The Topology of Representation Varieties

by

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Abstract

The crux of this thesis is the topology of representation varieties. To be more precise, let G be a complex reductive linear algebraic group and let $K \subset G$ be a maximal compact subgroup. Given a nilpotent group Γ generated by r elements, we consider the representation spaces $\mathrm{Hom}(\Gamma, G)$ and $\mathrm{Hom}(\Gamma, K)$ with the natural topology induced from an embedding into G^r and K^r respectively. Our main result shows that there is a strong deformation retraction of $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$. We also obtain a strong deformation retraction of the geometric invariant theory quotient $\mathrm{Hom}(\Gamma, G)/\!\!/G$ onto the ordinary quotient $\mathrm{Hom}(\Gamma, K)/K$. Using these deformations, we then describe the topology of these spaces.

Preface

This thesis is an amalgam of two manuscripts [7, 8] previously published in *Geometry & Topology* and the *Journal of Group Theory*. The introduction of this thesis is a modified and expanded version of the respective introductions of these manuscripts. The mathematical results found in [7] are the fruit of independent work by the author under the guidance of Alexandra Pettet, Lior Silberman and Juan Souto; they make up the second (and main) chapter of this thesis. The content of the third chapter can also be found in [8] and is the result of joint work with Lior Silberman. Given the nature of this work, there is no clear way to separate the individual contributions to this chapter.

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Acknowledgments

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Chapter 1

Introduction

This thesis is motivated by a desire to understand the space of representations of finitely generated groups into linear algebraic groups. In the classical case where the source is finite, this essentially reduces to the correspondence between linear representations and characters, due to Frobenius. On the other hand, when the source is infinite, the analogue of character theory is the parametrization of representations by geometric varieties. Although these so-called *representation varieties* are interesting in their own right, they arise concretely in various settings such as mathematical physics, symplectic geometry and gauge theory. There, one often encounters basic questions about their topology; for instance, one might need to count their connected components. In this context, the author's long-standing goal has been to understand the topology of these spaces by exploiting their algebro-geometric structure.

To be more precise, let Γ be a group generated by r elements and let G be a complex linear algebraic group such as $\mathrm{SL}_n\mathbb{C}$. Since a representation of Γ in G is uniquely determined by the image of a generating set, the space $\mathrm{Hom}(\Gamma, G)$ of homomorphisms $\rho\colon \Gamma \to G$ can be realized as an affine algebraic set, carved out of G^r by the (polynomial) relations of Γ. Indeed, if Γ has a presentation $\langle \gamma_1, \ldots, \gamma_r \,|\, R_1, R_2, R_3, \ldots \rangle$ with generators γ_i and relations R_j, then we can identify $\mathrm{Hom}(\Gamma, G)$ with the set of tuples $(g_1, \ldots, g_r) \in G^r$ satisfying the given relations. It is well known and otherwise easy to see that this geometric structure is independent of the chosen presentation of Γ.

Remark 1. It is interesting to note that, by the Hilbert Basis Theorem [28], there is a group Γ' with presentation $\langle \gamma_1, \ldots, \gamma_r \,|\, R'_1, R'_2, \ldots, R'_m \rangle$ such that the variety $\mathrm{Hom}(\Gamma, G)$ coincides with the variety $\mathrm{Hom}(\Gamma', G)$. In other words, we may always assume without loss of generality that our group Γ is finitely presented.

Example 1. When $\Gamma = \mathbb{Z}^r$, the space $\mathrm{Hom}(\mathbb{Z}^r, G)$ can be identified with the space

$$\{(g_1, g_2, \ldots, g_r) \in G^r : g_i g_j = g_j g_i \text{ for all } i \text{ and } j\} \tag{1.1}$$

of commuting r-tuples in G. The nice thing about this example is that it is already complicated enough to be interesting yet simple enough to be understood concretely.

As a complex variety, $\mathrm{Hom}(\Gamma, G)$ admits a natural Hausdorff topology obtained from an embedding into affine space. If K is a maximal compact subgroup of G, for instance $\mathrm{SU}_n \subset \mathrm{SL}_n\mathbb{C}$, we can then endow $\mathrm{Hom}(\Gamma, K) \subset \mathrm{Hom}(\Gamma, G)$ with the subspace topology. It is much easier to answer topological questions for the compact space $\mathrm{Hom}(\Gamma, K)$. Indeed, most of what is known about the algebraic topology of representation spaces stems from the work of Adem and Cohen [2] and its follow-ups in this compact setting. This comes from the fact that many tools from algebraic topology and the theory of Lie groups break down when G is not compact. The fascinating thing here is that, in some cases, these problems can be circumvented because the entire representation variety retracts onto the space of representations in a compact subgroup. In order to make this precise, recall that a strong deformation retraction of a space X onto a subspace $Y \subset X$ is a continuous map

$$f : X \times [0,1] \to X \tag{1.2}$$

such that

1. $f(x, 0) = x$ for all $x \in X$,
2. $f(x, 1) \in Y$ for all $x \in X$, and
3. $f(y, t) = y$ for all $y \in Y$ and $t \in [0, 1]$.

The main result of this thesis is the following:

Theorem I (Bergeron [7])**.** *Let Γ be a finitely generated nilpotent group and let G be the group of complex or real points of a (possibly disconnected) reductive linear algebraic group, defined over $\mathbb{R}$ in the latter case. If K is any maximal compact subgroup of G, then there is a (K−equivariant) strong deformation retraction of* $\mathrm{Hom}(\Gamma, G)$ *onto* $\mathrm{Hom}(\Gamma, K)$*. In particular,* $\mathrm{Hom}(\Gamma, G)$ *and* $\mathrm{Hom}(\Gamma, K)$ *are homotopy equivalent.*

The assumption that G be reductive in Theorem I is necessary. Combining ideas of Mal'cev and Dyer, one can produce a (nilpotent) lattice Γ in a unipotent Lie group U for which the space $\mathrm{Hom}(\Gamma, U)$ is disconnected. Viewing U as an algebraic group and keeping in mind that a maximal compact subgroup $K \subset U$ is necessarily trivial, we see that the inclusion $\mathrm{Hom}(\Gamma, K) \hookrightarrow \mathrm{Hom}(\Gamma, U)$ is not even a bijection of connected components. On the other hand, it goes without saying that Theorem I does not hold for arbitrary finitely generated groups Γ. Indeed, if $n \geq 3$ and $\Gamma \subset \mathrm{SL}_n\mathbb{R}$ is a cocompact lattice, then it was shown by Selberg [49] that the component of the tautological representation $\Gamma \hookrightarrow \mathrm{SL}_n\mathbb{R}$ consists of the conjugates of Γ and, in particular, is disjoint from $\mathrm{Hom}(\Gamma, \mathrm{SO}_n\mathbb{R})$. Once again the inclusion $\mathrm{Hom}(\Gamma, K) \hookrightarrow \mathrm{Hom}(\Gamma, G)$ is not a bijection of π_0.

It may also be worth mentioning that Theorem I has already found applications beyond the author's own work. For example, Higuera [27] made use of it in his PhD Thesis to extend his results on nilpotent representations in SU_2 to $\mathrm{SL}_2\mathbb{C}$, Adem and Cheng [1] used it to generalize their work on almost commuting matrices in unitary groups to general linear groups and, in a different direction, it has been used in the development of nilpotent K-theory by Adem, Gómez, Lind and Tillmann [5].

Before going any further, it should be noted that Theorem I had two predecessors. When Γ is an abelian group, it is due to Pettet and Souto [43] and when Γ is an *expanding* nilpotent group it is is an unpublished result of Silberman and Souto. In fact, our approach was the fruit of a successful attempt to replace the ad-hoc homotopy-theoretic methods used by these authors with completely different techniques coming from algebraic geometry, notably, from *Geometric Invariant Theory* (GIT). In particular, even in the case where $\Gamma = \mathbb{Z}^r$, the proof is entirely new.

Remark 2. As the reader may be aware, there is always a strong deformation retraction of G onto K given by the polar decomposition. As such, they may be wondering where the difficulty lies in the aforementioned results. Indeed, if $\Gamma = F_r$ is a free group of rank r, one immediately obtains from this polar deformation that

$$\mathrm{Hom}(F_r, G) = G^r \simeq K^r = \mathrm{Hom}(F_r, K). \tag{1.3}$$

Nevertheless, when Γ is not a free group, there is no guarantee that such a deformation of G onto K preserves the necessary relations. Indeed, the predecessors to Theorem I already have rather delicate proofs because of a result by Souto [53] showing that, for $n \geq 8$, there is no retraction of $\mathrm{SL}_n\mathbb{C}$ onto SU_n which preserves commutativity.

Let us recall at this point that G acts by conjugation on (the image of) representations and that, when $G = \mathrm{SL}_n\mathbb{C}$ or $G = \mathrm{GL}_n\mathbb{C}$, representations are in the same G-orbit whenever their images differ by a change of basis. As such, one often wishes to classify representations modulo this equivalence relation. Unfortunately, since G is not compact, the naive topological quotient $\mathrm{Hom}(\Gamma, G)/G$ may be in need of repair: the failure of some orbits to be closed prevents it from being Hausdorff. Consider, for instance, the action of $\mathrm{SL}_2\mathbb{C}$ on $\mathrm{Hom}(\mathbb{Z}, \mathrm{SL}_2\mathbb{C}) = \mathrm{SL}_2\mathbb{C}$ by conjugation. In this setting it is easy to see that the following orbit is not closed

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}^{-1} \xrightarrow{\lambda \to 0} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.4}$$

This is why we consider the GIT quotient $\mathrm{Hom}(\Gamma, G)/\!\!/G$ obtained from the ring of invariant functions instead. In purely topological terms, this so-called *character variety* may be constructed as the universal quotient in the category of Hausdorff spaces. One of the advantages of the algebraic techniques used to prove Theorem I is that they also yield an analogous deformation result for character varieties. This is the second main result of this thesis:

Theorem II (Bergeron [7]). *Let Γ be a finitely generated nilpotent group and let G be the group of complex or real points of a (possibly disconnected) reductive linear algebraic group, defined over $\mathbb{R}$ in the latter case. If $K \subset G$ is any maximal compact subgroup, then there is a strong deformation retraction of the character variety* $\mathrm{Hom}(\Gamma, G)/\!\!/G$ *onto* $\mathrm{Hom}(\Gamma, K)/K$.

When Γ is abelian, Theorem II is a recent result of Florentino and Lawton [24] (following up on their previous work for free groups in [23]) and we refer the reader to their paper for various applications. The main tools from geometric invariant theory used to prove Theorem I and Theorem II are the Kempf–Ness Theorem and the Neeman–Schwarz Theorem. More precisely, for Γ and G as in Theorem I, we embed $\mathrm{Hom}(\Gamma, G)$ into the complex vector space $\oplus_{j=1}^r M_n\mathbb{C}$ equipped with the Frobenius norm. Following Richardson, we then denote by $\mathcal{M}$ the set of representations that are of minimal norm within their G–orbit, the so-called *Kempf–Ness set* of $\mathrm{Hom}(\Gamma, G)$. Our main technical result characterizes this set when Γ is nilpotent:

Theorem III (Bergeron [7]). *Let $G \subset \mathrm{SL_n}\mathbb{C}$ be a complex reductive linear algebraic* *group and suppose that $K = G \cap \mathrm{SU}_n \subset G$ is a maximal compact subgroup. If Γ is a finitely generated nilpotent group and $\mathcal{M}$ denotes the Kempf–Ness set of the G–subvariety $\mathrm{Hom}(\Gamma, G) \subset \oplus_{j=1}^{r} M_n\mathbb{C}$ equipped with the Frobenius norm, then*

$$\mathcal{M} = \{\rho \in \mathrm{Hom}(\Gamma, G) : \rho(\Gamma) \textit{ consists of normal matrices}\}.$$

Here, it is worth noting that the nilpotency assumption on Γ is necessary; indeed, we shall see that Theorem III is already false for the solvable group $\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$. Since the Neeman–Schwarz Theorem provides us with a retraction of $\mathrm{Hom}(\Gamma, G)$ onto $\mathcal{M}$, in order to deduce Theorem I from Theorem III it remains to deform representations of normal matrices into representations of unitary matrices. This is accomplished by scaling eigenvalues. Finally, Theorem II is a consequence of Theorem III after applying a corollary of the Kempf–Ness Theorem.

The main motivation behind Theorem I and Theorem II is to understand the topology of representation spaces. This can now be effectively achieved in many ways. For instance, building on previous work of Pettet, Silberman and Souto, Lior Silberman and the author have used them to get a handle on various invariants. To be more precise, recall that a non-abelian finitely generated group Γ is s-step nilpotent if its lower central series, defined inductively by

$$\Gamma_{(1)} = \Gamma \quad \text{and} \quad \Gamma_{(i+1)} = [\Gamma, \Gamma_{(i)}], \tag{1.5}$$

has $\Gamma_{(s)}$ non-trivial but $\Gamma_{(s+1)} = \{e\}$. Notice that the epimorphism $\Gamma \to \Gamma/\Gamma_{(i)}$ induces an embedding

$$\mathrm{Hom}(\Gamma/\Gamma_{(i)}, G) \to \mathrm{Hom}(\Gamma, G) \tag{1.6}$$

which, for general groups Γ and G, is not even an open map. Nevertheless, using Theorem I, we will see that the following holds:

Theorem IV (Bergeron-Silberman [8]). *Let Γ be a finitely generated nilpotent group and let G be the group of complex points of a (possibly disconnected) reductive linear algebraic group. For all $i \geq 2$, the inclusion*

$$\mathrm{Hom}(\Gamma/\Gamma_{(i)}, G) \xrightarrow{\iota} \mathrm{Hom}(\Gamma, G)$$

is a homotopy equivalence onto the union of those components of the target intersecting the image of ι.

Let us now consider $\mathrm{Hom}(\Gamma, G)$ as a pointed space by taking the trivial representation as base point. From this point of view, Theorem IV implies that the connected components

$$\mathrm{Hom}(\Gamma, G)_{\mathbb{1}} \subset \mathrm{Hom}(\Gamma, G) \text{ and } \mathrm{Hom}(\Gamma/\Gamma_{(i)}, G)_{\mathbb{1}} \subset \mathrm{Hom}(\Gamma/\Gamma_{(i)}, G) \tag{1.7}$$

of the trivial representation are homotopy equivalent for all $i \geq 2$. This allows us to describe the homotopy type of the component of the trivial representation in terms of abelian representations:

$$\mathrm{Hom}(\Gamma, G)_{\mathbb{1}} \simeq \mathrm{Hom}(\mathbb{Z}^{\mathrm{rank}\, H_1(\Gamma;\mathbb{Z})}, G)_{\mathbb{1}}. \tag{1.8}$$

Applying Theorem I to reduce things to the compact case, we can then use the work of Baird [6], Gómez–Pettet–Souto [25] and Biswas–Lawton–Ramras [11] in this setting to compute its fundamental group and describe its cohomology as well as that of the corresponding component in the character variety.

