Abstract

This essay focuses on exploring dilatons as an alternate model to the Higgs mechanism. An introductory analysis to the Higgs mechanism, effective potential method, and dilatons is provided. Then, three different models are explored on how to obtain a light dilaton that emerges as a pseudo Goldstone boson because of the spontaneously broken approximate scale invariance. This light dilaton is shown to have properties that are, in general, similar to the Higgs boson with minor differences that can differentiate between the two models in collider experiments.
Preface

This dissertation is original, unpublished, independent work by the author, Hassan Saadi.
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Chapter 1

Introduction

The Standard Model (SM) in particle physics consists of three generations of quarks and leptons, four bosons, and the Higgs boson. The quarks are spin-$\frac{1}{2}$ particles with 6 flavors: up, down, charm, strange, top, and bottom

\[
\begin{pmatrix}
  u \\
  d \\
\end{pmatrix}_L, \quad \begin{pmatrix}
  c \\
  s \\
\end{pmatrix}_L, \quad \begin{pmatrix}
  t \\
  b \\
\end{pmatrix}_L
\] (1.1)

And the leptons also with 6 flavors: electron, electron neutrino, muon, muon neutrino, tau, and tau neutrino

\[
\begin{pmatrix}
  \nu_e \\
  e^- \\
\end{pmatrix}_L, \quad \begin{pmatrix}
  \nu_\mu \\
  \mu^- \\
\end{pmatrix}_L, \quad \begin{pmatrix}
  \nu_\tau \\
  \tau^- \\
\end{pmatrix}_L
\] (1.2)

where the subscript $L$ denotes left-handed components. The SM has been consistent with experiment. However, the SM does not explain some phenomena such as dark matter, dark energy, gravity, neutrino oscillations, and the hierarchy problem (why the Higgs’ mass is small and doesn’t receive radiative corrections). In this chapter, how the Higgs mechanism gives mass to the particles of SM is demonstrated. Then, the effective potential description is derived.
1.1 The Higgs Mechanism

Let’s define

\[ \psi_{L_e} = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \]  
(1.3)

\[ \psi_{L_q} = \begin{pmatrix} u \\ d \end{pmatrix}_L \]  
(1.4)

In the following equations, one generation of quarks and leptons is considered and the neutrino is massless. The generalization to any number of generations is straightforward.

The Lagrangian of the SM consists of four parts: gauge, fermion, Yukawa, and Higgs (or scalar)

\[ \mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{fermion} + \mathcal{L}_Y + \mathcal{L}_{Higgs} \]  
(1.5)

\[ \mathcal{L}_{gauge} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} (W^a_{\mu\nu})^2 - \frac{1}{2} \text{Tr} (G_{\mu\nu} G^{\mu\nu}) \]  
(1.6)

\[ \mathcal{L}_{fermion} = \bar{\psi}_{L_e} i\gamma^\mu \left( \partial_\mu - \frac{i}{2} g' B_\mu + \frac{i}{2} g a W^a_\mu \right) \psi_{L_e} + \bar{e}_R i\gamma^\mu \left( \partial_\mu - i g' B_\mu \right) e_R 
+ \bar{\psi}_{L_q} i\gamma^\mu \left( \partial_\mu + \frac{i}{6} g' B_\mu + \frac{i}{2} g a W^a_\mu + i g s \lambda a G^a_\mu \right) \psi_{L_q} 
+ \bar{u}_R i\gamma^\mu \left( \partial_\mu + \frac{2i}{3} g' B_\mu + i g s \lambda a G^a_\mu \right) u_R 
+ \bar{d}_R i\gamma^\mu \left( \partial_\mu - \frac{i}{3} g' B_\mu + i g s \lambda a G^a_\mu \right) d_R \]  
(1.7)

\[ \mathcal{L}_Y = -\zeta_e \left[ \bar{e}_R \phi^\dagger \psi_{L_e} + \bar{\psi}_{L_e} \phi e_R \right] - \zeta_d \left[ \bar{d}_R \phi^\dagger \psi_{L_q} + \bar{\psi}_{L_q} \phi d_R \right] 
- \zeta_u \left[ \bar{u}_R \phi^\dagger \psi_{L_q} + \bar{\psi}_{L_q} \phi u_R \right] \]  
(1.8)

\[ \mathcal{L}_{Higgs} = (D_\mu \phi)^\dagger (D_\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \]  
(1.9)
where

\[ B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \implies U(1)_Y \text{ gauge field} \quad (1.10) \]

\[ W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc}W^b_\mu W^c_\nu \implies SU(2)_L \text{ gauge fields} \quad (1.11) \]

\[ G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s\epsilon^{abc}G^b_\mu G^c_\nu \implies SU(3)_c \text{ gauge fields} \quad (1.12) \]

\[ \tau_a W^a_\mu = \begin{pmatrix} W^3_\mu & \sqrt{2}W^a_\mu \\ \sqrt{2}W^a_\mu & -W^3_\mu \end{pmatrix} \quad (1.13) \]

\[ D_\mu \phi = \left( \partial_\mu + \frac{i}{2}g' B_\mu + \frac{i}{2}g\tau_a W^a_\mu \right) \phi \quad (1.14) \]

\[ \phi \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad \tilde{\phi} = i\tau_2 \phi^* = \begin{pmatrix} \phi^0 \\ -\phi^+ \end{pmatrix} \quad (1.15) \]

and \( \zeta \) is Yukawa’s coupling, \( \epsilon^{abc} \) is an antisymmetric three-index symbol, \( \gamma^\mu \) are Dirac matrices, \( \tau_a \) are Pauli matrices, and \( \lambda_a \) are Gell-mann matrices listed in appendix A.

Now let’s modify the potential by assuming that \( \mu^2 < 0 \) so that we have a Mexican-hat potential. Spontaneous symmetry breaking occurs when the vacuum picks up an expectation value. Let’s choose the vacuum expectation value of the scalar field to be

\[ \langle \phi \rangle_0 = \begin{pmatrix} 0 \\ \frac{\nu}{\sqrt{2}} \end{pmatrix} \quad (1.16) \]

where \( \nu = \sqrt{-\mu^2/\lambda} \). When the vacuum picks up an expectation value it breaks the symmetries of \( SU(2)_L \) and \( U(1)_Y \), but it preserves \( U(1)_{EM} \) symmetry. \( \phi \) can be expressed in a new gauge \( \phi' \) without a phase as

\[ \phi(x) = \exp \left( \frac{i\pi^a \tau_a}{2\nu} \right) \begin{pmatrix} 0 \\ \frac{\nu+\rho}{\sqrt{2}} \end{pmatrix} \quad (1.17) \]

\[ \phi'(x) = \exp \left(-\frac{i\pi^a \tau_a}{2\nu}\right) \phi = \begin{pmatrix} 0 \\ \frac{\nu+\rho}{\sqrt{2}} \end{pmatrix} \quad (1.18) \]

where \( \rho(x) \) and \( \pi(x) \) are associated with the radial and angular axes respectively in the Mexican hat potential of the Higgs, \( \tau_a \) are Pauli matrices, and \( \pi_a \) are real fields (would-be
Goldstone). Expressing (1.14) after $\phi$ picking up an expectation value

$$D_\mu\phi = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -i\sqrt{2}gW_\mu^+\rho + \nu \\ \partial_\mu\rho - i(g'B_\mu - gW_\mu^3)\nu \end{array} \right)$$

(1.19)

Expanding the Higgs Lagrangian (1.9) about the minimum of the Higgs potential by using (1.13) and (1.19) yields

$$L_{\text{Higgs}} = \frac{g^2}{4}W_\mu^+W^\mu_\mu(\rho + \nu)^2 + \frac{1}{8}(g'B_\mu - gW_\mu^3)(g'B_\mu - gW_\mu^3)(\rho + \nu)^2 + \frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{1}{2}\rho^2(\nu + \rho)^2 - \frac{1}{4}\lambda(\nu + \rho)^4$$

(1.20)

We rotate $B_\mu$ and $W^3_\mu$ so that the photon is massless

$$\left( \begin{array}{c} Z^0_\mu \\ A_\mu \end{array} \right) = \frac{1}{\sin(\theta_W)} \left( \begin{array}{cc} -\cos(\theta_W) & \sin(\theta_W) \\ \sin(\theta_W) & \cos(\theta_W) \end{array} \right) \left( \begin{array}{c} B_\mu \\ W^3_\mu \end{array} \right)$$

(1.21)

where

$$\sin(\theta_W) = \frac{g'}{\sqrt{g^2 + g'^2}}$$

(1.22)

$$\cos(\theta_W) = \frac{g}{\sqrt{g^2 + g'^2}}$$

(1.23)

and the masses of the particle can be read off the Lagrangian (1.20) (Quigg, 2013)

$$M_Z = \frac{\nu}{2}\sqrt{g^2 + g'^2} \simeq 91.2 \text{ GeV}$$

(1.24)

$$M_{W^\pm} = \frac{1}{2}\nu \simeq 80.4 \text{ GeV}$$

(1.25)

$$\Rightarrow \frac{M_{W^\pm}}{M_Z} = \cos(\theta_W)$$

(1.26)

and $\nu = (G_F\sqrt{2})^{-1/2} \simeq 246 \text{ GeV}$, so $\langle \phi^0 \rangle = (G_F\sqrt{8})^{-1/2} \simeq 174 \text{ GeV}$ where $G_F = 1.663787(6) \times 10^{-5} \text{ GeV}^{-2}$ is Fermi constant and $\sin(\theta_W) = 0.2122$ is the weak mixing parameter. The Yukawa Lagrangian (1.8) becomes

$$L_Y = -\zeta_e \frac{\rho + \nu}{\sqrt{2}} \left[ \bar{e}_{Re}e_L + \bar{e}_{Le}e_R \right] - \zeta_d \frac{\rho + \nu}{\sqrt{2}} \left[ \bar{d}_{Rd} + \bar{d}_{Ld} \right] - \zeta_u \frac{\rho + \nu}{\sqrt{2}} \left[ \bar{u}_{Ru} + \bar{u}_{Lu} \right]$$

(1.27)
and the masses of the electron, up and down quarks can be read off

\[ m_e = \frac{\zeta e \nu}{\sqrt{2}} \quad (1.28) \]
\[ m_d = \frac{\zeta d \nu}{\sqrt{2}} \quad (1.29) \]
\[ m_u = \frac{\zeta u \nu}{\sqrt{2}} \quad (1.30) \]

It has been shown above how gauge bosons, leptons, and quarks acquire mass when the symmetry of the Higgs potential is broken when it picks up a vacuum expectation value. This mechanism is called Glashow-Salam-Weinberg model (GSW) when applied to the electroweak theory. Hence, the GSW model describes how the group \( SU(2)_L \otimes U(1)_Y \) breaks to the electromagnetic gauge group \( U(1)_{EM} \)

\[
SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \xrightarrow{\text{GSW \ U(1)_{EM}}} SU(3)_c \otimes U(1)_{EM} \quad (1.31)
\]

The original four generators \((\tau_a, Y)\) are broken, but the electric charge operator \( Q = \frac{1}{2}(\tau_3 + Y) \) is not. Hence, by Goldstone theorem the photon and the gluons are massless.

One of the problems with the Higgs model is that it’s not a dynamical model, that’s to say, it does not explain why the symmetry was broken in the first place. The Higgs model suffers from the hierarchy problem that implies that the square mass of the Higgs should be finely tuned (to cancel radiative quantum corrections) with the correct sign in order to produce the scale \( \nu \). Moreover, the SM does not specify the value of the Higgs boson mass \((M_H \simeq 126 \text{ GeV})\) exactly, it only narrows it to a certain interval. Hence, the Higgs mass is a free parameter.

1.2 The Effective Potential Description

The method of semi-classical approximation is used in the effective potential description where we expand around the classical solution and see how quantum fluctuations affect the model. In this section, the effective potential method is derived. Then, we are going to apply this method to a \( \phi^4 \) theory.
1.2.1 Formalism

Consider a scalar field theory

\[ Z = e^{iW(J)} = \int D\phi e^{i[S(\phi) + \int d^4x J(x) \phi(x)]} \]  \hspace{1cm} (1.32)

Expanding \( W \) in a Taylor series

\[ W = \sum_n \frac{1}{n!} \int d^4x_1...d^4x_n G^n(x_1...x_n) J(x_1)...J(x_n) \]  \hspace{1cm} (1.33)

Green functions can be obtained from \( W \) by differentiating \( W \) with respect to \( J(x) \); that’s to say

\[ \phi_c(x) \equiv \frac{\delta W}{\delta J(x)} = \frac{1}{Z} \int D\phi e^{i[S(\phi) + \int d^4x J(x) \phi(x)]} \phi(x) \]  \hspace{1cm} (1.34)

By performing a Legendre transformation, a functional effective action \( \Gamma(\phi_c) \) can be obtained from \( W \) as

\[ \Gamma(\phi_c) = W(J) - \int d^4x J(x) \phi_c(x) \]  \hspace{1cm} (1.35)

By substituting (1.34) in (1.35) to eliminate \( \phi_c \) we get

\[ \Gamma(\phi_c) = \int d^4x \left[ \frac{1}{2} (\partial \phi_c)^2 Z(\phi_c) - V_{\text{eff}}(\phi_c) + ... \right] \]  \hspace{1cm} (1.36)

where (...) means higher powers of derivatives. Note that

\[ \frac{\delta \Gamma(\phi_c)}{\delta \phi_c(y)} = \int d^4x \frac{\delta J(x)}{\delta \phi_c(y)} \frac{\delta W(J)}{\delta J(x)} - \int d^4x \frac{\delta J(x)}{\delta \phi_c(y)} \phi_c(x) - J(y) = -J(y) \]  \hspace{1cm} (1.37)

This is similar to the relation between free energy and energy in thermodynamics \( F = E - TS \). \( J \) and \( \phi \) are conjugates such as \( T \) and \( S \). We can deduce from (1.36) and (1.37) that

\[ \frac{dV_{\text{eff}}(\phi_c)}{d\phi_c} = J \]  \hspace{1cm} (1.38)

and if there is no external source

\[ \frac{dV_{\text{eff}}(\phi_c)}{d\phi_c} = 0 \]  \hspace{1cm} (1.39)

This means that the expectation value of \( \phi \) is determined by minimizing the effective potential in the absence of an external source. Hence, the effective potential method is
useful in studying symmetry breaking of the vacuum of a theory.

