Reparameterization of Special Contours for Boundary Value Problems in Queueing Systems

by

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Abstract

We present several tests to detect whether a simple reparameterization of the contour is possible for the Riemann-Hilbert Boundary Value Problem associated with the stationary distribution of a two dimensional queueing system. We cover several special cases, some of which have been previously covered using different methods, and present an expansion on the existing theory to the case of elliptical contours. Two fast heuristic tests are also presented. Additionally, we construct a formal BVP solution for queueing systems with homogeneous fundamental equations.

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Chapter 1

Introduction

1.1 Overview

In this thesis we develop several original tests for finding special cases for the contours employed in the Boundary Value Problem (BVP) approach to Queueing Systems, and extract a formal BVP solution for the stationary probability distribution on the state space of a queueing system that presents the case of a homogeneous equation.

Some of the special contours in this text have appeared in queueing BVPs before, including Fayolle and Iasnogorodski's 1979 paper [3], Cohen and Boxma's 1983 book [2], and Avrachenkov, Nain, and Yechali's 2014 paper [1]. Our goal in creating tests for other elegantly parametrizable contours is to allow further work to be done in this line, create easy tests to see what can be connected to the methods applied in similar problems, and hopefully to allow for analytic solutions of certain BVPs rather than the formal ones which are most commonly seen. We will address these new results for special contours in chapter 3. Chapter 4 details a formal solution for a broad class of queueing BVPs, those with homogeneous fundamental equations. The rest of chapter 1 provides a timeline for historically important texts in this field and a few recent queueing papers. Chapter 2 provides the mathematical background for the work contained in this thesis, and the BVP methods used in our approach.

1.2 Previous Work in the Field

The genesis of the BVP approach to queueing systems can be found in the 1979 paper authored by G. Fayolle and R. Iasnogorodski [3], who pioneered the method, and dealt with one case. Of particular interest here is that this paper involved a circular contour.

A text was published in 1988 by J. Cohen and O. Boxma [2], which dealt with several cases of queueing systems and explained the required background information for the Riemann-Hilbert BVPs in detail. Several cases were also treated in detail here, including the very important 'Join Shortest Queue' model, which covers one of the most common arrival disciplines.

In 1999, G. Fayolle, R. Iasnogorodski, and V. Malyshev published in their own book [4] a broad class of random walks and the associated queueing systems were dealt with via BVP methods, by transforming the initial problem into a Riemann-Hilbert-Carelman BVP with a shift, the 'shift' referring to an automorphism of the independent variable appearing in some of the terms of the boundary condition. These authors also developed another approach, using covering spaces for the algebraic curve defined by the kernel equation. Said kernel equation (in this thesis, equation (2.43)) is determined by the fundamental equation for the random walk, so this algebraic approach starts in the same way as the BVP approach. However, this algebraic approach has a disadvantage compared to BVP methods, because it only works for certain fundamental equations, depending on the properties of what they declare to be the group of the random walk, which is defined using Galois automorphisms related to the algebraic curve associated with the kernel equation.

More recent papers in BVP queueing include [12], by J. Resing and L. Örmeci, published in 2003, [8], by F. Guillemin, C. Knessl, and S. van Leeuwaarden, published in 2013, and [1], by K. Avrachenkov, P. Nain, and U. Yechiali, published in 2014. Avrachenkov et al. study a system where the state of the server is unknown to the queues and vice versa, so an empty server could take a job from either one of the orbiting queues, which retry jobs that were sent to wait, or from a new incoming job. Guillemin et al. model a chain of wireless transceivers whose transmissions can interfere with one another. The paper [8] posits a slightly different approach to queueing BVPs, but the paper [1], while having a very interesting setup for the fundamental equation, follows the BVP setup found in [3] quite closely.

Chapter 2

Background

2.1 Notation

In the interest of being unambiguous we will define our notations for several common operations:

- For vectors v and $w \in \mathbb{C}^n$ we define the exponentiation of vectors:

$$v^w = \prod_{i=1}^{n} v_i^{w_i}.$$

- For a boolean statement V, we denote the indicator function $\mathbb{1}$ as

$$\mathbb{1}_V = \begin{cases} 1 & \text{if } V \\ 0 & \text{if not } V \end{cases},$$

- For sets A and B, where A is a subset of an additive group G, and for an element $b \in G$, we denote cosets and set difference operation:

$$A + b = \{a + b : a \in A\},$$

$$A - b = \{a - b : a \in A\},$$

$$A \backslash B = \{e : e \in A, e \notin B\}.$$

- We define a neighborhood of a point z in $\mathbb{C}$ as a bounded simply connected open subset of $\mathbb{C}$ containing that point.

- We also denote for a set $A \subseteq \mathbb{C}$ the closure of A as $\overline{A}$, the interior of

A as int A, and the boundary of A as bdy A. Specifically:

$$\overline{A} = A \cup \{l : \exists (z_n)_{n=1}^{\infty} \subseteq A, \lim_{n\to\infty} z_n = l\}.$$
$$\text{int } A = \{z \in A : z \text{ is in a neighborhood that is a subset of } A\}.$$
$$\text{bdy } A = \overline{A} \setminus \text{int } A.$$

– For a closed simple contour $B \subset \mathbb{C}$, we will denote the interior of the contour under a positive (counter-clockwise) traversal as G_B.

– For a complex number $z = a + ib, a, b \in \mathbb{R}$, we denote the complex conjugate $\overline{z} = a - ib$.

2.2 Queueing Systems

2.2.1 Two Dimensional Queuing Systems

The application at hand is a queueing system wherein jobs arrive into various queues where they wait for service by a processor. A processor may send a job into a different queue, changing the job type, or remove it from the system. Each processor may only handle one type of job. The processors don't have all the information about the system; they can only tell if there is a job waiting in queue or not.

