Sup-norm problem of certain eigenfunctions on arithmetic hyperbolic manifolds

by

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Abstract

We prove a power saving over the local bound for the L^∞ norm of uniformly non-tempered Hecke-Maass forms on arithmetic hyperbolic manifolds of dimension 4 and 5. We use accidental isomorphism and use the Hecke theory of the corresponding groups to show that if the automorphic form is non-tempered at positive density of finite places then the Hecke eigenvalues are large; amplifying the saving coming from the non temperedness we get a power saving.

Preface

This thesis is an unpublished and original work of the author. The results in chapter 3 and 4 were obtained by the author of the thesis. The results in chapter 5 were obtained in collaboration with my supervisor Professor Lior Silberman.

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Chapter 1

Introduction

The eigenfunctions of the Laplace-Beltrami operator on a Riemannian manifold are very important to study and create links between various fields in mathematics such as spectral geometry, harmonic analysis, quantum mechanics, thermodynamics, or global analysis. The limiting behaviour of the eigenfunctions and the distribution of their mass play an important role in both physics and mathematics.

Let M be a compact Riemannian manifold of dimension n and ψ be a function on M which satisfies $\Delta\psi + \lambda\psi = 0$ and $||\psi||_{L^2(M)} = 1$, where Δ is the Laplace-Beltrami operator on M. By $||\psi||$ and $||\psi||_p$ we will denote $||\psi||_{L^\infty(M)}$ and $||\psi||_{L^p(M)}$ for $1 \leq p < \infty$ unless mentioned otherwise. It is known that (see e.g. [15],[52]) one can have a general bound

$$||\psi|| \ll \lambda^{\nu(M)} \text{ with } \nu(M) = \frac{\dim(M) - 1}{4}. \tag{1.0.1}$$

The above bound is sharp for round sphere $M = S^n$ or on a surface of revolution that is diffeomorphic to S^2, but is far from the true bound on flat tori. It is usually believed that if the geodesic flow on the unit cotangent bundle of M is chaotic then the bound (1.0.1) can be improved. One result of this kind is due to Bérard [3], who proves that if M has negative sectional curvature (so geodesic flow is ergodic) then one has,

$$||\psi|| \ll \frac{\lambda^{\nu(M)}}{\sqrt{\log\lambda}}.$$

Similar results on the assumption on the geodesic flow of the manifold (ergodicity, positive entropy etc.) can be found in [56] or [55]. In case of negatively curved manifold understanding the behaviour of $||\psi_\lambda||_p$ as $\lambda \to \infty$ is an important question regarding quantum chaos (see [45]).

The case of bounding sup norm of eigenfunctions on congruence quotients of manifolds is an interesting problem because due to automorphy and symmetries arising from underlying Hecke algebra one expects a power saving in (1.1). The first breakthrough was done by Iwaniec and Sarnak [27] who proved that for compact (see [5] for non-compact) arithmetic hyperbolic surface, such as quotient of $\mathcal{H}^2$ by the group of units in an order in a quaternion division algebra over $\mathbb{Q}$, for a Hecke-Maass eigenform ψ with Laplace eigenvalue λ the supnorm $||\psi|| \ll_\epsilon \lambda^{\frac{5}{24}+\epsilon}$. They also provided a lower bound of sup-norm, namely $||\psi_\lambda|| \gg \sqrt{\log\log\lambda}$ for infinitely many eigenfunctions. In fact, it has been conjectured by Hejhal-Rackner [23] (using a random wave model and numerical computation on large deviation support) that the bound $||\psi|| \ll \lambda^\epsilon$ holds (modified in non-compact case) on compact hyperbolic surface. This is compactible with the results of Iwaniec-Sarnak. Not only the eigenvalue aspect, but the volume of the underlying manifold (so-called level aspect) is also an important aspect to bound the eigenforms as the complexity of the chaotic system formed by the quantum eigenstates does increase in level direction, for instance, see [25], [26], [5].

There are several generalizations of Iwaniec-Sarnak type result in higher dimensional and higher rank arithmetic locally symmetric spaces, i.e. a power saving in the bound of L^∞ norm from the so called 'convexity' bound. In a letter to Morawetz Sarnak [44] proved that if $X = G/K$ is a locally symmetric space of dimension n and rank r and $\Omega \subseteq X$ compact then an L^2 normalized joint eigenfuction of the ring of invariant differential operators ψ should satisfy

$$||\psi|_\Omega|| \ll \lambda^{\frac{n-r}{4}}.$$

Note that this is the 'convexity' bound and the natural replacement of the bound in equation (1.0.1). Blomer, Harcos and Milićević [4] recently proved a strong supnorm bound for cuspidal eigenfunctions on arithmetic hyperbolic 3-manifolds. There are also results where higher rank eigenfunctions are proved to have saving fro the standard bound in their supnorm; for instance $\mathrm{SL}_3(\mathbb{Z})\backslash\mathrm{SL}_3(\mathbb{R})$ by Holowinsky, Ricotta and Royer [24], $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathrm{Sp}_4(\mathbb{R})$ by Blomer and Pohl [11] and $\mathrm{SL}_n(\mathbb{Z})\backslash P\mathrm{GL}_n(\mathbb{R})$ by Blomer and Maga [6] and [7]. Marshall [37] in his wonderful work has proved the saving in sup norm for eigenfunctions on symmetric spaces arising from a large class of semisimple Lie groups. Bounds for L^p norm of the eigenfunctions can be found in [36]. There are also results on bounding eigenfunctions on round spheres and arithmetic ellipsoids, namely, [8] and [9].

There also have been a lot of works regarding lower bounds of the eigenfunctions. The naive expectation of $\|\psi\| \ll \lambda^{\epsilon}$ which is suggested by the random wave model, does not hold in higher dimension and higher rank case. It has been shown in [39] that on any arithmetic hyperbolic 3-manifold of Maclachlan-Reid type there exists an infinite orthonormal family of cusp forms ψ with lower bound

$$\|\psi\| \gg_{\epsilon} \lambda^{1/4-\epsilon}.$$

Large values of GL_n Maass forms are also established in the recent work of [13].

In this thesis, we focus our attention to a particular class of eigenfunctions on arithmetic hyperbolic manifolds of dimension 4 and 5, i.e. non-compact congruence quotients of $\mathcal{H}^4$ and $\mathcal{H}^5$ respectively. In general, one can view the n dimensional hyperbolic space as a locally symmetric space arising from $\mathrm{SO}_{n,1}(\mathbb{R})$, precisely, $\mathcal{H}^n = \mathrm{SO}^+_{n,1}(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. But due to sporadic isogenies of special orthogonal groups one may view $\mathrm{SO}_{5,1}(\mathbb{R})$ as $\mathrm{SL}_2(\mathbb{H})$ and $\mathrm{SO}_{4,1}(\mathbb{R})$ as $\mathrm{Sp}^*_{1,1}(\mathbb{H})$ (precise description in chapter 3), where $\mathbb{H}$ is the division algebra Hamilton quaternions over real. Let D be the unique quaternion algebra over $\mathbb{Q}$ ramified at $\{2, \infty\}$ so that $\mathbb{H} = D \otimes_{\mathbb{Q}} \mathbb{R}$ is the division algebra of Hamilton's quaternions. Let $\mathbb{G}$ be the algebraic group defined over $\mathbb{Q}$ such that $\mathbb{G}(\mathbb{Q}) \cong \mathrm{GL}_2(D)$ (correspondingly $\mathrm{Sp}^*_{1,1}(D)$). The real points $\mathbb{G}(\mathbb{R})$ is a real Lie group which is isomorphic to $\mathrm{GL}_2(\mathbb{H})$ (correspondingly $\mathrm{Sp}^*_{1,1}(\mathbb{H})$) and acts on $\mathcal{H}^5$ (correspondingly $\mathcal{H}^4$) by isometries with respect to the usual hyperbolic metric. Let O be the Hurwitz maximal order in D. Let $\Gamma = \mathrm{SL}_2(O)$ (correspondingly $\mathrm{Sp}^*_{1,1}(O)$) be the subgroup which acts discretely on the corresponding hyperbolic space (our results in fact apply with D being any quaternion algebra ramified at infinity and O any maximal order in D, and Γ any congruence subgroup in $\mathrm{SL}_2(O)$, but we make a specific choice for ease of presentation). We consider the Hecke-Maass forms on $M = \Gamma\backslash\mathcal{H}^n$. These are the eigenfunctions of the Laplace-Beltrami operator. They are naturally also eigenfunctions of the Hecke operators arising from Γ which commute among themselves and also with Laplace operator; so one may choose a basis of the Hilbert space which is the cuspidal part of $L^2(\Gamma\backslash\mathcal{H}^n)$ consisting joint eigenfunctions of Laplace and Hecke operators and they will be denoted as Hecke-Maass cusp forms. We parametrize the eigenvalue as follows: let $-\Delta\phi = \lambda\phi$ be a Maass form on $\mathcal{H}^n$. As $\lambda \geq 0$ (the Laplace-Beltrami operator is non-positive in $L^2(M)$),

$$t = \sqrt{\lambda - \frac{(n-1)^2}{4}} \in \mathbb{R} \cup \left[-\frac{n-1}{2}, \frac{n-1}{2}\right] i.$$

Throughout the thesis we denote

$$T := \max(1, |t|) \asymp 1 + \sqrt{\lambda}.$$

Using the accidental isomorphism of the groups $\mathrm{SO}_{4,1}(\mathbb{R})$ and $\mathrm{SO}_{5,1}(\mathbb{R})$ with $\mathrm{Sp}^*_{1,1}(\mathbb{H})$ and $\mathrm{SL}_2(\mathbb{H})$ respectively we can explicitly work out the Hecke algebra for corresponding cases. The Hecke algebras in respective cases are basically Hecke algebras of GSp_4 and GL_4 at odd places. As the Hecke eigenfunctions preserve further symmetries we expect them not to grow very large. The Hecke-algebra eigenvalues at a prime p of a Hecke eigenfunction as described by a *spectral parameter*, a vector of complex numbers. We say an eigenfunction is η-non-tempered at p if it has a spectral parameter with real part at least η (for the definition of the spectral parameters see definition 9). We say that a sequence of eigenfunctions is *uniformly*-nontempered if there is a set P of primes of natural density $\delta > 0$ and a constant $\eta > 0$ such that for every eigenfunction in the sequence and for any $p \in P$, it is η-non-tempered at p. In this setting we will describe our main theorem which is an improvement of (1.0.1) in the eigenvalue aspect for non-tempered eigenforms.

Theorem 1.

(i) Suppose we have a sequence of uniformly nontempered L^2-normalized Hecke-Maass cuspidal eigenform on $\mathrm{Sp}^*_{1,1}(\mathcal{O})\backslash\mathcal{H}^4$. Let ϕ be an element in the se-

quence with Laplace eigenvalue λ. Then for $\Omega \in \mathrm{Sp}^*_{1,1}(\mathcal{O})\backslash\mathcal{H}^4$ compact,

$$\|\phi|_\Omega\| \ll_\epsilon \lambda^{\frac{3}{4}-\epsilon},$$

for some $\epsilon > 0$.

(ii) Suppose we have a sequence of uniformly nontempered L^2-normalized Hecke-Maass cuspidal eigenform on $\mathrm{SL}_2(\mathcal{O})\backslash\mathcal{H}^5$. Let ϕ be an element in the sequence with Laplace eigenvalue λ. Then for $\Omega \in \mathrm{SL}_2(\mathcal{O})\backslash\mathcal{H}^5$,

$$\|\phi|_\Omega\| \ll_\epsilon \lambda^{1-\epsilon},$$

for some $\epsilon > 0$.

Remark. (i) Note that in the theorem the exponents $\frac{3}{4}$ and 1 in the exponents are from 1.0.1 for respective dimensions.

(ii) While the Generalized Ramanujan Conjecture (GRC) predicts that the generic cuspidal representations of quasi-split groups (e.g. GL_n) should be tempered at all places [47], there actually exist uniformly non-tempered cusp forms in case of $\mathrm{SO}(4,1)$ and $\mathrm{SO}(5,1)$. It is also believed that a representation is non-tempered i.e. does not satisfy the GRC if it comes from a functorial lift from a smaller group. The most evident counter-exaples are the CAP (Cuspidal representation Associated to Parabolic) representation defined by Howe and Piatetski-Shapiro [22] and the Kurokawa lift for Sp_4 [32].

(iii) Pitale [41] has constructed an example of CAP representation through a lift from $\widetilde{\mathrm{SL}_2}$ (metaplectic) to $\mathrm{Spin}(1,4)$ which gives a counter-example to the GRC for $\mathrm{Spin}(1,4)$. Also recently in [40] an example is constructed of CAP representation through a lift from SL_2 to $\mathrm{Spin}(1,5)$ which gives a counter-example to the GRC for $\mathrm{Spin}(1,5)$. So our result automatically proves a power saving in the sup norm of the sequence of Hecke-Maass forms they have constructed (in their cases $\eta = \frac{1}{2}$) in both cases.

Our proof starts with a pre-trace formula. We define an automorphic kernel on

$M \times M$ by

$$K(P, Q) = \sum_{\gamma \in \Gamma} k(u(\gamma P, Q))$$

where u is a point pair invariant and and Γ is the corresponding discrete subgroup and k is a rapid decay smooth function on positive real numbers. Then we choose an orthonormal basis of cuspidal eigenfunction which contains our favourite ϕ and write K in this basis. After we apply suitable Hecke operators on both sides (making sure that each term in spectral side is positive) the problem reduces to a diophantine counting problem where we count, at least at identity, how many of γ in Hecke double coset lie in the maximal compact. Then we show that, if the eigenfunction is nontempered then there exists a Hecke operator whose eigenvalue for that eigenfunction is big enough than the Hecke operator returns to its maximal compact and that gives us a power saving.

In the first chapter we describe basic formulation of Clifford algebra and Vahlen matrices and also describe the $\mathrm{Spin}(1, n)$ groups. In the second chapter we describe notion of automorphic forms. In the third chapter we devote ourselves on describing Hecke theory for both $\mathrm{Spin}(1, 4)$ and $\mathrm{Spin}(1, 5)$ cases. In the fourth chapter we describe pre-trace formula and develop the Archimedean amplification and also prove the 'trivial bound'. In fifth chapter we give the prove of our main theorem.