While the results above indicate many similarities between representation spaces of abelian and non-abelian nilpotent groups, the latter have a much richer topology than the former. For example, recall that $\mathrm{Hom}(\mathbb{Z}^r, \mathrm{SL}_n\mathbb{C})$, $\mathrm{Hom}(\mathbb{Z}^r, \mathrm{Sp}_{2n}\mathbb{C})$ and the corresponding character varieties are connected for all values of r and n. Using our results, we are able to show that the situation for non-abelian nilpotent groups is completely different:

Theorem V (Bergeron-Silberman [8])**.** *Let G be the group of complex points of a reductive linear algebraic group. If Γ is a finitely generated nilpotent group which surjects onto a finite non-abelian subgroup of G, then* $\mathrm{Hom}(\Gamma, G)$ *and* $\mathrm{Hom}(\Gamma, G)/\!\!/G$ *are both disconnected topological spaces.*

One way to see Theorem V in action is to observe that non-abelian free nilpotent groups and discrete Heisenberg groups surject onto the non-abelian nilpotent group of order 8. Contrary to popular beliefs, this implies that their representation varieties are essentially always disconnected.

Example 2. Consider the discrete Heisenberg group

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\} \cong \langle x, y, z \,|\, [x, z] = [y, z] = e, [x, y] = z \rangle. \tag{1.9}$$

The phenomenon mentioned above already occurs when we consider representations of H_3 in SU_2. There, $\mathrm{Hom}(H_3, \mathrm{SU}_2)$ breaks up into two components. A simple computation shows that any representation $\rho : H_3 \to \mathrm{SU}_2$ which doesn't factor through the abelianization $H_3/[H_3, H_3] \cong \mathbb{Z}^2$ is conjugate to the quaternionic representation

$$x \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \, y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } z \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.10}$$

Since the stabilizer of such a non-abelian representation corresponds to $\{\pm \mathrm{Id}\} \subset \mathrm{SU}_2$, we see that the component of $\mathrm{Hom}(H_3, \mathrm{SU}_2)$ consisting of these representations is homeomorphic to $\mathrm{SU}_2 / \pm \mathrm{Id} \simeq \mathbb{RP}_3$. The other component of $\mathrm{Hom}(H_3, \mathrm{SU}_2)$ consisting of abelian representations is homeomorphic to $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{SU}_2)$.

Remark 3. The statements of Theorem IV and Theorem V remain true when G is replaced by a compact Lie group K. In fact, we prove them in this setting before obtaining the complex reductive case by applications of Theorem I and Theorem II.

We conclude this introduction with a brief outline of the thesis. The second chapter concerns the details behind Theorem I and Theorem II. In the third chapter, we prove Theorem III, Theorem IV and their many corrolaries. Finally, in the concluding chapter, we explore the meaning of these results within the field as a whole and describe our ongoing work on the many questions that arose along the way.

Chapter 2

The Topology of Nilpotent Representations in Reductive Groups and their Maximal Compact Subgroups

This chapter is a transcript of [7] consisting of independent work by the author.

2.1 Introduction

Let Γ be a group generated by r elements and let G be a complex linear algebraic group. Since a representation of Γ in G is uniquely determined by the image of a generating set, the space $\mathrm{Hom}(\Gamma, G)$ of homomorphisms $\rho : \Gamma \to G$ can be realized as an affine algebraic set carved out of G^r by the relations of Γ. It is well known and otherwise easy to see that this geometric structure is independent of the chosen presentation of Γ. As a complex variety, $\mathrm{Hom}(\Gamma, G)$ admits a natural Hausdorff topology obtained from an embedding into affine space. If K is a maximal compact subgroup of G, we can then endow $\mathrm{Hom}(\Gamma, K) \subset \mathrm{Hom}(\Gamma, G)$ with the subspace Hausdorff topology. Although in general these topological spaces may be quite different, in this thesis we show that when Γ is nilpotent and G is reductive they are actually homotopy equivalent:

Theorem I. *Let Γ be a finitely generated nilpotent group and let G be the group of complex or real points of a (possibly disconnected) reductive linear algebraic group, defined over $\mathbb{R}$*

in the latter case. If K is any maximal compact subgroup of G, then there is a (K-equivariant) strong deformation retraction of $\mathrm{Hom}(\Gamma, G)$ *onto* $\mathrm{Hom}(\Gamma, K)$. *In particular,* $\mathrm{Hom}(\Gamma, G)$ *and* $\mathrm{Hom}(\Gamma, K)$ *are homotopy equivalent.*

The assumption that G be reductive in Theorem I is necessary. Combining ideas of Mal'cev and Dyer, one can produce a (nilpotent) lattice Γ in a unipotent Lie group U for which the space $\mathrm{Hom}(\Gamma, U)$ is disconnected. Viewing U as an algebraic group and keeping in mind that a maximal compact subgroup $K \subset U$ is necessarily trivial, we see that the inclusion $\mathrm{Hom}(\Gamma, K) \hookrightarrow \mathrm{Hom}(\Gamma, U)$ is not even a bijection of π_0. A complete discussion of this example will be given in the Section 2.7. Similarly, Theorem I does not hold for arbitrary Γ: if $n \geq 3$ and $\Gamma \subset \mathrm{SL}_n\mathbb{R}$ is a cocompact lattice, then it was shown by Selberg [49] that the component of the tautological representation $\Gamma \hookrightarrow \mathrm{SL}_n\mathbb{R}$ consists of the conjugates of Γ and, in particular, is disjoint from $\mathrm{Hom}(\Gamma, \mathrm{SO}_n\mathbb{R})$. Once again the inclusion $\mathrm{Hom}(\Gamma, K) \hookrightarrow \mathrm{Hom}(\Gamma, G)$ is not a bijection of π_0.

There is also an interesting contrast between the real and complex cases of Theorem I. For instance, while nilpotent subgroups of $\mathrm{O}^+_{n,1}\mathbb{R}$ (the group of isometries of hyperbolic n-space) are all virtually abelian, this is far from being true in $\mathrm{GL}_n\mathbb{R}$. Nevertheless, $\mathrm{O}_n\mathbb{R}$ is a maximal compact subgroup of both $\mathrm{GL}_n\mathbb{R}$ and $\mathrm{O}^+_{n,1}\mathbb{R}$ so Theorem I shows that $\mathrm{Hom}(\Gamma, \mathrm{O}^+_{n,1}\mathbb{R})$ and $\mathrm{Hom}(\Gamma, \mathrm{GL}_n\mathbb{R})$ are homotopy equivalent. This kind of example does not occur in the complex case because complex reductive linear algebraic groups are determined by their maximal compact subgroups.

It should be mentioned at this point that Theorem I has two predecessors. When Γ is an abelian group, it is due to Pettet and Souto [43] and when Γ is an expanding nilpotent group (c.f. the Section 2.7) it is is due to Lior Silberman and Juan Souto. While these authors used homotopy-theoretic methods, in this thesis we rely on algebraic geometry instead. In particular, even in the case where $\Gamma = \mathbb{Z}^r$ our proof is completely new. One of the advantages of this approach, coming from geometric invariant theory, is that the corresponding result for the character variety $\mathrm{Hom}(\Gamma, G)/\!\!/G$ follows from the proof of Theorem I:

Theorem II. *Let Γ be a finitely generated nilpotent group and let G be the group of complex or real points of a (possibly disconnected) reductive linear algebraic group, defined over $\mathbb{R}$ in the latter case. If $K \subset G$ is any maximal compact subgroup, then there is a strong deformation retraction of the character variety* $\mathrm{Hom}(\Gamma, G)/\!\!/G$ *onto* $\mathrm{Hom}(\Gamma, K)/K$.

When Γ is abelian, Theorem II is a recent result of Florentino and Lawton [24] (following up on their previous work for free groups in [23]) and we refer the reader to their paper for various applications. The main tools from geometric invariant theory used to prove Theorem I and Theorem II are the Kempf-Ness Theorem and the Neeman-Schwarz Theorem. More precisely, for Γ and G as in Theorem I, we embed $\mathrm{Hom}(\Gamma, G)$ into the complex vector space $\oplus_{j=1}^{r} M_n\mathbb{C}$ equipped with the Frobenius norm. Following Richardson, we then denote by $\mathcal{M}$ the set of representations that are of minimal norm within their G-orbit, the so-called *Kempf-Ness set* of $\mathrm{Hom}(\Gamma, G)$. Our main technical result characterizes this set when Γ is nilpotent:

Theorem III. *Let $G \subset \mathrm{SL_n}\mathbb{C}$ be a complex reductive linear algebraic group and suppose that $K = G \cap \mathrm{SU}_n \subset G$ is a maximal compact subgroup. If Γ is a finitely generated nilpotent group and $\mathcal{M}$ denotes the Kempf-Ness set of the G-subvariety* $\mathrm{Hom}(\Gamma, G) \subset \oplus_{j=1}^{r} M_n\mathbb{C}$ *equipped with the Frobenius norm, then*

$$\mathcal{M} = \{\rho \in \mathrm{Hom}(\Gamma, G) : \rho(\Gamma) \textit{ consists of normal matrices}\}.$$

Here, it is worth noting that the nilpotency assumption on Γ is necessary; in Section 2.4.2 we show that Theorem III is false for the solvable group $\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$. Since the Neeman-Schwarz Theorem provides us with a retraction of $\mathrm{Hom}(\Gamma, G)$ onto $\mathcal{M}$, in order to deduce Theorem I from Theorem III it remains to deform representations of normal matrices into representations of unitary matrices. This is accomplished by scaling eigenvalues. Finally, Theorem II is a consequence of Theorem III after applying a corollary of the Kempf-Ness Theorem.

2.1.1 Historical remarks and applications

In coarsest terms, this thesis is concerned with the geometric classification of representations of finitely generated groups. Classically, this subject finds its roots in Poincaré's work on monodromy groups of linear homogeneous equations and geometric invariant theory. More recently, the study of representation varieties has had impacts in a variety of contexts and we mention here but a few.

From a first point of view, the geometry of representation varieties can be used to deduce algebraic information about the representation theory of a group. For instance, Lubotzky and Magid [35] obtained such information for the $\mathrm{GL}_n\mathbb{C}$ representations of a finitely generated group Γ by using Weil's results on the Zariski tangent space of $\mathrm{Hom}(\Gamma, \mathrm{GL}_n\mathbb{C})$.

Incidentally, these techniques were most effective when Γ was assumed to be nilpotent, partly because this ensured a special vanishing property of its first group cohomology.

One can also study representation varieties from a purely differential geometric point of view. Here, when Γ is the fundamental group of a smooth manifold M, $\mathrm{Hom}(\Gamma, K)$ can be identified with the space of pointed flat connections on principal K-bundles over M. These spaces lie at the intersection of various fields including gauge theory and symplectic geometry as illustrated for surfaces in Jeffrey's survey [30]. More precisely, there is a natural conjugation action of K on $\mathrm{Hom}(\Gamma, K)$ and the quotient $\mathrm{Hom}(\Gamma, K)/K$ corresponds to the moduli space of flat-connections on principal K-bundles over M. In the case where M is compact and Kähler, the geometric invariant theory quotient $\mathrm{Hom}(\Gamma, G)/\!\!/G$ corresponds to the moduli space of polystable G-bundles over M via the Narasimhan-Seshadri Theorem (see Narasimhan-Seshadri [40] and Simpson [52]). More generally, these character varieties have attracted a lot of attention lately as shown in Sikora's survey [51, Section 11] and the references therein.

More topologically, the work in supersymmetric Yang-Mills theory and mirror symmetry of Witten in [55] and [56] sparked an interest in the connected components of free abelian representation varieties as seen in Kac-Smilga [31] and Borel-Friedman-Morgan [13]. Here, there has been much recent development in the study of higher topological invariants stemming from work of Ádem and Cohen [2] in the compact case. Prior to Pettet and Souto's work in [43], most known topological invariants concerned spaces of representations into compact groups; it was their deformation retraction which allowed many of these results to be extended to representations into reductive groups. We briefly indicate how this strategy works in the nilpotent case, referring the reader to [43] for more applications of this type.

Suppose for the sake of concreteness that Γ is a torsion free nilpotent group and G is a complex reductive linear algebraic group with a given maximal compact subgroup $K \subset G$. Denote by $\mathrm{Hom}(\Gamma, K)_{\mathbb{1}}$ the connected component of $\mathrm{Hom}(\Gamma, K)$ containing the trivial representations. In [25], Gómez, Pettet and Souto compute that $\pi_1(\mathrm{Hom}(\mathbb{Z}^r, K)_{\mathbb{1}}) \cong \pi_1(K)^r$ and use the retraction constructed in [43] to conclude that $\pi_1(\mathrm{Hom}(\mathbb{Z}^r, G)_{\mathbb{1}}) \cong \pi_1(G)^r$. It is not hard to see in our case that $\mathrm{Hom}(\Gamma, K)_{\mathbb{1}}$ coincides with the representations factoring through the abelianization of Γ lying in the component of the trivial representation. These observations can be combined with Theorem I to compute that $\pi_1(\mathrm{Hom}(\Gamma, G)_{\mathbb{1}}) \cong \pi_1(G)^{\operatorname{rank} H_1(\Gamma;\mathbb{Z})}$. The topology of $\mathrm{Hom}(\Gamma, K)$ will be analyzed in greater detail in the third chapter.

2.1.2 Outline of the chapter

We begin in Section 2.2 by establishing some notation and refreshing the reader's memory with some basic facts about algebraic groups. Then, in Section 2.3, we introduce the main technical tools from Kempf-Ness theory that we shall need. After these preliminaries, we begin working with representation varieties in Section 2.4 where we prove Theorem III. Then, in Section 2.5, we show how eigenvalues can be scaled to complete the proof of Theorem I in the complex case. Finally, in Section 2.6, we show how the proof carries over to the real case and we prove Theorem II.

