeated until there are no positive dimensional connected subgroups R such that $\text{Stab}(x)^0 = R$ for any $x \in \tilde{X}$.

3.7 In the following paragraphs, we provide the details and proofs required to carry out the construction outlined in paragraphs 3.5 and 3.6. Suppose that R is a connected reductive subgroup of G, maximal in dimension with the property that $R = \text{Stab}(x)^0$ for some $x \in X$.

Lemma 3.7.1. $Z_R(X_{\text{aff}}^{ss}) = GX_{\text{aff}}^R \cap X_{\text{aff}}^{ss}$.

Proof. Suppose that $x \in X_{\text{aff}}^{ss\langle H\rangle}$, where $H^0 \in \langle R\rangle$. Then, for some $g \in G$, $\text{Stab}(x) = gHg^{-1}$. Therefore, $ghg^{-1}x = x$, for any $h \in H$. In other words, $g^{-1}x \in X_{\text{aff}}^H$. Thus, $x \in GX_{\text{aff}}^R \cap X_{\text{aff}}^{ss}$. Conversely, suppose that $x \in GX_{\text{aff}}^R \cap X_{\text{aff}}^{ss}$. By definition, $\text{Stab}(x) \in \langle H\rangle$, for some H where $H^0 \in \langle R\rangle$. It remains to show that Gx is closed. Since $\text{Stab}(x) \in \langle H\rangle$, it is of maximal dimension among stabilizer subgroups of G acting on X_{aff}. Thus, Gx is an orbit of minimal dimension and is hence closed in X_{aff}. □

Now, we prove that $G(X_{\text{aff}}^R)^{ss} = GX_{\text{aff}}^R \cap X_{\text{aff}}^{ss}$ is smooth in $\mathcal{C}_{\text{aff}} := \overline{G(X_{\text{aff}}^R)^{ss}}$ (where the closure is taken in X_{aff}).

Proposition 3.7.1. *Suppose that $y \in (\mathcal{C}_{\text{aff}})^{ss}$. Then,*

1. *$y \in G(X^R_{\text{aff}})^{ss}$. Hence, $\mathcal{C}^{ss}_{\text{aff}} = G(X^R_{\text{aff}})^{ss}$. That is, taking the closure of $G(X^R_{\text{aff}})^{ss}$ in X_{aff} does not introduce any more semistable points.*
2. *$G(X^R_{\text{aff}})^{ss}$ is smooth.*

Proof. 1. As in the proof of Lemma 3.7.1, we see that y is contained in an orbit of minimal dimension. Therefore, Gy is closed. Now by the Luna Slice Theorem, there is an (Zariski) open U containing y, such that for any $z \in U$, $\text{Stab}(z)$ is conjugate to a subgroup of $\text{Stab}(y)$. As $y \in \mathcal{C}^{ss}_{\text{aff}} = \overline{G(X^R_{\text{aff}})^{ss}}$, one may choose such a $z \in U$ to be contained in $G(X^R_{\text{aff}})^{ss}$. Then, $gRg^{-1} \subset \text{Stab}(z)$ for a suitable $g \in G$. In turn, we thus have that $tRt^{-1} \subset \text{Stab}(y)$ for suitable $t \in G$. Therefore, $y \in t.X^R$ and $y \in G(X^R_{\text{aff}})^{ss}$ as required. Since y is an arbitrary semi-stable point of $\overline{GX^R_{\text{aff}}}$, we have that $(\overline{GX^R_{\text{aff}}})^{ss} = G(X^R_{\text{aff}})^{ss}$.

