# Representation Rings of Semidirect Products of Tori by Finite Groups

by

Maxim Stykow

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# Abstract

This dissertation studies semidirect products of a torus by a finite group from the representation theory point of view. The finite group of greatest interest is the cyclic group of prime order. Such semidirect products occur in nature as isotropy groups of Lie groups acting on themselves by conjugation and as normalizers of maximal tori in reductive linear algebraic groups.

The main results of this dissertation are

- the calculation of the representation ring of such semidirect products as an algebra over the integers for certain special cases,
- the adaptation of an algorithm from invariant theory to find finite presentations of representation rings,
- the computation of the topological $K$-theory of the classifying space of certain semidirect products,
- the demonstration that the equivariant $K$-theory of the projective unitary group of degree 2 acting on itself by conjugation is not a free module over its representation ring.

# Preface

This dissertation is original, unpublished, independent work by me, Maxim Stykow, under the supervision of Dr. Alejandro Ádem. I also thank Dr. Man Chuen Cheng, Dr. Julia Gordon, Dr. Kee Lam, and Dr. Zinovy Reichstein for fruitful discussions.

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# Some Notation and Conventions

The symbol $C_n$ denotes the cyclic group of order $n$ and $\mathbb{Z}_p$ the $p$-adic integers. Given an $R$-module $M$ and an ideal $I \subset R$ $M_{\hat{I}} := \varprojlim M/I^n M$ denotes the $I$-adic completion of $M$.

The class number of the ring of algebraic integers in the $n$th cyclotomic field will be referred to simply as the class number of $n$.

If $S$ is a set, curly brackets $\{S\}$ indicate the ring or algebra generated by $S$ while pointy brackets $\langle S \rangle$ indicate the group generated by $S$.

When occurring between representations, dot $\cdot$ denotes tensor product and plus $+$ denotes direct sum.

The symbol $S^n$ denotes a sphere of dimension $n$. The following kinds of equivalences occur:

$\cong$ is an isomorphism of algebraic structures (such as groups, rings, representations) or Lie groups

$\approx$ is a homeomorphism of topological spaces

$\equiv$ is a congruence of integers modulo another fixed integer

A square $\square$ denotes the end of a proof. A triangle $\triangle$ denotes the end of an example.

# Dedication

*To Yogi Ramsuratkumar*

# Introduction

Consider a split extension of groups

$$1 \to T \to \Gamma \to G \to 1$$

or equivalently the semidirect product $\Gamma \cong T \rtimes G$ with $T = S^1 \times \cdots \times S^1$ a torus of rank $n$ and $G$ a finite group. Such semidirect products occur in nature as isotropy groups of Lie groups acting on themselves by conjugation, cf. chapter 4, and as normalizers of maximal tori in reductive linear algebraic groups; cf. [LMMR13].

The topic of this dissertation is the calculation of the representation ring of $\Gamma$ and the topological $K$-theory of the classifying space of $\Gamma$.

In the first chapter, algebraic tools necessary to study the representation theory of semidirect products and the $K$-theory of their classifying space are introduced. Key to our study is the action of $G$ on the character group $\widehat{T} \cong \mathbb{Z}^n$, a $\mathbb{Z}$-free $\mathbb{Z}G$-module known as a $\mathbb{Z}G$-lattice.

The second chapter studies in detail the case when $G = C_p$, the cyclic group of prime order $p$. After finding a presentation of the representation ring $R(\Gamma)$ as a $\mathbb{Z}$-algebra, we establish a link to the multiplicative invariant algebra of $G$ which enables us to find a finite presentation of $R(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}(\zeta)$ where $\zeta$ is a primitive $p$th root of unity. Some restrictions apply when the class number of $\mathbb{Z}[\zeta]$ is not trivial. The reason we begin with $C_p$ is that a) $T \rtimes C_p$ was the first such semidirect product encountered as an isotropy group and b) $\mathbb{Z}C_p$-lattices are well understood. We go on to discuss the cases of

$G = D_{2p}$, the dihedral group of order $2p$, and $G = S_n$, the symmetric group on $n$ letters. The main tools used to describe the various representation rings are Mackey's method of "little groups", the theory of $\mathbb{Z}G$-lattices, and several theorems of Reiner classifying $\mathbb{Z}G$-lattices.

Let $T_M$ denote a torus with character $\mathbb{Z}G$-lattice $M$ and $M^G$ the sublattice of $G$-invariants in $M$. The main result of the second chapter is

**Theorem 2.1.7.** *Let $G = C_p = \langle \sigma \mid \sigma^p = 1 \rangle$, where $p > 1$ is any prime, and let $\mathbb{Z}G$ and $IG$ be the group ring respectively augmentation ideal of $G$. If $M$ is the $\mathbb{Z}G$-lattice $\mathbb{Z}G^s \oplus IG^t$ and $\Gamma_M = T_M \rtimes G$ is the associated semidirect product, then*

$$R(\Gamma_M) \cong \mathbb{Z}\left\{\mathbf{x}^m : m \in M^G\right\} \otimes \mathbb{Z}\left\{\operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^m : [m] \in M/G\right\} \otimes \mathbb{Z}[v]/(v^p - 1)$$

*subject to the relations*

*(i)* $\operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^m = \mathbf{x}^m \cdot \Phi_p$ *for* $m \in M^G$ *and* $\Phi_p = 1 + v + \cdots + v^{p-1}$,

*(ii)* $\operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^m \cdot \operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^{m'} = \sum_{i=0}^{p-1} \operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^{m+\sigma^i m'}$,

*(iii)* $\mathbf{x}^m \cdot \mathbf{x}^{m'} = \mathbf{x}^{m+m'}$,

*(iv)* $\mathbf{x}^m \cdot v^j \cdot \operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^{m'} = \operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^{m+m'}$.

In the third chapter, we apply our results to the $K$-theory of the classifying space $B\Gamma$ of $\Gamma$ by using the Atiyah-Segal completion theorem. By passing to a subset of $T$ known as a $p$-discrete torus it is possible to circumvent the number theoretic issues encountered in chapter 2 and present a complete answer for the mod $p$ $K$-theory of $B\Gamma$ when $G = C_p$:

**Theorem 3.3.1.** *Let $G = C_p$, where $p > 1$ is any prime. If $T$ is a torus of rank $n$ on which $G$ acts by automorphisms, and $\Gamma = T \rtimes G$ is the associated semidirect product, then*

$$K^*(B\Gamma; C_p) \cong K^*(B\Gamma_M; C_p)$$

*where* $\Gamma_M = T_M \rtimes G$ *with* $M = \mathbb{Z}^r \oplus \mathbb{Z}G^s \oplus IG^t$*, for some integers* $r, s, t$ *obeying* $r + sp + t(p-1) = n$*.*

The fourth chapter discusses the equivariant $K$-theory $K_G^*(G)$ of a compact connected Lie group $G$ acting on itself by conjugation and shows that $K_G^*(G)$ is not a free module over the representation ring of $G$ when $G$ is the projective unitary group $\mathrm{PU}(2)$ of degree 2. It is then discussed how to extend this result to $\mathrm{PU}(3)$.

In the last chapter we mention natural ways by which the results of this work could be extended and outline the potentials and difficulties to do so.

An interesting parallel to our research has been pursued by Adem et al. who computed the cohomology of semidirect products known as crystallographic groups[1]; cf. [AP06] and [AGPP08]. In this case the torus is replaced by its character lattice $\widehat{T} \cong \mathbb{Z}^n$ to form $\mathbb{Z}^n \rtimes C_p$. The $K$-theory of the classifying space of $\mathbb{Z}^n \rtimes C_p$ was determined based on these computations; cf. [DL13]. In our work the situation is reversed: the $K$-theory of the classifying space of $T \rtimes C_p$ is computed while its cohomology remains unknown.

[1] Cf. [Cha86, p. 74] for a definition.

# Chapter 1

# Background and Preliminary Results

## 1.1 Representations of semidirect products

In this section we assume all groups to be locally compact and Hausdorff.

Let $G$ be a subgroup of a group $\Gamma$ and $A$ a normal subgroup of $\Gamma$ such that $A \cap G = \{1\}$ and $\Gamma = AG$. Then $\Gamma$ is known as the semidirect product of $A$ by $G$, denoted $A \rtimes G$. Conjugation of elements of $A$ by elements of $G$ yields an action of $G$ on $A$. Conversely, given any two groups $A$ and $G$ and a group homomorphism $\psi : G \to \mathrm{Aut}(A)$, we can construct a new group $A \rtimes_{\psi} G$ with underlying set $A \times G$ and multiplication given by

$$(a_1, h_1) \cdot (a_2, h_2) := (a_1\psi_{h_1}(a_2), h_1h_2). \qquad (1.1.0.1)$$

Then $A \rtimes_{\psi} G$ is the semidirect product of subgroups isomorphic to $A$ and $G$ in the sense just established. This dissertation is concerned with computing representation rings of semidirect products $\Gamma = A \rtimes G$ when $A$ is abelian.

**Definition 1.1.1.** Given a subgroup $H$ of a group $G$ and a continuous $H$-representation $V$ we define the induced representation $\mathrm{ind}_H^G V$ of $G$ to be the vector space $C^0_H(G,V)$ of all continuous $H$-maps from $G$ to $V$. A continuous $G$-action is given by $g \cdot f(x) = f(g^{-1}x)$; cf. [BtD85, III.6].

Note that since an $H$-map is determined by its values on a system of representatives of $G/H$,

$$\dim C^0_H(G,V) = |G/H| \cdot \dim V. \tag{1.1.1.1}$$

Note that $|G/H|$ need not be finite. There is also a restriction map which takes a $G$-representation $U$ and restricts the action to a subgroup $H$ of $G$, denoted by $\mathrm{res}^G_H U$. The following will be a useful formula when $H$ is a normal subgroup of finite index in $G$; cf., e.g., [BtD85, VI.7.4]:

$$\mathrm{res}^G_H \mathrm{ind}^G_H V \cong \bigoplus_{[g]\in G/H} V^g \tag{1.1.1.2}$$

where the $H$-representation $V^g$ is defined to be $V$ as a vector space with the $H$-action

$$h \cdot v = ghg^{-1} \cdot v.$$

Often all irreducible representations of $\Gamma = A \rtimes G$ can be obtained by inducing representations of the form $\chi \otimes \rho$ where $\chi$ is a character of $A$ and $\rho$ an irreducible representation of a subgroup of $G$. For example:

**Proposition 1.1.2** ([Seg68b, 3.12])**.** *Let $A \to \Gamma \to G$ be an extension of a nilpotent group $G$ by a topologically cyclic*[1] *group $A$. Then any irreducible representation of $\Gamma$ is monomial, i.e., induced from a one-dimensional representation of a subgroup of finite index.*

This scenario applies when $\Gamma$ is the semidirect product of a torus of rank $n$ by a cyclic group of prime order $p$.

Specifically which representations to induce was worked out by G. W. Mackey. The original idea goes back to physics Nobel prize winner E.

[1]Cf. 2.1.4.

Wigner's method of "little groups" by which he computed representations of the Poincaré group, the group of isometries of Minkowski spacetime; cf. [Wig39]. The Poincaré group can be seen as a group extension of the Lorentz group—a noncompact, nonabelian Lie group which is not connected.

Mackey extended Wigner's ideas to locally compact Hausdorff groups in general. The theorem below is an application of his imprimitivity theorem for infinite groups. Following [Mac52] we briefly describe the method:

Recall that $A$ is abelian and so its irreducible representations are one-dimensional. They are the elements of the character or Pontryagin dual group $\widehat{A} := \mathrm{Hom}(A, \mathbb{C}^*)$. The action of $G$ on $A$ induces an action on $\widehat{A}$.

**Definition 1.1.3** ([Sun96])**.** Recall that a Borel set is any set in a topological space that can be formed from open or equivalently closed sets through the operations of countable union, countable intersection, and relative complement. Say that a semidirect product $\Gamma = A \rtimes G$ is regular if the character group $\widehat{A}$ is a measure space containing a Borel set that meets almost[2] every $G$-orbit in exactly one point. Note that if $\widehat{A}$ is countable this condition is automatically satisfied.

Let $(\chi)_{[\chi]\in\hat{A}/G}$ be a system of representatives for the orbits of $G$ in $\widehat{A}$. For each $[\chi] \in \widehat{A}/G$, let $G_\chi$ be the isotropy group of $\chi$, cf. A.0.2.1, and let $\Gamma_\chi = A \rtimes G_\chi \subset \Gamma$. Also let $\rho$ be an irreducible representation of $G_\chi$ and let

$$\pi_1 : \Gamma_\chi \to A \qquad \pi_2 : \Gamma_\chi \to G_\chi$$

be the canonical projection maps. Extend $\chi$ to $\Gamma_\chi$ via $\pi_1$ and extend $\rho$ to $\Gamma_\chi$ via $\pi_2$. Finally, let $\theta_{\chi,\rho} = \mathrm{ind}_{\Gamma_\chi}^{\Gamma}(\chi \otimes \rho)$ be the induced representation of $\Gamma$.

**Theorem 1.1.4** (Method of "little groups")**.** *Let $\Gamma$ be the regular semidirect product of a normal abelian subgroup $A$ by a subgroup $G$ and let $V$ be any irreducible continuous representation of $\Gamma$. Then with the above notation*

[2]'Almost' here in the sense of measure theory, i.e., the measure of the subset of orbits not met is zero.

*(i)* $\theta_{\chi,\rho}$ *is irreducible,*

*(ii)* $\theta_{\chi,\rho} \cong \theta_{\chi',\rho'}$ *implies* $[\chi] = [\chi']$ *and* $\rho \cong \rho$*,*

*(iii)* $V$ *is isomorphic to* $\theta_{\chi,\rho}$ *for some* $\chi$ *and some* $\rho$*.*

This determines $R(\Gamma)$ as a free abelian group.

**Example 1.1.5.** Let the circle $S^1$ act on the additive group of complex numbers $\mathbb{C}$ by multiplication of complex numbers. Then $\Gamma = \mathbb{C} \rtimes S^1$ is the group of all Euclidean motions of the plane. The character group $\widehat{\mathbb{C}} = \mathrm{Hom}(\mathbb{C}, \mathbb{C}^*)$ is again the additive group of complex numbers. The orbits are the circles with center at the origin. Any ray from the origin is a Borel set meeting each orbit only once, making $\Gamma$ regular. Let $\{z_r\}_{r=0}^{\infty}$ be a system of representatives. If $r > 0$, the isotropy group is $S^1_{z_r} = \{1\}$ and if $r = 0$, $S^1_0 = S^1$. Applying 1.1.4, we see that in addition to the obvious one dimensional representations there is a continuum of infinite dimensional irreducible representations, one for each $r > 0$. These exhaust the irreducible representations of $\Gamma$. $\triangle$

**Example 1.1.6.** Think of the cyclic group $C_n = \langle \sigma \mid \sigma^n = 1 \rangle$ of order $n$ as the group of roots of unity on the unit circle and let $C_2 = \langle \tau \mid \tau^2 = 1 \rangle$ act on $C_n$ by complex conjugation. Then the dihedral group of order $2n$ can be defined as the semidirect product $D_{2n} := C_n \rtimes C_2$.

The character group $\widehat{C_n} = \langle v \mid v^n = 1 \rangle$ is isomorphic to $C_n$ and thus countable making the semidirect product regular. A system of representatives is given by taking roots of unity lying only on the upper semicircle with endpoints included.

If $n$ is odd, then $v^0 = 1$ has $C_2$-isotropy giving two one-dimensional representations $1, w$ of $D_{2n}$ with $w^2 = 1$. The remaining representatives $v, v^2, \ldots, v^{\frac{n-1}{2}}$ have trivial isotropy and give two-dimensional representations after induction. The representations

$$1, w, \mathrm{ind}_{C_n}^{D_{2n}} v, \ldots, \mathrm{ind}_{C_n}^{D_{2n}} v^{\frac{n-1}{2}}$$

exhaust all irreducible representations of $D_{2n}$ when $n$ is odd. Finally, when $n$ is even, $v^{n/2} = -1$ also has $C_2$-isotropy giving two more one-dimensional representations $-1, -w$ of $D_{2n}$ and the list

$$\pm 1, \pm w, \operatorname{ind}_{C_n}^{D_{2n}} v, \ldots, \operatorname{ind}_{C_n}^{D_{2n}} v^{\frac{n}{2}-1}$$

exhausts all irreducible representations.

The product structure can also be determined by techniques that will be introduced in chapter 2. For completeness, we note here that

$$w \cdot \operatorname{ind}_{C_n}^{D_{2n}} v^i \cong \operatorname{ind}_{C_n}^{D_{2n}} v^i$$

and

$$\operatorname{ind}_{C_n}^{D_{2n}} v^i \cdot \operatorname{ind}_{C_n}^{D_{2n}} v^j \cong \operatorname{ind}_{C_n}^{D_{2n}} (v^{i+j}) + \operatorname{ind}_{C_n}^{D_{2n}} (v^{i-j})$$

with $\operatorname{ind}_{C_n}^{D_{2n}}(1) \cong 1 + w$. △

## 1.2 The classification of $\mathbb{Z}G$-lattices

As we will see, semidirect products $T \rtimes G$ are determined by lattices equipped with a $G$-action. The following definitions and theorems from [CR62], [CR90], and [Lor05] make these notions precise.

Let $T = (\mathbb{R}/\mathbb{Z})^n$ be a torus of rank $n$ and let $\psi \in \operatorname{Aut}(T)$ be a smooth group automorphism of $T$. The differential $d\psi$ is an invertible linear transformation of the Lie algebra $\mathfrak{t} = \mathbb{R}^n$ of $T$ that commutes with the exponential map and must preserve the kernel $\mathbb{Z}^n$ of $\exp : \mathfrak{t} \to T$. By the naturality of the exponential map, cf., e.g., [BtD85, I.3], the assignment $\psi \mapsto d\psi|_{\mathbb{Z}^n}$ gives an isomorphism

$$\operatorname{Aut}(T) \cong \operatorname{GL}_n(\mathbb{Z}).$$

If $G$ is an arbitrary group, it follows that having a homomorphism $\phi : G \to \operatorname{Aut}(T)$ is the same as having a $\mathbb{Z}G$-lattice, that is, a $\mathbb{Z}G$-module which happens to be a finitely generated free $\mathbb{Z}$-module.

