Momentum-Space Classification of Topologically Stable Fermi Surfaces

by

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Abstract

The purpose of the present work is to derive a classification for topologically stable Fermi surfaces for translationally invariant systems with no electron-electron interactions. To derive such a classification we introduce the necessary concepts in condensed matter and electronic band theory as well as those in mathematics such as topological spaces, building up to topological K-theory and its connections with Fredholm operators. We further compute such classes when there is only translational invariance for dimensions $d = 1, 2, 3$ and discuss the inclusion of other symmetries.

Preface

Chapter 1 is an introduction to the subject of topological phases matter and topological Fermi surfaces. Chapter 2 is an introduction to Analysis, topological spaces, homotopy and K-theory. Chapter 3 describes the type of physical systems we wish to describe and introduces the basics if electronic band theory in condensed matter. Chapter 4 derives a classification of topologically stable Fermi surfaces from the material introduced in previous chapters. Finally, Chapter 5 is a summary of the present work.

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Chapter 1

Introduction

The study and classification of topological phases of matter began in 1980 with the unexpected experimental discovery by von Klitzing, Dorda and Pepper [27] of exact quantization of the Hall conductance of a two-dimensional electron gas in a strong magnetic field, in integer units of $e^2/\hbar$, known as the Integer Quantum Hall effect. Experimentally, it is the most precise measurement of quantization known to date, such that its used as the standard calibration for resistivity world wide since 1990. Laughlin [29] related this effect to gauge invariance but it was not until a couple of years later that Thouless, Kohomoto, Nightingale and Den Nijs [36] explained the origin of the exact quantization of the Kubo formula for the Hall conductance (using Berry's phase [8]) as a Chern number, a topological invariant discovered by the Chinese mathematician S.S Chern in 1946. This invariant is also known to physicists as the TKNN invariant, named after the aforementioned authors.

It was later realized that *gapped* systems such as the Integer Quantum Hall effect with different values for their topological invariants, represent different phases of matter, so-called *topological phases* of matter, which have very robust edge-states, that are insensitive to impurities and perturbations. This means that one can not transit from one state to another without closing the energy gap. Closing and reopening the energy gap is therefore tantamount to changing the value of a corresponding topological invariant, such as the TKNN invariant.

This discovery is a paradigm shifting breakthrough in condensed matter, for it was thought that phase transitions only occurred through symmetry breaking, according to the Landau paradigm[28]. Now it is known that there are different topological phases protected by a given symmetry [11], meaning that one cannot transit from one phase to other without breaking and restablishing the corresponding symmetry. Examples of such symmetries are time-reversal, parity and chirality among others.

Much later, in 2004 Kane and Mele [23] discovered a $\mathbb{Z}_2$- invariant for the three dimensional Quantum Spin Hall effect, in which the system is naturally time-reversal invariant due to spin-orbit coupling. In 2006 Bernveg,

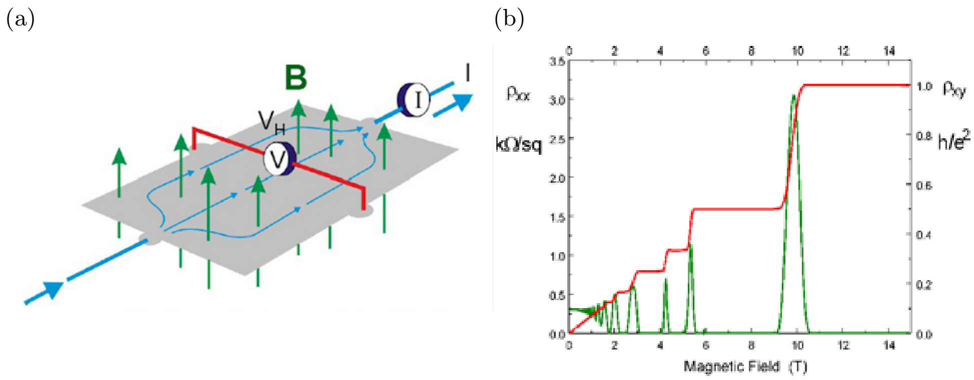

Figure 1.1: (a) Experimental setup for the Quantum Hall effect. (b) Plateus observed at integer values of the Hall resistivity

Hughes and Zhang [7] predicted such topological phenomena would be observable in Mercury Telluride potential wells, which indeed was observed experimentally.
Soon after this developments, there has been what can only be described as an explosion of research, both theoretically and experimentally to study systems in condensed matter which depend in one way or an other on topology, most notably gapped phases[18] but also topological defects[37], Majorana wires [13] and Fermi surfaces[21], to name a few. There are other kinds of Quantum Hall effects known as the Fractional and Anomalous Quantum Hall efects. The former as the name suggests has fractional Hall conductance. It is understood that electron-electron interactions play a fundamental role and distinguish it from the Integer case. The fractional case is much less understood, but has been studied using very deep mathematical ideas known as Chern-Simons topological quantum field theory. The anomalous Hall effect has a long and controversial history, however, recent efforts such as that of Haldane [17], have reinterpreted part of it in terms of Berry's curvature and the same invariants used for the integer case. This work in fact has surprising connections with the kind of systems we shall classify.

Chapter 2

Mathematical preamble

In this chapter we will introduce a vast array of mathematical definitions, objects and a few (but very powerful) theorems, most of which shall be employed for the construction of our classification of topologically stable Fermi surfaces. There are several reasons for doing this. On the one hand, it makes the present work sufficiently self-contained and easier to follow. This chapter is intended particularly for readers with a background in physics, most of which might not have studied these concepts, which is why we provide them in such detail. These concepts also play a fundamental role in our remarks on inconsistencies lurking in the existing literature of classifications, that is, in order to understand why some constructions are inconsistent it is a good idea to have the formal definitions at hand, for though some concepts are similar, the richness of the construction lies in their difference.
First we introduce concepts of analysis since they are more familiar to physicists in general, then we go through the topic of what is known to mathematicians as point-set topology and is not as common to encounter in a physicist's repertoire and finally to K-theory.
We assume that our readers are familiar with the basics of set-theory as well as finite-dimensional linear algebra, nevertheless we may reintroduce some of these notions as we see fit.

2.1 Analysis

The branch of mathematics known as analysis came into being with the work of Fourier on the now famous heat equation, with the byproduct invention of his transform and series in the late 18th century but it was not until the 19th century that it was understood as an independent field through the work of Weierstrass, Cauchy, Fredholm and others. In the 20th century it was the work of Hilbert and Von Neumann that connected analysis and quantum physics.
All of the definitions presented in this section can be found in [34], one of the standard texts in the literature of mathematical physics. Finally, we avoid unnecessary over-citation of [34].

2.1.1 Hilbert space

We begin by introducing the concepts of a Cauchy sequence and a complete metric space,

Definition 1 *A sequence $\{x_n\}$ of points in a metric space M is a* ***Cauchy sequence*** *if $\forall\, \epsilon > 0\ \exists\, N \equiv N(\epsilon) \in \mathbb{N}$ such that $\forall\, n, m > N,\ \|x_n - x_m\| < \epsilon$, where $\|\ \|$ denotes de norm of M.*

Definition 2 *A* ***complete metric space*** *M is a metric space for which every Cauchy sequence converges.*

With these at hand we can define a Hilbert space,

Definition 3 *A* ***Hilbert space*** *$\mathscr{H}$ is a vector space over a field $\mathbb{F}$ which is a also a complete metric space.*

We will exclusively focus on the case were the field $\mathbb{F} = \mathbb{C}$, the set of complex numbers, since (as we will later see) this choice is part of the foundation upon which quantum mechanics was built.
Elements of $\mathscr{H}$ will be denoted as $|\Phi\rangle$ using the Dirac bra-ket notation, $|\Phi\rangle$ being a ket. The bra $\langle\Phi| = |\Phi\rangle^\dagger$ is the *dual* element to $|\Phi\rangle$, where $\langle\Phi|\Psi\rangle \in \mathbb{F}$ denotes the inner product of $\mathscr{H}$.
We shall also restrict our attention to *separable* Hilbert spaces

Definition 4 *A Hilbert space $\mathscr{H}$ is* ***separable*** *iff it has a countable orthonormal basis $\{|\Phi_n\rangle\}_{n=1}^{\infty}$.*

Let us give an example of a complex separable Hilbert space.

Example 1 *Let $L^2([0, 2\pi])$ denote the set of square integrable functions $f : [0, 2\pi] \longrightarrow \mathbb{C}$, $\int_0^{2\pi} |f(x)|^2 dx < \infty$. $L^2([0, 2\pi])$ is a complex separable Hilbert space with orthonormal basis $|\Phi_n\rangle = (2\pi)^{-1/2} e^{inx}$.*

Note that a square integrable function $f(x)$ is viewed as a vector and following our convention, should be denoted as $|f\rangle$. Any square integrable function has a Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} (2\pi)^{-1/2} a_n e^{inx} \,, \tag{2.1}$$

$$a_n = (2\pi)^{-1/2} \int_0^{2\pi} f(x) e^{-inx} dx \,, \tag{2.2}$$

which is why L^2 (as is commonly abbreviated) has applications across a wide variety of fields of study.

2.1.2 Bounded and compact operators

Let us now study operators on our Hilbert space $\mathscr{H}$. The first kind of operators are *bounded* operators:

Definition 5 *An operator* $\mathcal{B} : \mathscr{H} \longrightarrow \mathscr{H}$ *is* ***bounded*** *if* $\exists\, c$ *such that* $\forall\, |\Phi\rangle \in \mathscr{H},\ \langle\Phi|\, \mathcal{B}^{\dagger}\mathcal{B}\, |\Phi\rangle \leq c\, \langle\Phi|\Phi\rangle$.

c is known as the operator norm of $\mathcal{B}$. The set of bounded operators of $\mathscr{H}$ is denoted as $\mathcal{B}(\mathscr{H})$.
We shall also define the set of *compact* operators:

Definition 6 $\mathcal{K}$ *is a* ***compact operator*** *iff* $\forall\, \{x_n\} \subset \mathscr{H}$ *bounded sequence,* $\mathcal{K}(\{x_n\})$ *has a convergent subsequence in* $\mathscr{H}$*. The set of compact operators is denoted as* $\mathcal{K}(\mathscr{H})$*.*

Compactness is a topological property, but we seem to have gotten away with defining a "compact" thing without defining topology. In reality it is that we have implicitly used the notion of the norm of a Hilbert space in our definition and so have used indirectly the topology defined by the norm! For completion and for later use we should define the Calkin algebra [24],

$$\mathcal{CAL}(\mathscr{H}) \equiv \mathcal{B}(\mathscr{H})/\mathcal{K}(\mathscr{H})\,. \tag{2.3}$$

where we have used the fact that $\mathcal{K}(\mathscr{H})$ is a two-sided ideal (see [3]) in $\mathcal{B}(\mathscr{H})$. $\mathcal{CAL}(\mathscr{H})$ is known as a C^*-algebra, which are widely used, especially in what is known as non-commutative geometry [35] and there is a non-commutative approach to problems in condensed matter. We will discuss some of these in later chapters.

2.1.3 Fredholm operators

Fredholm operators will play such an important role in our work that they should be discussed in greater generality. These operators are named after Swedish mathematician Erik Ivar Fredholm and arose in his work on integral equations which derived into what is known in analysis as the Fredholm alternative, which states the following:

Let $\mathcal{K}$ be a compact operator in $\mathscr{H}$ and $|\Phi\rangle \in \mathscr{H}$, then only one of the following possibilities occurs

- $\mathcal{K}\, |\Phi\rangle = \lambda\, |\Phi\rangle$,
- $(\mathcal{K} - \lambda I)^{-1}$ is bounded.

A generalization to Banach spaces was developed by Alexander Grothendieck in his doctoral work [16].
We are now in a position to define Fredholm operators, at least for a single Hilbert space, this will suffice for our purposes.

Definition 7 *An operator $\mathcal{H} : \mathscr{H} \longrightarrow \mathscr{H}$ is **Fredholm** if*

- *$dim\, Ker\, \mathcal{H}$ is finite.*
- *$dim\, Ker\, \mathcal{H}^\dagger$ is finite.*

Let us denote the set of Fredholm operators of $\mathscr{H}$ as it is done in [3],[4] as $\mathcal{F}(\mathscr{H})$. One of the classical results of Fredholm theory is that if $\mathcal{K}$ is compact then $I + \mathcal{K}$ is Fredholm.
We shall study many of the topological properties of the set of Fredholm operators and various corresponding subspaces. As we will see they play a prominent role in K-theory through the work of Sir Michael Atiyah and Isadore Singer [3][4] and their work on the index theorem for elliptic operators, but more on this later. They have become fundamental in many applications outside analysis such as geometry through Donaldson's work and in Conne's non-commutative geometry as well as having applications in physics, including condensed matter but we do not discuss their role in those subjects any further.

2.2 Topology

The subject of topology is one of the most incredible ones in mathematics, commonly described as the study of an object's "shape", independent of its geometrical properties but it is much deeper than this claim leads us to believe and fortunately in many important cases it is inextricably linked with geometry, algebra and analysis themselves. Developed as an independent field in mathematics until the early 20th century by Hausdorff, Poincaré and many others, it has increasingly played an important role in physics, in relation to statistical mechanics for classical and quantum systems, symmetries in quantum field theory and the subjects discussed in this thesis, in which we shall particularly show its connections with analysis through the work of Atiyah, Bott and Singer.
Readers can consult all the basic definitions of topology (topological space, continuous functions, compact spaces, etc) presented here in [31] and those of algebraic topology in [19] , which is the standard textbook reference on

these subjects. Both of them are excellent references. Once again, we generally avoid referencing them in the following sections.

Let us start by defining a *topological space* in a formal way

Definition 8 *A **topology** on a set* X *is a collection* $\mathscr{T}$ *of subsets of* X *having the following properties*

1. $\emptyset$ *and* X *are in* $\mathscr{T}$.

2. *The union of elements of any subcollection in* $\mathscr{T}$ *is in* $\mathscr{T}$.

3. *The intersection of the elements of any finite subcollection of* $\mathscr{T}$ *is in* $\mathscr{T}$.

Thus, providing X with such a $\mathscr{T}$ makes it a topological space.
A few comments are in order, first a set X may be provided with different topologies, something that we will encounter in the classification chapter for operator sets. However, there are some topologies which arise naturally (such as the topology induced by the metric of $\mathbb{R}^n$) and in those cases where there is no confusion we will denote the topological space simply as X, even though formally, we should write a topological space as a pair $(\mathrm{X}, \mathscr{T})$. Also note that by its definition $\mathscr{T} \subset \mathbb{P}(\mathrm{X})$, the *power set* of X.

