Combinatorial aspects of spatial mixing and new conditions for pressure representation

by

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Abstract

Over the last few decades, there has been a growing interest in a measure-theoretical property of Gibbs distributions known as *strong spatial mixing (SSM)*. SSM has connections with decay of correlations, uniqueness of equilibrium states, approximation algorithms for counting problems, and has been particularly useful for proving special representation formulas and the existence of efficient approximation algorithms for *(topological) pressure*. We look into conditions for the existence of Gibbs distributions satisfying SSM, with special emphasis in hard constrained models, and apply this for pressure representation and approximation techniques in $\mathbb{Z}^d$ lattice models.

Given a locally finite countable graph $\mathscr{G}$ and a finite graph H, we consider $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ the set of graph homomorphisms from $\mathscr{G}$ to H, and we study Gibbs measures supported on $\mathrm{Hom}(\mathscr{G},\mathrm{H})$. We develop some sufficient and other necessary conditions on $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ for the existence of Gibbs specifications satisfying SSM (with exponential decay). In particular, we introduce a new combinatorial condition on the support of Gibbs distributions called *topological strong spatial mixing (TSSM)*. We establish many useful properties of TSSM for studying SSM on systems with hard constraints, and we prove that TSSM combined with SSM is sufficient for having an efficient approximation algorithm for pressure. We also show that TSSM is, in fact, necessary for SSM to hold at high decay rate.

Later, we prove a new pressure representation theorem for nearest-neighbour Gibbs interactions on $\mathbb{Z}^d$ shift spaces, and apply this to obtain efficient approximation algorithms for pressure in the $\mathbb{Z}^2$ (ferromagnetic) Potts, (multi-type) Widom-Rowlinson, and hard-core lattice gas models. For Potts, the results apply to every inverse temperature except the critical. For Widom-Rowlinson and hard-core lattice gas, they apply to certain subsets of both the subcritical and supercritical regions. The main novelty of this work is in the latter, where SSM cannot hold.

Preface

The thesis is split in two main parts (Part I and Part II), based on the following three articles:

1. *Representation and poly-time approximation for pressure of $\mathbb{Z}^2$ lattice models in the non-uniqueness region*. Joint work with Stefan Adams, Brian Marcus, and Ronnie Pavlov. Journal of Statistical Physics, 162(4), 1031-1067. (See [1].)

2. *The topological strong spatial mixing property and new conditions for pressure approximation*. Accepted for publication in Ergodic Theory and Dynamical Systems. (See [15].)

3. *Strong spatial mixing in homomorphism spaces*. Joint work with Ronnie Pavlov. Submitted. (See [16].)

Part I (Chapter 2, Chapter 3, and Chapter 4) is mainly based on paper 2 and paper 3. Part II (Chapter 5, Chapter 6, and Chapter 7) is mainly based on paper 1 and paper 2.

Some of the writing of this thesis was done while the author was visiting the Simons Institute for the Theory of Computing at University of California, Berkeley.

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Nomenclature

$x, y, \ldots$	Vertices, p. 18.
$\sim, \sim_G$	Vertex adjacency relation (in graph G), p. 18.
$\mathrm{Loop}(G)$	Set of vertices with a loop, p. 18.
$A, B, \ldots$	Subsets of vertices in a board, p. 19.
$\mathrm{N}(x)$	Neighbourhood of vertex x, p. 19.
$\Delta(G)$	Maximum degree, p. 19.
$\mathscr{G}$	Board graph (simple, connected, locally finite), p. 19.
$\mathbb{Z}^d$	d-dimensional integer lattice, p. 19.
$\mathbb{T}_d$	d-regular tree, p. 19.
$\mathrm{dist}(x, y)$	Graph distance between x and y, p. 19.
∂A	Outer boundary of A, p. 20.
$\underline{\partial} A$	Inner boundary of A, p. 20.
$\mathscr{N}_n(A)$	n-neighbourhood of A, p. 20.
$\partial_n(A)$	n-boundary of A, p. 20.
$\preccurlyeq$	Lexicographic order in $\mathbb{Z}^d$, p. 20.
$\mathscr{P}$	Lexicographic past of $\mathbb{Z}^d$, p. 20.
B_n	n-block, p. 20.

$\mathrm{Q}_{y,z}$	Broken $[y,z]$-block, p. 20.
R_n	n-rhomboid, p. 21.
$\star$	Superindex for concepts of $\mathbb{Z}^d$ with diagonal edges, p. 21.
H	Constraint graph (finite, possibly with loops), p. 21.
$\mathrm{H}^{\circlearrowleft}$	Constraint graph with loops everywhere, p. 21.
K_n	Complete graph with n vertices, p. 22.
H_φ	Constraint graph of hard-core model, p. 22.
S_n	n-star graph, p. 22.
$\mathscr{A}$	Alphabet, p. 23.
Ω	Configuration space (shift space, homomorphism space, etc.), p. 23.
$\alpha, \beta, \dots$	Configurations, p. 23.
$\omega, \upsilon, \dots$	Points in Ω, p. 23.
$\mathscr{L}(\Omega)$	Language of Ω, p. 23.
$[\alpha]^\Omega$	Cylinder set in Ω for a configuration α, p. 23.
$\sqcup, \bigsqcup$	Disjoint union of sets, p. 23.
$\mathrm{Hom}(\mathscr{G}, \mathrm{H})$	Homomorphism space, p. 24.
σ, σ_x	Shift action in $\mathbb{Z}^d$, p. 24.
$\mathfrak{F}$	Family of finite configurations/constraints, p. 25.
$\mathfrak{E}_i$	Set of n.n. constraints in the ith direction, p. 25.
$\Omega_{\mathfrak{F}}$	Shift space induced by $\mathfrak{F}$, p. 25.
Ω_φ^d	Support of the $\mathbb{Z}^d$ hard-core lattice gas model, p. 25.
$\mathrm{O}(\omega)$	Orbit of $\omega \in \mathscr{A}^{\mathbb{Z}^d}$, p. 26.

$\mathscr{F}_A$	σ-algebra generated by cylinder sets with support A, p. 26.
μ, ν	Borel probability measures on $\mathscr{A}^{\mathscr{V}}$, p. 26.
$\mathrm{supp}(\mu)$	Support of μ, p. 26.
$\mathscr{M}_1(\Omega)$	Set of Borel probability measures with support contained in Ω, p. 26.
$\mathbb{A}, \mathbb{B}, \ldots$	Events, p. 27.
$\mathscr{M}_{1,\sigma}(\Omega)$	Subset of shift-invariant measures in $\mathscr{M}_{1,\sigma}(\Omega)$, p. 27.
δ_ω	δ-measure supported on ω, p. 27.
ν^ω	Shift-invariant measure induced by a periodic point ω, p. 27.
Φ	Interaction for configurations on vertices and edges, p. 28.
ϕ	Function on finitely many configurations inducing Φ, p. 28.
ξ	Generic inverse temperature, p. 29.
$\mathscr{H}_A^\Phi$	Hamiltonian function in A, p. 30.
Z_A^Φ	Partition function of A, p. 30.
$\pi_A^{(f)}$	Free-boundary probability measure on $\mathscr{A}^A$, p. 30.
$\mathscr{H}_{A,\omega}^\Phi$	Hamiltonian in A with ω-boundary, p. 30.
$\mathrm{Z}_{A,\omega}^\Phi$	Partition function of A with ω-boundary, p. 30.
π_A^ω	ω-boundary probability measure on $\mathscr{A}^A$, p. 30.
π	Nearest-neighbour Gibbs (Ω, Φ)-specification, p. 31.
$\mathcal{G}(\pi)$	Set of Gibbs measures for π, p. 31.
$\Sigma_B(\alpha_1, \alpha_2)$	Set of B-disagreement between α_1 and α_2, p. 34.
d_{TV}	Total variation distance, p. 35.
$\mathbb{P}, \mathbb{Q}$	Couplings, p. 35.

$\mathcal{D}_{\omega,x}$	Digraph of descending paths, p. 55.
$Q(\pi)$	SSM criterion, p. 56.
$p_{\mathrm{c}}(\mathbb{Z}^d)$	Threshold of $\mathbb{Z}^d$ Bernoulli site percolation, p. 56.
$c_\pi(\omega)$	Infimum of $\pi_A^\omega(\omega\|_{\{x\}})$ over all finite A and $x \in A$, p. 58.
$\underline{c}_\pi$	Infimum of $c_\pi(\omega)$ over all $\omega \in \Omega$, p. 58.
$\Phi_{\max}$	Greatest absolute value of Φ over all n.n. configurations, p. 58.
$\phi_\lambda^{\preceq}$	Constrained energy function for λ, p. 66.
$\preceq, \prec$	Linear order on V or $\mathscr{A}$ and coordinate-wise extension, p. 71.
ω_α	Unique maximal configuration for α, p. 71.
$\phi_\lambda^{\preceq}$	Constrained energy function for λ and $\preceq$, p. 72.
B^n	n-barbell graph, p. 82.
$h(\Omega)$	Topological entropy of Ω, p. 96.
$h(\mu)$	Measure-theoretic entropy of μ, p. 99.
$\mathscr{C}(\Omega)$	Set of continuous functions from Ω to $\mathbb{R}$, p. 99.
$\mathrm{P}_\Omega(f)$	Topological pressure of $f \in \mathscr{C}(\Omega)$, p. 99.
$\mathrm{P}_\Omega(\Phi)$	Pressure of interaction Φ, p. 100.
$\hat{\mathrm{Z}}^\Phi_{\mathrm{B}_n}$	Partition function of locally admissible configurations in B_n, p. 100.
A_Φ	Continuous function induced by Φ, p. 101.
p_μ	Limit of $p_{\mu,\mathscr{P}\cap\mathrm{B}_n}$ as $n \to \infty$, p. 103.
I_μ	Information function for μ, p. 103.
$\underline{c}_\mu$	Infimum of $p_{\mu,S}(\omega)$ over all S and ω, p. 104.
$\underline{c}_\mu^-(\nu)$	Infimum of $p_{\mu,S}(\omega)$ over all $S \Subset \mathscr{P}$ and $\omega \in \mathrm{supp}(\nu)$, p. 104.

$\pi_{y,z}(\omega)$	Abbreviation of $\pi_{\mathrm{Q}_{y,z}}(\omega)$, p. 106.
$\pi_n(\omega)$	Abbreviation of $\pi_{\vec{1}n,\vec{1}n}(\omega)$, p. 106.
$\hat{\pi}(\omega)$	Limit of $\pi_n(\omega)$ as $n \to \infty$, p. 106.
I_π	Modified information function for π, p. 106.
$\underline{c}_\pi(\nu)$	Infimum of c_π over all $\omega \in \mathrm{supp}(\nu)$, p. 107.
$\Lambda, \Delta, \ldots$	Subsets of vertices in $\mathbb{Z}^d$, p. 112.
π_β^{FP}	Potts model with inverse temperature β, p. 113.
$\beta_{\mathrm{c}}(q)$	Critical inverse temperature $\mathbb{Z}^2$ Potts model with q types, p. 113.
π_ζ^{WR}	Widom-Rowlinson model with inverse temperature ζ, p. 113.
$\pi_\lambda^{\mathrm{HC}}$	Hard-core lattice gas model with activity λ, p. 114.
ω_q	Fixed point $q^{\mathbb{Z}^d}$, p. 114.
$\omega^{(e)}/\omega^{(o)}$	$\mathbb{Z}^d$ hard-core lattice gas (even/odd) checkerboard points, p. 114.
$\varphi_{p,q,\Lambda}^{(i)}$	Free/wired ($i = 0/1$) bond random-cluster distribution, p. 116.
$\varphi_{p,q}^{(i)}$	Bond random-cluster limiting distributions, p. 116.
$p_{\mathrm{c}}(q)$	Critical value $\mathbb{Z}^2$ bond random-cluster model, p. 116.
$\mathbb{P}_{p,q,\Lambda}^{(1)}$	Edwards-Sokal coupling, p. 117.
p^*	Dual value of the bond-random cluster parameter p, p. 118.
$\psi_{p,q,\Lambda}^{(i)}$	Free/wired ($i = 0/1$) site random-cluster limiting distribution, p. 118.
$\partial_\downarrow \mathrm{Q}_{y,z}$	Past boundary of $\mathrm{Q}_{y,z}$, p. 123.
$\partial_\uparrow \mathrm{Q}_{y,z}$	Future boundary of $\mathrm{Q}_{y,z}$, p. 123.
$\leq_{\mathrm{D}}$	Stochastic dominance relation, p. 124.
$\varphi_{p,q}(x \leftrightarrow y)$	Two-point connectivity function, p. 125.

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To the memory of my dog Ron,
pure and beautiful stubbornness.

Chapter 1

Introduction

1.1 Preliminaries

The focus of this thesis is the study of *spin systems* through the view of two complementary areas: *statistical mechanics* and *symbolic dynamics*. Both share a common ground with different emphasis, namely

- the study of measures on graphs (typically, a lattice such as the *d-dimensional integer lattice* $\mathbb{Z}^d$) where vertices take values on a set of *symbols* with *hard constraints*, i.e. measures are not fully supported because some configurations of symbols are forbidden due to local restrictions; and
- the computation of *thermodynamic quantities*. In particular, we are interested in the development of techniques useful for representing and approximating *topological entropy* and its generalization, *(topological) pressure*.

Consequently, the main goal of this work is twofold. First, we aim to introduce combinatorial and measure-theoretic conditions, establish connections among them, and see how combinatorial aspects of hard constrained systems interact with measure-theoretic *mixing* properties. Secondly, we use these insights to improve results on representation and efficient approximation algorithms for pressure.

A classical example

A good example of a hard constrained system is the $\mathbb{Z}^d$ *hard-core lattice gas* model. In this model, every site $x \in \mathbb{Z}^d$ is identified with a random variable $\hat{\omega}(x)$ (a *spin*) that takes values in the set of symbols $\mathscr{A} = \{0,1\}$. In order to obtain a joint distribution $\hat{\omega} = (\hat{\omega}(x))_{x \in \mathbb{Z}^d}$ representing an interplay between spins, we consider an *interaction* or, more specifically, a *nearest-neighbour (n.n.) interaction* Φ. A n.n.

interaction Φ is a function that associates some weight to configurations of symbols supported on finite subsets of $\mathbb{Z}^d$, by considering local contributions in sites and bonds. In the case of the hard-core lattice gas model with *inverse temperature* $\xi > 0$, given a finite subset $A \subseteq \mathbb{Z}^d$ and a configuration $\omega \in \mathscr{A}^{\mathbb{Z}^d}$, the joint distribution π_A^ω on A with exterior $\omega|_{A^c}$ is given by

$$\pi_A^\omega(\hat{\omega}|_A = \alpha) = \mathbb{1}_{\{\alpha \in [\omega|_{A^c}]\}} \cdot \frac{\exp(-\xi \#_1(\alpha))}{Z_{A,\omega}^\Phi}, \tag{1.1}$$

where $\alpha \in \mathscr{A}^A$ and $\omega|_{A^c}$ denotes the restriction of ω to A^c. Here the n.n. interaction Φ implicitly determines $\#_1(\alpha)$ (the number of 1's in the configuration α), the set $[\omega|_{A^c}]$ (the set of "admissible" configurations), and the *partition function* $Z_{A,\omega}^\Phi$ (a very relevant normalizing factor). By "admissible" configurations we refer to α's such that the new configuration $\alpha\,\omega|_{A^c}$ obtained from ω after replacing $\omega|_A$ by α does not violate any constraint. In the hard-core lattice gas model case, where pairs of adjacent 1's are forbidden, these configurations can be understood as *independent sets*, a familiar concept in combinatorics and graph theory involving local constraints. Notice that ω (in particular, the *boundary configuration* $\omega|_{\partial A}$) has some influence over the distribution inside the volume A. A good part of our research has to do with the study of the influence that boundaries have in the distributions π_A^ω for general systems.

The extension of these distributions to the infinite volume $\mathbb{Z}^d$ is via *(nearest-neighbour) Gibbs measures*, which are a particular kind of Borel probability measure μ such that for every finite $A \subseteq \mathbb{Z}^d$ and $\omega \in \mathscr{A}^{\mathbb{Z}^d}$, $\mathbb{E}_\mu(\cdot|\mathscr{F}_{A^c})(\omega)$ coincides with π_A^ω μ-a.s. Here, $\mathbb{E}_\mu(\cdot|\mathscr{F}_{A^c})(\omega)$ denotes the conditional expectation with respect the product σ-algebra on $\mathscr{A}^{A^c}$. The collection $\pi = \{\pi_A^\omega\}_{A,\omega}$ is usually known as the *Gibbs specification* for Φ. Sometimes there is more than a single measure μ for a given Gibbs specification π. In that case we talk about the existence of a *phase transition* (see [35]), one of the central issues in statistical physics.

It is common to consider an inverse temperature parameter $\xi > 0$ to modulate the strength of interactions. Depending on ξ, the interaction $\xi\Phi$ can give rise to multiple Gibbs measures (the *supercritical regime*) or just a single one (the *subcritical regime*). The latter case, when there is no phase transition, is related with systems where the influence of boundaries decays with the distance. Several classical lattice models, like the hard-core lattice gas model, exhibit these two behaviours for different regimes of ξ. Other instances that exhibit this phenomenon are the *(ferromagnetic) Potts model* and the *(multitype) Widom-Rowlinson* model (see [1]).

Pressure and topological entropy

The pressure of an interaction is a crucial quantity studied in statistical mechanics and dynamical systems. In the former, it coincides (up to a sign) with the *specific Gibbs free energy* of a statistical mechanical system (e.g. [35, Part III] and [70, Chapter 3-4]). In the latter, it is a generalization of topological entropy and has many applications in a wide variety of classes of dynamical systems, ranging from symbolic to smooth systems (e.g. [13, 47, 76]).

The pressure of a $\mathbb{Z}^d$ lattice model with *translation-invariant* n.n. interaction Φ is defined as the asymptotic exponential growth rate of the partition function $\mathrm{Z}^{\Phi}_{\mathrm{B}_n}$:

$$\mathrm{P}(\Phi) := \lim_{n\to\infty} \frac{\log \mathrm{Z}^{\Phi}_{\mathrm{B}_n}}{|\mathrm{B}_n|}, \tag{1.2}$$

where $\mathrm{Z}^{\Phi}_{\mathrm{B}_n}$ denotes the partition function with *free boundary* (i.e. there is no interaction with the exterior) and $\mathrm{B}_n = [-n,n]^d \cap \mathbb{Z}^d$ is an increasing sequence of boxes that exhausts $\mathbb{Z}^d$ as $n \to \infty$

Roughly speaking, the pressure tries to capture the complexity of a given system by associating to it a nonnegative real number. This value can be represented in several ways: sometimes as a closed formula, other times as a limit and, in the cases of our interest, as the integral of a conditional probability distribution or as the output of an algorithm.

Often, it is a difficult task to compute it. When $d = 1$, there is a closed-form expression for $\mathrm{P}(\Phi)$ in terms of the largest eigenvalue of an adjacency matrix formed from Φ (see [52, p. 99]). In contrast, when $d \geq 2$ there are very few n.n. interactions Φ for which $\mathrm{P}(\Phi)$ is known exactly (e.g. *Ising model* [66], *dimer model* [46], *square ice-type model* [54]). In fact, there are computability constraints for approximating pressure that in general cannot be overcome. In [44] it was proven that, when $d \geq 2$, the set of numbers that can arise as topological entropies for $\mathbb{Z}^d$ *shifts of finite type ($\mathbb{Z}^d$ SFT)* coincide with the set of *right-recursive enumerable* numbers. When $d = 1$, the closed-form expression mentioned above always gives an *efficient* approximation algorithm.

A $\mathbb{Z}^d$ SFT Ω can be understood as the support of a lattice model or also as a particular case of a hard constrained $\mathbb{Z}^d$ lattice model where the interaction Φ is *uniform*, and it is the main object of study in the area known as *multidimensional symbolic dynamics*. In this case, the topological entropy coincides with the pressure and it is given by the formula

$$h(\Omega) = \lim_{n\to\infty} \frac{\log|\mathscr{L}_{\mathrm{B}_n}(\Omega)|}{|\mathrm{B}_n|}, \tag{1.3}$$

where $\mathscr{L}_{\mathrm{B}_n}(\Omega)$ is the set of all configurations with shape B_n that appear in Ω.

Despite the general computability constraints, one can still expect to be able to approximate efficiently the topological entropy and pressure of some systems (formally, to prove the existence of a *polynomial time approximation* algorithm), and also hope to delineate the general characteristics of systems where this is possible. Part of of this work has to do with developing approximation techniques and finding conditions for computing these quantities.

The process of calculating the pressure $\mathrm{P}(\Phi)$ (or more particularly, the topological entropy $h(\Omega)$) can be broken up into two steps: *representation* and *approximation*. Since $\mathrm{P}(\Phi)$ is usually defined in terms of integrals and/or limits that are very hard or too slow to compute directly, it is useful to develop alternative formulas to represent it in such a way that the new representation can be efficiently approximated. In this regard, of particular interest is a correlation decay property of Gibbs measures known as *strong spatial mixing (SSM)*, which is a strengthening of another property known as *weak spatial mixing (WSM)*.

Spatial mixing properties

Let $f: \mathbb{N} \to \mathbb{R}$ be a function such that $f(n) \searrow 0$ as $n \to \infty$. We say that a $\mathbb{Z}^d$ lattice model satisfies *WSM with decay function f* if for any finite $A \subseteq \mathbb{Z}^d$, $B \subseteq A$, $\beta \in \mathscr{A}^B$, and any pair of admissible configurations $\omega_1, \omega_2 \in \mathscr{A}^{\mathbb{Z}^d}$,

$$\left|\pi_A^{\omega_1}(\hat{\omega}|_B = \beta) - \pi_A^{\omega_2}(\hat{\omega}|_B = \beta)\right| \leq |B| f(\text{dist}(B, \partial A)), \tag{1.4}$$

i.e. given a set B, the influence on the distribution $\hat{\omega}|_B$ of the boundary configurations $\omega_1|_{\partial A}$ and $\omega_2|_{\partial A}$ decays with the graph distance from B to ∂A, according to f. In other words, WSM implies asymptotic independence between the configuration of a finite volume and the boundary configuration outside a large ball around this volume.

On the other hand, a $\mathbb{Z}^d$ lattice model satisfies *SSM with decay function f* if for any finite $A \subseteq \mathbb{Z}^d$, $B \subseteq A$, $\beta \in \mathscr{A}^B$, and any pair of admissible configurations $\omega_1, \omega_2 \in \mathscr{A}^{\mathbb{Z}^d}$,

$$\left|\pi_A^{\omega_1}(\hat{\omega}|_B = \beta) - \pi_A^{\omega_2}(\hat{\omega}|_B = \beta)\right| \leq |B| f(\text{dist}(B, \Sigma_{\partial A}(\omega_1, \omega_2))), \tag{1.5}$$

where $\Sigma_{\partial A}(\omega_1, \omega_2) = \{x \in \partial A : \omega_1(x) \neq \omega_2(x)\}$.

The main difference between SSM and WSM is that SSM considers the distance of B to only the sites in the boundary $\Sigma_{\partial A}(\omega_1, \omega_2)$ where ω_1 and ω_2 differ. In this case, we can still have a decay of correlation if B is close to ∂A but the disagreements between ω_1 and ω_2 are far apart.

WSM has direct connections with the nonexistence of phase transitions (see [78]). SSM is a stronger version of WSM and is related with the absence of *boundary phase transitions* [61].

There has been a growing interest (see [61, 79, 39, 33]) in SSM, due to its connections with *fully polynomial-time approximation schemes (FPTAS)* for counting problems which are #P-hard (e.g. approximation algorithms for counting independent sets [79]; see also [6, 31]), and mixing time of the *Glauber dynamics* in some particular systems (see [45, 28]). Of particular interest for us, is that it has also proven to be useful for pressure representation and approximation (see [32, 58, 15]).

Examples of systems that satisfy these properties in some regime include the Ising and Potts models (see [61, 40]), and even some cases where hard constraints are considered, such as the hard-core lattice gas model (see [79]) and *q-colourings* (see [33]).

Combinatorial mixing properties

When dealing with hard constraints, we face the problem that sometimes two or more configurations are not compatible. This is an extra difficulty when proving spatial mixing properties or a useful pressure representation, since many tools (probability couplings, equivalences, etc.) have been developed only for the non-constrained cases. By combinatorial mixing properties we refer to conditions on the support Ω of a hard constrained system that allow us to "glue" together configurations in an admissible way. Two relevant properties are *strong irreducibility* and the existence of a *safe symbol*.

A $\mathbb{Z}^d$ SFT Ω is *strongly irreducible with gap* $g \in \mathbb{N}$ if for any $A, B \subseteq \mathbb{Z}^d$ such that $\operatorname{dist}(A,B) \geq g$, and for every $\alpha \in \mathscr{L}_A(\Omega)$ and $\beta \in \mathscr{L}_B(\Omega)$,

$$\alpha \in \mathscr{L}_A(\Omega), \beta \in \mathscr{L}_B(\Omega) \implies \alpha\beta \in \mathscr{L}_{A \cup B}(\Omega), \tag{1.6}$$

where $\alpha\beta$ denotes the configuration obtained after combining the configurations α and β into a single one. In simple terms, a $\mathbb{Z}^d$ SFT Ω is strongly irreducible if any two admissible (partial) configurations can be glued together in a (full) configuration $\omega \in \Omega$, provided their supports are sufficiently far apart.

Given a $\mathbb{Z}^d$ SFT $\Omega \subseteq \mathscr{A}^{\mathbb{Z}^d}$, a safe symbol $s \in \mathscr{A}$ for Ω is a symbol that can be adjacent to any other symbol in $\mathscr{A}$ (think of 0 in the hard-core lattice gas model). This property allows us to replace any symbol by s and preserve admissibility of configurations while we modify them. The existence of a safe symbol has proven to be useful to develop pressure representation and approximation theorems (see [32, Assumption 1]). In addition, the existence of a safe symbol implies strong irreducibility.

Strong irreducibility and the existence of a safe symbol are qualitatively different properties. The former is a global condition that involves the lattice as a whole. On the other hand, the existence of a safe symbol is a local condition. An interesting family of examples is the case of proper colourings of the lattice (q-colourings), where given q colours the constraints impose that two adjacent vertices cannot have the same colour. It is easy to check that the underlying $\mathbb{Z}^d$ SFT does not contain any safe symbol. However, if we have enough colours the system does satisfy a weaker combinatorial condition introduced in [58] called *single site-fillability (SSF)*. SSF is a local condition that also implies strong irreducibility and has shown to be useful for having have pressure representation and approximation theorems (see [58]).

Part of the work presented here involves pushing this even further and looking for weaker combinatorial properties still useful for these purposes. Of particular interest is the study of characteristics that hard constraints should satisfy in order to be (to some extent) compatible with the SSM property.

Pressure representation

In [32, 58] it was proven that SSM (with decay function $f(n) = Ce^{-\gamma n}$) together with some extra combinatorial properties are enough for having a pressure representation and approximation algorithms for $\mathrm{P}(\Phi)$. The basic idea was motivated by the *Variational Principle* (see [47, Section 4.4]), given by the formula

$$\mathrm{P}(\Phi) = \sup_{\mu \in \mathscr{M}_{1,\sigma}(\Omega)} \left(h(\mu) + \int \mathrm{A}_\Phi d\mu \right), \tag{1.7}$$

where Ω is the support of the $\mathbb{Z}^d$ lattice model, $\mathscr{M}_{1,\sigma}(\Omega)$ denotes the set of all *shift-invariant* (or *stationary*) *Borel probability measures* μ supported on Ω, and A_Φ is an auxiliary function defined as $\mathrm{A}_\Phi(\omega) = -\Phi\left(\omega|_{\{\vec{0}\}}\right) - \sum_{i=1}^{d} \Phi\left(\omega|_{\{\vec{0},\vec{e}_i\}}\right)$, where $\vec{0}$ denotes the origin and $\{\vec{e}_i\}_{i=1}^{d}$ denotes the canonical basis of $\mathbb{Z}^d$. Here, the quantity $h(\mu)$ is the *measure-theoretic entropy* of μ. Any measure that achieves the supremum is known as an *equilibrium state*, which (under very mild conditions) coincide with a Gibbs measures for Φ.

Given an equilibrium state μ, we have that $\mathrm{P}(\Phi) = h(\mu) + \int \mathrm{A}_\Phi d\mu$. It is known that the measure-theoretic entropy $h(\mu)$ can be expressed as the integral with respect to μ of the *information function* I_μ. The information function is defined μ-a.s. as $\mathrm{I}_\mu(\omega) = -\log \mu\left(\hat{\omega}(\vec{0}) = \omega(\vec{0}) \middle| \omega|_{\mathscr{P}}\right)$, where $\mathscr{P}$ denotes the *lexicographic past* of the origin $\vec{0}$. In other words, given $\omega \in \mathscr{A}^{\mathbb{Z}^d}$, $\mathrm{I}_\mu(\omega)$ is the negative logarithm of the conditional probability of having the value $\omega(\vec{0})$ at the origin, given that every site in the lexicographic past coincides with ω. Then we can write $h(\mu) = \int \mathrm{I}_\mu d\mu$

and we have

$$\mathrm{P}(\Phi) = \int (\mathrm{I}_\mu + \mathrm{A}_\Phi) d\mu, \tag{1.8}$$

for any equilibrium state μ. The idea in [32, 58] was to represent $\mathrm{P}(\Phi)$ as the integral of the same integrand, but with respect to a simpler measure ν, i.e.

$$\mathrm{P}(\Phi) = \int (\mathrm{I}_\mu + \mathrm{A}_\Phi) d\nu. \tag{1.9}$$

This can be regarded as a *pressure representation*. A pressure representation becomes especially useful for approximating $\mathrm{P}(\Phi)$ in the case ν is a *periodic point* measure, i.e. a measure which assigns equal weight to each distinct translation of a given periodic configuration $\omega \in \mathscr{A}^{\mathbb{Z}^d}$. Then $\int (\mathrm{I}_\mu + \mathrm{A}_\Phi) d\nu$ becomes a finite sum. The terms in this sum corresponding to A_Φ are easy to compute. In this way, the problem of approximating $\mathrm{P}(\Phi)$ reduces to approximate I_μ on a single periodic configuration ω and its translates. We remark that to approximate $\mathrm{I}_\mu(\omega)$ is equivalent to just approximate $\mu\left(\hat{\omega}(\vec{0}) = \omega(\vec{0}) \middle| \omega|_{\mathscr{P}}\right)$, the conditional probability of a single site taking some particular value. Then this new way to express $\mathrm{P}(\Phi)$ by-passes the need of computing the integral with respect to μ. A pressure representation requires some assumptions on the measures μ and ν, and also on the support Ω. In general, it may fail. For example, when considering 3-*colourings* in $\mathbb{Z}^2$ (see [58]), the representation formula in Equation (1.9) is not always true.

Pressure approximation

Given a pressure representation as in Equation (1.9), the problem of approximating $\mathrm{P}(\Phi)$ reduces to know how to approximate I_μ efficiently. In [32] an approach was given to approximate $\mu\left(\hat{\omega}(\vec{0}) = \omega(\vec{0}) \middle| \omega|_{\mathscr{P}}\right)$, and therefore $\mathrm{I}_\mu(\omega)$, in polynomial time in some region of the subcritical regime for the $\mathbb{Z}^d$ hard-core lattice gas model. This approach was based on the *computational tree method* developed by Weitz in [79], which is a powerful technique in the *binary* case (i.e. $|\mathscr{A}| = 2$). In [58], an alternative method was develop for the $\mathbb{Z}^2$ case to approximate I_μ efficiently, based on the *transfer matrix method*. Both approaches rely in the assumption that the measures involved satisfy the SSM property (with exponential decay) plus some combinatorial property on the support. We review this approach and also deal with the (a priori, radically different) case when the SSM property fails.

1.2 Results overview

The results are divided in two main parts:

I. Development of a robust framework for working with hard constrained systems in general graphs, and study measure-theoretic correlation decay properties on them.

II. Study of techniques for representing and efficiently computing pressure in the uniqueness and non-uniqueness regions in $\mathbb{Z}^d$ lattice models.

Part I: Combinatorial aspects of the strong spatial mixing property

There is a good amount of literature devoted to study the regimes where SSM holds for particular systems. In particular, the Ising and Potts models [53, 80], the hard-core lattice gas model [79], and q-colourings [31] have received special attention. Given a locally finite countable graph $\mathscr{G}$ (the *board*), a finite set of symbols $\mathscr{A}$, some hard constraints, and an interaction Φ, we can define very general systems (Ω, Φ), where Ω is the *configuration space*, the subset of configurations not violating any constraint.

We develop a new framework for studying SSM in general hard constrained spin systems. This is done in part through the introduction of a combinatorial mixing property named *topological strong spatial mixing (TSSM)*. A configuration space Ω is *TSSM with gap* $g \in \mathbb{N}$ if for any subsets A, B, and S such that $\mathrm{dist}(A,B) \geq g$ (with respect to the graph distance in $\mathscr{G}$), and for every $\alpha \in \mathscr{L}_A(\Omega)$, $\beta \in \mathscr{L}_B(\Omega)$, and $\sigma \in \mathscr{L}_S(\Omega)$,

$$\alpha\sigma \in \mathscr{L}_{A \cup S}(\Omega), \sigma\beta \in \mathscr{L}_{S \cup B}(\Omega) \implies \alpha\sigma\beta \in \mathscr{L}_{A \cup S \cup B}(\Omega). \tag{1.10}$$

To develop the TSSM property we combine notions from combinatorics in lattice models (like the existence of a safe symbol and SSF) and from symbolic dynamics, in particular strong irreducibility. The result is a hybrid concept strictly weaker than all the combinatorial properties used in [32] and [58], and strictly stronger than strong irreducibility (notice that if we take $S = \emptyset$ in Equation (1.10) we recover the definition of strong irreducibility).

We prove that, under certain hypotheses, if a system satisfies SSM, then its support satisfies TSSM. As two of many applications, we show that the space of 4-colourings on the $\mathbb{Z}^2$ lattice does not satisfy SSM for any interaction Φ (see Proposition 3.5.6; it has been suggested in the literature [74] that a uniform Gibbs measure on this system should satisfy WSM), and that TSSM implies the existence of periodic points in $\mathbb{Z}^d$ shift spaces, for every $d \in \mathbb{N}$ (see Proposition 3.4.3). Later, in Part II, we use TSSM to generalize work in [32] and [58], giving conditions for simple representation and efficient approximation of pressure.

One of the main ideas is that in order to have a measure-theoretic correlation decay property like SSM, there may be some necessary combinatorial conditions on the configuration space. We also explore a complementary question: given a configuration space satisfying a particular combinatorial property, can we always find a Gibbs specification satisfying SSM supported on it? In general, if a configuration space just satisfies strong irreducibility, the answer is no. For example, the space of 4-colourings on $\mathbb{Z}^2$ satisfies strong irreducibility, but no measure supported on it can satisfy SSM.

In [17], Brightwell and Winkler did a complete study of the family of *dismantlable graphs*, including several interesting alternative characterizations. Among the equivalences discussed in that work, many involved a board $\mathscr{G}$, a finite graph H (the *constraint graph*, assumed to be dismantlable), and the set of all graph homomorphisms from $\mathscr{G}$ to H, which we denote here by $\mathrm{Hom}(\mathscr{G},\mathrm{H})$. We call such a set of graph homomorphisms a *homomorphism space*, and it can be understood as a configuration space Ω. In this context, we should interpret the set of vertices $\mathrm{V}(\mathrm{H})$ of H as the set of symbols $\mathscr{A}$ in some spin system living on vertices of $\mathscr{G}$. The adjacencies given by the set of edges $\mathrm{E}(\mathrm{H})$ of H indicate the pairs of symbols that are allowed to be next to each other in $\mathscr{G}$, and the edges that are missing can be seen as hard constraints in our system (i.e. pair of spin values that cannot be adjacent in $\mathscr{G}$). Examples of such systems are very common. If we consider $\mathscr{G} = \mathbb{Z}^2$ and H_φ, with $\mathrm{V}(\mathrm{H}_\varphi) = \{0,1\}$ and $\mathrm{E}(\mathrm{H}_\varphi)$ containing every edge but the loop connecting 1 with itself, then $\mathrm{Hom}(\mathbb{Z}^2,\mathrm{H}_\varphi)$ represents the support of the hard-core lattice gas model in $\mathbb{Z}^2$, i.e. the set of independent sets in the *square lattice*.

We are interested in combinatorial mixing properties that are satisfied by a homomorphism space $\mathrm{Hom}(\mathscr{G},\mathrm{H})$, i.e. properties that allow us to "glue" together partial configurations in $\mathscr{G}$. For example, the homomorphism space $\mathrm{Hom}(\mathbb{Z}^2,\mathrm{H})$, where $\mathrm{V}(\mathrm{H}) = \{0,1\}$ and H has a unique edge connecting 0 with 1, has only two elements, both checkerboard patterns of 0s and 1s. This homomorphism space lacks good combinatorial mixing properties since, for example, it is not possible to "glue" two 0s together which are separated by an odd distance horizontally or vertically. Note that this is not the case for $\mathrm{Hom}(\mathbb{Z}^2,\mathrm{H}_\varphi)$, where the only difference is that H_φ has in addition an edge connecting 0 with itself. A gluing property of particular interest is strong irreducibility. In [17], dismantlable graphs were characterized as the only graphs H such that $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ is strongly irreducible for every board $\mathscr{G}$.

In addition, we can consider a n.n interaction Φ that associates some "energy" to every vertex and edge of a constraint graph H. From this we can construct a *Gibbs* $(\mathscr{G},\mathrm{H},\Phi)$*-specification* π, which is an ensemble of probability measures supported in finite portions of $\mathscr{G}$. Specifications are a common framework for working with spin systems and defining Gibbs measures μ. From this point it is possible to start studying spatial mixing properties, which combine the geometry of $\mathscr{G}$, the structure of H, and the distributions induced by Φ. In [17], dismantlable graphs were also characterized as the only graphs H for which for every board $\mathscr{G}$ of bounded degree there exists a n.n. interaction Φ such that the Gibbs $(\mathscr{G},\mathrm{H},\Phi)$-specification π has no phase transition (i.e. there is a unique Gibbs measure for π).

We consider the problem of existence of strong spatial mixing measures supported on homomorphism spaces. First, we extend the results of Brightwell and Winkler on uniqueness, by characterizing dismantlable graphs as the only graphs H for which for every board $\mathscr{G}$ of bounded degree there exists a n.n. interaction Φ such that the Gibbs $(\mathscr{G},\mathrm{H},\Phi)$-specification π satisfies WSM (with exponential decay function; see Section 4.2). Then we give sufficient conditions (see Section 4.3) on H and $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ for the existence of Gibbs $(\mathscr{G},\mathrm{H},\Phi)$-specifications satisfying SSM (with exponential decay function). Since SSM implies WSM, a necessary condition for SSM to hold in every board $\mathscr{G}$ is that H is dismantlable. We exhibit examples showing that SSM is a strictly stronger property, in terms of combinatorial properties of H and $\mathrm{Hom}(\mathscr{G},\mathrm{H})$, than WSM. In particular, there exist dismantlable graphs H where SSM fails in $\mathrm{Hom}(\mathscr{G},\mathrm{H})$, for some boards $\mathscr{G}$.

To our knowledge, this is the first attempt to characterize the constraint graphs H that are suitable for SSM to hold for general boards $\mathscr{G}$. Related work was done in [27], where the family of dismantlable graphs was related with *rapid mixing* of the Glauber dynamics in finite boards with *free boundary conditions*. In contrast, we demonstrate that the role of boundary conditions cannot be ignored when we are looking for Gibbs specifications satisfying SSM in models with hard constraints.

Other combinatorial and measure-theoretic mixing properties have been considered in the literature of lattice models. We also explore the relationships between some of them. In Part II, we see how they are useful in some cases for representation and approximation. In particular, in the context of stationary $\mathbb{Z}^d$ Gibbs measures μ satisfying SSM, we establish many useful properties that show how TSSM on the support of μ is sufficient for a special pressure representation and efficient approximation algorithms.

Part II: Representation and poly-time approximation for pressure

The main focus of Part II is to find simple representations of pressure and use this to develop efficient (in principle) algorithms to approximate it, profiting from the measure-theoretical and combinatorial mixing properties studied in Part I.

There is much work in the literature on numerical approximations for pressure (see [7, 29]). We take a theoretical computer science point of view (see [50]): an algorithm for computing a real number r is said to be *poly-time* if for every $N \in \mathbb{N}$, the algorithm outputs an approximation r_N to r, which is guaranteed to be accurate within $\frac{1}{N}$ and takes time at most polynomial in N to compute. In that case, we say that r is *poly-time computable*. One of our goals is to prove the existence of poly-time algorithms for $\mathrm{P}(\Phi)$ under certain assumptions on Φ and the support Ω.

Following the works of Gamarnik and Katz [32], and Marcus and Pavlov [58], we provide extended versions of representation theorems of pressure in terms of conditional probabilities and also conditions for more general approximation algorithms. In [32], for obtaining such representation and approximation theorems, they assumed the existence of a safe symbol, which is a very strong combinatorial condition, together with SSM (and an exponential assumption on the decay function of SSM for algorithmic purposes). Later, in [58], these assumptions were replaced by more general and technical conditions in the case of representation, and the SSF property, which generalized the safe symbol case both in the representation and in the algorithmic results. Here, making use of the theoretical machinery developed in [58], we have relaxed those conditions even more by using the more general property of TSSM.

Next, we continue the development of representing the pressure with a simplified expression. We prove a new pressure representation theorem and different techniques to tackle the problem of pressure approximation in supercritical regimes, where long range correlations do not decay and therefore SSM cannot hold. We considered three classical lattice models: the *(ferromagnetic) Potts*, *(multitype) Widom-Rowlinson*, and hard-core lattice gas models. For Widom-Rowlinson and hard-core lattice gas, the techniques apply to certain subsets of both the subcritical and supercritical regions, where the main novelty is in the latter. In the case of the Potts model with q colours, we obtain a pressure representation for any $\beta > 0$ and a poly-time approximation algorithm for any $\beta \neq \beta_c(q)$, where $\beta_c = \log(1+\sqrt{q})$ denotes the critical inverse temperature.

To do this the main tools are coupling techniques in *Markov random fields* (see [11]) and the relation of these models (in particular, Potts and Widom-Rowlinson) with *random-cluster models* (see [42]). Two of the most exploited couplings were the van den Berg and Maes coupling for Markov random fields inducing paths of disagreement (see [11]), and the Edwards-Sokal coupling, relating the Potts model with the *bond random-cluster model* (see [42]).

The pressure representation theorems developed in [32] and [58] were given in terms of the information function I_μ, that depends on a stationary Gibbs measure μ for an interaction Φ. In order to have analogous representation formulas for supercritical regimes, we had to modify the representation formula by introducing a closely related function I_π, which depends only on the Gibbs specification $\{\pi_A^\omega\}_{A,\omega}$. This turned out to be necessary but at the same time natural, since pressure depends only on the interaction and not on any particular Gibbs measure μ.

The results hold in every dimension d. Among other conditions, these results require conditions on Ω and a convergence condition for certain sequences of finite-volume half-plane measures (different convergence conditions in the different results). In the case $d = 2$, if the convergence holds at exponential rate, then one obtains a poly-time algorithm for approximating $\mathrm{P}(\Phi)$. For $d > 2$, if the exponential convergence holds, one can deduce an algorithm for approximating $\mathrm{P}(\Phi)$ with sub-exponential but not polynomial rate. We remark that the finite-volume half-plane measures mentioned above typically are constant on their bottom boundaries and thus are related to *wetting models* (see [68, 73]).

In [32], the convergence condition is SSM of a Gibbs measure μ for the n.n. interaction Φ. This condition is known to imply that there is a unique Gibbs measure for Φ and thus can be applied only in the uniqueness (subcritical) region of a given lattice model. The convergence conditions in [58] are weaker but also apply primarily to this region. However, since our convergence condition depends only on the interaction, one might expect that the pressure representation and approximation results can apply in the non-uniqueness region as well. Indeed, they do. As illustrations, we apply these results to explicit subcritical and supercritical sub-regions of three main classical lattice models in $\mathbb{Z}^2$. In particular, for the pressure approximation results for these models, we establish the required exponential convergence conditions (see Section 6.5). However, we believe that our results are applicable to a much broader class of models, in particular satisfying weaker conditions on Ω. We remark that the SSM condition of [32] is a much stronger version of our condition, and so in this sense our results generalize some results of that paper (in particular, for the $\mathbb{Z}^2$ hard-core lattice gas model).

In the case of the 2-dimensional Potts model, we obtain a pressure representation and efficient pressure approximation for all $\beta \neq \beta_c(q)$, where $\beta_c(q) = \log(1+\sqrt{q})$ is the *critical value* which separates the uniqueness and non-uniqueness regions. Our proof in the non-uniqueness region generalizes a result from [22] for $q = 2$ (i.e. the Ising model) and we closely follow their proof, which relies heavily on a coupling with the bond random-cluster model and *planar duality*. For the uniqueness region, our result follows from [4].

For the Widom-Rowlinson and hard-core lattice gas models, our results are not as complete as in the Potts case, since the subcritical and supercritical regions for these two models haven't been completely determined. We also expect our results can be improved, because they only apply to proper subsets of the currently known uniqueness/non-uniqueness regions.

For the Widom-Rowlinson model, in the supercritical region, we use a variation of the disagreement percolation technique introduced in [11], combined with the connection between the Widom-Rowlinson model and the *site random-cluster model*. In the subcritical region, we apply directly the results in [11].

For the hard-core lattice gas model, in the supercritical region, we combine the coupling in [11] and a Peierls argument used by Dobrushin (see [26]). In the subcritical region, we use a recent result on SSM for the hard-core lattice gas model in $\mathbb{Z}^2$.

For the Potts model, we also extend the pressure representation, by a continuity argument, to give an expression for the pressure at criticality (see Corollary 13). It is of interest that there is an exact, explicit, but non-rigorous, formula for the pressure at criticality due to Baxter (see [8]). So, our rigorously obtained expression should agree with that formula, though we do not know how to prove this statement. It seems that Baxter's explicit expression gives a poly-time approximation algorithm, but we cannot justify that our expression is poly-time computable.

1.3 Thesis structure

The thesis is split in mainly two parts.

Part I consists of Chapter 2, Chapter 3, and Chapter 4.

In Chapter 2, we give the basic notions of graph theory (Section 2.1), configuration spaces (Section 2.2), and Gibbs measures (Section 2.3).

In Chapter 3, we define spatial mixing (Section 3.1) and combinatorial (Section 3.2) properties, we introduce the topological strong spatial mixing property (Section 3.3), establish properties and characterizations of it, and relationships to some measure-theoretic quantities (Section 3.4). Next, we provide connections between measure-theoretic and combinatorial mixing properties (Section 3.6); in particular, Theorem 3.6.5 provides evidence that TSSM is closely related with SSM. In Section 3.5, we give several examples illustrating different types of mixing properties in the context of $\mathbb{Z}^d$ SFTs.

In Chapter 4, we introduce the necessary background for studying homomorphism spaces and dismantlable graphs (Section 4.1). After that, we explore the connections between dismantlability and WSM (Section 4.2). Then we introduce the *unique maximal configuration (UMC) property* (Section 4.3) and show that this property is sufficient for having a Gibbs specification satisfying SSM with arbitrarily high exponential decay of correlations (Section 4.4). We introduce a fairly general family of graphs H, strictly contained in the family of dismantlable graphs, such that $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ satisfies the UMC property for every board $\mathscr{G}$ (Section 4.5). Theorem 4.6.3 provides a summary of relationships and implications among the properties studied (we encourage the reader to take a quick look to it before a more detailed reading). Later, we focus in the particular case where H is a (*looped*) tree T and conclude that the properties on T yielding WSM for some measure on $\mathrm{Hom}(\mathscr{G},T)$ coincide with those yielding SSM (Section 4.6). At the end of this chapter, we provide examples illustrating the qualitative difference between the combinatorial properties necessary for WSM and SSM to hold in homomorphism spaces (Section 4.7).

Part II consists of Chapter 5, Chapter 6, and Chapter 7.

In Chapter 5, we introduce topological entropy (Section 5.1) and topological pressure (Section 5.2). We discuss some variational principles and then we exhibit some pressure representation theorems (Section 5.3).

In Chapter 6, we review the three specific $\mathbb{Z}^d$ lattice models to which we apply our main results (Section 6.1), and we review the bond random-cluster (Section 6.2) model and site random-cluster model (Section 6.3), where Lemma 6.3.2 is a novel result concerning the latter. Next, we review some spatial mixing and stochastic dominance concepts (Section 6.4) and use this to help establish exponential convergence results (Section 6.5).

In Chapter 7, we combine the pressure representation theorems and exponential convergence results in order to obtain efficient approximation algorithms for pressure (Section 7.1).

Finally, in Chapter 8, we discuss possible future directions of research.

Part I

Combinatorial aspects of the strong spatial mixing property

Chapter 2

Basic definitions and notation

2.1 Graphs

A **graph** is an ordered pair $G = (V, E)$ (or $G = (V(G), E(G))$ if we want to emphasize the graph G), where V is a finite set or a countably infinite set of elements called **vertices**, and E is contained in the set of unordered pairs $\{\{x,y\} : x,y \in V\}$, whose elements we call **edges**. We denote $x \sim y$ (or $x \sim_G y$ if G is not clear from the context) whenever $\{x,y\} \in E$, and we say that x and y are **adjacent**. A vertex x is said to have a **loop** if $\{x,x\} \in E$. The **set of looped vertices** of a graph G will be denoted $\mathrm{Loop}(G) := \{x \in V : \{x,x\} \in E\}$ and G will be called **simple** if $\mathrm{Loop}(G) = \emptyset$. A graph G is **finite** if $|G| < \infty$, where $|G|$ denotes the cardinality of $V(G)$, and **infinite** otherwise.

Fix $n \in \mathbb{N}$. A **path (of length n)** in a graph G will be a finite sequence of distinct edges $\{x_0,x_1\},\{x_1,x_2\},\dots,\{x_{n-1},x_n\}$. A single vertex x will be considered to be a path of length 0. A **cycle (of length n)** will be a path such that $x_0 = x_n$. Notice that a loop is a cycle. A vertex y will be said to be **reachable** from another vertex x if there exists a path (of some length $n \geq 0$) such that $x_0 = x$ and $x_n = y$. For $A_1, A_2 \subseteq V$, a **path from A_1 to A_2** is any path whose first vertex is in A_1 and whose last vertex is in A_2. A graph will be said to be **connected** if every vertex is reachable from any other vertex, and a **tree** if it is connected and has no cycles. A graph which is a tree plus possibly some loops, will be called a **looped tree**.

For a vertex x, we define its **neighbourhood** $\mathrm{N}(x)$ as the set $\{y \in V : y \sim x\}$. A graph G will be called **locally finite** if $|\mathrm{N}(x)| < \infty$, for every $x \in V$. A locally finite graph will have **bounded degree** if $\Delta(G) := \sup_{x \in V} |\mathrm{N}(x)| < \infty$. In this case we say that G is of **maximum degree** $\Delta(G)$. Given $d \in \mathbb{N}$, a graph of maximum degree d is **d-regular** if $|\mathrm{N}(x)| = d$, for all $x \in V$.

We say that a graph $G' = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and

$E' \subseteq E$. For a subset of vertices $A \subseteq V$, we define the subgraph of G **induced by** A as $G[A] := (A, E[A])$, where $E[A] := \{\{x,y\} \in E : x,y \in A\}$.

2.1.1 Boards

A **board** $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ will be any simple, connected, locally finite graph with at least two vertices, where $\mathscr{V}$ is the set of vertices (or **sites**) and $\mathscr{E}$ is the set of edges (or **bonds**). We will use the letters x, y, etc. for denoting the vertices in a board. Boards and *configurations* on them (see Section 2.2) will be the main playground in this work.

Example 2.1.1. *Given $d \in \mathbb{N}$, we have the two following families of boards:*

- *The d-**dimensional integer lattice** $\mathbb{Z}^d$ (see Subsection 2.1.2).*
- *The d-**regular tree** $\mathbb{T}_d$ (also known as **Bethe lattice**).*

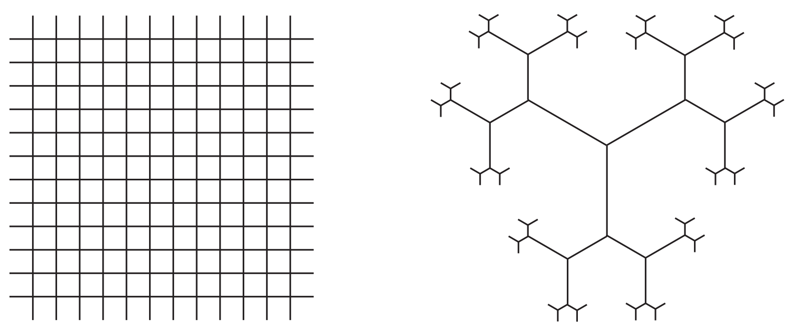

Figure 2.1: A sample of the boards $\mathbb{Z}^2$ and $\mathbb{T}_3$.

Given a board $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, we can define a natural distance function between vertices $x, y \in \mathscr{V}$, namely

$$\mathrm{dist}(x,y) := \min\{n : \exists \text{ a path of length } n \text{ s.t. } x = x_0 \text{ and } x_n = y\}, \tag{2.1}$$

which can be extended to sets $A_1, A_2 \subseteq \mathscr{V}$ as $\mathrm{dist}(A_1, A_2) = \min_{x \in A_1, y \in A_2} \mathrm{dist}(x,y)$.

We denote $B \Subset A$ whenever a finite set $B \subseteq \mathscr{V}$ is contained in a set $A \subseteq \mathscr{V}$. When denoting subsets of $\mathscr{V}$ that are singletons, brackets will usually be omitted, e.g. $\mathrm{dist}(x, A)$ will be regarded to be the same as $\mathrm{dist}(\{x\}, A)$. Notice that if $x \in$

A, then $\mathrm{dist}(x,A) = 0$. Given $A \subseteq \mathscr{V}$, we define its **(outer) boundary** as $\partial A := \{x \in \mathscr{V} : \mathrm{dist}(x,A) = 1\}$ and its **closure** as $\overline{A} := A \cup \partial A$. The **inner boundary** of A will be the set $\underline{\partial} A := \partial A^{\mathrm{c}}$, i.e. the set of sites x in A which are adjacent to some other site in A^{c}. Given $n \in \mathbb{N}$, we call $\mathscr{N}_n(A) := \{x \in \mathscr{V} : \mathrm{dist}(x,A) \leq n\}$ the ***n*-neighbourhood** of A, and $\partial_n A := \mathscr{N}_n(A) \setminus A$, the ***n*-boundary** of A. Notice that $\mathscr{N}_0(A) = A$, $\mathscr{N}_1(x) = \mathrm{N}(x) \cup \{x\}$, and $\partial_1(A) = \partial A$. For $A \Subset \mathscr{V}$, we define its **diameter** as $\mathrm{diam}(A) := \max_{x,y \in A} \mathrm{dist}(x,y)$.

2.1.2 The d-dimensional integer lattice

The board of main interest will be the d-dimensional integer lattice $\mathbb{Z}^d$. Given $d \in \mathbb{N}$, we define $\mathbb{Z}^d = \left(\mathscr{V}(\mathbb{Z}^d), \mathscr{E}(\mathbb{Z}^d)\right)$ to be the $2d$-regular (countably infinite) graph such that

$$\mathscr{V}(\mathbb{Z}^d) = \mathbb{Z}^d, \quad \text{and} \quad \mathscr{E}(\mathbb{Z}^d) = \Big\{\{x,y\} : x,y \in \mathbb{Z}^d, \|x-y\| = 1\Big\}, \tag{2.2}$$

with $\|x\| = \sum_{i=1}^{d} |x_i|$ the 1-norm, for $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$. In a slight abuse of notation, we will use $\mathbb{Z}^d$ to denote both the set of vertices and the board itself.

A natural order on $\mathbb{Z}^d$ is the so-called **lexicographic order**, where $y \preccurlyeq x$ (or equivalently, $x \succcurlyeq y$) iff $y = x$ or $y_i < x_i$, for the smallest $1 \leq i \leq d$ for which $y_i \neq x_i$. Considering this order, we define the **(lexicographic) past** of $\mathbb{Z}^d$ as

$$\mathscr{P} := \Big\{x \in \mathbb{Z}^d \setminus \{\vec{0}\} : x \preccurlyeq \vec{0}\Big\}, \tag{2.3}$$

where $\vec{0}$ denotes the **origin**. Given $y,z \in \mathbb{Z}^d$ such that $y,z \geq \vec{0}$ (here $\geq$ denotes the coordinate-wise comparison of vectors), we also define the $[y,z]$**-block** as the set

$$\mathrm{B}_{y,z} := \Big\{x \in \mathbb{Z}^d : -y \leq x \leq z\Big\}, \tag{2.4}$$

and the **broken** $[y,z]$**-block** as

$$\mathrm{Q}_{y,z} := \mathrm{B}_{y,z} \setminus \mathscr{P} = \Big\{x \succcurlyeq \vec{0} : -y \leq x \leq z\Big\}. \tag{2.5}$$

In addition, given $n \in \mathbb{N}$, we define the ***n*-block** as $\mathrm{B}_n := \mathrm{B}_{\vec{1}n,\vec{1}n}$ and the **broken *n*-block** as $\mathrm{Q}_n = \mathrm{B}_n \setminus \mathscr{P}$, where $\vec{1}$ denotes the vector $(1, \ldots, 1) \in \mathbb{Z}^d$. Notice that

$B_n = [-n,n]^d \cap \mathbb{Z}^d$. We also define the ***n*-rhomboid** as $R_n := \mathscr{N}_n(\{\vec{0}\})$.