**Example 1.2.1.** For any subgroup $H \subset G$ with $[H : G] < \infty$ we may form the $\mathbb{Z}G$-lattice

$$\mathbb{Z}[G/H] = \bigoplus_{[g] \in G/H} \mathbb{Z}gH$$

where the $G$-action is given by $g(g'H) = gg'H$ for $g, g' \in G$. Thus, $G$ permutes a $\mathbb{Z}$-basis of the $\mathbb{Z}G$-lattice $\mathbb{Z}[G/H]$. Lattices of this type are called permutation lattices. △

When $G$ is finite, the group ring $\mathbb{Z}G$, viewed as a module over itself via multiplication, is a special case of a permutation lattice, the so-called regular $\mathbb{Z}G$-lattice. Any lattice isomorphic to $\mathbb{Z}G^r$ for some $r$ is called a free $\mathbb{Z}G$-lattice. Direct summands of free lattices are called projective.

The group ring gives rise to another important $\mathbb{Z}G$-lattice,

$$IG = \ker\left(\mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} : \epsilon(g) = 1, g \in G\right) \tag{1.2.1.1}$$

known as the augmentation ideal.

Say a lattice is indecomposable if it cannot be written as a direct sum of nonzero lattices. For example, the permutation lattices $\mathbb{Z}[G/H]$ are all indecomposable; cf. [CR90, 32.14]. The set of isomorphism classes of indecomposable $\mathbb{Z}G$-lattices is usually infinite. The following result describes when this set is finite; cf., e.g., [CR90, 33.6]:

**Theorem 1.2.2.** *Let $G$ be a finite group. There are only finitely many indecomposable $\mathbb{Z}G$-lattices (up to isomorphism) precisely if, for each prime $p$ dividing the order $|G|$, the Sylow $p$-subgroups of $G$ are cyclic of order $p$ or $p^2$.*

Complete sets of indecomposable $\mathbb{Z}G$-lattices are only known in rare cases. One such case is the cyclic group $C_p = \langle \sigma \mid \sigma^p = 1 \rangle$ of prime order $p$. To state the main theorem, we remind the reader of the following number-theoretic facts and definitions.

Let $\zeta$ be a primitive $p$th root of unity and recall that the ring of algebraic integers[3] in the cyclotomic field $\mathbb{Q}(\zeta)$ is $\mathbb{Z}[\zeta]$; cf. [CR62, 21.13]. Note that $\mathbb{Q}(\zeta)$ is the field of fractions of $\mathbb{Z}[\zeta]$. In fact, an algebraic number field is always the field of fractions of its ring of algebraic integers; cf. [CR62, p. 104]. The ideal class group of $\mathbb{Q}(\zeta)$ is defined as the multiplicative group of fractional ideals of $\mathbb{Z}[\zeta]$ in $\mathbb{Q}(\zeta)$ modulo the principal ideals. The class number $h_p$ of $\mathbb{Z}[\zeta]$, henceforth referred to as the class number of $p$, is the size of the ideal class group which is always finite; cf. [CR62, 20.6].

**Theorem 1.2.3** (Diederichsen, Reiner [CR62, 74.3])**.** *Let $p > 1$ be any prime and let $h_p$ be the class number of $p$. Then there are $2h_p + 1$ non-isomorphic indecomposable $\mathbb{Z}C_p$-lattices and every $\mathbb{Z}C_p$-lattice $M$ is isomorphic to a direct sum of finitely many of these indecomposables.*

A complete set of non-isomorphic indecomposable $\mathbb{Z}C_p$-lattices is exhibited through the following constructions.

Recall that $\mathbb{Z}[\zeta]$ is a Dedekind domain[4] with $\mathbb{Z}$-basis $\{1, \zeta, \zeta^2, \ldots, \zeta^{p-2}\}$. So if $A$ is a fractional ideal in $\mathbb{Q}(\zeta)$, then $A$ has rank $p-1$ and we may turn it into a $\mathbb{Z}C_p$-lattice by defining

$$\sigma \cdot a := \zeta a$$

for all $a \in A$. By definition, then, two ideals are isomorphic as $\mathbb{Z}C_p$-modules if and only if they are isomorphic as $\mathbb{Z}[\zeta]$-modules. The latter occurs if and only if they are in the same ideal class; cf. [CR62, 22.2].

Next, consider the direct sum $A \oplus \mathbb{Z}y$ of a fractional ideal $A$ and the free $\mathbb{Z}$-module $\mathbb{Z}y$. Define a $C_p$-action by

$$\sigma \cdot a := \zeta a, \qquad a \in A, \qquad \sigma \cdot y := a_0 + y$$

[3]An element $\alpha$ of some finite field extension of $\mathbb{Q}$ is an algebraic integer if its unique monic irreducible polynomial has integral coefficients.

[4]More generally, the ring of algebraic integers of *any* algebraic number field is a Dedekind domain, that is, an integral domain in which every nonzero, proper ideal factors into a product of primes; cf. [CR62, p. 108].

for some fixed $a_0 \in A$, $a_0 \notin (\zeta - 1)A$. Let $\Phi_p(x) = 1 + x + \cdots + x^{p-1}$ be the $p$th cyclotomic polynomial in an indeterminate $x$. Then

$$\begin{aligned} \sigma^2 \cdot y &= \zeta(a_0 + y) = (\zeta + 1)a_0 + y \\ &\vdots \\ \sigma^p \cdot y &= \Phi_p(\zeta)a_0 + y = y \end{aligned}$$

as required and we denote this $\mathbb{Z}C_p$-lattice of rank $p$ by $(A, a_0)$. The isomorphism class of $(A, a_0)$ is independent of the choice of $a_0$; cf. [CR62, §74].

A complete set of non-isomorphic indecomposable $\mathbb{Z}C_p$-lattices is then given by

$$\{(A_1, a_1), \ldots, (A_{h_p}, a_{h_p}), A_1, \ldots, A_{h_p}, \mathbb{Z}\}$$

and any rank $n$ $\mathbb{Z}C_p$-lattice $M$ is isomorphic to a direct sum of these, i.e.,

$$M \cong (A_1, a_1) \oplus \cdots \oplus (A_s, a_s) \oplus A_{s+1} \oplus \cdots \oplus A_{s+t} \oplus \mathbb{Z}^r$$

where the isomorphism class of $M$ is determined by the integers $r, s, t$ obeying $r + sp + t(p-1) = n$ and the ideal class of each $A_i$.

Continuing with $G$ an arbitrary finite group, it is often useful to replace a $\mathbb{Z}G$-lattice $M$ by a related but better behaved lattice. Let

$$\overline{\phantom{M}} : M \to M/M^G$$

denote the canonical map, where $M^G$ is the sublattice of $G$-invariants in $M$. Note that $\overline{M}$ is a $\mathbb{Z}G$-lattice and the map $\overline{\phantom{M}}$ is $G$-equivariant so that we have an extension of $\mathbb{Z}G$-lattices

$$0 \to M^G \to M \to \overline{M} \to 0. \qquad (1.2.3.1)$$

Note that there is an isomorphism of isotropy groups

$$G_m \cong G_{\bar{m}} \tag{1.2.3.2}$$

for all $m \in M$. This follows from the fact that 1.2.3.1 tensored with $\mathbb{Q}$ is split exact. As a consequence, $\overline{M}^G = \{0\}$. We call a lattice with trivial sublattice of $G$-invariants effective and $\overline{M}$ is called the effective quotient of $M$.

Let $\mathbb{K}$ be a commutative ring. The group algebra $\mathbb{K}[M]$ contains a copy of $M$ as a subgroup of the group of multiplicative units of $\mathbb{K}[M]$ and this copy of $M$ forms a $\mathbb{K}$-basis of $\mathbb{K}[M]$.

Working inside $\mathbb{K}[M]$, we must pass from the additive notation of $M$ to a multiplicative notation. We will thus write the basis element of $\mathbb{K}[M]$ corresponding to the lattice element $m \in M$ as $\mathbf{x}^m$, so $\mathbf{x}^0 = 1$, $\mathbf{x}^{m+m'} = \mathbf{x}^m\mathbf{x}^{m'}$, and $\mathbf{x}^{-m} = (\mathbf{x}^m)^{-1}$.

A choice of $\mathbb{Z}$-basis $\{e_i : i = 1, \ldots, n\}$ of $M$ gives rise to a $\mathbb{K}$-algebra isomorphism of $\mathbb{K}[M]$ with the Laurent polynomial algebra $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ via $\mathbf{x}^{e_i} \mapsto x_i$. Under this isomorphism,

$$M \cong \mathbb{Z}^n \cong \langle x_1, \ldots, x_n \rangle. \tag{1.2.3.3}$$

The augmentation ideal of a lattice $M$ is, also cf. 1.2.1.1,

$$IM = \ker\left(\mathbb{K}[M] \xrightarrow{\epsilon} \mathbb{K} : \epsilon(\mathbf{x}^m) = 1, m \in M\right). \tag{1.2.3.4}$$

The action of $G$ on $M$ extends uniquely to a $\mathbb{K}$-algebra action via

$$g\left(\sum_{m \in M} k_m \mathbf{x}^m\right) = \sum_{m \in M} k_m \mathbf{x}^{gm}$$

and the ring of invariants $\mathbb{K}[M]^G$ is known as the multiplicative invariant algebra of $G$. In contrast, additive or algebraic invariant theory deals with the algebra of polynomial invariants $\mathrm{Sym}(V^*)^G \cong \mathbb{K}[x_1, \ldots, x_n]^G$ where $V$ is a free $\mathbb{K}$-module and $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ its dual module with basis $\{x_i : i = 1, \ldots, n\}$ and action extended to the symmetric algebra $\mathrm{Sym}(V^*)$. Note that every $\mathbb{Z}G$-lattice $M$ may be regarded as a free $\mathbb{Z}$-module to form $\mathrm{Sym}(M^*)$.

Since $G$ permutes the $\mathbb{K}$-basis $M$ of $\mathbb{K}[M]$, the invariant algebra $\mathbb{K}[M]^G$ has a $\mathbb{K}$-basis consisting of finite $G$-orbit sums

$$(\mathbf{x}^m)^* := \sum_{m' \in G(m)} \mathbf{x}^{m'} \tag{1.2.3.5}$$

where $G(m) = \{hm : h \in G\}$ is the $G$-orbit of $m$; cf. A.0.2.2. Thus,

$$\mathbb{K}[M]^G \cong \bigoplus_{m \in S} \mathbb{K}(\mathbf{x}^m)^* \tag{1.2.3.6}$$

where $S$ is a system of representatives of finite $G$-orbits and $\mathbb{K}(\mathbf{x}^m)^*$ is the free $\mathbb{K}$-algebra generated by $(\mathbf{x}^m)^*$. Extend $(\ \ )^*$ linearly to a map $\mathbb{K}[M] \to \mathbb{K}[M]^G$ and note that $(fg)^* = fg^*$ for all $f \in \mathbb{K}[M]^G$ and $g \in \mathbb{K}[M]$.

# Chapter 2

# Representation Rings of Semidirect Products

For the first two sections of this chapter, let $G = C_p = \langle \sigma \mid \sigma^p = 1 \rangle$, let $T = (S^1)^n$ be a torus of rank $n$, and consider the compact Lie group $\Gamma = T \rtimes_\phi G$ where $\phi : G \to \mathrm{Aut}(T)$ is an injective homomorphism; cf. 1.1. The purpose of this chapter is to compute the representation ring $R(\Gamma)$ for some special cases of $\phi$ and then the general case when $p$ is a prime of trivial class number. The third section will examine the situation when $G$ is the dihedral group $D_{2p}$ and the symmetric group $S_n$.

## 2.1 From representations to invariant theory

As explained in 1.1, given $T$ and $G$ the semidirect product of $T$ by $G$ is determined by the $G$-module $\phi$. By 1.2, such $G$-modules are $\mathbb{Z}G$-lattices, i.e., $G \subset \mathrm{GL}_n(\mathbb{Z})$ with $n = \operatorname{rank} T$. Isomorphic $\mathbb{Z}G$-lattices give isomorphic semidirect products. That is, if two $\mathbb{Z}G$-lattices $\phi$ and $\psi$ are intertwined by a matrix $U \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$\phi(g) = U^{-1}\psi(g)U \qquad \text{for all } g \in G$$

then

$$\begin{aligned} T \rtimes_\phi G &\to T \rtimes_\psi G \\ (x, g) &\to (UxU^{-1}, g) \end{aligned}$$

is an isomorphism.

Section 1.2 discussed the indecomposable $\mathbb{Z}G$-lattices. In particular, when the ideal class group is trivial there are only three non-isomorphic indecomposable $\mathbb{Z}G$-lattices. They are the trivial lattice $\mathbb{Z}$, the group ring lattice $\mathbb{Z}G$ of rank $p$, and the rank $p-1$ augmentation ideal $IG \cong \mathbb{Z}[\zeta]$; cf. 1.2.1.1.

Alternatively, we can identify $\mathbb{Z}$ with $\mathbb{Z}G^G$ and $IG$ with the effective quotient $\overline{\mathbb{Z}G}$ in the exact sequence, cf. 1.2.3.1,

$$0 \to \mathbb{Z}G^G \to \mathbb{Z}G \to \overline{\mathbb{Z}G} \to 0. \tag{2.1.0.1}$$

This is so because there is a ring homomorphism $\mathbb{Z}G \to \mathbb{Z}[\zeta]$ given by

$$\sum a_i \sigma^i \mapsto \sum a_i \zeta^i$$

with kernel generated by $\Phi_p(\sigma) = 1 + \sigma + \sigma^2 + \cdots + \sigma^{p-1}$. Indeed, $\Phi_p(\zeta) = 0$ and if $f(\sigma) \in \mathbb{Z}G$ with $f(\zeta) = 0$, then $f(x) = \Phi_p(x) \cdot g(x)$ for some $g(x) \in \mathbb{Z}[x]$. So $f(\sigma) = 0 \mod (\Phi_p(\sigma))$.

In matrix notation, the nontrivial lattices are given by

$$IG : \sigma \mapsto \begin{pmatrix} & & & & -1 \\ 1 & & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \vdots \\ & & & 1 & -1 \end{pmatrix} \in \mathrm{GL}_{p-1}(\mathbb{Z})$$

and

$$\mathbb{Z}G : \sigma \mapsto \begin{pmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix} \in \mathrm{GL}_p(\mathbb{Z}).$$

**Definition 2.1.1.** Let $T_M$ denote a torus with character $\mathbb{Z}G$-lattice $M$ and let $\Gamma_M := T_M \rtimes G$ be the associated semidirect product.

The representation rings $R(\Gamma_{\mathbb{Z}^r})$, $R(\Gamma_{\mathbb{Z}G})$, and $R(\Gamma_{IG})$ are important special cases which will be computed before moving on to the general torus $T$ made up of copies of $T_{\mathbb{Z}}$, $T_{\mathbb{Z}G}$, and $T_{IG}$ when $p$ is a prime of trivial class number.

**Proposition 2.1.2** ([BtD85, II.8.3])**.** *If $H$ is a locally compact abelian group and $\widehat{H}$ its character group, then there is a canoncial isomorphism of rings*

$$R(H) \cong \mathbb{Z}[\widehat{H}]$$

*between the representation ring of $H$ and the integral group ring on $\widehat{H}$. In particular,*

$$R(T_M) \cong \mathbb{Z}[M],$$

*the group algebra on the $\mathbb{Z}G$-lattice $M$; cf. 1.2.*

**Example 2.1.3.** By [BtD85, II.7.7], the representation ring of a product is the tensor product (over $\mathbb{Z}$) of the representation rings of the factors so that

$$R(\Gamma_{\mathbb{Z}^r}) = R(T_{\mathbb{Z}^r} \times G) \cong R(T_{\mathbb{Z}^r}) \otimes R(G) \cong \mathbb{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}][v]/(v^p - 1)$$

where $v$ is the character $v(\sigma^k) = e^{2\pi i k/p}$ generating all $p$ irreducible representations of $G$. △

To study the representation theory of compact but not necessarily connected Lie groups, we need an analog for what maximal tori are to compact connected Lie groups.

**Definition 2.1.4** ([Seg68b, 1.1])**.** A closed subgroup $C$ of a compact Lie group $H$ is called a Cartan subgroup if it is topologically cyclic, i.e., it contains an element called a generator whose powers are dense in the group, and has finite Weyl group $W_H(C) = N_H(C)/C$ with $N_H(C)$ the normalizer of $C$ in $H$.

