Homotopy colimits of classifying spaces of finite abelian groups

by

Cihan Okay

A thesis submitted in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

in

The Faculty of Graduate and Postdoctoral Studies
(Mathematics)

The University of British Columbia
(Vancouver)

April 2014

Abstract

The classifying space BG of a topological group G can be filtered by a sequence of subspaces $B(q,G)$, $q \geq 2$, using the descending central series of free groups. If G is finite, describing them as homotopy colimits is convenient when applying homotopy theoretic methods. In this thesis we introduce natural subspaces $B(q,G)_p \subset B(q,G)$ defined for a fixed prime p. We show that $B(q,G)$ is stably homotopy equivalent to a wedge sum of $B(q,G)_p$ as p runs over the primes dividing the order of G. Colimits of abelian groups play an important role in understanding the homotopy type of these spaces. Extraspecial p–groups are key examples, for which these colimits turn out to be finite. We prove that for extraspecial p–groups of rank $r \geq 4$ the space $B(2,G)$ does not have the homotopy type of a $K(\pi,1)$ space thus answering in a negative way a question which appears in [1]. Furthermore, we give a group theoretic condition, applicable to symmetric groups and general linear groups, which implies the space $B(2,G)$ not having the homotopy type of a $K(\pi,1)$ space. For a finite group G, we compute the complex K–theory of $B(2,G)$ modulo torsion.

Preface

This dissertation is original, independent work by the author, C. Okay. A version of this material is accepted for publication in Algebraic & Geometric Topology.

Table of Contents

Acknowledgements

I wish to thank Alejandro Adem, for his supervision and patience; Jeff Smith for pointing out Proposition 3.1.5 and useful conversations; Fred Cohen and Jose Gómez, for commenting on an early version of this manuscript. I would like to thank Ian Allison for his help with any sort of computer issues. I want to thank my family for their support.

Introduction

The classifying space BG is an important object in algebraic topology with applications to bundle theory and cohomology of groups. When G is discrete, it is homotopy equivalent to an Eilenberg–Maclane space of type $K(G,1)$. A key property of BG is that it classifies principal G–bundles; there is a bijection between the set of isomorphism classes of principal G–bundles over a CW–complex X and the set of homotopy classes of maps $X \to BG$. In this thesis we study certain subspaces of the classifying space BG.

In [1] a natural filtration of the classifying space BG of a topological group G is introduced:

$$B(2,G) \subset \cdots \subset B(q,G) \subset \cdots \subset B(\infty,G) = BG.$$

For a fixed $q \geq 2$, let $\Gamma^q(F_n)$ denote the q-th stage of the descending central series of the free group on n generators. Then $B(q,G)$ is the geometric realization of the simplicial space whose n-simplices are the spaces of homomorphisms $\mathrm{Hom}(F_n/\Gamma^q(F_n),G)$. In this thesis we study homotopy-theoretic properties of these spaces.

In Chapter 1 we introduce the spaces $B(q,G)$ and describe them as homotopy colimits. Furthermore, we define a subspace $B(q,G)_p \subset B(q,G)$ which captures information specific to a prime p. Let G be a finite group and p a prime dividing its order. Consider the free pro-p group P_n, the pro-p completion of the free group F_n. As noted in [1], the geometric realization of the simplicial set $\mathbf{n} \mapsto \mathrm{Hom}(P_n/\Gamma^q(P_n),G)$ gives a natural subspace of $B(q,G)$, and is denoted by $B(q,G)_p$. We prove that there is a stable homotopy equivalence:

Theorem. *Suppose that G is a finite group. There is a natural weak equivalence*

$$\bigvee_{p||G|} \Sigma B(q,G)_p \to \Sigma B(q,G) \quad \textit{for all } q \geq 2$$

induced by the inclusions $B(q,G)_p \to B(q,G)$.

Let $\mathcal{N}(q,G)$ denote the collection of subgroups of nilpotency class less than q and $G(q) = \operatorname*{colim}_{\mathcal{N}(q,G)} A$. The key observation in [1] is the following fibration

$$\operatorname*{hocolim}_{\mathcal{N}(q,G)} G(q)/A \to B(q,G) \to BG(q),$$

which can be constructed using the (homotopy) colimit description of $B(q,G)$. This raises the following question of whether they are actually homotopy equivalent appears in [1].

Question. *[1, page 15] If G is a finite group, are the spaces $B(q,G)$ Eilenberg-Mac Lane spaces of type $K(\pi,1)$?*

One of the objectives of this thesis is to show the existence of a certain class of groups for which $B(2,G)$ does not have the homotopy type of a $K(\pi,1)$ space. In Chapter 2 we address this question, and study the colimits $G(q)$ more closely. The following theorem is used to show that extraspecial p-groups are examples of such groups. Let $\widehat{G}$ denote the kernel of the map $G \times G \to G/[G,G]$ given by $(x,y) \mapsto xy[G,G]$.

Theorem. *Let S_p^r denote an extraspecial p-group of rank $2r \geq 4$ then*

$$\pi_1(B(2,S_p^r)) \cong \operatorname*{\mathit{colim}}_{\mathcal{N}(2,S_p^r)} A \cong \widehat{S_p^r}.$$

Indeed for the central products $D_8 \circ D_8$ and $D_8 \circ Q_8$, that is $r = 2$, the higher homotopy groups are given by

$$\pi_i(B(2, S_2^2)) \cong \pi_i(\bigvee^{151} S^2) \ \text{ for } i > 1.$$

The point here is to detect torsion elements in the kernel $T(2)$ of the natural map $\psi : G(2) \to G$. This is enough to conclude that the space $B(2, G)$ is not a $K(\pi, 1)$ space. We give a group theoretic condition, applicable to the class of symmetric groups Σ_k and general linear groups $\mathrm{GL}_n(\mathbb{F}_q)$ over a finite field of characteristic p. More precisely, the kernel $T(2)$ has a torsion element for symmetric groups Σ_k and general linear groups $\mathrm{GL}_n(\mathbb{F}_q)$ when $k \geq 8$ and $n \geq 4$. In particular for these groups $B(2, G)$ is not a $K(\pi, 1)$ space. To state the theorem we need a definition. We call a sequence of non-identity elements $\{g_i\}_{i=1}^{2r}$ a *symplectic sequence* if the following conditions are satisfied

1. $[g_i, g_{i+r}] \neq 1$ for $1 \leq i \leq r$ and $[g_i, g_j] = 1$ for any other pair,
2. $[g_i, g_{i+r}] = [g_j, g_{j+r}]$ for all $1 \leq i, j \leq r$.

Theorem. *Suppose that G is a finite group which has a symplectic sequence $\{g_i\}_{i=1}^{2r}$ for some $r \geq 2$. Then the kernel $T(2)$ has a torsion element of order p for each prime dividing the order of $c = [g_i, g_{i+r}]$. Moreover, the space $B(2, G)$ is not a $K(\pi, 1)$ space.*

Another natural question is to compute the complex K-theory of a homotopy colimit. The main tool to study a representable generalized cohomology theory of a homotopy colimit is the Bousfield-Kan spectral sequence whose E_2-term consists of the derived functors of the inverse limit functor. In Chapter 3 we address this problem for the homotopy colimit of classifying spaces of abelian subgroups of a finite group, that is, for $B(2, G)$. First we study higher limits of certain functors from the poset **dn** of proper subsets of $\{0, 1, \cdots, n\}$ ordered by inclusion to the category **Ab** of abelian groups.

Theorem. *Let* $R : \mathbf{dn} \to \mathbf{Ab}$ *be a pre-Mackey functor then* $\underset{\leftarrow}{lim}^s\, R^* = 0$ *for* $s > 0$.

This theorem can be used to conclude that the E_2–term of the spectral sequence consists of torsion groups. The Atiyah–Segal completion theorem along with this observation implies the following result.

Theorem. *There is an isomorphism*

$$\mathbb{Q} \otimes K^i(B(2,G)) \cong \begin{cases} \mathbb{Q} \oplus \bigoplus_{p||G|} \mathbb{Q}_p^{n_p} & \text{if } i = 0, \\ 0 & \text{if } i = 1, \end{cases}$$

where n_p *is the number of (non-identity) elements of order a power of* p *in* G.

Torsion groups can appear in $K^1(B(2,G))$. An explicit example is the case of S_2^2, there is an isomorphism

$$K^i(B(2,S_2^2)) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2^{31} & \text{if } i = 0, \\ (\mathbb{Z}/2)^9 & \text{if } i = 1. \end{cases}$$

In Chapter 4 we give more examples which illustrate the theorems. An appendix on some properties of simplicial sets is added at the end.

Chapter 1

Classifying spaces and homotopy colimits

1.1 Filtrations of classifying spaces

1.1.1 Preliminaries

Let Δ be the category whose objects are finite non-empty totally ordered sets $\mathbf{n}$, $n \geq 0$, with $n+1$ elements

$$0 \to 1 \to 2 \to \cdots \to n$$

and whose morphisms $\theta : \mathbf{m} \to \mathbf{n}$ are order preserving set maps or alternatively functors. The *nerve* of a small category $\mathbf{C}$ is the simplicial set

$$B\mathbf{C}_n = \mathrm{Hom}_{\mathbf{Cat}}(\mathbf{n}, \mathbf{C}).$$

For instance, the geometric realization of the nerve $BG_\bullet$ of a discrete group G (regarded as a category with one object) is the classifying space BG. An equivalent way of describing this space is as follows. The assignment

$$\mathbf{n} = (0 \xrightarrow{e_1} 1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} n) \mapsto F_n$$

where F_n is the free group generated by $\{e_1, e_2, ..., e_n\}$, defines a faithful functor $F : \Delta \to \mathbf{Grp}$ injective on objects. Then the classifying space of G is isomorphic to the simplicial set

$$\begin{aligned} BG_\bullet : \Delta^{op} &\to \mathbf{Set} \\ \mathbf{n} &\mapsto \mathrm{Hom}_{\mathbf{Grp}}(F_n, G). \end{aligned}$$

Note that this can be regarded as a functor $B : \mathbf{Grp} \to \mathbf{S}$ via the assignment $G \mapsto BG_\bullet$.

A generalization of this construction to certain quotients of free groups is studied in [1]. We recall the constructions and describe some alternative versions.

Definition 1.1.1. For a group Q define a chain of groups inductively: $\Gamma^1(Q) = Q$, $\Gamma^{q+1}(Q) = [\Gamma^q(Q), Q]$. The *descending central series* of Q is the normal series

$$1 \subset \cdots \subset \Gamma^{q+1}(Q) \subset \Gamma^q(Q) \subset \cdots \subset \Gamma^2(Q) \subset \Gamma^1(Q) = Q.$$

Now take Q to be the free group F_n. For $q > 0$, the natural maps $F_n \to F_n/\Gamma^q(F_n)$ induce inclusions of sets

$$\mathrm{Hom}_{\mathbf{Grp}}(F_n/\Gamma^q(F_n), G) \subset \mathrm{Hom}_{\mathbf{Grp}}(F_n, G).$$

Furthermore, the simplicial structure of $BG_\bullet$ induces a simplicial structure on the collection of these subsets.

Definition 1.1.2. Let G be a discrete group. We define a sequence of simplicial sets $B_\bullet(q,G)$, $q \geq 2$, by the assignment

$$\mathbf{n} \mapsto \mathrm{Hom}_{\mathbf{Grp}}(F_n/\Gamma^q(F_n), G)$$

and denote the geometric realization $|B(q,G)_\bullet|$ by $B(q,G)$. Define $B_\bullet(\infty,G) = BG_\bullet$ and $B(\infty,G) = BG$ by convention.

As in [1] this gives rise to a filtration of the classifying space of G

$$B(2,G) \subset B(3,G) \subset \cdots \subset B(q,G) \subset B(q+1,G) \subset \cdots \subset B(\infty,G) = BG.$$

This construction of a filtration can be generalized to any functor $P : \Delta \to \mathbf{Grp}$ with a natural transformation $\lambda : F \to P$, where F is the faithful functor $\mathbf{n} \mapsto F_n$ introduced at the beginning. Consider the functor $B_P : \mathbf{Grp} \to \mathbf{Top}$ defined by sending G to the geometric realization of the simplicial set defined by $n \mapsto \mathrm{Hom}(P(n), G)$, and the natural transformation $B_P \to B_F$ induced by λ. Choosing $P(n)$ as the quotients $F_n/\Gamma^q(F_n)$ recovers the spaces $B(q,G)$. In the next section we will consider another example of this construction where $P(n)$ is the pro-p completion P_n of the free group F_n and λ is the completion map $\lambda : F_n \to P_n$. One can also work in the category of topological groups and simplicial spaces. In this case the sets $\mathrm{Hom}(P(n), G)$ can be given a topology by the transformation λ induced from $\mathrm{Hom}(F_n, G)$ regarded as the n–fold product $\underbrace{G \times \cdots \times G}_{n \text{ times}}$. A useful property of these functors B_P is that they preserve finite products.

Proposition 1.1.3. *There is a natural homeomorphism $B_P(G_1 \times G_2) \to B_P(G_1) \times B_P(G_2)$ induced by the projections $\pi_i : G_1 \times G_2 \to G_i$.*

Proof. At the simplicial level there are natural maps

$$\mathrm{Hom}(P(n), G_1 \times G_2) \to \mathrm{Hom}(P(n), G_1) \times \mathrm{Hom}(P(n), G_2)$$

induced by the natural projections. This map is a bijection by the universal property of products, and compatible with the simplicial structure. The result follows from the fact that the geometric realization functor preserves products. $\square$

1.1.2 p-local version

We discuss an alternative construction which is obtained by replacing the free group F_n by its pro-p completion, namely the free pro-p-group P_n, see [9] for their basic properties. It can be defined using the *p-descending central series* of the free group

$$P_n = \lim_{\leftarrow} F_n/\Gamma_p^q(F_n)$$

where $\Gamma_p^q(F_n)$ are defined recursively:

$$\Gamma_p^1(F_n) = F_n \text{ and } \Gamma_p^{q+1}(F_n) = [\Gamma_p^q(F_n), F_n](\Gamma_p^q(F_n))^p.$$

The group P_n contains F_n as a dense subgroup. Observe that each quotient $F_n/\Gamma_p^q(F_n)$ is a finite p-group.

Let **TGrp** denote the category of topological groups. We consider continuous group homomorphisms $\phi \in \mathrm{Hom}_{\mathbf{TGrp}}(P_n, G)$ where G is a discrete group. The image of ϕ is a finite p-subgroup of G. This follows from the fact that P_n is compact and if K is a subgroup of P_n of finite index then $|P_n : K|$ is a power of p by [9, Lemma 1.18]. Then there exists $q \geq 1$ such that ϕ factors as

$$\begin{tikzcd} P_n \arrow[r, "\phi"] \arrow[d] & G \\ F_n/\Gamma_p^q(F_n). \arrow[ur] & \end{tikzcd}$$

This implies that $\mathrm{Hom}_{\mathbf{TGrp}}(P_n, G) \subset \mathrm{Hom}_{\mathbf{Grp}}(F_n, G)$ and also there is a simplicial structure induced from $BG_\bullet$.

