Abstract

This thesis is composed of three chapters. The first chapter argues that boom-bust behavior in asset prices can be explained by a model in which boundedly-rational agents learn the process for prices. The key feature of the model is that learning operates in both the demand for assets and the supply of credit. Interactions between agents on either side of the market create complementarities in their respective beliefs, providing an additional source of propagation. In contrast, the chapter shows why learning involving only one side on the market, the focus of most of the literature, cannot plausibly explain persistent and large price booms. Quantitatively, the model explains recent experiences in US housing markets. The full appreciation in US house prices in the 2000s can be generated from observed mortgage rate changes. The model also generates endogenous liberalizations in household lending conditions during price booms and replicates key volatilities of housing market variables at business cycle frequencies.

The second chapter presents a learning model in which households are endowed with recursive preferences. The chapter evaluates how the introduction of bounded rationality in beliefs effects the level of long run consumption risk in the economy. The chapter shows that structural learning frameworks currently found in the literature lead to a perception of low persistence in exogenous shocks, regardless of the underlying stochastic processes in the economy. Generating long run risk requires a preference for late resolution of uncertainty.

The third chapter provides an explanation for two features of the world saving distribution: (i) saving rates are significantly different across countries and they remain different for long periods of time; and (ii) some countries and regions have shown very sharp changes in their average saving rates over short periods of time. It formalizes a model of the world economy comprised of open economies inhabited by heterogeneous agents endowed with recursive preferences. The model can generate the time series behavior of saving observed in the data from measured productivity shocks. The model can also generate the sudden and long-lived increase in East Asian savings by incorporating shocks to societal aspiration.
Preface

Chapter 3 of this thesis is joint work with Prof. Amartya Lahiri at the Vancouver School of Economics. I was responsible for implementing the quantitative work in the chapter. The empirical and theoretical components of the chapter are joint work. The chapter was first drafted by Prof. Lahiri.
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Dedication

To Elyse (& Rosie)
Chapter 1

Can Learning Explain Boom-Bust Cycles in Asset Prices? An Application to the US Housing Boom

1.1 Introduction

Following the financial crisis of 2008 there has been an intense focus on the tendency of markets to generate boom-bust patterns in asset prices. Explaining these episodes poses a difficult question for researchers: how can large and persistent price growth be explained in the absence of large and persistent exogenous variation? Across a wide range of settings it has proven difficult to identify market fundamentals or frictions that can explain price booms as well as asset price volatility more generally. This chapter argues that boom-bust behavior in asset prices can be explained by a model in which boundedly rational agents learn the process for prices. The key feature of the model is that, in contrast with the literature, learning operates in both the demand for assets and the supply of credit. Propagation comes from the interaction between the two sets of agents in the model, which creates complementarities in their respective beliefs. The quantitative performance of the model is evaluated in the context of recent experiences in US housing markets. A single unanticipated mortgage rate drop, consistent with that observed in the early 2000s, generates 20 quarters of house price growth whilst capturing the full appreciation in US housing in the early 2000s.

The novel feature of this work is that it allows for learning in the credit supply problem facing lenders. This is in contrast to canonical asset pricing models with learning that restrict subjective beliefs to the demand side of the market. Models of bounded rationality allow for the possibility of feedback loops to exist between subjective beliefs and observed outcomes. In order for such environments to generate large and persistent asset price growth in response to a small set of shocks, beliefs need to exhibit two properties. First, subjective beliefs must be highly responsive to...
1.1. Introduction

observed shocks. The response of outcomes to shifts in beliefs must be of sufficient size to generate subsequent belief shifts. Second, the belief process itself must be sufficiently persistent to prevent the episode from dying out quickly. A contribution of this chapter is to show that in models with only demand side learning, there exists a trade-off between these two properties. In other words, increasing the elasticity of beliefs with respect to shocks comes at the cost of decreasing the persistence of these beliefs. As a result of this trade-off such models struggle to explain asset price booms.

Next, the chapter shows that this trade-off in the formation of the belief process can be broken by extending bounded rationality to credit suppliers. When learning about prices operates in both the demand side and credit supply side of the market there are complementarities in the beliefs of buyers and lenders. An increase in buyers’ price forecasts increases the capital gains they expect to receive on their assets, driving up demand. An increase in lenders’ price forecasts decreases the default rate they expect to face on their assets, leading to relaxed lending conditions. Each of these actions drive up prices, and through the learning mechanism further increase the price forecasts of each type of agent. The chapter shows that this complementarity loosens the trade-off between generating beliefs that are both persistent and responsive.

Finally, the chapter shows that this mechanism can quantitatively capture many properties of US housing markets. The full appreciation in US housing seen in the early 2000s can be explained with observed mortgage rate movements. The calibrated model is also shown to replicate key volatilities of housing market variables at business cycle frequencies. Furthermore, the chapter explains observed comovements in house prices and household leverage. The model developed here is able to endogenously generate substantial liberalizations in households’ borrowing environment concurrent with periods of prolonged price growth.

The chapter is structured as follows. Section 1.2 provides an overview of the literature in which this work is placed as well as a discussion of the recent experience in US housing markets. Section 1.3 presents the main model, outlining the microfoundations of agents’ beliefs, discussing the decision problem and optimality conditions of households and lenders, and finally providing an equilibrium for the model under learning. A discussion of the model’s calibration is to be found in section 1.4. Section 1.5 presents the analytical findings of the chapter and demonstrates how the presence of bounded rationality in both the demand for housing and the supply of credit breaks
a trade-off between the persistence of beliefs and the sensitivity of beliefs to shocks that exists in traditional learning models. Empirical findings are discussed in section 1.6.1 examines the effect of observed mortgage rate drops and highlights the model’s ability to capture much of the observed experience in the US housing market post-2000, while 1.6.2 shows the model’s performance in attempting to match business cycle moments of the US housing market.

1.2 Background & Related Literature

1.2.1 Background

Across almost all of the major urban centers in the United States price-rent ratios rose between 20 and 70 percent in the 10 years leading up to the market crash. The experience in US housing is widely seen as being central to the subsequent crisis in global financial markets and the ensuing recession. Housing wealth, about 80 percent of which is encompassed by the stock of owner-occupied homes, accounts for half of household net worth in the United States (Iacoviello (2011)) and residential investment has been a relatively large component of US GDP growth over the past 30 years (Wheaton and Nechayev (2010)). In spite of the prevalence and importance of housing booms, data on housing market fundamentals typically doesn’t display large volatilities. Consequently, standard frameworks struggle to explain large and persistent movements in house prices\(^1\) suggesting a role for an expectations-based explanation. The overweighting of recently observed information in the formation of beliefs, particularly beliefs about long horizon events, is documented in many settings\(^2\). An implication of such extrapolative behavior when applied to beliefs about price movements is that agents tend to underpredict price growth during inflationary periods and overpredict growth during deflations.

Figure 1.1 plots the 10-city composite of the Case-Shiller Home Price Index together with the prices of futures contracts that trade on this index on the Chicago Mercantile Exchange\(^3\). The

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1. Literature has arisen as a result that attempts to explain housing booms as arising from a variety of different and often conflicting mechanisms, including: varying supply-side constraints (see Gleaser et al. (2008), Bulusu, Duarte, and Vergaga-Alert (2013), Kiyotaki, Michaelides, and Nikolov (2011)), demand-side factors such as changes in credit conditions (Chu (2014), Favilukis, Ludvigson, and Nieuwerburgh (2010)), income instability (Nakajima (2011), Pastor and Veronesi (2006)), and social interaction mechanisms (Burnside, Eichenbaum, and Rebelo (2011)). Also see Justiniano, Primiceri, and Tambalotti (2015) for further discussion.


3. Note a version of this figure appears in Gelain and Lansing (2013).
1.2. Background & Related Literature

futures contracts provide a proxy for market expectations about future house prices, and they bear the evidence of extrapolative expectations. Following the crest in US house prices in 2006, forecasts substantially overpredicted the realized path of house prices for the best part of two years. After beliefs about future growth adjusted in late 2007, the forecasted series missed the turning point at the bottom of the market in 2009 and persistently underpredicted prices as inflation began in 2012. Similar evidence is documented in Case and Shiller (2003) and Piazzesi and Schneider (2009), who estimate household beliefs about price increases and find increasing levels of optimism throughout the run-up in house prices in the mid 2000s.

This evidence suggests a potential mechanism for resolving the observed boom in prices with the relative lack of variation in market fundamentals. If agents’ beliefs about future events, particularly in the long-run, are sufficiently responsive to shocks, booms may arise as self-confirming events: shocks to prices may shift the distribution of expected prices – and therefore expected capital gains, expected default rates, etc... – enough to generate large increases in demand and therefore subsequent price growth. In formulating a learning model in which boundedly rational agents recursively update beliefs about the process for house prices this chapter captures precisely these kind of dynamics.

\[\text{Gelain and Lansing (2013)}\]
1.2. Background & Related Literature

A key feature of the US house price boom was the large increase in credit provided to households. The left hand panel in figure 1.2 plots the All-Transaction House Price index from the Federal Housing Finance Agency together which three measures of net household liabilities as a fraction of GDP. The right hand panel shows net new borrowing by US households as a fraction of the market value of housing stock. As can be seen, the data reveals close comovement between prices and household credit during the late 1990s and early-mid 2000s. It seems reasonable, then, that a story of price booms in US housing markets should speak to this phenomenon. Under the learning framework developed in this chapter, a simple contracting mechanism between households and lenders endogenously generates substantial comovement in house prices and household leverage. This is a result of the sensitivity of credit supply to subjective beliefs. Shifts in the distribution of expected price growth cause lenders to significantly liberalize lending conditions to households, driving inflation in house prices and subsequent shifts in lenders’ beliefs.

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Box & Mendoza (2014) study the effect of credit market liberalizations in a learning framework. The authors are able to capture significant growth in land prices, however credit market liberalizations are exogenously imposed and the model does not allow for feedback between agents’ exogenous variables and beliefs.
1.2. Background & Related Literature

1.2.2 Related Literature

This work follows from a wide literature that models learning behavior of economic agents so as to amplify and propagate shocks in macro models. In such frameworks agents are assumed to be uninformed about some process or set of processes and hence hold subjective beliefs about their true law of motion. These beliefs are updated over time to account for new information via a learning rule such as Bayesian updating, least squares updating, or constant gain learning (under such an updating rule agents place a constant weight, called a gain, on new information). Typically, agents perceive that temporary shocks have long run effects through their influence on learned beliefs. Much of the initial interest in this work arose in the monetary policy and business cycle literature.

The ability of learning mechanisms to generate improved volatilities and comovements in macroeconomic environments has been mixed.\(^6\) The range of results found in this literature can in part be accounted for by the effect different learning mechanisms have on the microfoundations of agents’ decision problems. Preston (2005) argued that researchers implicitly impose an inconsistency upon agents’ beliefs when assuming them to be uninformed about the future evolution of their own choice variables, an assumption that is common in the learning literature. When agents are uninformed about the true law of motion of some variable(s) in learning models, they forecast future values of these variables using a perceived law of motion that they have estimated from previously observed data. Hence, if uninformed about the law of motion of one of their own control variables, the agent’s forecasts of this object are not constrained to be consistent with what optimal choices would be given its beliefs about the evolution of other variables in the model. Suppose, for example, that an agent is uninformed about the true process governing investment. In a learning framework, forecasted investment would then be given by the expected value of an estimated process for investment and not by the expected optimal response of investment to prices and exogenous shocks. Such models therefore implicitly assume that agents are either uninformed about their decision problem in the future or that agents predict they will be making suboptimal decisions in the future.\(^7\)

This work is extended by Adam and Marcet (2011) who formalize the concept of internal rationality. This stipulates that agents with subjective beliefs should make choices that are everywhere

\(^6\)See Williams (2003), Carceles-Poveda and Giannitsarou (2008), Huang, Liu, and Zha (2009), and Milani (2007).

\(^7\)Marshall and Shea (2013) provide an example of a learning model in housing where households are assumed to be uninformed about the evolution of their own consumption.
optimal conditional on these beliefs (i.e. on and off the equilibrium path). In restricting subjective beliefs to the space of prices observed in the housing market the model formulated in this chapter builds upon this work. There is growing evidence that such frameworks can improve the internal propagation mechanisms of models. Eusepi and Preston (2011) consider a real business cycle environment where agents are restricted to learning the parameters of wage and capital return functions via constant gain learning. The specification allows the consumption-saving decisions of households to be a discounted sum of subjective wage and rental rate forecasts. Relative to rational expectations the persistence of shocks and overall volatilities are substantially increased. Similar results are found in Branch and McGough (2011), while Sinha (2011) suggests that such learning specifications can improve the performance of business cycle models in matching financial moments.\footnote{Similar frameworks can be found in Adam, Marcet, and Nicolini (2013), Branch (2014), and Williams (2003).}

Asset pricing models with internally rational learning have achieved some success in explaining the observed volatility and persistence in asset prices (most notably stock prices and house prices) over the business cycle, however this research has thus far failed to provide a convincing explanation of price booms. Adam, Beutel, and Marcet (2014) propose a learning framework in which boundedly rational agents believe stock price growth to be governed by a simple linear hidden Markov model. A similar model is examined by Adam and Marcet (2010) with agents that are assumed to be uninformed about the evolution of stock returns instead of price growth. Both frameworks struggle to endogenously generate sequences of beliefs sufficient to yield price booms in general equilibrium settings.

Adam, Kuang, and Marcet (2011) implement a learning model, similar in spirit to that found in Adam, Beutel, and Marcet (2014), to try to explain joint house price-current account movements in the G7 over the 2000s. The authors find that reductions in interest rates in the early 2000s can generate substantial increases in house prices as households become increasingly optimistic about price appreciation, however the result is highly sensitive to the initial conditions of beliefs and price growth at the time shocks hit. Similar models are formulated by Gelain and Lansing (2013) and Granziera and Kozicki (2012). This chapter is also closely related to Kuang (2014) who considers a
1.3. A Model With Learning

Kiyotaki and More (1997) environment with learning. Internally-rational agents in the model form beliefs about the joint law of motion for collateral prices and collateral stock. The author shows that the specified process of expectation formation can generate boom-bust cycles in prices and credit. This chapter follows from Kuang (2014) in considering credit constraints to be an important channel through which learning can operate. This chapter goes further, however, in considering a role for internal rationality in the determination of loan-to-value ratios. It shows that under such conditions boom-bust cycles can be obtained even with the simple models of expectation formation found in Adam, Kuang, and Marcet (2011), Gelain and Lansing (2013), and Granziera and Kozicki (2012). The environment specified here also allows for belief heterogeneity between borrowers and lenders.

Boz and Mendoza (2014) argue that credit market liberalizations in the late 1990s lie at the root of the US housing boom. The authors propose a learning framework in which agents learn a two-state Markov-switching model for collateral constraints, however the model cannot explain the majority of the growth in US house prices.

This chapter also contributes to a literature which considers the role of bounded rationality in driving the credit supply choices of financial institutions. Luzzetti and Neumuller (2015) argue that the dynamics of household debt and bankruptcy can in part be explained when lenders learn the riskiness of the financial environment in which they operate. Similarly, Pancrazi and Pietrunti (2014) consider the role that boundedly-rational beliefs about prices on the part of lenders can play in determining debt and home equity extraction.

1.3 A Model With Learning

The housing market is modelled as an open economy environment. The key interactions in the model involve households who purchase and consume housing stock, and mortgage lenders who supply households with credit in return for claims on their housing. Housing stock is treated as

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10 Many of the papers in the learning literature make stronger assumptions about agents’ information structure. Gelain and Lansing (2013) study the behavior of US house prices under a form of learning where households make forecasts of a composite variable which is composed of price-rent ratios and consumption growth. Williams (2012) considers a model where agents learn the mean and standard deviation of stock returns. The model includes an occasionally-binding borrowing constraint about which the agent is uninformed and which it does not internalize in its decision problem. Branch, Petrosky-Nadeau, and Rocheteau (2014) examine US housing using a learning model with search and matching frictions in employment. Agents in the model believe that price growth will always exceed the level of growth extrapolated off of recent data.
an asset which, in addition to providing households with a source of capital gains, yields a flow of housing services and can be posted as collateral when borrowing. Household borrowing is subject to default. The chapter abstracts from modelling strategic behavior in the household’s default choice. While there is a rich literature that seeks to model the decision problem facing households when defaulting on residential mortgages, these issues are not central to the mechanism considered in the chapter. The default choice is modelled to reflect observed default patterns in the data. Lenders are assumed to have access to an outside source of funds (ie. international financial markets) and are owned by agents outside the housing market.

In modelling the lender’s problem the chapter does not seek to provide a complete description of mortgage financing. Instead, the model provides a stylized contracting problem between lenders and households in order to capture observed properties of mortgage default as well as measured correlations between default rates, price growth, and household leverage. Lenders set their supply of credit in response to expected default rates and the value of their collateral claims in the event of default. Household demand is driven by expected capital gains, expected default rates, and their credit constraint.

All agents are assumed to be uninformed about the determination of house prices, and therefore hold subjective beliefs about the evolution of these prices in the economy. Beliefs are homogeneous across lenders and households, and are updated continuously to account for new information. Stationary Bayesian updating of beliefs implies that learning follows the well-known constant gain algorithm. Expectations about price growth at long horizons heavily weight recent price observations. Importantly, the information structure respects the international rationality of agents’ decision problem. Household and lender choices are optimal responses to their subjective beliefs.

### 1.3.1 Household Problem

An individual household has preferences over consumption and housing services given by

$$E_0^\mathcal{P} \sum_{t=0}^{\infty} \beta^t u(c_t, h_t), \text{ where } u(c_t, h_t) = c_t^{\gamma_c} h_t^{1-\gamma_c}$$

(1.1)

where $\mathcal{P}$ denotes the household’s subjective beliefs. The household receives an exogenous endowment $y_t$ each period out of which it can accumulate housing stock $h_t$ and purchase consumption
1.3. A Model With Learning

goods (the numeraire in the economy). Housing stock has a price $q_t$ and depreciates at the rate $\delta_h$. The household also has access to a technology that allows it to convert consumption goods into non-housing capital one-for-one. Capital can be rented to builders, who are owned by the household, at a price $p_t$ and depreciates fully after one period. The household can also borrow from a credit supplier at a risk-free rate $R_t$ by posting its housing stock as collateral. All debt contracts are one period in length. The household’s borrowing is limited by a collateral constraint

$$R_t b_t \leq \theta_t E_t q_{t+1} h_t$$  \hspace{1cm} (1.2)

which is assumed to bind each period. The flow budget constraint is given by

$$c_t + q_t (h_t - (1 - \delta_h) h_{t-1}) + R_{t-1} b_{t-1} + k_t = y_t + b_t + p_t k_{t-1} + \Pi_t$$  \hspace{1cm} (1.3)

Where $\Pi_t$ denotes profits earned by builders. At the beginning of each period the household is able to decide whether or not it wants to default on its debt repayment $R_{t-1} b_{t-1}$. If it chooses not to repay then the lender confiscates the household’s housing stock $(1 - \delta_h) h_{t-1}$. The model abstracts from considerations of strategic default. Instead, households default so as to minimize their per-period repayment. However this process is subject to an idiosyncratic shock $\zeta$, such that the household chooses to default if

$$(1 - \delta_h) q_t h_{t-1} \leq \zeta_t R_{t-1} b_{t-1}, \quad \log \zeta_t \sim N(\log \bar{\zeta}, \sigma^2_{\zeta})$$  \hspace{1cm} (1.4)

The shock $\zeta$ can be thought of as capturing differences in liquidity across households. This formulation allows the model to reflect the observed fact that only a subset of households who go into negative equity end up having their mortgages foreclosed. The presence of default in the model implies that housing stock will be heterogeneous across households. Owing to the fact that per period utility in (1.1) is Cobb-Douglas, the model nevertheless aggregates to a representative household structure with stochastic default (the representative agent loses only a fraction of his/her housing stock each period as a result of default). The formulation of the default problem in (1.4) is necessary to smooth the default choice after aggregation to a representative consumer. The aggregation is shown in appendix B.
1.3. A Model With Learning

The household side of the model is completed by outlining the supply of new housing stock. Builders construct new housing stock from the physical capital they rent from households, according to a production function \( h_s = f(k) = Ak^\alpha \). Profits earned by builders are returned to the households. The builders’ profit maximization problem is given by

\[
\max_k \left\{ \Pi(q_t, p_t, \tilde{k}) = q_t A\tilde{k}^\alpha - p_t \tilde{k} \right\} \implies p_t = \alpha q_t \tilde{k}^{\alpha-1} \quad (1.5)
\]

1.3.2 Lender Problem

The representative lender chooses the amount of credit to supply to the household at the note rate \( R_t \). The lender is assumed to have access to an outside source of funds (ie. international financial markets) from which it can borrow. In the event of default, a household’s stock of housing is transferred to the lender. The lender can recoup the value of the collateral in the housing market, however it faces delay in doing so. In particular, the lender can only sell a unit of the foreclosed housing stock in its possession with probability \( \mu \) each period. This assumption captures the presence of delay in the foreclosure process and allows the lender’s choice to depend on its beliefs about long-horizon events. The lender’s valuation of housing is also subject to a stochastic markdown, \( \phi_t \). \( \phi_t \) is assumed to follow a random walk in logs (which bounds \( \phi_t > 0 \))

\[
\log \phi_t = \log \phi_{t-1} + \epsilon^d_t \quad (1.6)
\]

which is known to the lender. The calibration of this process allows the model to match observed correlations between lagged house prices and household leverage. The lender’s profits are given by

\[
\Pi^L|\mathcal{H}_t, \mathcal{P} = -\tilde{b}_t + \beta (1 - \text{Pr}(\text{default}|\mathcal{P})) \cdot R_t \tilde{b}_t + ... \\
\text{Pr}(\text{default}|\mathcal{P}) \cdot \left[ \sum_{j=1}^{\infty} \beta^j (1 - \delta_h)^j \mu (1 - \mu)^{j-1} E^P_t (\phi_{t+j} q_{t+j}) h_t \right] 
\]

\(^{11}\phi_t \) can be thought of as capturing non-monetary costs/benefits to liquidating foreclosed housing in a given period (ie. a liquidity value).
where \( \mathcal{P} \) denotes the lender’s beliefs about future prices. Normalizing \( \Pi^L \) by the value of household’s housing stock, the profit maximization problem can be written

\[
\max_{\theta} \left\{ -\frac{\partial}{\partial \Pi^L_t} \left( \frac{q_{t+1}}{q_t} \right) + \beta \left( 1 - \Pr(\text{default}|\mathcal{P}) \right) \cdot \theta \Pi^L_t \left( \frac{q_{t+1}}{q_t} \right) + \ldots \right. \\
\Pr(\text{default}|\mathcal{P}) \cdot \left[ \sum_{j=1}^{\infty} \beta^j (1 - \delta_h)^j \mu (1 - \mu)^{j-1} \Pi^P_t \left( \phi_{t+j} \frac{q_{t+1}}{q_t} \right) \right] \right\} \tag{1.8}
\]

\[
\Rightarrow \theta^* = \theta(\theta_t) \tag{1.9}
\]

Where \( \theta \) is the value of the debt repayment \( R_t b_t \) as a fraction of the expected value of the household’s housing stock, \( \Pi^P(q_{t+1} h_t) \). In other words, conditional on its observation of \( (q_t, h_t) \) and its beliefs \( \mathcal{P} \), the lender’s choice of credit supply \( \tilde{b} \) is equivalent to determining the credit constraint that the household faces in (1.2). The lender’s choice of \( \theta_t \) implies that the perceived probability of default is given by

\[
\Pr(\text{default}|\mathcal{P}) = \Pr \left[ \log \left( \frac{q_{t+1}}{q_t} \right) - \log(q_{t+1}) \leq \log \left( \frac{\theta_t}{1 - \delta h} \right) + \log \left( \Pi^P_t \left( \frac{q_{t+1}}{q_t} \right) \right) \right] \tag{1.10}
\]

The lender’s credit supply choice is determined by the probability of default and the value of its claims to the collateral in the event of default. As can be seen in (1.8), delay in the sale of foreclosure inventory implies that the value of the lender’s claim to the housing stock depends on long-run forecasts of price growth. This introduces a degree of convexity in the supply of credit with respect to the belief \( \theta_t \).

### 1.3.3 Learning & Subjective Beliefs

Subjective beliefs are homogeneous across all agents in the economy. In order to impose discipline in learning, beliefs are specified so as to respect the internal consistency of agents’ decision problems. More precisely, an agent \( i \)’s decision rule must be an optimal response to their beliefs \( \mathcal{P}^i \) everywhere on and off the equilibrium path. This restricts the range of variables over which agents can make subjective forecasts. When agents make forecasts of their choice variables using a perceived law

\footnote{As the household’s problem is calibrated to ensure that (1.2) binds in each period, the household will always be willing to accept this contract.}

\footnote{Alternatively, this could be achieved through the presence of a foreclosure cost, however the assumed cost function would need to be a function of expected forward prices, which is not intuitive.}
of motion which they estimate from past data (as is the case in learning models where agents are uninformed about the process governing the path of their own choice variables in the future, an assumption that is present in much of the literature), the forecasted values will not in general be optimal responses to forecasted values of the other variables in the system. In such a case the agent is implicitly assumed to either be uninformed about their own decision problem or to be forecasting that they will make suboptimal decisions at some future history\textsuperscript{14}. Internal consistency of this kind requires that agents only hold subjective beliefs about variables outside of their decision set. In the model specified here agents are assumed to be uninformed about the true process governing house prices in the economy. Agents meet the standard of Internal Rationality (Adam and Marcet (2011)) in that their decision rules will be optimal conditional on their subjective beliefs. Furthermore, as prices are an equilibrium object, this formulation allows for feedback between subjective beliefs, which will be extrapolated from price data as a result of learning, and realized prices.