In $\mathbb{Z}^d$ we can also define an alternative notion of adjacency and therefore, an alternative notion of boundary, inner boundary, closure, path, connectedness, etc., by replacing the 1-norm $\|\cdot\|$ with the ∞-norm $\|\cdot\|_\infty$, defined as $\|x\|_\infty = \max_{i=1,\ldots,d}|x_i|$, for $x \in \mathbb{Z}^d$. When referring to these notions with respect to the ∞-norm, we will always add a $\star$ superscript and talk about $\star$-adjacency $x \overset{\star}{\sim} y$, $\star$-boundary $\partial^\star A$, inner $\star$-boundary $\underline{\partial}^\star A$, $\star$-closure $\overline{A}^\star$, $\star$-path, $\star$-connectedness, etc. Notice that two vertices x and y are $\star$-adjacent if they are adjacent in a version of the d-dimensional integer lattice $\mathbb{Z}^d$ including in addition diagonal edges. We will denote this version of the lattice by $\mathbb{Z}^{d,\star}$.

2.1.3 Constraint graphs

A **constraint graph** $H = (V,E)$ is a finite graph, where loops are allowed. The main role of constraint graphs will be to prescribe adjacency rules for values of vertices in a given board (see Subsection 2.2.1). A difference between boards and constraint graphs is that the latter must be finite. Another one is that constraint graphs are allowed to have loops. We will denote by $H^{\circlearrowleft} = (V,E^{\circlearrowleft})$ the constraint graph obtained by adding loops to every vertex, i.e. $E^{\circlearrowleft} = E \cup \{\{u,u\} : u \in V\}$ and $\mathrm{Loop}(H^{\circlearrowleft}) = V$. We will use the letters u, v, etc. for denoting vertices in a constraint graph.

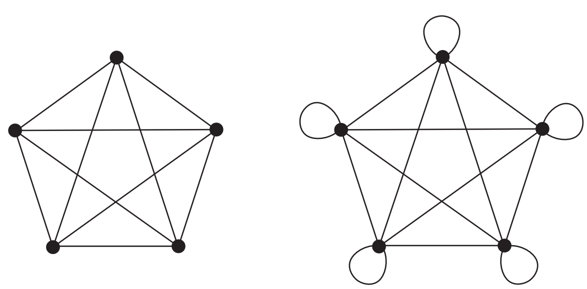

Figure 2.2: The graphs K_n and $K_n^{\circlearrowleft}$, for $n = 5$.

A constraint graph will be called **complete** if $u \sim v$, for every $u,v \in V$ such that $u \neq v$. The complete graph with n vertices will be denoted K_n (notice that $\mathrm{Loop}(K_n) = \emptyset$). A constraint graph will be called **loop-complete** if $u \sim v$, for every $u,v \in \mathrm{V}$. Notice that the loop-complete graph with n vertices is $\mathrm{K}_n^{\circlearrowleft}$. The graphs K_n and $\mathrm{K}_n^{\circlearrowleft}$ are very important examples (see Example 2.3.1) of constraint graphs, which relate to proper colourings of boards and unconstrained models, respectively. Other relevant examples are the following.

Example 2.1.2. *The constraint graph given by*

$$\mathrm{H}_{\varphi} := (\{0,1\},\{\{0,0\},\{0,1\}\})\,, \tag{2.6}$$

shown in Figure 2.1.2, is related to the **hard-core model** *(see Example 2.3.1).*

Figure 2.3: The graph H_{φ}.

*Given $n \in \mathbb{N}$, the n-***star graph** *is defined as*

$$\mathrm{S}_n = (\{0,1,\ldots,n\},\{\{0,1\},\ldots,\{0,n\}\})\,. \tag{2.7}$$

In addition, it will be useful to consider the constraint graphs

$$\mathrm{S}_n^{\mathrm{o}} = (\mathrm{V}(\mathrm{S}_n),\mathrm{E}(\mathrm{S}_n)\cup\{\{0,0\}\})\,, \tag{2.8}$$

and $\mathrm{S}_n^{\circlearrowleft}$ (see Figure 2.4). Notice that $\mathrm{H}_{\varphi} = \mathrm{S}_1^{\mathrm{o}}$.

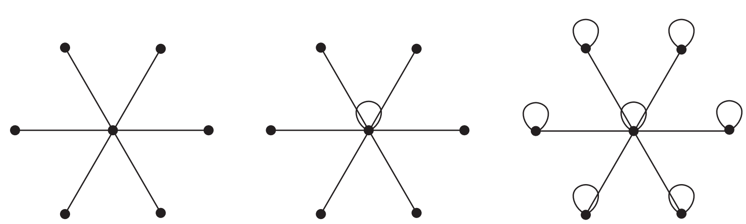

Figure 2.4: The graphs S_6, $\mathrm{S}_6^{\mathrm{o}}$, and $\mathrm{S}_6^{\circlearrowleft}$.

2.2 Configuration spaces

Fix a board $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ and let $\mathscr{A}$ be a finite set of **symbols** called **alphabet**. We endow $\mathscr{A}$ with the discrete topology and $\mathscr{A}^{\mathscr{V}}$ with the corresponding product topology. We will call any closed subset $\Omega \subseteq \mathscr{A}^{\mathscr{V}}$ a **configuration space**. The elements in Ω will be called **points** and will be denoted with the Greek letters ω, υ, etc.

Two families of configuration spaces considered here are *homomorphism spaces* (see Section 2.2.1) and $\mathbb{Z}^d$ *shift spaces* (see Section 2.2.2).

Given $A \subseteq \mathscr{V}$, a **configuration** will be any map $\alpha : A \to \mathscr{A}$ (or equivalently, $\alpha \in \mathscr{A}^A$), denoted with lowercase Greek letters α, β, etc. The set A is called the **shape** of α, and a configuration will be said to be **finite** if its shape is finite. Notice that, in particular, a point is a configuration with shape $\mathscr{V}$. For any configuration α with shape A and $B \subseteq A$, $\alpha|_B$ denotes the **sub-configuration** of α occupying B, i.e. the map from B to $\mathscr{A}$ obtained by restricting the domain of α to B. Given a symbol $a \in \mathscr{A}$, a^A will denote the configuration of all a's on A.

Given a configuration space Ω, a configuration $\alpha \in \mathscr{A}^A$ is said to be **globally admissible** if there exists $\omega \in \Omega$ such that $\omega|_A = \alpha$. The **language** $\mathscr{L}(\Omega)$ of Ω is the set of all finite globally admissible configurations, i.e.

$$\mathscr{L}(\Omega) := \bigcup_{A \Subset \mathscr{V}} \mathscr{L}_A(\Omega), \tag{2.9}$$

where $\mathscr{L}_A(\Omega) := \{\omega|_A : \omega \in \Omega\}$, for $A \subseteq \mathscr{V}$. Given $A \subseteq \mathscr{V}$ and a configuration $\alpha \in \mathscr{A}^A$, we define the **cylinder set**

$$[\alpha]^\Omega := \{\omega \in \Omega : \omega|_A = \alpha\}. \tag{2.10}$$

We will say that $[\alpha]^\Omega$ has **support** A. When omitting the superscript Ω, we will consider $[\alpha]$ to be the cylinder set for $\Omega = \mathscr{A}^{\mathscr{V}}$.

For A_1 and A_2 disjoint sets, $\alpha_1 \in \mathscr{A}^{A_1}$, and $\alpha_2 \in \mathscr{A}^{A_2}$, $\alpha_1\alpha_2$ will be the configuration on $A_1 \sqcup A_2$ (here $\sqcup$ denotes the disjoint union) defined by $(\alpha_1\alpha_2)|_{A_1} = \alpha_1$ and $(\alpha_1\alpha_2)|_{A_2} = \alpha_2$. In particular, $[\alpha_1\alpha_2]^\Omega = [\alpha_1]^\Omega \cap [\alpha_2]^\Omega$.

Notice that $\alpha \in \mathscr{A}^A$ is globally admissible iff $\alpha \in \mathscr{L}_A(\Omega)$ iff $[\alpha]^\Omega \neq \emptyset$.

2.2.1 Homomorphism spaces

A natural way to define a rich family of configuration spaces is by relating boards and constraint graphs via **graph homomorphisms**.

Definition 2.2.1. *A graph homomorphism $\alpha : G_1 \to G_2$ from a graph $G_1 = (V_1, E_1)$ to a graph $G_2 = (V_2, E_2)$ is a mapping $\alpha : V_1 \to V_2$ such that*

$$\{x,y\} \in E_1 \implies \{\alpha(x), \alpha(y)\} \in E_2. \tag{2.11}$$

Given two graphs G_1 and G_2, we will denote by $\mathrm{Hom}(G_1, G_2)$ the set of all graph homomorphisms $\alpha : G_1 \to G_2$ from G_1 to G_2.

Fix a board $\mathscr{G}$ and a constraint graph H. We will call **homomorphism space** the set $\mathrm{Hom}(\mathscr{G}, \mathrm{H})$. In this context, the graph homomorphisms that belong to $\Omega = \mathrm{Hom}(\mathscr{G}, \mathrm{H})$ can be understood as points in a configuration space contained in $\mathscr{A}^{\mathscr{V}}$, where $\mathscr{A} = \mathrm{V}(\mathrm{H})$. Notice that a point $\omega \in \Omega$ is a "colouring" of $\mathscr{V}$ with elements from $\mathrm{V}(\mathrm{H})$ such that $x \sim_{\mathscr{G}} y \implies \omega(x) \sim_{\mathrm{H}} \omega(y)$. In other words, ω is a colouring of $\mathscr{G}$ that respects the constraints imposed by H with respect to adjacency.

Example 2.2.1. *For $d \in \mathbb{N}$, two examples of homomorphism spaces are*

- $\mathrm{Hom}(\mathbb{Z}^d, \mathrm{H}_\varphi)$*, the set of elements in $\{0,1\}^{\mathbb{Z}^d}$ with no adjacent* 1*s, and*
- $\mathrm{Hom}(\mathbb{T}_d, \mathrm{K}_q)$*, the set of proper q-colourings of the d-regular tree.*

2.2.2 Shift spaces in $\mathbb{Z}^d$

Fix the board to be $\mathbb{Z}^d$, for some $d \in \mathbb{N}$. For an alphabet $\mathscr{A}$, we can define the **shift action** $\sigma : \mathbb{Z}^d \times \mathscr{A}^{\mathbb{Z}^d} \to \mathscr{A}^{\mathbb{Z}^d}$ given by $(x, \omega) \mapsto \sigma_x(\omega)$, where $x \in \mathbb{Z}^d$, $\omega \in \mathscr{A}^{\mathbb{Z}^d}$, and $(\sigma_x(\omega))(y) = \omega(x+y)$, for $y \in \mathbb{Z}^d$. In this case, we call $\mathscr{A}^{\mathbb{Z}^d}$ the **full shift** and we can consider the metric $m(\omega, \upsilon) := 2^{-\inf\{\|x\| : \omega(x) \neq \upsilon(x)\}}$, for $\omega, \upsilon \in \mathscr{A}^{\mathbb{Z}^d}$, which induces the product topology in $\mathscr{A}^{\mathbb{Z}^d}$, i.e. $(\mathscr{A}^{\mathbb{Z}^d}, m)$ is a *compact metric space*.

We can also extend the shift action σ to configurations with arbitrary shapes, i.e. given $\alpha \in \mathscr{A}^A$ and $x \in \mathbb{Z}^d$, we define $\sigma_x(\alpha) \in \mathscr{A}^{A-x}$ as the configuration such that $(\sigma_x(\alpha))(y) = \alpha(x+y)$, for $y \in A - x = \{y - x : y \in A\}$.

We are interested in configuration spaces $\Omega \subseteq \mathscr{A}^{\mathbb{Z}^d}$ that are **shift-invariant**, i.e. $\sigma_x(\Omega) = \{\sigma_x(\omega) : \omega \in \Omega\} = \Omega$, for all $x \in \mathbb{Z}^d$. In order to obtain such spaces, given a family of finite configurations $\mathfrak{F} \subseteq \bigcup_{A \Subset \mathbb{Z}^d} \mathscr{A}^A$, we define

$$\Omega_{\mathfrak{F}} := \left\{\omega \in \mathscr{A}^{\mathbb{Z}^d} : \sigma_x(\omega)|_A \notin \mathfrak{F}, \text{ for all } A \Subset \mathbb{Z}^d, \text{ for all } x \in \mathbb{Z}^d\right\}. \tag{2.12}$$

Here, $\Omega = \Omega_{\mathfrak{F}} \subseteq \mathscr{A}^{\mathbb{Z}^d}$ is called a $\mathbb{Z}^d$ **shift space** and it is the set of all points that do not contain a translation of an element from $\mathfrak{F}$ as a sub-configuration. Notice that a $\mathbb{Z}^d$ shift space Ω is always a shift-invariant set, i.e. $\sigma_x(\Omega) = \Omega$, for all $x \in \mathbb{Z}^d$. In fact, a subset $\Omega \subseteq \mathscr{A}^{\mathbb{Z}^d}$ is a $\mathbb{Z}^d$ shift space iff it is shift-invariant and closed in the product topology (see [56, Theorem 6.1.21] for a proof of the case $d = 1$. The general case is analogous). More than one family $\mathfrak{F}$ can define the same $\mathbb{Z}^d$ shift space Ω and, when Ω can be defined by a finite family $\mathfrak{F}$, it is said to be a $\mathbb{Z}^d$ **shift of finite type** (**$\mathbb{Z}^d$ SFT**). A $\mathbb{Z}^d$ SFT is a $\mathbb{Z}^d$ **nearest-neighbour (n.n.) SFT** if $\mathfrak{F}$ can be partitioned in sets $\{\mathfrak{E}_i\}_{i=1}^d$ such that each $\mathfrak{E}_i$ contains configurations only on shapes on edges of the form $\{\vec{0}, \vec{0} + \vec{e}_i\}$, where $\vec{e}_1, \dots, \vec{e}_d$ denote the canonical basis. We will mostly restrict our attention to $\mathbb{Z}^d$ n.n. SFTs. In such case, we call $\mathfrak{F} = \bigsqcup_{i=1}^d \mathfrak{E}_i$ a **set of n.n. constraints** and, given a set $A \subseteq \mathbb{Z}^d$ and a configuration $\alpha \in \mathscr{A}^A$, we say that α is **feasible for** $\mathfrak{F}$ if for every $x \in A$ and for every $i = 1, \dots, d$ such that $\{x, x + \vec{e}_i\} \subseteq A$, we have that $\sigma_{-x}\left(\alpha|_{\{x, x+\vec{e}_i\}}\right) \notin \mathfrak{E}_i$. Notice that the $\mathbb{Z}^d$ n.n. SFT induced by $\mathfrak{F}$ is the set $\Omega_{\mathfrak{F}} = \left\{\omega \in \mathscr{A}^{\mathbb{Z}^d} : \omega \text{ is feasible for } \mathfrak{F}\right\}$.

Notice that homomorphism spaces $\mathrm{Hom}(\mathbb{Z}^d, \mathrm{H})$ are a particular case of $\mathbb{Z}^d$ n.n. SFTs, where additional symmetries are preserved besides translations (in particular, any automorphism of $\mathbb{Z}^d$). For example, the homomorphism space $\Omega_\varphi^d = \mathrm{Hom}(\mathbb{Z}^d, \mathrm{H}_\varphi)$ can be also regarded as a $\mathbb{Z}^d$ n.n. SFT. When $d = 2$, Ω_φ^2 it is known as the **hard square shift**.

In some contexts, a feasible configuration $\alpha \in \mathscr{A}^A$ is said to be **locally admissible** and it is called globally admissible if in addition it extends to a point of Ω. A globally admissible configuration is always locally admissible, but the converse is false (see Section 3.5.7 for an example). In addition, notice that if a configuration $\alpha \in \mathscr{A}^A$ is globally (resp. locally) admissible, then $\alpha|_B$ is also globally (resp. locally) admissible, for any $B \subseteq A$.

A point $\omega \in \mathscr{A}^{\mathbb{Z}^d}$ is **periodic (of period k) in the ith direction** if $\sigma_{k\vec{e}_i}(\omega) = \omega$, for $k \in \mathbb{N}$, and **periodic** if it is periodic in the ith direction, for every $i = 1, \ldots, d$. A periodic point ω is a **fixed point** if $\sigma_x(\omega) = \omega$, for all $x \in \mathbb{Z}^d$. We define the **orbit** of a point ω as the set $\mathrm{O}(\omega) := \left\{\sigma_x(\omega) : x \in \mathbb{Z}^d\right\}$. Notice that a point ω is periodic iff $|\mathrm{O}(\omega)| < \infty$, and ω is a fixed point iff $|\mathrm{O}(\omega)| = 1$.

2.3 Gibbs measures

In this section we develop the Gibbs formalism, focusing on the case where the configuration space is a homomorphism space or/and a $\mathbb{Z}^d$ n.n. SFT. We will refer to the conjunction of these two classes as **invariant nearest-neighbour (n.n.) configuration spaces**.

2.3.1 Borel probability measures and Markov random fields

Given a set $A \subseteq \mathscr{V}$, we denote by $\mathscr{F}_A$ the σ-algebra generated by all the cylinder sets $[\alpha]$ with support A, and we equip $\mathscr{A}^{\mathscr{V}}$ with the σ-algebra $\mathscr{F} = \mathscr{F}_{\mathscr{V}}$. A **Borel probability measure** μ on $\mathscr{A}^{\mathscr{V}}$ is a measure such that $\mu(\mathscr{A}^{\mathscr{V}}) = 1$, determined by its values on cylinder sets of finite support.

For notational convenience, when measuring cylinder sets we just use the configuration α instead of $[\alpha]$. For instance, $\mu(\alpha_1\alpha_2|\beta)$ represents the conditional measure $\mu([\alpha_1] \cap [\alpha_2]|[\beta])$. Given $B \subseteq A \subseteq \mathscr{V}$ and a measure μ on $\mathscr{F}_A$, we denote by $\mu|_B$ the restriction (or projection or marginalization) of μ to $\mathscr{F}_B$.

The **support** $\mathrm{supp}(\mu)$ of a Borel probability measure μ is defined as the closed set

$$\mathrm{supp}(\mu) := \left\{\omega \in \mathscr{A}^{\mathscr{V}} : \mu(\,\omega|_A) > 0, \text{ for all } A \Subset \mathscr{V}\right\}. \tag{2.13}$$

Given a configuration space Ω, we will denote by $\mathscr{M}_1(\Omega)$ the set of Borel probability measures whose support $\mathrm{supp}(\mu)$ is contained in Ω.

A measure $\mu \in \mathscr{M}_1(\Omega)$ is a **Markov random field on $\mathscr{G}$ ($\mathscr{G}$-MRF)** if, for any $A \Subset \mathscr{V}$, $\alpha \in \mathscr{A}^A$, and $B \Subset \mathscr{V}$ such that $\partial A \subseteq B \subseteq \mathscr{V} \setminus A$, and any $\beta \in \mathscr{A}^B$ with $\mu(\beta) > 0$, it is the case that

$$\mu(\alpha|\beta) = \mu(\alpha|\,\beta|_{\partial A}). \tag{2.14}$$

In other words, an MRF is a measure where every finite configuration conditioned on a boundary configuration is independent of the configuration on the "exterior".

2.3.2 Shift-invariant measures

When $\mathscr{G} = \mathbb{Z}^d$, a measure μ on $\mathscr{A}^{\mathbb{Z}^d}$ such that $\mu(\sigma_x(\mathbb{A})) = \mu(\mathbb{A})$, for all measurable sets $\mathbb{A} \in \mathscr{F}$ and $x \in \mathbb{Z}^d$, is called **shift-invariant** (or **stationary**). In this context, the support $\operatorname{supp}(\mu)$ turns out to be always a $\mathbb{Z}^d$ shift space (a closed and shift-invariant subset of $\mathscr{A}^{\mathbb{Z}^d}$). Given a $\mathbb{Z}^d$ shift space Ω, $\mathscr{M}_{1,\sigma}(\Omega)$ denotes the set of shift-invariant measures contained in $\mathscr{M}_1(\Omega)$. A measure $\mu \in \mathscr{M}_{1,\sigma}(\Omega)$ is **ergodic** if for all shift-invariant $\mathbb{A} \in \mathscr{F}$ (i.e $\sigma_x(\mathbb{A}) = \mathbb{A}$, for all $x \in \mathbb{Z}^d$), it is the case that $\mu(\mathbb{A}) \in \{0,1\}$, and **measure-theoretic strong mixing**, if for every nonempty sets $A, B \Subset \mathbb{Z}^d$ and any $\alpha \in \mathscr{A}^A$, $\beta \in \mathscr{A}^B$,

$$\lim_{\|x\|\to\infty} \mu\left([\alpha] \cap \sigma_{-x}([\beta])\right) = \mu(\alpha)\,\mu(\beta), \tag{2.15}$$

i.e. for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\|x\| \geq n \implies \left|\mu\left([\alpha] \cap \sigma_{-x}([\beta])\right) - \mu(\alpha)\,\mu(\beta)\right| < \varepsilon. \tag{2.16}$$

If $\mu \in \mathscr{M}_{1,\sigma}(\Omega)$ is measure-theoretic strong mixing, then μ is ergodic (see [76] for these and related notions). Given any point $\omega \in \mathscr{A}^{\mathbb{Z}^d}$, we define the **$\delta$-measure supported on ω** as the measure

$$\delta_\omega(\mathbb{A}) = \begin{cases} 1 & \text{if } \omega \in \mathbb{A}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.17}$$

for any $\mathbb{A} \in \mathscr{F}$.

If ω is a periodic point with orbit $\mathrm{O}(\omega) = \{\omega_1, \dots, \omega_k\}$, we define ν^ω to be the shift-invariant Borel probability measure supported on $\mathrm{O}(\omega)$ given by

$$\nu^\omega := \frac{1}{k}\left(\delta_{\omega_1} + \cdots + \delta_{\omega_k}\right). \tag{2.18}$$

Gibbs measures will be the main class of Borel probability measures studied in this work. Before defining them, we need to introduce *nearest-neighbour interactions* and *Gibbs specifications*.

2.3.3 Nearest-neighbour interactions

Let Ω be a configuration space for a board $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. We define the **set of configurations on vertices** as $\mathscr{L}^v(\Omega) := \bigcup_{x \in \mathscr{V}} \mathscr{L}_{\{x\}}(\Omega)$, and the **set of configurations on edges** as $\mathscr{L}^e(\Omega) := \bigcup_{\{x,y\} \in \mathscr{E}} \mathscr{L}_{\{x,y\}}(\Omega)$. We will abbreviate

$$\mathscr{L}^{v,e}(\Omega) := \mathscr{L}^v(\Omega) \cup \mathscr{L}^e(\Omega). \tag{2.19}$$

A **nearest-neighbour (n.n.) interaction on** Ω will be any real-valued function

$$\Phi : \mathscr{L}^{v,e}(\Omega) \to \mathbb{R} \tag{2.20}$$

that evaluates configurations in $\mathscr{L}^{v,e}(\Omega)$. We will always assume that

$$\Phi_{\max} := \sup_{\alpha \in \mathscr{L}^{v,e}(\Omega)} |\Phi(\alpha)| < \infty. \tag{2.21}$$

Constrained energy functions

Given a constraint graph $\mathrm{H} = (\mathrm{V}, \mathrm{E})$, a **constrained energy function** will be any pair (H, ϕ) such that ϕ is a real-valued function $\phi : \mathrm{V} \cup \mathrm{E} \to \mathbb{R}$ from vertices and edges in H. Given a homomorphism space $\Omega = \mathrm{Hom}(\mathscr{G}, \mathrm{H})$, a constrained energy function ϕ induces naturally a n.n. interaction Φ in Ω by taking

$$\Phi(\alpha) = \begin{cases} \phi\left(\alpha(x)\right) & \text{if } \alpha \in \mathscr{L}_{\{x\}}(\Omega),\ x \in \mathscr{V}, \\ \phi\left(\{\alpha(x), \alpha(y)\}\right) & \text{if } \alpha \in \mathscr{L}_{\{x,y\}}(\Omega),\ \{x,y\} \in \mathscr{E}. \end{cases} \tag{2.22}$$

Example 2.3.1. *Let $q \in \mathbb{N}$ and $\xi > 0$. Many constrained energy functions represent well-known classical models.*

- *Ferromagnetic Potts $(\mathrm{K}_q^{\circlearrowleft}, \xi \phi^{\mathrm{FP}})$: $\phi^{\mathrm{FP}}\big|_{\mathrm{V}} \equiv 0$, $\phi^{\mathrm{FP}}(\{u,v\}) = -\mathbb{1}_{\{u=v\}}$.*

- *Anti-ferromagnetic Potts* $(\mathrm{K}_q^{\circlearrowleft}, \xi\phi^{\mathrm{AP}})$*:* $\phi^{\mathrm{AP}}\big|_{\mathrm{V}} \equiv 0$, $\phi^{\mathrm{AP}}(\{u,v\}) = -\mathbb{1}_{\{u \neq v\}}$.
- *Proper q-colourings* $(\mathrm{K}_q, \phi^{\mathrm{PC}})$*:* $\phi^{\mathrm{PC}}\big|_{\mathrm{V}\cup\mathrm{E}} \equiv 0$.
- *Hard-core* $(\mathrm{H}_\varphi, \xi\phi^{\mathrm{HC}})$*:* $\phi^{\mathrm{HC}}(0) = 0$, $\phi^{\mathrm{HC}}(1) = -1$, $\phi^{\mathrm{HC}}\big|_{\mathrm{E}} \equiv 0$.
- *Multi-type Widom-Rowlinson* $(\mathrm{S}_q^{\circlearrowleft}, \xi\phi^{\mathrm{WR}})$*:* $\phi^{\mathrm{WR}}(v) = -\mathbb{1}_{\{v \neq 0\}}$, $\phi^{\mathrm{WR}}\big|_{\mathrm{E}} \equiv 0$.

Usually, the parameter ξ *is referred to as the* **inverse temperature***.*

All the previous models are **isotropic**, i.e. when $\mathscr{G} = \mathbb{Z}^d$ they have the same interaction in every coordinate direction $\{\vec{0}, \vec{e}_i\}$, for $i = 1, \ldots, d$.

$\mathbb{Z}^d$ lattice energy functions

Given $d \in \mathbb{N}$ and the board $\mathscr{G} = \mathbb{Z}^d$, we can define a n.n. interaction that evaluates configurations on edges according to their orientation. Given a set of n.n. constraints $\mathfrak{F} = \bigsqcup_{i=1}^{d} \mathfrak{E}_i$, any real-valued function ϕ from $\mathscr{A}^{\{\vec{0}\}}$ and $\mathscr{A}^{\{\vec{0},\vec{e}_i\}} \setminus \mathfrak{E}_i$ $(i = 1, \ldots, d)$ will be a $\mathbb{Z}^d$ **lattice energy function** for $\mathfrak{F}$. A $\mathbb{Z}^d$ lattice energy function ϕ for $\mathfrak{F}$ induces naturally a n.n. interaction Φ on the $\mathbb{Z}^d$ n.n. SFT $\Omega_{\mathfrak{F}}$ by taking

$$\Phi(\alpha) = \begin{cases} \phi\left(\sigma_{-x}(\alpha)|_{\{\vec{0}\}}\right) & \text{if } \alpha \in \mathscr{L}_{\{x\}}(\Omega), \\ \phi\left(\sigma_{-x}(\alpha)|_{\{\vec{0},\vec{e}_i\}}\right) & \text{if } \alpha \in \mathscr{L}_{\{x,x+\vec{e}_i\}}(\Omega), \end{cases} \tag{2.23}$$

for $x \in \mathbb{Z}^d$ and $i = 1, \ldots, d$. Notice that in this case Φ is *shift-invariant* (i.e. the value of a configuration on an edge is the same for any translation of it), which is not a requirement in general. However, the shift-invariance of Φ fits naturally with the shift-invariance of $\Omega_{\mathfrak{F}}$, so we will usually assume this in the context of $\mathbb{Z}^d$ shift spaces. Clearly, a shift-invariant n.n. interaction is defined by only finitely many numbers (namely, the values defining ϕ).

2.3.4 Hamiltonian and partition function

Let $\Omega \subseteq \mathscr{A}^{\mathscr{V}}$ be an invariant n.n. configuration space (i.e. a homomorphism space or a $\mathbb{Z}^d$ n.n. SFT), and let Φ be a n.n. interaction. Given $A \Subset \mathscr{V}$, we define the

Hamiltonian in A as $\mathscr{H}_A^{\Phi} : \mathscr{L}_A(\Omega) \to \mathbb{R}$, where

$$\mathscr{H}_A^{\Phi}(\alpha) := \sum_{x \in A} \Phi\left(\alpha|_{\{x\}}\right) + \sum_{\{x,y\} \in \mathscr{E}[A]} \Phi\left(\alpha|_{\{x,y\}}\right), \tag{2.24}$$

for $\alpha \in \mathscr{L}_A(\Omega)$. We define the **partition function of** A as

$$\mathrm{Z}_A^{\Phi} := \sum_{\alpha \in \mathscr{L}_A(\Omega)} \exp\left(-\mathscr{H}_A^{\Phi}(\alpha)\right), \tag{2.25}$$

and the **free-boundary probability measure on** $\mathscr{A}^A$ given by

$$\pi_A^{(f)}(\alpha) := \begin{cases} \frac{\exp\left(-\mathscr{H}_A^{\Phi}(\alpha)\right)}{\mathrm{Z}_A^{\Phi}} & \text{if } \alpha \in \mathscr{L}_A(\Omega), \\ 0 & \text{otherwise.} \end{cases} \tag{2.26}$$

In addition, given $\omega \in \Omega$, we define the **ω-boundary Hamiltonian in** A as $\mathscr{H}_{A,\omega}^{\Phi} : \{\alpha \in \mathscr{L}_A(\Omega) : \alpha\, \omega|_{A^c} \in \Omega\} \to \mathbb{R}$, where

$$\mathscr{H}_{A,\omega}^{\Phi}(\alpha) = \sum_{x \in A} \Phi\left(\alpha|_{\{x\}}\right) + \sum_{\{x,y\} \in \mathscr{E}[A \cup \partial A] : \{x,y\} \cap A \neq \emptyset} \Phi\left((\alpha\omega|_{A^c})|_{\{x,y\}}\right), \tag{2.27}$$

and the **ω-boundary probability measure on** $\mathscr{A}^A$ given by

$$\pi_A^{\omega}(\alpha) := \begin{cases} \frac{\exp\left(-\mathscr{H}_{A,\omega}^{\Phi}(\alpha)\right)}{\mathrm{Z}_{A,\omega}^{\Phi}} & \text{if } \alpha\, \omega|_{A^c} \in \Omega, \\ 0 & \text{otherwise,} \end{cases} \tag{2.28}$$

where $\mathrm{Z}_{A,\omega}^{\Phi} := \sum_{\alpha : \alpha\, \omega|_{A^c} \in \Omega} \exp\left(-\mathscr{H}_{A,\omega}^{\Phi}(\alpha)\right)$.

Both, for $\pi_A^{(f)}$ and π_A^{ω}, given $B \subseteq A$ and $\beta \in \mathscr{A}^B$, we marginalize as follows:

$$\pi_A^*(\beta) = \sum_{\alpha \in \mathscr{A}^A :\, \alpha|_B = \beta} \pi_A^*(\alpha), \tag{2.29}$$

where $* = (f)$ or ω. Notice that π_A^* is a $\mathscr{G}[A]$-MRF (here we use the assumption that Ω is an invariant n.n. configuration space and Φ is a n.n. interaction).

2.3.5 Gibbs specifications

The collection $\pi = \{\pi_A^\omega : A \Subset \mathscr{V}, \omega \in \Omega\}$ will be called **nearest-neighbour (n.n.) Gibbs** (Ω, Φ)**-specification** (or just **Gibbs specification** if Ω and Φ are understood). If $\Phi \equiv 0$, we call the n.n. Gibbs $(\Omega, 0)$-specification π the **uniform Gibbs specification** on Ω (see the case of proper q-colourings in Example 2.3.1).

Notice that a n.n. Gibbs specification π for a shift-invariant n.n. interaction Φ on a $\mathbb{Z}^d$ n.n. SFT Ω is always *stationary*, in the sense that $\pi_{A-x}^{\sigma_x(\omega)}(\sigma_x(\alpha)) = \pi_A^\omega(\alpha)$, for every $\alpha \in \mathscr{A}^A$ and $x \in \mathbb{Z}^d$.

2.3.6 Nearest-neighbour Gibbs measures

A n.n. Gibbs (Ω, Φ)-specification π is regarded as a meaningful representation of an ideal physical situation where every finite volume A in the space is in thermodynamical equilibrium with the exterior. The extension of this idea to infinite volumes is via a particular class of Borel probability measures on Ω called *nearest-neighbour Gibbs measures*, which are MRFs specified by n.n. interactions.

Definition 2.3.1. *Given a n.n. Gibbs* (Ω, Φ)*-specification* π*, a* **nearest-neighbour (n.n.) Gibbs measure for** π *is any measure* $\mu \in \mathscr{M}_1(\Omega)$ *such that for any* $A \Subset \mathscr{V}$ *and* $\omega \in \Omega$ *with* $\mu(\omega|_{\partial A}) > 0$*, we have* $\mathrm{Z}_{A,\omega}^\Phi > 0$ *and*

$$\mathbb{E}_\mu(\mathbb{1}_{[\alpha]^\Omega} | \mathscr{F}_{A^c})(\omega) = \pi_A^\omega(\alpha) \quad \mu\text{-a.s.}, \tag{2.30}$$

for every $\alpha \in \mathscr{L}_A(\Omega)$.

Equation (2.30) is equivalent to what is known as the **Dobrushin-Lanford-Ruelle (DLR) equation**. It is stated only for cylinder events $[\alpha]^\Omega$ in A, but this is equivalent to the usual definition with general events $\mathbb{A} \in \mathscr{F}$ instead. Given a n.n. Gibbs (Ω, Φ)-specification π, we will denote $\mathcal{G}(\pi)$ the set of n.n. Gibbs measures for π. If $\Omega \neq \emptyset$, then $\mathcal{G}(\pi) \neq \emptyset$ (special case of a result in [26], see also [18] and [70, Theorem 3.7 and Theorem 4.2]). Notice that every n.n. Gibbs measure μ is a $\mathscr{G}$-MRF because the formula for π_A^ω only depends on $\omega|_{\partial A}$.

Often there are multiple n.n. Gibbs measures for a single π. This phenomenon is usually called a **phase transition**. There are several conditions that guarantee uniqueness of n.n. Gibbs measures (i.e. $|\mathcal{G}(\pi)| = 1$). Some of them fall into the category of *spatial mixing properties*, introduced in the next chapter.

In the case $\mathbb{Z}^d$ n.n. SFTs, a Gibbs measure for a stationary Gibbs specification π may or may not be shift-invariant, but $\mathcal{G}(\pi)$ must contain at least one shift-invariant measure (see [35, Corollary 5.16]).

Chapter 3

Mixing properties

In this chapter we introduce some mixing properties of measure-theoretic and combinatorial type for a Gibbs (Ω,Φ)-specification π. In general terms, a mixing property says that, either a measure or the support of it, does not have *strong long-range correlations*. This last aspect will be relevant for obtaining succinct representations of pressure, and when developing efficient algorithms for approximating it.

3.1 Spatial mixing properties

In the following, let $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function, referred as **decay function**, such that $f(n) \searrow 0$ as $n \to \infty$. We will loosely use the term "spatial mixing property" to refer to any measure-theoretical property satisfied by π defined via a *decay of correlation between events (or configurations)* with respect to the distance separating the shapes they are supported on.

The first property introduced here, *weak spatial mixing (WSM)*, has direct connections with the nonexistence of phase transitions and has been studied in several works, explicitly and implicitly (see [17, 78]). The next one, *strong spatial mixing (SSM)*, is a strengthening of WSM that also has connections with meaningful physical idealizations (see [61, 45, 28]) and has proven to be useful for developing approximation algorithms (see [79, 39, 33, 6, 31, 80]).

Definition 3.1.1. *A Gibbs (Ω,Φ)-specification π satisfies* **weak spatial mixing (WSM) with decay** f *if for any $A \Subset \mathcal{V}$, $B \subseteq A$, $\beta \in \mathscr{A}^B$, and $\omega_1, \omega_2 \in \Omega$,*

$$\left|\pi_A^{\omega_1}(\beta) - \pi_A^{\omega_2}(\beta)\right| \leq |B| f(\mathrm{dist}(B, \partial A)). \tag{3.1}$$

If a Gibbs (Ω,Φ)-specification π satisfies WSM, then there is a unique n.n. Gibbs measure μ for π (see [78, Proposition 2.2]).

We use the convention that $\mathrm{dist}(B,\emptyset)=\infty$. Given two configurations $\alpha_1 \in \mathscr{A}^{A_1}$, $\alpha_2 \in \mathscr{A}^{A_2}$, and $B \subseteq A_1 \cap A_2$, we define their **set of B-disagreement** as

$$\Sigma_B(\alpha_1,\alpha_2) := \{x \in B : \alpha_1(x) \neq \alpha_2(x)\}. \tag{3.2}$$

Considering this, we have the following definition, a priori stronger than WSM.

Definition 3.1.2. *A Gibbs (Ω,Φ)-specification π satisfies* **strong spatial mixing (SSM) with decay** f *if for any $A \Subset \mathscr{V}$, $B \subseteq A$, $\beta \in \mathscr{A}^B$, and $\omega_1,\omega_2 \in \Omega$,*

$$\left|\pi_A^{\omega_1}(\beta)-\pi_A^{\omega_2}(\beta)\right| \leq |B| f\left(\mathrm{dist}(B,\Sigma_{\partial A}(\omega_1,\omega_2))\right). \tag{3.3}$$

Notice that $\mathrm{dist}(B,\Sigma_{\partial A}(\omega_1,\omega_2)) \geq \mathrm{dist}(B,\partial A)$, so SSM implies WSM.

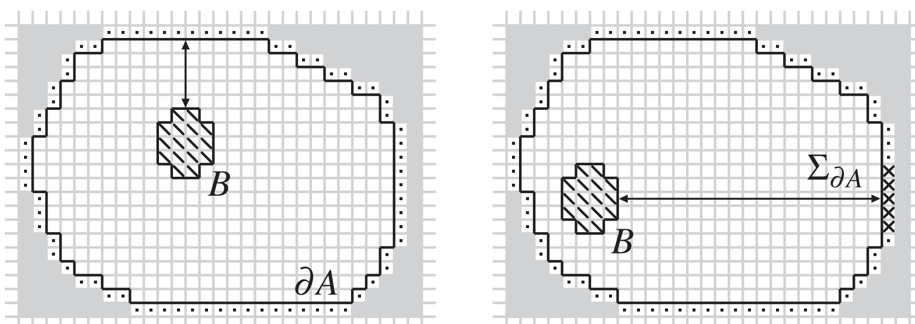

Figure 3.1: The weak and strong spatial mixing properties.

We will say that a Gibbs specification π satisfies SSM (resp. WSM) if it satisfies SSM (resp. WSM) with decay f, for some decay function f. For $\gamma > 0$, a Gibbs specification π satisfies **exponential SSM** (resp. **exponential WSM**) **with decay rate** γ if it satisfies SSM (resp. WSM) with decay function $f(n) = Ce^{-\gamma n}$, for some $C > 0$. We say that a Gibbs specification π satisfies **SSM for a class of sets** $\mathscr{C}$ if Definition 3.1.2 holds restricted to sets $A \in \mathscr{C}$, for $\mathscr{C}$ a (possibly infinite) family of finite sets.

Definition 3.1.3 (Total variation distance)**.** *Let K be a finite set and let X_1 and X_2 be two K-valued random variables with probability distributions ρ_1 and ρ_2, respectively. The* **total variation distance** *between X_1 and X_2 (or equivalently,* *between ρ_1 and ρ_2) is defined as*

$$d_{\mathrm{TV}}(\rho_1,\rho_2) := \frac{1}{2}\sum_{x\in X}|\rho_1(x)-\rho_2(x)|. \tag{3.4}$$

In the literature, it is also common to find the definition of WSM and SSM with the expression $\left|\pi_A^{\omega_1}(\beta)-\pi_A^{\omega_2}(\beta)\right|$ replaced by the total variation distance of $\pi_A^{\omega_1}\big|_B$ and $\pi_A^{\omega_2}\big|_B$. The definitions here are, a priori, slightly weaker (so the results where SSM is an assumption are also valid for this alternative definition), but sufficient for our purposes.

Definition 3.1.4. *A* **coupling** *of two probability measures ρ_1 on a finite set K_1 and ρ_2 on a finite set K_1, is a probability measure $\mathbb{P}$ on the set $K_1\times K_1$ such that, for any $\mathbb{A}\subseteq K_1$ and $\mathbb{B}\subseteq K_1$, we have that*

$$\mathbb{P}(\mathbb{A}\times K_1)=\rho_1(\mathbb{A}), \quad \textit{and} \quad \mathbb{P}(K_1\times\mathbb{B})=\rho_2(\mathbb{B}). \tag{3.5}$$

Given a finite set K and two K-valued random variables X_1 and X_2 with probability distributions ρ_1 and ρ_2, respectively, it is well-known that $d_{\mathrm{TV}}(\rho_1,\rho_2)$ is a lower bound on $\mathbb{P}(X_1\neq X_2)$ over all couplings $\mathbb{P}$ of ρ_1 and ρ_2 and that there is a coupling, called the **optimal coupling**, that achieves this lower bound.

Lemma 3.1.1 ([60, Lemma 2.3])**.** *Let π be a Gibbs (Ω,Φ)-specification and f a decay function such that for any $A\Subset\mathcal{V}$, $x\in A$, $\beta\in\mathscr{A}^{\{x\}}$, and $\omega_1,\omega_2\in\Omega$,*

$$\left|\pi_A^{\omega_1}(\beta)-\pi_A^{\omega_2}(\beta)\right| \leq f\left(\mathrm{dist}(x,\Sigma_{\partial A}(\omega_1,\omega_2))\right). \tag{3.6}$$

Then, π satisfies SSM with decay f.

Remark 1. *The proof of Lemma 3.1.1 given in [60] is for MRFs μ satisfying exponential SSM on $\mathscr{G}=\mathbb{Z}^d$, but its generalization to Gibbs specifications, arbitrary decay functions, and more general boards is direct. We do not know if there is an analogous lemma for WSM.*

Example 3.1.1. *Recall the constrained energy functions introduced in Example 2.3.1. The following Gibbs specifications satisfy exponential SSM (and therefore, WSM).*

- $(\mathrm{K}_q^{\circlearrowleft}, \xi\phi^{\mathrm{FP}})$ *on* $\mathscr{G} = \mathbb{Z}^2$*, for any* $q \in \mathbb{N}$ *and small enough* ξ *(see [1, 11]).*
- $(\mathrm{K}_q^{\circlearrowleft}, \xi\phi^{\mathrm{AP}})$ *on* $\mathscr{G} = \mathbb{Z}^2$*, for* $q \geq 6$ *and any* $\xi > 0$ *(see [40]).*
- $(\mathrm{K}_q, \phi^{\mathrm{PC}})$ *on* $\mathscr{G} = \mathbb{T}_d$*, for* $q \geq 1 + \delta^* d$*, where* $\delta^* = 1.763\ldots$ *is the unique solution to* $xe^{-1/x} = 1$ *(see [34, 39]).*
- $(\mathrm{H}_\varphi, \xi\phi^{\mathrm{HC}})$ *on* $\mathscr{G}$ *with* $\Delta(\mathscr{G}) \leq d$*, for any* ξ *such that* $e^\xi < \lambda_{\mathrm{c}}(d) := \frac{(d-1)^{(d-1)}}{(d-2)^d}$ *(see [79]).*
- $(\mathrm{S}_q^{\circlearrowleft}, \xi\phi^{\mathrm{WR}})$ *on* $\mathscr{G} = \mathbb{Z}^d$*, for any* $q \in \mathbb{N}$ *and small enough* ξ *(see [1, 11]).*

There are more general sufficient conditions for having exponential SSM (for instance, see Subsection 3.6.1 or the discussion in [60]).

3.2 Combinatorial mixing properties

In this section we consider combinatorial properties for a configuration space Ω. According to the way they are defined, we classify them as *global* or *local*.

3.2.1 Global properties

By a global property we understand any property of Ω that allow us to "glue" together globally admissible configurations in a single point $\omega \in \Omega$. The two properties introduced here have in common that they allow us to put together configurations provided they are separated enough.

Definition 3.2.1. *A* $\mathbb{Z}^d$ *shift space* Ω *is* **topologically mixing** *if for any* $A, B \Subset \mathbb{Z}^d$ *there exists a separation constant* $g_{A,B} \in \mathbb{N}$ *such that for every* $\alpha \in \mathscr{A}^A$*,* $\beta \in \mathscr{A}^B$*, and any* $x \in \mathbb{Z}^d$ *with* $\mathrm{dist}(A, x+B) \geq g_{A,B}$*,*

$$[\alpha]^\Omega, [\beta]^\Omega \neq \emptyset \implies [\alpha]^\Omega \cap \sigma_{-x}([\beta]^\Omega) \neq \emptyset. \tag{3.7}$$

Definition 3.2.2. *A configuration space* Ω *is* **strongly irreducible with gap** $g \in \mathbb{N}$ *if for any* $A, B \Subset \mathscr{V}$ *with* $\mathrm{dist}(A,B) \geq g$*, and for every* $\alpha \in \mathscr{A}^A$*,* $\beta \in \mathscr{A}^B$*,*

$$[\alpha]^\Omega, [\beta]^\Omega \neq \emptyset \implies [\alpha\beta]^\Omega \neq \emptyset. \tag{3.8}$$

In other words, in the context of $\mathbb{Z}^d$ shift spaces, strong irreducibility means that the separation constant $g_{A,B}$ from Definition 3.2.1 can be chosen to be uniform in A and B.

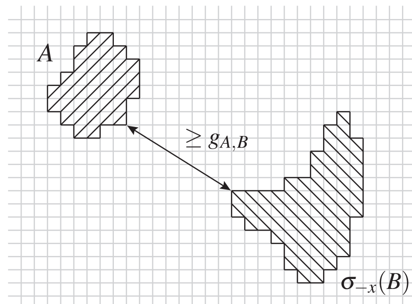

Figure 3.2: A topologically mixing $\mathbb{Z}^2$ shift space.

Remark 2. *Since a configuration space is a compact space, it does not make a difference if the shapes of A and B are allowed to be infinite in the definition of strong irreducibility.*

3.2.2 Local properties

Now we introduce two local properties concerning constraint graphs and n.n. constraints, which will later be shown to have implications on the global properties (like, for example, strong irreducibility) of invariant n.n. configuration spaces Ω (homomorphism spaces and $\mathbb{Z}^d$ n.n. SFTs, respectively).

The first property is the existence of a special symbol which can be adjacent to every other symbol (including itself).

Definition 3.2.3. *Given a constraint graph* H*, we say that* $s \in \mathrm{V}$ *is a* **safe symbol** *if* $\{s,v\} \in \mathrm{E}$*, for every* $v \in \mathrm{V}$*. Analogously, given an alphabet* $\mathscr{A}$*, a set of n.n. constraints* $\mathfrak{F}$*, and the corresponding* $\mathbb{Z}^d$ *n.n. SFT* $\Omega = \Omega_{\mathfrak{F}}$*, we say that* $s \in \mathscr{A}$ *is a safe symbol for* Ω *if* $s^{\{\vec{0}\}}\delta$ *is locally admissible for every configuration* $\delta \in \mathscr{A}^{\partial\{\vec{0}\}}$.

Example 3.2.1 (A $\mathbb{Z}^d$ n.n. SFT with a safe symbol)**.** *The constraint graph* H_φ *has a safe symbol (see Figure 2.1.2). In the support of the* $\mathbb{Z}^d$ *hard-core lattice gas model (the* $\mathbb{Z}^d$ *n.n. SFT* Ω^d_φ*),* 0 *is a safe symbol for every* d *(see [32]).*

The following property was introduced in [58] in the context of $\mathbb{Z}^d$ shift spaces, and here we proceed to adapt it to the case of homomorphism spaces.

Definition 3.2.4. *A homomorphism space* $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ *is* **single-site fillable (SSF)** *(or we say that it satisfies* **single-site fillability***) if for every site* $x \in \mathscr{V}$ *and* $B \subseteq \partial\{x\}$*, any graph homomorphism* $\beta : \mathscr{G}[B] \to \mathrm{H}$ *can be extended to a graph homomorphism* $\alpha : \mathscr{G}[B \cup \{x\}] \to \mathrm{H}$ *(i.e.* α *is such that* $\alpha|_B = \beta$*). Analogously, a* $\mathbb{Z}^d$ *n.n. SFT* Ω *is SSF if for some set of n.n. constraints* $\mathfrak{F}$ *such that* $\Omega = \Omega_{\mathfrak{F}}$*, for every* $\delta \in \mathscr{A}^{\partial\{\vec{0}\}}$*, there exists* $a \in \mathscr{A}$ *such that* $a^{\{\vec{0}\}}\delta$ *is locally admissible.*

Note 1. *An invariant n.n. configuration space* Ω *is SSF iff every locally admissible configuration is globally admissible (see [58] for the* $\mathbb{Z}^d$ *n.n. SFT case).*

In the definition of SSF above, the symbol a may depend on the configuration α. Clearly, a $\mathbb{Z}^d$ n.n. SFT containing a safe symbol satisfies SSF. Also, it is easy to check that a $\mathbb{Z}^d$ n.n. SFT Ω that satisfies SSF is strongly irreducible with gap $g = 2$ (see Proposition 3.3.3).

Example 3.2.2 (A $\mathbb{Z}^d$ n.n. SFT that satisfies SSF without a safe symbol)**.** *The* $\mathbb{Z}^d$ *q-colourings n.n. SFT* $\mathrm{Hom}(\mathbb{Z}^d,\mathrm{K}_q)$ *has no safe symbol for any* $q \geq 2$ *and* $d \geq 1$*. However,* $\mathrm{Hom}(\mathbb{Z}^d,\mathrm{K}_q)$ *satisfies SSF for* $q \geq 2d+1$ *(see [58]).*

3.3 Topological strong spatial mixing

Now we introduce a property which is somehow a hybrid between the global and local combinatorial properties from last section. Because of its close relationship with *topological Markov fields* (see [20]), we prefer to use the word "topological" for naming it. This condition will be relevant to give a partial characterization of systems that admit measures satisfying SSM, and also to generalize results related with pressure representation and approximation in Part II.

Definition 3.3.1. *A configuration space* Ω *is* **topologically strong spatial mixing (TSSM) with gap** $g \in \mathbb{N}$*, if for any* $A,B,S \Subset \mathscr{V}$ *with* $\mathrm{dist}(A,B) \geq g$*, and for every* $\alpha \in \mathscr{A}^A$*,* $\beta \in \mathscr{A}^B$*, and* $\sigma \in \mathscr{A}^S$*,*

$$[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset \implies [\alpha\sigma\beta]^\Omega \neq \emptyset. \tag{3.9}$$

Notice that TSSM implies strong irreducibility by taking $S = \emptyset$ in Definition 3.3.1. The difference here is that we allow an arbitrarily close globally admissible configuration on S in between two sufficiently separated globally admissible configurations, provided that each of the two configurations is compatible with the one on S individually. Clearly, TSSM with gap g implies TSSM with gap $g+1$. We will say that a configuration space satisfies TSSM (resp. strong irreducibility) if it satisfies TSSM (resp. strong irreducibility) with gap g, for some $g \in \mathbb{N}$.

It can be checked that for a $\mathbb{Z}^d$ n.n. SFT Ω,

$$\text{Safe symbol} \implies \text{SSF} \implies \text{TSSM} \implies \text{Strong irred.} \implies \text{Top. mixing}, \quad (3.10)$$

and all implications are strict. See Section 3.5 for examples that illustrate the differences among some of these conditions.

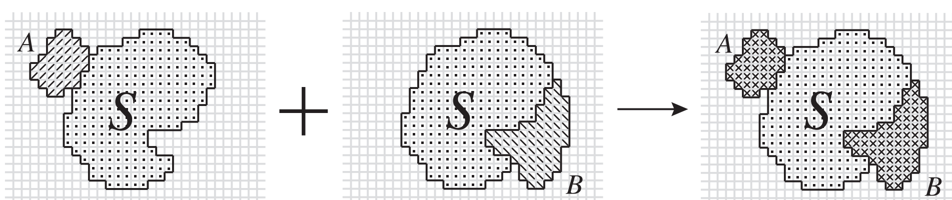

Figure 3.3: The topological strong spatial mixing property.

A useful tool when dealing with TSSM is the next lemma, which states that if we have the TSSM property when A and B are singletons, then we have it uniformly (in terms of separation distance) for any pair of finite sets A and B.

Lemma 3.3.1. *Let Ω be a configuration space and $g \in \mathbb{N}$ such that for any $x, y \in \mathcal{V}$ with $\mathrm{dist}(x,y) \geq g$ and $S \Subset \mathcal{V}$, we have that for every $\alpha \in \mathscr{A}^{\{x\}}$, $\beta \in \mathscr{A}^{\{y\}}$, and $\sigma \in \mathscr{A}^S$ with $[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset$, then $[\alpha\sigma\beta]^\Omega \neq \emptyset$. Then, Ω satisfies TSSM with gap g.*

Proof. We proceed by induction. The base case $|A| + |B| = 2$ is given by the hypothesis of the lemma. Now, let's suppose that the property is true for subsets $A, B \Subset \mathcal{V}$ such that $|A| + |B| \leq n$ and let's prove it for the case when $|A| + |B| = n+1$.

Given $A, B, S \Subset \mathcal{V}$ with $\mathrm{dist}(A,B) \geq g$ and $|A| + |B| = n+1$, and given $\alpha \in \mathscr{A}^A$,

$\beta \in \mathscr{A}^B$, and $\sigma \in \mathscr{A}^S$, we can write $A = \{x_1,\dots,x_k\}$ and $B = \{y_1,\dots,y_m\}$, where $|A| = k$, $|B| = m$, and $k+m = n+1$, for $k,m \geq 1$. Let's consider $A' = A \setminus \{x_k\}$ and $B' = B \setminus \{y_m\}$, possibly empty sets (but not both empty at the same time, since we can assume that $|A|+|B| > 2$). Similarly, let's consider the restrictions $\alpha' = \alpha|_{A'}$ and $\beta' = \beta|_{B'}$. By the induction hypothesis, we have that $[\alpha'\sigma\beta]^\Omega, [\alpha\sigma\beta']^\Omega \neq \emptyset$ (even in the case A' or B' being empty). Then, if we consider $\sigma' = \alpha'\sigma\beta'$ on $S' = A' \cup S \cup B'$, we can apply the property for singletons with $\alpha|_{\{x_k\}}$ and $\beta|_{\{y_m\}}$, and we conclude that $\emptyset \neq [\alpha|_{\{x_k\}}\, \sigma'\, \beta|_{\{y_m\}}]^\Omega = [\alpha\sigma\beta]^\Omega$. □

Proposition 3.3.2. *Let Ω be a configuration space. The two following properties are equivalent:*

1. *Ω satisfies TSSM with gap g.*
2. *For every $\Gamma \Subset \mathscr{V}$ and $\eta,\eta' \in \mathscr{L}_\Gamma(\Omega)$ with $\Sigma_\Gamma(\eta,\eta') = \{x_1,\dots,x_k\}$, there exists a sequence $\eta = \eta_1,\eta_2,\dots,\eta_{k+1} = \eta' \in \mathscr{L}_\Gamma(\Omega)$ such that $\Sigma_\Gamma(\eta_i,\eta_{i+1}) \subseteq \Sigma_\Gamma(\eta,\eta') \cap \mathscr{N}_g(x_i)$, for all $1 \leq i \leq k$.*

Proof.
$(\mathbf{1}) \implies (\mathbf{2})$. Take $\eta_1 = \eta$ and $\eta_{k+1} = \eta'$. Suppose that for some $i \leq k$ we have already constructed a sequence $\eta_1,\dots,\eta_i \in \mathscr{L}_A(\Omega)$ such that

$$\Sigma_A(\eta_j,\eta_{j+1}) \subseteq \Sigma_A(\eta,\eta') \cap \mathscr{N}_g(x_j), \text{for all } 1 \leq j < i, \text{ and} \tag{3.11}$$
$$\Sigma_A(\eta_j,\eta') \subseteq \{x_j,\dots,x_k\}, \text{for all } 1 \leq j \leq i. \tag{3.12}$$

The base case $i = 1$ is clear. Now, let's extend the sequence to $i+1$. Consider the sets $A_i = \{x_i\}$, $B_i = \Sigma_\Gamma(\eta_i,\eta') \setminus \mathscr{N}_g(x_i)$, and $S_i = \Gamma \setminus \Sigma_\Gamma(\eta_i,\eta')$. Take the configurations $\alpha_i \in \mathscr{A}^{A_i}$, $\beta_i \in \mathscr{A}^{B_i}$, and $\sigma_i \in \mathscr{A}^{S_i}$ defined as $\alpha_i := \eta'|_{A_i}$, $\beta_i := \eta_i|_{B_i}$, and $\sigma_i := \eta_i|_{S_i} = \eta'|_{S_i}$. Since $\operatorname{dist}(A_i,B_i) \geq g$, $\emptyset \neq [\eta']^\Omega \subseteq [\alpha_i\sigma_i]^\Omega$, and $\emptyset \neq [\eta_i]^\Omega \subseteq [\sigma_i\beta_i]^\Omega$, by TSSM, we can take $\omega_i \in [\alpha_i\sigma_i\beta_i]^\Omega$ and consider $\eta_{i+1} := \omega_i|_\Gamma \in \mathscr{L}_\Gamma(\Omega)$. Then, $\Sigma_\Gamma(\eta_{i+1},\eta') \subseteq \{x_{i+1},\dots,x_k\}$ and $\Sigma_\Gamma(\eta_i,\eta_{i+1}) \subseteq \Sigma_\Gamma(\eta,\eta') \cap \mathscr{N}_g(x_i)$, as we wanted. Iterating until $i = k$, we conclude.