2. It suffices to prove that $G(X^R_{\text{aff}})^{ss}$ is smooth for $x \in (X^R_{\text{aff}})^{ss} \subset G(X^R_{\text{aff}})^{ss}$. We have seen that such an x has a closed orbit, and so we apply the Luna Slice Theorem to both the G action on X_{aff} and the $N := N_G(R)$ action on X^R_{aff}. This provides us with étale slices S and S' respectively. Then since both X_{aff} and X^R_{aff} are smooth at x, we have the tangent space decompositions,

$$T_x(X_{\text{aff}}) = T_x(Gx) \oplus T_x(S)$$

$$T_x(X^R_{\text{aff}}) = T_x(Nx) \oplus T_x(S')$$

We have étale maps (defined at x), $\Psi_S : X^{ss}_{\text{aff}} \to G \times^{G_x} S$ and $\Psi_{S'} : (X^R_{\text{aff}})^{ss} \to N \times^{G_x} S'$, respectively. Furthermore, by smoothness at x, we have étale maps $\phi_S : S \to T_x(S)$ and $\phi_{S'} : S' \to T_x(S')$. Thus, X^{ss}_{aff} and $(X^R)^{ss}$ are étale equivalent (at x) to $G \times^{G_x} T_x(S)$ and $N \times^{G_x} T_x(S')$, respectively.

Now, we have that $T_x(X^R_{\text{aff}}) = T_x(X_{\text{aff}})^R = T_x(Gx)^R \oplus T_x(S)^R = T_x(Nx) \oplus T_x(S')$. On the other hand, $T_x(Gx)^R = T_x(Gx^R)$. $p \in (Gx)^R$ iff $p = gx$ for some $g \in G$ and $r.p = p$ for all $r \in R$. Therefore, $(Gx)^R = Nx$. So, $T_x(S)^R \cong T_x(S')$ as G_x-modules.

Consider now the following diagram:

$$
\begin{tikzcd}
N \times^{G_x} S' \arrow[r, "\Psi_{S'}"] \arrow[d, "\phi_{S'}"] & (X^R)^{ss} \\
N \times^{G_x} T_x(S') \arrow[d, hook, "i"] & \\
G \times^{G_x} T_x(S) & G \times^{G_x} S \arrow[l, "\phi_S"'] \arrow[r, "\Psi_S"] & X^{ss}
\end{tikzcd}
$$

where $\Psi_{S'}$, Ψ_S are étale at x, and ϕ_S, $\phi_{S'}$ are étale. $\Psi_{S'}$, $\phi_{S'}$ are N-equivariant, and Ψ_S, ϕ_S are G-equivariant. The inclusion i is induced by the inclusion of N-representations $T_x(S') \hookrightarrow T_x(S)$, and the inclusion $N \hookrightarrow G$. Hence, i is R-equivariant.

We would like to conclude that $G(X^R_{\text{aff}})^{ss}$ is smooth at x. We have $i : N \times^{G_x} T_x(S') \hookrightarrow G \times^{G_x} T_x(S)$. G acts naturally on the image $i(N \times^{G_x} T_x(S'))$, so that $G.i(N \times^{G_x} T_x(S')) = G \times^{G_x} T_x(S')$. Also, R acts trivially on $i(N \times^{G_x} T_x(S'))$. As R is connected and ϕ_S is étale, we have $\phi_S^{-1}(i(N \times^{G_x} T_x(S'))) = (G \times^{G_x} S)^R$. Indeed, recall that the fiber of an étale map over a point is set-theoretically finite. Thus by the connectedness of R, the preimages of R-fixed points are also R-fixed points. Also, the image of $(G \times^{G_x} S)^R$ under the étale map Ψ_S is $(X^{ss}_{\text{aff}})^R$. Finally, as the maps on the bottom row in the diagram are G-equivariant, we have $\phi_S^{-1}(G \times^{G_x} T_x(S')) = \phi_S^{-1}(G.i(N \times^{G_x} T_x(S'))) = G.(G \times^{G_x} S)^R$, and that $\Psi_S(G.(G \times^{G_x} S)^R) = G.(X^R_{\text{aff}})^{ss}$.

Since x is contained in the image of Ψ_S, we have that $G \times^{G_x} T_x(S')$ is étale equivalent to $G.(X^R_{\text{aff}})^{ss}$ at x. Finally, as $G \times^{G_x} T_x(S')$ is smooth, we have that $G.(X^R_{\text{aff}})^{ss}$ is smooth at x.