As has been widely studied in the learning literature, simple rule-of-thumb updating rules can capture the property that long-horizon price forecasts heavily weight recent price data, consistent with the evidence discussed in sections\textsuperscript{1} and \textsuperscript{2}. Such updating rules arise endogenously from Bayesian learning of parsimonious hidden state models. Agents in the model perceive that prices are generated by the following data generating process (DGP)

\[
\begin{align*}
\ln \frac{q_t}{q_{t-1}} &= \ln \omega_t + \epsilon_t^q \\
\ln \omega_t &= \ln \omega_{t-1} + \epsilon_t^\omega \\
\epsilon_t^q, \epsilon_t^\omega &\sim iid \sim N \left( 0, \begin{pmatrix} \sigma^2_q & 0 \\ 0 & \sigma^2_\omega \end{pmatrix} \right)
\end{align*}
\tag{1.11}
\]

where the persistent component of price growth, \(\log \omega_t\), is a hidden state variable. Agents observe the realization of \(\ln \frac{q_t}{q_{t-1}}\) and learn by updating beliefs about the distribution of \(\ln \omega_t\). The choice of perceived DGP follows a number of papers in the asset pricing learning literature, including Adam, Kuang, and Marcet (2011) and Adam, Beutel, and Marcet (2014). Optimal updating of (1.11) implies patterns of forecast errors in prices consistent the evidence discussed in section Ch1-sec:Background.

Under stationary Bayesian learning, the household’s posterior beliefs about \(\ln \omega_t\) are given by

\[
\ln \omega_t \sim N \left( \ln m_t, \sigma_0(\sigma_q, \sigma_\omega)^2 \right)
\tag{1.12}
\]

\textsuperscript{14}See Preston (2005) or Eusepi and Preston (2011) for discussion.
1.3. A Model With Learning

\[
\sigma_0^2 = -\sigma_0^2 + \sqrt{\sigma_0^4 + 4\sigma_q^2\sigma_\omega^2} \quad (1.13)
\]

The stationary Kalman filtering equations imply the constant gain algorithm for updating beliefs about the posterior mean, \( \ln m_t \)

\[
\ln m_t = \ln m_{t-1} + g(\sigma_q, \sigma_\omega) \cdot \left( \ln \frac{q_t}{q_{t-1}} - \ln m_{t-1} \right) \quad (1.14)
\]

\[
g(\sigma_q, \sigma_\omega) = \frac{\sigma_0(\sigma_q, \sigma_\omega)^2}{\sigma_q^2} \quad (1.15)
\]

Under constant gain updating (1.14), posterior beliefs will be a weighted average of past price growth observations. The gain parameter \( g \), which controls the weight agents place on new price data when forming beliefs, is equivalent to the inverse of the rate at which old observations are discounted over time.\(^{15}\)

Given this belief structure, \( \mathbb{E}_t^P (q_{t+j}/q_t) \) in (1.8) can be written as

\[
\mathbb{E}_t^P \left( \frac{q_{t+j}}{q_t} \right) = \exp \left( j \log(m_t) + \frac{1}{2} j^2 \sigma_0^2 \right) \cdot \exp \left( \frac{1}{2} j \sigma_q^2 \right) \exp \left( \frac{1}{2} \sigma_\omega^2 \sum_{s=1}^{j} s^2 \right) \quad (1.16)
\]

A shift in beliefs \( m_t \) influences the lenders’ supply of credit via its effect on perceived default probabilities and the expected values of the lenders’ claims on housing

\[
m_t \uparrow \implies \left\{ \begin{array}{c} \mathbb{P}_t^\text{P}(\text{Default}) \downarrow \\ \mathbb{E}_t^P \left( \Pi_{t+1}^{L,\text{default}} \right) \uparrow \end{array} \right\} \implies \tilde{b}_t, \theta_t \uparrow
\]

The household’s Euler equation for housing is given by

\[
q_t = \frac{u_h(t)}{u_c(t)} + \mathbb{E}_t^P \left[ \left( \beta(1-\delta) \cdot \frac{\lambda_{t+1}^i}{\lambda_t^i} \cdot \Gamma_{t+1} + \frac{\theta_t}{R} \right) \cdot q_{t+1} - \beta \theta_t (\mathbb{E}_t^P q_{t+1}) \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} \right] \quad (1.17)
\]

\(^{15}\)The constant gain makes this a model of perpetual learning. Even when the DGP (1.11) is correctly specified, discounting of past data implies that \( m_t \) will not converge in levels. In this case, however, beliefs should be ergodically distributed around the rational expectations equilibrium when the gain is small (see Evans and Honkapohja (2001)).
1.3. A Model With Learning

where \( \Gamma_{t+1} = 1 - \Pr(\text{default}|\mathcal{P}) \) in (1.10). Increases in the economy-wide posterior mean of the permanent component of price growth, \( \log m_t \), directly influence the household’s housing demand via three complementary channels: (i) increasing expected capital gains on housing, (ii) increasing the supply of credit available to the household, and (iii) decreasing the household’s expected default probability. As neither the lender nor the household understand the correct mapping between fundamentals and prices, it is assumed that neither agent is able to account for the effect of their actions on future beliefs. In other words, when forecasting future prices the agents also do not internalize the effect of future price movements on \( m_t \).

A few comments should be made at this point about the assumption that beliefs in the model are homogeneous across all agents. Heterogeneity could be incorporated into the model in a number of different ways: (i) households and lenders could be assumed to have the same perceived data generating process for prices (1.11) but differ with respect to rate at which they discount the past (ie. \( g \) varies between households and lenders), (ii) the perceived data generating process for prices could differ between households and lenders, or (iii) heterogeneity could be assumed in the beliefs amongst households and lenders. In the case of (i), the dynamics of the model would be equivalent to formulation considered here with a different overall gain. In the case of (ii), a general comment about the effect on the model’s dynamics cannot be made, however the available evidence does not support the conclusion that financial institutions and households hold structurally different beliefs about house prices. Finally, in the case of (iii), the model considered here abstracts from a transaction margin in housing and features an intentionally sparse contracting problem. This is done so as to retain a focus on aggregate price movements. As a result, it is not a good framework for considering heterogeneity in beliefs between households and lenders.

1.3.4 Equilibrium Under Learning

The equilibrium concept of the model is an Internally Rational Expectations Equilibrium (IREE), formalized by Adam and Marcet (2011). An Internally Rational Expectations Equilibrium for this

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16 This is commonly referred to as the anticipated utility assumption (see Kreps (1998) and Sargent (1999)). In practice this assumption does not have a significant effect on the results presented in section 1.6 and is made for computational ease.

17 See Cheng, Raina, and Xiong (2012)

18 Note, Burnside, Eichenbaum, and Rebelo (2011) consider a housing model with heterogeneity between households. One difficulty facing frameworks such as this is calibrating the distribution of prior beliefs across agents.
1.3. A Model With Learning

An economy is characterized by:

1. A probability measure $P^i$ representing an agent’s beliefs over $\Omega_s$, where $\Omega_s$ denotes the space of realizations of variables exogenous to an agent.

2. A sequence of equilibrium prices $\{p_t^*, q_t^*\}_{t=0}^\infty$ where $p_t^*, q_t^* : \Omega_S^t \to R^N$ for all $t$, all realizations in $\Omega_S$ almost surely in $P^i$.

3. A sequence of choice functions $\{c_t^i, h_t^i, b_t^i, \bar{k}_t^i, \bar{b}_t^i, \bar{h}_t^i\}_{t=0}^\infty$ that maximize agent $i$’s objective function conditional on $P^i$. All agents $i = 1, ..., I$ are internally rational.

The IREE is closely related to the Self-Confirming Equilibrium concept in Fudenberg and Levine (1998). The key difference is that in a self-confirming equilibrium an agent’s beliefs only need to be consistent with observations on the equilibrium path (ie. individuals do not observe behavior that contradicts their subjective beliefs). An IREE can also coincide with a rational expectations equilibrium. In particular, a rational expectations equilibrium is an IREE in which subjective beliefs $P^i$ coincide with the objective probability distribution. Under rational expectations agents infer the correct process for prices from their knowledge of the system.

Trivially, in order for the model to sustain an IREE in which $P^i$ deviates from the objective distribution, agents must not have access to the full information set available to agents in a rational expectations environment. In practice this implies that agents must be unaware of some equation(s) or identity(ies) in the rational expectations equilibrium. Agents in the learning model are assumed to be uninformed about the preferences and beliefs of other agents. Hence, they are uninformed about the mechanism that links prices to state variables (computationally, this is equivalent to a representative agent who does not have the market clearing condition for housing in their information set). When households and lenders enter the marketplace they are unaware of how the prices they observe relate to the fundamentals of the housing market. As a result agents hold a subjective belief about the evolution of house prices and make decisions taking as given the price prevailing in the market and their own beliefs. It is important to emphasize, however, that while agents are unaware of the market clearing condition for housing, the house price that realizes is the price that clears the housing market given agents’ beliefs and choices.

The model is closed by specifying the market clearing condition for housing. The solution to the household’s problem yields a housing demand equation $h_t^d(h^d_{t-1}, k_{t-1}, b_{t-1}, q_t, m_t, \theta_t, y_t|\mathcal{P})$. The
market clearing price $q_t^*$ is determined by the identity

$$h^d(h^d_{t-1}, k_{t-1}, b_{t-1}, q_t^*, m_t, \theta_t, y_t | P) - (1 - \delta_h)(1 - D_t)h^d_{t-1} = A k_{t-1}^\alpha + \mu(1 - \delta_h) \cdot (D_t h^d_{t-1} + F_{t-1})$$  

(1.18)

where $D_t$ is the proportion of households who default in period $t$ and $F_{t-1}$ is the inventory of foreclosed housing that the lender holds at the end of period $t - 1$. The left hand size of (1.18) denotes new purchases of housing after default choices are made. The amount of housing made available for sale is given by the sum of newly constructed housing and the proportion $\mu$ of the foreclosure stock the lender is able to liquidate. The law of motion for $F$ is given by

$$F_t = (1 - \mu)(1 - \delta_h)(F_{t-1} + D_t h_{t-1})$$  

(1.19)

The household’s problem is solved via a form of parameterized expectations using spectral methods. The details of the solution method can be found in appendix C. The lender’s decision rule is approximated by a simple interpolation of the solution to (1.8). Given these two approximations the market clearing prices can be solved for any state vector via (1.18).

### 1.4 Calibration

The complete set of calibration results can be found in table 1.1. Exogenous variation in the model comes from the endowment process $y_t$ and mortgage rate $R_t$. The endowment is estimated as a log AR(1) process using detrended wages and salaries compensation data from the Bureau of Economic Analysis (BEA). The mortgage rate series is taken from Freddie Mac’s 30-year fixed mortgage average for the United States. An Augmented Dickey-Fuller test on the series does not reject the hypothesis of a unit root, and the mortgage rate process is estimated as being a random walk in logs.

The delay in liquidating foreclosed housing, $\mu$, is set so that foreclosure stock as a fraction of total

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19 Appendix H lists the data sources used for this chapter.
20 The wages and salaries data is detrended using a bandpass filter. An AR(1) process is specified to limit the number of state variables in the model for computational ease.
21 The model presented in section 1.3 is a zero trend growth environment with a well-defined stationary steady state, hence the model is simulated with shocks to detrended incomes.
22 When simulating the model, shocks to $y$ and $R$ are correlated. The correlation is estimated from measured shocks in the data.
1.4. Calibration

housing in steady state equals its 1996 value in the National Delinquency Survey of the Mortgage Bankers Association. The determination of credit supply in the model implies a correlation between a weighted average of past prices \( m_t \) and household leverage. The length and volatility of the lenders’ markdown shock \( \{ \epsilon_s^t \}_{s=1}^T \) are chosen so as to match this correlation as measured from the data, as well as the level of loan-to-value (LTV) ratios on US mortgages. Using the Federal Housing Finance Agency’s (FHFA) All-Transaction House Prices Series for the United States, a sequence \( \hat{m}_t(g) \) is estimated using (1.14). Household leverage is measured using net mortgage liabilities from the Federal Reserve Financial Accounts as a fraction of the aggregate market value of non-farm residential homes, taken from Heathcote and Davis (2007). The level of the series is adjusted to match the mean LTV ratio on US mortgages in 1996 Q1, measured in the American Housing Survey (AHS). The parameters are set so that (i) the correlation between these two series matches the implied correlation of the lender’s choice of \( \theta_t \) with \( m_t \) in the model, and (ii) steady state \( \theta \) matches the LTV ratio in the US in 1996 Q1.

The parameters governing the liquidity shock, \( \zeta \) and \( \sigma_\zeta \), in part control the level of default in the model as well as its volatility. The pair \( (\zeta, \sigma_\zeta) \) are set so that (i) the level of default in steady state matches the aggregate delinquency rate on single-family residential mortgages in the US in 1996 Q1 (measured by the St. Louis Federal Reserve), and (ii) the elasticity of default with respect to \( \theta \) matches an estimated elasticity of the delinquency rate with respect to household leverage from 1992 Q1 to 2014 Q1.

The gain parameter determines both the size and persistence of the response of beliefs to changes in prices. It is therefore key in governing the dynamics of the model. The beliefs are calibrated so as to match forecast errors taken from the data as follows. The perceived DGP for house price growth (1.11) implies the relationship

\[
\text{Var} \left( \log \frac{q_t}{q_{t-1}} - \log \frac{q_{t-1}}{q_{t-2}} \right) = f (\sigma_q, \sigma_\omega) \tag{1.20}
\]

This identity, together with the identities (1.13) and (1.15) implies a relationship

\[
\sigma(g) = (\sigma_0(g), \sigma_q(g), \sigma_\omega(g)) \tag{1.21}
\]
Therefore, the choice of $g$ together with the variance in (1.20) implies the values of the prior variances $\sigma_q$ and $\sigma_\omega$. In order to choose $g$ the left hand side of (1.20) is measured from the FHFA house price series\(^{23}\) and the model is simulated over a grid of $g$ values (where for each $g$ the priors are set according to $\sigma(g)$) using shocks to $y_t$ and $R_t$ measured in the data. The chosen gain is that which minimizes the sum of squared errors between the vector of model-implied one-quarter-ahead forecast errors $\log \frac{q_t}{q_{t-1}} - \mathbb{E}^P_{t-1} \log \frac{q_t}{q_{t-1}}$ and a data analog of this series. Forecast error data is constructed using the prices of futures contracts on the S&P Case-Shiller home price index, which trade on the Chicago Mercantile Exchange. The futures prices can be thought of as a measure of the market’s expectations about house prices\(^{24}\). The calibrated gain is 0.014. This value is slightly smaller than the quarterly-implied gain parameter estimated in Adam, Kuang, and Marcet (2011) and sits within the range of values typically found in the learning literature\(^{25}\).

The parameter $\gamma_c$ in the utility function is equal to the consumption share of disposable income. This is set to 0.558 using BEA data on personal consumption expenditures\(^{26}\). The elasticity of housing supply in the model is given by $\alpha/(1 - \alpha)$. The parameter $\alpha$ is set using Saiz (2010), who provides estimates of local-level supply elasticities computed using data on land availability at the MSA level. The discount factor $\beta$ is set to 0.96 so that the borrowing constraint binds in each period. In order to ensure a stable solution to the household’s problem a compromise has to be made in the calibration of the depreciation rate, $\delta_h$, which is set to the relatively high value of 0.06.

It should be noted, however, that this compromise serves to dampen rather than accentuate price volatility during model simulations.

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\(^{23}\)The level of the house price series is set using the Census Bureau’s 2005 American Community Survey.


\(^{25}\)See Adam, Kuang, and Marcet (2011). It should be noted that the size of gains found in the learning literature vary with respect to the setting concerned. One should not necessarily expect estimated/calibrated gains to be the same in models where agents learn about asset prices as in models where they learn about economy wide output or wages for example, as it is reasonable to assume that agents’ information about these objects differ.

\(^{26}\)$\gamma_c$ is set equal to the 1999-2012 mean of the sum of the GDP shares for personal consumption expenditures on durables, nondurables, and services, minus the GDP share of personal consumption expenditures on housing services (imputed rental of owner-occupied non-farm housing).
1.5 Analytic Results

When both households and the suppliers of credit use subjective beliefs to forecast price movements, learning creates complementarities between the two sides of the market. This departure from standard demand-side learning frameworks bolsters the internal propagation mechanisms of the model. In order to demonstrate the difficulty in generating price booms with demand-side learning, consider the model outlined in Section 1.3 without the lender’s problem discussed in 1.3.2. Such a model is similar to the demand-side learning models of Adam, Kuang, and Marcet (2011); Adam, Beutel, and Marcet (2014); and Gelain and Lansing (2013)\(^{27}\) Self-confirming deviations in prices occur through a simple feedback mechanism\(^{28}\)

\[
q_t \uparrow \implies m_t \uparrow \implies E_t^P[\text{Capital Gain}] \implies q_{t+1} \uparrow \implies m_{t+1} \uparrow
\]

\(^{27}\)Such a setting can also be related to learning frameworks used to explain stock price volatilities. Winkler (2015) considers an environment in which both investors and firms learn the stock price of the firm. Investors are concerned about capital gains on their investments while firms have a debt financing constraint that depends upon their market value. This can be considered a decentralization of the setting considered here.

\(^{28}\)Note that shifts in expected capital gains operate on the household choice through two channels: (i) through changes in the expected resale value of housing stock, and (ii) through their effect on the household’s credit constraint.
1.5. Analytic Results

In order to generate large and persistent price growth without relying upon a rich set of shocks to fundamentals, such a mechanism requires subjective beliefs to exhibit two properties. First, beliefs must be sufficiently responsive to price changes that the resulting response of demand drives subsequent price increases. Second, the belief process \( m_t \) itself must be highly persistent. The trade-off between the two can be illustrated by deriving a law of motion for \( m_t \). Writing the household’s Euler equation (1.17) in simplified form

\[
q_t = \Theta_t + q_t m_t E^P_t [\rho_{t+1}] \tag{1.22}
\]

This implies

\[
q_t = \frac{\Theta_t}{1 - m_t E^P_t [\rho_{t+1}]} \tag{1.23}
\]

\[
\Rightarrow \log \left( \frac{q_t}{q_{t-1}} \right) = \log \left( \frac{\Theta_t}{\Theta_{t-1}} \right) + \log \left( \frac{1 - m_{t-1} E^P_{t-1} [\rho_t]}{1 - m_t E^P_t [\rho_{t+1}]} \right) \tag{1.24}
\]

Substituting (1.24) into (1.14) yields

\[
\log m_t = (1 - g) \log m_{t-1} + g \log \left( \frac{\Theta_t}{\Theta_{t-1}} \right) + g \log \left( \frac{1 - m_{t-1} E^P_{t-1} [\rho_t]}{1 - m_t E^P_t [\rho_{t+1}]} \right) \tag{1.25}
\]

linearizing this equation yields:

\[
\ln m_t \approx \frac{g}{1 - \rho - g\rho} \left\{ (1 - \bar{\rho}) \ln \left( \frac{\Theta_t}{\Theta_{t-1}} \right) + \bar{\rho} \bar{e}_t \ln \left( \frac{\rho_{t+1}}{\rho_t} \right) \right\} + \frac{\equiv^P}{1 - \rho - g\rho} \ln m_{t-1} \tag{1.26}
\]

Where

\[
\Theta_t = \frac{u_h(t)}{u_c(t)}
\]

\[
\rho_{t+1} = \left( \beta(1 - \delta) \frac{\lambda_{t+1}}{\lambda_t} \Gamma_{t+1} + \theta_t \frac{1}{R} \right) \cdot \Sigma_{t+1} - \beta \theta_t \left( \Sigma_{t+1} \frac{\lambda_{t+1}}{\lambda_t} \Gamma_{t+1} \right)
\]

\[
\Sigma_{t+1} = \exp \left( \sigma_0 \epsilon_{t+1} + \epsilon_t^q + \epsilon_{t+1}^c \right) \quad \epsilon_t \sim N(0, 1)
\]
Note that the persistence parameter $P$ is downward sloping with respect to $g$

$$\frac{dP}{dg} = \frac{-\bar{\rho}^2 + 2\bar{\rho} - 1}{(1 - \bar{\rho} - g\bar{\rho})^2} \begin{cases} < 0 \text{ if } \bar{\rho} \neq 1 \\ = 0 \text{ else} \end{cases} \tag{1.27}$$

The law of motion (1.26) makes clear that in the absence of highly persistent shocks or strong internal propagation mechanisms in the model, persistent growth in $m_t$ requires a relatively large value for $P$. Given (1.27), this can be achieved by lowering the value of the gain parameter. The gain parameter, however, determines the weight that agents place on new information when updating beliefs. Hence, a reduction in $g$ diminishes the responsiveness of beliefs to price changes. The trade-off is illustrated in figure [1.3], which shows the response of log prices to a wage shock in the demand-side learning environment. When the gain is low the shift in beliefs after the shock hits is insufficient for the effect of higher expected capital gains to outweigh the effect of the wage shock dying out. As a result, the shock does not propagate. In contrast, under high gain calibrations the shock is propagated through prices. However, owing to household beliefs placing a relatively larger weight on current shocks (represented through the $\Theta_t$ and $\rho_t$ terms in (1.26)), once the wage shock dies off $m_t$ is quick to readjust to fundamentals and $q_t$ returns to steady state faster than was the case under a low gain. This tight trade-off between the persistence and responsiveness of beliefs under demand-side learning implies that such frameworks can struggle to generate the kind of sustained price growth seen in the data without a similarly persistent set of shocks.

The credit supply problem in 1.3.2 introduces a complementary learning mechanism into the model. Shifts in expected log price growth, $\log m_t$, lead the lender to increase credit supply due to lower perceived default risk and higher expected payoffs in the event of default. As households are constrained this increases their demand for housing. As before, the shift in $m_t$ also pushes up the demand for housing through its effect on expected capital gains. Hence, shifts in subjective beliefs give rise to credit supply changes that complement the effect of demand-side learning on prices. The parallel channels through which learning operates in the full model can be illustrated.
as follows:

$$q \uparrow \Rightarrow m_t \uparrow \Rightarrow \begin{cases} \text{Household:} & \mathbb{E}_t^P[\text{Capital Gain}] \uparrow \\ \text{Lender:} & \text{Pr}_t^P(\text{Default}) \downarrow \\ & \mathbb{E}_t^P(\text{Default Value}) \uparrow \end{cases} \Rightarrow \theta_t \uparrow \Rightarrow \text{Demand} \uparrow \Rightarrow q \uparrow$$

Intuitively, the introduction of learning on the credit supply side should increase the persistence in the belief process. Following the an initial shock to prices and a shift in $m_t$, the subsequent change in prices should be greater as there is both a demand and credit supply response to the belief shift. This will imply that the change in $m_t$ in the second period following the shock will be greater than under demand-side learning and so on. As a result, for any level of responsiveness of beliefs to shocks the propagation cycle should be longer.

In order to explicitly show the influence of this parallel mechanism on the dynamics of household beliefs an approximate law of motion for $m_t$ is derived by combining (1.25) with the lender’s decision...
1.6 Quantitative Results

rule \( \theta(m_t) \). Taking a linear approximation yields

\[
\log m_t \approx \left\{ \begin{array}{l}
1 - g \left( 1 + \mu_1 + \mu_2 \frac{\partial \theta}{\partial m} \right) \\
1 - g \left( \mu_1 + \mu_2 \frac{\partial \theta}{\partial m} \right)
\end{array} \right. \log m_{t-1} - 1 + \frac{\partial g}{\partial m} \cdot \log m_{t-1}
\]

\[
\exp. \text{ capital gains+updating+credit supply} + \mathbb{E}_t \left[ D(g, \bar{\theta}, R, \bar{\rho}, \gamma_c) \cdot \left( \log \frac{q_t}{q_{t-2}} \right) \right]
\]

where \( \mu_1, \mu_2 < 0 \). The autoregressive coefficient \( P' \) now has the following properties

\[
\frac{dP'}{d\theta} > 0 ; \quad \frac{d\left( \text{abs} \left( \frac{dP' \gamma}{dg} \right) \right)}{d\theta} < 0
\]

While the autoregressive coefficient is decreasing in the gain as before, the effect of learning in credit supply on beliefs is clear. First, increasing the elasticity of credit supply with respect to \( m_t \) increases the persistence in \( m_t \) conditional on the gain. Second, as the elasticity of credit supply with respect to \( m_t \) increases the influence of the gain on persistence \( P' \) decreases. This is the key influence of the complementarity induced by two-sided learning. If lenders are sufficiently responsive to their beliefs in the full model, the trade-off in trying to generate beliefs that are both persistent and sensitive to price changes that exists in demand-side learning frameworks can be broken. Section 1.6 tests whether this is indeed the case in the calibrated model.