2.2 Algebraic groups

In this preliminary section, we refresh the reader's memory with some basic facts about linear algebraic groups, referring to Borel's book [12] for the details. In his own words: *According to one's taste about naturality and algebraic geometry, it is possible to give several definitions of linear algebraic groups.* Here, we will favour a concrete and slightly pedestrian approach since it is all that we shall need. There is only one possibly nonstandard result, Lemma 2.2.4.

2.2.1 A crash course

An affine algebraic group is an affine variety endowed with a group law for which the group operations are morphisms of varieties, i.e., polynomial maps. For our purposes, an algebraic group G shall mean the group of complex (or real) points of an affine algebraic group (defined over $\mathbb{R}$ in the latter case) and an affine variety shall mean an affine algebraic set, i.e., the zero locus of a family of complex polynomials. It turns out that affine algebraic groups are *linear* so we will always identify them with a Zariski closed subgroup of $\mathrm{SL_n}\mathbb{C}$.

Remark 4. We shall have the occasion to consider two different topologies on varieties and their subsets, the classical Hausdorff topology and the Zariski topology. Unless we specify otherwise, all references to topological concepts will refer to the Hausdorff topology. For instance, a *closed set* is closed in the Hausdorff topology and a *Zariski closed* set is closed in the Zariski topology. Nevertheless, since a complex algebraic group is connected in the Hausdorff topology if and only if it is connected in the Zariski topology, we shall make no such distinction when referring to their identity component.

Let $G \subset \mathrm{SL_n}\mathbb{C}$ be an algebraic group.

1. We say that a subgroup L is an *algebraic subgroup* of G if it is Zariski closed. Since an arbitrary subgroup H of G isn't necessarily algebraic, we often pass from H to its Zariski closure $\mathscr{A}(H)$. It turns out that $\mathscr{A}(H)$ is also a subgroup of G and, hence, the smallest algebraic subgroup of G containing H.

2. We say that G is *connected* if it is connected in the Zariski topology. Much of the behaviour of G is governed by the connected component of its identity element which we denote by G°. This is always a normal algebraic subgroup of finite index in G. The following Theorem due to Mostow [38] indicates one of the many roles played by G°; it shall be used several times in this thesis.

   Theorem 2.2.1 (Conjugacy Theorem)**.** *An algebraic group G contains a maximal compact subgroup and all such subgroups are conjugate by elements of G°.*

3. Lastly, since G is a smooth variety, it has a well defined tangent space at the identity that we denote by $\mathfrak{g}$. The *Lie algebra* of G is the set $\mathfrak{g}$ endowed with the usual Lie algebra structure given by the bracket operation on derivations.

2.2.2 Jordan decomposition

Let V be a finite dimensional complex vector space and recall the following elementary definitions from linear algebra; an endomorphism σ of V is said to be:

1. *Nilpotent* if $\sigma^n = 0$ for some $n \in \mathbb{N}$.

2. *Unipotent* if $\sigma - \mathrm{Id}$ is nilpotent.

3. *Semisimple* if V is spanned by eigenvectors of σ.

If $G \subset \mathrm{SL_n}\mathbb{C}$ is an algebraic group, we say that $g \in G$ is semisimple (resp. unipotent) if it is semisimple (resp. unipotent) as an endomorphism of $\mathbb{C}^n$. One can show that this definition does not depend on the chosen embedding of G in $\mathrm{SL_n}\mathbb{C}$. In fact, we have the following useful theorem:

Jordan Decomposition Theorem. *Let G be a linear algebraic group.*

1. *If $g \in G$ then there are unique elements g_s and g_u in G so that g_s is semisimple, g_u is unipotent and $g = g_s g_u = g_u g_s$.*

2. *If $\varphi : G \to G'$ is a morphism of algebraic groups, then $\varphi(g)_s = \varphi(g_s)$ and $\varphi(g)_u = \varphi(g_u)$.*

We denote the set of all unipotent (resp. semisimple) elements of G by G_u (resp. G_s). Although the set G_s is seldom Zariski closed, G_u is always a Zariski closed subset of G contained in G°. In fact, we say that an algebraic group G is *unipotent* if $G = G_u$. We can now define the class of algebraic groups that we will be most concerned with in this thesis:

Definition 2.2.2. The *unipotent radical* of an algebraic group G is its maximal connected normal unipotent subgroup. An algebraic group G is said to be *reductive* if its unipotent radical is trivial.

Example 3. The classical groups are all reductive, e.g., $\mathrm{SL_n}\mathbb{C}$, $\mathrm{GL}_n\mathbb{C}$ and $\mathrm{Sp_n}\mathbb{C}$.

2.2.3 Nilpotent algebraic groups

Let A and B be any subgroups of G and denote by $[A, B]$ the commutator subgroup of G abstractly generated by elements of the form $aba^{-1}b^{-1}$, $a \in A$ and $b \in B$. The *lower central series* of G is defined inductively by the rule

$$G =: G_{(0)} \trianglerighteq [G, G_{(0)}] =: G_{(1)} \trianglerighteq [G, G_{(1)}] =: G_{(2)} \trianglerighteq \ldots \trianglerighteq [G, G_{(n)}] =: G_{(n+1)} \trianglerighteq \ldots$$

and one says that G is *nilpotent* if for some $n \geq 0$ we have $G_{(n)} = \{e\}$. This definition makes sense in the context of algebraic groups since all of the groups in the lower central series are algebraic, i.e., Zariski closed. In fact, we have the following classification result which may be found in Borel [12, III.10.6]:

Proposition 2.2.3. *Let G be an algebraic group and recall that we denote its subset of semisimple (resp. unipotent) elements by G_s (resp. G_u). If G is connected and nilpotent, then G_s and G_u are algebraic subgroups of G and $G \cong G_s \times G_u$.*

We end this subsection with a structural result that will be useful later on. Although it may be known to the experts, we provide a proof since we were unable to find it in the literature.

Lemma 2.2.4. *If N is a (possibly disconnected) nilpotent reductive algebraic group, then*

1. *N consists of semisimple elements.*

2. N has a unique maximal compact subgroup.

3. The connected component N° is contained in the centre of N.

The following argument was shown to us by Lior Silberman.

Proof. Recall from Proposition 2.2.3 that the connected component of a nilpotent algebraic group admits a direct product decomposition $N^\circ = N_s^\circ \times N_u^\circ$. Since N° is reductive, N_u° must be trivial and $N^\circ = N_s^\circ$ consists of semisimple elements. The proof of claim (1) is then completed by observing that $N_u \subset N^\circ$. Moreover, assuming for the moment that N° is central in N, claim (2) follows at once from the Conjugacy Theorem 2.2.1.

It remains to prove that N° is central in N. To do so, it suffices to show that the adjoint action [12, Chapter I.3.13] of N on its Lie algebra $\mathfrak{n}$ is trivial. Seeking a contradiction, suppose that $Ad(g)$ acts nontrivially on $\mathfrak{n}$ for some $g \in N$. Since g is semisimple, the Jordan Decomposition Theorem ensures that $Ad(g)$ is semisimple and must therefore have an eigenvalue $\lambda \neq 1$. If $X \in \mathfrak{n}$ is an eigenvector of $Ad(g)$ corresponding to λ, we have that

$$[g, exp(tX)] = g e^{tX} g^{-1} e^{-tX} = e^{Ad(g)(tX)-tX} = e^{t(\lambda-1)X}.$$

As such, the iterated commutators $[g,[g,[g,\ldots,[g,[g,e^X]]\ldots]]]$ are all non-trivial and the lower central series of N does not terminate in finitely many steps. This is a contradiction. □

2.2.4 Complex reductive algebraic groups

One of the advantages of working over $\mathbb{C}$ is the strong relation between reductive groups and their maximal compact subgroups. We illustrate this by following the treatment in Schwarz [48].

Given a compact Lie group K, the Peter-Weyl Theorem provides a faithful embedding $K \hookrightarrow \mathrm{GL}_n\mathbb{R}$ for some n. Identifying K with its image realizes it as a real algebraic subgroup of $\mathrm{GL}_n\mathbb{R}$. We then define the *complexification* $G := K_{\mathbb{C}}$ to be the vanishing locus in $\mathrm{GL}_n\mathbb{C}$ of the ideal defining K. The group G is a complex algebraic group which is independent up to isomorphism of the embedding provided by the Peter-Weyl Theorem. The following properties of complexifications of compact Lie groups are well known (see, for instance, Onishchik and Vinberg [42, Chapter 5]):

1. The Lie algebra $\mathfrak{g}$ of G has a natural splitting in terms of the Lie algebra $\mathfrak{k}$ of K:

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}.$$

2. The group K is a maximal compact subgroup of G. Moreover, it is Zariski dense in G, i.e., $\mathscr{A}(K) = G$.

3. $G = K \cdot exp(i\mathfrak{k}) \cong K \times exp(i\mathfrak{k})$ where $\cong$ is a diffeomorphism. (Contracting $exp(i\mathfrak{k})$ is one way to see that $K \hookrightarrow G$ is a homotopy equivalence.)

Example 4. Viewing SU_n as a compact Lie group, we can realize it as a real algebraic subgroup of $\mathrm{GL}_{2n}\mathbb{R}$. In this case, the complexification of SU_n is isomorphic to $\mathrm{SL}_n\mathbb{C} \subset \mathrm{GL}_{2n}\mathbb{C}$.

Remark 5. The decomposition $G = K \cdot exp(i\mathfrak{k})$ is often called the *polar decomposition*. Indeed, for any linear realization $G \subset \mathrm{SL_n}\mathbb{C}$, it is a Theorem of Mostow [38] that for some $m \in \mathrm{SL_n}\mathbb{C}$ the corresponding polar decomposition of $mGm^{-1} \subset \mathrm{SL_n}\mathbb{C}$ coincides with the usual polar decomposition in $\mathrm{SL_n}\mathbb{C}$. We shall use this decomposition several times in the proof of our main theorem.

As one might suspect, the following theorem holds [42, Chapter 5, Section 2]:

Theorem 2.2.5. *A complex linear algebraic group is reductive if and only if it is the complexification of a compact Lie group.*

2.3 Algebraic actions

Let V be a complex vector space and suppose that the complex reductive algebraic group G is contained in $\mathrm{GL}(V)$. The goal of this preliminary section is to study the algebraic and topological structures of a variety $X \subset V$ that is stable under the action of $G \subset \mathrm{GL}(V)$. This will lead us to the main known result used in our proof: the Neeman–Schwarz Theorem.

In keeping with the general spirit of this thesis, recall that we favour a concrete approach to algebraic groups. As such, we also adopt the following concrete point of view for algebraic actions. A *G–variety* is a variety $X \subset V$ acted upon algebraically by $G \subset \mathrm{GL}(V)$. Recall that the *orbit* of $x \in X$, denoted $G \cdot x$, is the set of all $g \cdot x$ with $g \in G$. A *G–subvariety* of X is then a G–variety $Y \subset X \subset V$ such that $G \cdot y \subset Y$ for every $y \in Y$. If X and Y are G–varieties, a morphism $\alpha : X \to Y$ is *G–equivariant* if $\alpha(g \cdot x) = g \cdot \alpha(x)$.

Remark 6. There is no real loss of generality in our concrete approach to G–varieties. Indeed, any affine variety endowed with a reductive algebraic group action is equivariantly isomorphic to a closed G–subvariety of a finite dimensional complex vector space (see, for instance, Brion [15, Proposition 1.9]).

2.3.1 Geometric invariant theory

Once a variety has been equipped with an algebraic group action, one might be tempted to take quotients. Unfortunately, a naive construction of the quotient fails to produce a variety because the orbits of points are not always closed. Informally, geometric invariant theory remedies this situation by altering the quotient and gluing "bad orbits" together. What follows is intended to be a brief reminder of this construction; the reader unfamiliar with these concepts is invited to consult Brion [15] for the details.

Let X be a G–variety and recall that this endows the ring of regular functions $\mathbb{C}[X]$ with a natural action of G. One would expect that the ring of regular functions of the quotient of X by G should correspond to those functions in $\mathbb{C}[X]$ which are invariant under G. Indeed, we define the affine Geometric Invariant Theory (GIT) quotient as the corresponding affine variety which we denote by

$$X/\!\!/G := \mathrm{Specm}(\mathbb{C}[X]^G).$$

The topological space underlying $X/\!\!/G$ has a nice description of independent interest. The key observation here is that the closure of every G–orbit contains a unique closed orbit. Consequently, points of $X/\!\!/G$ correspond bijectively with the closed G–orbits. In fact, if we denote the set of closed G–orbits by $X/\!\!/_{top}\,G$ and consider the quotient map $\pi : X \to X/\!\!/_{top}\,G$ sending $x \in X$ to the unique closed orbit in $\overline{G \cdot x}$, then $X/\!\!/_{top}\,G$ (equipped with the quotient topology) is homeomorphic to $X/\!\!/G$.

Remark 7. This so-called topological Hilbert quotient $X/\!\!/_{top}\,G$ was investigated by Luna in [36] and [37] to understand the orbit space of real and complex algebraic group actions. In particular, he showed that $X/\!\!/_{top}\,G$ is the universal quotient in the category of Hausdorff spaces and in the category of complex analytic varieties.

Definition 2.3.1. Let X be a G–variety. A point $x \in X$ is said to be *polystable* if its orbit $G \cdot x \subset X$ is Zariski closed. Since $G \cdot x$ is Zariski open in its Zariski closure, this is equivalent to requiring that $G \cdot x$ be closed in the Hausdorff topology. We denote the set of polystable points of X by X^{ps}.

Example 5. Let us consider the simple case where G acts on itself by inner automorphisms. Since our definition of a G–variety requires an ambient vector space, we think of G as a G–subvariety of the complex vector space of $n \times n$ matrices $G \subset M_n\mathbb{C}$ where G acts on $M_n\mathbb{C}$ by conjugation. In this setting, it is well known that $x \in G$ is polystable if and only if x is semisimple.

Similarly, we can consider the diagonal action of G on G^r given by $g \cdot (x_1, \ldots, x_r) := (gx_1g^{-1}, \ldots, gx_rg^{-1})$. Here, Richardson [46, Theorem 3.6] generalized the previous characterization of polystability. The following theorem will play an important part in the proof of Theorem III in Section 2.4.2.