If $H$ is connected, the Cartan subgroups are the maximal tori. By Kronecker's theorem almost every element of a torus is a generator and topologically cyclic Lie groups are isomorphic to the direct product of a torus and a finite cyclic group; cf. [BtD85, I.4.13,14]. Conjugacy classes of Cartan subgroups are detected by cyclic subgroups of the group of components:

**Proposition 2.1.5** ([BtD85, IV.4.6])**.** *Let $H$ be a compact Lie group and $H_0$ the connected component of the identity. There is a bijection of conjugacy classes between Cartan subgroups of $H$ and cyclic subgroups of $H/H_0$.*

Now let $M$ be the $\mathbb{Z}G$-lattice $\mathbb{Z}G^s \oplus IG^t$. By 2.1.0.1, there is an exact sequence

$$0 \to M^G \xrightarrow{i} M \to \overline{M} \to 0$$

with $M^G \cong (\mathbb{Z}G^s)^G \cong \mathbb{Z}^s$ and $\overline{M} \cong IG^{s+t}$. We can view the injection $i$ as a map $T_{\mathbb{Z}^s} \to T_M$, $\mathbf{x}^m \mapsto \mathbf{x}^{i(m)}$ and, by abuse of notation, extend it to an injection

$$i : \Gamma_{\mathbb{Z}^s} = T_{\mathbb{Z}^s} \times G \to \Gamma_M. \qquad (2.1.5.1)$$

**Proposition 2.1.6.** *Let $G = C_p$, where $p > 1$ is any prime. If $M$ is the $\mathbb{Z}G$-lattice $\mathbb{Z}G^s \oplus IG^t$, then the Cartan subgroups of the associated semidirect product $\Gamma_M = T_M \rtimes G$ are conjugate to either $T_M$ or $\Gamma_{\mathbb{Z}^s}$ as per 2.1.5.1.*

*Proof.* Proposition 2.1.5 states that there are exactly two conjugacy classes of Cartan subgroups corresponding to all of $G$ and the trivial subgroup of

$G$. The subgroups $T_M$ and $\Gamma_{\mathbb{Z}^s}$ are topologically cyclic and not conjugate. We compute their normalizers and Weyl groups. By 1.1.0.1,

$$(t,g)\cdot(t',g')\cdot(t,g)^{-1} = (tg(t')g'(t^{-1}), g').$$

Then

$$\begin{aligned} N_\Gamma(\Gamma_{\mathbb{Z}^s}) &= \{(t,g) : tg'(t^{-1}) \in T_{\mathbb{Z}^s} \text{ for all } g' \in G\} \\ &\cong \{(A_1,\dots,A_s,B_1,\dots,B_t)\} \rtimes G \end{aligned}$$

where each $A_i \in T_{\mathbb{Z}G}$ is of the form $(x, \zeta x, \dots, \zeta^{p-1}x)$ with $x \in S^1$ and $\zeta^p = 1$ and where each $B_i \in T_{IG}$ is of the form $(\theta,\dots,\theta)$ with $\theta^p = 1$. It follows that $W_\Gamma(\Gamma_{\mathbb{Z}^s}) \cong C_p^{s+t}$ is finite. Also,

$$N_\Gamma(T_M) = \{(t,g) : g(t') \in T_M \text{ for all } t' \in T_M\} = \Gamma_M$$

so that $W_\Gamma(T_M) \cong G$ is finite. □

**Theorem 2.1.7.** *Let $G = C_p = \langle \sigma \mid \sigma^p = 1\rangle$, where $p > 1$ is any prime. If $M$ is the $\mathbb{Z}G$-lattice $\mathbb{Z}G^s \bigoplus IG^t$ and $\Gamma_M = T_M \rtimes G$ is the associated semidirect product, then*

$$R(\Gamma_M) \cong \mathbb{Z}[M^G] \otimes \mathbb{Z}\left\{\operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^m : [m] \in M/G\right\} \otimes \mathbb{Z}[v]/(v^p - 1)$$

*subject to the relations*

*(i)* $\operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^m = \mathbf{x}^m \cdot \Phi_p$ *for* $m \in M^G$ *and* $\Phi_p = 1 + v + \cdots + v^{p-1}$,

*(ii)* $\operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^m \cdot \operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^{m'} = \sum_{i=0}^{p-1} \operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^{m+\sigma^i m'}$,

*(iii)* $\mathbf{x}^m \cdot v^j \cdot \operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^{m'} = \operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^{m+m'}$.

*Proof.* Consider the character lattice $\widehat{T}_M \cong M$. Since $\widehat{T}_M$ is countable the semidirect product $\Gamma_M$ is regular; cf. 1.1.3. Since $p$ is prime, $M$ has two types of $G$-orbits. In multiplicative notation they are those with a single character $\mathbf{x}^m$, $m \in M^G$ having isotropy all of $G$, and those with $p$ characters $\{\mathbf{x}^m, \mathbf{x}^{\sigma(m)}, \ldots, \mathbf{x}^{\sigma^{p-1}(m)}\}$ with trivial isotropy.

Let $\widehat{G} = \langle v \mid v^p = 1 \rangle$ with $v$ as per 2.1.3. Then by 1.1.4, the following is a complete list of irreducible representations of $\Gamma_M$:

(i) $\mathbf{x}^m \cdot v^j$ for every $m \in M^G$ and $0 \leqslant j < p$,

(ii) $\operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^m$ where $m$ runs through a system of representatives for $M/G$, $m \notin M^G$.

To understand the product structure, note that the restriction map

$$\operatorname{res} : R(\Gamma_M) \to \prod_{[C]} R(C) \tag{2.1.7.1}$$

where $[C]$ runs through the finite set of conjugacy classes of Cartan subgroups of $\Gamma_M$ is injective because every element of a compact Lie group is contained in some Cartan subgroup; cf. [Seg68b, 1.2]. By 2.1.6, the only Cartan subgroups of $\Gamma_M$ up to conjugation are $T_M$ and $\Gamma_{\mathbb{Z}^s}$ so that 2.1.7.1 becomes

$$\begin{aligned} \operatorname{res} : R(\Gamma_M) &\to R(T_M) \times R(\Gamma_{\mathbb{Z}^s}) \\ V &\mapsto (\operatorname{res}_{T_M}^{\Gamma_M} V, \operatorname{res}_{\Gamma_{\mathbb{Z}^s}}^{\Gamma_M} V) \end{aligned}$$

To calculate the image of this map we use three facts:

1. By 2.1.3, $R(\Gamma_{\mathbb{Z}^s}) \cong R(T_{\mathbb{Z}^s}) \otimes \mathbb{Z}[v]/(v^p - 1)$ with a restriction map

$$i^* : R(T_M) \to R(T_{\mathbb{Z}^s})$$

given by 2.1.5.1.

2. By the restriction-induction formula, cf. 1.1.1.2 and 1.2.3.5,

$$\operatorname{res}_{T_M}^{\Gamma_M} \operatorname{ind}_{T_M}^{\Gamma_M} \mathbf{x}^m \cong \mathbf{x}^m + \mathbf{x}^{\sigma(m)} + \cdots + \mathbf{x}^{\sigma^{p-1}(m)} = (\mathbf{x}^m)^*$$

for every $[m] \in M/G$, $m \notin M^G$.

3. Combining these two we get

$$\operatorname{res}^{\Gamma_M}_{\Gamma_{\mathbb{Z}^s}} \operatorname{ind}^{\Gamma_M}_{T_M} \mathbf{x}^m \cong i^*\mathbf{x}^m \cdot \Phi_p$$

for every $[m] \in M/G$, $m \notin M^G$ and $\Phi_p = 1 + v + \cdots + v^{p-1}$.

The irreducible representations of $R(\Gamma_M)$ are thus mapped as follows:

$$\begin{aligned} \mathbf{x}^m \cdot v^j &\mapsto (\mathbf{x}^m, i^*\mathbf{x}^m \cdot v^j) & m &\in M^G \\ \operatorname{ind}^{\Gamma_M}_{T_M} \mathbf{x}^m &\mapsto ((\mathbf{x}^m)^*, i^*\mathbf{x}^m \cdot \Phi_p) & [m] &\in M/G, m \notin M^G. \end{aligned}$$

The product structure between the irreducibles is thus

$$\begin{aligned} v^p &= 1, \\ \operatorname{ind}^{\Gamma_M}_{T_M} \mathbf{x}^m \cdot \operatorname{ind}^{\Gamma_M}_{T_M} \mathbf{x}^{m'} &= \sum_{i=0}^{p-1} \operatorname{ind}^{\Gamma_M}_{T_M} \mathbf{x}^{m+\sigma^i m'}, \\ \mathbf{x}^m \cdot v^j \cdot \operatorname{ind}^{\Gamma_M}_{T_M} \mathbf{x}^{m'} &= \operatorname{ind}^{\Gamma_M}_{T_M} \mathbf{x}^{m+m'}. \end{aligned}$$

Note that $m + \sigma^i m'$ may be in $M^G$ but that $\operatorname{ind}^{\Gamma_M}_{T_M} \mathbf{x}^m = \mathbf{x}^m \cdot \Phi_p$ whenever $m \in M^G$. □

Another way to obtain relation $(ii)$ of 2.1.7 is to express the induced representations $\operatorname{ind}^{\Gamma_M}_{T_M} \mathbf{x}^m$ as matrix representations and then compute their Kronecker product; cf. [CR62, §§12A,D]. Also note that $\dim \mathbf{x}^m \cdot v^j = 1$ and that $\dim \operatorname{ind}^{\Gamma_M}_{T_M} \mathbf{x}^m = p$; cf. 1.1.1.1.

As mentioned in appendix B, $R(H)$ is a finitely generated ring for any compact Lie group $H$. Continuing with the notation $M = \mathbb{Z}G^s \oplus IG^t$, our next goal, then, is to find a finite presentation for $R(\Gamma_M)$, that is, to find finitely many generators and relations for $R(\Gamma_M)$. It follows from the proof of 2.1.7 that the restriction map

$$\mathrm{res}^{\Gamma_M}_{T_M} : R(\Gamma_M) \to R(T_M)$$

maps onto $R(T_M)^G \cong \mathbb{Z}[M]^G$ with kernel the ideal generated by $v-1$. Thus, if we can find finite generators of $\mathbb{Z}[M]^G$ we may hope to extract finite generators for $R(\Gamma_M)$. The next section, then, will discuss finite presentations of $\mathbb{Z}[M]^G$. Before moving on, we show that for this purpose we may assume $M$ to be a free $\mathbb{Z}G$-lattice, that is, $M \cong \mathbb{Z}G^r$ for some $r$; cf. 1.2.

**Lemma 2.1.8.** *Let $\mathbb{K}$ be a commutative ring and let $N$ be a $\mathbb{Z}G$-lattice with effective quotient $\bar{N}$; cf. 1.2.3.1. Then*

$$\mathbb{K}[\bar{N}]^G \cong \mathbb{K}[N]^G/(\mathbf{x}^m - 1 : m \in N^G)$$

*where it suffices to let $m$ run over a $\mathbb{Z}$-basis of $N^G$.*

*Proof.* The canonical map $\overline{\phantom{m}} : N \to \bar{N}$ extends to $\mathbb{K}[N]$:

$$\begin{aligned} \mathbb{K}[N] &\to \mathbb{K}[\bar{N}] \\ \mathbf{x}^m &\mapsto \mathbf{x}^{\bar{m}} \end{aligned}$$

It follows from 1.2.3.2 that there is a bijection of $G$-orbits $G(m) \cong G(\bar{m})$ so that the diagram

$$\begin{array}{ccc} \mathbb{K}[\bar{N}] & \xrightarrow{(\ )^*} & \mathbb{K}[\bar{N}]^G \\ {\scriptstyle -}\uparrow & & \uparrow{\scriptstyle -} \\ \mathbb{K}[N] & \xrightarrow{(\ )^*} & \mathbb{K}[N]^G \end{array}$$

commutes. This implies

$$\mathbb{K}[\bar{N}] \cong \overline{\mathbb{K}[N]}$$

since $\mathbb{K}[N]^G$ is $\mathbb{K}$-generated by finite $G$-orbit sums $(\mathbf{x}^m)^*$; cf. 1.2.3.6. Moreover, $(\mathbf{x}^{\bar{m}})^* = (\mathbf{x}^{\bar{n}})^*$ is equivalent to $(\mathbf{x}^m)^* = (\mathbf{x}^m)^*\mathbf{x}^p$ for some $p \in N^G$. The result follows. □

Since $IG \cong \overline{\mathbb{Z}G}$, cf. 2.1.0.1, the lemma implies

$$\mathbb{Z}[M]^G = \mathbb{Z}[\mathbb{Z}G^s \oplus IG^t]^G \cong \mathbb{Z}[\mathbb{Z}G^{s+t}]^G/(\mathbf{x}^m : m \in \left(\mathbb{Z}G^t\right)^G \cong \mathbb{Z}^t\right).$$

When $p$ is a prime of trivial class number, the representation ring of $\Gamma_M$ with $M$ *any* $\mathbb{Z}G$-lattice consists of pieces we have now computed:

**Proposition 2.1.9.** *Let $G = C_p$, where $p > 1$ is any prime. If $M$ is the $\mathbb{Z}G$-lattice $\mathbb{Z}^r \oplus \mathbb{Z}G^s \oplus IG^t$ and $\Gamma_M = T_M \rtimes G$ is the associated semidirect product, then*

$$R(\Gamma_M) \cong R(T_{\mathbb{Z}^r}) \otimes R\left(T_{\mathbb{Z}G^s \oplus IG^t} \rtimes G\right).$$

*Moreover, if $p$ is of class number one and $T$ is any torus of rank $n$ on which $G$ acts by automorphisms with $\Gamma = T \rtimes G$ the associated semidirect product, then*

$$R(\Gamma) \cong R(\Gamma_M)$$

*for some integers $r, s, t$ obeying $r + sp + t(p-1) = n$.*

*Proof.* Since

$$T_M \cong T_{\mathbb{Z}^r} \times T_{\mathbb{Z}G^s \oplus IG^t}$$

and the action on $\mathbb{Z}^r$ is trivial we have

$$T_M \rtimes G \cong T_{\mathbb{Z}^r} \times (T_{\mathbb{Z}G^s \oplus IG^t} \rtimes G).$$

The first claim now follows from the fact already used in 2.1.3 that the representation ring of a product is the tensor product of the representation rings of the factors. Next, let $M'$ be the $\mathbb{Z}G$-lattice given by the character group of a general rank $n$ torus $T$. Since the class number of $p$ is trivial, by 1.2.3 there is a $\mathbb{Z}G$-isomorphism

$$M = \mathbb{Z}^r \oplus \mathbb{Z}G^s \oplus IG^t \cong M'$$

for some $r$, $s$, $t$. It follows that $T'_M \cong T_M$ so that $R(\Gamma) \cong R(\Gamma_M)$. □

## 2.2 Rings of invariants

As mentioned towards the end of the last section, we are interested in finding a finite generating set for $\mathbb{Z}[M]^G$ when $M$ is a free $\mathbb{Z}G$-lattice. That this is possible is established by

**Lemma 2.2.1.** *Let $H$ be an arbitrary group and let $N$ be any $\mathbb{Z}H$-lattice. The multiplicative invariant algebra $\mathbb{K}[N]^H$ is a finitely generated $\mathbb{K}$-algebra for any commutative ring $\mathbb{K}$.*

*Proof.* It was shown in 1.2.3.6 that a $\mathbb{K}$-basis for $\mathbb{K}[N]^H$ is given by finite $H$-orbit sums on $N$. In particular, $\mathbb{K}[N]^H \subset \mathbb{K}[D]$, where $D$ is the subgroup of $N$ consisting of all elements having finitely many $H$-conjugates. But $N \cong \mathbb{Z}^n$ for some $n$, so $D \cong \mathbb{Z}^m$, for some $m \leqslant n$. Thus $D$ is finitely generated, and therefore $H$ acts as a finite group $\widetilde{H}$ on $D$ with $\mathbb{K}[N]^H \cong \mathbb{K}[D]^{\widetilde{H}}$. It follows from Noether's finiteness theorem, cf., e.g., [Bou64, V.1.2], that $\mathbb{K}[N]^H$ is a finitely generated $\mathbb{K}$-algebra. □

Continuing with the free $\mathbb{Z}G$-lattice $M = \mathbb{Z}G^r$ for some $r$, introduce the standard $\mathbb{Z}$-basis on $\mathbb{Z}G \cong \mathbb{Z}^p \cong \langle x_1, \ldots, x_n \rangle =: \langle \mathbf{x} \rangle$; cf. 1.2.3.3. Then

$$\mathbb{Z}[M]^G \cong \mathbb{Z}[\mathbf{x}_1^{\pm 1}, \ldots, \mathbf{x}_r^{\pm 1}]^G \cong \mathbb{Z}[\mathbf{x}_1, \ldots, \mathbf{x}_r]^G[s_p^{-1}(\mathbf{x}_1), \ldots, s_p^{-1}(\mathbf{x}_r)]$$

where $s_p(\mathbf{x}_i)$ is the $p$th elementary symmetric polynomial in the variables $\mathbf{x}_i$ and $G$ acts on $\mathbf{x}_i = (x_{i1}, \ldots, x_{ip})$ by cyclic permutation. The ring

$$\mathbb{Z}[\mathbf{x}_1, \ldots, \mathbf{x}_r]^G \cong \mathrm{Sym}\,(M^*)^G$$

is the additive invariant theory of $M$ regarded as a $\mathbb{Z}$-module; cf. 1.2. Permutation lattices are self-dual, i.e., $M^* \cong M$ so that we may drop the $(\ \ )^*$ in the above isomorphism; cf. [Lor05, 1.4.3].

After tensoring with a field of characteristic 0, there is an upper bound for the number of generators of the additive invariant theory; cf., e.g., [Stu08, 2.1.4]:

**Theorem 2.2.2** (Noether's degree bound)**.** *Let $\mathbb{K}$ be a field of characteristic 0 and let $V$ be an $n$-dimensional $\mathbb{K}$-representation of a finite group $H$. Then the invariant ring $\mathrm{Sym}(V^*)^H$ has an algebra basis consisting of at most $\binom{n+|H|}{n}$ invariants whose degree is bounded above by the group order $|H|$.*

Finding a minimal set of generators—called fundamental invariants—for $\mathbb{Z}[M]^G$ as a $\mathbb{Z}$-algebra and the algebraic relations, called syzygies, among them turns out to be rather complicated except for the following special case:

**Example 2.2.3.** Consider $R(\Gamma_{\mathbb{Z}G})$ and $R(\Gamma_{IG})$ when $p = 2$. In that case, $\mathbb{Z}[\mathbb{Z}G]^G \cong \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]^{S_2}$ where $G = C_2 = S_2$ is the symmetric group on two letters. By the fundamental theorem of symmetric polynomials, cf., e.g., [Stu08, 1.1.1]),

$$\mathbb{Z}[x_1, x_2]^G \cong \mathbb{Z}[s_1, s_2]$$

where $s_1 = x_1 + x_2$ and $s_2 = x_1 x_2$. Thus

$$\mathbb{Z}[\mathbb{Z}G]^G \cong \mathbb{Z}[x_1, x_2]^{S_2}[s_2^{-1}] \cong \mathbb{Z}[s_1, s_2][s_2^{-1}] \cong \mathbb{Z}[s_1, s_2^{\pm 1}].$$

The lattice of $G$-invariants, $\mathbb{Z}G^G$, is generated by $s_2^{\pm 1}$ and $s_1 = x_1^*$ so that by 2.1.7 and 2.1.8,

$$\begin{aligned} R(\Gamma_{\mathbb{Z}G}) &\cong \mathbb{Z}\left[\mathrm{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1, s_2^{\pm 1}, v\right] / \left(v^2 - 1, (v-1)\,\mathrm{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1\right), \\ R(\Gamma_{IG}) &\cong \mathbb{Z}\left[\mathrm{ind}_{T_{IG}}^{\Gamma_{IG}} x_1, v\right] / \left(v^2 - 1, (v-1)\,\mathrm{ind}_{T_{IG}}^{\Gamma_{IG}} x_1\right). \end{aligned}$$

Indeed, this calculation agrees with that for $R(\mathrm{O}(2n))$, $n = 1$ found in [BtD85, VI.7.7] since $\Gamma_{IG} \cong S^1 \rtimes C_2 \cong \mathrm{O}(2)$. △

The reason that in this example one is readily able to write down a minimal algebra basis and syzygies is that the invariant ring $\mathrm{Sym}(\mathbb{Z}G)^{S_n} \cong \mathbb{K}[x_1, \ldots, x_n]^{S_n}$ is classically known even when $\mathbb{K}$ is only an integral domain such as $\mathbb{Z}$—a fact we will make use of in 2.3.1 when discussing semidirect products between tori and the symmetric group $S_n$. However, most algorithms for computing $\mathrm{Sym}(V^*)^H$ require that $\mathbb{K}$ be a field. Information about $\mathbb{Z}[M]^H$ as a subring of $\mathbb{K}[M]^H$ may be obtained by carrying out "Galois descent" as done in [Len74] but not pursued in this thesis.

