UTILIZING THE EMPTY NUMBER LINE TO FACILITATE SENSE MAKING IN THE MENTAL MATH CLASSROOM

by

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Abstract

The purpose of this study was to explore the possibilities of the Empty Number Line to further develop and strengthen students’ numeracy skills, specific to addition and subtraction. The Empty Number Line (ENL) is a Dutch approach to developing numeracy and mental math skills in the elementary classroom. Internationally acknowledged, the Empty Number Line (ENL) boasts to solving computational tasks in a manner that builds on users intuitive understanding of number. Typically, this model is introduced in the primary years to support early numeracy development. However, this study set out to determine if the tool could be utilized to strengthen the computational skills and fundamental numerical understanding of a sample of 28 Grade 4 students, having no prior exposure to this model. Specifically, the researcher sought to determine what the ENL could reveal about students’ sense of number, while utilizing the tool in a manner that supported sense making and self generated strategies. Central to this study was to establish student opinion of this tool, in regards to its effectiveness and ease of use. Over a four week period, students were asked to commit to eight one hour blocks focusing on the exploration of this tool. Three strategies, stringing, bridging and splitting were presented. Via whole class lessons, independent tasks and group activities, students completed a variety of tasks by applying a presented strategy. Student samples and journals were analyzed to determine students performance and opinions over the four week period. Overall students responded very favourably to this tool, and the majority of students developed a good understanding of how to utilize the ENL. However, data unveiled much about students’ numerical capabilities, and in many cases highlighted gaps in children’s number sense. In addition, data analysis highlighted some important future considerations for those considering using this tool, including the
delivery of solutions as well as the challenge of applying splitting to subtraction tasks. To conclude, this study highlights advantages and disadvantages when using the ENL to solve 2 and 3 digit computation tasks, as well as considerations to educators intending to present this approach in future teaching endeavours.
Preface

This thesis is an original, unpublished, independent work by the author W. Macken. The fieldwork reported in Chapters 3–5, was approved by UBC Behavioural Research Ethics Board, Certificate Number H11-00192.
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1 Introduction

The Teaching of Number Sense

“Number, operation, and computation have a long and prominent history in the school mathematics curriculum. In addition, this area of mathematics, perhaps more than any other, is widely recognized and valued beyond the school setting” (National Council of Teachers of Mathematics, 2000, p. 5).

Approaches to the teaching of number have developed over the past two decades, towards our current emphasis on mathematical “reasoning rather than on speed and accuracy and conceptual understanding rather than procedures” (Yackel, Underwood & Elias, 2007. p. 351). Mathematics is now seen as a “sense-making activity” (Tsao, 2004. p. 71) which is a very different approach to what many current math educators experienced as students themselves. This may explain why, while the British Columbia Performance Standards recognize “A true sense of number goes well beyond the skills of simply counting, memorizing facts and the situational rote use of algorithms” (BC Ministry of Education, 2007. p.13), traditional teaching methods are still the dominant form when addressing numeracy in many mathematics classrooms. In order to develop computational skills in the classroom, a common practice for many teachers entails an explicit modelling of prescribed steps for students to follow, before giving students repeated opportunities to mirror these steps independently. However, it is questionable if students actually gain any real understanding of what they are doing or rather simply learn to survive by memorizing procedures. Real understanding can become an afterthought if students produce the desired answer. Such an approach is proving ineffective in helping students learn to cope with the quantitative demands of modern society. As Lynn Steen writes, While such skills may not have been deemed essential for all in past decades, society now demands much more numerical understanding of its citizens in order to actively participate in today's world.
Globally, educators across the world recognize that there is a need to adjust longstanding numerical education, in order to match these new developments and numerical needs in today's world. Revised curriculums have shifted their focus towards developing a deeper understanding of number, placing emphasis on sense making rather than on memorization of rote procedures. In order to address these needs, one component often implemented in the math classroom to develop and improve number sense is ‘mental math’. Mental mathematics is becoming increasingly more prevalent in mathematics classrooms and prescribed curriculums worldwide (Board of Studies New South Wales, 2002; Callingham, 2005; Cooper, Heirdsfield & Iron, 1996; Hartnett, 2007; Mardjetko & MacPherson, 2007). Sowder (1988) defines mental math as the process of carrying out arithmetical operations without the aid of external devices. However, mental math is not simply the quick fire recall of facts but rather deals with numbers holistically (MacLellan, 2001). Mental math requires more than memorization of a number of learned procedures, but rather a deeper knowledge of how numbers work (Hartnett, 2007; MacLellan 2001). In her work, MacLellan (2001) describes mental math strategies as “variable, flexible, creative and idiosyncratic” (p.146) and “demand the child to be actively thoughtful” (p.147).

In recent years, countries such as the UK and Wales have raised the importance of mental calculations in their math curriculum (Threlfall, 2002). Similarly, the National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics note that all students should conceptually understand the place-value structure of our number system, see relationships between numbers, be able to recognize and generate equivalent representations for numbers, and demonstrate mental computation fluency (NCTM, 2000). But how exactly the exploration and development of 'mental computation fluency' should unfold in the classroom is less clear. While mental math drills seem to be regularly encapsulated in the math classroom, do these types of activities provide students the opportunity to understand, develop and build on
these skills in a manner that will enhance their overall sense of number? One could argue quite easily that the popular “mad minute” drills are simply the quick fire recall of facts and overlook what MacLellan (2001) believes should be at the heart of mental math, children dealing with numbers holistically.

Thompson (1999a) claims that there are four reasons for developing mental calculation: (1) the most popular method of calculation among adults is mental, (2) mental work facilitates a deeper understanding of the number system, (3) problem solving skills develop with exposure to mental calculations, and (4) mental work supports later written calculations. Thompson's extensive research (1994, 1999a, 1999b, 1999c, 1999d) suggests the wide range of benefits of developing mental math skills within the classroom.

While this is the case, it appears many educators are still reluctant to fully embrace and explore mental math skills within the classroom in the manner it is intended. One possible reason for this is the longstanding dominance of algorithms in number sense education. Despite criticism from many, algorithms appear to remain the primary method explored to complete computation tasks in classrooms. Hartnett (2007) offers one possible reason for this loyalty to algorithms, being that current classroom teachers were themselves immersed in the world of rote procedures and memorization of facts, and therefore do not fully understand or value this recommended shift in teaching. However, the effectiveness of computational algorithms for teaching arithmetic has been under scrutiny for many years. In addition, it is argued that such a skill has little value outside the school community. In a study conducted by Northcote & McIntosh (1999), of 200 people over 24 hours, only 11.1% of calculations involved a written component. Researchers such as Kamii and Doninick (1997), Anghileri, Beishuizen and Van Putten (2002), not only question the benefits of algorithms but insist they can be harmful to the development of students’ number sense. Kamii and Dominick (1997) identifies two main reasons
why. Firstly, they believe algorithms “unteach place value” (p. 58) and secondly they do not support children’s natural way of thinking (from left to right) and instead call for students to abandon their own way of thinking. Along with many others in the field, Kamii and Dominick (1997) believe that teaching algorithms “make most children stop trying to make sense of numbers and lead them to focus on remembering only the steps” (p. 58). While mental math strategies are seen as useful, many teachers appear reluctant to allow this approach to replace the teaching of traditional algorithms. Because of this, as children advance into intermediate grades, mental math can simply become the visualization of formal written calculations that are performed mentally.

However, if math educators are committed to this new ‘sense-making’ mathematics, we need to seriously consider our approaches to teaching numeracy in classrooms. We need to examine the value of both mental math and algorithms and look at whether one can work alongside the other. If mental math methods do allow "number to be treated in a holistic way, as quantities rather than digits" as stated in QCA/Qualifications and Curriculum Authority (1999, p.14) can the two in fact work together. Algorithms, rather, ask students to look at digits in isolation, with no need to consider the quantity of each digit. This most certainly contradicts the approach to mental math as outlined by MacLellan (2001) who believes that numbers need to be dealt with holistically.

So what is the way forward? What can educators do to provide their students with opportunities to explore and develop their numeracy and mental math skills, while still giving teachers the proof and peace of mind that students leave with the basic skills necessary to perform computation successfully? One possibility may be found by looking to the Netherlands. In contrast to many countries, the Netherlands do not introduce column arithmetic until “Groep 5”, equivalent to the UK’s Year 4 or Canada's Grade 3. Rather, as pointed out by Bramald (2000)
they differentiate clearly between doing maths in your head (recalling number facts) and doing mental arithmetic with your head (using mental strategies). To support this process, the Dutch use a didactical model that is receiving attention globally (Bobis & Bobis, 2005; Bramald, 2000; Heuvel-Parhuizen, 2008; Klein, Beishuizen & Treffers, 1998), known as the empty number line (ENL).

The Empty Number Line

Simply put, the empty number line (ENL) is a line that is presented without markers or numbers. It acts as a visual representation for recording and sharing students’ thinking strategies, as well as providing a stage for students to think about number. The ENL focuses on the skill of drawing a visual representation of thinking strategies used to carry out processes such as addition or subtraction. The ENL can provide teachers with opportunities to model and discuss a variety of strategies and methods, while giving students the opportunity to create and develop their own mathematical thinking. The model allows users to develop relations between numbers but also visualize the operation, by creating a model of the operation using arrows to represent jumps (Beishuizen, 1993). It is believed that such an approach promotes a stronger sense of number among students. In addition, research suggests that the ENL supports our own natural ways of thinking about and working with number (Gravemeijer, 1994).

Purpose of Study

Motivation for this study arises from a deep concern for the development of number sense in today’s elementary classrooms. The purpose of this study is to add to the current body of research existing on the empty number line (Murphy, 2010; Klein et al., 1998; Van Den Heuvel-Panhuizen, 2008), and to determine whether the ENL may be an effective way to support and
develop students' own intuitive mental math strategies, which in turn will strengthen students number sense in a North American context. The ENL is a tool which research shows assists children in mental math development. While it is becoming more commonplace in Europe and Australia, neither the NCTM nor the BC curriculum points to its use. As an early primary teacher, I have supported young children's early mental math development through various methods. Prior to this study, I had utilized an empty number line with students within a Grade One classroom, limited to working with one digit numbers. However, due to the age of those students, I was unable to present the ENL model as an alternative to the standard algorithm. Based on my experiences, I believe continuing to "teach" conventional algorithms to students before they have fully explored and developed a strong sense of number, can be a detriment to their mental math strategies and overall sense of number. Although no one tool is expected to be a panacea, it seems that the ENL may provide a user-friendly mechanism to support students’ development of mental math and math sense making. As we seek to reform mathematics education, I believe this didactic tool could be deemed a worthwhile and effective approach to explore with students, in order to develop their number sense and mental math skills, in a manner which places emphasis on meaning and understanding rather than simply on implementing a number of learned steps, often without any understanding of what one is doing.

Thus, I was interested to learn how students utilized the ENL to explore and develop a variety of student-generated strategies, to successfully solve addition and subtraction tasks. The study examined in what ways and to what extent students used this tool to perform computation tasks with two and three digit numbers, and whether and how the use of the ENL impacted students’ understanding of number. In addition, an effort was made to investigate how students view this model, in regards to its effectiveness and ease of use, in comparison to other approaches they have used. Participants chosen were older than is the norm for the introduction of the ENL.
Typically the ENL is introduced to students in the early primary years before ongoing exposure to the standard algorithm. However, this study set out to introduce students to the tool in the latter part of Grade 4, after they had received extensive teaching of the algorithm. Apart from being able to obtain comparisons from students about both approaches, the intent for introducing the tool to such a sample was to determine if the ENL could offer those struggling with the algorithm a worthwhile alternative approach, and if such a tool has the ability to undo the results of teaching students rote algorithmic procedures.

**Research Questions:**

1. In what ways does the empty number line support students sense making and self-generated strategies when carrying out 2 and 3 digit computation tasks?

2. What does their use of ENL reveal about students' sense of number?

3. In what ways, if any, are students more or less effective in solving computation tasks, following exposure to ENL?

4. What opinions do students formulate about ENL with regards to its effectiveness and ease of use?

**Conclusion**

In Cuoco's work entitled “Habits of Mind”, he states, "Much more important than specific mathematical results are the habits of mind" (Cuoco, Goldenberg & Mark, 1996. p.375). Such an approach to math education involves the "process of creating, inventing, conjecturing and experimenting” as well as “encouraging false starts, calculations, experiments, and special cases" (p.376). It aims to help students "look for logical and heuristic connections between new ideas and old ones" (p.376). In my study, students were being asked to utilize a completely new
approach to solve computational tasks. The hope here was that students would build connections between their old approach, the standard algorithm, and this new method, the ENL. Could the ENL facilitate the processes outlined in Cuoco's work, in an effort to support student sense making and self-generated strategies?
2 Literature Review

Introduction

“The learner should never be told directly how to perform any operation in arithmetic….

Nothing gives scholars so much confidence in their own powers and stimulates them so much to use their own efforts as to allow them to pursue their own methods and to encourage them in them” (Colburn, 1912, p.463).

Central to math education lies the development of number sense. Students’ ability to compute numbers with speed and accuracy remains an important indicator for determining their level of skill in many math classrooms. While educators revise curriculums to raise standards of achievement, particularly in the area of number sense and specifically computation, alternate approaches to developing number skills, including the use of the Empty Number Line, now play a role in number sense instruction. However, while these new approaches have entered many classrooms, they have not become a replacement for other longstanding computational approaches, such as the traditional algorithm. Therefore I begin with an examination of the research on algorithms, to determine the need for additional methods to promote computational skills among our students.

The Influence of Algorithms

The standard algorithm was developed long ago to support everyday arithmetic in a manner that maximized speed and minimized fuss. However, the benefits of this approach – such as "their conciseness, their dependence on symbol manipulation and their generalizability – also constitute their major weaknesses" (Thompson, 1999d, p. 175). In addition, the excessive demands made on working memory can result in difficulty remembering the steps in the selected procedure or, if remembered, students may mix up their order. In 1979, Plunket (cited in
McIntosh, 1995) defined written algorithms as “standardized, contracted (summarizing several steps involving distributivity and associativity), effective, automatic, symbolic, general, and analytic (requiring numbers to be broken up and numbers to be dealt with separately)” (p.237). However, written algorithms can facilitate “cognitive passivity” and can be challenging to internalize because they are not presented in the way people naturally think about numbers (Reys, & Barger, 1994, p.31).

However, the standard algorithms have remained a prominent part of the math curriculum in the province of British Columbia and across North America throughout the 20th and into the 21st century, despite sense-making and mathematical understanding being deemed essential in today’s classrooms. Indeed, effectiveness of teaching arithmetic through algorithms has been scrutinized over the years by Kamii and Dominick (1997), Anghileri, Beishuizen and Van Putten (2002), and others who question the positive impact of such an approach. Rather, their research indicates teaching algorithms can be harmful to the development of students’ number sense. The work of Murray, Oliver and Human (1991) with young students in South Africa found that despite being taught the standard algorithm, students prefer to employ their own informal algorithm when allowed. In addition, the success rate when students were allowed to do so was significantly higher than when they used the standard algorithm.

In a similar vein, Narode, Board and Davenport (1993) conducted a year-long study of nineteen first, second and third-grade students, pre and post algorithmic instruction. Students were asked to carry out 2 and 3 digit addition and subtraction tasks embedded in problem solving stories. Before algorithmic instruction most students applied invented strategies, looking at numbers from left to right. Post algorithmic instruction, students’ approaches had changed, looking at numbers from right to left and showing a reluctance to use their invented strategies in lieu of an algorithmic approach. The researchers concluded that encouraging students to use only
one method (algorithmic) to solve problems, leads them to lose some of their capacity for flexible and creative thought. They become less willing to attempt problems in alternate ways, and they become afraid to take risks. Furthermore, there is a high probability that the students will lose conceptual knowledge in the process of gaining procedural knowledge (p. 260). This study reaffirms the research of Hope (1986) who found that many intermediate students chose to use an algorithm when it was not necessary, as in $100 - 99.95$. He concluded that such is the outcome when there is too much focus on procedures rather than on thinking.

For many decades, researchers (Ashlock, 2010; Brown & Burton, 1978) have identified a number of common mistakes children make when completing algorithms, which interestingly remain common in today's classrooms (Burns, 2000). Those in the field believe that this is because students are struggling to implement a number of learned steps to carry out the operation, rather than making sense of the numbers with which they are working (Kamii and Dominick, 1998). After an extensive review of literature, Cooper, Heirdsfield and Irons (1996) recommended that the amount of time teaching algorithms needs to be greatly reduced making time for students to engage in extensive exploration of mental mathematics. Thompson (1999a) supports such an approach stating that mental mathematics supports the development of a sound sense of number as well as problem solving skills. In their work, Kamii and Dominick (1997) assessed the number sense of groups of second, third and fourth grade students through a variety of computation tasks. The subjects were categorized based on their exposure to teaching of formal algorithms in school. Kamii and Dominick found that students who have not received any formal teaching of algorithms outperformed those who had received some or extensive amounts of formal instruction on algorithms. Another important finding was found in studying the incorrect answers of students. Those with no formal algorithmic instruction gave far more reasonable incorrect answers than those with formal algorithmic instruction, showing that their
"number sense and knowledge of place value" was more developed. This work supported the argument that algorithms unteach place value and encourage students to give up their own natural ways of thinking about number. Kamii and Dominick (2007) state that teaching algorithms redirects students’ attention "from trying to make sense of numbers to remembering procedures” (p.59). They concludes by saying, “Many educators recognize that children invent their own method for solving problems, but their goal is still to teach the conventional algorithms in the end.....the time has come to stop teaching the algorithms and, instead, encourage children to make the mental relationships necessary to build number sense” (p.60).