Chapter 2

Basic Notations

2.1 Clifford Algebras, Vahlen Matrices and Real Hyperbolic Spaces

In this section we will briefly describe a model of $k+1$-dimensional real hyperbolic space $\mathcal{H}^{k+1}$ following an approach of Vahlen [57]; for details reader may look at [18]. For integer $k > 0$ we construct $C_k := C_k(\mathbb{R})$ to be the Clifford algebra over $\mathbb{R}$ associated with the negative definite unit form $I_k = (-\delta_{ij})_{1\leq i,j\leq k}$. Let $i_1, \ldots, i_k$ be the standard basis of $\mathbb{R}^k$. We embed canonically $\mathbb{R}^k \hookrightarrow C_k$ we note that the elements i_n satisfy:

$$i_n^2 = -1, \;\; i_n i_m = -i_m i_n \;\; (m, n = 1, \ldots, k, m \neq n). \tag{2.1.1}$$

The Clifford algebra is an associative algebra generated by $i_1, \ldots, i_k$. Hence the 2^k elements $i_{n_1} \ldots i_{n_p}$ where $1 \leq n_1 \leq \cdots \leq n_p, 0 \leq p \leq k$ form a basis of C_k over the real. One may note $C_0 = \mathbb{R}, C_1 = \mathbb{C}$ and $C_3 = \mathbb{H}$ =Hamilton's quaternion. There are three involutions defined on C_k by means of,

$$\forall a = a_0 + \sum_{p=1}^{k} \sum_{1\leq n_1 \leq \cdots \leq n_p, 0\leq p\leq k} a_{n_1\ldots n_p} i_{n_1} \ldots i_{n_p} \in C_k$$

$$
\begin{cases}
a \mapsto a' := a = a_0 + \sum_{p=1}^{k} (-1)^p \sum_{1 \le n_1 \le \cdots \le n_p} a_{n_1 \ldots n_p} i_{n_1} \ldots i_{n_p}, \\
a \mapsto \bar{a} := a = a_0 + \sum_{p=1}^{k} (-1)^{\frac{p(p+1)}{2}} \sum_{1 \le n_1 \le \cdots \le n_p} a_{n_1 \ldots n_p} i_{n_1} \ldots i_{n_p}, \\
a \mapsto a^* := a_0 + \sum_{p=1}^{k} (-1)^{\frac{p(p-1)}{2}} \sum_{1 \le n_1 \le \cdots \le n_p} a_{n_1 \ldots n_p} i_{n_1} \ldots i_{n_p}.
\end{cases}
\tag{2.1.2}
$$

These maps satisfy

$$
\begin{cases}
\bar{x}' = x^*, \quad \forall x \in C_k, \\
\bar{x} = -x, x' = -x, x^* = x, \quad \forall x \in \mathbb{R}i_1 \oplus \cdots \oplus \mathbb{R}i_k, \\
(\bar{xy})\bar{y}\bar{x}, (xy)' = y'x', (xy)^* = y^*x^*, \quad \forall x, y \in C_k.
\end{cases}
\tag{2.1.3}
$$

Vahlen's model of $k+1-$dimensional hyperbolic space is constructed as follows. $\mathcal{H}^{k+1}$ is embedded in the $k+1-$dimensional subspace of C_k:

$$
V_k := \mathbb{R}.1 \oplus \mathbb{R}i_1 \oplus \cdots \oplus \mathbb{R}i_k.
$$

For all $v \in V_k$ we denote,

$$
\begin{aligned}
&\text{real part of } v = \mathfrak{R}(v_0 + v_1 i_1 + \ldots v_k i_k) = v_0, \\
&\text{trace of } v = tr(v) = v + \bar{v} = 2\mathfrak{R}(v) \\
&\text{norm of } v = \|v\|^2 = v.\bar{v}.
\end{aligned}
$$

We equip V_k with the quadratic form

$$
q_k(x) := \|x\|^2 \ (x \in V_k)
$$

and $\mathrm{SO}(q_k, \mathbb{R})$ be the special orthogonal group associated to q_k. The set underlying the model of $(k+1)-$dimensional hyperbolic space is the upper half-space

$$\mathcal{H}^{k+1} := \{x_0 + x_1 i_1 + \cdots + x_k i_k : x_0 \ldots, x_k \in \mathbb{R}, x_k > 0\}.$$

We define for $P := x_0 + x_1 i_1 + \cdots + x_k i_k \in \mathcal{H}^{k+1}$

$$\begin{cases} x(P) = x_0 + x_1 i_1 + \cdots + x_{k-} i_{k-1} \\ y(P) = x_k \\ r(P) = \|P\|^2 = \|x(P)\|^2 + y(P)^2. \end{cases} \tag{2.1.4}$$

We endow $\mathcal{H}^{k+1}$ with the Riemannian metric whose line element is

$$\mathrm{ds}^2 = \frac{\mathrm{dx}_0^2 + \cdots + \mathrm{dx}_k^2}{x_k^2}$$

and obtain a model of the $(k+1)-$dimensional hyperbolic space.

Definition 1. *For $P, Q \in \mathcal{H}^{k+1}$ we define a* point pair invariant *by*

$$u(P, Q) := \frac{\|z(P) - z(Q)\|^2 + (y(P) - y(Q))^2}{2y(P)y(Q)}, \tag{2.1.5}$$

and note that

$$\cosh(d(P, Q)) = 1 + 2u(P, Q), \tag{2.1.6}$$

where $d(P, Q)$ is the distance between P and Q coming from Riemannian metric.

Now we will describe the orientation preserving isometry group $\mathrm{Iso}^+(\mathcal{H}^{k+2})$ by means of a group of certain (2×2) matrices over C_k. See [57] for details.

Definition 2. *An element $0 \neq v \in C_k$ is called a* transformer *if there exists a linear automorphism $\phi_v : V_k \to V_k$ such that*

$$vx = \phi_v(x)v' \quad \forall x \in V_k.$$

Following the language of Maass we denote T_k to be the set of all transformers of $0 \neq v \in C_k$.

Definition 3. *For an integer $k \geq 0$ we define* Vahlen group SV_k *by*

$$SV_k := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{C}_k) : \alpha, \beta, \gamma, \delta \in T_k \cup \{0\}, \alpha\beta^*, \delta\gamma^* \in V_k, \alpha\delta^* - \beta\gamma^* = 1 \right\}.$$

One may check that $SV_0 = \mathrm{SL}_2(\mathbb{R})$, $SV_1 = \mathrm{SL}_2(\mathbb{C})$ and $SV_3 = \mathrm{SL}_2(\mathbb{H})$.

Proposition 1.

(1) *The group SV_k is generated by*

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \quad x \in V_k.$$

(2) *Suppose $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k$ and $P \in \mathcal{H}^{k+2}$, then $\gamma P + \delta \in T_k$ and*

$$\sigma P := (\alpha P + \beta)(\gamma P + \delta)^{-1} \in \mathcal{H}^{k+2}.$$

The above formula defines an action of SV_k on $\mathcal{H}^{k+2}$ by orientation preserving isomerism. The corresponding group homomorphism of SV_k into $\mathrm{Iso}^+(\mathcal{H}^{k+2})$ *induces an isomorphism of groups*

$$\mathrm{SO}^\circ_{k+2,1}(\mathbb{R}) \cong \mathrm{Iso}^+(\mathcal{H}^{k+2}) \cong SV_k/\{\pm I\}. \tag{2.1.7}$$

The action of SV_k on $\mathcal{H}^{k+2}$ is transitive on pairs of points with fixed hyperbolic distance.

Proof. See page 381 in [18]. As we can think $\mathcal{H}^n$ as the Minkowski space $(\mathbb{R}^{n+1}, q)$ where $q(x) = x_0^2 - x_1^2 - \cdots - x_n^2$ is a quadratic form of signature $(1, n)$, it is clear that

$$\mathrm{Iso}^+(\mathcal{H}^{k+2}) \cong \mathrm{SO}^\circ(\mathbb{R}^{k+1}, q) \cong \mathrm{SO}^\circ_{n,1}.$$

□

The map $P \mapsto \sigma P$ is so-called Möbius transformation, the coordinates of σP can be recovered from the Iwasawa decomposition of SV_2 or equivalently $\mathrm{SO}^\circ_{n,1}(\mathbb{R})$.

For $k \geq 0$, let $U_k \subset SV_k$ denote the stabilizer of $i_{k+1} \in \mathcal{H}^{k+2}$. Then we have,

Proposition 2. *U_k is a maximal compact subgroup of SV_k and*

(1)
$$U_k = \left\{ \begin{pmatrix} \alpha & -\beta' \\ \beta & \alpha' \end{pmatrix} \in SV_k : \|\alpha\|^2 + \|\beta\|^2 = 1 \right\}, \tag{2.1.8}$$

(2) $\mathrm{SO}(K+2) \cong U_k / \pm I$ *and*

$$f : SV_k/U_k \to \mathcal{H}^{k+2},\ f(vU_k) = vi_{k+1}$$

is an SV_k-equivariant isometry.

Proof. The description of U_k is clear from the definition of transformers. From 2.1.7 we may think

$$U_k/\pm I = \mathrm{Stab}_{SV_k/\pm I} i_{k+1} = \mathrm{Stab}_{\mathrm{SO}^{\circ}_{k+2,1}(\mathbb{R})} e_{k+2} = \mathrm{SO}(k+2) \times \mathrm{SO}(1) \cong \mathrm{SO}(k+2),$$

where $e_{k+2} = (0, \ldots, 0, 1)$. The rest follows easily. □

Note that the isomorphism 2.1.7 and above proposition recover the accidental isomorphisms between

$$\mathrm{SO}_{n,1}(\mathbb{R}) \cong P\mathrm{SL}_2(F)$$

where $F = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ for $n = 2, 3,$ and 5 respectively. In a similar fashion

$$U_0 = \mathrm{SO}(2), U_1 = SU(2), U_3 = \mathrm{Sp}(2).$$

From the Iwasawa decomposition of $\mathrm{SO}_{n,1}(\mathbb{R}) = NAK$ where $N \cong \mathbb{R}^n, A \cong \mathbb{R}^{\times}$ and $K = \mathrm{SO}(n)$ using above isomorphism one can deduce the Iwasawa decomposition of

$$SV_k/\pm I = N_k A_k U_k \tag{2.1.9}$$

where,

$$N_k = \left\{ n(x) := \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} | x \in V_k \right\} \text{ and } A_k = \left\{ a(y) := \begin{pmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{pmatrix} | y \in \mathbb{R}^+ \right\}.$$

Thus for any $P \in \mathbb{H}^{k+2} \cong SV_k/\pm IU_k$ we have that $P : (x, y) = (x(P), y(P)) \mapsto n(x(P))a(y(P))$ from 2.1.4. By doing a re-Iwasawa decomposition of σP for $\sigma \in SV_k$ and recalling the definition of point pair invariant from 2.1.7 one can easily deduce the following.

Proposition 3. *For $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k$ and $P = (x, y) \in \mathcal{H}^{k+2}$ we have,*

(1)

$$x(\sigma P) = \frac{(\alpha x + \beta)\overline{(\gamma x + \delta)} + \alpha\bar{\gamma}y^2}{\|\gamma x + \delta\|^2 + \|\gamma\|^2}$$

and

$$y(\sigma P) = \frac{y}{\|\gamma x + \delta\|^2 + \|\gamma\|^2}.$$

(2)

$$2u(i_{k+1}, \sigma i_{k+1}) + 1 = \|\alpha\|^2 + \|\beta\|^2 + \|\gamma\|^2 + \|\delta\|^2. \tag{2.1.10}$$

2.2 Arithmetic Subgroups

Let $C_k(\mathbb{Q})$ be the Clifford algebra over $\mathbb{Q}$. Then $C_k(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} = C_k(\mathbb{R})$ is a Clifford algebra over $\mathbb{R}$.

Definition 4. *A $\mathbb{Z}$-order $O \subset C_k(\mathbb{Q})$ is called* compatible *if it is stable under the involutions $^-$ and $'$ defined in 2.1.2. For a compatible order O let us also define,*

$$V_{k+1}(O) := O \cap V_{k+1},$$
$$SV_{k+1}(O) := \mathrm{Mat}_2(O) \cap SV_k(C_k(\mathbb{Q})).$$

It is clear that for any compatible $\mathbb{Z}$-order O then $V_{k+1}(O)$ is a lattice in V_{k+1} and as $SV_k(O)$ is the stabilizer of the lattice O^2 of SV_k action on $C_k(\mathbb{Q})^2$, $SV_k(O)$ is a discrete arithmetic subgroup of SV_k. The group $SV_k(O)$ can be thought as a higher dimensional Fuschian group which acts on $\mathcal{H}^{k+2}$ discontinuously. Let P be the standard minimal parabolic subgroup of SV_k. Then for any discrete subgroup $\Gamma < SV_k(C_k(\mathbb{Q}))$ we define the set $\Gamma\backslash SV_k(C_k(\mathbb{Q}))/P \cap SV_k(C_k(\mathbb{Q}))$ to be the set of

Γ cusps. It is well known that set of Γ cusps is finite (see proposition 15.6 in [10]). In particular, for $\Gamma = SV_k(O)$ we know that Γ is cofinite i.e.

$$\mathrm{Vol}(SV_k(O)\backslash\mathcal{H}^{k+2}) = \int_{\Gamma\backslash\mathcal{H}^{k+2}} \mathrm{d}\mu < \infty,$$

where $\mathrm{d}\mu$ is the usual $(k+2)$-dimensional hyperbolic volume measure $\frac{\mathrm{dxdy}}{y^{k+2}}$, for dx usual $(k+1)$-dimensional Lebesgue measure (see corollary 6.4 [17]).

