Structure and randomness in dynamical systems: different forms of equicontinuity and sensitivity

by

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Abstract

We study topological and measure theoretic forms of mean equicontinuity and mean sensitivity for dynamical systems. With this we characterize well known notions like systems with discrete spectrum, almost periodic functions, and subshifts with regular extensions. We also study the limit behaviour of $\mu$-equicontinuous cellular automata.

In this thesis we prove a conjecture from [55] (see Corollary 2.3.18); this was independently solved by Li-Tu-Ye in [45]. In Chapter 3 we answer questions from [8].
Preface

This thesis is based on three manuscripts written by the author Felipe Garcia-Ramos which were submitted for publication (one accepted to Ergodic Theory and Dynamical Systems): Chapter 2 is based on [22] and [20], Chapter 3 is based on [21]. The material in Chapters 2.1.1, 2.1.2, and 2.3.3 is part of a paper in preparation with Brian Marcus.
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Dedication

I dedicate this thesis to Sandra my sister.
Chapter 1

Introduction

The theory of dynamical systems emerged as the study of how time affected physical systems in the long term. In the mathematical theory, time can be replaced by an action on a space. The abstraction of dynamical systems has provided many benefits. It has provided a way to study spatial structure as well as time behaviour with the same tools, and it has provided interesting connections with many other fields such as, Functional Analysis, Information Theory, Thermodynamics, Statistical Physics, Number Theory, Differential Geometry and Group Theory.

The study of dynamical systems can be divided into Smooth Dynamics, Topological Dynamics and Ergodic Theory. The work of this thesis is related mostly to Topological Dynamics and Ergodic Theory (measurable dynamics). A topological dynamical system (TDS) is a compact metric set with a continuous action. A measure preserving system (MPS) is a probability space with a measure preserving action. Similar properties for dynamical systems can be described in a purely topological and in a purely measure theoretical way (e.g. mixing, entropy, discrete spectrum). During my PhD studies my main research interest was the following: the comparison of results involving similar measure theoretical and topological properties and the study of hybrid properties, that is, properties that use topology and measure theory. In particular I am interested in studying and comparing dichotomies between chaos/randomness/unpredictability and order/structure/rigidity.

Vancouver is a very rainy city. During my first year of Ph.D. I entertained myself by doing models of how raindrops behave on windows. I wanted the models to capture the two different phases that I could see. When there is very little rain, most of the raindrops don’t move; the system is very predictable. When it is raining hard, the system looks chaotic and unpredictable. I presented my first seminar talk on this topic and in [19] I prove that a certain kind of discrete systems can be decomposed into products of chaotic elements (shifts) and predictable elements (identities). After this I began studying different ways in which predictability and unpredictability could be expressed in topological and measure theoretical terms.
1.1 Chapter 2

A topological dynamical system (TDS) is a pair $\langle X, T \rangle$, where $X$ is a compact metric space (a.k.a. phase space), and $T : X \to X$ is a continuous transformation (we will later study more general dynamical systems). Many definitions stated in this section will be stated later to make the thesis easier to read.

Let $X = S^1 = [0, 1)$ be the unit circle with normalized angles. The map defined by $T_x := x + \theta \mod 1$ represents a rotation by $\theta$ (see Figure 1.1). We have that $\langle X, T \rangle$ is a TDS.

\begin{center}
\begin{tikzpicture}
  \draw (0,0) circle (1);
  \fill (1,0) circle (0.05) node[above] {$X$};
  \fill (0,0) circle (0.05) node[above] {$T_x$};
  \draw (0:1 cm) -- (0:0 cm) node[above] {$\theta$};
\end{tikzpicture}
\end{center}

Figure 1.1: Rotation

Within a small neighbourhood it is very easy to predict how every point in it will behave. This system is predictable. A way to define predictability on topological dynamical systems is with equicontinuity. A TDS is equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) \leq \delta$ then $d(T^i x, T^i y) \leq \varepsilon$ for all $i \in \mathbb{Z}_+$. In other words, points that start close will always stay close. The prototype for an equicontinuous TDS is a rotation on a compact abelian group (e.g. the circle rotation).

Again let $X = S^1 = [0, 1)$ be the unit circle with normalized angles. The transformation defined by $T_x := 2x \mod 1$ is a TDS known as the doubling map (see Figure 1.2). We do not have a good prediction of how points in a small neighbourhood behave. This map is chaotic. A weak form of chaos is called sensitivity (or sensitive dependence on initial conditions). A TDS is sensitive if there exists $\varepsilon > 0$ such that for every open set $A \subset X$ there exists $x, y \in A$ and $i \in \mathbb{Z}_+$ such that $d(T^i x, T^i y) > \varepsilon$. This means that, wherever you start, a small change can cause a significant difference in the future. Sensitive and equicontinuous systems are related.

A TDS is minimal if it has no proper subsystems ($Y \subset X$ s.t. $T(Y) \subset Y$ and $Y$ is closed).
Theorem 1.1.1 (Auslander-Yorke '80). *A minimal TDS is equicontinuous if and only if it is not sensitive.*

Note that there are non-minimal systems that are neither sensitive nor equicontinuous. This notion of equicontinuity is very strong. For example every non-periodic shift space is not equicontinuous. A weaker form of equicontinuity was introduced by Fomin in '51. A TDS \((X, T)\) is **mean equicontinuous** if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(d(x, y) \leq \delta\) then

\[
\limsup_{n} \frac{1}{n} \sum_{i=1}^{n} d(T^i x, T^i y) \leq \varepsilon.
\]

Every equicontinuous TDS is mean equicontinuous. There are many interesting mean equicontinuous systems that are not equicontinuous. We will see examples later.

We say \((X, T)\) is **mean sensitive** if there exists \(\varepsilon > 0\) such that for every open set \(U\) there exists \(x, y \in U\) such that

\[
\limsup_{n} \frac{1}{n} \sum_{i=1}^{n} d(T^i x, T^i y) > \varepsilon.
\]

**Theorem 1.1.2** (G. see Theorem 2.3.17). *A minimal TDS is mean sensitive if and only if it is not mean equicontinuous.*

We say \((X, \mu, T)\) is a measure preserving topological dynamical system (MP-TDS) if \(X\) is a compact metric space, \((X, \mu)\) is a Borel probability space, and \(T : X \to X\) is a continuous measure preserving transformation \((\mu(T^{-1}(A)) = \mu(A))\). We say \((X, \mu, T)\) and \((X', \mu', T')\) are isomorphic (measure theoretically) if there exists a measure preserving function \(f : X \to X'\) that is 1-1 almost
everywhere, \( f^{-1} \) is measure preserving and satisfies \( T' \circ f = f \circ T \). To every MP-TDS we can associate an operator (known as a Koopman operator) in \( L^2(X, \mu) \) (the range is \( \mathbb{C} \)) given by \( U_T f := f \circ T \). We say \((X, \mu, T)\) has discrete spectrum if \( U_T \) has discrete spectrum (i.e. basis of eigenfunctions). A MP-TDS has discrete spectrum if and only if it is isomorphic to a group rotation (thus isomorphic to an equicontinuous system). We may ask how much equicontinuity is preserved by the measure isomorphisms. In purely topological terms the answer is none. There exist topologically chaotic dynamical systems that have discrete spectrum when equipped with any ergodic Borel measure. In hybrid terms (topological and measure theoretical) we can say something. We say \((X, \mu, T)\) is \( \mu \)-mean equicontinuous if for every \( \tau > 0 \) there exists a set \( M \) such that \( \mu(M) > 1 - \tau \) and \( T \) restricted to \( M \) is mean equicontinuous.

**Theorem 1.1.3** (G. see Theorem 2.3.17). Let \((X, \mu, T)\) be an ergodic MP-TDS. We have that \((X, \mu, T)\) has discrete spectrum if and only if it is \( \mu \)-mean equicontinuous.

Some characterizations of dynamical systems with discrete spectrum are the following:

**Functional analysis:** Associated Koopman operator on \( L^2(X, \mu) \) has discrete spectrum (Halmos/von-Neumann).

**Algebraic:** Isomorphic to a rotation on a compact abelian group. (Halmos/von-Neumann).

**Information theory:** Zero sequence entropy for any sequence (Kushnirenko).

**Purely dynamical:** \( \mu \)-mean equicontinuous (G.)

We also have a dichotomy with sensitivity in the hybrid case. We say \((X, \mu, T)\) is \( \mu \)-mean sensitive if there exists \( \varepsilon > 0 \) such that for every Borel set of positive measure \( U \) there exists \( x, y \in U \) such that

\[
\limsup \frac{1}{n} \sum_{i=1}^{n} d(T^i x, T^i y) > \varepsilon
\]

**Theorem 1.1.4** (G. see Theorem 2.3.17). Let \((X, \mu, T)\) be an ergodic MP-TDS. We have that \((X, \mu, T)\) is \( \mu \)-mean sensitive if and only if it is not \( \mu \)-mean equicontinuous.

### 1.2 Chapter 3

So far we have only discussed invariant measures. When studying non-invariant measures one may ask what happens in the long term; in particular we may ask if \( T^n \mu \) or the Cesàro average converges in the weak topology (where \( T \mu \) is the push-forward of the measure). We will study this problem for a particular kind of TDS, cellular automata on subshifts (also known as shift endomorphisms).

Cellular automata (CA) are discrete systems that depend on local rules. Hedlund [31] characterized CA as dynamical systems: \( \phi : X \rightarrow X \) is a cellular automaton on the subshift \( X \) if and only if \( \phi \) is continuous (with respect to the Cantor product topology) and shift-commuting. This means that if \( \phi \) is a CA then \((X, \phi)\) is a topological dynamical system (TDS).
In [8] Blanchard-Tisseur showed that if $\mu$ is a shift-ergodic measure that gives full measure to equicontinuity points and $\phi$ is a CA then the weak limit of the Cesàro average of $\phi^n\mu$ exists; also they asked questions about the dynamical behaviour of the limit measure. In particular they asked when the limit measure is shift-ergodic, a measure of maximal entropy or $\phi$–ergodic (p. 581 in [8]). In Chapter 3 we show that these three conditions are very strong.

**Theorem** (see Sections 3.2 and 3.3). Let $X$ be an irreducible SFT, $(X, \phi)$ CA, and $\mu$ a shift-ergodic Borel probability measure that gives full measure to equicontinuity points. Let $\mu_\infty$ be the weak limit of the Cesàro average of $\phi^n\mu$. We have that
- $\mu_\infty$ is the measure of maximal entropy if and only if $\mu$ is the measure of maximal entropy and $\phi\mu = \mu = \mu_\infty$ (Theorem 3.2.18).
- $\mu_\infty$ is $\phi$–ergodic if and only if $(X, \mu_\infty, \phi)$ is isomorphic (measurably) to a cyclic permutation on a finite set (Corollary 3.3.4).

Furthermore if we assume $\phi$ is surjective then
- $\mu_\infty$ is $\sigma$–ergodic if and only if $\mu$ is $\phi$–invariant (Theorem 3.2.17).

### 1.3 Definitions

Let $\{G, +\}$ be a locally compact semigroup. A $G$–**measure preserving system** (MPS) is a quadruple $(X, \Sigma, \mu, T)$ where $(X, \Sigma, \mu)$ is a Lebesgue probability measure space and $T := \{T^j : j \in G\}$ is a $G$– measure preserving action on $X$. When it is not useful we will omit writing $\Sigma$. A $\mathbb{Z}_+-$measure preserving system can be represented with a Lebesgue probability measure space $(X, \mu)$ and a measure preserving transformation $T : (X, \mu) \rightarrow (X, \mu)$. For simplicity in this thesis we will assume $G = \mathbb{Z}_+^d$ even though many results hold for more general actions. Nonetheless the results are new even for $d = 1$. We say a subset is invariant if it is invariant under every $T^i$; we say $(X, \mu, T)$ is an **ergodic system** if it is a measure preserving system and $\mu$ is an ergodic measure (i.e. every invariant set has measure 0 or 1). The measurable sets with positive measure are denoted with $\Sigma^+$.

A measure preserving system $(X, \mu, T)$ generates a family of unitary linear operators (known as the Koopman operators) on the complex valued Hilbert space $L^2(X, \mu)$, $U_T^j : f \mapsto f \circ T^j$. We denote the inner product of $L^2(X, \mu)$ with $\langle \cdot, \cdot \rangle$.

A $G$–**topological dynamical system** (TDS) is a pair $(X, T)$, where $X$ is a compact metric space and $T := \{T^i : i \in G\}$ is a $G$– continuous action on $X$. Given a compact metric space $X$ and a Borel probability measure $\mu$ we denote the Borel sets with $B_X$ and the Borel sets with positive measure with $B^+_X$.

An important class of TDS are the subshifts. Let $A$ be a finite set. For $x \in A^G$ and $j \in G$ we use $x_j$ to denote the $i$th coordinate of $x$ and

$$\sigma := \left\{ \sigma^i : A^G \rightarrow A^G \mid x_{i+j} = (\sigma^i x)_j \text{ for all } x \in A^G \text{and } j \in G \right\}$$
to denote the \textbf{shift maps}. Using the product topology of discrete spaces, we have that $\mathcal{A}^G$ is a compact metrizable space. A subset $X \subset \mathcal{A}^G$ is a \textbf{subshift (or shift space)} if it is closed and $\sigma$–invariant; in this case $(X, \sigma)$ is a TDS.
Chapter 2

Mean sensitivity, mean equicontinuity and spectral properties

2.1 Measure theoretical results

In this section we define mean sensitive functions and we characterize almost periodic functions. This provides a characterization of discrete and continuous spectrum for measure preserving systems.

Definition 2.1.1. Let $S \subset G$. We denote with $F_n$ the $n$-cube $[0,n]^d$. We define lower density of $S$ as

$$D(S) := \liminf_{n \to \infty} \frac{|S \cap F_n|}{|F_n|},$$

and upper density of $S$ as

$$\overline{D}(S) := \limsup_{n \to \infty} \frac{|S \cap F_n|}{|F_n|}.$$

The following properties are easy to prove and will be used throughout the paper.

Lemma 2.1.2. Let $S, S' \subset G$, $i \in G$, and $F \subset G$ finite set. We have that

1. $D(S) = D(i + S)$ and $\overline{D}(S) = \overline{D}(i + S)$.
2. $D(S) + \overline{D}(S^c) = 1$.
3. If $D(S) + \overline{D}(S') > 1$ then $S \cap S' \neq \emptyset$.
4. $D(S) := \liminf_{n \to \infty} \frac{|S \cap F_n \setminus F|}{|F_n \setminus F|}$
5. $\overline{D}(S) := \limsup_{n \to \infty} \frac{|S \cap F_n \setminus F|}{|F_n \setminus F|}$

Let $(X, \mu, T)$ be a MPS and $B \in \Sigma^+$. We say $x \in X$ is a generic point for $B$ if

$$\lim_{n \to \infty} \frac{|i \in F_n : T^i x \in B|}{|F_n|} = \mu(B).$$
The pointwise ergodic theorem states that if \((X, \mu, T)\) is ergodic and \(B \in \Sigma^+\) then almost every point is a generic point for \(B\). This theorem was originally proved for \(\mathbb{Z}_+\)–systems by Birkhoff. It also holds if \(G = \mathbb{Z}_d\) [28]. An equivalent version of the pointwise ergodic theorem states that if \((X, \mu, T)\) is ergodic and \(f \in L^1(X, \mu)\) then

\[
\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{i \in F_n} f(T^i x) = \int f \, d\mu \text{ for a.e } x.
\]

### 2.1.1 Almost periodic and \(\mu\)–mean sensitive functions

**Definition 2.1.3.** Let \((X, \mu, T)\) be an ergodic system and \(f \in L^2(X, \mu)\).

- We say \(f\) is an **almost periodic function** (i.e. \(f \in H_{ap}\)) if \(cl(\{U^j f : j \in \mathbb{G}\}) \subset L^2(X, \mu)\) is compact.
- We say \(f \neq 0\) is an **eigenfunction** of \((X, \mu, T)\) if there exists \(w \in \mathbb{R}^d\) such that

\[
U^j(f) = e^{2\pi i \langle w, j \rangle} f \quad \forall j \in \mathbb{G}.
\]

In this case we say \(w\) is an **eigenvalue** of \((X, \mu, T)\).

**Remark 2.1.4.** Equivalently eigenfunctions can be described with characters. Note that every continuous homomorphism from \(\mathbb{G}\) to the unit circle on \(\mathbb{C}\) is of the form \(\chi(j) = e^{2\pi i \langle w, j \rangle}\).

When \(d = 1\), \(f\) is an eigenfunction of \((X, \mu, T)\) if and only if there exists \(\lambda \in \mathbb{C}\) such that \(|\lambda| = 1\) and

\[
U_T(f) = \lambda f.
\]

**Definition 2.1.5.** A subset \(S \subset \mathbb{G}\) is **syndetic** if there exists a finite set \(K \subset \mathbb{G}\) such that \(S + K = \mathbb{G}\).

The following results are well known (see for example [57]).

**Proposition 2.1.6.** Let \((X, \mu, T)\) be an ergodic system.

- \(H_{ap}\) is the closure of the set spanned by all the eigenfunctions of \(T\).
- \(f \in L^2(X, \mu)\) is almost periodic if and only if for every \(\varepsilon > 0\) there exists a syndetic set \(S \subset \mathbb{G}\) such that \(\int |f - U^j f| \, d\mu \leq \varepsilon\) for every \(j \in S\).
- The product of two almost periodic functions is almost periodic.
- If \(f\) is an eigenfunction then \(|f|\) is constant almost everywhere.

**Definition 2.1.7.** Let \((X, \mu, T)\) and \((X', \mu', T')\) be two MPS. We say they are **isomorphic (measurably)** if there exists an a.e. bijective and measure preserving function \(f : (X, \mu) \to (X', \mu')\) such that \(T' \circ f = f \circ T\) and the inverse is also measure preserving.
Definition 2.1.8. Let \((X, \mu, T)\) be a ergodic system.

- \((X, \mu, T)\) has \textbf{discrete spectrum} if there exists an orthonormal basis for \(L^2(X, \mu)\) which consists of eigenfunctions of \(U_T\).
- \((X, \mu, T)\) has \textbf{continuous spectrum} if the only eigenfunctions are the constant functions.
- \((X, \mu, T)\) is a \textbf{measurable isometry} if it is isomorphic to an isometric TDS (i.e. \(d(x, y) = d(T^i x, T^i y)\)).
- \((X, \mu, T)\) is \textbf{weakly mixing} if \(T \times T\) is ergodic.

The following two results are due to Halmos and Von-Neumann (see for example [59]).

Theorem 2.1.9. Let \((X, \mu, T)\) be an ergodic system. The following are equivalent:

- \((X, \mu, T)\) has discrete spectrum.
- \((X, \mu, T)\) is a measurable isometry.
- \(L^2(X, \mu) = H_{ap}\)

Theorem 2.1.10. Let \((X, \mu, T)\) be an ergodic system. The following are equivalent:

- \((X, \mu, T)\) has continuous spectrum.
- \((X, \mu, T)\) is weakly mixing.
- \(H_{ap}\) consists only of constant functions.

Definition 2.1.11. Let \(f \in L^2(X, \mu)\). We say an MPS \((X, \mu, T)\) is \(\mu-\text{mean sensitive}\) if there exists \(\varepsilon > 0\) such that for every \(A \in \Sigma^+\) there exists \(x, y \in A\) such that

\[
\limsup \frac{1}{|F_n|} \sum_{j \in F_n} |f(T^j x) - f(T^j y)|^2 > \varepsilon.
\]

If this happens we say \(f\) is \(\mu-\text{mean sensitive}\) \((f \in H_{ms})\) and \(\varepsilon\) is a \textbf{sensitivity constant}.

Remark 2.1.12. Let \((X, \mu, T)\) be an ergodic system and \(B \in \Sigma^+\). It is not difficult to see that \((X, T)\) is \(\mu-1_B-\text{mean sensitive}\) if and only if there exists \(\varepsilon > 0\) such that for every \(A \in \Sigma^+\) there exists \(x, y \in A\) such that

\[
\overline{D} \left\{ i \in \mathbb{G}: T^i x \in B \text{ and } T^i y \notin B \text{ or } T^i y \in B \text{ and } T^i x \notin B \right\} \geq \varepsilon.
\]

Lemma 2.1.13. Let \((X, \mu, T)\) be an ergodic MPS. We have that \(H_{ms} \subset L^2(X, \mu)\) is an open set.

Proof. Let \(f \in H_{ms}\). There exists \(\varepsilon > 0\) such that for every \(A \in \Sigma^+\) there exists \(x, y \in A\) such that

\[
\limsup \frac{1}{|F_n|} \sum_{j \in F_n} |f(T^j x) - f(T^j y)|^2 > \varepsilon.
\]
Assume that \( \int |f - g|^2 \, d\mu \leq (\varepsilon/4) \). By the pointwise ergodic theorem there exists \( Y \subset X \) with \( \mu(Y) = 1 \) and
\[
\lim \frac{1}{|F_n|} \sum_{j \in F_n} \left| f(T^j x) - g(T^j x) \right|^2 \leq (\varepsilon/4) \text{ for every } x \in Y.
\]
We have that for every \( A \in \Sigma^+ \) there exist \( x, y \in A \cap Y \) such that
\[
\limsup \frac{1}{|F_n|} \sum_{j \in F_n} \left| g(T^j x) - g(T^j y) \right|^2 \geq \limsup \frac{1}{|F_n|} \sum_{j \in F_n} \left| f(T^j x) - f(T^j y) \right|^2 > \varepsilon/2.
\]
This implies that \( H_{ms} \subset L^2(X,\mu) \) is an open set.