We assume that the arrival rate of jobs and the service rate of the processor are independent of the number of jobs waiting in the system, of one another, and of the current time. Specifically, we can treat the lengths of the queues as the state space of a time-independent Markov chain, with a discrete time variable, so that if we write down the length of queue i at time n as the random variable X_n^i, the following equations in the expected values of these variables represent the stated independence, given that $n > m + t$, and that $m, t \geq 0$, and that v, v_i are some nonnegative integers:

$$E\left[X_n^i | X_{m+t}^i = v_{m+t}\right] = E\left[X_n^i | X_m^i = v_m, \ldots, X_{m+t}^i = v_{m+t}\right], \qquad (2.1)$$

$$E\left[X_n^i | X_{m+t}^i = v\right] = E\left[X_{n+s}^i | X_{m+t+s}^i = v\right]. \qquad (2.2)$$

where $E[A|B]$ is the expected value of a function A given that B.

Equation (2.1) states that the information of how one got to the current state is unimportant for predicting the future behavior of the system, and equation (2.2) states that the amount of time that the system took to arrive at its current state is similarly unimportant. These conditions are necessary for the approaches to analyzing this system that appear in the next three subsections, as they rely heavily on the theory of Markov chains.

Since a negative queue length is not meaningful in most applications, we restrict X_n^i to be non-negative. The focus of our approach will be on systems that either start as or reduce to two coupled queues, and so in all the work that follows we will have $\mathbb{Z}_+ \times \mathbb{Z}_+$, all pairs of nonnegative integers, as our state space, and denote the Markov chain for the state of the random walk at time n as the vector of random variables $X_n = \left\langle X_n^1, X_n^2 \right\rangle$.

2.2.2 Jump Probability Representation for a Random Walk

Up to this point, our time variable could have taken any real value, but we want to work with a discrete time Markov chain, so we note that even in a continuous time Markov chain, two events happening at exactly the same time is analogous to randomly picking two real numbers and having them come out identical, a probabilistic impossibility which can be safely neglected. We subdivide the real line that is the domain of our time variable into intervals of equal size small enough that it is safe to say only one event will happen in each. In this way we will claim the random walk has bounded jumps, so the position in the state space for queue length will only ever increase or decrease by one at a time. This formulation of the problem allows us to work with a discrete space, discrete time, time independent Markov chain that tracks both queues, both of which are bounded to have non-negative queue lengths, which corresponds to a random walk in the quarter plane.

All the processors in our queueing system can only tell if their queue is empty or nonempty, which implies a condition referred to as spatial homogeneity. The state space can be divided up into regions where the jump probabilities at any given point in a given region is the same as any other point in the same region. These regions for the random walk in the quarter plane are depicted in figure 2.1, and are defined as follows:

$$S = \{1, 2, 3, \dots\}^2, \tag{2.3}$$
$$S' = \{1, 2, 3, \dots\} \times \{0\}, \tag{2.4}$$
$$S'' = \{0\} \times \{1, 2, 3, \dots\}, \tag{2.5}$$
$$S^0 = \{(0, 0)\}. \tag{2.6}$$

It follows that we can define p^r_{ij} as the probability of transitioning from state (a, b) to state $(a + i, b + j)$ first, for (a, b) in a given region r:

$$p^r_{i,j} = P\left(X^1_{t+1} - X^1_t = i \wedge X^2_{t+1} - X^2_t = j | (X^1_t, X^2_t) \in r\right). \tag{2.7}$$

Rather than write $p^S_{i,j}$ or $p^{S''}_{i,j}$, we will sometimes write $p_{i,j}$ or $p''_{i,j}$, and so on.

Spatial homogeneity is equivalent to saying that we could instead condition equation (2.7) on (X^1_t, X^2_t) being any specific element of a given region r, and it wouldn't change the result, which means that knowing the probability of being in any specific state is not needed to calculate $p^r_{i,j}$.

It is common in the literature to create a diagram that represents the transition probabilities from one state to another for the associated Markov chain for the queueing system. Since we have spatial homogeneity here, it is common to represent these transition probabilities with diagrams similar to figure 2.2, omitting those arrows that are associated with a jump probability of zero. Probabilities of zero commonly occur in this application, as queueing theorists tend to study special cases.

2.2.3 Stationary Distribution

Now that the rudiments are available, we can state the problem around which this thesis is focused. We want to find the long-run or stationary distribution over the state space, that is, the spatial distribution of the probability of the system being in a specific state after the queue has run

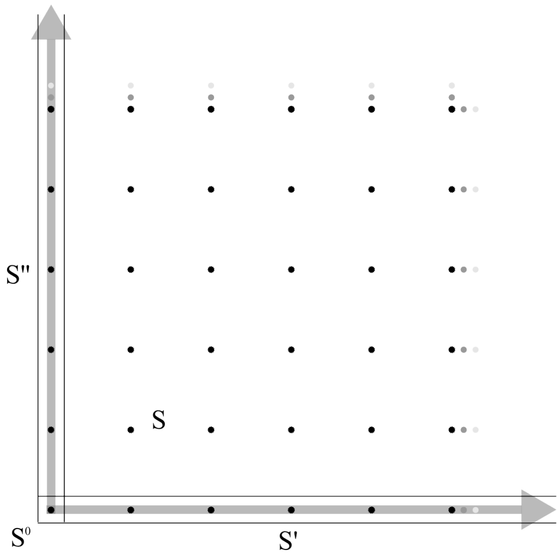

Figure 2.1: The regions S^r of the state space.

for an arbitrarily long period of time. We define the stationary probability for a point (i, j) in the state space as

$$\pi_{ij} = \lim_{n\to\infty} P(X_n = (i, j)). \tag{2.8}$$

The stationary distribution encoded in equation (2.8) is the most important definition in this section. It does not depend on the time variable n, and satisfies the property that for an arbitrary time t_0, if $P(X_{t_0} = (i, j)) = \pi_{ij}, \forall (i, j) \in \mathbb{Z}_+^2$ then $P(X_{t_0+1} = (i, j)) = \pi_{ij}, \forall (i, j) \in \mathbb{Z}_+^2$.