$(\mathbf{2}) \implies (\mathbf{1})$. Consider $x,y \in \mathscr{V}$ with $\operatorname{dist}(x,y) \geq g$, $S \Subset \mathscr{V}$, $\alpha \in \mathscr{A}^{\{x\}}$, $\beta \in \mathscr{A}^{\{y\}}$, and $\sigma \in \mathscr{A}^S$. Suppose that $[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset$. It suffices to prove that $[\alpha\sigma\beta]^\Omega \neq \emptyset$ and we conclude by applying Lemma 3.3.1. Let $\Gamma = S \cup \{x,y\}$ and take $\omega \in [\alpha\sigma]^\Omega$,

$\upsilon \in [\sigma\beta]^\Omega \neq \emptyset$, and let $\eta_1 = \omega|_\Gamma$, $\eta_3 = \upsilon|_\Gamma$. W.l.o.g., suppose that $\Sigma_\Gamma(\eta_1,\eta_3) = \{x,y\}$. By (2), there exists a configuration $\eta_2 \in \mathscr{L}_\Gamma(\Omega)$ such that $\Sigma_\Gamma(\eta_1,\eta_2) \subseteq \{y\}$ and $\Sigma_\Gamma(\eta_2,\eta_3) \subseteq \{x\}$. Therefore, $\eta_2|_S = \sigma$, $\eta_2|_{\{x\}} = \alpha_1|_{\{x\}} = \alpha$, and $\eta_2|_{\{y\}} = \eta_3|_{\{y\}} = \beta$, so $[\alpha\sigma\beta]^\Omega \neq \emptyset$. Notice that it is important that we took the order $x_1 = y$ and $x_2 = x$ in the previous proof. □

Remark 3. *Property (2) in Proposition 3.3.2 is a stronger version of the* **generalized pivot property** *(see [20]), related with* **connectedness of configurations' spaces** *(see [17]).*

Proposition 3.3.3. *If an invariant n.n. configuration space Ω satisfies SSF, then it satisfies TSSM with gap $g = 2$.*

Proof. Since Ω satisfies SSF, every locally admissible configuration is globally admissible. If we take $g = 2$, for all sets $A,B,S \Subset \mathbb{Z}^d$ such that $\text{dist}(A,B) \geq g$ and for every $\alpha \in \mathscr{A}^A$, $\beta \in \mathscr{A}^B$, and $\sigma \in \mathscr{A}^S$, if $[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset$, in particular we have that $\alpha\sigma$ and $\sigma\beta$ are locally admissible. Since $\text{dist}(A,B) \geq g = 2$, $\alpha\sigma\beta$ must be locally admissible, too. Then, by SSF, $\alpha\sigma\beta$ is globally admissible and, therefore, $[\alpha\sigma\beta]^\Omega \neq \emptyset$. □

3.4 TSSM in $\mathbb{Z}^d$ shift spaces

Definition 3.4.1. *Given a $\mathbb{Z}^d$ shift space Ω and $\Gamma \subseteq \mathbb{Z}^d$, a configuration $\eta \in \mathscr{A}^\Gamma$ is called a* **first offender for** Ω *if $\eta \notin \mathscr{L}(\Omega)$ but $\eta|_S \in \mathscr{L}(\Omega)$, for every $S \subsetneq \Gamma$. We define the* **set of first offenders of** Ω *as*

$$\mathscr{O}(\Omega) := \left\{\eta \in \cup_{\vec{0}\in\Gamma\Subset\mathbb{Z}^d}\mathscr{A}^\Gamma \middle| \eta \text{ is a first offender for } \Omega\right\}. \tag{3.13}$$

Note 2. *When $d = 1$, a similar notion of first offender can be found in [56, Exercise 1.3.8], where it is used to characterize a "minimal" family $\mathfrak{F}$ inducing a $\mathbb{Z}$ SFT Ω.*

Proposition 3.4.1. *Let Ω be a $\mathbb{Z}^d$ shift space. Then Ω satisfies TSSM iff $|\mathscr{O}(\Omega)| < \infty$.*

Proof. First, suppose that Ω satisfies TSSM with gap g, for some $g \in \mathbb{N}$, and take $\eta \in \mathscr{O}(\Omega)$ with shape $\Gamma \Subset \mathbb{Z}^d$ such that $\vec{0} \in \Gamma$. By way of contradiction, assume that

$\operatorname{diam}(\Gamma) \geq g$ and let $x,y \in \Gamma$ be such that $\operatorname{dist}(x,y) = \operatorname{diam}(\Gamma)$. W.l.o.g., assume that $x \neq y$. Then, for $S = \Gamma \setminus \{x,y\} \subsetneq \Gamma$ and since η is a first offender, we have that $[\eta|_{\{x\}}\, \eta|_S]^\Omega, [\eta|_S\, \eta|_{\{y\}}]^\Omega \neq \emptyset$. Since $\operatorname{dist}(x,y) \geq g$, by TSSM, we have that $[\eta]^\Omega = [\eta|_{\{x\}}\, \eta|_S\, \eta|_{\{y\}}]^\Omega \neq \emptyset$, which is a contradiction. Then, $\operatorname{diam}(\Gamma) < g$ and, since $\vec{0} \in \Gamma$, we have that $|\mathscr{O}(\Omega)| < \infty$.

Now, suppose that $|\mathscr{O}(\Omega)| < \infty$ and take

$$g^* = 1 + \max_{\vec{0} \in \Gamma \Subset \mathbb{Z}^d} \left\{ \operatorname{dist}(\vec{0},y) : y \in \Gamma, \mathscr{A}^\Gamma \cap \mathscr{O}(\Omega) \neq \emptyset \right\} < \infty, \tag{3.14}$$

which is well defined. Consider arbitrary $x,y \in \mathbb{Z}^d$ and $S \Subset \mathbb{Z}^d$, with $\operatorname{dist}(x,y) \geq g^*$, and take $\alpha \in \mathscr{A}^{\{x\}}$, $\sigma \in \mathscr{A}^S$, $\beta \in \mathscr{A}^{\{y\}}$ such that $[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset$. W.l.o.g., by shift-invariance, we can take $x = \vec{0}$. Now, by contradiction, assume that $[\alpha\sigma\beta]^\Omega = \emptyset$. Consider a minimal $S' \subseteq S$ such that $[\alpha\, \sigma|_{S'}\, \beta]^\Omega = \emptyset$ but $[\alpha\, \sigma|_{S''}\, \beta]^\Omega \neq \emptyset$, for all $S'' \subsetneq S'$ (this includes the case $S' = \emptyset$, where the condition over S'' is vacuously true). It is direct to check that $\alpha\, \sigma|_{S'}\, \beta$ is a first offender with shape $\Gamma = \{\vec{0}, y\} \cup S'$. Then, since $\operatorname{dist}(\vec{0},y) = \operatorname{dist}(x,y) \geq g^*$, we have a contradiction with the definition of g^*. Therefore, thanks to Lemma 3.3.1, Ω satisfies TSSM with gap g^*. □

Notice that $\Omega = \Omega_{\mathscr{O}(\Omega)}$. Considering this, we have the following corollary.

Corollary 1. *Let Ω be a $\mathbb{Z}^d$ shift space that satisfies TSSM. Then, Ω is a $\mathbb{Z}^d$ SFT.*

Note 3. *If $\Omega \subseteq \mathscr{A}^{\mathbb{Z}^d}$ is a $\mathbb{Z}^d$ shift space that satisfies TSSM with gap g, then it can be checked that Ω is a $\mathbb{Z}^d$ SFT that can be defined by a family of configurations $\mathfrak{F} \subseteq \mathscr{A}^{\mathscr{N}_g}$.*

The next lemma provides another characterization of TSSM for $\mathbb{Z}^d$ n.n. SFTs.

Lemma 3.4.2. *A $\mathbb{Z}^d$ n.n. SFT Ω satisfies TSSM with gap g iff for all $S \subseteq \mathrm{R}_g$,*

$$\forall (\alpha,\sigma,\beta) \in \mathscr{A}^{\{\vec{0}\}} \times \mathscr{A}^S \times \mathscr{A}^{\partial \mathrm{R}_g} : [\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset \implies [\alpha\sigma\beta]^\Omega \neq \emptyset. \tag{3.15}$$

Proof. Let's prove that if Ω satisfies Equation (3.15), then Ω satisfies TSSM with gap g. W.l.o.g., by Lemma 3.3.1 and shift-invariance, consider $x,y \in \mathbb{Z}^d$ with $\operatorname{dist}(x,y) \geq g$, $x = \vec{0}$, $S \Subset \mathbb{Z}^d$, and configurations $\alpha \in \mathscr{A}^{\{x\}}$, $\beta \in \mathscr{A}^{\{y\}}$, and $\sigma \in \mathscr{A}^S$ such that $[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset$. Take $\omega \in [\sigma\beta]^\Omega$ and consider $\alpha' = \alpha$, $\beta' = \omega|_{\underline{\partial}\mathrm{R}_g \setminus S}$, and $\sigma' = \sigma|_{\mathrm{R}_g \cap S}$. Then, $\emptyset \neq [\alpha\sigma]^\Omega \subseteq [\alpha'\sigma']^\Omega$ and $\emptyset \neq [\omega|_{\mathrm{R}_g}]^\Omega \subseteq [\sigma'\beta']^\Omega$, so $[\alpha'\sigma'\beta']^\Omega \neq \emptyset$, by Equation (3.15). Take $\upsilon \in [\alpha'\sigma'\beta']^\Omega$ and notice that $\upsilon|_{\underline{\partial}\mathrm{R}_g} = \omega|_{\underline{\partial}\mathrm{R}_g}$. Then, since Ω is a $\mathbb{Z}^d$ n.n. SFT, we conclude that $\tilde{\omega} = \upsilon|_{\mathrm{R}_g}\,\omega|_{\mathbb{Z}^d \setminus \mathrm{R}_g} \in [\alpha\sigma\beta]^\Omega$, so $[\alpha\sigma\beta]^\Omega \neq \emptyset$ and Ω satisfies TSSM. The converse is immediate. $\square$

3.4.1 Existence of periodic points

Proposition 3.4.3. *Let Ω be a nonempty $\mathbb{Z}^d$ shift space that satisfies TSSM with gap g. Then, Ω contains a periodic point of period $2g$ in every direction.*

Proof. Consider the hypercube $Q = [1,2g]^d \cap \mathbb{Z}^d$. Given $\ell \in \{0,1\}^d$, denote $Q(\ell) = g\ell + \left([1,g]^d \cap \mathbb{Z}^d\right) \subseteq Q$. Notice that $\mathbb{Z}^d = \bigsqcup_{x \in 2g\mathbb{Z}^d} (x+Q)$ and $Q = \bigsqcup_{\ell \in \{0,1\}^d} Q(\ell)$. Then

$$\mathbb{Z}^d = \bigsqcup_{\ell \in \{0,1\}^d} \bigsqcup_{x \in 2g\mathbb{Z}^d} (x + Q(\ell)) = \bigsqcup_{\ell \in \{0,1\}^d} W(\ell), \tag{3.16}$$

where $W(\ell) = \bigsqcup_{x \in 2g\mathbb{Z}^d} (x + Q(\ell))$. Notice that $\operatorname{dist}(x+Q(\ell), y+Q(\ell)) \geq g$, for all $\ell \in \{0,1\}^d$ and $x,y \in 2g\mathbb{Z}^d$ such that $x \neq y$.

Consider $\ell_0, \ell_1, \dots, \ell_{2^d-1}$ an arbitrary order in $\{0,1\}^d$. Let $\alpha_0 \in \mathscr{L}_{Q(\ell_0)}(\Omega)$ and $N \in \mathbb{N}$. By using repeatedly the TSSM property (in particular, strong irreducibility), we can construct a point $\omega_0^N \in \Omega$ with $\sigma_{2gx}(\omega_0^N)|_{Q(\ell_0)} = \alpha_0$, for all x such that $\|x\|_\infty \leq N$. By compactness of Ω, we can take the limit when $N \to \infty$ and obtain a point $\omega_0 \in \Omega$ such that $\sigma_{2gx}(\omega_0)|_{Q(\ell_0)} = \alpha_0$, for all $x \in \mathbb{Z}^d$.

Given $0 \leq k < 2^d - 1$, suppose that there exists a point $\omega_k \in \Omega$ and $\alpha_i \in \mathscr{L}_{Q(\ell_i)}(\Omega)$, for $i = 0,1,\dots,k$, such that $\sigma_{2gx}(\omega_k)|_{Q(\ell_i)} = \alpha_i$, for all $i \in \{0,1,\dots,k\}$ and $x \in \mathbb{Z}^d$. Notice that if $k = 2^d - 1$, the point ω_{2^d-1} is periodic of period $2g$ in every direction. Then, since we have already constructed ω_0, it suffices to prove that we can construct ω_{k+1} from ω_k.

Take $\alpha_{k+1} = \omega_k|_{Q(\ell_{k+1})}$. Notice that $\alpha_{k+1} = \sigma_{-2gx}(\omega_k)|_{2gx+Q(\ell_{k+1})}$ and that $\sigma_{-2gx}(\omega_k)$ has the same property of ω_k, i.e. $\sigma_{2gy}(\sigma_{-2gx}(\omega_k))|_{Q(\ell_i)} = \alpha_i$, for all $i = 0,1,\dots,k$ and $y \in \mathbb{Z}^d$.

Consider an arbitrary enumeration of $\mathbb{Z}^d = \{x_0, x_1, x_2, \dots\}$, with $x_0 = \vec{0}$. Let $\omega_{k+1}^0 = \omega_k$ and suppose that, given $m \in \mathbb{N}$, there is a point ω_{k+1}^m such that

- $\sigma_{2gx_j}(\omega_{k+1}^m)|_{Q(\ell_i)} = \alpha_i$, for all $i \in \{0,1,\dots,k\}$ and $j \in \mathbb{N}$, and

- $\sigma_{2gx_j}(\omega^m_{k+1})\big|_{Q(\ell_{k+1})} = \alpha_{k+1}$, for all $0 \le j \le m$.

Take $A = \bigsqcup_{j=0}^m 2gx_j + Q(\ell_{k+1})$, $B = 2gx_{m+1} + Q(\ell_{k+1})$, and $S = \bigsqcup_{r=0}^k W(\ell_r)$. Notice that $\mathrm{dist}(A,B) \ge g$. Then take $\omega^m_{k+1}\big|_A \in \mathscr{A}^A$, $\sigma_{-2gx_{m+1}}(\omega_k)\big|_B \in \mathscr{A}^B$, and $\omega_k|_S \in \mathscr{A}^S$. We have that $\omega^m_{k+1}\big|_A\ \omega_k|_S$ is globally admissible by the hypothesis of the existence of ω^m_{k+1} and $\omega_k|_S\ \sigma_{-2gx_{m+1}}(\omega_k)\big|_B$ is globally admissible thanks to the observation about $\sigma_{-2gx}(\omega_k)$. Then, $\omega^m_{k+1}\big|_A\ \omega_k|_S\ \sigma_{-2gx_{m+1}}(\omega_k)\big|_B$ is globally admissible by TSSM. Notice that here S is an infinite set and the TSSM property is for finite sets. This is not a problem since we can consider the finite set $S' = S \cap \mathrm{B}_n$ and take the limit $n \to \infty$ for obtaining the desired point, by compactness.

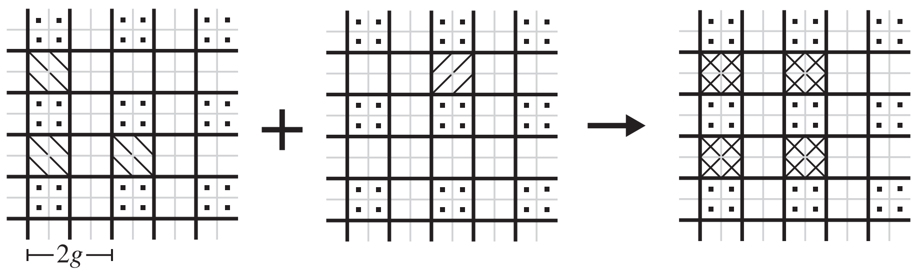

Figure 3.4: Construction of a periodic point using TSSM.

Now, notice that any extension of $\omega^m_{k+1}\big|_A\ \omega_k|_S\ \sigma_{-2gx_{m+1}}(\omega_k)\big|_B$ is a point with the properties of ω^{m+1}_{k+1}. Taking the limit $m \to \infty$, we obtain a point with the properties of ω_{k+1}. Since k was arbitrary, we can iterate the argument until $k = 2^d - 1$, for obtaining the point ω_{2^d-1} which is periodic of period $2g$ in every canonical direction. □

Note 4. *It is known that for $d = 1,2$ a non-empty strongly irreducible $\mathbb{Z}^d$ SFT contains a periodic point. The case $d = 1$ is easy once one knows how to represent a $\mathbb{Z}$ SFT as the space of infinite paths in a finite directed graph. For the case $d = 2$, see [77, 55]. The case $d \ge 3$ is still an open problem.*

Proposition 3.4.4. *Let Ω be a nonempty $\mathbb{Z}^d$ shift space that satisfies TSSM with gap g. Then, Ω contains a periodic point of periods $k_1 + g, \dots, k_d + g$ in the directions $\vec{e}_1, \dots, \vec{e}_d$, respectively, for every $k_i \ge g$. Moreover, the set of periodic points is dense in Ω.*

Proof. We can modify the proof of Proposition 3.4.3, replacing the hypercube Q by $\prod_{i=1}^{d}([1,k_i+g]\cap\mathbb{Z})$ and the sub-hypercube $Q(\ell_0)$ by $\prod_{i=1}^{d}([1,k_i]\cap\mathbb{Z})$. This gives the first part of the statement. For checking density of periodic points, notice that in the proof of Proposition 3.4.3, $\alpha_0 \in \mathscr{L}_{Q(\ell_0)}(\Omega)$ was arbitrary. □

Remark 4. *In particular, any globally admissible finite configuration with shape* $S \subseteq \prod_{i=1}^{d}([1,k_i]\cap\mathbb{Z})$ *with* $k_i \geq g$*, can be embedded in a periodic point of periods* $k_1+g,\ldots,k_d+g$ *in directions* $\vec{e}_1,\ldots,\vec{e}_d$*, respectively.*

It is well known that in the case of $\mathbb{Z}$ SFTs ($d=1$) the mixing hierarchy collapses, i.e. topologically mixing, strongly irreducible, and other intermediate properties, such as *block gluing* and *uniform filling*, are all equivalent (for example, see [14]). In the nearest-neighbour case, we extend this to TSSM.

Proposition 3.4.5. *A* $\mathbb{Z}$ *n.n. SFT* Ω *satisfies TSSM iff it is topologically mixing.*

Proof. We prove that if Ω is topologically mixing, then it satisfies TSSM. The other direction is obvious.

It is known that a topologically mixing $\mathbb{Z}$ n.n. SFT Ω is strongly irreducible with gap $g = g(\{0\},\{0\})$, where $g(\{0\},\{0\})$ is the gap according to Definition 3.2.1. Consider arbitrary $x,y \in \mathbb{Z}$ and $S \Subset \mathbb{Z}\setminus\{x,y\}$ with $\operatorname{dist}(x,y) \geq g$. Take $\alpha \in \mathscr{A}^{\{x\}}$, $\beta \in \mathscr{A}^{\{y\}}$, and $\sigma \in \mathscr{A}^{S}$ with $[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset$. W.l.o.g., by shift-invariance, assume that $x = 0 < y$.

First, consider the open interval (x,y) and suppose that $S\cap(x,y) = \emptyset$. By strong irreducibility, there is $\delta \in \mathscr{L}_{(x,y)\cap\mathbb{Z}}(\Omega)$ such that $[\alpha\delta\beta]^\Omega \neq \emptyset$. Consider $\omega \in [\alpha\sigma]^\Omega$, $\upsilon \in [\sigma\beta]^\Omega$, and $\tau \in [\alpha\delta\beta]^\Omega$. Then, $\omega|_{(-\infty,x]\cap\mathbb{Z}}\,\tau|_{(x,y)\cap\mathbb{Z}}\,\upsilon|_{[y,\infty)\cap\mathbb{Z}} \in [\alpha\sigma\beta]^\Omega$, so $[\alpha\sigma\beta]^\Omega \neq \emptyset$. Now, suppose that $S\cap(x,y) \neq \emptyset$. Take $r \in S\cap(x,y)$ and $\omega \in [\alpha\sigma]$, $\upsilon \in [\sigma\beta]$. Then, $\omega|_{(-\infty,r]\cap\mathbb{Z}}\,\upsilon|_{(r,\infty)\cap\mathbb{Z}} \in [\alpha\sigma\beta]^\Omega$, and $[\alpha\sigma\beta]^\Omega \neq \emptyset$. Finally, we conclude using Lemma 3.3.1. □

3.4.2 Algorithmic results

In general, given a $\mathbb{Z}^d$ n.n. SFT Ω for $d \geq 2$, it is **algorithmically undecidable** to know if a given configuration is in $\mathscr{L}(\Omega)$ or not (see [12, 69]). We present now some algorithmic results related with TSSM. First, a lemma.

Lemma 3.4.6. *Let Ω be a nonempty $\mathbb{Z}^d$ shift space that satisfies strong irreducibility with gap g. Then, for all $A \Subset \mathbb{Z}^d$, $\alpha \in \mathscr{A}^A$, and $\omega \in \Omega$, α is globally admissible iff there exists $\upsilon \in \Omega$ such that*

$$\upsilon|_A = \alpha, \quad \textit{and} \quad \upsilon|_{\mathbb{Z}^d \setminus \mathscr{N}_g(A)} = \omega|_{\mathbb{Z}^d \setminus \mathscr{N}_g(A)}. \tag{3.17}$$

Proof. This is a direct application of the definition of strong irreducibility for the configurations α and $\omega|_{\mathbb{Z}^d \setminus \mathscr{N}_g(A)}$, considering that $\operatorname{dist}\left(A, \mathbb{Z}^d \setminus \mathscr{N}_g(A)\right) \geq g$. □

Corollary 2. *Let $\Omega \subseteq \mathscr{A}^{\mathbb{Z}^d}$ be a nonempty $\mathbb{Z}^d$ n.n. SFT that satisfies TSSM with gap g. Then, there is an algorithm to check whether α belongs to $\mathscr{L}_A(\Omega)$ or not, for every $A \Subset \mathbb{Z}^d$ and $\alpha \in \mathscr{A}^A$, in time $|\mathscr{A}|^{O(|\partial_g A|)}$.*

Proof. By Proposition 3.4.3, there exists a periodic point in Ω of period $2g$ in every direction. Then, by checking all the possible configurations in $\mathscr{A}^{[0,2g]^d}$, we can find a valid periodic point $\overline{\omega}$ in time $O(|\mathscr{A}|^{(2g+1)^d} \cdot d(2g+1)^d|\mathscr{A}|^2)$. Given $\alpha \in \mathscr{A}^A$, by Lemma 3.4.6, we only need to check that α and $\overline{\omega}|_{\partial \mathscr{N}_g(A)}$ can be extended together to a locally admissible configuration on $\overline{\mathscr{N}_g(A)}$. It can be checked in time $O(d|A||\mathscr{A}|^2)$ whether α is locally admissible or not. On the other hand, it can be decided in time $O\left(|\mathscr{A}|^{|\partial_g A|} \cdot d|\partial_g A||\mathscr{A}|^2\right)$ if there exists a configuration $\sigma \in \mathscr{A}^{\partial_g A}$ such that $\alpha\sigma\, \overline{\omega}|_{\partial \mathscr{N}_g(A)}$ is locally admissible. This is enough for deciding if α is globally admissible or not. Thanks to the discrete isoperimetric inequality $|\partial A| \geq 2d|A|^{\frac{d-1}{d}}$ (this follows directly from the discrete Loomis and Whitney inequality [57]), we have that $|A| = |\mathscr{A}|^{O(|\partial A|)}$, and we conclude that the total time of the algorithm is $|\mathscr{A}|^{O(|\partial_g A|)}$. □

Remark 5. *It is worthwhile to point that, when $d \geq 3$, there are no known explicit bounds on the time for checking global admissibility in $\mathbb{Z}^d$ n.n. SFTs that only satisfy strong irreducibility.*

Corollary 3. *Let $\Omega \subseteq \mathscr{A}^{\mathbb{Z}^d}$ be a nonempty $\mathbb{Z}^d$ n.n. SFT that satisfies strong irreducibility with gap g_0. Then, for every $g \geq g_0$, there is an algorithm to check whether Ω satisfies TSSM with gap g or not, in time $e^{O(g^d \log|\mathscr{A}|)}$.*

Proof. Given the set of n.n. constraints $\mathfrak{F}$ such that $\Omega = \Omega_{\mathfrak{F}}$, the algorithm would be the following:

1. Look for the periodic point provided by Proposition 3.4.3. If such point does not exist, then Ω does not satisfy TSSM with gap g. If such point exists, let's denote it by $\overline{\omega}$. (This can be done in time $e^{O(g^d \log|\mathscr{A}|)}$.)

2. Fix a shape $S \subseteq \mathrm{R}_g \setminus \{\vec{0}\}$ and then fix configurations $\alpha \in \mathscr{A}^{\{\vec{0}\}}$, $\beta \in \mathscr{A}^{\underline{\partial}\mathrm{R}_g \setminus S}$, and $\sigma \in \mathscr{A}^S$.

   (a) Using strong irreducibility with gap g_0, check whether $[\alpha\sigma]^\Omega$, $[\sigma\beta]^\Omega$, and $[\alpha\sigma\beta]^\Omega$ are empty or not, by trying to embed $\alpha\sigma$, $\sigma\beta$, and $\alpha\sigma\beta$ in the periodic point $\overline{\omega}$ in a locally admissible way (as in Corollary 2). (This can be done in time $O(|\mathrm{R}_{g+g_0}|)e^{O(|\mathrm{R}_{g+g_0}|\log|\mathscr{A}|)} = e^{O(|\mathrm{R}_g|\log|\mathscr{A}|)}$.)

   (b) If $[\alpha\sigma]^\Omega = \emptyset$ or $[\sigma\beta]^\Omega = \emptyset$, continue.

   (c) If $[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset$, but $[\alpha\sigma\beta]^\Omega = \emptyset$, then Ω does not satisfy TSSM with gap g.

   (d) If all the cylinders are nonempty, continue.

3. If after checking all the configurations we have not found α, σ, and β such that $[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset$, but $[\alpha\sigma\beta]^\Omega = \emptyset$, then Ω satisfies TSSM with gap g (by Lemma 3.4.2).

Then, since $|\mathrm{R}_n| \leq (2n+1)^d$, the total time of this algorithm is $e^{O(g^d \log|\mathscr{A}|)}$. □

3.5 Examples: $\mathbb{Z}^d$ n.n. SFTs

In this chapter we exhibit examples of $\mathbb{Z}^d$ n.n. SFTs which illustrate some of the mixing properties discussed in the previous sections.

3.5.1 A strongly irreducible $\mathbb{Z}^2$ n.n. SFT that is not TSSM

Clearly, the SSF property implies strong irreducibility (a way to see this is through Proposition 3.3.3). As it is mentioned in Example 3.2.2, the q-colourings $\mathbb{Z}^2$ n.n. SFT $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_q)$ satisfies SSF iff $q \geq 5$. For the $\mathbb{Z}^2$ n.n. SFT $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_4)$, given $\delta \in {\mathrm{V}_4}^{\partial\{\vec{0}\}}$ (where $\mathrm{V}_4 := \mathrm{V}(\mathrm{K}_4) = \{1,2,3,4\}$) defined by $\delta(\vec{e}_1) = 1$, $\delta(\vec{e}_2) = 2$, $\delta(-\vec{e}_1) = 3$, and $\delta(-\vec{e}_2) = 4$, there is no $a \in {\mathrm{V}_4}^{\{\vec{0}\}}$ such that $a\delta$ remains locally admissible, so $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_4)$ does not satisfy SSF. However, inspired by the SSF property, we have the following definition.

Definition 3.5.1. *Given $N \in \mathbb{N}$, a $\mathbb{Z}^d$ n.n. SFT Ω satisfies N-**fillability** if, for every locally admissible configuration $\delta \in \mathscr{A}^T$, with $T \subseteq \mathbb{Z}^d \setminus [1,N]^d$, there exists $\alpha \in \mathscr{A}^{[1,N]^d \cap \mathbb{Z}^d}$ such that $\alpha\delta$ is locally admissible.*

Remark 6. *In the previous definition, since Ω is a $\mathbb{Z}^d$ n.n. SFT, it is equivalent to consider δ to have shape $T \subseteq \partial[1,N]^d \cap \mathbb{Z}^d$. In this sense, notice that 1-fillability coincides with the notion of SSF (which only considers locally admissible configurations on $\partial\{\vec{0}\}$).*

Lemma 3.5.1. *The $\mathbb{Z}^2$ n.n. SFT $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_4)$ satisfies 2-fillability.*

Proof. Consider an arbitrary locally admissible configuration $\delta \in \mathrm{V_4}^T$, with $T \subseteq \mathbb{Z}^d \setminus [1,2]^2$. We want to check if there is $\alpha \in \mathrm{V_4}^{[1,2]^2 \cap \mathbb{Z}^2}$ such that $\alpha\delta$ remains locally admissible. W.l.o.g., we can assume that $T = \partial[1,2]^2 \cap \mathbb{Z}^2$, which is the worst case. Given a locally admissible boundary $\delta \in \mathrm{V_4}^{\partial[1,2]^2 \cap \mathbb{Z}^2}$ and $x \in [1,2]^2 \cap \mathbb{Z}^2$, let's denote by W_x^δ the set of values $a \in \mathrm{V_4}^{\{x\}}$ such that $a\delta$ remains locally admissible. Notice that $|W_x^\delta| \geq 2$, for every $x \in [1,2]^2 \cap \mathbb{Z}^2$ and for every such δ. W.l.o.g., assume that $|W_x^\delta| = 2$, $W_{(1,1)}^\delta = \{1,2\}$, and consider $\alpha \in \mathrm{V_4}^{[1,2]^2 \cap \mathbb{Z}^2}$ to be defined.

First, suppose that $W_{(1,1)}^\delta \cap W_{(2,2)}^\delta \neq \emptyset$ or $W_{(2,1)}^\delta \cap W_{(1,2)}^\delta \neq \emptyset$. By the symmetries of $[1,2]^2 \cap \mathbb{Z}^2$ and the constraints, we may assume that $1 \in W_{(1,1)}^\delta \cap W_{(2,2)}^\delta$ and take $\alpha(1,1) = \alpha(2,2) = 1$, $\alpha(2,1) \in W_{(2,1)}^\delta \setminus \{1\}$, and $\alpha(1,2) \in W_{(1,2)}^\delta \setminus \{1\}$. It is easy to check that $\alpha\delta$ is locally admissible.

On the other hand, if $W_{(1,1)}^\delta \cap W_{(2,2)}^\delta = \emptyset$ and $W_{(2,1)}^\delta \cap W_{(1,2)}^\delta = \emptyset$, w.l.o.g. we have that $W_{(1,1)}^\delta = \{1,2\}$ and $W_{(2,2)}^\delta = \{3,4\}$. We consider two cases based on whether a diagonal and off-diagonal coincide or intersect in exactly one element:

- If $W_{(2,1)}^\delta = \{1,2\}$ and $W_{(1,2)}^\delta = \{3,4\}$, we can take $\alpha(1,1) = 1$, $\alpha(2,1) = 2$, $\alpha(1,2) = 3$, and $\alpha(2,2) = 4$.

- If $W_{(2,1)}^\delta = \{1,3\}$ and $W_{(1,2)}^\delta = \{2,4\}$, we can take $\alpha(1,1) = 1$, $\alpha(2,1) = 3$, $\alpha(1,2) = 2$, and $\alpha(2,2) = 4$.

In both cases it can be checked that $\alpha\delta$ is locally admissible. The remaining cases are analogous. $\square$

Definition 3.5.2. *Given $N \in \mathbb{N}$, a set $A \subseteq \mathbb{Z}^d$ is called an* N**-shape** *if it can be written as a union of translations of $[1,N]^d \cap \mathbb{Z}^d$, i.e. if there exists a set $\Gamma \subseteq \mathbb{Z}^d$ such that $A = \bigcup_{x\in\Gamma}\left(x + \left([1,N]^d \cap \mathbb{Z}^d\right)\right)$. A set is called a* **co-N-shape** *if it is the complement of an N-shape. Notice that every shape is a 1-shape and co-1-shape.*

Lemma 3.5.2. *If a $\mathbb{Z}^d$ n.n. SFT Ω satisfies N-fillability then, for any N-shape $A \subseteq \mathbb{Z}^d$ and every locally admissible configuration $\delta \in \mathscr{A}^T$, with $T \subseteq \mathbb{Z}^d \setminus A$, there exists $\alpha \in \mathscr{A}^A$ such that $\alpha\delta$ is locally admissible.*

Proof. Let $A \subseteq \mathbb{Z}^d$ be an N-shape and $\delta \in \mathscr{A}^T$, for $T \subseteq \mathbb{Z}^d \setminus A$, a locally admissible configuration. Consider a minimal $\Gamma \subseteq \mathbb{Z}^d$ such that $A = \bigcup_{x\in\Gamma}\left(x + \left([1,N]^d \cap \mathbb{Z}^d\right)\right)$, in the sense that $\bigcup_{x\in\Gamma'}\left(x + \left([1,N]^d \cap \mathbb{Z}^d\right)\right) \subsetneq A$, for every $\Gamma' \subsetneq \Gamma$. Take an arbitrary $x^* \in \Gamma$. By N-fillability, consider $\beta \in \mathscr{A}^{\left(x^* + \left([1,N]^d \cap \mathbb{Z}^d\right)\right)}$ such that $\beta\delta$ is locally admissible (notice that $T \subseteq \mathbb{Z}^d \setminus \left(x^* + [1,N]^d\right)$.

Now, take the set $A' = \bigcup_{x\in\Gamma\setminus\{x^*\}}\left(x + \left([1,N]^d \cap \mathbb{Z}^d\right)\right)$. Notice that A' is also an N-shape. By minimality of Γ, we have that $\emptyset \neq A \setminus A' \subseteq \left(x^* + \left([1,N]^d \cap \mathbb{Z}^d\right)\right)$. Define $\delta' = \beta|_{A\setminus A'}\delta$ and $T' = (A \setminus A') \cup T$. Then, A' is an N-shape and $\delta' \in \mathscr{A}^{T'}$ is a locally admissible configuration, with $T' \subseteq \mathbb{Z}^d \setminus A'$ as in the beginning, but $A' \subsetneq A$.

Now, given $M \in \mathbb{N}$ and iterating the previous argument, we can always find $\alpha \in \mathscr{A}^{A\cap \mathrm{B}_M}$ such that $\alpha\delta$ is locally admissible. Since M is arbitrary and $\mathscr{A}^{\mathbb{Z}^d}$ is a compact space, then there must exist $\alpha \in \mathscr{A}^A$ such that $\alpha\delta$ is locally admissible. $\square$

Definition 3.5.3. *Given $N \in \mathbb{N}$, a $\mathbb{Z}^d$ shift space Ω is said to be* N**-strongly irreducible with gap** *g if, for any $A, B \Subset \mathbb{Z}^d$ with $\mathrm{dist}(A,B) \geq g$ and such that $A \sqcup B$ is a co-N-shape, we have*

$$\forall(\alpha,\beta) \in \mathscr{A}^A \times \mathscr{A}^B : [\alpha]^\Omega, [\beta]^\Omega \neq \emptyset \implies [\alpha\beta]^\Omega \neq \emptyset. \tag{3.18}$$

Proposition 3.5.3. *If a $\mathbb{Z}^d$ n.n. SFT Ω satisfies N-fillability, then it is N-strongly irreducible with gap $g = 2$.*

Proof. Let $A, B \Subset \mathbb{Z}^d$ with $\mathrm{dist}(A, B) \geq 2$ and such that $A \sqcup B$ is a co-N-shape, and take $\alpha \in \mathscr{A}^A$, $\beta \in \mathscr{A}^B$ such that $[\alpha]^\Omega, [\beta]^\Omega \neq \emptyset$. Then consider $\delta = \alpha\beta \in \mathscr{A}^{A \sqcup B}$ and $W = (A \sqcup B)^{\mathrm{c}}$. Notice that δ is a locally admissible configuration (α and β are globally admissible and $\mathrm{dist}(A, B) \geq 2$), and W is an N-shape. Then, by Lemma 3.5.2, there exists $\eta \in \mathscr{A}^W$ such that $\omega = \eta\delta$ is locally admissible. Then, ω is a locally admissible point (therefore globally admissible) such that $\omega \in [\alpha\beta]^\Omega$. □

Proposition 3.5.4. *If a $\mathbb{Z}^d$ shift space Ω is N-strongly irreducible with gap g, then Ω is strongly irreducible with gap $g + 2N$.*

Proof. Let $A, B \Subset \mathbb{Z}^d$ with $\mathrm{dist}(A, B) \geq g + 2N$, and $\alpha \in \mathscr{A}^A$, $\beta \in \mathscr{A}^B$ such that $[\alpha]^\Omega, [\beta]^\Omega \neq \emptyset$. Consider the partition $\mathbb{Z}^d = \bigsqcup_{x \in N\mathbb{Z}^d} \left(x + \left([1, N]^d \cap \mathbb{Z}^d\right)\right)$, and the sets

$$S_1 := \left\{x \in N\mathbb{Z}^d : \left(x + [1, N]^d\right) \cap A \neq \emptyset\right\}, \tag{3.19}$$

$$S_2 := \left\{x \in N\mathbb{Z}^d : \left(x + [1, N]^d\right) \cap B \neq \emptyset\right\}. \tag{3.20}$$

Consider $A' := \bigsqcup_{x \in S_1} x + \left([1, N]^d \cap \mathbb{Z}^d\right)$ and $B' := \bigsqcup_{x \in S_2} x + \left([1, N]^d \cap \mathbb{Z}^d\right)$. Notice that $A \subseteq A'$ and $B \subseteq B'$. Take $\omega \in [\alpha]^\Omega$ and $\upsilon \in [\beta]^\Omega$, and consider the configurations $\alpha' = \omega|_{A'}$ and $\beta' = \upsilon|_{B'}$. Then, we have that $[\alpha']^\Omega, [\beta']^\Omega \neq \emptyset$, $A' \cup B'$ is a co-N-shape and $\mathrm{dist}(A', B') \geq \mathrm{dist}(A, B) - 2N \geq (g + 2N) - 2N = g$ so, by N-strong irreducibility, we conclude that $\emptyset \neq [\alpha'\beta']^\Omega \subseteq [\alpha\beta]^\Omega$. □

Corollary 4. *If a $\mathbb{Z}^d$ n.n. SFT Ω satisfies N-fillability, then it is strongly irreducible with gap* $2(N+1)$.

Corollary 5. *The $\mathbb{Z}^2$ n.n. SFT* $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_4)$ *is strongly irreducible with gap $g = 6$.*

We have concluded $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_q)$ is strongly irreducible iff $q \geq 4$ (the cases $k = 2, 3$ do not even satisfy the *D-condition* due to the existence of *frozen configurations*; see Definition 5.2.3 and [58]). On the other hand, $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_q)$ satisfies TSSM (in particular, SSF) iff $q \geq 5$. In particular, TSSM fails when $q = 4$, as the next result shows.

Proposition 3.5.5. *The $\mathbb{Z}^2$ n.n. SFT* $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_4)$ *does not satisfy TSSM.*

Proof. Take $g \in \mathbb{N}$, consider the sets $A = \{(-2g,0)\}$, $B = \{(2g,0)\}$, and $S = ([-2g,2g] \cap \mathbb{Z}) \times \{-1,1\}$, and the configurations $\alpha \in \mathrm{V_4}^A$, $\beta \in \mathrm{V_4}^B$, and $\sigma \in \mathrm{V_4}^S$ (see Figure 3.5) defined by $\alpha = 3^A$, $\beta = 4^B$, and

$$\sigma((i,j)) = \begin{cases} 1 & \text{if } (j=1 \text{ and } i \in 2\mathbb{Z}) \text{ or } (j=-1 \text{ and } i \notin 2\mathbb{Z}), \\ 2 & \text{if } (j=1 \text{ and } i \notin 2\mathbb{Z}) \text{ or } (j=-1 \text{ and } i \in 2\mathbb{Z}). \end{cases} \tag{3.21}$$

Then, if we denote $\Omega = \mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_4)$, it can be checked that $[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset$. However, for all $\omega \in [\alpha\sigma]^\Omega$ and $\upsilon \in [\sigma\beta]^\Omega$ we have that $\omega((0,0)) = 3 \neq 4 = \upsilon((0,0))$. Therefore, $[\alpha\sigma\beta]^\Omega = \emptyset$. Since g was arbitrary and $\mathrm{dist}(A,B) = 4g \geq g$, we conclude that Ω does not satisfy TSSM. $\square$

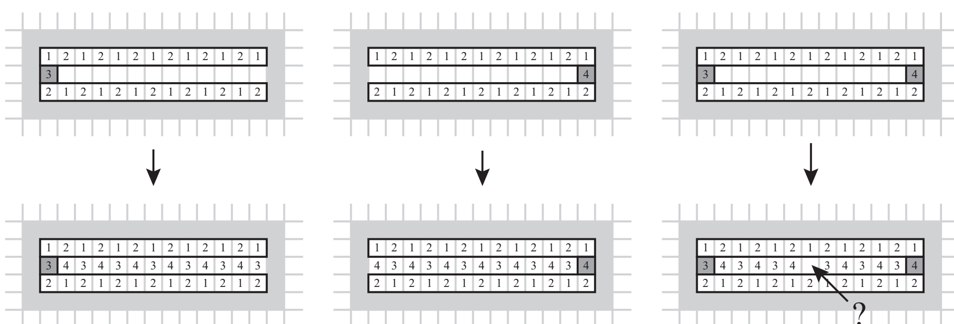

Figure 3.5: Proof that $\Omega = \mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_4)$ does not satisfy TSSM nor SSM.

A by-product of the construction from the previous counterexample is the following result, which also illustrates how TSSM is related with SSM.

Proposition 3.5.6. *Take* $\Omega = \mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_4)$ *and let* π *be any n.n. Gibbs* (Ω,Φ)*-specification. Then,* π *cannot satisfy SSM.*

Proof. Let's suppose that there is a n.n. Gibbs (Ω,Φ)-specification π for $\Omega = \mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_4)$ that satisfies SSM with decay function f. Take $n_0 \in \mathbb{N}$ such that $f(n) < 1$, for all $n \geq n_0$. Consider the set $\Gamma = ([-2n_0+1, 2n_0-1] \cap \mathbb{Z}) \times \{0\} \Subset \mathbb{Z}^2$ and its boundary $\partial\Gamma = ([-2n_0, 2n_0] \cap \mathbb{Z}) \times \{-1,1\} \cup \{(-2n_0,0)\} \cup \{(2n_0,0)\} = A \cup B \cup S$, where A, B, and S are as in Proposition 3.5.5. Take $\delta_1, \delta_2 \in \mathrm{V_4}^{\partial\Gamma}$ defined by $\delta_1|_S = \delta_2|_S = \sigma$ (where σ is also as in Proposition 3.5.5), $\delta_1|_A = \delta_1|_B = 3$, and

$\delta_2|_A = \delta_2|_B = 4$. It is easy to see that δ_1 and δ_2 are both globally admissible and, in particular, there exists $\omega_1, \omega_2 \in \Omega$ such that $\omega_i|_{\partial\Gamma} = \delta_i$, for $i = 1,2$. Now, if we consider the configuration $\beta = 3^{\{(0,0)\}}$, we have that

$$1 = |1-0| = \left|\pi_\Gamma^{\omega_1}(\beta) - \pi_\Gamma^{\omega_2}(\beta)\right| \leq f(2n_0) < 1, \tag{3.22}$$

which is a contradiction. Then, π cannot satisfy SSM. $\square$

Remark 7. *It has been suggested (see [74]) that the uniform Gibbs specification supported on* $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_4)$ *satisfies exponential WSM. Here we have proven that SSM is not possible for any n.n. Gibbs specification supported on* $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_4)$ *and for any rate, not necessarily exponential. The counterexample in Proposition 3.5.6 corresponds to a family of very particular shapes where SSM fails and not what we could call a "common shape" (like* B_n*, for example), but is enough for discarding the possibility of SSM if we stick to its definition. We also have to consider that this family of configurations (and other variations, with different colours and different narrow shapes) can appear as sub-configurations in more general shapes and still produce combinatorial long-range correlations.*

3.5.2 A TSSM $\mathbb{Z}^2$ n.n. SFT that is not SSF

The *Iceberg model* was considered in [19] as an example of a strongly irreducible $\mathbb{Z}^2$ n.n. SFT with multiple measures of maximal entropy. Given $M \geq 2$, and the alphabet $\mathscr{A}_M = \{-M, \dots, -1, +1, \dots, +M\}$, the **Iceberg model** $\mathscr{I}_M$ is defined as

$$\mathscr{I}_M := \left\{ \omega \in \mathscr{A}_M^{\mathbb{Z}^2} : \omega(x) \cdot \omega(x + \vec{e}_i) \geq -1, \text{ for all } x \in \mathbb{Z}^d, i = 1, \dots, d \right\}. \tag{3.23}$$

In the following, we show that for every $M \geq 2$, the Iceberg model satisfies TSSM, but not SSF. In particular, this provides an example of a $\mathbb{Z}^2$ n.n. SFT satisfying TSSM with multiple measures of maximal entropy.

It is easy to see that $\mathscr{I}_M$ does not satisfy SSF, since $+M$ and $-M$ cannot be at distance less than 3. In particular, we can take the configuration $\delta \in \mathscr{A}_M^{\partial\{\vec{0}\}}$ given by $\delta(\vec{e}_1) = \delta(\vec{e}_2) = +M$, and $\delta(-\vec{e}_1) = \delta(-\vec{e}_2) = -M$, which does not remain locally admissible for any $a \in \mathscr{A}_M^{\{\vec{0}\}}$. On the other hand, $\mathscr{I}_M$ satisfies TSSM, as it is shown in the next proposition.

Proposition 3.5.7. *For every $M \geq 2$, the Iceberg model $\mathscr{I}_M$ satisfies TSSM with gap $g = 3$.*

Proof. Consider Lemma 3.3.1 and take $x, y \in \mathbb{Z}^2$ with $\mathrm{dist}(x,y) \geq 3$ and $S \Subset \mathbb{Z}^2$. Given $\alpha \in \mathscr{A}_M^{\{x\}}$, and $\beta \in \mathscr{A}_M^{\{y\}}$, and $\sigma \in \mathscr{A}_M^S$, suppose that $[\alpha\sigma]^{\mathscr{I}_M}, [\sigma\beta]^{\mathscr{I}_M} \neq \emptyset$. Next, take $\omega \in [\sigma\beta]^{\mathscr{I}_M}$ and define a new point υ given by

$$\upsilon(z) = \begin{cases} \omega(z) & \text{if } z \in S \cup \{y\}, \\ +1 & \text{if } z \in (S \cup \{y\})^{\mathrm{c}} \text{ and } \omega(x) \in \{+1, \ldots, +M\}, \\ -1 & \text{if } z \in (S \cup \{y\})^{\mathrm{c}} \text{ and } \omega(x) \in \{-M, \ldots, -1\}. \end{cases} \tag{3.24}$$

It is not hard to see that υ is a valid point in $[\sigma\beta]^{\mathscr{I}_M}$. Now, let's construct a point $\tau \in [\alpha\sigma\beta]^{\mathscr{I}_M}$ from υ.

Case 1: $\alpha(x) = \pm 1$. W.l.o.g., suppose that $\alpha(x) = +1$. Now, since $[\alpha\sigma]^{\mathscr{I}_M} \neq \emptyset$, all the values of $\upsilon|_{\partial\{x\} \cap S}$ must belong to $\{-1, +1, \ldots, +M\}$. On the other hand, since $\partial\{x\} \setminus S \subseteq (S \cup \{y\})^{\mathrm{c}}$, all the values in $\upsilon|_{\partial\{x\} \setminus S}$ belong to $\{-1, +1\}$. Then, all the values in $\upsilon|_{\partial\{x\}}$ belong to $\{-1, +1, \ldots, +M\}$, and we can replace $\upsilon|_{\{x\}}$ by $+1$ in order to get a valid point τ from υ such that $\tau \in [\alpha\sigma\beta]^{\mathscr{I}_M}$.

Case 2: $\alpha(x) \neq \pm 1$. W.l.o.g., suppose that $\alpha(x) = +M$. Then, all the values in $\upsilon|_{\partial\{x\} \cap S}$ belong to $\{+1, \ldots, +M\}$. We claim that we can switch every -1 in $\partial\{x\} \setminus S$ to a $+1$. If it is not possible to do this for some site $x^* \in \partial\{x\} \setminus S$, then its neighbourhood $\partial\{x^*\}$ contains a site with value in $\{-M, \ldots, -2\}$ and, in particular, different from $+1$ and -1. Then, $\partial\{x^*\}$ necessarily intersects S (and not $\{y\}$, because $\mathrm{dist}(x,y) \geq 3$). Then, a site in $\partial\{x^*\} \cap S \neq \emptyset$ is fixed to some value in $\{-M, \ldots, -2\}$ and then the site x^* must take a value in $\{-M, \ldots, -1\}$, given σ. Therefore, x cannot take a value in $\{+2, \ldots, +M\}$, contradicting the fact that $[\alpha\sigma]^{\mathscr{I}_M} \neq \emptyset$. Therefore, we can set all the values in $\upsilon|_{\partial\{x\} \setminus S}$ to $+1$. Let's call that point υ'. Finally, if we replace $\upsilon'(x) = +1$ by $+M$, we obtain a valid point τ from υ' such that $\tau \in [\alpha\sigma\beta]^{\mathscr{I}_M}$.

Then, we conclude that $\mathscr{I}_M$ satisfies TSSM with gap $g = 3$, for every $M \geq 2$. □

Remark 8. *In particular, Proposition 3.5.7 provides an alternative way of checking* *the well-known fact that $\mathscr{I}_M$ is strongly irreducible.*

3.5.3 Arbitrarily large gap, arbitrarily high rate

Notice that the Iceberg model can be regarded as a $\mathbb{Z}^d$ shift space where two types (positives and negatives) coexist separated by a boundary of ± 1s. Now we introduce a variation of the Iceberg model that extends the idea of two types coexisting to an arbitrary number of them. First, we will see that this variation gives a family of $\mathbb{Z}^d$ n.n. SFTs satisfying TSSM with gap g but not $g-1$, for arbitrary $g \in \mathbb{N}$. Second, these models admits the existence of a n.n. Gibbs specification satisfying exponential SSM with arbitrarily high decay rate, showing in particular (as far as we know, for the first time) that there are systems that satisfy SSM and TSSM, without satisfying any of the other stronger combinatorial mixing properties, like having a safe symbol or satisfying SSF.

Given $g, d \in \mathbb{N}$, consider the alphabet $\mathscr{A}_g = \{0, 1, \ldots, g\}$ and the $\mathbb{Z}^d$ n.n. SFT defined by

$$\Omega_g^d := \left\{ \omega \in \mathscr{A}_g^{\mathbb{Z}^d} : |\omega(x) - \omega(x + \vec{e}_i)| \leq 1, \text{ for all } x \in \mathbb{Z}^d, i = 1, \ldots, d \right\}. \quad (3.25)$$

Notice that $\Omega_0^d = \left\{ 0^{\mathbb{Z}^d} \right\}$ (a fixed point) and $\Omega_1^d = \mathscr{A}_1^{\mathbb{Z}^d}$ (a 2 symbols full shift), so both satisfy TSSM with gap $g = 0$ and $g = 1$, respectively. Also, notice that 1 is a safe symbol for Ω_2^d (and therefore, by Proposition 3.3.3, it satisfies TSSM with gap $g = 2$).

Proposition 3.5.8. *The $\mathbb{Z}^d$ n.n. SFT Ω_g^d satisfies TSSM with gap g but not $g-1$.*

Proof. First, let's see that Ω_g^d does not satisfy TSSM with gap $g-1$. In fact, recall that TSSM with gap $g-1$ implies strong irreducibility with the same gap. However, if we consider two configurations on single sites with values 0 and g, respectively, they cannot appear in the same point if they are separated by a distance less or equal to $g-1$, since the value of consecutive sites can only increase or decrease by at most 1. Therefore, Ω_g^d is not TSSM with gap $g-1$.

In order to prove that Ω_g^d satisfies TSSM with gap g, by way of Lemma 3.3.1, let $x, y \in \mathbb{Z}^d$ with $\operatorname{dist}(x, y) \geq g$, and $S \Subset \mathbb{Z}^d \backslash \{x, y\}$. Given $\alpha \in \mathscr{A}_g^{\{x\}}$, $\beta \in \mathscr{A}_g^{\{y\}}$, and $\sigma \in \mathscr{A}_g^S$, suppose that $[\alpha\sigma]^{\Omega_g^d}, [\sigma\beta]^{\Omega_g^d} \neq \emptyset$. We want to prove that $[\alpha\sigma\beta]^{\Omega_g^d} \neq \emptyset$.

Since $[\sigma\beta]^{\Omega_g^d} \neq \emptyset$, we can consider a point $\omega \in [\sigma\beta]^{\Omega_g^d}$. If $\omega(x) = \alpha(x)$, we are done. W.l.o.g., suppose that $\omega(x) < \alpha(x)$ (the case $\omega(x) > \alpha(x)$ is analogous). We proceed by finding a *valid* point ω' such that $\omega'|_S = \sigma$, $\omega'(y) = \beta(y)$, and $\omega'(x) = \omega(x)+1$. Iterating this process $|\alpha(x) - \omega(x)|$ times, we conclude. Notice that the only obstruction to an increase by 1 of $\omega(x)$ are the values of neighbours of x strictly below $\omega(x)$.

We introduce an auxiliary **digraph of descending paths** $\mathcal{D}_{\omega,x} = (\mathcal{V}^g_{\omega,x}, \mathcal{E}^g_{\omega,x})$, where $\mathcal{V}^0_{\omega,x} = \{x\}$, $\mathcal{E}^0_{\omega,x} = \emptyset$, and, for $n \geq 1$,

$$\mathcal{V}^{n+1}_{\omega,x} = \mathcal{V}^n_{\omega,x} \cup \bigcup_{\substack{y' \in \partial \mathcal{V}^n_{\omega,x}, \\ \omega(y') = \omega(x) - n}} \{y'\}, \tag{3.26}$$

$$\mathcal{E}^{n+1}_{\omega,x} = \mathcal{E}^n_{\omega,x} \cup \bigcup_{\substack{y' \in \partial \mathcal{V}^n_{\omega,x}, \\ \omega(y') = \omega(x) - n}} \bigcup_{\substack{x' \in \mathcal{V}^n_{\omega,x}, \\ x' \sim y'}} \left\{(x', y')\right\}. \tag{3.27}$$

Notice that, since $\omega(x) < g$, the recurrence stabilizes for some $n < g$, i.e. $\mathcal{V}^n_{\omega,x} = \mathcal{V}^{g-1}_{\omega,x}$ and $\mathcal{E}^n_{\omega,x} = \mathcal{E}^{g-1}_{\omega,x}$, for every $n \geq g$. In particular, the sites that $\mathcal{D}_{\omega,x}$ reaches are sites at distance at most $g-1$ from x, and the site y cannot belong to the graph. Now, suppose that a site from S belongs to $\mathcal{D}_{\omega,x}$. If that is the case, the value at x of any point in $[\sigma]^{\Omega_g^d}$ would be forced to be at most $\omega(x)$ (since the graph is strictly decreasing from x to S), which contradicts the fact that $[\alpha\sigma]^{\Omega_g^d} \neq \emptyset$.

Then neither y nor any element of S belongs to $\mathcal{D}_{\omega,x}$, so if we modify the values of $\mathcal{D}_{\omega,x}$ in a valid way, we will still obtain a valid point ω' such that $\omega'|_S = \sigma$ and $\omega'(y) = \beta(y)$. Now, take the set $D = \mathcal{V}^g_{\omega,x} \subseteq \mathbb{Z}^d$ and consider the point ω' such that

$$\omega'\big|_D = \omega'\big|_D + 1^D, \quad \text{and} \quad \omega'\big|_{\mathbb{Z}^d \setminus D} = \omega|_{\mathbb{Z}^d \setminus D}, \tag{3.28}$$

where $\omega(D) + 1^D$ represents the configuration obtained from $\omega|_D$ after adding 1 in every site. We claim that ω' is a valid point. To see this, we only need to check that the difference between values in the endpoints of an arbitrary edge is at most 1. If both endpoints are in D or in $\mathbb{Z}^d \setminus D$, it is clear that the edge is valid since the original point ω was a valid point, and adding 1 to both endpoints does not affect the difference. If one endpoint is in $x_1 \in D$ and the other one is in $x_2 \in \mathbb{Z}^d \setminus D$, then $\omega(x_1) \leq \omega(x_2)$, necessarily (if not, $\omega(x_1) > \omega(x_2)$, and x_2 would be part of the digraph of descending paths). Since $|\omega(x_1)-\omega(x_2)| \leq 1$ and $\omega(x_1) \leq \omega(x_2)$, then $\omega(x_2)-\omega(x_1) \in \{0,1\}$. Therefore, $\omega'(x_1)-\omega'(x_2) = (\omega(x_1)+1)-\omega(x_2) = 1-(\omega(x_2)-\omega(x_1)) \in \{1,0\}$, so $|\omega'(x_1)-\omega'(x_2)| \leq 1$. Then, we conclude that $\omega' \in \Omega_g^d$ and $\omega'|_S = \sigma$, $\omega'(y) = \beta(y)$, and $\omega'(x) = \omega(x)+1$, as we wanted. □

The next result is a special case of a more general result, whose full proof will be given in the following chapters.

Proposition 3.5.9. *For any $g,d \in \mathbb{N}$ and $\gamma > 0$, there exists a n.n. Gibbs (Ω_g^d,Φ)-specification that satisfies exponential SSM with decay rate γ.*

Proof. This result follows from Proposition 4.4.2, Corollary 7, Proposition 4.6.2, and from noticing that Ω_g^d is a homomorphism space of the form $\mathrm{Hom}(Z^d,T)$, with T a looped tree. □

Note 5. *The ingredients of the proof of Proposition 3.5.9 can be easily adapted to the $\mathbb{Z}^d$ hard-core lattice gas model case.*

3.6 Relationship between spatial and combinatorial mixing properties

In this section we establish some connections between spatial and combinatorial mixing properties. In particular, we show that TSSM is a property that arises naturally for the support of Gibbs specifications satisfying (exponential) SSM, at least when the decay rate is high enough.