1.6 Quantitative Results

Following a number of papers in the literature the constant gain algorithm is modified for model simulations as follows:

\[
\ln m_t = \ln m_{t-1} + g(\sigma_q, \sigma_\omega) \cdot \left( \ln \frac{q_{t-1}}{q_{t-2}} - \ln m_{t-1} \right)
\]

\[
30 \text{Combing the (1.24) with (1.3) and (1.2) yields}
\]

\[
\log \frac{q_t}{q_{t-1}} = \mu_1 \log \frac{m_t}{m_{t-1}} + \mu_2 \bar{\theta} \log \frac{\theta_t}{\theta_{t-1}} + \mu_3 \log \frac{h_t}{h_{t-1}} + \mu_4 \log \frac{\Gamma_{t+1}}{\Gamma_t} + \mu_5 \log \frac{\Sigma_{t+1}}{\Sigma_t}
\]

\[
\mu_0 = 1 - \rho - \left( \frac{1 - \gamma_c}{\gamma_c} \right) \cdot \left( \bar{\theta} - \delta \right) ; \quad \mu_1 = \frac{1}{\mu_0} \cdot \left( \rho + \bar{\theta} \cdot \left( \frac{1 - \gamma_c}{\gamma_c} \right) \right) ; \quad \mu_2 = \frac{1}{\mu_0} \cdot \frac{1}{R} \cdot \left( \frac{1 - \gamma_c}{\gamma_c} \right)
\]

\[
\mu_3 = -\frac{1}{\mu_0} \cdot \left( 1 - \rho + \left( 1 - \bar{\theta} \right) \cdot \left( -\frac{1 - \gamma_c}{\gamma_c} \right) \right) ; \quad \mu_4 = \frac{1}{\mu_0} \cdot \left( \rho - \bar{\theta} \right) \mu_5 = \frac{\rho}{\mu_0}
\]
1.6. Quantitative Results

This assumption avoids simultaneity between the determination of prices and beliefs, and significantly speeds up the computation\textsuperscript{31}. Furthermore, in order to guarantee stability a constraint is imposed on the lenders credit supply choice when simulating the model: $\theta_t \leq \bar{\theta}$. This can be conceptualized as a regulatory constraint on LTV ratios. In practice, the model does not hit the constraint when simulating at business cycle frequencies. $\bar{\theta}$ is set equal to 1.05.

1.6.1 Interest Rates & Boom-Bust in House Prices

Figure 1.4: Response of Log House Prices to Interest Rate Drop

The chapter now considers the model’s ability to endogenously generate persistent growth in house prices consistent with the observed boom in US housing markets in the mid-2000s. Given the paucity of empirical evidence that identifies significant trends in housing market fundamentals during this period, it is desirable that models should be able to generate persistent price growth following a small set of shocks. This section evaluates the potential for interest rate movements in particular to generate boom-bust periods in the calibrated model.

In recent years there has been wide discussion about the extent to which monetary policy contributed to the 2008 financial crisis in general and to the house price boom more particularly. An argument commonly advanced in both the popular and academic literature is that persistently

\textsuperscript{31}See Adam, Kuang, and Marcet (2011) and Eusepi and Preston (2011).
low interest rates encouraged excessive borrowing through the early 2000s. The ensuing effect of the credit expansion on demand for housing may have in turn generated appreciation in house prices over a near-10-year period. A growing empirical literature links interest rate movements with periods of financial instability. Hott and Jokipii (2012) show that over the past 30 years, across a sample of 14 OECD countries, periods of low interest rates Granger-cause deviations of house prices from fundamentals-implied levels (the authors’ characterization of a bubble). In a similar vein, Ahearne et al. (2005) show that across advanced economies house price bubbles tend to be preceded by a period of loosening monetary policy.

The early 2000s saw a period of abrupt decreases in mortgage rates across the US economy. Beginning in late 2000 the 30-year conventional mortgage rate in the US began a 3% drop, and thereafter remained relatively low until 2006. The rate drop coincided with an acceleration in the aggregate house price index for the US. In the environment presented in section 1.3.1, such a rate decrease not only drops the borrowing costs that households face, but also serves to relax their credit constraints. As discussed in section 1.5, the learning framework considered here gives the model strong internal propagation mechanisms by allowing for potentially large persistence in subjective beliefs without sacrificing the responsiveness of these beliefs to new information.

In order to investigate whether the calibrated model can explain the pattern of house prices in the 2000s, the effect of an unanticipated drop in \( R \) is considered. The model is simulated from steady state with an initial interest rate set equal to the mean 30-year conventional mortgage rate in the US from 1996 Q1 to 2000 Q4. The drop in \( R \) is calibrated to match the mean US mortgage rate from 2001 Q1 to 2006 Q4.\(^{32}\) Figure 1.4 plots the response of log prices to the unanticipated rate drop, together with both the response of prices when learning is restricted to the demand side of the housing market\(^{33}\) and the actual path of the FHFA All-Transaction House Price Index. Under demand-side learning the shift in the distribution of expected capital gains is insufficient to subsequently generate large shifts in housing demand. As a result the propagation of the interest rate shock in house prices is negligible. In the full model the shift in beliefs \( m_t \) following the initial increase in prices generates an increase in credit supply relative to the market value of housing. As

\(^{32}\)This follows from an exercise carried out in Adam, Kuang, and Marcet (2011). A similar exercise can be found in Kuang (2014).

\(^{33}\)In the “demand-side learning model” considered from here on, only households are assumed to have the beliefs specified in section 1.3.3.
is clear in figure [1.4] this additional mechanism has a dramatic influence on the evolution of prices following the shock. The model can account for the full appreciation in US house prices in the early 2000s with total growth between 2001 Q1 and 2006 Q4 slightly overstating the level observed in the data. Furthermore, the model can explain much of the persistence in prices following 2001. Following 2001 Q1, price growth persists for 20 quarters in the model, compared with 22 quarters in the data series. Importantly, the model is also capable of capturing asymmetry in boom-bust cycles. Following the peak in the simulated price series in figure [1.4] prices collapse to the steady state level within 12 quarters.

Figure 1.5: Response of Log House Prices and Credit to Interest Rate Drop

(a) House Price and Absolute Value of the 1-Quarter-Ahead Forecast Error

(b) Leverage

Two comments should be made about this result. First, the literature on learning has not yet reached a settled view on how to appropriately calibrate gain parameters in constant gain learning. As a result a wide range of values are found in the literature and the estimated gain in this chapter sits in the high end of this spectrum. Second, as noted in section [1.4] the calibration of the model suffers from the necessity of imposing a relatively high value for $\delta_h$. In order to gauge the sensitivity of the price response to these assumptions, the previous exercise was carried out over a grid of $g$ and $\delta_h^{34}$. Figure A.1 in appendix A plots the maximum deviation of log prices

34 Note that for the different values of $g$ the priors and parameters of the lender’s problem were recalculated as in section [1.4]
1.6. Quantitative Results

from steady state following the $R$ drop as well as the length of the propagation period (the number of quarters of positive price growth following the shock) over $(g, \delta_h)$. Conditional on a given $\delta_h$, decreasing the gain parameter actually increases the size of price growth following the interest rate shock. Similarly, increasing the depreciation rate on housing serves to augment rather than dampen price growth in the model. In both cases, changes in $\delta_h$ or $g$ from the values listed in table 1.1 have an ambiguous effect on the length of the propagation period. However, the finding that price growth persists for many quarters following the interest rate shock is robust to perturbations of either parameter in the neighborhood of the calibrated values.

As discussed in section 1.1, the available data on price forecasts suggest that agents’ forecast errors tend to be largest following turning points in the price series. This is consistent with the model’s response to the $R$ shock. The left hand panel of figure 1.5 plots the absolute value of the forecast error of log price growth in the model following the unanticipated rate drop. The large spike in forecast errors at the point of the interest rate drop drives the initial appreciation in expected price growth. As beliefs adjust to higher prices the magnitude of the forecast errors decreases. This slows the growth in $m_t$, and as a result growth in $q_t$, until forecast errors go to zero and the model hits a turning point. The ensuing collapse in the simulated prices and beliefs $m_t$ is driven by a series of large negative forecast errors. The model also captures comovements in house prices and household leverage.

The right hand panel in figure 1.5 plots a simulated series of household leverage measures. Because debt contracts are one period in length and the framework does not feature an occasionally binding constraint, the model abstracts from some dynamics of household debt. In order to make the data and model series comparable, the right hand panel in figure 1.5 plots net new mortgage borrowing as a fraction of the market value of housing stock in both the data and the model. Consistent with evidence provided in figure 1.2, the model generates increases in credit supply and household leverage concurrent with the takeoff in house prices. Under demand-side learning the growth in leverage is negligible. By contrast the full model can generate almost a third of the growth in the leverage measure seen in the data. Because the lender’s problem is highly stylized, the leverage series tracks house prices closely following the shock. This is the result of the lenders’ choice being predominantly a function of $m_t$ through the lender’s credit supply problem

35Note, the series are normalized to 2001 Q1.
1.6. Quantitative Results

outlined in section 1.3.2. Nevertheless, the framework suggests a potentially powerful mechanism for endogenously generating liberalizations in credit markets during price booms.

Table 1.2: Business Cycle Moments

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Learning</th>
<th>Rational</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1978Q1 - 1978Q1</td>
<td>Demand-Side Learning</td>
<td>Full Learning Model</td>
</tr>
<tr>
<td></td>
<td>2014Q1 - 1990Q1</td>
<td>Learning</td>
<td></td>
</tr>
</tbody>
</table>

|                      |                 | 1.715          | 2.452          | 0.598          |
|                      |                 | 1.770          | 2.480          | 0.607          |
|                      |                 | 0.556          | 0.652          | 0.334          |
|                      |                 | 0.849          | 0.506          |                |
|                      |                 | 1.943          | 9 \times 10^{-4} |                |

Reported moments are for log values. Simulation moments are taken from simulated sample of size 50000 quarters, with the first 1000 quarters dropped as burn-in. Reported data moments of one-quarter-ahead forecast errors are from CME data sample running from 2007Q1 to 2011Q1. Wage data, y, is detrended using a bandpass filter with frequency range 1/32 to 1/8 cycles per quarter.

1.6.2 Capturing Cyclical Variation in Housing Markets

In order to test whether the results in section 1.6.1 come at the expense of the the model’s ability to capture normal cyclical variation in prices and forecast errors, several large-sample simulations are carried out. Table 1.2 shows moments for simulations of the full model as well as for the model when learning only takes place on the demand side of the housing market, and the model under rational expectations. Note that the presence of the mid-2000s price boom has a large effect on
measured volatilities. Table 1.2 therefore includes data moments for a pre-boom sample covering 1978Q1 - 1990Q1 in order to gauge moments during ‘normal’ periods. Under rational expectations neither prices nor the market value of housing display anything like the volatility seen in the data. The standard deviation of each of these series, relative to wages, is only about 20-25% of that seen in the pre-boom sample. Furthermore, under rational expectations the relative standard deviation of the one-quarter-ahead forecast error of house price growth is a quarter of that measured in the data.

Table 1.3: Business Cycle Correlations

<table>
<thead>
<tr>
<th></th>
<th>Data Learning</th>
<th>Demand-Side Learning</th>
<th>Full Learning Model</th>
<th>Rational Expectations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{corr}(\cdot, y))</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(q)</td>
<td>0.234</td>
<td>0.438</td>
<td>0.362</td>
<td>0.964</td>
</tr>
<tr>
<td>(q \cdot h)</td>
<td>0.268</td>
<td>0.437</td>
<td>0.362</td>
<td>0.964</td>
</tr>
<tr>
<td>(fe(q/q_{-1}))</td>
<td>-0.613</td>
<td>0.054</td>
<td>0.060</td>
<td>0.319</td>
</tr>
<tr>
<td>(F)</td>
<td>\cdot</td>
<td>0.062</td>
<td>0.260</td>
<td>-0.999</td>
</tr>
<tr>
<td>(\theta)</td>
<td>-0.143</td>
<td>\cdot</td>
<td>0.315</td>
<td>\approx 0</td>
</tr>
<tr>
<td>(\text{corr}(\cdot, R))</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(q)</td>
<td>-0.586</td>
<td>-0.770</td>
<td>-0.766</td>
<td>0.219</td>
</tr>
<tr>
<td>(q \cdot h)</td>
<td>-0.492</td>
<td>-0.772</td>
<td>-0.769</td>
<td>0.218</td>
</tr>
<tr>
<td>(fe(q/q_{-1}))</td>
<td>-0.433</td>
<td>-0.194</td>
<td>-0.237</td>
<td>0.113</td>
</tr>
<tr>
<td>(F)</td>
<td>\cdot</td>
<td>0.225</td>
<td>-0.506</td>
<td>-0.142</td>
</tr>
<tr>
<td>(\theta)</td>
<td>-0.869</td>
<td>\cdot</td>
<td>-0.694</td>
<td>-0.215</td>
</tr>
</tbody>
</table>

Reported moments are for log values. Simulation moments are taken from simulated sample of size 50000 quarters, with the first 1000 quarters dropped as burn-in. Reported data moments of one-quarter-ahead forecast errors are from CME data sample running from 2007Q1 to 2011Q1. Wage data, \(y\), is detrended using a bandpass filter with frequency range 1/32 to 1/8 cycles per quarter.

The introduction of bounded rationality on the demand side of the market significantly increases volatilities in the model, however the relative standard deviation of prices and the market value of housing remain 31% and 40% below their pre-boom data values. By contrast, the full two-sided
1.6. Quantitative Results

The model is able to capture almost all the observed volatility in prices and the vast majority the relative standard deviation in the market value of housing. The volatility of the housing stock is relatively low in the model. This is a result of the very simple framework governing the construction of new housing stock. As a result the model performs better in explaining price variation than it does variation in the market value of total housing stock. Introducing interaction between near-rational households and near-rational lenders similarly increases the volatility of forecast errors, with the relative standard deviation 17% higher under two-sided learning. While the model fails to capture the volatility in household leverage, the introduction of bounded rationality amongst lenders nevertheless produces an extreme increase in the volatility of $\theta$ relative to the rational expectations model (as a result there is also a large increase in the volatility of the foreclosure inventory).

In order to gauge the robustness of these findings, figure 1.6 plots the relative standard deviation of simulated prices and forecast errors in the model for different values of $g$. For each $g$, $\{\sigma_0, \sigma_q, \sigma_\omega, \mu, K\}$ is recalibrated in line with section 1.4. As can be seen, in the region of the value of $g$ found in section 1.4, the volatilities displayed by the model remain close to the data values listed in table 1.2. As is also clear in table 1.2, the model provides a closer fit to the third moments measured in the data. Neither the rational expectations model nor the demand-side learning model can produce the positive skew observed in house prices and household leverage. The model also matches the observed skewness in the forecast error series.

Table 1.3 shows correlations between the simulated data and the exogenous processes $y$ and $R$. Given the literature indicating that interest rates have been a key driver of house prices over the past 30 years (see Hott and Jokipiä (2012)) the correlations of simulated prices and forecast errors with $R$ are of particular interest. Both the rational expectations model and the model with only demand-side learning overstate the negative correlation between prices (and the market value of housing stock) and mortgage rates relative to the data. The introduction of learning amongst lenders has little effect on the measured value of this correlation. The full learning model also captures 55% of the observed negative correlation between $R$ and forecast errors, an improvement upon the demand-side learning model, and comes close to matching the observed correlation between $\theta$ and $R$. As can also be seen in table 1.3, the full model can also capture the observed correlation between house prices (and the market value of housing stock) and wages.
1.7 Conclusion

Figure 1.6: Standard Deviation of $q$ and $fe(q/q_{-1})$ Relative to $y$

Turning to the time-series properties of the model, figure [1.7] plots the periodogram of prices for the three models listed in tables [1.2] and [1.3]. Under rational expectations house prices fail to display the level of low-frequency variation seen in the data. This problem is alleviated through the introduction of bounded rationality in the model. Both the full model as well as the model with only demand-side learning can capture the bulk of the low frequency (ie. less than 0.2 cycles per quarter) variation in the data, however the full learning model provides a marginally better fit to the data spectrum. Autocorrelations for forecast errors and $\theta$ are shown in table [1.4]. The full learning model matches the first order autocorrelation in forecast errors, however it overstates persistence in the series at higher lags. The learning model also overstates persistence in household leverage, however it significantly outperforms the rational expectations model in this regard.

In sum, the model developed here weakly dominates the demand-side learning model along the dimensions discussed. The model captures almost all of the volatility in house prices over the business cycle whilst providing a closer fit to observed correlations with market fundamentals.

1.7 Conclusion

This chapter provides a framework for explaining asset price booms. A general equilibrium model with learning can quantitatively explain US house price growth in the 2000s and account for volatili-
1.7. Conclusion

Table 1.4: Autocorrelation

<table>
<thead>
<tr>
<th>ψₜ(fe(q/q₋₁))</th>
<th>Data*</th>
<th>Learning Demand-Side</th>
<th>Full Learning Model</th>
<th>Rational Expectations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.412</td>
<td>0.154</td>
<td>0.438</td>
<td>−4.27×10⁻⁴</td>
</tr>
<tr>
<td>2</td>
<td>-0.091</td>
<td>-0.037</td>
<td>0.156</td>
<td>−7.19×10⁻⁴</td>
</tr>
<tr>
<td>5</td>
<td>-0.068</td>
<td>-0.042</td>
<td>-0.054</td>
<td>−5.10×10⁻³</td>
</tr>
<tr>
<td>10</td>
<td>-0.282</td>
<td>-0.034</td>
<td>-0.081</td>
<td>−1.08×10⁻²</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ψₜ(θ)</th>
<th>Data*</th>
<th>Learning Demand-Side</th>
<th>Full Learning Model</th>
<th>Rational Expectations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.992</td>
<td>-</td>
<td>0.998</td>
<td>0.999</td>
</tr>
<tr>
<td>2</td>
<td>0.984</td>
<td>-</td>
<td>0.995</td>
<td>0.999</td>
</tr>
<tr>
<td>5</td>
<td>0.959</td>
<td>-</td>
<td>0.985</td>
<td>0.998</td>
</tr>
<tr>
<td>10</td>
<td>0.862</td>
<td>-</td>
<td>0.971</td>
<td>0.998</td>
</tr>
</tbody>
</table>

* autocorrelations for θ are taken on sample from 1978Q1-2014Q1

Reported moments are for log values. Simulation moments are taken from simulated sample of size 50000 quarters, with the first 1000 quarters dropped as burn-in. Reported data moments of one-quarter-ahead forecast errors are from CME data sample running from 2007Q1 to 2011Q1. Wage data, y, is detrended using a bandpass filter with frequency range 1/32 to 1/8 cycles per quarter.

ties in house prices and forecast errors when agents learn subjective beliefs about prices. The model departs from established frameworks in allowing for bounded rationality amongst both households and the suppliers of credit. In demand-side learning models there exists a trade-off between generating subjective beliefs that are persistent and at the same time highly responsive to innovations. The interaction between the demand and credit supply sides of the market under learning boost the persistence in beliefs, thereby breaking this trade-off. A single permanent and unanticipated decrease in mortgage rates, consistent with the observed drop in US interest rates in the early 2000s, produces 20 quarters of price growth whilst capturing the total growth in prices observed across the US. The model also outperforms both demand-side learning and rational expectations models in capturing key business cycle moments.
There are several further issues that can be explored using the framework developed here. In particular, the chapter has assumed homogeneous beliefs between households and credit suppliers, however this can be easily relaxed. The open economy assumption implies that interest rate movements in the model are exogenous. The model can be augmented to consider endogenous mortgage rates and the influence of movements in mortgage rate expectations amongst households. Similarly, a natural line of inquiry would be to augment the funding side of the lenders problem and consider the effects of uncertainty about financial risk.
Chapter 2

Learning in a Business Cycle Model with Recursive Preferences

2.1 Introduction

Learning and adaptive expectations has been increasingly employed in macroeconomics in order to address the failure of canonical models to generate rich amplification or propagation while capturing commonly-observed properties of household and investor expectations. While the process of updating beliefs can be an important source of propagation, the assumption that agents hold subjective beliefs alters their understanding of how observed variation relates to long run outcomes. A large body of work now exists that uses variation in expected long run consumption growth to explain key asset pricing moments. In canonical long run risk models, persistent and forecastable components in the consumption growth process generate large variation in discount factors under certain recursive preference structures. This chapter demonstrates that the introduction of adaptive learning can have a substantial effect on long run consumption risks generated in otherwise standard frameworks.

This chapter formulates a model in which agents learn by updating their beliefs about the structural parameters of the economy. As has been studied elsewhere in the learning literature, agents in the model are uninformed about the technology that generates rental rates of capital and wages. The learning framework respects the internal rationality of agent choice while allowing for feedback between boundedly-rational beliefs and equilibrium outcomes. Agents attempt to learn the process governing wages and rental rates, which are themselves functions of investment choices and therefore beliefs. In order for long run consumption risks to be priced, households in the model are also endowed with Epstein-Zin preferences.

Epstein-Zin preferences have become widely used in both macroeconomics and finance as they allow the rate at which households discount consumption across periods to explicitly be a function of
2.1. Introduction

expected long run consumption growth. As has been extensively studied in the literature, Epstein-Zin utility specifies preferences over the timing of the resolution of uncertainty. The combination of this preference with beliefs about the underlying stochastic processes in the economy generates long run consumption risk. When agents have a preference for the early resolution of uncertainty they have an aversion to variation in long run consumption growth. Environments in which shocks are permanent are therefore riskier as variation in current consumption growth is positively correlated with variation in growth over long horizons. Conversely, a preference for the late resolution of uncertainty generates more long run risk when shocks are temporary.

The chapter shows that a key implication of the class of learning models considered here is their effect on the perceived persistence of shocks. The propagation induced by feedback between boundedly-rational beliefs and outcomes increases overall volatility and variation in investment. However, unanticipated variation in wages and the rental rate of capital is perceived as being the result of shocks with extremely low persistence, regardless of whether the underlying shocks that cause it are themselves temporary or permanent. As a result, the generation of long run risks in the learning model requires a preference for the late resolution of uncertainty. The is in contrast to much of the macro finance literature studying long run risks, in which a preference for early resolution of uncertainty is assumed. Importantly, this result is shown to be true regardless of whether agents are able to anticipate the evolution of their own beliefs. Agents who internalize the effect of observed outcomes on the future evolution of beliefs can anticipate the long-run consequences of shocks. However, even when agents place a large weight on recent observations when updating beliefs, the perceived persistence in shocks is insufficient to generate long run risks with a preference for the early resolution of uncertainty.

The chapter is structured as follows. Section 2.2 discusses the literature in which this work is placed. The model together with the learning framework and equilibrium concept are described in section 2.3. Section 2.4 analyzes the model under rational expectations in order illustrate the role the preferences play in generating long run risks. The quantitative analysis of the full model is presented in section 2.5. The chapter concludes in section 2.6.

36 See Lochstoer and Kaltenbrunner (2010)
2.2 Related Literature

The use of learning to generate amplification and persistence in macro models has met with mixed success. Much of this literature has studied learning mechanisms in which households forecast future values of their own choice variables. Bullard and Duffy (2002), Carceles-Poveda and Giannitsarou (2008), Ellison and Perlman (2011), Evans and Honkapohja (2001), Williams (2003), and Zhang (2012) all consider variants of the so-called Euler equation or reduced form approach to learning. Under this framework households make forecasts using estimates of the law of motion for the complete vector of endogenous variables in the economy, typically under the assumption that households know the functional form of the solution to the rational expectations equilibrium.

These frameworks have been subject to two primary criticisms. First, in these models individuals are limited to only caring about one-period-ahead forecasts of the economy. In standard settings one-period-ahead forecasts of consumption, labour supply and investment will completely determine decisions when combined with the optimality conditions of the household’s decision problem. This fundamentally changes the nature of the household’s decision problem relative to rational expectations. Under rational expectations, households make consumption-savings decisions contingent on a forecasted sequence of prices. Branch and McGough (2011) show that lengthening the forecasting horizon of boundedly-rational households can increase the propagation of shocks in macro models. An additional criticism of the reduced form approach to learning is that it removes any conceptual distinction between the household’s beliefs and its decisions. Not only does this make models difficult to interpret, but it also yields consumption-savings decisions that are not internally consistent. Boundedly-rational households compute forecasts of future choice variables using their forecasting functions, however these forecasts are not restricted to be consistent with what optimal decisions will be given current beliefs about future prices and allocations. Agents are implicitly assumed to not use part of their own information set when making choices.

This chapter contributes to an area of the learning literature in which households are considered to be uninformed about some aspects of the underlying structure of the economy. This structural
2.2. Related Literature

approach to learning separates household decisions and beliefs, avoiding the consistency problems inherent in the reduced form approach, while also making the household decision problem depend upon long-run forecasts. Preston (2005) and Eusepi and Preston (2011) show that learning frameworks in which households forecast only the paths of future prices can substantially improve the internal propagation of shocks in standard business cycle models. The model presented in this chapter considers households who are uninformed about the formation of prices in the economy. Households attempt to learn the parameterization of the economy by estimating the relationship between prices and the system’s state variables in each period. The nature of the household’s decision problem is similar to rational expectations, the difference being that the perceived parameters underlying the economy differ from their true values. The household is therefore not limited to considering only one-period-ahead forecasts in this framework, and both decisions and the expected values of future choice variables are consistent with optimizing behavior given maintained beliefs.

This chapter also contributes to a literature which studies the effect of recursive preferences in macro models. Following the introduction of the long-run risk model of Bansal and Yaron (2004), recursive preferences have become popular in macroeconomics. The underlying appeal is twofold. First, they provide greater flexibility in that they separate risk aversion and the intertemporal elasticity of substitution. That is, separate parameters control how the agents treat uncertainty within and between periods, implying a preference for either early or late resolution of uncertainty. Second, under this class of preferences the household’s discount factor becomes a function of expected long-run consumption growth. Recursive preferences can therefore bolster propagation mechanisms in models where agents infer information about future consumption from current shocks or in which uncertainty varies over time. For example, Malkhozov and Shamloo (2010) show that a model with recursive preferences and news shocks replicates key macroeconomic data, Gourio (2013) uses variation in the probability of economic disaster together with recursive preferences to explain asset pricing moments, while Kuehn, Petrosky-Nadeau, and Zhang (2012) introduce recursive utility to a model with labour market frictions. Thus far, there has been limited study of whether models with non-rational expectations can generate the type of behavior proposed by Bansal and Yaron

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41 See Caldara et al. (2009) for discussion.
42 See Bansal and Yaron (2004) and Lochstoer and Kaltenbrunner (2010) for discussion.
Hirshleifer, Li, and Yu (2015) considers subjective expectations in a model with recursive preferences by introducing households that extrapolate future productivity growth from recent levels, matching key macroeconomic and financial moments in the process. This chapter in contrast considers an environment that allows for feedback between agents’ beliefs and observed outcomes, as they form beliefs about variables that are equilibrium objects. As discussed, this is in line with much of the recent learning literature.

