Voter Models and External Influence

by

Jimit Majmudar

B.Tech. (Hons.), Indian Institute of Technology Bombay, 2012

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in

THE COLLEGE OF GRADUATE STUDIES

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA
(Okanagan)
June 2016
© Jimit Majmudar, 2016
The undersigned certify that they have read, and recommend to the College of Graduate Studies for acceptance, a thesis entitled: VOTER MODELS AND EXTERNAL INFLUENCE submitted by Jimit Majmudar in partial fulfilment of the requirements of the degree of Master of Science

Rebecca C. Tyson, Unit 5 (Mathematics), University of British Columbia Okanagan
Supervisor, Professor

Bert O. Baumgaertner, Department of Philosophy, University of Idaho
Supervisory Committee Member, Professor

W. John Braun, Unit 5 (Mathematics), University of British Columbia Okanagan
Supervisory Committee Member, Professor

Stephen M. Krone, Department of Mathematics, University of Idaho
Supervisory Committee Member, Professor

Liane Gabora, Department of Psychology, University of British Columbia Okanagan
University Examiner, Professor

June 9, 2016
(Date Submitted to Grad Studies)
Abstract

Opinions, and subsequently opinion dynamics, depend not just on interactions among individuals, but also on external influences such as the mass media. The dependence on local interactions, however, has received considerably more attention. In this work, we extend the classical voter model, the biased voter model, and the threshold voter model to include external influences. We study the new models both analytically and computationally, and we show that some of the new models can be understood after employing diffusion approximations and mean field approximations. We derive results pertaining to the probability of reaching consensus on a particular opinion and also the expected consensus time for the different models. We find that although including an external influence leads a faster consensus in general, this effect is more pronounced in the classical voter model as compared to the threshold voter model. Some of our findings suggest the potential importance of “macro-level” phenomena such as the external influences as compared to the “micro-level” local interactions, in modelling opinion dynamics.
# Table of Contents

Abstract ................................................................. ii

Table of Contents ....................................................... iii

List of Tables ............................................................ v

List of Figures ........................................................... vi

Acknowledgements ........................................................ vii

Dedication ................................................................. viii

Chapter 1: Introduction .................................................. 1
  1.1 Background ......................................................... 1
  1.2 Contributions ...................................................... 1
  1.3 Organisation ....................................................... 2

Chapter 2: Preliminaries ................................................ 3
  2.1 Voter Model ......................................................... 3
  2.2 Biased Voter Model ................................................ 3
  2.3 Threshold Voter Model ............................................ 4
  2.4 Noisy Voter Model ................................................ 4
  2.5 Voter Models on Finite Graphs ................................... 4

Chapter 3: Models ......................................................... 6
  3.1 Jump Voter Model ................................................ 6
  3.2 Jump Voter Diffusion ............................................. 7
  3.3 Jump Biased Voter Model ....................................... 10
  3.4 Jump Biased Voter Diffusion ................................... 11
  3.5 Jump Threshold Voter Model ................................... 11

Chapter 4: Results ....................................................... 13
  4.1 Jump Voter Model ................................................ 14
    4.1.1 Comparison with Classical Voter Model .................. 17
  4.2 Jump Biased Voter Model ....................................... 22
# TABLE OF CONTENTS

4.3 Jump Threshold Voter Model ........................................... 25  
Chapter 5: Conclusion ......................................................... 28  
Bibliography ........................................................................... 31
List of Tables

Table 3.1  All model parameters and their brief meanings . . . . . . . . . . 12
Table 5.1  Overview of all models and their approximations (if any) . . . . . . 29
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Fixation probability for the jump voter model</td>
<td>16</td>
</tr>
<tr>
<td>4.2</td>
<td>Expected consensus time for the jump voter model</td>
<td>17</td>
</tr>
<tr>
<td>4.3</td>
<td>Comparison between the voter model and the jump voter model</td>
<td>19</td>
</tr>
<tr>
<td>4.4</td>
<td>Expected consensus time dependence on jump parameters for the jump voter model - part 1</td>
<td>20</td>
</tr>
<tr>
<td>4.5</td>
<td>Expected consensus time dependence on jump parameters for the jump voter model - part 2</td>
<td>21</td>
</tr>
<tr>
<td>4.6</td>
<td>Fixation probability for the jump biased voter model</td>
<td>23</td>
</tr>
<tr>
<td>4.7</td>
<td>Expected consensus time for the jump biased voter model</td>
<td>24</td>
</tr>
<tr>
<td>4.8</td>
<td>Fixation probability and expected consensus time for the jump threshold voter model</td>
<td>25</td>
</tr>
<tr>
<td>4.9</td>
<td>Spatial comparison between the jump voter model and the jump threshold voter model</td>
<td>26</td>
</tr>
</tbody>
</table>
Acknowledgements

I would like to begin by thanking my advisor, Prof. Rebecca Tyson, for reasons more than one - giving me the chance to do mathematics and the freedom to shape my research, putting me in touch with the right people, and so on; the list is quite long. I am fortunate to have had Prof. Tyson as my advisor. I am also thankful to my committee members, Prof. Steve Krone, Prof. Bert Baumgaertner, and Prof. John Braun, for their continued support. I am particularly grateful to Prof. Krone for the amount of time he has spent with me, and the patience he has had while I worked my way through some of the material. Without his belief in my mathematical abilities, this thesis would not have been possible.

I am incredibly grateful to my parents, Heena and Rankesh Majmudar, and my sister, Chaitasi Majmudar, for always being the solid anchor in my life. They have always been understanding and supportive of all my academic pursuits, and provided me counsel whenever I needed it. I am highly indebted to my mother, without whose self-sacrifices I would not have been able to get such good quality education.

I am also thankful to my friends - both at UBC Okanagan and from IIT Bombay. It has always been a pleasure to discuss research and life, in general, with Alireza, Mayang, and Joey. I am thankful to Chad for always being there whenever I needed help with some material. Indeed those late evening discussions with him have only fuelled my liking for mathematics. Last but not the least, Saurabh (Javelin) has been the one friend who I can talk to at any time about pretty much anything. Conversations with him almost always serve as pleasant reminders for my liking for fundamental research.

I am also thankful to Mandy Baumgaertner and Claudia Krone for graciously giving me space to live while I visited Moscow to work on my research. I will always remember the warm hospitality I received while there, and the laughs we shared.

Lastly, I would also like to give a special mention to some of my musical heroes - Guthrie Govan, Plini, and Warren Mendonsa. Apart from being my constant companion, their musical eloquence regularly teaches me a thing or two about creative expression.
Dedicated to my parents, Heena and Rankesh Majmudar, and my sister, Chaitasi Majmudar.
Chapter 1

Introduction

1.1 Background

Human opinions and collective human behaviours have now been studied for more than a century. Prominent historic examples range from studies of irrational crowd behaviours [Mac12, LB97], to social conformity [She36, Asc56], obedience [MF74], and herd mentality [Key06], to critical mass phenomena [Sch71]. Relatively recent is the mathematical analysis of these dynamics, which has been made possible due to seminal frameworks like the voter model [CS73, HL75], DeGroot learning [DeG74], and the naming game model [BFC05, DBBL06]. Considerable work has been done extending these models so as to incorporate a combination of diverse social phenomena [PB10, YAS11, YSO11, DGM14, FR14]. These advances, on just the quantitative side, have been made by mathematicians, physicists, and computer scientists, who, along with researchers of sociology, psychology, philosophy, politics, etc. make this field immensely interdisciplinary. For a detailed review of the origins and evolution of this domain, the reader is referred to the comprehensive review by Xia et al [XWX11].