From the theory of discrete space discrete time Markov chains [13] it is known that for stable stationary distribution to exist, a Markov chain must have all of the following properties:

- Irreducible: There is a nonzero probability of eventually getting from any one state to any other state.

  Irreducibility dictates that the Markov chain cannot get 'stuck' in a proper subset of the state space.

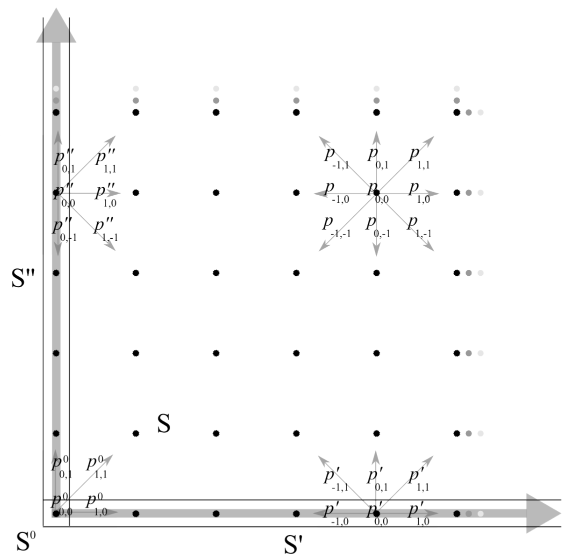

Figure 2.2: A common representation scheme for the jump probabilities for a random walk in the quarter plane.

- Aperiodic: The greatest common divisor of the set of the number of steps taken to go from any one specific state to itself with nonzero probability is one.

  Aperiodicity dictates that the state space cannot be divided into proper subsets such that the Markov chain always arrives in a specific subset given time arguments in arithmetic progression.

- Ergodic: The mean number of steps taken to return to any given state from itself is finite. [1]

We will primarily assume irreducibilty and aperiodicity for our purposes, although in the application of two-dimensional random walks, these

[1] In [4], Fayolle et al. characterize ergodicity with stochastic matrices: If P is the stochastic transition matrix for an irreducible aperiodic Markov chain, then the said Markov chain is ergodic if and only if there exists a row vector π with non-negative entries that sum to 1 such that $\pi \mathbf{P} = \pi$. This is in fact how stationary distributions are characterized later in this chapter. We feel that this alternate definition offers no positive characterization of what ergodicity actually means.

are easily characterized. For example, for there to be no chance of getting from any specific state to another, the queueing system would have to have at least one nonfunctioning queue, (a queue with no arrivals or no service,) or both queues would behave like one queue, with linked arrivals and service. An example case of a periodic Markov chain would be when $p^0_{1,0} = 1, p'_{-1,1} = 1, p''_{0,-1} = 1$, resulting in an endless cycle, where jobs are instantly served after arrival. In short, if our type of analysis is required, then the Markov chain is almost certainly irreducible and aperiodic.

Ergodicity is dealt with in [4, page 3] by looking at the mean jump sizes in the x and y directions: if we write

$$\begin{aligned} M_x &= \sum ip_{ij}, \quad M_y = \sum jp_{ij}, \\ M'_x &= \sum ip'_{ij}, \quad M'_y = \sum jp'_{ij}, \\ M''_x &= \sum ip''_{ij}, \quad M''_y = \sum jp''_{ij}, \end{aligned} \tag{2.9}$$

then, given that M_x and M_y are both not identically zero, the random walk is ergodic if and only if one of the following three sets of conditions holds:

Condition 1:

$$\begin{aligned} M_x < 0, M_y &< 0, \\ M_x M'_y - M_y M'_x &< 0, \\ M''_x M_y - M''_y M_x &< 0. \end{aligned} \tag{2.10}$$

Condition 2:

$$\begin{aligned} M_x < 0, M_y &\geq 0, \\ M''_x M_y - M''_y M_x &< 0. \end{aligned} \tag{2.11}$$

Condition 3:

$$\begin{aligned} M_x \geq 0, M_y < 0, \\ M_x M_y' - M_y M_x' < 0. \end{aligned} \tag{2.12}$$

These conditions essentially embody the idea that if a random walk tends to move away from the origin, it will grow past any bound, and will likely never return to any given finite state.

One needs to verify that the queueing system that one wants to model is representable with a irreducible, aperiodic, ergodic Markov chain before using the work that appears in the rest of the thesis. These conditions imply the existence of a unique stationary probability distribution for the Markov chain, and the existence of this distribution underpins most of the work that follows in this thesis.

Calculation of the stationary distribution is desirable as it allows the calculation of valuable statistics for the queueing system, and on its own gives how one should expect the queueing system to behave in the case where you have no information about the system's exact behavior at or before the current time, relying instead and solely on the jump probabilities to inform your expectations.

2.3 Generating Function Approach for Queueing Systems

2.3.1 Generating Functions

A generating function is a means of encoding a data set into a function, specifically, by making its Taylor coefficients store the data set. If we have f in the polynomial ring $\mathbb{R}(x,y)$ with $x, y \in \mathbb{C}$, and $f(x,y) = \sum_{i,j=0,0}^{\infty} a_{i,j} x^i y^j$, then f's derivatives contain the information to recover $a_{i,j}$, as

$$\frac{\partial^{m+n} f}{\partial^m x \partial^n y}(0,0) = m!n!a_{m,n}. \tag{2.13}$$

Thus, by solving for the function $f(x, y)$, either analytically or by numerically computing it and its derivatives, one can recover the data set the function encodes.