We remark that a theorem of Serre on topological groups says that any (abstract) group homomorphism from a finitely generated pro-p group to a finite group is continuous [9, Theorem 1.17]. This implies that when G is finite any (abstract) homomorphism $\phi : P_n \to G$ is continuous. More generally, as a consequence of [22, Theorem 1.13] if every subgroup of G is finitely generated then the image of an abstract homomorphism ϕ is a finite group, hence in this case ϕ is also continuous.

Definition 1.1.4. Let p be a prime integer and G a discrete group. We define a sequence of simplicial sets $B_\bullet(q, G)_p$, $q \geq 2$, by

$$\mathbf{n} \mapsto \mathrm{Hom}_{\mathbf{TGrp}}(P_n/\Gamma^q(P_n), G).$$

Define $B_\bullet(\infty, G)_p$ to be the simplicial set $\mathbf{n} \mapsto \mathrm{Hom}_{\mathbf{TGrp}}(P_n, G)$ and set $B(\infty, G)_p = |B_\bullet(\infty, G)_p|$.

Let us examine the simplicial set $B_\bullet(\infty, G)_p$. There is a natural map

$$\theta : \lim_{\substack{\rightarrow \\ q}} \mathrm{Hom}(F_n/\Gamma^q_p(F_n), G) \to \mathrm{Hom}(P_n, G)$$

induced by the projections $P_n \to F_n/\Gamma^q_p(F_n)$. This map is injective since it is induced by injective maps and surjective since $\phi : P_n \to G$ factors through $F_n/\Gamma^q_p(F_n) \to G$ for some $q \geq 1$, as observed above. Therefore θ is a bijection of sets.

Definition 1.1.5. Let p be a prime integer and G a discrete group. We define a sequence of simplicial sets $B_\bullet(q, G, p)$, $q \geq 2$, by

$$\mathbf{n} \mapsto \mathrm{Hom}_{\mathbf{Grp}}(F_n/\Gamma^q_p(F_n), G)$$

and denote the geometric realization $|B(q, G, p)_\bullet|$ by $B(q, G, p)$.

The facts that colimits in the category of simplicial sets can be constructed dimension-wise and the geometric realization functor commutes with colimits imply the following.

Proposition 1.1.6. *Suppose that G is a discrete group then θ induces a homeomorphism*

$$\lim_{\overrightarrow{q}} B(q, G, p) \to B(\infty, G)_p.$$

1.2 The case of finite groups

We restrict our attention to finite groups. Note that the following collections of subgroups have an initial object (the trivial subgroup), and they are closed under taking subgroups and the conjugation action of G. Define a collection of subgroups for $q \geq 2$

$$\mathcal{N}(q, G) = \{H \subset G | \, \Gamma^q(H) = 1\}.$$

These are nilpotent subgroups of G of class less than q. Observe that $\mathcal{N}(2, G)$ is the collection of abelian subgroups of G.

Denote the poset of p-subgroups of G by $\mathcal{S}_p(G)$ (including the trivial subgroup) and set

$$\mathcal{N}(q, G)_p = \mathcal{S}_p(G) \cap \mathcal{N}(q, G)$$

when $q = 2$ this is the collection of abelian p-subgroups of G.

1.2.1 Homotopy colimits

In [1] it is observed that there is a homeomorphism

$$\operatorname*{colim}_{\mathcal{N}(q,G)} BA \to B(q, G)$$

and furthermore the natural map

$$\operatorname*{hocolim}_{\mathcal{N}(q,G)} BA \to \operatorname*{colim}_{\mathcal{N}(q,G)} BA$$

turns out to be a weak equivalence. A similar statement holds for the spaces $B(q,G)_p$.

Proposition 1.2.1. *Suppose that G is a finite group. Then there is a homeomorphism*

$$B(q,G)_p \cong \underset{\mathcal{N}(q,G)_p}{\mathit{colim}}\, BP$$

and the natural map

$$\underset{\mathcal{N}(q,G)_p}{\mathit{hocolim}}\, BP \to \underset{\mathcal{N}(q,G)_p}{\mathit{colim}}\, BP$$

is a weak equivalence.

Proof. There is a natural bijection of sets

$$\underset{\mathcal{N}(q,G)_p}{\operatorname{colim}} \operatorname{Hom}_{\mathbf{Grp}}(F_n, P) \to \operatorname{Hom}_{\mathbf{Grp}}(P_n/\Gamma^q(P_n), G).$$

The image of a morphism $P_n/\Gamma^q(P_n) \to G$ is a p-group of nilpotency class less than q. Conversely, if $P \subset G$ is a p-group of nilpotency class less than q then $F_n \to P$ induces a map $P_n \to P$ by taking the pro-p completions, and factors through the quotient $P_n/\Gamma^q(P_n)$. This induces the desired homeomorphism $B(q,G)_p \cong \underset{\mathcal{N}(q,G)_p}{\operatorname{colim}} BP$. The natural map from the homotopy colimit to the ordinary colimit is a weak equivalence since the diagram of spaces is free (Appendix). □

Remark 1.2.2. For the homotopy colimits considered in this thesis, it is sufficient to consider the sub-poset determined by the intersections of the maximal objects of the relevant poset. More precisely, let $M_1, M_2, ..., M_k$ denote the maximal groups in the poset $\mathcal{N}(q,G)$ and $\mathcal{M}(q,G) = \{\cap_J M_i |\ J \subset \{1,2,...,k\}\}$. The inclusion map

$$\mathcal{M}(q,G) \to \mathcal{N}(q,G)$$

is *right cofinal* and hence induces a weak equivalence ([28, Proposition 3.10])

$$\operatorname*{hocolim}_{\mathcal{M}(q,G)} BM \to \operatorname*{hocolim}_{\mathcal{N}(q,G)} BA.$$

A similar observation holds for $\mathcal{N}(q,G)_p$.

Proposition 1.2.3. *Let Z denote the intersection of the maximal objects in $\mathcal{N}(q,G)$. Then there is a commutative diagram of fibrations*

$$\begin{tikzcd}
 & \operatorname*{hocolim}_{\mathcal{M}(q,G)} G/A \arrow[r, equal] \arrow[d] & \operatorname*{hocolim}_{\mathcal{M}(q,G)} G/A \arrow[d] \\
BZ \arrow[r] \arrow[d, equal] & B(q,G) \arrow[r] \arrow[d] & \operatorname*{hocolim}_{\mathcal{M}(q,G)} BA/Z \arrow[d] \\
BZ \arrow[r] & BG \arrow[r] & BG/Z.
\end{tikzcd}$$

Proof. Only the right hand column and the middle row fibrations are unclear. Both follows from an application of the Theorems of Puppe [14, Appendix HL]. The first one is a homotopy colimit of a diagram of fibrations over a fixed base and the second is a diagram of fibrations whose fibers are homotopy equivalent to BZ. We prove the first one, the other is similar to the fibration in the middle column which was discussed before. Note that every object in the poset contains Z, and the diagram

$$\begin{tikzcd}
BN \arrow[r] \arrow[d] & BM \arrow[d] \\
BN/Z \arrow[r] & BM/Z
\end{tikzcd}$$

induced by the inclusion $BN/Z \subset BM/Z$ is a pull-back diagram which is identity on the fibre BZ. The fiber of the homotopy colimit is the common value BZ by the Theorem of Puppe. $\square$

1.2.2 A stable decomposition of $B(q, G)$

Recall that a finite nilpotent group N is a direct product of its Sylow p-subgroups

$$N \cong \prod_{p||N|} N_{(p)}.$$

For a prime p dividing the order of the group G, and $N \in \mathcal{N}(q, G)$ define

$$\pi_p(N) = \begin{cases} N_{(p)} & \text{if } p||N| \\ 1 & \text{otherwise} . \end{cases}$$

Lemma 1.2.4. *The inclusion map $\iota_p : \mathcal{N}(q, G)_p \to \mathcal{N}(q, G)$ of posets induces a weak equivalence*

$$\underset{\mathcal{N}(q,G)_p}{hocolim}\, BP \to \underset{\mathcal{N}(q,G)}{hocolim}\, B\pi_p(A).$$

Proof. By freeness (Appendix) it is enough to consider ordinary colimits. Let $B : \mathcal{N}(q, G)_p \to \mathbf{Top}$ denote the functor $P \mapsto BP$. The map $\pi_p : \mathcal{N}(q, G) \to \mathcal{N}(q, G)_p$ defined by $A \mapsto \pi_p(A)$ induces an inverse to the map induced by ι_p on the colimit

$$\underset{\mathcal{N}(q,G)_p}{\operatorname{colim}}\, BP \to \underset{\mathcal{N}(q,G)}{\operatorname{colim}}\, B\pi_p(A).$$

This follows from the fact that $\pi_p \circ \iota_p$ and $\iota_p \circ \pi_p$ induce the identity on $BP \to B\pi_p(P) = BP$ and $B\pi_p(A) \to B\iota_p\pi_p(A) = B\pi_p(A)$. □

Theorem 1.2.5. *Suppose that G is a finite group. There is a natural weak equivalence*

$$\bigvee_{p||G|} \Sigma B(q, G)_p \to \Sigma B(q, G) \quad \textit{for all } q \geq 2$$

induced by the inclusions $B(q, G)_p \to B(q, G)$.

Proof. First note that the nerve of the poset $\mathcal{N}(q, G)$ is contractible since the trivial subgroup is an initial object. Note that for $A \in \mathcal{N}(q, G)$ each BA is pointed via the inclusion $B1 \to BA$.

Let $\mathcal{J}$ denote the pushout category $0 \leftarrow 01 \rightarrow 1$. Define a functor F from the product category $\mathcal{J} \times \mathcal{N}(q, G)$ to **Top** by

$$F((0, A)) = \text{pt},\ F((01, A)) = BA \ \text{ and } F((1, A)) = \text{pt}.$$

Commutativity of homotopy colimits ([5, 4.5.20]) implies that

$$\operatorname*{hocolim}_{\mathcal{J}} \operatorname*{hocolim}_{\mathcal{N}(q,G)} F \cong \operatorname*{hocolim}_{\mathcal{J} \times \mathcal{N}(q,G)} F \cong \operatorname*{hocolim}_{\mathcal{N}(q,G)} \operatorname*{hocolim}_{\mathcal{J}} F.$$

The nerve of $\mathcal{N}(q, G)$ is contractible, in which case we can identify the homotopy colimit over $\mathcal{J}$ as the suspension and conclude that the map

$$\Sigma(\operatorname*{hocolim}_{\mathcal{N}(q,G)} BA) \rightarrow \operatorname*{hocolim}_{\mathcal{N}(q,G)} \Sigma(BA) \tag{1.1}$$

is a homeomorphism. The natural map induced by the suspension of the inclusions $BA_{(p)} \rightarrow BA$

$$\bigvee_{p||A|} \Sigma BA_{(p)} \rightarrow \Sigma BA$$

is a weak equivalence. This follows from the splitting of suspension of products:

$$\Sigma(BA_{(p)} \times BA_{(q)}) \simeq \Sigma(BA_{(p)}) \vee \Sigma(BA_{(q)}) \vee \Sigma(BA_{(p)} \wedge BA_{(q)})$$

and from the equivalence $\Sigma(BA_{(p)} \wedge BA_{(q)}) \simeq \text{pt}$ when p and q are coprime. The latter follows from the Künneth theorem and the Hurewicz theorem by considering the homology isomorphism induced by the inclusion $BA_{(p)} \vee BA_{(q)} \rightarrow BA_{(p)} \times BA_{(q)}$.

Invariance of homotopy colimits under natural transformations which induce a weak equivalence on each object, the homeomorphism (1.1) and

Lemma 1.2.4 give the following weak equivalences

$$\begin{aligned}
\Sigma(\operatorname*{hocolim}_{\mathcal{N}(q,G)} BA) &\cong \operatorname*{hocolim}_{\mathcal{N}(q,G)} \Sigma(BA) \\
&\simeq \operatorname*{hocolim}_{\mathcal{N}(q,G)} \bigvee_{p||A|} \Sigma(BA_{(p)}) \\
&\simeq \bigvee_{p||G|} \Sigma(\operatorname*{hocolim}_{\mathcal{N}(q,G)} B\pi_p(A)) \\
&\simeq \bigvee_{p||G|} \Sigma(\operatorname*{hocolim}_{\mathcal{N}(q,G)_p} BP).
\end{aligned}$$

When commuting the wedge product with the homotopy colimit, again an argument using the commutativity of homotopy colimits can be used similar to the suspension case. Note that the wedge product is a homotopy colimit.

□

This stable equivalence immediately implies the following decomposition for a generalized cohomology theory.

Theorem 1.2.6. *There is an isomorphism*

$$\tilde{h}^*(B(q,G)) \cong \prod_{p||G|} \tilde{h}^*(B(q,G)_p) \quad \textit{for all } q \geq 2$$

where $\tilde{h}^*$ denotes a reduced cohomology theory.

Therefore one can study homological properties of $B(q,G)$ at a fixed prime. In particular, this theorem applies to the complex K-theory of $B(q,G)$ and each piece corresponding to $B(q,G)_p$ can be computed using the Bousfield-Kan spectral sequence [6].

1.2.3 Homotopy types of $B(q,G)$ and $B(q,G)_p$

We follow the discussion on the fundamental group of $B(q,G)$ from [1], similar properties are satisfied by $B(q,G)_p$. Let $\mathcal{N}$ be a collection of subgroups of a finite group G. We require that $\mathcal{N}$ is closed under taking subgroups. Note that the trivial group is the initial object. In particular, it can be taken as $\mathcal{N}(q,G)$ or $\mathcal{N}(q,G)_p$. The fundamental group of a homotopy colimit is described in [13, Corollary 5.1]. The trivial group is the initial object in $\mathcal{N}$ and its classifying space $B1$ is regarded as a base point of the spaces in the diagram. Therefore there is an isomorphism

$$\pi_1(\operatorname*{hocolim}_{\mathcal{N}} BA) \cong \operatorname*{colim}_{\mathcal{N}} A$$

where the colimit is in the category of groups. We will study colimits of (abelian) groups in the next section in more detail.