While refining the mechanistic description of how individuals communicate has received considerable attention [CD91, Lig94, HK02], the incorporation of external influences has received little modelling scrutiny, one reason for which seems to be the associated loss of analytical tractability. There is already evidence, however, that media can play an important role in opinion dynamics in many different contexts, for example, climate change [BCJ12], and electoral voting, to name a few. Furthermore, commercial relevance of this phenomenon can be found in quantitative marketing, when two brands compete to sell their respective products via advertising. A theoretical examination of the effect of external influence is thus important. Existing work, however, is either not amenable to detailed mathematical analysis [QCS14], or lacks generality [FR13].

1.2 Contributions

We address herein some of the above discussed shortcomings by considering a novel approach that incorporates an external influence that is general, but also maintains considerable mathematical tractability at the same time. We model the effect of external influence via sporadic events that make multiple agents flip their opinion simultaneously. To perform the mathematical analysis, we use the theory of jump diffusion processes, and therefore attach the word “jump” as an adjective in naming our models. Jump diffusion models are widely used in financial domains such as derivative pricing.
and risk management [Kou07], and also in physics, but have not before been used in the
domain of social dynamics. They were primarily introduced in finance because, more
often than not, the quantity to be modelled exhibited nonsmooth fluctuations that a
regular diffusion model failed to capture. Qualitatively speaking, in sociodynamics,
external influences can generate nonsmooth effects in the evolution of opinions, hence
our creation of these jump models. We explore the effects of including the jumps in
three well-established models: the classical voter model, the biased voter model, and
the threshold voter model.

1.3 Organisation

Chapter 2 discusses the well-established models that act as a basis for the jump
models. In chapter 3 we define all our models, followed by our results and their in-
terpretations in chapter 4. Lastly, we include a conclusion that contains a high-level
discussion of our approach and our key findings in chapter 5.
Chapter 2

Preliminaries

In this chapter, we review some of the existing models that inspired our models. Before doing that, we briefly present the terminology that we follow. By $\mathbb{Z}^d$, we mean the $d$-dimensional lattice of integer coordinates, and $G$ denotes an undirected finite graph where $S(G)$ denotes the set of nodes of $G$, and $k_x$ denotes the degree of node $x$. The cardinality of $S(G)$ is $N$, i.e., the graph consists of $N$ nodes. We interpret each node as an individual agent. A state is associated with each node, and we herein think of that state as the opinion of that node on a particular subject. We consider a binary opinion space, i.e., each node must have either opinion 0 or opinion 1 at any given time $t$, where $t \in [0, \infty)$ for continuous-time models and $t \in \{0, 1, 2, \ldots\}$ for discrete-time models. The state of the entire system at a time $t$ is given by $s(t)$, and the state at a specific node $x$ is given by $s_x(t)$. We denote by $\mathcal{N}(x)$ the set of neighbours of node $x$; for a lattice this set consists of its four four immediate neighbours (north, south, east, west), and for a graph it consists of all immediate nodes with which $x$ is connected. Moreover, we denote by $c(x, s)$ the flip rate, i.e., the rate at which the opinion at node $x$ flips when the process is in state $s$. Therefore, we have

$$\lim_{\Delta t \to 0} \frac{P(s_x(t + \Delta t) \neq s_x(t)|s(t) = s)}{\Delta t} = c(x, s)$$

(2.1)

2.1 Voter Model

The voter model [CS73, HL75] is a continuous-time Markov process on $\mathbb{Z}^d$,

$$c(x, s) = \# \{y \in \mathcal{N}(x) : s_y \neq s_x\}$$

(2.2)

This is a linear model in the sense that the rate of change of opinion at a node $x$ is a linear function of the count of opposing opinions in the neighbourhood of $x$. Intuitively, we may interpret the process as follows: There are independent Poisson clocks at each node. Whenever the clock at a node $x$ rings, the node scans its neighbourhood, uniformly selects one of its neighbours at random, and then adopts the opinion of that chosen neighbour.

2.2 Biased Voter Model

The biased voter model [Wil72] is a modification of the voter model where,
2.3 Threshold Voter Model

\[
c(x, s) = \begin{cases} 
  b \cdot \# \{ y \in \mathcal{N}(x) : s_y \neq s_x \}, & \text{if } s_x = 0 \\
  \# \{ y \in \mathcal{N}(x) : s_y \neq s_x \}, & \text{if } s_x = 1 
\end{cases} \quad (2.3)
\]

with \( b > 0 \). Notice that this is also a linear model. The difference between this model and the voter model is only that the transition rate here is adjusted by a bias factor \( b \) for a 0 \( \to \) 1 flip, without loss of generality (WLOG). Intuitively, this adjustment may make the opinion 0 more or less retentive, depending on the value of \( b \). That is, a 0 \( \to \) 1 flip would be more frequent than a 1 \( \to \) 0 flip if \( b > 1 \), and vice versa for \( b < 1 \).

2.3 Threshold Voter Model

The threshold voter model [Lig94] is a modification of the voter model where,

\[
c(x, s) = \begin{cases} 
  1, & \text{if } \# \{ y \in \mathcal{N}(x) : s_y \neq s_x \} \geq \theta \\
  0, & \text{if } \# \{ y \in \mathcal{N}(x) : s_y \neq s_x \} < \theta 
\end{cases} \quad (2.4)
\]

with \( \theta > 0 \). Flip rate (2.4) yields a nonlinear model in the sense that the rate of change of opinion at a node \( x \) is a nonlinear function of the count of opposing opinions in the neighbourhood of \( x \). Intuitively, we may again think of independent Poisson clocks at each node. When the clock at a node \( x \) rings, the node scans its neighbourhood, and updates its opinion if and only if there are at least \( \theta \) opposing opinions in its neighbourhood.

2.4 Noisy Voter Model

The noisy voter model [GM95] is a modification of the voter model where,

\[
c(x, s) = \begin{cases} 
  \# \{ y \in \mathcal{N}(x) : s_y \neq s_x \} + \beta, & \text{if } s_x = 0 \\
  \# \{ y \in \mathcal{N}(x) : s_y \neq s_x \} + \delta, & \text{if } s_x = 1 
\end{cases} \quad (2.5)
\]

with \( \beta > 0 \) and \( \delta > 0 \). Flip rate (2.5) yields again a linear model. Here the flipping rates are incremented by parameters \( \beta \) and \( \delta \), which are aptly called the noise parameters. Intuitively, these parameters inject an additional source of flipping that is independent of the states of the neighbouring sites. As an extreme case, in this model there is a possibility of an opinion change at a node \( x \) even in the situation when all its neighbours have the same opinion as that of \( x \).

