Spaces of homomorphisms and commuting orthogonal matrices

by

Galo Higuera Rojo

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Abstract

In this work we study the space of group homomorphisms $\mathrm{Hom}(\pi, G)$ for special choices of π and G. In the first part of this thesis, we enumerate and describe the path components for the spaces of ordered commuting k-tuples of orthogonal and special orthogonal matrices respectively. This corresponds to choosing $\pi = \mathbb{Z}^k$ and $G = O(n), SO(n)$. We also provide a lower bound on the number of components for the case $G = Spin(n)$ for sufficiently large n.

In the second part, we describe the space $\mathrm{Hom}(\Gamma, SU(2))$, where Γ is a group arising from a central extension of the form

$$0 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z}^k \to 0.$$

The description of this space is good enough that, using some known results, it allows us to compute its cohomology groups.

Preface

All of the results in this thesis are my own work under the supervision of Alejandro Adem. The paper “On the space of commuting orthogonal matrices” is based on the work done in the first chapter and has been published in [Roj13]. I had a lot of great conversations with Juan Souto and he gave me great ideas and suggestions on how to attack the problem.

For the second chapter, I mostly worked on my own with guidance of Alejandro but José Manuel Gómez was of great help when I was trying to understand the space of homomorphisms from abelian groups into $SU(2)$.

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Introduction

The spaces of group homomorphisms $\mathrm{Hom}(\pi, G)$ between a discrete group π and a Lie Group G are interesting for their relation with bundle theory and can be quite complicated. The topology on $\mathrm{Hom}(\pi, G)$ is given by considering it as a subspace of G^k, where k is the number of elements in a generating set of π. The group G acts naturally on these spaces by conjugation, the quotient space $\mathrm{Rep}(\pi, G) := \mathrm{Hom}(\pi, G)/G$ is called the representation space and it can be identified with the moduli space of flat $G-$bundles over a manifold with fundamental group equal to π. These moduli spaces of bundles are important in physics, they are related with a number of important quantum field theories, such as Yang-Mills and Chern-Simons theories.

In particular, when $\pi = \mathbb{Z}^k$, the space $\mathrm{Hom}(\mathbb{Z}^k, G)$ can be identified with the space

$$C_k(G) := \{(g_1, \ldots, g_k) \in G^n : g_i g_j = g_j g_i \text{ for all } i, j\},$$

of ordered commuting k-tuples in G. The fundamental group of $(\mathbb{S}^1)^k$ is $\mathbb{Z}^k$ so the space $R_k(G) := \mathrm{Rep}(\mathbb{Z}^k, G)$ can be identified with the moduli space of flat $G-$bundles over the $k-$torus.

The spaces $C_k(G)$ were studied from the homotopical viewpoint by Adem and Cohen in [AC07], where they find a stable splitting (after one suspension) of the space $C_k(G)$ for large class of Lie groups, including all closed subgroups of $GL(n,\mathbb{C})$. The splitting has the following form:

$$\Sigma(C_k(G)) \simeq \bigvee_{1\le j\le k} \Sigma\left(\bigvee_{\binom{k}{j}} C_j(G)/S_j(G)\right),$$

where $S_j(G)$ is the subspace of $C_j(G)$ of tuples with at least one entry equal to the identity of G. The quotients $C_j(G)/S_j(G)$ are highly singular spaces and are very complicated in general, so explicit computations using this technique have been limited. However, based on this splitting, Crabb in [Cra11] and independently Baird, Jeffrey and Selick in [BJS11] were able to compute the stable homotopy type of $C_k(SU(2))$ explicitly.

The spaces $C_k(G)$ are in general not connected, even if G is connected and simply connected. But note that if G is connected, then $R_k(G)$ and $C_k(G)$ have the same number of path connected components.

The number of path components of $C_k(G)$ is known for some special cases of G, for example, in [AC07] it was shown that $C_k(G)$ is connected when G is one of $U(q), SU(q)$ or $Sp(q)$. In [TGS08] Torres-Giese and Sjerve found the number of connected components of $C_k(SO(3))$, and in [ACG13] the calculations were done for the case when G is a central product of special unitary groups. In [KS00], it was shown that the space $C_3(\mathrm{Spin}(n))$ is disconnected for $n \geq 7$, previously, a wrong assumption about the connectivity of these spaces led to a miscalculation of the number of vacuum states in supersymmetric Yang-Mills theories over spatial $(\mathbb{S}^1)^3$ by Witten in [Wit82].

In the first chapter of this dissertation, we will find the number of connected components of $C_k(O(n))$ and $C_k(SO(n))$ for all k and n. Here, the group $O(n)$ is the group of orthogonal $n \times n$ real matrices, and $SO(n)$ is the subgroup of $O(n)$ of matrices with determinant equal to one. It is interesting to note that $O(n)$ is a disconnected group and that all of the Lie groups in the examples above are connected. However, even though $O(n)$ is not connected, our analysis will show that conjugating by an element of G preserves the connected components of $C_k(O(n))$, so our results effectively compute the connected components of $R_k(O(n))$ and $R_k(SO(n))$.

The proofs will be based in the approach described in Section 2.4, and although we use it here mainly to calculate connected components this technique has been used to obtain a variety of results about the topology of the spaces of commuting tuples, some examples are [Bai07], [GPS12] and [PS13].

We will also show that the components of $C_k(O(n))$ and $C_k(SO(n))$ stabilize for sufficiently large values of n, which allows us to calculate the components of $C_k(O)$ and $C_k(SO)$, where O and SO are the direct limits of $O(n)$ and $SO(n)$ respectively under the inclusion maps. At the end of the chapter, we use the calculations for $SO(n)$ and Stiefel-Whitney classes to find a lower bound to the number of components of $C_k(\mathrm{Spin}(n))$ (for a sufficiently large n), where $\mathrm{Spin}(n)$ is the spinor group of dimension n, which is the connected double cover of $SO(n)$.

Up to this point, we will have only been looking at spaces of homomorphisms fixing the source group to be $\mathbb{Z}^k$ and changing the target group. In the second chapter, we will explore what happens when we try to take a more general group as a source group. Namely, we study the spaces of homomorphisms $\mathrm{Hom}(\Gamma, SU(2))$, where Γ is a central extension of the form

$$0 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z}^n \to 0.$$

Notice that we have restricted the target group to be $SU(2)$, this was one of the first groups to be chosen as a target when the spaces of commuting tuples began to be studied systematically in [AC07]. Since we are generalizing the source group, we should look at a target group for which we know enough about the space of commuting tuples to help us understand the case with a different source group. As mentioned before, the stable homotopy type of $C_k(SU(2))$ was found in [Cra11] and [BJS11] so this makes $SU(2)$ an ideal candidate for our purposes. This is also what motivates the class of source groups we are considering, these extensions are close enough to the free abelian case so that we can use what we know about $C_k(SU(2))$ to give a description of $\mathrm{Hom}(\Gamma, SU(2))$ good enough to be able to compute its cohomology.

The way we go about analyzing this space is by splitting it into two subspaces and then studying each piece separately. We have a decomposition

$$\mathrm{Hom}(\Gamma, SU(2)) = \mathrm{Hom}(\Gamma/[\Gamma, \Gamma], SU(2)) \cup R_\Gamma,$$

where R_Γ is the space of all the homomorphisms that do not factor through the abelianization of Γ. The first piece of this decomposition is analyzed in Section 2.1 using results by Adem and Gómez in [AG11]. Section 2.3 deals with R_Γ in the case when $r = 1$ and finally, the general case for r is done in Section 2.4.

In [PS13] it was shown that if K is a maximal compact subgroup of G, then the space $C_k(G)$ deformation retracts to $C_k(K)$, this means that our results about the number of connected components of $C_k(O(n))$ and $C_k(SO(n))$ can be translated verbatim to $C_k(GL(n, \mathbb{R}))$ and $C_k(SL(n, \mathbb{R}))$ since $O(n)$ and $SO(n)$ are the maximal compact subgroups of $GL(n, \mathbb{R})$ and $SL(n, \mathbb{R})$ respectively. More generally, in [Ber13], Bergeron proves that, given any nilpotent group N then $\mathrm{Hom}(N, G)$ deformation retracts to $\mathrm{Hom}(N, K)$. Since the groups considered in Chapter 2 are nilpotent, this means that we can also translate the results in Chapter 2 about $\mathrm{Hom}(\Gamma, SU(2))$ to their corresponding counterparts for $\mathrm{Hom}(\Gamma, SL(2, \mathbb{C}))$.

Each chapter begins with a summary of the precise statements of the main results proved in that chapter, as well as how they are organized within the chapter. Some of the proofs in Chapter 1 rely heavily on a few well known facts of linear algebra, and for the sake of completeness and self-containment of this text I have added their simple proofs as an appendix.

I have also added a small epilogue with the concluding remarks, where we mention some of the possible directions in which one would want to generalize the results of this thesis and state some of the complications that may arise.

Notation: Throughout this dissertation, the symbol $\mathbb{Z}_2$ will represent the multiplicative group with two elements $\{1, -1\}$ and $\mathbb{Z}/2\mathbb{Z}$ will be the additive group of 2 elements $\{0, 1\}$. I will use $|A|$ to denote the cardinality of the set A and the symbol $\binom{n}{k}$ is the number of ways to choose k elements from a set containing n elements.

Chapter 1

Spaces of commuting matrices.

In this chapter we we will enumerate the path connected components of the space of ordered commuting k-tuples of orthogonal and special orthogonal matrices respectively. We will also provide a lower bound on the number of connected components of the space of commuting elements in the spinor groups.

The first section will serve as an introduction to the study of spaces of homomorphisms in general and will present some known results. The second section presents one of the current strategies used for the general study of the space of commuting elements in a Lie Group.

In Section 1.3, we will use the strategy described in Section 1.2 to prove the main theorem in this chapter, which calculates the number of connected components of the space $C_k(O(n))$. The theorem is:

Theorem 1.0.1. *For each $n, k \in \mathbb{N}$, the space $C_k(O(n))$ has*

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2^k}{n-2j}$$

connected components.

Note that the formula no longer depends on n when $n \geq 2^k - 1$ since the binomial coefficients vanish when $n-2j$ is large enough. So we get the following corollary:

Corollary 1.0.2. *If $n \geq 2^k - 1$ then $C_k(O(n))$ has 2^{2^k-1} connected components.*

In fact, in this range, the inclusion $O(n) \hookrightarrow O(n+1)$ will induce an isomorphism at the π_0 level. So if we let $O = \underrightarrow{\operatorname{colim}}\, O(n)$ be the infinite orthogonal group, then $C_k(O)$ has 2^{2^k-1} connected components.

Although the group $O(n)$ is not connected, our analysis will show that conjugating by an element of G acts as the identity on π_0. As a result, we have the following corollary:

Corollary 1.0.3. *$R_k(O(n))$ and $C_k(O(n))$ have the same number of components.*

So we effectively calculate the number of connected components of $R_k(O(n))$ as well.

At the end of the section, we also make some progress in the direction of computing higher invariants of the spaces $C_k(O(n))$, in the form of proving that they have $\binom{2^k}{n}$ components homeomorphic to $O(n)/\mathbb{Z}_2^n$.

The space $C_k(SO(n))$ is equal to the intersection of $C_k(O(n))$ with $SO(n)^k$. In Section 1.4 we use this fact and Theorem 1.0.1 to calculate the number of connected components of $C_k(SO(n))$:

Corollary 1.0.4. *If $n < 2^k$, the space $C_k(SO(n))$ has*

$$\frac{1}{2^k} \sum_{j=0}^{\frac{n-1}{2}} \binom{2^k}{n-2j}$$

components when n is odd, and it has

$$\frac{1}{2^k} \sum_{m=0}^{\frac{n}{2}} \left(\binom{2^k}{n-2j} + (-1)^{\frac{n-2j}{2}} \left(2^k - 1\right) \binom{2^{k-1}}{\frac{n-2j}{2}} \right).$$

components when n is even. If $n \geq 2^k - 1$ it has 2^{2^k-k-1} components.

Similar to the case of $O(n)$, if we let $SO = \underrightarrow{\operatorname{colim}} SO(n)$ we obtain that $C_k(SO)$ has 2^{2^k-k-1} components. Since $SO(n)$ is connected, Corollary 1.0.4 also computes the number of connected components of the representation spaces $R_k(SO(n))$.

To finish the section we extend the computation of the cohomology of $C_k(SO(3))$ done in [TGS08] to $C_k(O(3))$.

In [AC07] Adem and Cohen found lower bounds for the number of components of $\operatorname{Hom}(\pi, O(n))$ for a general discrete finitely generated group π. Their approach was to separate components using the Stiefel-Whitney classes. In the case $\pi = \mathbb{Z}^k$ this approach does not give a sharp result, in Section 1.5 we will see that the Stiefel-Whitney classes are not enough to identify the components of $C_k(O(n))$. In other words, we prove the following corollary:

Corollary 1.0.5. *Suppose $k \geq 3$ and $n \geq 2^k - 1$. There exist homomorphisms lying in different components of $C_k(O(n))$ with the same total Stiefel-Whitney class, and so the total Stiefel-Whitney class does not distinguish all the components of $C_k(O(n))$.*

Even though the Stiefel-Whitney classes fail to distinguish the components completely, in Section 1.6 we will see how the second Steifel-Whitney class helps to find a bound for the components of $C_k(\mathrm{Spin}(n))$. We will prove the following:

Corollary 1.0.6. *If $n \geq 2^k - 1$ then $C_k(Spin(n))$ has at least $2^{2^k-k-1-\binom{k}{2}}$ connected components.*

This bound is sharp enough to see that the space of commuting triples in $\mathrm{Spin}(n)$ is disconnected if $n \geq 7$. This fact first appeared in [KS00].

1.1 Background and preliminaries

Let π be a finitely generated group and G any Lie group. Consider the set $\mathrm{Hom}(\pi, G)$. This set can be given a topology as follows. Let $\mathbb{F}_k$ be the free group on k generators and let $f : \mathbb{F}_k \to \pi$ be any surjection. This induces an inclusion $f^* : \mathrm{Hom}(\pi, G) \to \mathrm{Hom}(\mathbb{F}_k, G) = G^k$ so we can give $\mathrm{Hom}(\pi, G)$ the subspace topology. It is easy to see that this topology does not depend on the choice of k or f.

In the next paragraph we explain the relation between these spaces and bundle theory; in particular, how the connected components of these spaces could help us understand principal G-bundles over $B\pi$.

Given $f \in \mathrm{Hom}(\pi, G)$ we can apply the classifying space functor to get a continuous map $Bf \in Map_*(B\pi, BG)$. It is known that the classifying space functor is continuous as long as the compactly generated topology is used on the source, and so we can pass to connected components and get a map $B_0 : \pi_0(\mathrm{Hom}(\pi, G)) \to [B\pi, BG]$. Now recall that $[B\pi, BG]$ classifies all principal G-bundles over $B\pi$, this means that understanding $\pi_0(\mathrm{Hom}(\pi, G))$ and the map B_0 may help us understand principal G-bundles over $B\pi$. This is of particular interest for the cases when $B\pi$ is a compact manifold, for example when $\pi = \mathbb{Z}^k$.

The spaces $\mathrm{Hom}(\pi, G)$ have a G action induced by the conjugation action on G. Since inner automorphisms induce mappings homotopic to the identity on BG we have that B_0 factors through $\mathrm{Rep}(\pi, G) := \mathrm{Hom}(\pi, G)/G$. Some people prefer to study $\mathrm{Rep}(\pi, G)$ instead but note that if G is connected then $\mathrm{Rep}(\pi, G)$ has the same number of components as $\mathrm{Hom}(\pi, G)$.

1.1.1 The space of commuting tuples

In the particular case when $\pi = \mathbb{Z}^k$ the space $\mathrm{Hom}(\mathbb{Z}^k, G)$ can be identified with the space $C_k(G)$ of ordered commuting k-tuples in G^k, i.e. the set

$$C_k(G) = \{(g_1, \dots, g_k) \in G^k : g_i g_j = g_j g_i \text{ for all } i \neq j\}.$$

We do this by identifying $f \in \mathrm{Hom}(\mathbb{Z}^k, G)$ with $(f(e_1), \dots, f(e_k)) \in C_k(G)$, where $\{e_1, \dots, e_k\}$ is the standard basis for $\mathbb{Z}^k$. In the sequel, we will identify these two spaces without explicitly mentioning it. In this case, $B\pi = (\mathbb{S}^1)^k$ and by studying $C_k(G)$ we will be getting information about principal G-bundles over the k-torus.

It is worth mentioning that the space $C_k(G)$ may have many components even when G is connected and simply connected. Such an example was given by Kac and Smilga in [KS00] where they showed that $C_3(\mathrm{Spin}(7))$ is not connected. Here is a condition which guarantees the connectedness of $\mathrm{Hom}(\mathbb{Z}^k, G)$:

Proposition 1.1.1. *If every abelian subgroup of G is contained in a maximal torus then $C_k(G)$ is path connected.*

This proposition was proved in [AC07]. We will give an alternate proof of this proposition in the following section as an example of an easy application of our approach.