**Theorem 2.1.14.** Let \((X,\mu,T)\) be an ergodic MPS and \( f \in L^2(X,\mu) \). The following are equivalent.

1) \( f \in H_{ap}^c \)

2) \( f \in H_{ms} \)

3) There exists \( \varepsilon > 0 \) such that for every \( A \in \Sigma^+ \) there exists \( x, y \in A \) such that
\[
\lim \frac{1}{|F_n|} \sum_{j \in F_n} \left| f(T^j x) - f(T^j y) \right|^2 > \varepsilon.
\]

Thus we have that \( H_{ap}^c = H_{ms} \).

**Proof.** 3) \( \Rightarrow \) 2)

Obvious (lim implies lim sup).

1) \( \Rightarrow \) 3)

Assume that the closure of \( \{U^j f : j \in \mathbb{G}\} \) is not compact. This implies it is not totally bounded, hence there exists \( \varepsilon > 0 \) such that for every \( N \in \mathbb{N} \) there exists \( S_N \subset \mathbb{G} \) with \( |S_N| = N \) such that
\[
\int \left| U^i f - U^j f \right|^2 \, d\mu \geq \varepsilon \text{ for every } i \neq j \in S_N. \tag{2.1.1}
\]

We want to show that \((X,T)\) is \( \mu - f \)-mean sensitive. Let \( A \in \Sigma^+ \) and \( N \) sufficiently large. There exist \( s \neq t \in S_N \) such that
\[
\mu(T^{-s} A \cap T^{-t} A) > 0.
\]

Let \( g(x) := \left| U^s f(x) - U^t f(x) \right|^2 \). By the pointwise ergodic theorem we have that there exists \( z \in T^{-s} A \cap T^{-t} A \) such that
\[
\lim \frac{1}{|F_n|} \sum_{j \in F_n} g(T^j z) = \int g \, d\mu \geq \varepsilon.
\]
Let \( p := T^s z \in A \) and \( q := T^t z \in A \). Since \( G \) is commutative we have that

\[
\lim \frac{1}{|F_n|} \sum_{j \in F_n} |f(T^j p) - f(T^j q)|^2
\]

\[
= \lim \frac{1}{|F_n|} \sum_{j \in F_n} |f(T^{j+s} z) - f(T^{j+t} z)|^2
\]

\[
= \lim \frac{1}{|F_n|} \sum_{j \in F_n} g(T^j z) > \varepsilon.
\]

We conclude that for every \( A \in \Sigma^+ \) there exist \( p, q \in A \) such that

\[
\lim \frac{1}{|F_n|} \sum_{j \in F_n} |f(T^j p) - f(T^j q)|^2 > \varepsilon.
\]

2) \( \Rightarrow \) 1)

Let \( g \) be an eigenfunction. This means there exists \( w \in \mathbb{R}^d \) such that \( U^j g = e^{2\pi i (w,j)} g \). Let \( \varepsilon > 0 \). There exists a set \( X_\varepsilon \in \Sigma^+ \) such that \( |g(x) - g(y)|^2 \leq \varepsilon \) for every \( x, y \in X_\varepsilon \). Since \( g \) is an eigenfunction we have that for every \( x, y \in X_\varepsilon \) and every \( j \in G \)

\[
|U^j g(x) - U^j g(y)|^2
\]

\[
= \left|e^{2\pi i (w,j)} (g(x) - g(y))\right|^2
\]

\[
= |g(x) - g(y)|^2 \leq \varepsilon.
\]

Thus \( g \in H^c_m \)

Using Lemma 2.1.13 we conclude that \( H_{ap} \subset H^c_m \).

**Proposition 2.1.15.** Let \((X, \mu, T)\) be an ergodic system. We have that

1) \((X, \mu, T)\) has discrete spectrum if and only if \( 1_B \in H_{ap} \) for every \( \{B, B^c\} \subset \Sigma^+ \).

2) \((X, \mu, T)\) is weakly mixing if and only if \( 1_B \notin H_{ap} \) for every \( \{B, B^c\} \subset \Sigma^+ \).

**Proof.** 1) \( \Rightarrow \)

Use Theorem 2.1.9.

1) \( \Leftarrow \)

Since \( H_{ap} \) is a closed subspace (Proposition 2.1.6) and indicator functions are dense in \( L^2(X, \mu) \)
we have that \( H_{ap} = L^2(X, \mu) \).

2) \( \Rightarrow \)

If \((X, \mu, T)\) is weakly mixing then \( H_{ap} \) contains only constant functions (Theorem 2.1.10) so \( 1_B \notin H_{ap} \) for every \( \{B, B^c\} \subset \Sigma^+ \).

2) \( \Leftarrow \)
Assume \((X, \mu, T)\) does not have continuous spectrum. This implies there exists a non-constant eigenfunction \(f\). This means there exists \(w \in \mathbb{R}^d\) such that
\[
U^j(f) = e^{2\pi i \langle w, j \rangle} f \quad \forall j \in G.
\]

By Proposition 2.1.6 we have that \(|f| = r\) is constant almost everywhere. This implies there exists \(A \subset \{x \in \mathbb{C} : |x| = r\}\) such that \(0 < \mu(H) < 1\) where \(H := f^{-1}(A)\). For every \(\varepsilon > 0\) there exists a syndetic set \(S \subset G\) such that
\[
\mu(H \triangle T^j H) \leq \varepsilon
\]
for every \(j \in S\). Using Proposition 2.1.6 we conclude \(1_H\) is almost periodic.

**Corollary 2.1.16.** Let \((X, \mu, T)\) be an ergodic system. We have that

1) \((X, \mu, T)\) has discrete spectrum if and only if \(1_B \in H_{ms}^c\) for every \(\{B, B^c\} \subset \Sigma^+\).

2) \((X, \mu, T)\) is weakly mixing if and only if \(1_B \notin H_{ms}^c\) for every \(\{B, B^c\} \subset \Sigma^+\).

### 2.1.2 \(\mu - f\)—Mean expansivity

**Definition 2.1.17.** Let \((X, \mu, T)\) be a measure preserving system and \(f : X \to \mathbb{C}\). We define the following pseudometric.
\[
d_f(x, y) = \limsup \frac{1}{|F_n|} \sum_{j \in F_n} \left| f(T^j x) - f(T^j y) \right|^2.
\]

The \(\varepsilon\)–closed balls of the pseudometric will be denoted with \(B^f_\varepsilon(x) := \{ y : d_f(x, y) \leq \varepsilon \}\).

**Definition 2.1.18.** Let \(f \in L^2(X, \mu)\). A system \((X, \mu, T)\) is \(\mu - f\)–mean expansive if there exists \(\varepsilon > 0\) such that \(\mu \times \mu \{(x, y) : d_f(x, y) > \varepsilon \} = 1\).

**Lemma 2.1.19.** Let \((X, \mu, T)\) be an ergodic system. Then \(g(x) := \mu(\{y : d_f(x, y) \leq \varepsilon\})\) is constant for almost every \(x \in X\) and equal to \(\mu \times \mu \{(x, y) : d_f(x, y) \leq \varepsilon\}\).

**Proof.** One can show that \(d_f(x, y)\) is \(\mu \times \mu\)–measurable. This means that \(\{(x, y) : d_f(x, y) \leq \varepsilon\}\) is \(\mu \times \mu\)–measurable for every \(\varepsilon > 0\). Using Fubini’s Theorem we obtain that
\[
\mu \times \mu \{(x, y) : d_f(x, y) \leq \varepsilon\} = \int_X \int_X 1_{\{(x, y) : d_f(x, y) \leq \varepsilon\}} d\mu(y) d\mu(x)
\]
\[
= \int_X \mu \{y : d_f(x, y) \leq \varepsilon\} d\mu(x).
\]
Since \( g \) is \( T \)-invariant we conclude that \( g(x) \) is constant for almost every \( x \in X \) and equal to 
\[ \mu \times \mu \{ (x, y) : d_f(x, y) < \varepsilon \} . \]

**Theorem 2.1.20.** Let \( (X, \mu, T) \) be an ergodic system and \( f \in L^2(X, \mu) \). The following are equivalent:

1) \( (X, T) \) is \( \mu - f \)-mean sensitive.  
2) \( (X, T) \) is \( \mu - f \)-mean expansive.  
3) There exists \( \varepsilon > 0 \) such that for almost every \( x \), \( \mu(B_{\varepsilon}^f(x)) = 0 \).

**Proof.** 2) \( \Rightarrow \) 1)  
Let \( A \in \Sigma^+ \). This means that \( A \times A \in \Sigma^+ \times \Sigma^+ \). By hypothesis we can find \( (x, y) \in A \times A \) such that \( d_f(x, y) \geq \varepsilon \).

1) \( \Rightarrow \) 3)  
Suppose \( (X, T) \) is \( \mu - f \)-mean sensitive (with \( \mu - f \)-mean sensitivity constant \( \varepsilon \)) and that 3) is not satisfied. This means there exists \( x \in X \) such that \( \{ x' : d_f(x, x') \leq \varepsilon/2 \} \in \Sigma^+ \). For any \( y, z \in \{ x' : d_f(x, x') \leq \varepsilon/2 \} \) we have that \( d_f(y, z) < \varepsilon \). This contradicts the assumption that \( (X, T) \) is \( \mu - f \)-mean sensitive.

3) \( \Rightarrow \) 2)  
Using Lemma 2.1.19 we obtain that \( \mu \times \mu \{ (x, y) : d_f(x, y) \leq \varepsilon \} = 0 \).

### 2.1.3 Sequence entropy

**Definition 2.1.21 ([44]).** Let \( (M, \mu, T) \) be a MPT. Given a finite measurable partition \( \mathcal{P} \) of \( X \) and \( S = \{ s_n \} \subset \mathbb{G} \) we define 
\[ h^S_\mu(\mathcal{P}, T) := \limsup_{n \to \infty} \frac{1}{n} H(\bigvee_{i=1}^{n} T^{-s_i} \mathcal{P}), \]
and the **sequence entropy of \( (X, \mu, T) \) with respect to \( S \) as \( h^S_\mu(T) := \sup_{\mathcal{P}} h^S_\mu(\mathcal{P}, T) \).** The system is said to be **\( \mu \)-null (or zero sequence entropy)** if \( h^S_\mu(T) = 0 \) for every \( S \subset \mathbb{G} \).

Kushnirenko’s theorem states that an ergodic system is \( \mu \)-null if and only if it has discrete spectrum [44].

**Theorem 2.1.22** (Kushnirenko ’67 [44]). Let \( (X, \mu, T) \) be an ergodic system and \( B \in \Sigma^+ \). Then there exists infinite \( S \subset \mathbb{G} \) so that \( h^S_\mu(\{ B, B^c \}, T) > 0 \) if and only if \( \text{cl}(\{ U^n 1_B : n \in \mathbb{G} \}) \subset L^2(X, \mu) \) is not compact. Thus an ergodic system is \( \mu \)-null if and only if it has discrete spectrum.

### 2.2 Topological results

The metric and \( \varepsilon \)-closed balls on a compact metric space \( X \) will be denoted by \( d \) and \( B_\varepsilon(x) \) respectively.
Mathematical definitions of chaos have been widely studied. Many of them require the system to be sensitive. A TDS \((X, T)\) is sensitive if there exists \(\varepsilon > 0\) such that for every open set \(A \subset X\) there exists \(x, y \in A\) and \(i \in G\) such that \(d(T^i x, T^i y) > \varepsilon\). On the other hand, equicontinuity represents predictable behaviour. A TDS is equicontinuous if \(T\) is an equicontinuous family. Auslander-Yorke showed that a minimal TDS is either sensitive or equicontinuous [4]. A problem with this classification is that equicontinuity is a strong property and not adequate for subshifts; it is not difficult to see that a subshift is equicontinuous if and only if it is finite.

The topological version of Halmos-Von Neumann Theorem states that for transitive TDS, equicontinuous maps can be characterized as those with topological discrete spectrum, i.e., the induced operator on \(C(X)\) has discrete spectrum (e.g. see [59]). It is easy to see that any equicontinuous TDS has zero topological entropy. Similar to (measure) sequence entropy one can define topological sequence entropy. A null system is a TDS that has zero topological sequence entropy. It is well known that equicontinuity implies nullness, but the converse is false [27]. Nevertheless, one can ask if there is a sense in which every null TDS is “nearly” equicontinuous. Indeed, in the minimal case there is. Any TDS has a unique maximal equicontinuous factor [3], and Huang-Li-Shao-Ye [34] showed that for any minimal null TDS \((X, T)\), the factor map from \(X\) to its maximal equicontinuous factor is 1-1 on a residual set (i.e., \((X, T)\) is almost automorphic). We strengthen this result for subshifts in Corollary 2.2.37, by showing that the factor map is 1-1 on a set of full Haar measure (i.e., \((X, T)\) is regular).

In order to establish Corollary 2.2.37, we introduce another weak topological form of equicontinuity that we call diam-mean equicontinuity (stronger than mean equicontinuity). We show that for minimal subshifts, nullness implies diam-mean equicontinuity (Corollary 2.2.36) and that an almost automorphic subshift is diam-mean equicontinuous if and only if it is regular (Theorem 2.2.24).

In conclusion for minimal subshifts we have the following implications:

\[
\begin{align*}
\text{Top. discrete spectrum} & = \text{equicontinuity} \\
& \implies \text{nullness} \\
& \implies \text{diam-mean equicontinuity} \\
& \implies \text{mean equicontinuity} \\
& \implies \mu - \text{mean equicontinuity (for every ergodic measure} \mu) \\
& = \text{every ergodic measure has discrete spectrum}
\end{align*}
\]

2.2.1 Mean equicontinuity and mean sensitivity

**Definition 2.2.1.** We define \(\Delta_\delta(x, y) := \{i \in G : d(T^ix, T^iy) > \delta\}\) and the Besicovitch pseudometric as \(d_\delta(x, y) := \inf \{\delta > 0 : D(\Delta_\delta(x, y)) < \delta\}\). By identifying points that are at pseudo-
distance zero we obtain a metric space \((X/d_b,d_b)\) that will be called the **Besicovitch space**. The projection \(f_b : (X,d) \to (X/d_b,d_b)\) will be called the **Besicovitch projection**. The \(\varepsilon\)-closed balls of the Besicovitch pseudometric will be denoted by \(B^b_\varepsilon(x)\).

One can check that in fact this is a pseudometric using that \(D(S) + D(S') \geq D(S \cup S')\).

**Definition 2.2.2.** Let \((X,T)\) be a TDS. We say \(x \in X\) is a **mean equicontinuity point** if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(y \in B_\delta(x)\) then

\[
d_b(x,y) < \varepsilon
\]

(equivalently \(D(i \in \mathbb{G} : d(T^i x, T^i y) \leq \varepsilon) \geq 1 - \varepsilon\)). We say \((X,T)\) is **mean equicontinuous** (or **mean-L-stable**) if every \(x \in X\) is a mean equicontinuity point. We say \((X,T)\) is **almost mean equicontinuous** if the set of mean equicontinuity points is residual.

Mean equicontinuous systems were introduced by Fomin [17]. They have been studied in [2], [52], [55] and [45].

In [7] equicontinuity with respect to the Besicovitch pseudometric (of the shift) was studied for cellular automata; this is a different property than mean equicontinuity.

If \(x \in X\) is a mean equicontinuity point then \(f_b\) is continuous at \(x\). This implies the Besicovitch projection is continuous if and only if \((X,T)\) is mean equicontinuous.

**Remark 2.2.3.** If \((X,T)\) is mean equicontinuous then \(f_b\) is continuous and hence \(f_b\) is uniformly continuous; this means that \((X,T)\) is uniformly mean equicontinuous i.e. for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(d(x,y) \leq \delta\) then \(D(i \in \mathbb{G} : d(T^i x, T^i y) \leq \varepsilon) \geq 1 - \varepsilon\).

**Lemma 2.2.4.** Let \((X,T)\) be a TDS. We have \(d_b\) is equivalent to the following pseudometric

\[
d_b'(x,y) := \limsup_{n \to \infty} \frac{1}{|F_n|} \sum_{i \in F_n} d(T^i x, T^i y).
\]

**Proof.** Without loss of generality assume the diameter of \(X\) is bounded by 1.

\(\Leftarrow\)

Let \(\varepsilon > 0\). Assume that \(\delta < \varepsilon/2\) and that

\[
\overline{D} \{ j \in \mathbb{G} : d(T^j x, T^j y) > \delta \} < \delta.
\]
This implies that
\[
\limsup \frac{1}{|F_n|} \sum_{j \in F_n} d(T_i x, T_i y) \\
\leq D \{ j \in G : d(T_i x, T_i y) > \delta \} + D \{ j \in G : d(T_i x, T_i y) \leq \delta \} \delta \\
\leq 2\delta \\
< \varepsilon.
\]

\[\Rightarrow\]

Let \( \varepsilon > 0 \). Assume that \( \delta \leq \varepsilon^2 \) and that
\[
\limsup \frac{1}{|F_n|} \sum_{j \in F_n} d(T_i x, T_i y) < \delta.
\]

Suppose that
\[
D \{ j \in G : d(T_i x, T_i y) > \varepsilon \} \geq \varepsilon.
\]

We have that
\[
\limsup \frac{1}{|F_n|} \sum_{i \in F_n} d(T_i x, T_i y) \\
\geq \varepsilon D \{ i \in G : d(T_i x, T_i y) > \varepsilon \} \\
\geq \varepsilon^2 \\
\geq \delta.
\]

A contradiction, hence
\[
D \{ j \in G : d(T_i x, T_i y) > \varepsilon \} < \varepsilon.
\]

It is well known that transitive equicontinuous systems are minimal. We give a similar result by weakening one hypothesis and strengthening the other.

**Definition 2.2.5.** A TDS \((X, T)\) is **strongly transitive** if for every open set \(U\) there exists a transitive point \(x \in U\) that returns to \(U\) with positive lower density.

**Theorem 2.2.6.** Every strongly transitive mean equicontinuous system is minimal.

**Proof.** Let \(x, y \in X\) and \(\varepsilon > 0\). Since the system is strongly transitive there exists a transitive point \(z \in B_{\varepsilon/2}(y)\) such that \(a := D \{ i : T^i z \in B_{\varepsilon/2}(y) \} > 0\). Since the system is mean equicontinuous there exists \(\delta > 0\) such that if \(w \in B_\delta(x)\) then \(d_\delta(x, w) \leq \min \{ \varepsilon/2, a \}\). There exists \(t_1 \in G\) such
that $T^{t_1}z \in B_\delta(x)$. By Lemma 2.1.2 (third bullet) there exists $t_2 \in G$ such that $T^{t_2}z \in B_{\epsilon/2}(y)$ and $d(T^{t_2}x, T^{t_2}z) \leq \epsilon/2$; thus $T^{t_2}x \in B_\epsilon(y)$. This means the system is minimal.

A similar result is known for null systems (Definition 2.2.33), i.e. every Banach transitive null system is minimal [34]. It is an open question whether every transitive recurrent null system is minimal [34][26].

Definition 2.2.7. We denote the set of mean equicontinuity points by $E^m$ and we define

$$E^m_\epsilon := \left\{ x \in X : \exists \delta > 0 \forall y, z \in B_\delta(x), \exists \delta \in G : d(T^iy, T^iz) \leq \epsilon \right\} \geq 1 - \epsilon.$$ 

Note that $E^m = \bigcap_{\epsilon > 0} E^m_\epsilon$.

Lemma 2.2.8. Let $(X, T)$ be a TDS. The sets $E^m$, $E^m_\epsilon$ are inversely invariant (i.e. $T^{-j}(E^m) \subseteq E^m$, $T^{-j}(E^m_\epsilon) \subseteq E^m_\epsilon$ for all $j \in G$) and $E^m_\epsilon$ is open.

Proof. Let $j \in G$, $\epsilon > 0$, and $x \in T^{-j}E^m_\epsilon$. There exists $\eta > 0$ such that if $d(T^jx, z) \leq \eta$ then $\exists \delta \in G : d(T^{i+j}x, T^{i+j}z) \leq \epsilon \geq 1 - \epsilon$. There exists $\delta > 0$ such that if $d(x, y) < \delta$ then $d(T^jx, T^jy) < \eta$ (and hence $\exists \delta \in G : d(T^{i+j}x, T^{i+j}y) \leq \epsilon \geq 1 - \epsilon$). We conclude that $x \in E^m_\epsilon$. This implies $E^m$ is also inversely invariant.

Let $x \in E^m_\epsilon$ and $\delta > 0$ be a constant that satisfies the property of the definition of $E^m_\epsilon$. If $d(x, w) < \delta/2$ then $w \in E^m_\epsilon$. Indeed if $y, z \in B_{\delta/2}(w)$ then $y, z \in B_\delta(x)$. \qed

Definition 2.2.9. A TDS $(X, T)$ is mean sensitive if there exists $\epsilon > 0$ such that for every open set $U$ there exist $x, y \in U$ such that

$$\exists \delta \in G : d(T^ix, T^iy) > \epsilon.$$ 

Other strong forms of sensitivity have been studied in [50] where $S$ was taken to be cofinite (complement is finite) or syndetic (bounded gaps).

Definition 2.2.10. Let $(X, T)$ be a TDS. We say $x \in X$ is a transitive point if $\{ T^ix : i \in G \}$ is dense. We say $(X, T)$ is transitive if $X$ contains a transitive point. If every $x \in X$ is transitive then we say the system is minimal.

It is not hard to see that mean sensitive systems have no mean equicontinuity points, as a matter of fact we have the following dichotomies.

Theorem 2.2.11. A transitive system is either almost mean equicontinuous or mean sensitive. A minimal system is either mean equicontinuous or mean sensitive.
Proof. First, we show that if \((X, T)\) is a transitive system then \(E^m_\epsilon\) is either empty or dense. Assume \(E^m_\epsilon\) is non-empty and not dense. Then \(U = X \setminus E^m_\epsilon\) is a non-empty open set. Since the system is transitive and \(E^m_\epsilon\) is open (Lemma 2.2.8) there exists \(t \in G\) such that \(U \cap T^{-t}(E^m_\epsilon)\) is non empty. By Lemma 2.2.8 we have that \(U \cap T^{-t}(E^m_\epsilon) \subset U \cap E^m_\epsilon = \emptyset\); a contradiction.