2.5 Voter Models on Finite Graphs

The voter model and its variants discussed in the previous sections have also been studied on finite lattices and finite graphs. The most pertinent reference point for that line of work is the paper by Sood et al [SR05]. They study discrete-time voter models
on a heterogeneous graph in the absence of degree correlations. They define degree-weighted moment, $\omega_1$, a generalisation of the notion of density as

$$\omega_1(t) \equiv \frac{1}{N\mu_1} \sum_{x \in S(G)} k_x s_x(t)$$

where $\mu_1$ is the first moment of the degree distribution, or

$$\mu_1 = \frac{1}{N} \sum_{x \in S(G)} k_x.$$

Moreover, let $\rho_k$ denote the subdensity of opinion 1 nodes that have degree $k$, given as

$$\rho_k(t) = \sum_{\substack{x \in S(G), \ k_x = k}} \frac{s_x(t)}{N_k}$$

where $N_k$ denotes the count of nodes that have degree $k$. They show that the voter model dynamics follow a two-time-scale approach to consensus. Initially, there is a rapid approach of all subdensities to $\omega_1(0)$, followed by which diffusive fluctuations drive the model to consensus, i.e., the state where either all nodes have opinion 0 or 1. Additionally, the time to consensus, $T_N$, is given analytically by

$$T_N(\omega) = N_{\text{eff}} \left[ (1 - \omega) \ln \frac{1}{1 - \omega} + \omega \ln \frac{1}{\omega} \right]$$

where $\omega \equiv \omega_1(0)$, and $N_{\text{eff}}$ denotes the effective population size defined as

$$N_{\text{eff}} \equiv \frac{N \mu_2^1}{\mu_2}$$

where $\mu_2$ denotes the second moment of the degree distribution. The study is further extended to the biased voter model on a heterogeneous graph, and it is demonstrated that in addition to the two-time-scale dynamics, the probability of single fitter agent to flip the entire population on its side is directly proportional to its degree $k$. 
Chapter 3

Models

For the subsequent models, we restrict our attention to graphs $G$ that are $k$–regular, where each node, as before, is considered to be an individual agent, and the links represent social connections between the agents. Also as before, consider a binary opinion space, i.e., a node $x \in S(G)$ must have either opinion 0 or opinion 1 at time $t$, which is denoted by $s_x(t)$. The adjacency matrix is denoted by $A$, with the element $A_{xy}$ being 1 if nodes $x$ and $y$ are connected, and 0 otherwise. The neighbourhood of a node consists of the immediate nodes with which it is connected. Consensus is said to be reached when either all nodes have opinion 0 or opinion 1, and it is assumed that the process spends forever in that state once it is attained. Consensus on opinion 1 is herein called fixation.

3.1 Jump Voter Model

The jump voter model is a discrete-time process that is updated according to the two rules given below. At each time step, one of the following occurs:

**Update Rule 1:** With probability $(1-p)$, a single node is randomly selected which then adopts the opinion of one of its neighbours chosen randomly.

**Update Rule 2:** With probability $p$, either a random number of 0 opinion nodes update their opinions to 1 or a random number of 1 opinion nodes update their opinions to 0. A signed form of this random variable ($Z$) follows the convention that it is negative in the case of the former update, and positive for the latter.

The first update rule captures the node-to-node interactions of the classical voter model. The second rule captures the more global external influence that makes several opinions flip simultaneously, a phenomenon that we call a jump. The flexibility granted by the model in terms of occasionality of the external influence is crucial here, because in certain application contexts the effect of this influence may be sporadic. For example, media attention in society is observed to be irregular, with this irregularity depending on, among other things, the nature of the topic. Call $p$ the jump probability, and note that when $p = 0$ the model reduces to a discrete-time version of the classical voter model on a graph structure. The jump voter model is a discrete state space Markov chain, and the total number of opinion 1 nodes in the graph at time step $t$, denoted by $X^N(t)$, can
be thought of as a global summary statistic of that Markov chain. More formally,

\[ X^N(t) = \sum_{x \in S(G)} s_x(t). \]

The random variable \( Z \) is independent of the state of the process, and we define the scaled jump \( Y \equiv Z/N \) whose mean is zero in this version of the model. A non-zero mean could be indicative of biases in the external influence, and we cover that case within the biased version of the jump voter model in Section 3.3. The variance of \( Z \) represents the strength of the external influence. (A brief summary of all model parameters and their brief meaning is provided in Table 3.1.) This model corresponds to model B in Table 5.1 that shows a global overview of all our models.

### 3.2 Jump Voter Diffusion

Diffusion approximations are an indispensable tool for inferring properties of discrete processes, albeit in the limiting case for large populations. We make use of a diffusion approximation here and derive a jump-diffusion process to which the jump voter model weakly converges. To proceed with this jump diffusion approximation, we will need the transition probabilities corresponding to the update rule 1:

- \( P[i \rightarrow i + 1] \equiv \) probability that a 0 \( \rightarrow \) 1 update happens at a certain time step when the count of nodes with opinion 1 is \( i \), and,
- \( P[i \rightarrow i - 1] \equiv \) probability that a 1 \( \rightarrow \) 0 update happens at a certain time step when the count of nodes with opinion 1 is \( i \).

Now,

\[ P[i \rightarrow i + 1] = P[(\text{selecting a node } x \text{ with opinion 0}) \cap (\text{selecting a } y \text{, in the neighbourhood of the node } x, \text{ with opinion 1})], \]

and,

\[ P[i \rightarrow i - 1] = P[(\text{selecting a node } x \text{ with opinion 1}) \cap (\text{selecting a } y \text{, in the neighbourhood of the node } x, \text{ with opinion 0})]. \]
3.2. Jump Voter Diffusion

Therefore, we have,

\[
P[i \rightarrow i + 1] = \sum_{x \in S(G), s_x = 0} \left( \frac{1}{N} \sum_{y \in S(G)} A_{xy} \frac{s_y}{k} \right)
\]

\[
P[i \rightarrow i - 1] = \sum_{x \in S(G), s_x = 1} \left( \frac{1}{N} \sum_{y \in S(G)} A_{xy} \frac{1 - s_y}{k} \right)
\]

(3.1)

We take a short detour here to explain the mean field approximation, in the spirit of that used by Sood et al [SR05]. This notion of mean field approximation is somewhat different from the one that is usually used. Typically, for a mean field approximation, one simply considers the “well-mixed” case where each node has links with every other node. For our mean field approximation, consider the scenario where the above process is run multiple times, but each time, before starting, we perform some rewiring so as to obtain a different regular graph. So, two nodes (say \(x\) and \(y\)) may be connected in certain runs (i.e. \(A_{xy} = 1\)), but may not be connected in other runs (i.e. \(A_{xy} = 0\)). In this case, in order to perform average calculations over all such realisations, we replace \(A_{xy}\) in (3.1) with \(E[A_{xy}]\). Alternatively, we may also try to find the probability that nodes \(x\) and \(y\) are connected, since

\[
E[A_{xy}] = 1 \cdot P[A_{xy} = 1] + 0 \cdot P[A_{xy} = 0].
\]

Considering all possible node pairs,

\[
E[\text{total number of links}] = \binom{N}{2} P[A_{xy} = 1].
\]

(3.2)

But the total number of links = \(\frac{Nk}{2}\). Therefore, comparing with equation (3.2), we get,

\[
\binom{N}{2} P[A_{xy} = 1] = \frac{Nk}{2}
\]

\[
\therefore P[A_{xy} = 1] = \frac{k}{N - 1} = E[A_{xy}].
\]

(3.3)

(We can check that the expression \(P[A_{xy} = 1] = \frac{k}{N - 1}\) indeed matches the “boundary cases”. When \(k = 0\) the probability of picking any two nodes that are connected is 0, and when \(k = N - 1\) we have a complete graph, and the same probability is 1; both of these cases are consistent with the relationship derived in equation (3.3).)