As a consequence of this proposition we have the following corollary.

Corollary 1.1.2. *For $G = SU(n), U(n), Sp(n)$ the spaces $C_k(G)$ are path connected.*

This follows from the known fact that maximal abelian subgroups in the given groups are precisely the maximal tori (see for example [BtD95]).

More recently, in [AG12], Adem and Gómez have described all compact simply connected Lie groups that have connected $C_k(G)$ for all k as the finite products of $Sp(n)$'s and $SU(m)$'s.

The final goal is to compute $\pi_0(C_k(G))$ for other kinds of groups G. This problem was solved for $G = SO(3)$ by Torres-Giese and Sjerve in [TGS08] and by Adem, Cohen and Gómez for central products of special unitary groups in [ACG13]. In the following sections we will present a complete solution to this problem in the case when $G = O(n), SO(n)$ generalizing the result in [TGS08], we also find a lower bound for the case $G = \mathrm{Spin}(n)$.

1.1.2 The case $G = O(n)$

Before restricting ourselves to studying $C_k(O(n))$ we state here what is known about the components of $\mathrm{Hom}(\pi, O(n))$ for a general finitely generated group π. These results are due to Adem and Cohen in [AC07] and they are obtained by separating components of $\mathrm{Hom}(\pi, O(n))$ using the first and second Stiefel-Whitney classes.

Theorem 1.1.3. *There is a decomposition of $Hom(\pi, O(n))$ into closed and open subspaces*

$$Hom(\pi, O(n)) = \coprod_{w \in H^1(\pi, \mathbb{F}_2)} Hom(\pi, O(n))_w$$

where $Hom(\pi, O(n))_w$ is the subset of $Hom(\pi, O(n))$ of homomorphisms with first Stiefel-Whitney class equal to w.

Theorem 1.1.4. *If π is a finitely generated discrete group and*

$$H^2(\pi/[\pi, \pi]; \mathbb{F}_2) \to H^2(\pi; \mathbb{F}_2)$$

is surjective, then

$$|\pi_0(Hom(\pi, O(n)))| \geq |H^1(\pi; \mathbb{F}_2)||H^2(\pi; \mathbb{F}_2)|.$$

We will show in Section 1.5 that not even the total Stiefel-Whitney class is able to identify the components of $C_k(O(n))$, which means that bounds obtained in this fashion cannot be sharp in general.

1.2 A general strategy to study $C_k(G)$

In this section we will introduce a general strategy to approach the problem of studying the spaces $C_k(G)$ for general G. This strategy has proven fruitful and has been used to obtain a variety of results; some examples are [Bai07], [GPS12] and [PS13].

The idea is to look at the following maps: for every abelian subgroup A of G we have maps

$$\Phi_A : G \times A^k \to C_k(G)$$

defined by

$$(g, a_1, \ldots, a_k) \mapsto (ga_1g^{-1}, \ldots, ga_kg^{-1}).$$

The space $G \times A^k$ has an $N(A)$-action given by

$$n \cdot (g, a_1, \ldots, a_k) = (gn^{-1}, na_1n^{-1}, \ldots, na_kn^{-1})$$

where $N(A)$ is the normalizer of A in G. The map Φ_A i s invariant on $N(A)$ orbits so it induces a map on the quotient:

$$\tilde{\Phi}_A : G \times_{N(A)} A^k \to C_k(G).$$

The space $G \times_{N(A)} A^k$ is homeomorphic to $G/A \times_{W(A)} A^k$, where $W(A) = N(A)/A$ is the Weyl group of A in G.

Let $\mathcal{S}_A = \mathrm{Im}(\Phi_A)$. Note that this is just the orbit of A^k under the conjugation action of G on $C_k(G)$. Then we have the following proposition, which is immediate from the definitions.

Proposition 1.2.1. *The following statements hold for all abelian subgroups A and A' in G:*

1. *If $A' \subset A$ then $\mathcal{S}_{A'} \subset \mathcal{S}_A$.*

2. $\mathcal{S}_{gAg^{-1}} = g\mathcal{S}_A g^{-1} = \mathcal{S}_A$.

3. *If $f \in C_k(G)$ and $A = Im(f)$ then $f \in \mathcal{S}_A$.*

From these properties we have the following direct corollary.

Corollary 1.2.2. *There is a decomposition*

$$C_k(G) = \bigcup_{A \in I} \mathcal{S}_A$$

where I is a set of representatives of conjugacy classes of maximal abelian subgroups of G.

This way we have obtained a decomposition of $C_k(G)$ into smaller spaces and we can use a "Mayer-Vietoris" approach to the problem by analyzing the smaller pieces and their intersections.

The relevant thing is that in general, even though the map $\tilde{\Phi}_A : G/A \times_{W(A)} A^k \to \mathcal{S}_A$ is not injective, we can use $\tilde{\Phi}_A$ to translate information about $G/A \times_{W(A)} A^k$ to information about $\mathcal{S}_A$, which has the advantage that the space $G/A \times_{W(A)} A^k$ may be much easier to understand than $C_k(G)$.

Another good thing is that in many cases, for example when G is compact, the set I can be taken finite which significantly simplifies the analysis. For a detailed description of the "many cases" of the previous sentence, one should look at [PS13].

This approach however does require some understanding of the abelian subgroups of G, or at least of the maximal conjugacy classes of abelian subgroups. In the next sections we will apply this to compute the connected components in the cases $G = O(n), SO(n)$, where we have a good understanding of conjugacy classes of maximal abelian subgroups, or a so called "normal form" for abelian subgroups. We should emphasize that even though this is mainly used here to compute connected components this strategy may be useful to compute higher invariants of these spaces. For example in Proposition 1.3.13 we use this approach to compute the homeomorphism type of some of the components of $C_k(O(n))$.

Note that if A is a maximal abelian subgroup then $G \times A^k$ is a manifold, on which path components and connected components agree. It follows that the path components and connected components of $\mathcal{S}_A$ agree. If G is compact this implies that the path components and connected components of $C_k(G)$ also agree.

As an example of an easy application of this approach we give an alternate proof of Proposition 1.1.1.

Proof of Proposition 1.1.1. The hypothesis clearly implies that G is connected. Since all maximal tori are conjugate and every abelian subgroup is contained in a maximal torus by hypothesis, we can take I from Corollary 1.2.2 to be $I = \{T\}$, where T is any maximal torus. Then $C_k(G) = \mathcal{S}_T = \Phi_T(G \times T^k)$ and so it is connected since $G \times T^k$ is connected. $\square$

1.3 The connected components of $C_k(O(n))$

In this section we will apply the strategy described in the previous section to enumerate the connected components of the space $C_k(O(n))$ for all k and n. Our goal is to prove Theorem 1.0.1, which we restate here:

Theorem 1.0.1. *For each $n, k \in \mathbb{N}$, the space $C_k(O(n))$ has*

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2^k}{n-2j}$$

connected components.

The strategy to prove Theorem 1.0.1 is to obtain a decomposition of $C_k(O(n))$ into closed and open disjoint subspaces, and then count the number of components of each of these subspaces. Theorem 1.0.1 follows directly from the next theorem:

Theorem 1.3.1. *There is a decomposition of $C_k(O(n))$ into closed and open subspaces:*

$$C_k(O(n)) = \coprod_{j=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{U}_j,$$

and for each j, the subspace $\mathcal{U}_j$ has $\binom{2^k}{n-2j}$ connected components.

We will construct the subspaces $\mathcal{U}_j$ using the techniques described in the previous section. The rest of this section is devoted to the proof of Theorem 1.3.1.

Consider the subgroup $A_j = (\mathbb{S}^1)^j \times (\mathbb{Z}_2)^{n-2j} \subset O(n)$ where the correspondence is

$$(\theta_1, \dots, \theta_j, t_1, \dots, t_{n-2j}) \leftrightarrow \begin{pmatrix} M_{\theta_1} & 0 & & \dots & & 0 \\ 0 & M_{\theta_2} & & & & \\ & & \ddots & & & \\ \vdots & & & M_{\theta_j} & & \vdots \\ & & & & t_1 & \\ 0 & & & & \ddots & 0 \\ & & \dots & & 0 & t_{n-2j} \end{pmatrix},$$

where

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is the 2×2 matrix corresponding to a clock-wise rotation of an angle θ.

For each $j = 1, \dots, \lfloor \frac{n}{2} \rfloor$ we have maps

$$\Phi_j : O(n) \times A_j^k \to C_k(O(n))$$

given by

$$\Phi_j(g, a_1, \dots, a_k) = (ga_1g^{-1}, \dots, ga_kg^{-1}).$$

It is well known that the A_j's are representatives of conjugacy classes of maximal abelian subgroups of $O(n)$. Or in other words, we have that if $\mathcal{S}_j = \mathrm{Im}(\Phi_j)$ then

$$C_k(O(n)) = \bigcup_{j=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{S}_j$$

The sets $\mathcal{S}_j$ are certainly not disjoint, but we will produce our sets $\mathcal{U}_j$ using the maps Φ_j.

Notation: For the remainder of this section we let $(e_1, \dots, e_n)$ be the standard ordered basis for $\mathbb{R}^n$. We will denote the span of the vectors $v_1, \dots, v_l$ by $\langle v_1, \dots, v_l \rangle$.

Definition 1.3.2. Given $a = (a_1, \dots, a_k) \in A_j^k$ we can write each coordinate like:

$$a_m = (\theta_{m,1}, \dots, \theta_{m,j}, t_{m,1} \dots, t_{m,n-2j}),$$

and given $i \in \{1, \dots, n-2j\}$ we let

$$r_i(a) := (t_{1,i}, \dots t_{k,i}) \in \mathbb{Z}_2^k.$$

The vector $r_i(a)$ is the vector of eigenvalues of the common eigenvector e_{2j+i} for the a_m's. We also define the following subset of A_j^k:

$$P_j := \{a \in A_j^k : r_i(a) \neq r_{i'}(a) \text{ whenever } i \neq i'\}$$

and we let

$$\mathcal{U}_j := \Phi_j(O(n) \times P_j).$$

The geometric description of the sets $\mathcal{U}_j$ is as follows: recall that every representation $\rho : A \to O(n)$ from a finitely generated abelian group A splits up as a direct sum of orientable 2-dimensional representations and line bundles. This follows from the fact that commuting complex matrices are simultaneously diagonalizable. The set $\mathcal{U}_j$ consists of the representations that have a direct sum decompositions in which j summands are 2-dimensional orientable representations but do not have a direct sum decompositions in which $j+1$ summands are 2-dimensional orientable representations.

These $\mathcal{U}_j$'s are the sets involved in Theorem 1.3.1. We shall prove that they are closed and disjoint, and that they have the claimed number of connected components.

Note that for every $g \in O(n)$ and every j we have $g\mathcal{U}_j g^{-1} = \mathcal{U}_j$ so these sets are invariant under the conjugation action on $C_k(O(n))$.

We begin the proof of Theorem 1.3.1 with the following proposition.

Proposition 1.3.3. *Let $a \in P_j$, $b \in P_{j'}$ and $g \in O(n)$ such that $gag^{-1} = b$. Then $j = j'$ and $\{r_1(a), \ldots, r_{n-2j}(a)\} = \{r_1(b), \ldots, r_{n-2j}(b)\}$.*

Since the proof is slightly technical we will first introduce some preliminary definitions and examples that will clarify the proof.

Definition 1.3.4. Let $a = (a_1, \ldots, a_k) \in A_j^k$, where each coordinate is

$$a_i = (\theta_{i,1}, \ldots, \theta_{i,j}, t_{i,1} \ldots, t_{i,n-2j}).$$

We will say a is *m-ordered* if $m \in \{0, \ldots, j\}$ is such that:

1. For every $l \in \{m+1, \ldots, j\}$ and $i \in \{1, \ldots, k\}$ it holds that $\theta_{i,l} \in \{0, \pi\}$.
2. For every $l \in \{1, \ldots, m\}$ there exists an $i \in \{1, \ldots, k\}$ such that $\theta_{i,l} \notin \{0, \pi\}$.

Part (1) of this definition means that the only nonzero entries of each a_i below row $2m$ are ± 1 in the diagonal. Part (2) of this definition means that the matrices $(a_1, \ldots, a_k)$ have no common eigenvectors in the span of $(e_1, \ldots, e_{2m})$. Note that an element of A_k^j is m-ordered for at most one $m \in \{0, \ldots, j\}$, but it may be that it is not m-ordered for any of them. Below are some examples to further clarify this definition:

Example 1.3.5. Let $k = 3$, $n = 5$ and $j = 2$. Let

$$a = ((\pi/2, 0, 1), (\pi/4, \pi, -1), (\pi/6, 0, 1)) \in A_2^3,$$

which corresponds to the triple of matrices

$$\left(\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & 0 & \\ & & 0 & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & & & \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & & \\ & & -1 & 0 & \\ & & 0 & -1 & \\ & & & & -1 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & & & \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & & & \\ & & 1 & 0 & \\ & & 0 & 1 & \\ & & & & 1 \end{pmatrix} \right).$$

The triple a is 1-ordered. Let

$$b = ((\pi, \pi/2, 1), (0, \pi/3, -1), (0, 0, -1)) \in A_2^3,$$

which corresponds to the triple of matrices

$$\left(\begin{pmatrix} -1 & 0 & & & \\ 0 & -1 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & \frac{1}{2} & \frac{-\sqrt{3}}{2} & \\ & & \frac{\sqrt{3}}{2} & \frac{1}{2} & \\ & & & & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & 1 & 0 & \\ & & 0 & 1 & \\ & & & & -1 \end{pmatrix} \right).$$

The triple b is not m-ordered for any m since e_1 and e_2 are common eigenvectors of the 3 matrices and there are nonzero entries away from the diagonal below their corresponding rows. Let

$$c = ((\pi/2, \pi, -1), (\pi, \pi/3, -1), (\pi/6, 0, 1)) \in A_2^3,$$

which corresponds to the triples of matrices:

$$\left(\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & -1 & 0 & \\ & & 0 & -1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & & & \\ 0 & -1 & & & \\ & & \frac{1}{2} & \frac{-\sqrt{3}}{2} & \\ & & \frac{\sqrt{3}}{2} & \frac{1}{2} & \\ & & & & -1 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & & & \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & & & \\ & & 1 & 0 & \\ & & 0 & 1 & \\ & & & & 1 \end{pmatrix} \right).$$

The triple c is 2-ordered.

Looking at b we see that not all elements $x \in A_j^k$ are m-ordered for some m. However, we can always conjugate x by a permutation matrix to get $x' \in A_j^k$ which is m-ordered for some m. In the case of the example above we can conjugate b to get the triple

$$b' = \left(\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & -1 & 0 & \\ & & 0 & -1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & & & \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & & & \\ & & 1 & 0 & \\ & & 0 & 1 & \\ & & & & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & 1 & 0 & \\ & & 0 & 1 & \\ & & & & -1 \end{pmatrix} \right),$$

which is 1-ordered. Notice that we only have to permute some of the first $2j$ vectors of the standard basis and so this process does not alter the value of any r_i, since these only involve the last $n - 2j$ rows of the matrices. This is summarized in the following lemma:

Lemma 1.3.6. *Given $a \in A_j^k$ there exists a permutation matrix $g \in O(n)$ such that $a' = gag^{-1} \in A_j^k$ is m-ordered for some m and $r_i(a) = r_i(a')$ for all $i \in \{1, \dots, n - 2j\}$.*

□

We need one more definition before going into the proof of Proposition 1.3.3:

Definition 1.3.7. Let $a = (a_1, \dots, a_k) \in A_j^k$ be m-ordered. Then we define the *eigenmatrix* of a to be the $(n - 2m) \times k$ matrix X with entries ± 1 determined by the equations:

$$a_i(e_{2m+l}) = X_{l,i} e_{2m+l}.$$

Example 1.3.8. If a, b' and c are as in Example 1.3.5 the the eigenmatrix of a is

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix},$$

the eigenmatrix of b' is

$$\begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix},$$

and the eigenmatrix of c is

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}.$$

Notice that if $a \in A_j^k$ is m-ordered and X is its eigenmatrix then the vectors $r_1(a), \ldots, r_{n-2j}(a)$ are the last $n-2j$ rows of X. Also notice that the first $2(j-m)$ rows of X appear by pairs since they come from rotations of angles 0 or π. If $a \in P_j$ then all of the $r_i(a)$ are different, which means that they are the only rows of X that appear an odd number of times. So we have the following lemma:

Lemma 1.3.9. *Let $a \in P_j$ be m-ordered and let X be its eigenmatrix, then the set $\{r_1(a), \ldots, r_{n-2j}(a)\}$ is the set of rows of X that appear an odd number of times in X.*

Now we are ready to prove Proposition 1.3.3.