The Case for Mental Math

In addition to diminishing support for teaching algorithms, a growing body of evidence emphasizes the importance of mental math in the mathematics curriculum. According to Straker, 1999 (as cited in Foxman & Beishuizen, 2002) “The ability to calculate mentally lies in the heart of numeracy” (p.41). In an effort to ensure math education provides the skills necessary for success in today’s society, many feel this current emphasis on mental math is a positive step forward. There is strong evidence to suggest that mental computation supports the development of number sense among students, allowing them to create strategies and carry out procedures in a manner that is meaningful and worthwhile (Cobb & Merkel, 1989; Klein & Beishuizen, 1994; Maclellan, 2001; Reys, 1984, 1985; Sowder, 1992 cited in Varol & Farran, 2007). Such an approach to number helps students appreciate the power of what they already know, as well as develop efficiency and flexibility when working with number.

The work of Reys (1985) suggests that if students are provided with an opportunity to engage in and develop these necessary skills daily, such skills become automatic leading to less of a need for paper and pencil type skills, and instead facilitate computational fluency by
developing efficiency, accuracy and flexibility. According to Mardjetko and Macpherson, (2007), who researched intermediate students stated that “students who have greater flexibility in their mental computation strategies provide a greater understanding of the underlying mathematical concepts and have higher success rates”(p.8). Mental math is also recognized as a valuable attribute that will contribute to success in written mathematics. The Cockcroft Report (DES 1982) states that “…it follows that the practice of mental methods of computation will also assist in the understanding and development of written methods” (Thompson, 1999a, p. 147). Thompson (1999a) adds that the development of mental math positively impacts problem solving skills.

However, mental math is a process that is viewed differently in the world of education. Some see it as basic skills, developed by structured teaching and practice, addressed in daily drills such as the popular practice “mad minute” (Shoecraft & Clukey, 1981). This view allows for direct teaching of valued practices in a rote manner, often relying on memorization without understanding, much the same as traditional algorithms (Varol & Farran, 2007). Others see mental math as “a higher order thinking process” where the process is as important as the product (Reys & Nohda, 1994. p.12). Those who hold this view recognize the need for instruction to be student driven as opposed to teacher directed and the need to provide students with the materials, time and support to produce student generated strategies. Research shows that students can indeed invent strategies of their own efficiently without the need for direct instruction (Carpenter, Franke, Jacobs, Fennema & Empson, 1998; Heirdsfield, 2000; Kamii, Lewis, & Jones, 1991; Kamii, Lewis & Livingston, 1993).

While many may agree that there is a need to develop both algorithmic and mental math skills, research concurs that the emphasis on learning algorithms negatively impacts a child’s own numerical thinking and mental math strategies. In his work, Bebout (1990) found that young
children have effective strategies for mental math appropriate to grade level. However, as children mature and are exposed to formal teaching of algorithms, mental math strategies decrease (Cooper, Heidsfield & Irons, 1996). While some may argue that the algorithm can indeed be used as a mental math strategy, Reys, Nohda and Emori (1995) found that such an approach to mental computation resulted in higher error rates. In addition this was the approach used least by higher performing students.

However, altering what can be seen as a very common, productive and teachable approach to computation in the classroom may appear a daunting and even unnecessary task for math educators. Already under pressure to cover a prescribed number of learning outcomes in what can be seen as a short time period, how can teachers facilitate exploration and development of numeracy and computational skills in an effective manner, while still supporting students’ understanding and mental math strategies? Some suggest that, in order to achieve this, we must look to approaches that support students’ natural way of thinking about number. One tool that boast of supporting this is the number line.

**Number Sense and the Number Line**

“Powerful ways of acting derive from powerful ways of seeing” (Marton, Runesson, & Tsui, 2004, p. 5). Mathematics is a cognitive thinking process that requires the dual coding of imagery and language. Imagery is fundamental to the process of thinking with numbers. The ability to create mental representations for mathematical concepts is directly related to success in mathematical reasoning and computation (Bell & Tuley 2003). One of the most important mental representations for the learning of mathematics is the number line (Moeller, Pixner, Kaufmann & Nuerk, 2009; Siegler & Booth, 2005). The acquisition of counting is eventually mapped onto an internal number line, which is related to an acquisition of addition and subtraction, as well as an
estimation of the magnitudes of numbers (Siegler & Booth, 2005).

A mental number line is an example of a mental model, or a way of internally representing an underlying concept, often in the form of a specific image, which provides a framework for problem solving that improves performance (Siegler & Booth, 2005; Siegler & Opfer, 2003). Dating back as early as 1880, there is evidence for a spatial representation of numbers akin to a mental number line (Galton, 1880). Subjects (adults with a mean age of 25.7 years) were asked to describe how they think of numbers and many described overt visual–spatial representations of numbers sometimes even with specific colors or characters associated with certain numbers. In most of these descriptions numbers were represented on a line—a mental number line (Gobel, Walsh & Rushworth, 2001). Forming a mental number line requires the ability to visualize and abstract a number line so that you can order numbers by quantity, locate any given number along a line, and generate any portion of the number line that may be required for problem solving (Gervasoni, 2005). Some of the central achievements of formal mathematics depend on the mapping between number and space and fundamental to this is the arrangement of numbers on a line (de Hevia & Spelke, 2008). Unlike other visual models utilized at school such as the hundred square, there are strong links between the number line and students’ own intuitive mental strategies (Bobis, 2007). Such a model gives students the freedom to work with numbers up to and greater than 100, in a manner that is conducive to students’ own natural ways of thinking about and working with number.

In order to reach American children who have not developed the "central conceptual structures on which success in arithmetic depends" (p.35), Griffin, Case and Siegler (1994) utilized a mental number line to provide students with hands on opportunities to explore and develop their sense of number. This program, named Rightstart, proved very successful for its participants and ensured that many students who would otherwise enter Grade One with
insufficient numerical skills to enjoy success, were indeed ready to successfully take on a Grade
One math program. They believe that this success was due to the fact that these activities were
congruent to the mental number line structure and aimed to represent this structure in a variety of
ways. Griffin et al. (1994) highlighted three important reasons for focusing on children’s
development of a mental number line: (1) it teaches children to respond to questions about
relative magnitude in the absence of any concrete sets of objects; (2) it teaches children the
increment rule, i.e., the addition and subtraction of one element alters the cardinal value of the set
by one unit, and therefore moves the value one unit up or down on the number line; and (3) it
teaches children that knowledge of relative position on the number line is useful for determining
relative quantity in various tasks, when it cannot be determined more directly (Gervasoni, 2005).

Learning the concept of number seems to be related to a child’s ability to generate a
mental number line (Dehaene, 1997 in Geary, 1996). However, Siegler and Opfer (2003) believe
that formal schooling must take place if a student is to learn to utilize the mental number line
efficiently. One way to help children develop a mental number line is to engage them in
activities involving an empty number line (ENL). A growing body of research suggests that this
ENL can address computational needs as the traditional algorithms does, but in a manner that
embraces the way students naturally think about number, while improving mental math skills.

The Empty Number Line (ENL)

The empty number line is a number line that is presented without markers or numbers and
acts as a visual representation for recording and sharing students’ thinking strategies. Instead of
using concrete manipulatives, which Uttal, Lui and DeLoache (2006) believe may “contribute to
the problem [of] focusing children’s attention on the characteristics of the object themselves,
rather than on what the objects are intended to represent” (p. 178), the empty number line offers
an alternate approach. The ENL focuses on the skill of drawing a visual representation of thinking strategies used to carry out processes such as addition or subtraction. One might view the ENL as a stepping stone for students, providing them with the necessary scaffolding when moving from a developed understanding of using concrete manipulatives towards being able to solve such tasks mentally.

The ENL is a strategy widely used in the Netherlands and is becoming more widespread in math education globally. The ideas of Dutch mathematician, Hans Freudenthal, informed this approach and now this pedagogical stance is common place in math classrooms across the Netherlands. He argued that “mathematics should be thought of as a human activity of ‘mathematizing’—not as a discipline of structures to be transmitted, discovered, or even constructed—but as schematizing, structuring, and modeling the world mathematically” (Fosnot, 2003, p. 9). While many educators worldwide recognize the value of planning problems as a means to stimulate thinking as well as offering opportunities for students to engage in rich discussion, implementing models such as the empty number line in classrooms demonstrates how Dutch educators have also “thought carefully about scaffolding and supporting development over time” (Fosnot, 2003, p. 6). They believe the ENL model allows students to utilize strategies of varying degrees of sophistication, while supporting and developing their current understanding of computational strategies. Students are not expected to work at the same rate or in the same manner as other students, and can develop their own understanding by creating individualistic strategies at a rate that works for them.

Introduced in the 1980’s, the intended use of the empty number line is as a flexible mental model to support mathematical processes such as addition and subtraction, rather than a measuring line from which the exact results of operations can be read. The Dutch emphasis is on children’s own informal methods, but the development of more sophisticated strategies is not left
to chance. A curriculum has been designed to support the development of strategies, incorporating the empty number line, which in time leads towards higher level strategies. Heuvel-Panhuizen (2008) considers the empty number line to be one of the most important didactical models for teaching calculations with numbers up to 100 and 1000. According to Beishuizen and Anghileri (1998), “The choice of the empty number line as a linear model of number representation up to 100 (instead of grouping models like arithmetic blocks) reflects the priority given to mental counting strategies as an informal knowledge base” (p. 525). In addition, the Dutch delay the teaching of standard algorithms to the post primary years, and instead utilize the empty number line to introduce and develop numeracy skills specific to addition and subtraction.

Researchers Beishuizen, Van Mulken, and Van Mulken (1997) have identified three basic strategies that users adopt when using the ENL: stringing, splitting and bridging. Stringing refers to keeping the first number intact while splitting the second number into tens and ones, which are then added or subtracted separately from the first number. This strategy is commonly referred to as N10. The second strategy, splitting, or commonly called 1010, involved both numbers being broken into tens and ones and processed separately to carry out the operation. Finally, bridging (A10) calls for the second number to be split to facilitate a bridge to the nearest decade, and from here the remainder is combined to the running total. While some research refers mainly to the ENL strategies of stringing and splitting, the work of Beishuizen et al (1997) includes bridging in this list. Within my study, students were made aware of the names of these three strategies, as well as the acronym for each.

However, flexible use of an empty number line is not always reflected in classrooms. Studies have highlighted some instructional pitfalls connected to the use of this model in mathematics. For example, teachers have been observed setting constraints on how students may utilize the number line, such as (1) requesting that the line be broken into decades before
beginning any calculation, and (2) applying confining specifics including adding hundreds first, followed by the tens and finally the ones (Bobis & Bobis, 2005; Heuvel-Panhuizen, 2008). Reviewing the use of the ENL in both the Netherlands and the UK, Murphy (2010) highlights a number of significant differences in the use of this model between the two countries, which greatly impacts the success each country has achieved. Examining the approach in a Dutch classroom, Murphy found children created and developed their own strategies while being supported by the common ideas accompanying the ENL. The Dutch approach to using the ENL was found to “pre-structure(d) cognition or to compel learning” (Murphy, 2010, p. 12). In contrast, use of the model in the UK consisted of teachers’ explicit teaching and modelling of strategies on the ENL. Students were then expected to carry out this taught strategy in their independent work. The explicit teaching of steps in the UK, as encouraged by their curriculum, might be considered similar to an algorithmic approach, contradicting the nature of mental math strategies (Murphy, 2010). Murphy believes that such an approach “could result in children’s passive learning of the strategies rather than the active use intended” (p.13). This concurs with the findings of Bobis and Bobis (2005) who conclude that if the empty number line is applied too rigidly, or is wrongly understood as a didactical model, it can be detrimental to the children’s understanding of and proficiency in mental calculation.

The Netherlands too, has had its difficulties in implementing the ENL as a tool. A study conducted wherein Heuvel-Panhuizen (2008) interviewed Dutch “Didactikids” who shared that they found using the empty number line constraining. These advanced math students were clearly becoming bored of the explicit use of the ENL and found it tedious and tiresome to practice using this method, particularly on occasions when they knew a faster way to carry out the prescribed task. The students in this study did not feel they had freedom with the ENL, but rather had to use it in a prescribed manner, which they felt only benefited those students who struggled with math.
This was very concerning for the researchers because the ENL is “intended as a flexible model that should give students a lot of freedom, and this includes both flexibility in the ways of recording results and flexibility in the jumps students make to solve problems” (Heuvel-Panhuizen, 2008, p. 15). In this work, Heuvel-Panhuizen warns about the danger of instrumentation with the ENL and states that "unfamiliarity with the number line's nature can initiate prescriptive use of it" (p.27). He believes that at no stage should the ENL take away from students opportunities to "find their own strategies, including shortcuts, and to come up with their own notations" (p.27). This is an important consideration when promoting the use of the ENL in classrooms and if this component is ignored it could lead to this tool being presented in a rote procedural manner much the same as the traditional algorithm.

The use of such didactical models in education can be a sensitive issue. Incorrect use can have a harmful effect and may actually become “anti-didactical”. Freudenthal (1973) defines the term anti-didactical as the tendency to take the scientific structure of the discipline as the guiding principle and present children with ready-made mathematics, rather than giving them the chance to develop mathematical concepts and methods themselves (Heuvel-Panhuizen, 2008, p. 17). To inform the current study, the research drew on the theoretical prospective of Freudenthal. Freudenthal's pedagogy sees the application of mathematics as a human activity. From this theoretical perspective, Freudenthal suggests that mathematics is the process of mathematizing. He considered this human activity an opportunity for participants to engage in the process of mathematics in a didactical manner which naturally leads to a result. He argued that schools traditionally teach in an anti-didactical manner, focusing on the teaching of the product rather than the process. Gravemeijer and Terwel (2000) summarize Freudenthal's educational goal as being "to make sure that the students experience objective mathematical knowledge” as a seamless extension of their everyday-life experience (p. 785).
Freudenthal identifies a number of components, which support his vision of mathematics in the education system. Firstly, he refers to guided reinvention, a process by which students are presented with opportunities to develop their own "mathematical knowledge store", allowing students to come to view this knowledge as their own rather than repetition of a teacher model (Gravemeijer & Terwel, 2000. p. 786). Secondly, Freudenthal recognizes necessary levels that must be addressed in the learning process. He believes that mathematizing should begin by presenting students with content that they view as realistic. From here, students should then be guided towards analyzing their own mathematical activity (Gravemeijer & Terwel, 2000. p. 787). Finally, didactical phenomenology addresses Freudenthal's emphasis on "the importance of a phenomenological embedding of mathematical objects" (Gravemeijer & Terwel, 2000. p. 787). Freudenthal recognizes that all students will not become mathematicians, but rather will call on their knowledge of mathematics to solve problems in everyday-life situations. He encourages educators to recognize this and move towards preparing students for such situations in their mathematics classrooms. I believe the ENL facilitates a move towards Freudenthal's vision of math education. His pedagogy was kept central while designing and implementing my study.

Conclusion

Research suggests that, while the traditional algorithm remains this longstanding method of teaching computation in classrooms, there is indeed a need to replace this as the primary method, and to look for alternate approaches that fit with today's educational goals in mathematics. The ENL facilitates the development of students’ own intuitive way of thinking about number, while engaging users in thinking and tinkering with numbers. While there is a growing body of research suggesting the benefits of the ENL in today’s mathematics classroom, the majority of research conducted on the ENL has taken place in Europe and Australia. Such
studies highlight the benefits of this tool. However, researchers do caution that if the ENL is presented in a rigid manner with explicit procedures and steps, it disables a student's ability to work with numbers in a flexible manner that supports authentic number sense development, much in the same way that a traditional algorithm does. However, there remains a need for further study in varied contexts, before such a model can be deemed worthwhile and beneficial for a wider audience. This study aimed to present the ENL in a didactic manner to students, giving users the freedom to utilize the tool in the manner in which the Dutch intended. Chapter 3 will give a detailed outline of my study, specifically describing the sample group, the concepts presented, as well as the methods of data collection.
3 Methods

Introduction

I have long concluded that it is very hard to say anything new that has not been said more eloquently elsewhere....I see working on education not in terms of an edifice of knowledge, adding new theorems to old, but rather as a journey of self discovery and development in which what others have learned has to be re-experienced by each traveller, re-learned, re-integrated and re-expressed in each generation (Mason, 1994. p. 177).

The aim of this study was to contribute to a growing body of research in mathematics education about the empty number line (ENL). Studies suggest that it is a highly effective computational tool which facilitates the development of number sense and mental math skills. My study sought to examine the benefits of the empty number line as a tool to support and develop students’ computation and mental math skills specific to addition and subtraction. In my study, students were asked to put aside the traditional algorithm, and embrace an alternative method of computation to solve 2 and 3 digit addition and subtraction questions presented through a variety of tasks. Student perception of the ENL was another important component of this research. My intention was to provide students with an opportunity to share their perspectives about this new tool, with regards to ease and effectiveness.