□

The above proof shows the following:

Corollary 3.7.1. *$G \times^N (X^R_{\text{aff}})^{ss} \to G(X^R_{\text{aff}})^{ss}$ is étale. In particular, $T_x(G(X^R_{\text{aff}})^{ss}) = T_x(Gx) \oplus T_x(X^R)$*

Proof. For $x \in (X^R_{\text{aff}})^{ss}$, we have seen in the proof of *2.* in Proposition 3.7.1, that there is an étale map $G.((G \times^{G_x} S)^R) \to G(X^R_{\text{aff}})^{ss}$ with x contained in the image.

Then, we have an étale map $G.(G \times^{G_x} S)^R) \to G \times^{G_x} T(S')$. We have étale maps $G \times^N (X^R_{\text{aff}})^{ss} \leftarrow G \times^N (N \times^{G_x} S') \to G \times^N (N \times^{G_x} T(S')) \cong$

$G \times^{G_x} T(S')$. It is easy to check that the above étale maps factors the nautral map $G \times^N (X^R_{\text{aff}})^{ss} \to G(X^R_{\text{aff}})^{ss}$.

Thus, $G \times^N (X^R_{\text{aff}})^{ss} \to G(X^R_{\text{aff}})^{ss}$ is étale.

□

Therefore, see that $Z_R(X^{ss}_{\text{aff}})$ is smooth, and thus so is $\mathbb{P}(Z_R(X^{ss}_{\text{aff}})) = G(X^R)^{ss}$. We have the following:

Lemma 3.7.2. *We have:*

1. $(\overline{G(X^R)^{ss}})_{\text{aff}} = \mathcal{C}_{\text{aff}}$
2. $\mathbb{P}(\mathcal{C}_{\text{aff}}) = \overline{\mathbb{P}(Z_R(X^{ss}_{\text{aff}}))} = \overline{G(X^R)^{ss}}$, *which we will denote as* $\mathcal{C} \subset X$.

Proof. Note that $G(X^R_{\text{aff}})^{ss}$ is $\mathbb{G}_{\text{m}}$-invariant, hence so is $\mathcal{C}_{\text{aff}}$. Therefore, $\mathcal{C}_{\text{aff}}$ is defined by a homogeneous ideal in X_{aff} and corresponds to a closed projective subvariety in X. Therefore, the result follows from the projective Nullstellensatz.

□

By this lemma, along with Proposition 3.7.1, we see that $\overline{G(X^R)^{ss}} \backslash G(X^R)^{ss}$ consists of only unstable points.

3.8 In this paragraph, we complete the construction of Kirwan's stable resolution.

Theorem 3.8.1. *In the notation above, Suppose that* $\sigma : X' \to X$ *is the blow-up of* X *along the center* $\overline{\mathbb{P}(Z_R(X^{ss}_{\text{aff}}))} = \overline{G(X^R)^{ss}}$ *in* X. *Then, no subgroup of* G *conjugate to* R *stabilizes any semi-stable point of* X'.

Proof. Suppose that y is a semi-stable point of X', which is fixed by a subgroup of the form $g^{-1}Rg$. Then, for all $r \in R$, $(g^{-1}rg)y = y$, which implies that $r(gy) = gy$. Hence, gy is a semi-stable point of X' fixed by R. Then, $\sigma(gy) = g\sigma(y) = gx$ is also fixed by R. Hence, $gx := x' \in X^R$ and is semi-stable in X. Thus, there exists a homogeneous G-invariant polynomial $f \in \mathcal{A}(X_{\text{aff}})^G$ such that $f(x') \neq 0$. That is, X_f is an affine open subvariety of X which contains x'. We have that $T_{x'}(X) = T_{x'}(G(X^R)^{ss}) \oplus \mathcal{N}_{x'} = T_{x'}(X_f) = T_{x'}(G(X^R_f)^{ss}) \oplus \mathcal{N}_{x'}$. Where $\mathcal{N}_{x'}$ is a G-invariant complement to $T_{x'}(\overline{G(X^R)^{ss}})$. We have seen that $T_{x'}(G(X^R_f)^{ss}) = T_{x'}(Gx') \oplus T_{x'}((X^{ss}_f)^R)$ (corollary 3.7.1), where $T_{x'}((X^{ss}_f)^R) = T_{x'}((X^{ss}_f))^R = T_{x'}(X')^R$. Therefore, the action of R on $\mathcal{N}_{x'}$ contains no fixed points.