If \(E^m_\epsilon\) is non-empty for every \(\epsilon > 0\) then we have that \(E^m = \cap_{n \geq 1} E^m_{1/n}\) is a residual set; hence the system is almost equicontinuous.

If there exists \(\epsilon > 0\) such that \(E^m_\epsilon\) is empty, then for any open ball \(U = B_{\delta}(x)\) there exist \(y, z \in B_{\delta}(x)\) such that \(D \{i \in G : d(T^i y, T^i z) \leq \epsilon\} \leq 1 - \epsilon\); this means that \(\overline{D} \{i \in G : d(T^i y, T^i z) > \epsilon\} > \epsilon\). It follows that \((X, T)\) is mean sensitive.

Now suppose \((X, T)\) is minimal and almost mean equicontinuous. For every \(x \in X\) and every \(\epsilon > 0\) there exists \(t \in G\) such that \(T^t x \in E^m_\epsilon\). Since \(E^m_\epsilon\) is inversely invariant \(x \in E^m_\epsilon\) and hence \(x \in E^m\).

An analogous result appeared in [45].

### 2.2.2 Diam-mean equicontinuity and diam-mean sensitivity

**Definition 2.2.12.** Let \(A \subset X\). We denote the diameter of \(A\) as \(diam(A)\).

**Definition 2.2.13.** Let \((X, T)\) be a TDS. We say \(x \in X\) is a **diam-mean equicontinuity point** if for every \(\epsilon > 0\) there exists \(\delta > 0\) such that

\[
\overline{D} \{i \in G : diam(T^i B_{\delta}(x)) > \epsilon\} < \epsilon.
\]

We say \((X, T)\) is **diam-mean equicontinuous** if every \(x \in X\) is a diam-mean equicontinuity point. We say \((X, T)\) is **almost diam-mean equicontinuous** if the set of diam-mean equicontinuity points is residual.

**Remark 2.2.14.** Equivalently \(x \in X\) is a diam-mean equicontinuity point if for every \(\epsilon > 0\) there exists \(\delta > 0\) such that \(\overline{D} \{i \in G : diam(T^i B_{\delta}(x)) \leq \epsilon\} \geq 1 - \epsilon\).

By adapting the proof that a continuous function on a compact space is uniformly continuous one can show a TDS is diam-mean equicontinuous if and only if it is uniformly diam-mean equicontinuous i.e. for every \(\epsilon > 0\) there exists \(\delta > 0\) such that \(\overline{D} \{i \in G : diam(T^i B_{\delta}(x)) > \epsilon\} < \epsilon\).

**Definition 2.2.15.** We denote the set of diam-mean equicontinuity points by \(E^w\) and we define

\[
E^w_\epsilon := \{x \in X : \exists \delta > 0, \text{ s.t. } \overline{D} \{i \in G : diam(T^i B_{\delta}(x)) \leq \epsilon\} \geq 1 - \epsilon\}.
\]

Note that \(E^w = \cap_{\epsilon > 0} E^w_\epsilon\).

**Lemma 2.2.16.** Let \((X, T)\) be a TDS. The sets \(E^w\) and \(E^w_\epsilon\) are inversely invariant (i.e. \(T^{-j}(E^w) \subseteq E^w\) and \(T^{-j}(E^w_\epsilon) \subseteq E^w_\epsilon\) for all \(j \in G\)) and \(E^w_\epsilon\) is open.
Proof. Let \( \varepsilon > 0 \) and \( T^j x \in E_\varepsilon^w \). There exists \( \eta > 0 \) and \( S \subset \mathbb{G} \) such that \( D(S) \geq 1 - \varepsilon \) and 
\[
d(T^{i+j} x, T^i z) \leq \varepsilon \quad \text{for every } i \in S \text{ and } z \in B_\eta(T^j x).
\]
There exists \( \delta > 0 \) such that if \( d(x, y) < \delta \) then 
\[
d(T^j x, T^j y) < \eta. \]
We conclude that \( x \in E_\varepsilon^w \). Thus \( T^{-1}(E^w) \subseteq E^w \).

Let \( x \in E_\varepsilon^w \) and take \( \delta > 0 \) a constant that satisfies the property of the definition of \( E_\varepsilon^w \). If 
\[
d(x, w) < \delta/2 \text{ then } w \in E_\varepsilon^w. \] Indeed if \( y, z \in B_{\delta/2}(w) \) then \( y, z \in B_\delta(x) \). \( \square \)

**Definition 2.2.17.** A TDS \((X, T)\) is **diam-mean sensitive** if there exists \( \varepsilon > 0 \) such that for every open set \( U \) we have
\[
\overline{D}\{i \in \mathbb{G} : \text{diam}(T^i U) > \varepsilon\} > \varepsilon.
\]

The proof of the following result is analogous to that of Theorem 2.2.11 (using \( E_\varepsilon^w \) instead of \( E_\varepsilon^m \)).

**Theorem 2.2.18.** A transitive system is either almost diam-mean equicontinuous or diam-mean sensitive. A minimal system is either diam-mean equicontinuous or diam-mean sensitive.

### 2.2.3 Almost automorphic systems

Let \((X_1, T_1)\) and \((X_2, T_2)\) be two TDSs (over the same group) and \( f : X_1 \to X_2 \) a continuous function such that \( f \circ T_1 = T_2 \circ f \).

If \( f \) is surjective we say \( f \) is a **factor map** and \((X_2, T_2)\) is a **factor** of \((X_1, T_1)\). If \( f \) is bijective we say \( f \) is a **conjugacy** and \((X_1, T_1)\) and \((X_2, T_2)\) are **conjugate** (topologically).

Any TDS \((X, T)\) has a unique (up to conjugacy) **maximal equicontinuous factor** i.e. an equicontinuous factor \( f_{eq} : (X, T) \to (X_{eq}, T_{eq}) \) such that if \( f_2 : (X, T) \to (X_2, T_2) \) is a factor map such that \((X_2, T_2)\) is equicontinuous then there exists a unique factor map \( g : (X_{eq}, T_{eq}) \to (X_2, T_2) \) such that \( g \circ f_{eq} = f_2 \). The equivalence relation whose equivalence classes are the fibers of \( f_{eq} \) is called the **equicontinuous structure relation**. This relation can be characterized using the regionally proximal relation. We say that \( x, y \in X \) are **regionally proximal** if there exist sequences \( \{t_n\}_{n=1}^\infty \subset \mathbb{G} \) and \( \{x^n\}_{n=1}^\infty, \{y^n\}_{n=1}^\infty \subset X \) such that
\[
\lim_{n \to \infty} x^n = x, \quad \lim_{n \to \infty} y^n = y, \quad \text{and} \quad \lim_{n \to \infty} d(T^{t_n} x^n, T^{t_n} y^n) = 0.
\]

The equicontinuous structure relation is the smallest closed equivalence relation containing the regional proximal relation ([3] Chapter 9).

For mean equicontinuous systems we can characterize the maximal equicontinuous factor map using the Besicovitch pseudometric (Definition 2.2.1).

**Proposition 2.2.19.** Let \((X, T)\) be mean equicontinuous. Then \( f_{eq} = f_b \).

*Proof.* We have that \( f_b \) is continuous and \((X/d_b, d_b, T)\) is equicontinuous so if \( f_{eq}(x) = f_{eq}(y) \) then \( f_b(x) = f_b(y) \). If \( f_b(x) = f_b(y) \), then \( d_b(x, y) = 0 \) so there exists a sequence \( \{t_n\} \) such that 
\[
\lim_{n \to \infty} d(T^{t_n} x, T^{t_n} y) = 0; \quad \text{hence } x \text{ and } y \text{ are regionally proximal. We conclude } f_{eq}(x) = f_{eq}(y). \] \( \square \)
A transitive equicontinuous system is conjugate to a system where $G$ acts as a translation on a compact metric abelian group. If $(X,T)$ is a transitive TDS we denote the maximal equicontinuous factor by $G_{eq}$ (since it is a group). The TDS $(G_{eq},T_{eq})$ has a unique ergodic invariant probability measure, the normalized Haar measure on $G_{eq}$; this measure has full support and will be denoted by $\nu_{eq}$.

**Definition 2.2.20.** We say a TDS is **almost automorphic** if it is an almost 1-1 extension of its maximal equicontinuous factor i.e. if $f^{-1}_{eq}f_{eq}(x) = \{x\}$ on a residual set. A transitive almost automorphic TDS is **regular** if

$$\nu_{eq}\{g \in G_{eq} : f^{-1}_{eq}(g) \text{ is a singleton}\} = 1.$$ 

It is not difficult to see that if there exists a transitive point $x \in X$ such that $f^{-1}_{eq}f_{eq}(x) = \{x\}$ then $(X,T)$ is almost automorphic. Transitive almost automorphic systems are minimal.

Two well known families of almost automorphic systems are the Sturmian subshifts (maximal equicontinuous factor is an irrational circle rotation) and Toeplitz subshifts (maximal equicontinuous factor is an odometer, see Section 4.4 for definition and examples).

An important class of TDS are the shift systems. Let $A$ be a compact metric space (with metric $d_A$). For $x \in A^G$ and $i \in \mathbb{G}$ we use $x_i$ to denote the $i$th coordinate of $x$ and

$$\sigma := \{\sigma^i : A^G \to A^G \mid x_{i+j} = (\sigma^i x)_j \text{ for all } x \in A^G \text{ and } j \in \mathbb{G}\}$$

to denote the shift maps. Using the (Cantor) product topology generated by the topology of $A$, we have that $A^G$ is a compact metrizable space. A subset $X \subset A^G$ is a **general shift system** if it is closed and $\sigma$–invariant; in this case $(X,\sigma)$ is a TDS. Every TDS is conjugate to a general shift system (by mapping every point to its orbit).

A general shift system is a subshift if $A$ is finite.

**Remark 2.2.21.** Let $X \subset A^G$ be a general shift system. We have that $(X,\sigma)$ is diam-mean equicontinuous if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$ there exists $S \subset \mathbb{G}$ with $D(S) \geq 1 - \varepsilon$, such that if $d(x,y) \leq \delta$ then $d_A(x,y) \leq \varepsilon$ for all $i \in S$.

**Definition 2.2.22.** Let $(X,\sigma)$ be a transitive general shift system. We define

$$D := \{g \in G_{eq} : \exists x, y \text{ such that } f_{eq}(x) = f_{eq}(y) = g \text{ and } x_0 \neq y_0\},$$

where $x_0$ and $y_0$ represent the 0th coordinates of $x$ and $y$ respectively.

If $(X,\sigma)$ is almost automorphic we have that $D$ is first category. If $A$ is finite then it is closed and nowhere dense. Also note that $(X,\sigma)$ is regular if and only if $\nu_{eq}(D) = 0$. 

20
Lemma 2.2.23. Let \((X, \sigma)\) be a transitive almost automorphic subshift and \(F \subset \mathbb{G}\) a finite set. Then for all \(g \in G_{eq}\), there exists \(k \in \mathbb{G}\) such that \(T_{eq}^{k+i}(g) \notin D\) for all \(i \in F\).

Proof. Let \(D' := \bigcup_{i \in F} T_{eq}^{-i}(D)\). This means that \(D'\) is also closed and nowhere dense. Thus there exists \(k\) such that \(T_{eq}^{k}(g) \in G_{eq} - D'\), hence \(T_{eq}^{k+i}(g) \notin D\) for all \(i \in F\).

Theorem 2.2.24. Let \((X, \sigma)\) be a transitive almost automorphic subshift. Then \((X, \sigma)\) is regular if and only if it is diam-mean equicontinuous.

Proof. Suppose that \(\nu_{eq}(D) = 0\). Hence for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(U := \{x \mid d(x, f_{eq}^{-1}(D)) \leq \delta\}\) then \(\nu(U) < \varepsilon\). Let \(x \in X\). We define \(S := \{i \in \mathbb{G} \mid x_i \notin U\}\); hence \(D(S) = 1 - \nu(U) \geq 1 - \varepsilon\) (using the pointwise ergodic theorem). We conclude \((X, \sigma)\) is diam-mean equicontinuous because if \(d(x, y) \leq \delta\) then \(d_A(x_i, y_i) \leq \varepsilon\) for all \(i \in S\).

Now suppose \((X, \sigma)\) is diam-mean equicontinuous. We have that \(f_b = f_{eq}\) (Proposition 2.2.19). Let \(\varepsilon > 0\) and \(\delta > 0\) given by Remark 2.2.21.

Let \(x \in X\). Note that \(\sigma^i(x) \in f_{eq}^{-1}(D)\) if and only if there exists \(y \in f_b^{-1} \circ f_b(x)\) such that \(y_i \neq x_i\).

Using Lemma 2.2.23 there exists \(k \in \mathbb{G}\) such that if \(f_{eq}(x) = f_{eq}(y)\) then \(d(\sigma^k y, \sigma^k x) \leq \delta\).

Since \((X, \sigma)\) is a diam-mean equicontinuous there exists \(S \subset \mathbb{G}\), with \(D(S) \geq 1 - \varepsilon\), such that if \(d(\sigma^k x, \sigma^k y) \leq \delta\) then \(d(x_i, y_i) \leq \varepsilon\) for all \(i \in S\).

This means that for every \(\varepsilon > 0\)

\[
\mathcal{D} \{i : \exists y \in f_b^{-1} \circ f_b(x) \text{ s.t. } d_A(x_i, y_i) > \varepsilon\} \leq \varepsilon.
\]

Using that

\[
\mathcal{D} = \bigcup_{n \in \mathbb{N}} \{g \in G_{eq} : \exists x, y \text{ s.t. } f_{eq}(x) = f_{eq}(y) = g \text{ and } d_A(x_0, y_0) > 1/n\},
\]

and the pointwise ergodic theorem we conclude that \(\nu_{eq}(D) = 0\).

\[
\square
\]

Definition 2.2.25. A TDS is **diam-mean equicontinuous** if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that for every \(x \in X\) we have \(\mathcal{D} \{i \in \mathbb{G} : \text{diam}(T^i B_\delta(x)) > \varepsilon\} < \varepsilon\).

It is clear that every diam-mean equicontinuous system is diam-mean equicontinuous. We do not know if the converse is true in general. We will show that under some conditions they are equivalent.

Definition 2.2.26. A TDS \((X, T)\) is **diam-mean sensitive** if there exists \(\varepsilon > 0\) such that for every open set \(U\) we have \(\mathcal{D} \{i \in \mathbb{G} : \text{diam}(T^i U)) > \varepsilon\} > \varepsilon\).

The proof of the following theorem is very similar to the proof of Theorem 2.2.11.
**Proposition 2.2.27.** A minimal TDS is either diam-mean equicontinuous or diam-mean sensitive.

**Definition 2.2.28.** Let \((X, T)\) be a TDS and \(x \in X\). We denote the orbit of \(x\) with \(\sigma_T(x)\).

**Proposition 2.2.29.** Let \((X, \sigma)\) be a transitive almost automorphic subshift. The function \(h : D^c \to A\) defined as \(h(g) = (f_{eq}^{-1}(g))_0\) is continuous and there exists \(y \in X\) such that
- \(y\) is transitive.
- \(f_{eq}^{-1}f_{eq}(y)\) is a singleton, in other words \(o_{\sigma_{eq}}(y) \cap D = \emptyset\).

**Proof.** This follows from Theorem 6.4 in [14].

**Lemma 2.2.30.** Let \((X, \sigma)\) be a transitive almost automorphic subshift and \(w = a_0...a_{n-1} \in A^n\) and \(U = \{x : x_0...x_{n-1} = w\} \subset X\) a non-empty set. There exists \(p \in U\) such that \(p' = f_{eq}(p)\) is generic for \(D\) (with respect to \(\nu_{eq}\)) and \(\sigma_{eq}^i p' \in D^c\) for \(i = 0, ..., n - 1\).

**Proof.** Let \(U_a = \{g \in X_{eq} : g \notin D, (f_{eq}^{-1}(g))_0 = a\}\) and \(h : D^c \to A\) the continuous function from Proposition 2.2.29. This implies that for every \(a, U_a\) is an open set. Hence \(\bigcap_{i=0}^{n-1} \sigma_{eq}^{-1}U_{a_i}\) is an open set.

Let \(y \in X\) be the point given by Proposition 2.2.29. Since \(U\) is non-empty and \(y\) transitive, there exists \(z \in o_{\sigma_{eq}}(y) \cap U\). Considering that \(o_{\sigma_{eq}}(y) \cap D = \emptyset\) we obtain \(f_{eq}(z) \in \bigcap_{i=0}^{n-1} \sigma_{eq}^{-1}U_{a_i}\). Thus \(\bigcap_{i=0}^{n-1} \sigma_{eq}^{-1}U_{a_i}\) is a non-empty open set. Since \(\nu_{eq}\) is fully supported it contains a generic point for \(D\).

**Proposition 2.2.31.** Let \((X, \sigma)\) be a minimal almost automorphic subshift. If \((X, \sigma)\) is not regular then it is diam-mean sensitive.

**Proof.** Assume \((X, \sigma)\) is not regular. This means that \(\nu_{eq}(D) > 0\).

Let \(w \in A^n\) and \(U = \{x : x_0...x_{n-1} = w\} \subset X\) non-empty (these sets form a base of the topology). Let \(p \in U\) be the point given by the previous lemma. Let \(S := \{i \in G : \sigma_{eq}^i p' \in D\}\). Since \(p'\) is generic for \(D\) we have that \(D(S) = \nu_{eq}(D)\). Furthermore for every \(i \in S\) there exists \(q \in X\) such that \(f_{eq}(p) = f_{eq}(q)\) and \(p_i \neq q_i\). Since \(\sigma_{eq}^i p' \in D^c\) for \(j = 0, ..., n - 1\) we have that \(q \in U\). Hence \((X, \sigma)\) is diam-mean sensitive.

**Corollary 2.2.32.** Let \((X, \sigma)\) be a minimal almost automorphic subshift. The following conditions are equivalent:

1) \((X, \sigma)\) is diam-mean equicontinuous.
2) \((X, \sigma)\) is not diam-mean sensitive.
3) \((X, \sigma)\) is regular.
4) \((X, \sigma)\) is diam-mean equicontinuous.
5) \((X, \sigma)\) is not diam-mean sensitive.
Proof. Apply Theorem 2.2.18 to get 1) ⇔ 2).
Applying Theorem 2.2.24 to get 2) ⇔ 3).
By definition 1) ⇒ 4).
Proposition 2.2.31 implies 5) ⇒ 3).
Proposition 2.2.27 implies 4) ⇔ 5).

2.2.4 Topological sequence entropy

Let $\mathcal{U}$ and $\mathcal{V}$ be two open covers of $X$. We define $\mathcal{U} \lor \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ and $N(\mathcal{U})$ as the minimum cardinality of a subcover of $\mathcal{U}$.

**Definition 2.2.33 ([27]).** Let $(X, T)$ be a TDS, $S = \{s_m\}_{m=1}^{\infty} \subset \mathbb{G}$, and $\mathcal{U}$ an open cover. We define

$$h^S_{\text{top}}(T, \mathcal{U}) := \lim_{n \to \infty} \sup \frac{1}{n} \log N(\lor_{m=1}^{n} T^{-s_m}(\mathcal{U})).$$

The **topological entropy along the sequence $S$** is defined by

$$h^S_{\text{top}}(T) := \sup_{\text{open covers } \mathcal{U}} h^S_{\text{top}}(T, \mathcal{U}).$$

A TDS is **null** if the topological entropy along every sequence is zero.

**Lemma 2.2.34.** Let $K$ be a finite set, $\varepsilon > 0$, and $h : K \to 2^\mathbb{G}$ such that $D(h(k)) > \varepsilon$ for every $k \in F$. There exist $K' \subset K$ and $i \in \mathbb{G}$ with $|K'| \geq \varepsilon |K|/2$ such that $i \in h(k)$ for every $k \in K'$.

**Proof.** There exists $n_0 \in \mathbb{N}$ such that

$$\frac{|h(k) \cap F_{n_0}|}{|F_{n_0}|} \geq \varepsilon/2 \quad \text{for every } k \in K.$$

This means that

$$\sum_{j \in F_{n_0}} |k \in K : j \subset h(k)| = \sum_{j \in F_{n_0}} |j \in F_{n_0} : j \in h(k)| \geq \frac{\varepsilon}{2} |K||F_{n_0}|.$$

Hence there exists $j \in F_{n_0}$ such that $|k \in K : j \subset h(k)| \geq \frac{\varepsilon}{2} |K|$. □

**Theorem 2.2.35.** Let $(X, T)$ be a TDS. If $(X, T)$ is **diam-mean sensitive** then there exists $S \subset \mathbb{G}$ such that $h^S_{\text{top}}(T) > 0$. 


Proof. Let \((X,T)\) be \text{diam-mean} sensitive with sensitive constant \(\varepsilon\). Let \(U := \{U_1,\ldots,U_N\}\) be a finite open cover with balls with diameter smaller than \(\varepsilon/2\).

We will define the sequence \(S^n = \{s_1,\ldots,s_n\}\) inductively with \(s_1 = 1\). For every \(n \in \mathbb{N}\) we define \(L_n := \left\{ \cap_{i=1}^{n} T^{-s_i} U_{v_i}, \neq \emptyset : v \in \{1,\ldots,N\}^n \right\}\) (\(N\) is the size of the cover). We denote by \(L_n' := \{A_k\}_{k \leq N(L_n)}\) a subcover of \(L_n\) of minimal cardinality. We define the function \(f : L_n' \to 2^G\) as follows; \(m \in f(A_k)\) if and only if there exists \(x, y \in A_k \setminus \cup_{j < k} \overline{A}_j\) such that \(d(T^m x, T^m y) > \varepsilon\).