3.6.1 SSM criterion

Let Ω be a $\mathbb{Z}^d$ n.n. SFT and Φ a shift-invariant n.n. interaction. For the n.n. Gibbs (Ω,Φ)-specification π, we define

$$Q(\pi) := \max_{\omega_1,\omega_2 \in \Omega} d_{\mathrm{TV}}\left(\pi_{\{\vec{0}\}}^{\omega_1}, \pi_{\{\vec{0}\}}^{\omega_2}\right). \tag{3.29}$$

In the following, let $p_c(\mathbb{Z}^d)$ to denote the **critical value** of *Bernoulli site percolation* on $\mathbb{Z}^d$ (see [41]). The following result is essentially in [11].

Theorem 3.6.1. *Let Ω be a $\mathbb{Z}^2$ n.n. SFT, Φ a shift-invariant n.n. interaction, and π the corresponding n.n. Gibbs (Ω,Φ)-specification. Suppose that Ω has a safe symbol. If $Q(\pi) < p_c(\mathbb{Z}^2)$, then π satisfies exponential SSM.*

Proof. Take $\mu \in \mathcal{G}(\pi)$. Since Ω has a safe symbol, μ is fully supported, i.e. $\mathrm{supp}(\mu) = \Omega$ (very special case of [70, Remark 1.14]). Given a $\mathbb{Z}^2$-MRF μ, define

$$Q(\mu) := \max_{\delta_1,\delta_2} d_{\mathrm{TV}}\left(\mu(\cdot|\delta_1)|_{\{\vec{0}\}}, \mu(\cdot|\delta_2)|_{\{\vec{0}\}}\right), \tag{3.30}$$

where δ_1 and δ_2 range over all configurations on $\partial\{\vec{0}\}$ such that $\mu(\delta_1)\mu(\delta_2) > 0$. Then, $Q(\mu) \leq Q(\pi) < p_c(\mathbb{Z}^2)$, so by [11, Theorem 1] and shift-invariance of Φ, μ satisfies exponential SSM as a MRF (see [59, Theorem 3.10]). Finally, since μ is fully supported, we can conclude that π satisfies exponential SSM. $\square$

3.6.2 Uniform bounds of conditional probabilities

We relate TSSM is closely related with bounds on elements of Gibbs specifications satisfying SSM. We have the following lemma.

Lemma 3.6.2. *Let $\mathcal{G}$ be a board of bounded degree Δ and π a Gibbs (Ω,Φ)-specification. Then, for every $\omega \in \Omega$ and $A \Subset \mathcal{V}$,*

$$\pi_A^{\omega}(\omega|_A) \geq \exp(-4\Delta|A|(\Phi_{\max} + \log|\mathscr{A}|)). \tag{3.31}$$

Proof. Notice that

$$\mathscr{H}_{A,\omega}^{\Phi}(\omega|_A) = \sum_{x \in A\cup\partial A} \Phi(\omega|_{\{x\}}) + \sum_{\{x,y\}\in\mathscr{E}[A\cup\partial A]:\{x,y\}\cap A\neq\emptyset} \Phi(\omega|_{\{x,y\}}) \tag{3.32}$$

$$\leq (|A| + \Delta|A|)\Phi_{\max} \leq 2\Delta|A|\Phi_{\max}, \tag{3.33}$$

where we use that $|\{\mathscr{E}[A\cup\partial A] : \{x,y\}\cap A \neq \emptyset\}| \leq \Delta|A|$. Similarly, $\mathscr{H}_{A,\omega}^{\Phi}(\omega|_A) \geq -2\Delta|A|\Phi_{\max}$, so $\mathrm{Z}_{A,\omega}^{\Phi} \leq |\mathscr{A}|^{|A|}\exp(2\Delta|A|\Phi_{\max})$. Therefore,

$$\pi_{A,\omega}^{\Phi}(\omega|_A) = \frac{1}{\mathrm{Z}_{A,\omega}^{\Phi}}\exp(-\mathscr{H}_{A,\omega}^{\Phi}(\omega|_A)) \tag{3.34}$$

$$\geq |\mathscr{A}|^{-|A|}\exp(-2\Delta|A|\Phi_{\max})\exp(-2\Delta|A|\Phi_{\max}) \tag{3.35}$$

$$\geq \exp(-4\Delta|A|(\Phi_{\max} + \log|\mathscr{A}|)). \tag{3.36}$$

□

Given a a Gibbs (Ω, Φ)-specification π, define the function $c_\pi : \Omega \to [0,1]$ given by

$$c_\pi(\omega) := \inf_{A \Subset \mathscr{V}} \inf_{x \in A} \pi_A^\omega(\omega|_{\{x\}}), \tag{3.37}$$

for $\omega \in \Omega$, and denote

$$\underline{c}_\pi := \inf_{\omega \in \Omega} c_\pi(\omega) \in [0,1]. \tag{3.38}$$

Notice that for a board $\mathscr{G}$ of bounded degree Δ we have that, for every $x \in \mathscr{V}$ and $g \in \mathbb{N}$, $|\mathscr{N}_g(x)| \leq \Delta^{g+1}$. Considering this, we have the next proposition.

Proposition 3.6.3. *Let π be a Gibbs (Ω, Φ)-specification, with $\mathscr{G}$ any board of bounded degree Δ. Suppose that Ω satisfies TSSM with gap g. Then,*

$$\underline{c}_\pi \geq \exp(-5\Delta^{1+g}(\Phi_{\max} + \log|\mathscr{A}|)) > 0. \tag{3.39}$$

Proof. Consider a point $\omega \in \Omega$, a set $A \Subset \mathscr{V}$, and a site $x \in A$. W.l.o.g., suppose that A is connected. If not, let C_x be the connected component of A containing x, and notice that $\pi_A^\omega(\omega|_{\{x\}}) = \pi_{C_x}^\omega(\omega|_{\{x\}})$.

Define the set $A_{g,x}$ to be the connected component of $A \cap \mathscr{N}_{g-1}(x)$ containing x, and let $B := A \cap \partial A_{g,x}$. Notice that $B \subseteq \mathscr{N}_g(x)$ and $|\mathscr{N}_g(x)| \leq \Delta^{g+1}$. First, assume that $B = \emptyset$. If this is the case, then $\partial A_{g,x} \subseteq A^c$, so $A \subseteq \mathscr{N}_{g-1}(x)$. Therefore, by Lemma 3.6.2,

$$\begin{aligned} \pi_A^\omega(\omega|_{\{x\}}) &\geq \pi_{A,\omega}^\Phi(\omega|_A) && (3.40)\\ &\geq \exp(-4\Delta|A|(\Phi_{\max} + \log|\mathscr{A}|)) && (3.41)\\ &\geq \exp(-4\Delta^{1+g}(\Phi_{\max} + \log|\mathscr{A}|)). && (3.42) \end{aligned}$$

where have used that $|A| \leq |\mathscr{N}_{g-1}(x)| \leq \Delta^g$.

On the other hand, suppose that $B \neq \emptyset$. By a counting argument, there must exist $\beta \in \mathscr{A}^B$ such that $\pi_A^\omega(\beta) \geq |\mathscr{A}|^{-|B|}$. In particular, this says that $[\beta\omega|_{A^c}]^\Omega \neq \emptyset$. Since $[\omega|_{A^c}\ \omega|_{\{x\}}]^\Omega \neq \emptyset$ and $\operatorname{dist}(x,B) \geq g$, we have $[\beta\ \omega|_{A^c}\ \omega|_{\{x\}}]^\Omega \neq \emptyset$, by TSSM.

Now, take $\omega' \in [\beta\ \omega|_{A^c}\ \omega|_{\{x\}}]^\Omega$. Then, by the MRF property,

$$\begin{aligned}
\pi_A^\omega(\omega|_{\{x\}}) &\geq \pi_A^\omega(\omega'|_{A_{g,x}}) && (3.43)\\
&\geq \pi_A^\omega(\omega'|_{A_{g,x}} \mid \omega'|_B)\pi_A^\omega(\omega'|_B) && (3.44)\\
&= \pi_{A_{g,x}}^{\omega'}(\omega'|_{A_{g,x}})\pi_A^\omega(\beta) && (3.45)\\
&\geq \exp(-4\Delta|A_{g,x}|(\Phi_{\max} + \log|\mathscr{A}|))|\mathscr{A}|^{-|B|}. && (3.46)
\end{aligned}$$

Notice that $|A_{g,x}| \leq \Delta^g$ and $|B| \leq \Delta^g$, so

$$\pi_A^\omega(\omega|_{\{x\}}) \geq e^{-4\Delta^{1+g}(\Phi_{\max}+\log|\mathscr{A}|)}|\mathscr{A}|^{-\Delta^g} \geq e^{-5\Delta^{1+g}(\Phi_{\max}+\log|\mathscr{A}|)}. \tag{3.47}$$

Since ω, A, and x were all arbitrary, we conclude. □

Proposition 3.6.4. *Let π be a Gibbs (Ω,Φ)-specification that satisfies SSM with decay function f and such that $\underline{c}_\pi > 0$. Then, Ω satisfies TSSM with gap $g = \min\{n \in \mathbb{N} : f(n) < \underline{c}_\pi\}$.*

Proof. Take $n_0 \in \mathbb{N}$ such that $f(n) < \underline{c}_\pi$, for all $n \geq n_0$ (recall that a decay function is decreasing). We claim that Ω satisfies TSSM with gap n_0. By contradiction, and considering Lemma 3.3.1, suppose that there exist $x,y \in \mathscr{V}$ with $\mathrm{dist}(x,y) \geq n_0$ and $S \Subset \mathscr{V}$, $\alpha \in \mathscr{A}^{\{x\}}$, $\beta \in \mathscr{A}^{\{y\}}$, and $\sigma \in \mathscr{A}^S$ such that $[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset$, but $[\alpha\sigma\beta]^\Omega = \emptyset$. Take $\omega_1 \in [\alpha\sigma]^\Omega$ and $\omega_2 \in [\sigma\beta]^\Omega$. Notice that $\omega_1 \neq \omega_2$ and $\omega_1|_S = \omega_2|_S = \sigma$. Consider the set $A = \mathscr{N}_{n_0}(x) \setminus S$. We have that $\pi_A^{\omega_1}(\alpha) \geq \underline{c}_\pi$ and $\pi_A^{\omega_2}(\alpha) = 0$. Then,

$$\begin{aligned}
\underline{c}_\pi \leq \pi_A^{\omega_1}(\alpha) &= |\pi_A^{\omega_1}(\alpha) - \pi_A^{\omega_2}(\alpha)| && (3.48)\\
&\leq f(\mathrm{dist}(x, \Sigma_{\partial A}(\omega_1,\omega_2))) \leq f(n_0) < \underline{c}_\pi, && (3.49)
\end{aligned}$$

which is a contradiction. □

Notice that no assumption on the degree of the board was necessary in the proof of Proposition 3.6.4.

Corollary 6. *Let π be Gibbs (Ω,Φ)-specification that satisfies SSM, with $\mathscr{G}$ a board of bounded degree. Then, Ω satisfies TSSM iff $\underline{c}_\pi > 0$.*

Proof. The result follows directly from Proposition 3.6.3 and Proposition 3.6.4. □

3.6.3 SSM with high decay rate

We have the following theorem.

Theorem 3.6.5. *Let $\Omega \subseteq \mathscr{A}^{\mathbb{Z}^2}$ be a $\mathbb{Z}^2$ n.n. SFT and let π be a n.n. Gibbs (Ω,Φ)-specification that satisfies exponential SSM with decay function $f(n) = Ce^{-\gamma n}$, where $\gamma > 4\log|\mathscr{A}|$. Then, Ω satisfies TSSM with gap*

$$g = \min\left\{n \in \mathbb{N} : \gamma - 4\log|\mathscr{A}| > \frac{1}{n}\log(4Cn)\right\}. \tag{3.50}$$

Proof. We will prove that $\underline{c}_\pi > 0$ and then conclude by Proposition 3.6.4. Take $\omega \in \Omega$, $A \Subset \mathbb{Z}^2$, and $x \in A$. Our goal is to bound $\pi_A^\omega(\omega|_{\{x\}})$ away from zero, uniformly in ω, A, and x.

Let $n \in \mathbb{N}$ be such that

$$\gamma - 4\log|\mathscr{A}| > \frac{1}{n}\log(4Cn), \tag{3.51}$$

and define $A_{n-1,x}$ to be the connected component of $A \cap \mathscr{N}_{n-1}(x)$ containing x, and $B := A \cap \partial A_{n-1,x}$. Notice that $B \subseteq A \cap \partial\mathscr{N}_{n-1}(x)$, $|A_{n-1,x}| \leq |\mathscr{N}_{n-1}(x)| \leq 2n^2$, and $|B| \leq |\partial\mathscr{N}_{n-1}(x)| = 4n$.

If $B = \emptyset$, then $\partial A_{n-1,x} \subseteq A^{\mathrm{c}}$, and by Lemma 3.6.2,

$$\begin{aligned}\pi_A^\omega(\omega|_{\{x\}}) &= \pi_{A_{n-1,x}}^\omega(\omega|_{\{x\}}) &(3.52)\\ &\geq \exp(-4\Delta|A_{n-1,x}|(\Phi_{\max} + \log|\mathscr{A}|)) &(3.53)\\ &\geq \exp(-32n^2(\Phi_{\max} + \log|\mathscr{A}|)). &(3.54)\end{aligned}$$

Now, let's suppose that $B \neq \emptyset$. By a counting argument, there must exist $\beta \in \mathscr{A}^B$ such that $\pi_A^\omega(\beta) \geq |\mathscr{A}|^{-|B|}$ and, in particular, $[\omega|_{A^{\mathrm{c}}}\beta]^\Omega \neq \emptyset$.

By contradiction, let's suppose that $[\omega|_{\{x\}}\,\omega|_{A^{\mathrm{c}}}\beta]^\Omega = \emptyset$. Then, $\pi_{A\setminus\{x\}}^\omega(\beta) = 0$. On the other hand, since $\pi_A^\omega(\beta) \geq |\mathscr{A}|^{-|B|}$, by taking weighted averages over configurations on $\{x\}$, there must exist $\alpha \in \mathscr{A}^{\{x\}}$ such that $\pi_A^\omega(\beta|\alpha) = \pi_{A\setminus\{x\}}^\upsilon(\beta) \geq$

$|\mathscr{A}|^{-|B|}$, for $\upsilon \in [\alpha\, \omega|_{A^c}\, \beta]^{\Omega} \neq \emptyset$. Notice that $x \notin B$ and $\mathrm{dist}(B, \Sigma_{\partial A \setminus \{x\}}(\omega, \upsilon)) \leq \mathrm{dist}(B, x) = n$. Then, we have that

$$|\mathscr{A}|^{-|B|} \leq \left|\pi^{\omega}_{A \setminus \{x\}}(\beta) - \pi^{\upsilon}_{A \setminus \{x\}}(\beta)\right| \leq |B| C e^{-\gamma n}, \tag{3.55}$$

by the MRF and SSM properties. Since $B \subseteq \partial \mathscr{N}_{n-1}(x)$, then $|B| \leq |\partial \mathscr{N}_{n-1}(x)|$, and

$$|\mathscr{A}|^{-|\partial \mathscr{N}_{n-1}(x)|} \leq |\mathscr{A}|^{-|B|} \leq |B| C e^{-\gamma n} \leq |\partial \mathscr{N}_{n-1}(x)| C e^{-\gamma n}. \tag{3.56}$$

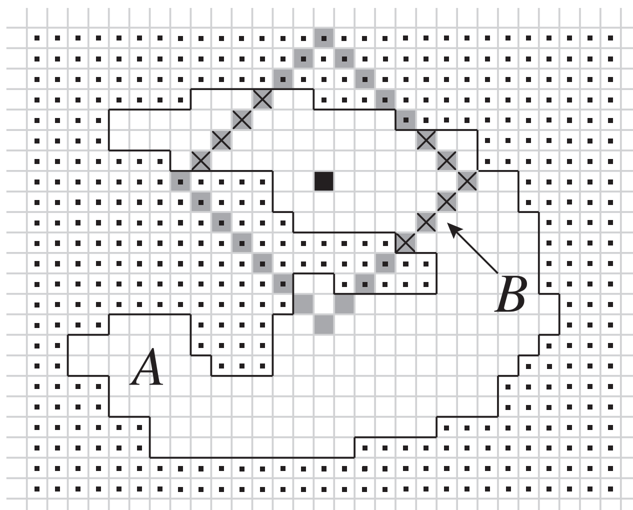

Figure 3.6: Representation of the proof of Theorem 3.6.5.

By taking logarithms, $-4n \log |\mathscr{A}| \leq \log(4n) + \log C - \gamma n$, so

$$\gamma \leq \frac{1}{n} \log(4Cn) + 4 \log |\mathscr{A}| = 4 \log |\mathscr{A}| + o(1), \tag{3.57}$$

which is a contradiction with the fact that $\gamma > 4 \log |\mathscr{A}|$ for sufficiently large n (notice that the difference between γ and $4 \log |\mathscr{A}|$ determines the size of $|B|$ and its distance to x). We conclude that $\left[\omega|_{\{x\}}\, \omega|_{A^c}\, \beta\right]^{\Omega} \neq \emptyset$. Therefore, by considering $\omega' \in [\omega|_{\{x\}}\, \omega|_{A^c}\, \beta]^{\Omega} \neq \emptyset$ and repeating the argument in the proof of Proposition

3.6.3, we have that

$$\pi_A^{\omega}(\omega|_{\{x\}}) \geq \pi_{A_{n,x}}^{\omega'}(\omega'|_{A_{n,x}})\pi_A^{\omega}(\beta) \tag{3.58}$$
$$\geq \exp(-4\Delta|A_{n,x}|(\Phi_{\max} + \log|\mathscr{A}|))|\mathscr{A}|^{-|B|} \tag{3.59}$$
$$\geq \exp(-32n^2(\Phi_{\max} + \log|\mathscr{A}|))|\mathscr{A}|^{-4n}. \tag{3.60}$$

Since this lower bound is positive and independent of ω, A, and x, taking the infimum we have that $\underline{c}_\pi \geq \exp(-32n^2(\Phi_{\max} + \log|\mathscr{A}|))|\mathscr{A}|^{-4n} > 0$ and, by Corollary 6, we conclude that Ω exhibits TSSM. □

Notice that Proposition 3.5.9 gives us an alternative way to prove TSSM for Ω_g^2, since the decay rate γ can be arbitrarily large (in particular, larger than $4\log(g+1)$) and Theorem 3.6.5 applies.

Remark 9. *Recall that TSSM implies strong irreducibility, so in view of the preceding result SSM with high exponential rate implies strong irreducibility. In general, it is not known whether SSM implies strong irreducibility.*

Note 6. *If π were a $\mathbb{Z}^d$ Gibbs (Ω,Φ)-specification satisfying SSM with decay function $f(n) = Ce^{-\gamma n^{d-1}}$, we could modify the previous proof to conclude that Ω exhibits TSSM for sufficiently large decay rate γ. The reason why exponential SSM is not enough in this proof for an arbitrary d, is that only in $\mathbb{Z}^2$ the boundary of a ball grows linearly with the radius. Consequently, the previous proof should work in any board where the boundary of neighbourhoods grows linearly with the radius, probably under a change of the bound for the decay rate γ.*

Chapter 4

SSM in homomorphism spaces

Given a board $\mathscr{G}$, a constraint graph H, and a constrained energy function ϕ, we are interested in studying Gibbs $(\mathrm{Hom}(\mathscr{G},\mathrm{H}),\Phi)$-specifications, where Φ denotes is the induced n.n. interaction. Whenever we talk about a Gibbs $(\mathscr{G},\mathrm{H},\phi)$-specification, we will understand that it is in the previous sense. Also, when talking about cylinder sets $[\alpha]^{\Omega}$, for $\Omega = \mathrm{Hom}(\mathscr{G},\mathrm{H})$, we will sometimes denote them by $[\alpha]^{\mathscr{G}}_{\mathrm{H}}$ in order to emphasize the board and constraint graph.

One of the main purposes in this chapter is to understand the combinatorial properties that constraint graphs H and homomorphism spaces $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ should satisfy in order to admit the existence of Gibbs $(\mathscr{G},\mathrm{H},\phi)$-specifications π with spatial mixing properties. The Gibbs $(\mathscr{G},\mathrm{H},\phi)$-specifications with a unique Gibbs measure have been already studied and, to some extent, characterized in [17]. We aim to develop a somewhat analogous framework and sufficiently general conditions under which Gibbs specifications satisfy SSM.

4.1 Dismantlable graphs and homomorphism spaces

The next definition is a structural description of a class of graphs introduced in [65], and heavily studied and characterized in [17].

Definition 4.1.1. *Given a constraint graph* H *and* $u,v \in \mathrm{V}$ *such that* $\mathrm{N}(u) \subseteq \mathrm{N}(v)$*, a* **fold** *is a homomorphism* $\alpha : \mathrm{H} \to \mathrm{H}[\mathrm{V} \setminus \{u\}]$ *such that* $\alpha(u) = v$ *and* $\alpha|_{\mathrm{V}\setminus\{u\}}$ *is the identity* $\mathrm{id}|_{\mathrm{V}\setminus\{u\}}$.

A constraint graph H *is* **dismantlable** *if there is a sequence of folds reducing* H *to a graph with a single vertex (with or without a loop).*

Notice that a fold $\alpha : \mathrm{H} \to \mathrm{H}[\mathrm{V} \setminus \{u\}]$ amounts to just removing u and edges containing it from the graph H, as long as a suitable vertex v exists which can "absorb" u.

The following proposition is a good example of the kind of results that we aim to achieve.

Proposition 4.1.1. *Let* H *be a constraint graph. Then,* H *is dismantlable iff for every board* $\mathcal{G}$ *of bounded degree and* $\gamma > 0$*, there exists a constrained energy function* ϕ *such that the Gibbs* $(\mathcal{G}, \mathrm{H}, \phi)$*-specification satisfies exponential WSM with decay rate* γ*.*

See Proposition 4.2.1 for a proof of this result. As we remark there, it is essentially due to Brightwell and Winkler [17]. One of our goals is to prove similar statements in which "WSM" is replaced by "SSM."

As in Proposition 4.1.1, it will be very common to have results where when a property is satisfied for every board $\mathcal{G}$, sometimes one can conclude facts about H. Another simple example is the following.

Proposition 4.1.2. *Let* H *be a constraint graph such that* $\mathrm{Hom}(\mathcal{G}, \mathrm{H})$ *satisfies SSF, for every board* $\mathcal{G}$*. Then,* H *has a safe symbol.*

Proof. Let $\mathcal{G} = \mathrm{S}_{|\mathrm{H}|}$ (the n-star graph with $n = |\mathrm{H}|$) and let $x \in \mathcal{V}$ to be the central vertex with boundary $\partial\{x\} = \left\{y_1, \dots, y_{|\mathrm{V}|}\right\}$. Write $\mathrm{V} = \{v_1, \cdots, v_{|\mathrm{V}|}\}$ and take $\beta \in \mathrm{V}^{\partial\{x\}}$ such that $\beta(y_i) = v_i$, for $1 \leq i \leq |\mathrm{V}|$. Then, by SSF, there exists a graph homomorphism $\alpha : \mathcal{G}[\partial\{x\} \cup \{x\}] \to \mathrm{H}$ such that $\alpha|_{\partial\{x\}} = \beta$. Since α is a graph homomorphism, $x \sim_{\mathcal{G}} y_i \implies \alpha(x) \sim_{\mathrm{H}} \alpha(y_i)$. Therefore, $\alpha(x) \sim_{\mathrm{H}} v_i$, for every i, so $\alpha(x)$ is a safe symbol for H. $\square$

Notice that the converse also holds, i.e. if a constraint graph H has a safe symbol, then $\mathrm{Hom}(\mathcal{G}, \mathrm{H})$ satisfies SSF, for every board $\mathcal{G}$.

Summarizing, given a homomorphism space $\mathrm{Hom}(\mathcal{G}, \mathrm{H})$, we have the following implications:

$$\begin{aligned} \text{H has a safe symbol} &\implies \mathrm{Hom}(\mathcal{G}, \mathrm{H}) \text{ satisfies SSF} && (4.1)\\ &\implies \mathrm{Hom}(\mathcal{G}, \mathrm{H}) \text{ satisfies TSSM} && (4.2)\\ &\implies \mathrm{Hom}(\mathcal{G}, \mathrm{H}) \text{ is strongly irreducible,} && (4.3) \end{aligned}$$

and all implications are strict in general (even if we fix $\mathcal{G}$ to be some particular board, like $\mathcal{G} = \mathbb{Z}^2$). See Section 3.5 and [58, 15] for examples that illustrate the differences among some of these conditions.

4.2 Dismantlable graphs and WSM

Dismantlable graphs are closely related with Gibbs measures, as illustrated by the following proposition, parts of which appeared in [17].

Proposition 4.2.1 ([17])**.** *Let* H *be a constraint graph. Then, the following are equivalent:*

1. H *is dismantlable.*
2. $\mathrm{Hom}(\mathcal{G},\mathrm{H})$ *is strongly irreducible with gap* $2|\mathrm{H}|+1$*, for every board* $\mathcal{G}$*.*
3. $\mathrm{Hom}(\mathcal{G},\mathrm{H})$ *is strongly irreducible with some gap, for every board* $\mathcal{G}$*.*
4. *For every board* $\mathcal{G}$ *of bounded degree, there exists a n.n. Gibbs* $(\mathcal{G},\mathrm{H},\phi)$*-specification that admits a unique n.n. Gibbs measure* μ*.*
5. *For every board* $\mathcal{G}$ *of bounded degree and* $\gamma > 0$*, there exists a n.n. Gibbs* $(\mathcal{G},\mathrm{H},\phi)$*-specification that satisfies exponential WSM with decay rate* γ*.*

The equivalence in Proposition 4.2.1 of (1), (2), (3), and (4) is proven in [17]. However, in that work the concept of WSM (a priori, stronger than uniqueness) is not considered. Since WSM implies uniqueness, (5) $\implies$ (4) is trivial. In the remaining part of this section, we introduce the necessary background to prove the missing implications, for which it is sufficient to show that (1) $\implies$ (5) (see Proposition 4.2.5). This and a subsequent proof (see Proposition 4.4.2) will have a similar structure to the proof of (1) $\implies$ (4) from [17, Theorem 7.2]. However, some coupling techniques will need to be modified, plus other combinatorial ideas need to be considered.

Given a dismantlable graph H, the only case where a sequence of folds reduces H to a vertex without a loop is when H is a set of isolated vertices without loops. In this case, we call H **trivial** (see [17, p. 6]). If $v^* \in \mathrm{V}$ has a loop and there is a sequence of folds reducing H to v^*, then we call v^* a **persistent** vertex of H.

Lemma 4.2.2 ([17, Lemma 5.2]). *Let* H *be a nontrivial dismantlable constraint graph and* v^* *a persistent vertex of* H. *Let* $\mathscr{G}$ *be a board,* $A \Subset \mathscr{V}$*, and* $\omega \in \mathrm{Hom}(\mathscr{G}, \mathrm{H})$. *Then there exists* $\upsilon \in \mathrm{Hom}(\mathscr{G}, \mathrm{H})$ *such that*

1. $\upsilon(x) = \omega(x)$*, for every* $x \in \mathscr{V} \setminus \mathscr{N}_{|\mathrm{H}|-2}(A)$,
2. $\upsilon(x) = v^*$*, for every* $x \in A$*, and*
3. $\omega^{-1}(v^*) \subseteq \upsilon^{-1}(v^*)$.

Now, given a constraint graph H, a persistent vertex v^*, and $\lambda > 1$, define ϕ_λ to be the constrained energy function given by

$$\phi_\lambda(v^*) = -\log\lambda, \quad \text{and} \quad \phi_\lambda|_{\mathrm{V}\setminus\{v^*\}\cup\mathrm{E}} \equiv 0. \tag{4.4}$$

We have the following lemma.

Lemma 4.2.3. *Let* H *be dismantlable constraint graph and let* $\mathscr{G}$ *be a board of bounded degree* Δ*. Given* $\lambda > 1$*, consider the Gibbs* $(\mathscr{G}, \mathrm{H}, \phi_\lambda)$*-specification* π*, a point* $\omega \in \mathrm{Hom}(\mathscr{G}, \mathrm{H})$*, and* $B \subseteq A \Subset \mathscr{V}$ *such that* $\mathscr{N}_{|\mathrm{H}|-2}(B) \subseteq A$*. Then, for any* $k \in \mathbb{N}$*,*

$$\pi_A^\omega\left(\{\alpha : |\{y \in B : \alpha(y) \neq v^*\}| \geq k\}\right) \leq |\mathrm{H}|^{|B|\Delta^{|\mathrm{H}|-1}} \lambda^{-k}. \tag{4.5}$$

Proof. Let's denote $\Omega = \mathrm{Hom}(\mathscr{G}, \mathrm{H})$ and $K = \mathscr{N}_{|\mathrm{H}|-2}(B)$. W.l.o.g., consider an arbitrary configuration $\alpha \in \mathrm{V}^A$ such that $\alpha\,\omega|_{A^c} \in \Omega$ (and, in particular, such that $\pi_A^\omega(\alpha) > 0$). Denote $\omega_1 = \alpha\,\omega|_{A^c}$. By Lemma 4.2.2, there exists $\omega_2 \in \Omega$ such that $\omega_1|_{\mathscr{V}\setminus K} = \omega_2|_{\mathscr{V}\setminus K}$, $\omega_2(x) = v^*$ for every $x \in B$, and $\omega_1^{-1}(v^*) \subseteq \omega_2^{-1}(v^*)$.

Notice that $\omega_1|_A\,\omega|_{A^c}, \omega_2|_A\,\omega|_{A^c} \in \Omega$ and $\omega_1|_{A\setminus K} = \omega_2|_{A\setminus K}$. Now, given some $k \leq |B|$, suppose that α is such that $|\{y \in B : \alpha(y) \neq v^*\}| \geq k$. Then, by the definition of Gibbs specification and the fact that $\omega_1^{-1}(v^*) \subseteq \omega_2^{-1}(v^*)$,

$$\frac{\pi_A^\omega\left(\omega_2|_K \,\middle|\, \omega_1|_{A\setminus K}\right)}{\pi_A^\omega\left(\omega_1|_K \,\middle|\, \omega_1|_{A\setminus K}\right)} = \frac{\pi_K^{\omega_1}\left(\omega_2|_K\right)}{\pi_K^{\omega_1}\left(\omega_1|_K\right)} \geq \lambda^k. \tag{4.6}$$

Therefore,

$$\pi_A^\omega\left(\alpha|_K \,\middle|\, \alpha|_{A\setminus K}\right) = \pi_A^\omega\left(\omega_1|_K \,\middle|\, \omega_1|_{A\setminus K}\right) \leq \lambda^{-k}. \tag{4.7}$$

Next, by taking weighted averages over all configurations $\beta \in \mathscr{L}_{A\setminus K}(\Omega)$ such that

$$\beta\ \alpha|_K\ \omega|_{A^c} \in \Omega, \tag{4.8}$$

we have

$$\pi_A^\omega(\alpha|_K) = \sum_\beta \pi_A^\omega(\alpha|_K|\beta)\,\pi_A^\omega(\beta) \le \sum_\beta \lambda^{-k}\pi_A^\omega(\beta) = \lambda^{-k}. \tag{4.9}$$

Notice that $|K| = \left|\mathscr{N}_{|\mathrm{H}|-2}(B)\right| \le |B|\Delta^{|\mathrm{H}|-1}$. In particular,

$$|\mathscr{L}_K(\Omega)| \le |\mathrm{H}|^{|B|\Delta^{|\mathrm{H}|-1}}. \tag{4.10}$$

Then, since α was arbitrary,

$$\begin{aligned}\pi_A^\omega(\{\alpha : |\{y \in B : \alpha(y) \neq v^*\}| \ge k\}) &\le \sum_{\substack{\alpha|_K : \alpha \in \mathscr{L}_A(\Omega), \\ |\{y\in B:\alpha(y)\neq v^*\}|\ge k}} \pi_A^\omega(\alpha|_K) &(4.11)\\ &\le |\mathrm{H}|^{|B|\Delta^{|\mathrm{H}|-1}}\lambda^{-k}. &(4.12)\end{aligned}$$

□

As mentioned before, we essentially use some coupling techniques from [17], with slight modifications. We will use the following theorem.

Theorem 4.2.4 ([11, Theorem 1])**.** *Given a n.n. Gibbs $(\mathscr{G},\mathrm{H},\phi)$-specification π, $A \Subset \mathscr{V}$, and $\omega_1,\omega_2 \in \mathrm{Hom}(\mathscr{G},\mathrm{H})$, there exists a coupling $((\alpha_1(x),\alpha_2(x)), x \in A)$ of $\pi_A^{\omega_1}$ and $\pi_A^{\omega_2}$ (whose distribution we denote by $\mathbb{P}_A^{\omega_1,\omega_2}$), such that for each $x \in A$, $\alpha_1(x) \neq \alpha_2(x)$ iff there is a* path of disagreement *from x to $\Sigma_{\partial A}(\omega_1,\omega_2)$, $\mathbb{P}_A^{\omega_1,\omega_2}$-a.s.*

In general, by a **path of disagreement** *from A to B, we mean that there is a path $\mathcal{P}$ from A to B such that $\alpha_1(y) \neq \alpha_2(y)$, for all $y \in \mathcal{P}$. We denote this event by $\left\{A \stackrel{\neq}{\longleftrightarrow} B\right\}$.*

Remark 10. *The result in [11, Theorem 1] is for MRFs, but here we state it for n.n. Gibbs specifications.*

Proposition 4.2.5. *Let $\mathscr{G}$ be a board of bounded degree Δ and H a dismantlable constraint graph. Then, for all $\gamma > 0$, there exists $\lambda_0 = \lambda_0(\gamma,|\mathrm{H}|,\Delta)$ such that for* *every $\lambda > \lambda_0$, the Gibbs $(\mathscr{G}, \mathrm{H}, \phi_\lambda)$-specification π satisfies exponential WSM with decay rate γ.*

Proof. Let $A \Subset \mathscr{V}$, $B \subseteq A$, $\beta \in \mathrm{V}^B$, and $\omega_1, \omega_2 \in \mathrm{Hom}(\mathscr{G}, \mathrm{H})$. W.l.o.g. (since C can be taken arbitrarily large in the desired decay function $Ce^{-\gamma n}$), we may suppose that

$$\mathrm{dist}(B, \partial A) = n > |\mathrm{H}| - 2. \tag{4.13}$$

By Theorem 4.2.4, we have

$$\begin{aligned}
\left|\pi_A^{\omega_1}(\beta) - \pi_A^{\omega_2}(\beta)\right| &= \left|\mathbb{P}_A^{\omega_1,\omega_2}(\alpha_1|_B = \beta) - \mathbb{P}_A^{\omega_1,\omega_2}(\alpha_2|_B = \beta)\right| && (4.14)\\
&\leq \mathbb{P}_A^{\omega_1,\omega_2}(\alpha_1|_B \neq \alpha_2|_B) && (4.15)\\
&\leq \sum_{x \in B} \mathbb{P}_A^{\omega_1,\omega_2}(\alpha_1(x) \neq \alpha_2(x)) && (4.16)\\
&= \sum_{x \in B} \mathbb{P}_A^{\omega_1,\omega_2}\left(x \xleftrightarrow{\neq} \Sigma_{\partial A}(\omega_1, \omega_2)\right) && (4.17)\\
&\leq \sum_{x \in B} \mathbb{P}_A^{\omega_1,\omega_2}\left(x \xleftrightarrow{\neq} \partial A\right) && (4.18)\\
&\leq \sum_{x \in B} \mathbb{P}_A^{\omega_1,\omega_2}\left(x \xleftrightarrow{\neq} \mathscr{N}_{|\mathrm{H}|-2}(\partial A)\right). && (4.19)
\end{aligned}$$

When considering a path of disagreement $\mathcal{P}$ from x to $\mathscr{N}_{|\mathrm{H}|-2}(\partial A)$, we can assume that $\mathscr{N}_{|\mathrm{H}|-2}(\mathcal{P}) \subseteq A$. Then, by Lemma 4.2.3,

$$\pi_A^{\omega}(\{\alpha : |\{y \in \mathcal{P} : \alpha(y) \neq v^*\}| \geq k\}) \leq |\mathrm{H}|^{|\mathcal{P}|\Delta^{|\mathrm{H}|-1}} \lambda^{-k}. \tag{4.20}$$

In addition, $|\mathcal{P}| \geq n - |\mathrm{H}| + 2$ and, for every $y \in \mathcal{P}$, we have that $\alpha_1(y) \neq \alpha_2(y)$, so $\alpha_1(y)$ and $\alpha_2(y)$ cannot be both v^* at the same time, and either $\alpha_1|_{\mathcal{P}}$ or $\alpha_1|_{\mathcal{P}}$ must have $|\mathcal{P}|/2$ sites different from v^*. In consequence,

$$\begin{aligned}
&\mathbb{P}_A^{\omega_1,\omega_2}\left(x \xleftrightarrow{\neq} \mathscr{N}_{|\mathrm{H}|-2}(\partial A)\right) && (4.21)\\
\leq &\sum_{k=n-|\mathrm{H}|+2}^{\infty} \sum_{|\mathcal{P}|=k} \pi_A^{\omega_1}\left(\left\{\alpha_1 : |\{y \in \mathcal{P} : \alpha_1(y) \neq v^*\}| \geq \frac{k}{2}\right\}\right) && (4.22)\\
&+ \sum_{k=n-|\mathrm{H}|+2}^{\infty} \sum_{|\mathcal{P}|=k} \pi_A^{\omega_2}\left(\left\{\alpha_2 : |\{y \in \mathcal{P} : \alpha_2(y) \neq v^*\}| \geq \frac{k}{2}\right\}\right)
\end{aligned}$$

$$\leq 2 \sum_{k=n-|\mathrm{H}|+2}^{\infty} \sum_{|\mathcal{P}|=k} |\mathrm{H}|^{k\Delta^{|\mathrm{H}|-1}} \lambda^{-\frac{k}{2}} \tag{4.23}$$

$$\leq 2 \sum_{k=n-|\mathrm{H}|+2}^{\infty} \Delta(\Delta-1)^k \left(\frac{|\mathrm{H}|^{\Delta^{|\mathrm{H}|-1}}}{\lambda^{1/2}} \right)^k \tag{4.24}$$

$$= 2\Delta \sum_{k=n-|\mathrm{H}|+2}^{\infty} \left(\frac{(\Delta-1)|\mathrm{H}|^{\Delta^{|\mathrm{H}|-1}}}{\lambda^{1/2}} \right)^k . \tag{4.25}$$

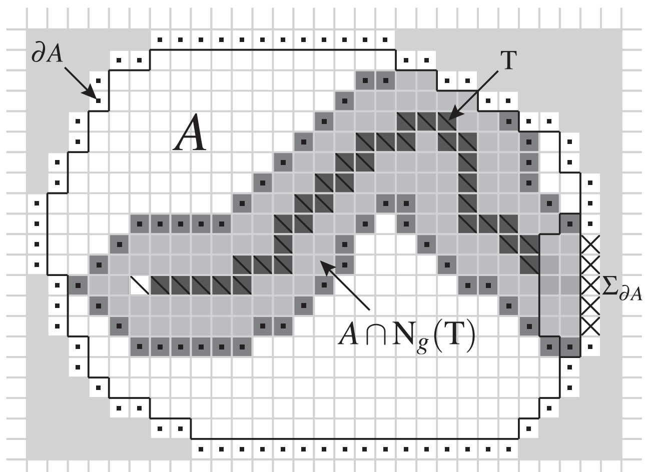

Figure 4.1: Decomposition in the proofs of Lemma 4.2.3 and Proposition 4.2.5.

Finally, we have

$$\left|\pi_A^{\omega_1}(\beta) - \pi_A^{\omega_2}(\beta)\right| \leq \sum_{x \in B} \mathbb{P}_A^{\omega_1,\omega_2} \left(x \xleftrightarrow{\neq} \mathcal{N}_{|\mathrm{H}|-2}(\partial A) \right) \tag{4.26}$$

$$\leq 2|B|\Delta \sum_{k=n-|\mathrm{H}|+2}^{\infty} \left(\frac{(\Delta-1)|\mathrm{H}|^{\Delta^{|\mathrm{H}|-1}}}{\lambda^{1/2}} \right)^k , \tag{4.27}$$

so, in order to have exponential decay, it suffices to take

$$(\Delta-1)^2 |\mathrm{H}|^{2\Delta^{|\mathrm{H}|-1}} < \lambda, \tag{4.28}$$

and we note that any decay rate γ is achievable by taking λ sufficiently large. □

Proof of Proposition 4.2.1. The implication $(1) \implies (5)$ follows from Proposition 4.2.5. Since WSM implies uniqueness, we have $(5) \implies (4)$. The implications $(4) \implies (3) \implies (2) \implies (1)$ can be found in [17, Theorem 4.1]. □

A priori, one would be tempted to think that the proof of Proposition 4.2.5 could give SSM instead of just WSM, since the coupling in Theorem 4.2.4 involves a path of disagreement from B to $\Sigma_{\partial A}(\omega_1, \omega_2)$ and not just to ∂A, just as in the definition of SSM. One of our motivations is to illustrate that this is not always the case (see the examples in Section 4.7), mainly due to combinatorial obstructions. We will see that in order to have an analogous result for the SSM property, H must satisfy even stronger conditions than dismantlability, which guarantees WSM by Proposition 4.2.1. One of the main issues is that our proof required that $\text{dist}(B, \partial A) > |\mathrm{H}| - 2$, so B cannot be arbitrarily close to ∂A, which is in opposition to the spirit of SSM.

Notice that, by Proposition 4.2.1, $\mathrm{Hom}(\mathscr{G}, \mathrm{H})$ is strongly irreducible for every board $\mathscr{G}$ iff H is dismantlable. Clearly, the forward direction still holds if "strongly irreducible" is replaced by "TSSM," since TSSM implies strongly irreducible. Later, we will address the question of whether the reverse direction holds with this replacement.

For a dismantlable constraint graph H and a particular or arbitrary board $\mathscr{G}$, we are interested in whether or not $\mathrm{Hom}(\mathscr{G}, \mathrm{H})$ is TSSM and whether or not there exists a Gibbs $(\mathscr{G}, \mathrm{H}, \phi)$-specification π that satisfies exponential SSM with arbitrarily high decay rate. Here we restate Theorem 3.6.5 in the context of homomorphism spaces, which shows that these two desired conclusions are related.

Theorem 4.2.6. *Let π be a Gibbs $(\mathbb{Z}^2, \mathrm{H}, \phi)$-specification that satisfies exponential SSM with decay rate $\gamma > 4\log|\mathrm{H}|$. Then,* $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{H})$ *satisfies TSSM.*

One of our main goals is to look for conditions on $\mathrm{Hom}(\mathscr{G}, \mathrm{H})$ suitable for having a Gibbs specification that satisfies SSM. SSM seems to be related with TSSM, as illustrated in the previous results. In the following section, we explore some other properties related with TSSM.

4.3 The unique maximal configuration property

Fix a constraint graph H and consider an arbitrary board $\mathscr{G}$. Given a linear order $\preceq$ on the set of vertices V, we consider the partial order (that, in a slight abuse of notation, we also denote by $\preceq$) on $\mathrm{V}^{\mathscr{V}}$ obtained by extending coordinate-wise the linear order $\preceq$ to subsets of $\mathscr{V}$, i.e. given $\alpha_1, \alpha_2 \in \mathrm{V}^A$, for some $A \subseteq \mathscr{V}$, we say that $\alpha_1 \preceq \alpha_2$ iff $\alpha_1(x) \preceq \alpha_2(x)$, for all $x \in A$. If $\alpha_1(x) \preceq \alpha_2(x)$ but $\alpha_1(x) \neq \alpha_2(x)$, we write $\alpha_1(x) \prec \alpha_2(x)$. In addition, if two vertices $u, v \in \mathrm{V}$ are such that $u \sim v$ and $u \preceq v$, we will denote this by $u \precsim v$.

Definition 4.3.1. *Given $g \in \mathbb{N}$, we say that* $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ *satisfies the* **unique maximal configuration (UMC) property with distance** *g if there exists a linear order $\preceq$ on* V *such that, for every $A \Subset \mathscr{V}$,*

(M1) for every $\alpha \in \mathscr{L}_A(\mathrm{Hom}(\mathscr{G},\mathrm{H}))$, there is a unique point $\omega_\alpha \in [\alpha]^{\mathscr{G}}_{\mathrm{H}}$ such that $\omega \preceq \omega_\alpha$, for every point $\omega \in [\alpha]^{\mathscr{G}}_{\mathrm{H}}$, and

(M2) for any two $\alpha_1, \alpha_2 \in \mathscr{L}_A(\mathrm{Hom}(\mathscr{G},\mathrm{H}))$, $\Sigma_{\mathscr{V}}(\omega_{\alpha_1}, \omega_{\alpha_2}) \subseteq \mathscr{N}_g(\Sigma_A(\alpha_1, \alpha_2))$.

If $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ satisfies the UMC property, then for any $\alpha \in \mathscr{L}_A(\mathrm{Hom}(\mathscr{G},\mathrm{H}))$ and $\beta = \alpha|_B$ with $B \subseteq A$, it is the case that $\omega_\alpha \preceq \omega_\beta$. This is natural, since we can see the configurations α and β as "restrictions" to be satisfied by ω_α and ω_β, respectively. In addition, observe that condition $(M2)$ in Definition 4.3.1 implies that $\Sigma_{\mathscr{V}}(\omega_\alpha, \omega_\beta) \subseteq \mathscr{N}_g(A \setminus B)$. In particular, by taking $B = \emptyset$, we see that if $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ satisfies the UMC property, then there must exist a greatest element $\omega_* \in \mathrm{Hom}(\mathscr{G},\mathrm{H})$ such that $\omega \preceq \omega_*$, for every $\omega \in \mathrm{Hom}(\mathscr{G},\mathrm{H})$, and $\Sigma_{\mathscr{V}}(\omega_\alpha, \omega_*) \subseteq \mathscr{N}_g(A)$, for every $\alpha \in \mathscr{L}_A(\mathrm{Hom}(\mathscr{G},\mathrm{H}))$.

The following proposition establishes that the UMC property is related with TSSM.

Proposition 4.3.1. *Suppose that* $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ *satisfies the UMC property with distance g. Then,* $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ *satisfies TSSM with gap* $2g+1$.

Proof. Consider a pair of sites $x, y \in \mathscr{V}$ with $\mathrm{dist}(x,y) \geq 2g+1$, a set $S \Subset \mathscr{V}$, and configurations $\alpha \in \mathrm{V}^{\{x\}}$, $\beta \in \mathrm{V}^{\{y\}}$, and $\sigma \in \mathrm{V}^S$ such that $[\alpha\sigma]^{\mathscr{G}}_{\mathrm{H}}, [\sigma\beta]^{\mathscr{G}}_{\mathrm{H}} \neq \emptyset$. Take the corresponding maximal configurations ω_σ, $\omega_{\alpha\sigma}$, and $\omega_{\sigma\beta}$. Notice that

$\Sigma_{\mathscr{V}}(\omega_{\alpha\sigma}, \omega_\sigma) \subseteq \mathscr{N}_g(x)$ and $\Sigma_{\mathscr{V}}(\omega_\sigma, \omega_{\sigma\beta}) \subseteq \mathscr{N}_g(y)$. Since $\mathrm{dist}(x,y) \geq 2g+1$, we can conclude that $\mathscr{N}_g(x)^c \cap \mathscr{N}_g(y) = \emptyset$ and $\mathscr{N}_g(x) \cap \mathscr{N}_g(y)^c = \emptyset$. Therefore,

$$\omega_\sigma|_{\mathscr{N}_g(x)^c \cap \mathscr{N}_g(y)^c} = \omega_{\alpha\sigma}|_{\mathscr{N}_g(x)^c \cap \mathscr{N}_g(y)^c} = \omega_{\sigma\beta}\big|_{\mathscr{N}_g(x)^c \cap \mathscr{N}_g(y)^c}, \tag{4.29}$$

and since $\mathrm{Hom}(\mathscr{G}, \mathrm{H})$ is a topological MRF (see [20]), we have

$$\omega_{\alpha\sigma}|_{\mathscr{N}_g(x)}\ \omega_\sigma|_{\mathscr{N}_g(x)^c \cap \mathscr{N}_g(y)^c}\ \omega_{\sigma\beta}\big|_{\mathscr{N}_g(y)} \in \mathrm{Hom}(\mathscr{G}, \mathrm{H}), \tag{4.30}$$

so $[\alpha\sigma\beta]_{\mathrm{H}}^{\mathscr{G}} \neq \emptyset$. Using Lemma 3.3.1, we conclude. □

4.4 SSM and the UMC property

Given a constraint graph H, a linear order $\preceq$ in $\mathrm{V} = \left\{v_1, \ldots, v_k, \ldots, v_{|\mathrm{H}|}\right\}$ such that $v_1 \prec \cdots \prec v_k \prec \cdots \prec v_{|\mathrm{H}|}$, and $\lambda > 1$, define $\phi_\lambda^{\preceq}$ to be the constrained energy function given by

$$\phi_\lambda^{\preceq}(v_k) = -k \log \lambda, \quad \text{and} \quad \phi_\lambda^{\preceq}|_{\mathrm{E}} \equiv 0. \tag{4.31}$$

We have the following lemma.

Lemma 4.4.1. *Suppose that* $\mathrm{Hom}(\mathscr{G}, \mathrm{H})$ *satisfies the UMC property with distance* g*, for* $\mathscr{G}$ *a board of bounded degree* Δ*. Given* $\lambda > 1$*, consider the Gibbs* $(\mathscr{G}, \mathrm{H}, \phi_\lambda^{\preceq})$*-specification* π*, a point* $\omega \in \mathrm{Hom}(\mathscr{G}, \mathrm{H})$*, and sets* $B \subseteq A \Subset \mathscr{V}$*. Then, for any* $k \in \mathbb{N}$*,*

$$\pi_A^\omega\left(\{\alpha : |\{y \in B : \alpha(y) \prec \omega_\delta(y)\}| \geq k\}\right) \leq |\mathrm{H}|^{|B|\Delta^{g+1}} \lambda^{-k}, \tag{4.32}$$

where $\delta = \omega|_{\partial A}$*.*

Proof. Let's denote $\Omega = \mathrm{Hom}(\mathscr{G}, \mathrm{H})$. Consider an arbitrary configuration $\alpha \in \mathrm{V}^A$ such that $\alpha\ \omega|_{A^c} \in \Omega$ (so, in particular, $\pi_A^\omega(\alpha) > 0$). Take the set $K = A \cap \mathscr{N}_g(B)$ and decompose its boundary K into the two subsets $A \cap \partial K$ and $\partial A \cap \partial K$. Consider $D = A \cap \partial K$ and name $\eta = \alpha|_D$. Since $\delta\eta \in \mathscr{L}_{\partial A \cup D}(\Omega)$, there exists a unique maximal configuration $\omega_{\delta\eta}$. Clearly, $\alpha(x) \preceq \omega_{\delta\eta}(x)$, for every $x \in A$. Moreover, $\omega_{\delta\eta}\big|_B = \omega_\delta|_B$, since $\Sigma_{\mathscr{V}}(\omega_{\delta\eta}, \omega_\delta) \subseteq \mathscr{N}_g(D)$ and $\mathrm{dist}(B, D) > g$, so $\mathscr{N}_g(D) \cap B = \emptyset$.

Now, suppose that α is such that $|\{y \in B : \alpha(y) \prec \omega_\delta(y)\}| \geq k$, for some $k \leq |B|$. Then, $\left|\{y \in B : \alpha(y) \prec \omega_{\delta\eta}(y)\}\right| \geq k$ and, by the (topological and measure-theoretical) MRF property,

$$\frac{\pi_A^\omega\left(\omega_{\delta\eta}\big|_K \middle| \alpha|_{A\setminus K}\right)}{\pi_A^\omega\left(\alpha|_K \middle| \alpha|_{A\setminus K}\right)} = \frac{\pi_K^{\omega_{\delta\eta}}\left(\omega_{\delta\eta}\big|_K\right)}{\pi_K^{\omega_{\delta\eta}}\left(\alpha|_K\right)} \geq \lambda^k. \tag{4.33}$$

Therefore,

$$\pi_A^\omega\left(\alpha|_K \middle| \alpha|_{A\setminus K}\right) \leq \lambda^{-k}. \tag{4.34}$$

Next, by taking weighted averages over all configurations $\beta \in \mathscr{L}_{A\setminus K}(\Omega)$ such that

$$\beta\ \alpha|_K\ \omega|_{A^c} \in \Omega, \tag{4.35}$$

we have

$$\pi_A^\omega(\alpha|_K) = \sum_\beta \pi_A^\omega(\alpha|_K|\beta)\,\pi_A^\omega(\beta) \leq \sum_\beta \lambda^{-k}\pi_A^\omega(\beta) = \lambda^{-k}. \tag{4.36}$$

Notice that $|K| \leq |\mathscr{N}_g(B)| \leq |B|\Delta^{g+1}$, so $|\mathscr{L}_K(\Omega)| \leq |\mathrm{H}|^{|B|\Delta^{g+1}}$. Then, since α was arbitrary,

$$\pi_A^\omega(\{\alpha : |\{y \in B : \alpha(y) \prec \omega_\delta(y)\}| \geq k\}) \leq \sum_{\substack{\alpha|_K : \alpha \in \mathscr{L}_A(\Omega), \\ |\{y \in B : \alpha(y) \prec \omega_\delta(y)\}| \geq k}} \pi_A^\omega(\alpha|_K) \tag{4.37}$$

$$\leq |\mathrm{H}|^{|B|\Delta^{g+1}}\lambda^{-k}. \tag{4.38}$$

□

Proposition 4.4.2. *Suppose that* $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ *satisfies the UMC property with distance g, for $\mathscr{G}$ a board of bounded degree Δ. Then, for all $\gamma > 0$, there exists $\lambda_0 = \lambda_0(\gamma, |\mathrm{H}|, \Delta, g)$ such that for every $\lambda > \lambda_0$, the Gibbs $(\mathscr{G}, \mathrm{H}, \phi_\lambda^{\preceq})$-specification π satisfies exponential SSM with decay rate γ.*

Proof. Let $A \Subset \mathscr{V}$, $x \in A$, $\beta \in \mathrm{V}^{\{x\}}$, and $\omega_1, \omega_2 \in \mathrm{Hom}(\mathscr{G},\mathrm{H})$. W.l.o.g., we may suppose that

$$\mathrm{dist}\,(x, \Sigma_{\partial A}(\omega_1, \omega_2)) = n > g. \tag{4.39}$$

By Theorem 4.2.4, and similarly to the proof of Proposition 4.2.5, we have

$$\begin{aligned}
\left|\pi_A^{\omega_1}(\beta)-\pi_A^{\omega_2}(\beta)\right| &\leq \mathbb{P}_A^{\omega_1,\omega_2}(\alpha_1(x)\neq\alpha_2(x)) &(4.40)\\
&= \mathbb{P}_A^{\omega_1,\omega_2}\left(x \overset{\neq}{\longleftrightarrow} \Sigma_{\partial A}(\omega_1,\omega_2)\right) &(4.41)\\
&\leq \mathbb{P}_A^{\omega_1,\omega_2}\left(x \overset{\neq}{\longleftrightarrow} \mathscr{N}_g(\Sigma_{\partial A}(\omega_1,\omega_2))\right). &(4.42)
\end{aligned}$$

When considering a path of disagreement $\mathcal{P}$ from x to $\mathscr{N}_g(\Sigma_{\partial A}(\omega_1,\omega_2))$, we can assume (by truncating if necessary) that $\mathcal{P}\subseteq A\setminus\mathscr{N}_g(\Sigma_{\partial A}(\omega_1,\omega_2))$ and $|\mathcal{P}|\geq n-g$. By the UMC property, if we take $\delta_1=\omega_1|_{\partial A}$ and $\delta_2=\omega_2|_{\partial A}$, we have $\Sigma_{\mathscr{V}}(\omega_{\delta_1},\omega_{\delta_2})\subseteq\mathscr{N}_g(\Sigma_{\partial A}(\omega_1,\omega_2))=\mathscr{N}_g(\Sigma_{\partial A}(\delta_1,\delta_2))$, so $\omega_{\delta_1}\big|_{\mathcal{P}}=\omega_{\delta_2}\big|_{\mathcal{P}}=:\theta\in\mathscr{L}_{\mathcal{P}}(\mathrm{Hom}(\mathscr{G},\mathrm{H}))$. Since $\mathcal{P}$ is a path of disagreement, for every $y\in\mathcal{P}$ we have $\alpha_1(y)\prec\alpha_2(y)\preceq\theta(y)$ or $\alpha_2(y)\prec\alpha_1(y)\preceq\theta(y)$. In consequence, using Lemma 4.4.1 yields

$$\begin{aligned}
&\mathbb{P}_A^{\omega_1,\omega_2}\left(x \overset{\neq}{\longleftrightarrow} \mathscr{N}_g(\Sigma_{\partial A}(\omega_1,\omega_2))\right) &(4.43)\\
&\leq \sum_{k=n-g}^{\infty}\sum_{|\mathcal{P}|=k}\mathbb{P}_A^{\omega_1,\omega_2}(\mathcal{P}\text{ is a path of disagr. from }x\text{ to }\mathscr{N}_g(\Sigma_{\partial A}(\delta_1,\delta_2))) &(4.44)\\
&\leq \sum_{k=n-g}^{\infty}\sum_{|\mathcal{P}|=k}\pi_A^{\omega_1}\left(\left\{\alpha_1:|\{y\in\mathcal{P}:\alpha_1(y)\prec\theta(y)\}|\geq\frac{k}{2}\right\}\right) &(4.45)\\
&\quad+\sum_{k=n-g}^{\infty}\sum_{|\mathcal{P}|=k}\pi_A^{\omega_2}\left(\left\{\alpha_2:|\{y\in\mathcal{P}:\alpha_2(y)\prec\theta(y)\}|\geq\frac{k}{2}\right\}\right)\\
&\leq 2\sum_{k=n-g}^{\infty}\sum_{|\mathcal{P}|=k}|\mathrm{H}|^{k\Delta^{g+1}}\lambda^{-\frac{k}{2}}\leq 2\Delta\sum_{k=n-g}^{\infty}\left(\frac{(\Delta-1)|\mathrm{H}|^{\Delta^{g+1}}}{\lambda^{1/2}}\right)^k. &(4.46)
\end{aligned}$$

Then, by Lemma 3.1.1, exponential SSM holds whenever

$$(\Delta-1)^2|\mathrm{H}|^{2\Delta^{g+1}}<\lambda, \tag{4.47}$$

and any decay rate γ may be achieved by taking λ large enough.