Proof of Proposition 1.3.3. Let $a = (a_1, \ldots, a_k) \in P_j$ and $b = (b_1, \ldots, b_k) \in P_{j'}$. By Lemma 1.3.6 we may assume without loss of generality that a is m-ordered and b is m'-ordered.

Let $E_i^{\pm}$ be the ± 1-eigenspace of a_i and let $F_i^{\pm}$ be the ± 1-eigenspace of b_i. Since $g a_i g^{-1} = b_i$ we know that $g(E_i^{\pm}) = F_i^{\pm}$ for all i. This implies that

$$g\left(\bigcap_{i=1}^{k}(E_i^+ \oplus E_i^-)\right) = \bigcap_{i=1}^{k}(F_i^+ \oplus F_i^-).$$

Since a is m-ordered we know that

$$\bigcap_{i=1}^{k}(E_i^+ \oplus E_i^-) = \langle e_{2m+1}, \ldots, e_n \rangle$$

and since b is m'-ordered we know that

$$\bigcap_{i=1}^{k}(F_i^+ \oplus F_i^-) = \langle e_{2m'+1}, \dots, e_n \rangle.$$

This implies that $m = m'$ and that g preserves $\langle e_{2m+1}, \dots, e_n \rangle$. In other words, g has the form:

$$\begin{pmatrix} \star & 0 \\ 0 & U \end{pmatrix}$$

where U is a $(n-2m) \times (n-2m)$ non-degenerate matrix.

Let X be the eigenmatrix of a and Y be the eigenmatrix of b. We claim that X and Y have the same rows up to permutation. This would finish the proof since we know by Lemma 1.3.9 that the sets $\{r_1(a), \dots, r_{n-2j}(a)\}$ and $\{r_1(b), \dots, r_{n-2j}(b)\}$ are the sets of rows that appear an odd number of times in X and Y respectively.

If $(u_{p,l})$ are the entries of U then for $l \in \{1, \dots, n-2m\}$ we have

$$ge_{2m+l} = \sum_{p=1}^{n-2m} u_{p,l} e_{2m+p}.$$

For $i \in \{1, \dots, k\}$ we can calculate:

$$b_i g e_{2m+l} = a_i \left(\sum_{p=1}^{n-2m} u_{p,l} e_{2m+p} \right) = \sum_{p=1}^{n-2m} u_{p,l} Y_{p,i} e_{2m+p}$$

$$g a_i e_{2m+l} = g(X_{l,i} e_{2m+l}) = \sum_{p=1}^{n-2m} u_{p,l} X_{l,i} e_{2m+p}.$$

Since $b_i g = g a_i$ then $u_{p,l} Y_{p,i} = u_{p,l} X_{l,i}$ for every p,i and l. In particular, if $u_{p,l} \neq 0$ then the p-th row of Y equals the l-th row of X. Since g is an isomorphism this implies that for every $p \in \{1, \dots, n-2m\}$ there is an $l \in \{1, \dots, n-2m\}$ such that $u_{l,p} \neq 0$, so each row on X appears at least once in Y.

Now suppose the rows $X_{l_1,\bullet}, \dots, X_{l_q,\bullet}$ are equal, with $l_i \neq l_{i'}$ if $i \neq i'$. We know that the dimension of $g(\langle e_{l_1}, \dots, e_{l_q} \rangle)$ is q, which means that the space generated by the columns $u_{\bullet,l_1}, \dots, u_{\bullet,l_q}$ has dimension q. This means that there are at least q values of p for which there is an r such that $u_{p,l_r} \neq 0$, *i.e.* the row $X_{l_1,\bullet}$ appears at least q times in X. Since X and Y have the same number of rows this shows that X and Y have the same rows up to permutation.

As we said before, by Lemma 1.3.9 the rows that appear an odd number of times in the matrix X are $\{r_1(a), \dots, r_{n-2j}(a)\}$. Similarly the rows that appear an odd number of times in Y are $\{r_1(b), \dots, r_{n-2j'}(b)\}$. We have proved that X and Y have the same rows so the proposition follows. $\square$

Proposition 1.3.10. *The spaces $\mathcal{U}_j$ are all closed and satisfy*

$$C_k(O(n)) = \coprod_{j=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{U}_j.$$

Proof. The condition $r_i(a) \neq r_{i'}(a)$ is closed in A_j^k for each pair (i, i'). So P_j is an intersection of closed subsets of A_j^k which is compact. This shows that P_j is compact. Since $\mathcal{U}_j = \Phi_j(O(n) \times P_j)$ then $\mathcal{U}_j$ is compact, and hence closed as required.

Now we will show that the $\mathcal{U}_j$'s cover $C_k(O(n))$. Since the A_j's are representatives of maximal conjugacy classes of abelian subgroups of $O(n)$ we can conjugate any commuting tuple into some A_j^k. And since the $\mathcal{U}_j$'s are invariant under conjugation it is enough to show that they cover every A_j^k . Suppose then that $a = (a_1, \dots, a_k) \in A_j^k$ with $a_i = (\theta_{i,1}, \dots, \theta_{i,j}, t_{i,1} \dots, t_{i,n-2j})$. If $(t_{1,i}, \dots t_{k,i}) \neq (t_{1,i'}, \dots t_{k,i'})$ whenever $i \neq i'$ then $a \in P_j$. Otherwise, there exist $i \neq i'$ such that $(t_{1,i}, \dots t_{k,i}) = (t_{1,i'}, \dots t_{k,i'})$, which means that we can conjugate a into A_{j+1}^k by simply reordering the rows to get $\theta_{i,j+1} = 0$ or π for all i. We can repeat the process until we find a number j' and a $g \in O(n)$ for which $gag^{-1} \in P_{j'}$. We finish this procedure because $P_{\lfloor \frac{n}{2} \rfloor} = A^k_{\lfloor \frac{n}{2} \rfloor}$. It follows that $a \in \mathcal{U}_{j'}$ and shows that

$$C_k(O(n)) = \bigcup_{j=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{U}_j.$$

To show that the union is disjoint suppose $x \in \mathcal{U}_j \cap \mathcal{U}_{j'}$. Then $x = gag^{-1} = hbh^{-1}$ for some $g, h \in O(n)$, $a \in P_j$ and $b \in P_{j'}$. This implies $h^{-1}ga(h^{-1}g)^{-1} = b$ and so $j = j'$ by Proposition 1.3.3.

□

Since there are finitely many $\mathcal{U}_j$'s it follows from Proposition 1.3.10 that each $\mathcal{U}_j$ is open and closed in $C_k(O(n))$ and so each $\mathcal{U}_j$ is a union of components. All that is left to do then is count the components of $\mathcal{U}_j$ for each j.

It is important in what follows to understand the action of the Weyl group $W(A_j) = N(A_j)/A_j$ on A_j by conjugation, where $N(A_j)$ is the normalizer of A_j in $O(n)$. We explain this action in the following paragraphs.

Recall that A_j admits a $N(A_j)$ conjugation action. Since A_j acts by the identity this induces an action of $W(A_j) = N(A_j)/A_j$ on A_j.

If we think about a matrix in $O(n)$ as a linear map on $\mathbb{R}^n$, then conjugating corresponds to changing the basis on which we write the map. If $(e_1, \ldots, e_n)$ is the standard basis for $\mathbb{R}^n$ and $c \in N(A_j)$ then ce_{2j+1} is a unit vector on which every element of A_j acts as ± 1, but the only unit vectors with that property are $\{\pm e_{2j+1}, \ldots, \pm e_n\}$ so ce_{2j+1} must be one of them. The same goes for $e_{2j+2}, \ldots, e_n$ and so c permutes the set $\{\pm e_{2j+1}, \ldots, \pm e_n\}$. A similar analysis gives that c maps the plane generated by $\{e_{2l-1}, e_{2l}\}$ isometrically to some other plane of that form for $l = 1, \ldots, \frac{j}{2}$. And so the action by conjugation on A_j is given in coordinates as above by:

$$c \cdot (\theta_1, \ldots, \theta_j, t_1, \ldots, t_{n-2j}) = ((-1)^{s_1}\theta_{\sigma(1)}, \ldots, (-1)^{s_j}\theta_{\sigma(j)}, t_{\xi(1)}, \ldots, t_{\xi(n-2j)})$$

where $\sigma \in \Sigma_j$ and $\xi \in \Sigma_{n-2j}$ are permutations and $s_l = 0, 1$. We now proceed to count the components of $\mathcal{U}_j$.

The action of $N(A_j)$ on A_j induces an action on $O(n) \times A_j^k$ given by

$$c \cdot (g, a_1, \ldots, a_k) = (gc^{-1}, ca_1c^{-1}, \ldots, ca_1c^{-1}).$$

Since A_j acts on the coordinate A_j^k by the identity this induces an action of $W(A_j) = N(A_j)/A_j$ on $O(n)/A_j \times A_j^k$.

From the paragraphs above it follows that if $c \in N(A_j)$ and $a \in A_j^k$ then

$$(r_1(cac^{-1}), \ldots, r_{n-2j}(cac^{-1})) = (r_{\xi(1)}(a), \ldots, r_{\xi(n-2j)}(a))$$

for some $\xi \in \Sigma_{n-2j}$ depending only on $[c] \in W(A_j)$. This implies that the $W(A_j)$ action restricts to P_j and to $O(n)/A_j \times P_j$. Recall that Φ_j is constant on the orbits of this action, so it induces a surjective map

$$\tilde{\Phi}_j : O(n)/A_j \times_{W(A_j)} P_j \to \mathcal{U}_j.$$

Lets call the projection maps

$$q : O(n) \times P_j \to O(n)/A_j \times P_j$$

and

$$p : O(n)/A_j \times P_j \to O(n)/A_j \times_{W(A_j)} P_j.$$

We want to count the components of $\mathcal{U}_j$ by counting the components of $O(n)/A_j \times_{W(A_j)} P_j$ and then showing that $\mathcal{U}_j$ has the same number of components.

Proposition 1.3.11. *The space $O(n)/A_j \times_{W(A_j)} P_j$ has $\binom{2^k}{n-2j}$ connected components, indexed by the subsets $\{r_1, \ldots, r_{n-2j}\} \subset \mathbb{Z}_2^k$ of size $n - 2j$. In fact, given $(g, a) \in O(n) \times P_j$ the connected component containing $q(p(g, a))$ is the one corresponding to the set $\{r_1(a), \ldots, r_{n-2j}(a)\}$.*

Proof. The set $L_j := \{(r_1, \ldots, r_{n-2j}) \in (\mathbb{Z}_2^k)^{n-2j} : r_m \neq r_l \text{ for all } m \neq l\}$ is a discrete set with $\frac{2^k!}{(2^k-n+2j)!}$ points, and $O(n)$ has 2 components. Since $P_j \cong (\mathbb{S}^1)^{jk} \times L_j$ then $O(n) \times P_j$ has exactly $2 \times \frac{2^k!}{(2^k-n+2j)!}$ components. It remains to see how q and p glue these components.

If $j = \frac{n}{2}$ then $P_j = A_j^k \cong (\mathbb{S}^1)^{jk}$ is connected. If $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$ then matrix which sends e_1 to $-e_1$ and is the identity on $\langle e_1 \rangle^\perp$ has determinant -1 and is in the normalizer of A_j. Then the set $O(n)/A_j \times_{W(A_j)} P_j$ is connected and so it has $\binom{2^k}{n-2j} = \binom{2^k}{0} = 1$ components as required.

If $j \neq \frac{n}{2}$ then A_j contains at least one element of determinant -1 and so $O(n)/A_j$ is connected. This means that p glues components by pairs and so $O(n)/A_j \times P_j$ has $\frac{2^k!}{(2^k-n+2j)!}$ components. Notice that these components are indexed by the set L_j. Recall that for all $a \in P_j$ and $[c] \in W(A_j)$ we have that $(r_1(a), \ldots, r_{n-2j}(a)) = (r_{\xi(1)}(cac^{-1}), \ldots, r_{\xi(n-2j)}(cac^{-1}))$ for some $\xi \in \Sigma_{n-2j}$. By definition of P_j this shows that each $W(A_j)$-orbit in $O(n)/A_j \times P_j$ intersects exactly $(n-2j)!$ connected components. This shows that $O(n)/A_j \times_{W(A_j)} P_j$ has $\frac{2^k!}{(2^k-n+2j)!(n-2j)!} = \binom{2^k}{n-2j}$ connected components as required, indexed by the subsets $\{r_1, \ldots, r_{n-2j}\} \subset \mathbb{Z}_2^k$ of size $n-2j$. $\square$

To finish the proof of our main theorem it remains to show the following proposition.

Proposition 1.3.12. *The subspace $\mathcal{U}_j$ has $\binom{2^k}{n-2j}$ connected components, indexed by the subsets $\{r_1, \ldots, r_{n-2j}\} \subset \mathbb{Z}_2^k$ of size $n-2j$.*

Proof. From Proposition 1.3.11 it is enough to show that if

$$(g, a), (h, b) \in O(n) \times P_j$$

are such that $\Phi_j(g, a) = \Phi_j(h, b)$, then

$$\{r_1(a), \ldots, r_{n-2j}(a)\} = \{r_1(b), \ldots, r_{n-2j}(b)\}$$

(i.e. the map $\tilde{\Phi}_j$ is injective on π_0). This is a direct application of Proposition 1.3.3 by recalling that by the definition of Φ_j the fact that $\Phi_j(g, a) = \Phi_j(h, b)$ implies that

$$h^{-1}ga(h^{-1}g)^{-1} = b.$$

$\square$

This completes the proof of Theorem 1.3.1 and in turn proves Theorem 1.0.1.

In conclusion, this analysis shows that the components of $C_k(O(n))$ are indexed by the subsets of $A \subset \mathbb{Z}_2^k$ such that $|A| \equiv n(mod2)$ and $|A| \leq n$. The inclusion $O(n) \hookrightarrow O(n+1)$ induces a map

$$\pi_0(C_k(O(n))) \to \pi_0(C_k(O(n+1))).$$

Using our description of the components this map is very easy to describe, it sends the component corresponding to $A \subset \mathbb{Z}_2^k$ to the component corresponding to the subset $A \triangle \{(1, \ldots, 1)\}$, where $\triangle$ denotes the symmetric difference of sets.

From this it follows easily that if $n \geq 2^k - 1$ the inclusion induces a bijection on π_0. This shows that

$$|\pi_0(C_k(O))| = 2^{2^k - 1}$$

as stated in Corollary 1.0.2.

Proposition 1.3.3 shows that conjugating by an element of $O(n)$ acts as the identity on $\pi_0(C_k(O(n)))$. This shows that $R_k(O(n))$ has the same number of connected components as $C_k(O(n))$. This fact was stated in Corollary 1.0.3.

We finish the section by identifying the homeomorphism type of some of the components of $C_k(O(n))$.

Proposition 1.3.13. *Each component of $\mathcal{U}_0$ is homeomorphic to $O(n)/\mathbb{Z}_2^n$.*

Proof. The space P_0 is discrete, since it is a subspace of $(A_0)^k$ which is discrete ($\cong \mathbb{Z}_2^{nk}$). Let $p \in P_0$. With arguments similar to the ones used in Proposition 1.3.3, it is easy to see that the stabilizer of p under the conjugation action of $O(n)$ is exactly A_0, the group with ± 1 as diagonal entries and zeros everywhere else. This group is clearly isomorphic to $\mathbb{Z}_2^n$.

The set $\Phi_0(O(n) \times \{p\})$ is the orbit of p in $C_k(O(n))$ under the conjugation action of $O(n)$ and so it is homeomorphic to $O(n)/Z(p) = O(n)/\mathbb{Z}_2^n$. All we have to argue now is that this is the full component of $C_k(O(n))$ corresponding to $\{r_1(p), \ldots, r_n(p)\}$. For this, we look at the following commuting diagram:

$$\begin{tikzcd} O(n) \times P_0 \arrow[r, "\Phi_0"] \arrow[d] & \mathcal{U}_0 \\ O(n)/A_0 \times_{W_0} P_0 \arrow[ur, "\tilde{\Phi}_0"'] & \end{tikzcd} .$$

We know that the map $\tilde{\Phi}_0$ sends components of $O(n)/A_0 \times_{W_0} P_0$ onto components of $\mathcal{U}_0$, so it is enough to show that the image of $O(n) \times \{p\}$ under the quotient map is a component of $O(n)/A_0 \times_{W_0} P_0$.