Elements of $\mathscr{T}$ (subsets of X) are called *open sets*. Alternatively we can define a *closed sets*, which are subsets of X (as a set) and give an equivalent definition of a topology in terms of closed sets, the relation between open and closed sets is that a set C is closed if $C^c = \mathrm{X}\backslash C$ is open. We present three examples of topological spaces

Example 2 *Let* $\mathrm{X} = \mathcal{B}(\mathscr{H})$*, the set of bounded operators on a Hilbert space* $\mathscr{H}$*, which we introduced earlier, with open sets* $U \in \mathscr{T}_{\mathcal{B}(\mathscr{H})}$*, which are arbitrary unions and finite intersections of sets of the form* $B_r(\mathcal{H}) = \{\mathcal{A} \in \mathcal{B}(\mathscr{H}) |\, \|\mathcal{A} - \mathcal{H}\| < r\}$. $B_r(\mathcal{H})$ *is called an open ball of radius* r *centered at the operator* $\mathcal{H}$.

This topology is known as the *metric topology* and can be generalized to any metric space, such as our well known acquaintance $\mathbb{R}^n$.

Example 3 *Let* $\mathrm{X} = \mathbb{R}^n$ *and let* $\mathscr{T} = \{U \subset \mathbb{R}^n \,|\, \mathbb{R}^n\backslash U \text{ is finite}\}$*. Then* $(\mathbb{R}^n, \mathscr{T})$ *is a topology in* $\mathbb{R}^n$*, known as the **cofinite topology*** $\mathscr{T}_{cofinite}$.

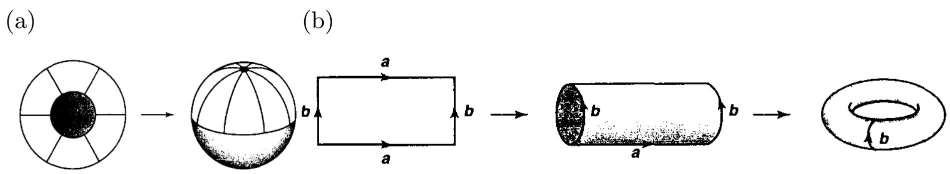

Figure 2.1: (a) How to construct an n-dimensional sphere from a an n-disk by quotienting the boundary of the disk to a point (b) Constructing an n-dimensional torus from an n-dimensional cube by identifying opposite edges. Figures taken from [31].

The only purpose of our last example is to show that a given set X can have different topologies. The last example we present will be very useful later on to define operations on topological spaces

Example 4 *Given a topological space* $(\mathrm{X}, \mathscr{T}_{\mathrm{X}})$*, we denote* X^* *a set of equivalence classes of elements of* X*, that is a partition of* X *into disjoint subsets whose union is* X*. Let* $P : \mathrm{X} \longrightarrow \mathrm{X}^*$ *be the function which takes an element* $x \in \mathrm{X}$ *to the corresponding subset which contains it. We can endow* X^**, with the* ***quotient topology*** $\mathscr{T}_q$*by asking a set* U *to be open in* $(\mathrm{X}^*, \mathscr{T}_q)$ *iff* $p^{-1}(U)$ *is open in* $(\mathrm{X}, \mathscr{T}_{\mathrm{X}})$*.* X^* *is known as a* ***quotient space****.*

Constructions of such quotient spaces are shown in 2.1a,2.1b.

Let us include the definition of *Hausdorff* topological space, which is the type of spaces we are used to

Definition 9 *A topological space* $(\mathrm{X}, \mathscr{T})$ *is* ***Hausdorff*** *if* $\forall\, x_1, x_2$ *pair of distinct points in* X*, there exists open sets* $U_1, U_2 \in \mathscr{T}$*, such that* $x_1 \in U_1$*,* $x_2 \in U_2$ *and* $U_1 \cap U_2 = \emptyset$*.*

This condition is necessary for sequence of points in a topological space to converge at a unique point, hence this condition is necessary for the concept of limit, fundamental to analysis. Indeed most spaces of interest, which arise from other areas in mathematics (and physics) satisfy this condition. A slightly weaker condition known as T_1 axiom, of the *separability* axioms, is the following

Definition 10 *A topological space* $(\mathrm{X}, \mathscr{T})$ *is* T_1 *if* $\forall\, x_1, x_2 \in \mathrm{X}$*,* $\exists\, U \in \mathscr{T}$ *such that* $x_1 \in U$ *but* $x_2 \in U^c$*.*

This definition will only be necessary to us later, when we introduce the concept of a *topological group*. Note that the Hausdorff condition is also a separability axiom, known as T_2.

2.2.1 Continuity, homeomorphisms

We are now in a position to introduce the notion of a *continuous function*

Definition 11 *A function $f : (\mathrm{X}, \mathscr{T}_{\mathrm{X}}) \longrightarrow (\mathrm{X}', \mathscr{T}_{\mathrm{X}'})$ is said to be* ***continuous*** *if $\forall\, U \in \mathscr{T}_{\mathrm{X}'}, f^{-1}(U) \in \mathscr{T}_{\mathrm{X}}$, where f^{-1} denotes the* ***inverse image*** *of f.*

In other words, the inverse image of an open set under f is open in X. The distinction between inverse image and inverse function is very important in this case. All functions have an inverse image (maybe empty) but not all functions have an inverse, let us give a simple example:

Example 5 *Let $f : \mathrm{X} \longrightarrow \{x_0\}$, then f is continuous for any topological space X since $f^{-1}(x_0) = \mathrm{X}$, which is open by definition. If X is different from a point, then $\nexists\, f^{-1}$ as a function, only as a relation.*

Also we can notice that the definition we have presented here is independent of whether our topological spaces are Hausdorff or not, since it plays no role in the definition. We had discussed that to have a notion of limit it was necessary for our topological space to be Hausdorff, making this independent definition of continuity at odds with the conventional way one is taught in a calculus or analysis course, were we define continuity in terms of limits but for more general spaces the notions of limit and continuity are independent!

This definition allows us to present the definition of a topological group

Definition 12 *A* ***topological group*** *G is a group that is also a topological space satisfying the following conditions*

1. *G is a T_1-space.*
2. *The function $(g, h) \mapsto gh$ is continuous.*
3. *The function $g \mapsto g^{-1}$ is continuous.*

Example 6 *Let $GL_n(\mathbb{C})$ denote the group of $n\,X\,n$ invertible matrices under multiplication with entries in $\mathbb{C}$, We can view $GL_n(\mathbb{C})$ as subspace of $\mathbb{C}^{2n}$ with the subspace topology under the induced metric topology of $\mathbb{C}^{2n}$. $GL_n(\mathbb{C})$ with this topology is naturally a topological group under matrix multiplication.*

This example is basically the whole reason why we went through the trouble of introducing the notion of a topological group and it will be fundamental later on when we study vector bundles and classifying spaces.

We can define *homeomorphisms* between topological spaces

Definition 13 *A bijective continuous function* $f : (\mathrm{X}, \mathscr{T}_{\mathrm{X}}) \longrightarrow (\mathrm{X}', \mathscr{T}_{\mathrm{X}'})$ *is a* ***homeomorphism*** *if* f^{-1} *is continuous.*

The condition for a bijection f to be a homeomorphism is equivalent to f being continuous and *open* (sends open sets in X to open sets in X′).
As far as it concerns topology, two spaces are indistinguishable if they are homeomorphic. Thus, topological spaces can in principle be completely classified up to homeomorphism (analogous notions exists for groups, geometries and other areas of mathematics). We state it as a principle since it is in general not possible to address whether two topological spaces are homeomorphic, nevertheless, there are other less general but still elucidating topological properties we can study.

2.2.2 Compact and connected spaces

To define *compact* spaces, we first need to introduce the notion of an open cover of a topological space, then we can immediately proceed to define a compact space

Definition 14 *An open cover* $\{A_\alpha\}$ *is a collection of elements in* $\mathscr{T}$ *whose union* $\bigcup_\alpha A_\alpha = \mathrm{X}$.

Definition 15 *A topological space* X *is said to be* ***compact*** *if every open cover* $\{A_\alpha\}$ *of* X*, has a finite subcover, which covers* X.

A subcover is simply a subset of a given cover. It is usually not simple to prove a space is compact, however the following theorem, which we state without proof, is very useful for some cases

Theorem 1 *Every closed and bounded subset of* $\mathbb{R}^n$ *is compact.*

Thus, the n-dimensional sphere $S^n = \{(x_1, .., x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{m=1}^{n+1} x_m^2 = 1\}$ is compact. In particular, the circle S^1 is compact and since the arbitrary cartesian product of compact spaces is compact (see [31], *Tychonoff's theorem*), the n-dimensional torus $\mathbb{T}^n = \prod^n S^1$ is also compact. This examples

are not only extremely common and simple but they will also play a fundamental role in what will follow since spheres are fundamental for K-theory and in general the Brillouin zone of the condensed matter systems we shall study is an n-dimensional torus.
Let us also introduce the notion of a connected space

Definition 16 *A topological space* X *is* ***connected*** *iff the only sets which are both open and closed are the total space* X *and the empty set* $\emptyset$*.*

This definition is very intuitive, for it simply means that a space is connected if we cannot separate it into two pieces which are disjoint. Connectedness will play a more subtle role but deep role in our further developments, particularly in homotopy theory.

2.2.3 Homotopy

From here on we will write *map* instead of continuous function. The notion of Homotopy has basically driven the field of algebraic topology since the 20th century, yet its definition is remarkably simple

Definition 17 *A* ***homotpy*** *is a family of maps* $f_t : \mathrm{X} \longrightarrow \mathrm{Y},\, t \in I$ *such that the underlying map* $F : \mathrm{X} \times I \longrightarrow \mathrm{Y}, F(x,t) = f_t(x)\ \forall\, x \in \mathrm{X},\, t \in I$ *is continuous, which we denote* $f_0 \simeq f_1$*.*

We say that two maps $f_0, f_1 : \mathrm{X} \longrightarrow \mathrm{Y}$ are *homotopic* if there is a homotopy $\{f_t\}$ connecting them. Also there is a notion of two topological spaces being homotopic

Definition 18 *A map* $f : \mathrm{X} \longrightarrow \mathrm{Y}$ *is called a* ***homotopy equivalence*** *or the same* ***homotpy type*** *if there exists* $g : \mathrm{Y} \longrightarrow \mathrm{X}$ *such that* $fg \simeq Id_{\mathrm{X}}$ *and* $gf \simeq Id_{\mathrm{Y}}$*. We denote this as* $\mathrm{X} \simeq \mathrm{Y}$*.*

Spaces which are homotopic to a point x_0 is called *null-homotopic*, examples of null-homotopic spaces are $\mathbb{R}^n, \mathbb{C}^n, D^n$ among many others. However, there are also plenty of spaces which are not homotopic to a point, such as spheres, tori and higher genus surfaces.
The set of all homotopy classes of maps $f : \mathrm{X} \longrightarrow \mathrm{Y}$ will be denoted as $[\mathrm{X}, \mathrm{Y}]$. It is a pretty big set! If we choose basepoints $x_0 \in \mathrm{X}, y_0 \in \mathrm{Y}$ making X, Y *pointed spaces*, then we shall denote the set of all basepoint-preserving-homotopy classes as $[\mathrm{X}, \mathrm{Y}]_*$. If X and Y satisfy a few conditions (See [19]) $[\mathrm{X}, \mathrm{Y}]_*$ is a group. If $\mathrm{X} = S^n$, then we write $[\mathrm{X}, \mathrm{Y}]_* = \pi_n(\mathrm{Y}, y_0)$, which mathematicians call the n-th homotpy group. $\pi_1(\mathrm{Y}, y_0) = [S^1, \mathrm{Y}]_*$ is known as the *fundamental group* and is the only one which may be non-Abelian (but not necessarily so)

Example 7 *$\pi_1(S^1) = \mathbb{Z}$, in general $\pi_n(S^n) = \mathbb{Z}$ and $\pi_i(S^n) = 0$ for $i < n$.*

Here we have abused of the fact that if the target space is connected then, then the choice of base-point is irrelevant up to isomorphism of the based homotopy classes, and since S^n is connected we can exclude the base-point in our notation for its homotopy groups.
Homotopy theory and its generalizations have driven algebraic topology the last 100 years or so. Homotopy groups provide more information than cohomology or homology groups (which we will only discuss a particular case of, namely K-theory) but they are in general much harder to compute, in fact even for simple enough spaces like spheres it is still open how to compute all their homotopy groups!
Applications of homotopy groups to physics are many. In quantum field theory they are used in the description of anomalies[32] and approaches to describe real space topological defects when there is symmetry breaking are usually classified via homotopy groups[9].

2.2.4 Suspension, loop space, wedge sum and smash product

There are a number of operations one can perform on one or more topological spaces, in fact there are many but we will only make use of four distinct ones in this work. The first one is the *suspension* of a topological space

Definition 19 *Let X be a topological space, the **suspension** of X, denoted as $S\mathrm{X}$, is constructed by taking the product $\mathrm{X} \times I$ and then quotienting $\mathrm{X} \times \{0\}$ to a single point (south pole of $S\mathrm{X}$) as well as quotienting $\mathrm{X} \times \{1\}$ also to a single, different point (north pole of $S\mathrm{X}$).*

The archetypal example of the use of suspension is with spheres where $SS^m = S^{m+1}$, where the equality is up to homeomorphism.
One can also suspend maps by $f \times Id_I : \mathrm{X} \times I \longrightarrow \mathrm{Y} \times I$ by making the same identification, yielding $Sf : S\mathrm{X} \longrightarrow S\mathrm{Y}$.
There is also the concept of a *reduced* suspension of a space

Definition 20 *The **reduced suspension** of a space X, denoted $\Sigma\mathrm{X}$, is constructed by first suspending, obtaining $S\mathrm{X}$ and then further quotienting the line $\{x_0\} \times I$, with $x_0 \in \mathrm{X}$ canonically identified with $\mathrm{X} \times \{\frac{1}{2}\}$.*

Reduced suspensions are particularly important for homotopy classes of pointed spaces and so called *loop spaces*

Definition 21 *Let* (X, x_0) *be a pointed space, the set* $\Omega\mathrm{X}$ *of loops* $S \longrightarrow (\mathrm{X}, x_0)$*, which is a subset of* X^I*, the set of maps* $I \longrightarrow \mathrm{X}$ *which naturally posses the* ***compact-open topology****.*

To see the connection between reduced suspensions and loop spaces, consider a map $f : \Sigma\mathrm{X} \longrightarrow \mathrm{Y}$. We can identify the image of $\{x\} \times I/(\{x\} \times \{0\} \sim \{x\} \times \{1\})$ under f with the image of $\{x\}$ under a map $\tilde{f} : \mathrm{X} \longrightarrow \Omega\mathrm{Y}$, $f(\{x\} \times I/(\{x\} \times \{0\} \sim \{x\} \times \{1\})) = \tilde{f}(x)$. Thus reaching the very important result

$$[\Sigma\mathrm{X}, \mathrm{Y}]_* \approx [\mathrm{X}, \Omega\mathrm{Y}]_*. \tag{2.4}$$

There is a much deeper connection between loop spaces and homotopy classes with cohomology but we shall only make reference to it in the particular case of K-theory, so the interested reader can explore this in [19].
We shall now define two more operations on topological spaces, but these ones are between pairs of such. They are known as the *wedge sum* and the *smash product*

Definition 22 *Given two topological spaces* X, Y *consider their disjoint union* $\mathrm{X} \sqcup \mathrm{Y}$*, and then quotient it by identifying a single point* $x_0 \in \mathrm{X}$ *with a single point* $y_0 \in \mathrm{Y}$*(*X *and* Y *viewed canonically in* $\mathrm{X} \sqcup \mathrm{Y}$*). This space is called the* ***wedge sum*** *of* X *and* Y*, denoted* $\mathrm{X} \vee \mathrm{Y}$*.*

Definition 23 *Given two topological spaces* X, Y*, the* ***smash product*** *of*X *and* Y *is the quotient* $\dfrac{\mathrm{X} \times \mathrm{Y}}{\mathrm{X} \vee \mathrm{Y}}$*, denoted as* $\mathrm{X} \wedge \mathrm{Y}$*.*

The archetypal example for the use of the smash product is the fact that $S^m \wedge S^n = S^{m+n}$, where once again, the equality is up to homeomorphism. Notice that both the smash product and the wedge sum are symmetric under the ordering of the pair of topological spaces upon which they are being employed. These operations are widely use in many of the fundamental constructions in algebraic topology, that is many spaces are at least homotopic to a mix of wedge sums and smash products of simpler ones. In our case, they are particularly necessary in connection with a reduced suspension for the following homotopy relation

Let X, Y be CW-complexes, then

$$\Sigma(\mathrm{X} \times \mathrm{Y}) \simeq \Sigma\mathrm{X} \vee \Sigma\mathrm{Y} \vee \Sigma(\mathrm{X} \wedge \mathrm{Y}). \tag{2.5}$$

The definition of CW-complexes and the proof of this proposition can be found in [19]. Just to remove the possible sensation of being cheated by this strange name (CW-complex), we mention that almost all nice spaces one can think of such as tori, spheres, higher genus surfaces, etc are CW-complexes, but we avoid giving the proper definition since it will not be necessary for what will follow.