Richardson's Theorem. *Let G be a complex reductive linear algebraic group. For every tuple $(x_1, \ldots, x_n) \in G^n$, denote by $\mathscr{A}(x_1, \ldots, x_n)$ the algebraic subgroup of G generated by the elements $\{x_1, \ldots, x_n\}$. If we consider the diagonal action of G on G^n by inner automorphisms, then $(x_1, \ldots, x_n) \in G^n$ is polystable if and only if $\mathscr{A}(x_1, \ldots, x_n)$ is reductive.*

2.3.2 The Kempf–Ness Theorem

Let us now consider the entire complex vector space V as a G–variety via the algebraic inclusion $G \subset \mathrm{GL}(V)$. If $K \subset G$ is a maximal compact subgroup, there is no loss of generality in assuming that V is equipped with a K–invariant Hermitian inner product. It turns out that the associated norm $\| \cdot \|$ on V sheds a lot of light on its polystable points. Indeed, one can determine if a vector $v \in V$ is polystable by understanding how its length changes as we move along its orbit $G \cdot v$. This variation in norm is captured by the *Kempf–Ness function* which is defined for every $v \in V$ by the rule

$$\Psi_v : G \to \mathbb{R} \ , \ g \mapsto \|g \cdot v\|^2. \tag{2.1}$$

Kempf and Ness [32, Theorem 0.2-0.3] used the convexity of these functions to formulate a simple characterization of polystability:

Kempf–Ness Theorem. *Let $G \subset \mathrm{GL}(V)$ and suppose that V is endowed with a K-invariant norm as above. If $X \subset V$ is a G–subvariety, then:*

1. *The point $x \in X$ is polystable if and only if Ψ_x has a critical point.*

2. *All critical points of Ψ_x are minima.*

Moreover, if $\Psi_x(e)$ is a minimum then $K \cdot x = \{y \in G \cdot x : \|y\| = \|x\|\}$.

With this theorem in mind, we can state the most important definition of this thesis. As above, let $G \subset \mathrm{GL}(V)$ where V is endowed with a K-invariant norm. Then, the *Kempf–Ness set* of V is defined as the $\mathcal{M}$inimal vectors

$$\mathcal{M}_V := \{v \in V : \|g \cdot v\| \geq \|v\| \text{ for all } g \in G\}. \tag{2.2}$$

Note that the Kempf–Ness Theorem ensures that $\mathcal{M}_V \subset V^{ps}$ and, although V^{ps} is an intrinsically defined subset of V, $\mathcal{M}_V$ depends on the choice of the invariant norm on V. If $X \subset V$ is a G–subvariety, we define the Kempf–Ness set of X as $\mathcal{M}_X := \mathcal{M}_V \cap X$.

The primary reason for calling the collection of minimal vectors the "Kempf–Ness set" is the following corollary [48, Corollary 4.7] of the Kempf–Ness Theorem. It allows GIT quotients to be topologically represented via ordinary quotients and will play a rôle in the proof of Theorem II.

Corollary 2.3.2. *The geometric invariant theory quotient $X/\!\!/G$ is homeomorphic (in the Hausdorff topology) to the ordinary quotient $\mathcal{M}_X/K$.*

Remark 8. Whenever the Kempf–Ness set under consideration is clear from the context we shall omit the subscript and refer to it simply as $\mathcal{M}$.

2.3.3 The Neeman–Schwarz Theorem

As above, let $G \subset \mathrm{GL}(V)$ act on the complex vector space V and suppose the latter is endowed with a $K-$invariant norm $\|\cdot\|$. The following theorem [48, Theorem 5.1] is the key known result used in the proof of our main theorems.

Neeman–Schwarz Theorem. *Let the complex vector space V be a G–variety endowed with a $K-$invariant norm. If $\mathcal{M}$ denotes the Kempf–Ness set of V, then there is a $K-$equivariant strong deformation retraction of V onto $\mathcal{M}$ which preserves G–orbits.*

More precisely, there is a map $\varphi : V \times [0,1] \to V$ with the following properties:

1. $\varphi_0 = \mathrm{Id}_V$, $\varphi_t|_{\mathcal{M}} = \mathrm{Id}_{\mathcal{M}}$, $\varphi_1(V) = \mathcal{M}$

2. $\varphi_t(v) \in G \cdot v$ *for* $0 \leq t < 1$

3. $\varphi_1(v) \in \overline{G \cdot v}$.

For our purposes, the following corollary shall be most important:

Corollary 2.3.3. *Let X be a G–subvariety of V and recall that $\mathcal{M}_X := \mathcal{M} \cap X$. Restricting φ, we obtain a K–equivariant strong deformation retraction of X onto $\mathcal{M}_X$.*

Remark 9. Initially, Neeman [41, Theorem 2.1] had proved that there was a strong deformation retraction of V onto a neighbourhood of $\mathcal{M}$. To do so, following a suggestion of Mumford, he considered the negative gradient flow of a function assigning to each $v \in V$ the "size" of the differential $(d\Psi_v)_e$. However, he believed that the local flow could be extended to a retraction of V onto $\mathcal{M}$ and conjectured a functional inequality that would allow him to do so. Later, he realized that this was in fact a special case of an inequality due to Łojasiewicz [34]. Finally a complete account of the proof of the Neeman–Schwarz Theorem was given by Schwarz in [48, Theorem 5.1].

Remark 10. Although we have been working over $\mathbb{C}$, Kempf–Ness theory was extended to the real setting by Richardson and Slodowy. For example, in the case of Richardson's Theorem it is shown in Richardson [46, Theorem 11.4] that $(x_1, \ldots, x_n) \in G(\mathbb{R})^n$ is polystable for the action of $G(\mathbb{R})$ if and only if $(x_1, \ldots, x_n) \in G^n$ is polystable for the action of G. For the Kempf–Ness Theorem, its Corollary and the Neeman–Schwarz Theorem, it is shown in Richardson–Slodowy [47] that the same results hold when one replaces G by its group of real points $G(\mathbb{R})$. This will be elaborated upon in Section 2.6 when we prove the real case of Theorem I.

2.4 The representation variety

Let Γ be a finitely generated group and let G be a complex reductive linear algebraic group. In a coarse sense, this thesis is concerned with the finite dimensional representation theory of Γ in G. When Γ is finite, this essentially reduces to the correspondence between linear representations and characters due to Frobenius. When Γ is infinite, the analogue of character theory is the parametrization of representations by geometric varieties. In this section, we study the topology of such varieties.

In order to endow the space $\mathrm{Hom}(\Gamma, G)$ of homomorphisms $\rho\colon \Gamma \to G$ with a geometric structure, it is convenient to work with a fixed generating set $\gamma_1, \ldots, \gamma_r$ of Γ. Since any representation $\rho\colon \Gamma \to G$ is uniquely determined by the image $\rho(\gamma_i)$ of its generators, the evaluation map

$$\mathrm{Hom}(\Gamma, G) \to G \times \ldots \times G = G^r,\ \rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_r)) \tag{2.3}$$

is a bijection between $\mathrm{Hom}(\Gamma, G)$ and a Zariski closed subset of G^r carved out by the relations of Γ. It is well known (see, for instance, Lubotzky and Magid [35]) and otherwise easy to see that the geometric structure induced on $\mathrm{Hom}(\Gamma, G)$ by this bijection doesn't depend on the chosen presentation of Γ.

Remark 11. With the hope that no confusion shall arise, we denote elements of $\mathrm{Hom}(\Gamma, G)$ by ρ when we consider them as homomorphisms and by tuples $(m_1, \ldots, m_r)$ when we identify them with points in G^r.

The action of G on itself by conjugation induces a diagonal action of G on the variety $\mathrm{Hom}(\Gamma, G) \subset G^r$. Explicitly, this action is given by the rule

$$G \times \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G),\ g \cdot (m_1, \ldots, m_r) := (gm_1g^{-1}, \ldots, gm_rg^{-1}).$$

Since two representations are in the same G–orbit when their image differs by a change of basis, one is often interested in classifying them modulo the action of G. This can be done by considering the geometric invariant theory quotient $\mathrm{Hom}(\Gamma, G)/\!\!/G$ which happens to parametrize isomorphism classes of completely reducible representations. Describing these varieties geometrically might be viewed as analogous to determining the characters of a finite group.

As an affine variety, $\mathrm{Hom}(\Gamma, G)$ also inherits a natural Hausdorff topology given from an embedding into affine space. In fact, it coincides with the subspace topology of $\mathrm{Hom}(\Gamma, G) \subset G^r$ when G is viewed as a Lie group. If $K \subset G$ is any subgroup, there is a natural inclusion $\mathrm{Hom}(\Gamma, K) \hookrightarrow \mathrm{Hom}(\Gamma, G)$ so we endow $\mathrm{Hom}(\Gamma, K)$ with the subspace topology. Theorem I states that under certain assumptions there is a deformation retraction of $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$. The goal of this section is to show how the algebro-geometric tools developed in the previous section can be used in this setting.

2.4.1 The Frobenius norm

Let G be a (possibly disconnected) complex reductive linear algebraic group and let $K \subset G$ be a maximal compact subgroup. In this subsection, we complete an intermediate step in the proof of Theorem I by showing how the Neeman–Schwarz Theorem can be applied to $\mathrm{Hom}(\Gamma, G)$. Keeping with the general spirit of this thesis, instead of working abstractly with our algebraic group G, we choose a matrix representation $G \hookrightarrow \mathrm{SL_n}\mathbb{C}$ for which $K = G \cap \mathrm{SU}_n$.

In order to apply the algebro-geometric results from Section 2.3, we need to endow $\mathrm{Hom}(\Gamma, G)$ with the structure of an affine G–variety. To do this, consider $G \subset \mathrm{SL_n}\mathbb{C}$ as a variety in the vector space $M_n\mathbb{C}$ of $n \times n$ complex matrices. Using the natural identifications arising from the evaluation map (2.3), we can realize $\mathrm{Hom}(\Gamma, G)$ as an embedded variety

$$\mathrm{Hom}(\Gamma, G) \subset G^r \subset \oplus_{j=1}^r M_n\mathbb{C}.$$

Once this has been done, the diagonal action of $G \subset \mathrm{SL_n}\mathbb{C}$ on $\oplus_{j=1}^r M_n\mathbb{C}$ by simultaneous conjugation endows $\mathrm{Hom}(\Gamma, G)$ with the structure of a G–subvariety of the vector space $\oplus_{j=1}^r M_n\mathbb{C} \cong \mathbb{C}^{rn^2}$.

In this setting, there is a prototypical $K-$invariant norm on $\oplus_{j=1}^r M_n\mathbb{C}$. Recall that the vector space $M_n\mathbb{C}$ admits a natural Hermitian inner product called the *Frobenius inner product* (see, for instance, Horn and Johnson [29, Chapter 5]). If we denote the adjoint of a matrix a by $a^* := \overline{a}^t$, then this inner product is defined by $\langle a, b\rangle := \mathrm{trace}(a^*b)$ for $a, b \in M_n\mathbb{C}$. It is easy to check that for any unitary matrix $u \in \mathrm{U}_n$ we have $\langle uau^{-1}, ubu^{-1}\rangle = \langle a, b\rangle$. The Frobenius inner product can then be extended "coordinate wise" to $\oplus_{j=1}^r M_n\mathbb{C}$ by the rule

$$\langle (a_1, \ldots, a_r), (b_1, \ldots, b_r)\rangle := \sum_{j=1}^r \langle a_j, b_j\rangle$$

and the associated norm on $\oplus_{j=1}^r M_n\mathbb{C}$ is unitary invariant (and therefore $G \cap \mathrm{SU}_n = K-$invariant).

Remark 12. We henceforth refer to the $K-$invariant norm described above as the Frobenius norm of the G–variety $\mathrm{Hom}(\Gamma, G) \subset \oplus_{j=1}^r M_n\mathbb{C}$.

Once it has been equipped with this additional structure, we can define the Kempf–Ness set $\mathcal{M}$ of $\mathrm{Hom}(\Gamma, G)$ as in Section 2.3.3 to be those representations that are of minimal norm within their $G-$orbit. The Neeman–Schwarz Theorem now provides us with a $K-$equivariant deformation retraction of $\mathrm{Hom}(\Gamma, G)$ onto $\mathcal{M}$.

2.4.2 The Kempf–Ness set

In order to fruitfully apply the Neeman–Schwarz Theorem to $\mathrm{Hom}(\Gamma, G)$, we need to understand the target of the retraction it provides. For a general group Γ, e.g., if Γ is a free group, explicitly determining the Kempf–Ness set of $\mathrm{Hom}(\Gamma, G)$ appears to be a hopeless task. This is why we henceforth let Γ be a finitely generated *nilpotent* group. In this case, the Kempf–Ness set admits a very nice description:

Theorem III. *Let $G \subset \mathrm{SL_n}\mathbb{C}$ be a complex reductive algebraic group and suppose that $K = G \cap \mathrm{SU}_n \subset G$ is a maximal compact subgroup. If Γ is a finitely generated nilpotent group and $\mathcal{M}$ denotes the Kempf–Ness set of the G–subvariety $\mathrm{Hom}(\Gamma, G) \subset \oplus_{j=1}^{r} M_n\mathbb{C}$ equipped with the Frobenius norm, then*

$$\mathcal{M} = \{\rho \in \mathrm{Hom}(\Gamma, G) : \rho(\Gamma) \text{ consists of normal matrices}\}.$$

The proof of Theorem III relies on two lemmas. Before diving into the heart of the matter, we refresh the reader's memory with some elementary linear algebra. Recall that a matrix $m \in M_n\mathbb{C}$ is *normal* if $mm^* = m^*m$ or, equivalently, V has an orthogonal basis of eigenvectors of m. The following well known[1] characterization of normal matrices follows from the unitary invariance of the Frobenius norm $\|\cdot\|$ of $M_n\mathbb{C}$ and the Schur Triangularization Theorem.

Lemma 2.4.1. *If $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of the matrix $m \in M_n\mathbb{C}$, then*

$$\|m\|^2 \geq \sum_{j=1}^{n} |\lambda_j|^2.$$

Moreover, equality holds if and only if m is a normal matrix. □

Our second lemma concerns the images of certain representations of Γ in G:

Lemma 2.4.2. *Let $G \subset \mathrm{SL_n}\mathbb{C}$ be a reductive algebraic group and suppose that $K = G \cap \mathrm{SU}_n$ is a maximal compact subgroup of G. If $N \subset G$ is a nilpotent reductive algebraic subgroup, then there is some $g \in G$ for which gNg^{-1} consists of normal matrices.*

Proof. Recall from Lemma 2.2.4 that a reductive nilpotent algebraic group N has a unique maximal compact subgroup C. Since C is contained in a maximal compact subgroup of G

[1] We invite the intrigued reader to consult Grone–Johnson–Sa–Wolkowicz [26] where this is the 53rd (out of 70!) characterization of normality.

and all such groups are conjugate in G, there is some $g \in G$ for which $gCg^{-1} \subset K$. We can therefore assume (after conjugating by g) that $N \subset G \subset \mathrm{SL_n}\mathbb{C}$ and $C \subset K \subset \mathrm{SU}_n$.