□

Notice that here λ_0 is defined in terms of γ, $|\mathrm{H}|$, Δ and g. In the WSM proof, g implicitly depended on $|\mathrm{H}|$, but here the two parameters could be, a priori, virtually independent.

4.5 UMC and chordal/tree decomposable graphs

4.5.1 Chordal/tree decompositions

Definition 4.5.1. *A simple constraint graph* H *is said to be* **chordal** *if all cycles of four or more vertices have a* **chord***, which is an edge that is not part of the cycle but connects two vertices of the cycle.*

Definition 4.5.2. *A* **perfect elimination ordering** *in a simple constraint graph* $\mathrm{H} = (\mathrm{V}, \mathrm{E})$ *is an ordering* $v_1, \ldots, v_n$ *of* V *such that* $\mathrm{H}[v_i \cup (\{v_{i+1}, \ldots, v_n\} \cap \mathrm{N}(v_i))]$ *is a complete graph, for every* $1 \leq i \leq n = |\mathrm{H}|$.

Proposition 4.5.1 ([30])**.** *A simple constraint graph* H *is chordal iff it has a perfect elimination ordering.*

Definition 4.5.3. *A constraint graph* $\mathrm{H} = (\mathrm{V}, \mathrm{E})$ *will be called* **loop-chordal** *if* $\mathrm{Loop}(\mathrm{H}) = \mathrm{V}$ *and* $\mathrm{H}' = (\mathrm{V}, \mathrm{E} \setminus \{\{v, v\} : v \in \mathrm{V}\})$ *(i.e.* H' *is the version of* H *without loops) is chordal.*

Proposition 4.5.2. *Given a loop-chordal constraint graph* $\mathrm{H} = (\mathrm{V}, \mathrm{E})$*, there exists an ordering* $v_1, \ldots, v_n$ *of* V *such that* $\mathrm{H}[v_i \cup (\{v_{i+1}, \ldots, v_n\} \cap \mathrm{N}(v_i))]$ *is a loop-complete graph, for every* $1 \leq i \leq n = |\mathrm{H}|$.

Proof. This follows immediately from Proposition 4.5.1. □

Proposition 4.5.2 can also be thought of as saying that a graph $G = (V, E)$ is loop-chordal iff $\mathrm{Loop}(G) = V$ and there exists an order $v_1 \prec \cdots \prec v_n$ such that

$$v_i \precsim v_j \wedge v_i \precsim v_k \implies v_j \sim v_k. \tag{4.48}$$

Proposition 4.5.3. *A connected loop-chordal graph* G *is dismantlable.*

Proof. Let $v_1, \ldots, v_n$ be the ordering of V given by Proposition 4.5.2 and take $v \in \mathrm{N}(v_1)$. Clearly, $v \in \{v_2, \ldots, v_n\}$ and then we have $G[v_1 \cup (\{v_2, \ldots, v_n\} \cap \mathrm{N}(v_1))] = G[\mathrm{N}(v_1)]$ is a loop-complete graph and $v \in \mathrm{N}(v_1)$. Therefore, $\mathrm{N}(v_1) \subseteq \mathrm{N}(v)$ and there is a fold from G to $G[V \setminus \{v_1\}]$. It can be checked that $G[V \setminus \{v_1\}]$ is also loop-chordal, so we apply the same argument to $G[V \setminus \{v_1\}]$ and so on, until we end with only one vertex (with a loop). $\square$

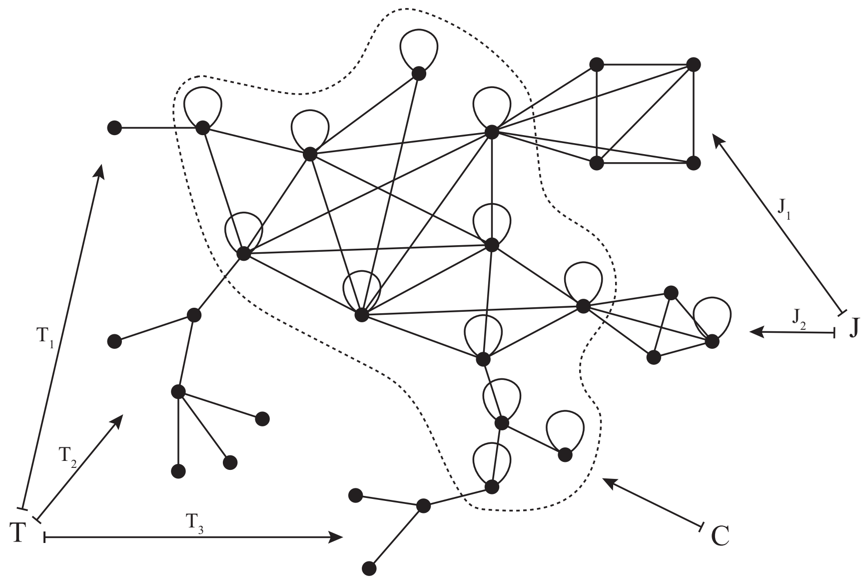

Figure 4.2: A chordal/tree decomposition.

We say that a constraint graph $\mathrm{H} = (\mathrm{V}, \mathrm{E})$ has a **chordal/tree decomposition** or is **chordal/tree decomposable** if we can write $\mathrm{V} = \mathrm{C} \sqcup \mathrm{T} \sqcup \mathrm{J}$ such that

1. $\mathrm{H}[\mathrm{C}]$ is a nonempty loop-chordal graph,

2. $\mathrm{T} = \mathrm{T}_1 \sqcup \cdots \sqcup \mathrm{T}_m$ and, for every $1 \leq j \leq m$, $\mathrm{H}[\mathrm{T}_j]$ is a tree such that there exist unique vertices $r_j \in \mathrm{T}_j$ (the root of T_j) and $c^j \in \mathrm{C}$ such that $\{r_j, c^j\} \in \mathrm{E}$,

3. $\mathrm{J} = \mathrm{J}_1 \sqcup \cdots \sqcup \mathrm{J}_n$ and, for every $1 \leq k \leq n$, $\mathrm{H}[\mathrm{J}_k]$ is a connected graph with a unique vertex $c^k \in \mathrm{C}$ such that $\{u, c^k\} \in \mathrm{E}$, for every $u \in \mathrm{J}_k$, and

4. $\mathrm{E}[\mathrm{T} : \mathrm{J}] = \mathrm{E}[\mathrm{T}_{j_1} : \mathrm{T}_{j_2}] = \mathrm{E}[\mathrm{J}_{k_1} : \mathrm{J}_{k_2}] = \emptyset$, for every $j_1 \neq j_2$ and $k_1 \neq k_2$.

Notice that, for every k, the vertex $c^k \in \mathrm{C}$ is a safe symbol for $\mathrm{H}[\{c^k\} \cup \mathrm{J}_k]$.

4.5.2 A natural linear order

Given a chordal/tree decomposable constraint graph H, we define a linear order $\preceq$ on V as follows:

- If $w \in \mathrm{J}$ and $t \in \mathrm{T}$, then $w \prec t$.
- If $t \in \mathrm{T}$ and $c \in \mathrm{C}$, then $t \prec c$.
- If $w \in \mathrm{J}_k$ and $w' \in \mathrm{J}_{k'}$, for some $1 \leq k < k' \leq n$, then $w \prec w'$.
- If $t \in \mathrm{T}_j$ and $t' \in \mathrm{T}_{j'}$, for some $1 \leq j < j' \leq m$, then $t \prec t'$.
- Given $1 \leq k \leq n$, we fix an arbitrary order in J_k.
- Given $1 \leq j \leq m$ and $t_1, t_2 \in \mathrm{T}_j$, then
  - if $\operatorname{dist}(t_1, r^j) < \operatorname{dist}(t_2, r^j)$, then $t_1 \prec t_2$,
  - if $\operatorname{dist}(t_1, r^j) > \operatorname{dist}(t_2, r^j)$, then $t_2 \prec t_1$, and
  - for each i, we arbitrarily order the set of vertices t with $\operatorname{dist}(t, r^j) = i$.
- If $c_1, c_2 \in \mathrm{C}$, then c_1 and c_2 are ordered according to Proposition 4.5.2.

Proposition 4.5.4. *If a constraint graph* H *has a chordal/tree decomposition, then* H *is dismantlable.*

Proof. W.l.o.g., suppose that $|\mathrm{H}| \geq 2$ (the case $|\mathrm{H}| = 1$ is trivial). Let $\mathcal{G}$ be an arbitrary board. In the following theorem (Theorem 4.5.6), it will be proven that if H is chordal/tree decomposable, then $\mathrm{Hom}(\mathcal{G}, \mathrm{H})$ satisfies the UMC property with distance $|\mathrm{H}| - 2$. Therefore, by Proposition 4.3.1, $\mathrm{Hom}(\mathcal{G}, \mathrm{H})$ satisfies TSSM with gap $2(|\mathrm{H}| - 2) + 1$ and, in particular, $\mathrm{Hom}(\mathcal{G}, \mathrm{H})$ is strongly irreducible with gap $2|\mathrm{H}| + 1$. Since the gap is independent of $\mathcal{G}$, we can apply Proposition 4.2.1 to conclude that H must be dismantlable. □

Proposition 4.5.5. *If a constraint graph* H *has a safe symbol, then* H *is chordal/tree decomposable.*

Proof. This follows trivially by considering $\mathrm{C} = \{s\}$, $\mathrm{T} = \emptyset$ and $\mathrm{J} = \mathrm{V} \setminus \{s\}$, with s a safe symbol for H. □

We show that chordal/tree decomposable graphs H induce combinatorial properties on homomorphism spaces $\mathrm{Hom}(\mathscr{G},\mathrm{H})$.

Theorem 4.5.6. *Let* H *be a chordal/tree decomposable constraint graph. Then,* $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ *has the UMC property with distance* $|\mathrm{H}|-2$*, for any board* $\mathscr{G}$.

Before proving Theorem 4.5.6, we introduce some useful tools. From now on, we fix $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ and $x_1,x_2,\dots$ to be an arbitrary order of $\mathscr{V}$. We also fix the linear order $\preceq$ on V as defined above. For $i \in \{1,\dots,|\mathrm{H}|\}$, define the sets $D_i := \{(\omega_1,\omega_2,x) \in \mathrm{Hom}(\mathscr{G},\mathrm{H}) \times \mathrm{Hom}(\mathscr{G},\mathrm{H}) \times \mathscr{G} : v_i = \omega_1(x) \prec \omega_2(x)\}$, and consider $D(\ell) = \bigcup_{i=1}^{\ell} D_i$, for $1 \leq \ell \leq |\mathrm{H}|$, and $D := D(|\mathrm{H}|)$. Notice that $D_{|\mathrm{H}|} = \emptyset$. In addition, given $(\omega_1,\omega_2,x) \in D$, define the set

$$\mathrm{N}^-(\omega_1,x) := \{y \in \mathrm{N}(x) : \omega_1(y) \prec \omega_1(x)\}, \tag{4.49}$$

and the partition

$$\mathrm{N}^-(\omega_1,x) = \mathrm{N}^-_{\prec}(\omega_1,\omega_2,x) \sqcup \mathrm{N}^-_{\succ}(\omega_1,\omega_2,x) \sqcup \mathrm{N}^-_{=}(\omega_1,\omega_2,x), \tag{4.50}$$

where

$$\mathrm{N}^-_{\prec}(\omega_1,\omega_2,x) := \{y \in \mathrm{N}^-(\omega_1,x) : \omega_1(y) \prec \omega_2(y)\}, \tag{4.51}$$

$$\mathrm{N}^-_{\succ}(\omega_1,\omega_2,x) := \{y \in \mathrm{N}^-(\omega_1,x) : \omega_1(y) \succ \omega_2(y)\}, \text{ and} \tag{4.52}$$

$$\mathrm{N}^-_{=}(\omega_1,\omega_2,x) := \{y \in \mathrm{N}^-(\omega_1,x) : \omega_1(y) = \omega_2(y)\}. \tag{4.53}$$

Let $\mathscr{P} : D \to D$ be the function that, given $(\omega_1,\omega_2,x) \in D$, returns

1. (ω_1,ω_2,y), if $\mathrm{N}^-_{\prec}(\omega_1,\omega_2,x) \neq \emptyset$,
2. (ω_2,ω_1,y), if $\mathrm{N}^-_{\prec}(\omega_1,\omega_2,x) = \emptyset$ and $\mathrm{N}^-_{\succ}(\omega_1,\omega_2,x) \neq \emptyset$, or
3. (ω_1,ω_2,x), if $\mathrm{N}^-_{\prec}(\omega_1,\omega_2,x) = \mathrm{N}^-_{\succ}(\omega_1,\omega_2,x) = \emptyset$,

where y is the minimal element in $\mathrm{N}^-_{\prec}(\omega_1,\omega_2,x)$ or $\mathrm{N}^-_{\succ}(\omega_1,\omega_2,x)$, respectively. Here the minimal element y is taken according to the previously fixed order of $\mathscr{V}$. We chose y to be minimal just to have $\mathscr{P}$ well-defined; it will not be otherwise relevant. Notice that if $(\omega_1,\omega_2,x) \in D_\ell$ and $\mathscr{P}(\omega_1,\omega_2,x) \neq (\omega_1,\omega_2,x)$, then $\mathscr{P}(\omega_1, \omega_2, x) \in D(\ell - 1)$. This implies that every element in D_1 must be a fixed point. Moreover, for every $(\omega_1, \omega_2, x) \in D$, the $(|\mathrm{H}| - 2)$-iteration of $\mathscr{P}$ is a fixed point (though not necessarily in D_1), i.e. $\mathscr{P}\left(\mathscr{P}^{|\mathrm{H}|-2}(\omega_1, \omega_2, x)\right) = \mathscr{P}^{|\mathrm{H}|-2}(\omega_1, \omega_2, x)$.

We have the following lemma.

Lemma 4.5.7. *Let $(\omega_1, \omega_2, x) \in D$ be such that $\mathscr{P}(\omega_1, \omega_2, x) = (\omega_1, \omega_2, x)$. Then, there exists $u \in \mathrm{V}$ such that $\omega_1(x) \prec u$, and the point $\tilde{\omega}_1$ defined as*

$$\tilde{\omega}_1(y) = \begin{cases} u & \text{if } y = x, \\ \omega_1(y) & \text{if } y \neq x, \end{cases} \tag{4.54}$$

is globally admissible. In particular, $\omega_1 \prec \tilde{\omega}_1$.

Proof. Notice that if (ω_1, ω_2, x) is a fixed point, $\mathrm{N}^-(\omega_1, x) = \mathrm{N}^-_{\equiv}(\omega_1, \omega_2, x)$. We have two cases:

Case 1: $\mathrm{N}^-(\omega_1, x) = \emptyset$. If this is the case, then $\omega_1(y) \succeq \omega_1(x)$, for all $y \in \mathrm{N}(x)$. Notice that $\omega_1(x) \prec \omega_2(x) \preceq v_{|\mathrm{H}|}$, so $\omega_1(x) \prec v_{|\mathrm{H}|}$. Then we have three sub-cases:

Case 1.a: $\omega_1(x) \in \mathrm{J}_k$ for some $1 \leq k \leq n$. Since $\{v, c^k\} \in E$ for all $v \in \mathrm{J}_k$, we can modify ω_1 at x in a valid way by replacing $\omega_1(x) \in \mathrm{J}_k$ with $u = c^k$.

Case 1.b: $\omega_1(x) \in \mathrm{T}_j$ for some $1 \leq j \leq m$. Since $\omega_1(y) \succeq \omega_1(x)$, for all $y \in \mathrm{N}(x)$, but $\omega_1(x) \in \mathrm{T}_j$ and T_j does not have loops, we have $\omega_1(y) \succ \omega_1(x)$, for all $y \in \mathrm{N}(x)$. Call $t = \omega_1(x)$. Then, there are three possibilities: $t = r^j$, $\mathrm{dist}(t, r^j) = 1$, or $\mathrm{dist}(t, r^j) > 1$.

If $t = r^j$, then $\omega_1(y) = c^j$ for all $y \in \mathrm{N}(x)$, where $c^j \succ r^j$ is the unique vertex in C connected with r^j. Since c^j must have a loop, we can replace $\omega_1(x)$ by $u = c^j \succ \omega_1(x)$ in ω_1.

If $\mathrm{dist}(t, r^j) = 1$, then $\omega_1(y) = r^j$ for all $y \in \mathrm{N}(x)$, and, similarly to the previous case, we can replace $\omega_1(x)$ by $u = c^j \succ \omega_1(x)$ in ω_1.

Finally, if $\mathrm{dist}(t, r^j) > 1$, then $\omega_1(y) = f \succ t$, for all $y \in \mathrm{N}(x)$, where $f \in \mathrm{T}_j$ is the parent of t in the r^j-rooted tree $\mathrm{H}[\mathrm{T}_j]$. Then, since $\mathrm{dist}(t, r^j) > 1$, there must exist $h \in \mathrm{T}_j$ that is the parent of f, so we can replace $\omega_1(x)$ by $u = h$ in ω_1.

Case 1.c: $\omega_1(x) \in \mathrm{C}$. If this is the case, and since $\omega_1(x) \prec v_{|\mathrm{H}|}$, there must exist $1 \leq i < |\mathrm{C}|$ such that $\omega_1(x) = c_i$ and $\mathrm{N}(c_i) \cap \{c_{i+1}, \ldots, c_{|\mathrm{C}|}\}$ is nonempty. Now, $\omega_1(y) \succeq \omega_1(x)$, for all $y \in \mathrm{N}(x)$, so $\omega_1(y) \in \{c_i\} \cup \big(\mathrm{N}(c_i) \cap \{c_{i+1}, \ldots, c_{|\mathrm{C}|}\}\big)$, for all $y \in \mathrm{N}(x)$. Since $\mathrm{H}[\{c_i\} \cup \big(\mathrm{N}(c_i) \cap \{c_{i+1}, \ldots, c_{|\mathrm{C}|}\}\big)]$ is a loop-complete graph with two or more elements, then we can replace $\omega_1(x)$ by any element $u \in \mathrm{N}(c_i) \cap \{c_{i+1}, \ldots, c_{|\mathrm{C}|}\}$ in ω_1.

Case 2: $\mathrm{N}^-(\omega_1, x) \neq \emptyset$. In this case, $\omega_1(x) \prec \omega_2(x)$ and, since

$$\mathrm{N}^-(\omega_1, x) \neq \emptyset \quad \text{and} \quad \mathrm{N}^-(\omega_1, x) = \mathrm{N}^-_{=}(\omega_1, \omega_2, x), \tag{4.55}$$

there must exist $y^* \in \mathrm{N}(x)$ such that $\omega_1(y^*) \prec \omega_1(x)$ and $\omega_1(y^*) = \omega_2(y^*)$. Notice that in this case, $\omega_1(x)$ cannot belong to T, because $\omega_1(x) \prec \omega_2(x)$ and both are connected to $\omega_1(y^*)$; this would imply that, for some $1 \leq j \leq m$, either (a) T_j does not induce a tree, or (b) more than one vertex in T_j is adjacent to a vertex in C. Therefore, we can assume that $\omega_1(x)$ belongs to C (and therefore, since $\omega_2(x) \succ \omega_1(x)$, also $\omega_2(x)$ belongs to C).

We are going to prove that, for every $y \in \mathrm{N}(x)$, we have $\omega_1(y) \sim \omega_2(x)$, so we can replace $\omega_1(x)$ by $\omega_2(x)$ in ω_1. Since $\omega_1(y) = \omega_2(y)$, for every $y \in \mathrm{N}^-(\omega_1, x)$, we only need to prove that $\omega_1(y) \sim \omega_2(x)$, for every $y \in \mathrm{N}(x)$ such that $\omega_1(y) \succeq \omega_1(x)$.

Take any $y \in \mathrm{N}^-_{=}(\omega_1, \omega_2, x)$. Then, $\omega_2(y) = \omega_1(y) \precsim \omega_1(x)$ and $\omega_2(y) \precsim \omega_2(x)$, so $\omega_1(x) \sim \omega_2(x)$, since H is loop-chordal. Consider now an arbitrary $y \in \mathrm{N}(x)$ such that $\omega_1(y) \succeq \omega_1(x)$. If $\omega_1(y) = \omega_1(x)$, we have $\omega_1(y) = \omega_1(x) \sim \omega_2(x)$, so we can assume that $\omega_1(y) \succ \omega_1(x)$. Then, $\omega_1(x) \precsim \omega_1(y)$ and $\omega_1(x) \precsim \omega_2(x)$, so $\omega_1(y) \sim \omega_2(x)$, again by loop-chordality of H. Then, $\omega_1(y) \sim \omega_2(x)$, for every $y \in \mathrm{N}(x)$, and we can replace $\omega_1(x)$ by $u = \omega_2(x)$ in ω_1, as desired. $\square$

Now we are in a good position to prove Theorem 4.5.6.

Proof of Theorem 4.5.6. Fix an arbitrary set $A \Subset \mathscr{V}$ and $\alpha \in \mathscr{L}_A(\mathrm{Hom}(\mathscr{G}, \mathrm{H}))$. We proceed to prove the conditions (M1) (i.e. existence and uniqueness of a maximal point ω_α) and $(M2)$.

Condition $(M1)$**.** Choose an ordering $x_1, x_2, \ldots$ of $\mathscr{V} \setminus A$ and, for $n \in \mathbb{N}$, define $A_n := A \cup \{x_1, \ldots, x_n\}$. Let $\alpha_0 := \alpha$ and suppose that, for a given n and all $0 < i \leq n$, we have already constructed a sequence $\alpha_i \in \mathscr{L}_{A_i}(\mathrm{Hom}(\mathscr{G},\mathrm{H}))$ such that $\alpha_i|_{A_{i-1}} = \alpha_{i-1}$ and $\beta(x_i) \preceq \alpha_i(x_i)$, for any $\beta \in \mathscr{L}_{A_i}(\mathrm{Hom}(\mathscr{G},\mathrm{H}))$ such that $\beta|_{A_{i-1}} = \alpha_{i-1}$.

Next, look for the globally admissible configuration α_{n+1} such that $\alpha_{n+1}|_{A_n} = \alpha_n$ and $\beta(x_{n+1}) \preceq \alpha_{n+1}(x_{n+1})$, for any $\beta \in \mathscr{L}_{A_{n+1}}(\mathrm{Hom}(\mathscr{G},\mathrm{H}))$ such that $\beta|_{A_n} = \alpha_n$. Iterating and by compactness of $\mathrm{Hom}(\mathscr{G},\mathrm{H})$, we conclude the existence of a unique point $\hat{\omega} \in \bigcap_{n\in\mathbb{N}}[\alpha_n]^{\mathscr{G}}_{\mathrm{H}}$.

We claim that $\hat{\omega}$ is independent of the ordering $x_1, x_2, \ldots$ of $\mathscr{V} \setminus A$.

By contradiction, suppose that given two orderings of $\mathscr{V} \setminus A$ we can obtain two different configurations $\hat{\omega}_1$ and $\hat{\omega}_2$ with the properties described above. Take $\bar{x} \in \mathscr{V} \setminus A$ such that $\hat{\omega}_1(\bar{x}) \neq \hat{\omega}_2(\bar{x})$. W.l.o.g., suppose that $\hat{\omega}_1(\bar{x}) \prec \hat{\omega}_2(\bar{x})$. Then we have that $(\hat{\omega}_1, \hat{\omega}_2, \bar{x}) \in D$ and $\mathscr{P}^{|\mathrm{H}|-2}(\hat{\omega}_1, \hat{\omega}_2, \bar{x})$ is a fixed point for $\mathscr{P}$. W.l.o.g., suppose that $\mathscr{P}^{|\mathrm{H}|-2}(\hat{\omega}_1, \hat{\omega}_2, \bar{x}) = (\hat{\omega}_1, \hat{\omega}_2, \tilde{x})$, where $\tilde{x} \in \mathscr{V} \setminus A$ (note that $\tilde{x}$ is not necessarily equal to $\bar{x}$). By an application of Lemma 4.5.7, $\hat{\omega}_1(\tilde{x})$ can be replaced in a valid way by a vertex $u \in \mathrm{V}$ such that $\hat{\omega}_1(\tilde{x}) \prec u$. If we let n be such that $\tilde{x} = x_n$ for the ordering corresponding to $\hat{\omega}_1$, we have a contradiction with the maximality of $\hat{\omega}_1$, since we could have chosen u instead of $\hat{\omega}_1(x_n)$ in the nth step of the construction of $\hat{\omega}_1$.

Therefore, there exists a particular $\hat{\omega}$ common to any ordering $x_1, x_2, \ldots$ of $\mathscr{V} \setminus A$. We claim that taking $\omega_\alpha = \hat{\omega}$ proves $(M1)$. In fact, suppose that there exists $\omega \in [\alpha]^{\mathscr{G}}_{\mathrm{H}}$ and $x^* \in \mathscr{V} \setminus A$ such that $\hat{\omega}(x^*) \prec \omega(x^*)$. We can always choose an ordering of $\mathscr{V} \setminus A$ such that $x_1 = x^*$. Then, according to such ordering, $\beta \preceq \hat{\omega}|_{A_1}$ for any $\beta \in \mathscr{L}_{A_1}(\mathrm{Hom}(\mathscr{G},\mathrm{H}))$ such that $\beta|_A = \alpha$. In particular, if we take $\beta = \alpha\ \omega|_{\{x^*\}}$, we have a contradiction.

Condition $(M2)$**.** Notice that if $(\omega_1', \omega_2', y) = \mathscr{P}(\omega_1, \omega_2, x)$, then $x = y$ or $x \sim y$. In addition, since the $(|\mathrm{H}|-2)$-iteration of $\mathscr{P}$ is a fixed point, if $(\omega_1', \omega_2', y) = \mathscr{P}^{|\mathrm{H}|-2}(\omega_1, \omega_2, x)$, then $\mathrm{dist}(x,y) \leq |\mathrm{H}|-2$. In order to prove condition $(M2)$, consider two configurations $\alpha_1, \alpha_2 \in \mathscr{L}_A(\mathrm{Hom}(\mathscr{G},\mathrm{H}))$, and the set $\Sigma_{\mathscr{V}}(\omega_{\alpha_1}, \omega_{\alpha_2})$. We want to prove that

$$\Sigma_{\mathscr{V}}(\omega_{\alpha_1}, \omega_{\alpha_2}) \subseteq \mathscr{N}_{|\mathrm{H}|-2}(\Sigma_A(\alpha_1, \alpha_2)). \tag{4.56}$$

W.l.o.g., suppose that $\alpha_1 \neq \alpha_2$ and take $x \in \Sigma_{\mathscr{V}}(\omega_{\alpha_1}, \omega_{\alpha_2}) \neq \emptyset$. It suffices to check that $\mathrm{dist}(x, \Sigma_A(\alpha_1, \alpha_2)) \leq |\mathrm{H}|-2$. Suppose for the sake of contradiction that

$\mathrm{dist}(x, \Sigma_A(\alpha_1, \alpha_2)) > |\mathrm{H}| - 2$ and let $(\omega'_{\alpha_1}, \omega'_{\alpha_2}, y) = \mathscr{P}^{|\mathrm{H}|-2}(\omega_{\alpha_1}, \omega_{\alpha_2}, x)$. Notice that, by definition of $\mathscr{P}$, y also belongs to $\Sigma_{\mathscr{V}}(\omega_{\alpha_1}, \omega_{\alpha_2})$, and since $\mathrm{dist}(x,y) \leq |\mathrm{H}|$, we have $y \notin \Sigma_A(\alpha_1, \alpha_2)$. Then, there are two possibilities: (a) $y \in A \setminus \Sigma_A(\alpha_1, \alpha_2)$, or (b) $y \in \mathscr{V} \setminus A$.

If $y \in A \setminus \Sigma_A(\alpha_1, \alpha_2)$, then $\omega_{\alpha_1}(y) = \alpha_1(y) = \alpha_2(y) = \omega_{\alpha_2}(y)$, and that contradicts the fact that $y \in \Sigma_{\mathscr{V}}(\omega_{\alpha_1}, \omega_{\alpha_2})$.

If $y \in \mathscr{V} \setminus A$ and, w.l.o.g., $\omega_{\alpha_1}(y) \prec \omega_{\alpha_2}(y)$, we can apply Lemma 4.5.7 to contradict the maximality of ω_{α_1}. $\square$

4.6 The looped tree case

A looped tree T will be called trivial if $|T| = 1$ and nontrivial if $|T| \geq 2$. We proceed to define a family of graphs that will be useful in future proofs.

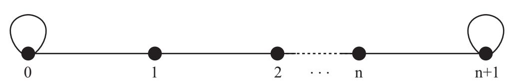

Figure 4.3: The n-barbell graph B^n.

Definition 4.6.1. *Given $n \in \mathbb{N}$, define the n-**barbell** graph as $B^n = (V(B^n), E(B^n))$, where*

$$V(B^n) = \{0, 1, \ldots, n, n+1\}, \tag{4.57}$$

and

$$E(B^n) = \{\{0,0\}, \{0,1\}, \ldots, \{n, n+1\}, \{n+1, n+1\}\}. \tag{4.58}$$

Notice that a looped tree with a safe symbol must be an n-star (see Equation (2.7)) with a loop at the central vertex, possibly along with other loops. The graph H_φ can be seen as a very particular case of a looped tree with a safe symbol. For more general looped trees, we have the next result.

Proposition 4.6.1. *Let T be a finite nontrivial looped tree. Then, the following are equivalent:*

(1) T is chordal/tree decomposable.

(2) T is dismantlable.

(3) $\mathrm{Loop}(T)$ *is connected in* T *and nonempty.*

Proof. We have the following implications.

(1) $\implies$ (2)**:** This follows from Theorem 4.6.3, which is for general constraint graphs.

(2) $\implies$ (3)**:** Assume that T is dismantlable. First, suppose $\mathrm{Loop}(T) = \emptyset$. Then, in any sequence of foldings of T, in the next to last step, we must end with a graph consisting of just two adjacent vertices v_{n-1} and v_n, without loops. However, this is a contradiction, because $\mathrm{N}(u) \subsetneq \mathrm{N}(v)$ and $\mathrm{N}(v) \subsetneq \mathrm{N}(u)$, so such graph cannot be folded into a single vertex. Therefore, $\mathrm{Loop}(T)$ is nonempty.

Next, suppose that $\mathrm{Loop}(T)$ is nonempty and not connected. Then, T must have an n-barbell as a subgraph, for some $n \geq 1$. Therefore, in any sequence of foldings of T, there must have been a vertex in the n-barbell that was folded first. Let's call such vertex u and take $v \in V$ with $\mathrm{N}(u) \subseteq \mathrm{N}(v)$. Then, v is another vertex in the n-barbell or it belongs to the complement. Notice that v cannot be in the n-barbell, because no neighbourhood of vertex in the n-barbell (even restricted to the barbell itself) contains the neighbourhood of another vertex in the n-barbell. On the other hand, v cannot be in the complement of n-barbell, because v would have to be connected to two or more vertices in the n-barbell (u and its neighbours), and that would create a cycle in T. Therefore, $\mathrm{Loop}(T)$ is connected.

(3) $\implies$ (1)**:** Define $\mathrm{C} := \mathrm{Loop}(T)$. Then C is connected in T and nonempty. Then, if we denote by T its complement $V \setminus \mathrm{C}$ and define $\mathrm{J} = \emptyset$, we have that V can be partitioned into the three subsets $\mathrm{C} \sqcup \mathrm{T} \sqcup \mathrm{J}$, which corresponds to a chordal/tree decomposition. $\square$

Corollary 7. *Let* T *be a finite nontrivial looped tree. Then, the following are equivalent:*

(1) T *is chordal/tree decomposable.*

(2) $\mathrm{Hom}(\mathscr{G}, T)$ *has the UMC property, for every board* $\mathscr{G}$.

(3) $\mathrm{Hom}(\mathscr{G}, T)$ *satisfies TSSM, for every board* $\mathscr{G}$.

(4) $\mathrm{Hom}(\mathscr{G}, T)$ *is strongly irreducible, for every board* $\mathscr{G}$.

(5) T is dismantlable.

Proof. By Theorem 4.6.3, we have $(1) \implies (2) \implies (3) \implies (4) \implies (5)$. The implication $(5) \implies (1)$ follows from Proposition 4.6.1. □

Sometimes, given a constraint graph H, if a property for homomorphism spaces holds for a certain distinguished board or family of boards, then the property holds for any board $\mathcal{G}$. For example, this is proven in [17] for a dismantlable graph H and the strong irreducibility property, when $\mathcal{G} \in \{\mathbb{T}_d\}_{d\in\mathbb{N}}$. The next result gives another example of this phenomenon.

Proposition 4.6.2. *Let T be a finite looped tree. Then, the following are equivalent:*

(1) $\mathrm{Hom}(\mathcal{G},T)$ *satisfies TSSM, for every board $\mathcal{G}$..*

(2) $\mathrm{Hom}(\mathbb{Z}^2,T)$ *satisfies TSSM.*

(3) There exist Gibbs $(\mathbb{Z}^2,T,\phi)$-specifications which satisfy exponential SSM with arbitrarily high decay rate.

Proof. We have the following implications.

$(1) \implies (2)$: Trivial.

$(2) \implies (1)$: Let's suppose that $\mathrm{Hom}(\mathbb{Z}^2,T)$ satisfies TSSM. If T is trivial, then $\mathrm{Hom}(\mathcal{G},T)$ is a single point or empty, depending on whether the unique vertex in T has a loop or not. In both cases, $\mathrm{Hom}(\mathcal{G},T)$ satisfies TSSM, for every board $\mathcal{G}$. If T is nontrivial, then $\mathrm{Loop}(T)$ must be nonempty. To see this, by contradiction, first suppose that T is nontrivial and $\mathrm{Loop}(T) = \emptyset$. Take an arbitrary vertex $u \in V(T)$ and a neighbour $v \in \mathrm{N}(u)$. Notice that $\mathrm{Hom}(\mathbb{Z}^2,T)$ is nonempty, since the point $\omega^{u,v}$ defined as

$$\omega^{u,v}(x) = \begin{cases} u & \text{if } x_1+x_2 = 0 \mod 2, \\ v & \text{if } x_1+x_2 = 1 \mod 2, \end{cases} \tag{4.59}$$

is globally admissible. Now, if we interchange the roles of u and v, and consider the (globally admissible) point $\omega^{v,u}$, we have $\omega^{u,v}((0,0)) = u$ and $\omega^{v,u}((2g+1,0)) = u$, for an arbitrary $g \in \mathbb{N}$. However, if this is the case, $\mathrm{Hom}(\mathbb{Z}^2,T)$ cannot be strongly irreducible with gap g, for any g (and therefore, cannot be TSSM), because

$$\left[\omega^{u,v}|_{(0,0)}\ \omega^{v,u}|_{(2g+1,0)}\right]_T^{\mathbb{Z}^2} = \emptyset. \tag{4.60}$$

A way to check this is by considering the fact that both T and $\mathbb{Z}^2$ are bipartite graphs. Therefore, we can assume that $\mathrm{Loop}(T) \neq \emptyset$.

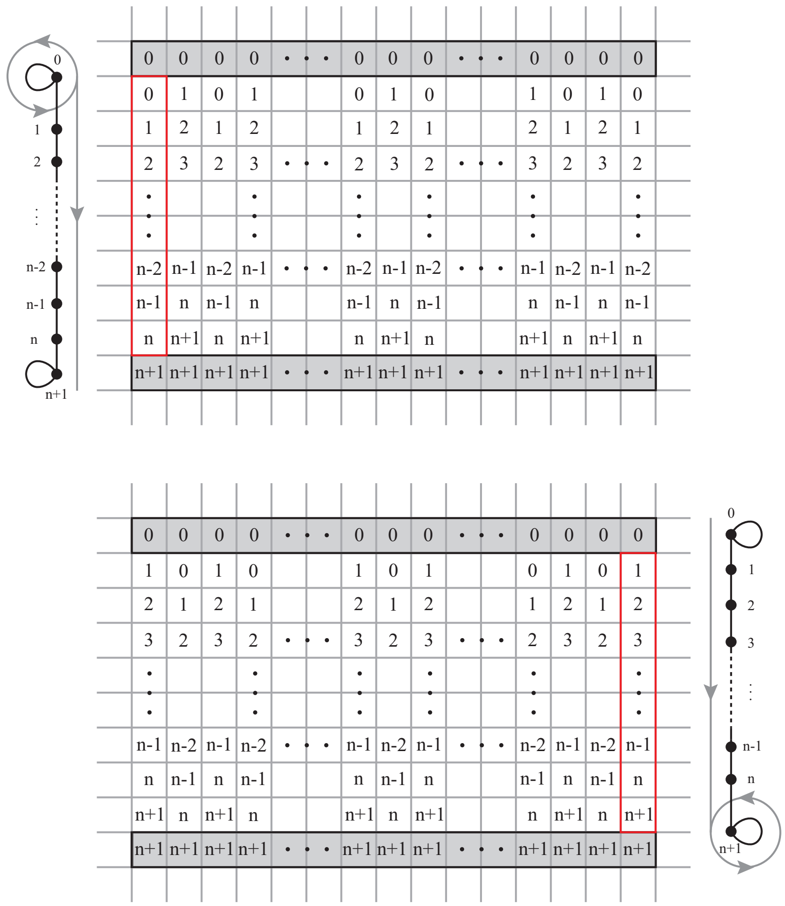

Figure 4.4: A "channel" in $\mathrm{Hom}(\mathbb{Z}^2, T)$ with two incompatible extremes.

Now, suppose that $\mathrm{Loop}(T) \neq \emptyset$ and $\mathrm{Loop}(T)$ is not connected in T. If this is the case, T must have an n-barbell as an induced subgraph, for some $n \geq 1$. Then, we would be able to construct configurations in $\mathscr{L}(\mathrm{Hom}(\mathbb{Z}^2, T))$ as shown in Figure 4.4. Note that vertices in the n-barbell can reach each other only through the path determined by the n-barbell, since T does not contain cycles. In Figure 4.4 are represented the cylinder sets $[\alpha\sigma]_T^{\mathbb{Z}^2}$ (top left), $[\sigma\beta]_T^{\mathbb{Z}^2}$ (top right) and $[\alpha\sigma\beta]_T^{\mathbb{Z}^2}$ (bottom), where:

1. α is the vertical left-hand side configuration in red, representing a sequence of nodes in the n-barbell that repeats 0 but not $n+1$,

2. β is the vertical right-hand side configuration in red, representing a sequence of nodes in the n-barbell that repeats $n+1$ but not 0, and

3. σ is the horizontal (top and bottom) configuration in black, representing loops on the vertices 0 and $n+1$, respectively.

It can be checked that $[\alpha\sigma]_T^{\mathbb{Z}^2}$ and $[\sigma\beta]_T^{\mathbb{Z}^2}$ are nonempty. However, the cylinder set $[\alpha\sigma\beta]_T^{\mathbb{Z}^2}$ is empty for every even separation distance between α and β, since α and β force incompatible alternating configurations inside the "channel" determined by σ. Therefore, $\mathrm{Hom}(\mathbb{Z}^2,T)$ cannot satisfy TSSM, which is a contradiction.

We conclude that $\mathrm{Loop}(T)$ is nonempty and connected in T, and by Proposition 4.6.1 T is chordal/tree decomposable. Finally, by Proposition 7, we conclude that $\mathrm{Hom}(\mathscr{G},T)$ satisfies TSSM, for every board $\mathscr{G}$.

$(3) \implies (2)$: This follows from Theorem 4.2.6.

$(1) \implies (3)$: Since $\mathrm{Hom}(\mathscr{G},T)$ satisfies TSSM for every board $\mathscr{G}$, $\mathrm{Hom}(\mathscr{G},T)$ has the UMC property for all board $\mathscr{G}$ (see Corollary 7). In particular, $\mathrm{Hom}(\mathbb{Z}^2,T)$ has the UMC property. Then, (3) follows from Proposition 4.4.2. □

We have the following summary.

Theorem 4.6.3. *Fix a constraint graph* H.

- *Then,*

$$\mathrm{H} \textit{ has a safe symbol} \iff \mathrm{Hom}(\mathscr{G},\mathrm{H}) \textit{ is SSF, for all } \mathscr{G}$$
$$\implies \mathrm{H} \textit{ is chordal/tree decomposable}$$

$$\begin{aligned}
&\Longrightarrow \mathrm{Hom}(\mathscr{G},\mathrm{H}) \textit{ has the UMC property, for all } \mathscr{G}\\
&\Longrightarrow \mathrm{Hom}(\mathscr{G},\mathrm{H}) \textit{ satisfies TSSM, for all } \mathscr{G}\\
&\Longrightarrow \mathrm{Hom}(\mathscr{G},\mathrm{H}) \textit{ is strongly irreducible, for all } \mathscr{G}\\
&\Longleftrightarrow \mathrm{H} \textit{ is dismantlable}.
\end{aligned}$$

- *Let* $\mathscr{G}$ *a fixed board. Then,*

$$\begin{aligned}
\mathrm{Hom}(\mathscr{G},\mathrm{H}) \textit{ has the UMC property} &\Longrightarrow \mathrm{Hom}(\mathscr{G},\mathrm{H}) \textit{ satisfies TSSM}\\
&\Longrightarrow \mathrm{Hom}(\mathscr{G},\mathrm{H}) \textit{ is strongly irreducible}.
\end{aligned}$$

- *Let* $\mathscr{G}$ *a fixed board with bounded degree. Then,*

$$\begin{aligned}
\mathrm{Hom}(\mathscr{G},\mathrm{H}) \textit{ has the UMC property} &\Longrightarrow \textit{For all } \gamma > 0\textit{, there exists a Gibbs}\\
&\qquad (\mathscr{G},\mathrm{H},\phi)\textit{-specification that satisfies}\\
&\qquad \textit{exponential SSM with decay rate } \gamma\\
&\qquad\qquad\qquad \Downarrow\\
\mathrm{H} \textit{ is dismantlable} &\Longrightarrow \textit{For all } \gamma > 0\textit{, there exists a Gibbs}\\
&\qquad (\mathscr{G},\mathrm{H},\phi)\textit{-specification that satisfies}\\
&\qquad \textit{exponential WSM with decay rate } \gamma.
\end{aligned}$$

Proof. The first chain of implications and equivalences follows from Proposition 4.1.2, Proposition 4.5.5, Theorem 4.5.6, Proposition 4.3.1, Equation (4.3), and Proposition 4.2.1. The second one, from Proposition 4.3.1 and Equation (4.3). The last one, from Proposition 4.4.2, Proposition 4.2.1, and the fact that SSM always implies WSM. □

4.7 Examples: Homomorphism spaces

Proposition 4.7.1. *There exists a homomorphism space* $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ *that satisfies TSSM but not the UMC property.*

Proof. The homomorphism space $\mathrm{Hom}(\mathbb{Z}^2,\mathrm{K}_5)$ satisfies TSSM (in fact, it satisfies

SSF) but not the UMC property. If $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_5)$ satisfies the UMC property, then there must exist an order $\prec$ and a greatest element $\omega_* \in \mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_5)$ according to such order (see Section 4.3). Denote $V(\mathrm{K}_5) = \{1, 2, \ldots, 5\}$ and, w.l.o.g., assume that $1 \prec 2 \prec \cdots \prec 5$. Then, because of the constraints imposed by K_5, there must exist $\bar{x} \in \mathbb{Z}^2$ such that $\omega_*(\bar{x}) \prec \omega_*(\bar{x} + (1,0))$. Now, consider the point $\tilde{\omega}$ such that $\tilde{\omega}(x) = \omega_*(x + (1,0))$ for every $x \in \mathbb{Z}^2$, i.e. a shifted version of ω_* (in particular, $\tilde{\omega}$ also belongs to $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{K}_5)$). Then $\omega(\bar{x}) \prec \omega_*(\bar{x} + (1,0)) = \tilde{\omega}(\bar{x})$, which contradicts the maximality of ω_*. $\square$

Note 7. *We are not aware of a homomorphism space* $\mathrm{Hom}(\mathscr{G}, \mathrm{H})$ *that satisfies the UMC property with* H *not a chordal/tree decomposable graph.*

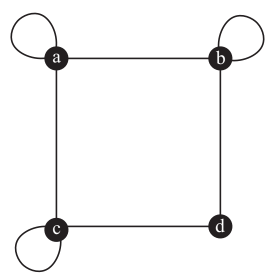

Figure 4.5: A dismantlable graph H such that $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{H})$ is not TSSM.

Proposition 4.7.2. *There exists a dismantlable graph* H *such that*

1. $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{H})$ *does not satisfy TSSM.*
2. *There is no constrained energy function* ϕ *such that the Gibbs* $(\mathbb{Z}^2, \mathrm{H}, \phi)$*-specification satisfies SSM.*
3. *For every board of bounded degree* $\mathscr{G}$ *and* $\gamma > 0$*, there exists a constrained energy function* ϕ *such that the Gibbs* $(\mathscr{G}, \mathrm{H}, \phi)$*-specification satisfies exponential WSM with decay rate* γ*.*

Proof. Consider the constraint graph $\mathrm{H} = (\mathrm{V}, \mathrm{E})$ given by $\mathrm{V} = \{a, b, c, d\}$, and

$$\mathrm{E} = \{\{a,a\}, \{b,b\}, \{c,c\}, \{a,b\}, \{a,c\}, \{b,d\}, \{c,d\}\}. \tag{4.61}$$

It is easy to check that H is dismantlable (see Figure 4.5).

By Proposition 4.2.1, we know that (3) holds. However, if we consider the configurations α and β (the pairs bb and cc in red, respectively) and the fixed configuration σ (the diagonal alternating configurations $adad\cdots$ in black) shown in Figure 4.6, we have that $[\alpha\sigma]^{\mathbb{Z}^2}_{\mathrm{H}}, [\sigma\beta]^{\mathbb{Z}^2}_{\mathrm{H}} \neq \emptyset$, but $[\alpha\sigma\beta]^{\mathbb{Z}^2}_{\mathrm{H}} = \emptyset$, and TSSM cannot hold. This construction works in a similar way to the construction in the proof ((2) $\implies$ (1)) of Proposition 4.6.2.

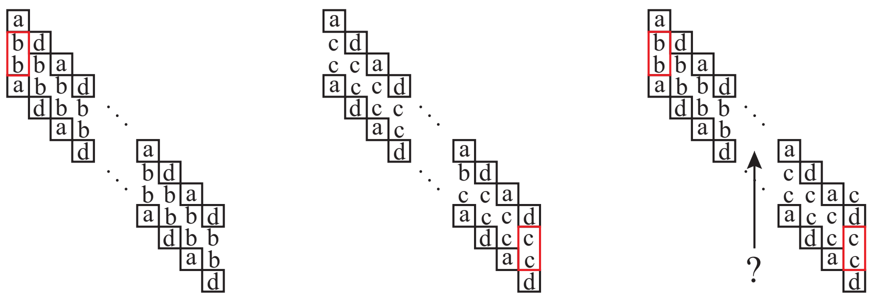

Figure 4.6: Two incompatible configurations α and β (both in red).

Now assume the existence of a Gibbs $(\mathbb{Z}^2, \mathrm{H}, \phi)$-specification π satisfying SSM with decay function f. Call $A \subseteq \mathbb{Z}^2$ the shape enclosed by the two diagonals made by alternating sequence of a's and d's shown in Figure 4.7 (in grey), and let x_l and x_r be the sites (in red) at the left and right extreme of A, respectively. If we denote by σ the boundary configuration of the a and d symbols on $\partial A \setminus \{x_r\}$, and $\alpha_1 = b^{\{x_r\}}$ and $\alpha_2 = c^{\{x_r\}}$, it can be checked that $[\sigma\alpha_1]^{\mathscr{G}}_{\mathrm{H}}, [\sigma\alpha_2]^{\mathscr{G}}_{\mathrm{H}} \neq \emptyset$. Then, take $\omega_1 \in [\sigma\alpha_1]^{\mathscr{G}}_{\mathrm{H}}$, $\omega_2 \in [\sigma\alpha_2]^{\mathscr{G}}_{\mathrm{H}}$, and call $B = \{x_l\}$ and $\beta = b^B$.

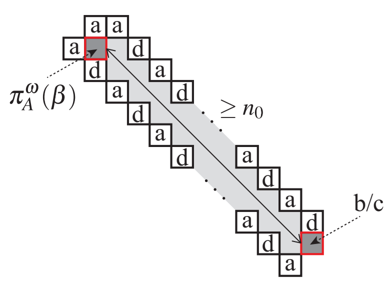

Figure 4.7: A scenario where SSM fails for any Gibbs $(\mathbb{Z}^2, \mathrm{H}, \phi)$-specification.

Notice that, similarly as before, the symbols b and c force repetitions of themselves, respectively, from x_r to x_l along A. Then, we have that $\pi_A^{\omega_1}(\beta) = 1$ and $\pi_A^{\omega_2}(\beta) = 0$. Now, since we can always take an arbitrarily long set A, suppose that $\mathrm{dist}(x_l, x_r) \geq n_0$, with n_0 such that $f(n_0) < 1$. Therefore,

$$1 = |1-0| = \left|\pi_A^{\omega_1}(\beta) - \pi_A^{\omega_2}(\beta)\right| \tag{4.62}$$

$$\leq |B| f\left(\mathrm{dist}(B, \Sigma_{\partial A}(\omega_1, \omega_2))\right) \leq f(n_0) < 1, \tag{4.63}$$

which is a contradiction. □

Proposition 4.7.3. *There exists a dismantlable graph* H *and a constant* $\gamma_0 > 0$*, such that*

1. *the set of constrained energy functions* ϕ *for which the Gibbs* $(\mathbb{Z}^2, \mathrm{H}, \phi)$*-specification satisfies exponential SSM is nonempty,*
2. *there is no constrained energy function* ϕ *for which the Gibbs* $(\mathbb{Z}^2, \mathrm{H}, \phi)$*-specification satisfies exponential SSM with decay rate greater than* γ_0*,*
3. $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{H})$ *satisfies SSF (in particular,* $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{H})$ *satisfies TSSM), and*
4. *for every* $\gamma > 0$*, there exists a constrained energy function* ϕ *for which the Gibbs* $(\mathbb{Z}^2, \mathrm{H}, \phi)$*-specification satisfies exponential WSM with decay rate* γ*.*

Moreover, there exists a family $\{\mathrm{H}^q\}_{q \in \mathbb{N}}$ *of dismantlable graphs with these properties such that* $|\mathrm{H}^q| \to \infty$ *as* $q \to \infty$*.*

Proof. By Theorem 3.6.1, we know that if π is such that $Q(\pi) < p_c(\mathbb{Z}^2)$, then π satisfies exponential SSM, where $p_c(\mathbb{Z}^2)$ denotes the critical value of Bernoulli site percolation on $\mathbb{Z}^2$ and $Q(\pi)$ is defined as

$$Q(\pi) = \max_{\omega_1, \omega_2} d_{\mathrm{TV}}\left(\pi_{\{\vec{0}\}}^{\omega_1}, \pi_{\{\vec{0}\}}^{\omega_2}\right). \tag{4.64}$$

Given $q \in \mathbb{N}$, consider the graph H^q as shown in Figure 4.8. The graph H^q consists of a complete graph K_{q+1} and two other extra vertices a and b, both adjacent to every vertex in the complete graph. In addition, a has a loop, and a and b are not adjacent. Notice that H^q is dismantlable (we can fold a into b and then we can fold every vertex in K_{q+1} into a), and a is a persistent vertex for H^q. Since H^q is dismantlable, by Proposition 4.2.1, for every $\gamma > 0$, there exists a constrained energy function ϕ for which the Gibbs $(\mathbb{Z}^2, \mathrm{H}, \phi)$-specification satisfies exponential WSM with decay rate γ.

Take π to be the uniform Gibbs specification on $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{H})$ (i.e. $\Phi \equiv 0$). The definition of $\pi^{\omega}_{\{\vec{0}\}}(u)$ implies that, whenever $\pi^{\omega}_{\{\vec{0}\}}(u) \neq 0$ for $u \in \mathrm{V}$,

$$\frac{1}{q+2} \leq \pi^{\omega}_{\{\vec{0}\}}(u) \leq \frac{1}{q-1}. \tag{4.65}$$

Notice that $\pi^{\omega}_{\{\vec{0}\}}(a) = 0$ iff b appears in $\omega|_{\partial\{\vec{0}\}}$. Similarly, $\pi^{\omega}_{\{\vec{0}\}}(b) = 0$ iff a or b appear in $\omega|_{\partial\{\vec{0}\}}$, and for $u \neq a, b$, $\pi^{\omega}_{\{\vec{0}\}}(u) = 0$ iff u appears in $\omega|_{\partial\{\vec{0}\}}$. Since $|\partial\{\vec{0}\}| = 4$, at most 8 terms vanish in the definition of $Q(\pi)$ (4 for each ω_i, $i = 1, 2$). Then, since $|\mathrm{H}^q| = q+3$ and $q \geq 1$,

$$d_{\mathrm{TV}}\left(\pi^{\omega_1}_{\{\vec{0}\}}, \pi^{\omega_2}_{\{\vec{0}\}}\right) = \frac{1}{2}\sum_{u \in \mathrm{V}} \left|\pi^{\omega_1}_{\{\vec{0}\}}(u) - \pi^{\omega_2}_{\{\vec{0}\}}(u)\right| \tag{4.66}$$

$$\leq \frac{1}{2}\left(8\frac{1}{q-1} + (q+3)\left|\frac{1}{q-1} - \frac{1}{q+2}\right|\right) \leq \frac{6}{q-1}. \tag{4.67}$$

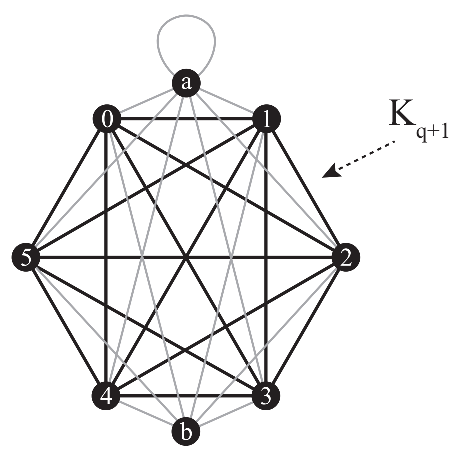

Figure 4.8: The graph H^q, for $q = 5$.

Then, if $q > 1 + \frac{6}{p_{\mathrm{c}}(\mathbb{Z}^2)}$, we have that $Q(\pi) < p_{\mathrm{c}}(\mathbb{Z}^2)$, so π satisfies exponential SSM. Since $p_{\mathrm{c}}(\mathbb{Z}^2) > 0.556$ (see [10, Theorem 1]), it suffices to take $q \geq 12$. In particular, the set of constrained energy functions Φ for which the Gibbs $(\mathbb{Z}^2, \mathrm{H}^q, \phi)$-specification satisfies exponential SSM is nonempty if $q > 12$.

Now, let π be an arbitrary Gibbs $(\mathbb{Z}^2, \mathrm{H}^q, \phi)$-specification that satisfies SSM with decay function $f(n) = Ce^{-\gamma n}$, for some C and γ that could depend on q and ϕ. For now, we fix q and an arbitrary ϕ.

Consider a configuration like the one shown in Figure 4.9. Define $\tilde{\mathrm{V}} = \mathrm{V} \setminus \{0, a, b\}$, $\tilde{\mathrm{E}} = \mathrm{E}[\tilde{\mathrm{V}}]$, and let $\tilde{\mathrm{H}}^q = \mathrm{H}^q[\tilde{\mathrm{V}}]$. Notice that $\tilde{\mathrm{H}}^q$ is isomorphic to K_q. Construct the auxiliary constrained energy function $\tilde{\phi} : \tilde{\mathrm{V}} \cup \tilde{\mathrm{E}} \to (-\infty, 0]$ given by $\tilde{\phi}(u) = \phi(u) + \phi(u, 0) + \phi(u, b)$, for every $u \in \tilde{\mathrm{V}}$ (representing the interaction with the "wall" $\cdots 0b0b \cdots$), and $\tilde{\phi} \equiv \phi|_{\tilde{\mathrm{E}}}$. The constrained energy function $\tilde{\phi}$ induces a Gibbs $(\mathbb{Z}, \tilde{\mathrm{H}}^q, \tilde{\phi})$-specification $\tilde{\pi}$ that inherits the exponential SSM property from π with the same decay function $f(n) = Ce^{-\gamma n}$. It follows that there is a unique (and therefore, stationary) n.n. Gibbs measure μ for $\tilde{\pi}$, which is a *Markov measure* with some symmetric $q \times q$ transition matrix M with zero diagonal (see [35, Theorem 10.21] and [21]).

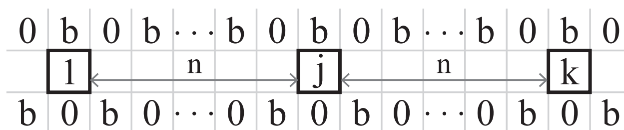

Figure 4.9: A Markov chain embedded in a $\mathbb{Z}^2$ Markov random field.

Let $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q$ be the eigenvalues of M. Since $\mathrm{tr}(M) = 0$, we have that $\sum_{i=1}^q \lambda_i = 0$. Let $\lambda_* = \max\{|\lambda_2|, |\lambda_q|\}$. Then, since $\lambda_1 = 1$, we have that $1 \leq \sum_{i=2}^q |\lambda_i| \leq (q-1)\lambda_*$. Therefore, $\lambda_* \geq \frac{1}{q-1}$.