For large enough \(N\), we may replace \(N - 1\) with \(N\) which gives us
3.2. Jump Voter Diffusion

\[ E[A_{xy}] \approx \frac{k}{N}. \quad (3.4) \]

Making use of the mean field approximation (i.e., substituting the expression for \( E[A_{xy}] \) from equation (3.4) in the equations in (3.1)), we obtain,

\[
P[i \rightarrow i + 1] \approx \left(1 - \frac{i}{N}\right) \left(\frac{i}{N}\right) \quad (3.5)
\]

\[
P[i \rightarrow i - 1] \approx \left(\frac{i}{N}\right) \left(1 - \frac{i}{N}\right).
\]

We scale the process by dividing by \( N \), and interpret the new state as density or proportion of nodes that have opinion 1. We also set a single time step \( \Delta t \) to be equal to \( \frac{1}{N^2} \). As \( N \) becomes large (i.e., the time step becomes small), the update rule 1 (which is the discrete version of the classical voter model) becomes a diffusion process whose drift \( \mu(x) \), using its standard definition [KT81], can be derived as follows.

\[
\mu(x) = \lim_{N \to \infty} \frac{N^2}{N^2} \cdot E \left[ \frac{X^N([N^2t] + 1) - X^N([N^2t])}{N} \middle| X([N^2t]) = i \right] = \lim_{N \to \infty} N^2 \cdot E \left[ \left(\frac{1}{N}\right) P[i \rightarrow i + 1] + \left(-\frac{1}{N}\right) P[i \rightarrow i - 1] \right].
\]

Substituting the probabilities from equation (3.5) into equation (3.6), we obtain \( \mu(x) = 0 \). Following similar arguments, the diffusion parameter is found to be,

\[
\sigma^2(x) = 2x(1 - x). \quad (3.6)
\]

The calculations so far give us a continuous approximation for the update rule 1. If we define \( \lambda \equiv N^2p \) and \( Y = Z/N \) (scaled jump), then for a small enough \( p \) and a large enough \( N \), the jump voter model (both update rule 1 and update rule 2) can be approximated by a superposition of the diffusion derived above and a compound Poisson process, i.e., a jump diffusion process given as,

\[
dX(t) = \sqrt{2X(t)(1 - X(t))}dW(t) + YdN(t) \quad (3.7)
\]

where \( W(t) \) represents a Wiener process, and \( N(t) \) represents a rate \( \lambda \) Poisson process. We denote by \( g(x) \) the probability density function of the jump random variable \( Y \), and also note that \( g(x) \) will have support \([-1, 1]\). The mean of \( Y \) is denoted by \( m \) (which is zero here), and the variance by \( v \). We call \( v \) the jump variance. The generator of a stochastic process is an operator that, intuitively, encodes important information about that process. Mathematically, (for a time-homogeneous process) it is defined as [Oks03]
3.3. Jump Biased Voter Model

\[
\mathcal{L} f(x) = \lim_{t \to 0^+} \frac{E[f(X(t))|X(0) = x] - f(x)}{t}.
\]

The generator of this process is the integro-differential operator \( \mathcal{L} \), where

\[
\mathcal{L} f(x) = x(1-x)f''(x) + \lambda \int_{-\infty}^{+\infty} [f(x-y) - f(x)]g(y)dy. \quad (3.8)
\]

The jump diffusion in equation 3.7 corresponds to model C in Table 5.1. Note here that as a result of using the mean field approximation, the jump voter diffusion given by equation 3.7 does not have a term containing the degree parameter \( k \). Therefore, based on this observation, we can say that quantities related to overall opinion density do not depend on the degree of the regular graph.

3.3 Jump Biased Voter Model

We now define a biased version of the jump voter model. For our purposes, we divide the biases into two categories: Biases in the external influence, and biases in the interactions between nodes. By the former, we mean that the external influence is skewed towards a particular opinion; by the latter, we mean that the agents of a particular opinion are less likely to interact with their neighbours and subsequently update their opinion. An example of bias in the external influence would be a biased media outlet. Bias in the node-to-node interactions, on the other hand, can stem from various sources. For example, global warming may bias the population towards an increased acceptance of climate change. In terms of modelling, we incorporate the biases by making the following adjustments to the two update rules:

- **External influence**: The jump random variable is now allowed to have a non-zero mean, i.e., \( m \neq 0 \). If \( m \in (0,1] \) then the external influence bias (or jump bias) favours opinion 0, whereas if \( m \in [-1,0) \) the external influence bias (or jump bias) favours opinion 1. Or,

- **Node-to-node interactions**: For mathematical simplicity, we assume, without loss of generality, that this bias \( b \) is in favour of opinion 1, thereby making an opinion change at a node from 1 to 0 less probable than before, while a change from 0 to 1 is more probable. More specifically, we define a bias parameter \( b \in (1, \infty) \), and adjust the probability of selecting a node as follows.

\[
P(\text{an opinion 1 node is selected for a potential update}) = \frac{i}{bN},
\]

\[
P(\text{an opinion 0 node is selected for a potential update}) = 1 - \frac{i}{bN}.
\]

(Note that the higher the value of bias \( b \), the lesser the probability of a change happening at an opinion 1 node, making that opinion more rententive.) Or,

- Both of the above.

The jump biased voter model corresponds to model E in Table 5.1.
3.4 Jump Biased Voter Diffusion

The revised transition probabilities, again incorporating the mean field approximation, are

\[ P[i \rightarrow i + 1] = \left( 1 - \frac{b^{-1} i}{N} \right) \frac{i}{N} \]
\[ P[i \rightarrow i - 1] = \frac{b^{-1} i}{N} \left( 1 - \frac{i}{N} \right) \]  

(3.9)

where \( b^{-1} = 1/b \) is the inverse bias. Applying the jump diffusion approximation, as before, gives the following stochastic differential equation corresponding to the jump biased voter model,

\[ dX(t) = X(t)(1 - b^{-1})dt + \sqrt{X(t)[(1 + b^{-1}) - 2b^{-1}X(t)]}dW(t) + YdN(t). \]  

(3.10)

Note the non-zero drift in equation (3.10) owing to the bias factor \( k \). The generator of this process is the integro-differential operator,

\[ \mathcal{L} f(x) = x(1 - b^{-1})f'(x) + \frac{x}{2}[(1 + b^{-1}) - 2b^{-1}x]f''(x) \]
\[ + \lambda \int_{-\infty}^{+\infty} [f(x - y) - f(x)]g(y)dy. \]  

(3.11)

The jump diffusion in equation (3.10) corresponds to model F in Table 5.1.