We can use our knowledge of how the stationary probability distribution must behave in different regions of the state space to define a functional equation in the generating functions for these distributions. The study of this functional equation will govern the rest of this thesis. Our approach here comes from the introduction of [4].

We denote the complex variables that our generating functions will take as arguments as $x, y \in \mathbb{C}$, defining for brevity $u = \langle x, y \rangle$, and $R = \{S, S', S'', S^0\}$.

First, we define the jump probability generating function over region r as

$$P_r(u) = E\left[u^{(X_{n+1}-X_n)}\mathbb{1}_{X_n \in r}\right]. \tag{2.14}$$

We may read equation (2.14) with the information that we have about our application, giving

$$P_r(u) = E\left[x^{\overbrace{(X^1_{n+1} - X^1_n)}^{\text{Change in } X^1 \text{ at } n}} y^{\overbrace{(X^2_{n+1} - X^2_n)}^{\text{Change in } X^2 \text{ at } n}} \middle| X_n \in r \right]$$

which, by the definition of expected value, and the definition in equation (2.7), implies that

$$P_r(u) = \sum_{i,j} p^r_{ij} x^i y^j. \tag{2.15}$$

The generating function for the stationary probability distribution in the region r is defined as follows:

$$\pi_r(u) = \lim_{n \to \infty} E\left[u^{X_n}\mathbb{1}_{\{X_n \in r\}}\right] = \sum_{(i,j) \in r} \pi_{ij} x^i y^j \tag{2.16}$$

We now start fresh and begin to construct our desired result, a relationship tying all of the relevant generating functions for the stationary distributions together through generating functions for the jump probabilities. Note that since $\{X_n\}_{n=1}^{\infty}$ is a Markov chain, the variables X_{n+1} and $X_{n+1} - X_n$ are independent, allowing us to factor $E\left[u^{X_{n+1}}\right]$ as follows:

$$E\left[u^{X_{n+1}}\right] = E\left[u^{X_{n+1}-X_n}u^{X_n}\right] = E\left[u^{X_{n+1}-X_n}\right] E\left[u^{X_n}\right], \tag{2.17}$$

Since the expected value is just a summation over the state space, and (X_n) is a Markov chain with each step forward only dependent on the current state, we can split up equation (2.17) over the different regions and into independent parts:

$$E\left[u^{X_{n+1}}\right] = \sum_{r\in R} E\left[u^{(X_{n+1}-X_n)}\mathbb{1}_{\{X_n\in r\}}\right] E\left[u^{X_n}\mathbb{1}_{\{X_n\in r\}}\right], \tag{2.18}$$

and by equation (2.14), we can simply identify $P_r(u)$, giving

$$E\left[u^{X_{n+1}}\right] = \sum_{r\in R} E\left[u^{X_n}\mathbb{1}_{\{X_n\in r\}}\right] P_r(u) \tag{2.19}$$

Now, all we need to do is note that

$$E\left[u^{X_{n+1}}\right] = \sum_{r\in R} E\left[u^{X_{n+1}}\mathbb{1}_{\{X_{n+1}\in r\}}\right], \tag{2.20}$$

and take the limit as $n \to \infty$ of both sides of equation (2.18) to get our desired relationship in the generating functions, applying equations (2.20) and (2.16) as needed:

$$\begin{aligned}\lim_{n\to\infty}\left[\sum_{r\in R} E\left[u^{X_{n+1}}\mathbb{1}_{\{X_{n+1}\in r\}}\right]\right] &= \lim_{n\to\infty}\left[\sum_{r\in R} E\left[u^{X_n}\mathbb{1}_{\{X_{n+1}\in r\}}\right]\right] P_r(u)\\ &= \sum_{r\in R}\pi^r(u) = \sum_{r\in R}\pi^r(u)P_r(u),\end{aligned} \tag{2.21}$$

which simplifies to

$$\sum_{r\in R} [1 - P_r(u)]\, \pi^r(u) = 0. \tag{2.22}$$

This equation (2.22) characterizes how the stationary distribution behaves for random walks in the quarter plane explicitly, and independently of the division the quarter plane into regions with distinct jump probabilities. However, equation (2.22) is rather abstract, and we will want to use the information that we have for our queueing system to make it more concrete, using the regions and jump probabilities as defined above. After specifying these, equation (2.22) is almost always factored as in the following equation (2.23), referred to as the fundamental equation (F.E.) for the random walk.

$$Q(x,y)\pi(x,y) = q_1(x,y)\pi_1(x) + q_2(x,y)\pi_2(y) + q_0(x,y)\pi_0. \tag{2.23}$$

We define the all of the functions present in equation 2.23 (with references to equation (2.22)) as follows:

$$\begin{aligned}
Q(x,y) &= \; xy\left(-1 + \sum_{\substack{1\geq i\geq -1\\ 1\geq j\geq -1}} p_{ij}x^iy^j\right) = -(1-P_S(u)), && (2.24)\\
q_1(x,y) &= \; x\left(1 - \sum_{\substack{1\geq i\geq -1\\ 1\geq j\geq 0}} p'_{ij}x^iy^j\right) = 1 - P_{S'}(u), && (2.25)\\
q_2(x,y) &= \; y\left(1 - \sum_{\substack{1\geq i\geq 0\\ 1\geq j\geq -1}} p''_{ij}x^iy^j\right) = 1 - P_{S''}(u), && (2.26)\\
q_0(x,y) &= \; 1 - \sum_{\substack{1\geq i\geq 0\\ 1\geq j\geq 0}} p^0_{ij}x^iy^j = 1 - P_{S^0}(u), && (2.27)\\
\pi(x,y) &= \; \sum_{i,j=1}^{\infty} \pi_{ij}x^{i-1}y^{j-1} = \pi_S(u), && (2.28)
\end{aligned}$$