Assume \(S^n\) is defined. Since \((X,T)\) is \text{diam-mean} sensitive we have that \(D(f(U)) > \varepsilon\) for every \(U \in L_n'\). By Lemma 2.2.34 there exists \(g \in G\) such that \(\frac{|\{U \in L_n', g \in f(U)\}|}{N(L_n)} > \varepsilon/2\); we define \(s_{n+1} := g\). The definition of \(f\) implies that \(\frac{N(L_{n+1})}{N(L_n)} > 1 + \varepsilon/2\). Let \(S^\infty := \bigcup_{n \in \mathbb{N}} S^n\). Since \(N(L_n) = N(\forall_{i \in S^n} T^{-s_i}(U))\), we conclude that \(h_{\text{log}}^{S^n}(T,U) > 0\).

\begin{corollary}
Let \((X,T)\) be a minimal TDS. If \((X,T)\) is null then it is \text{diam-mean} equicontinuous.
\end{corollary}

\begin{proof}
Apply Theorem 2.2.35 and Proposition 2.2.27.
\end{proof}

The converse of this result is not true (Example 2.2.44). We will see that an ergodic TDS is \(\mu\)-null if and only if it is \(\mu\)-mean equicontinuous. If \((X,T)\) is mean equicontinuous and \(\mu\) is an ergodic measure then \((X,T)\) is \(\mu\)-mean equicontinuous, and hence it has zero entropy. This implies that mean equicontinuous and diam-mean equicontinuous systems have zero topological entropy.

Surprisingly it was shown in [45] that transitive almost mean equicontinuous subshifts can have positive entropy.

\begin{definition}
Let \((X,T)\) be a TDS and \(\mu\) an invariant measure. We say \((X,T)\) is \text{mean distal} if \(d_0(x,y) > 0\) for every \(x \neq y \in X\), and we say \((X,T)\) is \(\mu\)-\text{tight} if there exists \(X'\) such that \(\mu(X') = 1\) and \(d_0(x,y) > 0\) for every \(x \neq y \in X'\).
\end{definition}

Mean distal systems were studied by Ornstein-Weiss in [51]. They showed that any tight measure preserving \(\mathbb{Z}\)-TDS has zero entropy (assuming the system has finite entropy). A \(\mathbb{Z}_+\)-TDS has zero topological entropy if and only if it is the factor of a mean distal TDS [15][51].

Let \((X,T)\) be a mean equicontinuous TDS. Since \(f_{eq} = f_b\) we have that if \((X,T)\) is mean distal then \(f_{eq}\) is 1-1 hence \((X,T)\) is equicontinuous. So mean equicontinuity and mean distality are both considered rigid properties, and a TDS satisfies both properties if and only if it is equicontinuous.
Proposition 2.2.39. A mean equicontinuous TDS is mean distal if and only if it is equicontinuous.

A measure theoretic version of mean distality was also defined. Let \((X, T)\) be a TDS and \(\mu\) an ergodic measure. The system is \(\mu\)-tight if there exists \(X'\) such that \(\mu(X') = 1\) and \(d_\phi(x, y) > 0\) for every \(x \neq y \in X'\). Let \((X, T)\) be a transitive almost automorphic diam-mean equicontinuous TDS and \(\mu\) an invariant measure. Using Theorem 2.2.24 it is not hard to see that \((X, T)\) is \(\mu\)-tight.

2.2.5 Counter-examples

The following example shows there are mean equicontinuous not diam-mean equicontinuous TDS. We do not have a transitive counterexample.

Example 2.2.40. Let \(X \subset \{0, 1\}^\mathbb{Z}_+\) be the subshift consisting of sequences that contain at most one 1. For every \(x, y \in X\), \(d_b(x, y) = 0\), so \((X, \sigma)\) is mean equicontinuous. Nonetheless, for every \(\varepsilon > 0\), \(D\{i \in \mathbb{Z}_+ : \exists x \in B_{\varepsilon}(0^\infty) \text{ s.t. } x_i = 1\} = 1\) so \(0^\infty\) is not a diam-mean equicontinuity point.

In [40] the relationship between independence and entropy was studied. We will make use of their characterization of null systems.

Definition 2.2.41. Let \((X, T)\) be a TDS, and \(A_1, A_2 \subset X\). We say \(S \subset \mathbb{G}\) is an independence set for \((A_1, A_2)\) if for every non-empty finite subset \(F \subset S\) we have
\[
\cap_{i \in F} T^{-i} A_v(i) \neq \emptyset
\]
for any \(v \in \{1, 2\}^F\).

Theorem 2.2.42 ([40]). Let \((X, T)\) be TDS. The system \((X, T)\) is not null if and only if there exists \(x, y \in X\) (with \(x \neq y\)), such that for all neighbourhoods \(U_x\) of \(x\) and \(U_y\) of \(y\) there exists an arbitrarily large finite independence set for \((U_x, U_y)\).

The following example shows there are transitive non-null mean equicontinuous systems.

Example 2.2.43. Let \(S = \{2^n\}_{n=1}^\infty\),
\[
Y := \left\{ x \in \{0, 1\}^\mathbb{Z}_+ : x_i = 0 \text{ if } i \notin S \right\},
\]
and \(X\) the shift-closure of \(Y\). For every \(x, y \in X\) we have \(d_b(x, y) = 0\), hence \((X, \sigma)\) is mean equicontinuous. Nonetheless, since \(S\) is an infinite independence set for \(\{x_0 = 0\}, \{x_0 = 1\}\) we conclude that \((X, \sigma)\) is not null.

A \(\mathbb{Z}_+\)-subshift is Toeplitz if and only if it is the orbit closure of a regularly recurrent point, i.e. \(x \in X\) such that for every \(j > 0\) there exists \(m > 0\) such that \(x_i = x_{j+i+m}\) for all \(i \in \mathbb{Z}_+\). Toeplitz subshifts are precisely the minimal subshifts that are almost 1-1 extensions of odometers (for \(\mathbb{Z}_+\)-actions see [14], for finitely generated discrete group actions see [11]).
Given a Toeplitz subshift and a regularly recurrent point \( x \in X \), there exists a set of pairwise disjoint arithmetic progressions \( \{ S_n \}_{n \in \mathbb{N}} \) (called the **periodic structure**) such that \( \cup_{n \in \mathbb{N}} S_n = \mathbb{Z}_+ \), \( x_i \) is constant for every \( i \in S_n \), and every \( S_n \) is maximal in the sense that there is no larger arithmetic progression where \( x_i \) is constant. Let \( x \) be a regularly recurrent point and \( X \) the orbit closure. We have that \( (X, \sigma) \) is regular if and only if \( \sum_{n \in \mathbb{N}} D(S_n) = 1 \) (see [14]).

**Example 2.2.44.** There exists a regular Toeplitz subshift (hence diam-mean equicontinuous) with positive sequence entropy.

**Proof.** For every \( n \in \mathbb{N} \) let \( w^n \) be a finite word that contains all binary words of size \( n \). We denote the concatenation of \( w^n \) by \( w^{n, \infty} \in \{0, 1\}^{\mathbb{Z}_+} \).

We define the sequence \( \{ j_n \} \subset \mathbb{N} \) inductively with \( j_1 = 0 \) and \( j_{n+1} := \min \{ \cup_{m \leq n} \cup_{k \in \mathbb{N}} \{ k2^m + j_m \} \}^c \).

Let \( x \in \{0, 1\}^{\mathbb{Z}_+} \) be the point such that for every \( n \in \mathbb{N} \) we have that \( x_{j_n + 2i^n} = w_i^{n, \infty} \) for all \( i \in \mathbb{Z}_+ \).

We define \( X \) as the orbit closure of \( x \). Since \( x \) is regularly recurrent we obtain that \( X \) is a Toeplitz subshift (hence almost automorphic). By using the condition for regularity using the periodic structure (see comment before this proposition) we obtain that \( (X, \sigma) \) is regular. Hence by Theorem 2.2.24 we get that \( (X, \sigma) \) is diam-mean equicontinuous.

On the other hand \( w^k \) contains all the binary words of size \( k \). This implies there exists arbitrarily long independence sets for \( (\{x_0 = 0\}, \{x_0 = 1\}) \). Using Theorem 2.2.42 we conclude \( (X, \sigma) \) is not null.

Another class of rigid TDS are the tame systems introduced in [42][25]. These systems were characterized in [40] similarly to Theorem 2.2.42 but with infinitely large independence sets. This means that Example 2.2.43 is also not tame (note it is not minimal). The example in Section 11 in [40] is a tame non-regular Toeplitz subshift; this means there are tame minimal systems that are not diam-mean equicontinuous.

### 2.3 Topological and measure theoretical

We remind the reader that given a metric space \( X \) and a Borel probability measure \( \mu \) we denote the Borel sets with \( \mathcal{B}_X \) and the Borel sets with positive measure with \( \mathcal{B}_X^+ \).

#### 2.3.1 \( \mu \)-Mean equicontinuity

**Definition 2.3.1.** Let \( (X, T) \) be a TDS and \( \mu \) a Borel probability measure. We say \( (X, T) \) is **\( \mu \)-equicontinuous** if for every \( \tau > 0 \) there exists a compact set \( M \subset X \) with \( \mu(M) \geq 1 - \tau \) such that for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( x, y \in M \) and \( d(x, y) \leq \delta \) then

\[
d(T^ix, T^iy) \leq \epsilon \quad \forall i \in G.
\]
This definition of \( \mu \)-equicontinuity was introduced in [35] and it was shown every ergodic \( \mu \)-equicontinuous TDS has discrete spectrum. We introduce a weaker notion and characterize discrete spectrum.

If \((X,T)\) is an equicontinuous TDS and \(\mu\) a Borel probability measure then \((X,T)\) is \(\mu\)-equicontinuous.

We will see that there exists a sensitive TDS on \(\{0,1\}^\mathbb{Z}\) such that \((X,T)\) is \(\mu\)-equicontinuous for every ergodic Markov chain \(\mu\) (Example 3.1.29).

**Definition 2.3.2.** Let \((X,T)\) be a TDS and \(\mu\) a Borel probability measure. We say \((X,T)\) is \(\mu\)-mean equicontinuous if for every \(\tau > 0\) there exists a compact set \(M \subset X\) with \(\mu(M) \geq 1 - \tau\) such that for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that whenever \(x,y \in M\) and \(d(x,y) \leq \delta\) then

\[ d_b(x,y) \leq \varepsilon. \]

**Remark 2.3.3.** Since

\[ d_b(x,y) = \inf \left\{ \delta > 0 : \limsup_{n \to \infty} \frac{|\{i \in \mathbb{G} \mid d(T^i x,T^i y) > \delta\} \cap F_n|}{|F_n|} < \delta \right\}, \]

and \(\mu\) is Borel, \(b_b(x,y)\) is a Borel function. This implies that for every \(\varepsilon > 0\) and every \(x \in X\) we have that \(B_b^\varepsilon(x)\) is \(\mu\)-measurable.

Measure theoretic forms of sensitivity for TDS have been studied in [24],[9], and [35]. In particular in [35] it was shown that ergodic TDS are either \(\mu\)-equicontinuous or \(\mu\)-sensitive.

**Definition 2.3.4.** A TDS \((X,T)\) is \(\mu\)-mean sensitive if there exists \(\varepsilon > 0\) such that for every \(A \in \mathcal{B}_X^+\) there exists \(x,y \in A\) such that \(d_b(x,y) > \varepsilon\) (and hence \(D\{i \in \mathbb{G} : d(T^i x,T^i y) > \varepsilon\} > \varepsilon\)). In this case we say \(\varepsilon\) is a \(\mu\)-mean sensitivity constant.

**Definition 2.3.5.** A TDS \((X,T)\) is \(\mu\)-mean expansive if there exists \(\varepsilon > 0\) such that \(\mu \times \mu\{ (x,y) : d_b(x,y) > \varepsilon\} = 1\).

The following fact is well known. We give a proof for completeness.

**Lemma 2.3.6.** Let \((Y,d_Y)\) be a metric space. Suppose that there is no uncountable set \(A \subset Y\) and \(\varepsilon > 0\) such that \(d_Y(x,y) > \varepsilon\) for every \(x,y \in A\) with \(x \neq y\), then \((Y,d_Y)\) is separable.

**Proof.** For every \(\varepsilon > 0\) rational we define:

\[ F_\varepsilon := \{ A \subset Y : d_Y(x,y) > \varepsilon \ \forall x \neq y \in A \}. \]

Using Zorn’s lemma we obtain that \(F_\varepsilon\) admits a maximal element \(M_\varepsilon\), which by hypothesis must be countable. Then \(M := \cup_{\varepsilon \in \mathbb{Q}_+} M_\varepsilon\) is also countable. We have that for every \(x \in X\) and \(\varepsilon > 0\) there exists \(y \in M\) such that \(d_Y(x,y) \leq \varepsilon\). \(\square\)
Theorem 2.3.7. Let $(X, \mu, T)$ be an ergodic TDS. The following are equivalent:

1) $(X, T)$ is $\mu$–mean sensitive.
2) $(X, T)$ is $\mu$–mean expansive.
3) There exists $\varepsilon > 0$ such that for almost every $x$, $\mu(B^b_\varepsilon(x)) = 0$.
4) $(X, T)$ is not $\mu$–mean equicontinuous.

Proof. 1) $\iff$ 2) $\iff$ 3)

The proof is similar to that of Theorem 2.1.20 but we use $d_b$ instead of $d_f$.

2) $\implies$ 4)

If $(X, T)$ is $\mu$–mean expansive then there exists $\varepsilon > 0$ such that $\mu \times \mu \{ (x, y) : d_b(x, y) > \varepsilon \} = 1$. Suppose $(X, T)$ is $\mu$–mean equicontinuous. This implies there exists a compact set $M$ such that $\mu(M) > 0$ and $f_b|_M$ is continuous (and hence uniformly continuous). This implies there exists $\delta > 0$ such that if $x, y \in M$ and $d(x, y) \leq \delta$ then $d_b(x, y) \leq \varepsilon/2$. We can cover $M$ with finitely many $\delta/2$–balls, this implies one of them must have positive measure. Using this and $\mu$–mean expansiveness we would obtain that there exists $p, q \in M$ such that $d(p, q) \leq \delta$ and $d_b(p, q) > \varepsilon$; a contradiction to the continuity of $f_b|_M$.

4) $\implies$ 3)

Suppose 3) is not satisfied. Note that Lemma 2.1.19 using $d_b$ instead of $d_f$ can be proved analogously. Using this we have that for every $n \in \mathbb{N}$ there exists a set of full measure $Y_n$ and $a_n > 0$ such that $\mu(B^{b}_{1/a_n}(x)) = a_n$ for all $x \in Y_n$. Let $Y := \cap_{n \in \mathbb{N}} Y_n$. If $(Y/d_b, d_b)$ is not separable then by Lemma 2.3.6 there exists $\varepsilon > 0$ and an uncountable set $A$ such that for every $x, y \in A$ such that $x \neq y$ we have that $B^b_\varepsilon(x) \cap B^b_\varepsilon(y) = \emptyset$. This is a contradiction, hence $(Y/d_b, d_b)$ is separable. Using Proposition 2.3.31 (which is proved later in the paper) we conclude $(X, T)$ is $\mu$–mean equicontinuous.

Definition 2.3.8. Let $(X, \mu, T)$ and $(X', \mu', T')$ be two MPS. We say $(X', \mu', T')$ is isomorphic to $(X, \mu, T)$ if there exist a bijective measure preserving function $f : (X, \mu) \to (X', \mu')$ such that $f^{-1}$ is measure preserving and $T \circ f = f \circ T'$.

Proposition 2.3.9. Let $(X, \mu, T)$ and $(X', \mu', T')$ be two ergodic TDSs. If $(X, \mu, T)$ is $\mu$–mean equicontinuous and $(X', \mu', T')$ isomorphic to $(X, \mu, T)$ then $(X', \mu', T')$ is $\mu$–mean equicontinuous.

Proof. We will denote by $d$ and $d'$ the metrics of $X$ and $X'$ respectively.

Suppose $(X', \mu', T')$ is not $\mu$–mean equicontinuous. By the previous Theorem we have that $(X', \mu', T')$ is $\mu$–mean expansive, i.e. there exists $\varepsilon > 0$ and a set $Y' \subset X' \times X$ such that for every $(x', y') \in Y'$ we have that $d_b(x', y') > \varepsilon$ and $\mu \times \mu(Y') = 1$.

Let $f : X \to X'$ be the measure preserving isomorphism. By Lusin’s Theorem we know that there exists a compact set $K \subset X$ such that $\mu(K) \geq 1 - \varepsilon/4$ and $f |_K$ is continuous and bijective. This implies there exists $\varepsilon_1 > 0$ such that if $f(x), f(y) \in f(K)$ and $d'(f(x), f(y)) > \varepsilon$
then $d(x, y) > \varepsilon_1$. This implies $\overline{D}\left\{ i \in \mathbb{G} : d'(T^i f(x), T^i f(y)) > \varepsilon \right\} > \varepsilon$. We define

$$E_1(x, y) := \{ i \in \mathbb{G} : T^i x, T^i y \in K \}$$
$$E_2(x, y) := \{ i \in \mathbb{G} : d(T^i x, T^i y) > \varepsilon_1 \}.$$

Using that $\mu(K) \geq 1 - \varepsilon/4$ and the ergodic theorem we have that for almost every $x, y \in X$ we have that $(f(x), f(y)) \in Y'$ and $\overline{D}(E_1(x, y)) \geq 1 - \varepsilon/2$. Since $\left\{ i \in \mathbb{G} : d'(T^i f(x), T^i f(y)) > \varepsilon \right\} \subset E_2(x, y)$ we have that $\overline{D}(E_2(x, y)) \geq \varepsilon$. This implies that for a.e. $x, y \in X$ we have that $d'(T^i x, T^i y) > \varepsilon_1$ for every $i \in E_1(x, y) \cap E_2(x, y)$, and that $\overline{D}(E_1(x, y) \cap E_2(x, y)) \geq \varepsilon/2$. Hence $(X, \mu, T)$ is $\mu$–mean expansive.

\end{proof}

2.3.2 $\mu$–Mean sensitive pairs

The notion of entropy pairs was introduced in [6]. Different kinds of pairs have been studied, in particular sequence entropy pairs in [36] and $\mu$–sensitive pairs [35]. In [35] $\mu$–sensitive pairs were used to characterize $\mu$–sensitivity; we introduce $\mu$–mean sensitive pairs to characterize $\mu$–mean sensitivity.

**Definition 2.3.10 ([36]).** Let $(X, \mu, T)$ be a measure preserving TDS. We say $(x, y)$ is a $\mu$–sequence entropy pair if for any finite partition $P$, such that there is no $P \in \mathcal{P}$ such that $x, y \in \overline{P}$, there exists $S \subset \mathbb{G}$ such that $h^S_\mu(S, T) > 0$.

The following result was proven for $\mathbb{Z}_+$–actions in ([36] Theorem 4.3), and was proved for more general actions in [41].

**Theorem 2.3.11.** An ergodic TDS $(X, \mu, T)$ is $\mu$–null if and only if there are no $\mu$–sequence entropy pairs.

**Definition 2.3.12.** We say $(x, y) \in X^2$ is a $\mu$–mean sensitive pair if $x \neq y$ and for all open neighbourhoods $U_x$ of $x$ and $U_y$ of $y$, there exists $\varepsilon > 0$ such that for every $A \in \mathcal{B}_X^+$ there exist $p, q \in X$ with $\overline{D}(i \in \mathbb{G} : T^ip \in U_x \text{ and } T^iq \in U_y) > \varepsilon$. We denote the set of $\mu$–mean sensitive pairs as $S^m_\mu(X, T)$.

**Theorem 2.3.13.** Let $(X, \mu, T)$ be an ergodic TDS. Then $S^m_\mu(X, T) \neq \emptyset$ if and only if $(X, T)$ is $\mu$–mean sensitive.

**Proof.** If $(x, y) \in S^m_\mu(X, T)$ then there exists open neighbourhoods $U_x$ of $x$ and $U_y$ of $y$ (with $d(U_x, U_y) > 0$) and $\varepsilon > 0$ such that for every $A \in \mathcal{B}_X^+$ there exist $p, q \in A$ and $S \subset \mathbb{G}$ with $\overline{D}(S) > \varepsilon$ such that $T^ip \in U_x$ and $T^iq \in U_y$ for every $i \in S$. This implies that $\overline{D}\left\{ i \in \mathbb{G} : d(T^ip, T^iq) \geq d(U_x, U_y) \right\} > \varepsilon$. Thus $(X, T)$ is $\mu$–mean sensitive.

Let $(X, T)$ be $\mu$–mean sensitive with sensitive constant $\varepsilon_0$ and $0 < \varepsilon < \varepsilon_0$. 29
For $\varepsilon > 0$ we define the compact set
\[ X^\varepsilon := \{ (x, y) \in X^2 \mid d(x, y) \geq \varepsilon \}. \]

Suppose that $S^m_\mu(X, T) = \emptyset$. In particular this implies that for every $(x, y) \in X^\varepsilon$ there exist open neighbourhoods of $x$ and $y$, $U_x$ and $U_y$, such that for every $\delta > 0$ there exists a set $A_{\delta}(x, y) \in B^+_X$ such that
\[ \overline{D} \{ i \in \mathbb{G} : (T^i p, T^i q) \in U_x \times U_y \} \leq \delta \]
for all $p, q \in A_{\delta}(x, y)$.

There exists a finite set of points $F \subset X^\varepsilon$ such that
\[ X^\varepsilon \subseteq \bigcup_{(x, y) \in F} U_x \times U_y. \]

Let $\delta = \varepsilon / |F|$. Since $\mu$ is ergodic for every $(x, y) \in F$ there exists $n(x, y) \in \mathbb{G}$ such that $A := \cap_{(x, y) \in F} T^{n(x, y)} A_{\delta}(x, y) \in B^+_X$. Thus for every $(x, y) \in F$
\[ \overline{D} \{ i \in \mathbb{G} : (T^i p, T^i q) \in U_x \times U_y \} \leq \delta, \]
for every $p, q \in A$.