2.3 Model

This section outlines the baseline model of the chapter. Households are endowed with Epstein-Zin preferences. The information structure is similar to the structural learning environments of Williams (2003) and Eusepi and Preston (2011). Individuals are uninformed about the formation of prices in the economy and hold boundedly-rational beliefs about their evolution over time. In keeping with much of the learning literature, beliefs are updated using a form of constant gain learning.

2.3.1 Household Problem

Households hold Epstein-Zin preferences over consumption $c_t$

$$V_t = \max_{\{c_t\}} \left\{ (1 - \beta)c_t^{1-\psi} + \beta \left( \mathbb{E}_t^P \left[ V_{t+1}^{1-\gamma} \right] \right)^{1-\gamma} \right\}^{1/(1-\psi)} \quad (2.1)$$

where $\mathbb{E}_t^P[\cdot]$ denotes the household’s expectations over its beliefs $\mathcal{P}$ at time $t$. Preferences of this type separate the agent’s preference for risk across time periods and preference for risk across states within a period. The parameters $\gamma$ and $\psi$ denote, respectively, the household’s coefficient of relative risk aversion and the inverse of the elasticity of intertemporal substitution (EIS). Together these two parameters govern the agent’s preference for the timing of the resolution of uncertainty. When $\gamma > \psi$ ($\gamma < \psi$) the agent prefers the early (late) resolution of uncertainty.

The rest of the household problem follows a standard real business cycle formulation. Agents accumulate capital $k_t$, which they rent to firms at the rate $r_t^k$ and inelastically supply 1 unit of

See Epstein and Zin (1989)
2.3. Model

labour to firms at wage \( w_t \). The agent chooses both consumption, \( c_t \), and investment, \( i_t \), and can also borrow \( a_{t+1} \) at the risk-free rate \( R^f_t \). The flow budget constraint of the household is given by

\[
 c_t + i_t + a_{t+1} = r^k_t k_t + w_t + R^f_{t-1} a_t \tag{2.2}
\]

The accumulation of capital is subject to convex adjustment costs, in particular

\[
 k_{t+1} = (1 - \delta)k_t + \phi \left( \frac{i_t}{k_t} \right) k_t \tag{2.3}
\]

where \( \phi(\cdot) \) is given by

\[
 \phi(X) = a_1 + \frac{a_2}{1 - 1/\zeta} X^{1-1/\zeta} \tag{2.4}
\]

The magnitude of the capital adjustment costs is decreasing in \( \zeta \). Following Lochstoer and Kaltenbrunner (2010) and Boldrin, Christiano, and Fisher (2001) the coefficients in the adjustment cost function \( a_1 \& a_2 \) are set so that the steady state of the model coincides with the case where no adjustment costs are present.\(^44\)

The optimality conditions for the household are given by

\[
 E_t^P \left[ M_{t+1} \cdot R^i_{t+1} \right] = 1 \tag{2.5}
\]

\[
 E_t^P \left[ M_{t+1} \cdot R^f_{t+1} \right] = 1 \tag{2.6}
\]

where \( M_{t+1} \) denotes the stochastic discount factor and is given by

\[
 M_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\psi} \left( \frac{V_{t+1}}{E_t^P (V_{t+1})^{1-\gamma}} \right)^{\psi-\gamma} \tag{2.7}
\]

The return on capital investment \( R^i_{t+1} \) in equation \(2.5\) is given by

\[
 R^i_{t+1} = \phi' \left( \frac{i_t}{k_t} \right) \left\{ \frac{k_{t+1}}{r^k_{t+1}} + \frac{1 - \delta + \phi(i_{t+1}/k_{t+1})}{\phi'(i_{t+1}/k_{t+1})} - \frac{i_{t+1}}{k_{t+1}} \right\} \tag{2.8}
\]

\(^44\) \( a_1 \& a_2 \) are set so that \( \phi(i_{ss}/k_{ss}) = i_{ss}/k_{ss} \) and \( \phi'(i_{ss}/k_{ss}) = 1 \). This implies \( a_1 = \frac{1}{1-\zeta} (\exp(\mu) + \delta - 1) \) and \( a_2 = (\exp(\mu) + \delta - 1)^{1/\zeta} \).
2.3. Model

2.3.2 Technology

The production technology is given by

\[ y_t = \exp(Z_t^{1-\alpha}) t^\alpha \]  

(2.9)

where \( Z_t \) is an exogenous TFP process. The technology implies

\[ r_t = \alpha \cdot y_t \]  

(2.10)

\[ w_t = (1 - \alpha) \cdot y_t \]  

(2.11)

Following Lochstoer and Kaltenbrunner (2010), the TFP process is given by

\[ Z_t = \exp(\mu t + \tilde{z}_t) \]  

(2.12)

\[ \tilde{z}_t = \rho \tilde{z}_{t-1} + \epsilon_t \]  

(2.13)

where \( \epsilon_t \sim N(0, \sigma^2_z) \) and \( \rho \leq 0 \). As has been studied extensively in the literature, the interaction between an agent’s expectation as to whether shocks are transitory or permanent (i.e. whether \( \rho < 1 \) or \( \rho = 1 \) in a rational expectations setting) and their preference over the timing of the resolution of uncertainty is key in generating long run risks. In the exposition \( \tilde{x}_t \equiv x_t / \exp(\mu t) \) denotes a variable normalized to account for the effect of trend growth.

2.3.3 Beliefs & Learning

Beliefs are assumed to be homogeneous across all agents in the economy. Similar to chapter I beliefs are specified so as to respect the internal consistency of the agent’s decision problem. Stated once again, this implies that an agent’s decision rule must everywhere be an optimal response to its beliefs \( \mathcal{P} \). As a result only variables outside the agent’s decision set can have their true data generating process excluded from the agent’s information set. As with the learning environment considered in Chapter I the model will feature an Internally Rational Expectations Equilibrium.

The assumed information structure is similar to the structural learning frameworks studied in Williams (2003) and Eusepi and Preston (2011). Agents are uninformed about the true process
governing prices $r_t$ and $w_t$. In particular, the technology (2.9) and the associated pricing functions (2.10) & (2.11) are assumed to be outside the information set of the agent. In order to make internally consistent choices agents must hold a belief about the behavior of the prices. The technology (2.9) implies that $r_t$ and $w_t$ are log-linear with respect to capital. In order to minimize the misspecification in beliefs, agents are also assumed to perceive this to be the case. Agents believe that $r_t$ and $w_t$ are governed by the model

$$
\begin{pmatrix}
\log r_t \\
\log w_t
\end{pmatrix}
= \begin{pmatrix}
\omega^r_0 \\
\omega^w_0 \\
\omega^r_1 \\
\omega^w_1
\end{pmatrix}
\cdot \log k_t
+ \begin{pmatrix}
\epsilon^r_t \\
\epsilon^w_t
\end{pmatrix}
$$

(2.14)

where $[\epsilon^r_t, \epsilon^w_t] \sim N(0, \Sigma)$. Note, that (2.14) implicitly assumes that agents are aware of the trend rate of growth in economy, $\mu$. Agents update their beliefs about the coefficients $\omega = [\omega^r_0, \omega^w_0, \omega^r_1, \omega^w_1]'$ in (2.14) each period to account for new information. The timing assumptions here are important.

The agent enters period $t$ having a belief about $\omega$, denoted $\hat{\omega}_{t-1} = [\hat{\omega}^r_{0,t-1}, \hat{\omega}^w_{0,t-1}, \hat{\omega}^r_{1,t-1}, \hat{\omega}^w_{1,t-1}]'$. It makes its consumption-investment decision in $t$ taking these values as given in (2.14) At the end of the period the household then updates its beliefs.

As in chapter 1, optimal Bayesian updating of the coefficients in (2.14) can be shown to converge to a simple constant gain algorithm when agents take $\omega_t$ to be an unobserved state vector following a random walk. That is, when agent’s believe

$$
\begin{pmatrix}
\log r_t \\
\log w_t
\end{pmatrix}
= \begin{pmatrix}
\omega^r_{0,t} \\
\omega^w_{0,t} \\
\omega^r_{1,t} \\
\omega^w_{1,t}
\end{pmatrix}
\cdot \log k_t
+ \begin{pmatrix}
\epsilon^r_t \\
\epsilon^w_t
\end{pmatrix}
$$

(2.15)

$$
\omega_t = \omega_{t-1} + \epsilon^\omega_t
$$

(2.16)

where $\epsilon^\omega_t \sim N(0, \sigma_\omega)$, then $E^P_t[\omega_t] = \hat{\omega}_{t-1}$ evolves according to

$$
\hat{\omega}_t = \hat{\omega}_{t-1} + g \cdot q_t(q_t q'_t)^{-1}(x_t - q_t \hat{\omega}_{t-1})
$$

(2.17)

Beliefs about the intercepts $\omega^r_0$ and $\omega^w_0$ can be thought of as reflecting uncertainty about long-run trends in technology. Beliefs about the slope coefficients $\omega^r_1$ and $\omega^w_1$ can be thought of as reflecting uncertainty about how observed prices correlate with capital.

The assumed belief structure in (2.14) departs from Eusepi and Preston (2011) in that it does not include a perceived law of motion for capital. Agents here understand that capital will evolve in line with optimal investment responses to observed prices and beliefs.
where $x_t = [\log r_t, \log \tilde{w}_t]'$ and

$$q_t = \begin{pmatrix}
1 & 0 & \log \tilde{k}_t & 0 \\
0 & 1 & 0 & \log \tilde{k}_t
\end{pmatrix} \quad (2.18)$$

The details are shown in Appendix E. Three further points need to be made at this juncture. First, the assumption that the agent updates its beliefs about $\omega$ after the observation of $r_t$ and $w_t$, and after having made its period $t$ choices, is made so as to avoid simultaneity in the determination of $\hat{\omega}_t$ and $x_t$. This assumption is common in the learning literature. Second, as the prices $r_t$ and $w_t$ are equilibrium objects they are themselves functions of $\hat{\omega}_t$. Thus the model allows for additional propagation mechanisms through the feedback between beliefs and outcomes. Finally, because $\hat{\omega}_t$ is a function of agent choices the model allows for the possibility that agents internalize the effect of their investment on future beliefs. In the quantitative evaluation that follows the model is analyzed both under the assumption that agents internalize the impact of their choice on their learning and that they do not.

### 2.3.4 Equilibrium Under Learning

The equilibrium of the model under learning is an Internally Rational Expectations Equilibrium characterized by

1. A probability measure $\mathcal{P}$ representing the homogeneous beliefs of agents over $\Omega_s$, where $\Omega_s$ denotes the space of realizations of variables exogenous to an agent.

2. A sequence of equilibrium prices $\{r_t^*, w_t^*, R_t^{f*}\}$ where $r_t^*, w_t^*, R_t^{f*} : \Omega_s^t \rightarrow \mathbb{R}_+^N \forall t$. Markets clear for all $t$, all realizations in $\Omega_s$ almost surely in $\mathcal{P}$.

3. A sequence of choice functions $\{e_t^*, i_t^*, a_t^*\}_{t=0}^{\infty}$ that maximize the objective function of the agents conditional on $\mathcal{P}$. All agents are internally rational.

In order for the model to sustain an IREE where beliefs differ from those in a rational expectations equilibrium, the agents cannot have access to the full information set available to agents under rational expectations. As outlined in section 2.3.3 this is achieved by removing the technology (2.9) as well as the functions (2.10) and (2.11) from the information set of agents in the learning model.

47See Eusepi and Preston (2011) and Adam, Kuang, and Marcet (2011) for similar examples.
2.4 Illustrating Long Run Risks Under Rational Expectations

Figure 2.1: Transitory & Permanent Shocks in Rational Expectations Model \((\gamma = 5, \psi = 1/1.5)\)

The model is solved via a form of parameterized expectations with the assumption of a representative agent. Both the value function of the representative agent and the euler equation for capital investment are approximated using the Chebyshev polynomials. The details of the solution method can be found in appendix F.

2.4 Illustrating Long Run Risks Under Rational Expectations

Before quantifying the learning model outlined in section 2.3 it is useful to make clear what is meant by ‘long run risk’ and to illustrate the use of Epstein-Zin preferences in generating them\(^{48}\). The form of the stochastic discount factor \((2.7)\) indicates how the use of Epstein-Zin preferences changes the consumption-saving decision relative to standard CRRA utility. The discount factor is composed of two components. The first is a function of one-period-ahead consumption growth and is identical to the discount factor under CRRA utility. The second is a function of unanticipated variation in the continuation utility. The value of Epstein-Zin preferences is precisely that they allow for long-horizon expectations about consumption growth to influence the pricing kernel separately from their effect on one-period consumption growth. As has been studied extensively in the literature\(^{49}\)

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\(^{48}\) Much of the analysis in this section follows that found in [Lochstoer and Kaltenbrunner (2010)](Lochstoer_Kaltenbrunner_2010).

\(^{49}\) See [Bansal and Yaron (2004)](Bansal_Yaron_2004) or [Lochstoer and Kaltenbrunner (2010)](Lochstoer_Kaltenbrunner_2010) for further discussion.
2.4. Illustrating Long Run Risks Under Rational Expectations

Figure 2.2: Transitory & Permanent Shocks in Rational Expectations Model ($\gamma = 5$, $\psi = 6$)

\[
m_t - \mathbb{E}^P_{t-1}[m_t] = -\gamma \left( \Delta \log c_t - \mathbb{E}^P_{t-1}[\Delta \log c_t] \right) - (\gamma - \psi) \left( \Delta vc_t - \mathbb{E}^P_{t-1}[\Delta vc_t] \right) \tag{2.19}
\]

where $m_t \equiv \log M_t$ and $vc \equiv \log \frac{V}{C}$. Unanticipated variation in the discount factor is decomposed into two sources: unanticipated consumption growth and unanticipated changes in the relative value function (which represents long run expected consumption growth). The variance of $m_t - \mathbb{E}^P_{t-1}[m_t]$ is commonly taken to be a measure of the level of overall perceived risk in the economy. Taking the second moment of (2.19) yields

\[
\text{var}(\hat{m}_t) = \gamma^2 \text{var}(\Delta \hat{\log} c_t) + (\gamma - \psi)^2 \text{var}(\Delta \hat{v}c_t) + 2\gamma(\gamma - \psi)\text{cov}(\Delta \hat{\log} c_t, \Delta \hat{v}c_t) \tag{2.20}
\]

where $\hat{x} \equiv x_t - \mathbb{E}^P_{t-1}[x_t]$. Overall risk is composed of short run risks, which are a function of realized consumption growth, and long run risks, which are a function expected long run consumption growth. Note that these long run consumption risks disappear when $\gamma = \psi$, as is the case with standard power utility (indeed, CRRA preferences can be viewed as a special case of Epstein-Zin preferences).

Long run risk in (2.20) is a function of agents’ beliefs about the underlying structure of the
2.4. Illustrating Long Run Risks Under Rational Expectations

The economy (which determine the variance and covariance terms) as well as their preferences. The parameters $\gamma$ and $\psi$ specify a preference for the timing of the resolution of uncertainty. When the coefficient of risk aversion exceeds the inverse of the elasticity of intertemporal substitution (when $\gamma > \psi$) agents are said to have a preference for the *early* resolution of uncertainty. Given the choice between two expected payoff-equivalent lotteries, one that offers a risky payoff for a certain number of periods and an certain payoff thereafter, and another that offers a certain payoff for the initial periods and a risky payoff thereafter, agents would prefer the former. Seen another way, when $\text{cov}(\Delta \log c_t, \Delta \bar{c}_t) > 0$ and shocks to realized consumption are correlated with variation in expected consumption growth, then the environment is riskier when $\gamma > \psi$ and agents experience greater variation in how they value consumption across states. Conversely agents have a preference for *late* resolution of uncertainty when $\gamma < \psi$ for the opposite reasons.

In order to make this distinction clear the model is solved and simulated under the assumption of rational expectations (ie. agents are informed about the true mapping between $\{r_t, w_t\}$ and the model’s state variables). Table 2.1 shows moments for two calibrations of the model’s preferences. In the first case ($\{\gamma, \psi\} = \{5, 1/1.5\}$) the agent has a preference for the early resolution of uncertainty and in the second ($\{\gamma, \psi\} = \{5, 6\}$) the agent has a preference for the late resolution of uncertainty. For each case the model is simulated using transitory shocks ($\rho = 0.95$) and then using permanent shocks ($\rho = 1$).

As can be seen in table 2.1 when $\gamma > \psi$ the level of risk $\sigma(m)$ is an order of magnitude larger when shocks are permanent instead of temporary. With permanent shocks, realized consumption growth is positively correlated with variation in expected long-run consumption growth ($V/C$). From (2.20), when $\gamma > \psi$ this positive correlation pushes up the level of risk. When agents have a preference for early resolution of uncertainty they consider the permanent shock environment to be riskier as they have an aversion to variation in expected consumption growth at long horizons relative to variation in realized consumption growth. Conversely, table 2.1 indicates that when agents have a preference for late resolution of uncertainty they find the transitory shock environment to be riskier for analogous reasons. When $\rho < 1$ shocks to realized consumption growth are

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50 The parameterization of the model is largely taken from standard values found in the literature, see Lochstoer and Kaltenbrunner (2010) or Aldrich and Kung (2011).

51 Note that in 2.1, the level of short run risk, $SRR$, is taken to be $\gamma \cdot \sigma(\Delta \log c)$ in following Lochstoer and Kaltenbrunner (2010). The level of long run risk is measured as the different between the total risk in the economy and $SRR$. 

---
negatively correlated with expected long-run consumption growth. With \( \gamma < \psi \) agents have a preference for variation in expected long-run consumption growth all things being equal, and so they find the transitory shock environment to be riskier. Mechanically, \( \text{cov}(\Delta \hat{\log} c_t, \Delta \hat{\nu}c_t) < 0 \) pushes up the level of risk when \( \gamma < \psi \). Note that results are shown for only one value of \( \rho < 1 \). Decreasing \( \rho \) will have a monotonic effect on \( \text{cov}(\Delta \hat{\log} c_t, \Delta \hat{\nu}c_t) \), so agents with a preference for late resolution of uncertainty will find environments with lower values of \( \rho \) to be riskier.

It should be noted that the riskiness of the two environments does not appear to have a qualitative effect on the financial moments in the model. From table 2.1 the Sharpe ratio of the realized risk premium \( (R^i - R^f) \) is approximately an order of magnitude larger with permanents shocks for each of the two sets of utility parameters. Similarly, for each set of utility parameters the Sharpe ratio of the expected risk premium \( \mathbb{E}[R^i - R^f] \) is approximately an order of magnitude larger when shocks are temporary.

Figures 2.1 and 2.2 show the response of consumption to shocks for each of the two parameterizations of the model under rational expectations. When \( \rho = 1 \), TFP shocks create an expectation of permanently higher consumption, hence for each of the two parameterizations a larger initial consumption response is seen in the permanent shock case. Similarly, table 2.1 shows that for each set of utility parameters, the use of permanent shocks increases the relative standard deviation of consumption growth and decreases the relative standard deviation of investment growth. Increasing \( \psi \) decreases the intertemporal elasticity of substitution and hence the desire to smooth consumption over time goes up. For each the two technology environments, figures 2.1 and 2.2 show a larger initial consumption when \( \psi \) is high, as well as a greater degree of consumption smoothing subsequently.

2.5 Quantifying the Learning Model

This section provides a quantitative evaluation of the model under learning. The goal of the section is to clarify the role of the learning mechanism, hence the parameters are not estimated. For the most part the parameterization is standard. The discount factor \( \beta \) is set equal to 0.998, the rate of trend growth \( \mu \) is set equal to 0.004, the depreciation rate of capital \( \delta \) is set equal to 0.021, and the capital share of output \( \alpha \) is set equal to 0.36. When simulating the model with
2.5. Quantifying the Learning Model

Table 2.1: Business Cycle Moments Under Rational Expectations

<table>
<thead>
<tr>
<th></th>
<th>(i): $\gamma = 5$, $\psi = 1/1.5$</th>
<th>(ii): $\gamma = 5$, $\psi = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Transitory</td>
<td>Permanent</td>
</tr>
<tr>
<td></td>
<td>Shocks</td>
<td>Shocks</td>
</tr>
<tr>
<td>$\sigma(\Delta \log c)/\sigma(\Delta \log y)$</td>
<td>0.347</td>
<td>0.546</td>
</tr>
<tr>
<td>$\sigma(\Delta \log i)/\sigma(\Delta \log y)$</td>
<td>2.237</td>
<td>1.846</td>
</tr>
<tr>
<td>$\mu(R_f)$</td>
<td>1.019</td>
<td>1.016</td>
</tr>
<tr>
<td>$\sigma(R_f)$</td>
<td>$2.15 \times 10^{-3}$</td>
<td>$5.65 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\mu(R^i - R_f)$</td>
<td>$6.51 \times 10^{-5}$</td>
<td>$5.62 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\sigma(R^i - R_f)$</td>
<td>$7.59 \times 10^{-3}$</td>
<td>$7.64 \times 10^{-3}$</td>
</tr>
<tr>
<td>$SR(\hat{R}^i - R_f)$</td>
<td>$8.38 \times 10^{-3}$</td>
<td>$0.074$</td>
</tr>
<tr>
<td>$\mu(\mathbb{E}[R^i - R_f])$</td>
<td>$4.18 \times 10^{-5}$</td>
<td>$5.44 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\sigma(\mathbb{E}[R^i - R_f])$</td>
<td>$6.15 \times 10^{-6}$</td>
<td>$3.78 \times 10^{-3}$</td>
</tr>
<tr>
<td>$SR(\mathbb{E}[R^i - R_f])$</td>
<td>$6.791$</td>
<td>$0.144$</td>
</tr>
<tr>
<td>$\sigma(m)$</td>
<td>0.916</td>
<td>12.734</td>
</tr>
<tr>
<td>$SRR$</td>
<td>0.022</td>
<td>0.035</td>
</tr>
<tr>
<td>$LRR(\Delta c)$</td>
<td>$-1.83 \times 10^{-2}$</td>
<td>$5.42 \times 10^{-2}$</td>
</tr>
<tr>
<td>$SRR(\text{risk})$</td>
<td>5.461</td>
<td>0.393</td>
</tr>
<tr>
<td>$LRR(\text{risk})$</td>
<td>-4.461</td>
<td>0.607</td>
</tr>
<tr>
<td>$\sigma(\log V/C)$</td>
<td>0.039</td>
<td>0.491</td>
</tr>
<tr>
<td>corr($\log(V/C), \Delta \log c$)</td>
<td>-0.052</td>
<td>0.485</td>
</tr>
<tr>
<td>corr($\Delta \log(V/C), \Delta \log c$)</td>
<td>-0.999</td>
<td>0.895</td>
</tr>
</tbody>
</table>

In each case in the above table the model is parameterized as follows: $\beta = 0.998$, $\alpha = 0.36$, $\delta = 0.021$, $\mu = 0.004$, $\zeta = 18$, $\sigma_z = 0.02$. When simulating the model with transitory shocks $\rho$ is set equal to 0.95. Interest rates are reported at annualized values, all other moments are computed for variables at quarterly frequency. In order to control for differences in consumption volatility across the models, the price of risk $\sigma(m)$ is normalized by $\sigma(\Delta \log c)$. The level of short run risk, $SRR$, is taken to be $\gamma \cdot \sigma(\Delta \log c)$ in keeping with Lochstoer and Kaltenbrunner (2010). The sample size is 110 000 periods with the first 10 000 periods dropped as a burn-in.
transitory shocks $\rho$ is set equal to 0.95. Capital adjustment costs are held low, with $\zeta = 18$. The standard deviation of the technology shocks is set equal to 2% across all of the specifications. As a baseline, the household’s prior belief about the shocks $\epsilon^r$ and $\epsilon^w$ is that they are uncorrelated with a standard deviation of 1% each.$^{52}$

2.5.1 The Effects of Learning & Long Run Risks

Figure [2.3] shows the response of consumption to both transitory and permanent technology shocks for the learning model as well as the model under the assumption of rational expectations. The gain parameter is set equal to 0.002 as in Eusepi and Preston (2011)$^{53}$ It is also assumed that agents do not internalize learning in their decision problem. It is immediately apparent that the impact effects in figure [2.3] are substantially different across the rational expectations and learning model. In particular, the initial response of consumption is considerably smaller in the learning model than under rational expectations. This is a result of the specification of beliefs in (2.14). While agents in the learning model understand that $r_t$ and $\tilde{w}_t$ are log-linear with respect to capital they do not observe the technology shocks $\tilde{z}_t$. From the standpoint of the agent, shocks to $\tilde{z}_t$ are perceived as being shocks to $\epsilon^r_t$ and $\epsilon^w_t$. As a result, both permanent as well as persistent but transitory shocks are perceived as being one-time shocks with no persistence. The income effect is therefore small and the agent invests almost all of the shock.

The second factor which is apparent in figure [2.3] is that learning appears to induce more propagation in the model. Agents in the model are updating their beliefs about the law of motion of equilibrium objects. As a result the model allows for feedback between $\{r_t, w_t\}$ and $\omega_t$. Following the initial realization of the shock agents update their beliefs about $\omega$, causing them to believe that there will be permanently higher prices $\{r_t, w_t\}$. As a result, income effects push up both consumption as well as investment (as does continued perceived shocks $\{\epsilon^r, \epsilon^w\}$). Investment continues to drive changes in rental rates and wages and the feedback between $\omega$ and $\{r_t, w_t\}$ persists.