3.5 Jump Threshold Voter Model

The jump threshold voter model is a discrete-time process that is updated according to the rules given below. At each time step, one of the following occurs:

**Update Rule 1:** With probability \((1 - p)\), a single node is randomly selected. If the number of opposing opinions in the neighbourhood of the selected node is greater than or equal to a threshold \( \theta \), then the opinion of the originally selected node is updated.

**Update Rule 2:** With probability \( p \), either a random number of 0 opinion nodes update their opinions to 1 or a random number of 1 opinion nodes update their opinions to 0. A signed form of this random variable \((Z)\) follows the convention that it is negative in the case of the former update, and positive for the latter.

Similar to the jump voter model, the first update rule captures the node-to-node interactions of the classical threshold voter model, whereas the second rule captures the external influence. Again as with the jump voter model, the mean of \( Z \) is zero in this
model. This model corresponds to model H in Table 5.1.

Table 3.1 summarises all the parameters discussed in this chapter.

**Table 3.1:** All model parameters and their brief meanings.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Brief Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>Total population.</td>
</tr>
<tr>
<td>$k$</td>
<td>Degree in a regular graph.</td>
</tr>
<tr>
<td>$p$</td>
<td>Probability of a jump occurring at a time step. Also called jump probability.</td>
</tr>
<tr>
<td>$Z$</td>
<td>Random variable that gives the jump or the external influence.</td>
</tr>
<tr>
<td>$m$</td>
<td>Jump bias, or external influence bias.</td>
</tr>
<tr>
<td>$v$</td>
<td>Jump variance, or strength of the external influence.</td>
</tr>
<tr>
<td>$b$</td>
<td>Bias in node-to-node interactions.</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Threshold parameter in the jump threshold voter model. An opinion update at a node, via node-to-node interactions, happens if and only if the number of opposing opinions in the node’s neighbourhood is greater than or equal to $\theta$.</td>
</tr>
</tbody>
</table>
Chapter 4

Results

If $X(t)$ is a jump diffusion with generator $\mathcal{L}$, Itô’s formula for jump processes [Phi90] implies that

$$M(t) \equiv f(X(t)) - \int_0^t \mathcal{L} f(X(s)) ds$$

(4.1)

is a martingale for any $C^2$ function $f(x)$. Application of the Optional Stopping Theorem (OST) to this martingale will result in the formulation of “boundary” value problems for both fixation probability and expected value of the consensus time for both the jump voter model and the jump biased voter model. The word “boundary” is in double quotes as this formulation does not lead to boundary value problems in the strict sense. While the solution of a traditional (one-dimensional) BVP is a function that obeys the governing equation in the region $C$, and whose value is known on $\partial C$, the solutions of the problems developed here will satisfy the governing equation in $C$, but their values will be known for the entire region $\mathbb{R} - C$. Qualitatively, this is because a jump diffusion has the potential to overshoot boundaries. Denoting by $\tau$ the time to consensus and using the OST, we obtain

$$E^x[f(X(\tau))] - E^x\left[\int_0^\tau \mathcal{L} f(X(s)) ds\right] = f(x).$$

(4.2)

(Here we use $E^x[\cdot]$ as shorthand for $E[\cdot | X_0 = x]$, and similarly $P^x[\cdot]$ for $P[\cdot | X_0 = x]$.)

The following two claims help us derive the fixation probability and consensus time boundary value problems. These claims are straightforward extensions of ideas common in a simple diffusion (i.e., a diffusion without any jumps) [Øks03]. Before proceeding, we quickly define the random times when the jump diffusion crosses the 0 boundary and the 1 boundary as follows.

$$T_0 \equiv \inf\{t \geq 0 : X(t) \in (-\infty, 0]\}$$

$$T_1 \equiv \inf\{t \geq 0 : X(t) \in [1, \infty]\}$$

Claim 4.1. If there exists a $C^2$ function $u : \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathcal{L}u(x) = 0$ for all $x \in (0, 1)$, and $u(x) = 0$ for all $x \in (-\infty, 0]$ and $u(x) = 1$ for all $x \in [1, \infty)$, then the function $u$ gives the fixation probability for the jump voter model and the jump biased voter model.

Proof. Since $u$ is a $C^2$ function, we can use equation (4.2) to get
4.1 Jump Voter Model

The expected hitting time for the jump voter model and the jump biased voter model solves the BVP

\[ \mathbb{E}^x[\tau] = \text{u}(x) \]

The second, and possibly more interesting, property of the jump voter model is the expected value of the consensus time. This is often referred to as the hitting time in the stochastic processes literature. Consensus time, here, is formally defined as

\[ \tau = \inf\{t \geq 0 : X(t) \in (-\infty, 0) \cup [1, \infty)\} \]

Claim 4.2. If there exists a \( C^2 \) function \( v : \mathbb{R} \rightarrow \mathbb{R} \) such that \( L^x v(x) = -1 \) for all \( x \in (0, 1) \), and \( v(x) = 0 \) otherwise, then the function \( v \) gives the expected consensus time for the jump voter model and the jump biased voter model.

Proof. Since \( v \) is a \( C^2 \) function, we can use equation (4.2) to get

\[ \mathbb{E}^x[v(X(\tau))] - \mathbb{E}^x[\int_0^\tau L^x v(X(s))ds] = v(x). \]

\[ \therefore \mathbb{E}^x[\tau] = \mathbb{E}^x[v(x)]. \]

The expected probability satisfies the following BVP

\[ L^x u(x) = 0 \quad \forall x \in (-\infty, 0) \cup [1, \infty) \]

4.1 Jump Voter Model

We begin by investigating the fixation probability and the expected consensus time for the jump voter diffusion. To determine the fixation probability, we use the generator of the jump voter diffusion from equation (3.8) in the BVP in (4.3), to get
4.1. Jump Voter Model

\[ (1-x)u''(x) + \lambda \int_{-\infty}^{+\infty} u(x-y)g(y)dy - \lambda u(x) = 0 \]  
(4.6)

\[ u(x) = 0, \ \forall x \in (-\infty, 0] \]

\[ u(x) = 1, \ \forall x \in [1, \infty). \]

It is quite challenging to find an analytical solution for a variable-coefficient integro-differential equation such as equation (4.6). Even a similar constant-coefficient equation requires imposing some structure on the function \( g(x) \) to make a closed-form solution possible [KW04]. We thus use numerical approaches to solve this problem.