Since M is *stochastic*, $M\vec{1} = \vec{1}$ and, since M is *primitive*, $\lambda_* < 1$ (see [63, Section 3.2]). W.l.o.g., suppose that $|\lambda_2| = \lambda_*$ and let $\vec{\ell}$ be the left eigenvector associated to λ_2 (i.e. $\vec{\ell}M = \lambda_2\vec{\ell}$). Then $\vec{\ell} \cdot \vec{1} = 0$, because $\lambda_2\vec{\ell} \cdot \vec{1} = (\vec{\ell} \cdot M) \cdot \vec{1} = \vec{\ell} \cdot (M \cdot \vec{1}) = \vec{\ell} \cdot \vec{1}$, so $(1 - \lambda_2)\vec{\ell} \cdot 1 = 0$. Then, $\vec{\ell} \in \langle \vec{e}_2 - \vec{e}_1, \vec{e}_3 - \vec{e}_1, \ldots, \vec{e}_q - \vec{e}_1 \rangle_{\mathbb{R}}$, so we can write $\vec{\ell} = \sum_{k=2}^q c_k(\vec{e}_k - \vec{e}_1)$, where $\{\vec{e}_k\}_{k=1}^q$ denotes the canonical basis of $\mathbb{R}^q$ and $c_k \in \mathbb{R}$. We conclude that $\lambda_*^n \vec{\ell} = \vec{\ell} \cdot M^n = \sum_{k=2}^q c_k(\vec{e}_k - \vec{e}_1) \cdot M^n = \sum_{k=2}^q c_k(M^n_{k\bullet} - M^n_{1\bullet})$, where $M^n_{i\bullet}$ is the vector given by the ith row of M.

Consider $j \in \{1,\dots,q\}$ such that $\vec{\ell}_j > 0$. Then, $\lambda_*^n = \sum_{k=2}^{q} \frac{c_k}{\vec{\ell}_j}(M_{kj}^n - M_{1j}^n)$ and

$$\left|M_{kj}^n - M_{1j}^n\right| = \left|\mu\left(j^{\{0\}}\middle|k^{\{-n\}}\right) - \mu\left(j^{\{0\}}\middle|1^{\{-n\}}\right)\right| \tag{4.68}$$

$$\leq \left|\mu\left(j^{\{0\}}\middle|k^{\{-n\}}\right) - \mu\left(j^{\{0\}}\middle|k^{\{-n\}},1^{\{n\}}\right)\right| \tag{4.69}$$

$$+\left|\mu\left(j^{\{0\}}\middle|k^{\{-n\}},1^{\{n\}}\right) - \mu\left(j^{\{0\}}\middle|1^{\{-n\}},k^{\{n\}}\right)\right| \tag{4.70}$$

$$+\left|\mu\left(j^{\{0\}}\middle|1^{\{-n\}},k^{\{n\}}\right) - \mu\left(j^{\{0\}}\middle|1^{\{-n\}}\right)\right| \tag{4.71}$$

$$\leq 3Ce^{-\gamma n}, \tag{4.72}$$

by the exponential SSM property of $\tilde{\pi}$ and using that $\mu\left(j^{\{0\}}\middle|k^{\{-n\}}\right)$ is a weighted average $\sum_{m\in\tilde{\mathrm{V}}} \mu\left(j^{\{0\}}\middle|k^{\{-n\}},m^{\{n\}}\right)\mu\left(m^{\{n\}}\middle|k^{\{-n\}}\right)$, along with a similar decomposition of $\mu\left(j^{\{0\}}\middle|1^{\{-n\}}\right)$. Therefore, $\lambda_*^n \leq 3C(q-1)\max_k \frac{|c_k|}{|\vec{\ell}_j|}e^{-\gamma n}$. By taking logarithms and letting $n \to \infty$, we conclude that $\gamma \leq -\log\lambda_* \leq \log(q-1)$. Then, since ϕ was arbitrary, there is no constrained energy function ϕ for which the Gibbs $(\mathbb{Z}^2, \mathrm{H}^q, \phi)$-specification satisfies exponential SSM with decay rate greater than $\gamma_0 := \log(q-1)$.

Finally, it is easy to see that if $q \geq 4$, $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{H})$ satisfies SSF. Therefore, by Proposition 3.3.3, $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{H}^q)$ satisfies TSSM (with gap $g = 2$). $\square$

Part II

Representation and poly-time approximation for pressure

Chapter 5

Entropy and pressure

In the context of $\mathbb{Z}^d$ shift spaces, we consider *(topological) pressure* and its particular case, *topological entropy*. The two appear in several subjects and both somehow try to capture the complexity of a given system by associating to it a nonnegative real number. Furthermore, and as an additional motivation, sometimes these quantities can help us to distinguish between systems that are not isomorphic (or *conjugate*) in the sense described below.

5.1 Topological entropy

A natural way to transform one $\mathbb{Z}^d$ shift space to another is via a particular class of maps compatible with the shift action called *sliding block codes*.

Definition 5.1.1. *A* **sliding block code** *between two $\mathbb{Z}^d$ shift spaces $\Omega_1 \subseteq \mathscr{A}_1^{\mathbb{Z}^d}$ and $\Omega_2 \subseteq \mathscr{A}_2^{\mathbb{Z}^d}$ is a map $J : \Omega_1 \to \Omega_2$ for which there is $N \in \mathbb{N}$ and $j : \mathscr{L}_{\mathrm{B}_N}(\Omega_1) \to \mathscr{A}_2$ such that*

$$J(\omega)(x) = j(\sigma_x(\omega)|_{\mathrm{B}_N}), \tag{5.1}$$

for all $x \in \mathbb{Z}^d$ and $\omega \in \Omega_1$. A **conjugacy** *is an invertible sliding block code, and two $\mathbb{Z}^d$ shift spaces Ω_1 and Ω_2 are said to be* **conjugate** *(denoted $\Omega_1 \cong \Omega_2$) if there is a conjugacy from one to the other.*

Example 5.1.1. *Given $N \in \mathbb{N}$ and a $\mathbb{Z}^d$ shift space $\Omega \subseteq \mathscr{A}^{\mathbb{Z}^d}$, a natural sliding block code is the* **higher block code** *$J_N : \Omega \to \left(\mathscr{A}^{\mathrm{B}_N}\right)^{\mathbb{Z}^d}$ defined by*

$$J_N(\omega)(x) = \sigma_x(\omega)|_{\mathrm{B}_N}. \tag{5.2}$$

We call the image $J_N(\Omega)$, a **higher block code representation** *of Ω. Notice that the alphabet of $J_N(\Omega)$ is a subset of $\mathscr{A}^{\mathrm{B}_N}$.*

Two $\mathbb{Z}^d$ shift spaces are often regarded as being the same if they are conjugate. Properties preserved by conjugacies are called **conjugacy invariants**. For example, the property of being a $\mathbb{Z}^d$ SFT is a conjugacy invariant: if a $\mathbb{Z}^d$ shift space Ω is conjugate to an SFT, then Ω itself is a $\mathbb{Z}^d$ SFT. The topologically mixing and strong irreducibility properties are invariants, too (see Subsection 3.2.1). Another important invariant, and one of the main objects of study in this work, is the following.

Definition 5.1.2. *The* **topological entropy** *of a $\mathbb{Z}^d$ shift space Ω is defined as*

$$h(\Omega) := \inf_n \frac{\log|\mathscr{L}_{\mathrm{B}_n}(\Omega)|}{|\mathrm{B}_n|} = \lim_{n\to\infty} \frac{\log|\mathscr{L}_{\mathrm{B}_n}(\Omega)|}{|\mathrm{B}_n|}. \tag{5.3}$$

Topological entropy is a conjugacy invariant, i.e. if $\Omega_1 \cong \Omega_2$, then $h(\Omega_1) = h(\Omega_2)$. The limit always exists because $\{|\mathscr{L}_{\mathrm{B}_n}(\Omega)|\}_n$ is a (coordinate-wise) sub-additive sequence and a well-known multidimensional extension of *Fekete's sub-additive lemma* applies (see [5]). Notice that topological entropy can be regarded as the exponential growth rate of globally admissible configurations on B_n.

It is important to point out that for every $\mathbb{Z}^d$ SFT there is a $\mathbb{Z}^d$ n.n. SFT higher block code representation, which makes $\mathbb{Z}^d$ n.n. SFTs a sufficiently rich family to study. For this reason, from now on we restrict our attention to $\mathbb{Z}^d$ n.n. SFTs.

Example 5.1.2 (TSSM is not a conjugacy invariant)**.** *Given $\mathscr{A} = \{0,1,2\}$ and the family of configurations $\mathfrak{F} = \{00,102,201\}$, consider the $\mathbb{Z}$ SFT $\Omega = \Omega_{\mathfrak{F}}$. It can be checked that Ω is strongly irreducible with gap $g = 3$. However, Ω is not TSSM. In fact, given $g \in \mathbb{N}$, consider $S = \{x \in \mathbb{Z} : 0 < x < 2g, x \text{ odd}\}$ and the configurations $\sigma = 0^S$, $\alpha = 1^{\{0\}}$, and $\beta = 2^{\{2g\}}$. Then, $[\alpha\sigma]^\Omega, [\sigma\beta]^\Omega \neq \emptyset$, because $\alpha\sigma$ can be extended with 1s in $\mathbb{Z}\setminus(S\cup\{0\})$ and $\sigma\beta$ can be extended with 2s in $\mathbb{Z}\setminus(S\cup\{2g\})$. However, $[\alpha\sigma\beta]^\Omega = \emptyset$, since the 1 in α forces any point in $[\alpha\sigma]^\Omega$ to have value 1 in $((0,2g)\cap\mathbb{Z})\setminus S$, but the 2 in β forces any point in $[\sigma\beta]^\Omega$ to have value 2 in $((0,2g)\cap\mathbb{Z})\setminus S$. Therefore, since g was arbitrary, Ω is not TSSM for any gap g.*

Now, if we define $\Omega' := \beta_1(\Omega)$, where β_1 is the higher block code with $N = 1$ (see Example 5.1.1), then Ω' is a $\mathbb{Z}$ n.n. SFT conjugate to Ω ($\Omega \cong \Omega'$), and therefore strongly irreducible (which is a conjugacy invariant). Then, by Proposition 3.4.5, we have that Ω' is TSSM, while Ω is not.

This example can be extended to any dimension d by considering the constraints $\mathfrak{F}$ in only one canonical direction. In other words, TSSM is not a conjugacy invariant for any d.

When Ω is a $\mathbb{Z}^d$ n.n. SFT, there is a simple algorithm for computing $h(\Omega)$ when $d = 1$, because $h(\Omega) = \log \lambda_M$, for λ_M the largest eigenvalue of the adjacency matrix M of the *edge shift representation* of Ω [58]. However, for $d \geq 2$, there is in general no known closed form for topological entropy. Only in a few specific cases a closed form is known (e.g. *dimer model* [46], *square ice-type model* [54]).

Example 5.1.3. *For $\Omega^1_\varphi = \mathrm{Hom}(\mathbb{Z}, \mathrm{H}_\varphi)$, it is easy to see that $h(\Omega^1_\varphi) = \log \varphi$, where $\varphi = \frac{1+\sqrt{5}}{2} = 1.61803\ldots$ is the* golden ratio*. On the other hand, for $d \geq 2$, no closed form is known for the value of $h(\Omega^d_\varphi)$.*

For $d \geq 2$, one can hope to approximate the value of the topological entropy of a $\mathbb{Z}^d$ n.n. SFT, whether by using its definition and truncating the limit or by alternative methods. A relevant fact to remember (see 3.4.2) is that, for $d \geq 2$, it is algorithmically undecidable to know if a given configuration is in $\mathscr{L}(\Omega)$ or not. In this sense, it is useful to define an alternative, still meaningful, set of configurations. Given a set of n.n. constraints $\mathfrak{F}$ and $A \Subset \mathbb{Z}^d$, we will denote $\mathscr{L}^{\text{l.a.}}_A(\mathfrak{F})$ the set of locally admissible configurations in A. Considering this, we have the following result.

Theorem 5.1.1 ([29, 44])**.** *Given a finite set of n.n. constraints $\mathfrak{F}$,*

$$h(\Omega_\mathfrak{F}) = \inf_n \frac{\log \left|\mathscr{L}^{\text{l.a.}}_{\mathrm{B}_n}(\mathfrak{F})\right|}{|\mathrm{B}_n|} = \lim_{n\to\infty} \frac{\log \left|\mathscr{L}^{\text{l.a.}}_{\mathrm{B}_n}(\mathfrak{F})\right|}{|\mathrm{B}_n|}, \tag{5.4}$$

i.e. the topological entropy $h(\Omega)$ of the $\mathbb{Z}^d$ n.n. SFT $\Omega = \Omega_\mathfrak{F}$ can be computed by counting locally admissible configuration rather than globally admissible ones.

Since counting locally admissible configurations is tractable, it can be said that Theorem 5.1.1 already provides an approximation algorithm for the topological entropy of a $\mathbb{Z}^d$ n.n. SFT. Formally, a real number h is **right recursively enumerable** if there is a **Turing machine** which, given an input $n \in \mathbb{N}$, computes a rational number $r(n) \geq h$ such that $r(n) \searrow h$ as $n \to \infty$. Given Theorem 5.1.1 and the fact that such limit is also an infimum, we can see that $h(\Omega)$ is right recursively enumerable for any $\mathbb{Z}^d$ n.n. SFT Ω. In fact, the converse is also true due to the following celebrated result from M. Hochman and T. Meyerovitch.

Theorem 5.1.2 ([44]). *The class of nonnegative real right recursively enumerable numbers is exactly the class of topological entropies of $\mathbb{Z}^d$ n.n. SFTs.*

A real number h is **computable** if there is a Turing machine which, given an input $n \in \mathbb{N}$, computes a rational number $r(n)$ such that $|h - r(n)| < \frac{1}{n}$. For example, every algebraic number is computable, since there are numerical methods for approximating the roots of an integer polynomial. This is a strictly stronger notion than right recursively enumerable (for more information, see [50]). It can be shown that, under extra assumptions (e.g. mixing properties) on a $\mathbb{Z}^d$ n.n. SFT Ω, the topological entropy $h(\Omega)$ turns out to be computable.

Theorem 5.1.3 ([44]). *If a $\mathbb{Z}^d$ n.n. SFT Ω is strongly irreducible, then $h(\Omega)$ is computable.*

Moreover, the difference $|h - r(n)|$ can be thought as a function of n, introducing a refinement of the classification of entropies by considering the speed of approximation. A relevant case for us is when the time to compute an approximation $r(n)$ such that $|h - r(n)| < \frac{1}{n}$ is bounded by a polynomial in $\frac{1}{n}$.

Example 5.1.4 ([67, 32]). *The topological entropy $h(\Omega_\varphi^2)$ of the hard square shift Ω_φ^2 is a computable number that can be approximated in polynomial time.*

In Example 5.1.4, which is basically a combinatorial result, the proofs from [67] and [32] are almost entirely based on probabilistic and measure-theoretic techniques. This motivates the following definition, which is a notion of entropy for shift-invariant Borel probability measures.

Definition 5.1.3. *The* **measure-theoretic entropy** *of $\mu \in \mathscr{M}_{1,\sigma}(\mathscr{A}^{\mathbb{Z}^d})$ is defined as*

$$h(\mu) := \lim_{n\to\infty} \frac{-1}{|\mathrm{B}_n|} \sum_{\alpha \in \mathscr{A}^{\mathrm{B}_n}} \mu(\alpha)\log(\mu(\alpha)), \tag{5.5}$$

where $0 \log 0 = 0$.

A fundamental relationship between topological and measure-theoretic entropy is the following.

Theorem 5.1.4 (Variational Principle [64])**.** *Given a $\mathbb{Z}^d$ n.n. SFT Ω,*

$$h(\Omega) = \sup_{\mu \in \mathscr{M}_{1,\sigma}(\Omega)} h(\mu) = \max_{\mu \in \mathscr{M}_{1,\sigma}(\Omega)} h(\mu). \tag{5.6}$$

Remark 11. *The measure(s) that achieve the maximum are called* **measures of maximal entropy (m.m.e.)** *for Ω. Notice that if μ is an m.m.e. for Ω, then $h(\Omega) = h(\mu)$.*

5.2 Topological pressure

Now we proceed to define *topological pressure* of a continuous function $f \in \mathscr{C}(\Omega)$, which can be regarded as a generalization of topological entropy.

Definition 5.2.1. *Given a $\mathbb{Z}^d$ n.n. SFT Ω and $f \in \mathscr{C}(\Omega)$, the* **topological pressure of f on Ω** *is*

$$\mathrm{P}_\Omega(f) := \sup_{\mu \in \mathscr{M}_{1,\sigma}(\Omega)} \left(h(\mu) + \int f d\mu \right). \tag{5.7}$$

In this case, the supremum is also always achieved and any measure μ which achieves the supremum is called an **equilibrium state** for Ω and f. Notice that in the special case when $f \equiv 0$, and thanks to Theorem 5.1.4, $\mathrm{P}(0)$ coincides with the topological entropy $h(\Omega)$, and equilibrium states are the same as measures of maximal entropy.

Note 8. *The preceding definition is a characterization of topological pressure in terms of a variational principle, but can also be regarded as its definition (see [70, Theorem 6.12]). Informally, topological pressure can be thought as an exponential growth rate, where the configurations are "weighted" by the given function f. This idea is formalized for a more particular case in the next paragraphs.*

We define the pressure of a shift-invariant n.n. interaction Φ on a $\mathbb{Z}^d$ n.n. SFT Ω.

Definition 5.2.2. *Given a $\mathbb{Z}^d$ n.n. SFT Ω and a shift-invariant n.n. interaction Φ on Ω, the* **pressure of** Φ *is defined as*

$$\mathrm{P}_\Omega(\Phi) := \lim_{n\to\infty} \frac{1}{|\mathrm{B}_n|} \log \mathrm{Z}^\Phi_{\mathrm{B}_n}. \tag{5.8}$$

The pressure coincides (up to a sign) with the *specific Gibbs free energy of* Φ, a more common name in statistical mechanics. We write $\mathrm{P}(\Phi)$ (resp. $\mathrm{P}(f)$) instead of $\mathrm{P}_\Omega(f)$ (resp. $\mathrm{P}_\Omega(f)$) if Ω is understood. If Φ is induced by a $\mathbb{Z}^d$ lattice energy function ϕ, we can also define an analogous version $\hat{\mathrm{Z}}^\Phi_{\mathrm{B}_n}$ of the partition function $\mathrm{Z}^\Phi_{\mathrm{B}_n}$ over locally admissible configurations in B_n rather than globally admissible ones, by extending $\mathscr{H}^\Phi_{\mathrm{B}_n}$ in the obvious way, i.e.

$$\hat{\mathrm{Z}}^\Phi_{\mathrm{B}_n} := \sum_{\alpha\in\mathscr{L}^{\mathrm{l.a.}}_{\mathrm{B}_n}(\mathfrak{F})} \exp\left(-\mathscr{H}^\Phi_{\mathrm{B}_n}(\alpha)\right). \tag{5.9}$$

Notice that $\hat{\mathrm{Z}}^\Phi_{\mathrm{B}_n} \geq \mathrm{Z}^\Phi_{\mathrm{B}_n}$. The following result (analogous to Theorem 5.1.1) states that in the normalized limit, both quantities coincide.

Theorem 5.2.1 ([70, Theorem 3.4], see also [29, Theorem 2.5])**.** *Given a set of n.n. constraints $\mathfrak{F}$, a $\mathbb{Z}^d$ lattice energy function ϕ, and the corresponding $\mathbb{Z}^d$ n.n. SFT $\Omega_{\mathfrak{F}}$ and n.n. interaction Φ on $\Omega_{\mathfrak{F}}$,*

$$\mathrm{P}_{\Omega_{\mathfrak{F}}}(\Phi) = \lim_{n\to\infty} \frac{1}{|\mathrm{B}_n|} \log \hat{\mathrm{Z}}^\Phi_{\mathrm{B}_n}. \tag{5.10}$$

One can ask whether is necessary to take increasing sequences of n-blocks B_n in the previous definitions instead of any other sequence of sets increasing to $\mathbb{Z}^d$. Given a sequence of sets $\{A_n\}_n$ with $A_n \Subset \mathbb{Z}^d$, we say that $\{A_n\}_n$ **tends to infinity in the sense of van Hove**, denoted $A_n \nearrow \infty$, if $|A_n| \to \infty$ as $n \to \infty$ and, for each $x \in \mathbb{Z}^d$,

$$\lim_{n\to\infty} \frac{|A_n \triangle (x + A_n)|}{|A_n|} = 0, \tag{5.11}$$

where $\triangle$ denotes the symmetric difference. It is well-known (see [70, Corollary 3.13]) that for any sequence such that $A_n \nearrow \infty$, we can replace B_n by A_n in the definition of pressure, i.e.

$$\mathrm{P}(\Phi) = \lim_{n\to\infty} \frac{1}{|A_n|} \log Z^{\Phi}_{A_n}. \tag{5.12}$$

In order to discuss connections between the definition of $\mathrm{P}(\Phi)$ and topological pressure for functions $f \in \mathscr{C}(\Omega)$, we need a mechanism for turning an interaction (which is a function on finite configurations) into a continuous function on the infinite configurations in Ω. We do this for the special case of n.n. interactions Φ as follows. Define the (continuous) function $\mathrm{A}_\Phi : \Omega \to \mathbb{R}$, given by

$$\mathrm{A}_\Phi(\omega) := -\Phi\left(\omega|_{\{\vec{0}\}}\right) - \sum_{i=1}^{d} \Phi\left(\omega|_{\{\vec{0},\vec{e}_i\}}\right). \tag{5.13}$$

A version of the Variational Principle states that the pressure of an interaction has a variational characterization in terms of shift-invariant measures. We state the variational principle below for the case of a n.n. interaction Φ.

Theorem 5.2.2 (Variational Principle [47, 64, 70])**.** *Given a n.n. interaction Φ on a $\mathbb{Z}^d$ n.n. SFT Ω, we have that $\mathrm{P}(\Phi) = \mathrm{P}(\mathrm{A}_\Phi)$, i.e.*

$$\mathrm{P}(\Phi) = \sup_{\mu \in \mathscr{M}_{1,\sigma}(\Omega)} \left(h(\mu) + \int \mathrm{A}_\Phi d\mu \right). \tag{5.14}$$

In particular, if μ is an equilibrium state for A_Φ, then

$$\mathrm{P}(\Phi) = h(\mu) + \int \mathrm{A}_\Phi d\mu. \tag{5.15}$$

The following property will be useful.

Definition 5.2.3. *A $\mathbb{Z}^d$ n.n. SFT Ω satisfies the* **D-condition** *if there exist sequences of finite subsets $\{A_n\}_n$, $\{S_n\}_n$ of $\mathbb{Z}^d$ such that $A_n \nearrow \infty$, $A_n \subseteq S_n$, $\frac{|A_n|}{|S_n|} \to 1$, and for any $\alpha \in \mathscr{L}_{A_n}(\Omega)$, and $\beta \in \mathscr{L}_B(\Omega)$, with $B \Subset S_n^{\mathrm{c}}$, we have that $[\alpha]^\Omega \cap [\beta]^\Omega \neq \emptyset$.*

Given a $\mathbb{Z}^d$ n.n. SFT Ω, a shift-invariant n.n. interaction Φ on Ω, and the corresponding Gibbs (Ω,Φ)-specification π, we can relate equilibrium states for A_Φ and shift-invariant n.n. Gibbs measures for π. In fact, if Ω satisfies the D-condition,

$$\mu \text{ is an equilibrium state for } \mathrm{A}_\Phi \text{ and } \Omega \iff \mu \in \mathcal{G}(\pi) \cap \mathscr{M}_{1,\sigma}(\Omega). \tag{5.16}$$

The "only if" direction is always true in the n.n. case (see [70, Theorem 3]). Another relevant consequence of the D-condition is the following.

Proposition 5.2.3 ([70, Remark 1.14])**.** *Let Ω be a $\mathbb{Z}^d$ n.n. SFT, Φ a shift-invariant n.n. interaction on Ω, and π the corresponding Gibbs (Ω,Φ)-specification. If Ω satisfies the D-condition, then* $\mathrm{supp}(\mu) = \Omega$*, for every $\mu \in \mathcal{G}(\pi)$.*

Note 9. *Notice that strong irreducibility implies the D-condition. In [70, Remark 1.14] a property even weaker than the D-condition is shown to imply* $\mathrm{supp}(\mu) = \Omega$*.*

5.3 Pressure representation

We fix the following elements:

- A $\mathbb{Z}^d$ n.n. SFT Ω that satisfies the D-condition,
- a shift-invariant n.n. interaction Φ on Ω,
- the corresponding n.n. Gibbs (Ω,Φ)-specification π, and
- $\mu \in \mathcal{G}(\pi) \cap \mathscr{M}_{1,\sigma}(\Omega)$.

Given a set $S \Subset \mathbb{Z}^d \backslash \{\vec{0}\}$, define $p_{\mu,S} : \Omega \to [0,1]$ to be

$$p_{\mu,S}(\omega) := \mu\left(\omega|_{\{\vec{0}\}} \middle| \omega|_S\right). \tag{5.17}$$

Notice that $p_{\mu,S}(\omega)$ is a value that depends only on $\omega|_{S\cup\{\vec{0}\}}$.

When dealing with pressure representation, it is useful to consider an order in the lattice. Recall the definitions of lexicographic order $\prec$ and past $\mathscr{P}$ from Subsection 2.1.2. Given $n \in \mathbb{N}$, define the set $\mathscr{P}_n := \mathscr{P} \cap \mathrm{B}_n$, and let

$$p_\mu(\omega) := \lim_{n\to\infty} p_{\mu,\mathscr{P}_n}(\omega), \tag{5.18}$$

which exists μ-a.s. by *Lévy's zero-one law* (see [47, Theorem 3.1.10]). In other words, p_μ is the μ-probability of taking the value $\omega(\vec{0})$ at the origin conditioned on boxes growing to the lexicographic past $\mathscr{P}$.

The **information function** I_μ is μ-a.s. defined as

$$\mathrm{I}_\mu(\omega) := -\log p_\mu(\omega). \tag{5.19}$$

It is well-known (see [35, p. 318, Equation (15.18)] or [51, Theorem 2.4, p. 283]) that the measure-theoretic entropy of μ (in fact, this is true for any shift-invariant measure, not necessarily an equilibrium state) can be expressed as

$$h(\mu) = \int \mathrm{I}_\mu d\mu. \tag{5.20}$$

Therefore, since μ is an equilibrium state for A_Φ, Equation (5.20) implies that

$$\mathrm{P}(\Phi) = \int \left(\mathrm{I}_\mu + \mathrm{A}_\Phi\right) d\mu, \tag{5.21}$$

so the pressure can be represented as the integral with respect to μ of a function determined by an equilibrium state μ and Φ.

5.3.1 A generalization of previous results

For certain classes of equilibrium states and Gibbs measures, sometimes there are even simpler representations for the pressure. A recent example of this was given by D. Gamarnik and D. Katz in [32, Theorem 1], who showed that if in addition π satisfies the SSM property and Ω has a safe symbol s, then

$$\mathrm{P}(\Phi) = \mathrm{I}_\mu\left(s^{\mathbb{Z}^d}\right) + \mathrm{A}_\Phi\left(s^{\mathbb{Z}^d}\right). \tag{5.22}$$

Here, $s^{\mathbb{Z}^d} \in \mathscr{A}^{\mathbb{Z}^d}$ is the fixed point which is s at every site of $\mathbb{Z}^d$. Notice that

$$\mathrm{I}_\mu\left(s^{\mathbb{Z}^d}\right) + \mathrm{A}_\Phi\left(s^{\mathbb{Z}^d}\right) = \int \left(\mathrm{I}_\mu + \mathrm{A}_\Phi\right) d\delta_{s^{\mathbb{Z}^d}}, \tag{5.23}$$

where $\delta_{s^{\mathbb{Z}^d}}$ is the δ-measure supported on $s^{\mathbb{Z}^d}$. They used this simple representation to give a polynomial time approximation algorithm for $\mathrm{P}(\Phi)$ in certain cases (the hard-core model, in particular; see Example 5.1.4). Later, B. Marcus and R. Pavlov [58] weakened the hypothesis and extended their results for pressure representation, obtaining the following corollary.

Corollary 8 ([58]). *Let Ω be a $\mathbb{Z}^d$ n.n. SFT, Φ a shift-invariant n.n. interaction on Ω, and π the corresponding n.n. Gibbs (Ω,Φ)-specification such that*

- *Ω satisfies SSF, and*
- *π satisfies SSM.*

Then,

$$\mathrm{P}(\Phi) = \int \left(\mathrm{I}_\mu + \mathrm{A}_\Phi\right) d\nu, \tag{5.24}$$

for every $\nu \in \mathscr{M}_{1,\sigma}(\Omega)$.

Corollary 8 relied on a more technical theorem from [58, Theorem 3.1]. To review this theorem, we first need a couple of definitions.

We denote

$$\lim_{S \to \mathscr{P}} p_{\mu,S}(\omega) \tag{5.25}$$

if there exists $L \in \mathbb{R}$ such that for all $\varepsilon > 0$, there is $n \in \mathbb{N}$ (that may depend on ω) such that

$$\mathscr{P}_n \subseteq S \Subset \mathscr{P} \implies |p_{\mu,S}(\omega) - L| < \varepsilon. \tag{5.26}$$

If such L exists, $L = p_\mu(\omega)$ by definition. In addition, given the equilibrium state μ, we define

$$\underline{c}_\mu := \inf_{\omega \in \Omega} \inf_{S \Subset \mathbb{Z}^d \setminus \{\vec{0}\}} p_{\mu,S}(\omega), \tag{5.27}$$

and, for $\nu \in \mathscr{M}_{1,\sigma}(\Omega)$,

$$\underline{c}_\mu^-(\nu) := \inf_{\omega \in \mathrm{supp}(\nu)} \inf_{S \Subset \mathscr{P}} p_{\mu,S}(\omega). \tag{5.28}$$

Recall from Equation 3.38 the definition of $\underline{c}_\pi$. It can be checked that $\underline{c}_\mu^-(\nu) \geq \underline{c}_\mu = \underline{c}_\pi$. Considering all this, we have the following theorem.

Theorem 5.3.1 ([58]). *Let Ω be a $\mathbb{Z}^d$ n.n. SFT and Φ a shift-invariant n.n. interaction on Ω. Consider μ an equilibrium state for A_Φ, and $\nu \in \mathscr{M}_{1,\sigma}(\Omega)$ such that*

A1. Ω *satisfies the D-condition,*

A2. $\lim_{S \to \mathscr{P}} p_{\mu,S}(\omega) = p_\mu(\omega)$ *uniformly over* $\omega \in \mathrm{supp}(\nu)$*, and*

A3. $\underline{c}_\mu^-(\nu) > 0$.

Then, $\mathrm{P}(\Phi) = \int (\mathrm{I}_\mu + \mathrm{A}_\Phi) d\nu$.

Considering Theorem 5.3.1 and the TSSM property, we have the following generalization of Corollary 8.

Corollary 9. *Let* Ω *be a* $\mathbb{Z}^d$ *n.n. SFT,* Φ *a shift-invariant n.n. interaction on* Ω*, and* π *the corresponding n.n. Gibbs* (Ω, Φ)*-specification such that*

- Ω *satisfies TSSM, and*
- π *satisfies SSM.*

Then,

$$\mathrm{P}(\Phi) = \int (\mathrm{I}_\mu + \mathrm{A}_\Phi) d\nu, \tag{5.29}$$

for every $\nu \in \mathscr{M}_{1,\sigma}(\Omega)$.

Proof. This follows from Theorem 5.3.1: (A1) is implied by TSSM, since TSSM implies the D-condition; (A2) is implied by SSM (see [58, Proposition 2.14]); and (A3) is implied by TSSM (see Proposition 3.6.3), considering that $\underline{c}_\mu^-(\nu) \geq \underline{c}_\pi$. □

Corollary 10. *Let* $\Omega \subseteq \mathscr{A}^{\mathbb{Z}^2}$ *be a* $\mathbb{Z}^2$ *n.n. SFT,* Φ *a shift-invariant n.n. interaction on* Ω*, and* π *the corresponding n.n. Gibbs* (Ω, Φ)*-specification. Suppose that* π *satisfies exponential SSM with decay rate* $\gamma > 4 \log |\mathscr{A}|$*. Then,* $\mathrm{P}(\Phi) = \int (\mathrm{I}_\mu + \mathrm{A}_\Phi) d\nu$*, for every* $\nu \in \mathscr{M}_{1,\sigma}(\Omega)$.

Proof. This follows from Theorem 3.6.5 and Corollary 9. □

Notice that, in contrast to preceding results, no mixing condition on the support is explicitly needed in Corollary 10.

5.3.2 The function $\hat{\pi}$ and a new pressure representation theorem

With the exception of Theorem 5.3.1, all the previous pressure representation results involved the SSM property, and therefore they are bound to fail when there is a phase transition (i.e. multiple equilibrium states). Here we show that in such case, and assuming some extra conditions, the pressure can still be represented as the integral of a function similar to $\mathrm{I}_\mu + \mathrm{A}_\Phi$, with respect to any shift-invariant measure ν. This will turn out to be useful for approximation of pressure when ν is an atomic measure supported on a periodic configuration (see Chapter 7).

Given a n.n. Gibbs (Ω,Φ)-specification π, for Ω a $\mathbb{Z}^d$ n.n. SFT and a shift-invariant n.n. interaction Φ on Ω, we introduce some useful functions from Ω to $\mathbb{R}$. First, given $\vec{0} \in A \Subset \mathbb{Z}^d$ and $\omega \in \Omega$, we define $\pi_A : \Omega \to [0,1]$ to be

$$\pi_A(\omega) := \pi_A^\omega(\omega|_{\{\vec{0}\}}). \tag{5.30}$$

Recall that, for $y,z \in \mathbb{Z}^d$ such that $y,z \geq \vec{0}$, we have defined the set $\mathrm{Q}_{y,z}$ as $\{x \succcurlyeq \vec{0} : -y \leq x \leq z\}$. Now, given $y,z \geq \vec{0}$ and $\omega \in \Omega$, define $\pi_{y,z}(\omega) := \pi_{\mathrm{Q}_{y,z}}(\omega)$ and, given $n \in \mathbb{N}$, abbreviate $\pi_n(\omega) := \pi_{\vec{1}n,\vec{1}n}(\omega)$. Considering this, we also define the limit $\hat{\pi}(\omega) := \lim_{n\to\infty} \pi_n(\omega)$, whenever it exists. If such limit exists, we will also denote $\mathrm{I}_\pi(\omega) := -\log \hat{\pi}(\omega)$, in a similar fashion as the information function I_μ.

It is not difficult to prove that under some mixing assumptions, namely the D-condition and the SSM property, one has that the original information function I_μ for an equilibrium state μ coincides with I_π in Ω. This new definition provides a generalization of previous results, since I_μ may not be defined in the same points as I_π.

We say that

$$\lim_{y,z\to\infty} \pi_{y,z}(\omega) = \hat{\pi}(\omega), \tag{5.31}$$

if for all $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that

$$y,z \geq \vec{1}k \implies |\pi_{y,z}(\omega) - \hat{\pi}(\omega)| < \varepsilon. \tag{5.32}$$

In addition, we introduce the bound

$$\underline{c}_\pi(\nu) := \inf_{\omega \in \mathrm{supp}(\nu)} \inf_{\vec{0} \in A \Subset \mathbb{Z}^d} \pi_A(\omega). \tag{5.33}$$

Notice that, by shift-invariance, $\underline{c}_\pi(\nu) = \inf_{\omega \in \mathrm{supp}(\nu)} \underline{c}_\pi(\omega)$. We leave it to the reader to verify that $\underline{c}_\pi(\nu) \geq \underline{c}_\mu(\nu)$, for any $\mu \in \mathcal{G}(\pi) \cap \mathscr{M}_{1,\sigma}(\Omega)$. In particular, by Proposition 3.6.3, if Ω satisfies TSSM, $\underline{c}_\pi(\nu) > 0$, for any $\nu \in \mathscr{M}_{1,\sigma}(\Omega)$.

Definition 5.3.1. *A $\mathbb{Z}^d$ n.n. SFT Ω satisfies the* **square block D-condition** *if there exists a sequence of integers $\{r_n\}_{n \geq 1}$ such that $\frac{r_n}{n} \to 0$ as $n \to \infty$ and, for any finite set $B \Subset \mathrm{B}_{n+r_n}^{\mathrm{c}}$, $\alpha \in \mathscr{L}_{\mathrm{B}_n}(\Omega)$ and $\beta \in \mathscr{L}_B(\Omega)$, we have that $[\alpha]^\Omega \cap [\beta]^\Omega \neq \emptyset$.*

This condition is a strengthened version of the D-condition. Notice that, as for the standard D-condition, strong irreducibility also implies the square block the D-condition.

The pressure representation results in [58, Theorems 3.1 and 3.6] are not adequate for the application to the specific models we consider here (see Section 6). Instead we will use the following result, whose proof is adapted from the proof of [58, Theorem 3.1], as well as an idea of [58, Theorem 3.6]. In contrast to the results of [58], this result makes assumptions on the Gibbs specification rather than an equilibrium state.

Theorem 5.3.2. *Let $\Omega \subseteq \mathscr{A}^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ n.n. SFT, Φ a shift-invariant n.n. interaction on Ω, and π the corresponding n.n. Gibbs (Ω, Φ)-specification. Suppose that, for $\nu \in \mathscr{M}_{1,\sigma}(\Omega)$,*

B1. *Ω satisfies the square block D-condition,*

B2. $\lim_{y,z \to \infty} \pi_{y,z}(\omega) = \hat{\pi}(\omega)$ *uniformly over $\omega \in \mathrm{supp}(\nu)$, and*

B3. $\underline{c}_\pi(\nu) > 0$.

Then,

$$\mathrm{P}(\Phi) = \int (\mathrm{I}_\pi + \mathrm{A}_\Phi) d\nu. \tag{5.34}$$

Proof. Given $n \in \mathbb{N}$, let r_n be as in the definition of the square block D-condition and consider the sets B_n and $\Lambda_n := \mathrm{B}_{n+r_n}$. We begin by proving that

$$\frac{1}{|\mathrm{B}_n|}(\log \mathrm{Z}^{\Phi}_{\mathrm{B}_n} + \log \pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n}) + \mathscr{H}^{\Phi}_{\mathrm{B}_n}(\omega|_{\mathrm{B}_n})) \to 0, \tag{5.35}$$

uniformly in $\omega \in \Omega$. For this, we will only use the square block D-condition. We fix $n \in \mathbb{N}$, $\omega \in \mathrm{supp}(\nu)$, and let $m_n := |\Lambda_n| - |\mathrm{B}_n|$. Let $C_d \geq 1$ be a constant such that for any $A \Subset \mathbb{Z}^d$, the total number of sites and bonds contained in $\overline{A}$ is bounded from above by $C_d|A|$. Then,

$$\begin{aligned}
\pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n}) &\geq \pi^{\omega}_{\Lambda_n}(\omega|_{\Lambda_n}) && (5.36)\\
&= \frac{\exp(-\mathscr{H}^{\Phi}_{\Lambda_n,\omega}(\omega|_{\Lambda_n}))}{\sum_{\alpha \in \mathscr{A}^{\Lambda_n} : \alpha\, \omega|_{\Lambda_n^c} \in \Omega} \exp(-\mathscr{H}^{\Phi}_{\Lambda_n,\omega}(\alpha))} && (5.37)\\
&\geq \frac{\exp(-\mathscr{H}^{\Phi}_{\mathrm{B}_n}(\omega|_{\mathrm{B}_n}) - C_d m_n \Phi_{\max})}{\sum_{\beta \in \mathscr{L}_{\mathrm{B}_n}(\Omega)} \exp(-\mathscr{H}^{\Phi}_{\mathrm{B}_n}(\beta))|\mathscr{A}|^{C_d m_n} \exp(C_d m_n \Phi_{\max})} && (5.38)\\
&= \frac{\exp(-\mathscr{H}^{\Phi}_{\mathrm{B}_n}(\omega|_{\mathrm{B}_n}))}{\mathrm{Z}^{\Phi}_{\mathrm{B}_n}} \exp(-C_d m_n (2\Phi_{\max} + \log|\mathscr{A}|)). && (5.39)
\end{aligned}$$

Now, if $\tau_{\max}$ achieves the maximum of $\pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n}\tau)$ over $\tau \in \mathscr{A}^{\Lambda_n \setminus \mathrm{B}_n}$, then

$$\begin{aligned}
\pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n}) &= \sum_{\tau \in \mathscr{A}^{\Lambda_n \setminus \mathrm{B}_n}} \pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n}\tau) && (5.40)\\
&\leq |\mathscr{A}|^{m_n} \pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n}\tau_{\max}) && (5.41)\\
&= |\mathscr{A}|^{m_n} \frac{\exp(-\mathscr{H}^{\Phi}_{\Lambda_n,\omega}(\omega|_{\mathrm{B}_n}\tau_{\max}))}{\mathrm{Z}^{\Phi}_{\Lambda_n,\omega}} && (5.42)\\
&\leq |\mathscr{A}|^{m_n} \frac{\exp(-\mathscr{H}^{\Phi}_{\mathrm{B}_n}(\omega|_{\mathrm{B}_n}) + C_d m_n \Phi_{\max})}{\sum_{\beta \in \mathscr{L}_{\mathrm{B}_n}(\Omega)} \exp(-\mathscr{H}^{\Phi}_{\mathrm{B}_n}(\beta)) \exp(-C_d m_n \Phi_{\max})} && (5.43)\\
&\leq \frac{\exp(-\mathscr{H}^{\Phi}_{\mathrm{B}_n}(\omega|_{\mathrm{B}_n}))}{\mathrm{Z}^{\Phi}_{\mathrm{B}_n}} \exp(C_d m_n (2\Phi_{\max} + \log|\mathscr{A}|)), && (5.44)
\end{aligned}$$

where the square block D-condition has been used in Equation (5.43). Therefore,

$$\theta^{-m_n} \leq \pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n}) \mathrm{Z}^{\Phi}_{\mathrm{B}_n} \exp(\mathscr{H}^{\Phi}_{\mathrm{B}_n}(\omega|_{\mathrm{B}_n})) \leq \theta^{m_n}, \tag{5.45}$$

where $\theta := \exp(C_d(2\Phi_{\max} + \log|\mathscr{A}|))$. Since $\frac{m_n}{|\mathrm{B}_n|} \to 0$, we have obtained Equation (5.35). We use Equation (5.35) to represent pressure:

$$\mathrm{P}(\Phi) = \lim_{n\to\infty} \frac{\log \mathrm{Z}^{\Phi}_{\mathrm{B}_n}}{|\mathrm{B}_n|} = \lim_{n\to\infty} \int \frac{\log \mathrm{Z}^{\Phi}_{\mathrm{B}_n}}{|\mathrm{B}_n|} d\nu \tag{5.46}$$

$$= \lim_{n\to\infty} \int \frac{-\log \pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n}) - \mathscr{H}^{\Phi}_{\mathrm{B}_n}(\omega|_{\mathrm{B}_n})}{|\mathrm{B}_n|} d\nu. \tag{5.47}$$

(Here the second equality comes from the fact that $\frac{\log \mathrm{Z}^{\Phi}_{\mathrm{B}_n}}{|\mathrm{B}_n|}$ is independent of ω, and the third from Equation (5.35).) Since ν is shift-invariant, it can be checked that

$$\lim_{n\to\infty} \int \frac{-\mathscr{H}^{\Phi}_{\mathrm{B}_n}(\omega|_{\mathrm{B}_n})}{|\mathrm{B}_n|} d\nu = \int \mathrm{A}_{\Phi} d\nu, \tag{5.48}$$

and so we can write

$$\mathrm{P}(\Phi) = \int \mathrm{A}_{\Phi} d\nu - \lim_{n\to\infty} \int \frac{\log \pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n})}{|\mathrm{B}_n|} d\nu. \tag{5.49}$$

It remains to show that

$$\lim_{n\to\infty} \int \frac{-\log \pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n})}{|\mathrm{B}_n|} d\nu = \int \mathrm{I}_{\pi} d\nu. \tag{5.50}$$

Fix $\omega \in \mathrm{supp}(\nu)$ and denote $c := \underline{c}_{\pi}(\nu)$. We will decompose $\pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n})$ as a product of conditional probabilities. By (B2), for any $\varepsilon > 0$, there exists $k = k(\varepsilon)$ so that for $y, z \geq \vec{1}k$, $|\pi_{y,z}(\omega) - \hat{\pi}(\omega)| < \varepsilon$ for all $\omega \in \mathrm{supp}(\nu)$. For $x \in \mathrm{B}_{n-1}$, we denote $\mathrm{B}^-_n(x) := \{y \in \mathrm{B}_{n-1} : y \prec x\}$. Then, we can decompose $\pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n})$ as

$$\pi^{\omega}_{\Lambda_n}(\omega|_{\mathrm{B}_n}) = \pi^{\omega}_{\Lambda_n}\left(\omega|_{\underline{\partial}\mathrm{B}_n}\right) \prod_{x\in \mathrm{B}_{n-1}} \pi^{\omega}_{\Lambda_n}\left(\omega|_{\{x\}} \middle| \omega|_{\mathrm{B}^-_n(x)\cup\underline{\partial}\mathrm{B}_n}\right) \tag{5.51}$$

$$= \pi^{\omega}_{\Lambda_n}\left(\omega|_{\underline{\partial}\mathrm{B}_n}\right) \prod_{x\in \mathrm{B}_{n-1}} \pi_{y(x),z(x)}(\sigma_x(\omega)), \tag{5.52}$$

where $y(x) := \vec{1}n + x$ and $z(x) := \vec{1}n - x$, thanks to the MRF property and stationarity of the Gibbs specification.

Let's denote $R_{n,k} := \mathrm{B}_n \setminus \mathrm{B}_{n-k}$. Then, $\mathrm{B}_n = \underline{\partial}\mathrm{B}_n \sqcup \mathrm{B}_{n-k-1} \sqcup R_{n-1,k}$, and we have

$$c^{|\underline{\partial}\mathrm{B}_n|+|R_{n-1,k}|} \prod_{x\in \mathrm{B}_{n-k-1}} \pi_{y(x),z(x)}(\sigma_x(\omega)) \leq \pi^{\omega}_{\Lambda_n}(\,\omega|_{\mathrm{B}_n}) \tag{5.53}$$

$$\leq \prod_{x\in \mathrm{B}_{n-k-1}} \pi_{y(x),z(x)}(\sigma_x(\omega)). \tag{5.54}$$

Taking $-\log(\cdot)$, we have that

$$0 \leq -\log \pi^{\omega}_{\Lambda_n}(\,\omega|_{\mathrm{B}_n}) - \sum_{x\in \mathrm{B}_{n-k-1}} -\log \pi_{y(x),z(x)}(\sigma_x(\omega)) \tag{5.55}$$

$$\leq (|\underline{\partial}\mathrm{B}_n| + |R_{n-1,k}|)\log\left(c^{-1}\right). \tag{5.56}$$

So, by the choice of k, for $x \in \mathrm{B}_{n-k-1}$,

$$\left|\pi_{y(x),z(x)}(\sigma_x(\omega)) - \hat{\pi}(\sigma_x(\omega))\right| < \varepsilon, \tag{5.57}$$

and since $\pi_{y(x),z(x)}(\sigma_x(\omega)), \hat{\pi}(\sigma_x(\omega)) \geq c > 0$, by the *Mean Value Theorem*,

$$\left|-\log \pi_{y(x),z(x)}(\sigma_x(\omega)) - \mathrm{I}_{\pi}(\sigma_x(\omega))\right| < \varepsilon c^{-1}. \tag{5.58}$$

It follows from (B2) that $\hat{\pi}$ is the uniform limit of continuous functions on $\mathrm{supp}(\nu)$. In addition, $\hat{\pi}(\omega) \geq c > 0$, for all $\omega \in \mathrm{supp}(\nu)$. Therefore, we can integrate with respect to ν and obtain

$$\left|\int -\log \pi_{y(x),z(x)}(\sigma_x(\omega))d\nu - \int \mathrm{I}_{\pi}(\omega)d\nu\right| < \varepsilon c^{-1}. \tag{5.59}$$

We now combine the previous equations to see that

$$\left|\int -\log \pi^{\omega}_{\Lambda_n}(\,\omega|_{\mathrm{B}_{n+1}})d\nu - \int \mathrm{I}_{\pi}(\omega)d\nu|\mathrm{B}_{n-k-1}|\right| \tag{5.60}$$

$$\leq |\mathrm{B}_{n-k-1}|\varepsilon c^{-1} + (|\underline{\partial}\mathrm{B}_n| + |R_{n-1,k}|)\log\left(c^{-1}\right). \tag{5.61}$$

Notice that, for a fixed k, $\lim_{n\to\infty} \frac{|\underline{\partial}\mathrm{B}_n|+|R_{n-1,k}|}{|\mathrm{B}_n|} = 0$ and $\lim_{n\to\infty} \frac{|B_{n-k-1}|}{|\mathrm{B}_n|} = 1$.

Therefore,

$$-\varepsilon c^{-1}+\int \mathrm{I}_\pi(\omega)d\nu \leq \liminf_{n\to\infty}\int \frac{-\log \pi^{\omega}_{\Lambda_n}(\,\omega|_{\mathrm{B}_n})}{|\mathrm{B}_n|}d\nu \tag{5.62}$$

$$\leq \limsup_{n\to\infty}\int \frac{-\log \pi^{\omega}_{\Lambda_n}(\,\omega|_{\mathrm{B}_n})}{|\mathrm{B}_n|}d\nu \tag{5.63}$$

$$\leq \int \mathrm{I}_\pi(\omega)d\nu+\varepsilon c^{-1}. \tag{5.64}$$

By letting $\varepsilon \to 0$, we see that

$$\lim_{n\to\infty}\int \frac{-\log \pi^{\omega}_{\Lambda_n}(\,\omega|_{\mathrm{B}_n})}{|\mathrm{B}_n|}d\nu = \int \mathrm{I}_\pi(\omega)d\nu, \tag{5.65}$$

completing the proof. □

Chapter 6

Classical lattice models and related properties

6.1 Three $\mathbb{Z}^d$ lattice models

In this chapter we introduce three families of classical lattice models (see also Example 2.3.1). The first one will be the *Potts model*, which can be regarded as a generalization of the *Ising model* by considering more than two types of *particles*. The second one, the *Widom-Rowlinson model*, is also a multi-type particle system but with hard-core exclusion between particles of different type. The third one is the classical *hard-core lattice gas model*.

Recall the definition of Gibbs $(\mathscr{G}, \mathrm{H}, \phi)$-specifications in the context of homomorphism spaces from Chapter 4, where $\mathscr{G}$ denotes a board, H a constraint graph, and ϕ a constrained energy function. In the context of $\mathbb{Z}^d$ lattice models, besides A, B, etc., we will sometimes use the uppercase Greek letters Λ, Δ, Θ to denote subsets of the lattice.

6.1.1 The (ferromagnetic) Potts model

Given $d, q \in \mathbb{N}$ and $\beta > 0$, the $\mathbb{Z}^d$ **(ferromagnetic) Potts model with** q **types and inverse temperature** β is given by the Gibbs $(\mathbb{Z}^d, \mathrm{K}_q^{\circlearrowleft}, \beta\phi^{\mathrm{FP}})$-specification π_β^{FP}, where

$$\phi^{\mathrm{FP}}\big|_{\mathrm{V}} \equiv 0, \quad \text{and} \quad \phi^{\mathrm{FP}}(\{u,v\}) = -\mathbb{1}_{\{u=v\}}. \tag{6.1}$$

We will denote the alphabet $\mathrm{V}(\mathrm{K}_q^{\circlearrowleft})$ of the Potts model with q types by $\mathscr{A}_{\mathrm{FP},q} = \{1, \ldots, q\}$, $\Omega_{\mathrm{FP}} = \mathrm{Hom}(\mathbb{Z}^d, \mathrm{K}_q^{\circlearrowleft})$ its support (assuming d and q are understood), and Φ_β the induced n.n. interaction.

The $\mathbb{Z}^d$ (ferromagnetic) Potts model is given by the Gibbs specification $\pi_\beta^{\mathrm{FP}} = \left\{\pi_{\beta,\Lambda}^{\omega} : \omega \in \Omega_{\mathrm{FP}}, \Lambda \Subset \mathbb{Z}^d\right\}$, where neighbouring sites preferably align to each other with the same type or “colour”.

Notice that when $q = 2$, we recover the classical **ferromagnetic Ising model**.

Theorem 6.1.1 ([9])**.** *For the $\mathbb{Z}^2$ (ferromagnetic) Potts model with q types and inverse temperature β, there exists a* **critical inverse temperature** $\beta_{\mathrm{c}}(q) := \log(1+\sqrt{q})$ *such that uniqueness of Gibbs measures holds for $\beta < \beta_{\mathrm{c}}(q)$ and for $\beta > \beta_{\mathrm{c}}(q)$ there is a phase transition.*

6.1.2 The (multi-type) Widom-Rowlinson model

Given $d, q \in \mathbb{N}$ and $\zeta > 0$, the $\mathbb{Z}^d$ **(multi-type) Widom-Rowlinson model with q types and activity ζ** is given by the the Gibbs $(\mathbb{Z}^d, \mathrm{S}_q^{\circlearrowleft}, -\log(\zeta)\phi^{\mathrm{WR}})$-specification π_ζ^{WR}, where

$$\phi^{\mathrm{WR}}(v) = -\mathbb{1}_{\{v \neq 0\}}, \quad \text{and} \quad \phi^{\mathrm{WR}}\big|_{\mathrm{E}} \equiv 0. \tag{6.2}$$

We will denote the alphabet $\mathrm{V}(\mathrm{S}_q^{\circlearrowleft})$ of the Widom-Rowlinson model with q types by $\mathscr{A}_{\mathrm{WR},q} = \{0, 1, \ldots, q\}$, $\Omega_{\mathrm{WR}} = \mathrm{Hom}(\mathbb{Z}^d, \mathrm{S}_q^{\circlearrowleft})$ its support (assuming d and q are understood), and Φ_ζ the induced n.n. interaction.

The $\mathbb{Z}^d$ (multi-type) Widom-Rowlinson model is given by the Gibbs specification $\pi_\zeta^{\mathrm{WR}} = \left\{\pi_{\zeta,\Lambda}^{\omega} : \omega \in \Omega_{\mathrm{WR}}, \Lambda \Subset \mathbb{Z}^d\right\}$, where neighbouring sites are forced to align to each other with the same type or “colour” or with the central symbol 0.

Theorem 6.1.2 ([71], see also [38])**.** *For the $\mathbb{Z}^2$ (multi-type) Widom-Rowlinson model with q types and activity ζ, uniqueness of Gibbs measures holds for sufficiently small ζ and there is a phase transition for sufficiently large ζ.*

6.1.3 The hard-core lattice gas model

Given $d \in \mathbb{N}$ and $\lambda > 0$, the $\mathbb{Z}^d$ **hard-core lattice gas model with activity λ** is given by the Gibbs $(\mathbb{Z}^d, \mathrm{H}_\varphi, \phi_\lambda^{\mathrm{HC}})$-specification $\pi_\lambda^{\mathrm{HC}}$, where

$$\phi^{\mathrm{HC}}(0) = 0, \quad \phi^{\mathrm{HC}}(1) = -1, \quad \text{and} \quad \phi^{\mathrm{HC}}\big|_{\mathrm{E}} \equiv 0. \tag{6.3}$$

We will denote the alphabet $\mathrm{V}(\mathrm{H}_\varphi)$ of the hard-core model by $\mathscr{A}_{\mathrm{HC}} = \{0,1\}$, $\Omega_{\mathrm{HC}} = \mathrm{Hom}(\mathbb{Z}^d, \mathrm{H}_\varphi)$ its support (assuming d is understood), and Φ_λ the induced n.n. interaction.

The $\mathbb{Z}^d$ hard-core lattice gas model is given by the Gibbs specification $\pi_\lambda^{\mathrm{HC}} = \left\{\pi_{\lambda,\Lambda}^{\omega} : \omega \in \Omega_{\mathrm{HC}}, \Lambda \Subset \mathbb{Z}^d\right\}$, where neighbouring sites cannot be both 1.

Theorem 6.1.3 ([36, Theorem 3.3]). *For the $\mathbb{Z}^2$ hard-core lattice gas model with activity λ, uniqueness of Gibbs measures holds for sufficiently small λ and there is a phase transition for sufficiently large λ.*

In fact, a major open problem is the uniqueness of the phase transition point for the $\mathbb{Z}^2$ hard-core lattice model.

For both the Potts and Widom-Rowlinson models we will also distinguish a particular type of particle or colour in the alphabet. W.l.o.g., we can take the type q in $\mathscr{A}_{\mathrm{FP},q}$ or $\mathscr{A}_{\mathrm{WR},q} \setminus \{0\}$, respectively. Given this colour, we will denote by ω_q the fixed point $q^{\mathbb{Z}^d}$. For the hard-core lattice gas model, we will consider the two special points $\omega^{(e)}$ and $\omega^{(o)}$, given by

$$\omega^{(e)}(x) := \begin{cases} 0 & \text{if } \sum_i x_i \text{ is even,} \\ 1 & \text{if } \sum_i x_i \text{ is odd,} \end{cases} \tag{6.4}$$

and $\omega^{(o)} = \sigma_{\vec{e}_1}(\omega^{(e)})$.

Notice that the Potts, Widom-Rowlinson, and hard-core lattice gas models have a safe symbol (any $a \in \mathscr{A}_{\mathrm{FP},q}$, $0 \in \mathscr{A}_{\mathrm{WR},q}$, and $0 \in \mathscr{A}_{\mathrm{HC}}$, respectively). In particular, we have that the supports of the three models satisfy the square block D-condition, and $\underline{c}_\pi > 0$ for π the corresponding n.n. Gibbs specification.

The Potts and Widom-Rowlinson models have interpretations in terms of a *random-cluster representation*. The Potts model is related to a *random-cluster model on bonds* (via the so-called *Edwards-Sokal coupling*), while the Widom-Rowlinson is naturally related to a *random-cluster model on sites*.