Role of the Researcher

My study was informed by action research and teacher-research. Action research presents an opportunity to shape and refine one’s teaching and to build on one’s successes (Beverly & Johnson, 1993). Kemmis and McTaggart(1988) believe action research involves various components including problem identification, data collection, reflection and analysis. This method involves exploring new ideas in practice, in an effort to improve or increase knowledge...
about teaching and learning (Kemmis & McTaggart, 1988). My ENL study was guided by action research in that this work materialized as a result of my concern with number sense development in today’s elementary classroom. Upon identifying that some students fail to perform computation tasks with consistent success, I decided to source and implement an alternate approach for such tasks, with the intention of using this data to determine if my approach could offer some guidance to improve and strengthen number sense instruction. However, a shortened timeframe precluded repeated analysis-reflection-action cycles, sometimes considered norm for action research.

Contrary to action research, teacher research aims to similarly inform practice, but does not necessarily involve cycles of testing, and allows for inquiry wherein in depth data analysis occurs upon completion of the intervention, as was the case in my study. Mohr (2004) defines teacher research as “a process that allowed teachers to build knowledge about their students’ learning and about their own practice” and believes that such research “could lead to new theories and revised practice in classrooms” (p. 4). In addition, teacher-research, now prominent in education, involves a dual, insider role for the researcher, in that it views the teacher as “decision maker, consultant, curriculum developer, analyst, activist, school leader,” (Cochran-Smith & Lytle, p.17) while concurrently enabling the ‘teacher as researcher’ to examine her practice to inform and shape future practices.

I began this study having some experience working with the ENL. This work, although limited to a grade one classroom, gave me some insight in how to present this tool to grade 4 students. I felt capable of carrying out a study aimed to determine if such a tool was conducive to strengthening and improving number sense instruction. Assuming the role of teacher researcher, I had the opportunity to research best practices when using the ENL, and create lessons and
activities that I felt reflected such practices within a grade four classroom. Through implementation of this unit, I experienced first hand the successes and challenges these students had with the ENL and from here was able to critically examine the manner in which the tool was presented to participants. In addition, delivering all lessons gave me an opportunity to reflect upon how this tool should be introduced, supporting my own personal growth and understanding of the ENL.

However, my dual role in this study limited the amount of time within each lesson that I was able to dedicate to solely observing and questioning students. Despite this, I feel sharing my time between teacher and researcher, provided not only data on how the students performed with and viewed the tool, but also offered some important considerations for the implementation and teaching of the ENL.

**Participants**

This study was conducted in a Grade 4 classroom in an urban setting, on the west coast of Canada. The sample can be classified as a convenience sample consisting of 28 students coming from a variety of backgrounds, in an independent school catering to a lower to middle class community. Different from other ENL research, my study introduced this method of computation to students at a Grade 4 level. Typically, the tool would be introduced in the early primary years prior to algorithmic instruction. However, I felt introducing the ENL to students following extensive algorithmic instruction could offer worthwhile comparisons between the ENL and the algorithm, which was an important part of this study. In addition, my aim was to see if the ENL was a worthwhile alternative for students struggling with ongoing successful application of the standard algorithm.
These participants in the sample had no prior exposure to the empty number line. Rather, the students’ computation instruction had focused significantly on traditional computational algorithms to perform such tasks over the previous 2.5 years. This exposure to the algorithm had developed students’ ability to perform computation at varying levels. According to the students' current classroom teacher, Miss Ryan, the majority were usually successful when performing addition and subtraction tasks independently using the algorithm. However, there was a portion of the class who still struggled to carry out such a task, and Miss Ryan noted that that consistent success with borrowing seemed to be the biggest challenge. Therefore, she deemed it necessary to maintain regular practice of these procedures in order for students to retain and successfully apply these steps.

Miss Ryan described the participants as energetic, and as a whole very willing to embrace and explore new things. In her experience, most students were open to sharing their ideas in the class setting, but she found engaging students in such an activity was challenging because of behavioural issues within the class. At times class members lacked basic classroom etiquette, struggling to take turns, listen to others and participate in a respectful way. Such issues often negatively impacted class discussions. Miss Ryan shared that she had done some work developing mental math skills earlier in the year. For instance, students had spent a little time looking at alternate computational strategies which were presented in their textbook. The main focus here appeared to be exploring the mental math strategy of front end addition, where tens and ones are added or subtracted separately, and then combined at the end.

**Instruments**

This study used mixed methods. Pre and post assessment data was analyzed through a quantitative lens. All other data, including student work samples, student reflection journals, as
well as transcriptions of conversations during lessons was treated as qualitative. No formal instruments such as standardized tests or published instructional materials were used for my study. Instead, I created task sheets and games, aimed specifically to facilitate student exploration of the ENL. I decided to create one formal assessment that was used pre and post study. Using the same assessment template facilitated a comparison between students’ pre and post assessment performance.

Procedure

For this study, students engaged in one 60 minute lesson twice weekly for 4 weeks, totaling eight lessons altogether. These lessons focused on exploring and developing mental computational strategies for 2- and 3-digit addition and subtraction. Students were introduced to the ENL as a model on which to explore and create a variety of mental math strategies. Lessons included classification and modelling of a number of effective approaches that can be applied to the ENL. However, rather than students simply repeating teacher-taught strategies, every effort was made to provide students with the opportunity to utilize this learning tool to develop students’ self-generated approaches. Lessons focused on encouraging students to explore and take risks with number using the ENL. Every effort was made to provide students with opportunities to develop and express their own thoughts and ideas about the material being presented. To facilitate this, I included student journaling in this study, which will be described later. Indeed, allowing students to recognize and build on what they naturally know about number as well as the value of self-generated strategies was a central tenet of this intervention.

Instructional Approach

A number of specific elements were considered central to the success of this study. Firstly,
I wanted my study to be student-centered with learning taking place through student discovery rather than teacher direction. Secondly, I designed the study to encourage exploration, and to create an environment where students would feel comfortable in making mistakes, recognizing such events as a necessary part of learning. In doing so, I hoped the lessons would have a positive effect on the students’ levels of confidence. I used tasks that facilitated group work and discussion whenever possible, usually in the form of games, in order to help make using the ENL more purposeful and meaningful for students, and to create a mathematics atmosphere filled with fun and excitement. These tasks and games will be described in detail later.

**Design of Study**

My intention when designing this study was to determine the benefits the ENL could offer users who were already familiar with algorithms for addition and subtraction. Since the students were in the latter part of Grade 4, and the performance of one digit computations limited to addition and subtraction (a task usually used to introduce ENL to primary students) would be considered to be well below grade level, I decided to introduce the ENL in the context of adding and subtracting two and three digit numbers.

This study was comprised of a number of phases, including pre-assessment /pre-teaching, teaching of stringing, teaching of bridging, teaching of splitting, as well as post-assessment. These phases will be outlined below.

**Pre-Assessment**

Before beginning any work on the ENL, all participants were asked to complete a pre-assessment. The pre-assessment consisted of ten computation tasks which I designed to test various aspects of addition and subtraction as shown in Table 3.1. See Appendix 1 for original
Table 3.1 Outline of Pre-assessment Items

<table>
<thead>
<tr>
<th>Task</th>
<th>Component Tested</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $73 + 49 = ____$</td>
<td>2 digit addition involving carrying presented in an equation</td>
</tr>
<tr>
<td>2. $345 + 278 = ____$</td>
<td>3 digit addition involving carrying presented in an equation</td>
</tr>
<tr>
<td>3. $76 - 48 = ____$</td>
<td>2 digit subtraction involving borrowing presented in an equation</td>
</tr>
<tr>
<td>4. $456 - 289 = ____$</td>
<td>3 digit subtraction involving borrowing presented in an equation</td>
</tr>
<tr>
<td>5. $228 + 494 + 659 = ____$</td>
<td>3 digit addition of 3 amounts presented in a word problem</td>
</tr>
<tr>
<td>6. $454 - 167 = ____$</td>
<td>3 digit subtraction involving borrowing presented in a word problem</td>
</tr>
<tr>
<td>7. Difference between 57 and 90 ie. 90 - 57</td>
<td>“Find the difference” word problem involving 2 digit numbers. Addition or subtractions could be applied to solve task.</td>
</tr>
<tr>
<td>8. Difference between 178 and 329 ie. 329 - 178</td>
<td>“Find the difference” word problem involving 2 digit numbers. Addition or subtraction could be applied to solve task.</td>
</tr>
<tr>
<td>9. $15.50 + $ 17.75 = ____</td>
<td>Addition of decimals presented in the context of money, embedded in a word problem</td>
</tr>
<tr>
<td>10. $12 °C − 15 °C = ____</td>
<td>2 digit subtraction presented in as word problem involving integers</td>
</tr>
</tbody>
</table>
The pre-assessment served to gather baseline data that addressed questions such as:

1. What were students’ preferred methods of computation when presented with 2 and 3 digit addition and subtraction tasks in the form of equations and word problems?
2. How accurate were students with their chosen method(s)?
3. What challenges, if any, did students have with this pre-assessment?
4. What can be determined about students’ number skills from this pre-assessment?

Such data was needed to compare student performance before and after the lessons about the ENL.

Launch of Study - ENL versus Algorithms

According to Spikell (1993), mathematics education must facilitate student engagement in exploring, conjecturing, and thinking, and aim to move past rote learning of rules and procedures. Before beginning any work using the ENL, I felt it was important to engage students in some discussion about the standard algorithm since I was asking them to put aside algorithms and embrace a new approach. Rather than simply telling students they must use the ENL, I described how the purpose of this study was to explore a new computational approach. As such, I let students know that their opinions regarding the ENL were a critical component of this study. Therefore, students were asked to make a sincere effort to put aside the standard algorithm for the duration of the study and attempt to use this new tool wholeheartedly. Furthermore, because this study was not simply about students' performance, but also about students’ perception of the ENL, this discussion also sought to establish the importance of student voice and opinion in this endeavour. By offering students this invitation, I hoped that they would recognize the importance of their honest feedback throughout the study.
**Pre-teaching for the ENL**

In order for the participants to feel comfortable, and hopefully successful, with the ENL, students needed to be able to fluently add 10 to and subtract 10 from a given number, as well as work with groups of 100 mentally when dealing with 3 digit numbers. Therefore, before exploring any ENL strategy with students, I invested one lesson in some pre-teaching. We began with tasks that required students to re-visit adding to and subtracting 10 from a given number both on and off the decade. Here I hoped that students would make the connection that, when adding or subtracting ten, the digit in the ones remained the same while the digit in the tens changed. Students engaged in a variety of tasks aimed at reinforcing this concept, such as working with place value pockets as well as with Kim Sutton's Random Number CD activities. I also directed student attention to the hundred square, which provided a systematic visual of numbers increasing and decreasing by ten as a user moved either up or down a column. Individual copies of the hundred square were available for all students. From here, I gave students tasks that involved adding or subtracting multiples of ten to or from a given number, such as 20, 30 or 40. When I felt students seemed comfortable mentally adding and subtracting multiples of 10 to a given number, we worked with groups of 100. Students were provided with a photocopied strip of hundred squares up to 500 for this work. Following this pre-teaching lesson, ENL work commenced.

**Design of Lessons**

As teacher-researcher, I designed all lessons, typically consisting of three parts; whole class instruction, individual task, group task/game. Each component will now be outlined.
Whole Class instruction/Exploration

When creating lesson outlines for this study, I gave the amount of instruction time and extent of modelling that students were to receive much consideration. The ENL is intended to be a didactic model and therefore lessons must communicate the flexible nature of this tool. As mentioned earlier, research (Murphy, 2010; Bobis & Bobis, 2005) has found that explicit, rigid teaching can actually have an antididactic effect, resulting in users following a set number of prescribed steps, much the same as they would using a standard algorithm. However, the ENL was a completely new approach for these Grade 4 students, meaning they had not received much of the ground work that ENL users are typically exposed to before beginning work with 2 and 3 digit computation. Another consideration in my study was the time constraints. In this context, the ENL was being explored during a four week period, rather than being explored and developed over the course of a school year, as is the case in the Dutch classroom. Rather, my study sought to determine the possibilities that exist when this tool is introduced at a later stage than is typical. Therefore, due to the short time frame, and the newness of the tool for students, I felt that a portion of lessons needed to be set aside for teaching and modelling, but that the flexible use of the ENL would be highlighted, explored and encouraged within instruction time.

Whole class lessons took place with students seated on the classroom carpet, and lasted approximately 8-10 minutes. During this time, I presented and explained a strategy. For example, I modeled the decomposition of numbers using base ten blocks (paper models) and these visuals were also used to support visual representations created on the ENL (See Appendix 1). Following my initial example, subsequent examples were presented with student volunteers suggesting the actions to be carried out. Following whole class instruction, I offered students an independent task to do.
Independent Tasks

I designed the independent tasks to provide students with the opportunity to practice a given strategy, while allowing for the incorporation of adaptations of a strategy that fits with students’ own exploration and thinking, as is the intention of this didactical tool. Tasks were kept short, usually consisting of four to six questions. During independent work, I was available to help those who needed it, and hundred squares up to 400 were available for students seeking support when adding or subtracting 10 or 100 to a number. Following some independent tasks, the class gathered again on the carpet to discuss the work that had just been completed. This allowed us time to talk about individual tasks as well as how to classify and compare the strategies the students had been implementing.

Group Tasks/Games

The concluding component of each lesson usually took the form of a game. The work of Owens (2005) found that the incorporation of games into the math classroom was an effective medium to initiate enjoyment while presenting participants with an opportunity to implement new learning in a more meaningful context. It was also an opportunity for students to offer and receive support from a peer. Therefore, I often paired students with this in mind. However, the work of Bragg (2006) cautions that while games can be well received by students, it can be problematic to incorporate games that are level-appropriate for all players. Within this study, my goal was that more able students would not simply play the game but also provide support for less able players if necessary. However, engaging all students was key as was ensuring that all participants felt challenged. Bragg (2006) suggests that adopting an open-ended approach to games may help overcome such a challenge, and this can be as simple as creating games with different levels. Such an option was built into some of the instructional games.
Exploration on the ENL

"Even in the most open classrooms, students' learning should not be accidental but should be the result of conscious attempts on the part of the teacher to bring about the intended learning outcomes." (Lo, Marton, Pang, & Pong, 2004, p.190). According to the work of Thompson (1999a), three main strategies have been identified as ways students typically use the ENL. For the purposes of this study, the three strategies were presented to students in the order of stringing, bridging and splitting.

Strategy One: Stringing (N10/ Sequencing/ Jump Method)

Researchers such as Beishuizen (1999) have identified stringing (N10) as one of the main strategies applied when using the ENL. Stringing refers to keeping the first number intact while breaking the second number into tens and ones, which are subsequently added or subtracted separately to/from the first number. Stringing is often described as the most “efficient strategy” (Beishuizen, 1999. p.159) as it often results in fewer steps than other known approaches, depending on the ability of the student. Research also indicated that this is the strategy most frequently preferred and used by stronger students (Beishuizen, Van Putten, Van Mulken, 1997).

As mentioned, lessons in my study, including those on stringing, usually consisted of three components: whole class lesson, independent task and group task/game. Rather than starting work with one digit numbers as would be the norm, I presented students with tasks containing two and three digit numbers. Appendix 2 provides a detailed summary of three stringing lessons. A brief outline of these lessons is provided below.

Lesson one focused on addition. The independent task consisted of six equations including both two and three digit numbers. Students could utilize a hundred square for support, and those
who finished the task quickly were provided an additional challenge in the form of a word problem. Upon completion of the independent task, students were gathered together to present the second task, a game called “Long Jump”. Students used two double decahedron dice to create two 2 digit numbers. Students combined their numbers using the ENL. The students with the greatest jump won the round and received a point. I determined the pairings somewhat strategically: those students who seemed less comfortable using an ENL model during the independent task were paired with students who worked with relative ease during the same task. These pairings provided less able students with further modelling as to how the ENL could be utilized, as well as peer support when creating their own visual on the line, if required.

Lesson two on stringing was identical to lesson one, with the substitution of subtraction for addition. Again, students were provided with the support of a hundred square if needed, and an extension of the independent task was available in the form of a word problem. Our game of “Long Jump” was repeated with the task requiring students to subtract the two 2 digit numbers from each other (See Appendix 3). The winner of the round remained the person with the greatest answer. I again implemented strategic pairing, based on observations made during the independent tasks.

Lesson three on stringing began with a review of both addition and subtraction using this approach. The first task presented in this lesson was “Name that Equation”. The objective of this task was to help students to see the various ways the ENL can be used, and to expose students to good modelling of how to apply stringing on the ENL. I began by providing students with a visual of a solution on the ENL, asking them to determine an equation to accompany it. This task required the participants to determine the two amounts being computed, as well as the form of computation applied. Students explored this task both during whole class instruction and independently. Following this, I presented students with another game aimed to further develop
their understanding of stringing on the ENL. In “Walk the Line”, students again worked in pairs, competing against each other to create the greatest jump. The additional challenge of this game, in comparison to “Name that Equation” (See Appendix 4) was that it required students to utilize addition and subtraction in each round. Here, students created three two digit numbers and recorded them on the game board. Students then had to utilize these numbers to create a three part equation involving addition and subtraction (_____ + _____ - _____ = ____). This required students to think more critically about the task and use estimation. Again, this game gave students an opportunity to work in pairs, providing modelling and peer support for students if necessary.

**Strategy Two: Bridging (A10) (Split Jump Method)**

The next strategy I presented to participants was bridging (also known as split jump or A10). This strategy builds somewhat on splitting (N10), while providing students with increased freedom in exploring and creating their own strategies. With bridging, the first number is kept whole and the second number is split into tens and ones. However, unlike stringing where the order of operation is determined, bridging offers students more flexibility. Typically with bridging, some ones are added first to bring the sum to the nearest decade, and from here the remainder of the number is added either in one final jump, or in several smaller jumps, depending on the level of competence of the student and the size of the quantity in question. Two bridging lessons, with lesson plans following a similar format to stringing lessons, are described briefly. (See Appendix 5 for details).