$$\pi_1(x) = \sum_{i=1}^{\infty} \pi_{i0} x^{i-1} = \pi_{S'}(u), \tag{2.29}$$

$$\pi_2(y) = \sum_{j=1}^{\infty} \pi_{0j} y^{j-1} = \pi_{S''}(u), \tag{2.30}$$

$$\pi_0 = \pi_{0,0} = \pi_{S^0}(u). \tag{2.31}$$

This fundamental equation is so named because solving for all of the unknown functions $(\pi(x, y), \pi_1(x), \pi_2(y), \pi_0,)$ in it allows recovery of the stationary distribution encoded in the generating functions. Chapter 4 is primarily focused on a formal integral solution to this equation, which is useful in numerical work.

In order to solve for the unknown functions in this equation, we will be using the theory of Riemann Hilbert boundary value problems, which require the following:

2.3.2 Indexes of Functions

If we have a complex function G defined over a simple closed piecewise smooth positively oriented contour L, the index χ of the function over the contour is a quantity we will define as

$$\chi = \frac{1}{2\pi} \arg_L G = \frac{1}{2\pi i} \int_L d(\mathrm{Log}(G(t)) = \frac{1}{2\pi i} \int_L \frac{G'(t)}{G(t)} dt, \tag{2.32}$$

assuming without loss of generality that 0 is in the interior of L. [4]

Interpreting the second formula above, this index counts the total number of times the function G goes over the branch cut of the logarithm under arguments coming from a positively oriented traversal of L, with a negative index corresponding to a net clockwise traversal in the image, and a positive index corresponding to a net counterclockwise traversal in the image.

Cauchy's argument principle [11] also grants another interpretation of the index, as the third formula in equation (2.32) gives the difference between the number of roots and poles of G inside L, with roots increasing the index and poles reducing it, with the multiplicity of both contributors being accounted for.

2.3.3 Riemann Hilbert BVPs

The Riemann Hilbert boundary value problem was used by Cohen and Boxma to analyze queueing systems in their book [2]. This particular BVP was also employed in the papers [1, 3, 8] to find formal solutions. The formulation given here was provided in [2].

The Riemann Hilbert boundary value problem is as follows: We wish to find a function Φ defined over the complex plane holomorphic on the interior and exterior of a given closed simple contour L excepting perhaps a pole of finite order at infinity. Additionally, Φ has limiting values coming from the inside and outside of L, labeled Φ^+ and Φ^- respectively.

Defining the notation G_A as the interior of a given closed simple contour A without including the contour itself, we can define these explicitly as

$$\Phi^+ : L \to \mathbb{C}, \quad \Phi^+(x) = \lim_{\substack{t \to x \\ t \in G_L}} \Phi(t), \tag{2.33}$$

$$\Phi^- : L \to \mathbb{C}, \quad \Phi^-(x) = \lim_{\substack{t \to x \\ t \notin G_L \cup L}} \Phi(t). \tag{2.34}$$

The values of Φ are unknown at the start, but if we have a relation between the limiting values Φ^+ and Φ^- given by

$$G(x)\Phi^-(x) + g(x) = \Phi^+(x), x \in L, \tag{2.35}$$

the Riemann Hilbert boundary value problem is solvable given the following conditions:

1. G is continuous on L,
2. G is nonzero on L,
3. g satisfies the Hölder condition $|g(t_1) - g(t_2)| \leq C|t_1 - t_2|^\mu$ on L, for real constants C and $0 \leq \mu < 1$.

If we define

$$\chi = \text{ the index of } G \text{ over } L, \tag{2.36}$$
$$\Gamma(x) = \frac{1}{2\pi i}\int_L \frac{\mathrm{Log}(t^{-\chi}G(t))dt}{t-x},$$

$$S^+(x) = e^{\Gamma(x)}, \qquad x \in G_L,$$
$$S^-(x) = x^{-\chi}e^{\Gamma(x)}, \quad x \notin G_L \cup L, \tag{2.37}$$

the solution to the homogeneous case of equation 2.35, where in the equation $g \equiv 0$, is given by

$$\Phi(x) = \begin{cases} S^+(x) & \text{if } x \in G_L, \\ S^-(x) & \text{if } x \notin G_L \cup L. \end{cases} \tag{2.38}$$

For the nonhomogeneous case, we have two types of solutions, both using the following definition:

$$\gamma(x) = \frac{1}{2\pi i}\int_L \frac{g(s)ds}{S^+(s)(s-x)}. \tag{2.39}$$

Which type of solution we use depends on the index:

- When $\chi \geq 0$:

$$\Phi(x) = \begin{cases} S^+(x)(\gamma(x) + P_\chi(x)) & \text{if } x \in G_L, \\ S^-(x)(\gamma(x) + P_\chi(x)) & \text{if } x \notin G_L \cup L. \end{cases} \tag{2.40}$$

  where P_χ is an arbitrary polynomial of degree χ over $\mathbb{C}$.

- When $\chi < 0$:

$$\Phi(x) = \begin{cases} S^+(x)\gamma(x) & \text{if } x \in G_L, \\ S^-(x)\gamma(x) & \text{if } x \notin G_L \cup L. \end{cases} \tag{2.41}$$

  but note that when $\chi \leq -2$, there are $-\chi - 1$ additional conditions that must be satisfied for the BVP to have a solution, given by

$$\int_L \frac{g(t)t^{k-1}dt}{S^+(t)} = 0, \text{ for } k \in \{1, 2, \ldots, -\chi - 1\}. \tag{2.42}$$

The Riemann Hilbert BVP turns out to be just the right tool for the last step in solving for one of the unknown functions in fundamental equation (2.23) in our application. We will use Φ to recover π_1 and π_2 individually and independently, by finding a G which satisfies the properties required by the BVP setup for each case, the construction of which will leave $g = 0$. For more details on the construction of the solution to this boundary value problem, see [2] or [4].