Note that all the elements of A_0^k are 0-ordered by default, and can be identified with their corresponding eigenmatrix. By definition, P_0 is precisely the subset of A_0^k consisting of elements that have eigenmatrix without repeating rows. The way in which $W(A_0)$ acts on A_0^k is by permuting the rows of the eigenmatrices, so we see that $W(A_0)$ acts freely on P_0, which is discrete. This implies that $O(n)/A_0 \times \{p\}$ is homeomorphic to its image in $O(n)/A_0 \times_{W(A_0)} P_0$ and that the image of $O(n)/A_0 \times \{p\}$ is one of the components of $O(n)/A_0 \times_{W(A_0)} P_0$. This finishes the proof.

$\square$

Proposition 1.3.14. *All of the components of $\mathcal{U}_{[\frac{n}{2}]}$ are homeomorphic to each other. In particular, they are all homeomorphic to the component containing the trivial homomorphism.*

Proof. Note that if n is even $\mathcal{U}_{[\frac{n}{2}]}$ has only one component so there is nothing to prove. If n is odd $\mathcal{U}_{[\frac{n}{2}]}$ has 2^k components indexed by the elements of $\mathbb{Z}_2^k$. In this case $A_{[\frac{n}{2}]} = P_{[\frac{n}{2}]} \cong (\mathbb{S}^1)^{[\frac{n}{2}]} \times \mathbb{Z}_2$. Let $t = (t_1, \ldots, t_k) \in \mathbb{Z}_2^k$. The map Φ_j restricts to a surjection

$$\Phi_j : O(n) \times (\mathbb{S}^1)^{[\frac{n}{2}]k} \times \{t\} \to K_t$$

where K_t is the component of $C_k(O(n))$ corresponding to t. Let $\underline{1} = (1, \ldots, 1) \in \mathbb{Z}_2^k$. We have a homeomorphism

$$f_t : O(n) \times (\mathbb{S}^1)^{k[\frac{n}{2}]} \times \{t\} \to O(n) \times (\mathbb{S}^1)^{k[\frac{n}{2}]} \times \{\underline{1}\}.$$
$$(g, (\theta_{ij}), t) \mapsto (g, (t_j\theta_{ij}), \underline{1})$$

This homeomorphism induces a homeomorphism between K_t and $K_{\underline{1}}$. $\square$

1.4 The connected components of $C_k(SO(n))$

In this section we will use what we know so far to count the components of the space $C_k(SO(n))$. Given our enumeration of the components of $C_k(O(n))$ we can reduce the problem to a combinatorial problem regarding matrices with ± 1 entries. We then proceed to solve the combinatorial problem by showing that the numbers we are looking for satisfy a recurrence relation. We will also calculate the cohomology ring of $C_k(O(3))$ with coefficients in a field with characteristic not equal to 2. This is a simple application of the results in [Bai07] and [TGS08].

Let's begin by seeing how to translate our problem to a combinatorial one. Note that

$$C_k(SO(n)) = SO(n)^k \cap C_k(O(n))$$

and since $SO(n)$ is closed and open in $O(n)$ then $C_k(SO(n))$ is a union of components of $C_k(O(n))$.

Let $\mathcal{B}_j = \mathcal{U}_j \cap SO(n)^k$, then it follows from Theorem 1.3.1 that

$$C_k(SO(n)) = \coprod_{j=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{B}_j,$$

and that each $\mathcal{B}_j$ is a union of components. Like before, it is enough to calculate the number of components of $\mathcal{B}_j$.

Similar to the case of $O(n)$ we do it by looking at $\tilde{\Phi}_j^{-1}(\mathcal{B}_j)$ which, as we know, has the same number of components than $\mathcal{B}_j$. Note that

$$a = (\theta_1, \ldots, \theta_j, t_1, \ldots, t_{n-2j}) \in A_j \cap SO(n)$$

if and only if

$$\prod_{i=1}^{n-2j} t_i = 1.$$

Recall that $P_j \cong (\mathbb{S}^1)^{kj} \times L_j$ where

$$L_j = \{(r_1, \ldots, r_{n-2j}) \in (\mathbb{Z}_2^k)^{n-2j} : r_m \neq r_l \text{ for all } m \neq l\}.$$

Then we have that $\Phi^{-1}(\mathcal{B}_j) \cong O(n) \times (\mathbb{S}^1)^{kj} \times M_j$, where M_j is the following set:

$$M_j := \{(r_1, \ldots, r_{n-2j}) \in L_j : \prod_{i=1}^{n-2j} t_{l,i} = 1 \text{ for all } l = 1, \ldots, k\}.$$

In the definition above, we must remember that $r_i = (t_{1,i}, \ldots, t_{k,i}) \in \mathbb{Z}_2^k$. Also, remember that the components of $O(n)/A_j \times_{W(A_j)} P_j$ are indexed by the sets $\{r_1, \ldots, r_{n-2j}\} \subset \mathbb{Z}_2^k$ of size $n - 2j$. The component corresponding to $\{r_1, \ldots, r_{n-2j}\}$ is in $\tilde{\Phi}_j^{-1}(\mathcal{B}_j)$ if and only if

$$\prod_{i=1}^{n-2j} t_{l,i} = 1$$

for all $l = 1, \dots, k$.

We are interested in the cardinality of M_j so let $(r_1, \dots, r_{n-2j}) \in M_j$ and consider putting the r_i's as rows in a matrix, this translates the problem of counting the components of $\tilde{\Phi}_j^{-1}(\mathcal{B}_j)$ to solving the following counting problem, in which $m = n - 2j$:

Question 1.4.1. *How many matrices of size $m \times k$ with entries in $\mathbb{Z}_2$ have all their rows different and the product of all the entries in any column equal to 1?*

Lets call the number of such matrices $f(m)$, recall that k is fixed from the start. Then the number components of $\mathcal{B}_j$ is $\frac{f(m)}{m!}$. This is because the components are indexed by the unordered set of rows not the ordered tuple of rows. We will now focus on finding $f(m)$ and begin by showing that f is defined recursively by:

Proposition 1.4.2. *The function f satisfies the following recurrence relation:*

$$f(m) = \frac{2^k!}{(2^k - (m-1))!} - (m-1) \times (2^k - (m-2)) \times f(m-2)$$

whenever $m \leq 2^k$. Clearly $f(m) = 0$ if $m > 2^k$.

Proof. We first choose the first $(m-1)$ rows of our matrix. We want them to be different so there are $\frac{2^k!}{(2^k-(m-1))!}$ ways of doing this. The property that the product of the entries in each column has to be 1 determines the last row uniquely. However, it may be that the last row turns out to be equal to some other row, so we must subtract these cases.

Note that if a matrix has exactly two rows equal and the product of all the entries of each column is 1, then the product of all the entries in each column of the non-repeating rows is 1 as well. So if we are counting the matrices which have exactly two rows equal, one of which is the last one, we have $f(m-2)$ choices for the non-repeated rows, then $2^k - (m-2)$ options for the row we are repeating and $(m-1)$ options for the position of the repeated row. $\square$

It is clear from the definition that $f(1) = 1$ (vector with all entries 1) and $f(0) = 1$ (the empty matrix). Since the recurrence relation is two-step these values determine $f(m)$ for all m. It is also clear from the definition (and the inductive formula) that $f(2) = 0$. Next we give formulas for $f(m)$ in the even and odd case.

Proposition 1.4.3. *If $m \leq 2^k$ is odd then*

$$f(m) = \frac{(2^k-1)!}{(2^k-m)!}$$

If $0 \leq m \leq 2^{k-1}$ then

$$f(2m) = \frac{1}{2^k}\left(\frac{2^k!}{(2^k-2m)!} + \frac{(-1)^m\left(2^k-1\right)2^{k-1}!(2m)!}{m!\,(2^{k-1}-m)!}\right)$$

The proof that these formulas satisfy the recurrence relation will be given at the end of the section in its own subsection. This is because the proof is long and purely computational, and it would distract us from our final goal of finding a formula for the components of $\mathcal{B}_j$.

As we said earlier we have the relation

$$|\pi_0(\mathcal{B}_j)| = \frac{f(n-2j)}{(n-2j)!}.$$

Using the explicit formula for f given in Proposition 1.4.3 we get an explicit formula for the number of components of $\mathcal{B}_j$.

Proposition 1.4.4. *If n is odd, then*

$$|\pi_0(\mathcal{B}_j)| = \frac{1}{2^k}\binom{2^k}{n-2j},$$

and if n is even, then

$$|\pi_0(\mathcal{B}_j)| = \frac{1}{2^k}\left(\binom{2^k}{n-2j} + (-1)^{\frac{n-2j}{2}}\left(2^k-1\right)\binom{2^{k-1}}{\frac{n-2j}{2}}\right).$$

Also recall that

$$C_k(SO(n)) = \coprod_{j=1}^{\lfloor \frac{n}{2} \rfloor} \mathcal{B}_j.$$

Summing over j we get the formulas for number of connected components of $C_k(SO(n))$ as stated in Corollary 1.0.4, which we will restate here.

Corollary 1.0.4. *If $n \leq 2^k$, the space $C_k(SO(n))$ has*

$$\frac{1}{2^k}\sum_{j=0}^{\frac{n-1}{2}}\binom{2^k}{n-2j}$$

components when n is odd, and it has

$$\frac{1}{2^k}\sum_{m=0}^{\frac{n}{2}}\left(\binom{2^k}{n-2j} + (-1)^{\frac{n-2j}{2}}\left(2^k-1\right)\binom{2^{k-1}}{\frac{n-2j}{2}}\right).$$

components when n is even. If $n \geq 2^k - 1$ it has 2^{2^k-k-1} components.

The fact that the number of components stabilizes after $n = 2^k - 1$ follows both from the fact that the ones for $O(n)$ stabilize at that point, and from the formulas given above. The value to which they stabilize is easily computed from either of the formulas.

The components of $C_k(SO(3))$ had been counted by Torres Giese and Sjerve in [TGS08] and independently by Adem, Cohen and Gómez in [ACG13] using two different approaches. Our formula in the case $n = 3$ reads:

$$
\begin{aligned}
|\pi_0(C_k(SO(3)))| &= \left(\binom{2^k}{3} + \binom{2^k}{1}\right)\frac{1}{2^k} \\
&= \binom{2^k}{3}\frac{1}{2^k} + 1 \\
&= \frac{(2^k-1)(2^{k-1}-1)}{3} + 1.
\end{aligned}
$$

Which is precisely the result in both references above. So our approach gives the known answer in the case $n = 3$.

Also in [TGS08], Torres-Giese and Sjerve calculated the homeomorphism type of the components of $C_k(SO(3))$ which do not contain the identity. In this case we only have $\mathcal{B}_0$ and $\mathcal{B}_1$, the latter being the component of the trivial homomorphism. The space $\mathcal{B}_0$ is a union of components of $\mathcal{U}_0$ and in Proposition 1.3.13 we found the homeomorphism type of the components of $\mathcal{U}_0$. It can easily be seen that our result for $n = 3$ agrees with the result in [TGS08]. The following example shows that we can now find the cohomology ring of $C_k(O(3))$ with coefficients in any field with characteristic $\neq 2$.

Example 1.4.5. The cohomology ring $H^*(C_k(O(3)); F)$. In this case we only have $\mathcal{U}_0$ and $\mathcal{U}_1$ which have $\binom{2^k}{3}$ and 2^k components respectively. Proposition 1.3.13 tells us that each component of $\mathcal{U}_0$ is homeomorphic to $O(3)/\mathbb{Z}_2^3$ and Proposition 1.3.14 tells us that all the components of $\mathcal{U}_1$ are homeomorphic to the component containing the element $\underline{1} = (1, \ldots, 1) \in C_k(O(3))$. Lets call that component $R_{k,3}$.

The rational cohomology groups of $R_{k,3}$ were computed in [TGS08] as well as the $mod - 2$ cohomology ring. The ring structure with coefficients in a field of characteristic relatively prime to $|W|$ can be computed using ([Bai07], Theorem 4.3):

Theorem 1.4.6. *Let G be a connected, compact Lie group, let T be a max-*

imal torus in G and let F be a field with characteristic relatively prime to $|W|$, where W is the Weyl Group. Then $H^*(R_{k,3}, F) \cong H^*(G/T \times T^k, F)^W$.

This theorem applies to connected groups, but note that $R_{k,3} \subset C_k(SO(3))$ so we can use $G = SO(3)$ to compute the cohomology. In this case $G/T \cong \mathbb{S}^2$ so

$$H^*(G/T \times T^k, F) \cong \Lambda_F(y_1, \ldots, y_k) \otimes \frac{F[x]}{< x^2 >}.$$

Where $|x| = 2$ and $|y_i| = 1$. The Weyl group is $W = \mathbb{Z}_2$ and acts by changing the sign on each y_i and x. So it follows that if F has characteristic $\neq 2$ we have

$$H^*(R_{k,3}, F) \cong (\Lambda_F^{even}(y_1, \ldots, y_k)) \oplus \left(\Lambda_F^{odd}(y_1, \ldots, y_k) \otimes x\right).$$

So the cohomology ring $H^*(C_k(O(3)), F)$ is the direct sum of 2^k copies of $H^*(R_{k,3}; F)$ and $\binom{2^k}{3}$ copies of the cohomology of $O(3)/\mathbb{Z}_2^3 \cong \mathbb{S}^3/Q_8$.

1.4.1 The proof of the recursive formula

In this subsection we will prove Proposition 1.4.3, we will show that the formulas given there satisfy the recurrence relation:

$$f(m+2) = \frac{2^k!}{(2^k - (m+1))!} - (m+1) \times (2^k - m) \times f(m), \tag{1.1}$$

described in Proposition 1.4.2 with initial values $f(1) = 1$ and $f(0) = 1$. Since this recurrence relation only depends on the terms two steps before there is a unique solution given those initial values. We will separate our analysis into two cases depending on the parity of m. We start with the odd case.

Proposition 1.4.7. *If $m \leq 2^k$ is odd, then*

$$f(m) = \frac{(2^k - 1)!}{(2^k - m)!}$$

satisfies the recursive equation (1.1) and has initial value $f(1) = 1$.

Proof. The fact that $f(1) = 1$ is obvious. Here are the calculations that show that the recurrence relation is satisfied:

$$\begin{aligned}
f(m+2) &= \frac{2^k!}{(2^k-(m+1))!} - (m+1) \times (2^k - m) \times f(m) \\
&= \frac{2^k!}{(2^k-(m+1))!} - (m+1) \times (2^k - m) \times \frac{(2^k-1)!}{(2^k-m)!} \\
&= \frac{(2^k-1)!}{(2^k-(m+1))!}(2^k-(m+1)) \\
&= \frac{(2^k-1)!}{(2^k-(m+2))!}
\end{aligned}$$

□

Now we move on to deal with the even case, which is a little more complicated.

Proposition 1.4.8. *If* $m \le 2^{k-1}$, *then*

$$f(2m) = \frac{1}{2^k}\left(\frac{2^k!}{(2^k-2m)!} + \frac{(-1)^m\left(2^k-1\right)2^{k-1}!(2m)!}{m!\,(2^{k-1}-m)!}\right)$$

satisfies the recurrence relation (1.1) and has initial value $f(0) = 1$.

Proof. First we check the initial value:

$$\begin{aligned}
f(0) &= \frac{1}{2^k}\left(\frac{2^k!}{2^k!} + \frac{(2^k-1)2^{k-1}!}{2^{k-1}!}\right) \\
&= \frac{1}{2^k}(2^k) \\
&= 1
\end{aligned}$$

as required.

We will prove the recurrence relation by parts. Let

$$g(m) = \frac{(-1)^m \left(2^k - 1\right) 2^{k-1}!(2m)!}{m!\,(2^{k-1} - m)!}$$

and

$$h(m) = \frac{2^k!}{(2^k - 2m)!}\ ,$$

then

$$f(2m) = \frac{1}{2^k}(h(m) + g(m))$$

Claim 1. The function g satisfies the recurrence relation:

$$g(m+1) = -(2m+1)(2^k - 2m)g(m).$$

Here is the calculation:

$$\begin{aligned}
g(m+1) &= \frac{(-1)^{m+1} \left(2^k - 1\right) 2^{k-1}!(2m+2)!}{(m+1)!\,(2^{k-1} - m - 1)!} \\
&= -\frac{(-1)^m \left(2^k - 1\right) 2^{k-1}!(2m)!(2m+1)(2m+2)(2^{k-1} - m)}{m!(m+1)\,(2^{k-1} - m - 1)!(2^{k-1} - m)} \\
&= -\frac{(-1)^m \left(2^k - 1\right) 2^{k-1}!(2m)!(2m+1)(2)(2^{k-1} - m)}{m!\,(2^{k-1} - m)!} \\
&= -g(m)(2m+1)(2^k - 2m).
\end{aligned}$$

Claim 2. The function h satisfies the recurrence relation:

$$h(m+1) = \frac{(2^k)(2^k!)}{(2^k - 2m - 1)!} - (2m+1)(2^k - 2m)h(m).$$

Here is the calculation:

$$
\begin{aligned}
h(m+1) &= \frac{2^k!}{(2^k - 2(m+1))!} \\
&= \frac{2^k!(2^k - 2m - 1)(2^k - 2m)}{(2^k - 2m)!} \\
&= \left(\frac{2^k!}{(2^k - 2m)!}\right)\left(2^k(2^k - 2m) - (2m+1)(2^k - 2m)\right) \\
&= \frac{(2^k)(2^k!)}{(2^k - 2m - 1)!} - (2m+1)(2^k - 2m)\left(\frac{2^k!}{(2^k - 2m)!}\right) \\
&= \frac{(2^k)(2^k!)}{(2^k - 2m - 1)!} - (2m+1)(2^k - 2m)h(m).
\end{aligned}
$$

And now we can finally prove the recurrence relation for f:

$$
\begin{aligned}
f(2m+2) &= \frac{1}{2^k}(h(m+1) + g(m+1)) \\
&= \frac{1}{2^k}\left(\frac{(2^k)(2^k!)}{(2^k - 2m - 1)!} - (2m+1)(2^k - 2m)\left(h(m) + g(m)\right)\right) \\
&= \frac{2^k!}{(2^k - 2m - 1)!} - (2m+1)(2^k - 2m)\left(\frac{1}{2^k}\left(h(m) + g(m)\right)\right) \\
&= \frac{2^k!}{(2^k - (2m+1))!} - (2m+1)(2^k - 2m)f(2m).
\end{aligned}
$$

$\square$

This completes the proof of Proposition 1.4.3.