For each $A \in \mathcal{N}$, there is a natural fibration $G/A \to BA \to BG$ and taking homotopy colimits one obtains a fibration

$$\operatorname*{hocolim}_{\mathcal{N}} G/A \to \operatorname*{hocolim}_{\mathcal{N}} BA \to BG. \tag{1.2}$$

This is a consequence of a Theorem of Puppe, see [14, Appendix HL]. The point is that each fibration has the same base space BG. Associated to this fibration there is an exact sequence of homotopy groups

$$\begin{aligned}1 \to \pi_1(\operatorname*{hocolim}_{\mathcal{N}} G/A) \to \pi_1(\operatorname*{hocolim}_{\mathcal{N}} BA) \xrightarrow{\psi} \pi_1(BG)\\ \to \pi_0(\operatorname*{hocolim}_{\mathcal{N}} G/A) \to 0\end{aligned} \tag{1.3}$$

where ψ is the natural map $\pi_1(\operatorname*{hocolim}_{\mathcal{N}} BA) \cong \operatorname*{colim}_{\mathcal{N}} A \to G$ induced by the inclusions $A \to G$ since the diagram

$$\begin{tikzcd} \operatorname*{hocolim}_{\mathcal{N}} BA \arrow[r] & BG \\ BA \arrow[u] \arrow[ur] & \end{tikzcd}$$

commutes for all $A \in \mathcal{N}$. Note that if $\mathcal{N}$ contains all cyclic subgroups of G

then $\operatorname*{hocolim}_{\mathcal{N}} G/A$ is connected. Since in this case, the commutativity of

$$
\begin{tikzcd}
\operatorname*{colim}_{\mathcal{N}} A \arrow[r, "\psi"] & G \\
\langle g \rangle \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

implies that ψ is surjective.

We point out here that for $\mathcal{N} = \mathcal{N}(q, G)$ the homotopy fibre $\operatorname*{hocolim}_{\mathcal{N}} G/A$ is homotopy equivalent to the pull-back of the universal principal G-bundle

$$
\begin{tikzcd}
E(q,G) \arrow[r] \arrow[d] & EG \arrow[d] \\
B(q,G) \arrow[r] & BG
\end{tikzcd}
$$

and this pull-back $E(q, G)$ can be defined as the geometric realization of a simplicial set as described in [1]. It can be identified as a colimit

$$E(q,G) \cong \operatorname*{colim}_{\mathcal{N}(q,G)} G \times_A EA$$

and there is a commutative diagram

$$
\begin{tikzcd}
\operatorname*{hocolim}_{\mathcal{N}(q,G)} G \times_A EA \arrow[r, "\sim"] \arrow[d, "\sim"] & \operatorname*{hocolim}_{\mathcal{N}(q,G)} G/A \arrow[d] \\
\operatorname*{colim}_{\mathcal{N}(q,G)} G \times_A EA \arrow[r] & \operatorname*{colim}_{\mathcal{N}(q,G)} G/A
\end{tikzcd}
$$

induced by the contractions $EA \to \mathrm{pt}$, and the weak equivalences as indicated. The exact sequence (1.3) becomes

$$0 \to T(q) \to G(q) \xrightarrow{\psi} G \to 0, \tag{1.4}$$

where $G(q) = \pi_1(B(q,G))$ and $T(q) = \pi_1(E(q,G))$.

Consider the universal cover $\widetilde{B(q,G)}$ of $B(q,G)$. Again using the Theorem of Puppe, it can be described as the homotopy fibre of the homotopy colimit of the natural fibrations $G(q)/A \to BA \to BG(q)$ ([1, Theorem 4.4])

$$\begin{array}{ccc}\widetilde{B(q,G)} \simeq \operatorname*{hocolim}_{\mathcal{N}(q,G)} G(q)/A & \longrightarrow & B(q,G) \\ & & \downarrow \\ & & BG(q).\end{array}$$

The question of $B(q,G)$ having the homotopy type of a $K(\pi,1)$ space is equivalent to asking whether the classifying space functor B commutes with colimits

$$B(q,G) = \operatorname*{colim}_{\mathcal{N}(q,G)} BA \to B(\operatorname*{colim}_{\mathcal{N}(q,G)} A).$$

The difference is measured by the simply connected, finite dimensional complex $\widetilde{B(q,G)}$, or equivalently by the values of the higher limits

$$H^i(\widetilde{B(q,G)};\mathbb{Z}) \cong \varprojlim_{\mathcal{N}(q,G)}{}^{i}\ \mathbb{Z}[G(q)/A]$$

as implied by the Bousfield-Kan spectral sequence.

Chapter 2

Finiteness of $G(2)$ and homotopy type of $B(2, G)$

The fundamental groups of $B(2, G)$ and $B(2, G)_p$ are isomorphic to the colimits of the collection of abelian subgroups and abelian p-subgroups of G, respectively. Therefore it is natural to study the colimit of a collection of abelian groups in greater generality to determine the homotopy properties of these spaces. Then we turn to a closer study of the colimit $G(2)$ of abelian subgroups. We give a group theoretic condition on G which implies the existence of torsion elements in the kernel $T(2)$ of the natural map $\psi : G(2) \to G$. This allows us to conclude that $B(2, G)$ is not a $K(\pi, 1)$ space for the groups satisfying this condition. Extraspecial p–groups of rank $r \geq 4$ are such examples. For further examples see Chapter 4.

2.1 Colimits of solvable groups

Let $\mathcal{A}$ denote a collection of solvable finite groups closed under taking subgroups. We consider the colimit of the groups in $\mathcal{A}$. This can be constructed as a quotient of the free product $\coprod$ of the groups in the collection

$$\operatorname*{colim}_{\mathcal{A}} A \cong (\coprod_{A \in \mathcal{A}} A)/\sim$$

by the normal subgroup generated by the relations $b \sim f(b)$ where $b \in B$ and $f : B \to A$ runs over the morphisms in $\mathcal{A}$. We need a definition from group theory.

Definition 2.1.1. A *chief* series of a group G is a series of normal subgroups

$$1 = N_0 \subset N_1 \subset ... \subset N_k = G$$

for which each factor N_{i+1}/N_i is a minimal (nontrivial) normal subgroup of G/N_i. Let $d(G)$ denote the number of indices $i = 1, 2, ..., k$ such that N_i/N_{i-1} has a complement in G/N_{i-1}

For $r > 0$ define a sequence of sub-collections $\mathcal{A}_r = \{A \in \mathcal{A} | \ d(A) \leq r\}$. The main result of this section is the following isomorphism of colimits, which reduces the collection $\mathcal{A}$ to the sub-collection $\mathcal{A}_2$.

Theorem 2.1.2. *Let $\mathcal{A}$ be a collection of solvable finite groups closed under taking subgroups. The natural map*

$$\underset{\mathcal{A}_2}{colim}\, A \to \underset{\mathcal{A}}{colim}\, A$$

induced by the inclusion map $\mathcal{A}_2 \to \mathcal{A}$ is an isomorphism.

The proof follows from the homotopy properties of a complex constructed from the cosets of proper subgroups of a group. We review some background first.

The coset poset

Consider the poset $\{xH | \ H \subsetneq G\}$ consisting of cosets of proper subgroups of G ordered by inclusion. The associated complex is denoted by $\mathcal{C}(G)$ and it is studied in [7]. This complex can be identified as a homotopy colimit

$$\mathcal{C}(G) \simeq \operatorname*{hocolim}_{H \in \mathcal{P}(G)} G/H,$$

where $\mathcal{P}(G) = \{H \subsetneq G\}$. The identification follows from the description of this homotopy colimit as the nerve of the transport category of the poset $\{H \subsetneq G\}$ which is precisely the nerve of the poset $\{xH |\ H \subsetneq G\}$.

When G is solvable, $\mathcal{C}(G)$ has the homotopy type of a bouquet of spheres of dimension $d(G)$.

Proposition 2.1.3. *[7, Proposition 11] Suppose that G is a solvable finite group and*

$$1 = N_0 \subset N_1 \subset ... \subset N_k = G$$

be a chief series then

$$\mathcal{C}(G) \simeq \bigvee^{n} S^{d(G)-1}$$

for some $n > 0$, where $d(G)$ is the number of indices $i = 1, 2, ..., k$ such that N_i/N_{i-1} has a complement in G/N_{i-1}.

We are interested only in the dimensions of the spheres. An immediate consequence of this result is the following.

Corollary 2.1.4. *Let G be a finite solvable group such that $d(G) \geq 3$, and $\mathcal{P}(G)$ denote the poset of proper subgroups $\{H \subsetneq G\}$. Then the natural map*

$$\beta : \operatorname*{colim}_{\mathcal{P}(G)} A \to G$$

is an isomorphism.

Proof. Observe that the fiber of the natural map $\operatorname*{hocolim}_{\mathcal{P}(G)} BA \to BG$ is the coset poset $\mathcal{C}(G) \simeq \operatorname*{hocolim}_{\mathcal{P}(G)} G/A$, and it is homotopy equivalent to $\bigvee^n S^{d(G)-1}$ by Proposition 2.1.3. The result follows immediately from the associated exact sequence of homotopy groups. ☐

Proof of Theorem 2.1.2. We have a sequence of inclusions

$$\mathcal{A}_2 \to \mathcal{A}_3 \to \ldots \to \mathcal{A}_i \to \mathcal{A}_{i+1} \to \ldots \to \mathcal{A}$$

which can be filtered further as follows

$$\mathcal{A}_i = \mathcal{A}_{i,0} \subset \mathcal{A}_{i,1} \subset \ldots \subset \mathcal{A}_{i,j} \subset \ldots \subset \mathcal{A}_{i+1}$$

where $\mathcal{A}_{i,j} = \{B \in \mathcal{A}_{i+1} | \ \mathcal{P}(B) \subseteq \mathcal{A}_{i,j-1}\}$ for $j > 0$. For $i \geq 2$ and $j \geq 0$, each of the maps

$$\Lambda : \operatorname*{colim}_{\mathcal{A}_{i,j}} A \to \operatorname*{colim}_{\mathcal{A}_{i,j+1}} A$$

has an inverse induced by the following compositions: Let $Q \in \mathcal{A}_{i,j+1}$

$$Q \xrightarrow{\beta^{-1}} \operatorname*{colim}_{\bar{\mathcal{P}}(Q)\cap\mathcal{A}_{i,j}} A \xrightarrow{\theta} \operatorname*{colim}_{\mathcal{A}_{i,j}} A$$

where $\bar{\mathcal{P}}(Q)$ is the poset of (all) subgroups of Q, β is the map in Corollary 2.1.4, and the map θ is induced by the inclusion $\bar{\mathcal{P}}(Q) \cap \mathcal{A}_{i,j} \to \mathcal{A}_{i,j}$. This is a consequence of the commutativity of

$$\begin{array}{ccc}
\operatorname*{colim}_{\mathcal{A}_{i,j}} A & \xrightarrow{\Lambda} & \operatorname*{colim}_{\mathcal{A}_{i,j+1}} A \\
\Big\uparrow{\scriptstyle\theta} & & \Big\uparrow{\scriptstyle\iota_Q} \\
\operatorname*{colim}_{\bar{\mathcal{P}}(Q)\cap\mathcal{A}_{i,j}} A & \xrightarrow[\cong]{\beta} & Q.
\end{array}$$

Hence each inclusion map $\mathcal{A}_{i,j} \to \mathcal{A}_{i,j+1}$ induces an isomorphism on the colimits. $\square$

2.2 Elementary properties of $G(2)$

Recall that the groups $G(q)$ are defined to be $\pi_1(B(q,G))$ and comes with a natural map $\psi : G(q) \to G$. For a fixed q it is described as the colimit of the collection $\mathcal{N}(q,G)$ of nilpotent subgroups of class less than q. The reduction theorem 2.1.2 simplifies this description.

Proposition 2.2.1. *$G(q)$ is isomorphic to the colimit of the groups in the sub-collection $\mathcal{N}_2 \subset \mathcal{N}(q,G)$ consisting of p–groups of rank at most 2 and products $P \times Q$ of a p–group with a q–group both of rank equal to 1, where p and q are distinct primes. Here the rank of a p–group is the rank of its Frattini quotient.*

Proof. By Theorem 2.1.2, the colimit is determined by the subgroups N such that $d(N) \leq 2$. Since N is nilpotent it is isomorphic to the product $\prod N_{(p)}$ of its Sylow p–subgroups. Note that for a p–group P the number $d(P)$ is at least the rank of the group, since the Frattini quotient $P \twoheadrightarrow P/\Phi(P)$ can be refined into a chief series. □

Also one can give a presentation of $G(2)$:

$$G(2) = \langle (g) |\ g \in G,\ \ (g)(h) = (gh) \text{ if } [g,h] = 1 \text{ in } G \rangle \tag{2.1}$$

as a quotient of the free group $F(G)$ generated on the set of elements of G. To see this, note that the homomorphism $F(G) \to G(2)$ induced by sending (g) to the image of g under the natural map $\langle g \rangle \to G(2)$ factors through the relations given in the presentation. There is also a map $G(2) \to F(G)/R$ induced by the inclusions $A \to F(G)/R$ of abelian subgroups A of G where R is the normal subgroup of $F(G)$ generated by the relations given in the presentation. These two maps are inverses of each other. In this presentation the natural map $G(2) \to G$ corresponds to $(g) \mapsto g$. By adding further relations to this presentation $G(q)$ can be described as a quotient of the free group as well. Then there is a sequence of surjective group homomorphisms

$$G(2) \to \cdots \to G(q-1) \to G(q) \to \cdots \to G(\infty) = G.$$

Proposition 2.2.1 suggests that the group $G(q)$ is closely related to the free product of the groups $G(q)_p = \pi_1(B(q,G)_p)$ which is a colimit over the collection $\mathcal{N}(q,G)_p$ of p–subgroups of nilpotency class less than q. There is a natural map $G(q)_p \to G(q)$ induced by the inclusion $\mathcal{N}(q,G)_p \subset \mathcal{N}(q,G)$ for each prime p which divides the order of the group. Furthermore 2.2.1 implies that the kernel of the map

$$\coprod_{p||G|} G(q)_p \to G(q)$$

is generated by the commutators of coprime order subgroups in G, hence is contained in the kernel of the natural map $\coprod G(q)_p \to \prod G(q)_p$. Therefore there exist a map $G(q) \to \prod G(q)_p$ which splits via the natural maps as shown in the following diagram

$$\begin{array}{ccc} \coprod_p G(q)_p & \longrightarrow & G(q) \\ \uparrow & & \downarrow \\ G(q)_p & \longleftarrow & \prod G(q)_p. \end{array} \tag{2.2}$$

Proposition 2.2.2. *The natural map $G(q)_p \to G(q)$ is an inclusion, and it splits.*

Proof. The splitting is given by the composition of the map $G(q) \to \prod G(q)_p$ in 2.2 with the natural projection $\prod G(q)_p \to G(q)_p$. □

There is a nicer interpretation of the splitting in the case of $G(2)$. First observe that sending an element to its n-th power defines a map of spaces

$$\omega_n : B(2,G) \to B(2,G)$$

at the simplicial level. This is induced by the map $(g_1, \cdots, g_k) \mapsto (g_1^n, \cdots, g_k^n)$ which is well–defined and respects the simplicial structure maps since g_i commutes with each other. The induced map on the fundamental groups will also be denoted by the same symbol. Writing the order $|G|$ of the group as a product of distinct primes $\prod p_i^{n_i}$ and defining $q_i = |G|/p_i^{n_i}$, one can see that the image of

$$\omega_{q_i} : G(2) \to G(2)$$

is isomorphic to $G(2)_{p_i}$. Moreover, the composition $\omega_{q_i'} \circ \omega_{q_i}$, where $q_i' \equiv 1/q_i \mod p_i^{n_i}$, restricted to $G(2)_{p_i}$ gives a splitting of the natural inclusion map $G(2)_{p_i} \to G(2)$. There are other useful properties one can deduce using the maps ω_n which we discuss next.