2.3 Automorphic Forms

Now let us fix some notations for this section. Let q_0 be the rational quadratic form defined by $q_0((x_1, x_2, \ldots, x_{n+1})) = -x_0^2 + x_1^2 + \cdots + x_n^2$. For any rational quadratic form q we have that its isometry group $\mathbb{G} = \mathrm{SO}(q)$, which is a connected and almost simple linear algebraic group over $\mathbb{Q}$. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. From the strong approximation theorem of $\mathrm{SO}(n, 1)$ (see theorem 104:4 [38]) we get

$$\mathbb{G}(\mathbb{A}) \cong \mathbb{G}(\mathbb{Q})\mathbb{G}^+(\mathbb{R})K_f, \text{ where } K_f = \prod_{p<\infty} \mathbb{G}(\mathbb{Z}_p). \tag{2.3.1}$$

For $q = q_0$ which is a quadratic form of signature $(n, 1)$, we let $G = \mathrm{SO}^\circ_{n,1}(\mathbb{R}) = \mathbb{G}^+(\mathbb{R})$ be a non-compact real Lie group with trivial center. $K \cong \mathrm{SO}(n)$ be a maximal compact subgroup in G. $S = G/K \cong \mathcal{H}^n$ be the n dimensional real hyperbolic space. G admits an Iwasawa decomposition of

$$G = NAK$$

where

$$N \cong \mathbb{R}^{n-1} \text{ and } A \cong \mathbb{R}^+.$$

Thus any point $P \in S = G/K \cong N \times A$ can be uniquely described as $(x(P), y(P)) := (x_1, \ldots x_{n-1}, y) \in \mathbb{R}^{n-1}\times \cong \mathbb{R}^+$.

There is a unique G invariant (up to scaling) Riemannian metric on $\mathcal{H}^n$ whose line element is

$$ds^2 = \frac{dx_1^2 + \cdots + dx_{n-1}^2 + dy^2}{y^2}$$

and obtain a model of the n-dimensional real hyperbolic space.

Proposition 4. *For $P, Q \in \mathcal{H}^n$ the point pair invariant defined in 2.1.5 gives that*

$$u(P,Q) := \frac{\|z(P) - z(Q)\|^2 + (y(P) - y(Q))^2}{2y(P)y(Q)}. \tag{2.3.2}$$

From the Riemannian metric on S as defined above, in the same coordinates the Laplace-Beltrami operator on $C^\infty(S)$ is given by

$$\Delta_n = -x_n^n \sum_{i=1}^{n} \frac{\partial}{\partial x_i} x_n^{2-n} \frac{\partial}{\partial x_i} \text{ for } (x, y) := (x_1, \dots x_n),$$

and this is a positive operator. Γ is a discrete subgroup of G acting on S properly discontinuously with finite covolume.

Definition 5. *A complex-valued fuction $\phi \in C^\infty(S)$ is called* automorphic form *with respect to Γ if ϕ satisfies following conditions:*

- $\Delta_n \phi = \lambda\phi$. *We define $\lambda = \frac{(n-1)^2}{4} + t^2$ for $t \in \mathbb{R} \cup i\left(-\frac{n-1}{2}, \frac{n-1}{2}\right)$.*
- $\phi(\gamma P) = \phi(P)$ *for all $P \in S$ and $\gamma \in \Gamma$.*
- ϕ *is of moderate growth.*

In sense of Maass we call the above functions as Maass forms which has a Fourier expansion (see [35])

$$\phi(P) = u(y) + y^{\frac{n-1}{2}} \sum_{m \in L\setminus\{0\}} a_m K_{it}(2\pi|m|y) e(\langle m, x\rangle), \tag{2.3.3}$$

where L is the dual lattice of $\Gamma \cap N \cong \mathbb{Z}^{n-1}$ in $\mathbb{R}^{n-1}$ with respect to standard inner product of the same and hence, $L \cong \mathbb{Z}^{n-1}$. Also $e(z) = \exp(2\pi i z)$, $u \in C^\infty(\mathbb{R}^+)$ and K_α is the modified Bessel function defined as,

$$K_\alpha(y) = \int_0^\infty \exp(-y\cosh(u))\cosh(\alpha u)du.$$

We let $u(y) = 0$ and therefore $\phi \in L^2(\Gamma\setminus S)$ for rapid decay of K-Bessel function; we normalize $\|\phi\|_{L^2} = 1$. We call such function as *Maass cusp forms*. In fact by a theorem of Harish-Chandra gives that (see [21] chapter 1 lemma 12),

Proposition 5. *If $\phi \in C^{\infty}(\Gamma\backslash S)$ is a cusp form then ϕ rapidly decays in every Siegel domain.*

The hyperbolic volume element on $\mathcal{H}^n$ is denoted as

$$\mu(P) = \frac{dx_1 \dots dx_{n-1} dy}{y^n}.$$

Definition 6. *On the space of automorphic form we define Petersson inner product as*

$$\langle f, g \rangle = \int_{\Gamma\backslash\mathcal{H}^n} f(P)\overline{g(P)} d\mu(P). \tag{2.3.4}$$

We would like adelize the automorphic form due to ease of explaining Hecke action. For a given congruence subgroup Γ there is an algebra of Hecke operators, which commute with the laplacian, acting on Maass forms. We will assume that our Maass form is an eigenform of full Hecke algebra (detailed description is in section 3). Given a Hecke-Maass eigen-cusp form ϕ, write $g = g_{\mathbb{Q}} g_{\infty} k_f$ where $g \in \mathbb{G}(\mathbb{A})$, $g_{\mathbb{Q}} \in \mathbb{G}(\mathbb{Q})$ and $k_f \in K_f$ and define $\Phi := \Phi_\phi : \mathbb{G}(\mathbb{A}) \to \mathbb{C}$ as $\Phi(g) = \phi(g_\infty)$. Φ satisfies the following:

- $\Phi(z\gamma g k_f k) = \Phi(g)$ for $(z, \gamma, g, k_f, k) \in \mathbb{A}^\times \times \mathbb{G}(\mathbb{Q}) \times \mathbb{G}(\mathbb{A}) \times K_f \times \mathrm{SO}(n)$,
- Φ has moderate growth.

Chapter 3

Hecke Theory

In this section we will describe the action of the Hecke algebra on automorphic forms on the congruence hyperbolic 4 and 5 manifolds only. We confine our discussion only to dimension 4 and 5 because we want to use the help of accidental isomorphism in those cases. We will give both classical and adelic viewpoint towards the Hecke theory. For standard details reader may look at Krieg's work [29] and [30]. For the purpose of the paper we will only discuss the Hecke theory for odd primes.

Let D be Hamilton's quaternions over $\mathbb{Q}$, that is the $\mathbb{Q}$-algebra spanned by $\{1, i, j, k\}$ subject to

$$i^2 = j^2 = k^2 = -1 \text{ and } ij = -k.$$

Let $\mathcal{O}$ be the Hurwitz order, defined by

$$\mathcal{O} = \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k + \mathbb{Z}\frac{1+i+j+k}{2}.$$

For a prime $p \leq \infty$ we define $D_p = D \otimes_{\mathbb{Q}} \mathbb{Q}_p$, i.e. $D_\infty = D \otimes \mathbb{R} = \mathbb{H}$ the usual Hamiltom quaternion, and for an odd prime p we have $D_p \cong \mathrm{Mat}_2(\mathbb{Q}_p)$ and $\mathcal{O}_p = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathrm{Mat}_2(\mathbb{Z}_p)$.

3.1 SO(4,1) Case

Let G be the $\mathbb{Q}$ algebraic group such that $G(\mathbb{Q}) = \mathrm{Sp}^*_{1,1}(D)$, where

$$\mathrm{Sp}^*_{1,1}(D) = \left\{ g \in \mathrm{GL}_2(D) : g^* \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} g = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\}.$$

and g^* is $\bar{g}^t$ entrywise quaternion conjugation. The first part of the following proposition will allow us to realize an automorphic form of $\mathrm{SO}_{4,1}(\mathbb{R})$ as an automorphic form on $G(\mathbb{R})$

Proposition 6. *(1) There is a* 2 *to* 1 *homomorphism between* $G(\mathbb{R}) = \mathrm{Sp}^*_{1,1}(\mathbb{H})$ *and* $\mathrm{SO}_{4,1}(\mathbb{R})$*.*

(2) For an odd prime p

$$G(\mathbb{Q}_p) \cong \mathrm{GSp}_4(\mathbb{Q}_p) \text{ and } G(\mathbb{Z}_p) \cong \mathrm{GSp}_4(\mathbb{Z}_p).$$

Proof.

(1) Let $g^\sigma = S g^* S^{-1}$, where $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Then for $g \in G(\mathbb{R})$ we have $g^\sigma g = I_2$. Let $G(\mathbb{R})$ acts on $\mathrm{Mat}_2(\mathbb{H})$ by $g.x = g^\sigma x g$. Let $\mathfrak{t}$ be the reduced trace on $\mathrm{Mat}_2(\mathbb{H})$ defined by

$$\mathfrak{t}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{1}{2}(a + \bar{a} + d + \bar{d}).$$

$(x, y) = \mathfrak{t}(xy)$ is a bilinear form on $\mathrm{Mat}_2(\mathbb{H}) \times \mathrm{Mat}_2(\mathbb{H})$. Note that

$$(g.x, g.y) = \mathfrak{t}(g^\sigma x g g^\sigma y g) = \mathfrak{t}(xy) = (x, y)$$

The 5 dimensional $\mathbb{R}$ vector space

$$V = \{x \in \mathrm{Mat}_2(\mathbb{H}) | x^\sigma = x \text{ and } (x, S) = 0\} = \left\{ \begin{pmatrix} a & b \\ -b & \bar{a} \end{pmatrix} \right\}$$

is stable under this action and has an orthogonal basis

$$\begin{pmatrix}1 & \\ & 1\end{pmatrix}, \begin{pmatrix}i & \\ & -i\end{pmatrix}, \begin{pmatrix}j & \\ & -j\end{pmatrix}, \begin{pmatrix}k & \\ & -k\end{pmatrix}, \begin{pmatrix} & 1\\ -1 & \end{pmatrix};$$

one can check it has the desired signature $(4, 1)$.

(2) Noting that for odd prime the division algebra and the Hurwitz order split over $\mathbb{Q}_p$ and $\mathbb{Z}_p$ respectively i.e.

$$D \otimes \mathbb{Q}_p = \mathrm{Mat}_2(\mathbb{Q}_p) \text{ and } \mathcal{O} \otimes \mathbb{Z}_p = \mathrm{Mat}_2(\mathbb{Z}_p)$$

the second part is immediate. □

Let us fix $\Gamma = \mathrm{Sp}^*_{1,1}(\mathcal{O})$. For an odd rational prime p let us fix π, a primitive quaternion integer (i.e. $\pi \notin n\mathcal{O}$ for any rational intger n) with $|\pi|^2 = p$. Let us define

$$M(n) = \{\gamma \in \mathrm{Mat}_2(\mathcal{O}) : \gamma^* S \gamma = nS\};$$

so $\Gamma = M(1)$. Also define

$$M_p = \cup_{m=0}^{\infty} M(p^m).$$

Following Shimura [51] we define the classical p-Hecke algebra $\mathcal{H}_p$ over Γ as the algebra generated by the double cosets

$$\{\Gamma M \Gamma : M \in M_p\}.$$

From theorem 7 [30] we have the following generating elements of $\mathcal{H}_p$

Proposition 7. *$\mathcal{H}_p$ is the polynomial ring over $\mathbb{Z}$ generated by the elements*

$$T(p) := \Gamma\begin{pmatrix}1 & \\ & p\end{pmatrix}\Gamma, S(p) := \Gamma\begin{pmatrix}\pi & \\ & \pi p\end{pmatrix}\Gamma, \text{ and } I(p) := p\Gamma,$$

which are algebraically independent.

To know how big the order of support of a Hecke operators is, we need to decompose the double cosets into single cosets. Next lemma is describes the single coset decomposition. For this purpose we will map the classical p-Hecke algebra to the canonical convolution p-adic Hecke algebra $H_p = H(\mathrm{GSp}_4(\mathbb{Q}_p), \mathrm{GSp}_4(\mathbb{Z}_p))$ which is the convolution algebra of $\mathrm{GSp}_4(\mathbb{Z}_p)$-biinvariant compactly supported function of $\mathrm{GSp}_4(\mathbb{Q}_p)$.

One may note that,

$$G(\mathbb{Z}[p^{-1}]) \cap G(\mathbb{Z}_p) = G(\mathbb{Z})$$

and

$$G(\mathbb{Q}_p) \cong G(\mathbb{Q})\mathrm{GSp}_4(\mathbb{Z}_p) \cong G(\mathbb{Z}[p^{-1}])G(\mathbb{Q}_p),$$

where $G(\mathbb{Z}) = \Gamma \cup \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \Gamma$. This shows that

$$\Gamma \backslash G(\mathbb{Z}[p^{-1}])/\Gamma \cong G(\mathbb{Z})\backslash G(\mathbb{Z}[p^{-1}])/G(\mathbb{Z}) \cong G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$$

and hence p-adic Hecke algebra is isomorphic to p-part of classical Hecke algebra.

Lemma 1. *With the notation above,*

$$H_p \cong \mathcal{H}_p,$$

where one can map

$$T(p) \mapsto \mathrm{Char}(K_p \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} K_p), S(p)a \mapsto \mathrm{Char}(K_p \begin{pmatrix} 1 & & & \\ & p & & \\ & & p & \\ & & & p^2 \end{pmatrix} K_p), \text{ and } I(p) \mapsto \mathrm{Char}(pK_p). \tag{3.1.1}$$

where $K_p = G(\mathbb{Z}_p) = \mathrm{GSp}_4(\mathbb{Z}_p)$ *with* $\mathrm{Vol}(K_p) = 1$ *under the usual Haar measure of* G_p.