The symmetric group $S_n$—when acting by permutation representations—is an example of a reflection group[1], a class of groups (really: representations!) for which the invariant ring is particularly nice; cf., e.g., [Stu08, 2.4.1]:

**Theorem 2.2.4** (Chevalley-Shephard-Todd)**.** *Let $V$ be a vector space. The invariant ring $\mathrm{Sym}(V^*)^H$ of a finite matrix group $H \subset \mathrm{GL}(V)$ is generated by $n$ algebraically independent homogeneous invariants if and only if $H$ is a reflection group.*

This fact can be illustrated with the representation rings of compact connected Lie groups $\mathfrak{G}$: in that case $R(\mathfrak{G})$ is isomorphic to $R(T)^W \cong \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^W$ with $T$ a maximal torus of $\mathfrak{G}$ of rank $n$ and $W$ the Weyl group of $T$, all of which are reflection groups; cf., e.g., [BtD85, VI.2.1].

**Example 2.2.5.** Consider the Lie group $\mathrm{U}(n)$. Its rank is $n$ and its Weyl group is the symmetric group $S_n$ acting by permutation of coordinates. Then

$$R(\mathrm{U}(n)) \cong \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^W \cong \mathbb{Z}[x_1, \ldots, x_n]^W[s_n^{-1}] \cong \mathbb{Z}[s_1, \ldots, s_n^{\pm 1}]$$

with $s_i$ the $i$th elementary symmetric polynomial. Chevalley-Shephard-Todd confirms the classic fact that $\{s_i : i = 1, \ldots, n\}$ forms a set of fundamental invariants with no nonzero syzygy. △

Unfortunately, the group $G = C_p$ under consideration is neither a reflection group nor does there seem to be any reason to believe that Galois

[1] A reflection group is a discrete group which is generated by a set of reflections of a finite-dimensional Euclidean space. Note that being a reflection group is not a property intrinsic to the group itself but rather to its action.

descent will be 'nice'. However, considering $\mathbb{K}[M]$ instead of $\mathbb{Z}[M]$ when $\mathbb{K}$ is a field amounts to tensoring $R(\Gamma)$ with $\mathbb{K}$ which is still valuable.

There exists a plethora of algorithms for computing fundamental invariants and syzygies over fields, many of which use Gröbner bases as basic building blocks; cf. [Stu08, ch. 2]. Since $G$ is finite abelian, the most promising candidate for such an algorithm is [Stu08, 2.7.3] which has been adapted for our purposes below. Before we can use it we need the following theorem. Recall that the exponent of a finite group $H$ is the smallest positive integer $n$ such that $x^n = 1$ for all $x \in H$.

**Theorem 2.2.6** ([CR62, 41.1])**.** *Let $n$ be the exponent of a finite group $H$. Then the cyclotomic field $\mathbb{Q}(\zeta)$ with $\zeta$ a primitive $n$th root of unity is a splitting field for $H$, i.e., each irreducible $\mathbb{C}H$-representation has a realization with matrices having entries in $\mathbb{Q}(\zeta)$.*

Clearly, $G = C_p$ has exponent $p$.

**Algorithm 2.2.7** (Invariant theory of the cyclic group of prime order)**.** Let $\mathbb{K} = \mathbb{Q}(\zeta)$ be a splitting field for the cyclic group $G = C_p = \langle \sigma \mid \sigma^p = 1 \rangle$ of prime order $p$ and let $M$ be a free $\mathbb{Z}G$-lattice of rank $r$. Then the following algorithm produces a finite algebra basis for

$$\mathrm{Sym}\,(\mathbb{K} \otimes M)^G \cong \mathbb{K}[\mathbf{x}_1, \ldots, \mathbf{x}_r]^G$$

with $\mathbf{x}_i$ the standard $\mathbb{K}$-basis of $\mathbb{K}G$.

1. Choose a diagonalizing matrix $M_G$ so that

$$M_G \sigma M_G^{-1} = \mathrm{diag}(1, \zeta, \ldots, \zeta^{p-1}).$$

2. Introduce new variables $\mathbf{y}_i = (y_1, \ldots, y_r) := M_G(\mathbf{x}_i)$.

3. For $a_i = (a_{i1}, \ldots, a_{ip}) \in \mathbb{N}^p$, $i = 1, \ldots, r$, consider the linear homoge-

neous congruence

$$\sum_{i=1}^{r} a_{i2} + 2a_{i3} + \cdots + (p-1)a_{ip} \equiv 0 \mod p.$$

Compute a finite generating set $\mathcal{H}$ for the solution monoid. Then

$$\begin{aligned}\mathbb{K}[\mathbf{x}_1, \ldots, \mathbf{x}_r]^G &\cong \mathbb{K}[\mathbf{y}_1, \ldots, \mathbf{y}_r]^{M_G G M_G^{-1}} \\ &\cong \mathbb{K}[\mathbf{y}_1^{a_1} \cdots \mathbf{y}_r^{a_r} : (a_1, \ldots, a_r) \in \mathcal{H}].\end{aligned}$$

4. To rewrite an invariant $f \in \mathbb{K}[\mathbf{x}_1, \ldots, \mathbf{x}_r]^G$ in terms of the $\mathbf{y}_1^{a_1} \cdots \mathbf{y}_r^{a_r}$ with $(a_1, \ldots, a_r) \in \mathcal{H}$, note that

$$f(\mathbf{x}) = f(M_G^{-1}\mathbf{y}) = g(\mathbf{y})$$

is an invariant polynomial in $\mathbf{y}$ and so can be expressed in the $\mathbf{y}_1^{a_1} \cdots \mathbf{y}_r^{a_r}$.

The generating set $\mathcal{H}$ in step 3 is the so-called Hilbert basis. Existence and uniqueness of the Hilbert basis for a solution monoid such as the one above can be established by algorithm [Stu08, 1.4.5]. The following examples are easy enough to do by hand. Let $R_{\mathbb{K}}(H) := R(H) \otimes_{\mathbb{Z}} \mathbb{K}$ for any locally compact Lie group $H$.

**Example 2.2.8.** Let us check that the above algorithm produces the same answer as example 2.2.3. A splitting field for $G = C_2$ is $\mathbb{Q}(-1) = \mathbb{Q}$. A matrix diagonalizing the generator $\sigma = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of $G$ in the desired way is $M_G = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Without loss of generality, we may thus return to working over $\mathbb{Z}$. The new variables are then $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$. The linear homogeneous congruence is

$$a_2 \equiv 0 \mod 2.$$

The solution monoid is generated by $\{(1,0),(0,2)\}$ so that

$$\mathbb{Z}[x_1,x_2]^G \cong \mathbb{Z}[y_1,y_2^2].$$

By step 4 of the algorithm or otherwise, we can write the generator $s_2 = x_1x_2$ of $\mathbb{Z}G^G$ as

$$4s_2 = y_1^2 - y_2^2$$

so that

$$\mathbb{Z}[IG] \cong \mathbb{Z}[y_1,y_2^2]^G[s_2^{-1}]/(s_2-1) \cong \mathbb{Z}[y_1].$$

Since $y_1 = x_1^*$, it then follows by 2.1.7 and 2.1.8 that

$$R(\Gamma_{\mathbb{Z}G}) \cong \mathbb{Z}\left[\operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1, s_2^{\pm 1}, v\right] / \left(v^2-1, (v-1)\operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1\right)$$
$$R(\Gamma_{IG}) \cong \mathbb{Z}\left[\operatorname{ind}_{T_{IG}}^{\Gamma_{IG}} x_1, v\right] / \left(v^2-1, (v-1)\operatorname{ind}_{T_{IG}}^{\Gamma_{IG}} x_1\right)$$

as before. △

**Example 2.2.9.** Continuing on from the previous example, we compute $R(\Gamma_{\mathbb{Z}G^2})$ when $G = C_2$. After diagonalizing the action on each summand $\mathbb{Z}G$ separately, i.e.,

$$y_{11} = x_{11} + x_{12}$$
$$y_{12} = x_{11} - x_{12}$$
$$y_{21} = x_{21} + x_{22}$$
$$y_{22} = x_{21} - x_{22}$$

we obtain the linear homogeneous congruence

$$a_{12} + a_{22} \equiv 0 \mod 2$$

having solution monoid generated by

$$\{(1,0,0,0),(0,0,1,0),(0,2,0,0),(0,0,0,2),(0,1,0,1)\}.$$

It follows that

$$\mathbb{Z}[\mathbf{x}_1,\mathbf{x}_2]^G \cong \mathbb{Z}[y_{11}, y_{21}, y_{12}^2, y_{22}^2, y_{12}y_{22}].$$

Note that these generators are *not* algebraically independent since

$$(y_{12}y_{22})^2 = y_{12}^2 y_{22}^2.$$

Since $y_{11} = x_{11}^*$, $y_{21} = x_{21}^*$, $4s_2(\mathbf{x}_1) = y_{11}^2 - y_{12}^2$, $4s_2(\mathbf{x}_1) = y_{21}^2 - y_{22}^2$, and $y_{12}y_{22} = (x_{11}x_{21})^* - (x_{11}x_{22})^*$, it follows by 2.1.7 that

$$R(\Gamma_{\mathbb{Z}G^2}) \cong \mathbb{Z}[\operatorname{ind} x_{11}, \operatorname{ind} x_{21}, \operatorname{ind} x_{11}x_{21} - \operatorname{ind} x_{11}x_{22}, s_2^{\pm 1}(\mathbf{x}_1), s_2^{\pm 1}(\mathbf{x}_2), v]$$

subject to the relations $v^2 = 1$, $v \cdot \operatorname{ind} \mathbf{x}^m = \operatorname{ind} \mathbf{x}^m$, and

$$(\operatorname{ind} x_{11}x_{21} - \operatorname{ind} x_{11}x_{22})^2 = \left((\operatorname{ind} x_{11})^2 - 4s_2(\mathbf{x}_1)\right)\left((\operatorname{ind} x_{11})^2 - 4s_2(\mathbf{x}_2)\right).$$

By 2.1.8, we then also obtain

$$\begin{aligned} R(\Gamma_{\mathbb{Z}G\oplus IG}) &\cong R(\Gamma_{\mathbb{Z}G^2})/(s_2(\mathbf{x}_2) - 1) \\ R(\Gamma_{IG^2}) &\cong R(\Gamma_{\mathbb{Z}G^2})/(s_2(\mathbf{x}_1) - 1, s_2(\mathbf{x}_2) - 1). \end{aligned}$$

$\triangle$

**Example 2.2.10.** Consider $R(\Gamma_{\mathbb{Z}G})$ and $R(\Gamma_{IG})$ when $p = 3$. A splitting field for $G = C_3$ is $\mathbb{K} := \mathbb{Q}(\zeta)$ with $\zeta = e^{2\pi i/3}$. After diagonalizing, the new variables are

$$\begin{aligned} y_1 &= x_1 + x_2 + x_3 \\ y_2 &= x_1 + \zeta^2 x_2 + \zeta x_3 \\ y_3 &= x_1 + \zeta x_2 + \zeta^2 x_3. \end{aligned}$$

The congruence relation is

$$a_2 + 2a_3 \equiv 0 \mod 3$$

and has solution monoid generated by $\{(1,0,0),(0,1,1),(0,3,0),(0,0,3)\}$. It follows that

$$\mathbb{K}[\mathbf{x}]^G \cong \mathbb{K}[y_1, y_2y_3, y_2^3, y_3^3].$$

In accordance with Chevalley-Shephard-Todd, note that the generating variables are *not* algebraically independent since

$$(y_2y_3)^3 = y_2^3 y_3^3. \tag{2.2.10.1}$$

That this is the only syzygy can be proved using Gröbner bases; cf. [Stu08, 2.5.3]. Using step 4 of 2.2.7,

$$\begin{aligned} s_1 &= y_1 \\ 3s_2 &= y_1^2 - y_2y_3 \\ 27s_3 &= y_1^3 + y_2^3 + y_3^3 - 3y_1y_2y_3 \\ 9(x_1^2x_2)^* &= y_1^3 + \zeta y_2^3 + \zeta^2 y_3^3. \end{aligned}$$

Changing variables, we obtain

$$\mathbb{K}[\mathbb{Z}G]^G \cong \mathbb{K}[s_1, s_2, (x_1^2x_2)^*, s_3^{\pm 1}]$$

and by 2.1.8,

$$R_{\mathbb{K}}(\Gamma_{\mathbb{Z}G}) \cong \mathbb{K}[\operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1, \operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1x_2, \operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1^2x_2, s_3^{\pm 1}, v]$$

subject to the relations $v^3 = 1$,

$$v \cdot \operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} \mathbf{x}^m = \operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} \mathbf{x}^m \text{ for } \mathbf{x}^m = x_1, x_1x_2, x_1^2x_2,$$

and the equivalent of 2.2.10.1 in these variables. Moreover by 2.1.7,

$$R_{\mathbb{K}}(\Gamma_{IG}) \cong R_{\mathbb{K}}(\Gamma_{\mathbb{Z}G})/(s_3 - 1).$$

△

**Example 2.2.11.** Let $p = 5$, $G = C_5$, and $\mathbb{K} = \mathbb{Q}(\zeta)$ with $\zeta = e^{2\pi i/5}$. By [Stu08, 2.7.1], $\mathbb{K}[\mathbf{x}]^G$ has the following minimal algebra basis consisting of 15 elements

$$\{y_1, y_2y_5, y_3y_4, y_2y_3^2, y_2^2y_4, y_3y_5^2, y_4^2y^5, y_2y_4^3, y_3^3y^5, y_4y_5^3, y_2^3y_3, y_2^5, y_3^5, y_4^5, y_5^5\}$$

with several obvious algebraic relations among them. △

## 2.3 Semidirect products by other groups

An analysis similar to that of the last sections can be carried out for cases other than $G = C_p$. However, as discussed in section 1.2 a complete reducibility theorem like 1.2.3 is only available in rare instances. For instance, Reiner states in [CR90, §34] that a complete classification theorem cannot be expected for cyclic $p$-groups of order $p^k$ when $k > 2$. When $k = 2$ there are precisely $4p + 1$ types of indecomposable $\mathbb{Z}C_{p^2}$-modules; cf. [CR90, 34.32]. Another example is the dihedral group $D_{2p}$ of order $2p$ with $p > 2$ a fixed prime discussed below. Some additional cases where integral representations have been calculated are listed on [CR90, p. 753].

So let

$$G := D_{2p} = C_p \rtimes C_2 = \langle \sigma \mid \sigma^p = 1 \rangle \rtimes \langle \tau \mid \tau^2 = 1 \rangle$$

the dihedral group of order $2p$ introduced in 1.1.6 and let $\zeta$ be a primitive $p$th root of unity. The 10 indecomposable types of $\mathbb{Z}G$-lattices are represented by the following list; cf. [CR90, 34.50]:

(i) $R := \mathbb{Z}[\zeta]$, $P := (1-\zeta)R$ on which $\sigma$ acts as multiplication by $\zeta$ and $\tau$ acts as complex conjugation.

(ii) $\mathbb{Z}$, $\mathbb{Z}'$, and $\mathbb{Z}C_2$ on which $\sigma$ acts trivially while $\tau$ acts as $+1$ on $\mathbb{Z}$ and as $-1$ on $\mathbb{Z}'$.

(iii) The two non-split extension $P \to X_0 \to \mathbb{Z}$, $R \to X_1 \to \mathbb{Z}'$, i.e., $X_0 \cong \mathrm{ind}_{C_2}^{G} \mathbb{Z}$ and $X_1 \cong \mathrm{ind}_{C_2}^{G} \mathbb{Z}'$.

(iv) The three non-split extensions $R \to Y_0 \to \mathbb{Z}C_2$, $P \to Y_1 \to \mathbb{Z}C_2$, $R \oplus P \to Y_2 \to \mathbb{Z}C_2$.