To launch our first lesson on bridging, I began by reminding students that the ENL is a flexible model, a tool that can be used in a variety of ways that best suits the user. From here, I presented a second strategy, bridging. Similar to the lesson on stringing, I modelled and
discussed the bridging strategy with the whole class on the carpet, encouraging student input during this presentation. For this initial lesson, rather than giving students a two digit equation to solve using bridging, I offered students a task that consisted of a partially completed bridging solution, asking them to determine the equation that accompanied this visual. This lesson on bridging was limited to students examining solutions modelling this strategy, in preparation for creating their own.

The second lesson required students to implement bridging in order to solve “Find the difference between” tasks. Following a whole class review of the strategy, I assigned students with an independent task. Next, I presented students a group task designed as a game. In “What's the Difference”, students used decahedron dice to generate two 2 digit numbers, and from here used the ENL to create a visual to unveil the difference between both quantities. The player who obtained the greatest difference won the round and was awarded a point. Points were recorded using a tally. Again, peer modelling and support was available in this format.

**Strategy Three: Splitting (1010)**

The final strategy presented to participants was splitting (1010). This strategy utilizes the place value oriented decomposition method, where tens and units are split off and handled separately. I allocated two lessons to the exploration of splitting, described briefly below. (See Appendix 6 for details).

As was typical in this study, I began the initial lesson with a whole class review of both strategies presented to date, and then presented students with the third and final strategy on the ENL in this study, splitting. I supported the explanation and modelling with base ten block visuals, encouraging input from the class throughout the demonstration. This initial splitting lesson was limited to the exploration of addition. Following our whole class lesson, I offered
students an independent task consisting of six addition equations. As in previous lessons, hundred squares were available to students. During the latter part of this lesson, students drew from their knowledge of two of the strategies presented to date, to complete a second task which presented participants with two addition equations with the instruction to solve each equation twice, once using the stringing approach and once with splitting.

The second lesson on splitting focused on applying ENL for subtraction. Following whole class modelling of several examples, students were asked to complete six subtraction equations utilizing this strategy. This independent task proved to be challenging for many students and afterwards, the class was brought back to the carpet to review and discuss the six equations assigned in this lesson. Following this second class gathering, the time allocated to this lesson was up and students were unable to complete a second activity as had been the custom in previous lessons.

**Written Reflections/Journaling**

Elbow (2004) claims that writing can be used “to help students comprehend and clarify concepts” (p. 13) and prompts students to answer the question “What do you have to say?” (p. 13). The purpose of writing within my study was simple: to provide students with an opportunity to reflect on what they had experienced, as well as to share their opinions about this new material. The writing tasks used in this study can be categorized as journal writing (expressing feelings and thoughts about the mathematical learning).

The use of writing in the math classroom is widely supported by research. Burns (2004) stated that “Writing in math class supports learning because it requires students to organize, clarify, and reflect on their ideas- all useful processes for making sense of mathematics” (p. 31). Central to this study was student opinions, and therefore providing a means in which all students
were able to communicate was important. Baxter, Woodward and Olson (2005) point out that one of the main challenges for all mathematics teachers is to find ways to include all students in meaningful communication. They found that, “students who did not actively participate in mathematics discussion did respond when asked to write about mathematical ideas” (p. 130). Via journaling, my hope was that students’ opinions would uncover some important considerations for educators implementing the ENL in their classrooms.

However, some research recognizes that math journals do have drawbacks. Stonewater (2002) reminds us that not all students experience success with math journals, concluding that "successful" writers are more likely to outperform "unsuccessful" writers in a variety of contexts, such as the ability to use appropriate mathematical language. In order to ensure all students felt able to share their experiences and opinions in this written form, I designed journal entry pages containing a variety of sentence starters. These were intended to support those students struggling to find words to communicate their thoughts and opinions. However, I stressed to students that these sentence starters were simply suggestions and did not necessarily need to be incorporated into their writing. An outlined summary of each of the journal entries presented to participants can be found in Appendix 7. A journal entry template including sentence starters can be found in Appendix 8.

All students were given four opportunities to share their personal opinions about the ENL tool in a journaling format. Journaling was kept simple. The intention of this journal was for students to share their thoughts as to the usefulness and effectiveness of the ENL as a means to carry out addition and subtraction, as well as to comment on a specific strategy. During a lesson, individual students may have been asked to elaborate on a comment made in class or to explain more explicitly an approach or strategy applied during tasks, either via journaling or orally.
Data Collection

The bulk of data for this study was collected in the natural classroom environment. Multiple sources of data were collected, in an effort to better understand the scope of happenings when utilizing the ENL within a classroom. Data analysis of both a quantitative and qualitative nature took place within this study.

However, prior to beginning any work on the ENL, two sources of non-classroom data were collected. Firstly, all participants were asked to complete a pre-assessment test which had two purposes: to measure student performance in computation tasks, specific to 2 and 3 digit addition and subtraction, and to establish student preferred methods when solving addition and subtraction tasks presented in a number sentence or word problem. In addition, I interviewed Miss Ryan in an effort to gain deeper insights into her students who were participating in this project. Miss Ryan was also asked about a number of issues such as current classroom practices regarding number sense, strengths and challenges within the group in relation to number sense, overall math performance and class dynamics.

Throughout the duration of this ENL unit, I collected a variety of data. Firstly, I collected and dated all student work completed. I also kept valuable artifacts from group tasks such as game boards and took some photos of student work. I recorded whole class lessons and discussions using audio equipment. I also kept observation notes to record informal student comments heard throughout the course of this study. These comments arose during a variety of situations including: comments made directly to me by a student, comments made during a group conversation that I was part of, and comments I overheard that were exchanged between two students. Student journaling was also collected and dated.

At the end of the study, each student was asked to complete a second assessment task. This assessment was to be used as comparative data, in an effort to measure student success with the
ENL. I administered all testing, and assigned all tasks.

**Data Analysis**

Upon completion of the pre assessment, student performance was marked, coded and categorized. A table was prepared showing students' method of choice in solving tasks, as well as their level of proficiency. To conclude the study, students were asked to complete this assessment for a second time, but were limited to solving all tasks on the ENL. Following this post-assessment, a table was prepared displaying the ENL strategy students applied to each task, as well as their success rate. Both pre and post assessment tables were compared, and a further table was created to display any changes in students’ success rate between these assessment tasks.

I collected and dated all student work samples, produced during both independent and group tasks. I coded and categorized all samples, in an effort to paint a picture of the work that students produced during this ENL study. Focus was not limited to the students’ level of accuracy, rather significant attention was placed on students’ approaches to each individual task. I noted and coded the ENL strategy applied, as well as students’ level of efficiency and individuality with each task.

In addition, I kept observation notes and audio recordings during the study. Audio recordings were limited to whole class lessons and discussions. Portions of these passages were transcribed, coded and categorized. I kept observation notes over the course of the study. I coded and categorized these notes.

From the large body of student work samples, the work of three students was selected to frame individual case studies. These particular cases illustrated different experiences using this tool, with regard to children’s approaches, levels of success as well as the challenges that arose for each of them.
In an effort to answer all research questions proposed, Chapter 4 will present a variety of data. To determine how the ENL facilitates student sense making and self-generated strategies, as well as what it reveals about students overall sense of number, I will refer to observation notes, audio recording and student work samples. With regards to determining if students are more or less effective with computational tasks following exposure to the ENL, I will analyze and compare pre and post test data. Finally, to establish how students felt about the ENL, I will refer to their journal entries.
4 Results

Purpose

"With the empty number line, one must look for flexible use for the indications of whether the model is used by students to express their own thinking. The focus will have to be on the quality of the solution procedures and on how these strategies come into being” (Gravemeijer, 1994, p.468).

My study aimed to encourage purposeful and sensible tinkering with number. Specifically, my study sought to determine if and how the ENL could support and develop students’ sense of number. My research questions inquired as to how effective students were at applying the ENL to solve computational tasks, in a manner that supported student sense making and self-generated strategies. Emphasis was also placed on determining if the ENL could provide insight into students’ own understanding and sense of number. Finally, obtaining student opinion of this new tool was sought throughout the course of my study.

This chapter aims to share findings that paint the big picture of my research study, to provide insights to educators considering the possibility of exploring the ENL model in future teaching practices. While at times the entire corpus of data will be referred to, specific cases will be looked at in detail. Included are case studies of three individual students who had very different experiences using this model. While there were commonalities within their experiences, each student offers a different lens with which to view how this model can be applied and utilized by students. When reviewing the work of participants, I paid close attention to the strategy students chose, and noted the variance of approaches when applying the same strategy to a given task. In addition, students’ overall level of accuracy was noted. Student work was analyzed, and used to determine the successes and challenges of this work. In addition, student opinion about
this tool was gathered from all participants and comments are presented in themes that reoccurred throughout this data.

**Background**

The Grade 4 participants in this study began this project after receiving a significant amount of instruction around number sense within a school setting. Computational instruction to date had focused primarily on developing the standard algorithmic approach. Typically, application of the ENL begins long before students enter Grade 4. In the work of Beishuizen and Anghileri (1998), we see that the Dutch begin using one digit numbers to explore addition and subtraction on the ENL. This work would begin in the equivalent to Kindergarten and Grade One. Only when this is understood, are students given a variety of tasks exposing them to the broad range of possibilities with the ENL when performing computation. However, due to the age of my participants (age 9 and 10) and the fact that they were now currently in the latter portion of Grade 4, I decided to proceed with this study differently. For the purpose of this study, students began by working with two and three digit numbers, while implementing a number of teacher presented strategies, in self-directed ways.

**Pre-assessment**

Before beginning the ENL unit, students completed a 'Pre-assessment' containing 10 computational tasks. This activity aimed to determine students preferred method when completing computational tasks involving addition and subtraction, as well as their level of success using their selected approach (See Table 4.1). Within this table, the number of correct solutions (C) and the number of incorrect solutions (IC) are shown in Column 3.
Table 4.1 Summary of the Pre-assessment

<table>
<thead>
<tr>
<th>Task</th>
<th>Description of Task</th>
<th>Student Performance (28 participants)</th>
<th>Method/ Strategy Used by Student (eg. Standard algorithm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 digit + w/ carrying: (73 + 49)</td>
<td>C : 25  IC : 3</td>
<td>algorithm: 28</td>
</tr>
<tr>
<td>2</td>
<td>3 digit + w/ carrying: (345 + 278)</td>
<td>C : 24  IC : 4</td>
<td>algorithm: 28</td>
</tr>
<tr>
<td>3</td>
<td>2 digit - w/ borrowing: (76 - 48)</td>
<td>C : 20  IC : 8</td>
<td>algorithms: 26 1010 strategy w/o ENL: 2 (unsuccessful)</td>
</tr>
<tr>
<td>4</td>
<td>3 digit - w/ borrowing (456 - 289)</td>
<td>C : 18  IC : 10</td>
<td>algorithms: 27 1010 strategy w/o ENL: 1 (unsuccessful)</td>
</tr>
<tr>
<td>5</td>
<td>3 digit + w/ carrying (228 + 494 + 659)</td>
<td>C : 15  IC : 13</td>
<td>algorithms: 27 2 step algorithm: 4 (1 unsuccessful) 1010 strategy w/o ENL: 1 (unsuccessful)</td>
</tr>
<tr>
<td>6</td>
<td>3 digit - w/ borrowing (454 - 167)</td>
<td>C : 22  IC : 6</td>
<td>algorithms: 27 1010 strategy w/o ENL: 1 (unsuccessful)</td>
</tr>
<tr>
<td>7</td>
<td>2 digit - w/ borrowing, money ($90 - $57)</td>
<td>C : 21  IC : 7</td>
<td>algorithms: 27 1010 strategy w/o ENL: 1 (unsuccessful)</td>
</tr>
<tr>
<td>8</td>
<td>3 digit - w/ borrowing (329 - 178)</td>
<td>C : 20  IC : 8</td>
<td>algorithms: 27 1010 strategy w/o ENL: 1 (unsuccessful)</td>
</tr>
<tr>
<td>9</td>
<td>+ with dollars and cent ($15.50 + $17.75)</td>
<td>C : 22  IC : 6</td>
<td>algorithms: 25  no work shown: 1 incomplete: 2</td>
</tr>
<tr>
<td>10</td>
<td>Word problem involving negative numbers (temp. 12<em>C, drops 15</em>C)</td>
<td>C : 7  IC : 21</td>
<td>Algorithm used by many unsuccessfully (12-15=3). Of those who were correct, the answer was simply given (no work shown) except in one case where a vertical number line was applied</td>
</tr>
</tbody>
</table>
The pre-assessment results show that algorithms were indeed the preferred method of choice of participants when presented with computation tasks specific to addition and subtraction in a testing context. It is important to note that pre-assessment items were presented as a horizontal number sentence or word problem, rather than in an algorithmic format. My reason for this was that I wanted to see what strategy the students would select to solve each task. Would students choose to rewrite the number sentence in the horizontal format typically associated with the paper and pencil algorithm, or would they use a different approach? In my conversations with Miss Ryan, she confirmed that most of her computation teaching focused on application of the standard algorithm, and alternative approaches were given little or no attention. Thus widespread use of algorithms in the pre-assessment was expected.

Upon review of student responses within the pre-assessment, a number of observations were made which I will outline below. Items 1-4 were classified as number sentences dealing with 2 and 3 digit addition and subtraction. In Table 4.2, we can see that a majority of students successfully completed addition with carrying using an algorithm. However, a sizeable group of students did not successfully complete subtraction with borrowing, despite Miss Ryan’s indication that a significant amount of class time was assigned to practicing this concept.

Table 4.2 Pre-Assessment Performance: Number Sentences (Task 1-4)

<table>
<thead>
<tr>
<th>Number Sentence Description</th>
<th>% Correct</th>
<th>% Incorrect</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 digit addition w/ carrying</td>
<td>89%</td>
<td>11%</td>
</tr>
<tr>
<td>3 digit addition w/ carrying</td>
<td>86%</td>
<td>14%</td>
</tr>
<tr>
<td>2 digit subtraction w/ borrowing</td>
<td>71%</td>
<td>29%</td>
</tr>
<tr>
<td>3 digit subtraction w/ borrowing</td>
<td>64%</td>
<td>36%</td>
</tr>
</tbody>
</table>
Reviewing the errors made in Tasks 1-4, some of the mistakes (see Fig 4.1) are similar to those that Kamii and Dominick (1997) classify as typical, such as “treating every column as a column of ones” (p. 6) and confusion causing carrying of the ones and carrying the tens. These results show that following 2.5 years of instruction focused on children’s use of the standard algorithm, this approach is still failing some users. However, this study aims to look beyond the analysis of performance alone. Pre-assessment data confirms that at least some still lack the basic understanding necessary to implement the algorithm with consistent success, suggesting significant gaps in some students understanding. Student work with the ENL will be closely examined to determine if it can shed any light on these numerical gaps.

Fig 4.1 Reviewing the errors made in Tasks 1-4

\[
\begin{align*}
73 + 49 &= \frac{73}{121} - \\
76 - 48 &= \frac{76}{151} - \\
456 - 289 &= \frac{456}{1411} - \\
345 + 278 &= \frac{345}{881} -
\end{align*}
\]

It is worth noting that two students did indeed attempt to use an alternative strategy to solve the subtraction tasks in this pre-assessment (See Fig. 4.2). These students made an attempt to incorporate a mental math strategy (front end addition) to complete some pre-test items. The classroom teacher confirmed the strategy had been presented in class in the manner it was outlined in their textbook. Instruction and exploration of this mental math strategy was described by the teacher as being brief. In Figure 4.2, we can see that students appear to lack a full
understanding of this strategy, and are unable to successfully utilize it in a more challenging situation involving borrowing. Application of the ENL will include a linear model for students to carry out such a strategy. The purpose of the line is to provide these students with a visual tool to support their mental math work and may facilitate those struggling to successfully apply such strategies mentally or in writing, as shown in Fig. 4.2.

**Analysis of Student Work**

As mentioned earlier, 28 students participated in this study, which resulted in a large body of data collected. To ascertain a more qualitative sense of the students’ use of the ENL, a more manageable corpus was needed. Thus the results reported here are based on an analysis of the work samples collected from three participants. Each of the three students had different experiences using this tool. Examining both their successes and challenges using the ENL provided thick descriptions of these students’ experiences and understanding of the ENL. Each is presented separately to point to the "stories behind the numbers, the quantitative….to see practical application in a local environment" (Cunningham, 2011, p. 6). These three participants were selected upon close examination of the contributions of all participants. Upon perusal of student work, I was able to uncover highlights and commonalities that occurred across the data
and from here select three case studies that provide evidence of the range of student performance using the ENL within my study.

The first case study reviews the work of Mary. Mary's results show how the ENL can be used in a flexible manner, to support individual thinking and numerical understanding. The second case study examines the work of Ruth. Ruth enjoyed success with the ENL, however, the manner in which she used the tool in her work samples was at times very lengthy. The final case study focuses on Alan, a child who, according to his teacher and his mother, has always struggled to develop his math and number sense skills. Alan's work samples show that he had real difficulty using both the standard algorithm and the ENL to perform computational tasks. In addition, analysis of Alan's ENL work samples reveals gaps in his sense of number.