Let ϕ be a Hecke-Maass cuspidal eigenform of $\Gamma\backslash\mathcal{H}^4$. Recall that there is an automorphic form Φ which we can produce from ϕ as described in the last paragraph of section 2. We consider the representation $\pi_\phi := \pi_\Phi$ of $G(\mathbb{A})$ on right translation of Φ. Let π be a irreducible component of π_Φ which is a cuspidal (as ϕ

and hence Φ is cuspidal) automorphic representation of $G(\mathbb{A})$. π has trivial central character as ϕ is invariant by the central action. Write $\pi = \otimes'_p \pi_p$ where π_p is representation of $G_p := \mathbb{G}(\mathbb{Q}_p)$. We note that, for an odd prime p, π_p is an irreducible unramified representation of G_p since K_p is the maximal compact subgroup of G_p. From [14] we know that there exists an unramified character χ of the Borel subgroup of G_p, unique up to the Weyl group orbit, such that π_p is isomorphic to the unique spherical constituent π_χ of the normalized induced representation $Ind_B^{G_p}(\chi)$. We will now decompose the double coset into single cosets to find the eigenvalues as a polynomial in the components of χ (see [53]).

In the following lemma we will describe the single coset decomposition for our case. Note that a single coset decomposition of a Hecke double coset is computationally hard problem in general. However for small groups one may try to invert the Satake map to get such decomposition, for the general inversion of Satake see [48]. For symplectic and general linear groups Hecke decomposition were done by several people, e.g. for the following decompositions one may look at [1], [43], or [49]

Lemma 2.

(1)

$$K_p \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} K_p = \bigcup \begin{pmatrix} p & & * & * \\ & p & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix} K_p \bigcup \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} K_p$$
$$\bigcup_{b \in \mathbb{Z}/p\mathbb{Z}} \begin{pmatrix} 1 & & * & * \\ -b & p & * & * \\ & & p & b \\ & & & 1 \end{pmatrix} K_p \bigcup \begin{pmatrix} p & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & p \end{pmatrix} K_p,$$

where in the first term the upper right corner has p^3 choices and third and fourth term the upper right corner have p choices each.

(2)

$$K_p\begin{pmatrix}1&&&\\&p&&\\&&p&\\&&&p^2\end{pmatrix}K_p=\bigcup_{b\in\mathbb{Z}/p\mathbb{Z}}\begin{pmatrix}1&&&\\-b&p&&\\&&p^2&bp\\&&&p\end{pmatrix}K_p\bigcup\begin{pmatrix}p&&&\\&1&&\\&&p&\\&&&p^2\end{pmatrix}K_p$$

$$\bigcup\begin{pmatrix}p^2&&*&*\\&p&*&*\\&&1&\\&&&p\end{pmatrix}K_p\bigcup_{b\in\mathbb{Z}/p\mathbb{Z}}\begin{pmatrix}p&&*&*\\-bp&p^2&*&*\\&&p&b\\&&&1\end{pmatrix}K_p$$

$$\bigcup\begin{pmatrix}p&&*&*\\&p&*&*\\&&p&\\&&&p\end{pmatrix}K_p,$$

where in the third and fourth term the upper right corners have p^3 choices and the fifth term the upper right corner has p^2-1 choices.

The following explicit Iwasawa decomposition and Satake Isomorphism for G_p may be found in the paper of Asgari-Schmidt [2]. G_p has an Iwasawa decomposition of the form $G_p = BK_p$ and $B = NA$ where

$$N=\left\{\begin{pmatrix}A&X\\&D\end{pmatrix}|A^{-1}=D^t \text{ and } X^t-X\right\}$$

is the unipotent radical and

$$A=\left\{a=\begin{pmatrix}a_1&&&\\&a_2&&\\&&a_1^{-1}a_0&\\&&&a_2^{-1}a_0\end{pmatrix}:a_i\in\mathbb{Q}_p^\times\right\}.$$

Let χ_0,χ_1,χ_2 be unramified characters on $\mathbb{Q}_p^\times$. We define a character on A by

$$\chi(a)=\chi_0(a_0)\chi_1(a_1)\chi_2(a_2). \tag{3.1.2}$$

We extend χ from A to $B = NA$ by setting χ to be trivial on N. Now let us define

$$I(\chi) := Ind_B^{G_p}(\chi) = \{f \in C_c(G_p) : f(nag) = \delta^{1/2}(a)\chi(a)f(g) \text{ for } (n, a, g) \in N{\times}A{\times}G_p\}$$

where δ, the modular function from the usual Haar measure on G_p defined by

$$\delta(a) = |a_0^{-3}a_1^2a_2^4|_p. \tag{3.1.3}$$

We will choose χ in such a way so that π_p becomes ismorphic to the unique spherical constituent π_χ of $I(\chi)$.

Let F_p be the unramified vector in the space π_χ with $F_p(e) = 1$. Then

$$F_p(nak) = \delta^{1/2}(a)\chi(a)$$

for $n \in N$, $a \in A$ and $k \in K_p$. Any $\phi \in H_p$ acts on F_p by convolution as following; for $h \in G_p$

$$(\phi * F_p)(h) := \int_{G_p} \phi(hg)F_p(g)dg.$$

Now let $\phi = \mathrm{Char}(K_pa_pK_p) \in H_p$ with $K_pa_pK_p = \coprod_i H_iK_p$ and also let $H_i = n_ia_i$ where $(n_ia_i) \in N \times A$. Then

$$\begin{aligned}(\phi * F_p)(e) &= \int_{G_p} \phi(g)F_p(g)dg \\ &= \int_{K_pa_pK_p} F_p(g)dg \\ &= \sum_i \int_{H_iK_p} F_p(g)dg \\ &= \sum_i F_p(n_ia_i) \\ &= \sum_i \delta^{1/2}(a_i)\chi(a_i).\end{aligned} \tag{3.1.4}$$

Along with above and lemma 2. we arrive at he following conclusion.

Proposition 8. *Suppose that π_χ has trivial central character. Then,* *(1)* $\mathrm{Char}(K_p \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} K_p) F_p = p^{3/2}\chi_0(p)[\chi_1(p)\chi_2(p) + 1 + \chi_1(p) + \chi_2(p)]F_p$,

(2) $\chi_0^2(p)\chi_1(p)\chi_2(p) = 1$,

(3) $\mathrm{Char}(K_p \begin{pmatrix} 1 & & & \\ & p & & \\ & & p & \\ & & & p^2 \end{pmatrix} K_p) F_p = [p^2\chi_0^2(p)\{\chi_1(p) + \chi_2(p) + \chi_1(p)\chi_2(p)(\chi_1(p) +$
$\chi_2(p))\} + (p^2 - 1)]F_p$.

Proof. As computed in 3.1.4 combining with first decomposition in lemma 2 we get that,

$$\mathrm{Char}(K_p \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} K_p) F_p = p^3 \delta^{1/2}(\begin{pmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{pmatrix}) \chi(\begin{pmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$+ \delta^{1/2}(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} \chi(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix}) + p\delta^{1/2}(\begin{pmatrix} 1 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix}) \chi(\begin{pmatrix} 1 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix})$$

$$+ \delta^{1/2}(\begin{pmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{pmatrix}) \chi(\begin{pmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{pmatrix})$$

From 3.1.2 and 3.1.3 we get that,

$$\begin{aligned}\mathrm{Char}(K_p\begin{pmatrix}1&&&\\&1&&\\&&p&\\&&&p\end{pmatrix}K_p)F_p &= [p^3p^{-3/2}\chi_0(p)\chi_1(p)\chi_2(p) + p^{3/2}\chi_0(p)\\ &\quad + p^2p^{-1/2}\chi_0(p)\chi_2(p) + pp^{1/2}\chi_0(p)\chi_1(p)]F_p\\ &= p^{3/2}\chi_0(p)[\chi_1(p)\chi_2(p) + 1 + \chi_1(p) + \chi_2(p)]F_p.\end{aligned}$$

This proves (1). Similary the second decomposition in lemma 2 gives

$$\begin{aligned}\mathrm{Char}(K_p\begin{pmatrix}1&&&\\&p&&\\&&p&\\&&&p^2\end{pmatrix}K_p)F_p &= p^2\chi_0^2(p)[\{\chi_1(p) + \chi_2(p) + \chi_1(p)\chi_2(p)(\chi_1(p) + \chi_2(p))\}\\ &\quad + (p^2-1)\chi_0^2(p)\chi_1(p)\chi_2(p)]F_p\end{aligned}$$

Now using $\phi = \mathrm{Char}(K_p pI_4 K_p)$ in 3.1.4 and assuming π_χ has trivial central character one gets

$$F_p = \mathrm{Char}(K_p pI_4K_p)F_p = \chi_0^2(p)\chi_1(p)\chi_2(p)F_p,$$

which gives (2) and (3). □

Now we will be proving our main lemma to get the amplification. On a separate note the component characters arising from the Borel subgroup are called the *Satake parameters* of the corresponding Hecke-Maass form, in particular, the eigenvalues of the same are polynomials in Satake parameters as shown above. Our goal is to show that if a representation is non-tempered then there exist at least one Hecke operator whose eigenvalue is much large than its support. This is usually called an amplification; the idea of amplication is inspired by [16]. For general amplification scheme one may look at [54].

Definition 7 (Temperedness)**.** *We say a Hecke Maass form is tempered at a place* $2 < p < \infty$ *if all its Satake parameters at p are unitary i.e.* $|\chi_i(p)| = 1$ *for all i.*

So in our setting where $G(\mathbb{Q}_p) = \mathrm{GSp}_4(\mathbb{Q}_p)$ temperdness is equivalent to that fact that χ_i's are unitary (see [46]). our definition is equivalent to the usual notion of temperedness, that if $\pi = \otimes_v \pi_v$ is the global cuspidal representation of a $\mathbb{Q}$ group G then π is tempered at the place $p < \infty$ if all the matrix coefficients of π_p lie in $L^{2+\epsilon}(G(\mathbb{Q}_p))$ for all $\epsilon > 0$.

Definition 8. *We say a Hecke-Maass form ϕ is η non-tempered at p if at least one corresponding Satake parameter at p has absolute value p^η i.e. there is a character χ_i such that $|\chi_i(p)| \geq p^\eta$.*

Now we will be proving our main amplification lemma. Let ϕ is the Hecke-Maass form in the question with

$$T(p)\phi = \lambda(p)\phi \text{ and } S(p)\phi = \mu(p)\phi.$$

Lemma 3. *Let ϕ be a Hecke-Maass of $\Gamma\backslash\mathcal{H}^4$ form such that ϕ is η non-tempered at positive density of odd primes p for some $\eta > 0$. Then, either*

$$|\lambda(p)| \gg p^{3/2+\eta/2} \text{ or } |\mu(p)| \gg p^{2+\eta}$$

as $p \to \infty$.

Proof. Combining lemma 1 and proposition 8 we get that,

$$\chi_0(p)^2\chi_1(p)\chi_2(p) = 1,$$

$$\begin{aligned}
p^{-3/2}\lambda(p) &= \chi_0(p)[\chi_1(p)\chi_2(p) + 1 + \chi_1(p) + \chi_2(p)] \\
&= \chi_0(p) + \chi_0(p)^{-1} + \chi_0(p)\chi_1(p) + \chi_0(p)\chi_2(p) \\
&= \chi_0(p)^2\chi_1(p)\chi_2(p)[\chi_0(p)^{-1} + \chi_0(p) + (\chi_0(p)\chi_1(p))^{-1} + (\chi_0(p)\chi_2(p))^{-1}]
\end{aligned}$$

and

$$\begin{aligned}
p^{-2}\mu(p) + 1 + p^{-2} &= \chi_0^2(p)[\chi_1(p) + \chi_2(p) + \chi_1(p)\chi_2(p)(\chi_1(p) + \chi_2(p)) + 2] \\
&= \chi_0^2(p)\chi_1(p) + \chi_0^2(p)\chi_2(p) + \chi_1(p) + \chi_2(p) + \chi_0(p)\chi_0(p)^{-1} + \chi_0^2(p)\chi_1(p)\chi_2(p).
\end{aligned}$$

Thus the polynomial equation

$$x^4 - p^{-3/2}\lambda(p)x^3 + (p^{-2}\mu(p) + 1 + p^{-2})x^2 - p^{-3/2}\lambda(p)x + 1 = 0 \tag{3.1.5}$$

has roots

$$\chi_0(p), \chi_0(p)^{-1}, \chi_0(p)\chi_1(p) \text{ and } \chi_0(p)\chi_2(p)$$

Claim: The equation 3.1.5 has a root $\nu(p)$ such that $|\nu(p)| \geq p^{\eta/2}$.

Note that if either $|\chi_0(p)| = p^{\pm\eta}$ or $|\chi_i(p)| = p^{\eta}$ for $i = 1, 2$ we are done. As the η non-temperedness forces at least one of the character to have magnitude $p^{\pm\eta}$, using symmetry of χ_1 and χ_2 the only remaining possible case is

$$|\chi_0(p)| = p^s, |\chi_1(p)| = p^{-\eta}, |\chi_2(p)| = p^{\eta-2s}.$$

If $s \notin \left(-\frac{\eta}{2}, \frac{\eta}{2}\right)$ then we have nothing to do, as either $|\chi_0(p)| \geq p^{\eta/2}$ or $|\chi_0(p)^{-1}| \geq p^{\eta/2}$. Now let $s \in \left(-\frac{\eta}{2}, \frac{\eta}{2}\right)$. Then

$$|\chi_0(p)\chi_2(p)| = p^{\eta-a} \geq p^{\eta/2}.$$

So the claim is proved.

Now using the root $\nu(p)$ in the claim the equation 3.1.5 gives

$$\begin{aligned}
&p^{-3/2}\lambda(p)(\nu(p)^3 + \nu(p)) = \nu(p)^4 + (p^{-2}\mu(p) + 1 + p^{-2})\nu(p)^2 + 1\\
&\implies p^{-3/2}|\lambda(p)|(|\nu(p)|^3 + |\nu(p)|) \geq |\nu(p)|^4 - 1 - |p^{-2}\mu(p) + 1 + p^{-2}||\nu(p)|^2\\
&\implies p^{-3/2}|\lambda(p)||\nu(p)|^3 + |p^{-2}\mu(p) + 1 + p^{-2}||\nu(p)|^2 \gg |\nu(p)|^4
\end{aligned}$$

so either

$$\begin{aligned}
&p^{-3/2}|\lambda(p)||\nu(p)|^3 \gg |\nu(p)|^4\\
&\implies |\lambda(p)| \gg p^{3/2+\eta/2}
\end{aligned}$$

or

$$\begin{aligned}
&|p^{-2}\mu(p) + 1 + p^{-2}||\nu(p)|^2 \gg |\nu(p)|^4\\
&\implies |p^{-2}\mu(p)| \gg p^{\eta}\\
&\implies |\mu(p)| \gg p^{2+\eta}.
\end{aligned}$$

This proves the lemma. $\square$

3.2 SO(5,1) Case

Let G be an algebraic group over $\mathbb{Q}$ such that $G(\mathbb{Q}) = \mathrm{GL}_2(D)$. The first part of the following proposition would let us realize a Maass form of hyperbolic 5 space as a form on $\mathrm{GL}_2(\mathbb{H})$.