In order to obtain the first order in quantum fluctuations we vary the generating functional $Z$ in (1.32) with respect to $\phi$ evaluated at $\phi_c$

$$\left. \frac{\delta Z}{\delta \phi} \right|_{\phi_c} = \frac{\delta}{\delta \phi(x)} \left[ S(\phi) + \int d^4y J(y)\phi(y) \right] \bigg|_{\phi_c} = -\partial^2 \phi_c(x) - V'(\phi_c(x)) + J(x) = 0 \quad (1.40)$$

Let’s write $\phi = \phi_c + \varphi$ in (1.32) and expand to second order in $\varphi$

$$Z = e^{(i/h)W(J)} = \int D\phi e^{(i/h)[S(\phi) + \int d^4x J(x)\phi(x)]}$$

$$\approx e^{(i/h)[S(\phi_c) + \int d^4x J(x)\phi_c(x)]} \int D\varphi e^{(i/h)\int d^4x \frac{1}{2}[(\partial \varphi)^2 - V''(\phi_c)\varphi^2]}$$

$$= e^{(i/h)[S(\phi_c) + \int d^4x J(x)\phi_c(x)]} \int D\varphi e^{(-i/h)\int d^4x \frac{1}{2}\varphi[(\partial \varphi)^2 + V''(\phi_c)]} \frac{1}{\sqrt{\det(\partial^2 + V''(\phi_c))}}$$

$$= e^{(i/h)[S(\phi_c) + \int d^4x J(x)\phi_c(x)]} - \frac{i}{2}Tr\left( \log[\partial^2 + V''(\phi_c)] \right)$$

Hence, we obtain

$$W(J) = S(\phi_c) + \int d^4x J(x)\phi_c(x) + \frac{i\hbar}{2}Tr\left( \log[\partial^2 + V''(\phi_c)] \right) + O(\hbar^2) \quad (1.42)$$

substituting this in (1.35) (the c subscript is dropped but implied for $\phi$)

$$\Gamma(\phi) = S(\phi) + \frac{i\hbar}{2}Tr\left( \log[\partial^2 + V''(\phi)] \right) + O(\hbar^2) \quad (1.43)$$

We are interested in cases when the vacuum expectation value is transitionally invariant. Therefore, $\phi$ will be independent of $x$, and the trace expression in momentum space becomes

$$Tr\left( \log[\partial^2 + V''(\phi)] \right) = \int d^4x \langle x | \log[\partial^2 + V''(\phi)] | x \rangle$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4} \langle x | k \rangle \langle k | \log[\partial^2 + V''(\phi)] | k \rangle \langle k | x \rangle$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4} \log[-k^2 + V''(\phi)] \quad (1.44)$$
The effective potential according to equation (1.36)

\[ V_{\text{eff}}(\phi) = V(\phi) - \frac{i\hbar}{2} \int \frac{d^4k}{(2\pi)^4} \log \left( \frac{k^2 - V''(\phi)}{k^2} \right) + O(\hbar^2) \]  

(1.45)

where the \(1/k^2\) is added to make the logarithm argument dimensionless because we need to add a term independent of \(\phi\). Note that the same expression holds for a fermion field but with the opposite sign for the integral. Moreover, expression (1.45) is quadratically divergent. So, three counterterms are introduced in the Lagrangian to make the model finite, two of them in the effective potential as (after converting to Euclidean space)

\[ V_{\text{eff}}(\phi) = V(\phi) + \frac{\Lambda^2}{32\pi^2} \left(V''(\phi) - \frac{(V''(\phi))^2}{64\pi^2} \left( \log \left( \frac{\Lambda^2}{V''(\phi)} \right) - \frac{1}{2} \right) \right) + \frac{1}{2} B\phi^2 + \frac{1}{4!} C\phi^4 + O(\hbar^2) \]  

(1.46)

Integrating this expression up to a cutoff \(k_E^2 = \Lambda^2\) yields (the integral is done in Appendix B)

\[ V_{\text{eff}}(\phi) = V(\phi) + \frac{\Lambda^2}{32\pi^2} \left(V''(\phi) - \frac{(V''(\phi))^2}{64\pi^2} \left( \log \left( \frac{\Lambda^2}{V''(\phi)} \right) - \frac{1}{2} \right) \right) + \frac{1}{2} B\phi^2 + \frac{1}{4!} C\phi^4 \]  

(1.47)

1.2.2 Renormalization Group

The relation between the bare and renormalized Green’s functions of scalar fields is

\[ G_B^{(n)}(p,m_B,\mu_B,g_B) = Z_n^2 G_p^{(n)}(p,m,\mu,g) \]  

(1.48)

The renormalized Green’s function is finite. Because we are working in the continuum approach of renormalization group, observables are independent of the scales that we define our renormalization quantities. This implies

\[ \mu \frac{d}{d\mu} G_B^{(n)} = 0 = Z_\phi^2 \left( \mu \frac{\partial}{\partial \mu} + \frac{n}{2} \frac{dZ_\phi}{d\mu} + \mu \frac{\partial g}{\partial \mu} + \mu \frac{\partial m}{\partial \mu} \right) G_p^{(n)} \]  

(1.49)

Define

\[ \gamma_\phi \equiv \frac{\mu}{Z_\phi} \frac{dZ_\phi}{d\mu} \]  

(1.50)

\[ \beta_g \equiv \mu \frac{\partial g}{\partial \mu} \]  

(1.51)

\[ \gamma_m \equiv \frac{\mu}{m} \frac{\partial m}{\partial \mu} \]  

(1.52)
and then we obtain
\[
\left( \mu \frac{\partial}{\partial \mu} + \frac{n}{2} \gamma_\phi + \beta \frac{\partial}{\partial g} + \gamma_m m \frac{\partial}{\partial m} \right) G^{(n)} = 0
\] (1.53)
which is the Callan-Symanzik equation.

The effective action can be written as
\[
\Gamma[\phi_{cl}] = \sum_{n=2}^{\infty} \frac{i}{n!} \int d^4 x_1 \ldots \int d^4 x_n \Gamma^{(n)}(x_1, \ldots, x_n) \phi_{cl}(x_1) \phi_{cl}(x_n) (1.54)
\]
And the relation between the renormalized and bare proper vertex functions is
\[
\Gamma^{(n)}(p, m, \mu, g) = Z_n^2 \Gamma_B^{(n)}(p, m_B, g_B) (1.55)
\]
and the Callan-Symanzik equation becomes
\[
\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + \gamma m m \frac{\partial}{\partial m} - n \gamma_\phi \phi_{cl}(x) \frac{\delta}{\delta \phi_{cl}(x)} \right) \Gamma^{(n)} = 0
\] (1.56)
By using equation (1.54), equation (1.56) becomes
\[
\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + \gamma m m \frac{\partial}{\partial m} - \gamma_\phi \phi_{cl}(x) \frac{\delta}{\delta \phi_{cl}(x)} \right) \Gamma[\phi_{cl}, \mu, g, m] = 0 (1.57)
\]
Now we apply (1.57) to (1.36) to get
\[
\left( - \frac{\partial}{\partial t} + \tilde{\beta} \frac{\partial}{\partial g} + 4 \tilde{\gamma} \phi_{cl}(x) \frac{\delta}{\delta \phi_{cl}(x)} \right) V^{(4)}(t, g) = 0 (1.58)
\]
\[
\left( - \frac{\partial}{\partial t} + \tilde{\beta} \frac{\partial}{\partial g} + 2 \tilde{\gamma} \phi_{cl}(x) \frac{\delta}{\delta \phi_{cl}(x)} \right) Z(t, g) = 0 (1.59)
\]
Define
\[
t = \ln \left( \frac{\phi_{cl}}{\mu} \right) (1.60)
\]
\[
\tilde{\beta} = \frac{\beta}{1 - \gamma_\phi} (1.61)
\]
\[
\tilde{\gamma} = \frac{\gamma_\phi}{1 - \gamma_\phi} (1.62)
\]
Hence, equations (1.58) and (1.59) for a $\phi^4$ massless theory become
\[
\left( - \frac{\partial}{\partial t} + \tilde{\beta} \frac{\partial}{\partial g} + 4 \tilde{\gamma} \right) V^{(4)}(t, g) = 0 (1.63)
\]
\[
\left( - \frac{\partial}{\partial t} + \tilde{\beta} \frac{\partial}{\partial g} + 2 \tilde{\gamma} \right) Z(t, g) = 0 (1.64)
\]
where \( V^{(4)} = \frac{\partial^4 V}{\partial \phi^4} \) and renormalization conditions

\[
V^{(4)}(0, g) = g, \quad Z(0, g) = 1 \tag{1.65}
\]

Combining (1.63) and (1.64), we get

\[
\bar{\gamma} = \frac{1}{2} \frac{\partial}{\partial t} Z(0, g) \tag{1.66}
\]

\[
\bar{\beta} = \frac{\partial}{\partial t} V^{(4)}(0, g) - 4\bar{\gamma} g \tag{1.67}
\]

Equations (1.63) and (1.64) are special cases of

\[
\left( -\frac{\partial}{\partial t} + \bar{\beta} \frac{\partial}{\partial g} + n\bar{\gamma} \right) F(t, g) = 0 \tag{1.68}
\]

Now \( g' \) is define as the solution of the ordinary differential equation

\[
\frac{dg'}{dt} = \bar{\beta}(g') \tag{1.69}
\]

with the boundary condition

\[
g'(0, g) = g \tag{1.70}
\]

The general solution to (1.68) is

\[
F(t, g) = f(g'(t, g)) \exp \left[ n \int_0^t dt \bar{\gamma}(g'(t, g)) \right] \tag{1.71}
\]

where \( f \) is an arbitrary function.

### 1.2.3 Application to \( \phi^4 \) Theory

If we have a potential with

\[
V(\phi) = \frac{\lambda}{4!} \phi^4 \tag{1.72}
\]

then, taking derivatives of the potential and then substituting in (1.47)

\[
V_{\text{eff}}(\phi) = \left( \frac{\Lambda^2}{64\pi^2} \lambda + \frac{1}{2} B \right) \phi^2 + \left( \frac{\lambda}{4!} + \frac{\lambda^2}{(16\pi)^2} (\log \frac{\lambda \phi^2}{2\Lambda^2} - \frac{1}{2}) + \frac{1}{4!} C \right) \phi^4 \tag{1.73}
\]
To cancel the first infinite term in (1.73) that comes from quantum fluctuations, we can impose that the renormalized mass should vanish. This implies

$$\frac{d^2V_{\text{eff}}}{d\phi_c^2}\bigg|_0 = 0 \implies B = -\lambda \frac{\Lambda^2}{32\pi^2} \tag{1.74}$$

However, we cannot do the same thing with $\frac{d^4V_{\text{eff}}}{d\phi_c^4} = 0$ because the logarithm function is not define at zero. In order to avoid this problem, $\phi$ is going to be evaluated at an arbitrary mass $M$

$$\frac{d^4V_{\text{eff}}}{d\phi_c^4}\bigg|_M = \lambda(M) \tag{1.75}$$

where the coupling constant $\lambda(M)$ (or $\lambda(\mu)$) depends on the choice of $M$. Imposing (1.75) we obtain

$$C = -\frac{3\lambda^2}{32\pi^2} \left( \log(\frac{\lambda M^2}{2\Lambda^2}) + \frac{11}{3} \right) \tag{1.76}$$

Substituting $B$ and $C$ in (1.73) we get

$$V_{\text{eff}}(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2\phi^4}{256\pi^2} \left( \log(\frac{\phi^2}{M^2}) - \frac{25}{6} \right) \tag{1.77}$$

where $\phi = \phi_c$ (Coleman and Weinberg, 1973).

Differentiating the potential in (1.77)

$$V^{(4)} = \lambda + \frac{3\lambda^2 t}{16\pi^2} \tag{1.78}$$

At one loop order, there is no correction to the wave function renormalization, so $Z = 1$, and from (1.66) we obtain $\bar{\gamma} = 0$, and from (1.67) we obtain

$$\bar{\beta} = \frac{3\lambda^2}{16\pi^2} \tag{1.79}$$

At one loop order, the differential equation (1.69) is

$$\frac{d\lambda'}{dt} = \frac{3\lambda'^2}{16\pi^2} \tag{1.80}$$

The solution to this differential equation is

$$\lambda' = \frac{\lambda}{1 - \frac{3\lambda t}{16\pi^2}} \tag{1.81}$$
and the effective potential becomes

\[ V^{(4)} = \frac{\lambda}{1 - \frac{3\lambda}{16\pi^2}} \]  \hspace{1cm} (1.82)

Note that to first order, equation (1.82) agrees with (1.78).
Chapter 2

Dilatons

In this chapter, an introduction to dilatons is provided with an application. Then, three models of dilatons are provided to explain the existence of a light Higgs boson in the SM.

2.1 Introduction

Scale invariance is a symmetry of spacetime when the coupling constant is dimensionless. Hence, a theory possessing this symmetry should be invariant when the energy or length scales change. Dilatations or scale transformations are defined as

$$\lambda : x^\mu \rightarrow e^{\lambda} x^\mu$$

(2.1)

where \(x\) is a spacetime point and \(\lambda\) is a real number. This transformation can be realized linearly on fields

$$\lambda : \phi(x) \rightarrow e^{\lambda d} \phi(e^{\lambda} x)$$

(2.2)

where \(d\) is a scaling dimension. The infinitesimal transformations are

$$\delta x^\mu = x^\mu \delta \lambda$$

(2.3)

$$\delta \phi = (d + x^\mu \partial_\mu) \phi \delta \lambda$$

(2.4)

According to Noether’s theorem, if there is a continuous global symmetry, then there exists a conserved current \(S^\mu\) associated with this symmetry such that \(\partial_\mu S^\mu = 0\). We can
define a physical energy-momentum tensor
\[ \Theta^{\mu\nu} = -2 \frac{\delta S_M}{\delta g_{\mu\nu}} \] (2.5)
(where \( S_M \) is the action for matter fields) that is symmetric and gauge invariant but it is different from the canonical energy-momentum tensor define as
\[ T^{\mu\nu} = \frac{\delta L}{\delta (\partial_\mu \phi^i)} \partial_\nu \phi^i - g^{\mu\nu} L \] (2.6)
where \( \phi^i \) here are the fields in the theory. It is different because the second tensor is not symmetric, gauge invariant, and traceless. However, it can be improved by adding some terms. That is why sometimes the physical energy-momentum tensor is called the improved energy-momentum tensor when dilatations are considered (Callan et al., 1970). The dilatation current is linked to \( \Theta^{\mu\nu} \) such that under scale transformation the trace of the physical energy-momentum is zero
\[ S^\mu = \Theta^{\mu\nu} x_\nu \rightarrow \partial_\mu S^\mu = \Theta^\mu_\mu = 0 \] (2.7)
To generalize the equations above, consider a Lagrangian in a basis of anomalous dimension eigen-operators
\[ \mathcal{L} = \sum_i g_i(\mu) O_i(x) \] (2.8)
where \([O_i] = d_i\) and transforms as
\[ O_i(x) \rightarrow e^{\lambda d_i} O_i(e^{\lambda} x) \] (2.9)
and \( \mu \) is a renormalization scale that transforms under scale transformations as
\[ \mu \rightarrow e^{-\lambda} \mu \] (2.10)
This yields
\[ \delta \mathcal{L} = \sum_i g_i(\mu)(d_i + x^\mu \partial_\mu) O_i(x) + \sum_i \beta_i(g) \frac{\partial \mathcal{L}}{\partial g_i} \] (2.11)
where \( \beta_i(g) = \mu \frac{\partial g_i(\mu)}{\partial \mu} \). From the definition of the scale current we obtain
\[ \partial_\mu S^\mu = \Theta^\mu_\mu = \sum_i g_i(\mu)(d_i - 4) O_i(x) + \sum_i \beta_i(g) \frac{\partial \mathcal{L}}{\partial g_i} \] (2.12)
Therefore, if \( d_i = 4 \) and \( \beta_i = 0 \), then there is scale invariance.