Being a reductive algebraic group, N is the complexification of C. Thus, it admits a polar decomposition (c.f. Section 2.2.4):

$$N = C \cdot exp(i\mathfrak{c}) \simeq C \times exp(i\mathfrak{c})$$

where $exp(i\mathfrak{c}) \subset N^\circ$ is central in N by Lemma 2.2.4. This shows that any matrix $m \in N$ can be written uniquely as a product $m = k \cdot h$ for some unitary $k \in C$ and Hermitian $h \in exp(i\mathfrak{c})$. Since unitary and Hermitian matrices are normal and since the product of two commuting normal matrices is normal, N consists of normal matrices. □

The first lemma is used to show that normal representations are critical points of the Kempf–Ness functions so they belong to the Kempf–Ness set. Then, the second lemma shows that polystable representations become normal after conjugation so the Kempf–Ness set consists exclusively of normal representations. More precisely:

Proof of Theorem III. Let $\rho \in \mathrm{Hom}(\Gamma, G)$ and consider the corresponding tuple

$$(m_1, \ldots, m_r) := (\rho(\gamma_1), \ldots, \rho(\gamma_r)) \subset G^r.$$

If m_j is a normal matrix for each $1 \leq j \leq r$, then Lemma 2.4.1 ensures that

$$\|(m_1, \ldots, m_r)\| \leq \|(gm_1g^{-1}, \ldots, gm_rg^{-1})\|$$

for any $g \in G$. Consequently, the representation ρ is in $\mathcal{M}$; this shows that

$$\mathcal{M} \supset \{\rho \colon \rho(\gamma_j) \text{ is normal for } 1 \leq j \leq r\} \supset \{\rho \colon \rho(\Gamma) \text{ consists of normal matrices}\}.$$

We now show that the three sets coincide. Seeking a contradiction, suppose that $\rho \in \mathcal{M} \subset \mathrm{Hom}(\Gamma, G)^{ps}$ is a representation whose image doesn't consist of normal matrices. Recall that Richardson's Theorem characterizes polystable points in G^r for the conjugation action of G as those tuples $(m_1, \ldots, m_r)$ for which $\mathscr{A}(m_1, \ldots, m_r) \subset G$ is reductive. Therefore, the Zariski closure of the image of our representation

$$N := \mathscr{A}(\rho(\Gamma)) = \mathscr{A}(\rho(\gamma_1), \ldots, \rho(\gamma_r))$$

is a reductive nilpotent algebraic subgroup of G. But then, Lemma 2.4.2 produces an element $g \in G$ for which gNg^{-1} consists of normal matrices and, by Lemma 2.4.1, this contradicts the assumption that ρ was of minimal norm within its G–orbit. □

Example 6. While it is more or less clear that if we replace Γ by a free group then $\mathcal{M}$ does not consist of normal representations (see, for instance, Florentino–Lawton [24, Remark 3.2]), it may be somewhat surprising that this already fails to be the case when Γ is solvable. To see this, consider the solvable group $\Gamma := \mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ generated by the elements $(1,0)$ and $(0,1)$. For this generating set, an obvious representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_2\mathbb{C})$ is given by mapping

$$(1,0) \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } (0,1) \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $|\lambda| \neq 1, 0$. Since both of these matrices are normal, it follows that ρ lies in the Kempf-Ness set. Nevertheless, their product is not a normal matrix! This example was shown to us by Lior Silberman.

2.5 The scaling operation

In the previous section, we determined the Kempf–Ness set of the representation variety $\mathrm{Hom}(\Gamma, G)$. To complete the proof of Theorem I in the complex case, we seek a deformation retraction of this set onto $\mathrm{Hom}(\Gamma, K)$. The goal of this section is to show that scaling eigenvalues does the trick.

2.5.1 Scaling eigenvalues

Following Pettet and Souto [43], we define an eigenvalue scaling operator on the semisimple elements of an algebraic group. Let $\mathbb{D}_n$ denote the diagonal subgroup of $\mathrm{SL_n}\mathbb{C}$, identified in the usual way with a subgroup of $(\mathbb{C}^\times)^n$. Consider the map

$$\tilde{\sigma} : [0,1] \times \mathbb{D}_n \to \mathbb{D}_n,\ (t, (\lambda_i)) \mapsto \tilde{\sigma}_t((\lambda_i)) := (e^{-t \log |\lambda_i|} \lambda_i)$$

where $\log(\cdot)$ denotes the standard real logarithm. Notice that $\tilde{\sigma}$ is continuous, equivariant under the action of the Weyl group, and that $\tilde{\sigma}_0(g) = g$ while $\tilde{\sigma}_1(g) \in \mathbb{D}_n \cap \mathrm{SU}_n$ for all $g \in \mathbb{D}_n$.

Given now a semisimple element $g \in \mathrm{SL_n}\mathbb{C}$, there is some $h \in \mathrm{SL_n}\mathbb{C}$ for which $hgh^{-1} \in$

$\mathbb{D}_n$. Since $\tilde{\sigma}$ is invariant under the Weyl group, we see that for all $t \in [0,1]$ the element

$$\sigma_t(g) := h^{-1}\tilde{\sigma}_t(hgh^{-1})h$$

is independent of the choice of h. In particular, we obtain a well defined *scaling map*

$$\sigma : [0,1] \times (\mathrm{SL_n}\mathbb{C})_s \to (\mathrm{SL_n}\mathbb{C})_s,\ (t,g) \mapsto \sigma_t(g) \tag{2.4}$$

where $(\mathrm{SL_n}\mathbb{C})_s$ denotes the set of semisimple elements in $\mathrm{SL_n}\mathbb{C}$. This map preserves commutativity and scales the eigenvalues of the semisimple elements of $\mathrm{SL_n}\mathbb{C}$ until they land in a subgroup conjugate to SU_n. Using this map, the following lemma is proved in [43, Section 8.3]:

Lemma 2.5.1 (Pettet–Souto)**.** *If G is any linear algebraic group and $K \subset G$ is a maximal compact subgroup, then there is a G–equivariant continuous map*

$$\sigma : [0,1] \times G_s \to G_s,\ (t,g) \mapsto \sigma_t(g)$$

which satisfies the following properties for all $g, g_1, g_2 \in G_s$:

1. *$\sigma_0(g) = g$ and $\sigma_1(g)$ is conjugate to an element of K.*
2. *$\forall\ 0 \leq t \leq 1$, σ_t fixes pointwise elements of G which are conjugate to elements of K.*
3. *$\forall\ 0 \leq t \leq 1$, if $g_1g_2 = g_2g_1$ then $\sigma_t(g_1g_2) = \sigma_t(g_1)\sigma_t(g_2) = \sigma_t(g_2)\sigma_t(g_1)$.*
4. *If G is defined over $\mathbb{R}$ then σ is equivariant under complex conjugation.*

2.5.2 Scaling the Kempf–Ness set

Let us now return to the setting of Theorem I where Γ is a finitely generated nilpotent group and G is a complex reductive algebraic group. In Section 2.4.1, given a maximal compact subgroup $K \subset G$, we chose a faithful representation $G \hookrightarrow \mathrm{SL}_n\mathbb{C}$ for which $K = G \cap \mathrm{SU}_n$ to endow $\mathrm{Hom}(\Gamma, G) \subset \oplus_{j=1}^r M_n\mathbb{C}$ with a $K-$invariant Frobenius norm. This allowed us to define the Kempf–Ness set $\mathcal{M}$ of $\mathrm{Hom}(\Gamma, G)$ as the set of representations of minimal norm within their $G-$orbit. In fact, Theorem III showed that

$$\mathcal{M} = \{\rho \in \mathrm{Hom}(\Gamma, G) : \rho(\Gamma) \text{ consists of normal matrices}\}.$$

The goal of this subsection is to use the scaling map σ from Lemma 2.5.1 to prove the following proposition:

Proposition 2.5.2. *Keeping the notation as above, there is a $K-$equivariant strong deformation retraction of the Kempf–Ness set $\mathcal{M}$ onto* $\mathrm{Hom}(\Gamma, K)$.

Before proving the proposition, we need to understand how the image of representations in the Kempf–Ness set behave under scaling; this is the content of the following lemma.

Lemma 2.5.3. *Let $N \subset \mathrm{SL_n}\mathbb{C}$ be a nilpotent reductive algebraic group. If σ_t denotes the scaling map from* Lemma 2.5.1*, then the restriction of σ_t to N is a group homomorphism.*

Proof. Recall from Lemma 2.2.4 that a nilpotent reductive algebraic group N consists of semisimple elements and has a unique maximal compact subgroup C. Moreover, it admits a polar decomposition

$$N = C \cdot exp(i\mathfrak{c}) \simeq C \times exp(i\mathfrak{c})$$

where $\mathfrak{c}$ denotes the Lie algebra of C and $exp(i\mathfrak{c}) \subset N^\circ$ is central in N by Lemma 2.2.4 again.

Let $m_1 = k_1 \cdot h_1$ and $m_2 = k_2 \cdot h_2$ denote the unique decomposition of elements $m_1, m_2 \in N$ into products of elements $k_1, k_2 \in C$ and $h_1, h_2 \in exp(i\mathfrak{c})$. Notice that since C is maximal compact it is fixed pointwise by σ_t. Then, since h_1 and h_2 are central in N, we have:

$$\sigma_t(m_1m_2) = \sigma_t(k_1h_1k_2h_2) = \sigma_t(k_1k_2h_1h_2) = \sigma_t(k_1k_2)\sigma_t(h_1)\sigma_t(h_2) = k_1k_2\sigma_t(h_1)\sigma_t(h_2)$$

$$= k_1\sigma_t(h_1)k_2\sigma_t(h_2) = \sigma_t(k_1)\sigma_t(h_1)\sigma_t(k_2)\sigma_t(h_2) = \sigma_t(k_1h_1)\sigma(k_2h_2) = \sigma_t(m_1)\sigma_t(m_2).$$

Here, we repeatedly use parts (2) and (3) of Lemma 2.5.1. □

Proof of Proposition 2.5.2. Let us denote by $\mathcal{N}$ the set of normal matrices in $\mathrm{SL_n}\mathbb{C}$ and consider the restriction of σ_t to $\mathcal{N}$. Observe that $\sigma_0(\mathcal{N}) = \mathcal{N}$ while $\sigma_1(\mathcal{N}) = \mathrm{SU}_n$ because a matrix is unitary if and only if it is normal and its eigenvalues all have unit norm. Since the representations ρ in the Kempf–Ness set $\mathcal{M}$ of $\mathrm{Hom}(\Gamma, G)$ are precisely those for which $N := \mathscr{A}(\rho(\Gamma)) \subset \mathcal{N}$ is a reductive nilpotent group, it follows from Lemma 2.5.3 that $\sigma_t \circ \rho$ is also in $\mathcal{M}$. Consequently, the map $\psi : \mathcal{M} \times [0,1] \to \mathcal{M}$ where $\psi(\rho, t) := \sigma_t \circ \rho$ is a strong deformation retraction with $\psi(\rho, 1)(\Gamma) \subset K = G \cap \mathrm{SU}_n$. □

2.6 The real and complex cases

The previous two sections contain a complete proof of Theorem I when G is a complex reductive linear algebraic group. To recapitulate, once $\mathrm{Hom}(\Gamma, G)$ has been endowed with the appropriate structure, the retraction proceeds essentially in two steps:

$$\mathrm{Hom}(\Gamma, G) \xrightarrow[Theorem]{Neeman-Schwarz} \text{Kempf–Ness Set} \xrightarrow[Eigenvalues]{Scaling} \mathrm{Hom}(\Gamma, K).$$

In this section, we use real Kempf–Ness theory to adapt these steps when G is replaced by its group of real points $G(\mathbb{R})$.

2.6.1 Real Kempf–Ness theory

To begin, we discuss the work of Richardson and Slodowy found in [47] where they develop real analogues of the main results in Kempf–Ness theory.

Let V be a finite dimensional complex vector space with real structure $V(\mathbb{R})$ and let H be a positive-definite Hermitian form on V. In this case, we always have $H(x,y) = S(x,y) + iA(x,y)$ where S (respectively A) is a symmetric (respectively alternating) real-valued $\mathbb{R}$-bilinear form on V. We say that the Hermitian form H is *compatible* with the $\mathbb{R}-$structure of V if A vanishes on $V(\mathbb{R}) \times V(\mathbb{R})$.

Example 7. The Frobenius inner product on $M_n\mathbb{C}$ is *compatible* with the $\mathbb{R}-$structure $M_n\mathbb{R}$.

Let us suppose now that the complex reductive algebraic group $G \subset \mathrm{GL}(V)$ is defined over $\mathbb{R}$ and denote its group of real points by $G(\mathbb{R})$. Since G is reductive, it follows from the work of Mostow [38] that, upon identifying $V \cong \mathbb{C}^m$ and $\mathrm{GL}(V) \cong \mathrm{GL}_m\mathbb{C}$, we can assume G to be stable under the involution $\theta(g) := (g^*)^{-1}$. In this case, the fixed points

$$K := G^\theta = \{g \in G : \theta(g) = g\}$$

form a maximal compact subgroup of G and the real fixed points $K(\mathbb{R}) = G(\mathbb{R})^\theta$ form a maximal compact subgroup of $G(\mathbb{R})$. One can show [47, Appendix 2] that the vector space V may always be equipped with a $K-$invariant Hermitian inner product which is compatible with the $\mathbb{R}-$structure $V(\mathbb{R})$. We denote the associated $K-$invariant norm on V by $\|\cdot\|$.