Proposition 9.

(1) There is an isomorphism between $G(\mathbb{R})/\mathbb{R} = P\mathrm{GL}_2(\mathbb{H}) = P\mathrm{SL}_2(\mathbb{H})$ *and* $\mathrm{SO}_{5,1}(\mathbb{R})$.

(2) For an odd prime p we have

$$G(\mathbb{Q}_p) \cong \mathrm{GL}_4(\mathbb{Q}_p) \text{ and } G(\mathbb{Z}_p) \cong \mathrm{GL}_4(\mathbb{Z}_p).$$

Proof. (1) It is enough to prove that $\mathrm{SO}_{5,1}(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{H})/\pm I$. The usual embedding of $\mathbb{H}$ to $\mathrm{Mat}_2(\mathbb{C})$ by

$$a + bi + cj + dk \mapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

can be characterized by as a subset of $\mathrm{Mat}_2(\mathbb{C})$

$$\mathbb{H} = \{x \in \mathrm{Mat}_2(CC) : \bar{x} = wxw^{-1}\}, \text{ where } w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

Thus we may identify

$$\mathrm{SL}_2(\mathbb{H}) = \{g \in \mathrm{SL}_4(\mathbb{C}) : \bar{g} = WgW^{-1}\}, \text{ where } W = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}.$$

Let e_1, e_2, e_3, e_4 be the standard basis of $\mathbb{C}^4$ and we give $\bigwedge^2 \mathbb{C}^4$ a $\mathbb{C}$-valued $\mathrm{SL}_4(\mathbb{C})$ invariant symmetric form

$$\langle x \wedge y, z \wedge w \rangle . e_1 \wedge e_2 \wedge e_3 \wedge e_4 = x \wedge y \wedge z \wedge w.$$

A 6-dimensional $\mathbb{R}$-subspace of $\bigwedge^2 \mathbb{C}^4$ stable under $SU(4)$ will be identied as the xed vectors of an $\mathbb{C}$-conjugate-linear isomorphism $\mathbb{C}^4 \to \mathbb{C}^4$ commuting with $\mathrm{SL}_2(\mathbb{H})$, on which $\langle , \rangle$ takes real values.

Let's define a conjugate linear map by

$$\begin{aligned} J : \bigwedge^2 \mathbb{C}^4 &\to \bigwedge^2 \mathbb{C}^4 \\ x \wedge y &\mapsto W\bar{x} \wedge W\bar{y}. \end{aligned}$$

Note that J commutes with the action of $\mathrm{SL}_2(\mathbb{H})$ as,

$$\begin{aligned} g.J(x \wedge y) &= gW\bar{x} \wedge gW\bar{y} \\ &= W\overline{W^{-1}\bar{g}Wx} \wedge W\overline{W^{-1}\bar{g}Wx} \\ &= W\overline{gx} \wedge W\overline{gy} \\ &= J(g.x \wedge y). \end{aligned}$$

Since,

$$We_1 = e_2, We_2 = -e_1, We_3 = e_4, We_4 = -e_3$$

we can readily have

$$J^2(e_i \wedge e_j) = e_i \wedge e_j \text{ for } i \neq j.$$

Since J is conjugate linear we have that $J^2 = 1$ on $\bigwedge^2 \mathbb{C}^4$. The $+1$ eigenspace has an orthogonal basis consisting

$$e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_2 - e_3 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_4,$$

$$ie_1 \wedge e_3 - ie_2 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3, ie_1 \wedge e_4 + ie_2 \wedge e_3$$

Computing $\langle , \rangle$ one can check that it has desired signature $(5, 1)$.

(2) Noting that for odd prime the divison algebra and the Hurwitz order split over $\mathbb{Q}_p$ and $\mathbb{Z}_p$ respectively i.e.

$$D \otimes \mathbb{Q}_p = \mathrm{Mat}_2(\mathbb{Q}_p) \text{ and } O \otimes \mathbb{Z}_p = \mathrm{Mat}_2(\mathbb{Z}_p)$$

the second part is immediate.

□

Let $\Gamma = \mathrm{SL}_2(O)$. For an odd rational prime p let us fix π, a primitive quaternion integer (i.e. $\pi \notin nO$ for any rational integer n) with $|\pi|^2 = p$. As we have previously discussed that there is a natural embedding of $\mathrm{Mat}_2(\mathbb{H})$ to $\mathrm{Mat}_4(\mathbb{C})$ and hence we define determinant of a matrix in $\mathrm{Mat}_2(\mathbb{H})$ by determinant of its image in $\mathrm{Mat}_4(\mathbb{C})$. Let us define

$$M(n) = \{\gamma \in \mathrm{Mat}_2(O) : \det(\gamma) = n\};$$

so $\Gamma = M(1)$. Also define

$$M_p = \cup_{m=0}^{\infty} M(p^m).$$

Following Shimura [51] we define the classical p-Hecke algebra $\mathcal{H}_p$ over Γ as the algebra generated by the double cosets

$$\{\Gamma M \Gamma : M \in M_p\}.$$

From theorem 3b of [30] we have the following generating elements of $\mathcal{H}_p$

Proposition 10. *$\mathcal{H}_p$ is the polynomial ring over $\mathbb{Z}$ generated by the elements*

$$T(p) := \Gamma \begin{pmatrix} \pi & \\ & 1 \end{pmatrix} \Gamma, S(p) := \Gamma \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma, T^*(p) = \Gamma \begin{pmatrix} p\pi & \\ & \pi \end{pmatrix} \Gamma \text{ and } I(p) := p\Gamma,$$

which are algebraically independent.

To know how big the order of support of a Hecke operators is, we need to decompose the double cosets into single cosets. Next lemma is describes the single coset decomposition. For this purpose we will map the classical p-Hecke algebra to the canonical convolution p-adic Hecke algebra $H_p = H(\mathrm{GL}_4(\mathbb{Q}_p), \mathrm{GL}_4(\mathbb{Z}_p))$ which is the convolution algebra of $\mathrm{GL}_4(\mathbb{Z}_p)$-biinvariant compactly supported function of $\mathrm{GL}_4(\mathbb{Q}_p)$.

One may note that,

$$G(\mathbb{Z}[p^{-1}]) \cap G(\mathbb{Z}_p) = G(\mathbb{Z})$$

and

$$G(\mathbb{Q}_p) \cong G(\mathbb{Q})\mathrm{GL}_4(\mathbb{Z}_p) \cong G(\mathbb{Z}[p^{-1}])G(\mathbb{Q}_p),$$

where $G(\mathbb{Z}) = \Gamma \cup \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\Gamma$. This shows that

$$\Gamma \backslash G(\mathbb{Z}[p^{-1}])/\Gamma \cong G(\mathbb{Z})\backslash G(\mathbb{Z}[p^{-1}])/G(\mathbb{Z}) \cong G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$$

and hence p-adic Hecke algebra is isomorphic to p-part of classical Hecke algebra.

Lemma 4. *With the notation above,*

$$H_p \cong \mathcal{H}_p,$$

where one can map

$$T(p) \mapsto \mathrm{Char}(K_p \begin{pmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} K_p), S(p)a \mapsto \mathrm{Char}(K_p \begin{pmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{pmatrix} K_p),$$
$$T^*(p) \mapsto \mathrm{Char}(K_p \begin{pmatrix} p & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix} K_p), \textit{ and } I(p) \mapsto \mathrm{Char}(pK_p). \tag{3.2.1}$$

where $K_p = G(\mathbb{Z}_p) = \mathrm{GL}_4(\mathbb{Z}_p)$ *with* $\mathrm{Vol}(K_p) = 1$ *under the usual Haar measure of* G_p.

Let ϕ be a Hecke-Maass cuspidal eigenform of $\Gamma\backslash\mathcal{H}^5$. Recall that there is an automorphic form Φ which we can produce from ϕ as described in the last paragraph of section 2. We consider the representation $\pi := \pi_\Phi$ of $G(\mathbb{A})$ on right translation of Φ. Note that π would be irreducible due to strong multiplicity one of $G(\mathbb{A})$ (see [12]) and cuspidal as ϕ is a cusp form. π has trivial central character as ϕ is invariant under central action. Write $\pi = \otimes'_p \pi_p$ where π_p is representation of $G_p := \mathbb{G}(\mathbb{Q}_p)$. We note that, for an odd prime p, π_p is an irreducible admissible representation of G_p since K_p is the maximal compact subgroup of G_p. From [14] we know that there exists an unramified character χ of the Borel subgroup of G_p, unique up to the Weyl group orbit, such that π_p is isomorphic to the unique spherical constituent π_χ of the normalized induced representation $Ind_B^{G_p}(\chi)$. We will now decompose the double coset into single cosets to find the eigenvalues as a polynomial in the components of χ (see [53]).

Lemma 5.

(i)

$$
\begin{aligned}
K_p\begin{pmatrix} p &&& \\ & p && \\ && p & \\ &&& 1 \end{pmatrix}K_p &= \bigsqcup_{x_{14},x_{24},x_{34}\in\mathbb{Z}_p/p\mathbb{Z}_p} \begin{pmatrix} p &&& \\ & p && \\ && p & \\ &&& 1 \end{pmatrix}\begin{pmatrix} 1 &&& p^{-1}x_{14} \\ & 1 && p^{-1}x_{24} \\ && 1 & p^{-1}x_{34} \\ &&& 1 \end{pmatrix}K_p \\
&\sqcup \bigsqcup_{x_{12}\in\mathbb{Z}_p/p\mathbb{Z}_p} \begin{pmatrix} p &&& \\ & 1 && \\ && p & \\ &&& p \end{pmatrix}\begin{pmatrix} 1 & p^{-1}x_{12} && \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}K_p \\
&\sqcup \bigsqcup_{x_{13},x_{23}\in\mathbb{Z}_p/p\mathbb{Z}_p} \begin{pmatrix} p &&& \\ & p && \\ && 1 & \\ &&& p \end{pmatrix}\begin{pmatrix} 1 && p^{-1}x_{13} & \\ & 1 & p^{-1}x_{23} & \\ && 1 & \\ &&& 1 \end{pmatrix}K_p \sqcup \begin{pmatrix} 1 &&& \\ & p && \\ && p & \\ &&& p \end{pmatrix}K_p.
\end{aligned}
$$

(ii)

$$K_p \begin{pmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} K_p = \bigsqcup_{x_{12},x_{13},x_{14} \in \mathbb{Z}_p/p\mathbb{Z}_p} \begin{pmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & p^{-1}x_{12} & p^{-1}x_{13} & p^{-1}x_{14} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} K_p$$

$$\sqcup \bigsqcup_{x_{34} \in \mathbb{Z}_p/p\mathbb{Z}_p} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & p^{-1}x_{34} \\ & & & 1 \end{pmatrix} K_p$$

$$\sqcup \bigsqcup_{x_{23},x_{24} \in \mathbb{Z}_p/p\mathbb{Z}_p} \begin{pmatrix} 1 & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & p^{-1}x_{23} & p^{-1}x_{24} \\ & & 1 & \\ & & & 1 \end{pmatrix} K_p$$

$$\sqcup \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{pmatrix} K_p.$$

(iii)

$$K_p\begin{pmatrix} p &&& \\ & p && \\ && 1 & \\ &&& 1\end{pmatrix}K_p = \bigsqcup_{x_{13},x_{14},x_{23},x_{24}\in\mathbb{Z}_p/p\mathbb{Z}_p}\begin{pmatrix} p &&& \\ & p && \\ && 1 & \\ &&& 1\end{pmatrix}\begin{pmatrix} 1 && p^{-1}x_{13} & p^{-1}x_{14} \\ & 1 & p^{-1}x_{23} & p^{-1}x_{24} \\ && 1 & \\ &&& 1\end{pmatrix}K_p$$

$$\sqcup \bigsqcup_{x_{24},x_{34}\in\mathbb{Z}_p/p\mathbb{Z}_p}\begin{pmatrix} 1 &&& \\ & p && \\ && p & \\ &&& 1\end{pmatrix}\begin{pmatrix} 1 &&& \\ & 1 && p^{-1}x_{24} \\ && 1 & p^{-1}x_{34} \\ &&& 1\end{pmatrix}K_p$$

$$\sqcup \bigsqcup_{x_{23}\in\mathbb{Z}_p/p\mathbb{Z}_p}\begin{pmatrix} 1 &&& \\ & p && \\ && 1 & \\ &&& p\end{pmatrix}\begin{pmatrix} 1 &&& \\ & 1 & p^{-1}x_{23} & \\ && 1 & \\ &&& 1\end{pmatrix}K_p$$

$$\sqcup \bigsqcup_{x_{12},x_{14},x_{34}\in\mathbb{Z}_p/p\mathbb{Z}_p}\begin{pmatrix} p &&& \\ & 1 && \\ && p & \\ &&& 1\end{pmatrix}\begin{pmatrix} 1 & p^{-1}x_{12} && p^{-1}x_{14} \\ & 1 && \\ && 1 & p^{-1}x_{34} \\ &&& 1\end{pmatrix}K_p$$

$$\sqcup \bigsqcup_{x_{12},x_{13}\in\mathbb{Z}_p/p\mathbb{Z}_p}\begin{pmatrix} p &&& \\ & 1 && \\ && 1 & \\ &&& p\end{pmatrix}\begin{pmatrix} 1 & p^{-1}x_{12} & p^{-1}x_{13} & \\ & 1 && \\ && 1 & \\ &&& 1\end{pmatrix}K_p$$

$$\sqcup \begin{pmatrix} 1 &&& \\ & 1 && \\ && p & \\ &&& p\end{pmatrix}K_p.$$

Proof. Fix $g = \begin{pmatrix} I_k & \\ & pI_{n-k}\end{pmatrix}$, for $0 \le k \le n$. We would like to compute the group

$P = gK_pg^{-1} \cap K_p$. We see that, if $p \in P$ is a typical element then,

$$p = \begin{pmatrix} A_{k\times k} & B_{k\times n-k} \\ pC_{n-k\times k} & D_{n-k\times n-k} \end{pmatrix} \overset{mod\ p}{\equiv} \begin{pmatrix} \bar{A}_{k\times k} & \bar{B}_{k\times n-k} \\ & \bar{D}_{n-k\times n-k} \end{pmatrix} = \bar{p},$$

where A, B, C, D are $\mathbb{Z}_p$ matrices of corresponding dimensions. Consider the Grassmannian $\mathcal{G}(k,n) = Gr(k, \mathbb{F}_p^n)$. Now $\mathrm{GL}_n(\mathbb{F}_p)$ acts on $\mathcal{G}(k,n)$ (by left multiplication) transitively. This action induces an action of $\mathrm{GL}_n(\mathbb{Z}_p)$, simply by left multiplication and reducing mod p. We note that, $\mathrm{Stab}_{\mathrm{GL}_n}(\langle e_1, e_2, \dots, e_k\rangle) = \bar{P}$, and so $\mathrm{GL}_n(\mathbb{F}_p)/\bar{P} \cong \mathcal{G}(k,n)$. Also we know $\mathcal{G}(k,n) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{F}_p^n)$ (Plucker embedding) is complete, hence $\bar{P}$ is parabolic.