2.4 The Beginning of the BVP Approach

2.4.1 The Kernel Method and Contours

Now equipped with information on the tool we will later use, we can begin work on the problem proper. The typical starting point that a great number of papers (for example, [1, 3, 5, 8]) use to tackle related problems is here referred to as the kernel method, wherein we restrict the values of x and y to the algebraic curve

$$Q(x, y) = 0. \tag{2.43}$$

This restriction does two jobs: First, the complexity of the fundamental equation (2.23) is reduced by eliminating one unknown function, leaving us with only two unknown functions, which is consistent with the BVP approach, as long as those functions satisfy the required properties. Second, it allows the construction of a contour for the BVP. If we factor Q as

$$Q(x, y) = a(x)y^2 + b(x)y + c(x), \tag{2.44}$$

we can apply the quadratic formula on the coefficients $a(x), b(x)$, and $c(x)$. We label the resulting two-branched function as

$$Y(x) = \frac{-b(x) \pm \sqrt{b^2(x) - 4a(x)c(x)}}{2a(x)}, \tag{2.45}$$

and label the individual branches Y^+ and Y^- corresponding to the choice of sign for the $\pm$ sign in the right hand side of equation (2.45) as

$$Y^+(x) = \frac{-b(x) + \sqrt{b^2(x) - 4a(x)c(x)}}{2a(x)}, \tag{2.46}$$

$$Y^-(x) = \frac{-b(x) - \sqrt{b^2(x) - 4a(x)c(x)}}{2a(x)}. \tag{2.47}$$

Often newcomers to this discipline wonder whether restricting the variables x and y by the relationship $Q(x, y) = 0$ implies that we have a restricted solution, and that perhaps what we have generated by the BVP is only valid on this algebraic curve. The proper way of thinking of this restriction is to recognize that the BVP only requires its boundary condition (equation (2.35)) to be met on the boundary, and that setting $Q(x, y) = 0$ gives us this boundary. Using $Q(x, y) = 0$ to meet the condition on the boundary is fine, since the BVP solution allows us to use the information on the boundary to move past the boundary by design. As long as the conditions are met the BVP gives us an unrestricted analytic solution.

In order to get a contour in the complex plane we investigate the branch points of the function Y, that is to say, the points where both branches are equal, or, equivalently where the discriminant $D(x) = b^2(x) - 4a(x)c(x) = 0$. The degree of $D(x)$ is either three or four [2] , and so there are at most four of these branch points, call them $x_i, i = 1..4$. If the degree of $D(x)$ is only three, we write $x_4 = \infty$. We know from the literature [4] that these x_i

[2]The degree of D is derived directly from the degrees of $a(x), b(x)$, and $c(x)$, which in turn are defined by the boundedness of the jumps in the queueing system's state space to one away in any given direction, which in turn is derived from our queueing system only having one job entering the system, switching queue, or leaving the system at any given time.

have to be real, with the exception of x_4 which is possibly ∞, and that they admit the ordering $0 \leq x_1 < x_2 \leq 1 \leq x_3 < x_4$, with $x_2 \neq x_3$. Restricted to the real line, $D(x)$ changes sign at each of its branch points, being negative between x_1 and x_2, and also between x_3 and x_4.

From here, there are two equivalent ways of proceeding. For a definition that allows easy numerical computation and parameterization we can define the contour L as the image of the line segment x_1x_2 under $Y^+(x)$, joined with the image of the line segment x_2x_1 under $Y^-(x)$:

$$L = \{Y^+(x) : x \in [x_1, x_2]\} \cup \{Y^-(x) : x \in [x_1, x_2]\} \tag{2.48}$$

Alternatively, we can define L as the limiting contour of the image under a branch of Y of a family of contours in the x-plane cut between x_1 and x_2, i.e. $\mathbb{C}\backslash x_1x_2$, that goes around the cut. This gives the same result as the above procedure, with the advantage that we can treat that branch of Y as analytic in that cut plane, and state L in concrete terms with respect to that branch of Y. This is mainly of use in proofs and analysis rather than numerical work. Looking ahead, and using a specific branch of Y given in the future equation (4.1.1), we can give the alternate characterization:

$$L = \lim_{\epsilon \downarrow 0} \{Y_0(x) : x \in \{[x_1, x_2] + \epsilon\} \cup \{[x_1, x_2] - \epsilon\}\} \tag{2.49}$$

Additionally, we define L^{ext} identically, except using x_3 and x_4 instead of x_1 and x_2.

While we remain on L, we know that $Q(x, y)$ is 0, allowing us to use the modified F.E. as a relation on that boundary. From this point, BVP approaches diverge in how they attack the problem, but most follow this contour and fundamental equation setup very closely.

2.4.2 Counterparts

We note that there is more than one way to factor $Q(x, y)$: the alternate factorization $Q(x, y) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y)$ allows us to proceed in exactly the same way as before, and define a new contour, a new discriminant, and so on. Rather than double the length of this thesis, we will introduce a table of counterparts (table 2.4.2) giving names for the constructions implied by the work we do following our admittedly arbitrary choice of factorization.