As before, let $\Gamma_M = T_M \rtimes G$ where $\widehat{T}_M = M$. Then

$$R(\Gamma_{\mathbb{Z}}) \cong R(T_{\mathbb{Z}}) \otimes R(G)$$

with $R(G)$ given by 1.1.6. Let $\widehat{C}_p = \langle v \mid v^p = 1 \rangle$, $\widehat{C}_2 = \langle w \mid w^2 = 1 \rangle$, and $M \cong \mathbb{Z}^n \cong \langle x_1, \ldots, x_n \rangle$ under a choice of $\mathbb{Z}$-basis; cf. 1.2.3.3. Based on Mackey's method, we determine $R(\Gamma_M)$ as abelian groups for some of the above $\mathbb{Z}G$-lattices:

- The irreducible representations of $\Gamma_{\mathbb{Z}'}$ are (i) those of $G$ and (ii) for each $i > 0$ and $0 \leqslant j < p$ an induced representation $\mathrm{ind}_{T_{\mathbb{Z}'} \rtimes C_p}^{\Gamma_{\mathbb{Z}'}} (x^i \cdot v^j)$.
- The irreducible representations of $\Gamma_{\mathbb{Z}H}$ are (i) those of $G$, (ii) the one-dimensional characters $(x_1 x_2)^i$, $i \in \mathbb{Z}$, and (iii) for each $i < j \in \mathbb{Z}$ and $0 \leqslant k < p$ an induced representation $\mathrm{ind}_{T_{\mathbb{Z}H} \rtimes C_p}^{\Gamma_{\mathbb{Z}H}} (x_1^i x_2^j \cdot v^k)$.

- The irreducible representations of $\Gamma_R$ are (i) those of $G$, (ii) induced representations $\mathrm{ind}_{T_R \rtimes C_2}^{\Gamma_R}(\chi \cdot w^i)$, $i = 0, 1$, for every complex conjugation invariant character $[\chi] \in \mathbb{Z}[\zeta]/G$, and (iii) $\mathrm{ind}_{T_R}^{\Gamma_R} \chi'$ for every other nonzero character $[\chi'] \in \mathbb{Z}[\zeta]/G$. For instance, when $p = 5$

$$\mathbb{Z}[\zeta] \cong \mathbb{Z}\{1, \zeta, \zeta^2, \zeta^3\}$$

has conjugation invariant subspaces $\mathbb{Z}\{1\}$, $\mathbb{Z}\{\zeta^2 + \zeta^3\}$, and their direct sum leading to induced representations

$$\mathrm{ind}_{T_R \rtimes C_2}^{\Gamma_R}(x_1^l \cdot w^i), \ \ \mathrm{ind}_{T_R \rtimes C_2}^{\Gamma_R}((x_3 x_4)^k \cdot w^i), \ \text{and} \ \mathrm{ind}_{T_R \rtimes C_2}^{\Gamma_R}(x_1^l (x_3 x_4)^k \cdot w^i).$$

A product structure between the listed irreducibles can be determined as in the last sections by using Cartan subgroups of $\Gamma_M$; cf. 2.1.7.1. By 2.1.5, we expect $T \rtimes D_{2p}$ to have three Cartan subgroups up to conjugation so this is a manageable task which could be pursued in future work.

Instead, we turn our attention to the symmetric group $S_n$ on $n$ letters with its natural permutation $\mathbb{Z}S_n$-lattice $M \cong \mathbb{Z}^n \cong \langle x_1, \ldots, x_n \rangle$ on which $S_n$ acts by sending $x_i$ to $x_{s(i)}$ for all $s \in S_n$; cf. 1.2.1.

**Theorem 2.3.1.** *Let $M \cong \mathbb{Z}^n \cong \langle x_1, \ldots, x_n \rangle$ be the natural permutation $\mathbb{Z}S_n$-lattice and let $T$ be a rank $n$ torus such that $\widehat{T} \cong M$. If $\Gamma = T \rtimes S_n$ is the associated semidirect product, then $R(\Gamma)$ is finitely generated as a $\mathbb{Z}$-algebra by the following representations:*

*(i) the irreducible representations of $S_n$,*

*(ii) the torus characters $s_n^{\pm 1}$.*

*(iii) for each $1 \leqslant i < n$ and for each irreducible representation $\rho$ of $S_{n-i} \times S_i$, an irreducible representation $\mathrm{ind}_{T \rtimes (S_{n-i} \times S_i)}^{\Gamma}(x_1 \cdots x_i \cdot \rho)$.*

*Proof.* By Mackey's method, the irreducible representations of $\Gamma$ are given by $\mathrm{ind}_{\Gamma_m}^{\Gamma}(\mathbf{x}^m \cdot \rho)$ for each $[m] \in M/G$ where $\Gamma_m = T \rtimes (S_n)_m$ and $\rho$ is an irreducible representation of the isotropy group $(S_n)_m$.

As in 2.1.7.1, the restriction map from $R(\Gamma)$ to the product of the representation rings of the Cartan subgroups of $\Gamma$ is an injection. One of these Cartan subgroups is $T$ with

$$\begin{aligned} \mathrm{res}^{\Gamma}_{T} &: R(\Gamma) \to R(T)^{S_n} \\ \mathrm{ind}^{\Gamma}_{\Gamma_m}(\mathbf{x}^m \cdot \rho) &\mapsto \mathrm{res}^{\Gamma}_{T}\,\mathrm{ind}^{\Gamma}_{\Gamma_m}(\mathbf{x}^m \cdot \rho) \cong \dim\rho \cdot (\mathbf{x}^m)^*. \end{aligned}$$

Since $R(T)^{S_n} \cong \mathbb{Z}[s_1, s_2, \ldots, s_{n-1}, s_n^{\pm 1}]$ with $s_i$ the $i$th elementary symmetric polynomial, it follows that we may take the generators of $R(\Gamma)$ to be those irreducible representations corresponding to $(\mathbf{x}^m)^* \in \{1, s_1, s_2, \ldots, s_{n-1}, s_n^{\pm 1}\}$ under the restriction map.

The preimages of $(\mathbf{x}^m)^* = s_n^j$, $j = -1, 0, 1$ are the representations $s_n^j \otimes \rho$ with $\rho$ an irreducible representation of $S_n$. Each other $s_i$ with $1 \leqslant i < n$ has preimage

$$\left\{\mathrm{ind}^{\Gamma}_{T \rtimes (S_{n-i} \times S_i)}(x_1 \cdots x_i \cdot \rho) : \rho \text{ an irreducible representation of } S_{n-i} \times S_i\right\}.$$

□

The product structure of $R(\Gamma)$ may be further studied by identifying all Cartan subgroups of $\Gamma$. By 2.1.5, we see that this task soon becomes difficult as $n$ grows.

**Example 2.3.2.** Consider $n = 3$, that is, $S_3 \cong D_6$ and let $T$ be as in 2.3.1. Using 2.1.5, we determine the Cartan subgroups of $\Gamma = T \rtimes S_3$ to be $T$,

$$\left\{ \begin{pmatrix} x & & \\ & x & \\ & & y \end{pmatrix} : x, y \in S^1 \right\} \times C_2 \cong (S^1)^2 \times C_2,$$

and

$$\left\{ \begin{pmatrix} z & & \\ & z & \\ & & z \end{pmatrix} : z \in S^1 \right\} \times C_3 \cong S^1 \times C_3.$$

It follows that $R(T \rtimes S_3)$ is generated by (also cf. 1.1.6)

$$\begin{aligned}
R(\Gamma) &\to R(T)^{S_3} \times R((S^1)^2 \times C_2) \times R(S^1 \times C_3) \\
&\cong \mathbb{Z}[s_1, s_2, s_3^{\pm 1}] \times \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, w]/(w^2 - 1) \times \mathbb{Z}[z^{\pm 1}, v]/(v^3 - 1) \\
\mathrm{ind}(x_1 \cdot w^i) &\mapsto (s_1, (2x + y)w^i, z(1 + v + v^2)) \\
\mathrm{ind}(x_1 x_2 \cdot w^i) &\mapsto (s_2, (2x^2 + y^2)w^i, z^2(1 + v + v^2)) \\
(x_1 x_2 x_3)^{\pm 1} &\mapsto (s_3, x^2 y, z^3)^{\pm 1} \\
w &\mapsto (1, w, 1) \\
\mathrm{ind}_{C_3}^{S_3} v &\mapsto (2, 1 + w, v + v^2).
\end{aligned}$$

We see from this that $\mathrm{ind}(\chi \cdot w) = w \cdot \mathrm{ind}(\chi)$ so that,

$$R(\Gamma) \cong \mathbb{Z}[\mathrm{ind}(x_1), \mathrm{ind}(x_1 x_2), (x_1 x_2 x_3)^{\pm 1}, w, \mathrm{ind}_{C_3}^{S_3} v]$$

subject to various relations. △

A similar approach may be used to find generators of $R_{\mathbb{K}}(T \rtimes G)$ whenever $G$ is a reflection group and $\mathbb{K}$ is a splitting field for $G$; cf. 2.2.4 and 2.2.6. In that case, a theorem of Reichstein, cf. [Rei03] and [DR09], asserts:

**Theorem 2.3.3.** *Let $G$ be a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$ and let $\mathbb{K}$ be a splitting field for $G$. Then the ring of multiplicative invariants $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^G$ has a finite SAGBI basis if and only if $G$ is a reflection group.*

Without going into detail, a finite SAGBI basis is in particular a finite collection of elements of $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^G$ that generates $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^G$ as a $\mathbb{K}$-algebra. The algorithm producing a SAGBI basis is known as the subduction algorithm. It follows that whenever $G$ is a finite reflection group, the subduction algorithm may be used to find finitely many generators of $R_{\mathbb{K}}(T \rtimes G)$.

# Chapter 3

# Applications to Lie Groups and $K$-Theory

Once again let $G = C_p = \langle \sigma \mid \sigma^p = 1 \rangle$ be the cyclic group of prime order $p$. In this chapter we apply the results of the previous chapter to classical Lie groups and the $K$-theory of the classifying space of the semidirect products $\Gamma = T \rtimes G$.

## 3.1 Tori of compact connected Lie groups

Denote by $T_{\mathfrak{G}}$ and $W_{\mathfrak{G}}$ a maximal torus respectively the Weyl group of a compact connected Lie group $\mathfrak{G}$. Consider the case when $\mathfrak{G}$ is one of $\mathrm{SU}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$, or $\mathrm{Sp}(n)$.[1] By [BtD85, IV.3], there are homeomorphisms

$$T_{\mathrm{U}(n)} \approx (S^1)^n \qquad T_{\mathrm{SU}(n)} \approx (S^1)^{n-1}$$

[1]There is sometimes confusion in the literature about the notation of this last group. Here $\mathrm{Sp}(n)$ denotes the compact symplectic group of isometries of the $p$-dimensional quarternion algebra $\mathbb{H}^p$ with its standard symplectic scalar product; cf. [BtD85, I.1.11].

and the inclusions $\mathrm{U}(n) \hookrightarrow \mathrm{SO}(2n) \hookrightarrow \mathrm{SO}(2n+1)$ and $\mathrm{U}(n) \hookrightarrow \mathrm{Sp}(n)$ induce homeomorphisms

$$T_{\mathrm{U}(n)} \approx T_{\mathrm{SO}(2n)} \approx T_{\mathrm{SO}(2n+1)} \text{ respectively } T_{\mathrm{U}(n)} \approx T_{\mathrm{Sp}(n)}.$$

Now fix a prime $p$ and let $S_p$ be the symmetric group on $p$ letters. Let $\mathrm{Wr}(p) = C_2^p \rtimes S_p$ be the wreath product of $C_2$ by $S_p$, and $\mathrm{SWr}(p)$ be the subgroup of $\mathrm{Wr}(p)$ consisting of even permutations. Then again by [BtD85, IV.3],

$$W_{\mathrm{U}(p)} \cong W_{\mathrm{SU}(p)} \cong S_p,\ W_{\mathrm{Sp}(p)} \cong W_{\mathrm{SO}(2p+1)} \cong \mathrm{Wr}(p), \text{ and } W_{\mathrm{SO}(2p)} \cong \mathrm{SWr}(p).$$

Up to conjugation, there is a unique subgroup $C_p \subset S_p \subset W_{\mathfrak{G}}$ for all $\mathfrak{G}$ and we can form the semidirect product $T_{\mathfrak{G}} \rtimes C_p$ as a subgroup of $T_{\mathfrak{G}} \rtimes W_{\mathfrak{G}}$.

**Proposition 3.1.1.** *Let $G = C_p$, where $p > 1$ is any prime. If $\mathfrak{G}$ is any of the Lie groups $\mathrm{SU}(p)$, $\mathrm{U}(p)$, $\mathrm{SO}(2p)$, $\mathrm{SO}(2p+1)$, or $\mathrm{Sp}(p)$ and $\Gamma_{\mathfrak{G}} = T_{\mathfrak{G}} \rtimes G$ as per above, then*

$$R(\Gamma_{\mathfrak{G}}) \cong \begin{cases} R(\Gamma_{IG}) & \text{if } \mathfrak{G} = \mathrm{SU}(p) \\ R(\Gamma_{\mathbb{Z}G}) & \text{else.} \end{cases}$$

*Proof.* The tori of $\mathrm{U}(p)$, $\mathrm{SO}(2p)$, $\mathrm{SO}(2p+1)$, and $\mathrm{Sp}(p)$ are all $G$-equivariantly isomorphic with $G$ acting by cyclic permutation. It follows that $T_{\mathfrak{G}}$ is $T_{\mathbb{Z}G}$ in all of these cases. Since $T_{\mathrm{SU}(p)}$ is $T_{\mathrm{U}(p)}$ modulo the diagonal, $T_{\mathrm{SU}(p)}$ is $G$-isomorphic to $T_{IG}$. □

These representation rings are thus known by 2.1.7. For example, 2.2.3 discussed the case of $T_{\mathrm{SU}(2)} \rtimes C_2 \cong \mathrm{O}(2)$.

Next, recall that the quotient of a Lie group $\mathfrak{G}$ by a closed normal subgroup $N$ is a Lie group; cf. [BtD85, I.4.4]. If $\mathfrak{G}$ is compact and connected then so is $\mathfrak{G}/N$, the latter being the continuous image of the quotient map $\pi : \mathfrak{G} \to \mathfrak{G}/N$. In particular, we can consider the center $Z(\mathfrak{G})$ of $\mathfrak{G}$ and form the projectivization $P\mathfrak{G} := \mathfrak{G}/Z(\mathfrak{G})$ of $\mathfrak{G}$.

**Proposition 3.1.2.** *Let $p > 1$ be a prime of class number one and let $\mathfrak{G}$ be any of the Lie groups $\mathrm{SU}(p)$, $\mathrm{SO}(2p)$, $\mathrm{SO}(2p+1)$, or $\mathrm{Sp}(p)$. Then*

$$R(\Gamma_{P\mathfrak{G}}) \cong R(\Gamma_{\mathfrak{G}}).$$

*Proof.* The group $\mathfrak{G}$ has finite center in all cases; cf. [BtD85, V, 7.13]. The quotient map $\pi$ is thus a finite covering space. In particular, the ranks of $P\mathfrak{G}$ and $\mathfrak{G}$ are the same and their Weyl groups are isomorphic. Since $p$ is of trivial class number there is only one nontrivial way in which $C_p \subset W_{P\mathfrak{G}}$ can act on $T_{P\mathfrak{G}}$ and this is the same way that $C_p \subset W_{\mathfrak{G}}$ acts on $T_{\mathfrak{G}}$. ☐

Recall that

**Lemma 3.1.3.** *The Lie groups $\mathrm{PSU}(n)$ and $\mathrm{PU}(n)$ are isomorphic.*

*Proof.* We have $Z(\mathrm{U}(n)) \cong S^1$, $Z(\mathrm{SU}(n)) = Z(\mathrm{U}(n)) \cap \mathrm{SU}(n) \cong C_n$, and

$$\mathrm{U}(n) \cong \mathrm{SU}(n)Z(\mathrm{U}(n)) := \{ab : a \in \mathrm{SU}(n), b \in Z(\mathrm{U}(n))\}.$$

By the second isomorphism theorem of group theory, cf., e.g., [DF04, 3.3.18],

$$\mathrm{PSU}(n) = \mathrm{SU}(n)/Z(\mathrm{U}(n)) \cap \mathrm{SU}(n) \cong \mathrm{SU}(n)Z(\mathrm{U}(n))/Z(\mathrm{U}(n)) \cong \mathrm{PU}(n).$$

☐

It follows that $R(\Gamma_{P\,\mathrm{U}(p)}) \cong R(\Gamma_{P\,\mathrm{SU}(p)}) \cong R(\Gamma_{\mathrm{SU}(p)})$. An addendum about the case of $p = 1$: since $\mathrm{U}(1) \approx \mathrm{SO}(2) \approx S^1$ has trivial Weyl group we get $R(\Gamma_{\mathrm{U}(1)}) \cong R(\Gamma_{\mathrm{SO}(2)}) \cong R(S^1) = \mathbb{Z}[x^{\pm 1}]$. The case of $\mathrm{Sp}(1) \approx \mathrm{SU}(2)$ is covered by 3.1.1. Finally, $\mathrm{SO}(3) \approx \mathrm{PSU}(2)$ is taken care of by 3.1.2.

## 3.2 Topological $K$-theory of the classifying space of semidirect products

By the Atiyah-Segal completion theorem, B.0.4, knowing the representation ring of a compact Lie group yields the complex $K$-theory of its classifying space[2]. Namely, all we need to do is complete the representation ring at its augmentation ideal. The following lemma will be helpful in doing so:

**Lemma 3.2.1** ([Spa81, 5.1.5])**.** *Let $R$ be a ring. If $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ are two epimorphisms of $R$-modules, then*

$$\ker(f_1 \otimes f_2) = \alpha(\ker f_1 \otimes V_2) + \beta(V_1 \otimes \ker f_2)$$

*where $\alpha : \ker(f_1) \otimes V_2 \to V_1 \otimes V_2$ and $\beta : V_1 \otimes \ker(f_2) \to V_1 \otimes V_2$ are the canonical maps.*

Note that if $V_1$ and $V_2$ are flat $R$-modules then the maps $\alpha$ and $\beta$ are inclusions. In the category of abelian groups, a $\mathbb{Z}$-module is flat if and only if it is torsion-free.