Consider a spin-half nucleon and a pseudoscalar meson interacting with the nucleon through Yukawa’s coupling. We are going to divide the Lagrangian into a scale-symmetric part and a scale-breaking part

\[
\mathcal{L} = \mathcal{L}_s + \mathcal{L}_B \tag{2.13}
\]

\[
\mathcal{L}_s = i \bar{\psi} \gamma_\mu \partial^\mu \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + g_0 \bar{\psi} \gamma_5 \psi \phi - \frac{\lambda_0}{4!} \phi^4 \tag{2.14}
\]

\[
\mathcal{L}_B = -m_0 \bar{\psi} \psi - \frac{1}{2} \mu_0^2 \phi^2 \tag{2.15}
\]

where the zero subscript means bare masses and coupling constants. According to (2.1) and (2.2) transformations we get

\[
\delta \psi = \left( \frac{3}{2} + x_\mu \partial^\mu \right) \psi \tag{2.16}
\]

\[
\delta \phi = (1 + x_\mu \partial^\mu) \phi \tag{2.17}
\]

\[
\delta \mathcal{L}_s = (4 + x_\mu \partial^\mu) \mathcal{L}_s \tag{2.18}
\]

Hence, according to (2.12), \( d_i = 4 \) and \( \beta_i = 0 \) for \( \mathcal{L}_s \), so it is scale invariant. However,

\[
\delta \mathcal{L}_B = -3 (x_\mu \partial^\mu) m_0 \bar{\psi} \psi - \frac{1}{2} (2 + x_\mu \partial^\mu) \mu_0^2 \phi^2 \tag{2.19}
\]

After integration by parts, we obtain

\[
\Delta = m_0 \bar{\psi} \psi + \mu_0^2 \phi^2 \rightarrow \partial_\mu S^\mu = \Delta \tag{2.20}
\]

In order to make the Lagrangian scale invariant, we can introduce a new field \( \chi(x) \) in order to capture this symmetry non-linearly. By (2.2), \( \chi \) should transform as

\[
\lambda : \chi(x) \rightarrow e^{\lambda} \chi(e^{\lambda} x) \tag{2.21}
\]

and

\[
\chi(x) = f e^{\sigma(x)/f} \tag{2.22}
\]

where \( \sigma \) is a new field, and \( f = \langle \chi \rangle \) is the order parameter where the scale symmetry breaks. \( \sigma \) transforms non-linearly as

\[
\frac{\sigma(x)}{f} \rightarrow \frac{\sigma(e^{\lambda} x)}{f} + \lambda \tag{2.23}
\]
In order to make a theory scale invariant now, we multiply the scale-breaking parts by appropriate powers of $e^{\sigma/f}$ and add a free $\sigma$ Lagrangian. For instance, the meson-nucleon Lagrangian (2.13) becomes (Coleman, 1985)

$$\mathcal{L} = \mathcal{L}_s - m_0 \bar{\psi} \psi e^{\sigma/f} - \frac{\mu_0^2}{2} \phi^2 e^{2\sigma/f} + \frac{f^2}{2} \partial_\mu e^{\sigma/f} \partial^\mu e^{\sigma/f}$$

$$= \mathcal{L}_s - m_0 \bar{\psi} \psi \frac{X}{f} - \frac{\mu_0^2}{2} \phi^2 \left(\frac{X}{f}\right)^2 + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi$$

(2.24)

In order to generalize this example to any Lagrangian, we modify

$$g_i (\mu) \rightarrow g_i \left(\mu \frac{X}{f}\right) \left(\frac{X}{f}\right)^{4-d_i}$$

(2.25)

Parameterizing $\chi$ in terms of the fluctuations around the vacuum expectation value as $\bar{\chi}(x) = \chi(x) - f$ yields

$$\mathcal{L}_\chi = \frac{1}{2} \partial_\mu \bar{\chi} \partial^\mu \bar{\chi} + \frac{\bar{\chi}}{f} \Theta^\mu_\mu + ...$$

(2.26)

### 2.2 Model 1: Distinguishing the Higgs Boson from the Dilaton at the Large Hadron Collider

In this section a model proposed by (Goldberger et al., 2008) is explored. The symmetry breaking of scale invariance is related to the breaking of gauge symmetry in the SM because the interactions in the SM are nearly conformal up to the QCD scale. A dilaton is a pseudo-Goldstone boson that emerges because scale invariance is broken. The Higgs and dilaton have similar properties because they are related to conformal symmetry. In this model, the electroweak symmetry breaking (EWSB) sector is strongly coupled; the conformal breaking scale $f$ might not be equal to $\nu$. So, the scale of spontaneous symmetry breaking $\Lambda_{CFT} \sim 4\pi f$ causes EWSB at a scale $\Lambda_{EW} \sim 4\pi \nu \leq \Lambda_{CFT}$. Notice that if the dilaton is realized non-linearly, then the dilaton and the Higgs couple to the SM at the tree level in the same way with the difference that in dilaton we replace $\nu$ with $f$. Hence, if $\nu$ and $f$ are close, then it would be hard to differentiate between a pseudo-dilaton and the Higgs. However, adding a term to the Lagrangian that explicitly breaks conformal symmetry would allow us to distinguish the dilaton from the Higgs.
2.2.1 Electroweak Sector

The electroweak chiral Lagrangian must respect two symmetries. The first symmetry is $SU(2)_L \otimes U(1)$ gauge symmetry. The second symmetry comes from experimental constraints that the Higgs sector respect approximately $SU(2)_L \otimes SU(2)_C$ symmetry where $SU(2)_C$ is custodial symmetry (Datta et al. (2009)). Accommodating $SU(2)_L \otimes SU(2)_C$ symmetry as (2, 2) requires a non-linear realization of the symmetry where $U(x)$ is a dimensionless unimodular matrix field that is unitary (compare this to (1.17))

$$U(x) = e^{i \sum_{a=1}^{3} \pi_a \tau_a}$$ (2.27)

with a covariant derivative

$$D_\mu U = \partial_\mu U + ig_1 B_\mu U \frac{T_3}{2} - \frac{1}{2} ig_2 W_\mu^a \tau_a U$$ (2.28)

The terms

$$T \equiv U \tau_3 U^\dagger, \quad V_\mu \equiv (D_\mu U) U^\dagger$$

$$W_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu + ig_2 [W_\mu, W_\nu]$$ (2.29)

respect $SU(2)_L$ covariance and $U(1)$ invariance with $T$, $V_\mu$, and $W_{\mu\nu}$ having dimensions of zero, one, and two respectively. The chiral Lagrangian is

$$\mathcal{L}_{\chi EW} = -\frac{1}{4} (B_{\mu\nu})^2 - \frac{1}{2} Tr(W_{\mu\nu}^2) + \frac{1}{4} \nu^2 Tr\left((D_\mu U) \dagger (D^\mu U)\right) + ...$$ (2.30)

where $B_{\mu\nu}$ as in (1.10). The first two terms have dimension four and are the kinetic terms. The third term has dimension two. The written terms in (2.30) are the only ones in the Lagrangian when the limit of $M_H \rightarrow \infty$ of the linear theory is considered at the tree level.

Yukawa’s Lagrangian become

$$\mathcal{L}_Y = -\bar{Q}_L m_q q_R - \bar{L}_L m_l l_R + h.c.$$ (2.31)

where $m_q/\nu$ and $m_l/\nu$ are Yukawa’s matrices for quarks and leptons respectively. Note that $m_{q,l}$ is a $2 \times 2$ diagonal matrix of $3 \times 3$ blocks, and in $m_l$ the lower block is zero.
Adding the fermion kinetic Lagrangian $\mathcal{L}_\psi$ to the picture, we obtain the full electroweak chiral Lagrangian at low energies

$$\mathcal{L} = \mathcal{L}_{\chi EW} + \mathcal{L}_Y + \mathcal{L}_\psi$$  \hspace{1cm} (2.32)

In order to make $\mathcal{L}_{\chi SM}$ scale invariance, we multiply the scale breaking part by appropriate powers of $\chi/f$ as in (2.24). $\mathcal{L}_{\chi SM}$ couples to fermions and gauge boson as in the SM with a minimal Higgs boson by replacing $\nu \to \nu\chi/f$ (expanding around $\langle \chi \rangle = f$)

$$\mathcal{L}_{\chi SM} = \left(\frac{2\bar{\chi}}{f} + \frac{\bar{\chi}^2}{f^2}\right) \left[M_W^2 W^+ \bar{W}^- + \frac{1}{2} M_Z^2 Z_\mu Z^\mu\right] + \frac{\bar{\chi}}{f} \sum_\psi M_\psi \bar{\psi} \psi$$  \hspace{1cm} (2.33)

### 2.2.2 Dilaton Self-coupling

If there are no terms that break the symmetry explicitly, then the dilaton self-interactions become

$$\mathcal{L}_\chi = \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{c_4}{(4\pi\chi)^4} (\partial_\mu \chi \partial^\mu \chi)^2 + ...$$  \hspace{1cm} (2.34)

where the constant $c_4 \sim \mathcal{O}(1)$ is determined by the underlying Conformal Field Theory (CFT). However, if there are terms that break the symmetry explicitly, then this effect can be incorporated in the Lagrangian by adding an operator $\mathcal{O}(x)$ with scaling dimension $\Delta_\mathcal{O} \neq 4$

$$\mathcal{L}_{CFT} \to \mathcal{L}_{CFT} + \lambda_\mathcal{O} \mathcal{O}(x)$$  \hspace{1cm} (2.35)

To realize this symmetry breaking in the low-energy effective theory, a spurion field is introduced that constrains the non-derivative interactions of $\chi(x)$ as

$$V(\chi) = \chi^4 \sum_{n=0}^{\infty} c_n (\Delta_\mathcal{O}) \left(\frac{\chi}{f}\right)^{n(\Delta_\mathcal{O} - 4)}$$  \hspace{1cm} (2.36)

where $c_n \sim \lambda_\mathcal{O}^n$ is determined by the underlying CFT. We are interested in the case where the explicit breaking of the conformal symmetry is small. This can be accomplished either by an operator $\mathcal{O}$ that is nearly marginal ($|\Delta_\mathcal{O} - 4| \ll 1$), or by a coefficient $\lambda_\mathcal{O}$ that is chosen to be small in units of $f$. For $m$ and $f$ fixed, the potential is

$$V(\chi) = \frac{1}{2} m^2 \chi^2 + \frac{\lambda}{3!} \frac{m^2}{f} \chi^3$$  \hspace{1cm} (2.37)
Note that the potential of the dilaton is determined in terms of \( m^2_\chi = \frac{d^2 V}{d\chi^2} > 0 \) and VEV \( \langle \chi \rangle = f \). To leading order, the cubic coupling is

\[
\lambda = \begin{cases} 
(\Delta_\sigma + 1) + O(\lambda_\sigma), & \text{when } \lambda_\sigma \ll 1 \\
5 + O(|\Delta_\sigma - 4|) & \text{when } |\Delta_\sigma - 4| \ll 1
\end{cases}
\]  

(2.38)

Conformal algebra and unitarity dictate \( \lambda > 2 \). For \( \lambda_\sigma \ll 1 \) and \( \Delta_\sigma = 2 \), the Higgs coupling is recovered. Moreover, the nearly marginal case can be realized by Coleman-Weinberg effective potential up to corrections as

\[
V(\chi) = \frac{1}{16} \frac{m^2}{f^2} \chi^4 \left[ 4 \ln \left( \frac{\chi}{f} \right) - 1 \right] + O(|\Delta_\sigma - 4|^2)
\]  

(2.39)

The couplings in (2.33) can receive radiative corrections of order \( m^2 \). For instance, because the top quark is heavy, it contributes \( \delta m^2 \sim \frac{m^2_t \Lambda^2}{16 \pi^2 f^2} \) where \( \Lambda \) is an ultraviolet (UV) cutoff. Moreover, the cubic coupling receives \( \frac{m^3_t \Lambda}{16 \pi^2 f^2} \). These radiative corrections vanish under the assumption that in the limit of \( \lambda_\sigma \to 0 \), the theory is scale invariant. In order to incorporate this idea in the low-energy model, we make the UV cutoff dependent on \( \chi \)

\[
\Lambda \to \Lambda \frac{\chi}{f}
\]  

(2.40)

This means that \( \chi \) serves as a conformal compensator as before.

2.2.3 Couplings to Massless Gauge Bosons

There are loop contributions \( H\gamma\gamma \) and \( Hgg \) because the Higgs couples to the massive gauge bosons and fermions. As mentioned above, dilatons couples to the SM like the Higgs; hence, there are dilaton couplings \( \chi\gamma\gamma \) and \( \chi gg \). These loop effects are dominated by heavy mass particles such as the top quark. In the SM, the Higgs couples to gluons as (the Higgs background is considered to be spacetime independent)

\[
\Gamma[A,H] = \frac{1}{4} \int_q \hat{G}^a_{\mu\nu}(-q) \Pi(q^2, H) \hat{G}^{a\mu\nu}(q) + ...
\]  

(2.41)

where \( \Pi(q^2, H) \) is the vacuum polarization function that includes the loop contributions from heavy particles. The one-loop contribution of fermions is

\[
\Pi(q^2, H) = \frac{1}{g^2(\mu)} - \frac{4}{(4\pi)^2} \sum_i \int_0^1 dx x(1 - x) \times \ln \left[ \frac{x(1 - x) + 2m^2_i H^\dagger H/\nu^2}{\mu^2} \right]
\]  

(2.42)
$q^2$ is considered to be of order $M_H^2$ when dealing with Higgs processes. For light particles, $m_i^2 \ll q^2 \approx M_H^2$. Hence, the vacuum polarization function is independent of the Higgs field, and light particles have negligible effect to $Hgg$ couplings. However, for heavy particles such as the top quark, $m_i \gg M_H$, the $q^2$ term is neglected and the Higgs couples to gluons through an operator

$$L_{hGG} = \frac{\alpha_s}{8\pi} \sum_i b_i^0 \frac{h \bar{h}}{\nu} (G_{\mu\nu}^a)^2$$

(2.43)

where we have performed an expansion $\ln\left(\frac{H^\dagger H}{\nu^2}\right) = \frac{1}{2} + \frac{h}{\nu} + ...$ and $G_{\mu\nu}^a$ is the canonically normalized gluon field strength. We sum over heavy particles only and $b_i^0$ is the contribution of each heavy particle to the one-loop QCD beta function $\beta_i(g) = \frac{b_i g^3}{16\pi^2}$. For instance, the top quark has $b_0 = \frac{2}{3}$.