Since $\varepsilon$ is smaller than a sensitive constant there exist $p, q \in A$ such that
\[ \overline{D} \{ i \in \mathbb{G} : (T^i p, T^i q) \in X^\varepsilon \} > \varepsilon, \]
and hence
\[ \overline{D} \{ i \in \mathbb{G} : (T^i p, T^i q) \in \bigcup_{(x, y) \in F} U_x \times U_y \} > \varepsilon. \]

We have a contradiction since this means there exists $(x', y') \in F$ such that
\[ \overline{D} \{ i : (T^i p, T^i q) \in U_{x'} \times U_{y'} \} > \varepsilon / |F| = \delta. \]

2.3.3 $\mu – f$–Mean equicontinuity

We will make use of $d_f$ and $B^f_\varepsilon$ (see Definition 2.1.17).

**Definition 2.3.14.** Let $(X, T)$ be a TDS $\mu$ a Borel probability measure and $f \in L^2(X, \mu)$. We say $(X, T)$ is $\mu – f$–mean equicontinuous if for every $\tau > 0$ there exists a compact set $M \subset X$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in M$ and $d(x, y) \leq \delta$ then
\[ d_f(x, y) \leq \varepsilon. \]

In this case we say $f$ is $\mu$–mean equicontinuous ($f \in H_{me}$).
Theorem 2.3.15. Let $(X, T)$ be a TDS, $\mu$ ergodic and $f \in L^2(X, \mu)$. The following are equivalent:

1) $(X, T)$ is $\mu - f$–mean sensitive.

2) $(X, T)$ is $\mu - f$–mean expansive.

3) There exists $\varepsilon > 0$ such that for almost every $x$, $\mu(B_{\varepsilon}^f(x)) = 0$.

4) $(X, T)$ is not $\mu - f$–mean equicontinuous.

Proof. 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3)

Given by Theorem 2.1.20.

2) $\Rightarrow$ 4) and 4) $\Rightarrow$ 3)

The proof is similar to the proof of Theorem 2.3.7 but we use $d_f$ instead of $d_b$. \hfill \Box

Using this and Theorem 2.1.14.

Corollary 2.3.16. Let $(X, \mu, T)$ be an ergodic TDS. Then $H_{ap} = H_{ms}^c = H_{me}$.

Given a measure preserving system $(X, \mu, T)$ and a non-trivial measurable partition $\{B, B^c\}$ we can associate a shift invariant measure $\mu_B$ on $\{0, 1\}^G$ as follows.

We define the function $\psi : X \to \{0, 1\}^G$ with $(\psi(x))_j = 0$ if and only if $T^i x \in B$. We define $\mu_B = \psi \mu$. We have that $(\{0, 1\}^G, \mu_B, \sigma)$ is a factor of $(X, \mu, T)$.

Theorem 2.3.17. Let $(X, \mu, T)$ be an ergodic TDS. The following conditions are equivalent:

1) $(X, T)$ is $\mu$–mean equicontinuous

2) $(X, T)$ is $\mu - 1_B$–mean equicontinuous for every $\{B, B^c\} \subset B_X^+.

3) $(X, T)$ is $\mu - f$–mean equicontinuous for every $f \in L^2$.

4) $(X, \mu, T)$ has discrete spectrum.

5) $(X, T)$ is not $\mu$–mean sensitive.

Proof. 2) $\Leftrightarrow$ 3)

Use Proposition 2.1.15 and Corollary 2.3.16

3) $\Leftrightarrow$ 4)

Use Theorem 2.1.14 and Corollary 2.3.16.

1) $\Leftrightarrow$ 5)

Theorem 2.3.7.

4) $\Rightarrow$ 1)

Any isometry is $\mu$–mean equicontinuous which is an isomorphism invariant property (use Proposition 2.3.9).

1) $\Rightarrow$ 2)

Using Proposition 2.3.9 we obtain that $(\{0, 1\}^G, \sigma)$ is $\mu_B$–mean equicontinuous for every $\{B, B^c\} \subset B_X^+$. This implies $(X, T)$ is $\mu - 1_B$–mean equicontinuous for every $\{B, B^c\} \subset B_X^+$. \hfill \Box
Corollary 2.3.18. Let \((X, T)\) be a mean equicontinuous TDS and \(\mu\) an ergodic probability measure. Then \((X, \mu, T)\) has discrete spectrum.

There is work in progress regarding topological \(f\)-mean equicontinuity and the relationship with weakly almost periodic functions.

2.3.4 \(\mu\)-Mean equicontinuity points

Definition 2.3.19. Let \((X, T)\) be a TDS. We define the orbit metric \(d_T\) on \(X\) as 
\[
d_T(x, y) := \sup_{i \in \mathbb{G}} \{d(T^i x, T^i y)\},
\]
and the orbit balls as 
\[
B_T^\varepsilon(x) = \{y \mid d_T(x, y) \leq \varepsilon\} = \{y \mid d(T^i x, T^i y) \leq \varepsilon \forall i \in \mathbb{G}\}.
\]

Let \((X, T)\) be a TDS. A point \(x \in X\) is an equicontinuity point of \(T\) if and only if for every \(\varepsilon > 0\) there exists \(\delta\) such that 
\[
B_\delta(x) \subset B_T^\varepsilon(x).
\]

Based on [24] we define \(\mu\)-equicontinuity points.

Definition 2.3.20. A point \(x\) is a \(\mu\)-equicontinuity point of \((X, T)\) if for all \(\varepsilon > 0\) we have 
\[
\lim_{\delta \to 0} \frac{\mu(B_T^\varepsilon(x) \cap B_\delta(x))}{\mu(B_\delta(x))} = 1.
\]

If \(x\) is an equicontinuity point in the support of \(\mu\) then \(x\) is a \(\mu\)-equicontinuity point.

We will use the Besicovitch pseudometric \(d_b\) and the projection to the Besicovitch space \(f_b\) (Definition 2.2.1).

Definition 2.3.21. A point \(x\) is a \(\mu\)-mean equicontinuity point of \((X, T)\) if for all \(\varepsilon > 0\) we have 
\[
\lim_{\delta \to 0} \frac{\mu(B_b^\varepsilon(x) \cap B_\delta(x))}{\mu(B_\delta(x))} = 1.
\]

We will show when \(\mu\)-(mean) equicontinuity can be described locally using \(\mu\)-(mean) equicontinuity points.

Definition 2.3.22. We say \((X, \mu)\) satisfies Lebesgue’s density theorem if for every Borel set \(A\) we have that 
\[
\lim_{\delta \to 0} \frac{\mu(A \cap B_\delta(x))}{\mu(B_\delta(x))} = 1 \quad \text{for a.e. } x \in A. \quad (2.3.1)
\]

The original Lebesgue’s density theorem applies to \(\mathbb{R}^d\) and the Lebesgue measure. If \(X \subset \mathbb{R}^d\) and \(\mu\) is a Borel probability measure then \((X, \mu)\) satisfies Lebesgue’s density theorem (see Remark 2.4 in [38]).
Theorem 2.3.23 (Levy’s zero-one law [16] pg. 262). Let $\Sigma$ be a sigma-algebra on a set $\Omega$ and $P$ a probability measure. Let $\{\mathcal{F}_n\} \subset \Sigma$ be a filtration of sigma-algebras, that is, a sequence of sigma-algebras $\{\mathcal{F}_n\}$, such that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ for all $i$. If $D$ is an event in $\mathcal{F}_\infty$ (the smallest sigma-algebra that contains every $\mathcal{F}_n$) then $P(D \mid \mathcal{F}_n) \to 1$ almost surely.

Corollary 2.3.24. Let $X$ be a Cantor space and $\mu$ a Borel probability measure. Then $(X, \mu)$ satisfies Lebesgue’s density theorem.

Proof. Let $n \in \mathbb{N}$, and $\mathcal{F}_n$ the smallest sigma-algebra that contains all the balls $\{B_{1/n'}(x)\}$ with $n' \leq n$. It is easy to see that $\mathcal{F}_\infty$ is the Borel sigma-algebra on $X$. The desired result is a direct application from Theorem 2.3.23. \qed

There exist Borel probability spaces that do not satisfy Lebesgue’s density theorem (e.g. Example 5.6 in [38]).

Definition 2.3.25. We say $(X, \mu)$ is Vitali (or satisfies Vitali’s covering theorem) if for every Borel set $A \subset M$ and every $N, \varepsilon > 0$, there exists a finite subset $F \subset A$ and $n_x \geq N$ (defined for every $x \in F$) such that $\{B_{1/n_x}(x)\}_{x \in F}$ are disjoint and $\mu(A \setminus \cup_{x \in F} B_{1/n_x}(x)) \leq \varepsilon$.

If $X \subset \mathbb{R}^d$ and $\mu$ is a Borel probability measure then $(X, \mu)$ is Vitali ([32] p.8). This result is known as Vitali’s covering theorem.

Using a clopen base it is not hard to see that if $X$ is a Cantor space and $\mu$ a Borel probability measure then $(X, \mu)$ satisfies Vitali’s covering theorem. For more conditions see Chapter 1 in [32].

The following theorems will be proved at the end of the subsection.

Theorem 2.3.26. Let $(X, T)$ be a TDS and $\mu$ a Borel probability measure. Consider the following properties:

1) $(X, T)$ is $\mu$-equicontinuous.
2) Almost every $x \in X$ is a $\mu$-equicontinuity point.
3) There exists $X' \subset X$ such that $\mu(X') = 1$ and $(X', d_T)$ is separable

We have that 1) $\iff$ 3).

If $(X, \mu)$ is Vitali then 2) $\Rightarrow$ 1).

If $(X, \mu)$ satisfies Lebesgue’s density theorem then 1) $\Rightarrow$ 2).

Theorem 2.3.27. Let $(X, T)$ be a TDS and $\mu$ a Borel probability measure. Consider the following properties:

1) $(X, T)$ is $\mu$-mean equicontinuous.
2) Almost every $x \in X$ is a $\mu$-mean equicontinuity point.
3) There exists $X' \subset X$ such that $\mu(X') = 1$ and $(X', d_b)$ is separable

We have that 1) $\iff$ 3).

If $(X, \mu)$ is Vitali then 2) $\Rightarrow$ 1).

If $(X, \mu)$ satisfies Lebesgue’s density theorem then 1) $\Rightarrow$ 2).
Definition 2.3.28. Given a TDS \((X,T)\) we denote the change of metric projection map as \(f_T : (X,d) \to (X,d_T)\).

The function \(f_T^{-1}\) is always continuous. A point \(x \in X\) is an equicontinuity point of \((X,T)\) if and only if it is a continuity point of \(f_T\). Thus \((X,T)\) is equicontinuous if and only if \(f_T\) is continuous.

We have already seen that \((X,T)\) is mean equicontinuous if and only if \(f_b\) is continuous.

Definition 2.3.29. Let \(X,Y\) be metric spaces and \(\mu\) a Borel probability measure on \(X\). A function \(f : X \to Y\) is \(\mu\)–Lusin (or Lusin measurable) if for every \(\tau > 0\) there exists a compact set \(M \subset X\) such that \(\mu(M) > 1 - \tau\) and \(f|_M\) is continuous.

This implies that a TDS \((X,T)\) is \(\mu\)–equicontinuous if and only if \(f_T\) is \(\mu\)–Lusin and \((X,T)\) is \(\mu\)–mean equicontinuous if and only if \(f_b\) is \(\mu\)–Lusin.

Every \(\mu\)–Lusin function is \(\mu\)–measurable. Lusin’s theorem states the converse is true if \(Y\) is separable (note that \((X,d_T)\) is not necessarily separable). This fact is generalized in the following theorem.

Theorem 2.3.30 ([56] pg. 145). Let \(f : X \to Y\) be a function and \(\mu\) a Borel probability measure on \(X\) such that there exists \(X' \subset X\) such that \(\mu(X') = 1\) and \(f(X')\) is separable. The function \(f\) is \(\mu\)–Lusin if and only if for every ball \(B\), \(f^{-1}(B)\) is \(\mu\)–measurable.

Proposition 2.3.31. Let \((X,T)\) be a TDS and \(\mu\) a Borel probability measure. We have that \((X,T)\) is \(\mu\)–mean equicontinuous if and only if there exists \(X' \subset X\) such that \(\mu(X') = 1\) and \((X'/d_b,d_b)\) is separable; and \((X,T)\) is \(\mu\)–equicontinuous if and only if there exists \(X' \subset X\) such that \(\mu(X') = 1\) and \((X',d_T)\) is separable

Proof. Define \(f := f_b\).

If there exists \(X' \subset X\) such that \(\mu(X') = 1\) and \((X'/d_b,d_b)\) is separable apply Theorem 2.3.30 to obtain that \(f_b\) is \(\mu\)–Lusin and hence \((X,T)\) is \(\mu\)–mean equicontinuous.

If \(f_b\) is \(\mu\)–Lusin it means that for every \(\kappa > 0\) there exists a compact set \(M_\kappa \subset X\) such that \(\mu(M_\kappa) > 1 - \kappa\) and \(f_b|_{M_\kappa}\) is continuous. This implies that \(X' = \cup_{n \in \mathbb{N}} M_1/n\) satisfies the desired conditions.

The other part of the result is proved analogously with \(f := f_T\). \(\square\)

We will define a measure theoretic notion of continuity point (\(\mu\)–continuity point) that satisfies the following property: \(x \in X\) is a \(\mu\)–equicontinuity point of \(T\) if and only if \(x\) is a \(\mu\)–continuity point of \(f_T\). We will show that if \((X,\mu)\) is Vitali and almost every \(x \in X\) is a \(\mu\)–continuity point of \(f\) then \(f\) is \(\mu\)–Lusin; we show the converse is true if \((X,\mu)\) satisfies Lebesgue’s density theorem (Theorem 2.3.35).

In this subsection \(X\) will denote a compact metric space, \(\mu\) a Borel probability measure on \(X\), and \(Y\) a metric space with metric \(d_Y\) (and balls \(B^Y_\varepsilon(y) = \{z \in Y : d_Y(y,z) \leq \varepsilon\}\)).
Definition 2.3.32. In some cases we can talk about the measure of not necessarily measurable sets. Let \( A \subset X \). We say \( A \) has full measure if \( A \) contains a measurable subset with measure one. We say \( \mu(A) < \tau \), if there exists a measurable set \( A' \supset A \) such that \( \mu(A') < \tau \).

Definition 2.3.33. A set \( A \subset X \) is \( \mu \)-measurable if \( A \) is in the the completion of \( \mu \).

The function \( f : X \rightarrow Y \) is \( \mu \)-measurable if for every Borel set \( D \subset Y \), we have that \( f^{-1}(D) \) is \( \mu \)-measurable.

A point \( x \in X \) is a \( \mu \)-continuity point if for every \( \varepsilon > 0 \) we have that \( 1_{f^{-1}(B_{\delta}^Y(f(x)))} \) is \( \mu \times \mu \)-measurable and
\[
\lim_{\delta \to 0} \frac{\mu \left[ f^{-1}(B_{\delta}^Y(f(x))) \cap B_{\delta}(x) \right]}{\mu(B_{\delta}(x))} = 1. \tag{2.3.2}
\]
If \( \mu \)-almost every \( x \in X \) is a \( \mu \)-continuity point we say \( f \) is \( \mu \)-continuous.

Lemma 2.3.34. Let \((X, \mu)\) be Vitali and \( f : X \rightarrow Y \) \( \mu \)-continuous. For every \( \varepsilon, \tau > 0 \) there exists a finite set \( F_{\varepsilon, \tau} \) and a function \( \delta(x) \) such that \( \{ B_{\delta}(x) \}_{x \in F_{\varepsilon, \tau}} \) are disjoint, and
\[
\mu(\bigcup_{x \in F_{\varepsilon, \tau}} [f^{-1}(B_{\varepsilon}^Y(f(x))) \cap B_{\delta}(x)]) > 1 - \tau.
\]

Proof. Let \( \tau > 0 \). For every \( \mu \)-equicontinuity point \( x \) and \( \varepsilon > 0 \) there exists \( \delta(x) > 0 \) such that \( \frac{\mu(f^{-1}(B_{\varepsilon}^Y(f(x))) \cap B_{\delta}(x))}{\mu(B_{\delta}(x))} > 1 - \tau \) for every \( \delta \leq \delta(x) \). Let
\[
A_i := \{ x \mid x \text{ is a } \mu \text{-continuity point, } 1/\delta(x) \leq i \}.
\]

Using the function \( f = 1_{f^{-1}(B_{\varepsilon}^Y(f(x)))}I_{B_{\delta}(x)} \) and Fubini’s theorem one can check that \( \frac{\mu(f^{-1}(B_{\varepsilon}^Y(f(x))) \cap B_{\delta}(x))}{\mu(B_{\delta}(x))} \) is a measurable function and hence the sets \( A_i \) are also \( \mu \)-measurable. Since \( \bigcup_{i \in \mathbb{N}} A_i \) contains the set of \( \mu \)-continuity points, there exists \( N \) such that \( \mu(X \setminus A_N) < \tau \). Since \((X, \mu)\) is Vitali there exists a finite set of points \( F_{\varepsilon, \tau} \subset A_N \) and a function \( \delta(x) \geq 1/N \) such that \( \{ B_{\delta}(x) \}_{x \in F_{\varepsilon, \tau}} \) are disjoint and \( \mu(\bigcup_{x \in F_{\varepsilon, \tau}} B_{\delta}(x)) > \mu(A_N) - \tau \). This implies \( \mu(\bigcup_{x \in F_{\varepsilon, \tau}} B_{\delta}(x)) > 1 - 2\tau \). Since \( F_{\varepsilon, \tau} \subset A_N \), we obtain
\[
\mu(\bigcup_{x \in F_{\varepsilon, \tau}} [f^{-1}(B_{\varepsilon}^Y(f(x))) \cap B_{\delta}(x)]) > 1 - 3\tau.
\]

Theorem 2.3.35. Let \( f : X \rightarrow Y \). Consider the following properties

1) \( f \) is \( \mu \)-Lusin.

2) \( f \) is \( \mu \)-continuous.

If \((X, \mu)\) is Vitali then 2) \( \implies \) 1).

If \((X, \mu)\) satisfies Lebesgue’s density theorem then 1) \( \implies \) 2).

Proof. 2) + Vitali \( \implies \) 1)
Let $\tau, \varepsilon > 0$. By Lemma 2.3.34 there exists a finite set of points $F_{\varepsilon, \tau \varepsilon}$ and a function $\delta(x)$ (defined on $F_{\varepsilon, \tau \varepsilon}$) such that

$$
\mu(\bigcup_{x \in F_{\varepsilon, \tau \varepsilon}} [f^{-1}(B^Y_\varepsilon(f(x))) \cap B_{\delta(x)}(x)]) > 1 - \tau \varepsilon
$$

and $\{B_{\delta(x)}(x)\}_{x \in F_{\varepsilon, \tau \varepsilon}}$ are disjoint. Let

$$
x_1, x_2 \in \bigcup_{x \in F_{\varepsilon, \tau \varepsilon}} G_x := \bigcup_{x \in F_{\varepsilon, \tau \varepsilon}} [f^{-1}(B^Y_\varepsilon(f(x))) \cap B_{\delta(x)}(x)].
$$

If $x_1$ and $x_2$ are sufficiently close then they will be in the same set $G_x$ and hence $f(x_1) \in B^Y_{\varepsilon/2}(f(x_2))$.

Let

$$
M := \bigcap_{\varepsilon \in \{1/2^n : n \in \mathbb{N}\}} \bigcup_{x \in F_{\varepsilon, \tau \varepsilon}} [f^{-1}(B^Y_\varepsilon(f(x))) \cap B_{\delta(x)}(x)].
$$

Then $f \mid_M$ is continuous and $\mu(M) > 1 - \tau$. The regularity of the Borel measure gives us the existence of the compact set.

1) $\Rightarrow$ 2) $LDT$ Since $f$ is $\mu$–Lusin we have that it is $\mu$–measurable.

Let $\tau > 0$. There exists a compact set $M \subset X$ such that $\mu(M) > 1 - \tau$ and $f \mid_M$ is continuous. Using Lebesgue’s density theorem we have

$$
\lim_{\delta \to 0} \frac{\mu(M \cap B_\delta(x))}{\mu(B_\delta(x))} = 1 \text{ for a.e. } x \in M.
$$

Let $\varepsilon > 0$. Since $f \mid_M$ is continuous then for sufficiently small $\delta$ and almost every $x \in M$, $M \cap B_{\delta}(x) \subset f^{-1}(B^Y_\varepsilon(f(x)))$, and so for almost every $x$

$$
\lim_{\delta \to 0} \frac{\mu(f^{-1}(B^Y_\varepsilon(f(x))) \cap B_\delta(x))}{\mu(B_\delta(x))} = 1.
$$

This implies the set of $\mu$–continuity points has measure larger than $1 - \tau$. We conclude the desired result.

Proof of Theorem 2.3.26. Take $f = f_T$, $(Y, d_Y) = (X, d_T)$, $B^Y_m = B^T_m$.

Note that orbit balls are countable intersections of balls, hence measurable.

3) $\Rightarrow$ 1)

Apply Theorem 2.3.30.

1) $\Rightarrow$ 3)

If $f_T$ is $\mu$–Lusin, then that for every $\tau > 0$ there exists a compact set $M_\tau \subset X$ such that $\mu(M_\tau) > 1 - \tau$ and $f_T \mid_{M_\tau}$ is continuous. This implies that $X' = \cup_{n \in \mathbb{N}} M_1/n$ satisfies $\mu(X') = 1$ and $(X', d_T)$ is separable.

For the other implications apply Theorem 2.3.35. 