**Proposition 3.2.2.** *Let $G = C_p$, where $p > 1$ is any prime. If $M$ is the $\mathbb{Z}G$-lattice $\mathbb{Z}^r \oplus \mathbb{Z}G^s \oplus IG^t$ and $\Gamma_M = T_M \rtimes G$ is the associated semidirect product, then*

$$K^*(B\Gamma_M) \cong R(T_{\mathbb{Z}^r})_{\hat{I_1}} \otimes R(T_{\mathbb{Z}G^s \oplus IG^t} \rtimes G)_{\hat{I_2}}$$

*with $I_1 = \ker(\dim : R(T_{\mathbb{Z}^r}) \to \mathbb{Z})$ and $I_2 = \ker(\dim : R(T_{\mathbb{Z}G^s \oplus IG^t} \rtimes G) \to \mathbb{Z})$. Moreover, if $p$ is of class number one and $T$ is any torus of rank $n$ on which $G$ acts by automorphisms with $\Gamma = T \rtimes G$ the associated semidirect product, then*

$$K^*(B\Gamma) \cong K^*(B\Gamma_M)$$

*for some integers $r$, $s$, $t$ obeying $r + sp + t(p-1) = n$.*

[2]cf. appendix B for a definition

*Proof.* By B.0.4, there is an isomorphism $K^*(B\Gamma) \cong R(\Gamma)_{\widehat{I\Gamma}}$ with $I\Gamma = \ker(\dim : R(\Gamma) \to \mathbb{Z})$. Let $p$ be of trivial class number. Then by 2.1.9, there exists a $\mathbb{Z}G$-lattice $M = \mathbb{Z}^r \oplus \mathbb{Z}G^s \oplus IG^t$ with integers $r, s, t$ obeying $r + sp + t(p-1) = n$ and

$$R(\Gamma) \cong R(\Gamma_M) \cong R(T_{\mathbb{Z}^r}) \otimes R(T_{\mathbb{Z}G^s \oplus IG^t} \rtimes G).$$

Since both $R(T_{\mathbb{Z}^r})$ and $R(T_{\mathbb{Z}G^s \oplus IG^t} \rtimes G)$ are $\mathbb{Z}$-torsion-free, it follows by 3.2.1 that

$$R(\Gamma_M)_{\widehat{I\Gamma}_M} \cong R(T_{\mathbb{Z}^r})_{\hat{I}_1} \otimes R(T_{\mathbb{Z}G^s \oplus IG^t} \rtimes G)_{\hat{I}_2}$$

with $I_1 = \ker(\dim : R(T_{\mathbb{Z}^r}) \to \mathbb{Z})$ and $I_2 = \ker(\dim : R(T_{\mathbb{Z}G^s \oplus IG^t} \rtimes G) \to \mathbb{Z})$. $\square$

Recall that as a consequence of the Hilbert basis theorem, cf., e.g., [AM69, 7.6], every finitely generated ring such as the representation ring of a compact Lie group is Noetherian. By [AM69, 10.12], completion of finitely generated modules over Noetherian rings preserves exact sequences and it follows that completion commutes with quotients. Moreover, if $\mathfrak{a} \subset \mathfrak{m} \subset A$ are two ideals in a Noetherian ring $A$, then $\mathfrak{a}_{\hat{\mathfrak{m}}} \cong \mathfrak{a}A_{\hat{m}}$.

Consequently, completion of $R(T_{\mathbb{Z}^r})$ and $R(T_{\mathbb{Z}G^s \oplus IG^t} \rtimes G)$ at their respective augmentation ideals is not difficult since our work thus far has enabled us to express these representation rings as finitely presented algebras, at least as long as we work over a splitting field for $G$.

We present some examples.

**Example 3.2.3.** Consider $K^*(B(T \rtimes C_2))$. By example 2.2.3 we have

$$R(\Gamma_{\mathbb{Z}G}) \cong \mathbb{Z}\left[\operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1, s_2^{\pm 1}, v\right] / \left(v^2 - 1, (v-1)\operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1\right),$$
$$R(\Gamma_{IG}) \cong \mathbb{Z}\left[\operatorname{ind}_{T_{IG}}^{\Gamma_{IG}} x_1, v\right] / \left(v^2 - 1, (v-1)\operatorname{ind}_{T_{IG}}^{\Gamma_{IG}} x_1\right).$$

Completing the former at $\big(c_1 := \operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1 - 2, c_2 := s_2 - 1, u := v - 1\big)$,

$$\begin{aligned} K^*(B\Gamma_{\mathbb{Z}G}) &\cong \mathbb{Z}[c_1+2, c_2+1, u+1]_{\widehat{(c_1,c_2,u)}} \Big/ (u^2+2u, u(c_1+2)) \\ &\cong \mathbb{Z}[c_1, c_2, u]_{\widehat{(c_1,c_2,u)}} \Big/ (u^2+2u, u(c_1+2)) \\ &\cong \mathbb{Z}[\![c_1, c_2, u]\!]/(u^2+2u, u(c_1+2)) \end{aligned}$$

and similarly

$$K^*(B\Gamma_{IG}) \cong \mathbb{Z}[\![c_1, u]\!]/(u^2+2u, u(c_1+2)).$$

A calculation based on the representation rings of 2.2.9 reveals

$$K^*(B\Gamma_{\mathbb{Z}G^2}; \mathbb{Q}) \cong \mathbb{Q}[\![c_1, c_2, d, e_1, e_2, u]\!]$$

subject to the relations $u^2 + 2u = 0$, $u(c_i + 2) = 0$, $ud = 0$ and

$$d^2 = (c_1^2 + 4c_1 - 4e_1)(c_2^2 + 4c_2 - 4e_2)$$

and similarly for $K^*(B\Gamma_{\mathbb{Z}G\oplus IG}; \mathbb{Q}) \cong \mathbb{Q}[\![c_1, c_2, d, e_1, u]\!]$ and $K^*(B\Gamma_{IG^2}; \mathbb{Q}) \cong \mathbb{Q}[\![c_1, c_2, d, u]\!]$. △

**Example 3.2.4.** Consider $K^*(B(T \rtimes C_3))$. Let $\zeta$ be a primitive third root of unity and let $\mathbb{K} = \mathbb{Q}(\zeta)$. Then by 2.2.10,

$$R_{\mathbb{K}}(\Gamma_{\mathbb{Z}G}) \cong \mathbb{K}[\operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1, \operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1x_2, \operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} x_1^2x_2, s_3^{\pm 1}, v]$$

subject to relations some of which are $v^3 = 1$ and $v \cdot \operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} \mathbf{x}^m = \operatorname{ind}_{T_{\mathbb{Z}G}}^{\Gamma_{\mathbb{Z}G}} \mathbf{x}^m$.

After a similar change of variables as in the previous example, the $K$-theory of $B\Gamma_{\mathbb{Z}G}$ with coefficients in $\mathbb{K}$ is

$$K^*(B\Gamma_{\mathbb{Z}G}; \mathbb{K}) \cong \mathbb{K}[\![c_1, c_2, c_3, d, u]\!]$$

subject to the relations $u^3 + 3u^2 + 3u = 0$, $u(c_i + 3) = 0$, and the equivalent of 2.2.10.1 in these variables. Similarly, $K^*(B\Gamma_{IG}; \mathbb{K}) \cong \mathbb{K}[\![c_1, c_2, c_3, u]\!]$. △

## 3.3 Results with mod $p$ coefficients

The primes $p$ which have a cyclotomic field $\mathbb{Q}(\zeta)$ of class number one are 2, 3, 5, 7, 11, 13, 17, 19; cf., e.g., [Was97, 11.1]. Proposition 3.2.2 can be extended to other primes $p$ by considering mod $p$ $K$-theory instead:

**Theorem 3.3.1.** *Let $G = C_p$, where $p > 1$ is any prime. If $T$ is a torus of rank $n$ on which $G$ acts by automorphisms, and $\Gamma = T \rtimes G$ is the associated semidirect product, then*

$$K^*(B\Gamma; C_p) \cong K^*(B\Gamma_M; C_p)$$

*with $M = \mathbb{Z}^r \oplus \mathbb{Z}G^s \oplus IG^t$, for some integers $r, s, t$ obeying $r + sp + t(p-1) = n$.*

Recall that by the universal coefficient theorem, C.0.5,

$$K^*(BH; C_p) \cong K^*(BH) \otimes C_p$$

for any compact Lie group $H$. The basic idea in proving this theorem is to introduce an object $C_p$-equivalent to the torus $T$ but that does not have any of the class number issues.

The class number of the $p$th cyclotomic field $\mathbb{Q}(\zeta)$ came into consideration in Reiner's theorem (1.2.3) since the ring of algebraic integers $\mathbb{Z}[\zeta]$ in $\mathbb{Q}(\zeta)$ is not a principal ideal domain in general. However, after $p$-adic completion $\mathbb{Q}_p(\zeta)$ has ring of algebraic integers $\mathbb{Z}_p[\zeta]$: a principal ideal domain; cf., e.g., [CR62, p. 121]. The $p$-adic version of Reiner's theorem is then the statement that *every* rank $n$ $\mathbb{Z}_pG$-lattice is isomorphic to $(\mathbb{Z}^r \oplus \mathbb{Z}G^s \oplus IG^t) \otimes \mathbb{Z}_p$ for some integers $r, s, t$ such that $r + sp + t(p-1) = n$.

It is then natural to ask which topological group has as Pontryagin dual group the $p$-adic integers $\mathbb{Z}_p$. The answer is given most directly by

**Theorem 3.3.2** (Pontryagin Duality Theorem)**.** *Let $H$ be a locally compact abelian group. Then the dual (character group) of $\widehat{H}$ is canonically isomorphic to $H$.*

See, e.g., [Rud62, 1.7.2] for a proof. The desired group is thus $\widehat{\mathbb{Z}}_p$. More explicitly:

**Definition 3.3.3.** Define the Prüfer group $C_{p^\infty}$ to be the colimit of the groups $C_{p^i}$ with respect to the homomorphisms $p : C_{p^i} \to C_{p^{i+1}}$ given by multiplication by $p$. Then $\widehat{\mathbb{Z}}_p \cong C_{p^\infty}$; cf., e.g., [MP12, 10.1.11]. A discrete group isomorphic to $C_{p^\infty}^n$ is known as a $p$-discrete torus of rank $n$ and denoted by $\breve{T}$.

Note that $\breve{T}$ is only locally compact since the Prüfer group is not finite. The $p$-discrete torus $\breve{T}$ can also be identified with the subgroup

$$\{t \in T : t^{p^i} = 1 \text{ for some } i \in \mathbb{N}\}$$

of a (regular) torus $T$. The natural map $\breve{T} \to T$ induces a $C_p$-equivalence of classifying spaces, that is, the induced map

$$H^*(BT; C_p) \to H^*(B\breve{T}; C_p)$$

is an isomorphism; cf. [DW94, 6.4].

One more ingredient is needed before we can prove 3.3.1:

**Theorem 3.3.4** (Leray-Serre-Atiyah-Hirzebruch spectral sequence)**.** *Let*

$$F \hookrightarrow E \to B$$

*be a fibration with $B$ a path-connected CW complex and let $h^*$ be a generalized cohomology theory. Then there exists a multiplicative spectral sequence*

$$E_2^{p,q} = H^p(B; h^q(F)) \Rightarrow h^{p+q}(E).$$

Note that $B$ is not required to be finite-dimensional. See, e.g., [Whi78, XIII.4.9* and the remarks after] for a proof. We are now ready for the

*Proof of 3.3.1.* The given semidirect product is a split extension of groups

$$1 \to T \to \Gamma \to G \to 1$$

inducing a fibration of classifying spaces

$$BT \to B\Gamma \to BG$$

with path-connected base. The Leray-Serre-Atiyah-Hirzebruch spectral sequence of this fibration in mod $p$ topological $K$-theory is

$$E_2^{i,j} = H^i(BG; K^j(BT; C_p)) \Rightarrow K^{i+j}(B\Gamma; C_p). \tag{3.3.4.1}$$

Also, the spectral sequence of the identity map $BT \to BT$ with fiber a point $*$ is

$$E_2^{i,j} = H^i(BT; K^j(*; C_p)) \Rightarrow K^{i+j}(BT; C_p). \tag{3.3.4.2}$$

The mod $p$ $K$-theory of a point is

$$K^j(*; C_p) = \begin{cases} 0 & \text{if } j \text{ is odd,} \\ C_p & \text{if } j \text{ is even.} \end{cases}$$

The associated $p$-discrete torus $\breve{T} \subset T$ inherits a $G$-action from $T$ making $H^i(BT; C_p) \cong H^i(B\breve{T}; C_p)$ a $G$-equivariant isomorphism. By the $p$-adic version of Reiner's theorem $\breve{T}$ must have Pontryagin dual $\mathbb{Z}_pG$-isomorphic to $M \otimes \mathbb{Z}_p$ with

$$M = \mathbb{Z}^r \oplus \mathbb{Z}G^s \oplus IG^t$$

for some integers $r, s, t$ such that $r + sp + t(p-1) = n$. But then there is also a $G$-equivariant isomorphism $H^i(B\breve{T}; C_p) \cong H^i(BT_M; C_p)$. It follows that there is an isomorphism of spectral sequences 3.3.4.2 leading to a $G$-equivariant isomorphism

$$K^*(BT; C_p) \cong K^*(BT_M; C_p).$$

This in turn induces an isomorphism of spectral sequences 3.3.4.1 leading to

$$K^*(B\Gamma; C_p) \cong K^*(B\Gamma_M; C_p).$$

□

# Chapter 4

# Adjoint Action and Equivariant $K$-Theory of Lie Groups

This chapter is concerned with computing the equivariant $K$-theory of compact connected Lie groups acting on themselves by conjugation. When the fundamental group of $G$ is torsion-free the answer is already known. To state it, we need the following

**Definition 4.0.5** ([Wei94, 8.8.1])**.** Fix a commutative ring $\mathbb{K}$ and let $R$ be a $\mathbb{K}$-algebra. Then the $R$-module $\Omega^1_{R/\mathbb{K}}$ of Kähler differentials of $R$ over $\mathbb{K}$ has the following presentation: there is one generator $dr$ for every $r \in R$ with $dk = 0$ if $k \in \mathbb{K}$. For each $r, s \in R$ there are two relations:

$$d(r+s) = dr + ds \qquad d(rs) = s \cdot dr + r \cdot ds.$$

Also let $\Omega^0_{R/\mathbb{K}} = R$ and $\Omega^i_{R/\mathbb{K}} = \bigwedge^i \Omega^1_{R/\mathbb{K}}$, the $i$th exterior power.

**Theorem 4.0.6** ([BZ00, 3.2])**.** *Let $G$ be a compact connected Lie group having torsion-free fundamental group and acting on itself by conjugation. Then there is an algebra homomorphism between the equivariant $K$-theory of $G$ and the ring of Kähler differentials of the representation ring over the* *integers. In symbols,*

$$K_G^*(G) \cong \Omega^*_{R(G)/\mathbb{Z}}. \tag{4.0.6.1}$$

This shows in particular that $K_G^*(G)$ is a free $R(G)$-module of rank

$$\sum_{k=0}^{\operatorname{rank} G} \binom{\operatorname{rank} G}{k} = 2^{\operatorname{rank} G}.$$

It is an open problem what the $K$-theory might be in case $\pi_1(G)$ has torsion. To show that 4.0.6.1 does not hold in the torsion case we inspect the projective unitary group $\mathrm{PU}(n) \cong \mathrm{PSU}(n) = \mathrm{SU}(n)/Z(\mathrm{SU}(n))$, cf. 3.1.3, having fundamental group isomorphic to

$$Z(\mathrm{SU}(n)) = \{\operatorname{diag}(\zeta, \ldots, \zeta) : \zeta^n = 1\} \cong C_n,$$

the center of $\mathrm{SU}(n)$. A maximal torus of $\mathrm{PU}(n)$ can be realized as the image of a maximal torus of $\mathrm{SU}(n)$ under the projection map

$$C_n \to \mathrm{SU}(n) \xrightarrow{\pi} \mathrm{PSU}(n) \cong \mathrm{PU}(n). \tag{4.0.6.2}$$

The long exact homotopy sequence applied to this fibration gives the isomorphism $\pi_1(\mathrm{PU}(n)) \cong C_n$. In the non-equivariant setting:

**Theorem 4.0.7** ([Pet67])**.** *Let* $\mathrm{PU}(p^r)$ *be the projective unitary group of degree* $p^r$ *with* $p$ *an odd prime. Then the* $K$*-theory and the cohomology of* $\mathrm{PU}(p^r)$ *are isomorphic as abelian groups.*

However, their approach in calculating the $K$-theory using the fibration

$$\mathrm{SU}(p^r) \to \mathrm{PU}(p^r) \to BC_{p^r}$$

and the Atiyah-Hirzebruch spectral sequence (3.3.4) does not generalize to the equivariant setting. Instead, we shall employ, cf. [Seg68a] and [Mat73],

**Theorem 4.0.8** (Equivariant Atiyah-Hirzebruch spectral sequence)**.** *Let $G$ be a compact Lie group and let $X$ be a finite $G$-CW complex. Then associated to the skeletal filtration*

$$X^0 \subset X^1 \subset \cdots \subset X^n \subset \cdots \subset X$$

*there exists a multiplicative spectral sequence*

$$\begin{aligned} E_2^{p,q} &\cong H_G^p(X;\mathcal{K}_G^q) \\ E_\infty^{p,q} &\cong K_{G,p}^{p+q}(X)/K_{G,p+1}^{p+q}(X) \end{aligned}$$

*where $K_{G,p}^n(X) := \ker\left(K_G^n(X) \to K_G^n(X^{p-1})\right)$ and where $\mathcal{K}_G^q$ is the coefficient system defined by $G/H \mapsto K_G^q(G/H)$; cf. appendix A.*

## 4.1 The projective unitary group of degree 2

In this section, let $G := \mathrm{PU}(2) \cong \mathrm{SU}(2)/C_2 \cong \mathrm{SO}(3)$ so that $\pi_1(G) \cong C_2$.