Because the Higgs and dilatons couple to massless gauge bosons, we can make the replacement

$$2 \frac{m_i^2}{\nu^2} H^\dagger H \rightarrow m_i^2 \chi^2$$

(2.44)

in (2.42) and the Lagrangian becomes

$$L_{\chi} = \left[ \frac{\alpha_{EM}}{8\pi} c_{EM} (F_{\mu\nu})^2 + \frac{\alpha_s}{8\pi} c_G (G_{\mu\nu})^2 \right] \ln\left(\frac{\chi}{f}\right)$$

(2.45)

where $c_{EM}$ and $c_G$ are the coefficients of the one-loop beta functions for the gauge couplings $e$ and $g_s$. As we have done before, we can split the sum to run over light and heavy particles by comparing them to the dilaton mass. Moreover, if we want to consider QCD to be conformal, then by conformal invariance

$$\sum_{\text{light}} b_0 + \sum_{\text{heavy}} b_0 = 0$$

(2.46)

Hence, the effective coupling becomes

$$L_{\chi gg} = -\frac{\alpha_s}{8\pi} b_{0}^{\text{light}} \frac{\bar{\chi}}{f} (G_{\mu\nu}^a)^2$$

(2.47)

where $b_{0}^{\text{light}} = -11 + \frac{2}{3} n_{\text{light}}$. Now, if the top quark is heavier than the dilaton mass, then $n_{\text{light}} = 5$, otherwise $n_{\text{light}} = 6$. Equation (2.47) shows a tenfold increase in the coupling.
strength compared to the Higgs. Therefore, the coupling of dilaton to gluons is enhanced, while to the photons is suppressed.

In the Higgs model, the Higgs might receive corrections from particles beyond the SM that are heavy with dimension-six operator as

$$\mathcal{L}_{hGG} \supset \frac{\alpha_s}{4\pi} c_{hg} H^\dagger H (G^a_{\mu\nu})^2$$

depending on the value of $c_{hg}$. On the other hand, in the dilaton model, corrections come from higher dimension operators as

$$\mathcal{L}_{\chi gg} \supset g_s^2 \frac{c_{\chi g}}{4\pi^2} D_\alpha G^a_{\mu\nu} D^\alpha G^{\mu\nu}$$

These operators are suppressed by powers of $m^2/f^2 \ll 1$ in comparison with conformal anomaly terms. Hence, it is hard to distinguish between the Higgs and the dilaton because it is not clear whether corrections are coming from heavy particles beyond the SM in the Higgs model or enhancement of the coupling strength in the dilaton model.

The low-energy Lagrangian of the dilaton below the scale $4\pi f$ is

$$\mathcal{L}_\chi = \frac{1}{2} \partial_\mu \bar{\chi} \partial^\mu \chi - \frac{1}{2} m^2 \bar{\chi} \chi + \frac{\lambda}{3!} \frac{m^2}{f} \bar{\chi}^3
+ \frac{\chi}{f} \sum_{\psi} M_\psi \bar{\psi} \psi + \left( \frac{2 \bar{\chi}}{f} + \frac{\chi^2}{f^2} \right) \left[ M_W^2 W_\mu^+ W^-\mu + \frac{1}{2} M_Z^2 Z_\mu Z^\mu \right]
+ \frac{\alpha_{EM}}{8\pi} c_{EM} \frac{\bar{\chi}}{f} (F_\mu\nu) F_\mu\nu
+ \frac{\alpha_s}{8\pi} c_G \frac{\bar{\chi}}{f} (G_\mu\nu)^2$$

If strong and electromagnetic interactions are considered in the conformal sector at high energies, then $c_{EM}$ and $c_G$ become

$$c_{EM} = \begin{cases} -\frac{17}{9} & \text{when } M_W < m < m_t, \\ -\frac{11}{3} & \text{when } m > m_t \end{cases}$$

$$c_G = 11 - \frac{2}{3} n_{light}$$

The branching ratios to $WW, ZZ$, and fermions in dilatons are the same as in the Higgs. The important parameters are $f$ and $m$, and the Lagrangian has three couplings: $\lambda$, $c_G$, and $c_{EM}$. 

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2.3 Model 2: Effective Theory of a Light Dilaton

In this section a model proposed by (Chacko and Mishra, 2013) is demonstrated. According to precision electroweak tests, the SM Higgs is the favored model. And it remains to solve the hierarchy problem. The simplest model of electroweak symmetry broken by strong dynamics is disfavored by experiment. A dilaton emerges in the low energy theory as a massless scalar when an exact conformal symmetry is spontaneously broken. We are interested in theories of electroweak symmetry breaking where there are operators that break conformal symmetry explicitly and increase in the infrared to become strong at the breaking scale. Hence, there does not exist an obvious reason for the existence of a light dilaton in the low energy theory.

2.3.1 Effective Theory of a Dilaton

A model for the dilaton is constructed based on a marginal operator that breaks conformal symmetry yielding a naturally light dilaton. In the beginning, we are going to work in the limit of exact conformal invariance, then we incorporate violating effects.

Effective Theory in the Limit of Exact Conformal Invariance

The theory is scale invariant in the absence of effects that break conformal symmetry. The Lagrangian has derivative terms

\[ \frac{1}{2} Z \partial_\mu \chi \partial^\mu \chi + \frac{c}{\chi^4} (\partial_\mu \chi \partial^\mu \chi)^2 + ... \]  

and the potential is

\[ V(\chi) = \frac{Z^2 \kappa_0}{4!} \chi^4 \]  

There is a preferred expectation value \( f = \langle \chi \rangle \) when there is a non-derivative term in the Lagrangian even when there are no terms that break conformal symmetry explicitly. The effective potential is obtained and used to find the dilaton mass. A mass-independent
scheme ($\overline{MS}$) is used. The coupling constants $c, Z^2\kappa_0$, etc and $Z$ are labeled by dimensionless $g_i$. We will begin with one loop analysis and then generalize to any number of loops.

At one loop, $g_i$ evolves to $g_i'$ as

$$g_i' = g_i - \frac{dg_i}{d\log \mu} \log \left( \frac{\chi}{f} \right)$$  \hspace{1cm} (2.55)

All $g_i'$s are considered zero but $Z$ and $Z^2\kappa_0$. After this modification, the dilaton potential becomes

$$V(\chi) = \left[ Z^2\kappa_0 - \frac{d(Z^2\kappa_0)}{d\log \mu} \log \left( \frac{\chi}{f} \right) \right] \frac{\chi^4}{4!}$$  \hspace{1cm} (2.56)

Now the potential does not have scale invariance. Note that the wave function renormalization does not exist at one loop order

$$\frac{d\log Z}{d\log \mu} = -2\gamma = 0$$  \hspace{1cm} (2.57)

From perturbation theory we get

$$\frac{d\kappa_0}{d\log \mu} = \frac{3\kappa_0^2}{16\pi^2}$$  \hspace{1cm} (2.58)

At this stage we can set $Z$ to one.

The Lagrangian can be made conformal by making the renormalization scale $\mu$ depends on $\tilde{\chi}$ as in (2.10). Therefore, making the coupling constants dimensionless, and the theory would look massless. The kinetic part becomes

$$\frac{1}{2} \tilde{Z}\partial_\mu \chi \partial^\mu \chi$$  \hspace{1cm} (2.59)

where $\tilde{Z}$ to one loop order is

$$\tilde{Z} = Z - \frac{dZ}{d\log \mu} \log \left( \frac{\mu}{f} \right)$$  \hspace{1cm} (2.60)

Note that $\tilde{Z}$ is equal to $Z$ since at one loop order there is no wave function renormalization. We can set $\tilde{Z}$ to one now. The potential becomes

$$V(\chi) = \left[ \kappa_0 - \frac{d(Z^2\kappa_0)}{d\log \mu} \log \left( \frac{\chi}{\mu} \right) \right] \frac{\chi^4}{4!}$$  \hspace{1cm} (2.61)
where $\tilde{\kappa}_0$ is

$$\tilde{\kappa}_0 = Z^2 \kappa_0 - \frac{d(Z^2 \kappa_0)}{d \log \mu} \log \left( \frac{\mu}{f} \right)$$  \hspace{1cm} (2.62)

We use Coleman-Weinberg equation in order to get the effective potential at one loop order

$$V_{eff} = V \pm \frac{1}{64\pi^2} \sum_i \frac{M_i^4}{4!} \left[ \log \left( \frac{\mu^2}{\frac{1}{2} \kappa_0 f^2} \right) - \frac{1}{2} \right]$$  \hspace{1cm} (2.63)

Hence, we obtain

$$V_{eff}(\chi_{cl}) = \kappa_0 - \frac{3\kappa_0^2}{32\pi^2} \left[ \log \left( \frac{\kappa_0}{2} \right) - \frac{1}{2} \right]$$  \hspace{1cm} (2.64)

Conformal invariance can be manifested by expressing

$$\hat{\kappa}_0 = \tilde{\kappa}_0 + \frac{3\kappa_0^2}{32\pi^2} \left[ \log \left( \frac{\kappa_0}{2} \right) - \frac{1}{2} \right]$$  \hspace{1cm} (2.65)

so that the potential is

$$V_{eff}(\chi_{cl}) = \frac{\hat{\kappa}_0}{4!} \chi_{cl}^4$$  \hspace{1cm} (2.66)

For $\hat{\kappa}_0 > 0$, conformal symmetry is not broken because the breaking scale $\langle \chi \rangle = f$ is driven to zero. For $\hat{\kappa}_0 < 0$, conformal symmetry is broken. And for $\hat{\kappa}_0 = 0$, there exist a stable minimum corresponding to a massless dilaton.

This result can be generalized to any number of loops by generalizing the coupling constants (2.55), potential (2.56), wave function renormalization (2.60), modified potential (2.61), and $\tilde{\kappa}_0$ in (2.62) as

$$g_i' = g_i + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{d^n g_i}{d \log \mu^n} \left[ \log \left( \frac{\mu}{f} \right) \right]^n$$  \hspace{1cm} (2.67)

$$V(\chi) = \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n (Z^2 \kappa_0)}{d \log \mu^n} \left[ \log \left( \frac{\mu}{f} \right) \right]^n \right\} \chi_{cl}^4$$  \hspace{1cm} (2.68)

$$\tilde{Z} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{d^m Z}{d \log \mu^m} \left[ \log \left( \frac{\mu}{f} \right) \right]^m$$  \hspace{1cm} (2.69)

$$V(\chi) = \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \tilde{\kappa}_0, n \left[ \log \left( \frac{\mu}{f} \right) \right]^n \right\} \chi_{cl}^4$$  \hspace{1cm} (2.70)

$$\tilde{\kappa}_{0,n} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{d^{n+m} (Z^2 \kappa_0)}{d \log \mu^{n+m}} \left[ \log \left( \frac{\mu}{f} \right) \right]^m$$  \hspace{1cm} (2.71)
respectively. The beta functions for all $g_i's$ and $\bar{\kappa}_{0,n}$ are zero because we have conformal invariance. In order to generalize the effective potential, it is easier to work in the method of renormalization group than work with counter terms in the Lagrangian method in section (1.2). According to (1.58), the Callan-Symanzik equation becomes

$$\left( \mu \frac{\partial}{\partial \mu} + \beta_i \frac{\partial}{\partial \bar{g}_i} - \gamma \chi_{cl} \frac{\partial}{\partial \chi_{cl}} \right) V_{eff}(\chi_{cl}, \bar{g}_i, \mu) = 0$$

(2.72)

Because we have conformal invariance, $\beta_i$ and $\gamma$ of $\chi$ are zero. Hence, (2.72) becomes

$$\mu \frac{\partial}{\partial \mu} V_{eff}(\chi_{cl}, \bar{g}_i, \mu) = 0$$

(2.73)

The effective potential is

$$V_{eff}(\chi_{cl}) = \hat{\kappa}_0 \frac{4!}{4!} \chi_{cl}^4$$

(2.74)

and it only has a stable minimum when $\hat{\kappa}_0 = 0$.

**Conformal Symmetry Violating Effects**

Now let us consider effects that break conformal symmetry explicitly. Consider an operator with scaling dimension $\Delta$ in the Lagrangian

$$\mathcal{L} = \mathcal{L}_{CFT} + \lambda_\mathcal{O} \mathcal{O}(x)$$

(2.75)

The dimensionless coupling constant becomes

$$\hat{\lambda}_\mathcal{O} = \lambda_\mathcal{O} \mu^{\Delta - 4}$$

(2.76)

Let us assume $\hat{\lambda}_\mathcal{O} \ll 1$ in order to be able to apply perturbation theory. $\hat{\lambda}_\mathcal{O}$ in this case satisfies the renormalization group differential equation

$$\frac{d \log \hat{\lambda}_\mathcal{O}}{d \log \mu} = -(4 - \Delta)$$

(2.77)

derived from (2.76). Under the transformation $x \to e^{-\omega}x$, the operator transforms as $\mathcal{O}(x) \to e^{\omega \Delta} \mathcal{O}(x)$. In order for the Lagrangian (2.75) to be scale invariant, $\lambda_\mathcal{O}$ must be promoted to a spurion as in (2.25) as

$$\lambda_\mathcal{O} \to e^{(4-\Delta)\omega} \lambda_\mathcal{O}$$

(2.78)
As before, we make the renormalization scale depends on the conformal compensator as
\[ \mu = \mu \chi. \]
The dilaton potential now is
\[
V(\chi) = \frac{Z^2 \kappa_0}{4!} \chi^4 - \sum_{n=1}^{\infty} \frac{Z^{2-n\epsilon} \kappa_n}{4!} \lambda_n \chi^{4-ne}
\]
where \( \epsilon = 4 - \Delta. \)

At one loop order (2.79) becomes
\[
V(\chi) = \frac{Z^2 \kappa_0}{4!} \chi^4 - \frac{Z^{\Delta/2} \kappa_1}{4!} \lambda \Delta \chi
\]
where \( \kappa_0 \) and \( \kappa_1 \) are coupling constants. From (2.55), we know how the coupling constants evolve in the renormalization group. Hence, modifying the potential to be
\[
V(\chi) = \left\{ \tilde{\kappa}_0 - \frac{d(Z^2 \kappa_0)}{d \log \mu} \log \left( \frac{\chi}{\mu} \right) \right\} \frac{\chi^4}{4!}
- \left\{ \tilde{\kappa}_1 - \frac{d(Z^{\Delta/2} \kappa_1)}{d \log \mu} \log \left( \frac{\chi}{\mu} \right) \right\} \frac{\lambda \Delta \chi}{4!}
\]
All \( g'_i \)s are considered zero but \( Z, Z^2 \kappa_0, \) and \( Z^{\Delta/2} \kappa_1. \) From perturbation theory we get
\[
\frac{d\kappa_0}{d \log \mu} = \frac{3\kappa_0^2}{16\pi^2} \quad (2.82)
\]
\[
\frac{d\kappa_1}{d \log \mu} = \frac{\Delta(\Delta - 1)\kappa_1 \kappa_0}{32\pi^2} \quad (2.83)
\]
In order to make coupling constants dimensionless, the coefficient of the dilaton kinetic term becomes
\[
\tilde{Z} = Z - \frac{dZ}{d \log \mu} \log \left( \frac{\mu}{f} \right)
\]
The potential is modified to
\[
V(\chi) = \left\{ \tilde{\kappa}_0 - \frac{d(Z^2 \kappa_0)}{d \log \mu} \log \left( \frac{\chi}{\mu} \right) \right\} \frac{\chi^4}{4!}
- \left\{ \tilde{\kappa}_1 - \frac{d(Z^{\Delta/2} \kappa_1)}{d \log \mu} \log \left( \frac{\chi}{\mu} \right) \right\} \frac{\lambda \Delta \chi}{4!}
\]
where
\[
\tilde{\kappa}_0 = Z^2 \kappa_0 - \frac{d(Z^2 \kappa_0)}{d \log \mu} \log \left( \frac{\mu}{f} \right)
\]
\[
\tilde{\kappa}_1 = Z^{\Delta/2} \kappa_1 - \frac{d(Z^{\Delta/2} \kappa_1)}{d \log \mu} \log \left( \frac{\mu}{f} \right)
\]
λ_\mathcal{O} has a mass dimension of 4−Δ because it is considered as a spurion. The beta functions for \( \tilde{\kappa}_0 \) and \( \tilde{\kappa}_1 \) vanish because of conformal invariance by construction. We use (2.63) to find the effective potential at one loop order

\[
V_{\text{eff}}(\chi_{cl}) = \frac{\tilde{\kappa}_0}{4!} \chi_{cl}^4 - \frac{\tilde{\kappa}_1}{4!} \lambda_\mathcal{O} \chi_{cl}^\Delta + \frac{\tilde{\kappa}_0}{4!} \left[ \frac{3\tilde{\kappa}_0}{32\pi^2} \chi_{cl}^4 - \frac{\lambda_\mathcal{O} \Delta (\Delta - 1) \tilde{\kappa}_1}{64\pi^2} \chi_{cl}^\Delta \right] \left( \Sigma - \frac{1}{2} \right) \tag{2.88}
\]

where

\[
\Sigma = \log \left[ \frac{\tilde{\kappa}_0}{2} - \frac{\tilde{\kappa}_1}{4!} \Delta (\Delta - 1) \lambda_\mathcal{O} \chi_{cl}^{\Delta-4} \right] \tag{2.89}
\]