Remark 1.4.9. In the odd case, there is a way to avoid the recurrence relation and calculate f directly as follows. Let D be the set of $m \times k$ matrices with entries in $\mathbb{Z}_2$ and that have all their rows different. Let $\rho : D \to \mathbb{Z}_2^k$ be the function that sends a matrix to the vector containing the product of its columns. Note that $f(m)$ is precisely the cardinality of $\rho^{-1}(1, \ldots, 1)$. If m is odd, then changing the signs of all the entries in a column of a matrix will change the sign of the product of that column. This implies that the cardinality of $\rho^{-1}(x)$ does not depend on $x \in \mathbb{Z}_2^k$ and so

$$\begin{aligned} f(m) &= \frac{|D|}{|\mathbb{Z}_2^k|} \\ &= \frac{2^k!}{2^k(2^k - m)!}, \end{aligned}$$

which is exactly the formula in Proposition 1.4.3. However, this doesn't work in the case when m is even since changing the sign of all the entries in one column does not change the sign of the product of the entries in the column. Hence, we are forced to solve the problem as we did, by solving the recurrence relation. The fact that, in this case, the fibres of ρ don't all have the same size is captured in the formula by the strange correction term that does not appear in the odd case, I have yet to find the combinatorial meaning of this term.

1.5 Bounds using Stiefel-Whitney classes cannot be sharp

In this section we will prove that the Stiefel-Whitney classes cannot distinguish all of the components of $C_k(O(n))$.

As described in Section 1.1 we have a map

$$B : C_k(O(n)) \to Map_*(B\mathbb{Z}^k, BO(n)).$$

The total Stiefel-Whitney class (SW class for short) is a locally constant function

$$w : Map_*(B\mathbb{Z}^k, BO(n)) \to H^*(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z}),$$

so we get a function

$$wB : C_k(O(n)) \to H^*(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z}).$$

This function is locally constant so it is constant on connected components. In [AC07] Adem and Cohen obtained the bounds stated in Theorem 1.1.4 for $\pi_0(C_k(O(n)))$ by analyzing the first and second SW classes of the elements in $C_k(O(n))$. A natural question following that is if the total SW class distinguishes all the components of $C_k(O(n))$, in other words, is the map wB_0 injective? We will use our calculations and some results from [ACC03] to explain why the answer of this question is "No".

Given a representation $\rho : \mathbb{Z}^k \to O(n)$ we will write $w(\rho)$ when we really mean $wB_0(\rho)$, this should not cause any confusion. Recall that the total SW class has the form

$$w(\rho) = 1 + w_1(\rho) + \cdots + w_n(\rho)$$

where $w_i(\rho) \in H^i(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$ is the i-th SW class of ρ. Also recall that any representation $\rho : \mathbb{Z}^k \to O(n)$ splits as a direct sum of 2-dimensional orientable representations and line bundles. In Lemma 3.1 of [ACC03] it is shown that twice any line bundle gives an $SO(2)$ representation and that $SO(2)$ representations have trivial total SW class.

Let $\rho \in C_k(O(n))$. We can write

$$\rho = \Theta_1 \oplus \cdots \oplus \Theta_j \oplus \theta_1 \oplus \cdots \oplus \theta_{n-2j}$$

where Θ_i is an $SO(2)$ representation and the θ_i's are non-repeating line bundles. This is the same as saying $\rho \in \mathcal{U}_j$. In fact, given the description of the components of $\mathcal{U}_j$ we know that the component in which ρ lies is indexed by the set $\{w_1(\theta_1), \ldots, w_1(\theta_{n-2j})\}$ under the bijection between $H^1(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$ and $\mathbb{Z}_2^k$ induced by the standard basis $\{e_1, ..., e_k\} \subset H^1(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$.

The total SW class splits sums as products, we know $w(\Theta_i) = 1$ and the only possible non-zero SW class of a line bundle is the first, so we have

$$w(\rho) = w(\theta_1) \cdots w(\theta_{n-2j}) = (1 + w_1(\theta_1)) \cdots (1 + w_1(\theta_{n-2j})).$$

The total SW class $w(\rho)$ is of the form $1 + \alpha$ for $\alpha \in H^{>0}(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$. We know that $H^*(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z}) \cong \Lambda_{\mathbb{Z}/2\mathbb{Z}}(e_1, \ldots, e_k)$ so $H^{>0}(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$ has cardinality 2^{2^k-1}. Corollary 1.0.2 tells us that if n is large enough then $C_k(O(n))$ has exactly the same number of components. So if we want to show that wB_0 is not injective it is enough to show that there exists an α such that $1+\alpha$ is not the SW class of any representation. The following follows from ([ACC03], Proposition 3.4):

Proposition 1.5.1. *Let $\rho : \mathbb{Z}^k \to O(n)$ be a sum of line bundles. If $w_1(\rho) = w_2(\rho) = 0$ then $w(\rho) = 1$.*

This tells us that if $k \geq 3$ then there are many $\alpha \in H^{>0}(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$ such that $1+\alpha$ is not the SW class of any representation. Hence we've shown the following:

Corollary 1.5.2. *If $k \geq 3$ and $n \geq 2^k - 1$ then the map wB_0 is not injective, or in other words, the total Stiefel-Whitney class does not distinguish all the components of $C_k(O(n))$.*

We can actually find explicit pairs of different components with equal SW class. By our description of components, this is the same as finding 2 different subsets $\{x_1, \ldots, x_i\}$ and $\{y_1, \ldots, y_{i'}\}$ of $H^1(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$ such that

$$\prod_{j=1}^{i}(1 + x_j) = \prod_{j=1}^{i'}(1 + y_j). \tag{1.2}$$

This correspondence occurs because we can easily find line bundles which have as first SW class any specified element in $H^1(\mathbb{Z}^k, \mathbb{Z}/2\mathbb{Z})$. So if we let $\theta_1, \ldots, \theta_i$ be line bundles such that $w_1(\theta_j) = x_j$ and $\theta'_1, \ldots, \theta'_{i'}$ be line bundles such that $w_1(\theta'_j) = y_j$ and construct

$$\rho = \theta_1 \oplus \cdots \oplus \theta_i$$

and similarly ρ', then ρ and ρ' lie in different components of $C_k(O(n))$ since the subsets $\{x_1, \ldots, x_i\}$ and $\{y_1, \ldots, y_{i'}\}$ are different, but $w(\rho) = w(\rho')$ by equation 1.2.

Note that if $x \in H^1(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$ then $x^2 = 0$ and so $(1+x)^2 = 1$. So it is actually enough to find a subset $A \subset H^1(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$ such that

$$\prod_{x \in A} (1+x) = 1.$$

And then for every $B \subset A$ the components corresponding to B and $A - B$ will have the same SW class. The following proposition shows that $A = H^1(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$ works.

Proposition 1.5.3. *If $k \geq 3$ then*

$$\prod_{x \in H^1(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})} (1+x) = 1.$$

Proof. The proof is by induction on k. Recall that we know that $H^1(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z}) \cong \Lambda_{\mathbb{Z}/2\mathbb{Z}}(e_1, \ldots, e_k)$. The case $k = 3$ is a direct computation, which we will omit.

Suppose the proposition is true for k and let $A = H^1(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z}) \cup \{0\}$, then:

$$\begin{aligned}\prod_{x \in H^1(\mathbb{Z}^{k+1};\mathbb{Z}/2\mathbb{Z})} (1+x) &= \prod_{x\in A}(1+x)\prod_{x\in A}(1+x+e_{k+1}) \\ &= \prod_{x\in A}(1+x+e_{k+1}) \\ &= \prod_{x\in A}(1+x) + e_{k+1}\sum_{x\in A}\left(\prod_{y\in A-\{x\}}(1+y)\right) \\ &= 1 + e_{k+1}\sum_{x\in A}(1+x) = 1.\end{aligned}$$

Where the last sum is equal to zero since each generator appears 2^{k-1} times and 1 appears 2^k times. □

This shows that if $k \geq 3$ and $n \geq 2^k$ then for every subset $B \subset H^1(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$ the components of $C_k(O(n))$ corresponding to B and to $H^1(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z}) - B$ have the same SW class.

1.6 A bound for the number of components of $C_k(\mathbf{Spin}(n))$

In his section we will use the second SW class and our results about the components of $C_k(SO(n))$ to find a lower bound for the number of connected components of $C_k(\mathrm{Spin}(n))$ for sufficiently large values of n. We will prove Corollary 1.0.6, which we restate here:

Corollary 1.0.6. *If $n \geq 2^k - 1$ then $C_k(Spin(n))$ has at least $2^{2^k-k-1-\binom{k}{2}}$ connected components.*

Here the Lie group $\mathrm{Spin}(n)$ is the universal cover of $SO(n)$, we will call that covering map $s : \mathrm{Spin}(n) \to SO(n)$.

In [KS00], Kac and Smilga showed that the space of commuting triples in $\mathrm{Spin}(n)$ is disconnected if $n \geq 7$. Notice that this fact can be deduced from our bound as well. So this can be consider a generalization of their result.

It is known that a homomorphism $\rho : \mathbb{Z}^k \to SO(n)$ lifts to $\mathrm{Spin}(n)$ if and only if $w_2(\rho) = 0$. In other words, in the following sequence:

$$C_k(\mathrm{Spin}(n)) \xrightarrow{s_*} C_k(SO(n)) \xrightarrow{w_2} H^2(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z}) \ ,$$

we have that $\mathrm{Im}(s_*) = w_2^{-1}(0)$. Let $K = w_2^{-1}(0)$. Since K is the continuous image of $C_k(\mathrm{Spin}(n))$ it follows that $|\pi_0(C_k(\mathrm{Spin}(n)))| \geq |\pi_0(K)|$. Corollary 1.0.6 then follows from the next proposition:

Proposition 1.6.1. *If $n \geq 2^k - 1$ then $|\pi_0(K)| \geq 2^{2^k - k - \binom{k}{2}}$.*

Proof. As we said before, w_2 is locally constant so it induces a map

$$w_2 : \pi_0(C_k(SO(n))) \to H^2(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$$

and $\pi_0(K)$ is the inverse image of 0 under this map.

The cohomology group $H^2(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space. If n is large enough we can give $\pi_0(C_k(SO(n)))$ a structure of a $\mathbb{Z}/2\mathbb{Z}$-vector space as described in Proposition 1.6.2 in such a way that w_2 is linear. Then

$$|\mathrm{Im}(w_2)| \times |\mathrm{Ker}(w_2)| = |\pi_0(C_k(SO(n)))|.$$

The proposition then follows from the inequality:

$$|\mathrm{Im}(w_2)| \leq |H^2(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})| = 2^{\binom{k}{2}},$$

and the equalities: $|\mathrm{Ker}(w_2)| = |\pi_0(K)|$, $|\pi_0(C_k(SO(n)))| = 2^{2^k - 1 - k}$. $\square$

Proposition 1.6.2. *If $n \geq 2^k - 1$ then $\pi_0(C_k(SO(n)))$ admits a structure of $\mathbb{Z}/2\mathbb{Z}$-vector space for which w_2 is linear.*

Proof. Recall that for $n \geq 2^k - 1$ the map $SO(n) \hookrightarrow SO(n+1)$ induces an isomorphism on π_0 and so we get an isomorphism between $\pi_0(C_k(SO(n)))$ and $\pi_0(C_k(SO))$. We will define the $\mathbb{Z}/2\mathbb{Z}$-vector space structure on $\pi_0(C_k(SO))$.

Given $x \in C_k(SO)$ we denote the component containing x by $[x]$. Let $\alpha, \beta \in C_k(SO)$. We know that α factors through some $SO(n)$ and β factors through some $SO(m)$ so we can define $\alpha + \beta$ to be the composition

$$\mathbb{Z}^k \xrightarrow{\alpha \oplus \beta} SO(n) \oplus SO(m) \longrightarrow SO(n+m) \longrightarrow SO$$

Given $[\alpha], [\beta] \in \pi_0(C_k(SO))$ we can define $[\alpha] + [\beta] := [\alpha + \beta]$. Strictly speaking $\alpha + \beta$ depends on the choice of m and n but by our description of the connected components of $\mathrm{Hom}(\mathbb{Z}^k, SO)$ we know that $[\alpha + \beta]$ only depends on the non repeating line bundles in the decomposition of $\alpha \oplus \beta$, and that does not depend on any choices so the operation is well defined.

Another way to think about the same operation is the symmetric difference of sets. We know that each component of $C_k(SO)$ corresponds uniquely to some subset of $\mathbb{Z}_2^k$. Given two such subsets $A, B \subset \mathbb{Z}_2^k$ we can define $A + B := A \triangle B$, where $\triangle$ represents the symmetric difference.

This gives $C_k(SO)$ the structure of an abelian group on which every element is 2 torsion. In other words, this gives $C_k(SO)$ the structure of a $\mathbb{Z}/2\mathbb{Z}$-vector space.

It remains to see that $w_2 : C_k(SO) \to H^2(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$ is linear.

Given $[\alpha], [\beta] \in \pi_0(C_k(SO))$ we know that

$$\begin{aligned} w_2([\alpha] + [\beta]) &= w_2([\alpha + \beta]) = w_2(\alpha \oplus \beta) \\ &= w_1(\alpha) w_1(\beta) + w_2(\alpha) + w_2(\beta) \\ &= w_2(\alpha) + w_2(\beta) = w_2([\alpha]) + w_2([\beta]) \end{aligned}$$

Here we remember that w_1 is zero for SO representations. This finishes the proof of Proposition 1.6.2 and completes the proof of Proposition 1.6.1.

Chapter 2

Spaces of homomorphisms from central extensions of free groups by free groups into $SU(2)$.

In the previous chapter we talked about spaces of the form $\mathrm{Hom}(\pi, G)$ for the particular case when $\pi = \mathbb{Z}^k$ and then we specialized to the cases $G = O(n), SO(n), \mathrm{Spin}(n)$. In this chapter, we will explore what happens when we try to take a slightly more general source group π. However, this comes at the cost of having to specialize G to a simpler group, in this case, $SU(2)$.

We will study the space of group homomorphisms $\mathrm{Hom}(\Gamma, SU(2))$, where Γ is a central extension of a free abelian group by a free abelian group. We provide a detailed analysis of this space capable of computing its cohomology and path components in terms of the cohomology class classifying the extension.

Historically, $SU(2)$ has been the first interesting candidate for explicit calculations when talking about spaces of homomorphisms, so it makes sense to start there when we are trying to generalize the source group. The cohomology of the space $C_k(SU(2)) = \mathrm{Hom}(\mathbb{Z}^k, SU(2))$ was first studied in [AC07] for $k = 2, 3$ and in [Cra11] and [BJS11] for general k. In [AC07] Adem and Gómez studied topological invariants of the space $\mathrm{Hom}(A, G)$, where A is any abelian group and G is in a class of compact Lie groups containing $SU(2)$, in particular they obtain the number of its connected components. We will use these results to do our analysis.