A natural question is to ask when $G(2)$ is equal to G. This can be answered by considering the automorphism ω_{-1} of $G(2)$ of order 2.

Proposition 2.2.3. *G is an abelian group if and only if the natural map $\psi : G(2) \to G$ is an isomorphism.*

Proof. Consider the diagram

$$\begin{array}{ccc} G(2) & \xrightarrow{(-1)} & G(2) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ G & & G. \end{array}$$

If ψ is an isomorphism then the composition $\psi \circ (-1) \circ \psi^{-1} : G \to G$ is an isomorphism, that is the inversion map $g \mapsto g^{-1}$ is an isomorphism. Then G is abelian. The converse is clear from the definition of $G(2)$. $\square$

Corollary 2.2.4. *The natural map $B(2, G) \to BG$ is a homotopy equivalence if and only if G is an abelian group.*

2.3 Finiteness of $G(2)$

We denote the commutator subgroup by G' and the quotient G/G' by G_{ab}. In this section we discuss certain class of (non-abelian) finite groups for which $G(2)$ is a finite group. First we introduce a finite group which is a sort of intermediary between $G(2)$ and G. Consider the group homomorphism

$$\widehat{\psi} : G(2) \to G \times G, \quad (g) \mapsto (g, g^{-1})$$

and denote its image by $\widehat{G}$. Clearly, this image is in the kernel of

$$\bar{m} : G \times G \to G_{ab}, \quad (g_1, g_2) \mapsto g_1 g_2 G',$$

and we claim that $\widehat{G} = \ker(\bar{m})$. First note that as a set $\ker(\bar{m})$ is the disjoint union $\coprod_{gG' \subset G} gG' \times g^{-1}G'$ hence has order $|G||G'|$, whereas $\widehat{G}$ is isomorphic to the quotient of $G(2)$ by the intersection $I = \ker(\psi) \cap \omega_{-1}(\ker(\psi))$. Therefore it suffices to show that the index $|\ker(\psi) : I|$ is equal to $|G'|$, which is also equal to the index $|\omega_{-1}(\ker(\psi)) : I|$. Observe that if $[g_1, g_2] \neq 1$ in G then $(g_1^{-1})(g_2^{-1})(g_2 g_1)$ is in the kernel of ψ but not contained in I. Therefore $(g_1)(g_2)(g_2 g_1)^{-1}$ is in $\omega_{-1}(\ker(\psi))$ and the subgroup generated by the images of such elements under $\widehat{\psi}$ contains G'. This proves the claim. Moreover there is a commutative diagram

$$\begin{tikzcd} G(2) \arrow[r, "\widehat{\psi}"] \arrow[dr, "\psi"'] & \widehat{G} \arrow[d, "\pi_1"] \\ & G, \end{tikzcd} \tag{2.3}$$

where π_1 is induced by projecting onto the first coordinate.

Proposition 2.3.1. *The commutator subgroup $(\widehat{G})'$ of $\widehat{G}$ is isomorphic to the pull–back of*

$$\begin{tikzcd} & G' \arrow[d] \\ G' \arrow[r] & G'/[G, G'] \end{tikzcd}$$

under the natural projections. Moreover, $G' \subset Z(G)$ if and only if $(\widehat{G})'$ is the diagonal subgroup $\Delta(G') = \{(g, g) | \ g \in G'\}$.

Proof. The commutator $(\widehat{G})'$ is a characteristic subgroup of $\widehat{G}$, hence normal in $G \times G$. Therefore any commutator of the form $[(t,1),[(g,g^{-1}),(h,h^{-1})]] = ([t,[g,h]],1)$ lies in $(\widehat{G})'$ for all $t, g, h \in G$. Furthermore, the group $\widehat{G}$ is invariant under the swap map $(x,y) \mapsto (y,x)$ which implies that the product $[G,G'] \times [G,G']$ is contained in $(\widehat{G})'$. There is a commutative diagram

$$\begin{array}{ccccc}
[G,G'] \times [G,G'] & \longrightarrow & (\widehat{G})' & \longrightarrow & (\widehat{Q})' \\
\| & & \downarrow & & \downarrow \\
[G,G'] \times [G,G'] & \longrightarrow & \widehat{G} & \longrightarrow & \widehat{Q}
\end{array}$$

where Q denotes the quotient $G/[G,G']$. Note that it suffices to prove the last part of the proposition for Q, which satisfies $Q' \subset Z(Q)$. Recall that $\widehat{Q}$ is generated by (g,g^{-1}), $g \in Q$, in $Q \times Q$. Then for all $g, h \in Q$ we have $[(g,g^{-1}),(h,h^{-1})] = ([g,h],[g^{-1},h^{-1}]) = ([g,h],[g,h])$ which proves one direction. For the converse, assume that $(\widehat{Q})' = \Delta(Q')$. The commutator $(\widehat{Q})'$ is normal in $Q \times Q$, i.e. $(1,g)(g',g')(1,g^{-1}) \in \Delta(Q')$ for all $g \in Q$ and $g' \in Q'$. $\square$

Proposition 2.3.2. *Let $\theta : G \to H$ be a surjective (injective) group homomorphism. Then there is a commutative diagram*

$$\begin{array}{ccccc}
\ker(\widehat{\psi}_G) & \longrightarrow & \widehat{G} & \longrightarrow & G \\
\downarrow & & \downarrow & & \downarrow \\
\ker(\widehat{\psi}_H) & \longrightarrow & \widehat{H} & \longrightarrow & H,
\end{array}$$

where all the vertical maps are surjective (injective).

Proof. Recall that $\widehat{G}$ is the kernel of $G \times G \to G/G'$ the composition of the abelianization map with the multiplication map. Therefore the conclusion of the proposition holds for the middle vertical arrow. The kernel $\ker \widehat{\psi}_G$ is the product $1 \times G'$ since $\widehat{\psi}_G$ is the projection π_1 onto the first factor. The first vertical arrow between the kernels is the map between the commutators $1 \times G' \to 1 \times H'$, hence it is surjective (injective) if the initial map $G \to H$

is surjective (injective). $\square$

In the next sections we will consider certain classes of p–groups where $\widehat{\psi}$ turns out to be an isomorphism. The following observation justifies our restriction to p–groups.

Proposition 2.3.3. *Assume G is a finite group. If $G(2)$ is a finite group then G is nilpotent, i.e. isomorphic to the product of its Sylow p–subgroups.*

Proof. Assume that $G(2)$ is finite, we will prove that G is nilpotent. Note by 2.2.3 each $G(2)_p$, $p||G|$, is also finite. If G is a p–group then there is nothing to prove. So assume G has composite order. In this case we will show G is isomorphic to the product of its Sylow p–subgroups. In other words, any two elements of coprime order commutes. Let p and q be distinct primes which divide the order of G. Assume the contrary that there are non–trivial elements g and h in G of order a power of p and q respectively such that $[g,h] \neq 1$. The product $(g)(h)$ has infinite order in the free product $\coprod_{p||G|} G(2)_p$ since it is cyclically reduced ([25, 1.3, Proposition 2]), where (g) and (h) denotes the corresponding generators in $G(2)_p$ and $G(2)_q$. Assume that its order in $G(2)$ is n. From 2.2 we obtain the diagram

$$\begin{tikzcd} G(2)_p \coprod G(2)_q \arrow[r] \arrow[dr, "\alpha"] & G(2) \arrow[d] \\ & G(2)_p \prod G(2)_q, \end{tikzcd}$$

where $(g)(h)$ maps to $((g),(h))$ in the product. Therefore the order n of the image of $(g)(h)$ in $G(2)$ is divisible by the order p of g and the order q of h. Now the product $((g)(h))^n$ can be written in $G(2)$ as a product of commutators as follows

$$\begin{aligned} \underbrace{(g)(h)\cdots(g)(h)}_{n} &= [(g),(h)][(h),(g)^2]\cdots[(h)^{n-1},(g)^n](g)^n(h)^n \\ &= [(g),(h)][(h),(g)^2]\cdots[(h)^{n-1}(g)^n] \end{aligned}$$

which is assumed to be in the kernel of $G(2)_p \coprod G(2)_q \to G(2)$. This kernel is also a subgroup of the kernel of α which is a free group whose basis consists of the commutators $[x, y]$ of elements in $G(2)_p$ and $G(2)_q$, see [25, 1.3, Proposition 4]. Therefore there is a consecutive pair of commutators which are inverses of each other. This implies $[(g), (h)] = 1$, hence $[g, h] = 1$, which is a contradiction. □

2.3.1 Extraspecial p–groups

We will restrict attention to central extensions of the form

$$\mathbb{Z}/p \lhd P \overset{\pi}{\to} V$$

where V is an elementary abelian p–group, which can also be regarded as a vector space over $\mathbb{Z}/p$. Let ρ denote a representative of the class of the extension. There is a bilinear form associated to ρ defined by $\mathfrak{b} = \rho + \rho^t$ where $\rho^t(x, y) = \rho(y, x)$. In this case $\mathfrak{b} : V \times V \to \mathbb{Z}/p$ is given by $\mathfrak{b}(v, w) = [\pi^{-1}(v), \pi^{-1}(w)]$. In particular, $\mathfrak{b}$ is *skew symmetric* $\mathfrak{b}(v, w) = -\mathfrak{b}(v, w)$. We further assume that this bilinear form is non-degenerate. This is equivalent to having $Z(P) = \mathbb{Z}/p$. Hence by standard results [12, Chapter I] in the theory of bilinear forms $(V, \mathfrak{b})$ is isometric to an orthogonal sum $r\mathbb{H}_{-1}$ of r copies of the hyperbolic plane. In particular the dimension of V is even, $\dim(V) = 2r$. A group of this type is called an *extraspecial p-group*. At a fixed r there are two types up to isomorphism. One can choose a *symplectic basis* $\{e_i\}_{1\le i\le 2r}$ where $\mathfrak{b}(e_i, e_{i+r}) = 1$ and for any other pair $\mathfrak{b}(e_i, e_j) = 0$.

Definition 2.3.4. Let $\mathfrak{b}$ be a bilinear form on a vector space V over a field of prime characteristic and $\mathcal{I}(\mathfrak{b})$ denote the collection of isotropic subspaces W, i.e. the restriction $\mathfrak{b}|_W$ is the zero form. Define the following group

$$V(\mathfrak{b}) = \operatorname*{colim}_{W\in\mathcal{I}} W$$

as a colimit in the category of groups. Here W is regarded as an elementary abelian p–group.

Note that by Proposition 1.2.3 there is a central extension

$$\mathbb{Z}/p \hookrightarrow P(2) \twoheadrightarrow V(\mathfrak{b}),$$

which will be used in the next result. A presentation of $V(\mathfrak{b})$ similar to 2.1 can be given as

$$\langle (v) \in V |\ (v)(w) = (v + w) \text{ if } \mathfrak{b}(v, w) = 0 \rangle.$$

The subgroup $\mathrm{Sp}(V, \mathfrak{b})$ of linear automorphisms $\mathrm{Aut}(V)$ which leaves $\mathfrak{b}$ invariant, i.e. $\alpha \in \mathrm{GL}(V)$ such that $\mathfrak{b}(\alpha(v), \alpha(w)) = \mathfrak{b}(v, w)$, acts on the group $V(\mathfrak{b})$ via the automorphism $(v) \mapsto (\alpha(v))$.

Theorem 2.3.5. *Let P be an extraspecial p-group. Then the map $\widehat{\psi} : P(2) \to \widehat{P}$ in 2.3 is an isomorphism if and only if $r \geq 2$.*

Proof. When $r = 1$ the group $V(\mathfrak{b})$ is the free product of its cyclic subgroups hence $P(2)$ is infinite. We assume $r \geq 2$, and observe that the conclusion of the theorem holds for P if and only if $V(\mathfrak{b})$ has order $|P|$. Choose a symplectic basis $\{e_i\}_{i=1}^{2r}$ for V. Given $i \neq j$ in $\{1, \cdots, r\}$ the following equations hold in $V(\mathfrak{b})$

$$\begin{aligned}
(e_i)(e_{i+r})(e_i)^{-1}(e_{i+r})^{-1} &= (e_i)(e_j - e_j)(e_{i+r})(e_i)^{-1}(e_{j+r} - e_{j+r})(e_{i+r})^{-1} \\
&= (e_j)(e_i)(-e_j + e_{i+r})(-e_i + e_{j+r})(e_{i+r})^{-1}(-e_{j+r}) \\
&= (e_j)(e_i)(-e_i + e_{j+r})(-e_j + e_{i+r})(e_{i+r})^{-1}(-e_{j+r}) \\
&= (e_j)(e_{j+r})(e_j)^{-1}(e_{j+r})^{-1},
\end{aligned}$$

where we make use of the identity $(v)^{-1} = (-v)$. The key step is the observation that $\mathfrak{b}(-e_j + e_{i+r}, -e_i + e_{j+r}) = -\mathfrak{b}(-e_i, e_{i+r}) + \mathfrak{b}(-e_j, e_{j+r}) = 0$. This shows, in particular, that the commutator $[(e_i), (e_{i+r})]$ is central. Here we use the fact that an element (v) for some $v \in V$ can be written as a product $\prod_{i=1}^{r}(\alpha_i e_i + \beta_i e_{i+r})$ in $V(\mathfrak{b})$.

Now any pair $v, w \in V$ with $\mathfrak{b}(v, w) \neq 0$ can be mapped to the hyperbolic pair e_i, e_{i+r} after renormalizing w to $\bar{w}$ such that $\mathfrak{b}(v, \bar{w}) = 1$ by sending $v, \bar{w}$ to e_i, e_{i+r}. By Witt's Lemma [3, pg. 81] the isometry $\langle v, \bar{w}\rangle \to \langle e_i, e_{i+r}\rangle$ extends to a map $\alpha \in \mathrm{Sp}(V, \mathfrak{b})$. Hence $[(v), (\bar{w})]$ is also central as well as the commutator $[(v), (w)]$ since $(w) = (\mathfrak{b}(v, w)\bar{w})$. We see that the commutator of $V(\mathfrak{b})$ is cyclic of order p and its abelianization is an elementary abelian p–group of rank $2r$. This proves that $V(\mathfrak{b})$ has the same order as P. $\square$

Observe that Proposition 2.3.1 can be used to prove that $\widehat{P}$ sits in a central extension

$$\mathbb{Z}/p \lhd \widehat{P} \to (\mathbb{Z}/p)^{2r-1} \times T \quad \text{if } p \text{ odd},$$

where $T \cong \mathbb{Z}/p^2$ if the exponent of P is p^2, and $T \cong (\mathbb{Z}/p)^2$ otherwise, and for $p = 2$

$$\mathbb{Z}/2 \lhd \widehat{P} \to (\mathbb{Z}/2)^{2r+1}.$$

We can extend our consideration to an arbitrary $\mathfrak{b}$ not necessarily non–degenerate without difficulty. Writing $\mathfrak{b} = \mathfrak{b}' \perp 0|_{V^\perp}$ where $\mathfrak{b}'$ is non–degenerate and $V^\perp$ denotes the radical of $\mathfrak{b}$, one has $V(\mathfrak{b}) \cong V(\mathfrak{b}') \times V^\perp$.