From now on, when talking about Gibbs specifications for the Potts, Widom-Rowlinson, and hard-core lattice gas models, we will distinguish them (in a slight abuse of notation) by the subindex corresponding to the parameter β, ζ, or λ of the model, i.e. $\pi_{\beta,\Lambda}^{\omega}$ should be understood as a probability measure in the Potts model,

$\pi^{\omega}_{\zeta,\Lambda}$ in the Widom-Rowlinson, and $\pi^{\omega}_{\lambda,\Lambda}$ in the hard-core lattice gas model, and π_β, π_ζ and π_λ will denote the corresponding Gibbs specifications. Also, we will write π^{β}_{Λ}, $\hat{\pi}^{\beta}$, and I^{β}_{π} for the functions π_Λ, $\hat{\pi}$, and I_π in the Potts model, and short-hand notations when $\Lambda = \mathrm{Q}_n$ or $\mathrm{Q}_{y,z}$ such as

$$\pi^{\beta}_{n}(\omega) = \pi^{\omega}_{\beta,\mathrm{Q}_n}(\,\omega|_{\{\vec{0}\}}). \tag{6.5}$$

The analogous notation will be used for the Widom-Rowlinson and hard-core lattice gas cases, but using the parameters ζ and λ, respectively.

6.2 The bond random-cluster model

We will make use of the *bond random-cluster model*. One the main results, Part 1 of Theorem 6.5.1, is proven using arguments based on this model. This model is a two-parameter family of *dependent bond percolation* models (on a finite graph). We are mainly interested in finite subgraphs of $\mathbb{Z}^2$.

Fix a set of sites Λ. Let $\mathscr{E}^0(\Lambda)$ denote the set of bonds with both endpoints in Λ (i.e. if $e = \{x,y\}$ is a bond, then both x and y belong to Λ), and $\mathscr{E}^1(\Lambda)$ the set of bonds with at least one endpoint in Λ. We will consider configurations $w \in \{0,1\}^{\mathscr{E}^i(\Lambda)}$, for $i = 0,1$. We speak of a bond e as being **open** if $w(e) = 1$, and as being **closed** if $w(e) = 0$.

We describe the model with boundary conditions indexed by $i = 0,1$. We will give special attention to the case $d = 2$. A set $A \subseteq \mathbb{Z}^2$ is **simply lattice-connected** if A and A^{c} are both connected.

Definition 6.2.1. *Given a finite simply lattice-connected set* Λ*, and parameters* $p \in [0,1]$ *and* $q > 0$*, we define the* **free** *(*$i = 0$*) and* **wired** *(*$i = 1$*)* **bond random-cluster** *distributions on* $\mathscr{E}^i(\Lambda)$ *(*$i = 0,1$*) as the measures* $\varphi^{(i)}_{p,q,\Lambda}$ *that to each* $w \in \{0,1\}^{\mathscr{E}^i(\Lambda)}$ *assigns probability proportional to*

$$\varphi^{(i)}_{p,q,\Lambda}(w) \propto \left\{\prod_{e\in\mathscr{E}^i(\Lambda)} p^{w(e)}(1-p)^{1-w(e)}\right\} q^{k^i_\Lambda(w)} = \left(\frac{p}{1-p}\right)^{\#_1(w)} q^{k^i_\Lambda(w)}, \tag{6.6}$$

where $\#_1(w)$ *is the number of open bonds in* w*, and* $k^0_\Lambda(w)$ *and* $k^1_\Lambda(w)$ *are the* *number of connected components (including isolated sites) in the graphs* $(\Lambda, \{e \in \mathscr{E}^0(\Lambda) : w(e) = 1\})$ *and* $(\mathbb{Z}^2, \mathscr{E}^0(\mathbb{Z}^2 \setminus \Lambda) \cup \{e \in \mathscr{E}^1(\Lambda) : w(e) = 1\})$*, respectively.*

Notice that when $q = 1$, we recover the ordinary *Bernoulli bond percolation measure* $\varphi_{p,\Lambda}$, while other choices of q lead to dependence between bonds. For given p and q, one can also define bond random-cluster measures $\varphi^{(i)}_{p,q}$ on $\mathbb{Z}^2$ as a limit of finite-volume measures $\varphi^{(i)}_{p,q,\Lambda}$ $(i = 0, 1)$.

Theorem 6.2.1 ([36, Lemma 6.8])**.** *For* $p \in [0,1]$ *and* $q \in \mathbb{N}$*, the limiting measures*

$$\varphi^{(i)}_{p,q} = \lim_{n\to\infty} \varphi^{(i)}_{p,q,\Lambda_n}, \qquad i \in \{0,1\}, \tag{6.7}$$

exist and are shift-invariant, where $\{\Lambda_n\}_n$ *is any increasing sequence of finite simply lattice-connected sets that exhausts* $\mathbb{Z}^2$*.*

General bond random-cluster measures on $\{0,1\}^{\mathbb{Z}^2}$ can be defined using an analogue of the DLR equation (see [42, Definition 4.29]). For $q \geq 1$, there is a value $p_{\mathrm{c}}(q)$ that delimits exactly the transition for existence of an *infinite open cluster* for these measures. It is known (see [42, p. 107] and [24]) that for $q \geq 1$ and $p \neq p_{\mathrm{c}}(q)$, there is a unique such measure which we denote by $\varphi_{p,q}$, and that coincides with $\varphi^{(0)}_{p,q}$ and $\varphi^{(1)}_{p,q}$. It was recently proven (see [9]) that $p_{\mathrm{c}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$, for every $q \geq 1$.

Let $p = 1 - e^{-\beta}$. The **free Edwards-Sokal coupling** $\mathbb{P}^{(0)}_{p,q,\Lambda}$ (see [42]) is a coupling between the free-boundary Potts measure $\pi^{(f)}_{\beta,\Lambda}$ and $\varphi^{(0)}_{p,q,\Lambda}$. The **wired Edwards-Sokal coupling** $\mathbb{P}^{(1)}_{p,q,\Lambda}$ is a coupling between $\pi^{\omega_q}_{\beta,\Lambda}$ and $\varphi^{(1)}_{p,q,\Lambda}$. Notice that $p_{\mathrm{c}}(q) = 1 - e^{-\beta_{\mathrm{c}}(q)}$.

These couplings are measures on pairs of site configurations and corresponding bond configurations. The projection to site configurations is the free-boundary/ω_q-boundary Potts measure, and the projection to bond configurations is the free/wired bond random-cluster measure, respectively.

Theorem 6.2.2 ([42, Theorem 1.13])**.** *Let* $\Lambda \Subset \mathbb{Z}^2$ *be a finite simply lattice-connected set,* $q \in \mathbb{N}$*, and let* $p \in [0,1]$ *and* $\beta > 0$ *be such that* $p = 1 - e^{-\beta}$*. Then:*

1. *Given* $w \in \{0,1\}^{\mathscr{E}^1(\Lambda)}$*, the conditional measure* $\mathbb{P}^{(1)}_{p,q,\Lambda}\left(\cdot \middle| {\mathscr{A}_{\mathrm{FP},q}}^{\Lambda} \times \{w\}\right)$ *on* ${\mathscr{A}_{\mathrm{FP},q}}^{\Lambda}$ *is obtained by putting random colours on entire clusters of* w *not*

connected with $\mathbb{Z}^2 \setminus \Lambda$ *(of which there are* $k^1_\Lambda(w) - 1$*) and colour q on the clusters connected with* $\mathbb{Z}^2 \setminus \Lambda$*. These colours are constant on given clusters, are independent between clusters, and the random ones are uniformly distributed on the set* $\mathscr{A}_{\mathrm{FP},q}$*.*

2. *Given* $\alpha \in \mathscr{A}_{\mathrm{FP},q}{}^{\Lambda}$*, the conditional measure* $\mathbb{P}^{(1)}_{p,q,\Lambda}\left(\cdot \,\middle|\, \{\alpha\} \times \{0,1\}^{\mathscr{E}^1(\Lambda)}\right)$ *on* $\{0,1\}^{\mathscr{E}^1(\Lambda)}$ *is obtained as follows. Consider the extended configuration* $\overline{\alpha} = \alpha q^{\partial\Lambda}$ *and an arbitrary bond* $e = \{x,y\} \in \mathscr{E}^1(\Lambda)$*. If* $\overline{\alpha}(x) \neq \overline{\alpha}(y)$*, we set* $w(e) = 0$*. If* $\overline{\alpha}(x) = \overline{\alpha}(y)$*, we set*

$$w(e) = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{otherwise,} \end{cases} \tag{6.8}$$

the values of different $w(e)$ *being (conditionally) independent random variables.*

The couplings can be used to relate probabilities and expectations for the Potts model to corresponding events and expectations in the associated bond random-cluster model. A main example is a relation between the *two-point correlation function* in the Potts model and the *connectivity function* in the bond random-cluster model (see [42, Theorem 1.16]).

By considering a displaced version of $\mathbb{Z}^2$, namely $\frac{1}{2}\vec{1} + \mathbb{Z}^2$ (the **dual lattice**), we can define a notion of duality for bond configurations w. Notice that every bond $e \in \mathscr{E}(\mathbb{Z}^2)$ (if we think of bonds as unitary vertical and horizontal straight segments) is intersected perpendicularly by one and only one **dual bond** $e^* \in \mathscr{E}(\frac{1}{2}\vec{1} + \mathbb{Z}^2)$, so there is a clear correspondence between $\mathscr{E}(\mathbb{Z}^2)$ and $\mathscr{E}(\frac{1}{2}\vec{1} + \mathbb{Z}^2)$. We are mainly interested in wired bond random-cluster distributions on the set of sites $\tilde{\mathrm{B}}_n := [-n+1, n]^2 \cap \mathbb{Z}^2$. Given $n \in \mathbb{N}$, if we consider the set of bonds $\mathscr{E}^1(\tilde{\mathrm{B}}_n)$, it is easy to check that there is a correspondence $e \mapsto e^*$ between this set and the set of bonds from $\frac{1}{2}\vec{1} + \mathbb{Z}^2$ with both endpoints in $[-n,n]^2 \cap \left(\frac{1}{2}\vec{1} + \mathbb{Z}^2\right)$, which can be identified with the set $\mathscr{E}^0(\mathrm{B}_n)$. Then, given a bond configuration $w \in \{0,1\}^{\mathscr{E}^1(\tilde{\mathrm{B}}_n)}$ we can associate a dual bond configuration $w^* \in \{0,1\}^{\mathscr{E}^0(\mathrm{B}_n)}$ such that $w^*(e^*) = 0$ iff $w(e) = 1$. Considering this, we have the following equality.

Proposition 6.2.3 ([42, Equation (6.12) and Theorem 6.13]). *Given $n \in \mathbb{N}$, $p \in [0,1]$, and $q \in \mathbb{N}$,*

$$\varphi^{(1)}_{p,q,\tilde{\mathrm{B}}_n}(w) = \varphi^{(0)}_{p^*,q,\mathrm{B}_n}(w^*), \tag{6.9}$$

for any bond configuration $w \in \{0,1\}^{\mathscr{E}^1(\tilde{\mathrm{B}}_n)}$, where $\tilde{\mathrm{B}}_n = [-n+1,n]^2 \cap \mathbb{Z}^2$ and $p^* \in [0,1]$ is the dual value of p, which is given by

$$\frac{p^*}{1-p^*} = \frac{q(1-p)}{p}. \tag{6.10}$$

The previous duality result can be generalized to more arbitrary shapes and it has also a counterpart from free-to-wired boundary conditions, instead of from wired-to-free.

The unique fixed point of the map $p \mapsto p^*$ defined by Equation (6.10) is $\frac{\sqrt{q}}{1+\sqrt{q}}$ and, as mentioned before, is known to coincide with the critical point $p_{\mathrm{c}}(q)$ for the existence of an infinite open cluster for the bond random-cluster model (see [9, Theorem 1]). It is easy to see that $p > p_{\mathrm{c}}(q)$ iff $p^* < p_{\mathrm{c}}(q)$.

6.3 The site random-cluster model

In a similar fashion to the bond random-cluster model, we can perturb Bernoulli site percolation, where the probability measure is changed in favour of configurations with many (for $q > 1$) or few (for $q < 1$) connected components. The resulting model is called the *site random-cluster model*.

Definition 6.3.1. *Given $\Lambda \Subset \mathbb{Z}^2$, and parameters $p \in [0,1]$ and $q > 0$, the* **wired site random-cluster** *distribution $\psi^{(1)}_{p,q,\Lambda}$ is the probability measure on $\{0,1\}^\Lambda$ which to each $\theta \in \{0,1\}^\Lambda$ assigns probability proportional to*

$$\psi^{(1)}_{p,q,\Lambda}(\theta) \propto \left\{\prod_{x\in\Lambda} p^{\theta(x)}(1-p)^{1-\theta(x)}\right\} q^{\kappa_\Lambda(\theta)} = \zeta^{\#_1(\theta)} q^{\kappa_\Lambda(\theta)}, \tag{6.11}$$

where $\zeta = \frac{p}{1-p}$, $\#_1(\theta)$ is the number of 1's in θ, and $\kappa_\Lambda(\theta)$ is the number of connected components in $\{x \in \Lambda : \theta(x) = 1\}$ that do not intersect $\underline{\partial}\Lambda$.

The **free site random-cluster measure** $\psi^{(0)}_{p,q,\Lambda}$ is defined as in Equation (6.11) by replacing $\kappa_\Lambda(\theta)$ with the total number of connected components in Λ. However, we will not require that measure in this work. In any case, taking $q = 1$ gives the ordinary Bernoulli site percolation $\psi_{p,\Lambda}$, while other choices of q lead to dependence between sites, similarly to the bond random-cluster model.

Proposition 6.3.1. *Given a set $\Lambda \Subset \mathbb{Z}^2$, and parameters $\zeta > 0$ and $q \in \mathbb{N}$, consider the (multi-type) Widom-Rowlinson model with q types distribution and monochromatic boundary condition $\pi^{\omega_q}_{\zeta,\Lambda}$. Now, let $f : \mathscr{A}_{\mathrm{WR},q}{}^\Lambda \to \{0,1\}^\Lambda$ be defined site-wise as*

$$(f(\alpha))(x) = \begin{cases} 0 & \text{if } \alpha(x) = 0, \\ 1 & \text{if } \alpha(x) \neq 0, \end{cases} \tag{6.12}$$

for $\alpha \in \mathscr{A}_{\mathrm{WR},q}{}^\Lambda$ and $x \in \Lambda$, and let $p = \frac{\zeta}{1+\zeta}$. Then,

$$f_*\pi^{\omega_q}_{\zeta,\Lambda} = \psi^{(1)}_{p,q,\Lambda}, \tag{6.13}$$

where $f_*\pi^{\omega_q}_{\zeta,\Lambda}(\cdot) := \pi^{\omega_q}_{\zeta,\Lambda}(f^{-1}(\cdot))$ denotes the push-forward measure *on $\{0,1\}^\Lambda$.*

The requirement that $\kappa_\Lambda(\cdot)$ does not count connected components that intersect the inner boundary of Λ in the site random-cluster model, corresponds to the fact that non 0 sites adjacent to the monochromatic boundary $\omega_q\big|_{\partial\Lambda}$ in the Widom-Rowlinson model must have the same colour q.

For $q = 2$, Proposition 6.3.1 is proven in [43, Lemma 5.1 (ii)], and the proof extends easily for general q. Proposition 6.3.1 can be regarded as a coupling between $\pi^{\omega_q}_{\zeta,\Lambda}$ and $\psi^{(1)}_{p,q,\Lambda}$, because a push-forward measure can be naturally coupled with the original measure.

It is important to notice that $\psi^{(1)}_{p,q,\Lambda}$ is itself not an MRF: given sites on a "ring" $\mathcal{C}$, the inside and outside of $\mathcal{C}$ are generally not conditionally independent, because knowledge of sites outside $\mathcal{C}$ could cause connected components of 1's in $\mathcal{C}$ to "amalgamate" into a single component, which would affect the conditional distribution of configurations inside $\mathcal{C}$ (the same for the bond random-cluster model $\varphi^{(1)}_{p,q,\Lambda}$). The following lemma shows that in certain situations, when conditioning on a cycle $\mathcal{C}$ labeled entirely by 1's, this kind of amalgamation does not occur.

Lemma 6.3.2. *Let $\emptyset \neq \Theta \subseteq \Lambda \Subset \mathbb{Z}^2$ be such that $\Lambda^c \cup \overline{\Theta}^\star$ is connected. Take $\Delta := \partial^\star \Theta \cap \Lambda$. Consider an event $\mathbb{A} \in \mathscr{F}_\Theta$ and a configuration $\tau \in \{0,1\}^\Sigma$, where $\Sigma \subseteq \Lambda \setminus \overline{\Theta}^\star$. Then,*

$$\psi^{(1)}_{p,q,\Lambda}(\mathbb{A}|1^\Delta \tau) = \psi^{(1)}_{p,q,\Lambda}(\mathbb{A}|1^\Delta 0^{\Lambda \setminus \overline{\Theta}^\star}). \tag{6.14}$$

Proof. W.l.o.g., we may assume that $\mathbb{A}$ is a cylinder event $[\theta]$ with $\theta \in \{0,1\}^\Theta$ (by linearity) and $\Sigma = \Lambda \setminus \overline{\Theta}^\star$ (by taking weighted averages).

Now, $\Sigma = \Lambda \setminus \overline{\Theta}^\star$ can be written as a disjoint union of $\star$-connected components $\Sigma = K_1 \sqcup \cdots \sqcup K_n$. For every i, $\partial^\star K_i \subseteq \Lambda^c \cup \overline{\Theta}^\star$ (in fact, $\partial^\star K_i \subseteq \Lambda^c \cup \Delta$). Since $\Lambda^c \cup \overline{\Theta}^\star$ is connected and Λ is finite, for every site in $\partial^\star K_i$ there is a path to infinity that does not intersect K_i.

Then, by application of a result of Kesten (see [48, Lemma 2.23]), $\partial^\star K_i$ is connected, for every i. In addition, we have that $\Lambda = \Theta \sqcup \Delta \sqcup \Sigma$ and $\partial^\star K_i \subseteq \Lambda^c \cup \Delta$.

We claim that

$$\kappa_\Lambda(\upsilon) = \kappa_\Lambda(\upsilon|_\Theta 1^\Delta 0^\Sigma) + \sum_{i=1}^{n} \kappa_{K_i}(\upsilon|_{K_i}) = \kappa_\Lambda(\upsilon|_\Theta 1^\Delta 0^\Sigma) + \kappa_\Sigma(\tau), \tag{6.15}$$

for any $\upsilon \in \{0,1\}^\Lambda$ such that $\upsilon|_\Delta = 1^\Delta$ and $\upsilon|_\Sigma = \tau$.

To see this, given such υ, we exhibit a bijection r between the connected components of υ that do not intersect $\underline{\partial}\Lambda$ and the union of: (a) the connected components of $\upsilon|_\Theta 1^\Delta 0^\Sigma$ that do not intersect $\underline{\partial}\Lambda$, and (b) the connected components of $\upsilon|_{K_i}$ that do not intersect $\underline{\partial} K_i$, for all i. Namely, if $C \subseteq \Lambda$ is a connected component of υ, then r is defined as follows

$$r(C) = \begin{cases} C \cap \overline{\Theta}^\star & \text{if } C \cap \overline{\Theta}^\star \neq \emptyset, \\ C & \text{if } C \subseteq \Sigma. \end{cases} \tag{6.16}$$

In order to see that r is well-defined, note that if C intersects $\overline{\Theta}^\star$ and Σ, the set $C \cap \overline{\Theta}^\star$ is still connected since $\partial^\star K_i$ is connected and $\upsilon|_\Delta = 1^\Delta$. To see that r is onto, observe that if C' is a connected component of $\upsilon|_\Theta 1^\Delta 0^\Sigma$, then there is a unique component C of υ such that $C \cap \overline{\Theta}^\star = C'$, due again to the fact that $\partial^\star K_i$ is connected. And r is clearly injective because two distinct connected components cannot intersect.

Finally, we conclude from Equation (6.15) that

$$\psi^{(1)}_{p,q,\Lambda}(\theta \mid 1^{\Delta}\tau) = \frac{\zeta^{\#_1(\theta 1^{\Delta}\tau)} q^{\kappa_\Lambda(\theta 1^{\Delta}\tau)}}{\sum_{\upsilon\in\{0,1\}^{\Lambda}:\, \upsilon|_{\Delta}=1^{\Delta},\, \upsilon|_{\Sigma}=\tau} \zeta^{\#_1(\upsilon)} q^{\kappa_\Lambda(\upsilon)}} \tag{6.17}$$

$$= \frac{\zeta^{\#_1(\theta 1^{\Delta})+\#_1(\tau)} q^{\kappa_\Lambda(\theta 1^{\Delta} 0^{\Sigma})+\kappa_\Sigma(\tau)}}{\sum_{\upsilon\in\{0,1\}^{\Lambda}:\, \upsilon|_{\Sigma}=\tau} \zeta^{\#_1(\upsilon(\Theta)1^{\Delta})+\#_1(\tau)} q^{\kappa_\Lambda(\upsilon(\Theta)1^{\Delta}0^{\Sigma})+\kappa_\Sigma(\tau)}} \tag{6.18}$$

$$= \frac{\zeta^{\#_1(\theta 1^{\Delta})} q^{\kappa_\Lambda(\theta 1^{\Delta} 0^{\Lambda\setminus\overline{\Theta}^{\star}})}}{\sum_{\tilde\theta\in\{0,1\}^{\Theta}} \zeta^{\#_1(\tilde\theta 1^{\Delta})} q^{\kappa_\Lambda(\tilde\theta 1^{\Delta} 0^{\Lambda\setminus\overline{\Theta}^{\star}})}} = \psi^{(1)}_{p,q,\Lambda}(\theta | 1^{\Delta} 0^{\Lambda\setminus\overline{\Theta}^{\star}}), \tag{6.19}$$

as we wanted. □

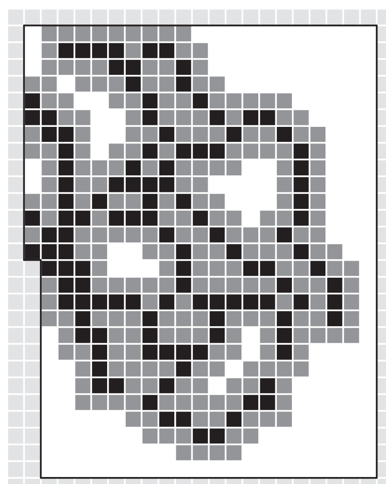

Figure 6.1: A $\star$-connected set Θ (in black), $\Delta = \partial^{\star}\Theta \cap \Lambda$ (in dark grey), and Λ^{c} (in light grey) for $\Lambda = \mathrm{Q}_{y,z}$.

Remark 12. *We claim that if $\emptyset \neq \Theta \subseteq \Lambda \Subset \mathbb{Z}^2$ are such that Λ^{c} is connected, Θ is $\star$-connected and $\overline{\Theta}^{\star} \cap \underline{\partial}\Lambda \neq \emptyset$, then $\Lambda^{c} \cup \overline{\Theta}^{\star}$ is connected, which is the main hypothesis of Lemma 6.3.2. This follows from the easy fact that the $\star$-closure of a $\star$-connected set is connected.*

6.4 Additional properties

6.4.1 Spatial mixing properties

We now introduce some extra concepts related with spatial mixing.

Definition 6.4.1. *([4, p. 445]) A $\mathbb{Z}^d$-MRF μ satisfies the* **ratio strong mixing property for a class of sets** $\mathscr{C}$ *if there exists $C,\gamma > 0$ such that for any $\Lambda \in \mathscr{C}$, any $\Delta,\Theta \subseteq \Lambda$ and $\delta \in \mathscr{A}^{\partial\Lambda}$ with $\mu(\delta) > 0$,*

$$\sup\left\{\left|\frac{\mu(\mathbb{A}\cap\mathbb{B}|\delta)}{\mu(\mathbb{A}|\delta)\mu(\mathbb{B}|\delta)} - 1\right| : \mathbb{A} \in \mathscr{F}_\Delta, \mathbb{B} \in \mathscr{F}_\Theta, \mu(\mathbb{A}|\delta)\mu(\mathbb{B}|\delta) > 0\right\} \leq C \sum_{x\in\Delta, y\in\Theta} e^{-\gamma \mathrm{dist}(x,y)}. \tag{6.20}$$

Proposition 6.4.1. *Let π be a n.n. Gibbs specification with $\Omega = \mathscr{A}^{\mathbb{Z}^2}$, the full shift. Consider $\mu \in \mathcal{G}(\pi)$ that satisfies the ratio strong mixing property for the class of finite simply lattice-connected sets. Then, π satisfies exponential SSM for the family of sets $\{\mathrm{Q}_{y,z}\}_{y,z\geq 0}$.*

Proof. Fix $y,z \geq 0$ and the corresponding set $A = \mathrm{Q}_{y,z} \Subset \mathbb{Z}^2$. Let $B \subseteq A$, $\beta \in \mathscr{A}^B$, and $\omega_1,\omega_2 \in \Omega$. Consider

1. the sets $\Theta := \Sigma_{\partial A}(\omega_1,\omega_2)$ and $\Lambda := A \cup \Theta$,
2. an arbitrary point $\tilde{\omega} \in \Omega$ such that $\tilde{\omega}|_{\partial A\setminus\Theta} = \omega_1|_{\partial A\setminus\Theta} (= \omega_2|_{\partial A\setminus\Theta})$,
3. the configuration $\tilde{\delta} = \tilde{\omega}|_{\partial\Lambda}$, and
4. the events $\mathbb{A} := [\beta] \in \mathscr{F}_B$ and $\mathbb{B}_i := [\omega_i|_\Theta] \in \mathscr{F}_\Theta$, for $i = 1,2$.

Notice that Λ is a finite simply lattice-connected set. Since $\mathrm{supp}(\mu) = \mathscr{A}^{\mathbb{Z}^d}$, we can be sure that $\mu(\omega_1|_{\partial A})\mu(\omega_2|_{\partial A})\mu(\tilde{\delta}) > 0$. Then,

$$\begin{aligned}
\left|\pi_A^{\omega_1}(\beta) - \pi_A^{\omega_2}(\beta)\right| &= |\mu(\beta|\,\omega_1|_{\partial A}) - \mu(\beta|\,\omega_2|_{\partial A})| && (6.21)\\
&= \left|\mu\left(\mathbb{A}\middle|[\tilde{\delta}]\cap\mathbb{B}_1\right) - \mu\left(\mathbb{A}\middle|[\tilde{\delta}]\cap\mathbb{B}_2\right)\right| && (6.22)\\
&= \left|\frac{\mu(\mathbb{A}\cap\mathbb{B}_1|\tilde{\delta})}{\mu(\mathbb{B}_1|\tilde{\delta})} - \mu(\mathbb{A}|\tilde{\delta}) + \mu(\mathbb{A}|\tilde{\delta}) - \frac{\mu(\mathbb{A}\cap\mathbb{B}_2|\tilde{\delta})}{\mu(\mathbb{B}_2|\tilde{\delta})}\right| && (6.23)
\end{aligned}$$

$$\leq \left| \frac{\mu(\mathbb{A} \cap \mathbb{B}_1 | \tilde{\delta})}{\mu(\mathbb{B}_1 | \tilde{\delta}) \mu(\mathbb{A} | \tilde{\delta})} - 1 \right| + \left| 1 - \frac{\mu(\mathbb{A} \cap \mathbb{B}_2 | \tilde{\delta})}{\mu(\mathbb{B}_2 | \tilde{\delta}) \mu(\mathbb{A} | \tilde{\delta})} \right| \tag{6.24}$$

$$\leq 2C \sum_{x \in B, y \in \Theta} e^{-\gamma \mathrm{dist}(x,y)} \tag{6.25}$$

$$\leq |B| 2C \sum_{y \in \Theta} e^{-\gamma \mathrm{dist}(B,y)}. \tag{6.26}$$

W.l.o.g., we can assume that $|\Theta| = 1$ (see Lemma 3.1.1). Therefore, by taking $C' = 2C$, we conclude that

$$\left| \pi_A^{\omega_1}(\beta) - \pi_A^{\omega_2}(\beta) \right| \leq |B| 2K \sum_{y \in \Theta} e^{-\gamma \mathrm{dist}(B,y)} = |B| C' e^{-\gamma \mathrm{dist}(B, \Sigma_{\partial A}(\omega_1, \omega_2))}. \tag{6.27}$$

□

Remark 13. *The proof of Proposition 6.4.1 seems to require some assumption on the support of μ (in particular, for the existence of $\tilde{\omega}$ in the enumerated item list above). Fully supported (i.e. $\mathrm{supp}(\mu) = \mathscr{A}^{\mathbb{Z}^2}$) suffices, and is the only case in which we will apply this result (see Corollary 11), but the conclusion probably holds under weaker assumptions.*

Given $y, z \geq 0$, we define the **past boundary of** $\mathrm{Q}_{y,z}$ as $\partial_{\downarrow} \mathrm{Q}_{y,z} := \partial \mathrm{Q}_{y,z} \cap \mathscr{P}$, i.e. the portion of the boundary of $\mathrm{Q}_{y,z}$ included in the past, and the **future boundary of** $\mathrm{Q}_{y,z}$ as the complement $\partial_{\uparrow} \mathrm{Q}_{y,z} := \partial \mathrm{Q}_{y,z} \setminus \mathscr{P}$. Clearly, $\partial \mathrm{Q}_{y,z} = \partial_{\downarrow} \mathrm{Q}_{y,z} \sqcup \partial_{\uparrow} \mathrm{Q}_{y,z}$.

Proposition 6.4.2. *Let π be a Gibbs specification satisfying exponential SSM with parameters $C, \gamma > 0$. Then, for all $n \in \mathbb{N}$, $y, z \geq \vec{1}n$, and $a \in \mathscr{A}$,*

$$\left| \pi_{\mathrm{Q}_n}^{\omega_1}(a^{\{\vec{0}\}}) - \pi_{\mathrm{Q}_{y,z}}^{\omega_2}(a^{\{\vec{0}\}}) \right| \leq C e^{-\gamma n}, \tag{6.28}$$

uniformly over $\omega_1, \omega_2 \in \Omega$ such that $\omega_1|_{\mathscr{P}} = \omega_2|_{\mathscr{P}}$.

Proof. Fix $n \in \mathbb{N}$, $y, z \geq \vec{1}n$, $a \in \mathscr{A}$, and $\omega_1, \omega_2 \in \Omega$ such that $\omega_1|_{\mathscr{P}} = \omega_2|_{\mathscr{P}}$. Then,

$$\left| \pi_{\mathrm{Q}_n}^{\omega_1}(a^{\{\vec{0}\}}) - \pi_{\mathrm{Q}_{y,z}}^{\omega_2}(a^{\{\vec{0}\}}) \right| \tag{6.29}$$

$$= \left| \pi_{\mathrm{Q}_n}^{\omega_1}(a^{\{\vec{0}\}}) - \sum_{\upsilon \in [\omega_2|_{\mathrm{Q}_n^c}]^{\Omega}} \pi_{\mathrm{Q}_n}^{\upsilon}(a^{\{\vec{0}\}}) \pi_{\mathrm{Q}_{y,z}}^{\omega_2}(\upsilon|_{\mathrm{Q}_{y,z} \setminus \mathrm{Q}_n}) \right| \tag{6.30}$$

$$\leq \sum_{\upsilon \in [\omega_2|_{\mathrm{Q}_n^c}]^\Omega} \left| \pi^{\omega_1}_{\mathrm{Q}_n}(a^{\{\vec{0}\}}) - \pi^{\upsilon}_{\mathrm{Q}_n}(a^{\{\vec{0}\}}) \right| \pi^{\omega_2}_{\mathrm{Q}_{y,z}}(\upsilon|_{\mathrm{Q}_{y,z}\setminus \mathrm{Q}_n}) \tag{6.31}$$

$$\leq \sum_{\upsilon \in \left[\omega_2|_{\mathrm{Q}_n^c}\right]^\Omega} Ce^{-\gamma n} \pi^{\omega_2}_{\mathrm{Q}_{y,z}}(\upsilon|_{\mathrm{Q}_{y,z}\setminus \mathrm{Q}_n}) = Ce^{-\gamma n}, \tag{6.32}$$

since for any $\upsilon \in \left[\omega_2|_{\mathrm{Q}_n^c}\right]^\Omega$, we have that $\Sigma_{\partial \mathrm{Q}_n}(\omega_1, \upsilon) \subseteq \partial_\uparrow \mathrm{Q}_n$, and

$$\mathrm{dist}(\vec{0}, \Sigma_{\partial \mathrm{Q}_n}(\omega_1, \upsilon)) \geq \mathrm{dist}\left(\vec{0}, \partial_\uparrow \mathrm{Q}_n\right) = n. \tag{6.33}$$

□

6.4.2 Stochastic dominance

Suppose that $\mathscr{A}$ is a finite linearly ordered alphabet. Then for any set L (in our context, usually a set of sites or bonds), $\mathscr{A}^L$ is equipped with a natural partial order $\preceq$ which is defined coordinate-wise: for $\theta_1, \theta_2 \in \mathscr{A}^L$, we write $\theta_1 \preceq \theta_2$ if $\theta_1(x) \preceq \theta_2(x)$ for every $x \in L$. A function $f : \mathscr{A}^L \to \mathbb{R}$ is said to be **increasing** if $f(\theta_1) \leq f(\theta_2)$ whenever $\theta_1 \preceq \theta_2$. An event $\mathbb{A}$ is said to be increasing if its indicator function $\mathbb{1}_{\mathbb{A}}$ is increasing.

Definition 6.4.2. *Let ρ_1 and ρ_2 be two probability measures on $\mathscr{A}^L$. We say that ρ_1 is* **stochastically dominated** *by ρ_2, writing $\rho_1 \leq_{\mathrm{D}} \rho_2$, if for every bounded increasing function $f : \mathscr{A}^L \to \mathbb{R}$ we have $\rho_1(f) \leq \rho_2(f)$, where $\rho(f)$ denotes the expected value $\mathbb{E}_\rho(f)$ of f according to the measure ρ.*

Recall from Section 6.2 the bond random-cluster model on finite subsets of $\mathbb{Z}^2$ with boundary conditions $i = 0, 1$, and the bond random-cluster measure $\varphi_{p,q}$ on $\mathbb{Z}^2$ (see page 116).

Theorem 6.4.3 ([36, Equation (29)])**.** *For any $p \in [0,1]$, $q \in \mathbb{N}$, and $\Delta \subseteq \Lambda \Subset \mathbb{Z}^2$,*

$$\varphi^{(0)}_{p,q,\Delta} \leq_{\mathrm{D}} \varphi^{(0)}_{p,q,\Lambda} \quad \text{and} \quad \varphi^{(1)}_{p,q,\Lambda} \leq_{\mathrm{D}} \varphi^{(1)}_{p,q,\Delta}. \tag{6.34}$$

In particular, if $p < p_{\mathrm{c}}(q)$, we have that, for any $\Lambda \Subset \mathbb{Z}^2$,

$$\varphi^{(0)}_{p,q,\Lambda} \leq_{\mathrm{D}} \varphi_{p,q} \leq_{\mathrm{D}} \varphi^{(1)}_{p,q,\Lambda}, \tag{6.35}$$

where $\leq_{\mathrm{D}}$ is with respect to the restriction of each measure to events on $\mathscr{E}^0(\Lambda)$.

Connectivity decay for the bond random-cluster model

The following result was a key element of the proof that $\beta_{\mathrm{c}}(q) = \log(1+\sqrt{q})$ is the critical inverse temperature for the Potts model. We will use this result in a crucial way.

Recall that for $p < p_{\mathrm{c}}(q)$, $\varphi_{p,q}$ is the unique bond random cluster measure with parameters p and q.

Theorem 6.4.4 ([9, Theorem 2])**.** *Let $q \geq 1$ and $p < p_{\mathrm{c}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$. Then, the* **two-point connectivity function** *decays exponentially, i.e. there exist constants $0 < C(p,q), c(p,q) < \infty$ such that for any $x, y \in \mathbb{Z}^2$,*

$$\varphi_{p,q}(x \leftrightarrow y) \leq C(p,q) e^{-c(p,q)\|x-y\|_2}, \tag{6.36}$$

where $\{x \leftrightarrow y\}$ is the event that the sites x and y are connected by an open path and $\|\cdot\|_2$ is the Euclidean norm.

Stochastic dominance for the site random-cluster model

Lemma 6.4.5. *Given $\Lambda \Subset \mathbb{Z}^d$ and parameters $p \in [0,1]$ and $q > 0$, we have that for any $x \in \Lambda$ and any $\tau \in \{0,1\}^{\Lambda \setminus \{x\}}$,*

$$p_1(q) \leq \psi^{(1)}_{p,q,\Lambda}(\theta(x) = 1 | \tau) \leq p_2(q), \tag{6.37}$$

where $p_1(q) := \frac{pq}{pq+(1-p)q^{2d}}$ and $p_2(q) := \frac{pq}{pq+(1-p)}$. In consequence,

$$\psi_{p_1(q),\Lambda} \leq_{\mathrm{D}} \psi^{(1)}_{p,q,\Lambda} \leq_{\mathrm{D}} \psi_{p_2(q),\Lambda}. \tag{6.38}$$

(Recall that $\psi_{p,\Lambda}$ denotes Bernoulli site percolation.)

Proof. This result is obtained by adapting the discussion on [42, p. 339] to the wired site random-cluster model case. See also [43, Lemma 5.4] for the case $q = 2$. $\square$

Stochastic dominance for the Potts model

As before, let $q \in \mathscr{A}_{\mathrm{FP},q}$ denote a fixed, but arbitrary, choice of a colour. Let $\Lambda \Subset \mathbb{Z}^d$ and consider $g : {\mathscr{A}_{\mathrm{FP},q}}^{\Lambda} \to \{+,-\}^{\Lambda}$ be defined by

$$(g(\alpha))(x) = \begin{cases} + & \text{if } \alpha(x) = q, \\ - & \text{if } \alpha(x) \neq q. \end{cases} \tag{6.39}$$

The function g makes the non-q colours indistinguishable and gives a *reduced model* (these models are sometimes called *fuzzy Potts models*). We say $\alpha \simeq \alpha'$ if $g(\alpha) = g(\alpha')$. This relation defines a partition of ${\mathscr{A}_{\mathrm{FP},q}}^{\Lambda}$, and unions of elements of this partition form a sub-algebra of ${\mathscr{A}_{\mathrm{FP},q}}^{\Lambda}$ which can be identified with the collection of all subsets of $\{+,-\}^{\Lambda}$. Let $\pi^{+}_{\beta,\Lambda} := g_* \pi^{\omega_q}_{\beta,\Lambda}$ be the push-forward measure, which is nothing more than the restriction (or projection) of $\pi^{\omega_q}_{\beta,\Lambda}$ to $\{+,-\}^{\Lambda}$. Chayes showed that the FKG property holds on events in this reduced model.

Proposition 6.4.6 ([23, Lemma on p. 211])**.** *For all $\beta > 0$ and $\Lambda \Subset \mathbb{Z}^2$, $\pi^{+}_{\beta,\Lambda}$ satisfies the following properties:*

1. *For increasing subsets $\mathbb{A}, \mathbb{B} \subseteq \{+,-\}^{\Lambda}$, $\pi^{+}_{\beta,\Lambda}(\mathbb{A}|\mathbb{B}) \geq \pi^{+}_{\beta,\Lambda}(\mathbb{A})$.*
2. *If $\mathbb{A}$ is decreasing and $\mathbb{B}$ is increasing, then $\pi^{+}_{\beta,\Lambda}(\mathbb{A}|\mathbb{B}) \leq \pi^{+}_{\beta,\Lambda}(\mathbb{A})$.*
3. *If $\Delta \subseteq \Lambda$ and $\mathbb{A}$ is an increasing subset of $\{+,-\}^{\Delta}$, then $\pi^{+}_{\beta,\Delta}(\mathbb{A}) \geq \pi^{+}_{\beta,\Lambda}(\mathbb{A})$.*

Proof.

1. This is contained in [23, Lemma on p. 211].
2. This is an immediate consequence of (1).
3. This is a standard consequence of (1): let $\mathbb{B} = [+^{\partial\Delta}]^{\Omega_{\mathrm{FP}}}$. Since $g^{-1}(\mathbb{B})$ is a single configuration, namely $q^{\partial\Delta}$, we obtain from the Markov property of $\pi^{\omega_q}_{\beta,\Lambda}$ that $\pi^{+}_{\beta,\Delta}(\mathbb{A}) = \pi^{+}_{\beta,\Lambda}(\mathbb{A}|\mathbb{B})$. From (1), we have $\pi^{+}_{\beta,\Lambda}(\mathbb{A}|\mathbb{B}) \geq \pi^{+}_{\beta,\Lambda}(\mathbb{A})$. Now, combine the previous two statements.

□

Remark 14. *The preceding result immediately applies to $\pi^{\omega_q}_{\beta,\Lambda}$ for events in ${\mathscr{A}_{\mathrm{FP},q}}^{\Lambda}$ that are measurable with respect to $\{+,-\}^{\Lambda}$, viewed as a sub-algebra of ${\mathscr{A}_{\mathrm{FP},q}}^{\Lambda}$.*

Volume monotonicity for the Widom-Rowlinson model with 2 types

For the classical Widom-Rowlinson model ($q = 2$), Higuchi and Takei showed that the FKG property holds. In particular,

Proposition 6.4.7 ([43, Lemma 2.3])**.** *Fix $q = 2$ and let $\Delta \subseteq \Lambda \Subset \mathbb{Z}^d$ and $\zeta > 0$. Then,*

$$\pi_{\Lambda}^{\zeta}(\omega_q) \leq \pi_{\Delta}^{\zeta}(\omega_q). \tag{6.40}$$

However, this kind of stochastic monotonicity can fail for general q (see [36, p. 60]).

6.5 Exponential convergence of π_n

In this section, we consider the Potts, Widom-Rowlinson, and hard-core lattice gas models and establish exponential convergence results that will lead to pressure representation and approximation algorithms for these lattice models.

Recall that for the Potts model, $\pi_{y,z}^{\beta}(\omega) = \pi_{\beta,\mathrm{Q}_{y,z}}^{\omega}(\omega|_{\{\vec{0}\}})$ and, in particular, $\pi_n^{\beta}(\omega) = \pi_{\beta,\mathrm{Q}_n}^{\omega}(\omega|_{\{\vec{0}\}})$, with similar notation for the Widom-Rowlinson and hard-core lattice gas models.

6.5.1 Exponential convergence in the Potts model

Theorem 6.5.1. *For the (ferromagnetic) Potts model with q types and inverse temperature $\beta > 0$, there exists a critical parameter $\beta_{\mathrm{c}}(q) > 0$ such that for $\beta \neq \beta_{\mathrm{c}}(q)$, there exists $C, \gamma > 0$ such that, for every $y, z \geq \vec{1}n$,*

$$\left| \pi_n^{\beta}(\omega_q) - \pi_{y,z}^{\beta}(\omega_q) \right| \leq Ce^{-\gamma n}. \tag{6.41}$$

Proof. In the supercritical region $\beta > \beta_{\mathrm{c}}(q)$, our proof very closely follows [22, Theorem 3], which treated the Ising model case. We fill in some details of their proof, adapting that proof in two ways: to a half-plane version of their result (the quantities in Equation (6.41) are effectively half-plane quantities) and to the general Potts case. For the subcritical region $\beta < \beta_{\mathrm{c}}(q)$, the proposition will follow easily from [4, Theorem 1.8 (ii)].

Part 1: $\beta > \beta_{\mathrm{c}}(q)$. Let $\mathcal{P}^{-\star}_{\partial \mathrm{Q}_n}$ denote the event that there is a $\star$-path of $-$ from $\vec{0}$ to $\partial \mathrm{Q}_n$, i.e. a path that runs along ordinary $\mathbb{Z}^2$ bonds and diagonal bonds where the colour at each site is *not* q (in our context below, the configuration on the past boundary of Q_n will be all q and thus a $\star$-path of $-$ from $\vec{0}$ to $\partial \mathrm{Q}_n$ cannot terminate on $\partial_{\downarrow}\mathrm{Q}_n$). Note that $\mathcal{P}^{-\star}_{\partial \mathrm{Q}_n}$ is an event that is measurable with respect to the sub-algebra $\{+,-\}^{\Lambda}$, for any finite set Λ containing Q_n, introduced in Section 6.4.2 (recall that this sub-algebra corresponds to the reduced Potts model).

By decomposing $\pi^{\beta}_{y,z}(\omega_q)$ into probabilities conditional on $\mathcal{P}^{-\star}_{\partial \mathrm{Q}_n}$ and $(\mathcal{P}^{-\star}_{\partial \mathrm{Q}_n})^{\mathrm{c}}$, we obtain

$$
\begin{aligned}
&\pi^{\beta}_{n}(\omega_q) - \pi^{\beta}_{y,z}(\omega_q) &(6.42)\\
= \ & \pi^{\omega_q}_{\beta,\mathrm{Q}_n}(q^{\{\vec{0}\}}) - \pi^{\omega_q}_{\beta,\mathrm{Q}_{y,z}}(q^{\{\vec{0}\}}) &(6.43)\\
= \ & \pi^{\omega_q}_{\beta,\mathrm{Q}_{y,z}}(\mathcal{P}^{-\star}_{\partial \mathrm{Q}_n}) \left(\pi^{\omega_q}_{\beta,\mathrm{Q}_n}(q^{\{\vec{0}\}}) - \pi^{\omega_q}_{\beta,\mathrm{Q}_{y,z}}(q^{\{\vec{0}\}} | \mathcal{P}^{-\star}_{\partial \mathrm{Q}_n})\right) \\
& + (1 - \pi^{\omega_q}_{\beta,\mathrm{Q}_{y,z}}(\mathcal{P}^{-\star}_{\partial \mathrm{Q}_n})) \left(\pi^{\omega_q}_{\beta,\mathrm{Q}_n}(q^{\{\vec{0}\}}) - \pi^{\omega_q}_{\beta,\mathrm{Q}_{y,z}}(q^{\{\vec{0}\}} | (\mathcal{P}^{-\star}_{\partial \mathrm{Q}_n})^{\mathrm{c}})\right). &(6.44)
\end{aligned}
$$

We claim that the expression in Equation (6.42) is nonnegative. To see this, observe that the events $[q^{\{\vec{0}\}}]$, $[q^{\partial \mathrm{Q}_n}]$, and $[q^{\partial \mathrm{Q}_{y,z}}]$ may be viewed as the events $[+^{\{\vec{0}\}}]$, $[+^{\partial \mathrm{Q}_n}]$, and $[+^{\partial \mathrm{Q}_{y,z}}]$ in the sub-algebra $\{+,-\}^{\mathrm{Q}_{y,z}}$ of the reduced model, as discussed in Section 6.4.2. Now, apply Proposition 6.4.6 (part 3) and Remark 14.

We next claim that

$$
\pi^{\omega_q}_{\beta,\mathrm{Q}_{y,z}}(q^{\{\vec{0}\}} | (\mathcal{P}^{-\star}_{\partial \mathrm{Q}_n})^{\mathrm{c}}) \geq \pi^{\omega_q}_{\beta,\mathrm{Q}_n}(q^{\{\vec{0}\}}). \tag{6.45}
$$

To be precise, first observe that $\omega \in (\mathcal{P}^{-\star}_{\partial \mathrm{Q}_n})^{\mathrm{c}}$ iff ω contains an all-q path in Q_n from $\partial \mathscr{P} \cap \{x_1 < 0\}$ to $\partial \mathscr{P} \cap \{x_1 > 0\}$. So, $(\mathcal{P}^{-\star}_{\partial \mathrm{Q}_n})^{\mathrm{c}}$ can be decomposed into a disjoint collection of events determined by the unique furthest such path from $\vec{0}$. Using the MRF property of Gibbs specifications, it follows that we can regard each of these events as an increasing event in $\{+,-\}^{\mathrm{Q}_m}$. Now, apply Proposition 6.4.6 and Remark 14. The reader may notice that here we have essentially used the *strong Markov property* (see [37, p. 1154]).

Thus, the expression in Equation (6.44) is nonpositive. This, together with the fact that $\pi^{\omega_q}_{\beta,\mathrm{Q}_n}(q^{\{\vec{0}\}}|\mathcal{P}^{-\star}_{\partial\mathrm{Q}_n}) = 0$, yields

$$0 \le \pi^{\beta}_n(\omega_q) - \pi^{\beta}_{y,z}(\omega_q) \le \pi^{\omega_q}_{\beta,\mathrm{Q}_{y,z}}(\mathcal{P}^{-\star}_{\partial\mathrm{Q}_n})\pi^{\omega_q}_{\beta,\mathrm{Q}_n}(q^{\{\vec{0}\}}) \le \pi^{\omega_q}_{\beta,\mathrm{Q}_{y,z}}(\mathcal{P}^{-\star}_{\partial\mathrm{Q}_n}). \tag{6.46}$$

So, it suffices to show that $\sup_{y,z\ge\vec{1}n}\pi^{\omega_q}_{\beta,\mathrm{Q}_{y,z}}(\mathcal{P}^{-\star}_{\partial\mathrm{Q}_n})$ decays exponentially in n. Fix $y,z \ge \vec{1}n$ and let $m > n$ such that $\vec{1}m \ge y,z$. By Proposition 6.4.6 (part 2 and part 3) and Remark 14,

$$\begin{aligned}\pi^{\omega_q}_{\beta,\mathrm{Q}_{y,z}}(\mathcal{P}^{-\star}_{\partial\mathrm{Q}_n}) &\le \pi^{\omega_q}_{\beta,\mathrm{Q}_m}(\mathcal{P}^{-\star}_{\partial\mathrm{Q}_n}) &(6.47)\\ &= \pi^{\omega_q}_{\beta,\mathrm{B}_m}(\mathcal{P}^{-\star}_{\partial\mathrm{Q}_n}|q^{\mathscr{P}}) \le \pi^{\omega_q}_{\beta,\mathrm{B}_m}(\mathcal{P}^{-\star}_{\partial\mathrm{Q}_n}) \le \pi^{\omega_q}_{\beta,\mathrm{B}_m}(\mathcal{P}^{-\star}_{\partial\mathrm{B}_n}). &(6.48)\end{aligned}$$

So, it suffices to show that $\sup_{m>n}\pi^{\omega_q}_{\beta,\mathrm{B}_m}(\mathcal{P}^{-\star}_{\partial\mathrm{B}_n})$ decays exponentially in n. Recall the Edwards-Sokal coupling $\mathbb{P}^{(1)}_{p,q,\mathrm{B}_m}$ for the Gibbs distribution and the corresponding bond random-cluster measure with wired boundary condition $\varphi^{(1)}_{p,q,\mathrm{B}_m}$ (see Section 6.2).

W.l.o.g., let's suppose that n is even, i.e. $n = 2k < m$, for some $k \in \mathbb{N}$. We consider the following two events in the bond random-cluster model, as in [23, Theorem 3]. Let $\mathcal{R}_n$ be the event of an open cycle in $\mathrm{B}_{2k}\setminus\mathrm{B}_k$ that surrounds B_k. Let $\mathcal{M}_{n,m}$ be the event in which there is an open path from some site in B_k to $\partial\mathrm{B}_m$. The joint occurrence of these two events forces the Potts event $(\mathcal{P}^{-\star}_{\partial\mathrm{B}_n})^{\mathrm{c}}$ in the coupling: $\mathcal{R}_n\cap\mathcal{M}_{n,m} \subseteq (\mathcal{P}^{-\star}_{\partial\mathrm{B}_n})^{\mathrm{c}}$ (here, technically, we are identifying these events with their inverse images of the projections in the coupling).

Then, by the coupling property,

$$\begin{aligned}\pi^{\omega_q}_{\beta,\mathrm{B}_m}\left((\mathcal{P}^{-\star}_{\partial\mathrm{B}_n})^{\mathrm{c}}\right) &= \mathbb{P}^{(1)}_{p,q,\mathrm{B}_m}\left((\mathcal{P}^{-\star}_{\partial\mathrm{B}_n})^{\mathrm{c}}\right) &(6.49)\\ &\ge \mathbb{P}^{(1)}_{p,q,\mathrm{B}_m}\left((\mathcal{P}^{-\star}_{\partial\mathrm{B}_n})^{\mathrm{c}}\middle|\mathcal{R}_n\cap\mathcal{M}_{n,m}\right)\mathbb{P}^{(1)}_{p,q,\mathrm{B}_m}(\mathcal{R}_n\cap\mathcal{M}_{n,m}) &(6.50)\\ &= \varphi^{(1)}_{p,q,\mathrm{B}_m}(\mathcal{R}_n\cap\mathcal{M}_{n,m}), &(6.51)\end{aligned}$$

so

$$\pi^{\omega_q}_{\beta,\mathrm{B}_m}(\mathcal{P}^{-\star}_{\partial\mathrm{B}_n}) \le 1 - \varphi^{(1)}_{p,q,\mathrm{B}_m}(\mathcal{R}_n\cap\mathcal{M}_{n,m}) \le \varphi^{(1)}_{p,q,\mathrm{B}_m}(\mathcal{R}^{\mathrm{c}}_n) + \varphi^{(1)}_{p,q,\mathrm{B}_m}(\mathcal{M}^{\mathrm{c}}_{n,m}). \tag{6.52}$$

Therefore,

$$\sup_{m>n} \pi^{\omega_q}_{\beta,\mathrm{B}_m}(\mathcal{P}^{-\star}_{\partial \mathrm{B}_n}) \leq \sup_{m>n} \varphi^{(1)}_{p,q,\mathrm{B}_m}(\mathcal{R}^{\mathrm{c}}_n) + \sup_{m>n} \varphi^{(1)}_{p,q,\mathrm{B}_m}(\mathcal{M}^{\mathrm{c}}_{n,m}). \tag{6.53}$$

The first term on the right hand side of Equation (6.53) is bounded from above as follows:

$$\begin{aligned} \varphi^{(1)}_{p,q,\mathrm{B}_m}(\mathcal{R}^{\mathrm{c}}_n) &\leq \varphi^{(1)}_{p,q,\tilde{\mathrm{B}}_{m+1}}(\mathcal{R}^{\mathrm{c}}_n) && (6.54)\\ &\leq \sum_{x\in\partial\mathrm{B}_k, y\in\underline{\partial}\mathrm{B}_{2k}} \varphi^{(0)}_{p^*,q,\mathrm{B}_{m+1}}(x\leftrightarrow y) && (6.55)\\ &\leq \sum_{x\in\partial\mathrm{B}_k, y\in\underline{\partial}\mathrm{B}_{2k}} \varphi_{p^*,q}(x\leftrightarrow y), && (6.56)\end{aligned}$$

where $\tilde{\mathrm{B}}_m = [-m+1,m]^2\cap\mathbb{Z}^2$ and p^* denotes the dual of p and the inequalities follow from Proposition 6.2.3 and Theorem 6.4.3.

If $p > p_{\mathrm{c}}(q)$, then $p^* < p_{\mathrm{c}}(q)$, and by Theorem 6.4.4, the first term on the right side of Equation (6.53) is upper bounded by $64C(p^*,q)n^2\exp(-c(p^*,q)n/4)$, since $|\partial\mathrm{B}_k||\underline{\partial}\mathrm{B}_{2k}| \leq 64n^2$ and $\|x-y\|_2 \geq k-1 \geq \frac{n}{4}$, for all $x\in\partial\mathrm{B}_k$ and $y\in\underline{\partial}\mathrm{B}_{2k}$. So, the first term on the right side of Equation (6.53) decays exponentially.

As for the second term, in order for $\mathcal{M}_{n,m}$ to fail to occur, there must be a closed cycle in $\mathrm{B}_m\setminus\mathrm{B}_k$ and in particular a closed path from $L_{m,n} := \mathrm{B}_m\setminus\mathrm{B}_k\cap\{x_1<0, x_2=0\}$ to $R_{m,n} := \mathrm{B}_m\setminus\mathrm{B}_k\cap\{x_1>0,x_2=0\}$ in B_m. Thus,

$$\begin{aligned} \varphi^{(1)}_{p,q,\mathrm{B}_m}(\mathcal{M}^{\mathrm{c}}_{n,m}) &\leq \varphi^{(1)}_{p,q,\tilde{\mathrm{B}}_{m+1}}(\mathcal{M}^{\mathrm{c}}_{n,m}) && (6.57)\\ &\leq \sum_{x\in L_{m,n}, y\in R_{m,n}} \varphi^{(0)}_{p^*,q,\mathrm{B}_{m+1}}(x\leftrightarrow y) && (6.58)\\ &\leq \sum_{x\in L_{m,n}, y\in R_{m,n}} \varphi_{p^*,q}(x\leftrightarrow y), && (6.59)\end{aligned}$$

where the last inequality follows by Proposition 6.2.3 and Proposition 6.4.3. By Theorem 6.4.4, this is less than

$$\sum_{i=n,j=n} C(p^*,q)e^{-c(p^*,q)(i+j)} \leq C(p^*,q)\left(e^{-c(p^*,q)n}\frac{1}{1-e^{-c(p^*,q)}}\right)^2 \tag{6.60}$$

$$= \frac{C(p^*,q)}{(1-e^{-c(p^*,q)})^2} e^{-2c(p^*,q)n}. \tag{6.61}$$

Thus, the 2nd term on the right side of Equation (6.53) also decays exponentially. Thus, $\sup_{m>n} \pi^{\omega_q}_{\beta,m}(\mathcal{P}^{-\star}_{\partial \mathrm{B}_n})$ decays exponentially in n. Thus, by Equation (6.47), $\sup_{m>n} \pi^{\omega_q}_{\beta,m}(\mathcal{P}^{-\star}_{\partial \mathrm{Q}_n})$ also decays exponentially in n, as desired.

Part 2: $\beta < \beta_c(q)$. Recall from Section 6.4 the notions of strong spatial mixing and ratio strong mixing property. We will use the following result.