In order to make the analysis easier, we are going to neglect the loop suppressed terms in (2.88) (including them does not change the conclusion). The minimum for the tree level potential is

\[
f^{(\Delta-4)} = \frac{4\tilde{\kappa}_0}{\tilde{\kappa}_1 \lambda_\mathcal{O} \Delta} \tag{2.90}
\]

Hence, the dilaton mass is

\[
V'_{\text{eff}}(\chi_{cl}) \bigg|_{\chi_{cl}=f} = m_\sigma^2 = \frac{\tilde{\kappa}_1}{4!} \lambda_\mathcal{O} (4−\Delta) f^{\Delta-2} = \frac{4\tilde{\kappa}_0}{4!} (4−\Delta) f^2 \tag{2.91}
\]

by using (2.90). If we have weak coupling in our CFT, then \( \tilde{\kappa}_0, \tilde{\kappa}_1 \ll (4\pi)^2, \tilde{\lambda}_\mathcal{O} \ll 1 \) implies \( \lambda_\mathcal{O} f^{\Delta-4} \ll 1 \), and loop suppressed terms in (2.88) are negligible in this regime. On the other hand, if we have strong coupling in our CFT, then \( \tilde{\kappa}_0, \tilde{\kappa}_1 \sim \mathcal{O}(4\pi)^2 \), and \( \tilde{\lambda}_\mathcal{O} \) to be small at the scale \( f \) does not hold according to equation (2.90). The mass of the dilaton from equation (2.91) is not light anymore. Hence, if there is a CFT that is strongly coupled and it is broken explicitly by a relevant operator that grows in the infrared, then there is no obvious reason for a light dilaton to exist.

Nevertheless, if \( (4−\Delta) \ll 1 \) in (2.91) even in the strongly coupled scenario, then the dilaton mass would become small in comparison with the strong coupling scale \( \Lambda \). Therefore, if a CFT is broken by a marginal operator, then a light dilaton would exist with a mass that scales as \( m_\sigma \sim \sqrt{4−\Delta} \).

This result can be generalized beyond small \( \lambda_\mathcal{O} \). Two things must be taken into consideration for this to happen. Firstly, equation (2.77) must be generalized to

\[
\frac{d \log \tilde{\lambda}_\mathcal{O}}{d \log \mu} = -g(\tilde{\lambda}_\mathcal{O}) \tag{2.92}
\]
where $g(\hat{\lambda}_\mathcal{O})$ is a polynomial in $\hat{\lambda}_\mathcal{O}$ in general as

$$g(\hat{\lambda}_\mathcal{O}) = \sum_{n=0}^{\infty} c_n \hat{\lambda}_\mathcal{O}^n$$  \hspace{1cm} (2.93)

Note that the lowest order in (2.93) matches (2.77) when $\hat{\lambda}_\mathcal{O} \ll 1$

$$g(\hat{\lambda}_\mathcal{O}) = c_0 = (4 - \Delta)$$  \hspace{1cm} (2.94)

In a strongly coupled CFT, $c_n \sim \mathcal{O}(1)$ for $n \geq 1$ generally. Secondly, higher order terms in (2.79) become important and must be considered. Integrating equation (2.92)

$$G(\hat{\lambda}_\mathcal{O}) = \exp \left( - \int \frac{d\hat{\lambda}_\mathcal{O}}{\hat{\lambda}_\mathcal{O}} \frac{1}{g(\hat{\lambda}_\mathcal{O})} \right)$$  \hspace{1cm} (2.95)

where $G(\hat{\lambda}_\mathcal{O}) \mu^{-1}$ is a renormalization group invariant. As in (2.78), we need to promote $G(\hat{\lambda}_\mathcal{O}) \mu^{-1}$ to make the Lagrangian (2.75) scale invariant as

$$\hat{\lambda}_\mathcal{O}(\mu) \rightarrow \hat{\lambda}_\mathcal{O}(\mu e^{-\omega})$$  \hspace{1cm} (2.96)

$$G(\hat{\lambda}_\mathcal{O}) \mu^{-1} \rightarrow e^{-\omega} G(\hat{\lambda}_\mathcal{O}) \mu^{-1}$$  \hspace{1cm} (2.97)

From (2.95), we can deduce that

$$\bar{\lambda}_\mathcal{O} = \hat{\lambda}_\mathcal{O}[1 + g(\hat{\lambda}_\mathcal{O}) \log \mu]$$  \hspace{1cm} (2.98)

Furthermore, using (2.55) and (2.97), we can define

$$\bar{\Omega}(\hat{\lambda}_\mathcal{O}, \chi/\mu) = \hat{\lambda}_\mathcal{O} \left[ 1 - g(\hat{\lambda}_\mathcal{O}) \log \left( \frac{\chi}{\mu} \right) \right]$$  \hspace{1cm} (2.99)

such that it is invariant under infinitesimal scale transformations. Now that $\bar{\Omega}$ is a polynomial in $\hat{\lambda}_\mathcal{O}$.

For $\mu \sim f$ the scale of symmetry breaking and $g(\hat{\lambda}_\mathcal{O}) \ll 1$, $\bar{\Omega}$ can be approximated as

$$\bar{\Omega}(\hat{\lambda}_\mathcal{O}, \chi/\mu) = \hat{\lambda}_\mathcal{O} \left( \frac{\chi}{\mu} \right)^{-g(\hat{\lambda}_\mathcal{O})}$$  \hspace{1cm} (2.100)

The potential becomes to leading order in $\bar{\Omega}$

$$V(\chi) = \frac{\chi^4}{4!} (\kappa_0 - \kappa_1 \bar{\Omega})$$  \hspace{1cm} (2.101)
and the dilaton mass is

\[ m_\sigma^2 = \frac{\kappa_0}{4!} g(\hat{\lambda}_O) f^2 \]  \hspace{1cm} (2.102)

There is similarity between the expressions of the dilaton mass in (2.91) and (2.102). The dilaton mass in (2.102) implies that the mass of the dilaton is determined by the scaling behavior of the operator \( O \) to be marginal at the breaking scale \( \mu = f \) with \( g(\hat{\lambda}_O) \ll 1 \). However, if a CFT is strongly coupled, then \( \hat{\lambda}_O \) and \( g(\hat{\lambda}_O) \) are not small. Therefore, the existence of a light dilaton in such models has some tuning. Including higher order corrections in \( \hat{\lambda}_O \) does not change this result. We can write

\[ g(\hat{\lambda}_O) = \epsilon + \delta g(\hat{\lambda}_O) \]  \hspace{1cm} (2.103)

where \( \delta g(\hat{\lambda}_O) \) captures higher order corrections. For further discussion on how to construct an effective dilaton theory that includes higher orders in \( \hat{\lambda}_O \) check (Chacko and Mishra, 2013).

### 2.3.2 Dilaton Interactions in a Conformal SM

When there is exact conformal invariance, the dilaton interactions with the SM is determined by realizing the symmetry non-linearly in the low energy effective theory. On the other hand, if there is an operator that breaks this symmetry explicitly, then we should expect deviations that correct the coupling of the dilaton to the SM. All gauge kinetic terms are expressed as

\[ -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \]  \hspace{1cm} (2.104)

In the limit of exact conformal symmetry, dilatons couple to the gauge bosons such as \( W \) as

\[ \left( \frac{\chi}{f} \right)^2 \frac{m_W^2}{g^2} W^+ W^- \]  \hspace{1cm} (2.105)

From (2.22) we have \( \chi = f e^{\frac{\sigma}{f}} \). We expand \( \chi \) to leading order in \( f^{-1} \) in terms of \( \sigma \) to obtain

\[ 2\frac{\sigma}{f} \frac{m_W^2}{g^2} W^+ W^- \]  \hspace{1cm} (2.106)
Now we include effects that break conformal symmetry. We will assume that $\epsilon > |\delta g(\hat{\lambda}_O)|$ at the breaking scale, and we will show that the same result holds when $\epsilon < |\delta g(\hat{\lambda}_O)|$.

The conformal compensator kinetic part becomes

$$\frac{1}{2} \left[ 1 + \sum_{n=1}^{\infty} \alpha_{\chi,n} \hat{\lambda}_O^n \chi(-ne) \right] \partial_\mu \chi \partial^\mu \chi$$

(2.107)

where $\alpha_{\chi,n} \sim \mathcal{O}(1)$ depend on the operator $\mathcal{O}$ and the particular CFT in use. The dilaton kinetic term becomes

$$\frac{1}{2} \left[ 1 + \sum_{n=1}^{\infty} \alpha_{\chi,n} \hat{\lambda}_O^n f(-ne) \right] \partial_\mu \sigma \partial^\mu \sigma$$

(2.108)

To make the dilaton kinetic term canonical, $\sigma$ is rescaled which changes the value of the breaking scale $f$ by an order in (2.106) without changing the form of the interaction. Therefore, corrections to the dilaton kinetic term does not change the form of interactions between the dilaton and the SM to leading order in $\frac{f}{\sigma}$.

The gauge kinetic term (2.104) becomes

$$-\frac{1}{4 g^2} \left[ 1 + \sum_{n=1}^{\infty} \alpha_{W,n} \hat{\lambda}_O^n \chi(-ne) \right] F_{\mu\nu} F^{\mu\nu}$$

(2.109)

where $\alpha_{W,n} \sim \mathcal{O}(1)$ are dimensionless. The physical gauge coupling is modified to be

$$\frac{1}{g^2} = \frac{1}{4 g^2} \left[ 1 + \sum_{n=1}^{\infty} \alpha_{W,n} \hat{\lambda}_O^n f(-ne) \right]$$

(2.110)

To leading order in $\sigma$ (2.109) is

$$\frac{\epsilon}{4 g^2} \frac{\sigma}{f} F_{\mu\nu} F^{\mu\nu}$$

(2.111)

where

$$\tilde{c}_W = \sum_{n=1}^{\infty} \frac{n \alpha_{W,n} \hat{\lambda}_O^n f(-ne)}{1 + \sum_{n=1}^{\infty} \alpha_{W,n} \hat{\lambda}_O^n f(-ne)}$$

(2.112)

In a strongly coupled CFT $\tilde{c}_W \sim \mathcal{O}(\hat{\lambda}_O f^{-\epsilon})$, making the coupling of the dilaton suppressed by $\epsilon \hat{\lambda}_O$, which is of order $\frac{m^2}{\Lambda^2}$. The gauge boson mass (2.105) becomes

$$\left( \frac{\chi}{f} \right)^2 \left[ 1 + \sum_{n=1}^{\infty} \beta_{W,n} \hat{\lambda}_O^n \chi(-ne) \right] \frac{m_W^2}{g^2} f W_\mu^+ W^-$$

(2.113)
where $\beta_{W,n} \sim O(1)$. Expanding (2.113) to leading order in $f^{-1}$ in terms of $f$

$$
\frac{\sigma m_W^2}{f \cdot g^2} [2 + \epsilon c_W] W^+ W^- 
$$
(2.114)

The physical mass of the $W$ boson is defined as

$$
m_W^2 = \hat{m}_W^2 \left[ 1 + \sum_{n=1}^{\infty} \beta_{W,n} \bar{\lambda}_O f^{(-ne)} \right] 
$$
(2.115)

where

$$
c_W = -\sum_{n=1}^{\infty} n \beta_{W,n} \bar{\lambda}_O f^{(-ne)} 
$$
(2.116)

Corrections to the dilaton coupling in this case is also of order $\epsilon \bar{\lambda}_O \sim \frac{m^2}{\chi^2}$.

If $\epsilon < |\delta g(\bar{\lambda}_O)|$ at the scale of symmetry breaking, then we replace $\bar{\lambda}_O \chi^{-\epsilon}$ in (2.107), (2.109), and (2.113) by $\Omega(\bar{\lambda}_O, \chi)$, which in this regime becomes

$$
\Omega(\bar{\lambda}_O, \chi) = \frac{\bar{\lambda}_O}{1 + c_1 \bar{\lambda}_O \log \chi} 
$$
(2.117)

and corrections to the dilaton couplings are suppressed by $c_1 \Omega^2(\bar{\lambda}_O, f)$ which is of order $\frac{m^2}{\chi^2}$ again.

The dilaton also couples to the massless gauge bosons, namely, photon and gluon. Note that conformal symmetry for massless bosons is not broken at the classical level like massive bosons $W$ and $Z$, only at the quantum level. Therefore, under $x \rightarrow e^{-\omega} x$ transformation, $F_{\mu\nu} F^{\mu\nu}$ becomes

$$
F_{\mu\nu} F^{\mu\nu}(x) \rightarrow e^{4\omega} (1 + \frac{b_<}{8\pi^2} g^2 \omega) F_{\mu\nu} F^{\mu\nu}(x) 
$$
(2.118)

where $\frac{b_<}{8\pi^2}$ is determined as (2.51). If we want this term to be conformally invariant, then the dilaton coupling should be

$$
\frac{b_<}{32\pi^2} \log \left( \frac{\chi}{f} \right) F_{\mu\nu} F^{\mu\nu} 
$$
(2.119)

which is similar to (2.45). Expanding this term to leading order in $f^{-1}$ in terms of $\sigma$

$$
\frac{b_< \sigma}{32\pi^2} F_{\mu\nu} F^{\mu\nu} 
$$
(2.120)
Comparing (2.106) and (2.120), we can infer that the dilaton couples more strongly to massive gauge bosons than to massless gauge bosons.