$^{52}$A similar parameterization is used by Lochstoer and Kaltenbrunner (2010) in a rational expectations model with recursive preferences.

$^{53}$Note that the gain parameter chosen here is an order of magnitude smaller than that estimated in chapter 1. One should not expect these to be similar. Chapter 1 considers agents how are forming beliefs about house price growth while this chapter considers agents forming beliefs about wages and rental rates. The assumption that agents discount the past at different rates in these two settings simply implies that they differently informed about these two sets of objects.

$^{54}$Agent beliefs in the learning model are therefore misspecified.
2.5. Quantifying the Learning Model

The impulse responses in figure 2.3 indicate the effect that learning should have on the model’s ability to generate long-run risks. Under the structural learning framework considered here, unanticipated variation in \( r_t \) and \( w_t \) is perceived as being the result of extremely short lived shocks, regardless of whether the shocks that actually give rise to it are themselves permanent or transitory. As a result, agents who have a preference for the early resolution of uncertainty will not find this be a risky environment and the model should not generate positive long run risks.

Table 2.2 shows moments from simulations of the learning model. Once again the gain is set equal to 0.002 and agents do not internalize learning in their decision problem. As in table 2.1 the model is simulated when agents have a preference for the early resolution of uncertainty \((\gamma = 5, \psi = 1/1.5)\) and when agents have a preference for the late resolution of uncertainty \((\gamma = 5, \psi = 6)\). The effect of learning on how agents perceive shocks can be seen by examining the moments of the value function. For both parameterizations of the preferences learning induces a large negative correlation between realized consumption growth and \( V/C \). As a result, agents who have a preference for the early resolution of uncertainty perceive the learning environment to be less risky. This is true when shocks are permanent or temporary as can be seen by comparing the normalized variances of the pricing kernel \( M \) to table 2.1. The large negative correlation between current consumption growth and expected consumption growth that is induced by learning implies that the model doesn’t generate positive long run risks when agents have a preference for early resolution of uncertainty.

As should now be expected the results are reversed when agents have a preference for late resolution of uncertainty. In this case agents like variation in long run consumption growth and hence an environment in which shocks are perceived as being highly transitory is relatively more risky. As a result the normalized variation in \( M \) is higher when \( \gamma = 5 \) and \( \psi = 6 \) than under rational expectations (see table 2.1), for both specifications of \( \tilde{z}_t \). The learning now does generate long run risk. In table 2.2 about 8% of the perceived risk in the economy is accounted for by long run risk when agents have a preference for late resolution of uncertainty.

For both parameterizations of the preferences learning implies higher volatility of the financial variables. The standard deviation of the risk-free rate as well as both the realized and expected risk premium are higher under learning, regardless of whether agents have a preference for early or late resolution of uncertainty. This is the result of the higher overall volatility that results from
2.5. Quantifying the Learning Model

learning. Similarly, the added propagation mechanisms that result from learning imply that the relative standard deviation of both consumption and investment growth are higher under learning for both parameterizations of the preferences. It is interesting to compare the dynamics of learning under the different risk preferences. Figure 2.4 plots the response of consumption to permanent and transitory shocks for each of the two parameterizations considered in table 2.2. Unsurprisingly, a lower intertemporal elasticity of substitution implies that the initial response of consumption to the shock is significantly lower when $\psi$ equals 6 instead of 1/1.5. It is interesting to see, however, that under learning the propagation of the shock and variation in consumption over time is high with low elasticity of substitution. Because agents view deviations of $r_t$ and $w_t$ from the expected values implied by (2.14) as the result of shocks with zero persistence, a desire for more consumption smoothing increases investment in response to observed shocks to $r_t$ and $w_t$ (this is reflected in the relative variances of consumption and investment growth in table 2.2 for the two parameterizations). This investment implies subsequent growth in both $r_t$ and $w_t$, and through updating of beliefs increases in $\omega_{j,t}$, yielding further investment and so on. A lower intertemporal elasticity of substitution therefore bolsters the propagation mechanisms created by learning.

A preference for late resolution of uncertainty requires that $\gamma < \psi$. Hence, this can be obtained either through a decrease in the intertemporal elasticity of substitution (an increase in $\psi$) or through a reduction in the coefficient of risk aversion. Table 2.3 shows simulation moments for two different specifications of the model with a preference for late resolution of uncertainty: (i) $(\gamma = 5, \psi = 6)$ and (ii) $(\gamma = 1/5, \psi = 1/1.5)$. The higher desire for consumption smoothing induced by a larger value for $\psi$ implies that the relative volatility of consumption growth is lower and that of investment growth is higher in (i) than it is in (ii) (for both permanent and transitory shocks to $\tilde{z}$). Differences in the volatility of the financial variables between the two parameterizations reflect differences in overall volatility. Examining the risk variables, the large desire for consumption smoothing when $\psi = 6$ implies a significantly higher normalized volatility of $M$. However, the relatively low coefficient of risk aversion in parameterization (ii) implies a lower price of short run risk. As a result, the proportion of overall risk accounted for by long run risks is an order of magnitude larger under parameterization (ii).

55 The corresponding impulse responses for the beliefs, $\omega_{j,t}$, can be found in appendix D figure D.2.
2.5. Quantifying the Learning Model

Figure 2.3: Influence of Learning ($\gamma = 5, \psi = 6$)

(a) Consumption: Transitory Shock

(b) Consumption: Permanent Shock

Figure 2.4: Effect of Preference for Timing of Risk in Learning Model

(a) Consumption: Transitory Shock

(b) Consumption: Permanent Shock
Table 2.2: Business Cycle Moments Under Learning \( (g = 0.002, \text{Learning not Internalized}) \)

<table>
<thead>
<tr>
<th></th>
<th>(i): ( \gamma = 5, \psi = 1/1.5 )</th>
<th>(ii): ( \gamma = 5, \psi = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Transitory Shocks</td>
<td>Permanent Shocks</td>
</tr>
<tr>
<td>( \sigma(\Delta c)/\sigma(\Delta y) )</td>
<td>0.271</td>
<td>0.287</td>
</tr>
<tr>
<td>( \sigma(\Delta i)/\sigma(\Delta y) )</td>
<td>2.359</td>
<td>2.357</td>
</tr>
<tr>
<td>( \mu(R^f) )</td>
<td>1.019</td>
<td>1.018</td>
</tr>
<tr>
<td>( \sigma(R^f) )</td>
<td>( 2.59 \times 10^{-3} )</td>
<td>( 4.84 \times 10^{-3} )</td>
</tr>
<tr>
<td>( \mu(R^i - R^f) )</td>
<td>( -1.47 \times 10^{-3} )</td>
<td>( -4.06 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \sigma(R^i - R^f) )</td>
<td>0.028</td>
<td>0.049</td>
</tr>
<tr>
<td>( \text{SR}(R^i - R^f) )</td>
<td>( -0.053 )</td>
<td>( -8.23 \times 10^{-3} )</td>
</tr>
<tr>
<td>( \mu(\mathbb{E}[R^i - R^f]) )</td>
<td>( -6.03 \times 10^{-4} )</td>
<td>( 5.09 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \sigma(\mathbb{E}[R^i - R^f]) )</td>
<td>( 2.41 \times 10^{-3} )</td>
<td>( 4.04 \times 10^{-3} )</td>
</tr>
<tr>
<td>( \text{SR}(\mathbb{E}[R^i - R^f]) )</td>
<td>( -0.250 )</td>
<td>0.126</td>
</tr>
<tr>
<td>( \sigma(m) )</td>
<td>0.751</td>
<td>0.858</td>
</tr>
<tr>
<td>( SRR )</td>
<td>0.017</td>
<td>0.018</td>
</tr>
<tr>
<td>( LRR(\Delta c) )</td>
<td>9.87 \times 10^{-3}</td>
<td>0.463</td>
</tr>
<tr>
<td>( SRR(\text{risk}) )</td>
<td>6.66</td>
<td>5.83</td>
</tr>
<tr>
<td>( LRR(\text{risk}) )</td>
<td>( -5.66 )</td>
<td>( -4.83 )</td>
</tr>
<tr>
<td>( \sigma(\log V/C) )</td>
<td>0.035</td>
<td>0.084</td>
</tr>
<tr>
<td>( \text{corr}(\log(V/C), \Delta c) )</td>
<td>( -0.081 )</td>
<td>( -0.083 )</td>
</tr>
<tr>
<td>( \text{cov}(\Delta \log(V/C), \Delta c) )</td>
<td>( -1.20 \times 10^{-5} )</td>
<td>( -1.22 \times 10^{-5} )</td>
</tr>
<tr>
<td>( \text{corr}(\Delta \log(V/C), \Delta c) )</td>
<td>( -0.999 )</td>
<td>( -0.997 )</td>
</tr>
</tbody>
</table>

In each case in the above table the model is parameterized as follows: \( \beta = 0.998, \alpha = 0.36, \delta = 0.021, \mu = 0.004, \zeta = 18, \sigma_z = 0.02 \). When simulating the model with transitory shocks \( \rho \) is set equal to 0.95. Interest rates are reported at annualized values, all other moments are computed for variables at quarterly frequency. In order to control for differences in consumption volatility across the models, the price of risk \( \sigma(m) \) is normalized by \( \sigma(\Delta \log c) \). The level of short run risk, \( SRR \), is taken to be \( \gamma \cdot \sigma(\Delta \log c) \) in keeping with Lochstoer and Kaltenbrunner (2010). The sample size is 110 000 periods with the first 10 000 periods dropped as a burn-in period.
2.5. Quantifying the Learning Model

Table 2.3: Business Cycle Moments Under Learning \((g = 0.002, \text{Learning not Internalized})\)

<table>
<thead>
<tr>
<th></th>
<th>(i): (\gamma = 5, \psi = 6)</th>
<th>(ii): (\gamma = 1/5, \psi = 1/1.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Transitory Shocks</td>
<td>Permanent Shocks</td>
</tr>
<tr>
<td>(\sigma(\Delta c)/\sigma(\Delta y))</td>
<td>0.153</td>
<td>0.193</td>
</tr>
<tr>
<td>(\sigma(\Delta i)/\sigma(\Delta y))</td>
<td>4.4897</td>
<td>4.784</td>
</tr>
<tr>
<td>(\mu(R^f))</td>
<td>1.111</td>
<td>1.087</td>
</tr>
<tr>
<td>(\sigma(R^f))</td>
<td>0.068</td>
<td>0.143</td>
</tr>
<tr>
<td>(\mu(R^i - R^f))</td>
<td>-0.023</td>
<td>-9.16 \times 10^{-3}</td>
</tr>
<tr>
<td>(\sigma(R^i - R^f))</td>
<td>0.065</td>
<td>0.127</td>
</tr>
<tr>
<td>(\text{SR}(R^i - R^f))</td>
<td>-0.352</td>
<td>-0.072</td>
</tr>
<tr>
<td>(\mu(\mathbb{E}[R^i - R^f]))</td>
<td>-0.021</td>
<td>-7.29 \times 10^{-4}</td>
</tr>
<tr>
<td>(\sigma(\mathbb{E}[R^i - R^f]))</td>
<td>0.020</td>
<td>0.049</td>
</tr>
<tr>
<td>(\text{SR}(\mathbb{E}[R^i - R^f]))</td>
<td>-1.018</td>
<td>-0.015</td>
</tr>
<tr>
<td>(\sigma(m))</td>
<td>5.46</td>
<td>5.42</td>
</tr>
<tr>
<td>(SRR)</td>
<td>9.87 \times 10^{-3}</td>
<td>0.012</td>
</tr>
<tr>
<td>(LRR(\Delta c))</td>
<td>0.463</td>
<td>0.424</td>
</tr>
<tr>
<td>(SRR(\text{/risk}))</td>
<td>0.915</td>
<td>0.922</td>
</tr>
<tr>
<td>(LRR(\text{/risk}))</td>
<td>0.085</td>
<td>0.078</td>
</tr>
<tr>
<td>(\sigma(\log V/C))</td>
<td>0.018</td>
<td>0.043</td>
</tr>
<tr>
<td>(\text{corr}(\log(V/C), \Delta c))</td>
<td>-0.105</td>
<td>-0.105</td>
</tr>
<tr>
<td>(\text{cov}(\Delta \log(V/C), \Delta c))</td>
<td>-1.79 \times 10^{-6}</td>
<td>-2.36 \times 10^{-6}</td>
</tr>
<tr>
<td>(\text{corr}(\Delta \log(V/C), \Delta c))</td>
<td>-0.941</td>
<td>-0.943</td>
</tr>
</tbody>
</table>

In each case in the above table the model is parameterized as follows: \(\beta = 0.998, \alpha = 0.36, \delta = 0.021, \mu = 0.004, \zeta = 18, \sigma_z = 0.02\). When simulating the model with transitory shocks \(\rho\) is set equal to 0.95. Interest rates are reported at annualized values, all other moments are computed for variables at quarterly frequency. In order to control for differences in consumption volatility across the models, the price of risk \(\sigma(m)\) is normalized by \(\sigma(\Delta \log c)\). The level of short run risk, \(SRR\), is taken to be \(\gamma \cdot \sigma(\Delta \log c)\) in keeping with Lochstoer and Kaltenbrunner (2010). The sample size is 110 000 periods with the first 10 000 periods dropped as a burn-in.
2.5. Quantifying the Learning Model

2.5.2 The Effect of the Gain

In order to evaluate how the parameterization of the learning process effects the results in section 2.5.1, the model is simulated over a range of values for the gain $g$. Figure 2.5 plots the response of consumption to transitory and permanent shocks with $g = 0.002$ and $g = 0.01$. The corresponding impulse responses for the beliefs $\omega_{t, t}$ can be found in appendix D figure D.3. The influence of the gain on the dynamics is familiar from chapter 1. An increased gain causes a larger revision to beliefs following the initial shock, causing a larger subsequent investment response. The feedback between beliefs and $\{r_t, w_t\}$ is augmented and hence there is more propagation of the shock with a higher gain. The increased elasticity of beliefs with respect to $\{r_t, w_t\}$ under the high gain specification also serves to decrease the persistence in the belief process and so as $g$ increases so too does the speed with which consumption returns to its long run path.

It may seem odd that an order of magnitude change in the gain has a much less pronounced effect on real dynamics than would be the case in the environment studied in chapter 1. There are a number of reasons for this. First, belief updating in (2.17) is a function of variation in the capital stock, which is relatively low. Second, the presence of adjustment costs dampens the response of investment to shifts in beliefs. Finally, the learning framework here is conceptually different as agents are updating beliefs about a set of four parameters and are essentially learning about both the means of $r_t$ and $w_t$ as well as their slopes.

Table 2.4 shows moments from simulations of the model when $g = 0.002$ and when $g = 0.01$. The table includes simulations for the two sets of utility parameters considered in table 2.2. As can be seen, the additional propagation provided by the increased gain is too small to have a significant effect on the relative volatilities of consumption and investment, or on the moments of the risk free rate and the risk premium.

Crucially, the change in the gain has little noticeable effect on the amount of long-run risk generated by the model. The intuition for this result follows from the discussion in 2.5.1. The agent does not anticipate the fact that it will update its beliefs in the future and so the change in the gain has no effect on the perceived relationship between realized consumption growth and expected long-run consumption growth. While variation in the gain parameter may change the

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56 The prior variances about the errors in 2.15 are held constant.
57 Note, there is a slight decrease in the sample correlation between $\Delta c$ and $\log V/C$ owing the revisions in beliefs.
2.5. Quantifying the Learning Model

Figure 2.5: Influence of $g$: Transitory & Permanent Shocks in Learning Model

overall propagation of shocks, the assumption that agents do not internalize learning when making their decision implies that from their perspective changes in the gain have no bearing on the future course of consumption. Thus the standard insight about long-run risk in rational expectations will not be obtained through a particular choice of $g$.

2.5.3 Internalizing Learning

A key effect of the kind of learning framework considered here is that in addition to creating additional propagation mechanisms in the model, it introduces strong assumptions about how agents perceive shocks on impact. Agents perceive unanticipated variation in wages and the rental rate of capital to be the result of shocks with zero persistence. As a result, recovering long run risks requires a strong preference for the late resolution of uncertainty. The chapter now relaxes the assumption made thus far that agents do not internalize their learning when making their choices.

Agents are now assumed to take account of future belief updating when making their consumption-saving decision. When the representative household observes unanticipated changes to the rental rate of capital or to wages it anticipates that these will result in persistent changes in its beliefs about $\omega_{i,t}$ and hence that there should long run effects on consumption.

The moments for the simulation are listed in tables 2.5 and 2.6. Table 2.5 reports moments for and therefore expected consumption growth at the end of the period.
2.5. Quantifying the Learning Model

Table 2.4: Business Cycle Moments Under Learning: Effect of $g$

<table>
<thead>
<tr>
<th></th>
<th>$\gamma = 5, \psi = 1/1.5$</th>
<th></th>
<th>$\gamma = 5, \psi = 6$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$g = 0.002$</td>
<td>$g = 0.01$</td>
<td>$g = 0.002$</td>
<td>$g = 0.01$</td>
</tr>
<tr>
<td>$\sigma(\Delta c)/\sigma(\Delta y)$</td>
<td>0.271</td>
<td>0.274</td>
<td>0.153</td>
<td>0.170</td>
</tr>
<tr>
<td>$\sigma(\Delta i)/\sigma(\Delta y)$</td>
<td>2.36</td>
<td>2.36</td>
<td>4.4897</td>
<td>4.483</td>
</tr>
<tr>
<td>$\mu(R^f)$</td>
<td>1.019</td>
<td>1.019</td>
<td>1.111</td>
<td>1.111</td>
</tr>
<tr>
<td>$\sigma(R^f)$</td>
<td>$2.59 \times 10^{-3}$</td>
<td>0.024</td>
<td>0.068</td>
<td>0.059</td>
</tr>
<tr>
<td>$\mu(R^i - R^f)$</td>
<td>$-1.47 \times 10^{-3}$</td>
<td>$-1.53 \times 10^{-3}$</td>
<td>$-0.023$</td>
<td>$-0.023$</td>
</tr>
<tr>
<td>$\sigma(R^i - R^f)$</td>
<td>0.028</td>
<td>0.026</td>
<td>0.065</td>
<td>0.058</td>
</tr>
<tr>
<td>$SR(R^i - R^f)$</td>
<td>-0.053</td>
<td>-0.059</td>
<td>-0.352</td>
<td>-0.402</td>
</tr>
<tr>
<td>$\mu(\mathbb{E}[R^i - R^f])$</td>
<td>$-6.03 \times 10^{-4}$</td>
<td>$-6.34 \times 10^{-4}$</td>
<td>$-0.021$</td>
<td>$-0.021$</td>
</tr>
<tr>
<td>$\sigma(\mathbb{E}[R^i - R^f])$</td>
<td>$2.41 \times 10^{-3}$</td>
<td>$2.33 \times 10^{-3}$</td>
<td>0.020</td>
<td>0.017</td>
</tr>
<tr>
<td>$SR(\mathbb{E}[R^i - R^f])$</td>
<td>-0.250</td>
<td>-0.273</td>
<td>-1.018</td>
<td>-1.279</td>
</tr>
<tr>
<td>$\sigma(m)$</td>
<td>0.751</td>
<td>1.190</td>
<td>5.46</td>
<td>5.36</td>
</tr>
<tr>
<td>$SRR$</td>
<td>0.017</td>
<td>0.018</td>
<td>$9.87 \times 10^{-3}$</td>
<td>0.011</td>
</tr>
<tr>
<td>$LRR(/ \Delta c)$</td>
<td>-4.25</td>
<td>-3.810</td>
<td>0.463</td>
<td>0.358</td>
</tr>
<tr>
<td>$SRR(/risk)$</td>
<td>6.66</td>
<td>4.199</td>
<td>0.915</td>
<td>0.933</td>
</tr>
<tr>
<td>$LRR(/risk)$</td>
<td>-5.66</td>
<td>-3.199</td>
<td>0.085</td>
<td>0.067</td>
</tr>
<tr>
<td>$\sigma(\log V/C)$</td>
<td>0.035</td>
<td>0.025</td>
<td>0.018</td>
<td>0.014</td>
</tr>
<tr>
<td>$corr(\log(V/C), \Delta c)$</td>
<td>-0.081</td>
<td>-0.211</td>
<td>-0.105</td>
<td>-0.230</td>
</tr>
<tr>
<td>$cov(\Delta \log(V/C), \Delta c)$</td>
<td>$-1.20 \times 10^{-5}$</td>
<td>$-1.18 \times 10^{-5}$</td>
<td>$-1.79 \times 10^{-6}$</td>
<td>$-1.80 \times 10^{-6}$</td>
</tr>
<tr>
<td>$corr(\Delta \log(V/C), \Delta c)$</td>
<td>-0.999</td>
<td>-0.982</td>
<td>-0.941</td>
<td>-0.881</td>
</tr>
</tbody>
</table>

Agents do not internalize learning in any of the models. In each case in the above table the model is parameterized as follows: $\beta = 0.998$, $\alpha = 0.36$, $\delta = 0.021$, $\mu = 0.004$, $\zeta = 18$, $\sigma_z = 0.02$. Shocks are transitory with $\rho$ set equal to 0.95. Interest rates are reported at annualized values, all other moments are computed for variables at quarterly frequency. In order to control for differences in consumption volatility across the models, the price of risk $\sigma(m)$ is normalized by $\sigma(\Delta \log c)$. The level of short run risk, $SRR$, is taken to be $\gamma \cdot \sigma(\Delta \log c)$ in keeping with Lochstoer and Kaltenbrunner (2010). The sample size is 110000 periods with the first 10000 periods dropped as a burn-in.
the model under the assumption that agents have a preference for early resolution of uncertainty \( (\gamma = 5, \psi = 1/1.5) \) while table 2.6 reports moments for the model under the assumption that agents have a preference for late resolution of uncertainty \( (\gamma = 5, \psi = 6) \). It is clear that even allowing for a large gain, the expected long run consumption growth following shocks that might result from internalizing learning is insufficient to change the qualitative properties of the model. Allowing agents to internalize learning slightly increases (decreases) the normalized volatility of \( M \) when agents have a preference for early (late) resolution of uncertainty. This is in line with the aforementioned intuition. When agents have a preference for early (late) resolution of uncertainty the environment becomes riskier (less risky) for them as they anticipate that observed shocks will result in variation in long-run consumption growth. Similarly, the measured level of long run risk relative to the overall variation in \( M \) is slightly increased (decreased) when agents have a preference for early (late) resolution of uncertainty.

Nevertheless, quantitatively these changes are small. Assuming that agents internalize updating of beliefs in their decision problem allows for a channel through which observed exogenous variation can have long-run effects on consumption. Quantitative evaluation suggests, however, that these anticipated long-run consumption effects are insufficient to yield positive long-run risks under the standard assumption that agents prefer early resolution of uncertainty.