2.3 K-theory

We finally arrive at what is the area of algebraic topology we will employ the most to classify our physical systems, K-theory. Topological K-theory was introduced by Sir Michael Atiyah and Friedrich Hirzebruch in the 1960's as an *extraordinary* cohomology theory, the word extraordinary having a precise meaning in this case. It was based mainly on a fundamental construction developed earlier by Alexander Grothendieck, known as Grothendieck's completion, obtaining an Abelian group out of a semi-group and it also depended crucially on Raoul Bott's periodicity theorem on homotopy groups of classical groups, which he had proved in 1957. Roughly speaking, a cohomology theory is a *functor* (generalization of function) from the category (a generalization of set) of topological spaces to the category of Abelian groups, assigning to a topological space X, a chain of groups, denoted $H^n(\mathrm{X}), n \in \mathbb{Z}$ with connecting homomorphisms, usually denoted d_n, $H^n(\mathrm{X}) \xrightarrow{d_n} H^{n+1}(\mathrm{X})$ and use this functor to study topological properties of X in terms of the algebraic properties of the groups $H^n(\mathrm{X})$ and sometimes the other way around! To portray a picture, we could say that Grothendieck's completion constructs the groups $K^n(\mathrm{X})$, where as Bott's periodicity theorem constructs the connecting homomorphisms, though in reality their roles are intertwined in a deeper way. K-theory has many mathematical applications, some of which can be studied in [24]. In our case we are interested in its applications to condensed matter systems, as first pioneered by Hořava [21] and Kitaev [25], but that will come later.

From now we start to develop the necessary tools to make such a construction, which can be studied more carefully in [20], which is incomplete work but it conveys very beautifully the geometric constructions and we shall employ it repeatedly or [24], which is a more complete reference, at the expense of developing the necessary ideas through more abstract notions of Banach categories.

2.3.1 Vector bundles and isomorphism classes

We will now study vector bundles by giving their precise definition and a few examples particularly relevant for physics. Let us begin

Definition 24 *An n-dimensional vector bundle is map $p : \mathrm{E} \longrightarrow \mathrm{X}$ satisfying the following conditions*

1. $p^{-1}(x) \approx \mathbb{F}^n \; \forall \, x \in \mathrm{X}$, *where $\mathbb{F}^n$ is an n-dimensional vector space over the field $\mathbb{F}$.*

2. *There is a cover $\{U_\alpha\}$ of X such that for each U_α, there exists*

$$h_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{F}^n \tag{2.6}$$
$$p^{-1}(x) \mapsto \{x\} \times \mathbb{F}^n \tag{2.7}$$

   h_α homeomerphism and with $h_\alpha(x)$ an isomorphism of vector spaces for each $x \in U_\alpha$.

The last condition is known as the *local trivialization condition*, in plain words it states that every vector bundle is locally isomorphic to the *trivial bundle* $\mathrm{X} \times \mathbb{F}^{\mathrm{n}}$. the space X is known as the *base space* of the bundle, E is called the *total space* and $p^{-1}(x)$ are called the *fibers* of the bundle. We shall often only write the total space E to denote the whole vector bundle (with projection map and base space included). Also notice that we are assuming that $\mathbb{F}^n$ is a topological vector space. We will limit our selves to the cases $\mathbb{F} = \mathbb{R}, \mathbb{C}\mathrm{or}\ \mathbb{H}$.

Example 8 *Let $\mathbb{F} = \mathbb{R}$, $n = 1$ and $\mathrm{X} = S$, a circle in $\mathbb{R}^2$. Let us construct our bundle E by taking $I \times \mathbb{R}$ and identifying $(0, t) \sim (1, -t)$. This identification gives as base space S^1, where as the total space E is homeomorphic to a Möbius strip without its boundary circle. This bundle is called a Möbius bundle.*

The Möbius strip and the corresponding Möbius transformations are fundamental to holomorphicity conditions in complex analysis, which is applied to study the properties of propagators and other operators in quantum mechanical systems. Let us present another example which is invaluable for physics.

Example 9 *Let $\mathrm{X} = S^m \subset \mathbb{R}^{m+1}$, $n = m$ and let $\mathrm{E} = \{(x, v) \in S^m \times \mathbb{R}^{m+1} \,|\, \langle x, v\rangle = 0$, where $\langle , \rangle$ denotes the inner product in $\mathbb{R}^{m+1}$ and we are viewing points x of S^m as unit vectors in $\mathbb{R}^{m+1}$. This bundle is known as the Tangent bundle of S^m, denoted TS^m.*

The construction of a tangent bundle can be generalized to manifolds, in classical statistical mechanics, sections of the tangent bundle constitute the phase space of a system! If we take our space-time to be a more general manifold M as in general relativity, TM and the exponential map are fundamental to study the properties of geodesics!
Let us present an example which is not directly relevant for physics but relevant for the whole subject of vector bundles, as we shall see it is quite trivial

Example 10 *The n-dimensional **trivial** bundle over a base space* X *is simply the product* $\mathrm{X} \times \mathbb{F}^n$*, commonly denoted as* ξ^n

The trivial bundle is indispensable for Grothendieck's completion and the whole area of topological K- theory.

We now introduce the notion of *vector bundle isomorphism*

Definition 25 *A homeomorphism* $F : \mathrm{E}_1 \longrightarrow \mathrm{E}_2$ *between vector bundles over the same base space* X *is a **vector bundle isomorphism** if it takes each fiber* $p_1^{-1}(x)$ *of* E_1 *to the corresponding fiber* $p_2^{-1}(x)$ *of* E_2 *by a linear isomorphism, for each* $x \in \mathrm{X}$*.*

We will usually regard two vector bundles E_1 and E_2 as the same if they are in the same isomorphism class, which we denote as $\mathrm{E}_1 \approx \mathrm{E}_2$. Isomorphism classes of n-dimensional vector bundles over a base space X will be denoted as $Vec^n_{\mathbb{F}}(\mathrm{X})$. We shall also repeatedly make use of the following result

Let $p : \mathrm{E} \longrightarrow \mathrm{X} \times I$ be a vector bundle over $\mathrm{X} \times I$, where X is compact Hausdorff, then the restrictions of p to $\mathrm{X} \times \{0\}$ and $\mathrm{X} \times \{1\}$ are isomorphic. Thus, vector bundles over compact Hausdorff spaces, which are connected connected by a homotopy are isomorphic.

This definitions and results will be used throughout the rest of our constructions, so one should keep them in mind as we go along.
We shall skip many of the basic constructions and properties of vector bundles such as *sections, clutching maps maps, inner products* and *cocycle conditions*, for we will not use them directly in the construction of our classification, however readers are encouraged to consult [20] for examples and the definitions of these constructions, since these provide much of the geometric richness of the subject.

2.3.2 Direct sum and tensor product of vector bundles

Given two vector bundles over the same base space X, $p_1 : \mathrm{E}_1 \longrightarrow \mathrm{X}\,, p_2 : \mathrm{E}_2 \longrightarrow \mathrm{X}$, we can define their direct sum in the following way

$$\mathrm{E}_1 \oplus \mathrm{E}_2 = \{(v_1, v_2) \in \mathrm{E}_1 \times \mathrm{E}_2 \,|\, p_1(v_1) = p_2(v_2)\} \tag{2.8}$$

Thus, the direct sum of two vector bundles is again a vector bundle, one where each fiber is the direct sum of the corresponding individual fibers $\mathrm{E}_1, \mathrm{E}_2$ at each point $x \in \mathrm{X}$.
Let us present the following fact

$$Vec^n_{\mathbb{F}}(\mathrm{X} \vee \mathrm{Y}) = Vec^n_{\mathbb{F}}(\mathrm{X}) \oplus Vec^n_{\mathbb{F}}(\mathrm{Y})\,. \tag{2.9}$$

where the direct sum of isomorphism classes is understood as isomorphism classes of direct sums of vector bundles. The most important application we shall employ for the direct sum of vector bundles is the following:

Let $p : \mathrm{E} \longrightarrow \mathrm{X}$ be an n-dimensional vector bundle with X compact Hausdorff, then $\exists\, p' : \mathrm{E}' \longrightarrow \mathrm{X}$ and m-dimensional vector bundle, for some $m \in \mathbb{N}$, such that $\mathrm{E} \oplus \mathrm{E}'$ is the trivial bundle ξ^{n+m} over X.

This result will also be fundamental for Grothendieck's completion.

Given an n_1-dimensonal vector bundle X, $p_1 : \mathrm{E}_1 \longrightarrow \mathrm{X}$ and an n_2-dimensional vector bundle $p_2 : \mathrm{E}_2 \longrightarrow \mathrm{X}$ over the same base space , we can define their tensor product in the following way:

Let $\mathrm{E}_1 \otimes \mathrm{E}_2$ be as set, the disjoint union of vector spaces $p_1^{-1}(x) \otimes p_2^{-1}(x)$ for each $x \in \mathrm{X}$. We now choose isomorphisms $h_i : p_i^{-1}(U) \longrightarrow U \times \mathbb{F}^n$ for each open set $U \subset \mathrm{X}$, such that E_1 and E_2 restricted to U are trivial. Then a topology $\mathscr{T}_{p_1^{-1}(U) \otimes p_2^{-1}(U)}$ on the set $p_1^{-1}(U) \otimes p_2^{-1}(U)$ is defined by making the fiber-wise tensor product $h_1 \otimes h_2 : p_1^{-1}(U) \otimes p_2^{-1}(U) \longrightarrow U \times (\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2})$ a homeomorphism. the topology $\mathscr{T}_{\mathrm{E}_1 \otimes \mathrm{E}_2}$ is the union of all $\mathscr{T}_{p_1^{-1}(U) \otimes p_2^{-1}(U)}$ for each U satisfying the conditions above.

2.3.3 Pullbacks and universal bundle

Given two topological spaces X, Y, a vector bundle over one of them and a map between them we can construct a vector bundle over the other base space in terms of the map and the bundle we already had

Definition 26 *Given a map $f : \mathrm{X} \longrightarrow \mathrm{Y}$ and a vector bundle $p : \mathrm{E}_1 \longrightarrow \mathrm{Y}$, then there exists a vector bundle $p^{'} : \mathrm{E}_1^{'} \longrightarrow \mathrm{X}$ with a map $f^{'} : \mathrm{E}^{'} \longrightarrow \mathrm{E}$ taking the fiber of $\mathrm{E}^{'}$ over each point $x \in \mathrm{X}$ by a linear isomorphism onto the fiber of E over $f(x)$. The bundle $\mathrm{E}^{'}$ is called the* ***pullback*** *of E by f.*

The pullback is unique up to isomorphism of vector bundles, so we can consider the map $f^* : Vec^n_{\mathbb{F}}(\mathrm{Y}) \longrightarrow Vec^n_{\mathbb{F}}(\mathrm{X})$, taking the isomorphism class of E to the isomorphism class of $\mathrm{E}^{'}$. We will write the pullback as $f^*(\mathrm{E})$ instead of E, making our notation clearer.

Example 11 *The tangent bundle TS^m is the pullback of the tangent bundle $T\mathbb{R}P^m$ under the quotient map $S^m \longrightarrow \mathbb{R}P^m$, identifying antipodal points in S^m.*

We enlist the properties which are satisfied by pullbacks

1. $(fg)^*(\mathrm{E}) \approx g^*\left(f^*(\mathrm{E})\right)$.
2. $Id^*(\mathrm{E}) \approx \mathrm{E}$.
3. $f^*(\mathrm{E}_1 \oplus \mathrm{E}_2) \approx f^*(\mathrm{E}_1) \oplus f^*(\mathrm{E}_2)$.
4. $f^*(\mathrm{E}_1 \otimes \mathrm{E}_2) \approx f^*(\mathrm{E}_1) \otimes f^*(\mathrm{E}_2)$.