Throughout this chapter, Γ will be a central extension of the form

$$0 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z}^k \to 0.$$

These extensions are classified by elements of $H^2(\mathbb{Z}^k; \mathbb{Z}^r)$, so by r-tuples of the form

$$\omega = (\omega_1, \dots, \omega_r) \in H^2(\mathbb{Z}^k; \mathbb{Z}^r) \cong \left(H^2(\mathbb{Z}^k; \mathbb{Z})\right)^r,$$

where

$$\omega_l = \sum_{1 \leq i < j \leq k} \beta^l_{i,j} e^*_i \wedge e^*_j \in H^2(\mathbb{Z}^k; \mathbb{Z}).$$

In this description, $\{e_i\}_{i=1}^k$ are the standard generators of $\mathbb{Z}^k$, $\{e^*_i\}_{i=1}^k$ are the dual generators of the cohomology ring $H^*(\mathbb{Z}^k; \mathbb{Z})$ and $\beta^l_{i,j} \in \mathbb{Z}$. The group Γ corresponding to the class ω is given in terms of generators and relations by

$$\Gamma = \langle E_1, \dots, E_k, X_1, \dots, X_r : [E_i, E_j] = \prod_{l=1}^{r} X_l^{\beta^l_{i,j}}, X_i \text{ is central}\rangle.$$

We chose this type of groups in an attempt to generalize the source group from the abelian case in a way where we can still exploit our knowledge of the spaces $C_k(SU(2))$. These groups proved to be good candidates as the next step following the understanding of the spaces of commuting tuples.

One fundamental difference between $\mathrm{Hom}(\Gamma, SU(2))$ and $C_k(SU(2))$ is that the latter is always path connected and the former may not. In fact, there are two kinds of elements of $\mathrm{Hom}(\Gamma, SU(2))$: the ones that factor though the abelianization of Γ and the ones that do not. This gives a decomposition into disjoint subspaces

$$\mathrm{Hom}(\Gamma, SU(2)) = \mathrm{Hom}(\Gamma/[\Gamma,\Gamma], SU(2)) \cup R_\Gamma,$$

where R_Γ is the space of all the homomorphisms that do not factor through the abelianization. The space $\mathrm{Hom}(\Gamma/[\Gamma,\Gamma], SU(2))$ is compact as it is a closed subspace of $SU(2)^{k+r}$, and it will be shown that R_Γ is also compact, so this is actually a decomposition as coproduct.

The abelianization of Γ can be obtained from the homology exact sequence of the extension

$$H_2(\mathbb{Z}^k) \xrightarrow{\phi} H_1(\mathbb{Z}^r) \longrightarrow H_1(\Gamma) \longrightarrow H_1(\mathbb{Z}^k) \longrightarrow 0\ .$$

Since the last nonzero map splits we see that

$$H_1(\Gamma) \cong \Gamma/[\Gamma,\Gamma] \cong \mathbb{Z}^k \oplus \mathrm{coker}(\phi),$$

where

$$\phi : H_2(\mathbb{Z}^k) \to H_1(\mathbb{Z}^r)$$

is given in matrix notation by:

$$\phi = \begin{pmatrix} \beta^1_{1,2} & \beta^1_{1,3} & \cdots & \beta^1_{k-1,k} \\ \vdots & & & \vdots \\ \beta^r_{1,2} & \beta^r_{1,3} & \cdots & \beta^r_{k-1,k} \end{pmatrix}.$$

The cokernel of this matrix can be computed using its Smith normal form to be

$$\mathrm{coker}(\phi) \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_t \oplus \mathbb{Z}^{r-t}.$$

Where

$$a_i = \frac{d_i(\phi)}{d_{i-1}(\phi)},$$

$d_i(\phi)$ is the greatest common divisor of the $i \times i$ minors of ϕ and t is the rank of ϕ viewed as a transformation between $\mathbb{Q}$-vector spaces.

In [AG11], Adem and Gómez determine the number of path connected components of $\mathrm{Hom}(\mathbb{Z}^m \oplus A, G)$ in terms of the orbits of the Weyl group action on $\mathrm{Hom}(A, T)$, where T is a maximal torus of G. In the case $G = SU(2)$ their techniques can be used to find the homeomorphism type of each of those components. This will be discussed in Section 2.1, where we prove the following theorem:

Theorem 2.0.3. *Let $A = \mathbb{Z}/q_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_n\mathbb{Z}$ be a finite group, where $q_1, \ldots, q_r$ are all even and $q_{r+1}, \ldots, q_n$ are all odd, then the space $Hom(\mathbb{Z}^k \oplus A, SU(2))$ has 2^r path connected components homeomorphic to $C_k(SU(2))$ and $\frac{q_1 q_2 \cdots q_n - 2^r}{2}$ path connected components homeomorphic to $\mathbb{S}^2 \times (\mathbb{S}^1)^k$.*

This theorem completely describes the subspace $\mathrm{Hom}(\Gamma/[\Gamma, \Gamma], SU(2))$. The rest of the chapter is devoted to the understanding of R_Γ. In Section 2.3 we explain that this subspace only depends on the reduction modulo 2 of ω. Namely, we prove:

Proposition 2.0.4. *Let $\overline{\Gamma}$ be the quotient of Γ induced by $\mathbb{Z}^r \to (\mathbb{Z}/2\mathbb{Z})^r$ as in the diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^r \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \mathbb{Z}^k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathbb{Z}/2\mathbb{Z})^r \arrow[r] & \overline{\Gamma} \arrow[r] & \mathbb{Z}^k \arrow[r] & 0.
\end{tikzcd}$$

Let $R_{\overline{\Gamma}}$ the subspace of $\mathrm{Hom}(\overline{\Gamma}, SU(2))$ of homomorphisms that do not factor though the abelianization of $\overline{\Gamma}$, then

$$R_\Gamma \cong R_{\overline{\Gamma}}.$$

In Section 2.3 we cover the case when the kernel of the extension has rank 1 ($r = 1$) and show that, in this case, R_Γ is either empty or 2^{k-2} copies of

$SO(3)$ depending on the rank of the reduction of ω modulo 2.

Section 2.4 deals with the case when the kernel has rank greater than 1 ($r > 1$). We explain how the space R_Γ is a disjoint union of subspaces homeomorphic to $R_{\Gamma^{(v)}}$, where $v \in (\mathbb{Z}/2\mathbb{Z})^r$ and $\Gamma^{(v)}$ is a group like the ones covered in Section 2.3. This means that R_Γ is a disjoint union of copies of $SO(3)$ in this general case as well. For the final statement we use an auxiliary function t which is defined in Section 2.4.

Theorem 2.0.5. *Let Γ be the central extension corresponding to the class $\omega \in H^2(\mathbb{Z}^k; \mathbb{Z}^r)$, then R_Γ is homeomorphic to $t(\omega) \times 2^{k-2}$ copies of $SO(3)$.*

Since the cohomologies of $C_k(SU(2))$, $\mathbb{S}^2 \times (\mathbb{S}^1)^k$ and $SO(3)$ are all known, and our description of $\mathrm{Hom}(\Gamma, SU(2))$ is in terms of these spaces, we can use these results to calculate the cohomology groups of $\mathrm{Hom}(\Gamma, SU(2))$ and, in particular, its connected components.

2.1 The spaces Hom$(\mathbb{Z}^n \oplus A, SU(2))$

In this section, we will use the results from [AG11] and [BJS11] to compute the cohomology of the spaces $\mathrm{Hom}(\mathbb{Z}^n \oplus A, SU(2))$, where A is a finite abelian group. When A is empty this is the space of ordered commuting $n-$tuples in $SU(2)$. In [AC07] Adem and Cohen found a stable splitting for the space of commuting tuples, and then Crabb in [Cra11] and Baird, Jeffrey and Selick in [BJS11] independently found the homotopy type of the direct summands. This description can be used to explicitly calculate the cohomology of the space of commuting elements in $SU(2)$. In [AG11], Adem and Gómez calculated the number of path-connected components of the space $\mathrm{Hom}(\mathbb{Z}^n \oplus A, G)$ for G in a class of compact lie groups containing $SU(2)$. In the case of $SU(2)$, we will show how their techniques can be used to calculate not only the number of path-connected components but also the cohomology groups of all these components.

We begin by recalling the known results for the case when A is empty. The stable splitting of the commuting $n-$tuples in $SU(2)$ given in [BJS11] is as follows:

$$\Sigma(C_n(SU(2))) \simeq \Sigma\left(\bigvee_{m=1}^{n}\binom{n}{m}\Sigma S(mL)\right), \tag{2.1}$$

where

$$\Sigma S(mL) \simeq \begin{cases} \mathbb{S}^3 & \text{if } m=1, \\ \mathbb{S}^2 \vee (\mathbb{R}P^4/\mathbb{R}P^2) & \text{if } m=2, \\ \Sigma\mathbb{R}P^2 \vee (\mathbb{R}P^{k+2}/\mathbb{R}P^{k-1}) & \text{if } m>2. \end{cases} \tag{2.2}$$

Although an explicit formula for the cohomology groups of $C_n(SU(2))$ was not given in [BJS11], one can directly get it from (2.1) and (2.2). The cohomology groups are:

$$H^k(C_n(SU(2));\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k=0, \\ 0 & \text{if } k=1, \\ \mathbb{Z}^{\binom{n}{2}} & \text{if } k=2, \\ \mathbb{Z}^n \oplus (\mathbb{Z}/2\mathbb{Z})^{2^n-1-n-\binom{n}{2}} & \text{if } k=3, \\ \mathbb{Z}^{\binom{n}{k}} \oplus (\mathbb{Z}/2\mathbb{Z})^{\binom{n}{k-1}+\binom{n}{k-2}} & \text{if } k\geq 4 \text{ even}, \\ \mathbb{Z}^{\binom{n}{k-2}} & \text{if } k\geq 5 \text{ odd}. \end{cases} \tag{2.3}$$

Now we move on to the case when A is not empty. For this case, we will use the approach that Adem and Gómez used in [AG11]. For this section only, we will use the shorthand notation $X := \mathrm{Hom}(\mathbb{Z}^n \oplus A, SU(2))$ and $G := SU(2)$.

If A has a set of generators of size i then there is an inclusion $X \subset \mathrm{Hom}(\mathbb{Z}^{n+i}, G)$ induced by the surjection $\mathbb{Z}^{n+i} \to \mathbb{Z}^n \oplus A$. The conjugation action of G on $\mathrm{Hom}(\mathbb{Z}^{n+i}, G)$ restricts to a G action on X. It was shown in [AG11] that the isotropy groups of the G-action on X are connected and of maximal rank, that is, for every $x \in X$ the isotropy group G_x is connected and we can find a maximal torus T_x in G such that $T_x \subset G_x$.

Pick a maximal torus T in G, let $N_G(T)$ be its normalizer and $W = N_G(T)/T$ be the Weyl group of G. Consider the map

$$\begin{aligned} \phi : G \times X^T &\to X \\ (g, x) &\mapsto gx. \end{aligned}$$

As shown in [AG11], the fact that G acts with maximal rank isotropy implies that this map is surjective. There is a right $N_G(T)$-action on $G \times X^T$ given by $(g, x) \cdot n = (gn, n^{-1}x)$ and ϕ is invariant under this action, so we get an induced map on the quotient

$$\begin{aligned} \varphi : G \times_{N_G(T)} X^T = G/T \times_W X^T &\to X \\ [g, x] &\mapsto gx. \end{aligned}$$

In this case $X^T = \mathrm{Hom}(\mathbb{Z}^n \oplus A, T) = T^n \times \mathrm{Hom}(A, T)$. The following lemma tells us that the map φ is injective in a large subset of its domain.

Lemma 2.1.1. *Let $f \in Hom(\mathbb{Z}^n \oplus A, T)$. Assume that there exists $x \in \mathbb{Z}^n \oplus A$ such that $f(x)$ is a regular*[1] *element in G, then $\varphi^{-1}(\varphi([g, f])) = \{[g, f]\}$ for all $g \in G$.*

Proof. Let $g \in G$ and assume there exists $[g', f'] \in G/T \times_W \mathrm{Hom}(\mathbb{Z}^n \oplus A, T)$ such that $\varphi([g', f']) = \varphi([g, f])$. This implies that $gfg^{-1} = g'f'g'^{-1}$ and, in particular, that $gf(x)g^{-1} = g'f'(x)g'^{-1}$, so

$$f(x) = (g^{-1}g')f'(x)(g^{-1}g')^{-1} \in (g^{-1}g')T(g^{-1}g')^{-1}.$$

Since $f(x)$ is regular this implies that $T = (g^{-1}g')T(g^{-1}g')^{-1}$ and hence $g^{-1}g' \in N_G(T)$, and since $(g, f) \cdot (g^{-1}g') = (g', f')$ this proves that $[g', f'] = [g, f]$ as required. $\square$

[1] Recall that an element in a Lie Group is regular if it is in exactly one maximal torus.

Remark 2.1.2. The Lemma is true for a general compact Lie group G, but note that in our case, $G = SU(2)$, the hypothesis on f can be stated as "the image of f is not contained in $\{\pm 1\}$" since ± 1 are the only 2 elements of $SU(2)$ that are not regular.

By [AG11, Corollary 3.4] the number of path connected components of X is equal to the number of different orbits of the action of W on $\mathrm{Hom}(A, T)$. Notice that $\mathrm{Hom}(A, T)$ is a finite set. If $o \subset \mathrm{Hom}(A, T)$ is a W-orbit we will denote the component corresponding to o as X_o. If $\underline{1}$ is the trivial homomorphism then $X_{\{\underline{1}\}}$ is homeomorphic to $C_n(SU(2))$.

In our case the $W = \mathbb{Z}/2\mathbb{Z}$-action on $T = \mathbb{S}^1$ is given by complex conjugation. The only fixed points of this action are $\{\pm 1\}$. Given $f \in \mathrm{Hom}(A, T)$ there are only two possibilities: either $W = \mathbb{Z}/2\mathbb{Z}$ acts trivially on f or it acts freely on f. Since the only fixed points of the W-action on T are $\{\pm 1\}$ then W will act trivially on f precisely when $f(A) \subset \{\pm 1\}$. In this case the W-orbit of f is only $\{f\}$ and we have the surjection

$$\varphi : G/T \times_W (T^n \times \{f\}) \to X_{\{f\}}$$

which is just a shift by f of

$$\varphi : G/T \times_W (T^n \times \{\underline{1}\}) \to X_{\{\underline{1}\}} \cong C_n(SU(2))$$

and so $X_{\{f\}}$ is homeomorphic to $C_n(SU(2))$.

If $f(A) \not\subset \{\pm 1\}$ then W acts freely on f and its orbit is $\{f, \bar{f}\}$. In this case, W acts freely on $G/T \times T^n \times \{f, \bar{f}\}$ interchainging the two components, hence $G/T \times_W (T^n \times \{f, \bar{f}\}) \cong G/T \times T^n$. Furthermore, since the image of f contains regular elements of G the map

$$\varphi : G/T \times_W (T^n \times \{f, \bar{f}\}) \to X_{\{f, \bar{f}\}}$$

is a homeomorphism by Lemma 2.1.1 together with the fact that φ is surjective. This shows that $X_{\{f,\bar{f}\}}$ is homeomorphic to $G/T \times T^n$. We can now substitute $G = SU(2)$ and $T = \mathbb{S}^1$ to get that $X_{\{f,\bar{f}\}}$ is homeomorphic to $\mathbb{S}^2 \times (\mathbb{S}^1)^n$. We have proved then the following lemma:

Lemma 2.1.3. *Let $T \subset SU(2)$ be a maximal torus, $W = \mathbb{Z}/2\mathbb{Z}$ the Weyl group of $SU(2)$ and $o \subset Hom(A, T)$ a W-orbit, then the component of $Hom(\mathbb{Z}^n \oplus A, SU(2))$ corresponding to o is is homeomorphic to $C_n(SU(2))$ if o is a point and it is homeomorphic to $\mathbb{S}^2 \times (\mathbb{S}^1)^n$ if o is not a point.*

The only thing left to determine is how many W-orbits there are of each type. We do this in the next Lemma.

Lemma 2.1.4. *Let $A = \mathbb{Z}/q_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_k\mathbb{Z}$, where $q_1, \ldots, q_r$ are all even and $q_{r+1}, \ldots, q_k$ are all odd, then $Hom(A, T)$ has 2^r trivial W-orbits and $\frac{q_1 q_2 \cdots q_k - 2^r}{2}$ nontrivial W-orbits.*

Proof. Since every orbit has either one or two points it is enough to find the fixed points of the W-action. Notice $\mathrm{Hom}(A, T) = \mathbb{Z}/q_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_k\mathbb{Z}$ where we identify $\mathbb{Z}/q\mathbb{Z} \subset \mathbb{S}^1 = T$ as the q-th roots of the unity. The W-action is complex conjugation and the only fixed points of this action are the ones on which every entry is either 1 or -1. The only entries that could have a value of -1 are the first r. This proves that there are exactly 2^r trivial orbits. The remaining $q_1 q_2 \cdots q_k - 2^r$ points are paired up in nontrivial orbits of size 2. $\square$

Putting together Lemmas 2.1.3 and 2.1.4 we get the main theorem of this section.