### 2.6 Conclusion

This chapter considered the implications of structural learning for generating long run consumption risk. As has been studied elsewhere in the literature on bounded rationality, learning mechanisms can provide substantial amplification and propagation mechanisms in otherwise standard frameworks. This chapter demonstrates that a consequence of internal rationality is that it can change the required preference for the timing of the resolution of uncertainty needed to generate long-run risk. Agents in the model attribute observed variation to extremely short-lived shocks. This is true even if agents can anticipate the effect that their decisions have on the path of future beliefs. Even when the underlying stochastic processes in the economy are subject to permanent shocks, boundedly-rational beliefs require a preference for the late resolution of uncertainty in order to generate long run risks. A key assumption in the learning model is that agents do not have infor-
2.6. Conclusion

Table 2.5: Business Cycle Moments Under Learning: Internalizing Learning ($\gamma = 5, \psi = 1/1.5$)

<table>
<thead>
<tr>
<th></th>
<th>$g = 0.002$</th>
<th>$g = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Don’t Internalize</td>
<td>Internalize</td>
</tr>
<tr>
<td>$\sigma(\Delta c)/\sigma(\Delta y)$</td>
<td>0.271</td>
<td>0.272</td>
</tr>
<tr>
<td>$\sigma(\Delta i)/\sigma(\Delta y)$</td>
<td>2.359</td>
<td>2.358</td>
</tr>
<tr>
<td>$\mu(R_f)$</td>
<td>1.019</td>
<td>0.019</td>
</tr>
<tr>
<td>$\sigma(R_f)$</td>
<td>$2.59 \times 10^{-3}$</td>
<td>0.026</td>
</tr>
<tr>
<td>$\mu(R_i - R_f)$</td>
<td>-1.47 $\times 10^{-3}$</td>
<td>-1.35 $\times 10^{-3}$</td>
</tr>
<tr>
<td>$\sigma(R_i - R_f)$</td>
<td>0.028</td>
<td>0.028</td>
</tr>
<tr>
<td>SR($R_i - R_f$)</td>
<td>-0.053</td>
<td>-0.049</td>
</tr>
<tr>
<td>$\mu(\mathbb{E}[R_i - R_f])$</td>
<td>-6.03 $\times 10^{-4}$</td>
<td>-4.86 $\times 10^{-4}$</td>
</tr>
<tr>
<td>$\sigma(\mathbb{E}[R_i - R_f])$</td>
<td>$2.41 \times 10^{-3}$</td>
<td>$2.40 \times 10^{-3}$</td>
</tr>
<tr>
<td>SR($\mathbb{E}[R_i - R_f]$)</td>
<td>-0.250</td>
<td>-0.203</td>
</tr>
<tr>
<td>$\sigma(m)$</td>
<td>0.751</td>
<td>0.795</td>
</tr>
<tr>
<td>SRR</td>
<td>0.017</td>
<td>0.018</td>
</tr>
<tr>
<td>$LRR(\Delta c)$</td>
<td>-4.25</td>
<td>-4.21</td>
</tr>
<tr>
<td>$SRR(\text{/risk})$</td>
<td>6.66</td>
<td>6.29</td>
</tr>
<tr>
<td>$LRR(\text{/risk})$</td>
<td>-5.66</td>
<td>-5.29</td>
</tr>
<tr>
<td>$\sigma(\log V/C)$</td>
<td>0.035</td>
<td>0.035</td>
</tr>
<tr>
<td>corr($\log(V/C), \Delta c$)</td>
<td>-0.081</td>
<td>-0.080</td>
</tr>
<tr>
<td>cov($\Delta \log(V/C), \Delta c$)</td>
<td>-1.20 $\times 10^{-5}$</td>
<td>-1.20 $\times 10^{-5}$</td>
</tr>
<tr>
<td>corr($\Delta \log(V/C), \Delta c$)</td>
<td>-0.999</td>
<td>-0.999</td>
</tr>
</tbody>
</table>

Agents have a preference for the early resolution of uncertainty in the above models ($\gamma = 5, \psi = 1/1.5$). In each case in the above table the model is parameterized as follows: $\beta = 0.998, \alpha = 0.36, \delta = 0.021, \mu = 0.004, \zeta = 18, \sigma_z = 0.02$. Shocks are permanent ($\rho = 1$). Interest rates are reported at annualized values, all other moments are computed for variables at quarterly frequency. In order to control for differences in consumption volatility across the models, the price of risk $\sigma(m)$ is normalized by $\sigma(\Delta \log c)$. The level of short run risk, $SRR$, is taken to be $\gamma \cdot \sigma(\Delta \log c)$ in keeping with [Lochstoer and Kaltenbrunner (2010)]. The sample size is 110 000 periods with the first 10 000 periods dropped as a burn-in.
2.6. Conclusion

Table 2.6: Business Cycle Moments Under Learning: Internalizing Learning $(\gamma = 5, \psi = 6)$

<table>
<thead>
<tr>
<th></th>
<th>$g = 0.002$</th>
<th>$g = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Don’t Internalize</td>
<td>Internalize</td>
</tr>
<tr>
<td>$\sigma(\Delta c)/\sigma(\Delta y)$</td>
<td>0.153</td>
<td>0.193</td>
</tr>
<tr>
<td>$\sigma(\Delta i)/\sigma(\Delta y)$</td>
<td>4.4897</td>
<td>0.478</td>
</tr>
<tr>
<td>$\mu(R^f)$</td>
<td>1.111</td>
<td>1.109</td>
</tr>
<tr>
<td>$\sigma(R^f)$</td>
<td>0.068</td>
<td>0.142</td>
</tr>
<tr>
<td>$\mu(R^i - R^f)$</td>
<td>-0.023</td>
<td>-9.16$x10^{-3}$</td>
</tr>
<tr>
<td>$\sigma(R^i - R^f)$</td>
<td>0.065</td>
<td>0.127</td>
</tr>
<tr>
<td>$SR(R^i - R^f)$</td>
<td>-0.352</td>
<td>-0.072</td>
</tr>
<tr>
<td>$\mu(\mathbb{E}[R^i - R^f])$</td>
<td>-0.021</td>
<td>-7.78$x10^{-4}$</td>
</tr>
<tr>
<td>$\sigma(\mathbb{E}[R^i - R^f])$</td>
<td>0.020</td>
<td>0.049</td>
</tr>
<tr>
<td>$SR(\mathbb{E}[R^i - R^f])$</td>
<td>-1.018</td>
<td>-0.016</td>
</tr>
<tr>
<td>$\sigma(m)$</td>
<td>5.46</td>
<td>5.42</td>
</tr>
<tr>
<td>$SRR$</td>
<td>9.87$x10^{-3}$</td>
<td>0.012</td>
</tr>
<tr>
<td>$LRR(\Delta c)$</td>
<td>0.463</td>
<td>0.423</td>
</tr>
<tr>
<td>$SRR(risk)$</td>
<td>0.915</td>
<td>0.916</td>
</tr>
<tr>
<td>$LRR(risk)$</td>
<td>0.085</td>
<td>0.084</td>
</tr>
<tr>
<td>$\sigma(\log V/C)$</td>
<td>0.018</td>
<td>0.043</td>
</tr>
<tr>
<td>$corr(\log(V/C), \Delta c)$</td>
<td>-0.105</td>
<td>-0.104</td>
</tr>
<tr>
<td>$cov(\Delta \log(V/C), \Delta c)$</td>
<td>-1.79$x10^{-6}$</td>
<td>-2.37$x10^{-6}$</td>
</tr>
<tr>
<td>$corr(\Delta \log(V/C), \Delta c)$</td>
<td>-0.941</td>
<td>-0.947</td>
</tr>
</tbody>
</table>

 Agents have a preference for the late resolution of uncertainty in the above models $(\gamma = 5, \psi = 6)$. In each case in the above table the model is parameterized as follows: $\beta = 0.998$, $\alpha = 0.36$, $\delta = 0.021$, $\mu = 0.004$, $\zeta = 18$, $\sigma_x = 0.02$. Shocks are permanent $(p = 1)$. Interest rates are reported at annualized values, all other moments are computed for variables at quarterly frequency. In order to control for differences in consumption volatility across the models, the price of risk $\sigma(m)$ is normalized by $\sigma(\Delta \log c)$. The level of short run risk, $SRR$, is taken to be $\gamma \cdot \sigma(\Delta \log c)$ in keeping with [Lochstoer and Kaltenbrunner](2010). The sample size is 110 000 periods with the first 10 000 periods dropped as a burn-in.
mation about the underlying shocks in the economy. The results indicate that in order to maintain preference for the early resolution of uncertainty in learning models of long-run risk, researchers may wish to confine themselves to environments where agents are able to observe and form beliefs about the sources of exogenous variation in the economy.
Chapter 3

Explaining World Savings

3.1 Introduction

Data on the world saving distribution reveals two key features: (i) cross-country differences in saving rates are significant and persistent; and (ii) some countries and regions have shown very sharp changes in their average saving rates over short periods of time. These facts are problematic for the standard model with time-additive preferences. Without equal rates of time preference, the asymptotic distribution of world wealth is typically degenerate under additively separable preferences.

While models with equal rates of time preference but cross-country differences in demographics and productivity have had some success in accounting for part of the cross-country dispersion in saving rates, a substantial amount of variation still remains unexplained in these models.

This chapter provides an alternative explanation for the observed saving patterns. It formalizes a model of the world economy that is comprised of open economies inhabited by infinitely-lived agents. The main point of departure from the standard neoclassical exogenous growth model is that it endows agents with recursive preferences. Specifically, it follows Farmer and Lahiri (2005) and uses a modified version of recursive preferences. The key implication of the Farmer-Lahiri specification is that it generates a determinate steady state wealth distribution within a growing world economy, a feature that typical models with recursive preferences cannot generate.

The problem with the standard recursive preference specification is that it is inconsistent with balanced growth. The existence of balanced growth requires homothetic preferences. However,

\footnote{The dynamic properties of models with recursive preferences and multiple agents were analyzed in a celebrated paper by Lucas and Stokey (1984). Assuming bounded utility, Lucas-Stokey studied an endowment economy. Consequently, the results of Lucas-Stokey cannot be directly applied to growing economies. There is a small literature on recursive preferences with unbounded aggregators. Boyd (1990) has developed a version of the contraction mapping theorem that can be used to generalize Lucas-Stokey's proof of existence of a utility function to the unbounded case. If preferences are time-separable, King, Plosser, and Rebelo (1988) showed that the period utility function must be homogenous in consumption and Dolmas (1996) has generalized their result to the case of recursive utility. Farmer and Lahiri (2005) have applied the Dolmas result to a multi-agent economy and have established that homogeneity rules out the assumption of increasing marginal impatience. Hence, the existence of an endogenous stationary wealth distribution is inconsistent with balanced growth.}
under homothetic preferences, the asymptotic wealth distribution in multi-agent environments is
either degenerate or reflects the initial wealth distribution.

One might of course consider the issue of balanced growth to be irrelevant to understanding sav-
ings behavior. However, the inconsistency of standard recursive preferences with balanced growth
is problematic if one’s goal is to explain saving rates. In particular, given the constancy of both
long run average growth rates and saving rates for most groups of countries (regions or continents),
understanding long run patterns of savings would appear to be intrinsically linked to long run
steady state dynamics. As is well known, the Kaldor growth facts are quite stark in suggesting
balanced growth to be a robust feature of long run growth. Hence, the chapter argues that any
model attempting an explanation of dispersions in long run saving rates across countries should be
consistent with balanced growth.

To overcome the problems with the standard recursive preference specification, the chapter
follows Farmer and Lahiri (2005) and constructs a model of recursive utility in which agents care
about relative consumption. It is assumed that preferences are described by an aggregator that
contains current consumption, future utility, and a time-varying factor that is external to the agent
but grows at the common growth rate in a balanced growth equilibrium. In assuming that agents
derive utility from their place in the consumption distribution, the preference structure will be
consistent with the well-known Easterlin Paradox. Following Easterlin (1974) a number of studies
have shown empirical measures of utility and satisfaction to be increasing in relative rather than
absolute wealth. Such evidence can be found in Easterlin (1995), Helliwell (2003), and Di Tella,
MacCulloch, and Oswald (2003).\footnote{Clark, Frijters, and Shields (2008) for further discussion.}

The assumed time dependence allows for preferences to exhibit increasing marginal impatience,
which is a necessary condition for a non-degenerate asymptotic wealth distribution. A positive
productivity shock in the model induces a rise in saving which ultimately reverts back to its prior
level due to the increasing marginal impatience of agents as their wealth rises relative to world
wealth, thereby preserving a determinate asymptotic wealth distribution. Equally important, this
specification implies that different preferences induce different consumption to wealth ratios of
different agents which, in turn, map into different steady state saving rates of different countries.

The chapter starts by demonstrating that the modified preferences of Farmer and Lahiri (2005)
3.1. Introduction

can account for not only the observed differences in average long run saving rates between regions for
the period 1970-2010 but also the time series behavior of region-specific saving rates. Strikingly, the
model achieves these two goals without appealing to either different externality factors or frictions
in factor and goods flows. This feature of the model is first established by examining a two-region,
heterogenous agent world economy without any impediments in factor or goods mobility across
regions. Importantly, this version of the model assumes that the external factor in preferences
is indexed to the common world per capita consumption level. Hence, all agents have the same
external factor. A calibrated version of this two-region model is shown to quantitatively match both
the average region-specific levels of savings between 1970-2010 as well as the time series movements
in saving rates in the two regions during this period. Encouragingly, the model can match these
facts despite allowing only two factors to vary across countries (one preference parameter and labor
productivity) and using a common world productivity process as the only exogenous driver of time
series fluctuations around the balanced growth path. The chapter then establishes the same results
for a richer five-region world economy.

Lastly, the chapter turns to saving miracles. In order to generate sudden switches in saving
rates, an additional modification is proposed. Specifically, the world is divided into three regions:
the G7, the Asian tigers, and Emerging countries. Crucially, the external factor is now allowed
to be different in levels for the three groups even though it is still constrained to have the same
growth rate. The basic idea behind this is that all societies have role models/peer groups that they
want to keep up with or imitate. This approach to explaining miracles amounts to a hypothesis
that these sudden transformations of economies occur due to changes in their aspirations. More
particularly, under the formalization considered here the steady state saving rates are functions
of the external factor in preferences which describes the benchmark relative to whom the country
evaluates itself. The model can generate the observed saving miracle of the Asian Tigers if their
benchmark external factor is switched in 1970 from the average consumption level of the developing
world to the G7 consumption level. The model predicts that saving rates rise towards the observed
levels in the data as the economy starts building its consumption towards its new desired level.
As consumption rises however, increasing marginal impatience starts to become stronger over time
which eventually induces the saving rates to come back down. The model predictions are shown to
match the facts quantitatively as well as qualitatively for the Asian Tigers.
This work is related to two different strands of literature. The first is the relatively large body of research focused on explaining the dispersion of saving rates across countries. Explanations for the observed variation in cross-country savings have typically focused on variations in per capita incomes, productivity growth, fertility rates or the age distribution of the population. Contributions along these lines can be found in Mankiw, Romer, and Weil (1992), Christiano (1989), Chen, Imrohoroglu, and Imrohoroglu (2006), Horioka and Terada-Hagiwara (2012), Loayza, Schmidt-Hebbel, and Serven (2000) and Tobing (2012). These papers typically find significant explanatory power for demographics and some explanatory power for per capita income (though the direction of causality there is somewhat unclear). However, a significant part of the saving variability in the data continues to remain unaccounted for.

This chapter is also related to the work on recursive preferences and stationary wealth distributions that goes back in its modern form to Lucas and Stokey (1984) and Epstein and Hynes (1983). Of particular relevance to this work are the contributions of Boyd (1990), Dolmas (1996) and Ben-Gad (1998) who focused on characterizing the stationary wealth distribution in growing economies. A second line of research has examined the implications of recursive preferences for stationary wealth distribution in growing economies. Also relevant to this work are the papers by Uzawa (1969), Mendoza (1991) and others who examined the effects of endogenously varying discount rates on the equilibrium dynamics of the neoclassical growth model.

The next section describes some of the key data features that motivate this study. Section 3.3 quickly reviews the key issues associated with recursive preferences under balanced growth as well as the “fix” to the problem suggested by Farmer and Lahiri (2005). Section 3.4 presents and develops the model. Section 3.5 discusses the calibration of the model and examines its quantitative fit to average saving rates in a two-region economy. Section 3.6 discusses miracles while the last section concludes.

3.2 Two Facts on Cross-Country Saving

There are two features of the data that should be highlighted. First, is the sustained differences in saving rates across groups of countries. To do this the countries are collected into three groups: the
G7, Emerging, and Sub-Saharan Africa. Figure 3.1 plots the savings rates of these three groups of countries between 1970 and 2010. The figure illustrates another key feature of the data. Savings rates are different for different countries for long periods of time. Further, they show little or no evidence of convergence.

Figure 3.1: Differences in Saving Rates Across Regions

While the overall pattern suggests that saving rates are persistent, the data does have another important aspect: in some countries saving rates show sudden and sharp swings over relatively short periods of time. Figure 3.2 highlights this by plotting the saving rates in the Asian Tigers between 1960 and 2010. Clearly, saving rates in the Asian economies increased very sharply from 1960 onward. In a short time period of 20 years starting in 1970, their saving rates rose by almost 15 percentage points. Since the mid-1980s, the average saving rate in the Asian tigers has exceeded the average saving rate in the G7 countries by over 10 percentage points.

This chapter argues that this data can be explained by allowing the rate of time-preference to vary across countries using a modified version of recursive preferences. In the standard model of recursive preferences studied by Lucas and Stokey (1984) and Epstein and Hynes (1983), agents

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The list of countries in each group is in the Appendix.
become less patient as they become richer in an absolute sense. This idea is adapted to the case of a growing economy with the assumption that agents become less patient as they become richer in a relative sense.

### 3.3 Recursive Preferences & Balanced Growth

As discussed in Section 3.1, a key goal of the work is to examine the ability of a modified version of recursive preferences to rationalize the cross-country saving facts. Before presenting the model it is worthwhile to review why a modification is needed at all. In a nutshell, the need to modify the standard recursive specification arises because of an interest in analyzing environments with steady state balanced growth. Balanced growth is not only one of the celebrated Kaldor facts but also happens to characterize the modern data aggregated by regions. This can be seen from Figure 3.3 below which shows the time series behavior of the consumption-to-output ratio for G7 and developing countries separately. The figure makes clear that on average, consumption and output tend to grow at similar rates over long periods of time. Hence, it should be desirable to retain balanced growth as a key feature in any model of world saving.

The baseline recursive preference structure, however, is not consistent with balanced growth. It should be noted that the celebrated Lucas and Stokey (1984) paper that studied recursive preferences in a heterogenous agent economy did not have long run growth. Hence, this inconsistency was not germane to their work.
To see this, consider the following recursive aggregator of preferences

\[ u_t = W(c_t, u_{t+1}) \]

With heterogenous agents, there exists an asymptotic stationary wealth distribution if along a steady state balanced growth path all agents equate

\[ W_{ii} \left( c^i, u^i \right) = W_{jj} \left( c^j, u^j \right) = 1/R \]

for all \( i, j \) \((3.1)\)

where \( c \) is consumption, \( u \) is utility and \( R \) is the steady state interest factor common to all agents. 

Dolmas (1996) showed that this can only occur if the aggregator \( W \) is homogenous of the form

\[ W \left( \lambda x, \lambda^\gamma y \right) = \lambda^\gamma W \left( x, y \right) \] \( (3.2) \)

When this homogeneity condition is satisfied \( W_u \) becomes a constant along a balanced growth path thereby making it possible for the endogenous rate of time preference to remain constant and equal to the constant interest rate along a balanced growth path. More fundamentally, \( W_u \) becomes a number that is independent of \( c \) and \( u \) in steady state.

While the condition above is intuitively obvious, Farmer and Lahiri (2005) showed that this homogeneity condition also implies that the stationary wealth distribution in a heterogenous agent
3.3. Recursive Preferences & Balanced Growth

economy is generically degenerate. It leads to a non-degenerate steady state wealth distribution only in the knife-edged case of $W^i = W^j$. But those are exactly the implications in the case of additively separable preferences. In other words, the key Farmer-Lahiri result is that recursive preferences do not add anything to our understanding of stationary wealth distributions beyond what we already know from additively separable preferences.

In the context of recursive preferences in environments without steady state growth, Lucas and Stokey (1984) proved the existence of a stationary wealth distribution as long as preferences exhibited increasing marginal impatience, i.e., agents became more impatient as they grew wealthier. Intuitively, rising impatience bounds the desire to accumulate assets by raising the desire to consume. The problem of the specification in equation (3.2) is that there is no force akin to the increasing marginal impatience of Lucas-Stokey that can endogenously equate it across agents. Consequently, the equilibrium has a knife-edge property to it.

Farmer and Lahiri (2005) showed however that the introduction of an externality into preferences could fix this problem. In particular, they considered preferences of the form

$$u^i = W^i (c^i_t, u^i_{t+1}, a_t)$$

where $a_t$ is a factor that is external to the individual. This could stand in for habits, the average consumption level of the economy, or any other factor provided it grows at the rate of steady state growth and, as formalized here, is external to the individual household. Farmer-Lahiri showed that as long as the aggregator $W$ was homogenous in all three arguments so that $W (\lambda x, \lambda y, \lambda z) = \lambda W$, an economy with heterogenous agents would give rise to an endogenously determined stationary wealth distribution with different agents choosing different saving levels in order to equate $W^i_u (\bar{c}^i, \bar{g}^i u^i, 1) = W^j_u (\bar{c}^j, \bar{g}^j u^j, 1)$ where $\bar{x} = \frac{\bar{a}}{\bar{g}}$ and $g$ denotes the steady state growth. Moreover, the homogeneity property also ensures that $W_u$ would be constant in steady state. Hence, this specification can generate steady state differences in saving rates across agents facing a common vector of prices. The rest of the chapter will examine the potential of these preferences to account for the disparity in saving rates across the world.
The environment is a world economy consisting of $N$ small economies. Each country $i$ is populated by $l_i$ agents and this measure remains constant over time. Introducing population growth into the model is a straightforward extension that does not change any fundamental result. With no loss of generality the world population is normalized to unity so that $\sum_i l_i = 1$. It is assumed that the world economy has an integrated market for goods and factors. Hence, there are no impediments to goods or factor flows throughout the world. This assumption makes the role of the preference structure in accounting for saving disparities particularly stark.

Agents globally are endowed with one unit of labor time which they supply inelastically to the market. It is assumed that all agents within a country have identical preferences, but that preferences of agents across countries may be different. The preferences of the representative agent in country $i$ are described by the recursive representation

$$u_{it} = \frac{c_{it} \zeta_{it}^{1-\theta_i}}{\theta_i} + \mathbb{E}_t \left[ \beta_i \frac{u_{it+1} \zeta_{it+1}^{1-\delta_i}}{\delta_i} \right]$$ (3.3)

where $c$ denotes consumption and $u$ denotes utility. This recursive preference specification is standard except for the argument $\zeta_i$ which stands for an externality in preferences. It is external to the individual but is indexed by $i$ since this externality parameter is allowed to vary across countries. This externality could represent a number of different things including external habits, relative consumption (“keeping up with the Jones’s”) etc. Allowing it to vary across agents implies, for example, that the relative consumption targets could vary across countries. Note that these preferences reduce to the standard additively separable across time specification in the special case where $\delta = 1$. Ceteris paribus, a higher $\delta$ makes agents more patient by raising the discount factor. It is easy to check that this aggregator is linearly homogenous, thereby satisfying the homogeneity and regularity conditions needed for the existence of a Balanced Growth Path (BGP) as shown in Farmer and Lahiri (2005).

Agents have two sources of income: wage income from working and capital income earned by renting out their capital to firms. Households save by accumulating capital. Income can be used
for either consumption or saving. The budget constraint for households is thus given by

\[ c_{it} + \iota_{it} = r_{it}k_{it} + w_{it} \]  \hspace{1cm} (3.4)

where \( k \) is the capital stock of household \( i \) at the beginning of period \( t \) while \( \iota \) denotes saving. \( r \) is the rental rate on capital from country \( i \) while \( w \) is the wage rate of labor from country \( i \). The capital stock of the household evolves according to the accumulation equation

\[ k_{it+1} = (1 - d) k_{it} + \iota_{it} , k_{i0} \text{ given for } i = 1, \ldots, N \]  \hspace{1cm} (3.5)

where \( d \) is the depreciation rate.

Agents maximize utility subject to equations (3.4) and (3.5). The first order condition describing the optimal consumption-saving plan is

\[
\left( \frac{c_{it}}{\zeta_{it}} \right)^{\theta_i - 1} = \beta_i E_t \left[ \left( \frac{c_{it+1}}{\zeta_{it+1}} \right)^{\theta_i - 1} \left( \frac{u_{it+1}}{\zeta_{it+1}} \right)^{\delta_i - 1} (1 + r_{it+1} - d) \right]
\]

There is a common world production technology which produces output according to

\[ Y_t = A_t K^\alpha \left[ \sum_{i=1}^{N} (\gamma_i l_i)^\eta \right]^{\frac{1-\alpha}{\eta}} \]

where \( K = \sum_{i=1}^{N} k_i l_i \) is capital and \( l_i \) denotes labor from country \( i \). \( \gamma_i \) is the labor productivity of workers from country \( i \). This is allowed to vary across countries but not across time. \( \eta \) controls the elasticity of substitution across the different types of labor.

\( A \) is the productivity of the technology and is given by

\[ A_t = e^{z_t} a_t^{1-\alpha} \]

where \( a \) and \( z \) are productivity processes described by

\[ a_t = g a_{t-1} \]  \hspace{1cm} (3.6)
3.4. The Model

\[ z_t = \rho z_{t-1} + \sigma \varepsilon_t \]  \hspace{1cm} (3.7)

Thus, \( a \) is the long run trend in TFP with \( g \) being the trend growth of productivity while \( z \) represents TFP fluctuations around the trend.

Optimality in factor markets dictates that factor prices must be given by:

\[ r_t = \alpha \frac{y_t}{k_t} \]  \hspace{1cm} (3.8)

\[ w_{it} = (1 - \alpha) \frac{y_t}{l_i} \left( \frac{\gamma_i l_i}{\eta} \right) \sum_{i=1}^{N} \left( \frac{\gamma_i l_i}{\eta} \right) \]  \hspace{1cm} (3.9)

where \( y \) and \( k \) denote per capita world output and capital, respectively. Note that since the world population has been normalized to unity, aggregate world output and capital are also per capita world output and capital: \( Y = y, K = k \). The wage rate is country-specific as labor productivity is different and different types of labor are not perfect substitutes. On the other hand, the rental rate is identical for the different types of capital since they are perfect substitutes in production and they are all equally productive.

Any world equilibrium in this economy must clear goods and factor markets:

\[ \sum_{i=1}^{N} l_i (c_{it} + \iota_{it}) = y_t \]  \hspace{1cm} (3.10)

\[ \sum_{i=1}^{N} l_i k_{it} = k_t \]  \hspace{1cm} (3.11)

\[ \sum_{i=1}^{N} l_i = 1 \]  \hspace{1cm} (3.12)

Equation (3.10) is the goods market clearing condition which dictates that the total demand for consumption and investment by the world must equal the world GDP. Equation (3.11) says that total world capital supply must equal capital demand while equation (3.12) is the corresponding world labor market clearing condition.
### 3.4. The Model

#### Definition 3.4.1
An equilibrium in this economy is a set of allocations \(\{c_{it}, k_{it}, \ell_{it}, y_t\}\) and prices \(\{w_{it}, r_t\}\) such that at each \(t\) (a) all households solve their optimization problem given prices; (b) firms maximize profits given prices; and (c) the allocations clear all markets.