The \( x \) domain \([0, 1]\) is discretised into \( m \) steps of size \( h \), such that \( x_0 = 0 \) and \( x_m = 1 \). The difference equation corresponding to equation (4.6) thus becomes

\[ jh(1 - jh) \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} + \lambda h \sum_{i=-\infty}^{+\infty} u_{j-i}g_i - \lambda u_j = 0, \ \forall j \in \{1, 2, \ldots, (m-1)\} \]
(4.7)

\[ u_j = 0, \ \forall j \in \{\ldots, -1, 0\} \]

\[ u_j = 1, \ \forall j \in \{m, m+1, \ldots\}. \]

The infinite summation term on the LHS can be recognised as a discrete convolution of two discrete functions \( u \) and \( g \). Using the commutativity property of convolution, the boundary conditions on \( u \), and the fact that \( g \) has \([-1, 1]\) support, the infinite summation can be truncated to obtain the simplified set of equations

\[ jh(1 - jh) \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} + \lambda \sum_{i=1}^{2m-1} u_{i}g_{j-i} - \lambda u_j = 0, \ \forall j \in \{1, 2, \ldots, (m-1)\} \]
(4.8)

\[ u_j = 1, \ \forall j \in \{m, (m+1), \ldots, (2m-1)\}. \]

These equations give a system of \( 2m - 1 \) equations with the same number of unknowns. The solution of equation (4.8) obtained numerically closely matches the results from Monte Carlo simulations (Figure 4.1).
4.1. Jump Voter Model

![Figure 4.1: Fixation probability for the jump voter model on a regular graph with $N = 500$, $k = 4$, $p = 1/(500 \times 10)$, $v = 0.04$ (see Table 3.1 for meaning of parameters). The black curve denotes the numerical solution of equation (4.6), black points denote simulation results based on update rules in Section 3.1, where each point is obtained by averaging over 1000 runs. The external influence, $Z$, has a truncated normal distribution. This match confirms our approximation of model B by model C, as shown in Table 5.1.]

Similarly, if we expand (4.5) using the generator in (3.8), we obtain

$$x(1-x)v''(x) + \lambda \int_{-\infty}^{+\infty} v(x-y)g(y)dy - \lambda v(x) = -1$$

$$v(x) = 0, \forall x \in (-\infty, 0] \cup [1, \infty).$$

As before, rather than trying to derive a closed-form solution, we approach this equation numerically. Discretising the integro-differential equation, and again applying simplification techniques similar to those used to derive equation (4.8), we obtain a system of $m - 1$ equations, that approximates equation (4.9),

$$j h (1 - j h) v_{j-1} - \frac{2 v_j + v_{j+1}}{h^2} + \lambda h \sum_{i=1}^{m-1} v_i g_{j-i} - \lambda v_j = -1, \forall j \in \{1, 2, ..., (m - 1)\}$$

$$v_0 = v_m = 0.$$  

The solution of equation (4.10) closely matches the results of the Monte Carlo sim-
4.1. Jump Voter Model

Simulations (Figure 4.2). Note that the solution is symmetric about \( x = 0.5 \), which makes qualitative sense, since the mean external influence is 0.

Figure 4.2: Expected consensus time for the jump voter model on a \( 25 \times 25 \) lattice, with \( p = 1/(625\times 5), v = 0.04 \) (see Table 3.1 for meaning of parameters). The black curve denotes the exact numerical solution of equation (4.9), black points denote simulations results based on update rules in Section 3.1, where each point is obtained by averaging over 2000 runs. The external influence, \( Z \), has a truncated normal distribution. This match confirms our approximation of model B by model C, as shown in Table 5.1.

4.1.1 Comparison with Classical Voter Model

Based on the previous work by Sood et al [SR05], we know that the diffusion approximation of the classical voter model (discrete-time version) is given by the diffusion process,

\[
dX(t) = \sqrt{2X(t)(1-X(t))}dW(t)
\]

whose generator is given by the differential operator,

\[
\mathcal{L} f(x) = x(1-x)f''(x).
\]

As mentioned earlier, if the jump probability is set to \( p = 0 \) in our jump voter model, we retrieve the classical voter model. We derive the expected consensus time solution corresponding to that case, and find that it matches with the results in Sood.
et al [SR05]. (Results not shown.) The two parameters, jump probability $p$ and jump variance $v$, together determine the overall impact of the external influence, and we collectively refer to them as jump parameters. We find that even for fairly small jump parameter values, the expected consensus time differs dramatically between the classical voter model and the jump voter model (Figure 4.3(b)).
4.1. Jump Voter Model

Figure 4.3: (a) Fixation probability and (b) expected consensus time comparison between the classical voter diffusion and the jump voter diffusion (and hence the classical voter model and the jump voter model), and (c) Difference in fixation probability between the classical voter diffusion and the jump voter diffusion (and hence the classical voter model and the jump voter model). \( N = 500 \) and for the jump voter model, \( p = 1/(500 \times 10) \), \( v = 0.03 \) (see Table 3.1 for meaning of parameters). This comparison corresponds to comparing model A and model D (or model B and model E), as shown in Table 5.1.

If we consider just the maximum value of consensus time (i.e. the consensus time when the initial density of 1s is 0.5) (Figure 4.3), we can plot it as a function of the two jump parameters. We see that the consensus time decreases rapidly as jumps are
4.1. Jump Voter Model

introduced.

Figure 4.4: Maximum consensus time as a function of jump variance $v$ and jump probability $p$, $N = 500$. (The maximum consensus time axis has been restricted to 100 time steps for diagrammatic clarity.)

We also isolate the dependence of the consensus time on the individual jump parameters. The surface in Figure 4.4 appears symmetric about $p = v$. However, examination of two curves on the surface (Figure 4.5) indicates that the symmetry is only approximate. The parameter $p$ has a slightly stronger effect on consensus time than parameter $v$. 
4.1. Jump Voter Model

Figure 4.5: Maximum consensus time for the jump voter diffusion as a function of the jump parameters individually, $N = 500$. (The maximum consensus time axis has been restricted to 10000 time steps for diagrammatic clarity.) The maximum consensus time decreases rapidly as $p$ and $v$ increase. For plotting the dependence on $p$, $v = 0.001$, and for plotting the dependence on $v$, $p = 0.001$ (see Table 3.1 for meaning of parameters). We notice that the jump probability ($p$) has a slightly stronger effect than the jump variance ($v$) on the consensus time.

We notice in Figure 4.3(b) that although our solution for the consensus time has properties that are qualitatively similar to those of a classical voter model, the quantitative difference between the two is considerable. We therefore make a few observations:

1. Since the mean external influence is 0, the effect of jumps on the fixation probability (the probability of reaching consensus on opinion 1), is mostly minimal (Figure 4.3(a) and Figure 4.3(c)). In other words, the tendency of the process to reach consensus at a particular state is mostly unaltered (except at very low initial minority densities) by the external influence due to symmetry in the latter.

2. Since jumps can have a significant impact on the consensus time of the process, the jumps may be included in opinion dynamics models based on the voter model. (Figure 4.3)

3. The dependence of consensus time on both jump parameters, jump variance and jump probability, is qualitatively very similar. (Figure 4.5)

We also note how the effect of the jumps begins to appear as the values of the jump parameters are gradually increased (Figure 4.4 and Figure 4.5). Consensus time
4.2 Jump Biased Voter Model

is decreasing in both jump parameters \( (v \text{ and } p) \). The rate of decrease of the consensus
time is very high at low jump parameter values, and drops as parameter values increase.
Therefore, it is primarily the presence of jumps that appears to be a key factor for the
consensus time. In other words, the voter model is very sensitive to the presence of
jumps as far as consensus time is concerned.