Theorem 2.1.5. *Let $A = \mathbb{Z}/q_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_k\mathbb{Z}$ be a finite group, where $q_1, \ldots, q_r$ are all even and $q_{r+1}, \ldots, q_k$ are all odd, then the space $Hom(\mathbb{Z}^n \oplus A, SU(2))$ has 2^r path connected components homeomorphic to $C_n(SU(2))$ and $\frac{q_1 q_2 \cdots q_k - 2^r}{2}$ path connected components homeomorphic to $\mathbb{S}^2 \times (\mathbb{S}^1)^n$.*

This theorem, together with the formulas (2.3), can calculate the cohomology groups of $\mathrm{Hom}(\mathbb{Z}^n \oplus A, SU(2))$ explicitly for all n and A. As a corollary, we know exactly how many connected components these spaces have.

Corollary 2.1.6. *Let $A = \mathbb{Z}/q_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_k\mathbb{Z}$, where $q_1, \dots, q_r$ are all even and $q_{r+1}, \dots, q_k$ are all odd, then the space $Hom(\mathbb{Z}^n \oplus A, SU(2))$ has $\frac{q_1 q_2 \cdots q_k + 2^r}{2}$ path connected components.*

This corollary could have been deduced from [AG11, Corollary 3.4] and a simple calculation like the one in Lemma 2.1.4, but Theorem 2.1.5 gives an explicit description of the homeomorphism type of every component.

2.2 The reduction modulo 2

In the last section we completely described the subspace of $\mathrm{Hom}(\Gamma, SU(2))$ of homomorphisms that factor through the abelianization of Γ, from now on we will be focused on describing R_Γ, the space of homomorphisms with non-abelian image.

In this section, we will show that if ω is the classifying class of the extension, then R_Γ only depends on the reduction of ω modulo 2. This follows from the easy but crucial fact about $SU(2)$, as we will explain.

Proposition 2.2.1. *The center of any non-abelian subgroup of $SU(2)$ is $\{\pm 1\}$.*

Recall that Γ is given in terms of generators and relations by:

$$\Gamma = \langle E_1, \dots, E_k, X_1, \dots, X_r : [E_i, E_j] = \prod_{l=1}^{r} X_l^{\beta_{i,j}^l}, X_i \text{ is central} \rangle.$$

Given a homomorphism $f : \Gamma \to SU(2)$ we let $A_i = f(E_i)$ and $B_i = f(X_i)$. From the relations in Γ we know that the matrices B_i are in the center of

$f(\Gamma)$. In the particular case when $f \in R_\Gamma$, we know that $f(\Gamma)$ is a non-abelian subgroup of $SU(2)$ and hence its center is $\{\pm 1\}$ by Proposition 2.2.1. This implies that if $f \in R_\Gamma$, then f factors through the quotient $\overline{\Gamma}$ induced by $\mathbb{Z}^r \to (\mathbb{Z}/2\mathbb{Z})^r$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^r \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \mathbb{Z}^k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathbb{Z}/2\mathbb{Z})^r \arrow[r] & \overline{\Gamma} \arrow[r] & \mathbb{Z}^k \arrow[r] & 0.
\end{tikzcd}$$

The cohomology class in $H^2(\mathbb{Z}^k; (\mathbb{Z}/2\mathbb{Z})^r)$ classifying $\overline{\Gamma}$ is simply the reduction of ω modulo 2. We summarize this in the following proposition:

Proposition 2.2.2. *Let $\overline{\Gamma}$ be the quotient of Γ induced by $\mathbb{Z}^r \to (\mathbb{Z}/2\mathbb{Z})^r$ and let $R_{\overline{\Gamma}}$ the subspace of $\mathrm{Hom}(\overline{\Gamma}, SU(2))$ of homomorphisms that do not factor though the abelianization of $\overline{\Gamma}$, then*

$$R_\Gamma \cong R_{\overline{\Gamma}}.$$

Note that this proposition is not saying that $\mathrm{Hom}(\Gamma, SU(2))$ is homeomorphic to $\mathrm{Hom}(\overline{\Gamma}, SU(2))$, since Γ and $\overline{\Gamma}$ may have different abelianizations. We will use $\overline{\omega}$ to denote the reduction of ω modulo 2. Proposition 2.2.2 has the two following direct corollaries.

Corollary 2.2.3. *If Γ' is the central extension corresponding to the class $\omega' \in H^2(\mathbb{Z}^k; \mathbb{Z}^r)$, and $\overline{\omega'} = \overline{\omega}$, then $R_{\Gamma'}$ is homeomorphic to R_Γ.*

Proof. This follows from the fact that $\overline{\Gamma} \cong \overline{\Gamma'}$ whenever $\overline{\omega} = \overline{\omega'}$. $\square$

Corollary 2.2.4. *If $\overline{\omega} = 0$, then R_Γ is empty.*

Proof. In this case, $\overline{\Gamma}$ is the trivial extension and hence abelian. This implies $R_{\overline{\Gamma}} = \emptyset$. $\square$

This means that for the purposes of studying R_Γ we can restrict ourselves to the simpler case of extensions of the form

$$0 \to (\mathbb{Z}/2\mathbb{Z})^r \to \overline{\Gamma} \to \mathbb{Z}^k \to 0.$$

2.3 Extensions with center of rank 1.

In this section we will finish the description of $\mathrm{Hom}(\Gamma, SU(2))$ for the special case when Γ is a central extension of the form

$$0 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^k \to 0.$$

The cohomology class of this extension has the form

$$\omega = \sum_{1 \leq i < j \leq k} \beta_{i,j} e_i^* \wedge e_j^* \in H^2(\mathbb{Z}^k; \mathbb{Z}).$$

Recall that we have the decomposition

$$\mathrm{Hom}(\Gamma, SU(2)) = \mathrm{Hom}(\Gamma/[\Gamma,\Gamma], SU(2)) \cup R_\Gamma,$$

and that by the previous section $R_\Gamma \cong R_{\overline{\Gamma}}$. We will begin by describing $\mathrm{Hom}(\Gamma/[\Gamma,\Gamma], SU(2))$ more explicitly.

The abelianization of Γ in this case is

$$\Gamma/[\Gamma,\Gamma] \cong \mathbb{Z}^k \oplus \mathbb{Z}/m\mathbb{Z},$$

where m is the maximum common divisor of the $\beta_{i,j}$'s. We can use Theorem 2.1.5 to give an explicit description of the space $\mathrm{Hom}(\Gamma/[\Gamma,\Gamma], SU(2))$. There are 3 cases that depend on the value of m:

Corollary 2.3.1. *If $m = 0$, then*

$$Hom(\Gamma/[\Gamma,\Gamma], SU(2)) \cong C_{k+1}(SU(2))$$

if $m \neq 0$ and m is even, then

$$Hom(\Gamma/[\Gamma,\Gamma], SU(2)) \cong C_k(SU(2)) \bigsqcup C_k(SU(2)) \bigsqcup \left(\bigsqcup_{i=1}^{\frac{m-2}{2}} \mathbb{S}^2 \times (\mathbb{S}^1)^k \right)$$

if m is odd, then

$$Hom(\Gamma/[\Gamma,\Gamma], SU(2)) \cong C_k(SU(2)) \bigsqcup \left(\bigsqcup_{i=1}^{\frac{m-1}{2}} \mathbb{S}^2 \times (\mathbb{S}^1)^k \right).$$

All that is left then is to describe R_Γ. We have seen in Proposition 2.2.2 that R_Γ is determined by reduction of ω modulo 2. So, by the the skew-symmetric bilinear form $\overline{\omega} \in H^2(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z})$, and such bilinear forms are determined by their rank. Furthermore, the following proposition follows directly from [RV96, Proposition 4.1]:

Proposition 2.3.2. *There exists a $\mathbb{Z}$-module basis $g_1, \ldots, g_k$ of $\mathbb{Z}^k$ such that*

$$\overline{\omega} = g_1^* \wedge g_2^* + \cdots + g_{2m-1}^* \wedge g_{2m}^*$$

where $2m \leq n$.

This means that by changing basis on $\mathbb{Z}^k$ we may assume that

$$\overline{\omega} = e_1^* \wedge e_2^* + \cdots + e_{2m-1}^* \wedge e_{2m}^*. \tag{2.4}$$

The number $2m$ is called the rank of $\overline{\omega}$.

We are now ready for the main theorem of this section.

Theorem 2.3.3. *Let $\overline{\omega}$ be the reduction of ω modulo 2 and let $2m$ be the rank of $\overline{\omega}$, then the space R_Γ is homeomorphic to the disjoint union of 2^{k-2} copies of $SO(3)$ if $m = 1$, and it is empty otherwise.*

Before proving Theorem 2.3.3 we must prove a preliminary result about the space of anti-commuting pairs in $SU(2)$.

Proposition 2.3.4. *The space*

$$Y := \{(A_1, A_2) \in SU(2) \times SU(2) : [A_1, A_2] = -Id\}$$

of anti-commuting pairs in $SU(2)$ is homeomorphic to $SO(3)$.

Proof. The pairs of anti-commuting elements in $SU(2)$ are actually unique up to conjugation, this can be seen using [BFM02, Proposition 4.1.1] or by an easy direct calculation. In other words, the (diagonal) conjugation action of $SU(2)$ on Y is transitive, which implies that

$$Y \cong SU(2)/Z$$

where Z is the stabilizer of any element $(A_1, A_2) \in Y$. This stabilizer must be $\{\pm Id\}$ by Proposition 2.2.1 since A_1 and A_2 do not commute, and therefore

$$Y \cong SU(2)/\{\pm 1\} \cong SO(3)$$

as claimed. □

Now we are ready to prove Theorem 2.3.3.

Proof of Theorem 2.3.3. We first prove that if $m \geq 2$ then R_Γ is empty. We may assume $\overline{\omega}$ is as in (2.4) and $m > 1$ (in particular this implies that $k \geq 4$). Assume for a contradiction that $f \in R_\Gamma$. Let $A_i = f(E_i)$, since we are assuming (2.4) we know that A_1 commutes with A_3 and A_4, and that these two do not commute with each other. By Proposition 2.2.1 this implies that A_1 is equal to $\pm Id$, however, $[A_1, A_2] = -Id$ which is a contradiction. This proves that if $m > 1$, then R_Γ is empty.

Now we assume that $m = 1$ and that $\overline{\omega} = e_1^* \wedge e_2^*$. Given $f \in R_\Gamma$ we let $A_i = f(E_i) \in SU(2)$, for each $i > 2$ we have that A_i commutes with A_1 and A_2, and these two do not commute, which means that $A_i = \pm Id$. In other words we have that:

$$
\begin{aligned}
R_\Gamma &\cong \{(A_1, A_2, \pm Id, \ldots, \pm Id) \in SU(2)^k : [A_1, A_2] = -Id\} \\
&\cong \{(A_1, A_2) \in SU(2)^2 : [A_1, A_2] = -Id\} \times \{\pm Id\}^{k-2} \\
&\cong \bigsqcup_{i=1}^{2^{k-2}} SO(3).
\end{aligned}
$$

The last homeomorphism follows from Proposition 2.3.4. $\square$

Notice that this in particular shows that R_Γ is compact, so we have that the decomposition

$$\mathrm{Hom}(\Gamma, SU(2)) \cong \mathrm{Hom}(\Gamma/[\Gamma,\Gamma], SU(2)) \sqcup R_\Gamma$$

is in fact a decomposition as a coproduct. This decomposition, Theorem 2.3.3, Corollary 2.3.1 and equation (2.3) are all the pieces we need to find the cohomology groups of the space $\mathrm{Hom}(\Gamma, SU(2))$ in the case we are considering now. For the easy particular case when $k = 2$ we can use all this to get a neat description of the space $\mathrm{Hom}(\Gamma, SU(2))$:

Corollary 2.3.5. *Let Γ be the extension corresponding to the cohomology class $\beta e_1^* \wedge e_2^* \in H^2(\mathbb{Z}^2; \mathbb{Z})$, then*

$$Hom(\Gamma, SU(2)) \cong \begin{cases} Hom(\Gamma/[\Gamma,\Gamma], SU(2)) & \text{if } \beta \text{ is even,} \\ Hom(\Gamma/[\Gamma,\Gamma], SU(2)) \bigsqcup SO(3) & \text{if } \beta \text{ is odd.} \end{cases}$$

Note that, in this case, the abelianization of Γ is simply $\mathbb{Z}^2 \oplus \mathbb{Z}/\beta\mathbb{Z}$, so we can use Corollary 2.3.1 and (2.3) to easily find the cohomology groups of $\mathrm{Hom}(\Gamma, SU(2))$ in this case. In particular, we can calculate the number of path connected components.

Corollary 2.3.6. *Let Γ be the extension corresponding to the cohomology class $\beta e_1^* \wedge e_2^* \in H^2(\mathbb{Z}^2; \mathbb{Z})$, then*

$$|\pi_0(Hom(\Gamma, SU(2)))| = \begin{cases} 1 & \text{if } \beta = 0, \\ \frac{\beta+2}{2} & \text{if } \beta \neq 0 \text{ is even}, \\ \frac{\beta-1}{2} + 2 & \text{if } \beta \text{ is odd}. \end{cases}$$

As an example, we calculate the cohomology groups of the space of homomorphisms from the Heisenberg group to $SU(2)$:

Example 2.3.7. The Heisenberg group H fits in the extension

$$0 \to \mathbb{Z} \to H \to \mathbb{Z}^2 \to 0$$

which corresponds to the class $e_1^* \wedge e_2^* \in H^2(\mathbb{Z}^2, \mathbb{Z})$. In terms of generators and relations it is given by:

$$H = \langle E_1, E_2, X : [E_1, E_2] = X, X \text{ is central} \rangle.$$

The abelianization of H is $\mathbb{Z}^2$ and so we have a decomposition

$$\mathrm{Hom}(H, SU(2)) \cong C_2(SU(2)) \bigsqcup SO(3).$$

The cohomology groups in this case are given by:

$$H^i(\mathrm{Hom}(H, SU(2)); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if} \quad i = 0, \\ 0 & \text{if} \quad i = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if} \quad i = 2, \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{if} \quad i = 3, \\ \mathbb{Z}/2\mathbb{Z} & \text{if} \quad i = 4, \\ 0 & \text{if} \quad i \geq 5. \end{cases}$$

The quotient $SU(2)/\{\pm Id\}$ is homeomorphic to $SO(3)$, and the quotient map $SU(2) \to SO(3)$ induces a map $\rho : \mathrm{Hom}(H, SU(2)) \to \mathrm{Hom}(\mathbb{Z}^2, SO(3))$. The reason ρ lands on $\mathrm{Hom}(\mathbb{Z}^2, SO(3))$ is because if $f \in R_H$ then $f(X) = \pm Id$, which implies $(\rho f)(X) = Id$ so ρf factors through the abelianization of H. It was independently shown in [ACG13], [Roj13] and [TGS08] that the space $\mathrm{Hom}(\mathbb{Z}^2, SO(3))$ has exactly two components, the component containing the trivial homomorphism and one component homeomoprhic to $SO(3)/(\mathbb{Z}/2\mathbb{Z})^2$. We know that $C_2(SU(2))$ is connected so $\rho(C_2(SU(2)))$ is contained in the component of $\mathrm{Hom}(\mathbb{Z}^2, SO(3))$ which contains the trivial homomorphism. The component of $\mathrm{Hom}(H, SU(2))$ that is homeomorphic to $SO(3)$ gets mapped under ρ to the component of $\mathrm{Hom}(\mathbb{Z}^2, SO(3))$ that is homeomorphic to $SO(3)/(\mathbb{Z}/2\mathbb{Z})^2$, ρ is simply the quotient map.