Corollary 2.3.6. *Let P be a (non–abelian) group which fits in a central extension $\mathbb{Z}/p \lhd P \to V$ where V is an elementary abelian p–group. Then $P(2) \cong \widehat{P}$ if and only if $\dim(\mathfrak{b}') \geq 4$, where $\mathfrak{b}$ is the bilinear form associated to the commutator and $\mathfrak{b}'$ is its non–degenerate part.*

2.4 Homotopy type of $B(2, G)$

In this section we generalize some ideas used in Theorem 2.3.5 to detect torsion in the kernel $T(2)$ of the natural map $\psi : G(2) \to G$. This enables us to decide whether $B(2,G)$ is a $K(\pi,1)$ space by the following result.

Proposition 2.4.1. *If the natural map $B(q,G) \to BG(q)$ is a homotopy equivalence then the kernel $T(q)$ of the natural map $G(q) \to G$ is torsion free.*

Proof. Recall that the fiber of the map $B(q,G) \to BG$ is a finite dimensional complex. If a finite dimensional complex is a $K(\pi,1)$ space then its fundmental group is torsion free. This follows from the fact that the cohomological dimension of π is less then or equal to its geometric dimension, and π is torsion free if its cohomological dimension is finite, see [8, Chapter VIII]. □

Definition 2.4.2. Let G be a finite group. We call a sequence of non-identity elements $\{g_i\}_{i=1}^{2r}$ a *symplectic sequence* if the following conditions are satisfied

1. $[g_i, g_{i+r}] \neq 1$ for $1 \leq i \leq r$ and $[g_i, g_j] = 1$ for any other pair,
2. $[g_i, g_{i+r}] = [g_j, g_{j+r}]$ for all $1 \leq i, j \leq r$.

The pair (g_i, g_{i+r}) is called a *symplectic pair*.

Denote the subgroup of G generated by the elements $\{g_i\}_{i=1}^{2r}$ by S. An immediate observation, which follows from commutator identities, is that the element $c = [g_i, g_{i+r}]$ is central in S, and its order n divides the orders of g_i for all i. Moreover the derived subgroup S' is cyclic of order n and there is a central extension

$$\mathbb{Z}/n \to S \to A,$$

where A is an abelian group of rank $2r$ generated by the images $\bar{g}_i$. The bilinear form $\mathfrak{b}$ associated to the extension class $\rho \in H^2(A; \mathbb{Z}/n)$ is non-zero on the images of the symplectic pairs $\mathfrak{b}(\bar{g}_i, \bar{g}_{i+r}) \neq 0$. Let p be a prime dividing n, and S_p^r denote an extraspecial p-group of rank $2r$ with a symplectic sequence $\{e_i\}_{i=1}^{2r}$, one can take a lift of a symplectic basis in the central quotient. We claim that the map $S \to S_p^r$ defined by $g_i \mapsto e_i$ is a surjective group homomorphism. First note that there is a surjective homomorphism $S \to S_p$ where S_p is the central quotient of S by the subgroup of index p in $\mathbb{Z}/n$. Hence it sits in a central extension $\mathbb{Z}/p \to S_p \to A$. The associated bilinear form is the composition of $\mathfrak{b}$ with the projection $\mathbb{Z}/n \to \mathbb{Z}/p$. With respect to this bilinear form the symplectic sequence $\{g_i\}$ in S maps to a symplectic sequence in S_p. We still denote this sequence by $\{g_i\}$, and its image in A by $\{\bar{g}_i\}$. This means that the restriction of the extension class $\rho \in H^2(A;\mathbb{Z}/p)$ corresponding to S_p to the subgroup generated by the image of a symplectic pair (g_i, g_{i+r}) in A is non-zero. Therefore, p divides the order of g_i for each $1 \leq i \leq 2r$. Choose a surjective homomorphism $\pi : A \to V = (\mathbb{Z}/p)^{2r}$. The induced map on $\pi^* : H^2(V;\mathbb{Z}/p) \to H^2(A;\mathbb{Z}/p)$ is also surjective since the p–rank of A is the same as the p–rank of V. Then there exists a class ρ_V mapping to ρ. The extension corresponding to ρ_V is an extraspecial p-group of rank $2r$, that is an extension $\mathbb{Z}/p \to S_p^r \to V$. Moreover, there is an isomorphism between the pull–back E in the diagram

$$\begin{array}{ccc} E & \longrightarrow & A \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ S_p^r & \longrightarrow & V \end{array}$$

and S_p since they have the same extension class. Hence we proved the following

Proposition 2.4.3. *Let $\{g_i\}_{i=1}^{2r}$ be a symplectic sequence in G, and S denote the subgroup generated by the elements g_i. Then there is a surjective homomorphism $\pi_p : S \to S_p^r$ which factors through the central quotient S_p for each prime p dividing the order of $c = [g_i, g_{i+r}]$. The induced map on the quotient $\pi : A \to V$ sends the image of the symplectic sequence $\{\bar{g}_i\}_{i=1}^{2r}$ onto a symplectic basis $\{e_i\}_{i=1}^{2r}$ of V where $e_i = \pi(\bar{g}_i)$.*

Lemma 2.4.4. *Suppose that $\{g_i\}_{i=1}^{2r}$ is a symplectic sequence in G and $r \geq 2$. Then the equation $[(g_i), (g_{i+r})] = [(g_j), (g_{j+r})]$ holds in $G(2)$ for all $1 \leq i, j \leq$*

r. Moreover, the element $t = [(g_i),(g_{i+r})]([g_i,g_{i+r}])^{-1}$ has finite order in $G(2)$.

Proof. The proof is mainly the same as in Theorem 2.3.5, a direct computation using the presentation of $G(2)$. For $1 \le i \ne j \le r$ the following equations hold in $G(2)$

$$\begin{aligned}
(g_i)(g_{i+r})(g_i)^{-1}(g_{i+r})^{-1} &= (g_i)(g_j g_j^{-1})(g_{i+r})(g_i)^{-1}(g_{j+r}g_{j+r}^{-1})(g_{i+r})^{-1} \\
&= (g_j)(g_i)(g_j^{-1}g_{i+r})(g_i^{-1}g_{j+r})(g_{i+r})^{-1}(g_{j+r}^{-1}) \\
&= (g_j)(g_i)(g_i^{-1}g_{j+r})(g_j^{-1}g_{i+r})(g_{i+r})^{-1}(g_{j+r}^{-1}) \\
&= (g_j)(g_{j+r})(g_j)^{-1}(g_{j+r})^{-1},
\end{aligned}$$

where we make use of the identity $(g)^{-1} = (g^{-1})$. The key step is the observation that $[g_j^{-1}g_{i+r}, g_i^{-1}g_{j+r}] = [g_j^{-1}, g_{j+r}][g_{i+r}, g_i^{-1}] = [g_j, g_{j+r}]^{-1}[g_i, g_{i+r}] = 1$. This shows, in particular, that the element $c = [(g_i),(g_{i+r})]$ commutes with $([g_i,g_{i+r}])^{-1}$. Hence $t^m = 1$ for large enough m. □

Theorem 2.4.5. *Suppose that G is a finite group which has a symplectic sequence $\{g_i\}_{i=1}^{2r}$ for some $r \ge 2$. Then the kernel $T(2)$ of the natural map $\psi : G(2) \to G$ has a torsion element of order p for each prime dividing the order of $c = [g_i, g_{i+r}]$. Moreover, the space $B(2,G)$ is not a $K(\pi,1)$ space, that is the natural map $B(2,G) \to BG(2)$ is not a homotopy equivalence.*

Proof. By Proposition 2.4.1 it suffices to show that the kernel $T(2)$ of ψ : $G(2) \to G$ has a torsion element. By Lemma 2.4.4 the element

$$t = [(g_i),(g_{i+r})]([g_i,g_{i+r}])^{-1} \in T(2)$$

is a torsion element, it suffices to show that $t \ne 1$. This will follow from the commutative diagram

$$\begin{tikzcd}
S_p^r & S \arrow[l, two heads, "\pi_p"'] \arrow[r, hook, "\iota"] & G \\
\widehat{S_p^r} \arrow[u, two heads] & \widehat{S} \arrow[l, two heads] \arrow[r, hook, "\hat{\iota}"] \arrow[u, two heads] & \widehat{G} \arrow[u, two heads] \\
S_p^r(2) \arrow[u, "\cong"] & S(2) \arrow[l, two heads] \arrow[r] \arrow[u, two heads, "\widehat{\psi}_S"] & G(2) \arrow[u, two heads, "\widehat{\psi}_G"]
\end{tikzcd}$$

where Proposition 2.3.2 implies the surjectivity (injectivity) of the relevant maps. Regarding $g_i \in G$ as an element of S we see that there is an element t_S in $S(2)$ which maps to t. Now the lower part of the diagram gives

$$\begin{tikzcd}
t_p & \hat{t}_S \arrow[l] \arrow[r] & \hat{t} \\
t_p \arrow[u, equal] & t_S \arrow[l] \arrow[r] \arrow[u, "\widehat{\psi}_S"'] & t \arrow[u, "\widehat{\psi}_G"']
\end{tikzcd}$$

where $t_p \neq 1$ since it generates the kernel of $\widehat{S_p^r} \to S_p^r$. Then $\hat{t}_S \neq 1$ which implies $\hat{t} \neq 1$ since $\hat{\iota}$ is injective. Therefore we obtain the desired result that $t \neq 1$, and also p divides the order of t. □

Corollary 2.4.6. *If G has a subgroup isomorphic to S_p^r for some $r \geq 2$ then $B(2,G)$ is <u>not</u> a $K(\pi,1)$ space.*

Remark 2.4.7. *Note that having a subgroup generated by a symplectic sequence is stronger than just having a sub–quotient isomorphic to S_p^r. Consider the group $D_8 \times D_8$ which surjects onto a S_2^2 but $B(2, D_8 \times D_8) \cong B(2, D_8) \times B(2, D_8)$ by Proposition 1.1.3, and it is a $K(\pi,1)$. The quotient must be a quotient of a subgroup S generated by a symplectic sequence in G i.e.*

$$S_p^r \twoheadleftarrow S \hookrightarrow G \quad \textit{where} \quad r \geq 2.$$

Chapter 3

Complex K–theory

3.1 Higher limits

As the higher limits appear in the E_2-term of the Bousfield-Kan spectral sequence, in this section we discuss some of their properties. The main theorem of this section is a vanishing result of the higher limits of the contravariant part of a pre-Mackey functor $R : \mathbf{dn} \to \mathbf{Ab}$ where $\mathbf{dn}$ denotes the poset of non-empty subsets of $\{1, ..., n\}$ ordered by reverse inclusion.

Let $\mathbf{C}$ be a small category and $F : \mathbf{C} \to \mathbf{Ab}$ be a *contravariant* functor from $\mathbf{C}$ to the category of abelian groups. $\mathbf{Ab}^{\mathbf{C}}$ denotes the category of contravariant functors $\mathbf{C} \to \mathbf{Ab}$. Observe that

$$\lim_{\leftarrow} F \cong \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, \lim_{\leftarrow} F) \cong \mathrm{Hom}_{\mathbf{Ab}^{\mathbf{C}}}(\underline{\mathbb{Z}}, F).$$

Definition 3.1.1. Derived functors of the inverse limit of $F : \mathbf{C} \to \mathbf{Ab}$ are defined by

$$\lim_{\leftarrow}{}^{i} F \equiv \mathrm{Ext}^{i}_{\mathbf{Ab}^{\mathbf{C}}}(\underline{\mathbb{Z}}, F).$$

A projective resolution of $\underline{\mathbb{Z}}$ in $\mathbf{Ab}^{\mathbf{C}}$ can be obtained in the following way, for details see [27]. First note that the functors $F_c : \mathbf{C} \to \mathbf{Ab}$ defined by

$F_c(c') = \mathbb{Z}\,\mathrm{Hom}_{\mathbf{C}}(c', c)$ are projective functors and by Yoneda's lemma

$$\mathrm{Hom}_{\mathbf{Ab}^{\mathbf{C}}}(F_c, F) \cong F(c).$$

Let $\mathbf{C} \setminus - : \mathbf{C} \to \mathbf{S}$ denote the functor which sends an object c of $\mathbf{C}$ to the nerve of the under category $\mathbf{C} \setminus c$. The nerve $B(\mathbf{C} \setminus c)$ is contractible since the object $c \to c$ is initial. Then we define a resolution $P_* \to \underline{\mathbb{Z}}$ of the constant functor as the composition $P_* = C_* \circ \mathbf{C} \setminus -$ where $C_* : \mathbf{S} \to \mathbf{Ab}$ is the functor which sends a simplicial set X to the associated chain complex $C_*(X) = \mathbb{Z}[X_*]$. At each degree n, there are isomorphisms

$$\mathrm{Hom}_{\mathbf{Ab}^{\mathbf{C}}}(P_n, F) \cong \mathrm{Hom}_{\mathbf{Ab}^{\mathbf{C}}}(\bigoplus_{x_0 \leftarrow \cdots \leftarrow x_n} F_{x_n}, F) \cong \prod_{x_0 \leftarrow x_1 \leftarrow \cdots \leftarrow x_n} F(x_n).$$

Differentials are induced by the simplicial maps between chains of morphisms. The following result describes them explicitly.

Lemma 3.1.2 ([23])**.** $\underset{\leftarrow}{lim}^i\, F \cong H^i(C^*(\mathbf{C}; F), \delta)$ *where*

$$C^n(\mathbf{C}; F) = \prod_{x_0 \leftarrow x_1 \leftarrow \cdots \leftarrow x_n} F(x_n)$$

for all $n \geq 0$ and where for $U \in C^n(\mathbf{C}; F)$

$$\delta(U)(x_0 \leftarrow x_1 \leftarrow \cdots \leftarrow x_n \overset{\phi}{\leftarrow} x_{n+1}) = \sum_{i=0}^{n} (-1)^i U(x_0 \leftarrow \cdots \leftarrow \hat{x}_i \leftarrow \cdots \leftarrow x_{n+1})$$
$$+(-1)^{n+1} F(\phi)(U(x_0 \leftarrow \cdots \leftarrow x_n)).$$

A useful property of higher limits, which allows the change of the indexing category, is the following.

Proposition 3.1.3. *[19, Lemma 3.1] Fix a small category $\mathbf{C}$ and a contravariant functor $F : \mathbf{C} \to \mathbf{Ab}$. Let $\mathbf{D}$ be a small category and assume* *that $g : \mathbf{D} \to \mathbf{C}$ has a left adjoint. Set $g^*F = F \circ g : \mathbf{D} \to \mathbf{Ab}$ then $H^*(\mathbf{C}; F) \cong H^*(\mathbf{D}; g^*F)$.*

Assume that the indexing category $\mathbf{C}$ is a finite partially ordered set (poset) whose morphisms are $i \to j$ whenever $i \leq j$. We consider an appropriate filtration of F as in [17].