Now we include violating effects of conformal invariance would yield couplings between the conformal compensator $\chi$ and the kinetic term as

$$-\frac{1}{4g^2}[1 + \sum_{n=1}^{\infty} \alpha_{A,n} \tilde{\lambda}_O^n \chi^{(-n\epsilon)}] F_{\mu\nu} F^{\mu\nu}$$  \hspace{1cm} (2.121)

The physical gauge coupling is modified to be

$$\frac{1}{g^2} = \frac{1}{\hat{g}^2} \left[ 1 + \sum_{n=1}^{\infty} \alpha_{A,n} \tilde{\lambda}_O^n f^{(-n\epsilon)} \right]$$  \hspace{1cm} (2.122)

Expand (2.121) in terms of $\sigma$, and adding the effect of (2.120), the dilaton couples to massless gauge bosons as

$$\sigma f \left[ \frac{b_<}{32\pi^2} + \frac{c_A}{4g^2} \epsilon \right] F_{\mu\nu} F^{\mu\nu}$$  \hspace{1cm} (2.123)

where $c_A$ is dimensionless

$$c_A = \frac{\sum_{n=1}^{\infty} n \alpha_{A,n} \tilde{\lambda}_O^n f^{-n\epsilon}}{1 + \sum_{n=1}^{\infty} \alpha_{A,n} \tilde{\lambda}_O^n f^{-n\epsilon}}$$  \hspace{1cm} (2.124)

In a strongly coupled CFT $c_A \sim \mathcal{O}(\tilde{\lambda}_O f^{-\epsilon}) \sim \tilde{\lambda}_O$ at the breaking scale $f$. The corrections are suppressed by $\frac{m^2}{\lambda^2}$ as before.

In the limit of exact conformal invariance, the dilaton couples to fermions as

$$\frac{\chi}{f} m_\psi \bar{\psi} \psi$$  \hspace{1cm} (2.125)

Expanding (2.125) in terms of $\sigma$ yields

$$\sigma \frac{m_\psi}{f} \bar{\psi} \psi$$  \hspace{1cm} (2.126)

Now including effects that break conformal symmetry would modify (2.125) as

$$\frac{\chi}{f} \left[ 1 + \sum_{n=1}^{\infty} \beta_{\psi,n} \tilde{\lambda}_O^n \chi^{(-n\epsilon)} \right] \hat{m}_\psi \bar{\psi} \psi$$  \hspace{1cm} (2.127)

where $\beta_{\psi,n}$ are dimensionless, and $\hat{m}_\psi$ is the fermion mass in the unperturbed theory. Note that we assumed that the approximate $U(3)$ flavor symmetry is not broken by the
operator $\mathcal{O}$; hence, ensuring that the dilaton couples diagonally in the mass basis. The fermion kinetic term also receives corrections of the form

$$\left[ 1 + \sum_{n=1}^{\infty} \alpha_{\psi,n} \bar{\lambda}^n \chi \chi (-nc) \right] \bar{\psi} \gamma^\mu \partial_\mu \psi$$

(2.128)

Combining (2.127) and (2.128), we obtain the correction to the dilaton coupling

$$\sigma \frac{m_\psi}{f} \left[ 1 + c_\psi \epsilon \right] \bar{\psi} \psi$$

(2.129)

In a strongly coupled CFT $c_\psi \sim \mathcal{O}(\bar{\lambda} f^{-\epsilon})$; hence, the corrections are suppressed by $m_\psi^2 / \Lambda^2$.

### 2.4 Model 3: A Very Light Dilaton

In this section a model proposed by (Grinstein and Uttayarat, 2011) is explored. As we have seen in section (1.2), quantum effects destroy scale invariance even when there is no explicit mass terms in the Lagrangian. The goal is to construct a model where the dilaton mass is small; that is to say, the mass can be arbitrarily small while the rest of the spectrum is kept constant and interacting. This can be achieved by starting with an interacting field theory that has a perturbative attractive infrared fixed point and scalars. By following a renormalization group trajectory that is approaching a fixed point, spontaneous symmetry breaking can be found. A specific model will be considered below where as Yang-Mills gauge coupling runs towards the Banks-Zaks IR-fixed point (Banks and Zaks, 1982), it drives Yukawa couplings and the scalar towards the values of non-trivial fixed point also. A non-trivial minimum for the scalar field may be developed from the relative values of the coupling constants. The parameters in the model can either cause spontaneous symmetry breaking of scale invariance or not. If the couplings are close to the boundary of these two regimes, then a light dilaton emerges in units of its decay constant. Note that the rest of the spectrum is interacting and robust under changes in the parameter space to keep the dilaton mass light.
2.4.1 The Model

In this model we consider a SU($N$) gauge theory that has two scalars and two spinors $\psi_i, \chi_k$ with $n_f = n_\chi + n_\psi = 2n_\chi$ flavors. Scalars are considered as singlets, and spinors as vector-like in the fundamental representation. Let us consider this Lagrangian

$$\mathcal{L} = -\frac{1}{2} Tr(F_{\mu\nu}F^{\mu\nu}) + \sum_{j=1}^{n_\chi} \bar{\psi}_j \mathcal{D}_j \psi_j + \sum_{k=1}^{n_\chi} \bar{\chi}_k \mathcal{D}_k \chi_k + \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2$$

$$- y_1 (\bar{\psi} \psi + \bar{\chi} \chi) \phi_1 - y_2 (\bar{\psi} \chi + h.c.) \phi_2 - \frac{1}{24} \lambda_1 \phi_1^4 - \frac{1}{24} \lambda_2 \phi_2^4 - \frac{1}{4} \lambda_3 \phi_1^2 \phi_2^2$$

(2.130)

This Lagrangian has scale invariance at the classical level, discrete symmetry

$$\phi_1 \rightarrow \phi_1, \quad \phi_2 \rightarrow -\phi_2, \quad \psi \rightarrow \psi, \quad \chi \rightarrow -\chi$$

(2.131)

and global transformations $\psi \rightarrow U \psi, \chi \rightarrow U \chi$. For $n_f$ small, the gauge coupling increases in the infrared similar to QCD. There are Goldstone bosons in the spectrum because the chiral symmetry $SU(n_f) \otimes SU(n_f)$ is spontaneously broken. On the other hand, we are interested in large $n_f$ where the gauge coupling decreases in the ultraviolet but does not increase in the infrared.

Fixed Point Structure

We consider large values of $n_f$ and $N$, two loop order for gauge coupling, and one loop order for Yukawa and scalar couplings. The $\beta$ functions are (Machacek and Vaughn, 1983)(Machacek and Vaughn, 1984)(Machacek and Vaughn, 1985)(Caswell, 1974)

$$\begin{align*}
(16\pi^2) \frac{\partial g}{\partial t} &= -\frac{\delta N}{3} g^3 + \frac{25N^2}{2} \frac{g^5}{16\pi^2} \\
(16\pi^2) \frac{\partial y_1}{\partial t} &= 4y_1 y_2^2 + 11N^2 y_1^3 - 3Ng^2 y_1 \\
(16\pi^2) \frac{\partial y_2}{\partial t} &= 3y_1^2 y_2 + 11N^2 y_2^3 - 3Ng^2 y_2 \\
(16\pi^2) \frac{\partial \lambda_1}{\partial t} &= 3\lambda_1^2 + 3\lambda_3^2 + 44N^2 \lambda_1 y_1^2 - 264N^2 y_1^4 \\
(16\pi^2) \frac{\partial \lambda_2}{\partial t} &= 3\lambda_2^2 + 3\lambda_3^2 + 44N^2 \lambda_2 y_2^2 - 264N^2 y_2^4 \\
(16\pi^2) \frac{\partial \lambda_3}{\partial t} &= \lambda_1 \lambda_3 + \lambda_2 \lambda_3 + 4\lambda_3^2 + 22N^2 \lambda_3 y_1^2 + 22N^2 \lambda_3 y_2^2 - 264N^2 y_1^2 y_2^2
\end{align*}$$

(2.132)
where \( n_f = \frac{11N}{2} (1 - \frac{\delta}{\Pi}) \) and terms of order \( O(\delta) \) are neglected except in \( \beta_g \). Calculation for the beta function of the gauge coupling is done in Appendix C.

We should show that there exist a non-trivial fixed point. In order for this to occur, the gauge coupling should start from a UV value that is smaller than the fixed point. Then, the last term in the beta functions of Yukawa couplings is negative and dominates until the positive non-linear terms start to dominate over the negative linear terms. The same process applies to the scalars. We find the zero of equations (2.132) to find the fixed point. Note that after finding the zero of the gauge coupling, it is used in the subsequent equations for Yukawa couplings, and then these are used for the scalar self-coupling. To leading order in \( N^{-1} \), the fixed point is at the zero of \( \beta \) functions

\[
g^2_* = 16\pi^2 \frac{2}{75} \frac{\delta}{N}, \quad y^2_* = y^2_* = \frac{3}{11} \frac{g^2_*}{N}, \quad \lambda_1 = \lambda_2 = \lambda_3 = \frac{18}{75} \frac{g^2_*}{N} \tag{2.133}
\]

Theories with scalars and fermions in 4 dimensions do not have non-trivial infrared fixed point. However, this is not the case here because the gauge coupling is driving the other couplings toward the fixed point.

**Vacuum Structure**

The potential has a trivial minimum classically, \( \langle \phi_1 \rangle = \langle \phi_2 \rangle = 0 \). On the other hand, including quantum effects changes the vacuum as discussed in section (1.2). The effective potential at one loop in the \( \overline{MS} \) scheme is

\[
V_{\text{eff}} = -\frac{1}{24} \lambda_1 \phi_1^4 - \frac{1}{24} \lambda_2 \phi_2^4 - \frac{1}{4} \lambda_3 \phi_1^2 \phi_2^2 \\
- \frac{11N^2 M_f^4}{(64\pi^2)} \left( \ln \frac{M_f^2}{2\mu^2} - \frac{3}{2} \right) - \frac{11N^2 M_f^4}{(64\pi^2)} \left( \ln \frac{M_f^2}{2\mu^2} - \frac{3}{2} \right) \\
+ \frac{M_s^4}{(64\pi^2)} \left( \ln \frac{M_s^2}{\mu^2} - \frac{3}{2} \right) + \frac{M_s^4}{(64\pi^2)} \left( \ln \frac{M_s^2}{\mu^2} - \frac{3}{2} \right) \tag{2.134}
\]
where

\[ M_{f\pm} = y_1 \phi_1 \pm y_2 \phi_2 \]
\[ M_{s\pm}^2 = \frac{(\lambda_1 + \lambda_3)\phi_1^2 + (\lambda_2 + \lambda_3)\phi_2^2}{4} \]
\[ \pm \sqrt{(\lambda_1 - \lambda_3)^2\phi_1^2 + (\lambda_2 - \lambda_3)^2\phi_2^2 - 2(\lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 - 7\lambda_3^2)\phi_1^2\phi_2^2} \] (2.135)

As it was discussed in (Coleman and Weinberg, 1973), including masses for the spinors and scalars does not change the conclusion. It is hard to look for a minimum for this potential. However, we can use discrete symmetry (2.131) of the vacuum to find a local minimum. Along \( \phi_2 = 0 \), the effective potential becomes

\[ V_{\text{eff}} = \frac{\lambda_1}{24} \phi_1^4 + \frac{(\lambda_1 \phi_1^2)^2}{256\pi^2} \left( \ln \left( \frac{\lambda_1}{2\mu^2} \right) - \frac{3}{2} \right) + \frac{(\lambda_3 \phi_1^2)^2}{256\pi^2} \left( \ln \left( \frac{\lambda_3}{2\mu^2} \right) - \frac{3}{2} \right) - \frac{22N^2y_1^4}{64\pi^2} \left( \ln \left( \frac{y_1^2\phi_1^2}{\mu^2} \right) - \frac{3}{2} \right) \] (2.136)

with an extremum

\[ \frac{\partial}{\partial \phi_1} V_{\text{eff}}(\langle \phi_1 \rangle) = 0 \]

\[ \Rightarrow -\frac{\lambda_1}{6} = \frac{\lambda_1^2}{64\pi^2} \left( \ln \left( \frac{\lambda_1}{2\mu^2} \right) - 1 \right) + \frac{\lambda_3^2}{64\pi^2} \left( \ln \left( \frac{\lambda_3}{2\mu^2} \right) - 1 \right) \]
\[ - \frac{88N^2y_1^4}{64\pi^2} \left( \ln \left( \frac{y_1^2\langle \phi_1 \rangle^2}{\mu^2} \right) - 1 \right) \] (2.137)

Note that we can have dimensional transmutation where a dimensionless parameter \( \lambda_1 \) is replaced by a dimensional vacuum expectation value \( \langle \phi_1 \rangle \). Large logarithms might spoil perturbative analysis in the effective potential. To avoid this problem, the condition \( \frac{\lambda_1}{16\pi^2} \ln \left( \frac{\langle \phi_1 \rangle^2}{\mu^2} \right) \ll 1 \) must hold. Hence, \( \lambda_1 \) can be expressed by using the last two terms in (2.137), and the condition becomes

\[ \frac{\lambda_1^2}{(16\pi^2)^2} \ln^2 \frac{\langle \phi_1 \rangle^2}{\mu^2} \ll 1 \] (2.138)

We need to show that the extremum that we found is a local minimum by checking that the eigenvalues of the mass matrix are both positive. Note that mixed derivatives terms vanish at \( \langle \phi_1 \rangle \) because we have discrete symmetry and because we are working along
\( \phi_2 = 0 \). The two eigenvalues are

\[
\frac{\partial^2}{\partial \phi_1^2} V_{\text{eff}}(\langle \phi_1 \rangle, 0) = \frac{\lambda_3^2 - 88N^2y_1^4}{32\pi^2} \langle \phi_1 \rangle^2 \quad (2.139)
\]

\[
\frac{\partial^2}{\partial \phi_2^2} V_{\text{eff}}(\langle \phi_1 \rangle, 0) = \frac{\lambda_3}{2} \langle \phi_1 \rangle^2 - \lambda_3 (\langle \phi_1 \rangle + 4\lambda_3) \langle \phi_1 \rangle^2 \left( \frac{\ln \frac{\lambda_3 \langle \phi_1 \rangle}{\mu^2}}{64\pi^2} + 1 \right)
- \frac{264N^2y_1^2y_2^2}{64\pi^2} \langle \phi_1 \rangle^2 \left( \ln \frac{y_1^2 \langle \phi_1 \rangle}{\mu^2} - \frac{1}{3} \right) \quad (2.140)
\]

The first eigenvalue is positive given

\[\varepsilon \equiv \lambda_3^2 - 88N^2y_1^4 > 0 \quad (2.141)\]

And the second eigenvalue is positive if the one loop terms are small in comparison with the tree level term. Note that the value of the effective potential at \( \langle \phi_1 \rangle \) is negative

\[V_{\text{eff}}(\langle \phi_1 \rangle) = -\frac{\lambda_3^2 - 88N^2y_1^4}{512\pi^2} \langle \phi_1 \rangle^4 = -\frac{\varepsilon}{512\pi^2} \langle \phi_1 \rangle^4 \quad (2.142)\]

The importance of the second scalar field \( \phi_2 \) is that it provides us with a non-trivial minimum in the effective potential by perturbative analysis. To make \( \varepsilon \) small, we should choose an expectation value such that it is smaller than a fixed renormalization scale \( \mu_0 \), \( \langle \phi_1 \rangle \ll \mu_0 \). The coupling constant will run until the value of the scale \( \mu \) starts to be close to the heaviest particle mass in the model. At this point \( \mu \lesssim \langle \phi_1 \rangle \), the trajectory is modified. The coupling constant can run far enough to get arbitrarily close to the infrared fixed point as long as we start with a small \( \langle \phi_1 \rangle \). Moreover, if there are other minima outside \( \phi_2 = 0 \) axis, then the same analysis applies to the global minimum. Note that scale invariance of the classical Lagrangian is broken by two effects. The first effect is the non-trivial minimum found at one loop order. The second effect is quantum fluctuations at one loop order too. If the first effect is dominant, then a pseudo Goldstone boson would appear because we have spontaneously broken approximate scale invariance. But if quantum fluctuations are dominant, then there would be no pseudo Goldstone boson.