Computing $K_G^*(G)$ via the equivariant Atiyah-Hirzebruch spectral sequence, 4.0.8, begins with a skeletal filtration of $G$ to determine the Bredon cohomology $H_G^p(G;\mathcal{K}_G^q)$; cf. appendix A. Since the category of $G$-vector bundles over the homogeneous space $G/H$ is equivalent to the category of $H$-modules, cf. [Seg68a], $K_G^q(G/H)$ vanishes for $q$ odd and is $R(H)$ for $q$ even. This shows that $H_G^p(G;\mathcal{K}_G^q) = 0$ for $q$ odd and $H_G^p(G,\mathcal{K}_G^q) = H_G^p(G;\mathcal{R})$ for $q$ even, where $\mathcal{R}$ is the coefficient system $G/H \mapsto R(H)$. By A.0.3.1, the Bredon cochain complex is then given by

$$C_G^n(G;\mathcal{R}) \cong \bigoplus_{[\alpha]_n} R(G_\alpha)$$

where $[\alpha]_n$ are the $n$-dimensional $G$-cells and $\alpha$ without brackets is a representative.

Let

$$T = \left\{ \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} : x \in S^1 \right\}$$

be a maximal torus of $G$ with square brackets denoting the image of a matrix under $\pi : \mathrm{SU}(2) \to \mathrm{PU}(2)$; cf. 4.0.6.2. Also let $\sigma = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ be the generator of the Weyl group $W \cong C_2 \cong \langle \sigma \mid \sigma^2 = 1 \rangle$ of $G$ acting by permuting the coordinates of $T$. Then the orbit space $G/G$ is homeomorphic to $T/W$; cf. [BtD85, IV.2.6]. It follows that under the conjugation action, a skeletal filtration of $G$ is equivalent to a skeletal filtration of $T$. To study the latter, use an explicit homeomorphism of $T$ with $S^1$ given by

$$\phi : \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \mapsto x^2 =: y$$

Since $W$ acts by permutation, $\sigma$ sends $y$ to $y^{-1}$. Fixed points of this action are $y = 1$ and $y = -1$. A fundamental domain for the action of $W$ on $T$ is thus the upper hemisphere of the circle $S^1$ with two 0-cells and one 1-cell.

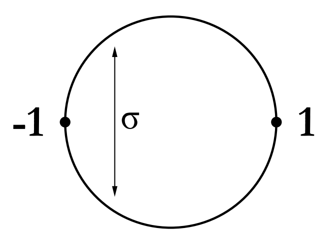


**Figure 4.1:** The action of $W$ on $T$.

We can now find the isotropy groups $G_\alpha$ and their representation rings. There are three types. First, the identity $\phi^{-1}(1) = I$ in $\mathrm{PU}(2)$ has isotropy the whole group $G$ with $R(G) \cong \mathbb{Z}[x + x^{-1}]$; cf. 2.2.5. Second, consider $\phi^{-1}(-1) = \left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right] = i \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G_{\phi^{-1}(-1)}$. Then we require

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{i.e.} \qquad \begin{bmatrix} a & -b \\ c & -d \end{bmatrix} = \begin{bmatrix} a & b \\ -c & -d \end{bmatrix}.$$

Solving this in PU(2) gives $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 0 & b \\ d & 0 \end{smallmatrix}\right]$. This means $G_{\phi^{-1}(-1)} \cong T \rtimes C_2$. By 3.1.2, 3.1.1, 2.2.3,

$$R(T \rtimes C_2) \cong \mathbb{Z}[x + x^{-1}, v]/\left(v^2 - 1, (v-1)(x + x^{-1})\right)$$

where $\dim(x + x^{-1}) = 2$ and $\dim v = 1$. Third, for any value other than $y = \pm 1$, $\phi^{-1}(y)$ is fixed only by $T$ and $R(T) \cong \mathbb{Z}[x^{\pm 1}]$.

It also follows from this $W$-CW structure that there is no cohomology $H^p_G(G; \mathcal{R})$ for $p > 1$ so that the spectral sequence has no nontrivial differentials and collapses at the $E_2$-page. We get

$$E^{p,q}_\infty \cong E^{p,q}_2 = \begin{cases} 0 & \text{if } q \text{ is odd;} \\ H^p_G(G; \mathcal{R}) & \text{if } q \text{ is even} \end{cases} \quad \Rightarrow \quad K^{p+q}_G(G).$$

Then

$$0 = E^{1,-1}_\infty \cong K^0_{G,1}(G)/K^0_{G,2}(G) = K^0_{G,1}(G)$$

implies

$$H^0_G(G; \mathcal{R}) \cong E^{0,0}_\infty \cong K^0_G(G)/K^0_{G,1}(G) = K^0_G(G)$$

and

$$0 = E^{0,1}_\infty \cong K^1_G(G)/K^1_{G,1}(G)$$

means

$$H^1_G(G; \mathcal{R}) \cong E^{1,0}_\infty \cong K^1_{G,1}(G) \cong K^1_G(G).$$

This can be summarized in the exact sequence

$$0 \to K^0_G(G) \to C^0_G(G; \mathcal{R}) \xrightarrow{\delta} C^1_G(G; \mathcal{R}) \to K^1_G(G) \to 0$$

where $C^*_G(G; \mathcal{R})$ is the Bredon cochain complex with differential $\delta$. By our computations thus far this yields

$$0 \to K^0_G(G) \to R(T \rtimes C_2) \oplus R(G) \xrightarrow{\delta} R(T) \to K^1_G(G) \to 0$$

with $\delta = i_2^* - i_1^*$ induced by the inclusions $i_1 : T \hookrightarrow T \rtimes_\psi C_2$, $i_2 : T \hookrightarrow G$; cf. A.0.3.2.

We can now compute the cohomology. Let

$$A = \sum a_i(x + x^{-1})^i + cv \in R(T \rtimes C_2)$$

and $B = \sum b_i(x + x^{-1})^i \in R(G)$ and suppose that $\delta(A, B) = 0$, that is,

$$\sum (a_i - b_i)(x + x^{-1})^i + c = 0.$$

Then $a_i = b_i$ for all $i \neq 0$ and $a_0 - b_0 + c = 0$ so that

$$\begin{aligned}(A, B) &= \left(\sum_i a_i(x + x^{-1})^i + cv, \sum_{i \neq 0} b_i(x + x^{-1})^i + b_0\right) \\ &= \left(\sum_i a_i(x + x^{-1})^i + cv, \sum_i a_i(x + x^{-1})^i + c\right) \\ &= \sum_i a_i(x + x^{-1}, x + x^{-1})^i + c(v, 1)\end{aligned}$$

so that

$$K_G^0(G) \cong \ker \delta \cong R(T \rtimes C_2).$$

Finally, the image of $\delta$ is clearly $\mathbb{Z}[x + x^{-1}] \cong R(G)$. So that, as an $R(G)$-module,

$$K_G^1(G) \cong R(T)/R(G).$$

**Proposition 4.1.1.** *$K_G^0(G)$ is not a free $R(G)$-module.*

*Proof.* We have

$$K_G^0(G) \cong R(T \rtimes C_2) \cong \mathbb{Z}[x + x^{-1}, v]/\big(v^2 - 1, (v - 1)(x + x^{-1})\big)$$

with $R(G) = \mathbb{Z}[x + x^{-1}]$ acting on $K_G^0(G)$ in the obvious way. Clearly, $R(T \rtimes C_2) \not\cong R(G)$. So suppose by contradiction that there is a basis $E$

of $R(T \rtimes C_2)$ as an $R(G)$-module with two different elements $\alpha \neq \beta \in E$. Then $(x + x^{-1}) \cdot \alpha = f$ and $(x + x^{-1}) \cdot \beta = g$ for some $f, g \in R(G)$ since multiplication by $x + x^{-1}$ kills all possible occurrences of $v$. But then

$$g \cdot (x + x^{-1}) \cdot \alpha - f \cdot (x + x^{-1}) \cdot \beta = 0.$$

A contradiction. □

**Proposition 4.1.2.** *$K^1_G(G)$ is a free $R(G)$-module of rank 1.*

*Proof.* We have $K^1_G(G) = R(T)/R(G) = \mathbb{Z}[x, x^{-1}]/\mathbb{Z}[x + x^{-1}]$ and claim that

$$\mathbb{Z}[x, x^{-1}] \cong \mathbb{Z}[x + x^{-1}]\{1, x\}.$$

To see this, note that $(x + x^{-1})x = x^2 + 1$ so that $x^2 = (x + x^{-1})x - 1$. Now argue inductively to see that all powers of $x$ are contained in $\mathbb{Z}[x+x^{-1}]\{1, x\}$. Likewise, $x^{-1} = (x + x^{-1}) - x$ and so again by induction or by swapping $x$ and $x^{-1}$ in the expressions for $x^n$ we see that all powers of $x^{-1}$ are contained in $\mathbb{Z}[x + x^{-1}]\{1, x\}$. The claim follows which finishes the proof. □

Another way to establish this last fact is to observe that $K^1_{\mathrm{PU}(2)}(\mathrm{PU}(2)) \cong K^1_{\mathrm{SU}(2)}(\mathrm{SU}(2))$ since $R(T_{\mathrm{PU}(2)})/R(\mathrm{PU}(2)) \cong R(T_{\mathrm{SU}(2)})/R(\mathrm{SU}(2))$. In summary,

$$K^*_G(G) \cong R(T \rtimes C_2) \oplus R(G)$$

In particular, we have shown that $K^*_G(G)$ is *not* a free $R(G)$-module of rank 2 as would have been the case had $G$ had torsion-free fundamental group; cf. 4.0.6.

## 4.2 The projective unitary group of degree 3

Inspired by the results of the last section, we attempt to compute in a similar fashion $K^*_G(G)$ for $G := \mathrm{PU}(3) \cong \mathrm{SU}(3)/C_3$ with $C_3$ generated by a primitive third root of unity $\zeta$. Let

$$T = \left\{ \begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix} : x_1 x_2 x_3 = 1, x_i \in S^1 \right\} \tag{4.2.0.1}$$

be a maximal torus of $G$ with square brackets denoting the image of a matrix under $\pi : \mathrm{SU}(3) \to \mathrm{PU}(3)$; cf. 4.0.6.2. Also let $W \cong S_3$ be the Weyl group of $G$.

Once again we find a skeletal filtration of $T$ rather than $G$. To do so, use the explicit homeomorphism of $T$ with $S^1 \times S^1$ given in additive notation $x_i = e^{2\pi i t_i}$, $t_i \in \mathbb{R}/\mathbb{Z}$, by

$$\begin{aligned} \phi : T &\to S^1 \times S^1 \\ [\mathrm{diag}(t_1, t_2, t_3)] &\mapsto (t_2 - t_1, 3t_1) =: (y_1, y_2) \end{aligned}$$

Let $\tau = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ and $\sigma = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]$ generate

$$W \cong S_3 = C_3 \rtimes C_2 = \langle \sigma \mid \sigma^3 = 1 \rangle \rtimes \langle \tau \mid \tau^2 = 1 \rangle.$$

Then the induced action of $W$ on $S^1 \times S^1$ is

$$\begin{aligned} \tau(y_1, y_2) &= (-y_1, 3y_1 + y_2) \\ \sigma(y_1, y_2) &= (y_1 + y_2, -3y_1 - 2y_2). \end{aligned}$$

A fundamental domain for this action is described by a triangle formed by the three line segments

$$\{(0, y_2)\},\ \{(y_1, 0) : -1/3 \leqslant y_1 \leqslant 0\},\ \{(y_1, -3y_1) : -1/3 \leqslant y_1 \leqslant 0\}$$

where the last two have been identified via $\sigma$; cf. top left square of figure 4.2.

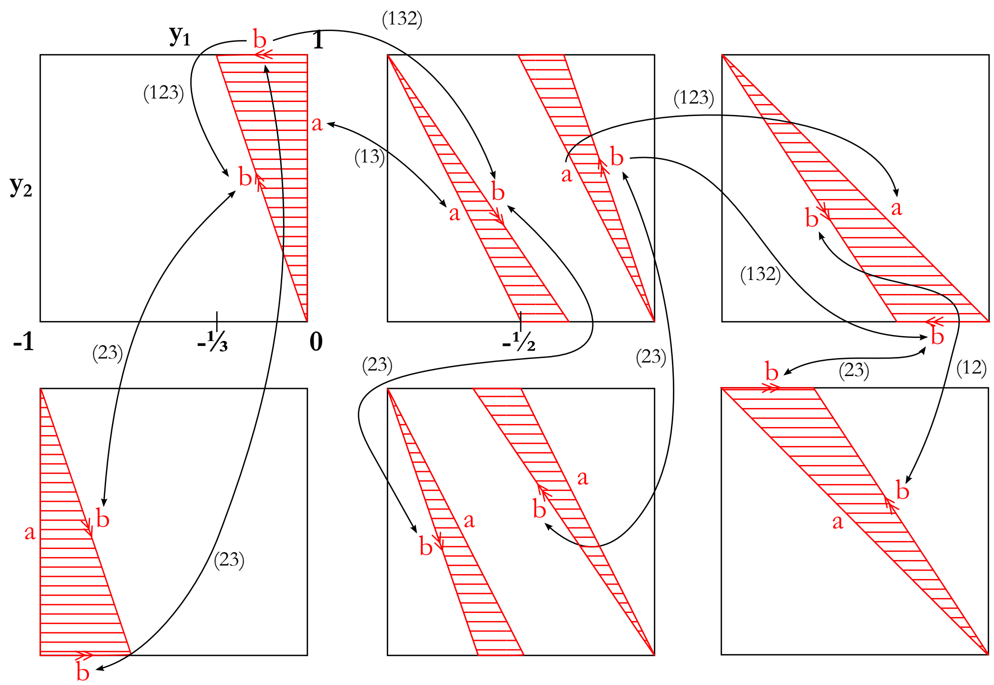


**Figure 4.2:** Translates of the fundamental domain of $T/W$ (top left square).

The $W$-CW complex $T$ thus has two 0-cells, two 1-cells, and one 2-cell. Because the fundamental domain is only 2-dimensional, there are no differentials on the $E_3$-page and the spectral sequence 4.0.8 collapses. It follows that

$$K^0_G(G)/H^2_G(G;\mathcal{R}) \cong H^0_G(G;\mathcal{R}) \quad \text{and} \quad K^1_G(G) \cong H^1_G(G;\mathcal{R}). \qquad (4.2.0.2)$$

Next we need to identify the isotropy groups $G_\alpha$ and their representation rings. This time there are four types. First, there is the 0-cell, $(-1/3, 0) \in S^1 \times S^1$. Let $\left[\begin{smallmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \\ z_7 & z_8 & z_9 \end{smallmatrix}\right] \in G_{\phi^{-1}(-1/3,0)}$ where $\phi^{-1}(-1/3,0) = \left[\begin{smallmatrix} \zeta & & \\ & 1 & \\ & & \zeta^2 \end{smallmatrix}\right]$. Solving

$$\begin{bmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \\ z_7 & z_8 & z_9 \end{bmatrix} \begin{bmatrix} \zeta & & \\ & 1 & \\ & & \zeta^2 \end{bmatrix} = \begin{bmatrix} \zeta & & \\ & 1 & \\ & & \zeta^2 \end{bmatrix} \begin{bmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \\ z_7 & z_8 & z_9 \end{bmatrix}$$

in $G$ gives

$$\begin{bmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \\ z_7 & z_8 & z_9 \end{bmatrix} = \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_5 & 0 \\ 0 & 0 & z_9 \end{bmatrix}, \begin{bmatrix} 0 & z_2 & 0 \\ 0 & 0 & z_6 \\ z_7 & 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 0 & z_3 \\ z_4 & 0 & 0 \\ 0 & z_8 & 0 \end{bmatrix}.$$

This means that $G_{\phi^{-1}(-1/3,0)}$ is isomorphic to the semidirect product

$$T \rtimes C_3 \subset T \rtimes W.$$

Since $p = 3$ is of trivial class number, it follows by 3.1.2 and 3.1.1 that the representation ring of $T \rtimes C_3$ is given by 2.1.7 with $M = \widehat{T} \cong IC_3$.

Second, there is the vertex at the origin corresponding to the identity $\phi^{-1}(0,0) = I$ in $G$. It has isotropy all of $G$. Because $G$ is compact and connected, we can express $R(G)$ as $R(T)^W$.

Third, the edge labeled $a$ in figure 4.2 has isotropy group

$$G_{\phi^{-1}(a)} = \left\{ \begin{bmatrix} z_1 & z_2 & 0 \\ z_4 & z_5 & 0 \\ 0 & 0 & z_9 \end{bmatrix} : z_9 \cdot \det \begin{pmatrix} z_1 & z_2 \\ z_4 & z_5 \end{pmatrix} = 1, \begin{pmatrix} z_1 & z_2 \\ z_4 & z_5 \end{pmatrix} \in \mathrm{U}(2) \right\}$$

which is compact and connected and has Weyl group $\langle \tau : \tau^2 = 1 \rangle = C_2 \subset W$ so that

$$R(G_{\phi^{-1}(a)}) \cong R(T)^{C_2}. \tag{4.2.0.3}$$

Diagonalizing

$$\begin{pmatrix} z_1 & z_2 \\ z_4 & z_5 \end{pmatrix} = U \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} U^{-1}$$

with $U$ unitary, the connectedness can be observed by defining a path

$$p(t) = \left[\begin{array}{c|c} U & \\ \hline & 1 \end{array}\right] \begin{bmatrix} e^{i(1-t)\theta_1} & 0 & 0 \\ 0 & e^{i(1-t)\theta_2} & 0 \\ 0 & 0 & z_9 e^{it\theta_1} e^{it\theta_2} \end{bmatrix} \left[\begin{array}{c|c} U^{-1} & \\ \hline & 1 \end{array}\right]$$

within $G_{\phi^{-1}(a)}$ connecting any element to the identity.

Lastly, the edge labeled $b$ in figure 4.2 as well as the 2-cell have isotropy the torus $T$. The Bredon cochain complex thus becomes

$$0 \to R(T \rtimes C_3) \oplus R(G) \xrightarrow{\delta_1} R(T)^{C_2} \oplus R(T) \xrightarrow{\delta_2} R(T) \to 0$$

with

$$\delta_1(A, B) = (0, i_1^* A - i_2^* B) \quad \text{and} \quad \delta_2(A, B) = i_3^* A - \sigma^* B + B$$

induced by the inclusions $i_1 : T \hookrightarrow T \rtimes C_3$, $i_2 : T \hookrightarrow G$, $i_3 : T \hookrightarrow G_{\phi^{-1}(a)}$; cf. A.0.3.2.