The (complex) Kempf–Ness set $\mathcal{M}$ of V for the action of G can now be defined as in Section 2.3.2 and the (real) Kempf–Ness set $\mathcal{M}(\mathbb{R})$ of $V(\mathbb{R})$ for the action of $G(\mathbb{R})$ can be defined by the analogous rule

$$\mathcal{M}(\mathbb{R}) := \{v \in V(\mathbb{R}) : \|g \cdot v\| \geq \|v\| \text{ for all } g \in G(\mathbb{R})\}. \tag{2.5}$$

The following lemma is an immediate consequence of [47, Lemma 8.1]:

Lemma 2.6.1. $\mathcal{M}(\mathbb{R}) = \mathcal{M} \cap V(\mathbb{R})$.

At this point we can state the two only results of Richardson and Slodowy that we shall need. The first is an analogue of Corollary 2.3.2 [47, Theorem 7.7]:

Theorem 2.6.2. *Keeping the notation as above, let X be a closed $G(\mathbb{R})-$stable subset of $V(\mathbb{R})$. If $\mathcal{M}_X(\mathbb{R}) := \mathcal{M}(\mathbb{R}) \cap X$, then there is a homeomorphism $\mathcal{M}_X(\mathbb{R})/K(\mathbb{R}) \cong X/\!\!/_{top}\, G(\mathbb{R})$.*

Remark 13. Here, $X/\!\!/_{top}\, G(\mathbb{R})$ denotes the *topological Hilbert quotient* (c.f. Section 2.3.1 and [47, Section 7.2]).

The second is an analogue of the Neeman–Schwarz Theorem [47, Theorem 9.1]:

Theorem 2.6.3. *Keeping the notation as above, there is a continuous $K(\mathbb{R})-$equivariant strong deformation retraction $\varphi : V(\mathbb{R}) \times [0,1] \to V(\mathbb{R})$ of $V(\mathbb{R})$ onto $\mathcal{M}(\mathbb{R})$. Moreover, the deformation is along orbits of $G(\mathbb{R})$, that is, $\varphi_t(v) \subset G(\mathbb{R}) \cdot v$ for $0 \leq t < 1$ and $\varphi_1(v) \in \overline{G(\mathbb{R}) \cdot v}$.*

Remark 14. As in the complex case, if X is a closed $G(\mathbb{R})$-stable subset of $V(\mathbb{R})$, then the restriction of φ to $X \times [0,1]$ is a deformation retraction of X onto $\mathcal{M}_X(\mathbb{R}) := \mathcal{M}(\mathbb{R}) \cap X$.

2.6.2 Proofs of Theorem I and Theorem II

We are now ready to complete the proofs of our main theorems. We begin with the proof of the real case of Theorem I:

Theorem I. *Let Γ be a finitely generated nilpotent group and let G be the group of complex or real points of a reductive linear algebraic group defined over $\mathbb{R}$. If K is a maximal compact subgroup of G, then there is a $K-$equivariant strong deformation retraction of $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$.*

Proof of the real case. Let $G \subset \mathrm{SL_n}\mathbb{C}$ be a complex reductive algebraic group defined over $\mathbb{R}$ and denote its group of $\mathbb{R}-$points by $G(\mathbb{R})$. In view of the Conjugacy Theorem 2.2.1, it is no loss of generality to assume that $K = G \cap \mathrm{SU}_n$ is a maximal compact subgroup of G for which $K(\mathbb{R}) = K \cap G(\mathbb{R})$ is the maximal compact subgroup of $G(\mathbb{R})$ in the statement of Theorem I. We can then proceed essentially as in Section 2.4.1. If Γ is generated by r elements, we embed $\mathrm{Hom}(\Gamma, G(\mathbb{R})) \subset \oplus_{j=1}^{r} M_n\mathbb{R}$ and Theorem 2.6.3 provides us with a $K(\mathbb{R})-$equivariant strong deformation retraction of $\mathrm{Hom}(\Gamma, G(\mathbb{R}))$ onto its Kempf–Ness set $\mathcal{M}(\mathbb{R})$. Here, Lemma 2.6.1 and Theorem III imply that

$$\mathcal{M}(\mathbb{R}) = \{\rho \in \mathrm{Hom}(\Gamma, G(\mathbb{R})) : \rho(\Gamma) \text{ consists of normal matrices } \}.$$

The proof is then completed by Lemma 2.5.1 and Proposition 2.5.2 which show that scaling eigenvalues induces a $K(\mathbb{R})-$equivariant retraction of $\mathcal{M}(\mathbb{R})$ onto $\mathrm{Hom}(\Gamma, K(\mathbb{R}))$. □

Finally, we prove the analogous theorem for character varieties:

Theorem II. *Let Γ be a finitely generated nilpotent group and let G be the group of complex or real points of a reductive linear algebraic group, defined over $\mathbb{R}$ in the latter case. If $K \subset G$ is any maximal compact subgroup, then there is a strong deformation retraction of the character variety $\mathrm{Hom}(\Gamma, G)/\!\!/G$ onto $\mathrm{Hom}(\Gamma, K)/K$.*

Proof. Let $G \subset \mathrm{SL}_n\mathbb{C}$ be a complex reductive linear algebraic group and equip $\mathrm{Hom}(\Gamma, G) \subset G^r$ with the Frobenius norm as in Section 2.4.1. If we let $\mathcal{M}$ denote its Kempf–Ness set, then Corollary 2.3.2 shows that there is a homeomorphism $\mathrm{Hom}(\Gamma, G)/\!\!/G \cong \mathcal{M}/K$. Proposition 2.5.2 then produces a K–equivariant strong deformation retraction of $\mathcal{M}$ onto $\mathrm{Hom}(\Gamma, K)$ which induces a strong deformation retraction of $\mathcal{M}/K$ onto $\mathrm{Hom}(\Gamma, K)/K$. If G is defined over $\mathbb{R}$, the argument above can be carried out for its group of real points $G(\mathbb{R})$ by viewing $\mathrm{Hom}(\Gamma, G(\mathbb{R}))/\!\!/_{top}\, G(\mathbb{R})$ as a topological Hilbert quotient (c.f. Section 2.3.1 and Richardson–Slodowy [47, Section 7.2]) and invoking Theorem 2.6.2 instead of Corollary 2.3.2. □

2.7 Expanding nilpotent groups

In this last section of the chapter, we complete the comment made in the introduction on the necessity of the assumption that G be reductive in Theorem I. The following content was shown to us by Lior Silberman and Juan Souto. We refer the reader to Raghunathan [44, Chapter 2] for generalities on nilpotent lattices in unipotent Lie groups.

Recall from the work of Mal'cev that every finitely generated torsion free nilpotent group admits a canonical completion to a unipotent Lie group which is usually called its *Mal'cev completion.* More precisely, one has the following theorem:

Theorem 2.7.1 (Mal'cev)**.** *Let Γ be a finitely generated torsion free nilpotent group. Then, there is a canonical unipotent Lie group U for which $\Gamma \subset U$ is a lattice. Moreover, if U' is any unipotent Lie group, then any homomorphism $\Gamma \to U'$ extends uniquely to a homomorphism $U \to U'$.*

Our goal is to show that one can choose Γ and U in a way that makes the conclusion of Theorem I fail when we consider the space of representations $\mathrm{Hom}(\Gamma, U)$. It turns out that the key property of $\Gamma \subset U$ in this setting is the "richness" of its automorphism group as described in the following definition.

Definition 2.7.2. A finitely generated torsion free nilpotent group Γ is said to be *expanding* if the Lie algebra $\mathfrak{u}$ of its Mal'cev completion U admits a semisimple automorphism, all of whose eigenvalues have norm larger than 1.

Example 8. Free nilpotent groups and Heisenberg groups are expanding.

The first non-expanding torsion free nilpotent group was implicitly produced by Dixmier and Lister in [20]. Later, Dyer [22] constructed a unipotent group U with a unipotent automorphism group $\mathrm{Aut}(U)$. Our interest in Dyer's counterexample stems from the fact that, being unipotent, $\mathrm{Aut}(U) \subset \mathrm{Hom}(U, U)$ is a closed subset. Since it is also an open subset, we can conclude that $\mathrm{Hom}(U, U)$ is a disconnected topological space. Moreover, since the Lie algebra of U has rational structure constants, there is a (nilpotent) lattice $\Gamma \subset U$ and the results of Mal'cev mentioned above ensure that $\mathrm{Hom}(\Gamma, U) = \mathrm{Hom}(U, U)$. However, since the maximal compact subgroup K of a unipotent group U is always trivial, $\mathrm{Hom}(\Gamma, K)$ consists of a single point. In particular, $\mathrm{Hom}(\Gamma, K)$ is not homotopy equivalent to $\mathrm{Hom}(\Gamma, U)$.

On the other hand, when Γ is an expanding nilpotent group the situation is entirely different. If we let U be the Mal'cev completion of Γ and U' be any unipotent Lie group, the representation space $\mathrm{Hom}(\Gamma, U') = \mathrm{Hom}(U, U')$ is contractible. Indeed, since $\mathrm{Aut}(U)$ is an algebraic group, it has finitely many connected components. If σ is an expanding automorphism of Γ, it follows that $\sigma^l \in \mathrm{Aut}(U)^\circ$ for some positive integer l. We can now choose a one-parameter-subgroup of automorphisms of Γ containing σ^l and reversing the associated flow induces the desired contraction. This is the observation that allowed Silberman and Souto to prove Theorem I for expanding nilpotent groups.

Chapter 3

A Note on Nilpotent Representations

This chapter is a transcript of [8] consisting of joint work by the author with Lior Silberman.

3.1 Introduction

Let G be the group of complex points of an affine algebraic group. When Γ is a finitely generated group, one may parametrize the homomorphisms from Γ to G by the images of a finite generating set. This realizes $\mathrm{Hom}(\Gamma, G)$ as an (affine) algebraic set, carved out of a finite product of copies of G by the relations of Γ. As a complex variety, $\mathrm{Hom}(\Gamma, G)$ admits a natural Hausdorff topology obtained from an embedding into affine space and it is easy to see (and well-known) that the analytic space structure on $\mathrm{Hom}(\Gamma, G)$ is independent of the chosen presentation of Γ. Here, we will only consider the case where G is reductive though, in principle, the questions we address below can be asked without this assumption.

These spaces of homomorphisms are of classical interest (see Lubotzky–Magid [35] and the references therein) and their algebraic topology has been the subject of much recent scrutiny (see, for instance, [3–6, 18, 19, 25]), stemming in part from the work of Ádem and Cohen [2]. In this context, it was recently shown by the first named author [7] that if Γ is nilpotent and K is a maximal compact subgroup of G, then there is a strong deformation retraction of $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$. This result was first established by homotopy-theoretic methods for Γ abelian by Pettet and Souto [43] and for Γ expanding nilpotent by Souto and the second named author. The result for arbitrary nilpotent groups was obtained in [7] by replacing these earlier approaches with algebro-geometric methods. Nevertheless, the machinery developed by Pettet–Souto and its followups is very well posed to the study of topological invariants. Accordingly, the goal of this note is to combine these topological and algebro-geometric tools to obtain topological information about representation spaces of nilpotent groups.

From now on, fix a non-abelian finitely generated s-step nilpotent group Γ. Recall that this means that the lower central series, defined inductively by

$$\Gamma_{(1)} = \Gamma, \qquad \Gamma_{(i+1)} = [\Gamma, \Gamma_{(i)}]$$

has $\Gamma_{(s)}$ non-trivial but $\Gamma_{(s+1)} = \{e\}$. The epimorphism $\Gamma \to \Gamma/\Gamma_{(i)}$ induces an embedding

$$\mathrm{Hom}(\Gamma/\Gamma_{(i)}, G) \to \mathrm{Hom}(\Gamma, G)$$

which (for general groups Γ and G) is not even an open map. Nevertheless, we will show:

Theorem IV. *Let Γ be a finitely generated nilpotent group and let G be the group of complex points of a (possibly disconnected) reductive algebraic group. For all $i \geq 2$, the inclusion*

$$\mathrm{Hom}(\Gamma/\Gamma_{(i)}, G) \xrightarrow{\iota} \mathrm{Hom}(\Gamma, G)$$

is a homotopy equivalence onto the union of those components of the target intersecting the image of ι.

Consider $\mathrm{Hom}(\Gamma, G)$ as a based space by taking the trivial representation as the base point. In this case, Theorem IV implies that the connected components

$$\mathrm{Hom}(\Gamma, G)_{\mathbb{1}} \subset \mathrm{Hom}(\Gamma, G) \text{ and } \mathrm{Hom}(\Gamma/\Gamma_{(i)}, G)_{\mathbb{1}} \subset \mathrm{Hom}(\Gamma/\Gamma_{(i)}, G)$$

of the trivial representation are homotopy equivalent for all $i \geq 2$. Using this, we will describe the homotopy type of the component of the trivial representation in terms of abelian representations:

Corollary 3.1.1. *For Γ and G as in* Theorem IV*, there is a homotopy equivalence*

$$\mathrm{Hom}(\Gamma, G)_{\mathbb{1}} \simeq \mathrm{Hom}(\mathbb{Z}^{\operatorname{rank} H_1(\Gamma;\mathbb{Z})}, G)_{\mathbb{1}}.$$

To introduce the other space we study, note first that the action of G on itself by conjugation induces an action on $\mathrm{Hom}(\Gamma, G)$ and conjugate homomorphisms are often considered equivalent (this is the usual notion of equivalence of representations in $\mathrm{GL_n}\mathbb{C}$). Accordingly one often wishes to understand the associated quotient but, unfortunately, the naive topological quotient is not a nice space: it need not even be Hausdorff. In order to "repair" this space, we use the affine geometric invariant theory quotient $\mathrm{Hom}(\Gamma, G)/\!\!/G$ instead. This so-called *character variety* is usually endowed with the structure of an affine variety but, for our purposes, it may be constructed topologically as the universal quotient in the category of Hausdorff spaces (see Brion–Schwarz [16]). The systematic study of the topology of theses spaces has seen much recent development (see, for instance, [10, 11, 23, 24, 33]). Concentrating on the component of the trivial representation, we will use Corollary 3.1.1 to prove:

Corollary 3.1.2. *Let* Γ *be a finitely generated nilpotent group and let* G *be the group of complex points of a reductive algebraic group. Then*

1. $\pi_1(\mathrm{Hom}(\Gamma, G)_{\mathbb{1}}) \cong \pi_1(G)^{\mathrm{rank}\, H_1(\Gamma;\mathbb{Z})}$,
2. $\pi_1((\mathrm{Hom}(\Gamma, G)/\!\!/G)_{\mathbb{1}}) \cong \pi_1(G/[G, G])^{\mathrm{rank}\, H_1(\Gamma;\mathbb{Z})}$.