However, as since x' is a smooth point of $\overline{G(X^R)^{ss}}$, the fiber $\sigma^{-1}(x')$ is R-equivariantly isomorphic to $\mathbb{P}(\mathcal{N}_{x'})$. Now, $gy \in \sigma^{-1}(x')$ is an R-fixed point in $\mathbb{P}(\mathcal{N}_{x'})$. By Lemma 3.5.1, there exists an R-fixed point in $\mathcal{N}_{x'}$, which is a contradiction.

□

Now, suppose that for a positive dimensional connected reductive subgroup R, $x \in \mathbb{P}(Z_R(X^{ss}_{\text{aff}}))$, and that x' is a semi-stable point of X' with $\sigma(x') = x$. Clearly, $\text{Stab}_G(x') \subset G_x$. Since x' is semi-stable, Proposition 3.7.1 implies that $\text{Stab}_G(x')^0 \subsetneq G_x^0 = R$. Since $\text{Stab}_G(x')^0$ is a connected proper subgroup of G_x^0, we have that $\dim(\text{Stab}_G(x')^0) \lneq \dim(G_x^0)$. Therefore, for $x \in X$, we in general have that $\dim(\text{Stab}_G(x')^0) \leq \dim(G_x^0)$; equality may only occur when either x is not contained in the center of the blow up, or if $0 \leq \dim(G_x^0) < \dim(R)$.

3.9 Therefore, to conclude the construction of the stable resolution, we may perform the following algorithm:

1. Given a linearized projective G-variety X with X^{ss} smooth, consider the set of Luna strata $\mathcal{L}(X^{ss}_{\text{aff}})$, and the set of corresponding reductive subgroups $\mathcal{R}(X^{ss}_{\text{aff}})$. These sets are finite in cardinality, so we may define $d_X = \max_{R_i \in \mathcal{R}(X^{ss}_{\text{aff}})}(\dim(R_i))$. Furthermore, there are only finitely many $R_i \in \mathcal{R}(X^{ss}_{\text{aff}})$ with $\dim(R_i) = d_X$.

2. For a reductive subgroup $R < G$ such that $(X^{ss}_{\text{aff}})^R \neq \emptyset$ and $\dim(R) = d_X$, we must have that R is conjugate to a subgroup in $\mathcal{R}(X^{ss}_{\text{aff}})$ by maximality. Consider $Z_R(X^{ss}_{\text{aff}})$ as defined in paragraph 3.5. $\mathbb{P}(Z_R(X^{ss}_{\text{aff}})) \subset X^{ss}$ is smooth by Proposition 3.7.1. $\mathcal{C} = \overline{\mathbb{P}(Z_R(X^{ss}_{\text{aff}}))} = \overline{G(X^{R^0})^{ss}}$ (see Lemma 3.7.2) is such that $\mathcal{C} \setminus \mathbb{P}(Z_R(X^{ss}_{\text{aff}}))$ consists only of unstable points.

3. Consider the blow up $X' \to X$ centered at $\mathcal{C}$. By 2., $(X')^{ss}$ is smooth. By paragraph 3.8, $\mathcal{R}((X')^{ss}_{\text{aff}})$ consists of subgroups whose dimensions are less than or equal to the dimension of those in $\mathcal{R}(X^{ss}_{\text{aff}})$. Furthermore, the number of subgroups with $dim(R) = d_X$ is strictly smaller. Indeed, no conjugate of R is contained in $\mathcal{R}((X')^{ss}_{\text{aff}})$. By 1., there are only finitely many reductive stabilizer subgroups R of X^{ss}_{aff} such that $\dim(R) = d_X$. Iterate steps 1-3 until there are no longer any such subgroups.

4. When step 3 is completed, we will have a linearized projective G-variety X' such that $(X')^{ss}$ is smooth, and $d_{X'} \lneq d_X$. Repeat steps 1-4 until we have a linearized projective G-variety $\tilde{X}$ such that $d_{\tilde{X}} = 0$. By Lemma 3.4.1, $\tilde{X} \to X$ is a stable resolution of X.