Now that we have seen pullbacks and their properties, we can study the universal bundle, we first introduce some terminology and construct such classifying spaces

Definition 27 *A* ***Grassman manifold*** *$G_k(\mathbb{F}^n)$ is the set of k-dimensional $\mathbb{F}$-planes which pass through the origin of $\mathbb{F}^n$.*

We can define an action of the n-dimensional general $\mathbb{F}$-linear group $GL_n(\mathbb{F})$ on $G_k(\mathbb{F}^n)$ as

$$GL_n(\mathbb{F}) \times G_k(\mathbb{F}^n) \longrightarrow G_k(\mathbb{F}^n) \tag{2.10}$$

$$(A, \quad \{e_1, ..., e_k\}) \mapsto span\{Ae_1, ...Ae_k\}. \tag{2.11}$$

However, if $A \in GL_k(\mathbb{F}) \times GL_{n-k}(\mathbb{F} \subset GL_n(\mathbb{F})$ then $span\{Ae_1, ...Ae_k\} = span\{e_1, ..., e_k\}$, leaving any k-plane fixed. Thus $GL_k(\mathbb{F}) \times GL_{n-k}(\mathbb{F})$ is the *isotropy subgroup* of the action. $G_k(\mathbb{F}^n)$ is an *homogeneous space* since for each pair $x, y \in G_k(\mathbb{F}^n)$ there is an $A \in GL_n(\mathbb{F})$ such that $Ay = x$.
Define the partition of $GL_n(\mathbb{F})$ into *left cosets* $GL_n(\mathbb{F})/GL_k(\mathbb{F}) \times GL_{n-k}(\mathbb{F}) = \{g\,(GL_k(\mathbb{F}) \times GL_{n-k}(\mathbb{F})) \mid g \in GL_n(\mathbb{F})\}$. We can give a bijection between $G_k(\mathbb{F}^n)$ and $GL_n(\mathbb{F})/GL_k(\mathbb{F}) \times GL_{n-k}$ by identifying $G_k(\mathbb{F}^n) \ni y \Leftrightarrow A\,(GL_k(\mathbb{F}) \times GL_{n-k}(\mathbb{F})) \in$

$GL_n(\mathbb{F})/GL_k(\mathbb{F}) \times GL_{n-k}(\mathbb{F})$ such that $(A, y) = x$, where $x \in G_k(\mathbb{F}^n)$ arbitrary but fixed.
Thus, viewing $GL_n(\mathbb{F})$ as a topological group, we endow $G_k(\mathbb{F}^n) = GL_n(\mathbb{F})/GL_k(\mathbb{F}) \times GL_{n-k}(\mathbb{F})$ with the quotient topology.

Since we can view the action of $GL_n(\mathbb{F})$ as rotations of the k-planes, we lose nothing by contracting it to its maximal unitary subgroup $U(n)$ for $\mathbb{F} = \mathbb{C}$, $O(n)$ and $Sp(n)$ for $\mathbb{F} = \mathbb{R}, \mathbb{H}$ respectively. Thus $G_k(\mathbb{F}^n)$ is the quotient subgroup of a compact Hausdorff space by a compact Hausdorff subspace, hence it is compact Hausdorff.
The inclusions $\mathbb{F} \subset \mathbb{F}^2 \subset \mathbb{F}^3 \subset ....\mathbb{F}^n \subset ....$ induce the natural inclusions $G_k(\mathbb{F}^K) \subset G_k(\mathbb{F}^{k+1}) \subset G_k(\mathbb{F}^{k+2})........$, hence when we take the limit $G_k(\mathbb{F}^\infty) = \bigcup_n G_k(\mathbb{F}^n)$ and give it the *direct limit topology*

Definition 28 *Given a union of topological spaces* $\mathrm{X}_\infty = \bigcup_n \mathrm{X}_n$*, a set* U *is open in* X_∞ *iff* $U \cap \mathrm{X}_n$ *is open in* $\mathrm{X}_n \,\forall\, n$*.*

We are finally in a position to construct a bundle $p : \mathrm{E}_k \longrightarrow G_k(\mathbb{F}^\infty)$ where $E_k = G_k(\mathbb{F}^\infty) \times \mathbb{F}^\infty$ and $p(x, v) = x \,\forall\, v \in x$. Then we have the extraordinary proposition, whose proof we must also omit

For every vector bundle L over a compact Hausdorff space X, there exists a map $f : \mathrm{X} \longrightarrow G_k(\mathbb{F}^\infty)$ such that $L \approx f^*(E_k)$. In other terms, $[X, G_k(\mathbb{F}^\infty)] \approx Vec^k_{\mathbb{F}}(\mathrm{X})$.

A similar but more general idea is that of a *classifying space*, which we shall use later, for our purposes, a classifying space F is a topological space which we can use homotopy classes and loop spaces from a space X to F to construct cohomology groups of X for a given cohomology theory, but we shall see this later.

2.3.4 Semi-group and its Grothendieck completion

Given $[\mathrm{E}_n] \in Vec^n_{\mathbb{F}}(\mathrm{X})$, $[\mathrm{E}_m] \in Vec^m_{\mathbb{F}}(\mathrm{X})$ we can define a direct sum of isomorphism classes as $[\mathrm{E}_n] \oplus [\mathrm{E}_m] = [\mathrm{E}_n \oplus \mathrm{E}_m] \in Vec^{n+m}_{\mathbb{F}}(\mathrm{X})$. Thus, we have a natural inclusion of $Vec^n_{\mathbb{F}}(\mathrm{X}) \subset Vec^{n+1}_{\mathbb{F}}(\mathrm{X})$, $[\mathrm{E}] \mapsto [\mathrm{E} \oplus \xi]$, making the union $\bigcup_n Vec^n_{\mathbb{F}}(\mathrm{X})$ denoted $Vec_{\mathbb{F}}(\mathrm{X})$ a *semi-group*, with identity ξ^0, given that it satisfies the the axioms of a group except the inverse axiom.

We can further define an equivalence relation between n-dimensional vector bundles known as *stable equivalence*

$$\mathrm{E}_1 \sim_s \mathrm{E}_2 \Leftrightarrow \mathrm{E}_1 \oplus \xi^m \approx \mathrm{E}_2 \oplus \xi^m \text{ for some } m \in \mathbb{Z}^+. \tag{2.12}$$

Notice that we loose some information under this equivalence relation, as in the following example

Example 12 *Let our vector bundle be TS^n, the tangent bundle of the n-dimensional sphere, then $TS^n \sim_s \xi^n$, for we can always add the* ***normal bundle*** *NS^n, which is the one-dimensional vector bundle consisting of the position vector x of each point in S^n. By construction $NS^n \approx S^n \times \xi$ and also $TS^n \oplus NS^n \approx \xi^{n+1}$. Thus*

$$\begin{aligned} TS^n \oplus \xi &\approx \xi^{n+1}, \\ &= \xi^n \oplus \xi \,. \end{aligned} \tag{2.13}$$

Thus stable equivalence is not without its consequences as we can see from this example, general relativity and classical statistical mechanics would be destroyed by taking this equivalence.
If X is compact Hausdorff, then $\sim_s$ satisfies the *cancelation property*, that is

$$\mathrm{E}_1 \oplus \mathrm{E}_3 \sim_s \mathrm{E}_2 \oplus \mathrm{E}_3 \Rightarrow \mathrm{E}_1 \sim_s \mathrm{E}_2, \tag{2.14}$$

since on a compact Hausdorff base there is always a bundle L_{E_3} such that $\mathrm{E}_3 \oplus L_{\mathrm{E}_3} \approx \xi^{n+m}$.
There is a universal way of constructing an Abelian group G out of a semi-group $(S, *)$ and an equivalence relation $\sim$ in S. Let $(a, b), (c, d) \in S \times S$, then we construct the equivalence relation

$$(a, b) \sim_{Grothendieck} (c, d) \Leftrightarrow a * d \sim c * b \,. \tag{2.15}$$

Example 13 *Let $(S, *) = (\mathbb{N} \cup \{0\}, +)$ and define the equivalence relation $\sim$ to be equality, then*

$$(a, b) \sim_{Grothendieck} (c, d) \Longleftrightarrow a + d = b + c \,. \tag{2.16}$$

We denote the pair (a, b) as $a - d$, and so the Grothendieck completion of $(\mathbb{N} \cup \{0\}, +)$ is $(\mathbb{Z}, +)$, the integers.

We can hence construct an Abelian group from $Vec_{\mathbb{F}}(\mathrm{X})$, $\sim_s$ and the cancelation property for a compact Hausdorff space as follows:

Let $([\mathrm{E}],[\mathrm{E}^{'}]) \in Vec_{\mathbb{F}}(\mathrm{X}) \times Vec_{\mathbb{F}}(\mathrm{X})$, we define an equivalence relation

$$([\mathrm{E}_1],[\mathrm{E}_2]) \sim_{Grothendieck} ([\mathrm{E}_3],[\mathrm{E}_4)] \Longleftrightarrow [\mathrm{E}_1] \oplus [\mathrm{E}_4] \sim_s [\mathrm{E}_2] \oplus [\mathrm{E}_3]\,. \tag{2.17}$$

We thus, denote elements of the Grothendieck completion of $(Vec_{\mathbb{F}}(\mathrm{X}), \oplus)$ as $[\mathrm{E}_1] - [\mathrm{E}_2]$ and, at long last

Definition 29 *the Grothendieck completion of the semi-group of stable isomorphism classes of vector bundles over a compact Hausdorff space* X *is* $K_{\mathbb{F}}(\mathrm{X})$*, the* K*-group of* X.

Given two elements $[\mathrm{E}_1]-[\mathrm{E}_2]$, $[\mathrm{E}_3]-[\mathrm{E}_4]$ of $K_{\mathrm{F}}(\mathrm{X})$, we have besides the addition, a multiplicative structure induced by teh tensor product of vector bundles. The way to realize this operations is as follows

$$\begin{aligned}
([\mathrm{E}_1] - [\mathrm{E}_2]) + ([\mathrm{E}_3] - [\mathrm{E}_4]) &= ([\mathrm{E}_1] \oplus [\mathrm{E}_3]) - ([\mathrm{E}_2] \oplus [\mathrm{E}_4])\,, && (2.18)\\
([\mathrm{E}_1] - [\mathrm{E}_2])([\mathrm{E}_3] - [\mathrm{E}_4]) &= ([\mathrm{E}_1] \otimes \mathrm{E}_3]) - ([\mathrm{E}_1] \otimes [\mathrm{E}_4]) - ([\mathrm{E}_2] \otimes \mathrm{E}_3]) \\
&+ ([\mathrm{E}_2] \otimes [\mathrm{E}_4])\,. && (2.19)
\end{aligned}$$

As it can be seen from its definition, for a point $\{x_0\}$ we obtain

$$K_{\mathbb{F}}(\{x_0\}) \approx \mathbb{Z}\,. \tag{2.20}$$

Thus K-theory can at best be an *extraordinary* cohomology theory, since it does not satisfy the *dimension axiom* [19], in which all cohomology groups $H^n(\{x_0\}) = 0\,, \forall\, nn \neq 0$.

Definition 30 *We can define a homorphism induced by the projection of* X *into* $\{x_0\}$ *[24]*

$$K(\mathrm{X}) \to K(\{x_0\}) \approx \mathbb{Z}\,. \tag{2.21}$$

The kernel of this homomorphism is called the ***reduced*** $\boldsymbol{K}$***-theory*** *of* X*, denoted as* $\tilde{K_{\mathbb{F}}}(\mathrm{X})$.

Reduced K-theory plays a central role in our classification of topological Fermi surfaces, though through a rather indirect route. It is also fundamental for Bott periodicity, which we will discuss in the next section.

2.3.5 K-groups and Bott periodicity

As we stated before, a cohomology theory on a space X gives a sequence of Abelian groups, but so far we have only constructed one such group, namely $K(\mathrm{X})$. Thus, without introducing properly the motivation for such a definition we construct the higher K-groups

Definition 31 $K^n_{\mathbb{F}}(\mathrm{X}) \equiv \tilde{K}_{\mathbb{F}}(S^n\mathrm{X}_+)\,\forall\, n \leq 0$, *where once again* S^n *denotes suspending* X_+ n *times and* X_+ *denotes the union* $\mathrm{X} \cup \{+\}$, *where* $\{+\}$ *is a disjoint base-point.*

Roughly speaking, this definition is given because we can construct long exact sequences of these K-groups through a sequence of spaces built out of unions of cones of both a space X and a closed subspace Y, and then iterating these as the new X and Y's, so that the next space in the sequence is built out of the previous two. We also use quotients of these, which yield $S^n\mathrm{X}_+$ and $S^n\mathrm{X}/\mathrm{Y}_+$ (The exact sequences induced by the inclusion and quotient maps). We especially recommend chapter II of [20] for a much more detailed discussion.
As one can see from our definition, we used the suspensions of X_+ and not simply X, this is obviously because otherwise we would not have the required long exact sequence. It has important consequences when computing these K-groups, for example

$$S\mathrm{X}_+ \simeq S\mathrm{X} \vee S^1\,. \tag{2.22}$$

So that using our definition we have

$$K_{\mathbb{F}}{}^{-1}(\mathrm{X}) \quad \approx \quad \tilde{K}_{\mathbb{F}}(S\mathrm{X}) \oplus \tilde{K}_{\mathbb{F}}(S^1)\,. \tag{2.23}$$

If $\mathbb{F} = \mathbb{C}$ then, since (see below for this notation) $\tilde{K}(S^1) \approx 0$ this means that $K^{-1}(\mathrm{X})$ coincides with the reduced K-group $\tilde{K}^{-1}(\mathrm{X})$, where as this is very different for $\mathbb{F} = \mathbb{R}$, where

$$KO^{-1}(\mathrm{X}) \approx \tilde{KO}(\mathrm{X}) \oplus \mathbb{Z}_2\,. \tag{2.24}$$

Hence , we have constructed half a cohomology theory, since we are still missing a definition for $n > 0$. For this we shall make use of a very deep result in topology known as Bott periodicity, developed by Raoul Bott in the end of the 1950's.
Starting from here it is necessary to differentiate the choice of field $\mathbb{F}$. If $\mathbb{F} = \mathbb{C}$, the K-group. simply as $K(\mathrm{X})$ (In some old references it may be

denoted as $KU(\mathrm{X})$ instead) and $KO(\mathrm{X})$ for $\mathbb{F}=\mathbb{R}$. We shall ignore the case $\mathbb{F}=\mathbb{H}$ since it shall not be directly related to our classification.
We begin by presenting the original statement of Bott periodicity, which concerns homotopy groups of classical spaces.
Given the inclusions $U(1)\subset U(2)\subset .....\subset U$, where again $U=\bigcup_n U(n)$ is the direct limit, then

$$\pi_{n+2}(U)\approx\pi_n(U)\ \forall\, n \tag{2.25}$$

Analogously, the inclusions $O(1)\subset O(2)\subset .....\subset O$, $O=\bigcup_n O(n)$ generates

$$\pi_{n+8}(O)\approx\pi_n(O)\ \forall\, n \tag{2.26}$$

In terms of loop spaces we obtain

$$\Omega^2 U \simeq U\,, \tag{2.27}$$
$$\Omega^8 O \simeq O\,. \tag{2.28}$$

Now its connection with K-theory comes about because the homotopy classes of a compact Hausdorff space X to a classifying space, either BO, BU for O and U respectively satisfy

$$K(\mathrm{X}) \approx [\mathrm{X}, BU\times\mathbb{Z}]\,, \tag{2.29}$$
$$KO(\mathrm{X}) \approx [\mathrm{X}, BO\times\mathbb{Z}]\,. \tag{2.30}$$

Thus the periodicity in loop spaces induces a periodicity in K-groups

$$K^{-2}(\mathrm{X}) \approx K(\mathrm{X})\,, \tag{2.31}$$
$$KO^{-8}(\mathrm{X}) \approx KO(\mathrm{X})\,. \tag{2.32}$$

Allowing us to define groups for positive n as $mod\,2$ for $K(\mathrm{X})$ and $mod\,8$ for $KO(\mathrm{X})$.
One can express complex and real K-groups as homotopy classes of classical symmetric spaces, whose full study was originally by H.Cartan. We present these spaces in **??**

Proving this results is quite a feat, but all details, specially for classifying spaces can be seen in [24]. We should mention at this point that there is a connection between Bott periodicity and *Clifford algebras*, but since we will not use them directly, we defer them to be studied again in [24], however, we will review a little about them when we criticize the existing literature on the classifications.

n	Classifying Space (KO)	Classifying Space (K)
0	$(O/(O \times O)) \times \mathbb{Z}$	$(U/(U \times U)) \times \mathbb{Z}$
-1	O	U
-2	O/U	
-3	U/Sp	
-4	$(Sp/(Sp \times Sp)) \times \mathbb{Z}$	
-5	Sp	
-6	Sp/U	
-7	U/O	

Table 2.1: Classifying spaces for real and complex K-theory. 8 classifying spaces for KO, where $Sp = \bigcup_n Sp(n)$, is the direct limit of $Sp(n)$'s, where $Sp(n)$ is the n-dimensional **Symplectic group**, the unitary group of $GL_n(\mathbb{H})$. Complex K-theory only has 2 classiying spaces.