Corollary 3.1.3. *Let* G *be the group of complex points of a connected reductive algebraic group, let* $T \subset G$ *be a maximal algebraic torus and let* W *be the Weyl group of* G*. If* Γ *is a finitely generated nilpotent group and* F *is a field of characteristic* 0 *or relatively prime to the order of* W*, then:*

1. $H^*(\mathrm{Hom}(\Gamma, G)_{\mathbb{1}}; F) \cong H^*(G/T \times T^{\mathrm{rank}\, H_1(\Gamma;\mathbb{Z})}; F)^W$,
2. $H^*((\mathrm{Hom}(\Gamma, G)/\!\!/G)_{\mathbb{1}}; F) \cong H^*(T^{\mathrm{rank}\, H_1(\Gamma;\mathbb{Z})}; F)^W$.

While the results above indicate many similarities between representation spaces of abelian and non-abelian nilpotent groups, the latter have a much richer topology than the former. For instance, recall that for a connected semisimple group S, the variety $\mathrm{Hom}(\mathbb{Z}^2, S)$ is irreducible and thus connected [45]. Moreover, $\mathrm{Hom}(\mathbb{Z}^r, \mathrm{SL}_n\mathbb{C})$, $\mathrm{Hom}(\mathbb{Z}^r, \mathrm{Sp_n}\mathbb{C})$ and the corresponding character varieties are connected for all values of r and n. The situation for non-abelian nilpotent groups is markedly different:

Theorem V. *Let* G *be the group of complex points of a (possibly disconnected) reductive algebraic group. If* Γ *is a finitely generated nilpotent group which surjects onto a finite non-abelian subgroup of* G*, then* $\mathrm{Hom}(\Gamma, G)$ *and* $\mathrm{Hom}(\Gamma, G)/\!\!/G$ *are both disconnected topological spaces.*

Since non-abelian free nilpotent groups and Heisenberg groups surject onto the non-abelian nilpotent group of order 8, this implies:

Corollary 3.1.4. *Let Γ be a non-abelian free nilpotent group or a Heisenberg group. If G is the group of complex points of a reductive algebraic group, then $\mathrm{Hom}(\Gamma, G)$ and $\mathrm{Hom}(\Gamma, G)/\!\!/G$ are connected if and only if G is an algebraic torus.*

Remark 15. All of the preceding statements remain true when G is replaced by a compact Lie group K. In fact, we will prove most of them in this setting before obtaining the complex reductive case via a homotopy equivalence.

Outline of the chapter. We begin Section 3.2 by describing compact representation spaces using a fibre bundle. Then, in Section 3.3, we use this bundle to prove Theorem IV and Theorem V along with their various corollaries.

3.2 An interesting bundle

The goal of this section is to prove the following key proposition:

Proposition 3.2.1. *Let K be a (possibly disconnected) compact Lie group. If Γ is an s-step nilpotent group with $s \geq 2$, then the set of abelian groups*

$$\mathcal{F} := \{\rho(\Gamma_{(s)}) \subset K : \rho \in \mathrm{Hom}(\Gamma, K)\}$$

admits a homogeneous manifold structure with finitely many connected components for which the projection map

$$p : \mathrm{Hom}(\Gamma, K) \to \mathcal{F},\ p(\rho) = \rho(\Gamma_{(s)}) \tag{3.1}$$

is a locally trivial fibre bundle.

The proof of Proposition 3.2.1 relies on the following lemma:

Lemma 3.2.2. *For all $m \in \mathbb{N}$ there is an $O = O(m) \in \mathbb{N}$ such that, if $N \subset \mathrm{SU}_m$ is an s-step nilpotent group with $s \geq 2$, then $N_{(s)}$ is an abelian subgroup of SU_m of order bounded by O.*

Proof. Recall that $N_{(s)}$ is an abelian subgroup of SU_m contained in the centre of N. As such, there is a direct sum decomposition $\mathbb{C}^m = V_1 \oplus \ldots \oplus V_r$ and r characters $\chi_1, \ldots, \chi_r$:

$N_{(s)} \to \mathbb{C}^\times$ such that $\chi_i \neq \chi_j$ for all $i \neq j$ and $\gamma(v) = \chi_i(\gamma) \cdot v$ for all $\gamma \in N_{(s)}$ and $v \in V_i$. Moreover, for all $g \in N$, $\gamma \in N_{(s)}$ and $v \in V_i$, we have

$$\gamma(g(v)) = g(\gamma(v)) = g(\chi_i(\gamma) \cdot v) = \chi_i(\gamma) \cdot g(v).$$

This allows us to consider the restrictions of the determinant homomorphism

$$\det_i : N \to \mathbb{C}^\times, \ \ \det_i(g) := \det(g|_{V_i})$$

where, since $\mathbb{C}^\times$ is abelian and $s \geq 2$, the subgroup $N_{(s)}$ must be contained in $\ker(\det_i)$. This means that for all i and all $\gamma \in N_{(s)}$, we have

$$\det_i(\gamma) = \chi_i(\gamma)^{\dim V_i} = 1,$$

so $\chi_i(\gamma)$ is always a root of unity of order bounded by m. Consequently, $N_{(s)}$ is conjugate in SU_m to a subset of those diagonal matrices whose diagonal elements are roots of unity of order bounded by m. This completes the proof since the order of this finite set does not depend on s. □

Proof of Proposition 3.2.1. Choose a faithful embedding of K into SU_m. By Lemma 3.2.2, there is a constant $O \in \mathbb{N}$ uniformly bounding the order of abelian subgroups of K occurring as the image of $\Gamma_{(s)}$ under homomorphisms $\rho : \Gamma \to K$. In order to give $\mathcal{F}$ a homogeneous manifold structure, we first consider the slightly larger set

$$\tilde{\mathcal{F}} := \{A \subset K : A \text{ is an abelian subgroup of order bounded by } O\}.$$

Observe that K° (the identity component of K) acts by conjugation on $\tilde{\mathcal{F}}$ with closed stabilizers. As such, we can endow $\tilde{\mathcal{F}}$ with the orbifold structure with respect to which each K°-orbit is a connected homogeneous K°-manifold (see Onishchik-Vinberg [42]). Concretely, if we define the "connected normalizer" as $N_{K^\circ}(H) := N_K(H) \cap K^\circ$, then the connected component of $H \in \tilde{\mathcal{F}}$ is identified with $K^\circ / N_{K^\circ}(H)$. Having a topology on each K°-orbit, we endow $\tilde{\mathcal{F}}$ with the disjoint union topology. Since K is a compact Lie group, there are only finitely many conjugacy classes of abelian subgroups of K of order bounded by O (c.f. Mostow [39, Lemma 1]) and, in particular, $\tilde{\mathcal{F}}$ has only finitely many connected components.

A homomorphism $\rho : \Gamma_{(s)} \to K$ need not extend to the full group Γ so the map

$$p : \mathrm{Hom}(\Gamma, K) \to \tilde{\mathcal{F}},\ p(\rho) = \rho(\Gamma_{(s)})$$

may not be surjective. Accordingly, we denote $\mathcal{F} := p(\mathrm{Hom}(\Gamma, K))$ and observe by K°-equivariance of p that it is a union of connected components of $\tilde{\mathcal{F}}$. Let $\mathcal{Z} \subset \mathcal{F}$ denote the connected component of a finite abelian subgroup $H \in \mathcal{F}$ and let $\mathcal{H} := p^{-1}(\mathcal{Z}) \subset \mathrm{Hom}(\Gamma, K)$. We can then identify the restriction of p to $\mathcal{H}$ with the twisted product

$$(K^\circ \times \mathcal{H})/N_{K^\circ}(H) \to K^\circ/N_{K^\circ}(H)$$

where $N_{K^\circ}(H)$ acts on K° (resp. $\mathcal{H}$) by right multiplication (resp. conjugation). Observing that $p^{-1}(H)$ is closed in $\mathrm{Hom}(\Gamma, K)$, it follows that $K^\circ \cdot p^{-1}(H) = \mathcal{H}$ is closed and therefore also open (since $\mathcal{F}$ has only finitely many components). This shows that p is a locally trivial fibre bundle. □

3.3 Proofs of the main results

Let G be the group of complex points of a (possibly disconnected) reductive algebraic group and recall that such a G necessarily arises as the complexification of a (possibly disconnected) compact Lie group K. In this section, we use Proposition 3.2.1 to prove the results mentioned in the introduction. In most cases, we prove a corresponding statement with K in lieu of G before obtaining the claimed result. We refer the reader to Onishchick–Vinberg [42] for basic facts about Lie groups and complex algebraic groups.

Let Γ be an s-step nilpotent group with $s \geq 2$ and recall that, for all i, the epimorphism $\Gamma \to \Gamma/\Gamma_{(i)}$ induces an embedding $\mathrm{Hom}(\Gamma/\Gamma_{(i)}, K) \to \mathrm{Hom}(\Gamma, K)$. Often, we shall abuse notation and identify $\mathrm{Hom}(\Gamma/\Gamma_{(i)}, K)$ with its image under this embedding. As a first consequence of Proposition 3.2.1 we obtain:

Proposition 3.3.1. *Let K be a (possibly disconnected) compact Lie group. If Γ is a finitely generated nilpotent group then, for all $i \geq 2$, the inclusion*

$$\mathrm{Hom}(\Gamma/\Gamma_{(i)}, K) \xrightarrow{\iota} \mathrm{Hom}(\Gamma, K)$$

is a homeomorphism onto the union of those components of the target intersecting the image of ι.

Proof. We proceed by induction on the nilpotence step of Γ. Recall from Proposition 3.2.1 that

$$p : \mathrm{Hom}(\Gamma, K) \to \mathcal{F},\ p(\rho) = \rho(\Gamma_{(s)})$$

is a locally trivial bundle. If Γ is 2-step nilpotent, then the image of ι consists of all representations factoring through the abelianization of Γ, that is those such that $\rho(\Gamma_{(2)}) = \{e_K\}$. Since e_K is fixed by the conjugation action of K, the subgroup $\{e_K\} \in \mathcal{F}$ is an isolated point in the given topology. Thus, for any $\rho \in p^{-1}(e_K)$, the full connected component of ρ (which is path-connected) has trivial restriction to $\Gamma_{(2)}$ and we see that $p^{-1}(\{e_K\})$ is the union of the connected components it intersects, completing the proof in this case.

Suppose now that Γ is s-step nilpotent. If $i = s$, the same argument as for the base case applies. Otherwise, $i < s$ and then

$$\Gamma/\Gamma_{(i)} \cong (\Gamma/\Gamma_{(s)})/(\Gamma_{(i)}/\Gamma_{(s)})$$

where the nilpotence step of $(\Gamma/\Gamma_{(s)})$ is $s - 1$. As such, the induction hypothesis implies that each of the following two embeddings

$$\mathrm{Hom}(\Gamma/\Gamma_{(i)}, K) \to \mathrm{Hom}(\Gamma/\Gamma_{(s)}, K) \to \mathrm{Hom}(\Gamma, K)$$

is a homeomorphisms onto those components of the target intersecting its image and, consequently, that the same holds for their composition. □

Proof of Theorem IV. The theorem follows at once by [7, Theorem I]. □

We can now prove:

Corollary 3.3.2. *If* Γ *and* K *are as in* Proposition 3.3.1*, then there is a homeomorphism*

$$\mathrm{Hom}(\Gamma, K)_{\mathbb{1}} \cong \mathrm{Hom}(\mathbb{Z}^{rank\, H_1(\Gamma;\mathbb{Z})}, K)_{\mathbb{1}}.$$

Proof. By Proposition 3.3.1, we have a homeomorphism

$$\mathrm{Hom}(\Gamma, K)_{\mathbb{1}} \cong \mathrm{Hom}(H_1(\Gamma;\mathbb{Z}), K)_{\mathbb{1}}.$$

Since $H_1(\Gamma;\mathbb{Z}) = \Gamma/[\Gamma,\Gamma]$ is a finitely generated abelian group, we may identify $H_1(\Gamma;\mathbb{Z})$

with $\mathbb{Z}^r \oplus A$ where $r := \operatorname{rank} H_1(\Gamma;\mathbb{Z})$ and A is a finite abelian group. At this point we would like to show that $\operatorname{Hom}(\mathbb{Z}^r \oplus A, K)_{\mathbb{1}} = \operatorname{Hom}(\mathbb{Z}^r, K)_{\mathbb{1}}$. Seeking a contradiction, suppose that $\rho_0 \in \operatorname{Hom}(\mathbb{Z}^r \oplus A, K)_{\mathbb{1}}$ maps A non-trivially into K. By assumption, there is a continuous path of representations $[0,1] \mapsto \rho_t$ starting at ρ_0 and ending at the trivial representation $\rho_1 = \mathbb{1}$. But now, this path induces a continuous deformation in $\operatorname{Hom}(A, K)$ of the representation $\rho_0|_A$ to the trivial representation. This is impossible since Lie groups contain no small subgroups (c.f. Bredon [14, Corollary 0.4.5]). □

Proof of Corollary I. The corollary follows at once by Theorem I. □

Using this, we immediately obtain:

Corollary II. *Let G be the group of complex points of a reductive algebraic group. If Γ is a finitely generated nilpotent group, then:*

1. $\pi_1(\operatorname{Hom}(\Gamma, G)_{\mathbb{1}}) \cong \pi_1(G)^{rank\, H_1(\Gamma;\mathbb{Z})}$, *and*
2. $\pi_1((\operatorname{Hom}(\Gamma, G)/\!\!/G)_{\mathbb{1}}) \cong \pi_1(G/[G,G])^{rank\, H_1(\Gamma;\mathbb{Z})}$.

Proof. The two formulas follow at once from Corollary I by the main results of Gómez–Pettet–Souto [25] and Biswas–Lawton–Ramras [11]. □

In order to prove our second corollary, we need the following:

Lemma 3.3.3. *If K is a compact Lie group and Γ is a finitely generated nilpotent group, then* $\operatorname{Hom}(\Gamma, K)_{\mathbb{1}}/K = (\operatorname{Hom}(\Gamma, K)/K)_{\mathbb{1}}$. *In particular,* $\operatorname{Hom}(\Gamma, K)$ *is connected if and only if* $\operatorname{Hom}(\Gamma, K)/K$ *is connected.*

Proof. Recall from Corollary 3.3.2 that any $\rho \in \operatorname{Hom}(\Gamma, K)_{\mathbb{1}}$ factors through the torsion free part of $H_1(\Gamma;\mathbb{Z})$. As such, by Baird [6, Lemma 4.2], $\rho \in \operatorname{Hom}(\Gamma, K)_{\mathbb{1}}$ if and only if there is a torus $T \subset K$ such that $\rho(\Gamma) \subset T$. Since this property is preserved under conjugation by elements of K, it follows that $(\operatorname{Hom}(\Gamma, K)/K)_{\mathbb{1}}$ coincides with the quotient $\operatorname{Hom}(\Gamma, K)_{\mathbb{1}}/K$. □

We can now prove the cohomological formulas mentioned in the introduction.