Now we will try to list out the elements of $\mathcal{G}(k)$ by so-called Plucker matrices through the Plucker embedding. Hence finding a set of representatives for KgK/K suffices finding set of representatives for $K/gKg^{-1} \cap K$ suffices finding 'good set' of representatives for $\mathcal{G}(k,n)$. We are providing proof for $n = 4$ and $k = 2$; proof in general case would follow similarly.

Theory of Plucker matrices tells us that any $t \in \mathcal{G}(2,4)$ are 2×4 matrices with rank = 2. We may embed the 2×4 matrices in $\mathrm{GL}_4(\mathbb{F}_p)$ by completing empty rows in obvious manner. Therefore they would look like (in certain base) as follows:

$$\begin{gathered} \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}, \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{gathered} \tag{3.2.2}$$

We use a known fact as follows:

Let H be a subgroup of G. Let $a \in G$. Suppose that $[a^{-1}Ha \cap H : H] < \infty$ a then,

$$HaH/H \cong H/a^{-1}Ha \cap H.$$

That is obvious as HaH has a transitive right action of H. The stabilizer of HaH for this action is $H \cap a^{-1}Ha$. In fact the representatives of HaH/H can be obtained from representatives of $H/a^{-1}Ha \cap H$. Now suppose that $H = K_p$ and $a = \mathrm{diag}(p, p, 1, 1)$, then using the isomorphism one may deduce the decomposition. For example, the first element in (3.2.2) would be mapped to the first coset in the decomposition (3). □

From now on by G_p and K_p we will mean $\mathrm{GL}_4(\mathbb{Q}_p)$ and $\mathrm{GL}_4(\mathbb{Z}_p)$ respectively. There is an Iwasawa decomposition of $G_p = BK_p$ where $B = NA$;

$$N = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} | * \in \mathbb{Q}_p \right\}$$

is the unipotent radical and A is the maximal torus i.e.

$$A = \left\{ a = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} | a_i \in \mathbb{Q}_p^\times \right\}.$$

Given four unramified characters $\chi_1, \chi_2, \chi_3, \chi_4$ of $\mathbb{Q}_p^\times$ we define a character χ of A by defining

$$\chi(a) = \chi_1(a_1)\chi_2(a_2)\chi_3(a_3)\chi_4(a_4). \tag{3.2.3}$$

We extend χ from A to B by setting χ to be N invariant. The unramified principal series representation corresponding to χ is $I(\chi)$ which is

$$I(\chi) := Ind_B^{G_p}(\chi) = \{f \in C_c(G_p) : f(nag) = \delta^{1/2}(a)\chi(a)f(g) \text{ for } (n, a, g) \in N{\times}A{\times}G_p\}$$

where δ, the modular function from the usual Haar measure on G_p defined by

$$\delta(a) = |a_1^3 a_2 a_3^{-1} a_4^{-3}|_p. \tag{3.2.4}$$

We will choose χ in such a way so that π_p becomes isomorphic to the unique spherical constituent π_χ of $I(\chi)$.

Let F_p be the unramified vector in the space π_χ with $F_p(e) = 1$. Then

$$F_p(nak) = \delta^{1/2}(a)\chi(a)$$

for $n \in N$, $a \in A$ and $k \in K_p$. Any $\phi \in H_p$ acts on F_p by convolution as following; for $h \in G_p$

$$(\phi * F_p)(h) := \int_{G_p} \phi(hg)F_p(g)dg.$$

Now let $\phi = \mathrm{Char}(K_p a_p K_p) \in H_p$ with $K_p a_p K_p = \coprod_i H_i K_p$ and also let $H_i = n_i a_i$ where $(n_i a_i) \in N \times A$. Then

$$\begin{aligned}(\phi * F_p)(e) &= \int_{G_p} \phi(g)F_p(g)dg \\ &= \int_{K_p a_p K_p} F_p(g)dg \\ &= \sum_i \int_{H_i K_p} F_p(g)dg \\ &= \sum_i F_p(n_i a_i) \\ &= \sum_i \delta^{1/2}(a_i)\chi(a_i).\end{aligned} \tag{3.2.5}$$

Along with above and lemma 5 we arrive at he following conclusion.

Proposition 11. *Suppose that π_χ has trivial central character. Then,*

(i) $\chi_1(p)\chi_2(p)\chi_3(p)\chi_4(p) = 1$

(ii) $\mathrm{Char}(K_p \begin{pmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} K_p)F_p = p^{3/2}(\chi_1(p) + \chi_2(p) + \chi_3(p) + \chi_4(p))F_p$

(iii) $\mathrm{Char}(K_p \begin{pmatrix} p & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix} K_p)F_p = p^{3/2}(\chi_1(p)^{-1} + \chi_2(p)^{-1} + \chi_3(p)^{-1} + \chi_4(p)^{-1})F_p$

(iv)

$$(K_p\begin{pmatrix}p&&&\\&p&&\\&&1&\\&&&1\end{pmatrix}K_p)F_p = p^2(\chi_1(p)\chi_2(p)+\chi_2(p)\chi_3(p)+\chi_3(p)\chi_4(p)+\chi_4(p)\chi_1(p)$$
$$+\chi_1(p)\chi_3(p)+\chi_2(p)\chi_4(p))F_p$$

.

Proof. As π_χ has trivial central character (1) is immediate from 3.2.5. We are giving a proof of (3); (2) and (4) would be similar.

Note that the first decomposition in lemma 5 can also be wriiten as

$$\bigsqcup_{x_{14},x_{24},x_{34}\in\mathbb{Z}_p/p\mathbb{Z}_p}\begin{pmatrix}1&&&x_{14}\\&1&&x_{24}\\&&1&x_{34}\\&&&1\end{pmatrix}\begin{pmatrix}p&&&\\&p&&\\&&p&\\&&&1\end{pmatrix}K_p\sqcup\bigsqcup_{x_{12}\in\mathbb{Z}_p/p\mathbb{Z}_p}\begin{pmatrix}1&x_{12}&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}p&&&\\&1&&\\&&p&\\&&&p\end{pmatrix}K_p$$

$$\sqcup\bigsqcup_{x_{13},x_{23}\in\mathbb{Z}_p/p\mathbb{Z}_p}\begin{pmatrix}1&&x_{13}&\\&1&x_{23}&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}p&&&\\&p&&\\&&1&\\&&&p\end{pmatrix}K_p\sqcup\begin{pmatrix}1&&&\\&p&&\\&&p&\\&&&p\end{pmatrix}K_p.$$

From 3.2.5 we get that

$$\mathrm{Char}(K_p\begin{pmatrix}p&&&\\&p&&\\&&p&\\&&&1\end{pmatrix}K_p)F_p = [p^3p^{-3/2}\chi_1(p)\chi_2(p)\chi_3(p) + pp^{1/2}\chi_1(p)\chi_3(p)\chi_4(p)$$
$$p^2p^{-1/2}\chi_1(p)\chi_2(p)\chi_4(p) + p^{3/2}\chi_2(p)\chi_3(p)\chi_4(p)]F_p$$
$$= p^{3/2}\chi_1(p)\chi_2(p)\chi_3(p)\chi_4(p)(\chi_1(p)^{-1}+\chi_2(p)^{-1}+\chi_3(p)^{-1}+\chi_4(p)^{-1})F_p$$
$$= (\chi_1(p)^{-1}+\chi_2(p)^{-1}+\chi_3(p)^{-1}+\chi_4(p)^{-1})F_p.$$

□

As in the previous section here also we will use non-temperedness to have an amplification. We will show that if a representation is non-tempered then there exist at least one Hecke operator whose eigenvalue is much large than its support. We have similar definition of nontemperedness here as well.

Definition 9. *We say a Hecke-Maass form ϕ is η non-tempered at p if at least one corresponding Satake parameter at p has absolute value p^{η} i.e. there is a character χ_i such that $|\chi_i(p)| \geq p^{\eta}$.*

Now we will be proving our main amplification lemma. Let ϕ is the Hecke-Maass form in the question with

$$T(p)\phi = \lambda(p)\phi, \text{ and } S(p)\phi = \mu(p)\phi.$$

Note that $T^*(p)$ is the adjoint Hecke operator of $T(p)$ with respect to the Petersson inner product defined in 2.3.4. So that gives us $T^*(p)\phi = \overline{\lambda(p)}\phi$.

Lemma 6. *Let ϕ be a Hecke-Maass of $\Gamma\backslash\mathcal{H}^5$ form such that ϕ is η non-tempered at positive density of odd primes p for some $\eta > 0$. Then, either*

$$|\lambda(p)| \gg p^{3/2+\eta/2} \text{ or } |\mu(p)| \gg p^{2+\eta}$$

as $p \to \infty$.

Proof. Note that from lemma 4 and proposition 11 the polynomial equation

$$x^4 - p^{-3/2}\lambda(p)x^3 + p^{-2}\mu(p) - p^{-3/2}\overline{\lambda(p)}x + 1 = 0 \tag{3.2.6}$$

has roots $\chi_i(p)$ for $i = 1, 2, 3, 4$. As at least one of χ_i is η away from unitary axis and $\prod_i \chi_i(p) = 1$, there is an i, say $i = 1$, such that $|\chi_1(p)| = p^{\eta}$. Now letting

$x = \chi_1(p)$ in equation (3.2.6) we get that,

$$\begin{aligned}
&|p^{-3/2}\lambda(p)x^3 + p^{-3/2}\overline{\lambda(p)}x| = |x^4 + p^{-2}\mu(p)x^2 + 1| \\
\implies &p^{-3/2}|\lambda(p)(|x|^3 + |x|) \geq |x|^4 - p^{-2}|\mu(p)||x|^2 - 1 \\
\implies &|p^{-3/2}\lambda(p)|p^{3\eta} + p^{2\eta}|p^{-2}\mu(p) \gg p^{4\eta} \\
&\text{either} \\
\implies &|\lambda(p)| \gg p^{3/2+\eta} \\
&\text{or} \\
\implies &|\mu(p)| \gg p^{2+2\eta}.
\end{aligned}$$

This proves the lemma. $\square$

Chapter 4

Amplified Pre-trace Formula

In this chapter we will construct an identity which will allow us to reduce our problem of estimating a Maass form in $\mathcal{H}^n$ to a diophantine analysis problem. The pre-trace formula is the beginning of general Arthur-Selberg's Trace formula, see [50]. For that, first we need an explicit Selberg/Harish-Chandra spherical transform pair (h, k). For dimension 2 i.e. on upper half the following proposition can be found in [20]. where as the idea of amplification is inspired by [16]. For general amplification scheme one may look at [54].

4.1 Spherical Transform and Automorphic Kernel

Proposition 12. *Let (h, k) be the Selberg/Harish-Chandra spherical transform pair in $\mathcal{H}^n$ and u is the point pair invariant defined in 2.3.2. Then*

$$h(t) = \frac{(n-2)(2\pi)^{\frac{n-1}{2}}}{2\Gamma\left(\frac{n+1}{2}\right)} \int_{-\infty}^{+\infty} e^{i\alpha t} \int_{v} (u-v)^{\left(\frac{n-3}{2}\right)} k(u)\, du\, d\alpha, \tag{4.1.1}$$

where $v = \frac{(e^{\alpha}-1)^2}{2e^{\alpha}}$.