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Proof of Theorem 2.3.27. Analogous to the proof of Theorem 2.3.26 but using $f_b$ instead of $f_T$. $\Box$
Chapter 3

Limit behaviour of $\mu$–equicontinuous cellular automata

Cellular automata (CA) are discrete systems that depend on local rules. Hedlund [31] characterized CA using dynamical properties: $\phi : \{1, 2, \ldots, n\}^\mathbb{Z} \to \{1, 2, \ldots, n\}^\mathbb{Z}$ is a cellular automaton if and only if it is continuous (with respect to the Cantor product topology) and shift-commuting. This means every CA is a $\mathbb{Z}_+$–topological dynamical system, i.e. a continuous transformation $\phi$ on a compact metric space $X$. The dynamical behaviour of these systems can range from very predictable to very chaotic. There are different classifications of cellular automata and TDS using equicontinuity and sensitivity (see [1] and [43]).

Equicontinuity is a very strong property particularly for cellular automata; different attempts have been made to define weaker but similar properties. Using shift-ergodic probability measures, Gilman [24][23] introduced the weaker concept of $\mu$–equicontinuity points for cellular automata.

The study of long term behaviour is a main topic of interest of dynamical systems/ergodic theory. Long term behaviour can be studied for points, sets, or measures. In particular one may ask if the orbit of a measure converges weakly. Limit behaviour of measures under CA have been studied mainly for two subclasses, linear and $\mu$–equicontinuous.

In [46] Lind studied the limit behaviour of the CA on the binary full-shift defined by adding the value of two consecutive positions mod 2. He concluded that the weak limit of the Cesaro average of every Bernoulli measure is the uniform Bernoulli measure. This result has been generalized to other linear expansive CA, and it has been shown that the Cesaro weak limit of an ergodic Markov measure is the uniform Bernoulli (or in a more general setting the measure of maximal entropy)[54][48].

In [8] Blanchard and Tisseur studied CA and measures that give equicontinuity points full measure. These CA are $\mu$–equicontinuous but $\mu$–equicontinuous CA may not have any equicontinuity point (like in Example 3.1.29). They showed that the Cesaro weak limit exists and they asked questions about the dynamical behaviour of the limit measure. In particular they asked when
the limit measure is shift-ergodic, a measure of maximal entropy or $\phi$-ergodic. We address those question in this thesis; we show that those three conditions are very strong.

We characterize $\mu$-equicontinuity on shifts of finite type using locally periodic behaviour; $\mu$ – LEP (Proposition 3.1.21). We present a natural generalization of Blanchard-Tisseur’s result (Theorem 3.2.8). Let $\phi$ be a CA and $\mu$ a $\sigma$–ergodic measure that gives equicontinuity points full measure. We show the limit measure is of maximal entropy (Theorem 3.2.18) if and only if $\phi$ is surjective and the original measure is the measure of maximal entropy. We show that if $\phi$ is surjective then the limit measure is shift-ergodic if and only $\mu$ is $\phi$–invariant (Theorem 3.2.17). Finally we show that if the limit measure is ergodic with respect to $\phi$ then the system is isomorphic (measurably) to a cyclic permutation on a finite set (Corollary 3.3.4). Some of our results hold for CA on multidimensional subshifts, in the cases where they don’t we present weaker analogous results. We also present several results for $\mu$ – LEP and $\mu$–equicontinuous systems, which may not have any equicontinuity points (like in Example 3.1.29).

3.1 Equicontinuity and local periodicity

3.1.1 Definitions

The $n$–window, $W_n \subset \mathbb{Z}^d$ is defined as the cube of radius $n – 1$ centred at the origin ($W_1$ contains only the origin); a window is an $n$–window for some $n$. For any set $W \subset \mathbb{Z}^d$ and $x \in \mathcal{A}^{\mathbb{Z}^d}$, $x_W \in \mathcal{A}^W$ is the restriction of $x$ to $W$. We will endow $\mathcal{A}^{\mathbb{Z}^d}$ with the product topology; this is the same topology obtained by the metric given by $d(x, y) = \frac{1}{m}$, where $m$ is the largest integer such that $x_{W_m} = y_{W_m}$. We denote the balls with $B_{1/m}(x) = \{z \mid d(x, z) \leq \frac{1}{n}\}$.

We define the full $\mathcal{A}$-shift as the metric space $\mathcal{A}^{\mathbb{Z}^d}$. For $i \in \mathbb{Z}^d$ we will use $\sigma_i : \mathcal{A}^{\mathbb{Z}^d} \rightarrow \mathcal{A}^{\mathbb{Z}^d}$ to denote the shift maps (the maps that satisfy $x_{i+j} = (\sigma_i x)_j$ for all $x \in \mathcal{A}$ and $i, j \in \mathbb{Z}^d$). The algebra of sets generated by balls and their shifts is called the cylinder sets. In this chapter a subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a subshift (or shift space) if it is closed and $\sigma_i$–invariant for all $i \in \mathbb{Z}^d$. If $d = 1$ we say the space is 1D. A cellular automaton (CA) is a pair $(X, \phi)$ where $X$ is a subshift and $\phi(\cdot) : X \rightarrow X$ is a continuous $\sigma$–commuting map, i.e. $\phi$ commutes with all the $\mathbb{Z}^d$ shifts. We say $(X, \phi)$ is a 1D CA if $X$ is a 1D subshift. In most of the literature cellular automata is studied only on full-shifts. In our definition a 1D subshift itself is a CA. Cellular automata of this kind are also known as shift endomorphisms.

A one sided subshift is a set $X \subset \mathcal{A}^\mathbb{N}$ that is closed and $\sigma$–invariant (i.e $\sigma(X) \subset X$).

The following theorem was established in [31] for 1D CA on full-shifts. The same result holds for CA on higher dimensional subshifts.

**Theorem 3.1.1** (Curtis-Hedlund-Lyndon). Let $X$ be a shift space, and $\phi(\cdot) : X \rightarrow X$ a function. The map $\phi$ is a CA if and only if there exists a non-negative integer $n$, and a function $\phi(\cdot)$ :
\[ A^n \rightarrow A, \text{ such that } (\phi(x))_i = \phi[\sigma_i x]_W \] (note the use of ( ), and [ ] to distinguish between the two functions that are related). The radius of the CA is the smallest possible \( n \).

**Definition 3.1.2.** We say \( X \subset A^\mathbb{Z}^d \) is a **shift of finite type (SFT)** if there exists \( n \in \mathbb{N} \) and a finite list of forbidden patterns \( \{B_i\} \subset A^n \) such that \( x \in X \) if and only if none of the elements of \( \{B_i\} \) appear in \( x \).

**Example 3.1.3.** The 1D SFT obtained by the forbidden word \( \{11\} \) (i.e. the set of doubly infinite sequences that never have two 1’s together) is commonly known as the golden mean shift.

For more information on subshifts and SFTs see [47].

### 3.1.2 Topological equicontinuity and local periodicity

We remind the reader of the following definition. We introduce new notation.

**Definition 3.1.4.** Given a CA \((X, \phi)\), we define the **orbit metric** \( d_\phi \) on \( X \) as \( d_\phi(x, y) := \sup_{i \geq 0} \{d(\phi^i x, \phi^i y)\} \), and the **orbit balls** as

\[ O_{1/m}(x) := \left\{ y \mid d_\phi(x, y) \leq \frac{1}{m} \right\} = \left\{ y \mid (\phi^i(x))_j = (\phi^i(y))_j \forall i \in \mathbb{N}, j \in W_m \right\}. \]

Note that in this chapter we use \( O \) instead of \( B^\phi \) or \( B^T \). This is to make notation easier as we will use other superscripts.

A point \( x \) is an **equicontinuity point** of \( \phi \) if for all \( m \in \mathbb{N} \) there exists \( n \in \mathbb{N} \) such that \( B_{1/n}(x) \subset O_{1/m}(x) \). The transformation \( \phi \) is **equicontinuous** if every \( x \in X \) is an equicontinuity point.

We have that \( \phi \) is equicontinuous if and only if the family \( \{\phi^i\}_{i \in \mathbb{N}} \) is equicontinuous.

If \( X \) is a subshift with dense periodic points, then a CA \((X, \phi)\) is equicontinuous if and only if it is eventually periodic (i.e. there exists \( p \) and \( p' \) such that \( \phi^{p+p'} = \phi^p \))[43][18].

A weaker notion of periodicity that is related to equicontinuity is local periodicity.

**Definition 3.1.5.** Let \((X, \phi)\) be a CA. The set of **m-locally periodic points** of \( \phi \), \( LP_m(\phi) \), is the set of points \( x \) such that \( (\phi^i x)_W \) is periodic; the set of **locally periodic** points is defined as \( LP(\phi) := \cap LP_m(\phi) \). Similarly, the set of **m-locally eventually periodic points** of \( \phi \), \( LEP_m(\phi) \), is the set of points \( x \) such that \( (\phi^i x)_W \) is eventually periodic; the set of **locally eventually periodic points** is defined as \( LEP(\phi) := \cap LEP_m(\phi) \).

A transformation \( \phi \) is **locally eventually periodic (LEP)** if \( LEP(\phi) = X \) and **locally periodic (LP)** if \( LP(\phi) = X \).

For \( x \in LEP_m(\phi) \) the smallest period will be denoted as \( p_m(x) \), and then the smallest preperiod as \( pp_m(x) \).
It is easy to see that $x \in LEP(\phi)$ if and only if $(\phi^n x)_i$ is eventually periodic for all $i \in \mathbb{Z}$.

**Proposition 3.1.6.** Let $(X, \phi)$ be a CA. If $(X, \phi)$ is equicontinuous then it is LEP.

**Proof.** Let $m \in \mathbb{N}$. Equicontinuity implies uniform equicontinuity, hence there exists $n \in \mathbb{N}$ such that if $y \in B_{1/n}(x)$ then $y \in O_{1/m}(x)$.

Let $x \in X$. There exists $j > j'$ such that $(\phi^j x)_i = (\phi^{j'} x)_i$ for $|i| \leq n$. Thus $\phi^j x \in B_{1/n}(\phi^{j'} x)$ and hence $\phi^j x \in O_{1/m}(\phi^{j'} x)$. This implies the orbit ball is eventually periodic so $x \in LEP_m(\phi)$; hence $\phi$ is LEP.

The converse is not true.

**Example 3.1.7.** Let $X \subset \{0, 1\}^\mathbb{Z}$ be the subshift that contains the points that contain at most one 1. We have that $(X, \sigma)$ is LEP but $0^\infty$ is not an equicontinuity point. Hence LEP does not imply equicontinuity.

Nonetheless we will see that LP implies equicontinuity.

The following is an unpublished result by Chandgotia [10].

**Proposition 3.1.8.** Let $(X, \sigma)$ be a one-sided subshift. If $X$ has infinitely many periodic points then $X$ contains a non-periodic point.

**Proof.** For $x \in \mathcal{A}^\mathbb{N}$ a periodic point, the set $\mathcal{B}_r(x)$ denotes all the possible words of size $r$ that appear in $x$; we also define $\mathcal{B}(x) := \cup \mathcal{B}_r(x)$; the minimal period of the word $w \in \mathcal{B}(x)$ in $x$ is denoted by $p_x(w)$. We say $X$ has bounded periodic words if for all $w \in \cup_{x \text{ periodic}} \mathcal{B}(x)$ there exists $n_w$ such that $p_x(w) \leq n_w$ for all periodic points $x$.

If $x \in \mathcal{A}^\mathbb{N}$ is periodic with minimal period $n$, then $|\mathcal{B}_n(x)| = n$ and $|\mathcal{B}_r(x)| > r$ for $1 \leq r < n$.

Suppose $X$ is not a subshift with bounded periodic words. So there exists $w \in \mathcal{B}(x)$ and $x^n \in X$ a sequence of periodic points such that $p_{x^n}(w) \geq n$ and $x^n$ begins with $w$. Any limit point of the sequence $x^n$ contains $w$ only once, hence it is not periodic.

Now suppose $X$ has bounded periodic words. Let $x^{1,n}$ be a sequence of periodic points such that $p(x^{1,n}) > n$ and $\mathcal{B}_1(x^{1,n})$ is constant. Inductively, define $x^{i,n}$ to be a subsequence of $x^{i-1,n}$ such that $\mathcal{B}_i(x^{i,n})$ is constant. We can then find a sequence of points $y^n \in X$ such that $p(y^n) > n$ (and hence $\lim_{n \to \infty} |\mathcal{B}_r(y^n)| > r$ for all $r$) and the sequence of sets $\mathcal{B}_r(y^n)$ is eventually constant for all $r$. Let $y$ be a subsequential limit of $y^n$ and $w \in \lim_{n \to \infty} \mathcal{B}_r(y^n)$. There exists $n_w$ such that $p_{y^n}(w) \leq n_w$ for all $n$, so $w \in \mathcal{B}_r(y^n)$. This means that $|\mathcal{B}_r(y)| \geq \lim_{n \to \infty} |\mathcal{B}_r(y^n)| > r$ for all $r$ and hence $y$ is not periodic.

**Corollary 3.1.9.** Let $X$ be a one-sided subshift that contains only $\sigma$—periodic points. Then $X$ is finite and hence $(X, \sigma)$ is equicontinuous.

**Proposition 3.1.10.** Let $(X, \phi)$ be a CA. If $(X, \phi)$ is LP then it is equicontinuous.
Proof. Let $X_j = \{ y \in A^\mathbb{N} | y_i = (\phi^i x)_j \text{ for some } x \in X \}$. We have that $(X_j, \sigma)$ is a one-sided subshift that contains only periodic points, hence there are only finitely many. This means $\phi$ is equicontinuous.

A point $x$ is **recurrent** if for every open neighbourhood $U$ the orbit of $x$ under $\phi$ intersects $U$ infinitely often; the set of recurrent points is denoted by $R(\phi)$. The following lemma will be useful later.

**Lemma 3.1.11.** Let $(X, \phi)$ be a CA. Then $R(\phi) \cap LEP(\phi) = LP(\phi)$.

**Proof.** Using the definitions it is easy to see that $LP(\phi) \subset R(\phi) \cap LEP(\phi)$.

Let $x \in R(\phi) \cap LEP(\phi)$, $m \in \mathbb{N}$, and $q := pp_m(x)$ (see Definition 3.1.5). Suppose $x \notin LP(\phi)$. This means that $q > 0$, $(\phi^{i+q} x)_{W_m}$ is periodic for $i \geq 0$, and $(\phi^{i-1} x)_{W_m} \neq (\phi^{i+pm(x)-1} x)_{W_m}$. Using the continuity of $\phi$, we know there exists $m' \geq m$ such that for every $y \in B_{1/m'}(\phi^{i-1} x)$, $(\phi^{i+q} x)_{W_m} = (\phi^{i+q} y)_{W_m}$ for $0 \leq i \leq p_m(x)$. Using the fact that $\phi^{i-1} x$ is recurrent we obtain that there exists $N > pp_m(x) + p_m(x)$ such that $\phi^N x \in B_{1/m'}(\phi^{i-1} x)$. This means that $(\phi^N x)_{W_m} \neq (\phi^{i+pm(x)-1} x)_{W_m}$ and $(\phi^{N+i} x)_{W_m} = (\phi^{i+pm(x)+i-1} x)_{W_m}$ for $0 < i \leq p_m(x)$. This is a contradiction since the second condition and the fact that $p_m(x)$ is the smallest period implies that there exists $j > 0$ such that $N = q + jp_m(x) - 1$, and hence $(\phi^N x)_{W_m} = (\phi^{q+pm(x)-1} x)_{W_m}$.

### 3.1.3 Measure theoretical equicontinuity and local periodicity

We will use $\mu$ to denote Borel probability measures on $X$. These do not need to be invariant under $\phi$. For the ergodic theory point of view of $\mu$–equicontinuity see Section 4.

We repeat the definition of $\mu$–equicontinuity points.

The concept of $\mu$–equicontinuity points first appeared in [24] [23], and it was used to classify cellular automata using Bernoulli measures. We remind the reader of the definition in this chapter’s notation.

**Definition 3.1.12.** Let $(X, \phi)$ be a CA and $\mu$ a Borel probability measure on $X$. A point $x \in X$ is a **$\mu$–equicontinuity** point of $\phi$ if for all $m \in \mathbb{N}$, one has

$$\lim_{n \to \infty} \frac{\mu(B_{1/n}(x) \cap O_{1/m}(x))}{\mu(B_{1/n}(x))} = 1.$$  

The following result is a consequence of Corollary 2.3.24, Theorem 2.3.26 and Proposition 2.3.31.

**Theorem 3.1.13** ([20]). Let $(X, \phi)$ be a CA and $\mu$ a Borel probability measure. The following are equivalent:

1) $(X, \phi)$ is $\mu$–equicontinuous

2) Almost every $x \in X$ is a $\mu$–equicontinuity point.

3) There exists $X' \subset X$ such that $X'$ is $d_\phi$–separable and $\mu(X') = 1$. 

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Definition 3.1.14. When \((X, \phi)\) is \(\mu\)-equicontinuous we say \(\mu\) is \(\phi\)-equicontinuous.

Definition 3.1.15. Let \((X, \phi)\) be a CA. If \(\mu(\text{LEP}(\phi)) = 1\), we say \((X, \phi)\) is \(\mu\)-locally eventually periodic (\(\mu\)-LEP) and \(\mu\) is \(\phi\)-locally eventually periodic (\(\phi\)-LEP). We define \(\mu\)-LP and \(\phi\)-LP analogously.

The concept of \(\mu\)-LEP was motivated by Proposition 5.2 in [24]. In [20] the property of LEP is defined and studied for general TDS.

Lemma 3.1.16. Let \(m \in \mathbb{N}\) and \(\varepsilon > 0\). If \((X, \phi)\) is \(\mu\)-LEP then there exist positive integers \(p^n \varepsilon\) and \(pp^n \varepsilon\) such that \(\mu(Y^m \varepsilon) > 1 - \varepsilon\), where

\[
Y^m \varepsilon := \{x \mid x \in \text{LEP}(\phi), \ \text{with} \ p_m(x) \leq p^n \varepsilon \ \text{and} \ pp_m(x) \leq pp^n \varepsilon\}.
\]

Proof. Let

\[
Y := \bigcup_{s,k \in \mathbb{N}} \{x \mid x \in \text{LEP}(\phi), \ \text{with} \ p_m(x) \leq s \ \text{and} \ pp_m(x) \leq k\}.
\]

Since \((X, \phi)\) is \(\mu\)-LEP we have that \(\mu(Y) = 1\). Monotonicity of the measure gives the desired result.

Definition 3.1.17. Given a \(\mu\)-LEP transformation \(\phi\) and \(m, \varepsilon > 0\), we will use \(p^n \varepsilon\) and \(pp^n \varepsilon\) to denote a particular choice of integers that satisfy the conditions of the previous lemma and that satisfy that \(p^n \varepsilon \to \infty\) and \(pp^n \varepsilon \to \infty\), as \(\varepsilon \to 0\).

Definition 3.1.18. Given a subshift \(X\), we denote the \(\sigma\)-periodic points with \(P_X(\sigma)\).

The following result was proved in [23] when \(X\) is a full shift, but the same result holds when \(X\) is an SFT.

Lemma 3.1.19. Let \(X\) be a 1D SFT with forbidden words of size \(q\), \((X, \phi)\) a CA with radius \(r\). If there is a point \(x\) and an integer \(m \neq 0\) such that \(O_{1/i}(x) \cap \sigma^{-m}O_{1/i}(x) \neq \emptyset\) with \(i \geq q, r\) then \(O_{1/i}(x) \cap P_X(\sigma) \neq \emptyset\).

Proposition 3.1.20. Let \((X, \phi)\) be a CA. If \((X, \phi)\) is \(\mu\)-LEP then it is \(\mu\)-equicontinuous.

Proof. Let \(\varepsilon > 0\) and \(M := \cap_{m \in \mathbb{N}} Y^m \varepsilon /2m\) (hence \(\mu(M) > 1 - \varepsilon\)).

Let \(m \in \mathbb{N}\). There exists \(K\) such that if \(d(x, y) < 1/K\) then \(d(\phi^i x, \phi^i y) < 1/m\) for \(0 \leq i \leq \left(p^n \varepsilon /2m\right)^2 + pp^n \varepsilon /2m\). Let \(x, y \in Y^m \varepsilon /2m\), \(d(x, y) < 1/K\), \(p = p_m(x)p_m(y)\) and \(i \geq p + pp^n \varepsilon /2m\). We can express \(i = pp^n \varepsilon /2m + j \cdot p + k\) with \(j \in \mathbb{N}\) and \(k \leq p\). Using the fact that \(x, y \in Y^m \varepsilon /2m\) and \(pp^n \varepsilon /2m + k \leq \left(p^n \varepsilon /2m\right)^2 + pp^n \varepsilon /2m\) we obtain

\[
d(\phi^i x, \phi^i y) \leq d(\phi^i x, \phi^{pp^n \varepsilon + k} x) + d(\phi^{pp^n \varepsilon + k} x, \phi^{pp^n \varepsilon + k} y) + d(\phi^{pp^n \varepsilon + k} y, \phi^i y)
\]

\[
\leq 3/m.
\]
This means that $B_{1/K}(x) \cap Y_{\epsilon/2}^m \subset O_{3/m}(x)$ and hence $\phi$ is $\mu$–equicontinuous.

We obtain the converse with an extra hypothesis.

**Proposition 3.1.21.** Let $X$ be a 1D SFT, $(X, \phi)$ a CA, and $\mu$ a $\sigma$–invariant probability measure on $X$. Then $(X, \phi)$ is $\mu$–equicontinuous if and only if it is $\mu$–LEP.

**Proof.** Let $X$ be a 1D SFT with forbidden words of size $q$, $(X, \phi)$ a CA with radius $r$ and $p \geq q, r$. If $x$ is a $\mu$–equicontinuous point then

$$\lim_{n \to \infty} \frac{\mu(B_{1/n}(x) \cap O_{1/p}(x))}{\mu(B_{1/n}(x))} = 1.$$ 

Using $\mu(O_{1/p}(x)) > 0$ and Poincare’s recurrence theorem we obtain that

$$\{y \mid \sigma^i(y) \in O_{1/p}(x) \text{ i.o.}\}$$

is not empty. Using $p \geq q, r$ and Lemma 3.1.19 we conclude that every orbit ball with positive measure contains a $\sigma$–periodic point and hence $(\phi^j x)_j$ is eventually periodic for $-p \leq j \leq p$. The reverse implication is obtained with Proposition 3.1.20.