Overall, the jumps have a key role in driving the dynamics. Moreover, the jumps also
introduce little skew in addition to that inherently present due to the initial densities.

\section{4.2 Jump Biased Voter Model}

In this section, we turn to investigating the fixation probability and the consensus
time for the jump biased voter diffusion. This model, discussed in Section 3.3, incor-
porates a bias in the external influence and/or a bias in the node-to-node interactions.
The fixation probability BVP based on the generator in equation (3.11) is,

\begin{equation}
\begin{aligned}
x(1-b^{-1})u'(x) + \frac{x}{2N}[(1+b^{-1}) - 2b^{-1}x]u''(x) + \\
\lambda \int_{-\infty}^{+\infty} u(x-y)g(y)dy - \lambda u(x) = 0
\end{aligned}
\end{equation}

\( u(x) = 0, \forall x \in (-\infty, 0] \)

\( u(x) = 1, \forall x \in [1, \infty) \).

We solve this problem numerically, and again find close match between the solution
of equation (4.13) and Monte Carlo simulations. (Figure 4.6)
4.2. Jump Biased Voter Model

Figure 4.6: Fixation probability for the jump biased voter model on a regular graph with 
\( N = 500, k = 100, p = 1/(500 \times 5), m = 0.1, v = 0.03, b = 1.1 \) (see Table 3.1 for meaning of 
parameters). The black curve denotes the numerical solution of equation (4.13), black points de-
ote simulation results based on update rules in Section 3.1 including the adjustments discussed 
in Section 3.3. Each point is obtained by averaging over 1000 runs. The external influence, \( Z \), 
has a truncated normal distribution. This biased model corresponds to the case where the ex-
ternal influence biases opinions towards 0, whereas the bias in node-to-node interactions makes 
1 retentive. This match confirms our approximation of model E by model F, as shown in Table 
5.1.

The consensus time BVP is,

\[
x(1 - b^{-1})v'(x) + \frac{x}{2N} [(1+b^{-1}) - 2b^{-1}x]v''(x) + \\
\lambda \int_{-\infty}^{+\infty} f(x-y)g(y)dy - \lambda v(x) = -1 
\]

(4.14)

The numerical solution of equation (4.14), along with the results from Monte Carlo 
simulations, is shown in Figure 4.7.
4.2. Jump Biased Voter Model

Figure 4.7: Expected consensus time for the jump biased voter model on a regular graph with $N = 500, k = 50, p = 1/(500 \times 2), m = 0.1, v = 0.04, b = 1.1$ (see Table 3.1 for meaning of parameters). The black curve denotes the numerical solution of equation (4.14), black points denote simulation results based on update rules in Section 3 including the adjustments discussed in Section 3.3. Each point is obtained by averaging over 100 runs. The external influence, $Z$, has a truncated normal distribution. This biased model corresponds to the case where the external influence biases opinions towards 0, whereas the bias in node-to-node interactions makes 1 retentive. This match confirms our approximation of model E by model F, as shown in Table 5.1.

We make some observations for the fixation probability and consensus time results for the jump biased voter model. Firstly, there is an increase in the fixation probability across all initial conditions (Figure 4.6). This implies that when the two biases are comparable, the jump bias seems to have a weaker effect than the bias embedded in the node-to-node interactions. Therefore, we see a net bias towards 1, which is the bias in the node-to-node interactions. Secondly, the expected consensus time for the jump biased voter model is reduced across all initial conditions, and it no longer remains a symmetric function (Figure 4.7). (The reduction for initial densities greater than 0.5 is greater than the reduction for the densities lesser than 0.5.) The asymmetry may be explained as follows: First, consider the case where the initial density of 1s is greater than 0.5. In the absence of a net bias, the process for this case would terminate at opinion 1 more than 50% of the times. With the introduction of a net bias towards 1, this tendency will be further reinforced, resulting in an increase in the relative number of instances where the process terminates at 1. This behaviour can be thought of as a reduction in the “distance” to termination. Moreover, the net bias can also be thought
4.3 Jump Threshold Voter Model

In this section, we study the fixation probability and the consensus time for the jump threshold voter model. The results are obtained through Monte Carlo simulations, and are shown in Figure 4.8.

![Figure 4.8: (a) Fixation probability and (b) expected consensus time for the jump threshold voter model in comparison with the jump voter model and the threshold voter model, on a 25x25 lattice with \( p = \frac{1}{625 \times 10} \), \( v = 0.03 \), \( \theta = 2 \) (see Table 3.1 for meaning of parameters). Each point is based on 1000 runs, and error bars indicate standard error of the mean. The external influence, \( Z \), has a truncated normal distribution. The comparison corresponds to comparing model B, model G, and model H, as shown in Table 5.1.](image)

Based on the results in Figure 4.8, we make the general observation here that the fixation probability and the consensus time for the threshold voter model and the jump threshold voter model become noticeably different only when the minority opinion at
4.3. Jump Threshold Voter Model

the start of the process is greater than approximately 0.4. For the current choice of model parameters, we may think of the initial density value of 0.4 as a “critical” density in the sense that the behaviour of the jump threshold voter model begins to differ from the threshold voter model, if the initial density is higher than the critical density.

For another comparison – between jump threshold voter model and jump voter model – the two models were run in parallel and the spatial results are shown in Figure 4.9. This direction of spatial analysis was motivated by the current understanding of spatial differences between the threshold voter model and the voter model.

![Figure 4.9: Snapshots of evolution of (a) the jump threshold voter model, and (b) the jump voter model on a 100 × 100 lattice, with \( p = 1/(10000 \times 5) \), \( v = 0.03 \), \( \theta = 2 \) (see Table 3.1 for meaning of parameters). Initial density of 1s is 0.5 for both models. The numbers in the top right corner denote the time step for the respective panel. The external influence, \( Z \), has a truncated normal distribution. This comparison corresponds to comparing model B and model H, as shown in Table 5.1.](image)

It is known that the evolution of clusters in the threshold voter model is characterised by motion by mean curvature [CFL09, DC07]. As a result, a random or disordered initial opinion distribution is rapidly arranged in the form of clusters or blobs first, and the subsequent dynamics is governed by boundary forces akin to surface tension. That is, convex shaped regions of any arbitrarily shaped opinion clusters will get encroached upon by the opposing opinion. Spatially, the introduction of jumps to this model then plays the role of disrupting the clustering sporadically. This can be seen in Figure 4.9(a).
4.3. Jump Threshold Voter Model

The jump threshold voter model is similar, for the fixation probability and the consensus time, to the threshold voter model at low initial minority densities. As noted before, the differences between the threshold voter model and the jump threshold voter model become prominent only at initial minority densities greater than the critical density 0.4. The fixation probability for the threshold voter model exhibits behaviour similar to a step function (Figure 4.8), where consensus is almost always reached on the opinion in majority at the start of the process. The threshold voter model thus amplifies the advantage held by a particular opinion type due to a higher initial density. Comparing the fixation probability in this case to that for the jump threshold voter model, we notice that the probabilities become less extreme at initial minority densities greater than the critical density. Therefore, at initial minority densities greater than the critical density, the jumps help in moderating the advantage amplification inherent to the threshold voter model. Next we assess the effect of the jumps on the consensus time. The jumps have an effect of increasing variability in the opinion density. This effect appears more pronounced at initial minority densities higher than the critical density, as we get a quicker consensus in that regime.