This work was mostly done in the late 1950's and early 1960's. When we derive our classification of topologically stable Fermi surfaces, we will present generalizations of this work due to Atiyah and Singer.
As a reminder, the references we have employed through this chapter are [34], [31], [19], [20] and [24] and should be consulted for more details.

Chapter 3

Physical setup

The most general area of physics one can fit to the kind of systems we shall study is condensed matter physics. It was formerly known as solid state physics but over the past decades the field has widen its gaze. Condensed matter can be portrayed as studying thermal, optical and magneto-electric properties of materials through modeling the behavior of the electrons and atoms (or ions) which constitute the materials. It was in the 1930's that Felix Bloch, Eugene Wigner, Léon Brillouin and others revolutionized the theory of solids in terms of the periodic structure of crystals and quantum mechanics. Nowadays it is perhaps the largest field in terms of its working force and applications. All modern electronic systems which surround us are in their majority understood by the theoretical developments of the 30's, 40's and 50's. We will use these building blocks to study properties of exotic states of matter, particularly Fermi surfaces of such states. Such materials, as of yet, have no general application but perhaps will be of great applicability in the future, since as it can be deduced from our previous discussion, topological structures are very robust, stable against continuous deformations, so we can conceive that topological properties of systems are inherently robust. Nonetheless we should not get ahead of ourselves, so far these states arise (as far as experiment goes) in particular crystal structures and at extremely low temperatures, often with the need of a strong magnetic field such as is the case for the quantum hall effect, discussed in the introduction.
There are many references for condensed matter physics such as[1], [9],[26] we shall mostly follow [1].

3.1 Free n electrons in a box

Let us study a system of non-interacting N electrons confined to a n-dimesional box of volume V. Since electrons are non interacting we can understand this problem by solving a single electron system and copy it N times. Thus, we begin with a single electron Hilbert space $\mathscr{H}_{s-e}$, and its state vector $|\psi_{s-e}(\vec{r},t)\rangle$, which satisfies the Schödinger equation

$$\mathcal{H}_{s-e}(t)\,|\psi_{s-e}(\vec{r},t)\rangle = i\hbar\frac{\partial}{\partial t}\,|\psi_{s-e}(\vec{r},t)\rangle\ , \tag{3.1}$$

where $\mathcal{H}_{s-e}(t)$ is the single electron Hamiltonian operator and $\hbar$ is the reduced Planck's constant. We will assume that our Hamiltonian is time-independent or equivalently that our system lies in a stationary state, thus

$$|\psi_{s-e}(\vec{r},t)\rangle = |\psi_{s-e}(\vec{r})\rangle\, e^{i\frac{\epsilon}{\hbar}t}\,, \tag{3.2}$$

$$\mathcal{H}_{s-e}\,|\psi_{s-e}(\vec{r})\rangle = \epsilon\,|\psi_{s-e}(\vec{r})\rangle\ . \tag{3.3}$$

For a one particle Hamiltonian, $\mathcal{H}_{s-e} = \frac{\mathcal{P}^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2$, so that the solution to the Schrödinger equation, ignoring boundary conditions is

$$|\psi_{s-e,\vec{k}}(\vec{r})\rangle = V^{-\frac{1}{2}}\,|e^{i\vec{k}\cdot\vec{r}}\rangle\ , \tag{3.4}$$

$$\epsilon \equiv \epsilon(\vec{k}) = \hbar^2\vec{k}\cdot\vec{k}\,/2m\,, \tag{3.5}$$

$$\hbar\vec{k} = \vec{p}. \tag{3.6}$$

The solution $e^{i\vec{k}\cdot\vec{r}}$ is known as the *plane wave* solution and $\vec{k}$ is called the *wave vector*. The name plane wave comes from the fact that the solution is constant for all planes orthogonal to $\vec{k}$ and periodic along lines parallel to it.

Some remarks are in order. First, when writing equations in quantum mechanics momentum, position and the wave vector can be referring to operators and/or vectors, so its important to distinguish from context what mathematical object we are employing. Secondly, the $V^{-\frac{1}{2}}$ factor is a *normalizing factor*, so that the probability of finding an electron with wave vector $\vec{k}$ always adds up to 1.

$$\int_V d\vec{r}\,\langle\vec{r},\psi_{s-e,\vec{k}}(\vec{r})\rangle\,\langle\psi_{s-e,\vec{k}}(\vec{r}),\vec{r}\rangle = 1\,. \tag{3.7}$$

Here we should stop to highlight an obscure mathematical detail which is not mentioned in most textbooks on quantum mechanics. The single particle Hilbert space is isomorphic to L^2, with basis plane waves $|\psi(\vec{k})\rangle$, but we saw in the previous chapter that L^2 is separable, so it has a countable basis. This is at odds with what is usually taught in introductory textbooks on quantum mechanics where we are told that our Hilbert space has as basis $\{|\vec{r}\rangle\ ,\vec{r}\in\mathbb{R}^3\}$, which, by the nature of $\mathbb{R}$, is uncountable. The correct interpretation is that the physical Hilbert space is not a Hilbert space persé as mathematicians define it but what is called by mathematicians themselves as a *rigged* Hilbert space. That is a Hilbert space with additional structure. One of the motivations for physicists to employ a rigged Hilbert space is so that experimental setups are better described in terms of $\vec{r}$ for systems of a small number of particles. For condensed matter systems, the position of a single particle is in general not relevant observable, thus, we shall continue to employ a separable Hilbert space and use all the standard mathematical properties mathematicians have derived for them.

We now have to make a choice of boundary conditions, for our purposes it suffices to choose *periodic boundary conditions*, so that

$$|\psi_{s-e,\vec{k}}(\vec{r})\rangle = |\psi_{s-e,\vec{k}}(\vec{r}+n^i\vec{l}_i)\rangle \tag{3.8}$$

where $\vec{l}_i$ is $L = V^{\frac{1}{3}}$ in the i-th component and zero elsewhere, $n^i \in \mathbb{Z}$ and we have used Einstein's summation convention. This conditions are equivalent to gluing the opposite sides of our box, making it $\mathbb{T}^3$ a 3-dimensional torus. This gluing is not realizable in $\mathbb{R}^3$ but as we will see later, this will not be a bad representation of our systems because of their periodicity.
This choice of boundary conditions has a powerful implication since

$$\begin{aligned} e^{i\vec{k}\cdot n^i\vec{l}_i} &= 1\,, && (3.9)\\ \vec{k}\cdot n^i\vec{l}_i &= = 2\pi m\,, m\in\mathbb{Z}\,, && (3.10)\\ & && (3.11)\end{aligned}$$

and so

$$\vec{k} = \frac{2\pi}{L}\vec{m}\,,\, \vec{m}\in\mathbb{Z}^3\,. \tag{3.12}$$

Momentum and energy are quantized! Or, said differently, we can only have discrete, whole values for momentum in terms if $\hbar$ and L. Given that we are dealing with electrons, which obey Fermi-Dirac statistics and at its core Pauli's *exclusion principle*

No two fermions may occupy the same one particle state $|\psi_{s-e}\rangle$. Thus for every wave vector $\vec{k}$ there are 2 electrons, one with spin $\frac{\hbar}{2}$ and one with $-\frac{\hbar}{2}$.

Thus, we can construct an N electron system by filling states corresponding to the lowest energy level ($\vec{k}=0$), 2 electrons for each $\vec{k}$ until we have used up our N electrons, filling a region in $\vec{k}$-space. In our case, since

$\epsilon(\vec{k}) \sim \vec{k}^2$ and typically $N \sim 10^{23}$, our region is indistinguishable from the closed disk $B_{k_F}(0)$, of radius k_F centered at $0 \in \vec{k}$-space. The boundary of this disk $\partial B_{k_F}(0) = S_F$ is called the *Fermi sphere*.
Notice that we have used two facts, one is that $\vec{k}$ is quantized so that k_F is not arbitrarily small, in fact it may be quite large since for our case

$$k_F = (\frac{3\pi^2 N}{V})^{\frac{1}{3}} \,. \tag{3.13}$$

The other fact is that though it is discrete we are approximating it by a continuum of points. This will be used extensively, particularly to study topological properties of such continuum approximation. It would perhaps be interesting to study in a more rigorous manner this continuum limit, using a tool called *persistent homology* [38] but we will not discuss this any further.
For more general systems there is an analogous construction of filling a region in $\vec{k}$-space, its boundary known as the *Fermi surface*. In general it does not yield a sphere but a surface with a much richer topological nature. It may not even be a connected surface nor its components posses the same dimensions. Materials which posses a Fermi surface behave like *metals*, were as those where the filled region has no boundary, e.g no Fermi surface are known as *insulators*. We shall present later on a more general and useful definition of a Fermi surface.

3.2 Bravais lattice

To describe the periodic structure of a crystal, it is necessary to define a *Bravais lattice*

Definition 32 *A **Bravais lattice** consists or all points with position vectors of the form*

$$\vec{R} = n^i \vec{a}_i \,, \{\vec{a}_i\} \;\; L.I, \;\; n^i \in \mathbb{Z} \,. \tag{3.14}$$

L.I stands for linearly independent.

One can see examples of Bravais lattices in 3.1a,3.1b. Real crystals are, of course, not infinite and there is a very important area of study of surface effects and edge states, which are in fact particularly important for topological phases of matter, such as the IQHE. Nevertheless since we have around $N \sim 10^{26} - 10^{27}$ ions in a crystal it is a very good first approximation, particularly to describe what is known as the bulk of the crystal. If we consider

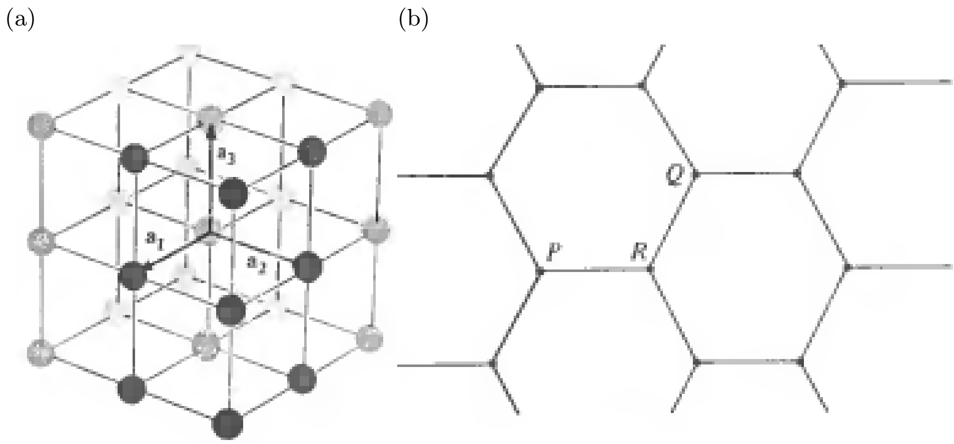

Figure 3.1: (a) A 3-dimensional cubic Bravais lattice (b) A two-dimensional Honeycomb or Hexagonal lattice. Figures taken from [1].

periodic boundary conditions , then there is no necessity for an infinite array, restricting the n^i's to be bounded.
We will now define a primitive cell which is the most common way of partitioning a Bravais lattice

Definition 33 *The* ***Wigner-Seitz primitve cell*** *is the region in* $\mathbb{R}^d$ *which is closest to a specific point in the Bravais lattice. Since the array is periodic all points in the lattice have the same Wigner-Seitz primitive cell.*

Hence, $\mathbb{R}^d$ can be completely foliated by the Wigner-Seitz cell. We shall not make much use of the Wigner-Seitz cell of the direct Bravais lattice of our crystal, but as we will see, a similar construction will be the basis of our classification.

3.2.1 Reciprocal lattice

The reciprocal lattice is also a fundamental construction underlying the study of crystals. Even though it is constructed from the direct Bravais lattice, it is this structure which is employed in the analysis of materials and ironically not the direct lattice (so in that sense the name is misleading but there is perhaps no better alternative).

Definition 34 *The* ***reciprocal lattice*** *is set of wave vectors* $\{\vec{K}\}$*, such* *that*

$$e^{\vec{K}\cdot\vec{R}} \;=\; 1\,, \tag{3.15}$$

Where $\vec{R} = n^i \vec{a}_i$ *is an element of the Bravais lattice, as defined above.*

Notice that we have defined the reciprocal lattice $\{\vec{K}\}$ in terms of a choice of direct lattice $\{\vec{R}\}$. Since the only allowed coefficients of the generators of the direct lattice are integers, each element $\vec{K} = k^j \vec{b}_j$ of the reciprocal lattice must satisfy

$$\vec{K}\cdot\vec{R} = n^i s^j (\vec{a}_i \cdot \vec{b}_j) = 2\pi m,\; m \in \mathbb{Z}\,, \forall\, \vec{R}. \tag{3.16}$$

The simplest way to construct such a lattice is to choose $\vec{b}_j$ so that

$$\vec{a}_i \cdot \vec{b}_j \;=\; 2\pi \delta_{ij}\,, \tag{3.17}$$

$$s^j \;\in \mathbb{Z}\,. \tag{3.18}$$

Hence our reciprocal lattice is also a Bravais lattice!

The Wigner-Seitz cell of the reciprocal lattice is called the *first Brillouin zone* and we shall denote it by X. We will see in our classification that the Brillouin zone will play the role of our topological base space! We will see in Bloch's theorem that for periodic boundary conditions the Brillouin zone is always a torus.