**Particle Spectrum**

If \( \langle \phi_1 \rangle = \langle \phi_2 \rangle = 0 \), then all the particles are massless because the symmetry is not broken. Calculations are performed by considering one loop order so to examine the invariance of
physical quantities in the renormalization group flow.

The fermion self-energy at large $N$ is dominated by the gauge interaction. To one loop order, the fermion self-energy is

$$i\Sigma(\not{p}) = i(Am + B\not{p})$$

$$A = \frac{g^2}{16\pi^2} N \left( -3 \ln \frac{y_1^2 \nu^2}{\mu^2} + 4 \right), \quad B = 1$$

in Landau gauge. Therefore, the masses of $\psi$ and $\chi$ are

$$M_\psi(\mu) = M_\chi(\mu) = y_1\nu \left[ 1 - \frac{g^2}{16\pi^2} N \frac{3}{2} \ln \frac{y_1^2 \nu^2}{\mu^2} - 4 \right]$$

Derivation of the renormalized mass in the $\overline{MS}$ scheme for fermions is explained in Appendix D. The masses of scalar fields $\phi_1$ and $\phi_2$ are

$$M_{\phi_i}^2 = \frac{\lambda_1 \nu^2}{2} + \frac{3\lambda_1^2 \nu^2}{64\pi^2} \left( \ln \frac{\lambda_1 \nu^2}{2\mu^2} - \frac{5}{3} + \frac{2\pi}{3\sqrt{3}} \right) + \frac{3\lambda_3^2 \nu^2}{64\pi^2} \left( \ln \frac{\lambda_3 \nu^2}{2\mu^2} - \frac{1}{3} - \frac{2\lambda_1}{3\lambda_3} \right)$$

$$+ \frac{22N^2 y_1^2}{16\pi^2} \left[ \frac{y_1^2 \nu^2}{12} - \lambda_1 \nu^2 \right] - 3 \int_{0}^{1} dx \left( \frac{x(1-x)}{2} \lambda_1 \nu^2 \right) \ln \left( 1 - x(1-x) \frac{\lambda_1}{2y_1^2} \right)$$

$$= \frac{3\lambda_1^2 \nu^2}{64\pi^2} \left( -2 + \frac{2\pi}{3\sqrt{3}} \right) + \frac{3\lambda_3^2 \nu^2}{64\pi^2} \left( -2 - \frac{2\lambda_1}{3\lambda_3} \right) + \frac{22N^2 y_1^2}{16\pi^2} \left[ -2 \left( \frac{y_1^2 \nu^2}{12} - \lambda_1 \nu^2 \right) \right]$$

$$\approx \frac{\lambda_3^2}{32\pi^2} - \frac{88N^2 y_1^4}{32\pi^2} \nu^2 = \frac{\varepsilon}{32\pi^2} \nu^2$$
\[ M^2_{\phi_2} = \frac{\lambda_3 \nu^2}{2} + \frac{\lambda_1 \lambda_3 \nu^2}{64 \pi^2} \left( \ln \frac{\lambda_1 \nu^2}{2 \mu^2} - 1 \right) + \frac{\lambda_2 \lambda_3 \nu^2}{64 \pi^2} \left( \ln \frac{\lambda_3 \nu^2}{2 \mu^2} - 1 \right) + \frac{\lambda_3^2 \nu^2}{16 \pi^2} \left( \ln \frac{\lambda_3 \nu^2}{2 \mu^2} + \int_0^1 dx \ln \left( x^2 + (1 - x) \frac{\lambda_3}{\nu^2} \right) \right) + \frac{22 N^2 y_2^2}{16 \pi^2} \left[ y_1^2 \nu^2 - \frac{\lambda_3 \nu^2}{12} - 3 \left( y_1^2 \nu^2 - \frac{\lambda_3 \nu^2}{12} \right) \left( \ln \frac{y_1^2 \nu^2}{\mu^2} \right) \right] - 3 \int_0^1 dx \left( y_1^2 \nu^2 - \frac{x(1 - x)}{2} \lambda_3 \nu^2 \right) \ln \left( 1 - x(1 - x) \frac{\lambda_3}{2y_1^2} \right) \approx \frac{\lambda_3 \nu^2}{2} + \frac{\lambda_2 \lambda_3 \nu^2}{64 \pi^2} \left( \ln \frac{\lambda_3 \nu^2}{2 \mu^2} - 1 \right) + \frac{\lambda_3^2 \nu^2}{16 \pi^2} \left( \ln \frac{\lambda_3 \nu^2}{2 \mu^2} - 2 \right) + \frac{22 N^2 y_2^2}{16 \pi^2} \left[ y_1^2 \nu^2 - \frac{\lambda_3 \nu^2}{12} - 3 \left( y_1^2 \nu^2 - \frac{\lambda_3 \nu^2}{12} \right) \left( \ln \frac{y_1^2 \nu^2}{\mu^2} \right) \right] - 3 \int_0^1 dx \left( y_1^2 \nu^2 - \frac{x(1 - x)}{2} \lambda_3 \nu^2 \right) \ln \left( 1 - x(1 - x) \frac{\lambda_3}{2y_1^2} \right) \] (2.147)

The first lines in (2.146) and (2.147) are the pole mass expressions at one loop order. In the second line in (2.146), equation (2.137) is used. And in the other lines, the approximation that \( \lambda_1 \) is small at the scale \( \mu_0 \) and the condition (2.138) are used. The vacuum expectation value \( \nu \) has the anomalous dimension of \( \phi_1 \)

\[ \frac{\partial \nu}{\partial t} = \gamma_{\phi_1} \nu = - \frac{11 N^2 y_2^2}{16 \pi^2} \nu \] (2.148)

where \( t = \ln \left( \frac{\mu}{\mu_0} \right) \) as it is defined in (1.60) with a minus sign difference in convention.

Using \( \beta \) functions in (2.132) and the above expression we get

\[ \frac{\partial M_\psi}{\partial t} = \frac{\partial M_\chi}{\partial t} = \frac{\partial M_{\phi_1}}{\partial t} = \frac{\partial M_{\phi_2}}{\partial t} = 0 \] (2.149)

Hence, the masses are RG-invariant at two loops order.

### 2.4.2 Dilaton

**Dilatation Current**

The dilatation current, \( D^\mu \) defined in (2.7) corresponds to the improved energy momentum tensor defined in (2.5). As mentioned in section (2.1), the canonical energy momentum
tensor can be improved by adding terms. Therefore, the improved energy momentum tensor from Lagrangian (2.130) and equation (2.6) is

\[ \Theta^{\mu\nu} = -F^a_{\mu\lambda} F_{\nu\lambda} + \frac{1}{2} \bar{\chi} i (\gamma^\mu D^\nu + \gamma^\nu D^\mu) \chi + \frac{1}{2} \bar{\psi} i (\gamma^\mu D^\nu + \gamma^\nu D^\mu) \psi + \partial^\mu \phi_i \partial^\nu \phi_i - \frac{1}{2} \kappa (\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi_i^2 - g^{\mu\nu} L \] (2.150)

where the term proportional to \( \kappa \) is the improvement term (Coleman and Jackiw, 1971).

The integral of this term vanishes because it is a total derivative with \( \kappa = \frac{1}{3} \). At the classical level, this current is conserved; hence, we have scale invariance. However, at the quantum level it is not, and we get a trace anomaly (Coleman and Jackiw, 1971)(Adler et al., 1977)

\[ \Theta^{\mu\mu}_{\mu} = \gamma_{\phi_1} \phi_1 \partial^2 \phi_1 + (4 \gamma_{\phi_1} \lambda_1 - \beta_{\lambda_1}) \frac{\phi_1^4}{24} + ... \] (2.151)

**Dilaton**

The dilaton state \( |\sigma\rangle \) can be created by acting on the vacuum with the spontaneously broken dilatation current. Similar to PCAC, we define a dilaton mass \( M_\sigma \) and a decay constant \( f_\sigma \) as

\[ \langle 0 | \partial_\mu D^\mu | \sigma \rangle = \langle 0 | \Theta^\mu_{\mu} | \sigma \rangle_{x=0} = -f_\sigma M_\sigma^2 \] (2.152)

Let us consider the energy momentum tensor

\[ \langle 0 | \Theta^{\mu\nu}(x) | \sigma \rangle = \frac{f_\sigma}{3} (p^\mu p^\nu - g^{\mu\nu} p^2) e^{ip.x} \] (2.153)

In (2.153), the momentum is defined on-shell \( p^2 = M_\sigma^2 \).

The goal is to identify \( M_\sigma \) and \( f_\sigma \) with a state in the spectrum that we have for our dilaton model in the previous section. As explained before, if we have approximate symmetry, we expect the dilaton to be light, and it couples to the stress energy tensor. Notice that \( M_{\phi_1} \) fits this description because it is lighter than \( M_{\phi_2} \). In order to show that it couples to the stress energy tensor, as before, we chose to expand the fields at \( \langle \phi_1 \rangle = \nu \) and \( \langle \phi_2 \rangle = 0 \), and we notice that \( \phi_1 \) is the only linear field term. Shifting the field \( \phi_1 \) in the Lagrangian
(2.150), the terms proportional to $p^\mu p^\nu$ contribute as
\[
\Theta^\mu\nu = -\frac{1}{3} \nu \partial^\mu \partial^\nu \phi_1 + ...
\]  
(2.154)

Comparing (2.153) and (2.154), we identify $f_\sigma = \nu$ and $M_\sigma = M_{\phi_1}$. Moreover, the anomaly in (2.151) also supports this identification as (after shifting $\phi_1$)
\[
\Theta^\mu_\mu = \gamma_{\phi_1} \nu \partial^2 \phi_1 + (4 \gamma_{\phi_1} \lambda_1 - \beta_{\lambda_1}) \frac{\nu^2 \phi_1}{6} + ...
\]  
(2.155)

Hence, to lowest order we obtain
\[
\langle 0 | \Theta^\mu_\mu | \sigma \rangle_{x=0} = -\gamma_{\phi_1} \nu p^2 - \frac{\lambda_1^2 + \lambda_2^2}{32 \pi^2} - \frac{88 N^2 y_1^4}{22} \nu_3 + ...
\]  
(2.156)

This result matches the definition in (2.152) with the identifications
\[
f_\sigma = \nu , \quad M_\sigma^2 = M_{\phi_1}^2 = \frac{\varepsilon}{32 \pi^2} \nu^2
\]  
(2.157)

which are RG-invariant and $\lambda_1^2 \nu^3$ and $\gamma_{\phi_1} \nu M_\sigma^2$ terms were dropped for consistency.

### 2.4.3 Phase Structure

As mentioned before, a condition was set for $\lambda_1$ to be small in order to have dimensional transmutation. Then, the model has a non-trivial minimum given that condition (2.141) is satisfied. The two conditions previously mentioned are not satisfied near the infrared fixed point. On the other hand, if we start at a point at large RG-time $t$ and then follow the flow to the infrared, then there always exist points that are arbitrarily close to the infrared fixed point where spontaneous symmetry breaking occurs. So we choose at a renormalization scale $\mu_0$ coupling constants that provide a non-trivial minimum and satisfy $\langle \phi_1 \rangle \ll \mu_0$. In the mass independent scheme, the coupling constants will run towards the infrared fixed point. The smaller $\langle \phi_1 \rangle$, the closer we get to the infrared fixed point.

Since we are close to the infrared fixed point, massive particles are integrated out, Yukawa and self-couplings are also not important. Hence, the beta function is governed by massless Yang-Mills vectors, and increases as $\mu$ is decreased below $\langle \phi_1 \rangle$ in the form
\[
g^2(\mu) \approx \frac{g_s^2}{1 + \frac{g_s^2}{16 \pi^2} \frac{22 N}{3} \ln \frac{\mu}{\langle \phi_1 \rangle}}
\]  
(2.158)
The effective theory is similar to the theory of glueballs with mass

\[ M_g \sim \langle \phi_1 \rangle e^{-\frac{3}{2\pi} \frac{16\pi^2}{g^2}} = \langle \phi_1 \rangle e^{-\frac{225}{449}} \]  

(2.159)

Hence, now we have \( n_f \) massive fermions, two massive scalars, glueballs with mass \( M_g \) in (2.159), and the dilaton is identified with the lighter scalar with mass (2.157). However, when \( \varepsilon \) is not small, then perturbation theory fails, and the true vacuum exists at \( \langle \phi_1 \rangle = \langle \phi_2 \rangle = 0 \) where no spontaneous symmetry breaking happens; and hence, all particles are massless.