Corollary III. *Let G be the group of complex points of a connected reductive algebraic group, let $T \subset G$ be a maximal algebraic torus and let W be the Weyl group of G. If Γ is a finitely generated nilpotent group and F is a field of characteristic 0 or relatively prime to the order of W, then:*

1. $H^*(\mathrm{Hom}(\Gamma, G)_{\mathbb{1}}; F) \cong H^*(G/T \times T^{\,rank\, H_1(\Gamma;\mathbb{Z})}; F)^W$, *and*
2. $H^*((\mathrm{Hom}(\Gamma, G)/\!\!/G)_{\mathbb{1}}; F) \cong H^*(T^{\,rank\, H_1(\Gamma;\mathbb{Z})}; F)^W$.

Proof of Corollary III. Following Pettet–Souto [43, Corollary 1.5], let $K \subset G$ be a maximal compact subgroup such that $T_K := T \cap K$ is a maximal torus in K. Notice that, for any $r \in \mathbb{N}$,

$$K/T_K \times T^r \to G/T \times T^r$$

is a W-equivariant homotopy equivalence and, in particular, that

$$H^*(K/T_K \times T^r)^W \cong H^*(G/T \times T^r)^W. \tag{3.2}$$

Here, it follows from Baird [6, Theorem 4.3] that the left hand side of the equation is isomorphic to $H^*(\mathrm{Hom}(\mathbb{Z}^r, K)_{\mathbb{1}})$. Now, letting $r := \mathrm{rank}\, H_1(\Gamma; \mathbb{Z})$, our first formula follows at once from the homotopy equivalences

$$\mathrm{Hom}(\Gamma, G)_{\mathbb{1}} \simeq \mathrm{Hom}(\mathbb{Z}^r, G)_{\mathbb{1}} \simeq \mathrm{Hom}(\mathbb{Z}^r, K)_{\mathbb{1}}$$

provided by Corollary I and Theorem I. Finally, it is also due to Baird [6, Remark 4] that $\mathrm{Hom}(\mathbb{Z}^r, K)_{\mathbb{1}}/K \cong T_K^r/W$ so our second formula follows from the homotopy equivalence and homeomorphisms

$$(\mathrm{Hom}(\Gamma, G)/\!\!/G)_{\mathbb{1}} \simeq (\mathrm{Hom}(\Gamma, K)/K)_{\mathbb{1}} \cong \mathrm{Hom}(\Gamma, K)_{\mathbb{1}}/K \cong \mathrm{Hom}(\mathbb{Z}^r, K)_{\mathbb{1}}/K.$$

provided by Theorem II, Lemma 3.3.3 and Corollary 3.3.2. □

Remark 16. The homotopy types of distinct components of representation spaces are typically different. For instance, if we take Γ to be the discrete Heisenberg group $H_3(\mathbb{Z})$, then $\mathrm{Hom}(\Gamma, \mathrm{SL}_2\,\mathbb{C})$ decomposes into a simply-connected component and a non simply-connected component. In fact, this phenomenon already occurs for Γ abelian as illustrated in Gómez–Adem [4] and Gómez–Pettet–Souto [25].

Theorem V. *Let G be the group of complex points of a reductive algebraic group. If Γ is a finitely generated nilpotent group which surjects onto a finite non-abelian subgroup of G, then* $\mathrm{Hom}(\Gamma, G)$ *and* $\mathrm{Hom}(\Gamma, G)/\!\!/G$ *are both disconnected.*

Proof. Let $\psi : \Gamma \to N$ be a surjective homomorphism onto a finite non-abelian subgroup of G and let K be a maximal compact subgroup of G containing N. Notice in particular that $\psi \in \mathrm{Hom}(\Gamma, K) \subset \mathrm{Hom}(\Gamma, G)$. Since $\mathrm{Hom}(\Gamma, K) \simeq \mathrm{Hom}(\Gamma, G)$ and $\mathrm{Hom}(\Gamma, K)/K \simeq \mathrm{Hom}(\Gamma, G)/\!\!/G$ by Theorem I and Theorem II, and since $\mathrm{Hom}(\Gamma, K)$ is disconnected if and only if $\mathrm{Hom}(\Gamma, K)/K$ is disconnected by Lemma 3.3.3, it suffices to prove that $\mathrm{Hom}(\Gamma, K)$ is disconnected.

Seeking a contradiction, suppose that $\mathrm{Hom}(\Gamma, K)$ is connected and recall from Proposition 3.3.1 that, in this case, $\mathrm{Hom}(\Gamma/\Gamma_{(i)}, K)$ is connected for all $i \geq 2$. Choose a minimal $s \in \mathbb{N}$ with the property that $\psi(\Gamma_{(s+1)}) = e_K$ and denote the s-step nilpotent group $\Gamma/\Gamma_{(s+1)}$ by $\hat{\Gamma}$. If we consider the fibre bundle (c.f. Proposition 3.2.1)

$$p : \mathrm{Hom}(\hat{\Gamma}, K) \to \mathcal{F},\, p(\rho) = \rho(\hat{\Gamma}_{(s)}),$$

then $p(\psi) = \psi(\hat{\Gamma}_{(s)}) \neq e_K$. As such, by Proposition 3.3.1 and our assumptions,

$$\psi \notin \mathrm{Hom}(\hat{\Gamma}, K)_{\mathbb{1}} \cong \mathrm{Hom}(\Gamma/\Gamma_{(s+1)}, K)_{\mathbb{1}} \cong \mathrm{Hom}(\Gamma, K)_{\mathbb{1}} \cong \mathrm{Hom}(\Gamma, K)$$

and this contradiction completes the proof. □

Corollary IV. *Let Γ be a non-abelian free nilpotent group or a Heisenberg group. If G is the group of complex points of a reductive algebraic group, then* $\mathrm{Hom}(\Gamma, G)$ *and* $\mathrm{Hom}(\Gamma, G)/\!\!/G$ *are connected if and only if G is an algebraic torus.*

Proof. If G is disconnected or not simply-connected then [43, Corollary 1.3], Theorem I, Theorem II and Lemma 3.3.3 show that $\mathrm{Hom}(H_1(\Gamma; \mathbb{Z}), G)$ and $\mathrm{Hom}(H_1(\Gamma; \mathbb{Z}), G)/\!\!/G$ are disconnected. As such, it suffices to consider the case where G is simply-connected. Notice that such a G contains a subgroup isomorphic to $\mathrm{SL}_2\,\mathbb{C}$ and, since $\mathrm{SL}_2\,\mathbb{C}$ contains a copy of the non-abelian group Q of order 8 generated by the matrices

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so does G. Since Q is a $\mathbb{Z}/2\mathbb{Z}$ central extension of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, it follows that if Γ is either a non-abelian free nilpotent group or a Heisenberg group, then Γ surjects onto Q. The claim now follows from Theorem V. □

Chapter 4

Conclusion

In some sense, this thesis came to be through the author's obsession with the work of Pettet and Souto [43] where it was first established that, for G a complex or real reductive linear algebraic group and $K \subset G$ a maximal compact subgroup, there is a strong deformation retraction of $\mathrm{Hom}(\mathbb{Z}^r, G)$ onto $\mathrm{Hom}(\mathbb{Z}^r, K)$. Prior to their result, most known topological invariants in this realm concerned spaces of representations into compact groups; it was their deformation retraction which allowed many of these results to be extended to representations into reductive groups. This implicitly asked the following:

Question 1. Given a real or complex reductive linear algebraic group G and a maximal compact subgroup $K \subset G$, what are the classes of finitely generated groups Γ for which there is a deformation retraction of $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$?

Starting with abelian groups, there were then many natural directions in which one might try to generalize Pettet and Souto's result. Having established in the guise of Theorem I that nilpotent groups also offer a positive answer to Question 1, solvable groups became the next natural class to investigate. In this context, although the key technical step in the proof of Theorem I breaks down (c.f. Example 6), we suspect that such a retraction still exists. Indeed, together with Lior Silberman, we have the following tantalizing partial result:

Proposition I (Bergeron-Silberman)**.** *Let Γ be a finitely generated virtually solvable group, let G be the group of complex points of a (possibly disconnected) reductive linear algebraic group, and let $K \subset G$ be a maximal compact subgroup. There is a constant $O = O(G)$ such that if $\mathcal{M}$ denotes the Kempf-Ness set of the G-subvariety* $\mathrm{Hom}(\Gamma, G) \subset \oplus_{j=1}^{r} M_n\mathbb{C}$

with respect to a $K-$invariant norm, then the image of any representation in $\mathcal{M}$ consists of semisimple elements and contains a normal abelian subgroup of index bounded by O.

Proof. Recall from the Kempf-Ness Theorem that $\mathcal{M}$ is always contained in the set of polystable points of $\mathrm{Hom}(\Gamma, G)$. Moreover, recall from Richardson's Theorem that if $\rho \in \mathrm{Hom}(\Gamma, G)$ corresponds to the tuple $(\rho(\gamma_1), \ldots, \rho(\gamma_r))$, it is polystable if and only if the algebraic subgroup of G generated by the elements $\{\rho(\gamma_1), \ldots, \rho(\gamma_r)\}$ is reductive. We may therefore assume without loss of generality that the Zariski closure of the image of any representation in $\mathcal{M}$ is a reductive solvable subgroup $S \subset G$. It then follows from work of Mal'cev [54, Lemma 3.5] that such an S contains a normal abelian subgroup of index bounded by a constant O depending only on G. Finally, since the connected abelian group S° is an algebraic torus and all unipotent elements of S must be contained in this identity component, we conclude that S consists of semisimple elements. □

Interestingly, recalling from the Neeman-Schwarz Theorem that there is a deformation of $\mathrm{Hom}(\Gamma, G)$ onto $\mathcal{M}$, this already implies that when Γ is virtually solvable the connected component of the trivial representation $\mathrm{Hom}(\Gamma, G)_{\mathbb{1}} \subset \mathrm{Hom}(\Gamma, G)$ has the homotopy type of the space of representations of a free-abelian group. Given that the latter's topology is relatively well understood, this paves the way for computations of various homotopy invariants. Nevertheless, in order to understand why Proposition I is a truly exciting development, it is worth recalling the broad lines of Pettet and Souto's initial argument in the abelian case. Their retraction proceeds in two steps. They first construct a commutativity-preserving deformation of G onto its set of semisimple elements G_s in order to induce a deformation of $\mathrm{Hom}(\mathbb{Z}^r, G)$ onto the set $\mathrm{Hom}(\mathbb{Z}^r, G_s)$ of representations whose images consist of semisimple elements. Then, using homotopy-theoretic tools, they manage to deform $\mathrm{Hom}(\mathbb{Z}^r, G_s)$ onto $\mathrm{Hom}(\Gamma, K)$. Although it remains unclear whether the first step of their proof is adaptable to the virtually abelian case, it would seem that the homotopy-theoretic techniques they developed (further generalized in unpublished work by Silberman and Souto for expanding nilpotent group representations) can be modified to treat this virtually abelian case. Once combined with Proposition I, this would add virtually solvable groups to the class of positive answers for Question 1.

This first exploration in the case of nilpotent and solvable groups is also intriguing in that it generalizes Pettet and Souto's result from the point of view of both Lie theory and geometric group theory. Pushing further in the latter direction, one can consider Question 1 in the context of right-angled Artin groups (c.f. [17]) or more general amalgamations of abelian groups. Although the author does not believe that there is a deformation retraction from $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$ for this class of groups, there is still much to be said on this topic which we are currently investigating with Juan Souto. Indeed, we have recently shown that when the underlying graph of a right-angled Artin groups Γ is a tree then there is a deformation of $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$. In fact, we prove a more general statement for tree-like amalgamations of abelian groups. To be precise, recall that a *finite tree of free abelian groups* $\mathcal{T}$ consists of a finite tree T, a free abelian group T_v for every vertex v of T, and a free abelian group T_e for every edge e of T, together with an injective homomorphisms $T_e \hookrightarrow T_v$ whenever the vertex v is an endpoint of the edge e. We can then consider the fundamental group $A_{\mathcal{T}}$ of the tree of free abelian groups $\mathcal{T}$ in the sense of Bass and Serre [50]. For instance, if T consists of a single edge e joining a vertex v to a vertex w then $A_{\mathcal{T}} \cong T_v *_{T_e} T_w$ is the amalgamated product of T_v and T_w along T_e. What we prove in this context is the following:

Theorem VI (Bergeron-Souto [9])**.** *Let $A_{\mathcal{T}}$ be the fundamental group of a finite tree of free abelian groups and let G be the group of complex or real points of a (possibly disconnected) reductive linear algebraic group, defined over $\mathbb{R}$ in the latter case. If $K \subset G$ is any maximal compact subgroup, then there is a strong deformation retraction of* $\mathrm{Hom}(A_{\mathcal{T}}, G)$ *onto* $\mathrm{Hom}(A_{\mathcal{T}}, K)$*.*

One amusing consequence of Theorem VI is that it implies without much work that a similar statement holds for right-angled Artin groups defined by a disjoint union of trees and triangles. This class of groups may seem a bit artificial at first sight but it turns out that a result of Droms [21] characterizes them as those right-angled Artin groups being the fundamental groups of 3-manifolds. It should also be pointed out that, as far as the author can tell, no such retraction should exist for a large class of right-angled Artin groups. More concretely, let us conclude with the following question which the author suspects to have a positive solution:

Question 2. For every $m \in \mathbb{N}$ with $m \geq 5$ is there some $n \in \mathbb{N}$ such that, if A_Γ is the right-angled Artin group determined by an m-cycle, then there is no strong deformation retraction of $\mathrm{Hom}(A_\Gamma, \mathrm{SL}_n\mathbb{C})$ onto $\mathrm{Hom}(A_\Gamma, \mathrm{SU}_n)$?

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