2.4 The general case.

We will now go back to the general case and investigate $\mathrm{Hom}(\Gamma, SU(2))$ when Γ is a central extension of the form

$$0 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z}^k \to 0$$

with classifying cohomology class

$$\omega = (\omega_1, \dots, \omega_r) \in H^2(\mathbb{Z}^k; \mathbb{Z}^r) \cong \left(H^2(\mathbb{Z}^k; \mathbb{Z})\right)^r,$$

where

$$\omega_l = \sum_{1 \le i < j \le k} \beta^l_{i,j} e^*_i \wedge e^*_j \in H^2(\mathbb{Z}^k; \mathbb{Z}).$$

Here, $\{e_i\}_{i=1}^k$ are the standard generators of $\mathbb{Z}^k$, $\{e^*_i\}_{i=1}^k$ are the generators of the cohomology ring $H^*(\mathbb{Z}^k; \mathbb{Z})$ and $\beta^l_{i,j} \in \mathbb{Z}$. Such group Γ is given in terms of generators and relations by

$$\Gamma = \langle E_1, \dots, E_k, X_1, \dots, X_r : [E_i, E_j] = \prod_{l=1}^{r} X_l^{\beta^l_{i,j}}, X_i \text{ is central} \rangle.$$

As before, we have a decomposition into subspaces

$$\mathrm{Hom}(\Gamma, SU(2)) = \mathrm{Hom}(\Gamma/[\Gamma,\Gamma], SU(2)) \cup R_\Gamma,$$

and it is enough to understand each of those subspaces. As mentioned in the beginning of this chapter, the abelianization of Γ can be obtained via the exact sequence

$$H_2(\mathbb{Z}^k) \xrightarrow{\phi} H_1(\mathbb{Z}^r) \longrightarrow H_1(\Gamma) \longrightarrow H_1(\mathbb{Z}^k) \longrightarrow 0 \ ,$$

we obtain that $H_1(\Gamma) \cong \Gamma/[\Gamma,\Gamma] \cong \mathbb{Z}^k \oplus \mathrm{coker}(\phi)$, where

$$\phi : H_2(\mathbb{Z}^k) \to H_1(\mathbb{Z}^r)$$

is given in matrix notation by:

$$\phi = \begin{pmatrix} \beta^1_{1,2} & \beta^1_{1,3} & \cdots & \beta^1_{k-1,k} \\ \vdots & & & \vdots \\ \beta^r_{1,2} & \beta^r_{1,3} & \cdots & \beta^r_{k-1,k} \end{pmatrix}.$$

The cokernel of this matrix can be computed using its Smith normal form to be

$$\mathrm{coker}(\phi) \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_t \oplus \mathbb{Z}^{r-t}.$$

Where

$$a_i = \frac{d_i(\phi)}{d_{i-1}(\phi)},$$

$d_i(\phi)$ is the greatest common divisor of the $i \times i$ minors of ϕ and t is the rank of ϕ viewed as a transformation between $\mathbb{Q}$-vector spaces. After calculating the abelianization, we can use the results of Section 2.1 to describe $\mathrm{Hom}(\Gamma/[\Gamma,\Gamma], SU(2))$. All that is left to do is to describe R_Γ.

We have seen in Section 2.2 that R_Γ is homeomorphic to $R_{\overline{\Gamma}}$, where $\overline{\Gamma}$ is the reduction of Γ induced by $\mathbb{Z}^r \to (\mathbb{Z}/2\mathbb{Z})^r$. We can break down $R_{\overline{\Gamma}}$ into pieces by considering the restriction map

$$res : \mathrm{Hom}(\overline{\Gamma}, SU(2)) \to \mathrm{Hom}((\mathbb{Z}/2\mathbb{Z})^r, SU(2))$$

induced by the inclusion $(\mathbb{Z}/2\mathbb{Z})^r \subset \overline{\Gamma}$. Since the only elements of order 2 in $SU(2)$ are $\pm Id$ the space $\mathrm{Hom}((\mathbb{Z}/2\mathbb{Z})^r, SU(2))$ can be identified with $\mathrm{Hom}((\mathbb{Z}/2\mathbb{Z})^r, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^r$. Since this is a discrete set, the inverse images under res of the elements in $(\mathbb{Z}/2\mathbb{Z})^r$ induce a decomposition of the space $\mathrm{Hom}(\overline{\Gamma}, SU(2))$ into closed and open subspaces. This decomposition induces a decomposition of $R_{\overline{\Gamma}}$ into closed and open subspaces by defining, for each $v \in (\mathbb{Z}/2\mathbb{Z})^r$, the subsets

$$R^{(v)}_{\overline{\Gamma}} := R_{\overline{\Gamma}} \cap res^{-1}(v).$$

We have argued then the following Proposition:

Proposition 2.4.1. *There is a disjoint union decomposition*

$$R_{\overline{\Gamma}} = \bigsqcup_{v \in (\mathbb{Z}/2\mathbb{Z})^r} R_{\overline{\Gamma}}^{(v)}.$$

The description of these subsets can be made explicit as follows: a homomorphism $f \in \mathrm{Hom}(\overline{\Gamma}, SU(2))$ is a choice of matrices A_i for $i \in \{1, \ldots, k\}$ and $B_i = \pm Id$ for $i \in \{1, \ldots, r\}$ such that for all i, j:

$$[A_i, A_j] = \prod_{l=1}^{r} B_l^{\beta_{i,j}^l} \tag{2.5}$$

For every $v = (v_1, \ldots, v_r) \in (\mathbb{Z}/2\mathbb{Z})^r$ we can describe $R_{\overline{\Gamma}}^{(v)}$ by

$$R_{\overline{\Gamma}}^{(v)} = \{f \in R_{\overline{\Gamma}} : B_i = (-1)^{v_i} Id\}.$$

Given Proposition 2.4.1, to understand $R_{\overline{\Gamma}}$, we only need to understand $R_{\overline{\Gamma}}^{(v)}$ for each $v \in (\mathbb{Z}/2\mathbb{Z})^r$. Recall that we can view v as a homomorphism

$$v : (\mathbb{Z}/2\mathbb{Z})^r \to \mathbb{Z}/2\mathbb{Z}$$

so v induces a map

$$v_* : H^2(\mathbb{Z}^k; (\mathbb{Z}/2\mathbb{Z})^r) \to H^2(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z}).$$

Definition 2.4.2. Given $v \in (\mathbb{Z}/2\mathbb{Z})^r$ we define $\Gamma^{(v)}$ to be the extension corresponding to the cohomology class

$$\omega^{(v)} := v_*(\overline{\omega}) \in H^2(\mathbb{Z}^k; \mathbb{Z}/2\mathbb{Z}).$$

Explicitly, the cohomology class $\omega^{(v)}$ is given by

$$\omega^{(v)} = \sum_{1 \le i < j \le n} \beta_{i,j}^{(v)} e_i^* \wedge e_j^* \in H^2(\mathbb{Z}^n; \mathbb{Z}/2\mathbb{Z}),$$

where the coefficients $\beta_{i,j}^{(v)}$ are given in terms of the coefficients of $\overline{\omega}$, and v as

$$\beta_{i,j}^{(v)} = \sum_{l=1}^{r} v_l \overline{\beta_{i,j}^l} \in \mathbb{Z}/2\mathbb{Z}.$$

Note that if $f \in R_{\overline{\Gamma}}^{(v)}$, then the relations (2.5) simplify to:

$$\begin{aligned}
[A_i, A_i] &= \prod_{l=1}^{r} (-1)^{v_l \beta_{i,j}^l} Id \\
&= (-1)^{\sum_{l=1}^{r} v_l \beta_{i,j}^l} Id \\
&= (-1)^{\beta_{i,j}^{(v)}} Id,
\end{aligned}$$

which are precisely the relations required for a homomorphism in $R_{\Gamma^{(v)}}$. This proves:

Theorem 2.4.3. *If $\Gamma^{(v)}$ is as in Definition 2.4.2 then $R_{\overline{\Gamma}}^{(v)}$ is homeomorphic to $R_{\Gamma^{(v)}}$.*

Also, note that $R_{\Gamma^{(v)}}$ can be calculated using Theorem 2.3.3 since $\Gamma^{(v)}$ is a central extension with kernel of rank 1. The space $R_{\Gamma^{(v)}}$ will be homeomorphic to 2^{k-2} copies of $SO(3)$ if the rank of $\omega^{(v)}$ is 2 and it will be empty otherwise.

Let $T(\omega) : (\mathbb{Z}/2\mathbb{Z})^r \to 2\mathbb{N}$ be the function defined as:

$$T(\omega)(v) := \mathrm{rank}(v_*(\overline{\omega})) = \mathrm{rank}(\omega^{(v)}).$$

And let $t : H^2(\mathbb{Z}^k; \mathbb{Z}^r) \to \mathbb{N}$ be the function defined as:

$$t(\omega) = |T(\omega)^{-1}(2)|.$$

By putting together Theorems 2.3.3, 2.4.3, and Proposition 2.4.1 we finally obtain:

Theorem 2.4.4. *The space R_Γ is homeomorphic to $t(\omega) \times 2^{k-2}$ copies of $SO(3)$.*

Note that in particular this means that R_Γ is compact, so the decomposition

$$\mathrm{Hom}(\Gamma, SU(2)) = \mathrm{Hom}(\Gamma/[\Gamma,\Gamma], SU(2)) \sqcup R_\Gamma$$

is a coproduct. Our description of the cofactors is all in terms of $C_k(SU(2))$, $SO(3)$, and $\mathbb{S}^2 \times (\mathbb{S}^1)^k$, and the cohomology groups of all of these is known, so we are able to compute the cohomology groups of the space $\mathrm{Hom}(\Gamma, SU(2))$.

Concluding remarks.

In this short epilogue I will mention some of the possible directions in which one might try to extend this work, and I will explain some of the complications that arise.

The first thing one might try to do is get more information about the higher invariants for $C_k(O(n))$. One can try to do something similar to what we did to find the connected components, since we have a decomposition of $C_k(O(n))$ as a disjoint union of the $\mathcal{U}_j$'s, we can concentrate on understanding each $\mathcal{U}_j$ independently. We can then look at the surjective maps

$$\tilde{\Phi}_j : O(n)/A_j \times_{W(A_j)} P_j \to \mathcal{U}_j$$

to try to get information about $\mathcal{U}_j$. The first step would be to understand the value of the invariant we are looking for on $O(n)/A_j \times_{W(A_j)} P_j$. I anticipate this can potentially be done, since the space $O(n) \times P_j$ is understood and we understand the $W(A_j)$ action on it. What I think will be the harder problem is to know what $\tilde{\Phi}_j$ does to the invariant. This was the hard part in the case of the connected components as well. However, I do think this approach can work since one can get a good idea of the subspace where $\tilde{\Phi}_j$ is not injective and maybe, with a careful analysis of this subspace and depending on the invariant we are looking for, that could be enough. Notice that since $C_k(SO(n))$ is just a union of components of $C_k(O(n))$, and we have determined exactly which ones, so anything we can get for $O(n)$ we will have for $SO(n)$ as well.

Another thing one might try to do is get the exact number of components of $C_k(Spin(n))$. For this one could try first to describe the components of $C_k(SO(n))$ which have second Stiefel-Whitney class equal to zero. I think this is a problem that is combinatorial in nature. And secondly, one would need to understand how many components of $C_k(Spin(n))$ are mapped to each of those components under the mapped induced by the covering map.

There are two directions one could try to go when trying to extend the results in Chapter 2. The first one is to try to extend the target group. Good candidates for this might be $SU(p)$ for some prime p. The big difference here is that Proposition 2.2.1 is not true anymore for $p > 2$, and that simple fact was used all over the chapter to analyze R_Γ. One possibility to compensate this could be to stratify the space R_Γ by the rank of the centralizer of the image of the homomorphisms and analyze the strata one by one. I must admit I do not know how hard this could get.

The second direction is to try to generalize the source group. Good candidates for this are groups that are obtained by a sequence of extensions by free abelian groups. For example, one first step could be to look at the space $\mathrm{Hom}(\Gamma', SU(2))$ where Γ' fits in an extension of the form

$$0 \to \mathbb{Z}^n \to \Gamma' \to \Gamma \to 0$$

where Γ is a group like the ones considered in Chapter 2. An even less restrictive option would be to try to study $\mathrm{Hom}(N, SU(2))$ for a nilpotent group N, and someone more ambitious might want to try to directly look at $\mathrm{Hom}(N, SU(p))$. The idea here would be to somehow induct on the degree of nilpotency of the group, but I don't have a concrete idea of how this might work.

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Appendix A

Maximal abelian subgroups of $O(n)$.

In this short appendix we explain how every abelian subgroup $A \subset O(n)$ can be conjugated inside one of the form $A_j = (\mathbb{S}^1)^j \times (\mathbb{Z}_2)^{n-2j} \subset O(n)$. This proves that the subgroups A_j are representatives of the conjugacy classes of maximal abelian subgroups of $O(n)$ as required for the work in Chapter 1. This is purely a fact of linear algebra and it follows from the so called "normal form" for orthogonal matrices. I provide these proofs for completeness and state the results in ways that are useful for the applications in Chapter 1, but none of this is new in any way. Similar results can be found in various books in linear algebra such as [Gan77] or [Gal00].

We will use the following well known fact about abelian subgroups of $U(n)$

Proposition A.0.5. *If $A \subset U(n)$ is an abelian subgroup then all the elements of A can be simultaneously diagonalized by an orthonormal basis.*

Given $a \in O(n)$ we know we can diagonalize a over the complex numbers. Given an eigenvalue λ of a let a_λ be the eigenspace of a corresponding to λ. If λ is a non-real eigenvalue of a, then $\overline{\lambda}$ is also an eigenvalue of a and has the same multiplicity as λ. If $v \in a_\lambda$, then $\overline{v} \in a_{\overline{\lambda}}$. This means we can chose an orthonormal basis of $\{v_1, \overline{v}_1, \ldots, \overline{v}_j, x_1, \ldots, x_{n-2j}\}$ where the pairs $v_l, \overline{v_l}$ correspond to pairs of non-real eigenvalues $\lambda_l, \overline{\lambda}_l$ and the x_l's are real eigenvectors for the real eigenvalues. From this basis we can construct a new real orthonormal basis by letting $y_l = \frac{v_l - \overline{v}_l}{\sqrt{2}i}$ and $y_l' = \frac{v_l + \overline{v_l}}{\sqrt{2}}$for $l = 1, \ldots, j$. This gives a new set of real vectors $\{y_1, y_1' \ldots, y_j', x_1, \ldots, x_{n-2j}\}$ and it is routine to check they are orthonormal and so they form an orthonormal basis for $\mathbb{R}^n$.

If $\lambda_l = \cos\theta_l + i \sin\theta_l$ we have the following

$$
\begin{aligned}
ay_l &= \frac{av_l - a\overline{v}_l}{\sqrt{2}i} \\
&= \frac{(\cos\theta_l + i\sin\theta_i)v_l - (\cos\theta_l - i\sin\theta_i)\overline{v}_l}{\sqrt{2}i} \\
&= \cos\theta_l(\frac{v_l - \overline{v}_l}{\sqrt{2}i}) + \sin\theta_l(\frac{v_l + \overline{v}_l}{\sqrt{2}}) \\
&= \cos\theta_l y_l + \sin\theta_l y_l'
\end{aligned}
$$

Similarly we can show $ay_l' = -\sin\theta_l y_l + \cos\theta_l y_l'$. Which shows that if we let $p \in O(n)$ be the matrix which has as columns $(y_1, y_1' \ldots, y_j', x_1, \ldots, x_{n-2j})$ then

$$
p^{-1}ap = \begin{pmatrix}
\cos\theta_1 & -\sin\theta_1 & & \cdots & & & & \\
\sin\theta_1 & \cos\theta_1 & & & & & 0 & \\
& & \ddots & & & & & \\
\vdots & & & \cos\theta_j & -\sin\theta_j & & & \\
& & & \sin\theta_j & \cos\theta_j & & & \\
& 0 & & & & \pm 1 & & \\
& & & & & & \ddots & \\
& & \cdots & & & & & \pm 1
\end{pmatrix}
$$

This is known as the canonical form of an orthogonal matrix. Notice that we got p out of the diagonalization of a over $\mathbb{C}$. We have then the following fact.

Proposition A.0.6. *If $A \subset O(n)$ is an abelian subgroup then there is an orthogonal decomposition $W_1 \oplus \cdots \oplus W_j \oplus V_1 \oplus \cdots \oplus V_{n-2j} \cong \mathbb{R}^n$ such that every W_l has dimension 2, every V_l has dimension one, every element of A acts on each W_l by a rotation (possibly trivial) and every element of A acts by ± 1 on each V_l.*

Proof. By Corollary A.0.5 we can find an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{C}^n$ which diagonalizes all the elements in A. In other words each v_i is an eigenvector for each $a \in A$. Since $A \subset O(n)$ we can further assume that the basis has the form $\{v_1, \overline{v}_1, \ldots, v_j, \overline{v}_j, x_1, \ldots, x_{n-2j}\}$ where $x_1, \ldots, x_{n-2j}$ are all real vectors. We constuct the basis $\{y_1, y_1', \ldots, y_j, y_j', x_1, \ldots, x_{n-2j}\}$ as avobe and let $W_l = \langle y_l, y_l' \rangle$ and $V_l = \langle x_l \rangle$.

□

Consider the subroup $A_j = (\mathbb{S}^1)^j \times (\mathbb{Z}_2)^{n-2j} \subset O(n)$, where the inclusion is given as in Section 1.3. From Proposition A.0.6 we get the following two Corollaries.

Corollary A.0.7. *For every abelian subgroup $A \subset O(n)$ there exists $p \in O(n)$ and $j \in \mathbb{N}$ such that $p^{-1}Ap \subset A_j$.*

Corollary A.0.8. *If $(a_1, ..., a_k) \in C_k(O(n))$, then there exists $p \in O(n)$ and $j \in \mathbb{N}$ such that $p^{-1}a_l p \in A_j$ for all $l = 1, \ldots, k$.*

Proof. Let $A = \langle a_1, ..., a_k \rangle$ and apply Corollary A.0.7. □