Case Study One: Mary

Background

Mary's teacher, Miss Ryan, described her as a capable mathematician, able to quickly pick up new math concepts. She was a keen participant in this study, and frequently took part in discussions and lessons throughout.

Pre-assessment Performance

Mary received a perfect score on her pre-assessment, the only student in the study to do so. She completed all 10 of the tasks using the standard algorithmic method, and was one of the first to complete this assessment. Thus, she presented as capable and efficient with 2-digit and 3 digit computation, whether given in the form of a number sentence or word problem.
The ENL Study: Mary's Performance

Mary took easily to the ENL, and by the end of the first lesson, showed she was already very comfortable using this model. This was evident via the speed and accuracy with which she completed tasks during lessons, and her level of comfort was confirmed in her journal responses. Several themes which emerged from Mary’s data are reported.

Flexibility and the ENL

Mary's work samples illustrated her flexibility with the new model, which appeared consistently throughout the course of the study. An example that illustrated such flexibility is present in Mary’s solution to the equation 235-57 (See Fig. 4.3). Here, Mary decomposed 57 into pieces that fit well with this specific equation. Mary began by placing 235 to the right of the ENL, suggesting that she knew she would be moving from right to left, as is the case with subtraction. Her initial step was to subtract 30 from 235, reaching 205 on the ENL. However, rather than continuing to subtract the final two groups of ten, Mary chose to move to the ones, subtracting five from 205, arriving at 200 on the ENL. Mary then returned to the tens, subtracting the remaining two groups of ten in one jump of twenty, reaching 180. Finally, she returned to the ones, and subtracted the final two ones from 180, reaching the correct answer of 178. This solution showed that Mary was able to use ENL in a manner that corresponded with her understanding of and comfort level with number. Mary displayed a high level of thinking, as indicated by the arrangement of the steps she took. Firstly, there was evidence to support the fact that Mary considered both amounts before deciding on her procedure. Rather than simply placing 235 on the ENL, and focusing her attention solely on the second amount 57, Mary decided to subtract 35 from 235, reducing the number to a manageable 200, and then proceeded accordingly using a combination of remaining tens and ones to arrive at the correct solution. She did not
restrict herself to dealing solely with all tens before allowing herself to move to the ones. Rather, she moved back and forth from the tens to the ones, in a manner she felt worked well with this particular equation. I had not yet modelled such sophisticated solutions, which supports the conjecture that Mary had taken to using this tool in a manner that worked well with her own intuitive way of thinking about number.

Fig. 4.3 Mary’s solution of the equation 235 – 57

![Figure 4.3](image)

**Compatibility and the ENL**

There are many examples throughout Mary's data that illustrated her strong sense of number. These examples, and the strategies she applied, confirm her ability to take risks and explore new approaches when working with number. This was evident in Mary’s solution to word problem requiring 178 to be taken away from 329. In this solution ((See Fig. 4.4), she proceeded to decompose 178 into chunks of 100, 29 and 49. Here, Mary was able to determine that decomposing 78 into 29 and 49 was beneficial, in that it facilitated creating a running total that was manageable to work with. Throughout such examples Mary chose to work in chunks that appear more compatible with each other, and as a result these “smart” choices allowed her to reduce the memory load needed as determined by the length of her solution. In such examples we saw that Mary’s strong sense of number and her ENL work are mutually supportive of each other.
**Challenge with Subtracting and Splitting**

It is interesting to note that, while she had a strong sense of number, Mary experienced difficulty applying splitting to subtraction tasks on the ENL. In the question, 54-25 Mary made the same error that was frequently made by the majority of participants in this study. She began by placing 50 on the ENL and successfully subtracted 20 from this amount. However, when she moved to the ones, she subtracted the 4 ones and then subtracted 5 ones (See Fig. 4.5). Most students, including Mary, failed to recognize that when using the splitting strategy, the ones from the first amount needed to be combined to the running total, and only the second group of ones needed to be subtracted. However, as you can see in Fig. 4.5, Mary was able to solve the same equation correctly using the stringing approach. In her journal entries, Mary shared her thoughts about splitting. She writes, "Splitting is very hard and confusing. I discovered that subtraction with the 1010 strategy (splitting) does not work well, especially when you need to borrow." Similar comments were voiced via journaling by several participants in the study. Here, Mary is referring to the challenge faced when asked to decompose both numbers as is called for in splitting. The question 61 - 23 is a good illustration of this confusion. Applying stringing to this equation, a student would typically begin by placing 61 on the ENL and from here subtract 20
and three from 61. Applying splitting to the same equation, a student would typically begin working with the tens, by placing 60 on the ENL and subtracting 20. Following this, attention would move to the ones. (The difference with splitting being the user deals solely with both groups of ten before moving to the ones). In an equation such as 24 -12, it is relatively easy to see 4 - 2 was called for. However, in equations such as 61 - 23, it is harder to determine 1 - 3 is called for. Many students failed to see that in the equation 61 - 23 the ones in the first quantity show what you have, and the ones in the second quantity express what you must subtract. Instead many students, Mary included, chose to subtract both groups of ones from the running total. Here, the confusion appeared to arise from the students’ inability to recognize the relationship between the ones in the minuend and those in the subtrahend.

Fig 4.5 Mary solves 54-25 applying splitting and stringing

Mary's Post-assessment Performance

Mary solved 8 out of 10 tasks correctly in the post-test. As directed, she utilized the ENL to solve each equation. Her preferred method appeared to be stringing. Her two errors in the post-assessment were relatively minor but arose in two of the three (out of ten) tasks where she applied splitting. In the first case (Figure 4.6) 345 + 278, where she erred while using the splitting strategy, Mary's work shows she was able to decompose both quantities and successfully place
and label these chunks on the ENL. However, she made a computational error when adding 40 to 
578, placing 696 on the ENL. Her second error (See Figure 4.7) in solving 76 - 48 showed some 
confusion and crossover in the strategy applied. Here it appears Mary chose to begin by applying 
stringing and placing the first quantity 76 (in its entirety) on the ENL. From here 40 is subtracted 
in one jump, followed by subtracting six and eight. Here Mary failed to recognize that the six 
one in 76 had already been accounted for. Quite possibly, her intention may have been to begin 
with 70 on the ENL and to proceed with splitting, however, this was not what actually occurred. 
While it is not clear if it was Mary's intention to switch strategies midway through her solution 
(stringing to splitting), it is interesting to note that she repeated the same splitting error she 
consistently made during ENL lessons, namely subtracting both groups of ones (six and eight) 
from the running total in the equation 76 - 48.

Fig. 4.6 Mary solves 345 + 278
Overall, Mary's work in the post-assessment showed that she continued to demonstrate a personal sense of decomposition, utilizing numbers in a more sophisticated manner than her peers. While she did not maintain her perfect score from the pre-test to the post-test, her work samples indicated that she demonstrated a clear understanding of how to apply the ENL tool to computational tasks in an effective manner that complimented her own higher level thinking of number. In addition, Mary's feedback about this tool, in both her journal and during dialogue with me, was very positive. She wrote, "I discovered that using the ENL was easier for me." She also expressed her preference for stringing, stating this strategy "works very well."

In summary, we can see Mary performed well with this tool. While her application of the splitting strategy on the ENL remained in its primitive stages during the study, overall her application of this new tool was effective and her opinion was favorable, especially considering the time frame of the study.

**Case Study Two: Ruth**

**Background**

Miss Ryan described Ruth as a very diligent and hard-working student. Ruth was excited to be a participant of this study, and liked the idea of sharing her thoughts and ideas about the
ENL material presented. At the outset of the study, Miss Ryan indicated that Ruth was fully meeting grade level expectations in number sense, and performed well on classroom-based tasks and assessments. In addition, Miss Ryan identified Ruth as a quieter member of the class, who appeared shy when asked to participate orally in class and who, at times was hesitant to ask for help when she needed it.

**Pre-assessment Performance**

We can see from Ruth's pre-assessment that she has developed her addition and subtraction skills sufficiently using the algorithm, scoring eight out of ten in this assessment. Her errors included failing to complete the second step in a task, and incorrectly solving task ten involving negative numbers. These errors did not involve unsuccessful application of an algorithm per se, and so Ruth’s pre-assessment results confirmed her teacher’s sense that, for this grade at this time of year, Ruth was meeting expectations specific to number sense.

**The ENL Study: Ruth's Performance**

As mentioned, Ruth was enthusiastic about participating in this project. She applied herself fully to lessons and discussions, and worked hard when given individual tasks and activities to complete. Two issues rose upon closer examination of Ruth's performance in the study. The first is the delivery of her solutions. The second deals with her challenge with the application of splitting.

**Delivery of Solutions**

From the initial stages of the project, Ruth’s work samples consisted of solutions that, while correct, were lengthy. For example, in the case of $68 + 29$, Ruth proceeded by placing $68$
on the ENL and taking two jumps of ten and nine jumps of one (See Fig. 4.8). Similarly, when solving 74 - 35 (See Fig. 4.9). Ruth proceeded by taking three jumps of ten, as well as five jumps of one. As such, her solutions ranged from eight to twelve jumps before landing on a final value. Thus, in these early stages, although Ruth availed herself of the flexibility of the ENL model, she chose jumps of 10 and 1 only, which did not appear to optimize the mental math facility ENL claims to support. Producing such lengthy solutions for relatively easy equations leaves to question whether the ENL is a worthwhile alternative to the algorithm for Ruth. It is noteworthy that Ruth kept track of such solutions and while lengthy, Ruth's solutions were error free.

Fig. 4.8 Ruth solves 68 + 29

Fig. 4.9 Ruth solves 74-35
Therefore, the delivery of Ruth's solutions became a point of concern, as my intent was to support Ruth using the ENL model in an effective manner that complimented and strengthened her own intuitive way of thinking about number, whereas Ruth had returned to counting on or back in ones and tens. As identified by Gravemeijer and Terwel (2000) Ruth needs to be guided towards the analysis of her own mathematical thinking. It is important to note that during the study, Ruth was asked to identify ways to shorten her solutions, and with some initial support from me, Ruth was able to identify how she could make her solutions more manageable and efficient. This occurred approximately midway through the study. Here, Ruth was given the opportunity to review and reflect upon the quality of her solutions and asked if she could see possible ways of shortening solutions that still felt manageable. Specifically, Ruth was asked to do this by simply taking fewer jumps. Ruth and I reviewed work samples of some of her most lengthy solutions and with some support, she was able to determine where jumps could be combined. She was then able to apply this more comprehensive approach to future tasks. Upon examination of work samples produced by Ruth following my intervention, her solutions appeared more efficient (i.e. less lengthy). For instance, in the solution she produced for 76 - 48, Ruth made two jumps only, breaking 48 into 40 and 8. Similarly, in a word problem involving 454 -167, Ruth limited her jumps to one jump of 100, followed by a jump of 60, and ended with a jump of 7 (See Figure 4.10).
Fig 4.10 Ruth solves 454 - 167

To conclude, we can see that Ruth's initial work samples show that she was using the model with limited flexibility resulting in long, drawn out solutions at times. While these initial samples demonstrated Ruth’s understanding of place value, through her compartmentalizing of numbers into groups of 10 and one, she did not demonstrate a personal sense of decomposing numbers. Ideally her work with the ENL would facilitate the development of her mental math skills but her solutions containing 8 to 12 jumps did not support this. However, Ruth made progress, becoming more confident and comfortable using the ENL, and with some interventions, her solutions became more comparable to effective counting strategies that facilitate the development of her number sense. Ruth’s contributions illustrated one student’s gradual progression of understanding when using this ENL tool to solve various computational tasks.

**Challenge with Subtraction and Splitting**

Another matter that arose for Ruth during this study was the challenge she, along with other participants including Mary, had with subtraction on the ENL, particularly when applying the splitting strategy. This was an area of frustration for many students, and Ruth was among those that voiced concerns, both in her journals and in conversation with me. As mentioned, subtraction was initially introduced when working with stringing on the ENL. Ruth was, for the most part, successful with subtraction when using stringing, with the exception of one question
involving a three digit number. However, in the journal entry that followed this particular task, Ruth expressed her difficulty with subtraction on the ENL, writing, "I still don't understand how to subtract with the ENL." When students were introduced to splitting, Ruth expressed concern with using this new strategy when subtraction was involved. She wrote, "I still don't understand how to subtract without borrowing in splitting. I prefer stringing if we subtract because it is less confusing." Thus, it appears that at this juncture she was feeling more comfortable with subtraction and stringing, although not so with the newer strategy: "If we could, I'd like more practice at splitting". It is important to note that via journaling, Ruth was able to pinpoint exactly what was confusing her in regards to subtraction and splitting, namely borrowing. Her ability to identify the specifics of what she found puzzling points to a metaunderstanding of her work with this ENL tool and its applied strategies.

Her frustration with splitting was apparent on many occasions. On the task page titled ‘ENL: Splitting,’ assigned in week two, Ruth applied the splitting strategy when adding. However, she reverted to stringing when completing the two subtraction tasks, despite the fact that the focus was on splitting. This evidence suggests she was more comfortable using stringing when subtracting, possibly because stringing was more compatible with her current level of number sense or that she simply had more practice with stringing.

Likewise on another task page requiring students to apply both stringing and splitting, Ruth found splitting when subtracting a challenge. In this task, assigned in week three of the study, students were asked to solve a given question twice, first using stringing and then applying splitting. Ruth successfully solved both subtraction tasks using the stringing approach, but when asked to apply splitting, she did not attempt to provide a solution for either. Even though Ruth easily solved the equation 54-25 using stringing, by taking a jump of 20 and 5, she made no
attempt to use splitting as requested and instead simply wrote the words "too confusing" in lieu of a solution, despite having already determined the answer.

Ruth's Post-Assessment Performance

Ruth scored eight out of ten in her post-assessment, the same score she produced on her pre-assessment. Interestingly, she made errors in the same two questions as in the pre-test. However, her performance in the post-assessment raised questions about her ability to use the ENL efficiently. While the majority of solutions were indeed correct, the number of steps Ruth used in two cases remained large, making solutions lengthy. In the first equation $73 + 49$, Ruth's jumps were limited to single jumps of ten or one, with the exception of one jump of two. This means that Ruth took twelve jumps to reach the correct solution for an equation that she should be working towards solving mentally. While jumping in groups of ten may be perceived as mental work, such a process could result in memory overload when dealing with numbers of higher value. However, she solved eight out of ten equations in an efficient manner, with the majority of solutions involving only two or three jumps (albeit two of these solutions were incorrect).

Such lengthy solutions, as seen in Ruth's initial work samples, are not ideal. However, Ruth's final work samples suggest she has a developing understanding of how the ENL can be applied to solve computational tasks, specific to addition and subtraction.

Case Study Two: Alan

Background

Alan is a shy member of this Grade 4 class. According to his teacher, Miss Ryan, and his mother, he struggles in math and lacks confidence in this area. He loves reading, and is reading
and comprehending a little beyond his grade level. His written work is considered to be at grade level. His mother, an elementary school teacher herself, is aware of Alan's difficulties in mathematics. She has recently made a point of putting aside time on a regular basis to review some key concepts with Alan, focusing primarily on number sense. At the time of data collection, her extra support of Alan’s math was in its early stages.

**Review of Pre-assessment Performance**

Alan scored four out of ten on his pre-assessment, which focused on addition and subtraction of two and three digit numbers. This was the lowest score in the class. As mentioned, each task was presented in the form of a number sentence, and like the majority of students, Alan chose to rewrite each into the vertical format and complete each equation with a standard algorithm. The pre-assessment showed that while Alan had some success using this method, he was inconsistent in his ability to perform such operations, particularly when borrowing was involved. In addition, the errors he made varied, highlighting a variety of misconceptions in Alan's understanding of algorithms. In Question five for example (228 + 494 + 659), Alan made a classic error (Kamii & Dominick, 1997) and carried the wrong digit when working in the tens column (See Figure 4.11). In his response to Question six (454 – 167: embedded in a word problem), Alan made a completely different type of error (See Figure 4.12). His confusion seemed to lay in his lack of understanding when applying borrowing to the standard algorithm. In this example, when borrowing from the tens, Alan increased the five tens to six, rather than decreasing them to four. However, all other steps in the equation were correctly completed. Such an error raises concerns as to Alan's level of understanding of the standard algorithm, as well as of the process of subtraction. Questions seven ($90 - $57: embedded in a word problem) and eight (329 - 178; embedded in a word problem) were also incorrect, and the errors here could be attributed to his
inability to determine the operation needed when the equation is embedded in math language.

Alan's performance in this pre-assessment suggest a limited understanding of the algorithm, a limited proficiency when carrying out his chosen method of computation (i.e. the standard algorithm) and a weaker overall sense of number. Alan's inability to successfully carry the correct digit when adding, along with misconceptions about how borrowing works, point to gaps in Alan’s understanding of place value. In addition, some of Alan's incorrect answers are significantly larger or smaller than the actual answer, and therefore suggest his inability to determine reasonableness of his answers.

Fig. 4.11 Alan’s pre-assessment solution to $228 + 494 + 659$

What is the total sum in all three triangles?

\[
\begin{align*}
228 \\
+ 494 \\
+ 659 \\
\hline
2291
\end{align*}
\]
The Study - Alan's Performance

Alan's work samples from the ENL lessons showed that Alan was unable to successfully solve the tasks presented in class. While these samples demonstrated that he gained some understanding of how this tool can be adopted and applied to computational tasks, he found it very challenging to successfully complete equations using this approach. I often provided one-to-one support to ensure Alan began a task, and he frequently requested help and support throughout independent work periods. Despite his struggles, Alan’s work samples did shed light on some of the challenges and gaps he has with number. Below we will take a closer look at Alan's performance on individual task pages.