That is, the above procedure provides a proof of Theorem **(A)** stated in the introduction (chapter 1).

3.10 In this next paragraph, we conclude the section with a few remarks.

1. In [3], Kirwan proves that blowing up X, in the way described in paragraph 3.6 under the condition that the center of the blow-up is smooth. However, $\mathcal{C} = \overline{\mathbb{P}(Z_R(X^{ss}_{\text{aff}}))}$ may be singular. Thus, Kirwan first performs a resolution of singularities on $\mathcal{C}$ before blowing up along it. Fortunately, one does not need to know the explicit form of the resolution of singularities of $\mathcal{C}$ to continue with Kirwan's procedure. This is due to the fact that $\mathbb{P}(Z_R(X^{ss}_{\text{aff}}))$ is smooth. Thus, the singularities of $\mathcal{C}$ lie strictly on the boundary of $\mathcal{C}$, which consists only of unstable points.

2. We may take Y to be the resulting blow-up space obtained by resolving the singularities of $\mathcal{C}$ in X. The linearization of Y may be taken to be the one described in paragraph 3.3. The result is that one has a sequence of blow-ups X_k $(:= Y) \to X_{k-1} \to .. \to X_0$ $(:= X)$ such that the center of each blow-up $X_i \to X_{i-1}$ is contained in the unstable locus of X_{i-1}. Therefore, the composition σ of the above blow-ups induces an isomorphism $\sigma : Y^{ss} \xrightarrow{\sim} X^{ss}$. Furthermore, as each blow-up is equivariant, σ also induces an isomorphism along $\mathbb{P}(Z_R(Y^{ss}_{\text{aff}}))$ and $\mathbb{P}(Z_R(X^{ss}_{\text{aff}}))$. Then, Kirwan considers blowing up Y along $\mathbb{P}(Z_R(Y^{ss}_{\text{aff}})) \cong \mathbb{P}(Z_R(X^{ss}_{\text{aff}}))$, where now $\overline{\mathbb{P}(Z_R(Y^{ss}_{\text{aff}}))}$ is smooth in Y.

3. Therefore, if X is smooth, and one performs blow-ups along smooth centers $\mathcal{C}$, then one may construct a stable resolution $\tilde{X} \to X$ such that $\tilde{X}$ is smooth (so that not just $\tilde{X}^{ss}$ is guaranteed to be smooth).

Chapter 4

Abelianization Procedure

In the last chapter, we provided a proof of Theorem **(A)** as presented in the introduction (see paragraph 3.9). In this chapter, we present a proof of Theorem **(B)**.

4.1 An important consequence of Proposition 3.4.1 is that once one has obtained a stable resolution $\tilde{X} \to X$, then the composition $\tilde{X}' \to \tilde{X} \to X$ is also a stable resolution of X (for a suitable linearization of $\tilde{X}'$) for any G-equivariant birational morphism $\tilde{X}' \to \tilde{X}$. Even more, if $Y \to \tilde{X}$ is any G-equivariant map, then Proposition 3.4.1 says that Y also has the property that, for a suitable linearization of Y, all points of Y are either properly stable, or unstable. Thus, if $\tilde{X} \to X$ is a stable model of X, then one cannot lose the property of being a stable model by resolving $\tilde{X}$ further.

4.2 In [7] Reichstein and Youssin define a type of resolution of generically free G-variety X: it is a sequence of blow-ups $\pi : X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \dots X_1 \xrightarrow{\pi_1} X$ so that X_n is in *standard form* with respect to a certan divisor D of X_n. The significance of this is that there are choices of D so that the stabilizers in X_n will be "tame" stabilizers.

Definition 4.2.1 (G-variety in standard form [7]). A generically free G-variety X is said to be in standard form with respect to a divisor D if:

1. X is smooth, and D is a normal crossing divisor on X.
2. The action of G on $X \setminus D$ is free
3. For each irreducible component D_i of D, for any $g \in G$, we either have $gD_i = D_i$, of $gD_i \cap D_i = \emptyset$

4.3 One of the main theorems in [7] is the following theorem. We state it here for reference (Theorem 3.2):

Theorem 4.3.1. *Let X be a smooth G-variety and $Y \subsetneq X$ be a closed G-invariant subvariety such that the action of G on $X \setminus Y$ is free. Then, there is a sequence of equivariant blow-ups:*

$$\pi : X_n \to X_{n-1} \cdots \to X$$

with smooth G-invariant centers $C_i \subset X_i$ such that X_n is in standard form with respect to the the divisor $D := E_n \cup \pi^{-1}(Y)$, where E_n is the exceptional divisor of π.