From this picture we can infer that there are two phases at a fixed renormalization scale \( \mu_0 \). In the first region, \( \varepsilon \) is not small, \( \langle \phi_1 \rangle = 0 \), particles are massless, and RG trajectories flow into the infrared fixed point. In the second region, \( \varepsilon \) is small, \( \langle \phi_1 \rangle = \nu \), particles have masses, and RG trajectories get close but do not end at the infrared fixed point. Adding relevant perturbations in the Lagrangian breaks scale invariance explicitly. Therefore, if these mass perturbations are smaller than the masses we obtained in this section, then qualitatively nothing changes, and quantitatively the corrections are small. On the other hand, if these mass perturbations are large enough, then a pseudo Goldstone boson will not exist.
Chapter 3

Conclusion

In this essay, an introduction to the Higgs mechanism, effective potential method, and dilatons is given. Various dilaton models are explored. In model 1, a pseudo-dilaton couples to the SM energy momentum tensor with strength proportional to $f^{-1}$. It was shown that the dilaton coupling to gluons is enhanced making it possible to differentiate it from the Higgs boson in collider physics experiments. In model 2, an effective theory of a light dilaton is constructed that includes explicit conformal symmetry breaking effects to any number of loops. Then, by considering a marginal operator that breaks the symmetry, it is shown that the existence of a light dilaton in a strongly coupled CFT is associated with some tuning. Finally, corrections to the dilaton interactions with a conformal SM is explained. In model 3, a $SU(N)$ gauge theory with two spinors and scalars is considered. Then, a non-trivial minimum in the effective potential is found, and dimensional transmutation happens. Hence, scale invariance is broken. Provided that certain conditions are satisfied on some parameters, one of the scalars is identified with a light dilaton.
Bibliography


Appendices

A Dirac and Gell-Mann Matrices

Dirac matrices satisfy anticommutation relations

\[ \{ \gamma^{\mu}, \gamma^{\nu} \} \equiv \gamma^{\mu} \gamma^{\nu} + \gamma^{\nu} \gamma^{\mu} = 2 \eta^{\mu\nu} \quad (A.1) \]

where

\[ \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (A.2) \]

Dirac matrices in Dirac representation are

\[ \gamma^{0} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^{j} = \begin{pmatrix} 0 & \sigma_{j} \\ -\sigma_{j} & 0 \end{pmatrix} \quad (A.3) \]

where \( I \) is the 2 \times 2 identity matrix, and \( \sigma_{j} \) are Pauli 2 \times 2 matrices

\[ \sigma_{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (A.4) \]

Pauli matrices satisfy commutation and anti-commutation relations

\[ [\sigma_{i}, \sigma_{j}] = 2i \epsilon^{ijk} \sigma_{k} \quad (A.5) \]

\[ \{ \sigma_{i}, \sigma_{j} \} = 2 \delta_{ij} I \quad (A.6) \]
where $\varepsilon^{ijk}$ is

$$
\varepsilon^{ijk} = \begin{cases} 
  +1, & \text{for even permutations of 123} \\
  -1, & \text{for odd permutations of 123} \\
  0, & \text{otherwise}
\end{cases}
$$

We can define $\gamma^5$ as

$$
\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_5 \\
= \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma
$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol defined as

$$
\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} 
  +1, & \text{for even permutations of 0123} \\
  -1, & \text{for odd permutations of 0123} \\
  0, & \text{otherwise}
\end{cases}
$$

and

$$
\{\gamma^5, \gamma^\mu\} = 0 \\
\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
$$

Dirac matrices in the Weyl representation are the same for $\gamma^j$ but different for $\gamma^0$ and $\gamma^5$

$$
\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}
$$

Gell-Mann matrices form a Lie algebra that satisfy the commutation relations

$$
[T^a, T^b] = i f^{abc} T^c
$$
where $f^{abc}$ are the structure constants. For $SU(3)$, Gell-Mann matrices in the basis $T^a = \frac{1}{2} \lambda^a$ are

\[
\begin{align*}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda^6 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix},
\end{align*}
\]

(A.14)

Note that there exist a Cartan subalgebra because $\lambda^3$ and $\lambda^8$ commute. This implies that there are two quantum numbers that are observable simultaneously; isospin $I_3 = \frac{1}{2} \lambda_3$ and hypercharge $Y = \frac{1}{\sqrt{3}} \lambda_8$.

Generally, equation (A.13) holds for any representation matrices $T^a$ for a gauge group $G$. The quadratic Casimir operator in an irreducible representation $R$ is defined as

\[
C_2(R) \equiv T^a T^a
\]

(A.15)

where summation over $a$ is implied and it is a multiple of the identity. One can choose to normalize the structure constants as

\[
\sum_{c,d} f^{acd} f^{bcd} = N \delta^{ab} = C_2(G) \delta^{ab}
\]

(A.16)

where $C_2(G)$ is the Casimir operator in the adjoint representation. The generators in the fundamental representation are normalized as

\[
Tr(T^a T^b) = \frac{1}{2} \delta^{ab}
\]

(A.17)

The Dynkin index is defined as

\[
Tr_R(T^a T^b) \equiv S_2(R) \delta^{ab}
\]

(A.18)
Setting $a = b$ in equation (A.18) and summing over $a$ we obtain

$$d(G)S_2(R) = d(R)C_2(R)$$  \hspace{1cm} (A.19)

where $d(R)$ is the representation dimension ($d(\text{fundamental}) = N$ and $d(\text{adjoint}) = N^2 - 1$), $d(G)$ is the Lie group dimension. For instance, $SU(N)$ has a dimension $d(SU(N)) = N^2 - 1$, $C_2(\text{fundamental}) = \frac{N^2-1}{2N}$, and $C_2(G) = C_2(\text{adjoint}) = N$. Note that in the adjoint representation $d(R) = d(G)$; hence, Dynkin index and the quadratic Casimir operator are the same in this case $S_2(R) = C_2(G) = N$. In the fundamental representation we have $S_2(R) = \frac{1}{2}$ from equation (A.17). Note that $C_2(F)$ and $C_2(S)$ are the Casimir operators for fermion and scalar representations, respectively, $S_2(F)$ and $S_2(S)$ are the Dynkin indices for fermion and scalar representations, respectively.

### B Integral with a Cutoff

In order to do the integral with a cutoff in (1.46), this identity should be used

$$\int d^4kG(k^2) = \pi^2 \int_0^\Lambda_0^2 dk^2G(k^2)$$  \hspace{1cm} (B.1)

Now we have

$$\frac{\hbar}{2} \int \frac{d^4k_E}{(2\pi)^4} \left[ \log(k_E^2 + V''(\phi)) - \log(k_E^2) \right] = \frac{\hbar}{2^5\pi^2} \int_0^{\Lambda^2} dk^2_E \left[ \log(k^2_E + V''(\phi)) - \log(k^2_E) \right] k^2_E$$  \hspace{1cm} (B.2)

Let $k^2_E = x$ and integrate by parts to get

$$\int_0^\Lambda x \log(x + V''(\phi))dx = \frac{\Lambda^4}{2} \log(\Lambda^2 - \frac{1}{2}) + V''(\phi)\Lambda^2 - \frac{(V''(\phi))^2}{2} \log(\Lambda^2) + \frac{(V''(\phi))^2}{2} \log(V''(\phi) - \frac{1}{2})$$  \hspace{1cm} (B.3)

$$\int_0^\Lambda x \log(x) = \frac{\Lambda^4}{2} \log(\Lambda^2 - \frac{1}{2})$$  \hspace{1cm} (B.4)

Hence, the integral in (B.2) becomes

$$\frac{\hbar}{32\pi^2} V''(\phi)\Lambda^2 + \frac{\hbar}{64\pi^2} (V''(\phi))^2 \left[ \log \left( \frac{V''(\phi)}{\Lambda^2} \right) - 1 \right]$$  \hspace{1cm} (B.5)
Performing the same analysis in the $\overline{MS}$ scheme where equation (1.45) is

$$V_{\text{eff}}(\phi) = V(\phi) - \frac{i\hbar}{2} \text{Tr} \left( \log[\partial^2 + V''(\phi)] \right)$$  \hspace{1cm} \text{(B.6)}$$

Going to the momentum space and to the Euclidean space, we obtain

$$\int \frac{d^Dp}{(2\pi)^D} \ln(-p^2 + V''(\phi)) = i \int \frac{d^DP_E}{(2\pi)^D} \ln(p_E^2 + V''(\phi))$$

$$= -i \frac{\partial}{\partial \alpha} \left. \int \frac{d^Dp_E}{(2\pi)^D} \frac{1}{(p_E^2 + V''(\phi))^\alpha} \right|_{\alpha=0}$$

$$= -i \frac{\partial}{\partial \alpha} \left. \left( \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(\alpha - \frac{D}{2})}{\Gamma(\alpha)} \frac{1}{(V''(\phi))^{\alpha - D/2}} \right) \right|_{\alpha=0}$$

$$= -i \frac{\Gamma(-D/2)}{(4\pi)^{D/2}} (V''(\phi))^{D/2}$$  \hspace{1cm} \text{(B.7)}$$

Where $D = 4 - 2\epsilon$ is the number of dimension. To obtain ordinary dimension, we introduce an arbitrary scale $\tilde{\mu}$, and the effective potential becomes

$$V_{\text{eff}}(\phi) = -\frac{1}{2} \tilde{\mu}^{4-D} V''(\phi)^{D/2} + V(\phi)$$

$$= -\frac{1}{2(4\pi)^2} (V''(\phi))^2 \Gamma(-2 + \epsilon) \left( \frac{V''(\phi)}{2\pi \tilde{\mu}^2} \right)^{-\epsilon} + V(\phi)$$  \hspace{1cm} \text{(B.8)}$$

In the limit of $\epsilon \to 0$, $\Gamma(-2 + \epsilon) = \frac{1}{2} \left( \frac{1}{\epsilon} - \gamma + \frac{3}{2} \right)$. Hence, (B.8) becomes

$$V_{\text{eff}}(\phi) = -\frac{1}{64\pi^2} (V''(\phi))^2 \left( \frac{1}{\epsilon} - \gamma + \frac{3}{2} + \log \frac{4\pi \tilde{\mu}^2}{V''(\phi)} \right) + V(\phi)$$

$$= -\frac{1}{64\pi^2} (V''(\phi))^2 \left( \frac{3}{2} + \log \frac{\tilde{\Lambda}^2}{V''(\phi)} \right) + V(\phi)$$  \hspace{1cm} \text{(B.9)}$$

where

$$\tilde{\Lambda} = \sqrt{4\pi} e^{-\frac{3}{2} + \frac{1}{2} \epsilon} \tilde{\mu}$$  \hspace{1cm} \text{(B.10)}$$
C Beta Function for the Gauge Coupling

According to (Machacek and Vaughn, 1983), the beta function at two loop order for the gauge field is

\[
\beta(g) = -\frac{g^3}{(4\pi)^2}b_0 - \frac{g^5}{(4\pi)^4}b_1
\] (C.1)

\[
b_0 = \frac{11}{3}C_2(G) - \frac{4}{3}\kappa S_2(F) - \frac{1}{6}S_2(S) + \frac{2\kappa}{(4\pi)^2}Y_4(F)
\] (C.2)

\[
b_1 = \frac{34}{3}[C_2(G)]^2 - \kappa[4C_2(F) + \frac{20}{3}C_2(G)]S_2(F)
- [2C_2(S) + \frac{1}{3}C_2(G)]S(F)
\] (C.3)

From Appendix A, \(SU(N)\) has

\[
C_2(G) = N \quad , \quad S_2(F) = n_f \quad , \quad C_2(F) = \frac{N^2 - 1}{2N} \quad , \quad S_2(S) = 0 \quad , \quad C_2(S) = 0
\] (C.4)

with \(\kappa = \frac{1}{2}, 1\) for two-component fermions or four-component fermions respectively. Hence, \(b_0\) becomes

\[
b_0 = \frac{11}{3}N - \frac{4}{3}\frac{1}{2}n_f = \frac{N\delta}{3}
\] (C.5)

with \(\delta = 11(1 - \frac{2n_f}{11N})\). According to (Caswell, 1974), a gauge theory is not expected to flip the sign of the beta function and approach zero to be valid in perturbation theory. Fixing the lowest order term \(b_0 = 0\) implies \(n_f = \frac{11N}{2}\). Using this in \(b_1\) yields

\[
b_1 = \frac{34}{3}N^2 - \frac{1}{2}[4\left(\frac{N^2 - 1}{2N}\right) + \frac{20N}{3}]n_f
\]

\[
= \frac{34}{3}N^2 - \left[\frac{1}{N}(N^2 - 1) + \frac{10}{3}N\right]11N
\]

\[
= \frac{34}{3}N^2 - \frac{11}{2}N^2 + \frac{11}{2} - \frac{55}{3}N^2 \simeq -\frac{25}{2}N^2
\] (C.6)
D The Renormalized Mass in Fermions

The form of the self-energy loop graph by using Feynman parameters is

\[ i\Sigma_2(\frac{i}{p}) = (ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i(k + m)}{k^2 - m^2 + i\varepsilon} \frac{\gamma^\mu}{(k - p)^2 + i\varepsilon} \]

\[ = e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \left[ \frac{2k - 4m}{(k^2 - m^2)(1 - x) + (p - k)^2x + i\varepsilon} \right] \]  

(D.1)

Completing the square and shifting \( k \to k + px \) yields

\[ i\Sigma_2(\frac{i}{p}) = 2e^2 \int dx \int \frac{d^4k}{(2\pi)^4} \frac{x_p - 2m}{(k^2 - (1 - x)(m^2 - p^2x) + i\varepsilon)^2} \]

(D.2)

We regularize this integral by using dimensional regularization in \( d = 4 - \varepsilon \) dimensions.

The self-energy becomes

\[ \Sigma_2(\frac{i}{p}) = -ie^2\mu^{4-d} \int_0^1 [(d - 2)x_p - dm] \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - (1 - x)(m^2 - p^2x) + i\varepsilon^2} \]

\[ = \frac{\alpha}{4\pi} \int_0^1 dx [(2 - \varepsilon)x_p - (4 - \varepsilon)m] \left[ \frac{2}{\varepsilon} + \ln \left( \frac{\tilde{\mu}^2}{(1 - x)(m^2 - p^2x)} \right) \right] \]

(D.3)

where \( \tilde{\mu}^2 \equiv 4\pi e^{-\gamma_E} \mu^2 \). The renormalized and bare Green's functions are related as

\[ iG_R(\frac{i}{p}) = \frac{1}{1 + \delta_2} iG_{bare}(\frac{i}{p}) = \frac{1}{1 + \delta_2} \frac{i}{p - m_0 + \Sigma_2(\frac{i}{p}) + ...} \]

\[ = \frac{i}{p - m_0 + \delta_2 p - m_0\delta_2 + \Sigma_2(\frac{i}{p}) + ...} \]

(D.4)

Using \( m_0 = m_R + m_R\delta_m \) it becomes

\[ iG_R(\frac{i}{p}) = \frac{i}{p - m_R + \Sigma_R(\frac{i}{p})} \]

(D.5)

where \( \Sigma_R(\frac{i}{p}) = \Sigma_2(\frac{i}{p}) + \delta_2 p - (\delta_m + \delta_2)m_R + O(e^4) \). In the modified minimal subtraction (MS), the \( \gamma_E \) and \( \ln(4\pi) \) factors are removed. Expanding (D.3) in \( \tilde{\mu}^2 \) we obtain the counter terms

\[ \delta_2 = -\frac{\alpha}{4\pi} \left( \frac{2}{\varepsilon} + \ln(4\pi e^{-\gamma_E}) \right) \]

\[ \delta_m = -\frac{3\alpha}{4\pi} \left( \frac{2}{\varepsilon} + \ln(4\pi e^{-\gamma_E}) \right) \]
Hence, the self-energy becomes UV finite in the form
\[
\Sigma_R(p) = \frac{\alpha}{2\pi} \int_0^1 dx \left\{ (x p - 2m_R) \left[ \ln \frac{\mu^2}{(1-x)(m_R^2 - p^2 x)} \right] - x p + m_R \right\} \tag{D.8}
\]

We can relate the pole mass (which is related to the physical mass) and the renormalized mass by looking for a pole in the propagator of equation (D.5) at \( p = m_P \). This condition gives
\[
m_P - m_R + \Sigma_R(m_P) = 0 \tag{D.9}
\]

To leading order, we take \( m_P = m_R \) and do the integral by parts in (D.8) to obtain
\[
m_R = m_P + \Sigma_R(m_p) = m_p \left[ 1 - \frac{\alpha}{4\pi} \left( 4 + 3 \ln \frac{\mu^2}{m_P^2} \right) + \mathcal{O}(\alpha^2) \right] \tag{D.10}
\]

Note that the renormalized mass in the \( \overline{MS} \) scheme depends on the arbitrary scale \( \mu \).