Table 2.1: Legend and Table of Counterparts

$a(x)$ path	Counterpart	c.f.
$a(x)$	$\tilde{a}(y)$	(2.44)
$b(x)$	$\tilde{b}(y)$	(2.44)
$c(x)$	$\tilde{c}(y)$	(2.44)
$Y(x)$	$X(y)$	(2.45)
$Y^+(x)$	$X^+(y)$	(2.46)
$Y^-(x)$	$X^-(y)$	(2.47)
$D(x)$	$\tilde{D}(y)$	§2.4.1
x_i	y_i	§2.4.1
L	M	(2.45)
y_0	x_0	(3.4)
m_r	$\tilde{m}_r$	(3.11)
y_{max}	x_{max}	(3.11)
m_i	$\tilde{m}_i$	(3.11)
f	$\tilde{f}$	(3.11)
h	$\tilde{h}$	(3.11)
t, u	$\tilde{t}, \tilde{u}$	(3.16)
$Y_0(x)$	$X_0(y)$	§4.1.1
$Y_1(x)$	$X_1(y)$	§4.1.1

Object	c.f.
$\pi^r_{i,j}, \pi_{i,j}, \pi'_{0,j}$, etc.	(2.15)
$\pi(x,y)$	(2.31)
$Q(x,y)$	(2.31)
$\pi_1(x)$	(2.31)
$q_1(x,y)$	(2.31)
$\pi_2(y)$	(2.31)
$q_2(x,y)$	(2.31)
π_0	(2.31)
$q_0(x,y)$	(2.31)

From here, the path forward forks. Both chapters 3 and 4 use the setup that appears here, but they proceed independently of one another. A small discussion of the overlap (or the current triviality thereof) will appear in the conclusion, after the work has been discussed.

Chapter 3

Special Contours

In this section we will examine special cases of contours as used in the BVP analysis of queueing systems, wherein a simple reparameterization of the contour may be created. We analyze these special cases in the hopes that they may be leveraged to find explicit solutions for the BVP integral.

3.1 Circles

3.1.1 The Case for a Circle with a Center on the Origin

This case has shown up in at least two papers [1, 6], where the contour for the BVP is a circle. Also of interest is the characterization appearing in ([4], 111), wherein it is shown that the contour is a circle when

$$\begin{vmatrix} p_{1,-1} & p_{1,0} & p_{1,1} \\ p_{0,-1} & p_{0,0} & p_{0,1} \\ p_{-1,-1} & p_{-1,0} & p_{-1,1} \end{vmatrix} = 0. \tag{3.1}$$

In this chapter, we will focus on techniques that are extensible into other contours.

One characterization of a circle in the complex plane is $\{z \ : \ |z| = r\}$, where r is a positive constant. We will try to find queueing BVP contours that are circles by setting

$$|Y(x)| = r \tag{3.2}$$

for either branch.

[2]Parts of this chapter were presented as a part of the 2014 CORS Annual General Conference, under the talk title 'Special contours in queueing BVPs'.

Developing this, we recall that the discriminant has negative value over the line segment in question, so the branches of $Y(x)$ are complex conjugates of one another, seeing as $a(x), b(x)$, and $c(x)$ are polynomials with real coefficients. This result allows the following computation for either branch, (with the notation $Y^{\pm}$ being either branch of $Y(x)$, it doesn't matter which,)

$$\begin{aligned}
|Y^{\pm}(x)| &= \sqrt{Y^{+}(x)Y^{-}(x)} \\
&= \sqrt{\frac{-b(x)+\sqrt{b^2(x)-4a(x)c(x)}}{2a(x)} \cdot \frac{-b(x)-\sqrt{b^2(x)-4a(x)c(x)}}{2a(x)}} \\
&= \sqrt{\frac{b^2(x)-b^2(x)+4a(x)c(x)}{4a^2(x)}} = \sqrt{\frac{c(x)}{a(x)}}.
\end{aligned} \tag{3.3}$$

Since equation (3.3) is just an identity coming from the form of Y, it holds regardless of whether or not the contour is a circle. So $c(x)/a(x)$ is a constant iff L is a circle with a center at the origin.

3.1.2 The Case for a Circle with a Center on the Real axis

Our first generalization is to account for circles that do not have a center at the origin. In our application, the only restriction on where the contour must lie comes from the fact that the branches of $Y(x)$ are conjugates of one another- as the top part and the bottom part of the contour are the same distance away from the real axis, if the contour is a circle, the center must lie on the real axis.

Moreover in this case, this center can easily be determined by taking $Y(x_1)$ and $Y(x_2)$ to find the ends of the contour and setting

$$y_0 = \frac{Y(x_2)+Y(x_1)}{2}. \tag{3.4}$$

To characterize a contour that is a circle with center y_0 in the complex plane, we can set

$$|Y^{\pm}(x) - y_0| = r. \tag{3.5}$$

Lemma. Independent of the shape of the contour,

$$\left|Y^{\pm}(x) - y_0\right|^2 = \frac{c(x) + y_0 b(x)}{a(x)} + y_0^2$$

for real y_0.

We may expand $|Y^{\pm}(x) - y_0|$:

$$\begin{aligned} \left|Y^{\pm}(x) - y_0\right|^2 &= (Y^{+}(x) - y_0)(Y^{-}(x) - \overline{y_0}) \\ &= Y^{+}(x)Y^{-}(x) - y_0 Y^{-}(x) - y_0 Y^{+}(x) + y_0^2, \end{aligned} \tag{3.6}$$

since $y_0 = \overline{y_0}$. Now, we can rewrite the right hand side of equation (3.6) as follows:

$$\underbrace{Y^{+}(x)Y^{-}(x)}_{\text{See ((3.3))}} - y_0(\underbrace{Y^{-}(x) + Y^{+}(x)}_{\text{Real part of } 2Y}) + y_0^2, \tag{3.7}$$

so equation (3.7) is equal to

$$\left|Y^{\pm}(x) - y_0\right|^2 = \frac{c(x)}{a(x)} - 2y_0 \frac{-b(x)}{2a(x)} + y_0^2, \tag{3.8}$$

which simplifies to the statement of the lemma. □

In summary, if the contour is a circle with center y_0 and radius r, then

$$\frac{y_0 b(x) + c(x)}{a(x)} = r^2 - {y_0}^2. \tag{3.9}$$

If the contour is a circle, equation (3.9) will hold as designed. In the interest of establishing equivalence, say we start with equation (3.9) assigning arbitrary real y_0 and r. That would imply that the circle characterization would hold for that y_0 and $|r|$, the squaring step being reversible because we know what sign a modulus has to take. So equation (3.9) holds for any y_0 and r iff the contour is a circle with those parameters.