4.4 Let X be a smooth linearized generically free G-variety. Then, we may apply Theorem 4.3.1 and obtain a $X_n \to X$ such that X_n is in standard form with respect to some divisor D. We show in this paragraph that for a suitable linearization of X_n, all *stable* points have abelian stabilizers. Thus, we define $X_{\text{ab}} := X_n$, where X_{ab} implicitly depends on the divisor D. X_{ab} is linearized following Proposition 3.4.1 (as usual).

Corollary. *If $x \in X^{ss}_{ab}$ is stable, then G_x is an abelian subgroup of G.*

Proof. Note that since x is stable, G_x is a reductive subgroup of G. We have that the action of G on $X_{\text{ab}} \setminus D$ is free, so all such points have trivial stabilizers. It suffices to consider $x \in X^{ss}_{\text{ab}} \cap D$. Suppose that x is contained in components $D_1, \dots, D_k$ components of D. By property 3. in the definition of a G-variety in standard form, we have that each D_i is G_x-invariant, and so is $W = D_1 \cap \cdots \cap D_k$. Then, we have the tangent space decompositions $T_x(X_{\text{ab}}) = T_x(D_i) \oplus V_i$, where V_i is G_x-invariant. Hence, V_i is 1-dimensional, and G_x acts on V_i by a character χ_i.

Furthermore, we have then that $T_x(X_{\text{ab}}) = T_x(W) \oplus (\bigoplus_{i=1}^m V_i)$ by the normal crossing property 1. in Definition 4.2.1. This is a G_x-invariant decomposition, and G_x acts on $V := \bigoplus_{i=1}^k V_i$ by the characters $(\chi_1, \dots, \chi_k)$. That is, we have a homomorphism $\chi : G_x \to \mathbb{G}_m^k$. If χ has trivial kernel, then $\text{Stab}_x(G)$ is necessarily abelian.

Suppose that $\ker(\chi)$ is non-trivial. Then, $x \in (X^{ss}_{\text{ab}})^{\ker(\chi)}$. In fact, as x is semi-stable, we may find a G-invariant affine open subset $(X_{\text{ab}})_f$ so that $x \in (X_{\text{ab}})_f^{\ker(\chi)} \subset X^{ss}_{\text{ab}}$. As the action of G on $X_{\text{ab}} \setminus D$ is free, we actually have that $(X_{\mathrm{ab}})_f^{\ker(\chi)}$ is contained in D. As $(X_{\mathrm{ab}})_f$ is smooth and affine, $(X_{\mathrm{ab}})_f^{\ker(\chi)}$ is smooth. Thus, $(X_{\mathrm{ab}})_f^{\ker(\chi)}$ is entirely contained in a single component D_i for some i.

We have that $T_x((X_{\mathrm{ab}})_f^{\ker(\chi)}) = T_x((X_{\mathrm{ab}})_f)^{\ker(\chi)} \subset T_x(D_i)$. However, by definition $V_i \subset T_x((X_{\mathrm{ab}})_f)^{\ker(\chi)}$. As V_i and $T_x(D_i)$ are direct sum complements of each other, we have a contradiction. Therefore, $\ker(\chi) = \{1_{G_x}\}$, and G_x is finite and abelian.