Confusion Distinguishing Digits

Task One consisted of six equations, focusing on addition using the stringing approach. Alan had some success with this task. With all six equations, he was able to plot the first number on the number line. In addition, he identified that jumping in multiples of 10 was an appropriate first step when applying this strategy, and proceeded to do so in all six cases. However, he consistently applied an incorrect number of jumps of 10. Repeatedly (four out of six questions), upon plotting the first addend on the ENL, Alan proceeded to jump in multiples of ten that was
one more than the number of tens shown in the second addend. Also, in these same four items, he did not show jumps for the ones in the second addend. Thus he did not reach the correct solution.

In the case of $67 + 45$, he attempted to add 40 to 67 unsuccessfully before moving to the next task (See Fig. 4.13). While, in this example, the correct number of tens was added to the first number, as mentioned, this was not consistent in his work. In the case of $59 + 34$, he added 40 to 59. In the case $68 + 29$, 30 was combined to 68. While this may have been an attempt to round up to 30, before subtracting one to reach 29, it did not appear from my interaction with or observation of Alan that this was his intention. Finally, when completing $235 + 67$, an attempt was made to add 70 to 235. Upon closer examination of Alan’s samples, we can see that he may have been focusing on the ones which coincidentally happened to be a digit larger that the tens (i.e. in the case of 34, he added four groups of ten, in the case of 67, he added 7 groups of ten), which somewhat correspond with his confusion in the pre-assessment where he carried the ones rather than the tens.

Fig. 4.13 Alan’s solves $67 + 45$
Subtraction work samples in future lessons also indicate this. In Task Two (a worksheet containing six questions), which focused on subtraction using the stringing strategy, Alan's inability to determine the correct number of tens and ones within a given number was repeated. When completing the question 57 - 13, Alan proceeded to carry out 57 - 30 (See Fig. 4.14). Here, it appears he looked to the ones to determine the number of tens in the given amount, and when these jumps of ten were applied, he chose to conclude work on the task. This is also true of questions 74 - 35 and 62 - 25. This second question will be looked at in more detail in the next section. Such jumps suggest confusion with the value of digits based on position.

**Challenge with Skip Counting**

As mentioned earlier, Alan’s challenges with place value were apparent in his use of the ENL when solving the equation 62-25 (See Fig. 4.15). However, this and other work samples also indicate that he struggled to skip count. Alan chose to take five jumps of ten, rather than taking the required two when working on subtracting 25. Alan's initial steps on this task were correct. He accurately plotted 62 on the ENL, placing it to the right of the line, as was the suggestion made to the class for subtraction. The jumps he took moved from right to left, indicating the sum is decreasing, appropriate moves when the equation calls for subtraction. While his jumps of ten went backwards, the amounts recorded on the line increased as he moved
from right to left, namely 62 to 72, 82, 92, 102 and finally 1002. Thus although he was moving in the right direction, he is skip counting by 10 in ascending order. When Alan reached 102, he recorded 1002 rather than 112, suggesting perhaps that he was unfamiliar with representing 112, such that he wrote 1002. Alternatively he may have thought jumping by 10 required an additional zero to be added to 102, i.e. 1002.

Fig. 4.15 Alan solves 62 - 25

Fig. 4.16 Alan solves 120 - 24

Additional examples on this Task Two page provide further insight into what appears to be Alan’s difficulty working with numbers greater than 100. To solve 120 - 24 (See Fig. 4.16), Alan chose to subtract four groups of 10 (possibly reading 4 ones as 4 tens), showing the running total decreasing from 120 to 020, to 10 and ending on 9. This example highlighted Alan's lack of understanding of place value, and his difficulty counting backwards in tens when needing to
bridge through 100. Alan chose to subtract from the hundreds rather than the tens, and when the hundreds were reduced to 0, moved to subtract from the tens, and when depleted moved to the ones. Alan seemed to have little understanding that the place of the number determines its value, and tried without success, to subtract the required amount using any means or process he could. Thus, this particular task page points to Alan's limited understanding of number and the large gaps he has in his comprehension of place value as well as his ability to count forwards or backwards by tens. Lacking such a basic understanding of place value will most definitely impact Alan's mathematical success. The ENL, while it has not been of particular help for developing Alan's computational skills up to this point, did shed some light on Alan's challenges with number. Many examples indicate that Alan was not aware of key concepts that are usually considered to be automatic by Grade 4. Awareness of reasonableness of answers in both addition and subtraction did not appear to be in place. Examples showing the sum of an addition equation to be less than one of the addends were disconcerting at this stage in Alan's mathematical learning. These challenges were evident in subtraction tasks as well.

**Post-assessment Performance**

It is important to note that in the post-assessment, which presented tasks identical to those in the pre-assessment, Alan's score decreased from 4/10 to 1/10. The sole difference between these tests was that students were asked to utilize the ENL to complete all equations. With this additional request, Alan's success rate dropped, solving only one word problem (i.e. Question 7: $90 - $57) successfully. Here, he managed to subtract one jump of 50 as well as a jump of seven successfully. However, this was the only solution in the post-assessment, where Alan did not limit his work to simply working with the tens, along with one solution where he attempted to work with groups of hundreds and tens but did so unsuccessfully. In all other solutions, Alan
simply combined jumps of 10 or 20 to the first number, even with number sentences containing the subtraction sign.

In summary, while the ENL did not prove effective in developing Alan's sense of number within this limited timeframe, we have seen how it did provide a more detailed analysis of some of the challenges he was having in mathematics, making the reasons for his errors were more apparent. Alan's inability to decompose numbers and correctly identify the number of tens and ones in a given number was apparent throughout his ENL work. He successfully plotted the first quantity on the line, and then in each case made an attempt to add groups of ten to the first quantity. In two cases, he chose to proceed by taking jumps of 20, but this did not take him towards the correct solution. In another task (345 + 278), he began by taking one jump of 200. He then moved to the tens and added two jumps of ten. While Alan took reasonable jumps working towards reaching the correct solution, he chose to stop work at this point. In this solution, it is important to point out that Alan was unable to correctly record the running total as he moved through his solution. While some numbers were added to the ENL, these numbers did not correspond correctly to the work taking place, but rather illustrated that Alan was unable to consistently add groups of 10 or 100 mentally to a given number with success. Such work samples in the post-assessment offer possible reasons for his computational struggles to date, and provide data that could guide his mathematical instruction in the future.
To conclude, we can see that each of the three case studies illustrates a very different experience with the tool. One factor that greatly contributed to student success was their understanding of number. Students who had a strong understanding of place value, could effectively count in multiples of ten both on and off the decade, as well as effectively decompose numbers in a variety of ways appeared to perform better with the ENL, within this short time frame. Work samples show the range of possibilities with the ENL, as well as the range of support students may need to work towards successful application of the tool.

**Analysis of Student Opinion**

This study also sought student opinion about the ENL and the strategies presented with regards to their effectiveness and ease of use. Through class discussion and journaling, students were given the opportunity to share their thoughts about the ENL model presented in this study. It was stressed to students how important their opinions were and that full honesty was key, regardless if feedback was positive or negative. Journaling opportunities were given periodically throughout the course of the project, resulting in three entries in total. However, inclusion of reflective comments, both in oral or in written form, were encouraged throughout the project. The
open-ended journaling did generate some interesting and useful dialogue from students about the tool. Journal entries of all 28 students were read and emergent themes are presented.

**Initial Impressions - First Journal Entry**

Based on the opening journal entry following the introduction of our first strategy stringing, initial impressions from students were positive. Comments included that the ENL was "fun and really easy" (Kevin) and an "easier way to add numbers" (Barry). Dennis wrote it was "a useful strategy to answer equations." Kim identified one of the benefits she felt the ENL brought when wrote "When I played with an ENL I was really good at it. I now understand that you can add ones differently like 7, you don't need to put all ones. You can make groups that equals 7." Harry felt it was "easier than algorithms because you can choose to do whatever strategy you want." Ava claimed it "helps me think more." "Easier" was a word commonly used by students at the beginning stages of this study. Although few students elaborated as to what made ENL easier, journal entries confirmed that many students felt they understood how to utilize and apply this tool at the initial stages of this unit.

However, some other students responded less favourably to the model in their first journal entries. Vick journaled that it "started off easy then it was kind of confusing. The confusing part was jumping in tens". Here we can see that Vick was able to identify exactly what he found challenging. A number of students expressed the need for more practice, so they could feel more comfortable and competent with the model, without pointing specifically to what they found hard. Claire commented "I'd like more practice doing the ENL. I found it a little challenging".

While there were no major concerns voiced by students following the introduction of our first strategy, stringing, some students did mention minor challenges and the benefit of more help.
Praise for the ENL

As students developed their understanding of the ENL, they indicated that a number of benefits arose when using this tool. Harry, for example, wrote in his second journal entry that "the ENL is helpful because it takes away the stress of carrying over and borrowing.” This benefit was mentioned by several participants and appeared to be a real bonus for students, as most had identified borrowing and carrying as a challenging feature when using the conventional algorithms.

In addition, journaling revealed that a group of students felt it was a benefit to create visual representations to solve equations, as is the process with the ENL. Alan, a child diagnosed with Autism Spectrum Disorder (ASD) who sometimes has real anxiety about math, told me, "I like making pictures of my thinking". Another student shared "the number line makes it easier to check over your work and see if you have made a mistake". Kalan, who was initially hesitant about the ENL as he felt it was a longer way of solving an equation, became more positive about the tool in his second journal entry. He stated that "this can be very helpful for my sister". He was now taking fewer jumps to reach his solution and seemed to be enjoying using this tool more as the unit progressed. When asked to elaborate on why the ENL would be helpful for his sister, he responded that "putting jumps on a line, it was easier to see what you were doing. She is still learning how to add small numbers and I think having a picture of what she was doing would be helpful for her, ’cause she likes drawing pictures."

As familiarity with this tool grew, some students began to think about the possibilities when using this tool. In his second journal entry, Steven recognized that the ENL could be "helpful for high numbers”. Mags "wonder[ed] if you can do times tables on the ENL". Such questions indicate that students are beginning to consider the use and value of this model, and recognize its versatility to perform a range of number sense tasks.
Criticisms and Challenges

As instructed, students also used journaling to voice their concerns about the ENL. Firstly, some students were concerned by the amount of time it took them to solve each equation using this approach. One student stated (in the second journal entry), "I think it just slows me down in math because you have to write a little bit more than you have to write in algorithms". A second child wrote, "I think the ENL was not helpful to me because it takes longer than just borrowing. I understand the ENL but it's just it takes longer". In addition, Kalan, a capable mathematician, wrote the following: "I felt bored because it took more steps than the way I do it. Next time I would not want to use it because I have a way that is easier and faster". These comments were made midway through the unit, at a stage when students had not yet been exposed to all the possibilities using the ENL. Much time up to this point had been spent introducing students to the basics as to how this tool worked, rather than exploring more sophisticated and flexible strategies. However, there had been no explicit limitations set as to how the tool was to be used.

As the unit progressed, I sought to illustrate to participants that the expectation was not to reiterate my approaches, but rather explore strategies of their own.

While some felt this tool was manageable but lengthy, other students in the class identified a different challenge. In her second journal entry, Tracey wrote that "it seems a bit easier but the thing that is not helpful is jumping by 30's or 20's. I still need a bit more practice". While some students displayed competence jumping in multiples greater than ten, it appeared that this was still a challenge to other members of the class. Vick, as mentioned earlier, expressed some difficulty with this portion of the task. In his initial journal entry, he wrote "the confusing part was jumping in tens," in reference to working with multiples of 10 (e.g. 20, 30). However, as work on the ENL continued, Vick expressed that he found such work a lot more manageable simply by having an opportunity to practice applying the strategy in class. It is important to
remember using the traditional algorithm does not require students to think in multiples of 10's but rather deals with 10's as single digits. Therefore, developing and maintaining computational fluency when adding and subtracting groups of 10 to quantities off the decade was not automatic for all students, despite the fact that they were competent applying the algorithm.

Strategic Preferences on the ENL

Three strategies were presented during the course of this study. Students used journaling opportunities to reflect upon and critique these strategies, and the group members seemed overall to be united in their thoughts and opinions about each strategy.

Stringing, the first strategy presented in this unit, was well received by the majority of participants. Reflecting upon this strategy, Mags simply stated "Stringing works very well." Grace wrote "I like it way better because you don't have to carry or borrow. And it is way easier to use than the old fashioned way." Barry wrote "it was an easier way to add numbers." Challenges with stringing were limited to students expressing some difficulty jumping in multiples of 10, off the decade. By this, I mean counting in 10’s beginning at a number that is not a multiple of ten, e.g. 14, 24, 34. Work samples showed that, when given the choice, most students selected to use this strategy, and journaling confirmed it was the preferred strategy for these students.

Bridging, the second strategy presented in the unit, received mixed reviews from students. This strategy required students to use the ENL model to bridge a gap between two numbers, as a means to determine the difference between two quantities. John stated in his final journal entry, "I was good at using the ENL because I knew how to add, subtract and use stringing and bridging". However, it appeared that students found this strategy a little more challenging than stringing. Claire wrote "I'm good at adding but I would like help at subtraction. I think I need a
little bit more practice so I understand it." Harry too, recognized that this strategy was not as manageable for him and wrote "the ENL is helpful because it takes away the stress for carrying over and borrowing. I'd like more practice for bridging. Next time, I would stick to the N10 strategy (stringing)." Student comments regarding the specifics as to why they held or formed a particular opinion about a strategy were limited, and instead journal entries mostly revealed in general terms whether they liked or disliked the strategy in question.

The final strategy presented in this unit was splitting. My initial observations of student behavior during lessons, along with students’ comments, indicated that this strategy proved challenging for many students, especially when used in subtraction. This was confirmed from the journal entries following the lessons.

Jack writes, "Stringing I was able to understand but splitting I still don't understand. Next time I would like to have more practice at splitting. Splitting was easy for addition but confusing for subtraction." Garry was able to pinpoint more what he found challenging about splitting. He writes that "stringing is easier than splitting because in splitting you have to break it up and it has more steps." Tom, expressed the same concerns, stating "I think that stringing is easy for addition and subtraction because you put the whole number down, but in splitting you put the tens down and then the ones. Splitting in addition is easy, in subtraction it gets harder." Here we can see that the decomposition of both numbers necessary in splitting, but not in stringing, was proving unpopular or less helpful for students. This strategy appeared to be especially difficult when students worked on tasks requiring subtraction. The main challenge for students arose with regard to the ones. In subtraction tasks, students assumed both sets of ones needed to be subtracted, whereas in actual fact the first group of ones were to be combined to the tens total before subtracting the second group of tens. This also proved “messy” on the ENL visual, when ENL work changed direction midway when combine the first group of ones and switched direction
again to subtract the second set of ones. Mags noted this difficulty writing, "Stringing works very well. Splitting is hard and confusing. I discovered that subtracting with the 1010 strategy (splitting) does not work well especially when you need to borrow". Some children simply expressed frustration with this strategy. Kim shares, "I tried to get the right answer in subtracting but I couldn't do it." Similarly, Ann wrote, "Splitting was hard to understand. I prefer learning stringing more than learning something really hard."

In conclusion, via my observations, as well as from student comments and journaling entries, it appeared that stringing was the preferred strategy utilized by students within this study. While participants identified the partial benefits of other strategies presented, this was clearly the strategy students felt most comfortable and competent applying. In fact, many voiced their preference for it over the standard algorithm.

**Post-assessment - Whole Class Results**

To conclude this study, 27 of the 28 students completed a post-assessment. As indicated, the post-assessment was identical to the pre-assessment, except students were directed to use the ENL to complete all tasks, using strategies of their choosing. I examined this data to determine student level of success in various tasks, as well as how they utilized the tool to solve tasks. The specific strategy used was noted along with the range of solutions with each individual task, in an effort to determine commonalities.

To begin, let us look at student performance in the post-assessment, specific to the number of correct answers achieved using ENL. Table 4.3 shows a comparison of pre and post assessment scores. In six out of 10 tasks, students’ performance dropped using the Empty Number Line, the difference ranging from a drop of 3 from pre-assessment to post-assessment (as in Task 1) to a drop of 8 (as in Task 2). However, scores did rise in the post-assessment with
Task 3 (2 digit subtraction), Task 7 (2 digit subtraction), Task 8 (3 digit subtraction) and Task 10 (2 digit subtraction involving negative numbers). As we can see, all tasks here called for subtraction. In other words, scores rose from pre-assessment to post-assessment in four out of five tasks that called for subtraction. Upon review of solutions in these four specific tasks, stringing was utilized in all correct solutions, which suggested that this strategy may be an effective and worthwhile approach for students when subtraction is needed. As students mentioned in journaling activities, stringing eliminated borrowing when subtracting, which they felt was a challenge when using the standard algorithm. Another observation upon review of the post-assessment is that there are four tasks that achieved more than 20 correct solutions. Those were Task 1(22 C), Task 3 (24 C), Task 7 (24 C) and Task 8 (21 C). It is interesting to note that three out of four of these tasks limited students to working with 2 digit numbers, supporting the claim that this tool may be most effective and beneficial to students when working with quantities of this size.