**Proposition 4.2.1.** *$H^0_G(G; \mathcal{R})$ is not a free $R(G)$-module.*

*Proof.* By the proof of 2.1.7, we know that $i_1^*$ surjects onto $R(T)^{C_3}$. By identifying $R(G)$ with $R(T)^W \subset R(T)^{C_3}$, we can interpret the kernel of $\delta_1$ to be those elements of $R(T \rtimes C_3)$ which map to $R(T)^W$ under $i_1^*$. The $R(G)$-module structure on $R(T \rtimes C_3)$—and hence also on $H^0_G(G; \mathcal{R})$—is given by writing a $W$-invariant $f \in R(G) \cong R(T)^W$ as a linear combination of $C_3$-orbit sums $\sum a_i \mathbf{x}^{m_i}$ and identifying $\mathbf{x}^{m_i}$ with $\operatorname{ind} \mathbf{x}^{m_i} \in R(T \rtimes C_3)$ when $m_i \neq 0$; cf. 2.1.7. Let $f \in R(G)$ be any linear combination of $C_3$-orbit sums without constant term. Then by 2.1.7, $f(v-1) = 0$ where $v - 1 \in H^0_G(G; \mathcal{R})$ since $i_1^*(v - 1) = 0$. That is, $H^0_G(G; \mathcal{R})$ has $R(G)$-torsion and cannot be free. □

By 4.2.0.2, $K^0_G(G)/H^2(G; \mathcal{R}) \cong H^0(G; \mathcal{R})$, that is, there is an extension of $R(G)$-modules

$$0 \to H^2(G;\mathcal{R}) \hookrightarrow K_G^0(G) \xrightarrow{\pi} H^0(G;\mathcal{R}) \to 0.$$

Since $\pi$ is surjective, there is a virtual bundle $E := [L] - [1] \in K_G^0(G)$ such that $\pi(E) = v - 1$. Note that

$$\pi(f \cdot E) = f\pi(E) = f(v-1) = 0$$

so that $f \cdot E \in H^2(G;\mathcal{R}) \cong R(T)/\operatorname{im}\delta_2$. Unfortunately, I do not know at this point how to prove that $f \cdot E = 0$, i.e., that $E$ is an $R(G)$-torsion element of $K_G^0(G)$ which would show that $K_G^0(G)$ is not a free $R(G)$-module. In the torsion-free fundamental group case one can use

**Theorem 4.2.2** ([Pit72])**.** *Let $H$ be a compact connected Lie group with torsion-free fundamental group and $T$ a maximal torus. Then $R(T)$ is a free $R(H)$-module of rank the order of the Weyl group $W$.*

More still, there exists an explicit algorithm by which to write down an $R(H)$-basis known as Steinberg basis for $R(T)^{\widetilde{W}}$ for $\widetilde{W}$ any reflection subgroup of $W$ with $T$ and $W$ the torus respectively Weyl group of a compact connected Lie group $H$ with torsion-free fundamental group; cf. [Ste75]. This enables one to compute Bredon cohomology groups as free $R(H)$-modules. It is an open problem what happens when there is torsion in the fundamental group.

What is known concerning $G := \mathrm{PU}(p)$ with $p$ a fixed prime is that it has a 0-cell

$$\begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{p-1} \end{bmatrix}$$

with isotropy isomorphic to $T \rtimes C_p$ with $C_p \subset W \cong S_p$. The representation ring of $T \rtimes C_p$ is known by 3.1.2, at least as long as $p$ is of trivial class number. By using algorithms computing Weyl invariants, it is also possible to compute $R_{\mathbb{K}}(G_\alpha) \cong R_{\mathbb{K}}(T)^{W_\alpha}$ for $W_\alpha \subset W$ the Weyl group of a connected isotropy group $G_\alpha$ and $\mathbb{K}$ a splitting field for $W_\alpha$; cf. 2.2.6. For instance, using the coordinates $y_1, y_2$ on the torus of PU(3) above and since $W$ is a reflection group we could run the subduction algorithm to obtain a finite SAGBI basis for $R_{\mathbb{K}}(G) \cong R_{\mathbb{K}}(T)^W$ with $\mathbb{K}$ a splitting field for $S_3$; cf. 2.3.3.

# Chapter 5

# Further Directions

At the end of the last chapter we discussed the possibility of computing $K^*_{\mathrm{PU}(3)}(\mathrm{PU}(3))$ explicitly. Even if that fails, it would be tempting to show that the equivariant $K$-theory of $\mathrm{PU}(p)$ for $p$ any prime is not a free $R(\mathrm{PU}(p))$-module. It is not known, however, whether the spectral sequence 4.0.8 for $\mathrm{PU}(p)$ collapses when $p > 3$: an area for future research. Examples of other compact connected Lie groups with fundamental groups having torsion and thus lending themselves to the kind of investigations pursued in that chapter are

$$\mathrm{PSO}(2p+1) \cong \mathrm{PO}(2p+1) \cong \mathrm{SO}(2p+1), \quad \mathrm{SO}(2p), \quad \mathrm{PSp}(p)$$

all with fundamental group $C_2$, and

$$\mathrm{PSO}(4p), \quad \mathrm{PSO}(4p+2)$$

with fundamental group $C_2 \oplus C_2$ respectively $C_4$. For these, an entirely different approach may be necessary.

Another area for further research concerns the cohomology of the classifying space of semidirect products $T \rtimes C_p$. By using the Leray-Serre-Atiyah-Hirzebruch spectral sequence, 3.3.4, it may be possible to make deductions.

Note, however, that this spectral sequence is usually more effective in going from cohomology to $K$-theory and not the other way around. For instance, as mentioned in the introduction this is how the $K$-theory of the classifying space of the crystallographic groups $\mathbb{Z}^n \rtimes C_p$ was computed. Perhaps there are also more direct ways to attack the cohomology of $B(T \rtimes C_p)$.

More work can also be done in the direction of obtaining $R(T \rtimes C_p)$—and hence the $K$-theory with integer coefficients instead of $C_p$ coefficients—when $p$ is not of class number one. The obstacle here seems to be the class number quickly becoming very large. For instance, the class number of the cyclotomic field $\mathbb{Q}(\zeta)$ with $\zeta$ a primitive 41st root of unity is already 121. It is unknown whether all these different ideal classes also give rise to different representation rings. To begin with, it would be interesting to exhibit an ideal class giving rise to a $\mathbb{Z}C_p$-lattice not isomorphic to one of the standard ones.

Likewise, it may be possible to use techniques such as Galois descent mentioned before 2.2.4 or an algorithm that does not require a splitting field $\mathbb{K}$ to obtain the integral representation ring $R(T \rtimes C_p)$ instead of the weaker $R(T \rtimes C_p) \otimes \mathbb{K}$. That is, is there a way to calculate $\mathbb{Z}[M]^{C_p}$ with $M$ a $\mathbb{Z}C_p$-lattice without needing to pass to a splitting field of $C_p$?

Lastly, it would be interesting to study the representation theory of $T \rtimes G$ when $G$ is any other finite group acting through smooth automorphisms on a rank $n$ torus $T$. This has been started for $G = D_{2p}$ and $G = S_n$ in section 2.3. In particular, we have seen that the multiplicative invariant algebra $\mathbb{Z}[M]^G$ with $M$ a $\mathbb{Z}G$-lattice appears via Cartan subgroups. If $G$ is a reflection group, we can find a finite basis for this invariant ring tensored by a splitting field by the subduction algorithm. As discussed, the difficulty in determining $R(T_M \rtimes G)$ for $M$ an arbitrary $\mathbb{Z}G$-lattice is in a) providing a complete reducibility theorem for $\mathbb{Z}G$-lattices into indecomposables the way Reiner's theorem did for $G = C_p$, b) understanding the product structure via Cartan subgroups and c) finding an algorithm to produce a finite presentation for the invariant ring $\mathbb{Z}[M]^G$.

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# Appendix A

# Equivariant algebraic topology

In this appendix, we review the basics of the study of topological spaces with group actions. The references are [May96], [Bre67], and [Lüc89].

Let $G$ be a topological group, that is, a group which is also a topological space in which the group operations are continuous maps. Requiring that $G$ also be a smooth manifold with smooth group operations gives a Lie group. The objects of study in equivariant algebraic topology are $G$-spaces and $G$-maps[1], that is, topological spaces $X$ equipped with a continuous action $G \times X \to X$ by $G$ and maps $f : X \to Y$ between them such that $f(gx) = gf(x)$. For $x \in X$,

$$G_x = \{g \in G : gx = x\} \tag{A.0.2.1}$$

is called the isotropy group of $x$ and

$$G(x) = \{gx : g \in G\} \tag{A.0.2.2}$$

is called the $G$-orbit of $x$.

A $G$-CW complex is a $G$-space $X$ with a decomposition $X = \operatorname{colim} X^k$

[1] Also known as $G$-equivariant maps.

such that

$$X^0 = \coprod_H G/H, \quad X^{n+1} = X^n \cup_{\varphi_n} \left( \coprod_H D^{n+1} \times G/H \right) \tag{A.0.2.3}$$

where $D^{n+1} \times G/H$ are called $G$-cells and $\varphi_n$ is made up of attaching $G$-maps $\varphi_{n,H} : S^n \times G/H \to X^n$. For compact Lie groups, the situation is particularly nice:

**Theorem A.0.3** ([Mat71, 4.4])**.** *Let $G$ be a compact Lie group. Then any compact smooth $G$-manifold has a finite $G$-CW complex structure inducing a triangulation on the orbit space.*

There are several generalized equivariant cohomology theories, equivariant $K$-theory reviewed in the next appendix being one of them. Another prominent one which we will make use of is Bredon cohomology defined next.

Let $\mathcal{O}_G$ be the orbit category of $G$, that is, the category with an object $G/H$ for every subgroup $H \subset G$ and morphisms the $G$-maps $G/H \to G/K$ between them. Let $R$ be a commutative ring. A coefficient system in the category of $R$-modules is defined to be a contravariant functor from $\mathcal{O}_G$ to the category of $R$-modules. Let $X$ be a $G$-CW complex. We have a coefficient system

$$\underline{C}_n(X)(G/H) = H_n\left((X^n)^H, (X^{n-1})^H\right)$$

where the right term is ordinary homology. The connecting homomorphisms of the triples $\left((X^n)^H, (X^{n-1})^H, (X^{n-2})^H\right)$ specify a map

$$d : \underline{C}_n(X) \to \underline{C}_{n-1}(X)$$

of coefficient systems such that $d^2 = 0$. Given another coefficient system $M$, define a cochain complex $(C^*_G(X; M), \delta)$, by

$$C^n_G(X; M) = \mathrm{Hom}_{\mathcal{O}_G}(\underline{C}_n(X), M) \text{ and } \delta = \mathrm{Hom}_{\mathcal{O}_G}(d, \mathrm{id})$$

where $\mathrm{Hom}_{\mathcal{O}_G}(M', M)$ denotes the abelian group of morphisms, i.e., natural transformations of coefficient systems $M' \to M$. The Bredon cohomology of $X$ with coefficients in $M$, $H^*_G(X; M)$, is defined to be the cohomology of this cochain complex.

Explicitly, there is an isomorphism

$$C^n_G(X; M) \cong \bigoplus_{[\alpha]_n} M(G/G_\alpha) \tag{A.0.3.1}$$

where $[\alpha]_n$ are the $n$-dimensional $G$-cells of $X$ and $\alpha$ without brackets is a representative. With this description, if $x \in C^n_G(X; M)$ and $[\beta]_{n+1}$ are the $(n+1)$-cells, then

$$\delta(X)_\beta = \sum_{[\alpha]_n} [\alpha : \beta] M(\alpha, \beta)(x_\alpha) \in M(G/G_\beta) \tag{A.0.3.2}$$

where $[\alpha, \beta]$ is the degree of the map induced by the attaching map of the cell $\beta$, cf. A.0.2.3 and [AGP02, 7.3.6]. If $[\alpha, \beta] \neq 0$, there is a $G$-map $i_{\alpha,\beta} : G/G_\alpha \to G/G_\beta$ and $M(\alpha, \beta)$ is defined as $M(i_{\alpha,\beta}) : M(G/G_\alpha) \to M(G/G_\beta)$.

# Appendix B

# Equivariant $K$-theory

In this appendix, we review the basics of representation theory of compact Lie groups and $K$-theory of $G$-spaces. The references are [AS69], [Seg68a], and [Seg68b].

A representation of a compact Lie group $G$ on a finite-dimensional complex vector space $V$ is a continuous homomorphism $G \to \mathrm{Aut}(V)$. The character of a representation $V$ of $G$ is the smooth function $\chi_V : G \to \mathbb{C}$ given by $g \mapsto \mathrm{Tr}(l_g)$ where $\mathrm{Tr}(l_g)$ is the trace of the linear map $l_g : V \to V, v \mapsto gv$.

The character ring $R(G)$ is the free abelian group generated by the irreducible characters of $G$ with ring structure given by $\chi_V \cdot \chi_W = \chi_{V \otimes W}$. By [Seg68b, 3.3], $R(G)$ is a finitely generated ring. It can also be constructed as the Grothendieck ring of the symmetric monoidal[1] abelian category of finite-dimensional representations of $G$ (under $\oplus$ and $\otimes$).

A $G$-vector bundle over a $G$-space $X$ is a $G$-space $E$ with a $G$-map $p : E \to X$ such that

(i) $p : E \to X$ is an ordinary complex vector bundle;

(ii) for each $g \in G$ and $x \in X$ the map $g : E_x \to E_{gx}$ is a vector space homomorphism.

[1] A symmetric monoidal category is a category with a product operation for which the product is "as commutative as possible"; cf. [MP12, 16.3].

If $X$ is a compact $G$-space, $G$-equivariant $K$-theory $K_G^*(X)$ is defined as the Grothendieck ring of the symmetric monoidal abelian category of $G$-vector bundles over $X$ under direct sum and tensor product. The natural map $X \to *$ induces a homomorphism $K_G^*(*) \to K_G^*(X)$ which turns $K_G^*(X)$ into a $K_G^*(*)$-algebra. Recall that $K_G^*(*) \cong R(G)$ since $G$-vector bundles over a point are representations of $G$.

Let $IG = \ker(\dim : R(G) \to \mathbb{Z})$ be the augmentation ideal in $R(G)$ and let $BG$ be the classifying space of $G$, that is, the quotient of a weakly contractible space $EG$ by a free action of $G$. The space $EG$ is constructed as the infinite join of $G$ with itself; cf., e.g., [Ros94, 5.1.15]. Denote by

$$X \times_G EG := (X \times EG)/G$$

the quotient by the diagonal action also known as the Borel construction.

**Theorem B.0.4** (Completion Theorem)**.** *Let $G$ be a compact Lie group and let $X$ be a finite $G$-CW complex. Then the map $\pi : X \times EG \to X$ induces an isomorphism of pro-algebras*[2]

$$\pi^* : K_G^*(X)_{\widehat{IG}} \to K^*(X \times_G EG)$$

*where the term on the right is defined as $\varprojlim K_G^*(X \times EG^n)$ for a suitable filtration of $EG$. In particular, if $X = *$ one obtains*

$$R(G)_{\widehat{IG}} \cong K^*(BG),$$

*the ordinary complex $K$-theory of $BG$.*

[2] A pro-object of a category $C$ is a cofiltered limit of objects of $C$. For instance, a pro-(finite group) is an inverse limit of an inverse system of finite groups.

# Appendix C

# $K$-theory with coefficients

While the reader may be versed in ordinary cohomology with coefficients, this may not be so for other generalized cohomology theories such as $K$-theory used in this thesis. Following [Ada74, 200ff.] we recall the main definitions and theorems.

Given an abelian group $A$, construct a free resolution

$$0 \to R \xrightarrow{i} F \xrightarrow{\pi} A \to 0.$$

Let $\mathbb{S} = \{\Sigma^n S^0\}$ denote the sphere suspension spectrum and take wedge sums such that

$$\pi_0\left(\bigvee_{a\in\mathcal{A}} \mathbb{S}\right) = R \qquad \text{and} \qquad \pi_0\left(\bigvee_{b\in\mathcal{B}} \mathbb{S}\right) = F.$$

Also take a map $f : \bigvee_{a\in\mathcal{A}} \mathbb{S} \to \bigvee_{b\in\mathcal{B}} \mathbb{S}$ inducing $i$. The homotopy cofiber (mapping cone) of $f$

$$M_A = \left(\bigvee_{b\in\mathcal{B}} \mathbb{S}\right) \cup_f C\left(\bigvee_{a\in\mathcal{A}} \mathbb{S}\right)$$

is then a Moore spectrum of type $A$, that is,

$$\begin{aligned} \pi_r(M_A) &= 0 \text{ for } r < 0, \\ \pi_0(M_A) = H_0(M_A) &= A \text{ for } r = 0, \\ H_r(M_A) &= 0 \text{ for } r > 0. \end{aligned}$$

Finally, given a spectrum $E$ define the corresponding spectrum with coefficients in $A$ to be the smash product $E \wedge M_A$. Thus, for instance,

$$K^*(-;A) := [-, B\,\mathrm{U} \wedge M_A]$$

defines complex $K$-theory with coefficients in $A$. $K$-theory with coefficients is related to the usual $K$-theory by

**Theorem C.0.5** (Universal coefficient theorem for generalized cohomology)**.** *Let $h^*$ be a generalized cohomology theory. Then there exists an exact sequence*

$$0 \to h^n(X) \otimes A \to h^n(X;A) \to \mathrm{Tor}_1^{\mathbb{Z}}(h^{n+1}(X), A) \to 0.$$

It follows that

$$K^*(-;A) \cong K^*(-) \otimes A$$

for all torsion-free abelian groups $A$. More still, since by B.0.4 $K^1(BG) = 0$ for any compact Lie group $G$ we have

$$K^*(BG;A) \cong K^*(BG) \otimes A$$

for *any* abelian group $A$.