$\square$

4.5 Let us illustrate the above theorem (and corollary) when one has a stable resolution $\sigma : \tilde{X} \to X$, where X is a generically free G-linearized projective variety with $X^{ps} \neq \emptyset$. Note then that $\tilde{X}$ is also a generically free G-variety, and we of course have that $\tilde{X}^{ss} = \tilde{X}^{ps}$ (containing points with finite stabilizer subgroups). By paragraph 3.10, we can arrange $\tilde{X}$ to be smooth by constructing a stable resolution $\tilde{X} \to X$ by blowing up along smooth centers. Alternatively, one may equivariantly resolve the singularities of any stable resolution $\tilde{X} \to X$ to arrive at a smooth $\tilde{X}_{\mathrm{sm}}$ that is a stable resolution of X (see the remark in paragraph 4.1). Therefore, let us assume that we have constructed a stable resolution $\tilde{X} \to X$ such that $\tilde{X}$ is smooth. Applying Theorem 4.3.1 to $\tilde{X}$, we obtain a smooth $\tilde{X}_{\mathrm{ab}}$ in standard form with respect to some divisor D, with an equivariant birational morphism $\tilde{X}_{\mathrm{ab}} \to \tilde{X}$. $\tilde{X}_{\mathrm{ab}}$ contains only properly stable and unstable points, so that $\tilde{X}_{\mathrm{ab}} \to X$ is a stable model of X. Furthermore, every $x \in \tilde{X}_{\mathrm{ab}}^{ps}$ is so that G_x is a *finite* abelian subgroup of G.

4.6 In Theorem 4.3.1, the centers of blow up contain only points of Y and its preimages (see the proof in [7], Theorem 3.2). If G acts freely on X^{ps}, then we may take $Y = X \setminus X^{ps}$. The result of this choice of Y will be that $\rho : X_{\mathrm{ab}} \to X$ is an isomorphism along X^{ps}. Therefore, we see that if G acts freely on X^{ps}, then we may find a (smooth) stable resolution $\tilde{X}$ and $\tilde{X}_{\mathrm{ab}}$ as above, such that $\tilde{X}_{ab} \to X$ is an isomorphism along X^{ps}. Thus, we have in this case that $\tilde{X}_{\mathrm{ab}} \to X$ is a still a stable resolution if $\tilde{X} \to X$ is a stable resolution. Note that if G acts freely on X^{ps}, then it acts freely on $\tilde{X}^{ps}$.

Remark 4.6.1. Note that in the statement of Theorem 4.3.1, X is required to be a generically free G-variety. It is necessary to impose a condition on the stabilizer in general position of X, as $X_{\mathrm{ab}} \to X$ is a birational map. Indeed, if X has a non-abelian stabilizer in general position, then such a birational map is impossible. X_{ab} is a G-variety with an open set containing points with abelian stabilizer, and thus cannot have a non-abelian stabilizer in general position. An example of such an X is the space of cubic curves on $\mathbb{P}^2$, with the natural action of PGL_3.

Remark 4.6.2. A stronger form of the above corollary is contained in [7] as Theorem 4.1. The authors prove that if X is in standard form with respect to a divisor D, then G_x is isomorphic to a semidirect product of a unipotent group, and a diagonalizable group. In particular, If G_x is reductive, then it is a abelian. Note that the statement of this result does not involve any linearizations. Therefore, the claim that stable points of X have abelian stabilizers is independent of a choice of linearization of X.

Remark 4.6.3. In the case, where G is a finite group, the above corollary was independently proved by Batyrev [1] and Borisov-Gunnels [2].

Bibliography

[1] V. V. Batyrev. Canonical abelianization of finite group actions. *ArXiv Mathematics e-prints*, Sept. 2000.

[2] L. A. Borisov and P. E. Gunnells. Wonderful blowups associated to group actions. *Selecta Math. (N.S.)*, 8(3):373–379, 2002.

[3] F. C. Kirwan. Partial desingularisations of quotients of nonsingular varieties and their betti numbers. *Ann. Of Math. (2)*, 122(1):45–85, 1985.

[4] J. Kuttler and Z. Reichstein. Is the luna stratification intrinsic? *Ann. Inst. Fourier*, 58(2):689–721, 2008.

[5] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory, 3rd Edition.* Springer-Verlag, Berlin, 1994.

[6] Z. Reichstein. Stability and equivariant maps. *Invent. Math.*, 96(2):349–383, 1989.

[7] Z. Reichstein and B. Youssin. Essential dimensions of algebraic groups and a resolution theorem for G-varieties. *Canad. J. Math.*, 52(5):1018–1056, 2000.