Definition 3.1.4. A *height function* on a poset is a strictly increasing map of posets $ht : \mathbf{C} \to \mathbb{Z}$.

A convenient height function can be defined by $ht(A) = -\dim(|\mathbf{C}_{\geq A}|)$ where $\mathbf{C}_{\geq A}$ is the subposet of $\mathbf{C}$ consisting of elements $C \geq A$. Let $N = \dim(|\mathbf{C}|)$ then it is immediate that $\lim\limits_{\leftarrow}{}^{i} F$ vanishes for $i > N$. The functor F can be filtered in such a way that the associated quotient functors are concentrated at a single height. Define a sequence of subfunctors

$$F_N \subset \cdots \subset F_2 \subset F_1 \subset F_0$$

by $F_i(A) = 0$ if $ht(A) > -i$ and $F_i(A) = F(A)$ otherwise. This induces a decreasing filtration on the cochain complexes hence there is an associated spectral sequence whose E_0-term is

$$E_0^{i,j} = C^{i+j}(\mathbf{C}; F_i/F_{i+1}) = \prod_{A \in \mathbf{C} | ht(A) = -i} \mathrm{Hom}_{\mathbb{Z}}(C_{i+j}(|\mathbf{C}_{\geq A}|, |\mathbf{C}_{>A}|), F(A))$$

with differentials $d_0 : C^k(\mathbf{C}; F_i/F_{i+1}) \to C^{k+1}(\mathbf{C}; F_i/F_{i+1})$. E_1-term is the cohomology of the pair $(|\mathbf{C}_{\geq A}|, |\mathbf{C}_{>A}|)$ in coefficients $F(A)$ that is

$$E_1^{i,j} = \prod_{A \in \mathbf{C} | ht(A) = -i} H^{i+j}((|\mathbf{C}_{\geq A}|, |\mathbf{C}_{>A}|); F(A)). \tag{3.1}$$

Fix A and let $A' > A$ such that $ht(A') = ht(A) + 1$, then differential d_1 can be described as the composition

$$\begin{aligned} H^k((|\mathbf{C}_{\geq A'}|, |\mathbf{C}_{>A'}|); F(A')) &\overset{F(A\leq A')}{\to} H^k((|\mathbf{C}_{>A}|, |\mathbf{C}_{>A,ht\geq ht(A)+2}|); F(A)) \\ &\overset{\partial}{\to} H^{k+1}((|\mathbf{C}_{\geq A}|, |\mathbf{C}_{>A}|); F(A)) \end{aligned}$$

where ∂ is the boundary map associated to the triple

$$(|\mathbf{C}_{\geq A}|, |\mathbf{C}_{>A}|, |\mathbf{C}_{>A,ht\geq ht(A)+2}|).$$

We now consider higher limits over specific kinds of diagrams. Let $\mathbf{dn}$ be the category associated to the non-degenerate simplexes of the standard n-simplex, the objects are increasing sequences of numbers $\sigma_k = [i_0 < \cdots < i_k]$ where $0 \leq i_j \leq n$ and the morphisms are generated by the face maps $d_j([i_0 < \cdots < i_k]) = [i_0 < \cdots < \hat{i_j} < \cdots < i_k]$ for $0 \leq j \leq k$.

Proposition 3.1.5. *For any contravariant functor $F : \mathbf{dn} \to \mathbf{Ab}$ the spectral sequence (3.1) collapses onto the horizontal axis hence gives a long exact sequence*

$$0 \to \prod_{\sigma_0} F(\sigma_0) \to \prod_{\sigma_1} F(\sigma_1) \to \cdots \to \prod_{\sigma_{n-1}} F(\sigma_{n-1}) \to F(\sigma_n) \to 0 \qquad (3.2)$$

and where for $U \in C^{k-1}_{\mathbf{dn}}(F) = \prod_{\sigma_{k-1}} F(\sigma_{k-1})$,

$$\delta^{k-1}(U)(\sigma_k) = \sum_{j=0}^{k} (-1)^{k-j} F(d_j) U(d_j(\sigma_k)).$$

Proof. For a simplex $\sigma_k \in \mathbf{dn}$, let $\mathbf{dn}_{\geq k}$ be the poset of simplices $\sigma \geq \sigma_k$ and $\mathbf{dn}_{>k}$ denote the poset of simplices $\sigma > \sigma_k$. Observe that the pair $(|\mathbf{dn}_{\geq k}|, |\mathbf{dn}_{>k}|)$ is homeomorphic to $(|\Delta^k|, |\partial\Delta^k|)$ where Δ^k is the standard

k-simplex with boundary $\partial\Delta^k$. The spectral sequence of the filtration has

$$E_1^{k,j} = \prod_{\sigma_k} H^{k+j}((|\mathbf{dn}_{\geq k}|, |\mathbf{dn}_{>k}|); F(\sigma_k)).$$

Therefore E_1-term vanishes unless $j = 0$, and otherwise

$$E_1^{k,0} = \prod_{\sigma_k} H^k((|\mathbf{dn}_{\geq k}|, |\mathbf{dn}_{>k}|); F(\sigma_k)) = \prod_{\sigma_k} F(\sigma_k)$$

hence collapses onto the horizontal axes, resulting in a long exact sequence. The differential is induced by the alternating sum of the face maps $d_j(\sigma_k)$ for $0 \geq j \geq k$. $\square$

In some cases, it is possible to show that the higher limits vanish. The next theorem is an illustration of this instance. First let us recall the definition of a *pre-Mackey functor* in the sense of Dress [10]. Let $M : \mathbf{C} \to \mathbf{D}$ be a bifunctor, that is a pair of functors (M^*, M_*) such that M^* is contravariant, M_* is covariant and both coincide on objects.

Definition 3.1.6. A *pre-Mackey functor* is a bifunctor $M : \mathbf{C} \to \mathbf{D}$ such that for any pull-back diagram

$$\begin{tikzcd} X_1 \arrow[r, "\beta_1"] \arrow[d, "\alpha_1"'] & X_2 \arrow[d, "\beta_2"] \\ X_3 \arrow[r, "\alpha_2"] & X_4 \end{tikzcd}$$

in $\mathbf{C}$ the diagram

$$\begin{tikzcd} M(X_1) \arrow[r, "M_*(\beta_1)"] & M(X_2) \\ M(X_3) \arrow[u, "M^*(\alpha_1)"] \arrow[r, "M_*(\alpha_2)"] & M(X_4) \arrow[u, "M^*(\beta_2)"'] \end{tikzcd}$$

commutes.

We give a direct proof of the following theorem, which can also be deduced from an application of [19, Theorem 5.15] to the category $\mathbf{dn}$.

Theorem 3.1.7. *Let $R : \mathbf{dn} \to \mathbf{Ab}$ be a pre-Mackey functor then $\underleftarrow{lim}^s\, R^* = 0$ for $s > 0$.*

Proof. We do induction on n. The case $n = 0$ is trivial since the complex (3.2) is concentrated at degree 0. There are inclusions of categories

$$\iota_0 : \mathbf{dn-1} \to \mathbf{dn}$$

defined by $[i_1 < \ldots < i_k] \mapsto [i_1 + 1 < \ldots < i_k + 1]$ and

$$\iota_1 : \mathbf{dn-1} \to \mathbf{dn}$$

where $[i_1 < \ldots < i_k] \mapsto [0 < i_1+1 < \ldots < i_k+1]$. Let $C^*_{\mathbf{dn}}$ denote the complex $C^*_{\mathbf{dn}}(R)$ in (3.2) that is $C^k_{\mathbf{dn}} = \prod_{\sigma_k} R(\sigma_k)$. Define a filtration $C^*_{-1} \subset C^*_0 \subset C^*_{\mathbf{dn}}$ such that

$$C^k_0 = \prod_{\sigma_k = [0 < i_1 < \ldots < i_k]} R(\sigma_k)$$

and $C^k_{-1} = C^k_0$ for $k > 0$ and $C^0_{-1} = 0$. Then by induction $H^i(C^*_{\mathbf{dn}}/C^*_0) = H^i(C^*_{\mathbf{dn-1}}(R \circ \iota_0)) = 0$ for $i > 0$ and $H^i(C^*_{-1}) = H^{i-1}(C^*_{\mathbf{dn-1}}(R \circ \iota_1)) = 0$ for $i > 1$. And short exact sequences associated to the filtration $C^*_{-1} \subset C^*_0 \subset C^*_{\mathbf{dn}}$ implies that it suffices to prove $H^1(C^*_0) = 0$.

For a simplex $\sigma = [0 < i < \ldots < j]$ and morphisms $\alpha_i : \sigma \to [0 < i]$ and $\alpha_j : \sigma \to [0 < j]$ in $\mathbf{dn}$, set $\phi_\sigma = R_*(\alpha_j) R^*(\alpha_i)$.

For a fixed i, define a map $\Phi_i : R([0 < i]) \to R([0])$ by

$$\Phi_i = \sum_{\sigma = [0 < i < \ldots < j]} (-1)^{|\sigma| - 1} R_*(d^j_1) \phi_\sigma \quad \text{where} \quad d^j_1 : [0 < j] \to [0],$$

where $|\sigma|$ denotes the dimension of the simplex σ. Combining these maps for all $1 \leq i \leq n$ define $\Phi : \prod_{l=1}^{n} R([0<l]) \to R([0])$ by

$$\Phi = \sum_{i=1}^{n} \Phi_i \pi_i,$$

where π_i denotes the natural projection $\prod_{l=1}^{n} R([0<l]) \to R([0<i])$ to the i^{th} factor.

We need to show that if U is in the kernel of δ^1, then $\delta^0\Phi(U) = U$ or equivalently $R^*(d_1^i)\Phi(U) = \pi_i(U)$ for all $1 \leq i \leq n$. Fix k, and compute the composition

$$\begin{aligned} R^*(d_1^k)\Phi &= \sum_{i=1}^{n} \sum_{\sigma=[0<i<...<j]} (-1)^{|\sigma|-1} R^*(d_1^k) R_*(d_1^j) \phi_\sigma \\ &= \pi_k + \sum_{i=1}^{n} \left(\sum_{\sigma | k \in \sigma \neq [0<k]} (-1)^{|\sigma|-1} R^*(d_1^k) R_*(d_1^j) \phi_\sigma \right. \\ &\qquad \left. + \sum_{\sigma | k \notin \sigma} (-1)^{|\sigma|-1} R^*(d_1^k) R_*(d_1^j) \phi_\sigma \right) \end{aligned}$$

in the second line first use the identity $R^*(d_1^k)R_*(d_1^j)\phi_{[0<k]} = \pi_k$ then separate the summation into two parts such that it runs over the simplices which contain $\{k\}$ and do not contain $\{k\}$. Suppose that $\sigma = [0 < i < ... < j]$ does not contain $\{k\}$ and $\overline{\sigma}$ denote the simplex obtained from σ by adjoining $\{k\}$. There are three possibilities for $\overline{\sigma}$:

$$\overline{\sigma} = \begin{cases} [0 < i < ... < k < ... < j] & \text{if } i < k < j, \\ [0 < i < ... < j < k] & \text{if } j < k, \\ [0 < k < i < ... < j] & \text{if } k < i. \end{cases}$$

The corresponding terms of the first two cases cancel out when σ and $\overline{\sigma}$ are paired off in the summation, since the following diagram commutes

$$\begin{array}{ccccc}
R(\overline{\sigma}) & \longleftarrow & R([0<i]) & \xrightarrow{R^*(\alpha_i)} & R(\sigma) \\
\big\downarrow & & & & \big\downarrow{\scriptstyle R_*(\alpha_j)} \\
R([0<k]) & \xleftarrow{R^*(d_1^k)} & R([0]) & \xleftarrow{R_*(d_1^j)} & R([0<j])
\end{array}$$

and the corresponding terms become equal with opposite signs. So the equation simplifies to

$$R^*(d_1^k)\Phi \;=\; \pi_k + \sum_{\sigma=[0<k<i<\ldots<j]} (-1)^{|\sigma|-1}(R^*(d_1^k)R_*(d_1^j)\phi_\sigma - R^*(d_1^k)R_*(d_1^j)\phi_{d_1\sigma})$$

where the sum runs over the simplices starting with $\{0 < k\}$. The maps $\sigma \xrightarrow{\theta} [0 < k < i]$, $\sigma \xrightarrow{\vartheta} [0 < k]$ and also the two face maps $[0 < i] \xleftarrow{d_1} [0 < k < i] \xrightarrow{d_2} [0 < k]$ in our indexing category **dn** give a commutative diagram

$$\begin{array}{ccccc}
R([0<i]) & \xrightarrow{R^*(d_1)} & R([0<k<i]) & \xrightarrow{R^*(\theta)} & R(\sigma) \\
\big\downarrow & & & & \big\downarrow{\scriptstyle R_*(\vartheta)} \\
R(d_1\sigma) & \longrightarrow & R([0<j]) & \xrightarrow{R^*(d_1^k)R_*(d_1^j)} & R([0<k]).
\end{array}$$

Hence we can write

$$R^*(d_1^k)\Phi \;=\; \pi_k + \sum_{\sigma=[0<k<i<\ldots<j]} (-1)^{|\sigma|-1} R_*(\vartheta)R^*(\theta)\left(R^*(d_2) - R^*(d_1)\right).$$

Now suppose that U is in the kernel of $\delta^1 : C_0^1 \to C_0^2$, the summation above vanishes since $\left(R^*(d_2) - R^*(d_1)\right)(U) = 0$. Therefore $R^*(d_1^k)\Phi(U) = \pi_k(U)$, which implies that $H^1(C_0^*) = 0$. $\square$

3.1.1 Higher limits of the representation ring functor

Let G be a finite group and p be a prime dividing its order. Let $M_0, ..., M_n$ denote the maximal abelian p-subgroups of G. Consider the contravariant functor $R : \mathbf{dn} \to \mathbf{Ab}$ defined by $\sigma \mapsto R(M_\sigma)$ where $M_\sigma = \cap_{i\in\sigma} M_i$ and $R(H)$ is the Grothendieck ring of complex representations of H. Note that $\mathbb{Q} \otimes R$ is a pre-Mackey functor: If $\alpha : \sigma \to \sigma'$ set $A = \cap_{i\in\sigma} M_i$ and $B = \cap_{i\in\sigma'} M_i$ then $(\mathbb{Q} \otimes R^*(\alpha), \mathbb{Q} \otimes R_*(\alpha)) = (\mathrm{res}_{B,A}, \frac{1}{|B:A|}\mathrm{ind}_{A,B})$. Hence Theorem 3.1.7 implies the following.