Proposition 3.1.21 shows that $\mu$–equicontinuity and $\mu$ – LEP are equivalent if $X$ is a 1D SFT and $\mu$ a $\sigma$–invariant measure. We do not know if this result holds for cellular automata on multidimensional SFTs. We can show a weaker result (Proposition 3.1.25) by strengthening the $\mu$–equicontinuity hypothesis.

**Definition 3.1.22.** Let $(X, \rho)$ be a metric space, and $A \subset X$. The closure of $A$ is denoted with $\text{cl}_\rho(A)$.

Recall that $d_\phi$ denotes the orbit metric (Definition 3.1.4).

**Lemma 3.1.23.** Let $(X, \phi)$ be a CA. If $\mu(\text{cl}_{d_\phi}(P_X(\sigma))) = 1$, then $(X, \phi)$ is $\mu$–LEP.

**Proof.** Using the fact that the $\phi$–image of a $\sigma$–periodic point is $\sigma$–periodic with at most the same period one can see that any point in $P_X(\sigma)$ is eventually periodic for $\phi$. Let $O_{1/m}$ be an orbit ball of size $m$. This means that if $O_{1/m} \cap P_X(\sigma) \neq \emptyset$ and $x \in O_{1/m}$ then $x \in \text{LEP}_m(\phi)$. Hence if $\mu(\text{cl}_{d_\phi}(P_X(\sigma))) = 1$ then $\phi$ is $\mu$–LEP.

We represent the change of metric identity map with $f_\phi : (X, d) \to (X, d_\phi)$. A point $x \in X$ is an equicontinuity point of $\phi$ if and only if it is a continuity point of $f_\phi$. Hence $\phi$ is equicontinuous if and only if $f_\phi$ is continuous.

**Definition 3.1.24.** Let $(X, \phi)$ be a CA. We denote the set of equicontinuity points with $\text{EQ}(\phi)$. 44
In [8] $\sigma$–ergodic measures that give full measure to the equicontinuity points of a CA were studied. As a consequence of Lemma 3.1 (in that paper) we can see that if $X$ is a 1D subshift, $(X, \phi)$ a CA, and $\mu$ a $\sigma$–ergodic probability measure on $X$ with $\mu(\text{EQ}(\phi)) = 1$ then $(X, \phi)$ is $\mu – \text{LEP}$.

If we assume the subshift has dense periodic points (or more generally are dense in a set of full measure) then we obtain that result for multidimensional subshifts.

**Proposition 3.1.25.** Let $(X, \phi)$ be a CA and $\mu$ a measure such that $\mu(\text{cl}_{d}(P_{X}(\sigma))) = 1$. If $\mu(\text{EQ}(\phi)) = 1$ then $(X, \phi)$ is $\mu – \text{LEP}$.

*Proof.* Since equicontinuity points have full measure, then $f_{\phi}$ is continuous on a set of full measure. This means that for almost every $x \in \text{cl}_{d}(P_{X}(\sigma))$, we have $x \in \text{cl}_{d_{\phi}}(P_{X}(\sigma))$. So $\mu(\text{cl}_{d_{\phi}}(P_{X}(\sigma))) = \mu(\text{cl}_{d}(P_{X}(\sigma))) = 1$. Using Lemma 3.1.23 we conclude that $(X, \phi)$ is $\mu – \text{LEP}$. \qed

We already noted that if $X$ is a subshift with dense $\sigma$–periodic points, and $(X, \phi)$ is an equicontinuous CA then $(X, \phi)$ is eventually periodic. Proposition 3.1.25 is a measure theoretic analogous result.

Now we present some examples.

**Example 3.1.26.** Let $x$ be a non eventually $\sigma$–periodic point and let $\mu$ be the delta measure supported on $\{x\}$. Then $\phi = \sigma$ is $\mu$–equicontinuous but not $\mu$-LEP.

**Definition 3.1.27.** A 1D CA $(X, \phi)$ has **right radius 0** if there exist $L \in \mathbb{N}$ and a function $\phi[:]: \mathcal{A}^{L} \to \mathcal{A}$, such that $(\phi(x))_{i} = \phi[x_{-L}...x_{i-1}x_{i}]$

**Example 3.1.28 ([24]).** Let $X = \{-1,0,1\}^{\mathbb{Z}}$, and $(X, \phi)$ a radius 1 CA with right radius 0 defined as follows: $\phi[11] = 1$, $\phi[10] = 1$, $\phi[1-1] = 0$, $\phi[\alpha 1] = 0$ if $a \neq 1$, $\phi[ab] = b$ if $a, b \neq 1$. The reader can picture a 1 moving to the right until it encounters a $-1$ and converts into a 0. It is easy to see that this CA does not contain equicontinuity points. For Bernoulli measures this CA is $\mu$–equicontinuous when $\mu(-1) > \mu(1)$.

We define the set $E_{i} := \left\{ x \in \{0,1\}^{\mathbb{Z}} \mid x_{j} = 1 \text{ for } 0 \leq j \leq i \right\}$.

**Example 3.1.29.** There exists a CA on the full 2-shift with no equicontinuity points that is $\mu$–equicontinuous for every $\sigma$–invariant measure that satisfies $\sum_{i \geq 0} \mu(E_{i}) < \infty$ (in particular every non-trivial ergodic Markov chain).

*Proof.* On $\{0,1\}^{\mathbb{Z}}$ we define $\phi(x) = y$ as $y_{i} = x_{i-1}x_{i-2}$.

One can check that for every $i > 0$, $(\phi^{i}x)_{0} = \prod_{j=-i}^{-2i} x_{j}$. Let $a \in \{0,1\}$. If there exists $n > 0$ such that $x_{i} = a$ for $i \leq -n$, then $(\phi^{i}x)_{0} = a$ for $i \geq 2n$. This means that for every ball $B$ there exists $x, y \in B$ such that $O_{0}(x) \neq O_{0}(y)$, so the CA has no equicontinuity points. Let
\( \mathcal{E}_i := \{ x \mid x_j = 1 \text{ for } -2i \leq j \leq -i \} \). We have that \( \sum_{i \geq 0} \mu(\mathcal{E}_i) = \sum_{i \geq 0} \mu(E_i) < \infty \). By the Borel-Cantelli Lemma we have that \( \mu(\mathcal{E}_i \text{ infinitely often}) = 0 \). This means that the probability that \((\phi^i x)_0\) has infinitely many ones is zero; since the same argument can be given for \((\phi^i x)_m\) we conclude \( \phi \) is \( \mu - \text{LEP} \) and hence \( \mu - \text{equicontinuous} \).

For every non-trivial ergodic Markov chain \( \mu(E_i) \) decreases exponentially so \( \sum_{i \geq 0} \mu(E_i) < \infty \). By the Borel-Cantelli Lemma we have that \( \mu(\mathcal{E}_i \text{ infinitely often}) = 0 \).

Note that in the trivial case (i.e. when \( \mu(1) = 1 \) or 0), the hypothesis is not satisfied but we also conclude \( \phi \) is \( \mu - \text{equicontinuous} \).

Q: Does there exists a CA with no equicontinuity points that is \( \mu - \text{equicontinuous} \) for every \( \sigma \)-invariant \( \mu \)?

For more examples of \( \mu - \text{equicontinous} \) CA see [58].

The following diagrams illustrate how the different properties relate on the topological and measure theoretical level.

**Topological**

\[
LP \Rightarrow \text{Equicontinuous} \Rightarrow \text{LEP}
\]

**Measure theoretical**

\[
\mu - LP \Rightarrow \mu - \text{LEP} \Rightarrow \mu - \text{equicontinuous}
\]

If \( X \) is a 1D SFT and \( \mu \sigma \)-invariant then

\[
\mu - \text{LEP} = \mu - \text{equicontinuous}
\]

### 3.2 Weak convergence

A sequence of measures \( \mu_n \) (on \( X \)) **converges weakly** to \( \mu_\infty \) (denoted as \( \mu_n \to w \mu_\infty \)) if for every continuous function \( f : X \to \mathbb{R} \),

\[
\int f d\mu_n \to \int f d\mu_\infty.
\]

This form of convergence is called weak convergence in the Probability literature and weak* convergence in the Functional Analysis literature.

One can study limit behaviour of dynamical systems by studying the long term behaviour of \( \phi^n \mu \) (\( \phi \mu \) is the push-forward of the measure) or of its Cesaro averages: \( \mu_n^\circ := \frac{1}{n} \sum_{i=1}^{n} \phi^i(\mu) \). In particular we may ask if \( \phi^n \mu \) or \( \mu_n^\circ \) converges weakly, and which are the properties of the limit measure.

**Theorem 3.2.1** (Portmanteau [5] pg. 15). We have \( \mu_n \to w \mu_\infty \) if and only if for every open set \( U \), \( \mu_\infty(U) \leq \liminf \mu_n(U) \) if and only if \( \mu_n(E) \to \mu_\infty(E) \) for every set \( E \) with zero boundary measure.
We will see that orbit balls form a weak convergence determining class when the limit measure is $\phi-$ equicontinuous (Lemma 3.2.2). Note that even when $\mu$ is $\phi -$ LEP, the measure of the boundary of an orbit ball is not necessarily zero. For example, one can check that the orbit balls of Example 3.1.28 are each contained in their own boundary.

**Lemma 3.2.2.** Let $(X, \phi)$ be a CA, $\mu_n$ be a sequence of measures, and $\mu_\infty$ a $\phi-$equicontinuous measure. If for every orbit ball $A$ we have that $\mu_n(A) \to \mu_\infty(A)$ then $\mu_n \to_w \mu_\infty$. Also, if $\mu$ and $\mu'$ are $\phi-$equicontinuous and $\mu(A) = \mu'(A)$ for every orbit ball $A$ then $\mu = \mu'$.

**Proof.** If we have two orbit balls $O$ and $O'$, then either $O \cap O' = \emptyset$ or one is contained in the other. This implies we have convergence for finite unions of orbit balls (since they can be written as unions of disjoint orbit balls). Let $U$ be an open set. We have that $U$ is the countable union of balls. From Theorem 3.1.13 we know there exists a $d_\phi-$separable set $X'$ such that $\mu_\infty(X') = 1$. This means that for every $\delta > 0$ there exist a finite number of orbit balls $O_1/i$ such that $\bigcup_{i=1}^N O_1/i \subset U$ and $\mu_\infty(U) \leq \mu_\infty(\bigcup_{i=1}^N O_1/i) + \delta$. We have that

$$\mu_\infty(U) - \delta \leq \mu_\infty(\bigcup_{i=1}^N O_1/i) = \lim_{n} \mu_n(\bigcup_{i=1}^N O_1/i) \leq \lim \inf_{n} \mu_n(U),$$

and hence

$$\mu_\infty(U) \leq \lim \inf_{n} \mu_n(U).$$

Therefore $\mu_n \to_w \mu_\infty$.

Now suppose $\mu(A) = \mu'(A)$ for every orbit ball $A$. Let $X'$ be the $d_\phi-$separable set with $\mu(X') = 1$. This means $\mu(X') = \mu'(X') = 1$ (that is because $X'$ is equal to the union of countably many orbit balls). Thus $\mu$ and $\mu'$ agree on a $\pi-$system (family closed under finite intersections) that generate the Borel sigma algebra (intersected with $X'$); we conclude $\mu = \mu'$.

**Definition 3.2.3.** For $x \in \text{LEP}_m(\phi)$ we define

$$O_{1/m}^{-q}(x) := \left\{ y \mid \exists i \in \mathbb{N} \text{ s.t. } \phi^{ip_m(x)+q}y \in O_{1/m}(x) \right\}.$$

**Definition 3.2.4.** We denote the Cesaro average of $\phi^i \mu$ with $\mu_n^c$, i.e. $\mu_n^c := \frac{1}{n} \sum_{i=1}^n \phi^i(\mu)$.

In the following proposition we show Cesaro convergence holds for orbit balls. Note in the proof that the convergence is actually stronger than Cesaro; there is convergence along periodic subsequences.

**Lemma 3.2.5.** Let $(X, \phi)$ be a $\mu -$ LEP CA, and $m \in \mathbb{N}$. If $x \in \text{LP}_m(\phi)$ then

$$\mu_n^c(O_{1/m}(x)) \to \frac{1}{p_m(x)} \sum_{q=0}^{p_m(x)-1} \mu(O_{1/m}^{-q}(x)).$$

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Furthermore if $x \notin LP_m(\phi)$ then $\mu_n^c(O_{1/m}(x)) \to 0$.

**Proof.** We have

\[ \cup_{n \geq 0} \phi^{-p_m(x)n-q}(O_{1/m}(x)) = O_{1/m}^{-q}(x). \]

For every $0 \leq q < p_m(x)$ and $n \geq 0$

\[ \phi^{-p_m(x)n-q}(O_{1/m}(x)) \subset \phi^{-p_m(x)(n+1)-q}(O_{1/m}(x)). \]

This implies $\mu(\phi^{-p_m(x)n-q}(O_{1/m}(x)))$ is non-decreasing and

\[ \lim_{n \to \infty} \mu(\phi^{-p_m(x)n-q}(O_{1/m}(x))) = \mu(O_{1/m}^{-q}(x)). \] (3.2.1)

Since we have convergence along periodic subsequences we have that

\[ \lim_{n \to \infty} \mu_n^c(O_{1/m}(x)) = \frac{1}{p_m(x)} \sum_{q=0}^{p_m(x)-1} \mu(O_{1/m}^{-q}(x)). \]

Let $\varepsilon > 0$ and $x \notin LP_m(\phi)$. If $np^r > pp_m^m$, then

\[ \phi^{-p^r n-s}(O_{1/m}(x)) \cap Y_m = \emptyset, \]

so $\mu(\phi^{-p^r n-s}(O_{1/m}(x))) < \varepsilon$. \hfill \Box

**Proposition 3.2.6.** Let $(X, \phi)$ be a $\mu$–LEP CA. If $\phi \mu = \mu$ then $(X, \phi)$ is $\mu$–LP.

**Proof.** Using the invariance of $\mu$ and Poincare’s recurrence theorem we obtain that the set of recurrent points has full measure, i.e. $\mu(R(\phi)) = 1$. By Lemma 3.1.11 we have that

\[ \mu(LP(\phi)) = \mu(LEP(\phi) \cap R(\phi)) = 1. \] \hfill \Box

**Remark 3.2.7.** Let $B$ be a finite union of balls (thus $B$ is compact). If $B = \cup_{i=1}^\infty B^i$, where $\{B^i\}$ is a disjoint family of balls, then there exists $K$ such that $B = \cup_{i=1}^K B^i$. From this fact we get that any premeasure on the algebra generated by the balls can be extended to a measure on the Borel sigma-algebra.

The existence of a limit measure in the following result is a natural generalization of a result for 1D CA in [8] ($X$ is multidimensional and $\phi$ may not have any equicontinuity points). Here we also show that the limit measure is $\phi – LP$. 48
**Theorem 3.2.8.** Let \((X, \phi)\) be a \(\mu -\) LEP CA. The sequence of measures \(\mu_n^c\) converges weakly to a \(\phi -\) LP measure \(\mu_\infty\).

**Proof.** Let \(B_{1/m}\) be a ball. For every \(\varepsilon > 0\) and \(n \in \mathbb{N}\) we have that

\[
\left| \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{1/m} \cap Y^m_\varepsilon)) - \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{1/m})) \right| \leq n\varepsilon / n = \varepsilon.
\]

Consequently

\[
\lim_{\varepsilon \to 0} \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{1/m} \cap Y^m_\varepsilon)) = \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{1/m}))
\]

uniformly on \(n\).

On the other hand for every \(\varepsilon > 0\) there exists a finite set of disjoint orbit balls \(\{O_{1/m_k}(x_k)\}\) such that the \(x_k\) are LEP and \(B_{1/m} \cap Y^m_\varepsilon = \bigcup_{k=1}^{K} O_{1/m_k}(x_k)\). This implies that

\[
\frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{1/m} \cap Y^m_\varepsilon)) = \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(\bigcup_{k=1}^{K} O_{1/m_k}(x_k))).
\]

By Lemma 3.2.5 \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(\bigcup_{k=1}^{K} O_{1/m_k}(x_k)))\) exists.

Let

\[
F(n, \varepsilon) := \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{1/m} \cap Y^m_\varepsilon)).
\]

We have shown that

\[
\lim_{n \to \infty} F(n, \varepsilon) \text{ exists, and}
\]

\[
\lim_{\varepsilon \to 0} F(n, \varepsilon) = \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{1/m})) \text{ uniformly on } n.
\]

Thus we obtain that \(\lim_{n \to \infty} \lim_{\varepsilon \to 0} F(n, \varepsilon) \text{ exists, and}

\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{1/m} \cap Y^m_\varepsilon)) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mu(\phi^{-i}(B_{1/m})) = \lim_{n \to \infty} \mu_n^c(B_{1/m}).
\]

The proof of the previous statement is common in analysis (see for example Theorem 1 in [37]).

We define \(\mu_\infty\) as the measure that satisfies \(\mu_\infty(B_{1/m}) = \lim_{n \to \infty} \mu_n^c(B_{1/m})\) (see Remark 3.2.7) for every ball \(B_{1/m}\). Every open set \(U\) can be approximated by a finite disjoint union of balls. This
implies \( \mu_\infty(U) \leq \lim \inf \mu_n^c(U) \), and hence \( \mu_n^c \to^w \mu_\infty \).

Since \( \phi^{-1}(\text{LEP}(\phi)) = \text{LEP}(\phi) \) we have that \( \mu_n^c \) is \( \phi - \text{LEP} \). For every \( m \in \mathbb{N} \) and \( \varepsilon > 0 \) we have that \( Y_\varepsilon^m \) is a finite union of orbit balls and hence,

\[
\mu_\infty(Y_\varepsilon^m) = \lim \mu_n^c(Y_\varepsilon^m) \geq 1 - \varepsilon.
\]

Since \( Y_\varepsilon^m \subset \text{LEP}(\phi) \) we obtain that \( \mu_\infty \) is \( \phi - \text{LEP} \). Considering that \( \phi \mu_\infty = \mu_\infty \) and Proposition 3.2.6 we obtain that \((X, \phi)\) is \( \mu - \text{LP} \).

There is a more general definition of \( \mu - \text{LEP} \) for topological dynamical systems (see [20]). It is possible to check that the previous result holds for topological dynamical systems on zero dimensional spaces.

**Definition 3.2.9.** From now on we will use \( \mu_\infty \) to denote the weak limit of \( \mu_n^c \).

**Proposition 3.2.10.** Let \((X, \phi)\) be a \( \mu - \text{LEP} \) CA. Then \( \phi^n \mu \to^w \mu_\infty \) if and only if \( \phi^n \mu(O) \to \mu_\infty(O) \) for all orbit balls \( O \), and \( \mu_n^c \to^w \mu_\infty \) if and only if \( \mu_n^c(O) \to \mu_\infty(O) \) for all orbit balls \( O \).

**Proof.** Assume that \( \phi^n \mu \to^w \mu_\infty \).

Let \( m \in \mathbb{N} \) and \( x \in \text{LEP}(\phi) \). Take \( \varepsilon > 0 \) so that \( p_\varepsilon^m > p_m(x) \).

Let \( O_{1/m}(x) = \cap_{i \in \mathbb{N}} \phi^{-i}G_i \), where every \( G_i \) is a ball. There exists \( k_1 \) such that

\[
\left| \mu_\infty(\cap_{i=1}^k \phi^{-i}G_i) - \mu_\infty(O_{1/m}(x)) \right| \leq \varepsilon \text{ for } k \geq k_1.
\]

Fix \( k \geq 2p_\varepsilon^m, k_1 \). If \( n \geq pp_\varepsilon^m \) then

\[
\phi^{-n}(\cap_{i=1}^k \phi^{-i}G_i) \cap Y_\varepsilon^m = \phi^{-n}(O_{1/m}(x)) \cap Y_\varepsilon^m.
\]

Since \( \mu(Y_\varepsilon^m) > 1 - \varepsilon \) we obtain

\[
\left| \phi^n \mu(\cap_{i=1}^k \phi^{-i}G_i) - \phi^n \mu(O_{1/m}(x)) \right| \leq \varepsilon \text{ for } n \geq pp_\varepsilon^m.
\]

Since \( \cap_{i=1}^k \phi^{-i}G_i \) has no boundary, there exists \( N \) such that

\[
\left| \phi^n \mu(\cap_{i=1}^k \phi^{-i}G_i) - \mu_\infty(\cap_{i=1}^k \phi^{-i}G_i) \right| \leq \varepsilon \text{ for } n \geq N.
\]

Using the inequalities we obtain

\[
\left| \phi^n \mu(O_{1/m}(x)) - \mu_\infty(O_{1/m}(x)) \right| \leq 3\varepsilon \text{ for } n \geq N, pp_\varepsilon^m.
\]

Hence \( \phi^n \mu(O_{1/m}(x)) \to \mu_\infty(O_{1/m}(x)) \).
The other direction is a corollary of Lemma 3.2.2. The proof for \( \mu_n^c \) is analogous.