The model presented here departs from a standard neoclassical framework along two dimensions. First, it allows for what is essentially a form of external consumption habits. Second, it allows for preference heterogeneity. Heterogeneity in the parameters in (3.3) is essential for generating a determinate asymptotic saving distribution. In order to make this clear it is worth sketching out a brief description of how the recursive specification works in steady state. Let \(\tilde{x} = \frac{x}{\delta}\). In steady state, the rate of time preference for agent \(i\) is given by

\[
W_u^i = \beta_i (g\tilde{u})^{\delta_i - 1}
\]

while steady state normalized utility is

\[
\tilde{u}^i = \left(\frac{R\beta_i}{g}\right)^{1-\delta_i}
\]

where \(R = 1 + r - d\). Using the definition of the aggregator, a steady state expression for normalized consumption follows:

\[
c^i = \left[\theta_i \tilde{u}^i - \frac{\theta_i \beta_i}{\delta_i} (\tilde{u}^i g)^{\delta_i}\right]^{\frac{1}{\delta_i}}
\]

Hence, each \(\tilde{u}\) maps into a different \(\tilde{c}\). In this set-up, different \(\delta'\)s and \(\beta'\)s imply different steady state \(\tilde{u}'\)s. The rate of time preference \(W_u\) is equated across agents by different steady state \(\tilde{c}'\)s and \(\tilde{u}'\)s. Hence, different \(\delta'\)s and \(\beta'\)s across agents induce a dispersion in steady state saving rates across agents. \(^{62}\)

For a growing economy characterized by agents with such heterogenous preferences, Farmer and Lahiri (2005) used the results of Lucas and Stokey (1984) to prove that there exists a unique convergent path to a unique steady state with a stationary distribution of saving rates provided \(c\) and \(u\) are both “non-inferior”\(^{63}\) and preferences display increasing marginal impatience, i.e., \(W_u\) is

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\(^{62}\)In the absence of heterogeneity in the preference parameters savings would be asymptotically equated across agents. The effect of reference-dependence in preferences in this case would be seen only in the dynamics of savings behavior.

\(^{63}\)c and u are non-inferior if \(c < c'\) and \(u > u' \implies \frac{W_u^i(c,u)}{W_u^i(c',u')} > \frac{W_u^i(c',u')}{W_u^i(c',u')}, i = 1, 2.\)
3.5 Quantifying the Model

The model is now taken to the data. Before proceeding to the quantification there are some measurement issues that need to be addressed. Specifically, measures for saving, the capital stock and the share of income accruing to capital by region are needed. The savings rate for each region $i$ is constructed from the relation

$$S_i^i = \frac{1 - C_i^i - I_i^i}{Y_i^i}$$

where $C_i^i$, $I_i^i$ and $Y_i^i$ denote aggregate consumption, investment and output of region $i$. For each region their aggregate output, consumption, capital stock, workers and investment are computed by summing across all countries in the region.

Capital stocks for each country in the sample are constructed using the perpetual inventory method. For the capital share numbers a number of alternative approaches were tried ranging from a constant 1/3 share of capital for all countries to the numbers computed by Caselli and Feyrer (2007) as well as those from Bernanke and Gurkaynak (2002). Since the results are robust to these alternative approaches, in the following the capital share is set to a common 1/3 for all countries.

To compute productivity in each region the production function is used to get:

$$Productivity_t = \frac{y_t}{k_t^\alpha \left( \sum_{i=1}^{N} (\gamma_i L_{it})^\eta \right)^{1/\eta}}$$

The series is detrended using a linear trend and the linear trend is set equal to $a_{t-1}^{1-\alpha}$. The detrended series is set equal to $z_t$, which in turn is used to estimate

$$z_t = \hat{\rho} z_{t-1} + \hat{\epsilon}_t$$

The computed residuals $\hat{\epsilon}_t$ are used as shocks to the model.

The first important choice to make is the choice of the externality parameter. For the baseline
3.5. Quantifying the Model

case this is set to

\[ \tilde{\zeta}_{it} = \tilde{c}_t \text{ for all } i \text{ and } t \]

where \( \tilde{c}_t \) denotes the average per capita world consumption level. This is the most neutral starting point where there is a common reference consumption target for all countries. Later this assumption will be relaxed to explore its potential in explaining saving miracles.

The world is divided into three groups: G7 economies, Asian tigers, and Emerging economies. The Emerging economies are deliberately separated from the Asian Tigers since the Tigers are analyzed separately. Details regarding the countries in each region, the data and the series construction are contained in Appendix G.

A two-country version of the model presented above is first examined. To align the two-country model to the data the world is normalized to only comprise the G7 and Emerging economies. We calibrate the model to match data on two regions (the G7 and Emerging economies) in 1970.

The calibration strategy is to first set the vector \((\beta_{G7}, \beta_{Emg}, \theta_{G7}, \theta_{Emg}, \eta)\) exogenously. For the baseline parameterization these parameters are set to be identical across countries so as to retain the focus of the analysis on recursive preferences and the key aspect of heterogeneity in preferences emanating from the non-additively separable component of preferences. Accordingly, we set \( \beta_{G7} = \beta_{Emg} = 0.97 \) and \( \theta_{G7} = \theta_{Emg} = 0.8 \). Lastly, \( \eta \) is set equal to 1, which implies that efficiency adjusted labor from different countries are perfect substitutes. \( \beta = 0.97 \) is relatively standard number for the discount factor in cross-country studies with additively separable preferences.

The regional \( l'_i \)s are chosen to match the relative labour shares in 1970 which were 0.205 and 0.795 for the G7 and the emerging economies, respectively. The vector \((\delta_{G7}, \delta_{Emg}, \gamma_{G7}, \gamma_{Emg})\) is then set so that the steady state values of the regional saving rates and relative world capital shares are equal to those observed in 1970. Note that the procedure targets the regional saving rates and world capital shares. Hence, four parameters are chosen. The parameter choices and the data targets for 1970 are summarized in Table 3.1 below:

To illustrate the mechanics of the model the response of the saving rates of the two regions to a one standard deviation positive shock to the common productivity process \( z \) is plotted. Figure 3.4 shows that the two regions respond differently to the same productivity shock. Specifically, the

\[ ^{64} \text{Below a richer five-country world is calibrated by splitting the Emerging economies into four groups. The two-}
\text{country specification is first presented in order to build intuition.} \]
3.5. Quantifying the Model

Table 3.1: Parameterization of Baseline Model

<table>
<thead>
<tr>
<th></th>
<th>Data moments (1970)</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>saving rate</td>
<td>world capital share</td>
</tr>
<tr>
<td>G7</td>
<td>0.247</td>
<td>0.698</td>
</tr>
<tr>
<td>Emerging economies</td>
<td>0.171</td>
<td>0.302</td>
</tr>
</tbody>
</table>

$(\beta_{G7}, \beta_{Emg}, \theta_{G7}, \theta_{Emg}, \eta) = (0.98, 0.98, 0.8, 0.8, 1)$

This table summarizes the regional data targets and the corresponding parameter choices to enable the baseline model to match those moments.

The saving rate in the G7 rises less and declines faster than in the emerging economies. The reason for this is the increasing marginal impatience that is built into these preferences. The high consumption region (the G7) is also more impatient and hence responds less and adjusts downwards faster in response to the same increase in the real interest rate relative to the emerging economies. The stationary world wealth distribution is non-degenerate precisely due to this feature of preferences.

Figure 3.4: Impulse Response of Savings to Productivity

Note: The graph depicts the response of saving rates in the two regions to a positive one standard deviation shock to the common productivity process $z_t$.

So, how well does the model explain world saving behavior? This is examined next by generating the saving from the model in response to the measured productivity shocks in the data between 1970 and 2010. Recall that the model was parameterized to mimic data in 1970. All those parameters are kept fixed across time for these simulations. Figure 3.5 shows the model and data generated saving rates for the G7 and the emerging economies. Panel (a) shows the time series of savings for the two regions while panel (b) of Figure 3.5 plots the difference between saving rates of the two
3.5. Quantifying the Model

regions in the model and the data. The figures make clear that the model tracks the trend behavior of saving rates in the two regions quite well. This shows up clearly in panel (b) where the differences in saving rates between the regions in the model closely tracks the data. The noteworthy aspect of this is that, despite the fact that the model has not been calibrated to match the time series behavior of the regional saving rates, it tracks turning points in the saving rates of both regions quite closely with just one common exogenous productivity process.

![Figure 3.5: Saving Rates: Data & Model](image)

(a) time series of saving rates
(b) difference in saving rates

Notes: Panel (a) shows the saving rates in the G7 and emerging economies in the model and the data between 1970 and 2010. Panel (b) shows the difference between the saving rates of the G7 and emerging economies in the model and the data during this period.

Table 3.2 also shows that the baseline model performs quite well in reproducing the correlation of saving rates between the two regions. Additionally, the model generated saving rates also correlate quite strongly with the actual saving rates for each region individually. How sensitive are the simulation results to the assumed parameter values for \( \theta \), a parameter for which direct estimates don’t exist? Table 3.2 suggests that the results of the baseline model are quite robust to changing \( \theta \). Neither the volatility nor the correlation statistics change too much in response to changing the baseline symmetric specification \((\theta_{G7}, \theta_{Dev}) = (0.8,0.8)\) to an asymmetric one.

Overall, these results suggest that the model does a good job of accounting for saving behavior over time both within regions as well as differences across regions of the world.
3.5. Quantifying the Model

Table 3.2: Comparing Savings in the Model and the Data

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(θ_{G7}, θ_{Dev})</td>
<td>(0.8, 0.8)</td>
</tr>
<tr>
<td>std dev(s_{G7})</td>
<td>0.981</td>
<td>2.694</td>
</tr>
<tr>
<td>std dev(s_{Dev})</td>
<td>1.099</td>
<td>2.940</td>
</tr>
<tr>
<td>corr(s_{G7}, s_{Dev})</td>
<td>0.662</td>
<td>0.973</td>
</tr>
<tr>
<td>corr(s_{G7, model}, s_{G7, data})</td>
<td>0.655</td>
<td>0.653</td>
</tr>
<tr>
<td>corr(s_{Dev, model}, s_{Dev, data})</td>
<td>0.572</td>
<td>0.568</td>
</tr>
</tbody>
</table>

Note: The table reports moments of saving rates in the data and the model for the period 1970-2010.

3.5.1 Adjustment Costs

The feature of the data that the model does not reproduce as well is the volatility of saving. In the data saving rates are much smoother than in the model. Table 3.2 summarizes the data and model moments of saving rates in the two regions. The table makes clear that under the baseline parameterization, the volatility of saving rates in the model is three-fold larger than in the data. This is mainly due to the fact that investment is too volatile in the model relative to the data. As has been noted by countless authors, this problem is easy to address by introducing adjustment costs on investment. Adjustment costs are introduced into the model by modifying the capital accumulation equation in each country as

\[
k_{it+1} = (1 - d) k_{it} + \ell_{it} - k_{it} \phi \left( \frac{\ell_{it}}{k_{it}} \right), \quad k_{i0} \text{ given for } i = 1, \ldots, N
\]

where \( \phi' > 0, \phi'' > 0 \). In calibration considered here the adjustment cost function is given by the quadratic form:

\[
\phi \left( \frac{\ell}{k} \right) = \frac{b}{2} \left( \frac{\ell}{k} + 1 - d - g \right)^2
\]

This specification implies that adjustment costs are zero along any deterministic steady state growth path. Moreover, the model reduces to one with no adjustment costs when \( b = 0 \). A crucial assumption underlying this formulation is that the adjustment cost function is common across countries, i.e., there is a common \( b \). Allowing \( b \) to be country-specific would simply improve the potential fit of the model.
3.5. Quantifying the Model

The parameter $b$ is calibrated to match the standard deviation of $\sum_i \omega_i s_i$, where $\omega_i = \frac{t_{it} k_{it}}{\sum_i t_{it} k_{it}}$ is the world capital share of country $i$. Hence, $b$ is picked to match the volatility of world savings. In the two-country world analyzed above, the implied $b$ is 0.47. The rest of the parameter configuration for the model under adjustment costs is summarized in Table 3.3 below:

<table>
<thead>
<tr>
<th>Parameters</th>
<th>G7</th>
<th>Emerging economies</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_i$</td>
<td>0.978</td>
<td>0.969</td>
</tr>
<tr>
<td>$\gamma_i$</td>
<td>2.803</td>
<td>0.535</td>
</tr>
<tr>
<td>$b$</td>
<td>0.47</td>
<td>0.47</td>
</tr>
</tbody>
</table>

$(\beta_{G7}, \beta_{Emg}, \theta_{G7}, \theta_{Emg}, \eta) = (0.98, 0.98, 0.8, 0.8, 1)$

This table summarizes the parameter choices to enable the model with adjustment costs to match the targeted data moments.

How does the model’s predictions for the regional saving rates change with adjustment costs? Figure 3.6 shows the predicted saving rates for the two economies:

Figure 3.6: Saving Rates Under Adjustment Costs: Data & Model

Notes: The figure shows the saving rates in the G7 and emerging economies in the model with adjustment costs and the data between 1970 and 2010.

Figure 3.6 shows that the introduction of adjustment costs allows the model to match the volatility of country-specific saving rates without changing the fit of the average level of saving. What is possibly somewhat surprising is that the model is able to match the individual saving...
3.5. Quantifying the Model

volatilities even with a common adjustment cost function. This reflects the fact that the primary
difference in saving behavior across countries is the level of savings rather than their volatility which
tends to be relatively similar.

3.5.2 A Five-Region Extension

The model presented above aggregated the world economy into two broad regions – the G7 and
Emerging economies. However, there is a lot of dispersion in saving behavior across the Emerging
group which comprises countries from sub-Saharan Africa, south Asia as well as Latin America.
Can the model capture the saving heterogeneity across such a wide spectrum of countries? This is
answered this by expanding the two-region model to a five-region economy. This done by breaking
up the Emerging economy group into four groups of countries: sub-Saharan Africa, Latin America,
Middle East and North Africa, and Developing Asia. The data appendix provides the list of
countries in each group.

As before $\delta$ and $\gamma$ are allowed to vary across regions. These parameters are calibrated to match
the saving rates in each region in 1970. The resulting parameter configuration for the 5-country
world economy is given in Table 3.4. The adjustment cost parameter $b$ is set so that the standard
deviation of capital share-weighted world saving in the model matches the data value. The model
is solved to a 2nd order approximation.

Table 3.4: Parameterization of 5-Region Model With Adjustment Costs

<table>
<thead>
<tr>
<th>1970 data targets</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Saving rate</td>
</tr>
<tr>
<td>G7</td>
<td>0.247</td>
</tr>
<tr>
<td>Sub-Saharan Africa</td>
<td>0.126</td>
</tr>
<tr>
<td>Latin America</td>
<td>0.147</td>
</tr>
<tr>
<td>Middle East and North Africa</td>
<td>0.317</td>
</tr>
<tr>
<td>Developing Asia</td>
<td>0.165</td>
</tr>
</tbody>
</table>

$(\beta, \theta, \eta, b) = (0.98, 0.87, 1, 0.39)$

This table summarizes the parameter choices to enable the five region model with
adjustment costs to match the targeted data moments.

How well does the model perform in matching the time series behavior of regional saving rates
in the data? The particular interest of this work lies in examining the ability of the model to match
the variation of saving rates across regions. Table 3.5 compares the key moments of interest in
3.6 Saving Miracles

The second motivation of this work was the observation that some countries have shown sudden and dramatic increases in their average saving rates. We showed an example of this in Figure 3.2 which depicted the sharp pickup in the average saving rate of the Asian Tigers. The chapter now turns to demonstrating how the model can accommodate these dramatic saving miracles. The principal idea is that saving behavior is dictated by our goals which, in turn, is often determined by our position relative to a comparison group. If a society begins to aspire to have the wealth levels of a much richer comparison group then its saving levels have to respond to achieve that new goal. A key feature of the recursive preference structure that has been formalized here is the presence of
3.6. Saving Miracles

relative consumption. In the model presented in the previous section the relative consumption level in preferences was just the world average per capita consumption. In this section the consequences of a country changing its relative comparison group from the average world level to a richer cohort will be examined. Could such a change generate an increase in saving rates similar in magnitude to the rise in Asian savings we saw in Figure 3.2? It should be clarified at the outset that this chapter is not building a theory of aspirations. Rather, it is quantitatively exploring the dynamic general equilibrium consequences of a change in aspirations.

Consider a world economy comprising three regions. Let the two regions now be the G7, Emerging economies, and the Asian Tigers. In other words, the world economy is now being explained from the two-region perspective of the previous section, with the Asian Tigers also included. Recall that at each date $t$ the externality in preferences for each $i$ is denoted by $\zeta_i$. Let average per capita world consumption be

$$C = l_{G7}c_{G7} + l_{Emg}c_{Emg} + l_{Tigers}c_{Tigers}$$

where $c_i$ is the per capita consumption of region $i$. Consider two regimes:

Regime 1: $\zeta_{G7} = \zeta_{Emg} = \zeta_{Tigers}$

Regime 2: $\zeta_{G7} = \zeta_{Emg} = C$ ; $\zeta_{Tigers} = \zeta_{G7}$

Under Regime 1 all three regions value their own consumption relative to the world per capita consumption level. Under Regime 2 however, the Asian Tigers switch their comparison group to the G7 while the other two regions continue to use the world average as the relevant consumption comparison group. We consider an environment where at some date $t^*$, the regime switches from Regime 1 to Regime 2. Given that per capita consumption is higher in the G7, such a regime switch represents a switch to a higher aspiration level for the Asian Tigers.

In the context of the model, could such a regime switch account for the almost 15 percentage point increase in the saving rate of the Asian Tigers since 1970? To answer this question the model is calibrated by choosing parameter values such that the model reproduces the steady state saving rates, world capital shares and world labor shares of the three regions in 1960 under the
maintained assumption of the world being in Regime 1. The model is then perturbed with a regime switch to Regime 2 in 1970, where the reference consumption level for the Tigers increases to the per capita consumption of the G7 economies. Keeping the parameters underlying the initial calibration unchanged, the estimated productivity process from 1960 to 2010 is then fed into the model and simulate the equilibrium paths for the three economies. The model generated saving rates are then compared with their data counterparts.

To clarify the role of the regime switch in generating the saving increase, Figure 3.7 shows the saving rates of the G7 and the Tigers when all productivity movements are shut down and only the regime switch in 1970 is allowed. Since the world economy is in steady state in 1960, saving rates are constant till 1970 when the regime switch occurs. From that date onward savings of the Tigers rises while the G7 saving rate initially declines before recovering towards its original level.

Figure 3.7: Response of Savings to Regime Switch

Note: The graph depicts the response of saving rates in the two regions when the only shock is a change in the reference consumption level of the Asian Tigers to per capita consumption of the G7 in 1970.

What are the predicted saving rates of the model when the measured productivity shocks are incorporated? Panel (a) of Figure 3.8 shows the simulated path of savings under the estimated productivity process between 1960 and 2010 with a regime switch from Regime 1 to Regime 2 in 1970. As a point of contrast, panel (b) of Figure 3.8 plots the saving rates under the assumption

65The decline in the G7 saving on impact occurs due to the fall in the interest rate that is induced by the rise in the desired savings of the Tigers. Note that the saving rate of the Emerging economies is not plotted here in order to keep the graph uncluttered.
of no change in regime. The fit in Panel (a) is considered to be quite remarkable in terms of how well the model-generated saving rates track the actual saving rates.

![Figure 3.8: Saving Miracles: Data & Model](image)

(a) saving rates under regime switch  
(b) saving rates without regime switch  
Notes: Panel (a) shows the saving rates in the G7 and the Asian Tiger economies in the model under a switch in the comparison group for the Tigers with the data between 1970 and 2010. Panel (b) shows the saving rates of the G7 and the Asian Tigers in the model without any switch in regime.

In summary, the switch to a higher aspiration is key for the model to reproduce the sharp increase in the saving rates of the Asian Tigers. This result is indicative of the power of the aspiration mechanism to explain the rapid growth of savings in Asia.

### 3.7 Conclusion

The variation in saving behavior across countries has long been a puzzle and a challenge to explain for standard neoclassical models. This chapter has explored the explanatory potential of recursive preferences and preference heterogeneity in jointly accounting for the cross-country saving data. The model presented here used a preference specification that displays a form of relative consumption. Specifically, agents of a country derive utility from consumption relative to the consumption of a reference group. The specification implies that when countries are poor they display high patience and high saving rates. As their consumption gets closer to the levels of their reference group however they become more impatient, a property that Lucas and Stokey (1984) called “increasing marginal impatience”. This feature of preferences keeps the wealth distribution from becoming degenerate even when preferences are heterogenous across countries.
3.7. Conclusion

These preferences were applied to a multi-country world economy model with free capital flows across countries and the model was calibrated to match the long-run differences in saving rates across countries. Using only productivity shocks to a common world production technology as an exogenous driver, it was then shown that the calibrated model can also account for the short-run differences in saving rates across countries.

In addition, it was also shown that a change in the aspirations of societies, as captured by a change in the reference consumption basket they use to value their own utility, can account for sudden and sharp changes in saving rates. Thus, the model can account for the rapid increase in Asian saving rates and its overall behavior between 1960 and 2010 by allowing for a change in the reference basket being used by the Asian economies from the average world consumption level to the G7 consumption level in 1970. Intuitively, a higher reference consumption level induces greater saving as accumulating greater wealth is the only way to achieve a higher steady state consumption.

The results in this chapter indicate that this class of models has great potential in also helping us understand changes in the wealth distribution within countries over time. Wealth evolves as a function of saving. Accounting for differential saving rates is thus key to explaining wealth distributions and changes therein. We hope to address this issue in future work.
Adam, Klaus, Johannes Beutel, and Albert Marcet. 2014. “Stock Price Booms and Expected Capital Gains.” Working Papers 14-12, University of Mannheim, Department of Economics.


Biblilography


Appendix A

Chapter 1, Additional Figures

Figure A.1: Maximum Log Price Deviation and Length of Propagation Following $R$ Drop

(a) Maximum Log Price Deviation

(b) Length of Propagation Following $R$ Drop
Appendix B

Chapter 1, Aggregation of Household Problem to Representative Agent

This appendix outlines the aggregation of the household problem in section 1.3.1 to a representative agent. Assume that all notation is in terms of individual household \(i\) in a heterogeneous agents economy. All agents are assumed to have the same information, preferences, credit constraint, etc... Assume that the households have access to a complete set of Arrow-Debreu securities in consumption goods. Also assume that the household has access to a security that is indexed to house prices (in other words, a security that delivers in units of housing). Assume that for both of these securities the household is exempt from obligations/payments if it defaults. For ease of notation, physical capital \(k_t\) is omitted from the exposition, but its presence does not alter the steps outlined below.

- The budget constraint and credit constraint of of household \(i\) are

\[
\begin{align*}
    c_t^i + q_t h_t^i + Rb_{t-1}^i + p_t^c e^c_t + p_t^h e^h_t &= y_t^i + (1-\delta)h_{t-1}^i + b_t^i + e^c_{t-1} + q_t e^h_{t-1} \\
    b_t^i &\leq \left( \frac{\theta}{R} \right) \tilde{E}_t[q_{t+1}]h_t^i \equiv \tilde{Q}_t h_t^i
\end{align*}
\]

these imply

\[
\begin{align*}
    c_t^i + (q_t - \tilde{Q}_t) h_t^i &= \tilde{y}_t^i + \left\{ (1-\delta)q_t - R\tilde{Q}_{t-1} \right\} h_{t-1}^i \\
    \text{where} \\
    \tilde{y}_t^i &= y_t^i - p_t^c e^c_t - p_t^h e^h_t + q_t e^h_{t-1} + e^c_{t-1}
\end{align*}
\]

- Let

\[
1 - \Gamma_t = \Pr \left( \frac{q_t}{q_{t-1}} < \zeta_t \frac{\theta}{1-\delta} \right) = \Pr \left( \text{Default} \left| \frac{q_t}{q_{t-1}} \right. \right)
\]

- If the household defaults, it loses its stock of housing and no longer has to repay its debt \(b_{t-1}\)

- The household’s maximization problem can be written as follows

\[
\max \tilde{E}_t \left\{ \sum_t \beta^t \left( u(c_t^i, h_t^i) + \lambda_t^i \left( \begin{array}{c}
    y_t^i + (1-\delta)q_t \Gamma_t h_{t-1}^i + b_t^i + \Gamma_t e^c_{t-1} + q_t \Gamma_t e^h_{t-1} - c_t^i - q_t h_{t-1}^i - R\Gamma_t b_{t-1}^i - p_t^c e^c_t - p_t^h e^h_t \\
    + p_t^i \left( \theta \tilde{E}_t[q_{t+1}] h_t^i - Rb_t^i \right)
  \end{array} \right) \right) \right\}
\]
Appendix B. Chapter 1, Aggregation of Household Problem to Representative Agent

The first order conditions for this problem are

\[ p^c_t = \beta \cdot \mathbb{E}_t^P \left( \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} \right) \] (B.7)

\[ p^h_t = \beta \cdot \mathbb{E}_t^P \left( \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} q_{t+1} \right) \] (B.8)

\[ \lambda_t^i = u_c(c_t^i, h_t^i) \equiv u_c(t) \] (B.9)

\[ \mu_t^i = \frac{\lambda_t^i}{R} - \beta \lambda_{t+1}^i \Gamma_{t+1} \] (B.10)

\[ q_t \lambda_t^i = u_h(t) + \mu_t^i q_{t+1} + \beta (1 - \delta) \lambda_{t+1}^i q_{t+1} \Gamma_{t+1} \] (B.11)

\[ \Rightarrow q_t = \frac{u_h(t)}{u_c(t)} + \mathbb{E}_t^P \left[ \left( \beta (1 - \delta) \cdot \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} \right) \cdot q_{t+1} - \beta \theta \left( \mathbb{E}_t^P q_{t+1} \right) \lambda_{t+1}^i \Gamma_{t+1} \right] \] (B.12)

From the securities market:

\[ p^c_t = \beta \cdot \mathbb{E}_t^P \left[ \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} \right] = \beta \cdot \mathbb{E}_t^P \left[ \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} \right] \] (B.13)

\[ p^h_t = \beta \cdot \mathbb{E}_t^P \left[ \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} q_{t+1} \right] = \beta \cdot \mathbb{E}_t^P \left[ \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} q_{t+1} \right] \] (B.14)

– So these markets imply that the price of consumption and default-weighted housing is the same for all agents.