Corollary 3.1.8. *If $R : \mathbf{dn} \to \mathbf{Ab}$ defined as above then $\lim\limits_{\leftarrow}{}^s \, \mathbb{Q} \otimes R^* = 0$ for $s > 0$.*

Proof. We give an alternative proof which is specific to the representation ring functor. Let $U : \mathbf{Grp} \to \mathbf{Set}$ denote the forgetful functor which sends a group to the underlying set. The isomorphism $\mathbb{C}\otimes R(M_\sigma) \cong \mathrm{Hom}_{\mathbf{Set}}(M_\sigma, \mathbb{C}) = H^0(U(M_\sigma); \mathbb{C}))$ induces an isomorphism $\lim\limits_{\leftarrow}{}^s \, \mathbb{C}\otimes R(M_\sigma) \cong \lim\limits_{\leftarrow}{}^s \, H^0(U(M_\sigma); \mathbb{C})$. Note that $U(M_\sigma)$ is a finite set of points. Consider the space $\underset{\mathbf{dn}}{\mathrm{hocolim}}\, U(M_\sigma)$ and the associated cohomology spectral sequence

$$E_2^{s,t} = \lim_{\underset{\mathbf{dn}}{\leftarrow}}{}^s \, H^t(U(M_\sigma); \mathbb{C}) \Rightarrow H^{s+t}(\underset{\mathbf{dn}}{\mathrm{hocolim}}\, U(M_\sigma); \mathbb{C}).$$

Now, the diagram $\mathbf{dn} \to \mathbf{Top}$ defined by $\sigma \mapsto U(M_\sigma)$ is free (Appendix), consists of inclusions of finite sets. Therefore the natural map $\underset{\mathbf{dn}}{\mathrm{hocolim}}\, U(M_\sigma) \to \underset{\mathbf{dn}}{\mathrm{colim}}\, U(M_\sigma) = U(G)$ is a homotopy equivalence, note that the latter is a finite set. The spectral sequence collapses at E_2-page and gives $\lim\limits_{\leftarrow}{}^s \, H^0(U(M_\sigma); \mathbb{C}) \cong H^s(\underset{\mathbf{dn}}{\mathrm{hocolim}}\, U(M_\sigma); \mathbb{C})$ which vanishes for $s > 0$. □

3.2 K-theory of $B(2,G)$

In this section, we assume that G is a finite group. We study the complex K-theory of $B(q,G)$ when $q=2$. Theorem 1.2.6 gives a decomposition

$$\tilde{K}^*(B(2,G)) \cong \bigoplus_{p||G|} \tilde{K}^*(B(2,G)_p),$$

of the reduced K-theory of $B(2,G)$. For each such prime p recall the fibration $\operatorname*{hocolim}_{\mathcal{N}(q,G)_p} G/P \to B(q,G)_p \to BG$ from §1.2. Furthermore, the Borel construction commutes with homotopy colimits and gives a weak equivalence

$$B(q,G)_p \simeq (\operatorname*{hocolim}_{\mathcal{N}(q,G)_p} G/P) \times_G EG.$$

There is the Atiyah-Segal completion theorem [4] which relates the equivariant K-theory of a G-space X to the complex K-theory of its Borel construction $X \times_G EG$.

Equivariant K-theory

Complex equivariant K-theory is a $\mathbb{Z}/2$-graded cohomology theory defined for compact spaces using G-equivariant complex vector bundles [24]. Recall that

$$K^n_G(G/H) \cong \begin{cases} R(H) & \text{if } n=0, \\ 0 & \text{if } n=1, \end{cases}$$

where $H \subseteq G$ and recall that $R(H)$ denotes the Grothendieck ring of complex representations of H. Note that in particular, $K^0_G(\mathrm{pt}) \cong R(G)$ and $K^*_G(X)$ is an $R(G)$-module via the natural map $X \to \mathrm{pt}$.

Let X be a compact G-space and $I(G)$ denote the kernel of the augmentation map $R(G) \xrightarrow{\varepsilon} \mathbb{Z}$. The Atiyah-Segal completion theorem [4] states that $K^*(X \times_G EG)$ is the completion of $K^*_G(X)$ at the augmentation ideal $I(G)$.

In particular, taking X to be a point this theorem implies that

$$K^n(BG) \cong \begin{cases} R(G)^\wedge & n=0, \\ 0 & n=1. \end{cases}$$

The completion $R(G)^\wedge$ can be described using the restriction maps to a Sylow p-subgroup for each p dividing the order of the group G. Let $I_p(G)$ denote the quotient of $I(G)$ by the kernel of the restriction map $I(G) \to I(G_{(p)})$ to a Sylow p-subgroup. There are isomorphisms of abelian groups

$$K^n(BG) \cong \begin{cases} \mathbb{Z} \oplus \bigoplus_{p \mid \, |G|} \mathbb{Z}_p \otimes I_p(G) & n=0, \\ 0 & n=1, \end{cases}$$

see [21]. Note that if G is a nilpotent group, equivalently $G \cong \prod_{p \mid \, |G|} G_{(p)}$, the restriction map $I(G) \to I(G_{(p)})$ is surjective and

$$\tilde{K}^0(BG) \cong I(G)^\wedge \cong \bigoplus_{p \mid \, |G|} \mathbb{Z}_p \otimes I(G_{(p)}). \tag{3.3}$$

We also use the fact that the completion of $R(H)$ with respect to $I(H)$ is isomorphic to the completion with respect to $I(G)$ as an $R(G)$-module via the restriction map $R(G) \to R(H)$.

3.2.1 K-theory of $B(2,G)_p$

The E_2-page of the equivariant version of the Bousfield-Kan spectral sequence [20] for the equivariant K-theory of $\operatorname*{hocolim}_{\mathcal{N}(q,G)_p} G/P$ is given by

$$E_2^{s,t} \cong \lim_{\substack{\leftarrow \\ \mathcal{N}(q,G)_p}}{}^{s}\ K_G^t(G/P) \cong \begin{cases} \lim_{\substack{\leftarrow \\ \mathcal{N}(q,G)_p}}{}^{s}\ R(P) & \text{if } t \text{ is even}, \\ 0 & \text{if } t \text{ is odd}, \end{cases} \tag{3.4}$$

and the spectral sequence converges to $K_G^*(\operatorname*{hocolim}_{\mathcal{N}(q,G)_p} G/P)$ whose completion is $K^*(\operatorname*{hocolim}_{\mathcal{N}(q,G)_p} BP)$. We will split off the trivial part of $E_2^{s,t}$ and consider

$$\tilde{E}_2^{s,t} \cong \underset{\mathcal{N}(q,G)_p}{\underset{\leftarrow}{\lim}}{}^s \; I(P) \quad \text{for } t \text{ even.}$$

Note that the splitting $R(P) \cong \mathbb{Z} \oplus I(P)$ gives $\underset{\mathcal{N}(q,G)_p}{\underset{\leftarrow}{\lim}}{}^s \; R(P) \cong \underset{\mathcal{N}(q,G)_p}{\underset{\leftarrow}{\lim}}{}^s \; I(P)$ for $s > 0$ since the nerve of $\mathcal{N}(q,G)_p$ is contractible. The following result describes the terms of this spectral sequence for the case $q = 2$.

Lemma 3.2.1. *Let $R : \mathcal{N}(2,G)_p \to \mathbf{Ab}$ be the representation ring functor defined by $P \mapsto R(P)$ then $\underset{\leftarrow}{\lim}{}^s \; R$ are torsion groups for $s > 0$. Furthermore, $\underset{\leftarrow}{\lim}{}^0 \; R \cong \mathbb{Z}^{n_p+1}$ where n_p is the number of (non-identity) elements of order a power of p in G.*

Proof. Let M_i for $0 \leq i \leq n$ denote the maximal abelian p-subgroups of G. Define a functor $g : \mathbf{dn} \to \mathcal{N}(2,G)_p$ by sending a simplex $[i_0 < i_1 < ... < i_k]$ to the intersection $\bigcap_{j=0}^k M_{i_j}$. Then g has a left adjoint, namely the functor $P \mapsto \bigcap_{M_i : P \subset M_i} M_i$ sending an abelian p-subgroup to the intersection of maximal abelian p-subgroups containing it. By Proposition 3.1.3 we can replace the indexing category $\mathcal{N}(2,G)_p$ by $\mathbf{dn}$ and calculate the higher limits over the category $\mathbf{dn}$. Then the first part of the theorem follows from Corollary 3.1.8.

For the second part note that $\underset{\leftarrow}{\lim}{}^0 \; R = \underset{\leftarrow}{\lim} \; R$ is a submodule of a free $\mathbb{Z}$-module, hence it is free. Then it suffices to calculate its rank. Tensoring with $\mathbb{C}$ gives an isomorphism

$$\mathbb{C} \otimes \underset{\mathcal{N}(2,G)_p}{\underset{\leftarrow}{\lim}}{}^0 \; R(P) \cong \mathrm{Hom}_{\mathbf{Set}}(\cup_{\mathcal{N}(2,G)_p} P, \mathbb{C}).$$

Recall that $\mathcal{N}(2,G)_p$ is the collection of abelian p-subgroups of G. The result follows from the identity $|\cup_{\mathcal{N}(2,G)_p} P| = n_p + 1$ where n_p is the number of (non-identity) elements of order a power of p in G. □

Theorem 3.2.2. *There is an isomorphism*

$$\mathbb{Q} \otimes K^i(B(2,G)) \cong \begin{cases} \mathbb{Q} \oplus \bigoplus_{p||G|} \mathbb{Q}_p^{n_p} & \text{if } i = 0, \\ 0 & \text{if } i = 1, \end{cases}$$

where n_p is the number of (non-identity) elements of order a power of p in G.

Proof. By the decomposition in Theorem 1.2.6 we work at a fixed prime p dividing the order of G. Then $K^*(B(2,G)_p)$ is isomorphic to the completion of $K_G^*(\operatorname{hocolim}_{\mathcal{N}(2,G)_p} G/P)$ at the augmentation ideal $I(G)$ by the Atiyah-Segal completion theorem. Since $R(G)$ is a Noetherian ring the completion $-\otimes_{R(G)} R(G)^\wedge$ is exact on finitely generated $R(G)$-modules, hence commutes with taking homology. Moreover, (3.3) gives

$$R(P) \otimes_{R(G)} R(G)^\wedge \cong R(P)^\wedge \cong \mathbb{Z} \oplus I(P) \otimes \mathbb{Z}_p.$$

This isomorphism induces

$$\tilde{E}_r^{s,t} \otimes_{I(G)} I(G)^\wedge \cong \tilde{E}_r^{s,t} \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \text{for } r \geq 2,$$

which gives an isomorphism between the abutments. For t even and $s > 0$, $\tilde{E}_2^{s,t}$ are torsion by Lemma 3.2.1. After tensoring with $\mathbb{Q}$, the spectral sequence collapses onto the vertical axis at the E_2-page and $\mathbb{Q} \otimes \tilde{K}^0(B(2,G)_p) \cong \mathbb{Q} \otimes \mathbb{Z}_p^{n_p}$ whereas $\mathbb{Q} \otimes K^1(B(2,G)_p)$ vanishes. □

Chapter 4

Examples

4.1 Homotopy type of $B(2, G)$

Extraspecial p–groups S_p^r are key examples we considered in the previous sections, and they turn out to be useful in determining torsion in the kernel of $\psi : G(2) \to G$ for other classes of groups. We start with a discussion of some properties of $B(2, G)$ for the extraspecial p–groups of rank 4, and then we discuss torsion in $T(2)$ for symmetric groups and general linear groups.

We want to determine the universal cover $\widetilde{B(2, S_p^2)}$ of $B(2, S_p^2)$, which is homotopy equivalent to the universal cover Y of $E(2, S_p^2)$. Recall that there is an extension $\mathbb{Z}/p \lhd S_p^2 \to V = (\mathbb{Z}/p)^4$. As we know from the description of the poset $\mathcal{I}(\mathfrak{b})$ of isotropic subspaces of V the maximal ones has dimension 2. Cohomology groups of the universal cover Y can be calculated from the chain complex

$$\prod_{W_0 \leftarrow W_1 \leftarrow W_2} \mathbb{Z}[V(\mathfrak{b})/W_2] \to \prod_{W_0 \leftarrow W_1} \mathbb{Z}[V(\mathfrak{b})/W_1] \to \prod_{W_0} \mathbb{Z}[V(\mathfrak{b})/W_0] \qquad (4.1)$$

where $W_i \in \mathcal{I}(\mathfrak{b})$ are of dimension $0 \le i \le 2$. The space Y is a simply connected 2–dimensional complex, in particular its cohomology is torsion free. Therefore there is an isomorphism

$$\pi_i(B(2,S^2_p)) \cong \pi_i(\bigvee^d S^2) \ \text{ for } i>1 \text{ for some } d \geq 0.$$

The dimension d can be computed from the Euler characteristic, for example in the case of $p=2$ we have

$$\begin{aligned}
\chi(\widetilde{B(2,S^2_2)}) &= \frac{64}{2}|\{0 \subset W_1 \subset W_2\}| - \frac{64}{2|W_1|}|\{W_1 \subset W_2\}| + \frac{64}{2|W_2|}|\{W_2\}| \\
&= \frac{64}{2}45 - \left(\frac{64}{4}45 + \frac{64}{2}30\right) + \left(\frac{64}{8}15 + \frac{64}{4}15 + \frac{64}{2}\right) \\
&= 152,
\end{aligned}$$

therefore $d = 151$. In 4.1 the group $V(\mathfrak{b}) \cong (\mathbb{Z}/2)^5$ has order $64/2$ (§2.3.1), and the number of flags can be counted inductively. See [26] for a counting of the number of flags in the symplectic space $2\mathbb{H} = ((\mathbb{Z}/2)^4, \mathfrak{b})$.

Next we give some class of groups which satisfy the conclusion of Theorem 2.4.5, and therefore contain torsion in the kernel of the natural map $\psi : G(2) \to G$. We refer the reader to the relevant sections in [2] for the group theoretic facts used below.

1. The general linear group $\mathrm{GL}_n(\mathbb{F}_q)$, $n \geq 4$, over the finite field $\mathbb{F}_q$ of characteristic p has a symplectic sequence given by the elementary matrices

   $$\{g_1 = E_{12}, g_2 = E_{13}, g_3 = E_{2n}, g_4 = E_{3n}\}$$

   where E_{ij} has 1's on the diagonal and in the (i,j)-slot. It follows easily from the commutation relations of elementary matrices that this sequnce satisfies the required commutation relations of a symplectic sequence. The commutator $E_{1n} = [g_1, g_3]$ has order q, hence by Theorem 2.4.5 the kernel of $\mathrm{GL}_n(\mathbb{F}_q)(2) \to \mathrm{GL}_n(\mathbb{F}_q)$ has a torsion element of order p.