Overall, for initial minority densities lower than the critical density, the clustering effect is too strong to be disrupted by the jumps. However, once the initial minority density exceeds the critical density, the disrupting effects of the jumps begin to counter the clustering effect inherent to the threshold voter model.

Jumps expedite the threshold voter model less than the voter model. We next compare the consensus time reduction caused by the jumps in threshold voter model with that in the voter model. We notice, based on comparing Figure 4.3 and Figure 4.8, that the jumps reduce the voter model consensus time significantly more than the threshold voter model consensus time. This suggests that the threshold voter model is comparatively more robust to an external influence. This is again attributable to motion by mean curvature in the threshold voter model. Consider (WLOG) the extreme case, for both jump voter model and jump threshold voter model, where the jump causes a \(0 \rightarrow 1\) flip at a node in the interior of a \(0\) cluster. In the jump voter model, there is a non-zero probability that the \(1\) in the interior can in turn cause a \(0 \rightarrow 1\) flip at one of its neighbouring nodes. But in the threshold voter model, the probability that this opinion \(1\) node can flip any of its opinion \(0\) neighbours is zero. Therefore, the cluster patterns of the voter model would tend to more vulnerable to jumps as compared to those of the threshold voter model. This serves as a possible explanation for the higher robustness of the threshold voter model to jumps with regards to the consensus time. This phenomenon can also be observed in Figure 4.9: Notice the similarity in the cluster pattern in the second and fourth panels of Figure 4.9(a) despite the occurrence of a jump in the third panel. On the contrary, no such trend is seen in Figure 4.9(b).
Chapter 5

Conclusion

In this work, I have developed extensions of the classical voter model, the biased voter model, and the threshold voter model, in order to incorporate an external influence that causes many opinions to shift in the same direction simultaneously. This type of influence could occur, for example, through mass media. I approximated the extensions of the classical voter model and the biased voter model, i.e., the jump voter model and the jump biased voter model, by means of jump diffusion processes. This approach allowed me to analytically determine the probability of reaching consensus on opinion 1 (fixation probability), and the consensus time. For the extension of the threshold voter model, the jump threshold voter model, I chiefly relied on simulations to determine fixation probability and consensus time. Most existing literature on opinion dynamics only studies opinion evolution under influences internal to the system. This work provides a systematic study of opinion dynamics under an additional external influence. A summary of all the models with their diffusion approximations (if any), along with relevant existing models is shown in Table 5.1.

The noisy voter model is another contemporary variant of the voter model, and it is instructive to compare the same with the jump voter model as both the models introduce an additional source of opinion flipping. In the (discrete-time) noisy voter model, an agent is chosen, following which, with some probability the agent adopts one of its neighbours’ opinion, and with the remaining probability flips its opinion irrespective of its neighbourhood. In the noisy voter model, a spontaneous $0 \rightarrow 1$ or $1 \rightarrow 0$ flip happens only at one site at a given time. Our jump voter model can be thought of as a “noisier” voter model where the spontaneous flips happen simultaneously at multiple sites. Another subtle difference is the connection between the opinion density at a given time, and the spontaneous flip that occurs at that time. In a noisy voter model, a spontaneous $0 \rightarrow 1$ flip, for example, is positively correlated with the density of 0s at that point of time. This correlation is because agents with opinion 0 will have a relatively higher probability of getting sampled. However, in our jump voter model, there is no explicit or implicit connection between the opinion density and the opinion flipping due to the jumps. Therefore, our model may be used over the noisy voter model in scenarios where a larger external influence (or noise) is required, and where the external influence is completely independent of the state of the system.

The diffusion approximations for the voter model and the biased voter model do not depend on the degree of the regular graph, and we note that this property is retained even for the jump voter model and the jump biased voter model. This means that the fixation probability and consensus time results for the jump voter model and the jump biased
Table 5.1: A global overview of the models formulated in this work, along with relevant existing models. Also shown are the approximations used to derive theoretical results, wherever possible. Note that the jump version models (column 2) are defined on a regular graph, meaning that the degree $k$ is a parameter among those models. Their approximate models (column 3), derived using diffusion approximation and mean field approximation, have no degree dependence. In the comparison graphs (column 4), black dots are obtained from the jump versions of the models and the black curves are obtained from their corresponding approximate models.

<table>
<thead>
<tr>
<th>Existing Model</th>
<th>Formulated Jump Version Model</th>
<th>Approximate Model</th>
<th>Comparison of Jump Model with its Approximate Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voter Model (Model A)</td>
<td>Jump Voter Model (Model B) (On any regular graph)</td>
<td>Jump Voter Diffusion (Model C) (No degree dependence)</td>
<td>(See Figure 4.1)</td>
</tr>
<tr>
<td>Biased Voter Model (Model D)</td>
<td>Jump Biased Voter Model (Model E) (On any regular graph)</td>
<td>Jump Biased Voter Diffusion (Model F) (No degree dependence)</td>
<td>(See Figure 4.6)</td>
</tr>
<tr>
<td>Threshold Voter Model (Model G)</td>
<td>Jump Threshold Voter Model (Model H)</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
voter model (discussed in Chapter 4) will remain true for all regular graphs ranging from a complete graph (where \(k = N - 1\)) to a lattice (where \(k = 4\)) to even a cycle graph (where \(k = 2\)). Thus, in a society where everyone has the same neighbourhood size, fixation probability and consensus time are independent of the neighbourhood size as long as agents update their opinions by randomly sampling from their neighbourhood.

Another key observation is that the jumps expedite consensus in both the voter model and the threshold voter model, but more so in the voter model. Thus, in a society where agents update their opinion if and only if the pressure from their neighbourhood is enough (based on a threshold parameter), external influence has a lesser effect on consensus time than in a society where agents update their opinions by randomly sampling from their neighbourhood. In a real-world system, agents most likely update their opinions by a combination of the two update rules, i.e., opposing pressure from the neighbourhood and random sampling. Since the jumps expedite consensus in both the models, we may expect them to also have a similar effect in a combination of the two models which might better reflect reality.

This work opens up multiple interesting directions that may be further explored. The domain of network science is currently expanding very rapidly, and one natural extension of our work is to study our models on a heterogeneous graph structure such as a scale-free network. Such work could lead to pragmatic insights since scale-free networks have been shown to be ubiquitous in various real-world social systems [BA99].

For the jump threshold voter model, this work mainly relied on simulation analyses, which may be extended further to rigorous mathematical analyses. More generally, for all the newly introduced models (the jump voter model, the jump biased voter model, and the jump threshold voter model) using duality might provide a detailed understanding of those models, as the technique of duality has been impactful in the study of interacting particle systems.
Bibliography


Chapter 5. Bibliography


Chapter 5. Bibliography


[She36] Muzafer Sherif. The psychology of social norms. 1936. → pages 1