Rewrite the budget constraint, and combine with the assumption that the borrowing constraint binds

\[ c_t^i + q_t h_t^i + R \tilde{Q}_{t-1} h_{t-1}^i = \tilde{y}_t^i + (1 - \delta) q_t h_{t-1}^i + \tilde{Q}_t h_t^i \] (B.15)

\[ \Rightarrow c_t^i + (q_t - \tilde{Q}_t) h_t^i = \tilde{y}_t^i + (1 - \delta) q_t (1 - R \tilde{Q}_{t-1}) h_{t-1}^i \] (B.16)
Appendix B. Chapter I, Aggregation of Household Problem to Representative Agent

- From the household Euler

\[
q_t = \frac{1 - \gamma c}{\gamma c} \cdot \frac{c_t^i}{h_t^i} + \beta (1 - \delta) \mathbb{E}_t^P \left[ \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} q_{t+1} \right] - \beta \mathbb{E}_t^P q_{t+1} \mathbb{E}_t^P \left[ \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} \right] \quad \text{common across agents}
\]

\[
\Rightarrow \frac{c_t^i}{h_t^i} = \frac{1 - \gamma c}{\gamma c} \left\{ q_t - \tilde{Q}_t \beta (1 - \delta) \mathbb{E}_t^P \left[ \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} q_{t+1} \right] + \beta \mathbb{E}_t^P q_{t+1} \mathbb{E}_t^P \left[ \frac{\lambda_{t+1}^i}{\lambda_t^i} \Gamma_{t+1} \right] \right\}
\]

\[
\equiv B_t, \text{ common across agents}
\]

\[
\Rightarrow c_t^i = B_t h_t^i \quad \text{(B.17)}
\]

- Substitute this into the modified budget constraint \((B.16)\)

\[
c_t^i + (q_t - \tilde{Q}_t) h_t^i = \tilde{y}_t^i + \left( (1 - \delta) q_t - R \tilde{Q}_{t-1} \right) h_{t-1}^i
\]

\[
\Rightarrow c_t^i + (q_t - \tilde{Q}_t) B_t^{-1} c_t^i = \tilde{y}_t^i + \left( (1 - \delta) q_t - R \tilde{Q}_{t-1} \right) h_{t-1}^i
\]

\[
\Rightarrow \left( 1 + (q_t - \tilde{Q}_t) B_t^{-1} \right) c_t^i = \tilde{y}_t^i + \left( (1 - \delta) q_t - R \tilde{Q}_{t-1} \right) \sum h_{t-1}^i
\]

\[
\Rightarrow \left( 1 + (q_t - \tilde{Q}_t) B_t^{-1} \right) \sum c_t^i = \sum \tilde{y}_t^i + \left( (1 - \delta) q_t - R \tilde{Q}_{t-1} \right) \sum h_{t-1}^i \quad \text{(B.18)}
\]

- Note that

\[
\sum_i c_t^i = C_t
\]

\[
\sum_i h_t^i = H_t
\]

\[
\sum_i \tilde{y}_t^i = \sum_i \left( y_t^i - p_t^i c_t^i - p_t^h e_t^i + q_t e_{ht-1}^i + e_{ct-1}^i \right)
\]

\[
= Y_t - p_t^i \sum_i e_{ct}^i - p_t^h \sum_i e_{ht}^i + q_t \sum_i e_{ht-1}^i + \sum_i e_{ct-1}^i
\]

where the zeros are the result of market clearing in securities

\[
\Rightarrow \left( 1 + (q_t - \tilde{Q}_t) B_t^{-1} \right) C_t = Y_t + \left( (1 - \delta) q_t - R \tilde{Q}_{t-1} \right) H_{t-1} \quad \text{(B.19)}
\]

which is equivalent to a representative agents’ choice
Appendix C

Chapter [1], Solving the Household’s Problem

The solution to the household problem is computed using a parameterized expectations algorithm (PEA). Rewriting the household Euler (1.17) in simplified notation

\[ q_t - u_h(t) = E_t^P [\Omega(q_{t+1}, c_{t+1}, h_{t+1}) | c_t, h_t, b_t, q_t, m_t, \theta_t, y_t, R_t] \] (C.1)

The expectation on the right hand side is approximated as

\[ \exp (e(X_t, \gamma)) \] (C.2)

The basis functions for \( e(\cdot) \) are the Chebyshev polynomials. The vector of state variables for the household, \( X_t \), is \( (h_{t-1}, k_{t-1}, q_t, m_t, \theta_t, y_t, R_t) \). Define \( \gamma \) to be a coefficient vector. The order of the approximation along each dimension is given by \( (r_h, r_k, r_q, r_m, r_\theta, r_y, r_\theta) \), hence \( \gamma \) is an \( r_h \times r_k \times r_q \times r_m \times r_\theta \times r_y \times r_\theta \) vector. Let \( \psi_i(x) \) denote the Chebyshev polynomial of order \( i \) evaluated at \( x \). Because the Chebyshev polynomials are defined on \([-1, 1]\) the state variables are rescaled according to

\[ \tilde{x} = 2 \frac{x - \bar{x}}{\bar{x} - \bar{x}} - 1 \in [-1, 1] \]

The approximating function \( e(X_t, \gamma) \) can be written

\[ e((h, k, q, m, \theta, y, R), \gamma) = T(h, k, q, m, \theta, y, R) \gamma \]

\[ = (T_h(h) \otimes T_k(k) \otimes T_q(q) \otimes T_m(m) \otimes T_\theta(\theta) \otimes T_y(y) \otimes T_R(R)) \gamma \] (C.3)

where \( T_x(x) = (\psi_0(\tilde{x}), \psi_1(\tilde{x}), ..., \psi_{r_x}(\tilde{x})) \)

The coefficient vector \( \gamma \) is solved by appealing to the Chebyshev interpolation theorem and setting the approximating function \( e(\cdot) \) to be exact at the Chebyshev nodes. \( \gamma \) therefore solves the system

\[ \exp (E_t^P [\Omega(\cdot)]) - e((h, k, q, m, \theta, y, R), \gamma) = 0 \ \forall \ \text{vectors of nodes } (h, k, q, m, \theta, y, R) \] (C.4)

which is an \( r_h \times r_k \times r_q \times r_m \times r_\theta \times r_y \times r_\theta \) system of equations with \( r_h \times r_k \times r_q \times r_m \times r_\theta \times r_y \times r_\theta \) unknowns. (C.4) is solved by an iterative procedure. Let \( \hat{x} \) denote the \( r_x \)-vector of Chebyshev

\[ 66\text{In order to limit the size of the space the following approximation is applied } b_t \approx \frac{\theta_t}{m_t} q_t m_t h_t. \]
nodes along dimension $x$ and $\hat{X} = \hat{h} \times \hat{k} \times \hat{q} \times \hat{m} \times \hat{\theta} \times \hat{y} \times \hat{R}$

1. Start from an initial guess $\gamma_0$

2. Compute the vector $e(\cdot, \gamma_0)$ for all Chebyshev nodes $\hat{h} \times \hat{k} \times \hat{q} \times \hat{m} \times \hat{\theta} \times \hat{y} \times \hat{R}$ using (C.3)

$$e(\hat{h} \times \hat{k} \times \hat{q} \times \hat{m} \times \hat{\theta} \times \hat{y} \times \hat{R}, \gamma_0) = T(\hat{h}, \hat{k}, \hat{q}, \hat{m}, \hat{\theta}, \hat{y}, \hat{R}) \cdot \gamma_0$$

$$\equiv B \cdot \gamma_0$$

3. Given $e(X, \gamma_0)$ one can back out a corresponding vector of choice variables $(c_t, h_t, b_t, k_t)$ from the model equations outlined in section 1.3.1, call this $\upsilon(X, \gamma_0)$. Compute the expectation $E_{P_t}[\Omega(\cdot)]$ at all the Chebyshev nodes using a Gaussian quadrature, call this vector $Y(\hat{X}, \gamma_0)$,

$$Y(\hat{X}, \gamma_0) = \int \Omega(q_{t+1}, c_{t+1}(v(\hat{X}, \gamma_0)), h_{t+1}(v(\hat{X}, \gamma_0)) | \hat{X}, v(\hat{X}, \gamma_0)) dF(y_{t+1}, q_{t+1}|q_t, y_t, P)$$

4. Compute

$$\gamma^* = (B'B)^{-1} B' \log Y(\hat{X}, \gamma_0)$$

5. Update the guess

$$\gamma_1 = \omega \gamma^* + (1 - \omega) \gamma_0$$

for $\omega \in (0, 1)$

6. Repeat 1-5 until convergence

---

67 The procedure fails to converge when $\gamma_1 = \gamma^*$, necessitating this step.

68 Further details on such methods can be found in Christiano and Fisher (2000).
Appendix D

Chapter 2, Additional Figures

Figure D.1: Effect of Internalizing Learning

permanent shock to $c$: $\gamma = 5, \psi = 6, g = 0.01$
Figure D.2: Effect of Preference for Timing of Risk in Learning Model

(a) ω response to transitory shock, $\gamma = 5, \psi = 1/1.5$

(b) ω response to transitory shock, $\gamma = 5, \psi = 6$

(c) ω response to permanent shock, $\gamma = 5, \psi = 1/1.5$

(d) ω response to permanent shock, $\gamma = 5, \psi = 6$
Figure D.3: Influence of $g$: Transitory & Permanent Shocks in Learning Model

(a) $\omega : g = .002$, transitory shocks
(b) $\omega : g = .01$, transitory shocks
(c) $\omega : g = .002$, permanent shocks
(d) $\omega : g = .01$, permanent shocks
Appendix E

Chapter 2, Learning

This appendix outlines the derivation of the updating rule for beliefs (2.17). Write the household’s belief as a linear hidden state model

$$
\begin{pmatrix}
\log r_t \\
\log \tilde{w}_t
\end{pmatrix}
= 
\begin{pmatrix}
\omega_{0,t}^r \\
\omega_{0,t}^w
\end{pmatrix}
+ 
\begin{pmatrix}
\omega_{1,t}^r \\
\omega_{1,t}^w
\end{pmatrix} \cdot \log \tilde{k}_t + \epsilon_t^z
$$
(E.1)

$$
\omega_t = \omega_{t-1} + \epsilon_t^r
$$
(E.2)

where $\epsilon \sim N(0, \Sigma^z)$ and $\epsilon^w \sim N(0, \Sigma^w)$. Equation (E.1) is the observation equation and (E.2) the state equation. As in section 2.3.3 Write $x_t = [\log r_t, \log \tilde{w}_t]'$ and

$$q_t = 
\begin{pmatrix}
1 & 0 & \log \tilde{k}_t & 0 \\
0 & 1 & 0 & \log \tilde{k}_t
\end{pmatrix}
$$

Then optimum Bayesian updating is given by the Kalman filtering equations

$$
\hat{\omega}_t = \hat{\omega}_{t-1} + K_t(z_t - q_t \hat{\omega}_{t-1})
$$
(E.3)

$$K_t = P_t q_t'(q_t P_t q_t' + \Sigma^z)^{-1}
$$
(E.4)

$$P_{t+1} = P_t - P_t q_t'(q_t P_t q_t' + \Sigma^z)^{-1} q_t P_t + \Sigma^w
$$
(E.5)

Assume that as $t \to \infty$

$$q_t P_t q_t' + \Sigma^z \approx \Sigma^z
$$
(E.6)

$$q_t'(\Sigma^z)^{-1} q_t \to \Gamma
$$
(E.7)

Then, assuming $\Sigma^\omega = \tilde{\gamma}^2 \Gamma^{-1}$ and $P_t \to P$ as $t \to \infty$, where $P = \tilde{\gamma} \cdot \Gamma^{-1}$, then (E.3) and (E.4) can be rewritten

$$\hat{\omega}_t = \hat{\omega}_{t-1} + \tilde{\gamma} \cdot q'(qq')^{-1}(\Sigma^z)^{-1}(x_t - q_t \hat{\omega}_{t-1})
$$
(E.8)

If $\epsilon_t^r$ and $\epsilon_t^w$ are uncorrelated then (E.8) can be rewritten

$$\hat{\omega}_t = \hat{\omega}_{t-1} + g \cdot q'(qq')^{-1}(x_t - q_t \hat{\omega}_{t-1})
$$
(E.9)
Appendix F

Chapter 2, Solution Method

The solution to the household problem is computed using a parameterized expectations algorithm (PEA). Rewriting the household Euler (2.5) and value function (2.1) in simplified notation

\[
c_t^{-\psi} = \mathbb{E}_t^P \left[ \Omega_e (c_{t+1}, V_{t+1}) | k_t, \epsilon_t^r, \epsilon_t^w, \omega_{0,t-1}^r, \omega_{1,t-1}^r, \omega_{0,t-1}^w, \omega_{1,t-1}^w \right] \tag{F.1}
\]

\[
V_t = \Omega_v (k_t, \epsilon_t^r, \epsilon_t^w, \omega_{0,t-1}^r, \omega_{1,t-1}^r, \omega_{0,t-1}^w, \omega_{1,t-1}^w) \tag{F.2}
\]

The expectation on the right hand side is approximated as

\[
\exp \left( e(X_t, \gamma_e) \right) \tag{F.3}
\]

The basis functions for \( e(\cdot) \) are the Chebyshev polynomials. Similarly approximate the value function as

\[
v(X_t, \gamma_v) \tag{F.4}
\]

The vector of state variables for the household, \( X_t \), is \( (k_t, \epsilon_t^r, \epsilon_t^w, \omega_{i,t-1}) \). Define \( \gamma_e \) and \( \gamma_v \) to be coefficient vectors. The order of the approximation along each dimension is given by \( (r_k, r_{\epsilon,r}, r_{\epsilon,w}, r_{\omega,r,0}, r_{\omega,r,1}, r_{\omega,w,0}, r_{\omega,w,1}) \), hence \( \gamma_j \) is an \( (r_k \times r_{\epsilon,r} \times r_{\epsilon,w} \times r_{\omega,r,0} \times r_{\omega,r,1} \times r_{\omega,w,0} \times r_{\omega,w,1}) \) vector. Let \( \psi_i(x) \) denote the Chebyshev polynomial of order \( i \) evaluated at \( x \). Because the Chebyshev polynomials are defined on \( [-1, 1] \) the state variables are rescaled according to

\[
\tilde{x} = 2 \frac{x - \bar{x}}{x - \bar{x}} - 1 \in [-1, 1]
\]

The approximating function \( e(X_t, \gamma) \) can be written

\[
e \left( (k, \epsilon^r, \epsilon^w, \omega_{0}^r, \omega_{1}^r, \omega_{0}^w, \omega_{1}^w), \gamma_e \right) = T(k, \epsilon^r, \epsilon^w, \omega_{0}^r, \omega_{1}^r, \omega_{0}^w, \omega_{1}^w) \gamma_e \tag{F.5}
\]

\[
v \left( (k, \epsilon^r, \epsilon^w, \omega_{0}^r, \omega_{1}^r, \omega_{0}^w, \omega_{1}^w), \gamma_v \right) = T(k, \epsilon^r, \epsilon^w, \omega_{0}^r, \omega_{1}^r, \omega_{0}^w, \omega_{1}^w) \gamma_v \tag{F.6}
\]

where \( T_x(x) = (\psi_0(\tilde{x}), \psi_1(\tilde{x}), ..., \psi_{r_x}(\tilde{x})) \)

The coefficient vectors \( \gamma_e \) and \( \gamma_v \) are solved by appealing to the Chebyshev interpolation theorem and setting the approximating functions \( e(\cdot) \) \( v(\cdot) \) to be exact at the Chebyshev nodes. \( \gamma_e \) and \( \gamma_v \)
therefore solve the system

\[
\begin{align*}
\exp \left( \mathbb{E}^P_t [\Omega(\cdot)] \right) - e \left( ((k, e^r, e^w, \omega^r_0, \omega^w_0, \omega^r_1, \omega^w_1, \gamma_e) = 0 \right. \\
\Omega_v(\cdot) - v \left( ((k, e^r, e^w, \omega^r_0, \omega^w_0, \omega^r_1, \omega^w_1, \gamma_v) = 0 \right.
\end{align*}
\] (F.7)

which is a \( 2 \times r_k \times r_{e,r} \times r_{e,w} \times r_{r,0} \times r_{r,1} \times r_{w,0} \times r_{w,1} \) system of equations with \( 2 \times r_k \times r_{e,r} \times r_{e,w} \times r_{r,0} \times r_{r,1} \times r_{w,0} \times r_{w,1} \) unknowns. (F.7) is solved by an iterative procedure. Let \( \hat{x} \) denote the \( r_x \)-vector of Chebyshev nodes along dimension \( x \) and \( \hat{X} = \hat{k} \times \hat{e}^r \times \hat{e}^w \times \hat{\omega}^r_0 \times \hat{\omega}^r_1 \times \hat{\omega}^w_0 \times \hat{\omega}^w_1 \) unknowns. (F.7) is solved by an iterative procedure. Let \( \hat{x} \) denote the \( r_x \)-vector of Chebyshev nodes along dimension \( x \) and \( \hat{X} = \hat{k} \times \hat{e}^r \times \hat{e}^w \times \hat{\omega}^r_0 \times \hat{\omega}^r_1 \times \hat{\omega}^w_0 \times \hat{\omega}^w_1 \)

1. Start from an initial guess \( \{ \gamma_{e,0}, \gamma_{v,0} \} \)

2. Compute the vectors \( e(\cdot, \gamma_{e,0}) \) and \( v(\cdot, \gamma_{v,0}) \) for all Chebyshev nodes \( \hat{X} \) using (F.5)

\[
\begin{align*}
e \left( \hat{X}, \gamma_{e,0} \right) &= T(\hat{X}) \cdot \gamma_{e,0} \\
&\equiv B \cdot \gamma_{e,0} \\
v \left( \hat{X}, \gamma_{v,0} \right) &= T(\hat{X}) \cdot \gamma_{v,0} \\
&\equiv B \cdot \gamma_{v,0}
\end{align*}
\]

3. Given \( e(X, \gamma_{e,0}) \) & \( v(X, \gamma_{v,0}) \) one can back out the corresponding choice \( c_t \) from the model equations outlined in section 2.3.1, call this \( c(X, \gamma_0) \). Compute the expectations \( \mathbb{E}^P_t [\Omega(\cdot)] \) and \( \mathbb{E}^P_t [\Omega(\cdot)^{1-\gamma}] \) at all the Chebyshev nodes using a Gaussian quadrature, call these vectors \( Y(\hat{X}, \gamma_0) \) and \( RV(\hat{X}, \gamma_0) \)

\[
\begin{align*}
Y(\hat{X}, \gamma_0) &= \int \Omega_e \left( c(\hat{X}', \gamma_0), v(\hat{X}', \gamma_0)|\hat{X}', \gamma_0 \right) \cdot dF(\hat{X}'|\hat{X}, c(\hat{X}, \gamma_0), \mathcal{P}) \\
RV(\hat{X}, \gamma_0) &= \int \Omega_v \left( c(\hat{X}', \gamma_0), v(\hat{X}', \gamma_0)|\hat{X}', \gamma_0 \right)^{1-\gamma} \cdot dF(\hat{X}'|\hat{X}, c(\hat{X}, \gamma_0), \mathcal{P})
\end{align*}
\]

4. Compute

\[
\begin{align*}
\gamma_e^* &= (B'B)^{-1} B' \log Y(\hat{X}, \gamma_0) \\
\gamma_v^* &= (B'B)^{-1} B' \left( (1 - \beta)c(\hat{X}, \gamma_0)^{1-\psi} + \beta RV^{1-\gamma} \right)^{\frac{1}{1-\psi}}
\end{align*}
\]

5. Update the guess

\[
\begin{align*}
\gamma_{e,1} &= \omega \gamma_e^* + (1 - \omega) \gamma_{e,0} \\
\gamma_{v,1} &= \omega \gamma_v^* + (1 - \omega) \gamma_{v,0}
\end{align*}
\]

for \( \omega \in (0, 1] \)

---

69The procedure can fail to converge when \( \gamma_1 = \gamma^* \), necessitating this step.
6. Repeat 1-5 until convergence\textsuperscript{70}

\textsuperscript{70}Further details on such methods can be found in Christiano and Fisher (2000).
# Appendix G

## Chapter 3, Data & Calibration Notes

### G.1 Countries

Table G.1: List of Countries in Regions, 3-Region Model

<table>
<thead>
<tr>
<th>Region</th>
<th>Countries</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asian Tigers</td>
<td>Hong Kong, Indonesia, Korea, Malaysia, Philippines, Singapore, Taiwan, Thailand, Vietnam</td>
</tr>
<tr>
<td>Emerging</td>
<td>Afghanistan, Angola, Albania, Algeria, Argentina, Armenia, Bahamas, Bahrain, Bangladesh, Barbados, Belize, Benin, Bhutan, Bolivia, Botswana, Brazil, Brunei, Bulgaria, Burkina Faso, Burundi, Cambodia, Cameroon, Cape Verde, Central African Republic, Chad, Chile, China, Colombia, Comoros, Democratic Republic of Congo, Republic of Congo, Costa Rica, Cote d’Ivoire, Dominican Republic, Ecuador, Egypt, El Salvador, Equatorial Guinea, Ethiopia, Fiji, Gabon, Gambia, Ghana, Guatemala, Guinea, Guinea-Bissau, Guyana, Haiti, Honduras, Hungary, India, Iran, Iraq, Jamaica, Kenya, Laos, Lebanon, Lesotho, Liberia, Madagascar, Malawi, Maldives, Mali, Maritius, Mauritania, Mexico, Mongolia, Morocco, Mozambique, Namibia, Nepal, Nicaragua, Niger, Nigeria, Oman, Pakistan, Panama, Papua New Guinea, Paraguay, Peru, Poland, Romania, Rwanda, Senegal, Sierra Leone, Solomon Islands, South Africa, Sri Lanka, Sudan, Suriname, Swaziland, Syria, Tanzania, Trinidad Tobago, Togo, Tunisia, Turkey, Uganda, Uruguay, Venezuela, Zambia, Zimbabwe</td>
</tr>
<tr>
<td>G7</td>
<td>Canada, France, Germany, Italy, Japan, United Kingdom, United States</td>
</tr>
</tbody>
</table>
### Table G.2: List of Countries in Regions, 5-Region Model

<table>
<thead>
<tr>
<th>Region</th>
<th>Countries</th>
</tr>
</thead>
<tbody>
<tr>
<td>Developing Asia</td>
<td>Afghanistan, Bangladesh, Bhutan, Brunei, Cambodia, China, Fiji, India, Indonesia, Malaysia, Maldives, Nepal, Pakistan, Papua New Guinea, Philippines, Solomon Islands, Sri Lanka, Thailand, Vietnam</td>
</tr>
<tr>
<td>G7</td>
<td>Canada, France, Germany, Italy, Japan, United Kingdom, United States</td>
</tr>
<tr>
<td>Latin America &amp; Caribbean</td>
<td>Argentina, Bahamas, Barbados, Belize, Bolivia, Brazil, Chile, Colombia, Costa Rica, Dominican Republic, Ecuador, El Salvador, Guatemala, Guyana, Haiti, Honduras, Jamaica, Mexico, Nicaragua, Panama, Paraguay, Peru, Suriname, Trinidad &amp; Tobago, Uruguay, Venezuela</td>
</tr>
<tr>
<td>Middle East &amp; Northern Africa</td>
<td>Algeria, Bahrain, Egypt, Iran, Iraq, Jordan, Lebanon, Mauritania, Morocco, Oman, Sudan, Syria, Tunisia</td>
</tr>
</tbody>
</table>
G.2 Calibration & Data Computation

The model is calibrated to match data on two regions over the period 1970. The vector \((\beta_1, \beta_2, \theta_1, \theta_2, \eta)\) is set exogenously, while \(l_i\) are set to match relative labour shares in at the start of the sample. \((\delta_1, \delta_2, \gamma_1, \gamma_2)\) is then set so that the steady state values of the regional saving rates and relative capital shares are equal to those observed in the data at the start of the sample. We set the capital share of output for each country as \(\alpha_i = 0.33\).

The saving rate of country is computed as

\[
1 - \text{consumption share of rgdp per capita} - \text{investment share of rgdp per capita}
\]

The savings rates of each region are averaged across regions to obtain regional saving rates (averages are unweighted).

The capital stock of each country in the sample is constructed using the perpetual inventory method. Assuming a depreciation rate of 0.06 for each country and using the growth rate of real gdp per capita the steady state relationship is given by

\[
k_{i1970} = \frac{i_{i1970}}{(dep + g_{rgdp})}
\]

In computing the capital stock numbers it is assumed that depreciation \((dep)\) is 0.06 for all countries. The growth rate \(g_{rgdp}\) is measured as the average growth rate of rgdp per worker in the first ten periods of the sample.

Labour supply as measured by the number of workers, is computed as

\[
lab = \frac{\text{real gdp per capita} \times \text{population}}{\text{real gdp per worker}}
\]

The output and labour supply of each country are summed across regions to obtain output per worker and capital per worker for each region.

TFP shocks are measures from the data. For each region, \(prod_t = \frac{y_t}{k_t}\) was computed and the series was detrended using a linear trend. The linear trend was set equal to \(a_t\). The detrended series was then set equal to \(z_t\) and estimate \(z_t = \hat{\rho}z_{t-1} + \hat{\epsilon}_t\). The residuals \(\hat{\epsilon}_t\) are used as shocks in the simulated model.
Appendix H

Data Appendix

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Board of Governors of the Federal Reserve System (US), Households and Nonprofit Organizations; Credit Market Instruments; Liability, level [HSTCMDODNS], retrieved from FRED, Federal Reserve Bank of St. Louis, https://research.stlouisfed.org/fred2/series/HSTCMDODNS/, August 1, 2015.  

Board of Governors of the Federal Reserve System (US), Households; Owner’s Equity in Real Estate, Level [OEHRENWBSHNO], retrieved from FRED, Federal Reserve Bank of St. Louis,
Appendix H. Data Appendix