3.3 Bloch's theorem

Consider an electron in a crystal array described by a Bravais lattice $\{\vec{R}\}$. Because of the periodicity of the lattice, the potential in the Schrödinger equation satisfies

$$U(\vec{r} + \vec{R}) = U(\vec{r})\,, \forall\, \{\vec{R}\}. \tag{3.19}$$

For such systems we have *Bloch's theorem*

Bloch 1 *Given the one-electron Schrödinger equation with a periodic potential and periodic boundary conditions*

$$\mathcal{H}(\vec{r})\,|\varphi(\vec{r})\rangle = \varepsilon\,|\varphi(\vec{r})\rangle\;, \tag{3.20}$$

For each eigenvector of $\mathcal{H}(\vec{r})$ *there exists* $\vec{k}$ *such that*

$$|\varphi_n(\vec{r} + \vec{R})\rangle = e^{i\vec{k}\cdot\vec{R}}\,|\varphi_n(\vec{r})\rangle\;. \tag{3.21}$$

where $\vec{k}$ *is known as the* ***crystal momentum****.*

Here we can deduce the fact that our Brillouin zone is topologically a torus, constructing the reciprocal lattice $\{\vec{K}\}$ corresponding to the direct lattice $\{\vec{R}\}$ and multipliying by $1 = e^{\vec{K}\cdot\vec{R}}$, we have

$$\begin{aligned} |\varphi_n(\vec{r}+\vec{R})\rangle &= e^{i\vec{k}\cdot\vec{R}}\,|\varphi_n(\vec{r})\rangle \ , \\ &= e^{i\vec{K}\cdot\vec{R}}e^{i\vec{k}\cdot\vec{R}}\,|\varphi_n(\vec{r})\rangle \ , \\ &= e^{i(\vec{k}+\vec{K})\cdot\vec{R}}\,|\varphi_n(\vec{r})\rangle \ . \end{aligned} \tag{3.22}$$

Thus, for each $\vec{k} \in \mathrm{X}$, $\vec{k} + a\vec{K}$ labels the same quantum state for $a \in \mathbb{Z}$, Since the Brillouin zone translated by the reciprocal lattice foliates $\vec{k}$-space, we can and should restrict ourselves to points in X.
We shall hence forth, restate the problem at hand in manner that exploits this periodicity to maximum. Define the *Bloch Hamiltonian* (also known as the crystal momentum Hamiltonian) as

$$\mathcal{H}(\vec{k}) = \sum_{\vec{r}} e^{-i\vec{k}\cdot\vec{r}}\mathcal{H}(\vec{r})e^{i\vec{k}\cdot\vec{r}} , \tag{3.23}$$

with corresponding eigenvectors

$$|\Phi_n(\vec{k})\rangle = \sum_{\vec{r}} e^{-i\vec{k}\cdot\vec{r}}\,|\varphi_{n,\vec{k}}(\vec{r})\rangle \ , \tag{3.24}$$

where $\displaystyle\sum_{\vec{r}}$ should properly be something similar to $\displaystyle\frac{1}{V}\int d\vec{r}$ but we avoid this to simplify notation.
Together, they satisfy

$$\mathcal{H}(\vec{k})\,|\Phi_n(\vec{k})\rangle = \varepsilon_n(\vec{k})\,|\Phi_n(\vec{k})\rangle \ . \tag{3.25}$$

Where $\{\varepsilon_n(\vec{k})\}$ are known as *energy bands* and are assumed to be continuous functions of $\vec{k}$, this assumption will be fundamental for our construction. Energy bands acquired their name from the *tight-binding approximation*, where the potential that the electrons are subject of is assumed to be that of a single ion and is related to the atomic levels s, p, d and so on. We will not have much to say about the tight-binding approximation other than that it is a very successful model and has many applications.
A few but very important remarks are in order. Since X is a torus, which is compact and $\varepsilon_n(\vec{k})$ is continuous, $\varepsilon_n(\vec{k})$ must be bounded! Nevertheless, $\varepsilon_n(\vec{k})$ is a countable set for each $\vec{k}$ and bounded operators on a Hilbert space are locally bounded functions because of the norm of the Hilbert

space. The crystal momentum $\vec{k}$ plays the analogous role of the momentum operator $\vec{p}$ but they are not equivalent, since in general the eigenstates of the Hamiltonian are not eigenstates of momentum.

3.4 Fermi surface

When the system under study is in a *metallic phase*, it means that some of the energy bands are partially filled. For these partially filled bands, there will be a surface embedded in X, separating the filled $\vec{k}$ from those which are not.

The derivations we have employed so far assume zero temperature. Experimentally it is possible to describe systems at very low temperatures to a good approximation using this schemes. In those cases one can experimentally control the chemical potential of the system, which is the equivalent of the Fermi energy ε_F, and so it is equivalent to setting the Fermi energy of the system.

The Fermi surface are the set of points $\vec{k}$ in X such that at least $\exists\, m \in \mathbb{N}$, so that $\varepsilon_m(\vec{k}) = \varepsilon_F$. There may be more than one such m for a given $\vec{k}$, these different m's are called the *branches* of the Fermi surface, the Fermi surface being the union of all such branches.

In mathematical terms we can define the momentum-space propagator (or parametrix) of a constant energy ε-manifold as

$$\begin{aligned} \mathrm{G}(\vec{k},\varepsilon)\,|\Phi_n(\vec{k})\rangle &= \frac{1}{\varepsilon_n(\vec{k})-\varepsilon}\,|\Phi_n(\vec{k})\rangle\ , && (3.26)\\ (\mathcal{H}(\vec{k})-\varepsilon I)\mathrm{G}(\vec{k},\varepsilon) &= I\,, && (3.27) \end{aligned}$$

where the last equation holds except at the poles of $\mathrm{G}(\vec{k},\varepsilon)$, which corresponds to the kernel of $(\mathcal{H}(\vec{k})-\varepsilon I)$.

Thus, the Fermi surface of system corresponds to the set of points of its Brilluoin zone such that

$$(\mathcal{H}(\vec{k})-\varepsilon_F I) = 0\,, \tag{3.28}$$

or equivalently the collection of all $\vec{k} \in$ X for which $\mathrm{G}(\vec{k},\varepsilon_F)$ has a pole singularity. Hence our operator of interest is $(\mathcal{H}(\vec{k})-\varepsilon_F I)$ and we wish to study its kernel, which has a physical interpretation. An important remark we can make at this point is that this is consistent with the freedom of choosing the zero of the energy scale in quantum mechanics, for ε_F has itself a geometric interpretation, since it tells us if a $\vec{k} \in$ X are *occupied* or not. Any rescaling of the energy can change the numerical value of ε_F but cannot change whether a given $\vec{k}$ is occupied or not! Nor the degeneracy depends on such value. We have the freedom to choose our energy scale such that $\varepsilon_F = 0$ and we shall only write $\mathcal{H}(\vec{k})$ instead. This definitions are useful for they generalize in some cases where there is electron-electron interactions and a $\vec{k}$- space but not a Brillouin zone. Non the less, we will restrict our selves to the framework we have elucidated here, where there are no electron-electron interactions and hence, we have a well defined Brillouin zone with a topology in the continuum limit approximation.

Chapter 4

Classification

We are now ready to develop a classification of the different topologically stable Fermi surfaces which can arise in metallic phases of condensed matter systems. Let us refrain from discussing results obtained in our construction until 4.4. First, we need to introduce some more mathematical concepts which are necessary but the basic tools for their constructions were introduced in 2. We will also employ the conventions of [34], [3], [4] and [33].

Let us remind ourselves of the topology we will be applying to our operators

Definition 35 *Let $\mathcal{B}(\mathscr{H})$ be the Banach algebra of bounded operators on a separable Hilbert space $\mathscr{H}$. The Norm topology of $\mathcal{B}(\mathscr{H})$ is the one induced by the norm on $\mathscr{H}$, where for $\mathcal{T} \in \mathcal{B}(\mathscr{H})$, it is given by*

$$\|\mathcal{T}\|_{\mathcal{B}(\mathscr{H})} = sup_{v\in\mathscr{H}} \frac{\|\mathcal{T}v\|_{\mathscr{H}}}{\|v\|_{\mathscr{H}}}, \tag{4.1}$$

and the norm topology is just as defined in 2.

One of the key properties we shall use of the norm topology for $\mathcal{B}(\mathscr{H})$ is that the composition of operators

$$\mathcal{B}(\mathscr{H}) \times \mathcal{B}(\mathscr{H}) \longrightarrow \mathcal{B}(\mathscr{H}), (\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A}\mathcal{B} \tag{4.2}$$

is continuous.
There are other types of topologies for $\mathcal{B}(\mathscr{H})$ such as the so-called *weak* and *strong* operator topologies. We should mention that the norm topology is the strongest topology for $\mathcal{B}(\mathscr{H})$, it is also more natural than the other two from a classical picture point of view. We should also mention that in [15] they employ a different topology which is for more general operators than bounded operators, called the *compact-open* topology and it is from their point of view a more natural topology for operators on the Hilbert space under study. That being said, in condensed matter systems with no electron-electron interaction, were we have Bloch's theorem, the eigenvalues

of our Hamiltonian are assumed to be continuous over the Brillouin zone X, which is compact. Therefore, the eigenvalues of our Hamiltonian (energy bands $\varepsilon_n(\vec{k})$) must be bounded, a fact whose satisfaction is not evidently clear by the choice of topology in [15].

4.1 K-theory and Fredholm operators

In what follows we shall relate K-theory and Fredholm operators, essentially following the work of Atiyah [3]. Let us endow the set of Fredholm operators $\mathcal{F}(\mathscr{H})$, defined as a set in 2 by first defining a topology on the Calkin algebra $\mathcal{CAL}(\mathscr{H})$ by giving $\mathcal{B}(\mathscr{H})$ the norm topology and quotienting by set of compact operators $\mathcal{K}(\mathscr{H})$. Then, we restrict to the set of *invertible* operators in $\mathcal{CAL}(\mathscr{H})$, denoted as $\mathcal{CAL}^*(\mathscr{H})$ and consider the projection

$$\pi : \mathcal{B}(\mathscr{H}) \longrightarrow \mathcal{CAL}(\mathscr{H})\,. \tag{4.3}$$

Then the set of Fredholm operators is $\pi^{-1}\left(\mathcal{CAL}^*(\mathscr{H})\right)$. Since $\mathcal{F}(\mathscr{H})$ is open in $\mathcal{B}(\mathscr{H})$, the composition of Fredholm operators is Fredholm (induced by the norm topology on $\mathcal{B}(\mathscr{H})$), that adjoints of Fredholm operators are Fredholm, since every open set in $\mathcal{B}(\mathscr{H})$ is open in $\mathcal{B}(\mathscr{H})$ and the adjoint relation is defined in terms of the norm on $\mathscr{H}$ and, finally that the addition of a compact operator and a Fredholm operator is again Fredholm.
Associated to a given Fredholm operator $\mathcal{H}$ is its *index*

$$Index\,\mathcal{F} = dim\ker\mathcal{F} - dim\ker\mathcal{F}^{\dagger}\,, \tag{4.4}$$

The index satisfies the following very useful properties

$$\begin{aligned} Index\,\mathcal{F}\circ\mathcal{F}' &= Index\,\mathcal{F} + Index\,\mathcal{F}'\,, && (4.5)\\ Index\,\mathcal{F}+\mathcal{K} &= = Index\,\mathcal{F}\,, && (4.6)\end{aligned}$$

for $\mathcal{F}, \prime\mathcal{F}$ Fredholm and $\mathcal{K}$ compact.
If we now study a family of Fredholm operators $\mathcal{F}(x)$, where $x \in$ X and X is compact, it happens that the dimension of the kernel of $\mathcal{F}(x)$ is only semi-continuous with x, jumping in $\mathbb{N}$, however its index is a locally constant function (may change for different path components of X).
We can generalize the index function in order to describe elements in $K(\mathrm{X})$. First, given an ONB $\{e_0, e_1, ...\}$ of $\mathscr{H}$, we define the following subspaces,

together with their corresponding projection operator as

$$\begin{aligned}
\mathscr{H}_n &= \{span\{e_i, e_{i+1}, ...\},\, i \geq n\}\ , && (4.7)\\
\mathscr{H}_n^{\perp} &= \mathscr{H}/\mathscr{H}_n\,, && (4.8)\\
\mathcal{P}_n :\ \mathscr{H} &\longrightarrow \mathscr{H}_n\,, && (4.9)\\
Index\,\mathcal{P}_n &= 0\,. && (4.10)
\end{aligned}$$

The last equation being true since $\mathcal{P}_n$ is self-adjoint. Thus, for any continuous Fredholm family $\mathcal{F}(x)$ we can define

$$\begin{aligned}
\mathcal{F}_n(x) &= \mathcal{P}_n \circ \mathcal{F}\,, && (4.11)\\
Index\,\mathcal{F}_n(x) &= Index\,\mathcal{F}(x)\,. && (4.12)
\end{aligned}$$

Since $\mathcal{F}_n(x)(\mathscr{H}) = \mathscr{H}_n\ \forall x$ and $dim \ker \mathcal{F}_n^{\dagger}(x) = dim \mathscr{H}_n^{\perp} = n$ for all $x \in \mathrm{X}$ for sufficiently large n by construction (fact proven at the beginning of [3]), we obtain

$$Index\,\mathcal{F}(x) = d - n\,, \tag{4.13}$$

where $d = dim \ker \mathcal{F}_n(x)$ and now it is only d which is an unknown.
Consider a family of vector spaces

$$V(x) = \ker \mathcal{F}_n(x)\,. \tag{4.14}$$

and consider the topological space

$$\mathrm{E} = \bigcup_{x \in \mathrm{X}} V(x) \subset \mathrm{X} \times \mathscr{H}\,. \tag{4.15}$$

E is a locally trivial vector bundle with base space X!
Imagine that we now wish to study homotopy classes of families of Fredholm operators, parametrized by a compact space, then we have the following incredible theorem due to Atiyah

Atiyah-Fredholm 1 *Let* X *be a compact space and let* $[\mathrm{X}, \mathcal{F}(\mathscr{H})]$ *denote the set of homotopy classes of continuous maps* $\mathcal{F} : \mathrm{X} \longrightarrow \mathcal{F}(\mathscr{H})$*. Then the original index map* $\mathcal{F}(x) \mapsto Index\,\mathcal{F}(x)$ *induces an* ***isomorphism***

$$\begin{aligned}
Index :\ [\mathrm{X}, \mathcal{F}(\mathscr{H})] &\longrightarrow K(\mathrm{X})\,, && (4.16)\\
Index\,F &= [\ker \mathcal{F}_n] - [\mathrm{X} \times \mathscr{H}_n^{\perp}]\,. && (4.17)
\end{aligned}$$

Thus, $\mathcal{F}(\mathscr{H}) \simeq BU \times \mathbb{Z}$*.*

We should clarify that, though the construction seems to depend on a choice of orthonormal basis for $\mathscr{H}$ and a choice of n for the subspaces $\mathscr{H}_n$, however it can be showed that

$$\ker \mathcal{F}_{n+1} \approx \mathcal{F}_n \oplus \xi \,, \tag{4.18}$$
$$\ker \mathcal{F}^\dagger_{n+1} \approx \mathcal{F}^\dagger_n \oplus \xi \,, \tag{4.19}$$

were the trivial line bundle ξ is generated by the extra element in the kernel, e_n. Also the former property it was only used as having any ONB, hence it is independent of the choice of ONB.