Theorem 6.5.2 ([4, Theorem 1.8 (ii)])**.** *For the $\mathbb{Z}^2$ (ferromagnetic) Potts model with q types and inverse temperature β, if $0 < \beta < \beta_c(q)$ and exponential decay of the two-point connectivity function holds for the corresponding bond random-cluster model, then the unique $\mu \in \mathcal{G}(\pi^{\mathrm{FP}}_{\beta})$ satisfies the ratio strong mixing property for the class of finite simply lattice-connected sets.*

Corollary 11. *For the $\mathbb{Z}^2$ Potts model with q types and inverse temperature $0 < \beta < \beta_c(q)$, the Gibbs specification $\pi^{\mathrm{FP}}_{\beta}$ satisfies exponential SSM for the family of sets $\{\mathrm{Q}_{y,z}\}_{y,z\geq 0}$.*

Proof. This follows immediately from Theorem 6.4.4, Theorem 6.5.2, and Proposition 6.4.1. □

Then, since exponential SSM holds for the class of finite simply lattice-connected sets when $\beta < \beta_c(q)$, the desired result follows directly from Proposition 6.4.2. This completes the proof of Theorem 6.5.1. □

6.5.2 Exponential convergence in the Widom-Rowlinson model

Recall that for Bernoulli site percolation in $\mathbb{Z}^2$ there exists a critical value $p_c(\mathbb{Z}^2)$ (or just p_c), known as the **percolation threshold**, such that for $p < p_c$, there is no infinite cluster of 1's $\psi_{p,\mathbb{Z}^2}$-almost surely and, for $p > p_c$, there is such a (unique) cluster $\psi_{p,\mathbb{Z}^2}$-a.s. Similarly, one can define an analogous parameter $p_c^\star$ for the $\mathbb{Z}^{2,\star}$ lattice, which satisfies $p_c + p_c^\star = 1$ (see [72]).

Theorem 6.5.3. *For the Widom-Rowlinson model with q types and activity ζ, there exist two critical parameters $0 < \zeta_1(q) < \zeta_2(q)$ such that for $\zeta < \zeta_1(q)$ or $\zeta >$*

$\zeta_2(q)$, there exists $C, \gamma > 0$ such that, for every $y, z \geq \vec{1}n$,

$$\left|\pi_n^{\zeta}(\omega_q) - \pi_{y,z}^{\zeta}(\omega_q)\right| \leq Ce^{-\gamma n}. \tag{6.62}$$

Proof. As in Theorem 6.5.1, we split the proof in two parts.

Part 1: $\zeta > \zeta_2(q) := q^3\left(\frac{p_c}{1-p_c}\right)$**.** Fix $n \in \mathbb{N}$ and $y, z \geq \vec{1}n$. Notice that, due to the hard constraints of the Widom-Rowlinson model, and recalling Proposition 6.3.1,

$$\pi_{y,z}^{\zeta}(\omega_q) = \pi_{\zeta,\mathrm{Q}_{y,z}}^{\omega_q}(q^{\{\vec{0}\}}) = \psi_{p,q,\mathrm{Q}_{y,z}}^{(1)}(1^{\{\vec{0}\}}), \tag{6.63}$$

where $p = \frac{\zeta}{1+\zeta}$, and the same holds for $\pi_n^{\zeta}(\omega_q)$. Then, it suffices to prove that

$$\left|\psi_{p,q,\mathrm{Q}_n}^{(1)}(1^{\{\vec{0}\}}) - \psi_{p,q,\mathrm{Q}_{y,z}}^{(1)}(1^{\{\vec{0}\}})\right| \leq Ce^{-\gamma n}, \tag{6.64}$$

for some $C, \gamma > 0$.

Notice that $\vec{0} \in \mathrm{Q}_n \subseteq \mathrm{Q}_{y,z} =: \Lambda$. Fix any ordering on the set $\overline{\Lambda}$. From now on, when we talk about comparing sites in $\overline{\Lambda}$, it is assumed we are speaking of this ordering. For convenience, we will extend configurations on Q_n and Λ to configurations on $\overline{\Lambda}$ by appending $1^{\overline{\Lambda}\setminus\mathrm{Q}_n}$ and $1^{\partial\Lambda}$, respectively.

We proceed to define a coupling $\mathbb{P}_{n,y,z}$ of $\psi_{p,q,\mathrm{Q}_n}^{(1)}$ and $\psi_{p,q,\Lambda}^{(1)}$, defined on pairs of configurations $(\theta_1, \theta_2) \in \{0,1\}^{\overline{\Lambda}} \times \{0,1\}^{\overline{\Lambda}}$. The coupling is defined one site at a time, using values from previously defined sites.

We use (τ_1^t, τ_2^t) to denote the (incomplete) configurations on $\overline{\Lambda} \times \overline{\Lambda}$ at step $t = 0, 1, \ldots, |\mathrm{Q}_n|$. We therefore begin with $\tau_1^0 = 1^{\overline{\Lambda}\setminus\mathrm{Q}_n}$ and $\tau_2^0 = 1^{\partial\Lambda}$. Next, we set $\tau_1^1 = \tau_1^0$ and form τ_2^1 by extending τ_2^0 to $\overline{\Lambda} \setminus \mathrm{Q}_n$, choosing randomly according to the distribution $\psi_{p,q,\Lambda}^{(1)}\left(\cdot \middle| 1^{\partial\Lambda}\right)$. At this point of the construction, both τ_1^1 and τ_2^1 have shape $\overline{\Lambda} \setminus \mathrm{Q}_n$. In the end, $(\tau_1^{|\mathrm{Q}_n|}, \tau_2^{|\mathrm{Q}_n|})$ will give as a result a pair (θ_1, θ_2), both with shape $\overline{\Lambda}$.

At any step t, we use W^t to denote the set of sites in $\overline{\Lambda}$ on which τ_1^t and τ_2^t have already received values in previous steps. In particular, $W^1 = \overline{\Lambda} \setminus \mathrm{Q}_n$. At an arbitrary step t of the construction, we choose the next site x^{t+1} on which to assign values in τ_1^{t+1} and τ_2^{t+1} as follows:

(i) If possible, take x^{t+1} to be the first site in $\partial^\star W^t$ that is $\star$-adjacent to a site

$y \in W^t$ for which $(\tau_1^t(y), \tau_2^t(y)) \neq (1,1)$.

(ii) Otherwise, just take x^{t+1} to be the smallest site in $\partial^\star W^t$.

Notice that at any step t, W^t is a $\star$-connected set, and that it it always possible to find the next site x^{t+1} for any $t < |\mathrm{Q}_n|$ (i.e. the two rules above give a well defined procedure).

Now we are ready to augment the coupling from W^t to $W^t \cup \{x^{t+1}\}$ by assigning $\tau_1^{t+1}\big|_{\{x^{t+1}\}}$ and $\tau_2^{t+1}\big|_{\{x^{t+1}\}}$ according to an optimal coupling of $\psi^{(1)}_{p,q,\mathrm{Q}_n}(\cdot|\tau_1^t)\Big|_{\{x^{t+1}\}}$ and $\psi^{(1)}_{p,q,\mathrm{Q}_{y,z}}(\cdot \mid \tau_2^t)\Big|_{\{x^{t+1}\}}$, i.e. a coupling which minimizes the probability that, given (τ_1^t, τ_2^t), $\theta_1(x^{t+1}) \neq \theta_2(x^{t+1})$. Since $\mathbb{P}_{n,y,z}$ is defined site-wise, and at each step is assigned according to $\psi^{(1)}_{p,q,\mathrm{Q}_n}(\cdot|\tau_1^t)$ in the first coordinate and $\psi^{(1)}_{p,q,\mathrm{Q}_{y,z}}(\cdot \mid \tau_2^t)$ in the second, the reader may check that it is indeed a coupling of $\psi^{(1)}_{p,q,\mathrm{Q}_n}$ and $\psi^{(1)}_{p,q,\mathrm{Q}_{y,z}}$. The key property of $\mathbb{P}_{n,y,z}$ is the following.

Lemma 6.5.4. *$\theta_1(\vec{0}) \neq \theta_2(\vec{0})$ $\mathbb{P}_{n,y,z}$-a.s. iff there exists a path $\mathcal{P}$ of $\star$-adjacent sites from $\vec{0}$ to $\partial \mathrm{Q}_n$, such that for each site $y \in \mathcal{P}$, $(\theta_1(y), \theta_2(y)) \neq (1,1)$.*

Proof. Suppose, for a contradiction, that $\theta_1(\vec{0}) \neq \theta_2(\vec{0})$ and that there exists no such path. This implies that there exists a cycle $\mathcal{C}$ surrounding $\vec{0}$ (when we include the past boundary as part of $\mathcal{C}$) and contained in $\overline{\mathrm{Q}}_n$ such that for all $y \in \mathcal{C}$, $(\theta_1(y), \theta_2(y)) = (1,1)$. Define by I the simply lattice-$\star$-connected set of sites in the interior of $\mathcal{C}$ and, let's say that at time t_0, x^{t_0} was the first site within I defined according to the site-by-site evolution of $\mathbb{P}_{n,y,z}$. Then, $(\tau_1^{t_0}(x^{t_0}), \tau_2^{t_0}(x^{t_0}))$ cannot have been defined according to rule (i) since all sites $\star$-adjacent to x^{t_0} are either in I (and therefore not yet defined by definition of x^{t_0}), or in $\mathcal{C}$ (and therefore either not yet defined or sites at which θ_1 and θ_2 are both 1).

Therefore, $(\theta_1(x^{t_0}), \theta_2(x^{t_0}))$ was defined according to rule (ii). We therefore define the set $D := \overline{\Lambda} \setminus W^{t_0-1} \supseteq I$, and note that $\vec{0}$ and x^{t_0} belong to the same $\star$-connected component Θ of D. We also know that $\tau_1^{t_0-1}\Big|_{\partial^\star D} = \tau_2^{t_0-1}\Big|_{\partial^\star D} = 1^{\partial^\star D}$, otherwise some unassigned site in D would be $\star$-adjacent to a 0 in either $\tau_1^{t_0-1}\Big|_{\partial^\star D}$ or $\tau_2^{t_0-1}\Big|_{\partial^\star D}$, and so rule (i) would be applied instead. We may now apply Lemma 6.3.2 (combined with Remark 12) to Θ and Λ in order to see that

$\psi^{(1)}_{p,q,\mathrm{Q}_n}(\theta_1|_\Theta|\tau_1^{t_0-1})$ and $\psi^{(1)}_{p,q,\mathrm{Q}_{y,z}}(\theta_2|_\Theta|\tau_2^{t_0-1})$ are identical. This means that the optimal coupling according to which $\tau_1^{t_0}(x^{t_0})$ and $\tau_2^{t_0}(x^{t_0})$ are assigned is supported on the diagonal, and so $\tau_1^{t_0}(x^{t_0}) = \tau_2^{t_0}(x^{t_0})$, $\mathbb{P}_{n,y,z}$-almost surely. This will not change the conditions under which we applied Lemma 6.3.2, and so inductively, the same will be true for each site in I as it is assigned, including $\vec{0}$. We have shown that $\theta_1(\vec{0}) = \theta_2(\vec{0})$, $\mathbb{P}_{n,y,z}$-almost surely, regardless of when $\vec{0}$ is assigned in the site-by-site evolution of $\mathbb{P}_{n,y,z}$. This is a contradiction, and so our original assumption was incorrect, implying that the desired path $\mathcal{P}$ exists. $\square$

Given an arbitrary time t, let

$$\rho_1^t(\cdot) := \psi^{(1)}_{p,q,\mathrm{Q}_n}(\cdot|\tau_i^{t-1})\Big|_{\{x^t\}} \quad \text{and} \quad \rho_2^t(\cdot) := \psi^{(1)}_{p,q,\Lambda}(\cdot|\tau_i^{t-1})\Big|_{\{x^t\}} \tag{6.65}$$

be the two corresponding probability measures defined on the set $\{0,1\}^{\{x^t\}}$. Note that at any step within the site-by-site definition of $\mathbb{P}_{n,y,z}$, Lemma 6.4.5 implies that $\frac{\zeta}{\zeta+q^3} \le \rho_i^t(1)$, where $\zeta = \frac{p}{1-p}$ and $i = 1,2$. Now, w.l.o.g., suppose that $\rho_2^t(0) \ge \rho_1^t(0)$. Then, an optimal coupling $\mathbb{Q}^t$ of ρ_1^t and ρ_2^t will assign $\mathbb{Q}^t(\{(0,0)\}) = \rho_1^t(0)$, $\mathbb{Q}^t(\{(0,1)\}) = 0$, $\mathbb{Q}^t(\{(1,0)\}) = \rho_2^t(0) - \rho_1^t(0)$, and $\mathbb{Q}^t(\{(1,1)\}) = 1 - \rho_2^t(0)$. Therefore,

$$\mathbb{Q}^t(\{(1,1)\}^{\mathrm{c}}) = \rho_2^t(0) \le \frac{q^3}{\zeta+q^3}. \tag{6.66}$$

Next, define the map $h : \{0,1\}^{\mathrm{Q}_n} \times \{0,1\}^{\mathrm{Q}_n} \to \{0,1\}^{\mathrm{Q}_n}$ given by

$$(h(\theta_1,\theta_2))(x) = \begin{cases} 1 & \text{if } (\theta_1(x),\theta_2(x)) \ne (1,1), \\ 0 & \text{if } (\theta_1(x),\theta_2(x)) = (1,1). \end{cases} \tag{6.67}$$

By Equation (6.66), $h_*\mathbb{P}_{n,y,z}$ (the push-forward measure) can be coupled against an i.i.d. measure on $\{0,1\}^{\mathrm{Q}_n}$ which assigns 1 with probability $\frac{q^3}{\zeta+q^3}$ and 0 with probability $\frac{\zeta}{\zeta+q^3}$, and that the former is stochastically dominated by the latter. This, together with Lemma 6.5.4, yields

$$\left|\psi^{(1)}_{p,q,\mathrm{Q}_n}(1^{\{\vec{0}\}}) - \psi^{(1)}_{p,q,\mathrm{Q}_{y,z}}(1^{\{\vec{0}\}})\right| \le \mathbb{P}_{n,y,z}(\theta_1(\vec{0}) \ne \theta_2(\vec{0})) \tag{6.68}$$

$$\le \psi_{\frac{q^3}{\zeta+q^3},\mathrm{Q}_n}(\vec{0} \overset{\star}{\longleftrightarrow} \partial\mathrm{Q}_n), \tag{6.69}$$

where $\{\vec{0} \overset{\star}{\longleftrightarrow} \partial Q_n\}$ denotes the event of an open $\star$-path from $\vec{0}$ to ∂Q_n. Since we have assumed $\zeta > q^3\left(\frac{p_c}{1-p_c}\right)$ and $p_c + p_c^\star = 1$, we have $\frac{q^3}{\zeta+q^3} < p_c^\star$. It follows by [2, 62] that the expression in Equation (6.69) decays exponentially in n. This completes the proof.

Part 2: $\zeta < \zeta_1(q) := \frac{1}{q}\left(\frac{p_c}{1-p_c}\right)$. Observe that, by virtue of Proposition 6.4.2, it suffices to prove that π_ζ^{WR} satisfies exponential SSM. By considering all cases of nearest-neighbour configurations at the origin, one can compute

$$Q(\pi_\zeta^{\mathrm{WR}}) = \max_{\omega_1,\omega_2\in\Omega_{\mathrm{WR}}} d_{\mathrm{TV}}(\pi^{\omega_1}_{\zeta,\{\vec{0}\}}, \pi^{\omega_2}_{\zeta,\{\vec{0}\}}) = \frac{q\zeta}{1+q\zeta}. \tag{6.70}$$

By Theorem 3.6.1, we obtain exponential SSM when

$$\zeta < \frac{1}{q}\left(\frac{p_c}{1-p_c}\right) = \zeta_1(q). \tag{6.71}$$

Uniqueness of Gibbs states in this same region was mentioned in [38, p. 40], by appealing to [11, Theorem 1] (which is the crux of Theorem 3.6.1). □

Remark 15. *In the case $q = 2$, it is possible to give an alternative proof of Theorem 6.5.3 (Part 1) using the framework of the proof of Theorem 6.5.1 (Part 1). The arguments through Equation (6.47) go through, with an appropriate redefinition of events and use of Proposition 6.4.7 for stochastic dominance. One can then apply Lemma 6.4.5 to give estimates based on the site random-cluster model. (In contrast to Theorem 6.5.1 (Part 1), this does not require the use of planar duality). So far, this approach is limited to $q = 2$ because we do not know appropriate versions of Proposition 6.4.7 for $q > 2$.*

6.5.3 Exponential convergence in the hard-core lattice gas model

Our argument again relies on proving exponential convergence for conditional measures with respect to certain "extremal" boundaries on Q_n, but these now will consist of alternating 0 and 1 symbols rather than a single symbol (recall from Section 6.1 that $\omega^{(o)}$ is defined as the configuration of 1's in all even sites and 0 in all odd sites).

Theorem 6.5.5. *For the $\mathbb{Z}^2$ hard-core lattice gas model with activity λ, there exist two critical parameters $0 < \lambda_1 < \lambda_2$ such that for any $0 < \lambda < \lambda_1$ or $\lambda > \lambda_2$, there exist $C, \gamma > 0$ such that, for every $y, z \geq \vec{1}n$,*

$$\left|\pi_n^\lambda(\omega^{(o)}) - \pi_{y,z}^\lambda(\omega^{(o)})\right| \leq Ce^{-\gamma n}. \tag{6.72}$$

Proof. As in the previous two theorems, we consider two cases.

Part 1: $\lambda > \lambda_2 := 468$. Our proof essentially combines the disagreement percolation techniques of [11] and the proof of non-uniqueness of equilibrium states for the hard-core lattice gas model due to Dobrushin (see [26]). We need enough details not technically contained in either proof that we present a mostly self-contained argument here. From [11, Theorem 1] and an averaging argument (as in the proof of Proposition 6.4.2) on $\partial_\uparrow \mathrm{Q}_n$ induced by a boundary condition on $\mathrm{Q}_{y,z}$, we know that for any $y, z \geq \vec{1}n$,

$$\left|\pi_n^\lambda(\omega^{(o)}) - \pi_{y,z}^\lambda(\omega^{(o)})\right| \leq \mathbb{P}_{n,y,z}\left(\vec{0} \stackrel{\neq}{\longleftrightarrow} \partial_\uparrow \mathrm{Q}_n\right) \tag{6.73}$$

for a certain coupling $\mathbb{P}_{n,y,z}$ of $\pi_{\lambda,\mathrm{Q}_n}^{\omega^{(o)}}$ and $\pi_{\lambda,\mathrm{Q}_{y,z}}^{\omega^{(o)}}\Big|_{\mathrm{Q}_n}$. We do not need the structure of $\mathbb{P}_{n,y,z}$ here, but instead note the following: a path of disagreement for the boundaries $\omega^{(o)}\big|_{\partial \mathrm{Q}_n}$ and $\omega^{(o)}\big|_{\partial \mathrm{Q}_{y,z}}$ implies that in one of the configurations, all entries on the path will be "out of phase" with respect to $\omega^{(o)}$, i.e. that all entries along the path will have 1 at every odd site and 0 at every even site rather than the opposite alternating pattern of $\omega^{(o)}$. Then, if we denote by $\mathscr{T}_n$ the event that there is a path $\mathcal{P}$ from $\vec{0}$ to $\partial_\uparrow \mathrm{Q}_n$ with 1 at every odd site and 0 at every even site, it is clear that

$$\mathbb{P}_{n,y,z}\left(\vec{0} \stackrel{\neq}{\longleftrightarrow} \partial_\uparrow \mathrm{Q}_n\right) \leq \pi_{\lambda,\mathrm{Q}_n}^{\omega^{(o)}}(\mathscr{T}_n) + \pi_{\lambda,\mathrm{Q}_{y,z}}^{\omega^{(o)}}(\mathscr{T}_n). \tag{6.74}$$

Since $y, z \geq \vec{1}n$ are arbitrary (in particular, y and z can be chosen to be $\vec{1}n$), it suffices to prove that $\sup_{y,z \geq \vec{1}n} \pi_{\lambda,\mathrm{Q}_{y,z}}^{\omega^{(o)}}(\mathscr{T}_n)$ decays exponentially with n. Define the set

$$\Theta_{y,z} = \{\theta \in \{0,1\}^{\overline{S}^\star_{y,z}} : \theta \text{ is feasible and } \theta|_{\partial^\star \mathrm{Q}_{y,z}} = \omega^{(o)}|_{\partial^\star \mathrm{Q}_{y,z}}\}. \tag{6.75}$$

For any $\theta \in \Theta_{y,z}$, we define $\Sigma_{\vec{0}}(\theta)$ to be the connected component of

$$\Sigma_{\mathrm{Q}_{y,z}}(\theta, \omega^{(o)}) = \{x \in \mathrm{Q}_{y,z} : \theta(x) \neq \omega^{(o)}(x)\} \tag{6.76}$$

containing the origin $\vec{0}$. Since $\mathscr{T}_n \subseteq \{\Sigma_{\vec{0}}(\theta) \cap \partial_{\uparrow}\mathrm{Q}_n \neq \emptyset\}$, our proof will then be complete if we can show that there exist $C, \gamma > 0$ so that, for any n and $y, z \geq \vec{1}n$,

$$\pi^{\omega^{(o)}}_{\lambda, \mathrm{Q}_{y,z}}(\Sigma_{\vec{0}}(\theta) \cap \partial_{\uparrow}\mathrm{Q}_n \neq \emptyset) \leq Ce^{-\gamma n}. \tag{6.77}$$

To prove this, we use a *Peierls argument*, similar to [26].

Fix any $y, z \geq \vec{1}n$ and for any $\theta \in \Theta_{y,z}$, define $\Sigma_{\vec{0}}(\theta)$ as above, and let $K(\theta)$ to be the connected component of $\{x \in \overline{S}^{\star}_{y,z} : \theta(x) = \omega^{(o)}(x)\}$ containing $\partial^{\star}\mathrm{Q}_{y,z}$. Clearly, $\Sigma_{\vec{0}}(\theta)$ and $K(\theta)$ are disjoint, $K(\theta) \neq \emptyset$ and, provided $\theta(\vec{0}) = 0$, $\Sigma_{\vec{0}}(\theta) \neq \emptyset$. Then, define $\Gamma(\theta) := \Sigma_{\vec{0}}(\theta) \cap \partial K(\theta) \subseteq \mathrm{Q}_{y,z}$. We note that for any $\theta \in \Theta_{y,z}$ with $\theta(\vec{0}) = 0$, we have that $\theta|_{\Gamma(\theta)} = 0^{\Gamma(\theta)}$, since adjacent sites in $\Sigma_{\vec{0}}(\theta)$ and $K(\theta)$ must have the same symbol by definition of $\Sigma_{\vec{0}}(\theta)$, and adjacent 1 symbols are forbidden in the hard-core lattice gas model. Therefore, every $x \in \Gamma(\theta)$ is even.

We need the concept of *inner external boundary* for a connected set $\Sigma \Subset \mathbb{Z}^2$. The **inner external boundary** of Σ is defined to be the inner boundary of the simply lattice-connected set consisting of the union of Σ and the union of all the finite components of $\mathbb{Z}^2 \setminus \Sigma$. Intuitively, the inner external boundary of Σ is the inner boundary of the set Σ obtained after "filling in the holes" of Σ. Notice that the set $\Gamma(\theta)$ corresponds exactly to the inner external boundary of $\Sigma_{\vec{0}}(\theta)$. In addition, by [25, Lemma 2.1 (i)], we know that the inner external boundary of a finite connected set (more generally a finite $\star$-connected set) is $\star$-connected. Thus, $\Gamma(\theta) \subseteq \mathrm{Q}_{y,z}$ is a $\star$-connected set $\mathcal{C}^{\star}$ that consists only of even sites and contains the origin $\vec{0}$, for any $\theta \in \Theta_{y,z}$ with $\theta(\vec{0}) = 0$.

Then, for $\mathcal{C}^{\star} \subseteq \mathrm{Q}_{y,z}$, we define the event $E_{\mathcal{C}^{\star}} := \{\theta \in \Theta_{y,z} : \Gamma(\theta) = \mathcal{C}^{\star}\}$, and will bound from above $\pi^{\omega^{(o)}}_{\lambda, \mathrm{Q}_{y,z}}(E_{\mathcal{C}^{\star}})$, for every $\mathcal{C}^{\star}$ such that $E_{\mathcal{C}^{\star}}$ is nonempty. We make some more notation: for every such a set $\mathcal{C}^{\star}$, define $O(\mathcal{C}^{\star})$ (for 'outside') as the connected component of $(\mathcal{C}^{\star})^{\mathrm{c}}$ containing $\partial^{\star}\mathrm{Q}_{y,z}$, and define $I(\mathcal{C}^{\star})$ (for 'inside') as $\mathrm{Q}_{y,z} \setminus (\mathcal{C}^{\star} \cup O(\mathcal{C}^{\star}))$. Then $\mathcal{C}^{\star}$, $I(\mathcal{C}^{\star})$, and $O(\mathcal{C}^{\star})$ form a partition of $\overline{\mathrm{Q}}^{\star}_{y,z}$. We note that there cannot be a pair of adjacent sites from $I(\mathcal{C}^{\star})$ and $O(\mathcal{C}^{\star})$ respectively, since they would then be in the same connected component of $(\mathcal{C}^\star)^{\mathrm{c}}$. We also note that for every $\theta \in E_{\mathcal{C}^\star}$, $\mathcal{C}^\star \subseteq \Sigma_{\vec{0}}(\theta) \subseteq \mathcal{C}^\star \cup I(\mathcal{C}^\star)$ and $K(\theta) \subseteq O(\mathcal{C}^\star)$, though the sets need not be equal, since $\Sigma_{\vec{0}}(\theta)$ or $K(\theta)$ could contain "holes" which are "filled in" in $I(\mathcal{C}^\star)$ and $O(\mathcal{C}^\star)$, respectively.

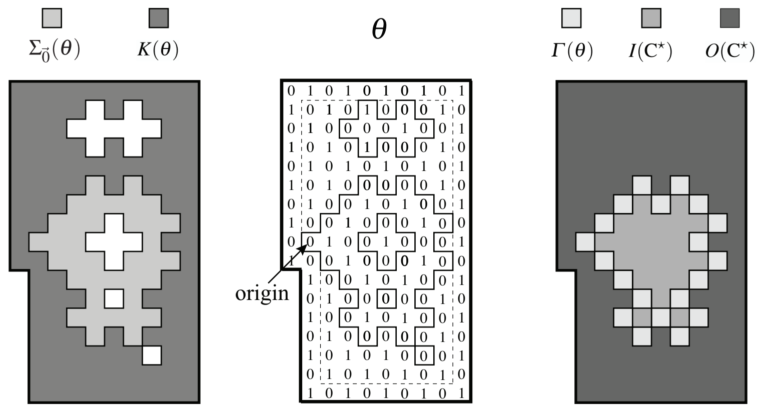

Figure 6.2: A configuration $\theta \in E_{\mathcal{C}^\star}$. On the left, the associated sets $\Sigma_{\vec{0}}(\theta)$ and $K(\theta)$. On the right, the sets $I(\mathcal{C}^\star)$ and $O(\mathcal{C}^\star)$ for $\Gamma(\theta) = \mathcal{C}^\star$.

Choose any set $\mathcal{C}^\star$ such that $E_{\mathcal{C}^\star} \neq \emptyset$. For each $\theta \in E_{\mathcal{C}^\star}$ and $x \in \mathcal{C}^\star$, using the definition of $\mathcal{C}^\star$ and the fact that $K(\theta) \subseteq O(\mathcal{C}^\star)$, there exists $x_0 \in \{\vec{e}_1, -\vec{e}_1, \vec{e}_2, -\vec{e}_2\}$ for which $x - x_0 \in O(\mathcal{C}^\star)$. Fix an x_0 which is associated to at least $|\mathcal{C}^\star|/4$ of the sites in $\mathcal{C}^\star$ in this way. Then, we define a function $s : E_{\mathcal{C}^\star} \to \{0,1\}^{\overline{\mathrm{Q}}^\star_{y,z}}$ that, given $\theta \in E_{\mathcal{C}^\star}$, defines a new configuration $s(\theta)$ such that

$$(s(\theta))(x) = \begin{cases} \theta(x - x_0) & \text{if } x \in I(\mathcal{C}^\star), \\ \theta(x) & \text{if } x \in O(\mathcal{C}^\star), \\ 1 & \text{if } x \in \mathcal{C}^\star \text{ and } x - x_0 \in O(\mathcal{C}^\star), \\ 0 & \text{if } x \in \mathcal{C}^\star \text{ and } x - x_0 \in I(\mathcal{C}^\star). \end{cases} \tag{6.78}$$

Informally, we move all 1 symbols inside $I(\mathcal{C}^\star)$ in the x_0-direction by 1 unit (even if those symbols were not part of $\Sigma_{\vec{0}}(\theta)$), add new 1 symbols at some sites in $\mathcal{C}^\star$, and leave everything in $O(\mathcal{C}^\star)$ unchanged.

It should be clear that $s(\theta)$ has at least $|\mathcal{C}^\star|/4$ more 1 symbols than θ did. We make the following two claims: s is injective on $E_{\mathcal{C}^\star}$, and for every $\theta \in E_{\mathcal{C}^\star}$, $s(\theta) \in \Theta_{y,z}$. If these claims are true, then clearly $\pi^{\omega^{(o)}}_{\lambda,\mathrm{Q}_{y,z}}(s(E_{\mathcal{C}^\star})) \geq \lambda^{|\mathcal{C}^\star|/4}\pi^{\omega^{(o)}}_{\lambda,\mathrm{Q}_{y,z}}(E_{\mathcal{C}^\star})$, implying that

$$\pi^{\omega^{(o)}}_{\lambda,\mathrm{Q}_{y,z}}(E_{\mathcal{C}^\star}) \leq \lambda^{-|\mathcal{C}^\star|/4}. \tag{6.79}$$

Firstly, we show that s is injective. Suppose that $\theta_1 \neq \theta_2$, for $\theta_1, \theta_2 \in E_{\mathcal{C}^\star}$. Then there is a site x at which $\theta_1(x) \neq \theta_2(x)$. If $x \in O(\mathcal{C}^\star)$, then $(s(\theta_1))(x) = \theta_1(x) \neq \theta_2(x) = (s(\theta_2))(x)$ and so $s(\theta_1) \neq s(\theta_2)$. If $x \in I(\mathcal{C}^\star)$, then $(s(\theta_1))(x+x_0) = \theta_1(x) \neq \theta_2(x) = (s(\theta_2))(x+x_0)$, and again $s(\theta_1) \neq s(\theta_2)$. Finally, we note that x cannot be in $\mathcal{C}^\star$, since at all sites in $\mathcal{C}^\star$, both θ_1 and θ_2 must have 0 symbols.

Secondly, we show that for any $\theta \in E_{\mathcal{C}^\star}$, $s(\theta)$ is feasible. All that must be shown is that $s(\theta)$ does not contain adjacent 1 symbols. We break 1 symbols in $s(\theta)$ into three categories:

1. *shifted*, meaning that the 1 symbol came from shifting a 1 symbol at a site in $I(\mathcal{C}^\star)$ in the x_0-direction,
2. *new*, meaning that the 1 symbol was placed at a site $x \in \mathcal{C}^\star$ such that $x - x_0 \in O(\mathcal{C}^\star)$, or
3. *untouched*, meaning that the 1 symbol was at a site in $O(\mathcal{C}^\star)$ ($\supseteq \partial^\star \mathrm{Q}_{y,z}$).

Note that untouched 1 symbols cannot be adjacent to $\mathcal{C}^\star$: θ contains all 0 symbols on $\mathcal{C}^\star$, and so since $\mathcal{C}^\star \subseteq \Sigma_{\vec{0}}(\theta)$, a 1 symbol adjacent to a symbol in $\mathcal{C}^\star$ would be in $\Sigma_{\vec{0}}(\theta)$ as well, a contradiction since $\Sigma_{\vec{0}}(\theta) \subseteq \mathcal{C}^\star \cup I(\mathcal{C}^\star)$, and so $\Sigma_{\vec{0}}(\theta)$ and $O(\mathcal{C}^\star)$ are disjoint.

Clearly, shifted 1 symbols cannot be adjacent to each other, since there were no adjacent 1 symbols in θ in the first place. All new 1's were placed at sites in $\mathcal{C}^\star$, and all sites in $\mathcal{C}^\star$ are even, so new 1 symbols cannot be adjacent to each other. Untouched 1's cannot be adjacent for the same reason as shifted 1's. We now address the possibility of adjacent 1 symbols in $s(\theta)$ from different categories. A shifted or new 1 in $s(\theta)$ is on a site in $\mathcal{C}^\star \cup I(\mathcal{C}^\star)$, and an untouched 1 cannot be adjacent to a site in $\mathcal{C}^\star$ as explained above, and also cannot be adjacent to a site in $I(\mathcal{C}^\star)$, since $I(\mathcal{C}^\star)$ and $O(\mathcal{C}^\star)$ do not contain adjacent sites. Therefore, shifted or new 1's cannot be adjacent to untouched 1's. The only remaining case which we need to rule out is a new 1 adjacent to a shifted 1. Suppose that $(s(\theta))(x)$ is a new 1 and $(s(\theta))(x')$ is a shifted 1. Then by definition, $x'-x_0 \in I(\mathcal{C}^\star)$ and $x-x_0 \in O(\mathcal{C}^\star)$. We know that $I(\mathcal{C}^\star)$ and $O(\mathcal{C}^\star)$ do not contain adjacent sites, so $x-x_0$ and $x'-x_0$ are not adjacent, implying that x and x' are not adjacent. We've then shown that $s(\theta)$ is feasible and then, since $\partial^\star \mathrm{Q}_{y,z} \subseteq O(\mathcal{C}^\star)$, $s(\theta) \in \Theta_{y,z}$, completing the proof of Equation (6.79).

Recall that every set $\mathcal{C}^\star$ which we are considering is $\star$-connected, occupies only even sites, and contains the origin $\vec{0}$. Then, given $k \in \mathbb{N}$, it is direct to see that the number of such $\mathcal{C}^\star$ with $|\mathcal{C}^\star| = k$ is less than or equal to $k \cdot t(k)$, where $t(k)$ denotes the number of *site animals* (see [49] for the definition) of size k (the first k factor comes from the fact that site animals are defined up to translation, and here given a site animal of size k, exactly k translations of it will contain the origin $\vec{0}$). We know that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $t(k) \leq C_\varepsilon(\delta+\varepsilon)^k$ for every k, where $\delta := \lim_{k\to\infty} (t(k))^{1/k} \leq 4.649551$ (see [49]).

If $\Sigma_{\vec{0}}(\theta) \cap \partial_\uparrow \mathrm{Q}_n \neq \emptyset$, then $\Sigma_{\vec{0}}(\theta)$ has to intersect the top, right, or bottom boundary of Q_n. W.l.o.g., we may assume that $\Sigma_{\vec{0}}(\theta)$ intersects the bottom boundary of Q_n. Then, every horizontal segment in the bottom half of Q_n must intersect $\Sigma_{\vec{0}}(\theta)$ and, therefore, at least one element of its inner external boundary, namely $\Gamma(\theta)$. Then,

$$\Sigma_{\vec{0}}(\theta) \cap \partial_\uparrow \mathrm{Q}_n \neq \emptyset \implies |\Gamma(\theta)| \geq n. \tag{6.80}$$

Therefore, taking an arbitrary $\varepsilon > 0$, we may bound $\pi^{\omega^{(o)}}_{\lambda,\mathrm{Q}_{y,z}}(\Sigma_{\vec{0}}(\theta) \cap \partial_\uparrow \mathrm{Q}_n \neq \emptyset)$ from above:

$$\pi^{\omega^{(o)}}_{\lambda,\mathrm{Q}_{y,z}}(\Sigma_{\vec{0}}(\theta) \cap \partial_\uparrow \mathrm{Q}_n \neq \emptyset) \leq \sum_{\mathcal{C}^\star : |\mathcal{C}^\star| \geq n} \lambda^{-|\mathcal{C}^\star|/4} \leq \sum_{k=n}^{\infty} k C_\varepsilon (\delta+\varepsilon)^k \cdot \lambda^{-k/4}, \tag{6.81}$$

which decays exponentially in n as long as $\lambda > (\delta+\varepsilon)^4$, independently of y and z. Since ε was arbitrary, $\lambda > 468 > \delta^4$ suffices for justifying Equation (6.77), completing the proof.

Part 2: $\lambda < \lambda_1 := 2.48$. It is known (see [75]) that when $d = 2$ and $\lambda < 2.48$, $\pi^{\mathrm{HC}}_\lambda$ satisfies exponential SSM. Then, by applying Proposition 6.4.2, we conclude. □

Chapter 7

Algorithmic implications

In this chapter we combine previous results (in particular, pressure representation theorems) in order to compute pressure efficiently.

By a **poly-time approximation algorithm** to compute a number r, we mean an algorithm that, given $N \in \mathbb{N}$, produces an estimate r_N such that $|r - r_N| < \frac{1}{N}$ and the time to compute r_N is polynomial in N. In that case, we regard this algorithm as an efficient way to approximate r and we say that r is **poly-time computable**.

One of our goals is to prove that, under certain assumptions on Φ and the support Ω, $\mathrm{P}_\Omega(\Phi)$ is poly-time computable.

7.1 Previous results

First, we give some previously known algorithmic results related with pressure approximation.

Proposition 7.1.1 ([32])**.** *Consider the $\mathbb{Z}^d$ hard-core lattice gas model with activity λ. If*

$$\lambda < \lambda_c(\mathbb{T}_{2d}) := \frac{(2d-1)^{2d-1}}{(2d-2)^{2d}}, \tag{7.1}$$

then, there is an algorithm to compute $\mathrm{P}(\Phi_\lambda)$ to within $\frac{1}{N}$ in time $\mathrm{poly}(N)$.

Note 10. *The value $\lambda_c(\mathbb{T}_{2d})$ corresponds to the critical activity of the hard-core model in the $2d$-regular tree $\mathbb{T}_{2d}$. This model satisfies exponential SSM if $\lambda < \lambda_c(\mathbb{T}_{2d})$. It is also known that the partition function of the hard-core model with $\lambda < \lambda_c(\mathbb{T}_{2d})$ in any finite board $\mathcal{G}$ of maximum degree $\Delta(\mathcal{G}) \leq 2d$ can be efficiently approximated (for these and more results, see the fundamental work of D. Weitz in [79]).*

Proposition 7.1.2 ([58, Proposition 4.1])**.** *Let Ω be a $\mathbb{Z}^d$ n.n. SFT, Φ a shift-invariant n.n. interaction on Ω, and π the corresponding n.n. Gibbs (Ω, Φ)-specification such that*

- *Ω satisfies SSF, and*
- *π satisfies exponential SSM.*

Then, there is an algorithm to compute $\mathrm{P}_\Omega(\Phi)$ to within $\frac{1}{N}$ in time $e^{O\left((\log N)^{d-1}\right)}$. In particular, if $d = 2$, $\mathrm{P}_\Omega(\Phi)$ is poly-time computable.

7.2 New results

The following result is based on a slight modification of the approach used to prove Proposition 7.1.2, but we include here the whole proof for completeness.

Proposition 7.2.1. *Let Ω be a $\mathbb{Z}^d$ n.n. SFT, Φ a shift-invariant n.n. interaction on Ω, and π the corresponding n.n. Gibbs (Ω, Φ)-specification such that*

- *Ω satisfies TSSM, and*
- *π satisfies exponential SSM.*

Then, there is an algorithm to compute $\mathrm{P}_\Omega(\Phi)$ to within $\frac{1}{N}$ in time $e^{O\left((\log N)^{d-1}\right)}$. In particular, if $d = 2$, $\mathrm{P}_\Omega(\Phi)$ is poly-time computable.

Proof. Given $\varepsilon > 0$ and the values of the n.n. interaction Φ, the algorithm would be the following:

1. Look for a periodic point $\overline{\omega} \in \Omega$, provided by Proposition 3.4.3. W.l.o.g., $\overline{\omega}$ has period $2g$ in every coordinate direction, for some $g \in \mathbb{N}$. This step does not need the gap of TSSM explicitly, and it does not depend on ε.
2. Take $\nu^{\overline{\omega}}$ the shift-invariant atomic measure supported on the orbit of $\overline{\omega}$. From Corollary 9, we have that

$$\mathrm{P}(\Phi) = \int \left(\mathrm{I}_\mu + \mathrm{A}_\Phi\right) d\nu \tag{7.2}$$

$$= \frac{1}{(2g)^d} \sum_{x \in [1,2g]^d \cap \mathbb{Z}^d} \left(-\log p_\mu(\sigma_x(\overline{\omega})) + \mathrm{A}_\Phi(\sigma_x(\overline{\omega})) \right), \tag{7.3}$$

for any μ an equilibrium state for Φ. We need to compute the desired approximations of $p_\mu(\omega)$, for all $\omega = \sigma_x(\overline{\omega})$ and $x \in [1,2g]^d \cap \mathbb{Z}^d$. We may assume $x = \vec{0}$ (the proof is the same for all x).

3. For $n = 1, 2, \ldots$, consider the sets $W_n = \mathrm{R}_n \setminus \mathscr{P}_n$ and $\partial W_n = S_n \sqcup V_n$, where $S_n = \partial W_n \cap \mathscr{P}$ and $V_n = \partial W_n \setminus \mathscr{P}$.

4. Represent $p_\mu(\omega)$ as a weighted average, using the MRF property,

$$p_\mu(\omega) = \sum_{\delta \in \mathscr{A}^{V_n} :\ \mu(\omega|_{S_n}\delta) > 0} \mu\left(\omega|_{\{\vec{0}\}} \middle| \omega|_{S_n} \delta \right) \mu(\delta). \tag{7.4}$$

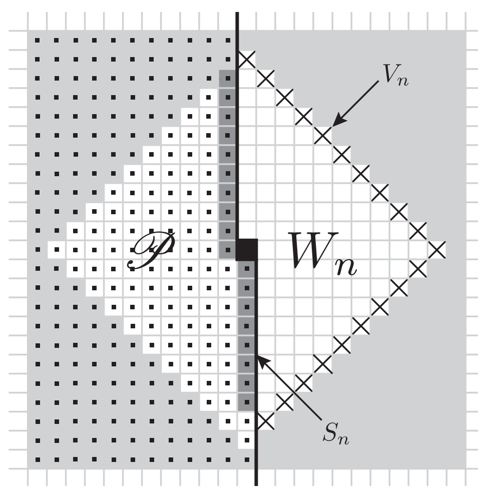

Figure 7.1: Decomposition in the proof of Proposition 7.2.1.

5. Take $\overline{\delta} \in \arg\max_\delta \mu\left(\omega|_{\{\vec{0}\}} \middle| \omega|_{S_n} \delta \right)$ and $\underline{\delta} \in \arg\min_\delta \mu\left(\omega|_{\{\vec{0}\}} \middle| \omega|_{S_n} \delta \right)$, over all $\delta \in \mathscr{A}^{V_n}$ such that $\mu(\omega|_{S_n}\delta) > 0$ (or, since TSSM implies the D-condition, such that $\omega|_{S_n}\delta \in \mathscr{L}(\Omega)$). Then,

$$\mu\left(\omega|_{\{\vec{0}\}} \middle| \omega|_{S_n} \underline{\delta} \right) \leq p_\mu(\omega) \leq \mu\left(\omega|_{\{\vec{0}\}} \middle| \omega|_{S_n} \overline{\delta} \right). \tag{7.5}$$

6. By exponential SSM, there are constants $C,\gamma > 0$ such that these upper and lower bounds on $p_\mu(\omega)$ differ by at most $Ce^{-\gamma n}$. Taking logarithms and considering that $\mu\left(\omega|_{\{\vec{0}\}}\middle|\omega|_{S_n}\underline{\delta}\right) \geq \underline{c}_\mu > 0$, a direct application of the Mean Value Theorem gives sequences of upper and lower bounds on $\log p_\mu(\omega)$ with accuracy $e^{-\Omega(n)}$, that is less than ε for sufficiently large n.

For $\delta \in \mathscr{A}^{V_n}$, the time to compute $\mu\left(\omega|_{\{\vec{0}\}}\middle|\omega|_{S_n}\delta\right)$ is $e^{O(n^{d-1})}$, because this is the ratio of two probabilities of configurations of size $O(n^{d-1})$, each of which can be computed using the *transfer matrix method* from [60, Lemma 4.8] in time $e^{O(n^{d-1})}$. Thanks to Corollary 2, the necessary time to check if $\omega|_{S_n}\delta \in \mathscr{L}(\Omega)$ or not is $e^{O(n^{d-1})}$. Since $\left|\mathscr{A}^{V_n}\right| = e^{O(n^{d-1})}$, the total time to compute the upper and lower bounds is $e^{O(n^{d-1})}e^{O(n^{d-1})} = e^{O(n^{d-1})}$. □

Remark 16. *In the previous algorithm it is not necessary to know explicitly the gap g of TSSM or the constants $C,\gamma > 0$ of the decay function $f(n) = Ce^{-\gamma n}$ from exponential SSM.*

Corollary 12. *Let $\Omega \subseteq \mathscr{A}^{\mathbb{Z}^2}$ be a $\mathbb{Z}^2$ n.n. SFT, Φ a shift-invariant n.n. interaction on Ω, and π the corresponding n.n. Gibbs (Ω,Φ)-specification. Suppose that π satisfies exponential SSM with decay rate $\gamma > 4\log|\mathscr{A}|$. Then, $\mathrm{P}_\Omega(\Phi)$ is poly-time computable.*

Proof. This follows from Theorem 3.6.5 and Proposition 7.2.1. □

Notice that, in contrast to preceding results, no mixing condition on the support is explicitly needed in Corollary 12.

Now we present a new algorithmic result for pressure representation, specially useful when SSM fails. We make heavy use of the representation and convergence results from the previous chapters.

Theorem 7.2.2. *Let Ω be a $\mathbb{Z}^d$ n.n. SFT that satisfies the square block D-condition, Φ a shift-invariant n.n. interaction on Ω, and π the corresponding n.n. Gibbs (Ω,Φ)-specification. Let $\overline{\omega} \in \Omega$ be a periodic point such that $\underline{c}_\pi(\nu^{\overline{\omega}}) > 0$. In addition, suppose that there exists $C,\gamma > 0$ such that, for every $y,z \geq \vec{1}n$,*

$$|\pi_n(\omega) - \pi_{y,z}(\omega)| \leq Ce^{-\gamma n} \text{ over } \omega \in \mathrm{O}(\overline{\omega}). \tag{7.6}$$

Then,

$$\mathrm{P}_\Omega(\Phi) = \frac{1}{|\mathrm{O}(\overline{\omega})|} \sum_{\omega \in \mathrm{O}(\overline{\omega})} \mathrm{I}_\pi(\omega) + \mathrm{A}_\Phi(\omega), \tag{7.7}$$

and, when $d = 2$, $\mathrm{P}_\Omega(\Phi)$ is poly-time computable.

Proof. Notice that $\mathrm{supp}(\nu^{\overline{\omega}}) = \mathrm{O}(\overline{\omega}) \subseteq \Omega$, since Ω is shift-invariant and $\overline{\omega} \in \Omega$. Now, since $|\pi_n(\omega) - \pi_{y,z}(\omega)| \leq Ce^{-\gamma n}$ over $\omega \in \mathrm{supp}(\nu^{\overline{\omega}})$, we can easily conclude that $\lim_{y,z\to\infty} \pi_{y,z}(\omega) = \hat{\pi}(\omega)$ uniformly over $\omega \in \mathrm{supp}(\nu^{\overline{\omega}})$. This, combined with Ω satisfying the square block D-condition and $\underline{c}_\pi(\nu^{\overline{\omega}}) > 0$, gives us

$$\mathrm{P}_\Omega(\Phi) = \int (\mathrm{I}_\pi + \mathrm{A}_\Phi) d\nu^{\overline{\omega}} = \frac{1}{|\mathrm{supp}(\nu^{\overline{\omega}})|} \sum_{\omega \in \mathrm{supp}(\nu^{\overline{\omega}})} \mathrm{I}_\pi(\omega) + \mathrm{A}_\Phi(\omega), \tag{7.8}$$

thanks to Theorem 5.3.2.

For the algorithm, it suffices to show that there is a poly-time algorithm to compute $\hat{\pi}(\omega)$, for any $\omega \in \mathrm{O}(\overline{\omega})$.

By Equation (7.6), there exist $C, \gamma > 0$ such that $|\pi_n(\omega) - \hat{\pi}(\omega)| < Ce^{-\gamma n}$. Since $|\partial \mathrm{Q}_n|$ is linear in n when $d = 2$, by the modified transfer matrix method approach from [60, Lemma 4.8], we can compute $\pi_n(\omega)$ in exponential time $Ke^{\rho n}$ for some $K, \rho > 0$. Combining the exponential time to compute $\pi_n(\omega)$ for the exponential decay of $|\pi_n(\omega) - \hat{\pi}(\omega)|$, we get a poly-time approximation algorithm to compute $\mathrm{P}_\Omega(\Phi)$: namely, given $N \in \mathbb{N}$, let n be the smallest integer such that $Ce^{-\gamma(n+1)} < \frac{1}{N}$. Then, $\pi_{n+1}(\omega)$ is within $\frac{1}{N}$ of $\hat{\pi}(\omega)$ and since $\frac{1}{N} \leq Ce^{-\gamma n}$, the time to compute $\pi_{n+1}(\omega)$ is at most

$$Ke^{\rho(n+1)} = (Ke^{\rho} C^{\rho/\gamma}) \frac{1}{(Ce^{-\gamma n})^{\rho/\gamma}} \leq (Ke^{\rho} C^{\rho/\gamma}) N^{\rho/\gamma}, \tag{7.9}$$

which is a polynomial in N. □

Corollary 13. *The following holds:*

1. *For the $\mathbb{Z}^2$ (ferromagnetic) Potts model with q types and inverse temperature $\beta > 0$,*

$$\mathrm{P}_{\Omega_{\mathrm{FP}}}(\Phi_\beta) = \mathrm{I}_\pi^\beta(\omega_q) + 2\beta. \tag{7.10}$$

2. *For the $\mathbb{Z}^2$ (multi-type) Widom-Rowlinson model with q types and activity $\zeta \in (0, \zeta_1(q)) \cup (\zeta_2(q), \infty)$,*

$$\mathrm{P}_{\Omega_{\mathrm{WR}}}(\Phi_\zeta) = \mathrm{I}_\pi^\zeta(\omega_q) + \log \zeta, \tag{7.11}$$

where $\zeta_1(q) := \frac{1}{q}\left(\frac{p_c}{1-p_c}\right)$ and $\zeta_2(q) := q^3\left(\frac{p_c}{1-p_c}\right)$.

3. *For the $\mathbb{Z}^2$ hard-core lattice gas model with activity $\lambda \in (0, \lambda_1) \cup (\lambda_2, \infty)$,*

$$\mathrm{P}_{\Omega_{\mathrm{HC}}}(\Phi_\lambda) = \frac{1}{2}\mathrm{I}_\pi^\lambda(\omega^{(o)}) + \frac{1}{2}\log \lambda, \tag{7.12}$$

where $\lambda_1 = 2.48$ and $\lambda_2 = 468$.

Moreover, for the three models in the corresponding regions (except in the case when $\beta = \beta_c(q)$ in the Potts model), there is a poly-time approximation algorithm for pressure, where the polynomial involved depends on the parameters of the models.

Proof. The representation of the pressure given in the previous statement for the $\mathbb{Z}^2$ Potts model with q types and inverse temperature $\beta \neq \beta_c(q)$, the $\mathbb{Z}^2$ Widom-Rowlinson model with q types and activity $\zeta \in (0, \zeta_1(q)) \cup (\zeta_2(q), \infty)$, and the $\mathbb{Z}^2$ hard-core lattice gas model with activity $\lambda \in (0, \lambda_1) \cup (\lambda_2, \infty)$, is a direct consequence of Theorem 7.2.2, by virtue of the following facts:

- Recall that the corresponding $\mathbb{Z}^2$ n.n. SFT Ω for the Potts, Widom-Rowlinson, and hard-core lattice gas model has a safe symbol, respectively, so Ω satisfies the square block D-condition and $\underline{c}_\pi(\nu) > 0$, for any shift-invariant ν with $\mathrm{supp}(\nu) \subseteq \Omega$, in each case.

- If we consider the δ-measure $\nu^{\omega_q} = \delta_{\omega_q}$, both in the Potts and Widom-Rowlinson cases (in a slight abuse of notation, since the Potts and Widom-Rowlinson σ-algebras are defined in different alphabets), and the measure $\nu = \nu^{\omega^{(o)}} = \frac{1}{2}\delta_{\omega^{(e)}} + \frac{1}{2}\delta_{\omega^{(o)}}$ in the hard-core lattice gas case, we have that in all three models, for the range of parameters specified, except for when $\beta = \beta_c(q)$ in the Potts model, there exists $C, \gamma > 0$ such that, for every

$y,z \geq \vec{1}n$,

$$|\pi_n(\omega) - \pi_{y,z}(\omega)| \leq Ce^{-\gamma n}, \text{ over } \omega \in \mathrm{supp}(\nu), \tag{7.13}$$

thanks to Theorem 6.5.1, Theorem 6.5.3, and Theorem 6.5.5, respectively. (Notice that $\mathrm{I}_\pi^\lambda(\omega^{(e)}) = \mathrm{A}_\Phi(\omega^{(e)}) = 0$.)

This proves Equation (7.10), Equation (7.11), and Equation (7.12), except in the Potts case when $\beta = \beta_c(q)$. To establish this case, first note that it is easy to prove that $\mathrm{P}(\Phi_\beta)$ is continuous with respect to β. Second, if $\beta_1 \leq \beta_2$, then $\pi_n^{\beta_1}(\omega_q) \leq \pi_n^{\beta_2}(\omega_q)$. This follows by the Edwards-Sokal coupling (see Theorem 6.2.2) and the comparison inequalities for the bond random-cluster model (see [3, Theorem 4.1]).

As an exercise in analysis, it is not difficult to prove that if $a_{m,n} \geq 0$, and each $a_{m+1,n} \leq a_{m,n}$ and $a_{m,n+1} \leq a_{m,n}$, then $\lim_m \lim_n a_{m,n} = \lim_n \lim_m a_{m,n} = a$, for some $a \geq 0$.

Now, consider the sequence $a_{m,n} := \pi_n^{\beta_c(q)+\frac{1}{m}}(\omega_q)$. By stochastic dominance (see Proposition 6.4.6), $a_{m,n}$ is decreasing in n. By the previous discussion (i.e. Edwards-Sokal coupling and comparison inequalities), it is also decreasing in m. Therefore, and since $a_{m,n} \geq 0$, we conclude that $\lim_m \lim_n a_{m,n} = \lim_n \lim_m a_{m,n} = a$, for some a. Then, we have that

$$\mathrm{P}(\Phi_{\beta_c(q)}) = \lim_m \mathrm{P}(\Phi_{\beta_c(q)+\frac{1}{m}}) \tag{7.14}$$

$$= \lim_m -\log \lim_n \pi_n^{\beta_c(q)+\frac{1}{m}}(\omega_q) + 2\left(\beta_c(q) + \frac{1}{m}\right) \tag{7.15}$$

$$= -\log \lim_m \lim_n \pi_n^{\beta_c(q)+\frac{1}{m}}(\omega_q) + 2\beta_c(q) \tag{7.16}$$

$$= -\log \lim_n \lim_m \pi_n^{\beta_c(q)+\frac{1}{m}}(\omega_q) + 2\beta_c(q) \tag{7.17}$$

$$= -\log \lim_n \pi_n^{\beta_c(q)}(\omega_q) + 2\beta_c(q) \tag{7.18}$$

$$= \hat{I}_\pi^{\beta_c(q)}(\omega_q) + 2\beta_c(q). \tag{7.19}$$

(To prove that $\lim_m \pi_n^{\beta_c(q)+\frac{1}{m}}(\omega_q) = \pi_n^{\beta_c(q)}(\omega_q)$ is straightforward.) Finally, the algorithmic implications are also a direct application of Theorem 7.2.2. □

Remark 17. *The algorithm in Theorem 7.2.2 seems to require explicit bounds on the constants C and γ, so that given $N \in \mathbb{N}$, we can find an explicit n such that $Ce^{-\gamma(n+1)} < \frac{1}{N}$. Without such bounds, while there exists a poly-time approximation algorithm, we do not always know how to exhibit an explicit algorithm. However, for all three models, for regions sufficiently deep within the supercritical region (i.e. β, ζ, or λ sufficiently large), one can find crude, but adequate, estimates on C and γ and thus can exhibit a poly-time approximation algorithm. This is the case for the hard-core lattice gas model, where our proof does allow an explicit estimate of the constants for any $\lambda > 468$. On the other hand, in the regions specified in Corollary 13 within the subcritical region, all three models satisfy exponential SSM and then using [60, Corollary 4.7], one can, in principle, exhibit a poly-time approximation algorithm (even without estimates on C and γ).*

Chapter 8

Conclusion

The current plan is to extend our research in the following directions:

1. We would like to develop a characterization of constraint graphs H for which $\mathrm{Hom}(\mathscr{G},\mathrm{H})$ satisfies TSSM, for every board $\mathscr{G}$. In addition, it would be interesting to understand the constraint graphs H for which, for every board of bounded degree $\mathscr{G}$, there always exist a constrained energy function ϕ such that the Gibbs $(\mathscr{G},\mathrm{H},\phi)$-specification satisfies (exponential) SSM.

2. We would like to make progress in determining regimes where SSM holds in classical models. There is already some progress in this direction (see [79, 39]), but not everything is completely understood. For example, it is not known whether the uniform 5-colourings model in $\mathbb{Z}^2$ satisfies SSM or not.

3. In [32], Gamarnik and Katz gave a representation theorem and an approximation algorithm for *surface pressure*, which is a first order correction of pressure. Their result relied on the existence of a (unique) Gibbs measure satisfying SSM and a safe symbol. We plan to generalize their result, by relaxing the combinatorial hypothesis and also extending both representation and approximation of surface pressure to the supercritical regime in $\mathbb{Z}^d$ lattice models.

4. We would like to develop new pressure approximation techniques, by finding more connections between pressure representation/approximation and recent algorithmic developments for approximate counting. In particular, we would like to explore alternative tools outside the scope of SSM and also SSM for the case of non-binary models, where computational tree methods are not well-understood.

5. It is not known any hard constrained model satisfying SSM, but not TSSM. In which cases is TSSM a necessary condition for SSM to hold? In which cases is TSSM a sufficient condition for the existence of a measure with SSM?

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