**Definition 3.2.11.** For \( a \in \mathbb{R} \) and \( E \subset X \) Borel, we define

\[
A_a^E := \left\{ y : \lim_{n \to \infty} \frac{1}{|W_n|} \sum_{i \in W_n} 1_E(\sigma^i(y)) = a \right\}.
\]

Note that if \( \mu \) is \( \sigma \)-ergodic then by the pointwise ergodic theorem

\[
\mu(A_a^E) = \begin{cases} 1 & \text{if } a = \mu(E) \\ 0 & \text{otherwise} \end{cases}.
\]

**Lemma 3.2.12.** Let \( \mu \) be a \( \sigma \)-ergodic measure, \( (X, \phi) \) a \( \mu \)-LEP CA, \( m \in \mathbb{N} \), and \( x \in L P_m(\phi) \). If for every \( 0 \leq q < p_m(x) \) there exists \( N_q \in \mathbb{N} \) such that

\[
\mu(\phi^{p_m(x) n + q}(O_{1/m}(x))) = \mu(O_{1/m}^{-q}(x)) \quad \text{for all } n \geq N_q
\]

then

\[
\mu_n^c(\mathcal{A}_{O_{1/m}(x)}^{A_a^E}) \to \mu_\infty(\mathcal{A}_{O_{1/m}(x)}^{A_a^E}) = \frac{1}{p_m(x)} \sum_{q=0}^{p_m(x)-1} \mu(\mathcal{A}_{O_{1/m}(x)}^{A_a^{O_{1/m}^{-q}(x)}}).
\]

**Proof.** By hypothesis we have that for \( n \geq N_q \)

\[
\phi^{p_m(x) n + q}(O_{1/m}(x)) = \mu(O_{1/m}^{-q}(x)).
\]

Note that since \( \mu \) is \( \sigma \)-ergodic then \( \phi^n \mu \) is \( \sigma \)-ergodic for every \( n \geq 1 \). Let \( 0 \leq q < p_m(x) \).

If \( n \geq N_q \) then

\[
\phi^{p_m(x) n + q}(\mu(O_{1/m}(x))) = \begin{cases} 1 & \text{if } a = \phi^{p_m(x) n + q}(\mu(O_{1/m}(x))) \\ 0 & \text{otherwise} \end{cases}.
\]

\[
= \begin{cases} 1 & \text{if } a = \mu(O_{1/m}^{-q}(x)) \\ 0 & \text{otherwise} \end{cases}.
\]

\[
= \mu(O_{1/m}^{-q}(x)).
\]

This implies we have convergence along periodic subsequences. Thus

\[
\mu_n^c(\mathcal{A}_{O_{1/m}(x)}^{A_a^E}) \to \mu_\infty(\mathcal{A}_{O_{1/m}(x)}^{A_a^E}) = \frac{1}{p_m(x)} \sum_{q=0}^{p_m(x)-1} \mu(\mathcal{A}_{O_{1/m}(x)}^{A_a^{O_{1/m}^{-q}(x)}}).
\]
Using the fact that \( \mu(\phi^{-p_m(x)}n^{-q}(O_{1/m}(x))) \rightarrow \mu(O_{1/m}^{\infty}(x)) \) one can check that one of the hypothesis of this result is always satisfied if we assume the CA is \( \mu - LP \).

**Lemma 3.2.13.** Let \( (X, \phi) \) be a \( \mu - LP \) CA, \( m \in \mathbb{N} \), and \( x \in LP_m(\phi) \). Then for every \( 0 \leq q < p_m(x) \) we have
\[
\mu(\phi^{-p_m(x)i-q}(O_{1/m}(x))) = \mu(O_{1/m}^{-q}(x)) \text{ for all } i \in \mathbb{N}.
\]

**Proof.** This comes from the fact that \( LP(\phi) \cap \phi^{-p_m(x)i-q}(O_{1/m}(x)) = LP(\phi) \cap O_{1/m}^{-q}(x) \) for all \( i \in \mathbb{N} \).

**Theorem 3.2.14.** Let \( \mu \) be a \( \sigma \)-ergodic measure and \( (X, \phi) \) a \( \mu - LP \) CA. Then \( \mu_\infty \) is \( \sigma \)-ergodic if and only if \( \mu \) is \( \phi \)-invariant.

**Proof.** If \( \phi \mu = \mu \) then \( \mu = \mu_\infty \) and hence it is \( \sigma \)-ergodic.

Suppose \( \mu_\infty \) is \( \sigma \)-ergodic and \( \mu \) is not \( \phi \)-invariant. By Lemma 3.2.2 we know there exists \( x \in LP(\phi) \) and \( m \in \mathbb{N} \) such that
\[
\mu(O_{1/m}(x)) \neq \mu(\phi^{-1}O_{1/m}(x)).
\]

We have that
\[
O_{1/m}(x) \cap LP(\phi) = O_{1/m}^0(x) \cap LP(\phi), \quad \text{and}
\]
\[
\phi^{-1}O_{1/m}(x) \cap LP(\phi) = O_{1/m}^{-1}(x) \cap LP(\phi).
\]

Since \( (X, \phi) \) is \( \mu - LP \) we obtain that
\[
\mu(O_{1/m}^0(x)) \neq \mu(O_{1/m}^{-1}(x)).
\]

Using Lemma 3.2.12 and Lemma 3.2.13 we get
\[
\mu_\infty(A_{a}^{O_{1/m}(x)}) = \frac{1}{p_m(x)} \sum_{q=0}^{p_m(x)-1} \mu(A_{a}^{O_{1/m}^{-q}(x)}).
\]

We reach a contradiction because \( A_{\mu(O_{1/m}(x))}^{O_{1/m}(x)} \) is \( \sigma \)-invariant, \( \mu_\infty \) is \( \sigma \)-ergodic but
\[
\frac{1}{p_m(x)} \leq \mu_\infty(A_{\mu(O_{1/m}(x))}^{O_{1/m}(x)}) \leq \frac{p_m(x)-1}{p_m(x)}.
\]

Every subshift has at least one **measure of maximal entropy** (MME), i.e. a measure whose entropy is the same as the topological entropy of the subshift. A 1D subshift is **irreducible** if
for every pair of balls \( U, V \) there exists \( j \in \mathbb{Z} \) such that \( \sigma^j U \cap V \neq \emptyset \). A well known result of Shannon and Parry state that every irreducible 1D SFT admits a unique MME, and it always has full support [53]. The MME of a fullshift is the uniform Bernoulli measure.

Note that we are only discussing measures of maximal entropy with respect to the shift not to \( \phi \).

**Theorem 3.2.15** (Coven-Paul [12]). Let \( X \) be a 1D irreducible SFT with a unique MME, and \((X, \phi)\) a CA. Then \((X, \phi)\) is surjective if and only if it preserves the MME. In particular if \( \phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}} \) is a CA, then \((X, \phi)\) is surjective if and only if it preserves the uniform Bernoulli measure.

**Lemma 3.2.16.** Let \((X, \phi)\) be CA. Assume that \( \phi \) preserves a measure with full support. If \( x \) is an equicontinuity point then \( x \) is recurrent \((x \in R(\phi))\).

**Proof.** Let \( x \) be an equicontinuity point and \( m \in \mathbb{N} \). There exists \( n \geq 2m \) such that \( B_{1/n}(x) \subset O_{1/2m}(x) \). We have that \( \phi \) preserves a fully supported measure. Using Poincare’s recurrence theorem we conclude there exists \( j \in \mathbb{N} \) such that \( \phi^j B_{1/n}(x) \cap B_{1/n}(x) \neq \emptyset \). This implies that \( \phi^j x \in B_{1/m}(x) \), and thus \( x \in R(\phi) \).

In [8] it was asked under which conditions the limit measure, under a 1D CA, of \( \sigma \)-ergodic measures that give full measure to equicontinuity points i.e. \( \mu(EQ(\phi)) = 1 \), is \( \sigma \)-ergodic, a measure of maximal entropy or \( \phi \)-ergodic. We address those questions (the first two in this section and the last one in the next section).

**Theorem 3.2.17.** Let \( X \) be a 1D irreducible SFT, \((X, \phi)\) a surjective CA, and \( \mu \) a \( \sigma \)-ergodic measure with \( \mu(EQ(\phi)) = 1 \). Then \( \mu_\infty \) is \( \sigma \)-ergodic if and only if \( \mu \) is \( \phi \)-invariant.

**Proof.** If \( \mu = \phi \mu \), then \( \mu_\infty = \mu \) is \( \sigma \)-ergodic.

Since \( \phi \) is surjective by Shannon-Parry and Coven-Paul we obtain that \( \phi \) preserves a fully supported measure. By Proposition 3.1.25 we have that \( \mu(\text{LEP}(\phi)) = 1 \). Using Lemma 3.2.16 and Lemma 3.1.11 we get that \( \mu(\text{LP}(\phi)) = \mu(\text{LEP}(\phi) \cap R(\phi)) = 1 \); and hence \((X, \phi)\) is \( \mu - \text{LP} \). Using Theorem 3.2.14 we obtain \( \mu = \phi \mu \).

The proof of the following result is similar.

**Theorem 3.2.18.** Let \( X \) be a 1D irreducible SFT, \((X, \phi)\) a CA, and \( \mu \) a \( \sigma \)-ergodic measure with \( \mu(EQ(\phi)) = 1 \). Then \( \mu_\infty \) is the MME if and only if \( \mu \) is the MME and \((X, \phi)\) is surjective.

**Proof.** If \( \mu \) is the MME and \( \mu = \phi \mu \), then \( \mu_\infty = \mu \) is the MME.

Assume \( \mu_\infty \) is the MME; hence it is \( \sigma \)-ergodic and has full support. By Proposition 3.1.25 we have that \( \mu(\text{LEP}(\phi)) = 1 \). Using Lemma 3.2.16 and Lemma 3.1.11 we get that \( \mu(\text{LP}(\phi)) = \mu(\text{LEP}(\phi) \cap R(\phi)) = 1 \); and hence \((X, \phi)\) is \( \mu - \text{LP} \). Using Theorem 3.2.14 we obtain \( \mu = \phi \mu = \mu_\infty \).

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There is interest in dynamical systems such that the orbit of the measure will converge in some sense to a measure of maximal entropy or an equilibrium measure (for example see Section 4.4 of [39]). It has been shown that the Markov measures under 1D linear permutative CA converge in Cesaro sense to the measure of maximal entropy (e.g. [46][54][48]). Theorem 3.2.18 shows that if a measure $\mu$ is not the measure of maximal entropy and the equicontinuity points have full measure then the limit measure will not be the measure of maximal entropy.

Under some conditions these results hold for multidimensional subshifts.

One of the implications of the mentioned theorem by Coven-Paul has been generalized. Let $X$ be a subshift with a unique MME, and $(X, \phi)$ a CA. If $(X, \phi)$ is surjective then it preserves the MME (Theorem 3.3 in [49]). The proof of the following results are almost the same as the proofs for Theorems 3.2.18 and 3.2.17.

**Theorem 3.2.19.** Let $X$ be a subshift with dense periodic points and a unique and fully supported MME, $(X, \phi)$ a surjective CA, and $\mu$ a $\sigma$-ergodic measure with $\mu(EQ(\phi)) = 1$. Then $\mu_\infty$ is $\sigma$-ergodic if and only if $\mu$ is $\phi$-invariant.

**Theorem 3.2.20.** Let $X$ be a subshift with dense periodic points and a unique and fully supported MME, $(X, \phi)$ a CA, and $\mu$ a $\sigma$-ergodic measure with $\mu(EQ(\phi)) = 1$. Then $\mu_\infty$ is the MME if and only if $\mu$ is the MME and $\mu = \phi \mu$.

For $\mu - LEP$ systems we can show sufficient conditions for the $\sigma$-ergodicity of $\mu_\infty$.

**Lemma 3.2.21.** Let $\mu$ be a $\phi - LEP$ measure. Then $\mu_\infty$ is $\sigma$-ergodic if and only if for every $x \in LP(\phi)$.

$$\mu_\infty(A_a^{O_{1/m}(x)}) = \begin{cases} 1 & \text{if } a = \mu_\infty(O_{1/m}(x)) \\ 0 & \text{otherwise} \end{cases}.$$  

**Proof.** The $\Rightarrow$ implication is given by the pointwise ergodic theorem.

If the equation is satisfied then the pointwise ergodic theorem conclusion holds for all sets of the form $O_{1/m}(x)$ with $x \in LP_m(\phi)$. By Proposition 3.2.6 $\mu_\infty(LP(\phi)) = 1$. Since $\{O_{1/m}(x) \mid m \in \mathbb{N} \text{ and } x \in LP_m(\phi)\}$ generates the Borel sigma algebra (intersected with $LP(\phi)$), we conclude $\mu_\infty$ is $\sigma$-ergodic (see [59] pg.41.)

**Theorem 3.2.22.** Let $\mu$ be a $\phi - LEP$, $\sigma$-ergodic measure. If for every orbit ball $O$, with $\mu_\infty(O) > 0$, there exists $N_O$ such that

$$\phi^n \mu(O) = \mu_\infty(O) \text{ for } n \geq N_O,$$  

then $\mu_\infty$ is $\sigma$-ergodic.

**Proof.** Let $m \in \mathbb{N}$, $x \in LP_m(\phi)$ and $0 \leq q < p_m(x)$. From the proof of equation (3.2.1) of Lemma 3.2.5 one can see that for every $0 \leq q < p_m(x)$, $\phi^{p_m(x)n+q} \mu(O_{1/m}(x))$ is non-decreasing and
converges (as \( n \to \infty \)). Using this and the hypothesis we have that there exists \( N \) such that

\[
\phi^{pm^{n+q}} \mu(O_{1/m}(x)) = \mu_{\infty}(O_{1/m}(x)) \text{ for } n \geq N.
\]

This implies

\[
\mu(O_{1/m}^{-q}(x)) = \mu_{\infty}(O_{1/m}(x)).
\]

Using Lemma 3.2.12 we obtain

\[
\mu_{\infty}(A_{a}^{O_{1/m}(x)}) = \frac{1}{pm(x)} \sum_{r=0}^{p_{m}(x)-1} \mu(A_{a}^{O_{1/m}^{-r}(x)}) \text{ for every } a \in \mathbb{R}.
\]

This implies

\[
\mu_{\infty}(A_{a}^{O_{1/m}(x)}) = \mu(A_{a}^{O_{1/m}^{-q}(x)}) \text{ for every } a \in \mathbb{R}.
\]

Using the \( \sigma \)-ergodicity of \( \mu \) we get

\[
\mu(A_{a}^{O_{1/m}^{-q}(x)}) = \begin{cases} 
1 & \text{if } a = \mu(O_{1/m}^{-q}(x)) \\
0 & \text{otherwise}
\end{cases}.
\]

Hence, we obtain

\[
\mu_{\infty}(A_{a}^{O_{1/m}(x)}) = \begin{cases} 
1 & \text{if } a = \mu_{\infty}(O_{1/m}(x)) \\
0 & \text{otherwise}
\end{cases}.
\]

Using the previous lemma we conclude that \( \mu_{\infty} \) is \( \sigma \)-ergodic.

\[\Box\]

### 3.3 \( \phi \)-Ergodicity

In this section we are interested in measure preserving topological dynamical systems.

We will now remind the reader of some definitions involving measure preserving systems. We say \((X, T, \mu)\) is a measure preserving transformation if \((X, \mu)\) is Lebesgue probability space, \(T : X \to X\) is measurable and \(T \mu = \mu\). When we say \( \mu \) is ergodic we also assume it is invariant under \( T \).

Two measure preserving transformations \((X_{1}, T_{1}, \mu_{1})\) and \((X_{2}, T_{2}, \mu_{2})\) are isomorphic (measurably) if there exists an invertible measure preserving transformation \(f : (X_{1}, \mu_{1}) \to (X_{2}, \mu_{2})\), such that the inverse is measure preserving and \(T_{2} \circ f = f \circ T_{1}\).

The spectral theory for dynamical systems (TDS and measure preserving transformations) is useful for studying rigid transformations. We will give the definitions and state the most important results. (For more details and proofs see [59]).

A measure preserving transformation \( T \) on a measure space \((M, \mu)\) generates a unitary linear operator on the Hilbert space \(L^{2}(M, \mu)\), by \(U_{T} : f \mapsto f \circ T\), known as the Koopman operator. The
spectrum of the Koopman operator is called the spectrum of the measure preserving transformation. The spectrum is pure point or discrete if there exists an orthonormal basis for $L^2(M, \mu)$ which consists of eigenfunctions of the Koopman operator. The spectrum is rational if the eigenvalues are complex roots of unity. Classical results by Halmos and Von Neumann state that two ergodic measure preserving transformation with discrete spectrum have the same group of eigenvalues if and only if they are isomorphic, and that an ergodic measure preserving transformation has pure point spectrum if and only if it is isomorphic to a rotation on a compact metric group. The eigenfunctions of a rotation on a compact group are generated by the characters of the group. Discrete spectrum can be characterized for topological dynamical systems using a weak forms of $\mu$–equicontinuity [22].

**Example 3.3.1.** Let $S = (s_0, s_1, \ldots)$ be a finite or infinite sequence of integers larger or equal than 1. The $S$–adic odometer is the $+(1,0,\ldots)$ (with carrying) map defined on the compact set $D = \prod_{i \geq 0} \mathbb{Z}_{s_i}$ (for a survey on odometers see [14]).

These transformations are also called adding machines. An ergodic measure preserving transformation has discrete rational spectrum if and only if it is isomorphic to an odometer. Any odometer can be embedded in a CA [13].

**Theorem 3.3.2.** Let $(X, \phi)$ be a CA and $\mu$ an invariant probability measure. If $(X, \phi)$ is $\mu – LEP$ then $(X, \phi, \mu)$ has discrete rational spectrum.

**Proof.** Using Proposition 3.2.6 we have that $(X, \phi)$ is $\mu – LP$.

Let $m \in \mathbb{N}$, $y \in LP(\phi)$ and $i = \sqrt{-1}$. We define $\lambda_{m,y} := e^{2\pi i / p_m(y)} \in \mathbb{C}$ and

$$f_{m,y,k} := \sum_{j=0}^{p_m(y)-1} \lambda_{m,y}^{j\cdot k} \cdot 1_{O_{1/m}(\phi^j y)} \in L^2(X, \mu).$$

Using the fact that $(X, \phi)$ is $\mu – LP$, $y \in LP(\phi)$ we have that $O_{1/m}(\phi^{p_m(y)} y) = O_{1/m}(y)$ and

$$U_\phi 1_{O_{1/m}(\phi^j y)} = \begin{cases} 1_{O_{1/m}(\phi^{j+1} y)} & \text{if } 1 \leq j < p_m(y) - 1 \\ 1_{O_{1/m}(\phi^{p_m(y)} y)} & \text{if } j = 0 \end{cases}.$$

This implies that

$$U_\phi f_{m,y,k} = \sum_{j=0}^{p_m(x)-1} \lambda_{m,y}^{j\cdot k} \cdot 1_{B_m^x(\phi^{j+1} y)}$$

$$= \sum_{j=0}^{p_m(y)-1} \lambda_{m,y}^{(j+1)\cdot k} \cdot 1_{B_m^y(\phi^j y)}$$

$$= \lambda_{m,y}^k f_{m,y,k}.$$
Thus \( f_{m,y,k} \) is an eigenfunction corresponding to the eigenvalue \( \lambda_{m,y}^k \), which is a complex root of unity.

Considering that
\[
\sum_{k=0}^{p_m(y)-1} \lambda_{m,y}^{j-k} = \begin{cases} 0 & \text{if } j > 0 \\ p_m(y) & \text{if } j = 0 \end{cases},
\]
we obtain
\[
\frac{1}{p_m(y)} \sum_{k=0}^{p_m(y)-1} f_{m,y,k} = 1_{O_m(y)}.
\]
This means \( 1_{O_m(y)} \in \text{Span} \{ f_{m,y,k} \}_{k} \).

Let \( n \in \mathbb{N} \), and \( x \in X \). Since \( (X, \phi) \) is \( \mu \)-LP and \( X \) is a Cantor set there exists a sequence \( \{y_i\} \subset \text{LP}(\phi) \) such that \( O_n(y_i) \) are disjoint and \( \mu(B_n(x)) = \mu(\cup O_n(y_i)) \). This means \( 1_{B_n(x)} \) can be approximated in \( L^2 \) by elements in
\[
\text{Span} \{ 1_{O_m(y)} : m \in \mathbb{N} \text{ and } y \in \text{LP}(\phi) \}.
\]
Since the closure of \( \text{Span} \{ 1_{B_n(x)} : n \in \mathbb{N}, x \in X \} \) is \( L^2(X, \mu) \) we conclude the closure of \( \text{Span} \{ f_{m,y,k} \}_{m,y,k} \) is \( L^2(X, \mu) \).

This implies that every ergodic \( \mu \)-LEP CA is isomorphic to an odometer.

We obtain a stronger result if the measure is \( \sigma \)-invariant.

**Proposition 3.3.3.** Let \( (X, \phi) \) be a \( \mu \)-ergodic, \( \mu \)-LEP CA. If \( \mu \) is \( \sigma \)-invariant then \( (X, \phi, \mu) \) is isomorphic to a cyclic permutation on a finite set.

**Proof.** By Proposition 3.2.6 \( \mu(\text{LP}(\phi)) = 1 \). Since \( \mu \)-LEP CA are \( \mu \)-equicontinuous, there exists \( x \in \text{LP}(\phi) \) such that \( \mu(O_1(x)) > 0 \). Let \( O_1^\infty \) be the \( \phi \)-orbit of \( O_1(x) \) (the orbit ball centred at the origin). We have that \( \mu(O_1^\infty) > 0 \). Since \( \phi(O_1^\infty) = O_1^\infty \) and \( \mu \) is \( \phi \)-ergodic, then \( \mu(O_1^\infty) = 1 \).

We have that \( p_1(O_1^\infty) = \{p_1(x)\} \). This implies the 0th column of almost every point is periodic with period \( p_1(x) \). Since \( \mu \) is \( \sigma \)-invariant we have that almost every point is periodic (with period \( p_1(x) \)).

Using this and other assumptions we can characterize when a limit measure of a \( \mu \)-equicontinuous CA is \( \phi \)-ergodic.

**Corollary 3.3.4.** Let \( X \) be a 1D SFT, \( \mu \) be a \( \sigma \)-invariant measure, \( (X, \phi) \) be a \( \mu \)-equicontinuous CA. We have that \( (X, \phi, \mu_\infty) \) is isomorphic to a cyclic permutation on a finite set if and only if \( \mu_\infty \) is \( \phi \)-ergodic.
Bibliography


[57] T. Tao. Compact systems.  