Let us now move on to study a particular kind of Fredholm operators, known as skew-adjoint Fredholm operators, denoted as $\hat{\mathcal{F}}(\mathscr{H})$. The topology of these operators was first studied by Atiyah and Singer [4]. Skew-adjoint Fredholm operators are a subset of Fredholm operators $\mathcal{F}$ with $Index\, F = 0$ $= BU$ by the preceding theorem.

First we split $\hat{\mathcal{F}}(\mathscr{H})$ into 3 disjoint components

- $\hat{\mathcal{F}}_+(\mathscr{H}) = \{i\varepsilon_n \leq 0\,,$ only for finite number of n's .}
- $\hat{\mathcal{F}}_-(\mathscr{H}) = \{i\varepsilon_n \geq 0\,,$only for finite number of n's .}
- $\hat{\mathcal{F}}_*(\mathscr{H}) = \hat{\mathcal{F}}(\mathscr{H})/\left(\hat{\mathcal{F}}_+(\mathscr{H}) \cup \hat{\mathcal{F}}_-(\mathscr{H})\right)$

Elements of $\hat{\mathcal{F}}_+(\mathscr{H})$ are called *essentially positive* and elements of $\hat{\mathcal{F}}_-(\mathscr{H})$ are called *essentially negative*. Equivalently we can say that $\hat{\mathcal{F}}_+(\mathscr{H})$ and $\hat{\mathcal{F}}_-(\mathscr{H})$ have invariant subspaces of codimension n, where either the eigenvalues are always positive or always negative.
We shall now prove that $\hat{\mathcal{F}}_+(\mathscr{H})$ and $\hat{\mathcal{F}}_-(\mathscr{H})$ are null-homotopic . Consider $\mathcal{H} \in \hat{\mathcal{F}}_+(\mathscr{H})$ and the following homotopy

$$\mathcal{H}_t = (1-t)\mathcal{H} - itI\,, \tag{4.20}$$
$$\mathcal{H}^\dagger_t = -\mathcal{H}_t\,. \tag{4.21}$$
$$\tag{4.22}$$

Since $\mathcal{H}_t$ is skew-adjoint for all t, it has an ONB in $\mathscr{H}$, written in terms of $\{|\varphi_n\rangle\}$, ONB associated to $\mathcal{H}$, we can see that

$$\mathcal{H}_t\,|\varphi_n\rangle = [(1-t)\varepsilon_n - it]\,|\varphi_n\rangle\ . \tag{4.23}$$

Thus,

$$i\varepsilon_n(t) = (1-t)i\varepsilon_n + t \geq 0\,, \tag{4.24}$$

and so, for every t, $\mathcal{H}_t \in \hat{\mathcal{F}}_+(\mathscr{H})$, thus there is path from every operator $\mathcal{H} \in \hat{\mathcal{F}}_+(\mathscr{H})$ to $-iI \in \hat{\mathcal{F}}_+(\mathscr{H})$, so we can finally conclude that $\hat{\mathcal{F}}_+(\mathscr{H})$ is null-homotopic. Interchanging itI for $-itI$ we obtain an equivalent retraction of $\hat{\mathcal{F}}_-(\mathscr{H})$ to iI.
The remaing non-trivial component of $\hat{\mathcal{F}}(\mathscr{H})$, $\hat{\mathcal{F}}_*(\mathscr{H})$ is of great importance to us because of the following theorem

Atiyah-Singer-Skew 1 *The map*

$$\alpha : \hat{\mathcal{F}}_*(\mathscr{H}) \longrightarrow \Omega\left(\mathcal{F}(\mathscr{H}), I\right) \tag{4.25}$$

$$A \mapsto \left(I\cos\pi t + A\sin\pi t\,, t\in[0,1]\right)\,, \tag{4.26}$$

Is a homotopy equivalence.

At first sight, the map given above is not even a loop since it starts at I and ends at $-I$, however since $GL(\mathscr{H})$, the set of invertible operators of our separable Hilbert space is null-homotopic, we can (through a very intricate way) deform this path into a loop in $\mathcal{F}(\mathscr{H})$. The original proof of this is due to Atiyah and Singer [4]. There is no simple way to summarize this proof without taking too great a detour of our interest, nevertheless we suggest our readers to attend to the the lecture notes of Dan. S. Freed [14]. We will only give a few remarks and examples concerning this proof, which are employed in these notes.

First, because of the previously sketched theorem of Atiyah, we know that $\mathcal{F}(\mathscr{H}) \simeq U/\left(U\times U\right)\times\mathbb{Z}$, then, the above theorem would imply that $\hat{\mathcal{F}}_*(\mathscr{H}) \simeq U$, since $\Omega\left(\mathcal{F}(\mathscr{H}), I\right) \simeq \Omega\left(U/\left(U\times U\right)\times\mathbb{Z}\right) \simeq U$. Notice that $\pi_0\left(\mathcal{F}(\mathscr{H})\right) = \mathbb{Z}$ with each component determined by the index, thus $\Omega\left(\mathcal{F}(\mathscr{H}), I\right)$ are simply loops in the component containing I, that being Fredholm operators of index 0. Hence we only need to proove $\hat{\mathcal{F}}_*(\mathscr{H}) \simeq U$. Now we retract $\hat{\mathcal{F}}_*(\mathscr{H})$ to $\hat{F}_*(\mathscr{H})$, where $H \in \hat{F}_*(\mathscr{H})$ satisfies

- H is Fredholm.
- $\|H\| = 1$.
- $Index\, H = 0$.
- $H^\dagger = -H$.

- H is neither essentially positive nor essentially negative.

The deformation retraction is given by

$$\mathcal{H}_t = \mathcal{H}\left[(1-t) + \frac{t}{\|\mathcal{H}\|}\right] , \tag{4.27}$$

which is well defined since $\|\mathcal{H}\| \neq 0$ because $\mathcal{H}$ is Fredholm. Now let us consider a new subset of $\mathcal{B}(\mathscr{H})$, namely

$$GL^{\mathcal{K}}(\mathscr{H}) := \{\mathcal{P} \in GL(\mathscr{H}) \,|\, \mathcal{P} - I \text{ compact}\}$$

We then retract $GL^{\mathcal{K}}(\mathscr{H})$ to $U^{\mathcal{K}}(\mathscr{H})$ using the same homotopy we used for $\hat{F}_*(\mathscr{H})$ and $\hat{\mathcal{F}}_*(\mathscr{H})$. One can prove (using methods for infinite dimensional manifolds developed by Pais) that $U^{\mathcal{K}}(\mathscr{H})$ is homotopic to U, the classifying space for $K^{-1}(\,)$. This is our connection to $\Omega\left(\mathcal{F}_(\mathscr{H}), I\right) \simeq U$ and so, we only need to prove $\hat{F}_*(\mathscr{H}) \simeq U^{\mathcal{K}}(\mathscr{H})$ via

$$\begin{aligned} \beta : \hat{\mathcal{F}}_*(\mathscr{H}) \quad &\longrightarrow \quad -U^{\mathcal{K}}(\mathscr{H}) \\ J \quad &\mapsto \quad e^{\pi J} . \end{aligned} \tag{4.28}$$

and we can conclude. We will do no such thing except give an example. Let $\mathcal{T} = \mathcal{P} + I$ ($\mathcal{P} \in U^{\mathcal{K}}(\mathscr{H})$) compact, furthermore assume $\mathcal{T}$ has *finite rank*, (a special subset), meaning $\dim \mathscr{H}/\mathcal{T}(\mathscr{H}) = \infty$, then naming $\mathscr{L} = \ker T$ we have

$$\mathscr{H} = \mathscr{L} \oplus \mathscr{L}^{\perp} . \tag{4.29}$$

Suppose $Q \in \beta^{-1}(\mathcal{P})$, so that $\mathcal{P} = e^{\pi Q}$. Note that the exponential for infinite dimensional operators is not injective. At $\mathscr{L}^{\perp}$, which is by construction finite dimensional, the restriction $Q|_{\mathscr{L}^{\perp}}$ is in one to one correspondence with $\mathcal{P}|_{\mathscr{L}^{\perp}}$, since $\|Q\| = 1$ and $Q = -Q^{\dagger}$, then the eigenvalues of Q must lie in $[-i, i]$, which maps almost homeomorphically to $S^1 \subset \mathbb{C}$, except at the end points, both of which go to $-i$. With this in mind one can prove that Q splits $L = L_+ \oplus L_-$, where $Q(L_+) = iL_+$ and $Q(L_-) = -iL_-$. Thus, $\beta^{-1}(\mathcal{P})$ is homeomorphic to $U(L)/\left(U(L_+) \times U(L_-)\right)$, however, due to Kuiper's theorem all three spaces at hand are infinite dimensional Hilbert spaces, so all of them are contractible. and thus we can retract $\beta^{-1}(\mathcal{P})$ to $\mathcal{P}$, viewing $U^{\mathcal{K}}(\mathscr{H})$ as "base space" for $\hat{\mathcal{F}}_*(\mathscr{H})$ and β as a projection. This is the general strategy employed, where it is shown that an inverse image of β retracts to an element of $U^{\mathcal{K}}(\mathscr{H})$, however, we must be careful for the inverse image β^{-1} is not always homeomorphic to $U(L)/\left(U(L_+) \times U(L_-)\right)$ (This was only true for $\mathcal{P}$ finite rank).

With this we have portrayed the path to proving both the theorem of Atiyah and the theorem of Atiyah-Singer for skew-adjoint Fredholm operators. We will now proceed to show the connection between Fredholm operators and Fermi surfaces.

4.2 K-theory and Fermi surfaces

Up to now, we have introduced, independently, many mathematical concepts leading up to K-theory and the basic condensed matter concepts of electronic transport theory in systems with discrete translational invariance, that is the Brillouin zone X and Bloch's theorem. Furthermore, with a few exceptions, we have not made any connections between K-theory and Fermi surfaces, though we have claimed that we would classify topologically stable Fermi surfaces. We will immediately amend this. We will reintroduce many of the necessary physical and mathematical concepts in a summary leading up to the classification. This will be useful for readers who already knew the concepts presented in the previous sections and chapters and are only interested in the connections and results.

Let X be the Brillouin zone of a non-interacting fermionic system. For each crystal momentum $\vec{k} \in \mathrm{X}$ there is a Bloch Hamiltonian operator $\mathcal{H}(\vec{k})$ acting on a complex separable Hilbert space $\mathscr{H}$. $\mathcal{H}(\vec{k})$ is hermitian [5], furthermore, there exists a complete orthonormal basis $\{|\Phi_n(\vec{k})\rangle\}_{n=1}^{\infty}$ of $\mathscr{H}$, such that

$$\mathcal{H}(\vec{k})\,|\Phi_n(\vec{k})\rangle \quad = \quad \varepsilon_n(\vec{k})\,|\Phi_n(\vec{k})\rangle \;, \tag{4.30}$$

where the $\vec{k}$-dependent eigenvalues $\varepsilon_n(\vec{k})$ are known as energy bands.
Define the momentum-space propagator (or parametrix) of a constant energy ε-manifold as

$$\mathrm{G}(\vec{k},\varepsilon)\,|\Phi_n(\vec{k})\rangle \quad = \quad \frac{1}{\varepsilon_n(\vec{k})-\varepsilon}\,|\Phi_n(\vec{k})\rangle \;, \tag{4.31}$$

$$(\mathcal{H}(\vec{k})-\varepsilon I)\mathrm{G}(\vec{k},\varepsilon) \quad = \quad I\,, \tag{4.32}$$

where the last equation holds except at the poles of $\mathrm{G}(\vec{k},\varepsilon)$, which corresponds to the kernel of $(\mathcal{H}(\vec{k})-\varepsilon I)$ (see chapter 3).
The Fermi surface [1] of a non-interacting fermionic system corresponds to the set of points of its Brilluoin zone for which one or more energy bands are equal to the Fermi energy ε_F

$$(\mathcal{H}(\vec{k})-\varepsilon_F I) \quad = \quad 0\,, \tag{4.33}$$

or equivalently the collection of all $\vec{k} \in \mathrm{X}$ for which $\mathrm{G}(\vec{k}, \varepsilon_F)$ has a pole singularity. Hence our operator of interest is $(\mathcal{H}(\vec{k}) - \varepsilon_F I)$ and we wish to study its kernel, which has a physical interpretation [1]. We have the freedom to choose our energy scale such that $\varepsilon_F = 0$ and we shall only write $\mathcal{H}(\vec{k})$ instead. If we restrict ourselves to the n-th energy band $\varepsilon_n(\vec{k})$, the corresponding surface embedded in the Brillouin zone is called the n-th *branch* of the Fermi surface, the Fermi surface being the union of all such branches.

We shall only consider Fermi surfaces which consist of a finite number of branches, implaying

$$dim\, Ker\, \mathcal{H}(\vec{k}) < \infty \,. \tag{4.34}$$

As in [25], we shall also make the physical assumption that a Hamiltonian describing this type of system is *bounded*. This is a generalization of models with a finite number of energy bands, which are inherently bounded operators. Thus, we allow an infinite number of bands and as we shall see, this infinity will be of utmost importance for the physical interpretation of our results.Enlisting the assumed physical properties of our Bloch Hamiltonian in mathematical terms,

- $\mathcal{H}(\vec{k})$ is *bounded*,
- $dim\, Ker\, \mathcal{H}(\vec{k}) < \infty$,
- $\mathcal{H}(\vec{k})$ is *self-adjoint*.

Operators satisfying these conditions are known in the mathematical literature as *self-adjoint Fredholm* operators [4],[33], where the set of Fredholm operators [3] are bounded operators in $\mathscr{H}$ with finite kernel and cokernel 2. We shall denote the space of all complex Fredholm operators $\mathcal{F}(\mathscr{H})$ and self-adjoint Fredholm operators as $\mathcal{F}^{sa}(\mathscr{H})$. Let us endow $\mathcal{F}(\mathscr{H})$ with the *norm* topology [2], as we did in 4.1.

This choice of topology comes at a price, for we proved in 4.1 that operators which are *essentially negative* (finite number of conduction bands, $\varepsilon_n(\vec{k}) > 0$) or *essentially positive* (finite number of valence bands, $\varepsilon_n(\vec{k}) < 0$) are topologically trivial in the norm topology, so we must assume

- $\mathcal{H}(\vec{k})$ is neither *essentially positive* nor *essentially negative.*

The former assumption adds an infinite number of trivial core bands and is common to all references we have cited here, that attempt at a classification.