2. Next we consider the n–fold wreath product $W_p(n) = \underbrace{\mathbb{Z}/p \wr \cdots \wr \mathbb{Z}/p}_{n}$ for $n \geq 3$. Regarding linear transformations of an n dimensional vector space V over $\mathbb{F}_p$ as the permutations of a set of cardinality p^n gives an inclusion

$$\iota : \mathrm{GL}_n(p) \to \Sigma_{p^n}.$$

Consider the case of $n = 4$, and the images of the elements in $\{g_i\}_{i=1}^4$ under this embedding. The elements in V fixed by all g_i for $1 \leq i \leq 4$ is given by the subspace $\{(x, 0, 0, 0)|\ x \in \mathbb{F}_p\}$ whose cardinality is p. Hence the images land inside the symmetric group $\Sigma_{p^4-p} \subset \Sigma_{p^4}$ under the inclusion ι. The p–Sylow subgroup $S = \prod^{p-1}(W_p(3) \times W_p(2) \times W_p(1))$ of Σ_{p^4-p} contains the image of the symplectic sequence. Therefore there is a p torsion in the kernel of $\psi_S : S(2) \to S$. Note that the isomorphism $(G\times H)(2) \cong G(2)\times H(2)$ for the product of any given two groups G and H induces an isomorphism $\ker \psi_{G\times H} \cong \ker \psi_G \times \ker \psi_H$. Since S is a product of wreath products, the kernel $\ker \psi_{W_p(3)}$ which corresponds to the largest wreath product contains a p torsion. This torsion element comes from the symplectic sequence $\{\pi(g_i)\}_{i=1}^4$ where $\pi : S \to W_p(3)$ is the natural projection.

3. Sylow p–subgroups of the symmetric group Σ_n on n letters is described by the p–adic expansion of n. For example $\mathrm{Syl}_p(\Sigma_{p^k})$ is the k–fold wreath product $\wr^k \mathbb{Z}/p$. The previous example implies that Σ_n with $n = \sum_{i=1}^m \alpha_i p^i$ where $0 \leq \alpha_i \leq p-1$ contains a p torsion in $\ker \psi_{\Sigma_n}$ when $m \geq 3$. This is a consequence of the inclusion $\wr^m \mathbb{Z}/p \subset \Sigma_n$.

   One can say the same thing for the alternating group A_n. For an odd prime p both have the same Sylow subgroups $\mathrm{Syl}_p(A_n) = \mathrm{Syl}_p(\Sigma_n)$, and the case $p = 2$ can be settled by the isomorphism $\mathrm{GL}_4(\mathbb{F}_2) \cong A_8$.

4.2 K–theory

We consider the class of groups called TC–groups, these groups are studied in [1]. A group G is called a *transitively commutative group (TC-group)* if commutation is a transitive relation for non-central elements that is $[x,y] = 1 = [y,z]$ implies $[x,z] = 1$ for all $x,y,z \in G - Z(G)$. Any pair of maximal abelian subgroups of G intersect at the center $Z(G)$.

We want to discuss the complex K–theory of $B(2,G)$ for this class of groups. Let $M_1, \cdots, M_k$ denote the set of maximal abelian subgroups of G. Higher limits of the representation ring functor $R : \mathcal{N}(2,G) \to \mathbf{Ab}$ fit into an exact sequence

$$0 \to \underset{\leftarrow}{\lim}^0 R \to R(Z(G)) \oplus \bigoplus_{i=1}^{k} R(M_i) \xrightarrow{\theta} \bigoplus_{i=1}^{k} R(Z(G)) \to \underset{\leftarrow}{\lim}^1 R \to 0$$

where $\theta(z, m_1, ..., m_k) = (z - \mathrm{res}_{M_1,Z(G)} m_1, ..., z - \mathrm{res}_{M_k,Z(G)} m_k)$. The map θ is surjective since the restriction maps $R(M_i) \to R(Z(G))$ are surjective. Therefore no higher limit occurs, i.e. $\underset{\leftarrow}{\lim}^1 R = 0$. The same conclusion holds for the restriction of the functor R to $\mathcal{N}(2,G)_p \subset \mathcal{N}(2,G)$. Therefore the spectral sequence in (3.4) collapses at the E_2-page and after completion

$$K^i(B(2,G)) \cong \begin{cases} \mathbb{Z} \oplus \bigoplus_{p||G|} \mathbb{Z}_p^{n_p} & \text{if } i = 0, \\ 0 & \text{if } i = 1. \end{cases}$$

For extraspecial p–groups of rank $2r \geq 4$, non-vanishing higher limits of the representation ring functor occur in the Bousfield-Kan spectral sequence. In the case of S_2^2 a calculation in GAP [15] shows that

$$\underset{\substack{\leftarrow \\ \mathcal{N}(2,S_2^2)}}{\lim}^i R \cong \begin{cases} \mathbb{Z}^{32} & i = 0, \\ (\mathbb{Z}/2)^9 & i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the spectral sequence in (3.4) collapses at the E_2-page and after completion

$$K^i(B(2, S_2^2)) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2^{31} & \text{if } i = 0, \\ (\mathbb{Z}/2)^9 & \text{if } i = 1. \end{cases}$$

At the prime $p = 3$ on the other hand one has

$$K^i(B(2, S_3^2)) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_3^{242} & \text{if } i = 0, \\ (\mathbb{Z}/3)^{177} & \text{if } i = 1. \end{cases}$$

Note that the K-theory computation also confirms that $B(S_p^2, 2)$, for $p = 2$ and 3, is not homotopy equivalent to $B(\widehat{S_p^2})$, as the latter has $K^1(B(\widehat{S_p^2})) = 0$ as a consequence of the Atiyah-Segal completion theorem.

Bibliography

[1] A. ADEM, F.R. COHEN, E. TORRES-GIESE: *Commuting elements, simplicial spaces and filtrations of classifying spaces*, Math. Proc. Cam. Phil. Soc. Vol. 152 (2011), 91-114.

[2] A. ADEM, R. J. MILGRAM: *Cohomology of finite groups*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 309. Springer-Verlag, Berlin, 2004.

[3] M. ASCHBACHER: *Finite group theory*, Cambridge Studies in Advanced Mathematics, 10. Cambridge University Press, Cambridge, 1986.

[4] M. F. ATIYAH, G.B. SEGAL: *Equivariant K-Theory and Completion*, J. Differential Geom. Volume 3, Number 1-2 (1969), 1-18.

[5] D.J. BENSON, S.D. SMITH: *Classifying spaces of sporadic groups*, Mathematical Surveys and Monographs, 147. American Mathematical Society, Providence, RI, 2008.

[6] A.K. BOUSFIELD, D.M. KAN: *Homotopy Limits, Completions and Localizations*, Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972.

[7] K. BROWN: *The coset poset and probabilistic zeta function of a finite group*, J. Algebra 225 (2000), no. 2, 989-1012.

[8] K. Brown: *Cohomology of groups*, Graduate Texts in Mathematics, 87. Springer-Verlag, New York-Berlin, 1982.

[9] J.D. Dixon, M.P.F. Du Sautoy, A. Mann, D. Segal: *Analytic Pro-p Groups*, Cambridge Studies in Advanced Mathematics, 61. Cambridge University Press, Cambridge, 1999.

[10] A. W. M. Dress: *Contributions to the theory of induced representations*, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 183-240. Lecture Notes in Math., Vol. 342, Springer, Berlin, 1973.

[11] W. G. Dwyer: *Classifying spaces and homology decompositions*, Topology 36 (1997), no. 4, 783-804.

[12] R. Elman, N. Karpenko, A. Merkurjev: *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications, 56. American Mathematical Society, Providence, RI, 2008.

[13] E. D. Farjoun: *Fundamental group of homotopy colimits*, Adv. Math. 182 (2004), no. 1, 127.

[14] E. D. Farjoun: *Cellular spaces, Nullspaces and Homotopy localization*, Springer LNM 1622 (1996)

[15] The GAP Group: *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008, (`http://www.gap-system.org`).

[16] P. G. Goerss, J. F. Jardine: *Simplicial homotopy theory*, Progress in Mathematics, 174. Birkhuser Verlag, Basel, 1999.

[17] J. Grodal: *Higher limits via subgroup complexes*, Ann. of Math. (2) 155 (2002), no. 2, 405-457.

[18] P.S. Hirschhorn: *Model categories and their localizations*, Mathematical Surveys and Monographs, 99. American Mathematical Society, Providence, RI, 2003.

[19] S. Jackowski, J. McClure: *Homotopy decomposition of classifying spaces via elementary abelian subgroups.*, Topology 31 (1992), no. 1, 113132.

[20] C. N. Lee: *A homotopy decomposition for the classifying space of virtually torsion-free groups and applications.*, Math. Proc. Cambridge Philos. Soc. 120 (1996), no. 4, 663-686.

[21] W. Lück: *Rational computations of the topological K-theory of classifying spaces of discrete groups* (English summary), J. Reine Angew. Math. 611 (2007), 163-187.

[22] N. Nikolov, D. Segal: *Generators and commutators in finite groups; abstract quotients of compact groups.* (English summary), Invent. Math. 190 (2012), no. 3, 513-602.

[23] B. Oliver: *Higher Limits Via Steinberg Representations*, Comm. Algebra 22 (1994), no. 4, 138-1393.

[24] G. Segal: *Equivariant K-Theory*, Inst. Hautes tudes Sci. Publ. Math. No. 34 1968 129-151.

[25] J. P. Serre: *Trees*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[26] S. D. Smith: *Subgroup complexes* Mathematical Surveys and Monographs, 179. American Mathematical Society, Providence, RI, 2011.

[27] P. Webb: *An introduction to the representations and cohomology of categories*, Group representation theory, 149-173, EPFL Press, Lausanne, 2007.

[28] V. Welker, G. Ziegler, R. T. Živaljević: *Homotopy colimits—comparison lemmas for combinatorial applications*, J. Reine Angew. Math. 509 (1999), 117-149.

Appendix

We review some definitions about simplicial sets especially homotopy colimits. We refer the reader to [16] and [11].

The category of simplicial sets $\mathbf{S}$ and the category of topological spaces $\mathbf{Top}$ are closely related. Given T a topological space there is a functor $\mathbf{Top} \to \mathbf{S}$ called the *singular set* defined by

$$\mathrm{Sing}(T): \quad \mathbf{n} \mapsto \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, T)$$

where $|\Delta^n|$ is the topological standard n-simplex. Conversely a simplicial set X gives rise to a topological space via the *geometric realization* functor given by the difference cokernel

$$|X| = \lim_{\rightarrow} \left\{ \coprod_{\theta:\mathbf{m}\to\mathbf{n}} X_n \times |\Delta^m| \rightrightarrows \coprod_{\mathbf{n}} X_n \times |\Delta^n| \right\}$$

where $(x,p) \mapsto (x, \theta_*(p))$ and $(x,p) \mapsto (\theta^*(x), p)$, see [28]. There is an adjoint relation between these two functors

$$|.| : \mathbf{S} \rightleftarrows \mathbf{Top} : \mathrm{Sing}.$$

Note that $|.|$ preserves all colimits in $\mathbf{S}$ since it has a right adjoint.

Let $\mathbf{I}$ be a small category, and $i \in \mathbf{I}$ an object, the under category $\mathbf{I} \setminus i$ is the category whose objects are the maps $i \to i_0$ and morphisms from $i \to i_0$ to $i \to i_1$ are commutative diagrams

$$\begin{tikzcd} i \arrow[r] \arrow[dr] & i_0 \arrow[d] \\ & i_1. \end{tikzcd}$$

Let $F : \mathbf{I} \to \mathbf{Top}$ be a functor. The *homotopy colimit* of F is the topological space described as a difference cokernel

$$\text{hocolim } F = \lim_{\rightarrow}\left\{\coprod_{f:j\to i} B(\mathbf{I}\setminus i)\times F(j) \rightrightarrows \coprod_{i} B(\mathbf{I}\setminus i)\times F(i)\right\}$$

where $(b,y)\mapsto (B(f)(b),y)$ and $(b,y)\mapsto (b,f(y))$, see [28]. If one wants to work in the category of simplicial sets, given $F:\mathbf{I}\to\mathbf{S}$ one can use the simplicial set $B_\bullet(\mathbf{I}\setminus i)$ instead.

From the definition we see that there is a natural map to the ordinary colimit

$$\text{hocolim } F \to \text{colim } F$$

obtained by collapsing the nerves $B(\mathbf{I}\setminus i)\to \text{pt}$. This map is not usually a weak equivalence. In general, it is a weak equivalence if the diagram $F:\mathbf{I}\to\mathbf{S}$ is a *free* diagram ([14, Appendix HC]). A diagram of sets $\mathbf{I}\to\mathbf{Set}$ is free if it is of the form $\coprod_i F^i$ for a collection of subobjects of $\mathbf{I}$ where each diagram $F^i:\mathbf{I}\to\mathbf{Set}$ is defined by $F^i(j)=\mathrm{Hom}_{\mathbf{I}}(i,j)$. A diagram of simplicial sets is free if in each dimension it gives a free diagram of sets. More precisely, a diagram $F:\mathbf{I}\to\mathbf{S}$ is free if there exists a sequence $\{S^0,S^1,\cdots\}$ of diagrams $S^n:\mathbf{I}^\delta\to\mathbf{Set}$ such that $S^n(i)\subset F(i)_n$, $s^i(S^n(i))\subset S^{n+1}(i)$ for each degeneracy map s^i, and for $\sigma\in F(i)_n$ there exists a morphism $\gamma:j\to i$ in $\mathbf{I}$ and $\tau\in S^n(j)$ such that $F(\gamma)(\tau)=\sigma$ ([18, 14.8.4]). The morphism γ and the n–simplex τ are supposed to be unique. Here $\mathbf{I}^\delta$ denotes the discrete category which consists of the same objects as $\mathbf{I}$ and only the identity as morphisms. In particular, a poset of simplicial sets which is closed under taking subsets and partially ordered by inclusion gives a free diagram of simplicial sets. Let $F:\mathbf{I}\to\mathbf{S}$ be such a diagram. In this case set $S^n(i)=F(i)_n-\cup_{\gamma:j\lneq i}F(\gamma)(F(j)_n)$. Then given $\sigma\in F(i)_n$ choose j such that $F(j)=\cap_{\sigma\in F(k)_n}F(k)$, such a j exists since we assume the diagram is closed under intersections. The unique morphism $\gamma:j\to i$ gives $F(\gamma)(\tau)=\tau$. Throughout the thesis we use diagrams indexed by a collection of groups closed under intersections, and after applying the classifying space functor this gives a diagram of simplicial sets closed under intersection. Hence those kind of diagrams are free.

A standard example of a homotopy colimit is the Borel construction. Let X be a G-space and consider $\mathbf{G}$ the category associated to the group then

$$\text{hocolim } F \simeq X \times_G EG$$

where $F : \mathbf{G} \to \mathbf{Top}$ sends the single object to X. The most important property of homotopy colimits is that a natural transformation $F \to F'$ of functors such that $F(i) \to F'(i)$ is a weak equivalence for all $i \in \mathbf{I}$ induces a weak equivalence

$$\text{hocolim } F \to \text{hocolim } F'.$$

Another useful property we use in the thesis is the commutativity of homotopy colimits. We refer the reader to [28] for further properties of homotopy colimits.