Table 4.3 Quantitative Comparison of Pre and Post Assessment Scores

<table>
<thead>
<tr>
<th>Task</th>
<th>Description of Task</th>
<th>Pre-assessment Performance</th>
<th>Post-assessment Performance</th>
</tr>
</thead>
</table>
| 1    | 2 digit addition w/ carrying: (73 + 49) | C : 25  
IC : 3 | C : 22  
IC : 5 |
| 2    | 3 digit addition w/ carrying: (345 + 278) | C : 24  
IC : 4 | C : 16  
IC : 11 |
| 3    | 2 digit subtraction w/ borrowing: (76-48) | C : 20  
IC : 8 | C : 24  
IC : 3 |
| 4    | 3 digit subtraction w/ borrowing: (456 - 289) | C : 18  
IC : 10 | C : 12  
IC : 15 |
<table>
<thead>
<tr>
<th>Task</th>
<th>Description of Task</th>
<th>Pre-assessment Performance</th>
<th>Post-assessment Performance</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 *</td>
<td>3 digit addition with carrying: (228 + 494 + 659)</td>
<td>C : 15&lt;br&gt;IC : 13</td>
<td>C : 10&lt;br&gt;IC : 17</td>
</tr>
<tr>
<td>6 *</td>
<td>3 digit subtraction with borrowing: (454 - 167)</td>
<td>C : 22&lt;br&gt;IC : 6</td>
<td>C : 17&lt;br&gt;IC : 10</td>
</tr>
<tr>
<td>7 *</td>
<td>2 digit subtraction with borrowing money: ($90 - $57)</td>
<td>C : 21&lt;br&gt;IC : 7</td>
<td>C : 24&lt;br&gt;IC : 3</td>
</tr>
<tr>
<td>8 *</td>
<td>3 digit subtraction with borrowing: (329 - 178)</td>
<td>C : 20&lt;br&gt;IC : 8</td>
<td>C : 21&lt;br&gt;IC : 6</td>
</tr>
<tr>
<td>9 *</td>
<td>Addition with dollars and cent: ($15.50 + $17.75)</td>
<td>C : 22&lt;br&gt;IC : 6</td>
<td>C : 15&lt;br&gt;IC : 12</td>
</tr>
<tr>
<td>10 *</td>
<td>Negative numbers: (temp.12° C, drops 15° C)</td>
<td>C : 7&lt;br&gt;IC : 21</td>
<td>C : 14&lt;br&gt;IC : 13</td>
</tr>
</tbody>
</table>

* Items 5-10 were word problems

**Student Strategies in the Post-assessment**

As mentioned, stringing was the strategy of choice by an overwhelming majority, used to solve over 90% of correct solutions. This strategy did appear in every lesson, either as the focus (as was the case in the first week) or in review. Being the first strategy presented to students, three classes were dedicated to its exploration. Review of work to date did occur at the beginning of every class, and as stringing was the first strategy presented, it was therefore mentioned most and frequently compared to other strategies introduced in the study.

Splitting, the strategy that seemed to challenge students most during this study was applied successfully by only seven students in the post-assessment. Of these seven students, only one student applied splitting successfully in more than one task. Upon review of data, there appears to
be no commonalities in the tasks they chose to apply splitting, making it difficult to establish their reasoning as to why they selected this particular strategy for a specific task. However, these students chose to apply the splitting approach to tasks involving addition, with the exception of one student who successfully applied splitting to a 3 digit subtraction task.

**Conclusion**

In this chapter, results upon application of the ENL in a Grade 4 classroom were presented. Student performance varied with this tool, although the majority of students demonstrate successful application of ENL to two and three digit computation. There was evidence of students’ ability to utilize this tool with flexibility, and higher level thinking was demonstrated by some. However, challenges utilizing this tool did arise; most notably, students had difficulties when working with larger quantities (3-digit) as well as when completing subtraction tasks, particularly when applying a splitting strategy. Indeed, while most student feedback regarding the ENL was positive, there was a common consensus that subtraction and splitting was not easily accomplished using this tool.

In conclusion, results of this study highlight the possibilities when utilizing this ENL model to solve computational tasks. These student experiences shed light on successes and challenges that can arise when using this model for the first time with Grade 4 students well versed in the standard algorithm. Chapter 5 will present some important issues for those considering implementation of the ENL.
5 Discussion

The Changing Face of Mathematics in this New Age

Central to math education must be the constant desire to prepare students with the quantitative and higher order thinking skills deemed necessary in today’s society. Practices and approaches must be guided by the mathematical skills necessary for this new age. As we continue to learn more about what facilitates successful learning, we need to critically examine and assess the effectiveness of different pedagogical representations and procedures used in the classroom.

The current BC Mathematics curriculum describes number sense as having the ability to be flexible with numbers, intuitive about numbers, along with being computationally fluent. Looking forward, the new BC curriculum draft ‘Transforming Curriculum & Assessment’ is moving towards an approach that places increased importance on the development of critical and creative thinking skills. In the domain of mathematics, attention is being placed on ensuring students “can use their mathematical knowledge, skills, and habits of mind to problem solve in a variety of ways and in different contexts” and “use flexible, effective, and accurate strategies to analyze and solve increasingly complex problems” (British Columbia Ministry of Education, 2013).

This study sought to attain an understanding of the ENL, a computational approach that requires users to view and utilize numbers in a manner that differs significantly from the standard algorithm. Research suggests the ENL may be a worthwhile computational approach to implement, as it is well suited to support and facilitate sense making within numeracy lessons, while building on the students’ own natural number sense and mental math skills.

This study set out to examine the use of the empty number line both as an effective means to perform computation, and also as support for students developing mental math skills and
overall sense of number. The aim of this study was to provide a detailed account of how Grade 4 students utilized the ENL in a didactical manner to perform computation tasks. Four research questions were formulated to guide this study.

RQ 1: In what ways does the empty number line support students sense making and self-generated strategies when carrying out 2 and 3 digit computation tasks?

RQ 2: What does their use of ENL reveal about students' sense of number?

RQ 3: In what ways, if any, are students more or less effective in solving computation tasks, following exposure to ENL?

RQ 4: What opinions do students formulate about ENL with regards to its effectiveness and ease of use?

To maintain continuity, discussion will be presented in a similar format to the results. Firstly, I will reflect upon conclusions based on the pre-assessment (RQ 3). Next I will discuss conclusions from both individual cases as well as a cross-case comparison (RQ 1 & 2). Finally, I will consider overall results from the whole class, including trends and themes that evolved (RQ 1, 2 & 3). In addition, I will discuss student opinions of this tool, gathered via student journaling activities and from my informal conversations with students (RQ 4). Following this, I will discuss considerations for this study along with considerations for future study. Finally, I will outline my recommendations drawn from the implementation of ENL in this study.

**Pre-Assessment**

A main point drawn from examination of the pre-assessment was that the algorithm, as predicted, was indeed the preferred choice of computation for participants. Considering that these participants had been exposed to the standard algorithm for approximately 2.5 years, and the
tasks presented on the pre-assessment were considered at or below grade level expectations, errors point to the ineffectiveness of the paper and pencil algorithmic method for some participants. Indeed, the errors made were consistent with what researchers such as Kamii and Ashlock describe as classic algorithmic mistakes. The errors found in the pre-assessment in the current study appear to support Kamii and Dominick’s (1997) claim that users of the algorithm, who lack understanding of the process, simply carry out a number of prescribed steps with little understanding of what they are actually doing. In their research on algorithms, Kamii and Dominick (1997) point to the contradictory nature of math educational practices, wherein teachers encourage students to invent their own methods for problem solving, while at the same time teaching the rote procedure that is the algorithm. The errors these Grade 4 students continued to make on computational tasks support the study's endeavour to introduce ENL as an alternative method for consideration.

Case Studies

In Chapter 4, I presented the results of three case studies, to illustrate the range of experiences with the tool and provide important considerations regarding these participants’ use of the ENL in this context. Each case will be discussed separately.

Case Study: Mary

Findings from Mary's case study highlighted possibilities for flexibility with the ENL, demonstrating that this tool supports a child’s individual numerical thinking and sense making. It appeared as though the ENL provided Mary with an opportunity to structure and shape her solutions in a way that complimented her current level of number sense. The "freedom to develop their own solution procedures…..and … the development of more sophisticated strategies"
(Selter, 1994 in Klein et al. 1998, p. 447), which is central to the Dutch theory RME (Realistic Mathematics Education), was evident throughout Mary's solutions. Hence, her case adds further support for this claim.

While Mary's work samples over the course of this study were impressive, and showed the possibilities of the ENL, there was a drop in her score from pre-assessment (10/10) to post-assessment (8/10). While one could doubt the effectiveness of the ENL due to these results, there are a few points to take into consideration. Firstly, Mary was introduced to algorithms almost three years ago, and she has had substantial practice perfecting this procedural approach to computation. At the same time, we must remember that the ENL is a new approach for participants such as Mary, and exposure to this method was limited to four weeks. Thus, in this short timeframe, to successfully apply strategies to the majority of tasks on the ENL (with the exception of splitting and subtraction), indicate that children, like Mary, who engage in higher level thinking are able to produce quality solutions. Mary’s strong sense of number served her well, allowing her to work in a creative and effective manner during this study.

**Case Study: Ruth**

In the case of Ruth, there was ample evidence that the ENL provides users the opportunity to work with numbers in chunks they find manageable. Interestingly, Ruth's case study indicates that in the short term at least, this may mean children produce lengthier solutions than warranted. Gravemeijer (1994), cited in Klein et al. (1998), highlights the importance of using the ENL "in a global, flexible manner" (p. 446). Freudenthal (1991) believes the ENL "compels the children to build on intuitive strategies towards a more abbreviated abstract use" (Murphy, 2010, p.12). While Ruth appeared to incorporate her own intuitive understanding of number within her solutions, having such extensive multi-step solutions for two digit addition and subtraction
equations was not ideal in moving her towards Freudenthal's goal, and more specifically towards the development and strengthening of her mental math skills. While this model provides the user the freedom to create their own solutions, "employing this model should also foster the development of more sophisticated strategies" (Klein et al. 1998, p.446). Therefore, in cases such as Ruth, teachers may need to intervene in order to ensure children move towards using the ENL in a more flexible manner, as intended.

As identified by Gravemeijer and Terwel (2000), Ruth needed to be guided towards analyzing her own mathematical thinking. With some initial support, Ruth was able to reflect upon the quality of her solutions and identify how she could make solutions more manageable and efficient. Progress was made in this area. Ruth's final work samples show an understanding of how the ENL can be applied to solve computational tasks, (eg. addition and subtraction) and demonstrate she could effectively do so. Indeed post-assessment shows that Ruth applied abbreviated solutions to eight of ten tasks (using the stringing approach), indicating her ability to address the efficiency of her solutions.

**Case Study: Alan**

The case of Alan, who struggled significantly with this new approach, highlights the importance of ensuring entry level tasks correlated with ones overall sense of number. Alan’s work samples reveal that he lacked the background knowledge necessary to utilize the tool with ease, specifically his poor understanding of place value up to 1000, as well as his inability to count by tens and hundreds both on and off the decade with consistent success. Based on the school age of participants and curriculum expectations for Grade 4 students, my ENL unit focused on two-digit and three-digit computation. However, in the case of Alan, this entry point proved too challenging and evidence suggests he may instead benefit from using the ENL with
smaller quantities. To develop readers, we as educators recognize the importance of providing learners with “good fit” books, in order to develop fluency and facilitate comprehension. If the text is too hard, or too easy, the reader can become disengaged and the progress made may be minimized. The same appears true of computation. Here, Alan is working on ENL tasks which prove to be beyond his level of ability and comfort, and therefore it appears an easier entry point is necessary if Alan is to develop a full understanding of this process.

Alan's work samples prove very insightful, and beneficial in identifying some of his challenges with number. His samples suggest he may be struggling to develop his logico mathematical knowledge, one of three types of knowledge identified by Piaget(1967/1971/1945/1962) cited in Kamii and Dominick (1997). It refers to the formation of mental relationships, which can be pointed out to an individual but ultimately relies on that individual making these connections to achieve logico mathematical knowledge. A student's understanding that the combination of two quantities results in a larger amount is an example of forming a mental relationship that contributes to logico mathematical knowledge. Alan's work samples clearly show that he had some real struggles with the material presented in this study. Firstly, we saw in Alan's pre-assessment that he has some holes in his understanding of the standard algorithm. He was inconsistent in his ability to carry the correct digit when adding, and showed confusion when borrowing was required. These misconceptions should be deemed concerning when you consider the amount of exposure Alan has had to this approach. These common mistakes with the standard algorithm, as recognized by researchers such as Ashlock (2010), and Brown and Burton (1978), raise the question as to Alan's level of understanding of the procedure he was attempting to carry out.

Alan's performance with the ENL during the study was poor. Factors contributing to this include a lack of pre-requisite skills (including an understanding of place value, ability to jump in
10's from a given number), an unsuitable entry point, as well as insufficient time to master presented ENL material. Time was limited within this study, and therefore I was unable to address all of Alan’s challenges. However, the ENL did prove very informative in identifying Alan’s misconceptions about number – for example, Alan had confusion distinguishing the value of digits in two and three digit numbers. In the area of number, Beishuizen et al. (1997) recognizes the ENL is a method that provides a transparent and natural lens in which to view students’ work with numbers, making identification of students’ strengths and challenges with number more apparent.

**Themes Identified from this Study**

Based on data collected from all 28 participants, I identified a number of commonalities.

**Students' Strategy of Choice**

Findings from my study clearly showed that stringing was the strategy of choice for participants. A possible reason for this may be students simply found this strategy easier to comprehend and employ. We must also remember that this was the first strategy presented, and it was addressed in some form in all lessons that followed, so students were exposed to it more often than the other strategies. However, student preference for stringing has been demonstrated in other studies as well. For instance, the work of Blote, Klein and Beishuizen (2000) found that stringing was the strategy of choice for the Dutch second grade students with whom they worked. Beishuizen (1999) noted that "many better pupils prefer the more effective strategy (stringing), while many weaker pupils chose 1010 (splitting) as the 'easier' procedure at first sight" (p. 159). According to Varol and Farran (2007), splitting (1010) is a more suitable strategy for equations such as '74-69'. They believe that if stringing (N10) is applied to such a task, students may be at a
disadvantage due to the increased "workload on the memory and the chance that errors occur" (p.90). However, from the data collected during my study, I was unable to determine if this was the case.

**Splitting, Subtraction and the ENL**

Of the three strategies presented in this study, students struggled most with splitting. Splitting proved especially challenging when it was applied to subtraction tasks. Specifically, challenges arose as students subtracted both groups of ones from the running total. While participants were able to successfully implement splitting to addition tasks, challenges and errors consistently arose when subtraction and splitting were required. Whenever they were given the option of which strategy to employ to complete tasks, the majority of participants chose to implement stringing. This finding raises concern about the utilization of this approach with subtraction tasks, and my data suggests that the ENL representation is problematic when such an approach is applied. This is consistent with the findings of Plunkett (1979), which pointed out that challenges with splitting are common, particularly when subtraction involves regrouping (although Plunkett's work did not involve application of splitting on the ENL, but rather as a linear equation). Beishuizen et al. (1997) note that while the splitting procedure may appear easier to implement at first glance due to its alignment with our place value number system, this is often not the case. The challenge with splitting and subtraction does not stem from the decomposition of numbers but rather on the ability to accurately recombine numbers (Beishuizen et al, 1997). However, the manner in which students were expected to maneuver their visual representation when applying splitting to subtraction (moving right on the ENL, then left, and then back to right) did not make sense to participants, and therefore suggests that such a process is not intuitive, and therefore not didactical for users as is the intent of the ENL model. However,
in his work, Beishuizen still recommends both strategies, N10 and 1010, as being important and of possible benefit to students. Beishuizen et al. (1997) suggest the implementation of the N10 strategy (stringing) occurs first and once this has been mastered, students should be presented with the second. While my study did introduce strategies in this order, time was limited for exploration of each strategy. If students had had more time to reflect upon and practice strategies presented, they may have performed better and responded more favourably to the splitting strategy.

**Student Opinion of the ENL**

This study sought to obtain student opinion of the ENL, and as we saw initial impressions about the tool were generally favorable: students enjoyed the flexibility and choice available in working on tasks, as well as the presence of a visual to support and illustrate their thinking. However, other initial comments were not positive, commenting that ENL was a longer and therefore an unhelpful computational tool. These comments are consistent with the findings of van den Heuvel-Panhuizen (2008) who found that capable students sometimes resent the model because they view it as a lengthier process than other approaches, such as the conventional algorithm. This then highlights the importance of ensuring flexible use of the model and facilitating the development of strategies suited to individual student ability.

Such comments as “it just slows me down” raised concerns as to whether this approach was beneficial for all students. However, as I continued to encourage and reference more flexible, higher level thinking, particularly during the latter part of this study, I began to see more efficient sophisticated strategies emerge. With this development, students began to express more favourable comments about the ENL, including those students who were somewhat negative towards this tool at the beginning of the study.
Considerations from the Study

Upon reflection, I recognize that there were elements in the design and implementation of this study, that if altered could further enhance the quality of students performance when using the ENL.

Firstly, my data suggests providing multiple entry points when introducing this model to intermediate students may be advantageous. As mentioned, due to the age of students and their grade level, I decided to launch work on the ENL with the application of computing two digit numbers. Based on the age and experience of participants, working with one digit numbers did not seem fitting. Consequently, students began working on the ENL with two digit numbers without having had the opportunity to tinker with the model using smaller quantities. While most were capable of working with this adapted approach, a few participants may have performed better if they had been provided with a simpler entry point.