Proof. Let $\hat{h}$ be the spherical transform of h in $\mathcal{H}^n$. Recall from 2.1.6 that the point-pair invariant u and distance $d(P) := d(P, e)$ of $P = (x, y)$ from origin $e = (0, 1)$ is related by,

$$u = \frac{\|x\|^2 + (y-1)^2}{2y} = \sinh^2\left(\frac{d}{2}\right).$$

Then by definition of spherical transform and letting $\hat{h}(d) = k(u)$ we have,

$$\begin{aligned}
h(t) &= \int_{\mathcal{H}^n} \hat{h}(d(P))y(P)^{\frac{n-1}{2}+it} d\mu(P) \\
&= \int_0^\infty y^{\frac{n-1}{2}+it} \int_{\mathbb{R}^{n-1}} \hat{h}(d) \frac{d^{n-1}xdy}{y^n} \\
&= \int_0^\infty y^{-\frac{n+1}{2}+it} \int_0^\infty k\left(\frac{r^2+(y-1)^2}{2y}\right) drdy \int_{\|x\|=r} d^{n-1}x \\
&= Vol(S^{n-2}) \int_0^\infty y^{-\frac{n+1}{2}+it} \int_0^\infty r^{n-2} k\left(\frac{r^2+(y-1)^2}{2y}\right) drdy.
\end{aligned}$$

Now letting $v = \frac{(y-1)^2}{2y}$ and $y \mapsto e^\alpha$, as $u = v + \frac{r^2}{2y}$ we get,

$$\begin{aligned}
h(t) &= \mathrm{Vol}(S^{n-2}) \int_0^\infty y^{-\frac{n+1}{2}+it} \int_v^\infty (2y(u-v))^{\frac{n-3}{2}} k(u) y du dy \\
&= 2^{\frac{n-3}{2}} \mathrm{Vol}(S^{n-2}) y^{it} \int_v^\infty (u-v)^{\frac{n-3}{2}} k(u) du \frac{dy}{y} \\
&= \frac{(n-2)(2\pi)^{\frac{n-1}{2}}}{2\Gamma\left(\frac{n+1}{2}\right)} \int_{-\infty}^{+\infty} e^{i\alpha t} \int_v^\infty (u-v)^{\frac{n-3}{2}} k(u) du d\alpha.
\end{aligned}$$

□

Note that we can describe 4.1.1 in a three step relation as following,

$$\begin{cases}
h(t) = \displaystyle\int_{-\infty}^{+\infty} g(\alpha) e^{i\alpha t} d\alpha \\
v = \dfrac{1}{2}\sinh^2\left(\dfrac{\alpha}{2}\right) \Leftrightarrow \alpha = \log(1 + v + \sqrt{v^2+2v}) \\
g(\alpha) = \dfrac{(n-2)(2\pi)^{\frac{n-1}{2}}}{2\Gamma\left(\frac{n+1}{2}\right)} \displaystyle\int_v^\infty (u-v)^{\left(\frac{n-3}{2}\right)} k(u) du.
\end{cases} \tag{4.1.2}$$

For the pre trace formula we need a basis of the Hilbert space $L^2(\Gamma\backslash S)$ which with Petersson inner product defined in 2.3.4 decomposes into (see [34], [33]),

$$L^2(\Gamma\backslash S) = L^2_{\mathrm{res}}(\Gamma\backslash S) \oplus L^2_{\mathrm{cusp}}(\Gamma\backslash S) \oplus L^2_{\mathrm{cont}}(\Gamma\backslash S),$$

where L^2_{cusp} is the closure of the space generated by the cusp forms, L^2_{res} is the residual Eisenstein series; those two spaces combine to L^2_{disc} correspond to discrete spectrum, L^2_{cont} is the continuous spectrum.

Let there be m cusps in $\Gamma\backslash S$. The Eisenstein series $E_k(P,s)$ corresponding to k'th cusp is holomorphic at $\Re(s) = 1/2$. Let $\{\phi\}_{i=0}^{\infty}$ be an orthonormal basis (containing ϕ the mass form we want to estimate) consisting of joint eigenfunction of Laplacian and full Hecke algebra. Then from the spectral theorem we can say the following (see [34]).

Proposition 13. *Let $f \in C^{\infty}(\Gamma\backslash S)$.Then f has the following spectral expansion*

$$f(P) = \sum_{i=0}^{\infty} \langle f, \phi_i \rangle \phi_i(P) + \sum_{k=1}^{m} \frac{1}{4\pi} \int_{-\infty}^{+\infty} \langle f, E_k(\cdot, 1/2 + it) \rangle E_k(P, 1/2 + it) dt, \quad (4.1.3)$$

which converges in C^{∞}-topology.

Let us choose f to be our automorphic kernel in P variable, namely for some smooth $k \in \mathcal{S}(\mathbb{R}_{\geq 0})$,

$$K(P, Q) = \sum_{\gamma \in \Gamma} k(u(\gamma P, Q)).$$

After doing a similar computation of respective inner products as in 7.4 [28] we get

$$K(P, Q) = \sum_{i=0}^{\infty} \phi_i(P) \overline{\phi_i(Q)} h(t_j) + + \sum_{k=1}^{m} \frac{1}{4\pi} \int_{-\infty}^{+\infty} E_k(P, 1/2 + it) \overline{E_k(Q, 1/2 + it)} h(t) dt, \quad (4.1.4)$$

where h is the Selberg/Harish-Chandra tranform of k as in 4.1.2. The above equation is called pre-trace formula.

4.2 Archimedean Amplification and Bounding k

We will use a specific choice for the spherical transform pair (h,k) which will be constructed so as to emphasize the contribution of the term ϕ in the basis. So that the function $h(t)$ becomes concentrated on the the spectral parameter T of ϕ. Inspired by [27], we consider

$$g(x) = \frac{A\cos(Tx)}{\cosh(Ax)}, \tag{4.2.1}$$

For some $A > 2$ large constant, h and k will be constructed as in 4.1.2.

Lemma 7. *For above choice of g,*

(1) *the function $h(t)$ is even, holomorphic, and rapidly decaying in the strip $|\Im(t)| < A$. It satisfies the bound*

$$h(t) > \begin{cases} 0, & t \in \mathbb{R} \cup (-A,A)i \\ 1/8, & t \in \mathbb{R} \cup (-A/2,A/2)i \text{ and } ||t| - T| < A/2. \end{cases} \tag{4.2.2}$$

(2) *The Selberg/Harish-Chandra transform $k(u)$ of $h(t)$, as in 4.1.2, satisfies the bound*

$$k(u) \ll_A \min\left(\frac{T^{[\frac{n}{2}]}}{(1+u)^A u^{[\frac{n}{2}]}}, T^{n-1}\right) \tag{4.2.3}$$

Proof. The function $g(x)$ defined in 4.2.1 is even and satisfies the bound

$$g(x) \ll_A e^{A|x|},$$

hence it Fourier transformation $h(t)$ in 4.1.2 is even and holomorphic in the strip

$|\Im(t)| < A$ by Morera's theorem. We first calculate the Fourier transform of sech$x$

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\, \mathrm{sech}(\pi x)\, e^{ikx} &= 2\int_{-\infty}^{\infty} dx \frac{e^{ikx}}{e^{\pi x} + e^{-\pi x}} \\
&= 2\int_{-\infty}^{0} dx \frac{e^{ikx}}{e^{\pi x} + e^{-\pi x}} + 2\int_{0}^{\infty} dx \frac{e^{ikx}}{e^{\pi x} + e^{-\pi x}} \\
&= 2\sum_{m=0}^{\infty} (-1)^m \left[\int_0^{\infty} dx\, e^{-[(2m+1)\pi + ik]x} + \int_0^{\infty} dx\, e^{-[(2m+1)\pi - ik]x}\right] \\
&= 2\sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{(2m+1)\pi - ik} + \frac{1}{(2m+1)\pi + ik}\right] \\
&= 4\pi \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)}{(2m+1)^2\pi^2 + k^2} \\
&= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{(-1)^m (2m+1)}{\left(m+\frac{1}{2}\right)^2 + \left(\frac{k}{2\pi}\right)^2}
\end{aligned}$$

By the residue theorem, the sum is equal to the negative sum of the residues at the non-integer poles of

$$\pi \csc(\pi z) \frac{1}{2\pi} \frac{2z+1}{\left(z+\frac{1}{2}\right)^2 + \left(\frac{k}{2\pi}\right)^2}$$

which are at $z_{\pm} = -\frac{1}{2} \pm i\frac{k}{2\pi}$. The sum is therefore

$$-\frac{1}{2}\csc(\pi z_+) - \frac{1}{2}\csc(\pi z_-) = -\Re\left[\frac{1}{\sin \pi\left(-\frac{1}{2} + i\frac{k}{2\pi}\right)}\right] = \mathrm{sech}\left(\frac{k}{2}\right)$$

By this reasoning, the Fourier transform of sech$x$ is $\pi\,\mathrm{sech}\left(\frac{\pi k}{2}\right)$. Thus we calculate

$$\begin{aligned}
h(t) &= \frac{1}{4}\mathrm{sech}\left(\frac{\pi(t+T)}{2A}\right) + \frac{1}{4}\mathrm{sech}\left(\frac{\pi(t-T)}{2A}\right) \\
&= \frac{\cosh\left(\frac{\pi t}{2A}\right)\cosh\left(\frac{\pi T}{2A}\right)}{\cosh\left(\frac{\pi t}{A}\right) + \cosh\left(\frac{\pi T}{A}\right)},
\end{aligned} \tag{4.2.4}$$

which shows that $h(t)$ is rapidly decaying in the strip $|\Im(t)| < A$.

Using this we will prove 4.2.2. Assume first that $t \in \mathbb{R}$. From (4.8) it is clear that $h(t) > 0$, because both terms are positive. Moreover, if $|t| \in (T - \frac{A}{2}, T + \frac{A}{2})$ then one of these terms exceeds $\frac{1}{4}\operatorname{sech}\left(\frac{\pi}{4}\right) > \frac{1}{8}$, so that $h(t) > \frac{1}{8}$. Assume now that $t \in (-A, A)i$. From (4.8) it is clear that $h(t) > 0$, because $\cosh\left(\frac{\pi t}{2A}\right) > 0$ and $\cosh\left(\frac{\pi t}{A}\right) > -1$. Moreover, if $t \in (-\frac{A}{2}, \frac{A}{2})i$ then $\cosh\left(\frac{\pi t}{2A}\right) > \cos\left(\frac{\pi}{4}\right)$ and $\cosh\left(\frac{\pi t}{A}\right) \leq 1$, so that

$$h(t) > \frac{\cos\left(\frac{\pi}{4}\right)\cosh\left(\frac{\pi T}{2A}\right)}{1 + \cosh\left(\frac{\pi T}{A}\right)} = \frac{\cos\left(\frac{\pi}{4}\right)}{2\cosh\left(\frac{\pi T}{2A}\right)}.$$

If, in addition, $|t| \in (T - \frac{A}{2}, T + \frac{A}{2})$, then $T = T - |t| + |t| < A/2 + A/2 = A$, whence

$$h(t) > \frac{\cos\left(\frac{\pi}{4}\right)}{2\cosh\left(\frac{\pi}{2}\right)} > \frac{1}{8}.$$

We will now prove 4.2.3. Recalling 4.1.2 et us define

$$\frac{(n-2)(2\pi)^{\frac{n-1}{2}}}{2\Gamma\left(\frac{n+1}{2}\right)} q(v) = g(\alpha).$$

Hence we get,

$$q(v) = \int_0^\infty x^{\frac{n-3}{2}} k(v + x) dx.$$

For $v = 0, 1$ according to $n \equiv 1, 0 \pmod 2$,

$$\int_u^\infty \frac{q^{\left(\frac{n+1-v}{2}\right)}(v)}{\sqrt{(v-u)^v}} dv = \int_u^\infty \frac{1}{\sqrt{(v-u)^v}} \int_0^\infty x^{\frac{n-3}{2}} k^{\left(\frac{n+1-v}{2}\right)}(x+v) dx dv$$

by integration by parts and using rapid decay of k,

$$= \frac{(-1)^{\frac{n+1-v}{2}}}{2^v} \Gamma\left(\frac{n-1}{2}\right) \int_0^\infty \int_0^\infty \frac{k'(u + x + v)}{\sqrt{(vx)^v}} dx dv.$$

Thus integrating left side using polar cordinate we get,

$$k(u) = \frac{(-1)^{\frac{n+1-v}{2}}}{2^v \pi \Gamma\left(\frac{n-1}{2}\right)} \int_u^\infty \frac{q^{\left(\frac{n+1-v}{2}\right)}(v)}{\sqrt{(v-u)^v}} dv. \tag{4.2.5}$$

Now note that from 4.1.2 and 4.2.1 we have precise bounds of following,

$$\alpha = \log(1 + u + \sqrt{u^2 + 2u}) \leq \sqrt{u^2 + 2u},$$

$$\cosh(Bx) \ll (1 + u)^B,$$

$$g^{(m)}(\alpha) \ll \frac{T^m}{(1+u)^{C_1}} \text{ for some constant } C_1 > 0,$$

$$\frac{d^m\alpha}{du^m} \ll \frac{(1+u)^{m-1}}{(\sqrt{u^2+2u})^{2m-1}}.$$

Thus we get the derivative bound,

$$\begin{aligned}
|q^{(m)}(u)| &\ll |g^{(m)}(\alpha)\left(\frac{d\alpha}{du}\right)^m| + |g'(\alpha)\frac{d^m\alpha}{du^m}| \\
&\ll \frac{1}{(1+u)^{C_2}}\left(\frac{T^m}{(\sqrt{u^2+2u})^m} + \frac{T|\sin(T\alpha)|}{(\sqrt{u^2+2u})^{2m-1}}\right) \\
&\ll \frac{1}{(1+u)^{C_2}}\left(\frac{T^m}{(\sqrt{u^2+2u})^m} + T\frac{min((T\alpha)^{m-1}, (T\alpha)^{2m-1})}{(\sqrt{u^2+2u})^{2m-1}}\right) \\
&\ll \frac{1}{(1+u)^A}\min\left(\frac{T^m}{u^m}, T^{2m}\right),
\end{aligned}$$

For some large enough A.