[1] No rescaling of the energy will change this kernel

[2] In [15] they employ instead the *compact-open* topology for *gapped* systems

Assuming an infinite number of valence bands is new, however, the behavior of bands below (or above) the Fermi energy is physically relevant for the behavior of our Fermi surface only when they cross, adding a branch to it, making them irrelevant otherwise. Let us denote the subset of operators that satisfy all our assumptions $\mathcal{F}_*^{sa}(\mathscr{H})$.
$\mathcal{H}(\vec{k})$ is a continuous function of $\vec{k}$ and we wish to put in the same equivalence relation all systems which can be *adiabatically evolved* 3 into one another [6] or equivalently, $\forall\ \mathcal{H}_0, \mathcal{H}_1 : \mathrm{X} \to \mathcal{F}_*^{sa}(\mathscr{H})$ we have

$$\begin{aligned} \mathcal{H}_0 \sim \mathcal{H}_1 \Longleftrightarrow \exists\, g : \mathrm{X} \times I \to \mathcal{F}_*^{sa}(\mathscr{H}) \ \textit{continuous}, \\ g(\vec{k}, 0) = \mathcal{H}_0(\vec{k}),\ \ g(\vec{k}, 1) = \mathcal{H}_1(\vec{k})\ \forall\ \vec{k} \in \mathrm{X}\,. \end{aligned} \tag{4.35}$$

Thus, all types of Fermi surfaces separated by this equivalence relation are given by the set of *homotopies* $[\mathrm{X}, \mathcal{F}_*^{sa}(\mathscr{H})]$.
The set of skew-adjoint Fredholm operators $\hat{\mathcal{F}}(\mathscr{H}) \equiv i\mathcal{F}^{sa}(\mathscr{H})$ is trivially *homeomorphic* to $\mathcal{F}^{sa}(\mathscr{H})$ in the norm topology, and so are $\mathcal{F}_*^{sa}(\mathscr{H})$ and $\hat{\mathcal{F}}_*(\mathscr{H})$, the corresponding non-trivial component of $\hat{\mathcal{F}}(\mathscr{H})$. As we portrayed in **??**, Atiyah and Singer proved in [4] that $\hat{\mathcal{F}}_*(\mathscr{H}) \simeq \Omega\mathcal{F}(\mathscr{H})$, the *loop* space of the set of Fredholm operators $\mathcal{F}(\mathscr{H})$. Combining this with Atiyah's proof [3] that $\mathcal{F}(\mathscr{H})$ is a *classifying* space for complex K-theory and the *suspension isomorphism* [19] we get

$$\begin{aligned} [\mathrm{X}, \mathcal{F}_*^{sa}(\mathscr{H})] &\approx [\mathrm{X}, \hat{\mathcal{F}}_*(\mathscr{H})]\,, \\ &\approx [\mathrm{X}, \Omega\mathcal{F}(\mathscr{H})]\,, \\ &\approx [S\mathrm{X}\,, \mathcal{F}(\mathscr{H})]_*\,, \\ &\approx K^{-1}(\mathrm{X})\,, \end{aligned} \tag{4.36}$$

where $(2)K^{-1}(\mathrm{X}) \approx \tilde{K}(S\mathrm{X})$ denotes the *Grothendieck completion* [20], of the semi-group $Vect_{\mathbb{C}}^{s}(S\mathrm{X})$ of *stable* isomorphism classes of complex *vector bundles* over $S\mathrm{X}$, the suspension of our Brillouin zone. It is an abelian group and its elements are not vector bundles but *virtual* vector bundles!
Notice that Grothendieck's completion popped out naturally from our construction!

4.3 Results

We will now compute $K^{-1}(\mathrm{X})$ for $\mathrm{X} = \mathbb{T}^d$, $d = 1, 2$ and 3. To do this, we shall employ the formula seen in 2

$$\Sigma(\mathrm{X} \times \mathrm{Y}) \simeq \Sigma\mathrm{X} \vee \Sigma\mathrm{Y} \vee \Sigma(\mathrm{X} \wedge \mathrm{Y}). \tag{4.37}$$

Thus for $d = 1$

$$\begin{aligned} K^{-1}(S^1) &\approx \tilde{K}(SS^1)\,, \\ &\approx \tilde{K}(S^2)\,, \\ &\approx \mathbb{Z}. \end{aligned} \tag{4.38}$$

For $d = 2$

$$\begin{aligned} K^{-1}(\mathbb{T}^2) &\approx \tilde{K}(S\mathbb{T}^2)\,, \\ &\approx \tilde{K}\left(S(S^1 \times S^1)\right)\,, \\ &\approx \tilde{K}\left(SS^1 \vee SS^1 \vee S(S^1 \wedge S^1)\right)\,, \\ &\approx \tilde{K}\left(SS^1\right) \oplus \tilde{K}\left(SS^1\right) \oplus \tilde{K}\left(S(S^1 \wedge S^1)\right)\,, \\ &\approx \tilde{K}\left(S^2\right) \oplus \tilde{K}\left(S^2\right) \oplus \tilde{K}\left(S(S^2)\right)\,, \\ &\approx \tilde{K}(S^2) \oplus \tilde{K}(S^2) \oplus \tilde{K}(S^3)\,, \\ &\approx \mathbb{Z} \oplus \mathbb{Z}\,. \end{aligned} \tag{4.39}$$

For $d = 3$, we have a more complicated computation since

$$\begin{aligned} K^{-1}(\mathbb{T}^3) &\approx \tilde{K}(S\mathbb{T}^3)\,, \\ &\approx \tilde{K}\left(S(S^1 \times S^1 \times S^1)\right)\,, \\ &\approx \tilde{K}\left(SS^1 \vee S\mathbb{T}^2 \vee S(S^1 \wedge \mathbb{T}^2)\right)\,, \\ &\approx \tilde{K}\left(S^2\right) \oplus \tilde{K}\left(S\mathbb{T}^2\right) \oplus \tilde{K}\left(S(S^1 \wedge \mathbb{T}^2)\right)\,. \end{aligned} \tag{4.40}$$

Now,

$$S^1 \wedge \mathbb{T}^2 = \mathbb{T}^3 / \left(S^1 \wedge \mathbb{T}^2\right)\,. \tag{4.41}$$

In [15], theorem 11.8, it is shown that

$$\mathbb{T}^3 \simeq S^1 \vee S^1 \vee S^1 \vee S^2 \vee S^2 \vee S^2 \vee S^3\,. \tag{4.42}$$

Thus,

$$\tilde{K}\left(S(S^1 \wedge \mathbb{T}^2)\right) \approx \mathbb{Z}\,, \tag{4.43}$$

which yields

$$\begin{aligned} K^{-1}(\mathbb{T}^3) &\approx \tilde{K}\left(S^0\right) \oplus \tilde{K}\left(S^2\right) \oplus \tilde{K}\left(S\mathbb{T}^2\right)\,, \\ &\approx \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}. \end{aligned} \tag{4.44}$$

Equations 4.38, 4.39 and 4.44 materialize the fruits of our labor. We will proceed to discuss part of the physical interpretation of these results in our final section.

4.4 Discussion

What is the physical interpretation of 4.38, 4.39 and 4.44 ? For $d = 1$, a class in $K^{-1}(S^1) \approx \pi_1(\mathcal{F}^{sa}(\mathscr{H})$ is determined by the *spectral flow* [33]. This invariant has been used to study the so called *chiral anomaly* in quantum field theory [32] but to our knowledge, had not been used previously to study Fermi surfaces. Quantum anomalies are, in short, the breaking of symmetries of the classical system by its mere quantization, never the less, they are independent of the choice of quantization procedure and play a fundamental role in the standard model of particle physics. For an accessible introduction to anomalies, we refer to [22]. We will defer to discuss further interpretation of the spectral flow for higher dimensions but will comment that it is very likely to be related to a *chiral current*, which is an effect that in principle, could have already been detected in experiments.
Our results differ from those obtained in [30] and those presented in [12]. We believe that this is because they restrict themselves to lower dimensional sphere surrounding a single path-component of the Fermi surface. The reasons why they employ such a restriction is due to so called *nesting instabilities*. This instabilities are related to the Fermion doubling theorem [10]. However, this types of argument do not seem to apply precisely when there is chiral symmetry-breaking, which is exactly the result we obtained with the spectral flow invariant! Also, such arguments apply to lattice models where the field equations are discretized and their is a problem when taking the continuum limit. This lattice argument applies to tight-binding models where the number of energy bands is finite but in our case we have no discretization and the number of energy bands is infinite as expected of the full single particle Hilbert space.Our work is thus in the same rationale as [17], where we allow for all possible topologies for the Fermi surface and where chirality plays an important role. This is also what [21] seems to have had in mind as a generalization of his work. At the level of the derivation of the classification, in which we obtained $K^{-1}(\mathrm{X})$ as the group, we could generalize this approach in the spirit of [25], [15] and [35] by including *symmetries*, such as *time reversal* or *reflection symmetry*, which are both operators acting on $\mathscr{H}$. As it is very well explained in [15], care must be taken for there is an ambiguity in the choice of representation for these symmetries, and to take it into account it is necessary to modify the isomorphism classes of our vector bundles, yielding a different kind of K-theory for each symmetry and each choice of its representation! Thus, for the case of time reversal symmetry where the representation θ is *anti-unitary*, we would expect to modify our isomorphism classes of vector bundles to isomorphism classes of *Real*

vector bundles, in the sense of [2], were as for reflection symmetry, where $\mathcal{R}$ is unitary, we would expect what is known as $\mathbb{Z}_2$-vector bundles (also see [2]. The corresponding K-theory is known as KR- theory or Real K-theory and for $\mathbb{Z}_2$-vector bundles is called $K_{\mathbb{Z}_2}$- theory or $\mathbb{Z}_2$-equivariant K-theory. It is of great importance to developed a framework to encompass all possible symmetries and their representations as the one employed in [15] for gapped systems, however, our classification does not readily admit such a generalization, for the topology employed in [15] is the *Compact-Open topology* and Fredholm operators are trivial in this topology! Non the less, there is no immediate contradiction between our classification and that of [15], as their framework was developed to study gapped systems, where there is no Fermi surface. This generalization is of great interest but will be deferred to future work.

Chapter 5

Conclusions

In this thesis, we made a great effort to introduce a vast array of physical and mathematical concepts from the ground up, in order to later employ in an integral manner for the development of a classification of topologically stable Fermi surfaces.

We first introduced our readers to the subject with a brief recount of the history of topological phases of matter, starting with the discovery of Von Klitzing and his collaborators of the exact quantization in the integer quantum Hall effect in the early 80's, up to its interpretation as the Chern number of a line bundle over the Brillouin zone due to the TKNN group, the $\mathbb{Z}_2$ topological invariant of Kane and Mele in the spin quantum Hall effect and the explosion of research that developed there after. We later restricted to the literature of topological properties of Fermi surfaces, where the earliest work is due to Hořava and generalizations developed there after, together with some of the criticisms on the general use of K-theory in condensed matter systems due to Thiang.

We plunged in deeply into the mathematical aspects, starting at the root of the matter with the introduction of analysis and the different kinds of operators such as bounded, compact and Fredholm operators, then proceeding to topological spaces, moving through what is denoted as point-set topology (continuity, compactness, connected, quotient and Hausdorff spaces, together with topological groups), then to algebraic topology (homotopy, loop-spaces, suspensions, homotopy groups, cohomology) and all the way up to vector bundles,isomorphism classes of vector bundles Grothendieck's completion, K-theory and Bott periodicity, as well as the powerful connection it has with Fredholm operators and its self-adjoint subset, discovered and developed by Atiyah and Singer.

Thereafter, we introduced the basic concepts of condensed matter systems for N electrons in a box, arriving at the Fermi sphere, and then studying systems with discrete translational invariance due to a periodic array of ions (or possibly something else) called lattices, then the reciprocal lattice in crystal momentum space and the first Brillouin zone. We later introduced Bloch's theorem and used it to describe the notions of band theory of electrons, and

Fermi surfaces, which are the generalization of the Fermi sphere for discrete translational invariant systems with no electron-electron interactions. We also discussed adiabatic evolution and its interpretation for gapless systems. We then proceeded to make the connection between the physical subjects we introduced and the mathematics, by showing that the Hamiltonian of a discrete translational invariant system with no electron-electron interactions (also called Bloch Hamiltonian) and taking out of it the Fermi energy yields a self-adjoint Fredholm operator, whose kernel has a precise physical interpretation concerning the Fermi surface of our system. Using the fact that the Bloch Hamiltonian is parametrized by the Brillouin zone of the system and the connection shown by Atiyah and Singer between self-adjoint Fredholm operators, Loop spaces and K-theory, we arrive at our classification where it is the elements of $K^{-1}(\mathrm{X})$, which constitute the distinct kinds of topologically stable Fermi surfaces, such distinct classes cannot be connected by adiabatic evolution or small perturbations, in the appropriate sense.
Once we had our classification for the case with no symmetries, we computed some K-groups of systems with dimension $d = 1, 2$ and 3, and we presented the results in **??**. After surveying the literature, we discussed the $d = 1$ case, were the topological invariant which determines the class of a Fermi surface is known as the spectral flow. The spectral flow had been used previously by Semenoff and Niemi and also by Haldane to describe what is known as chiral anomaly. Anomalies are of fundamental importance in quantum field theory and our work shows connections between anomalies and Fermi surfaces.
We also discussed briefly why our results differ from those presented by Ching-Kai and other sources, were the arguments presented there seem to be more restricted in the topological nature allowed for their surfaces and their arguments do not seem to apply as it was precisely the connection with chiral symmetry breaking which renders them inadequate on physical grounds.
We further discussed how to incorporate symmetries into our classification, we believe they should yield different kinds of vector bundle isomorphisms, such as those of Real vector bundles or $\mathbb{Z}_2$- vector bundles, yielding for every choice of representation of a symmetry a different kind of K-theory, in the spirit of the framework developed by Freed and Moore for topological phases of gapped systems. It is however, not possible to directly extend our framework to the one employed there, and we have pointed out were the troubles lie.
Thus, we have developed a classification of these strange phases of matter, employing the sophisticated mathematical framework of K-theory, arriving at unexpected connections with the profound concepts of quantum anoma-

lies. We have also pointed out what future work would have to surmount in order to have a classification which is fully consistent with the different symmetries and their representations. We should keep in mind however that we have specified our limitations and assumptions in our derivation, such as ignoring electron-electron interactions which may be relevant for many systems, as well as working in the zero temperature approximation. None the less, for many systems these approximations are shown by experiments to be good.

We hope to develop elsewhere a classification consistent with the one developed by Freed and Moore, and to have a better connection with experimental efforts, a fact which every physicist must always keep in mind, our classification yields results of the physical kind that have already been observed in experiments and so, such a connections seem a reasonable expectation for the near future.

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