In addition, the manner in which I introduced and implemented the tool within this study proved challenging at times. For the purposes of this study, I worked alone with a class of 28 participants for a one-hour period twice weekly for four consecutive weeks. As identified in my results, some students found aspects of this tool challenging, and could have benefited greatly from some small group instruction or some additional practice in a specific area. However, opportunities to pull students aside to break a concept down further, or provide additional support, which is often common practice in classrooms when specific students are having difficulty with a concept were not always possible. Upon reflection, I feel it may have been more beneficial for students if work was carried out in smaller groups, rather than as an entire class. This would have allowed closer monitoring of the work taking place, as well as more regular opportunities to interact with students about the work, providing opportunities for student to
share their thinking and choice of strategy during tasks.

Furthermore, the timeline of this study did not lend itself well to providing students with ongoing opportunities to review the content being presented. Reference to the ENL was limited to my visits, to the best of my knowledge, meaning opportunities for revisiting, reviewing and engaging in ongoing practice were not available to students outside the twice weekly block. In a typical classroom, the introduction of this tool would not be so compartmentalized but would be revisited several times within the week, both formally and informally, over an extended period of time before the introduction of another strategy. The work of Mighton (2007) highlights the importance of repetition and mastery before moving forward with new learning, and this was not always possible within the context of this study. Mighton states that "full understanding of a concept rarely emerges without a great deal of effort and practice" (p.67), sourcing the work of the great mathematician John Von Neumann to support his claim that "learning mathematics is a matter of getting used to things" (p.67). Therefore, I believe a four week time frame is too short a window to introduce three ENL strategies to students, regardless of their age. I suggest that students’ performance determine how long a specific strategy is explored, as well as when a new strategy is introduced. In addition, I feel that ongoing exposure to the content, in this case work on the ENL, is imperative, particularly for those students finding it challenging. To facilitate this, my two one-hour periods working with the students weekly, may have been better utilized if broken into shorter, more regular blocks.

**Considerations for Future Study**

Upon final review of data, my study offers some important considerations for future research into the implementation of the ENL. To begin, data confirms that while students were correctly utilizing the standard algorithm to solve computational tasks, a sense of number was
underdeveloped in some Grade 4 participants. Students’ challenges, including their inability to jump in multiples of 10 from a given number and their difficulty decomposing numbers, suggested that there were gaps, in some cases significant, in their numerical understanding. This supports the work of Kamii and Dominick (1997) who found that students can often complete the algorithm without any real understanding of what they are doing.

This study facilitated the comparison of students’ application of algorithms in the pre-assessment to their generated ENL solutions in the post-assessment. These two approaches require students to engage in two very different types of mathematical activities. The ENL requires conceptual thought and an understanding of the process if students are going to invent new methods of computation. The standard algorithm, on the other hand, involves a more rote application of rules and less comprehension of what one is doing (Mighton, 2007, p. 163). In contrast to what is revealed through the use of the standard algorithm, I found student work on the ENL to be very helpful in showing aspects of students’ thinking and understanding of number. My data also suggests that the ENL was a more transparent model than the standard algorithm as I was able to easily identify particular students’ challenges and misconceptions with number. Sowder (1998) states that "correct answers are not safe indicators of good thinking," and that "teachers must examine more than answers and must demand from students more than answers" (p.227). Further research in this area is suggested to determine if indeed the ENL is as valuable as I predict, in that it offers teachers ongoing informal assessment of students abilities to compute and tinker with number.

Upon review of the ENL work samples, I observed that students appeared to be far more successful using the ENL when working with quantities limited to two-digit numbers. This was especially evident in the two digit subtraction tasks, where student performance increased from pre to post assessment in both cases. Difficulties arose when children were given tasks involving
three digit numbers, or three quantities to combine. For Task 2 (345 + 278) in the post-assessment, only 16 were successful when application of the ENL was required, whereas in the pre-assessment 24 students were successful in the same task through using the standard algorithm. Such data suggest that the ENL may be more suited to single and double-digit numbers and limited to working with two given quantities. Researchers such as Murphy (2010) argue that the ENL work serves as a foundation for number sense development before the introduction to the standard algorithm. Murphy (2010) goes on to say that the ENL gives users the "opportunity to use the functional, common regularities of number and operations that build on counting and intuitive structures of number" (p.12). When this is developed, the introduction of algorithms could be embraced with a deeper understanding, allowing students to utilize the standard algorithm in a less mechanical way.

Analysis of student samples highlight some important considerations when implementing the ENL. Data shows that students were most successful when applying the ENL when working with two digit numbers, proving most effective when subtraction and stringing was utilized. Equations requiring the combination of three or more quantities, or working with three digit numbers, proved most challenging. Further research is suggested to determine how both approaches, the algorithm and the ENL, should be implemented and interwoven in an effort to best serve a child in their number sense development, when working with larger numbers.

As mentioned, challenges with this tool did arise. Primarily, participants found splitting difficult and raised the question as to whether this approach should be infused with subtraction. The visual representation of subtraction and splitting appears to be counterintuitive for students, resulting in subtraction of both groups of ones. Research into whether this resulted due to a conceptual weakness with the ENL model would be a most worthwhile investigation in the future.
Conclusion

In today's world of education, governments and educational authorities highlight the need for math education to be more meaningful for our students, encouraging integration of real life application and inquiry-based learning. In saying this, the educator remains responsible for ensuring students leave each grade with a solid understanding of the prescribed learning outcomes, wherein number and operations remains central in elementary mathematics.

As mentioned, the conventional paper and pencil algorithm remains the longstanding approach to addressing computation fluency in today's classroom. However, criticism for this mechanical approach to working with number remains. This study set out to examine the possibilities of utilizing a different approach to developing computational fluency and an overall deeper sense of number with students. The ENL, a pedagogic tool originating in the Netherlands, is considered by some to be a tool that builds on children's intuitive and mental math strategies. It supports children in constructing sequential calculation strategies that build on counting and the intuitive structures of number sense.

Findings from this study do not point to the implementation of a single approach. Rather, comparisons between pre and post assessment results in the current study, suggest that as the complexity of addition and subtraction tasks increase, the algorithm appeared to allow some students to carry out these tasks more successfully than their use of the ENL. As educators, we must think carefully about when algorithms are introduced in the classroom. At what stage do they belong in a child's number sense journey? Similar to the approach in the Netherlands, I believe a child should have a grounded understanding of number and place value and be able to utilize this understanding to successfully complete simple tasks mentally before being exposed to paper and pencil algorithmic teaching (Murphy, 2010).
Work on the ENL proved valuable in providing informal assessment of students’ number sense skills and raised the question, if students struggle to add or subtract 10 to a number, failing to see how this task can be completed mentally, are they ready for 2 digit addition and subtraction using the algorithm? I believe educators need to take time to closely evaluate student sense of number, and this study suggests using ENL can be one way to achieve this.

We as educators need to recognize that priority must be given to developing a solid sense of number, rather than focusing on mastering a number of prescribed steps. Priority must also be given to allowing students to tinker with number, working towards developing a sense of number necessary to succeed in an ever-increasing numerate society. We must seek out possibilities that provide students with opportunities to think critically and creatively in this area, in an effort to facilitate a meaningful understanding of number among all students.

To conclude, this study highlights advantages and disadvantages when utilizing the ENL to solve 2 and 3 digit computation tasks. Overall, students performed well, and were optimistic and enthusiastic about this new approach to computation. Data from this study illustrates that this tool provides students with the freedom to express and share their thinking in a transparent manner, proving beneficial to both students and teachers. I believe the ENL offers worthwhile possibilities for educators wanting to develop number sense and computational skills among students in a manner that draws and builds on students’ intuitive ways of thinking about number. Based on this, I recommend that the ENL should fall under the umbrella of number sense instruction in the elementary classroom. With the recent launch of the BC Math Curriculum (2013), now placing more emphasis on creative and critical thinking skills, using an inquiry based model, I believe this didactic tool can be even more fitting and effective in developing students numeracy skills, in a more personalized manner.
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Appendix 1 Empty Number Line Visual (with base ten blocks)
## Appendix 2 Stringing Lessons Summary

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Description of Lesson</th>
<th>Materials Used</th>
</tr>
</thead>
</table>
| **One** | *Whole class lesson:*  
  - ENL introduced as a computation tool  
  - Strategy 1: Stringing presented  
  - Stringing modeled for 2 digit addition using paper base 10's and 1's blocks (several examples with student input)  
  - Stringing modeled for 3 digit addition (two examples)  
  *Independent task:*  
  - Student asked to complete 'ENL Task 1 Worksheet' containing 6 number sentences (hundred squares available for those still finding adding ten mentally a challenge)  
  - Students given the option of applying this strategy to problem solving situations on a second task page  
  *Group task/game:*  
  In pairs, students given opportunity to apply new strategy (stringing) to real life situation - 'Long Jump Game'. Each student rolls two 2 digit numbers and combines them on the ENL. Student with the greatest jump wins that round. | Images of 100's, 10's and 1's  
  ENL Task 1 worksheet  
  Long Jump Game board  
  Double dodecahedron dice |
| **Two** | *Whole class lesson:*  
  - Review of stringing and addition  
  - Introduction of stringing and subtraction using paper base 10's and 1's blocks (several examples with student input)  
  - Stringing modeled for 3 digit subtraction (two examples)  
  *Independent task:*  
  - Student asked to complete 'ENL Task 2 Worksheet' containing 6 number sentences (hundred squares available for those still finding adding ten mentally a challenge)  
  - Students given the option of applying this strategy to problem solving situations on a second task page  
  *Group task/game:*  
  In pairs, students given opportunity to apply new strategy (stringing) to real life situation in a game 'Long Jump'. The variation here is that students must subtract numbers rather than combining them. | Images of 100's, 10's and 1's  
  ENL Task 2 worksheet  
  Long Jump Game board  
  Double dodecahedron dice |
<table>
<thead>
<tr>
<th>Lesson</th>
<th>Description of Lesson</th>
<th>Materials Used</th>
</tr>
</thead>
</table>
| Three  | *Whole class lesson:*  
- Review of addition/subtraction using stringing on the ENL.  
- Two examples of each (+,-) are completed using both 2 and 3 digit numbers, with input from students  
- Whole class presented with 'Name that Equation' task.  
Here the ENL is complete (using the N10 strategy) but no equation is given. Students must work out what equation the visual is showing.  
*Groups task/game*  
- In pairs, students are asked to apply stringing to carry addition and subtraction in a game 'Walk the Line'. Here, each player must create three 2 digit numbers using a deck of cards. These three amounts must then be placed in an equation i.e. ____+____-______ = ______. Each player must decide what order to place the numbers in to make the greatest sum, therefore walking the farthest down the line. All work is shown on an ENL. | Images of 100's, 10's and 1's  
Walk the Line game board  
Pack of cards (Ace to 9 only) |
Appendix 3 Long Jump Game

**LONG JUMP CHALLENGE**

How to play: Player 1 rolls the place value dice and plots this 2-digit number on the line. Player 1 rolls again and adds this 2-digit number to the total. Now player 2 does the same. The player with the longest jump wins that point.

Player 1: ___________ Player 2: ___________

Player 1 Jump One

Player 2 Jump One

Player 1 Jump Two

Player 2 Jump Two

Point to Player ___________ Point to Player ___________
Bridging and the Empty Number Line

Read each visual carefully. Record an equation to go with each equation.

The equation is
\[ 78 + 23 = 111 \]

The equation is
\[ 94 - 36 = 58 \]

The equation is
\[ 104 - 29 = 75 \]

The equation is
\[ 212 - 27 = 175 \]
## Appendix 5 Bridging Lessons Summary

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Description of Lesson</th>
<th>Materials Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td><em>Whole class lesson:</em></td>
<td>Images of 100's, 10's and 1's</td>
</tr>
<tr>
<td></td>
<td>Reinforced the ENL is a flexible model (many different ways to use it)</td>
<td>ENL task page 'Bridging and the ENL'</td>
</tr>
<tr>
<td></td>
<td>Bridging (A10) presented to class as one possible strategy to solve computation tasks. Modeled and discussed. Several examples completed with student participation.</td>
<td></td>
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<tr>
<td></td>
<td>Class presented with partially completed visuals of the ENL (modeling bridging) and asked to determine the equation to accompany each visual. Several examples completed with student participation.</td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>Independent task:</em> Students are asked to complete task page 'Bridging and the ENL'. Here, students are given a partially completed visual and asked to determine a number equation to accompany the visual.</td>
<td></td>
</tr>
<tr>
<td>Two</td>
<td><em>Whole class lesson:</em></td>
<td>Images of 100's, 10's and 1's</td>
</tr>
<tr>
<td></td>
<td>Bridging(A10) reviewed and students asked to differentiate between bridging and stringing</td>
<td>ENL task page 'Find the difference between'</td>
</tr>
<tr>
<td></td>
<td>Question presented to class: Find the difference between 41 and 23 (Class generate possible methods)</td>
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<tr>
<td></td>
<td>Retaining with wording 'Find the difference between' other examples modeled with student participation</td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>Independent task:</em> Student asked to complete ENL task page 'Find the difference' containing word problems that lend themselves well to this strategy</td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>Group task/game:</em> In pairs, students given opportunity to practice this strategy(bridging) to real life situation in a game 'What's the Difference'. Each player must generate two 2 digit numbers using a double decahedron dice. Using an ENL the players must then determine the difference between these two numbers on an ENL. The player who's numbers have the greatest difference wins that round, gaining one point.</td>
<td>'What's the Difference' game board Double decahedron dice</td>
</tr>
</tbody>
</table>
### Appendix 6 Splitting Lessons Summary

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<thead>
<tr>
<th>Lesson</th>
<th>Description of Lesson</th>
<th>Materials Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td><em>Whole class lesson:</em></td>
<td>Images of 100's, 10's and 1's</td>
</tr>
<tr>
<td></td>
<td>- Reinforced the ENL is a flexible model (many different ways to use it)</td>
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<tr>
<td></td>
<td>- Another strategy splitting (1010) presented</td>
<td>ENL task page 'Bridging and the ENL'</td>
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<tr>
<td></td>
<td>- Splitting modeled for 2 digit addition using paper base 10's and 1's blocks (several examples with student input)</td>
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<td></td>
<td>- Stringing modeled for 3 digit addition (two examples)</td>
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<td></td>
<td><em>Independent task:</em></td>
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<td></td>
<td>- Student asked to complete 'ENL Task 1 Worksheet' containing 6 number sentences (Hundred squares available for those still finding adding ten mentally a challenge)</td>
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<td></td>
<td>- Students given the option of applying this strategy to problem solving situations on a second task page</td>
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<tr>
<td></td>
<td>- Stringing (1010) presented as another possible strategy to solve computation tasks on ENL. Modeled/ discussed. Several examples complete with student input.</td>
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</tr>
<tr>
<td></td>
<td>- Class presented with partially completed visuals of the ENL (modeling bridging) and asked to determine the equation to accompany each visual.</td>
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<tr>
<td></td>
<td><em>Group task/game:</em></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Students are asked to complete task page 'Bridging and the ENL'. Here, students are given a partially completed visual and asked to determine the number equation to accompany the visual.</td>
<td></td>
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</table>
### Splitting (1010) Unit

<table>
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<tr>
<th>Lesson</th>
<th>Description of Lesson</th>
<th>Materials Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two</td>
<td><em>Whole class lesson:</em> - Splitting (1010) reviewed and students asked to differentiate between bridging and stringing - Question presented: Find the difference between 41 and 23 (Class generate possible methods) - More 'Find the difference between' questioned modeled with student participation <em>Independent task:</em> - Student asked to complete ENL task page 'Find the difference' containing word problems that lend themselves well to this strategy <em>Group task/game:</em> In pairs, students given opportunity to practice this strategy (bridging) to real life situation in a game 'What's the Difference'. Each player must generate two 2 digit numbers using a double decahedron dice. Using an ENL the players must then determine the difference between these two numbers on an ENL. The player who rolls numbers with the greatest difference wins that round, gaining one point.</td>
<td>Images of 100's, 10's and 1's ENL task page 'Find the Difference' ENL task page 'What's the difference problems' Double decahedron dice</td>
</tr>
</tbody>
</table>
Appendix 7 Student Journaling Summary

<table>
<thead>
<tr>
<th>Summary of Journal Entries</th>
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<tbody>
<tr>
<td><strong>Journal Entry 1:</strong></td>
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<tr>
<td>Concluding the second ENL lesson, students were asked to share their first impressions of the ENL. To date, students had only explored one strategy, stringing (N10).</td>
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<td><strong>Journal Entry 2:</strong></td>
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<td>Following lesson two on bridging (A10), students were asked to share their thought about this strategy.</td>
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<tr>
<td><strong>Journal Entry 3:</strong></td>
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<tr>
<td>Upon completion of two lessons exploring splitting (1010), students were asked to complete another journal entry focusing on their impression of splitting.</td>
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</tbody>
</table>
Appendix 8 Student Journaling Template

Name: __________________________ Date: __________________________

ENL - Your Thoughts

Prompts:
I think......... I know/don't know.......... I noticed that...... I wonder if........
I was surprised that.... I was able to....... I was good at....... I have learned
I now understand... I still don't understand... Next time, I would...
I'd like more practice at .. I'm proud that...... It has been helpful/unhelpful
Some questions I have are..