Now for odd n from 4.2.5 we get that,

$$k(u) \ll |q^{(\frac{n-1}{2})}(u)| \ll \min\left(\frac{T^{\frac{n-1}{2}}}{(1+u)^A u^{\frac{n-1}{2}}}, T^{n-1}\right).$$

And for even n we divide the integral in 4.2.5 in two parts and estimate as following,

$$\begin{aligned}
k(u) &\ll \int_0^\eta \frac{|q^{(\frac{n}{2})}(u+v)|}{\sqrt{v}}dv + \frac{|q^{(\frac{n}{2}-1)}(u+\eta)|}{\sqrt{\eta}} + \int_\eta^\infty \frac{|q^{(\frac{n}{2}-1)}(u+v)|}{\sqrt{v^3}}dv \\
&\ll \min\left(\frac{T^{n/2}}{(1+u)^A u^{n/2}}, \frac{T^n}{1+u}\right)\sqrt{\eta} + \frac{T^{(n-2)/2}}{(1+u+v)^A(u+v)^{n/2}\sqrt{\eta}} \\
&\ll \min\left(\frac{T^{n/2}}{(1+u)^A u^{n/2}}, \frac{T^n}{1+u}\right)\sqrt{\eta} + \frac{T^{(n-2)/2}}{(1+u)^A u^{(n-2)/2}\sqrt{\eta}}.
\end{aligned}$$

Now choosing $\sqrt{\eta} = T^{-1}\sqrt{u}$ we get that,

$$k(u) \ll \min\left(\frac{T^{\frac{n}{2}}}{(1+u)^A u^{\frac{n}{2}}}, T^{n-1}\right),$$

which proves our claim. $\square$

Chapter 5

Diophantine Analysis and Bounds

In this chapter we fix the dimension n of the arithmetic hyperbolic space to be 4 and 5. By Γ we would mean $\mathrm{Sp}^*_{1,1}(\mathcal{O})$ or $\mathrm{SL}_2(\mathcal{O})$ in case of $\mathcal{H}^4$ and $\mathcal{H}^5$ respectively, where $\mathcal{O}$ is Hurwitz integer quaternions. Recall the equation 4.1.4 of pre-trace formula. Let $\{\phi_j\}$ with Laplace eigenvalue λ_j be orthonormal basis of $L^2_{disc}(\Gamma\backslash\mathcal{H}^n)$ which contains ϕ the Maass cusp form we want to bound and E_k be the Eisenstein series in the continuous spectrum, and $K(P,Q) = \sum_{\gamma\in\Gamma} k(u(\gamma P, Q))$ be the automorphic kernel. Then,

$$\sum_{i=0}^{\infty}\phi_i(P)\overline{\phi_i(Q)}h(t_j)+\sum_{k=1}^{m}\frac{1}{4\pi}\int_{-\infty}^{+\infty}E_k(P,1/2+it)\overline{E_k(Q,1/2+it)}h(t)dt = \sum_{\gamma\in\Gamma}k(u(\gamma P,Q)).$$

Let T be a Hecke operator on $\Gamma\backslash\mathcal{H}^n$ which can be thought as a Γ double coset $\Gamma a\Gamma$, such that ϕ_j's and E_k's are Hecke eigenfunction with

$$T\phi_j(P) = \sum_{\gamma\in\Gamma\backslash\Gamma a\Gamma}\phi(\gamma P) = \lambda_j\phi_j(P) \text{ and } TE_k(P,\cdot) = \sum_{\gamma\in\Gamma\backslash\Gamma a\Gamma}E_k(\gamma P,\cdot) = \Lambda_j E_k(P,\cdot).$$

Let T^* be the adjoint Hecke operator of T with respect to the Petersson inner product. We now shall apply $T\circ T^*$ on the P variable of the pre-trace formula. And then letting $P = Q$ we get.

$$\sum_{i=0}^{\infty} h(t_j)|\lambda_j|^2|\phi_i(P)|^2 + \sum_{k=1}^{m} \frac{1}{4\pi} \int_{-\infty}^{+\infty} h(t)|\Lambda_k|^2|E_k(P, 1/2 + it)|^2 = \sum_{\gamma \in \Gamma a \Gamma} k(u(\gamma P, P)). \tag{5.0.1}$$

Recalling lower bound of $h(t)$ from 4.2.2 and noting the positivity of each term in LHS of (5.1) we conclude that,

$$\begin{aligned} |\lambda|^2|\phi(P)|^2 &\ll \sum_{\gamma \in \Gamma a \Gamma} k(u(\gamma P, P)) \\ &= \sum_{\substack{\gamma \in \Gamma a \Gamma \\ u(\gamma P, P) < \delta}} k(u(\gamma P, P)) + \sum_{\substack{\gamma \in \Gamma a \Gamma \\ u(\gamma P, P) \geq \delta}} k(u(\gamma P, P)) \end{aligned} \tag{5.0.2}$$

for some $\delta > 0$ to be fixed latter. We fix a compact Ω where we bound the L^∞ norm, in the fundamental domain and let $P \in \Omega$. Let us call $e = (0, 1) \in \Gamma\backslash\mathcal{H}^n$ and as the ambient group acts isometrically on this space there exist a g such that $g.e = P$. Hence,

$$\delta > u(\gamma P, P) = u(g^{-1}\gamma g.e, e)$$

which implies $g^{-1}\gamma g \in B_\delta(K)$ where K is the maximal compact in respective cases. Note that saying $P \in \Omega$ compact in fundamental domain is equivalent to saying $g \in \Omega'$ for some compact $\Omega' \subset G$.

Lemma 8. *For* $e = (0, 1)$

$$u(ge, e) = \frac{1}{2}\|g\|_{HS}^2 - \sqrt{\det(g)}, \tag{5.0.3}$$

where for a matrix $A \in \mathrm{Mat}_n(\mathbb{H})$ *Hilbert-Schmidt norm of A is defined by*

$$\|A\|_{HS}^2 = \sum_{i,j} |A_{ij}|^2.$$

Proof. We give a proof in $\mathrm{SL}_2(\mathbb{H})$ case; $\mathrm{Sp}_{1,1}^*(\mathbb{H})$ case would be similar. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in $\mathrm{SL}_2(\mathbb{H})$. From Iwasawa decomposition of the corresponding group it can be seen that,

$$g(0,1) = \left(\frac{b\bar{d} + a\bar{c}}{|c|^2 + |d|^2}, \frac{\sqrt{\det(g)}}{|c|^2 + |d|^2} \right).$$

Therefore from 2.3.2 we calculate,

$$\begin{aligned}
u(ge,e) &= \frac{|b\bar{d} + a\bar{c}|^2 + (|c|^2 + |d|^2 - \det(g))^2}{2(|c|^2 + |d|^2)} \\
&= \frac{|b|^2|d|^2 + |a|^2|c|^2 + 2\Re(a\bar{c}d\bar{b} + (|c|^2 + |d|^2)^2 - 2(|c|^2 + |d|^2)\sqrt{\det(g)} + \det(g)}{2(|c|^2 + |d|^2)} \\
&= \frac{|b|^2|d|^2 + |a|^2|c|^2 + |c|^2|d|^2 + |a|^2|b|^2 + (|c|^2 + |d|^2)^2}{2(|c|^2 + |d|^2)} - \sqrt{\det(g)} \\
&= \frac{(|a|^2 + |b|^2 + |c|^2 + |d|^2)(|c|^2 + |d|^2)}{2(|c|^2 + |d|^2)} - \sqrt{\det(g)} \\
&= \frac{1}{2}||g||_{HS}^2 - \sqrt{\det(g)}
\end{aligned}$$

□

Definition 10. *Let T be a Hecke operator defined by the double coset $\Gamma a \Gamma$. we define* support *of T by*

$$\mathrm{Supp}(T) = \{\gamma \in \Gamma \backslash \Gamma a \Gamma\}.$$

If T' is another Hecke operator then clearly

$$\mathrm{Supp}(TT') \subseteq \{tt' | t \in \mathrm{Supp}(T), t' \in \mathrm{Supp}(T')\}$$

and hence $|\mathrm{Supp}(TT')| \leq |\mathrm{Supp}(T)||\mathrm{Supp}(T')|$. In particular we have the following lemma.

Lemma 9.

$$|\mathrm{Supp}(T \circ T^*)| \leq |\mathrm{Supp}(T)|^2. \tag{5.0.4}$$

Combining lemma 1 and lemma 2 one can compute the size of the support of the chosen generators in proposition 6 of the corresponding Hecke algebra; namely for odd prime p,

$$\mathrm{Supp}(T(p)) \asymp p^3 \text{ and } \mathrm{Supp}(S(p)) \asymp p^4. \tag{5.0.5}$$

Also Combining lemma 4 and lemma 5 one can compute the size of the support of the chosen generators in proposition 10 of the corresponding Hecke algebra; namely for odd prime p,; namely for odd prime p,

$$\mathrm{Supp}(T(p)) = \mathrm{Supp}(T^*(p)) \asymp p^3 = \text{ and } \mathrm{Supp}(S(p)) \asymp p^4. \tag{5.0.6}$$

Proof of theorem 1*:* As the argument for the saving in the both dimensions are same we will give the proof in dimension 5 only.

Let the Hecke Maass form ϕ is η non-tempered at $\nu > 0$ density of primes. So for L sufficiently large in the interval $[L, 2L]$ there will be $\asymp \frac{\nu L}{\log L}$ primes p where ϕ is non-tempered by the prime number theorem. We assume L to be sufficiently large so that $\frac{\nu L}{\log L} > 1$. Recalling lemma lemma 6 it is clear that at each place there is a Hecke operator $H = H_p$ with $H\phi = \lambda\phi$ such that

$$|\lambda|^2 \gg \mathrm{Supp}(H)p^{\eta}.$$

In (5.0.1) we use $T = H \circ H^*$ and so $\lambda = |\nu|^2$. Let us assume $H = T(p)$ defined in proposition 10, if $T(p)$ does not have required property we will choose

$$\Gamma\begin{pmatrix} \pi^2 & \\ & 1 \end{pmatrix}\Gamma = pT^2(p) + S(p)$$

which has the required property. As proof in both cases are same we will prove for $T(p)$ only.

First we will bound the second summand in (5.2). Let us define

$$M(t) = M(P, t) := \#\{\gamma \in \mathrm{Supp}(T) : u(\gamma P, P) < t\}.$$

and for $g \in \Omega'$ we get from lemma 8 that,

$$\begin{aligned}
&\|g^{-1}\gamma g\|_{HS}^2 - \det(\gamma)^{1/2} < t \\
\implies &\|\gamma\|_{HS}^2 \ll_{\Omega'} p^k + t \\
\implies &|\gamma_{ij}| \ll_{\Omega} \sqrt{p^k + t} \\
\implies &\#\gamma \in M(t) \ll_{\Omega} L^{8k} + L^8
\end{aligned}$$

for some absolute positive integer k. Now from 4.2.3 integrating by parts we get that,

$$\begin{aligned}
\sum_{\substack{\gamma \in \mathrm{Supp}(T) \\ u(\gamma P, P) \geq \delta}} k(u(\gamma P, P)) &\ll \int_{\delta}^{\infty} \frac{T^{[\frac{n}{2}]}}{(1+u)^A u^{[\frac{n}{2}]}} dM(u) \\
&\ll T^{[\frac{n}{2}]} \left(L^{8k} \delta^{-[\frac{n}{2}]+1} + \delta^{-[\frac{n}{2}]+9} \right).
\end{aligned} \tag{5.0.7}$$

Now we will bound the first summand.

Lemma 10. *Let T be a p-Hecke operator on $\Gamma \backslash \mathcal{H}^5$ defined by $\Gamma \begin{pmatrix} \pi & \\ & 1 \end{pmatrix} \Gamma$ for some primitive quaternion π with norm p for an odd prime p. Then for $g \in G$ and K maximal compact*

$$|\{\gamma \in \Gamma \begin{pmatrix} \pi & \\ & 1 \end{pmatrix} \Gamma : g^{-1}\gamma g \in B_\delta(K)\}| \ll_{\Omega'} L^{3/2+\epsilon}, \tag{5.0.8}$$

for sufficiently small δ.

Proof. Now if $Q = (gg^*)^{-1}$, then

$$g^{-1}\gamma g \in B_\delta(K) \leftrightarrow \gamma^* Q \gamma = \det(\gamma)^{1/2} Q + O(\det(\gamma)^{1/2}\delta).$$

So the first column of Γ satisfies a quadratic form of 8 variables with $O(1)$ coefficients, therefore number of solutions will be $\ll L^{3/2+\epsilon}$ (see e.g. [6] Lemma 8a). Fixing the first column the other column can be obtained in $\ll L^{\epsilon}$ ways. □

Again by 4.2.3, (5.4) and (5.7) we get,

$$\sum_{\substack{\gamma\in \mathrm{Supp}(T)\\ u(\gamma P,P)<\delta}} k(u(\gamma P, P)) \ll T^{n-1}|\{\gamma \in \Gamma a \Gamma : g^{-1}\gamma g \in B_\delta(K)\}|$$

$$\ll_\Omega T^{n-1}\mathrm{Supp}(T)$$

$$\ll T^{n-1}|\lambda|^2 L^{-\eta}.$$

Choosing $\delta = \min\left(T^{-1/20}, \frac{1}{2}\right)$ and $L = T^{1/20k}$ and from (5.2) we get

$$|\phi(P)|^2 \ll_\Omega T^{n-1-\epsilon}.$$

□

Chapter 6

Concluding Remarks

(i) It seems natural to expect the techniques of this thesis would generalize to other groups, and in particular to other hyperbolic space. To do this would require developing an explicit Hecke theory, perhaps generalizing Krieg's work by starting from a $\mathbb{Z}$-order in the Clifford algebra over $\mathbb{Q}$. The diophantine analysis would also be more difficult as the maximal compact subgroup $\mathrm{SO}(n)$ would be larger.

(ii) The main Theorem is stated for the supremum of the eigenfunctions on a fixed compact set Ω. A similar bound probably does not hold in the cusp due to the behaviour of the K-Bessel function in its transition region (see [5]). In the case of GL_3 this is discussed in [13]. However, a typical no Siegel zero result of the Rankin-Selberg L-function should give supnorm bound near cusps.

(iii) We have already discussed the examples of CAP representaiton in $\mathcal{H}^4$ and $\mathcal{H}^5$ such that they are tempered in infinite place and nontempered in every finite place. In the similar fashion one may ask for example of CAP representation for $\mathrm{Spin}(n, 1)$ groups so that they will be nontempered Hecke-Maass form whose power saving in the L^∞ norms can be proved in similar manner.

(iv) As similar in [45], one may use the sup norm bounds of the Maass form to bound the corresponding L-function which is expected to have an integral representation with respect to the Maass form, at the critical line. Eventually the result obtained in theorem 1 gives a subconvexity bound towards the Generlized Lindelöf Hypothesis of the corresponding L-function, i.e. beats the convexity bound which is the estimate derived from the functional equation by the Lindelöf-Phragmen principle.

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