3.2 Ellipses

The characterization of a circle in the previous section (equation (3.5)) generalizes to a very similar characterization with ellipses, so we will extend our work to include this case. The locus characterization of an ellipse is that for any point y on the ellipse, the sum of the lengths of the lines from y to the foci f_1 and f_2 is a constant, h. In the complex plane, this characterization is written as

$$|y - f_1| + |y - f_2| = h.$$

As before, the symmetry along the real axis strongly restricts which ellipses might appear in our application. There are two cases:

- The major axis of the ellipse is on the real line, and we write the center of the ellipse as y_0, and the foci as $y_0 + f$ and $y_0 - f$, for a real y_0 and a real f.
- The minor axis of the ellipse is on the real line, and we write the center of the ellipse as y_0, and the foci as $y_0 + f$ and $y_0 - f$, for a real y_0 and a strictly imaginary f.

In both of these cases we can write the equation corresponding to the ellipse as follows:

$$|y - y_0 - f| + |y - y_0 + f| = h. \tag{3.10}$$

3.2.1 Determining Parameters

In order for the tests we develop to be a usable tool, we need to be able to determine what y_0, f, and h should be before we start applying the equations in the next sections 3.2.2 and 3.2.3. Fortunately, this is easy enough, after computing the length of the axes of the ellipse, noting here that m_r is the length of the axis of the ellipse parallel to the real axis, and

m_i is the length of the axis of the ellipse parallel to the imaginary axis:

$$\begin{aligned}
y_0 &= \frac{Y^+(x_2) + Y^+(x_1)}{2}, \\
m_r &= Y^+(x_2) - Y^+(x_1), \\
y_{max} &= Y^+(z) \text{ where } \left[\frac{d}{dx}\frac{\sqrt{4a(x)c(x) - b^2(x)}}{2a(x)}\right]_{x=z\in\mathbb{R}} = 0, \\
m_i &= 2|\mathfrak{Im}(y_{max})|, \\
f &= \frac{\sqrt{m_r^2 - m_i^2}}{2}, \\
h &= \max(m_r, m_i).
\end{aligned} \tag{3.11}$$

Note that the case where $m_r = m_i$ is in fact the degenerate case where the 'ellipse' is a circle. The constant h defined here is twice what r would be in section 3.1, but otherwise what follows is identical to the stated result, allowing the results of the following sections to be applied freely without testing for the cases from section 3.1.

Parameterizations

If we have a contour L that is an ellipse of one of the forms admissible to our application, with y_0, f and h as defined as in equation (3.11), it can be endowed with the parameterization

$$L = \left\{ \frac{m_r}{2}\cos(t) + \frac{im_i}{2}\sin(t) + y_0 \mid t \in [0, 2\pi] \right\}. \tag{3.12}$$

The choice of notation for m_r and m_i allows equation (3.12) to work for both cases. Sines and cosines behave extremely well in symbolic integration and will hopefully allow non-formal, non-numerical solutions to be derived for specific queueing BVPs in future work.

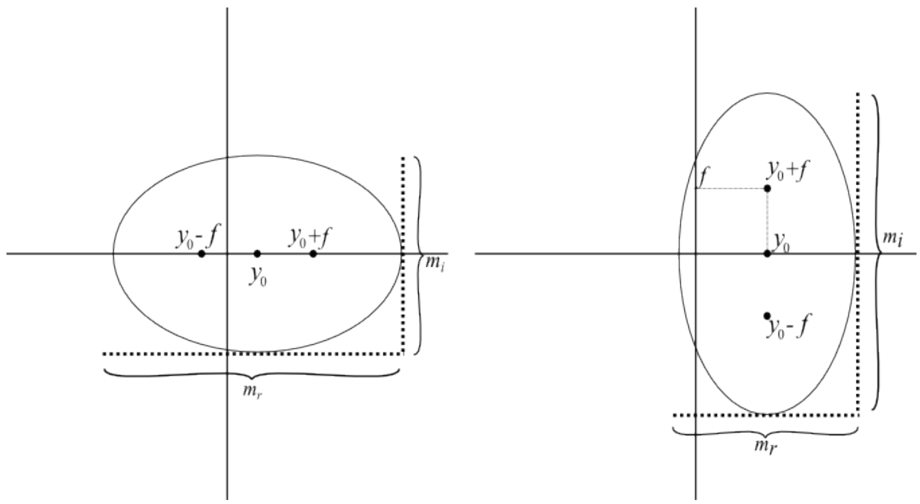

Figure 3.1: The two cases for an elliptical contour.
On the left, the major axis is on the real axis. On the right the minor axis is on the real axis.

3.2.2 The Case for an Ellipse with a Major Axis on the Real Axis

In the first case, the case for an ellipse with a major axis on the real axis, we can expand the focus equations as follows:

$$\underbrace{|Y(x) - y_0 - f|}_{t} + \underbrace{|Y(x) - y_0 + f|}_{u} = h \tag{3.13}$$

which expands to

$$4t^2u^2 = \left(h^2 - t^2 - u^2\right)^2. \tag{3.14}$$

Using the substitution coming from the lemma above, we replace t^2 and u^2 with

$$t^2 = |Y(x) - (y_0 + f)|^2 = \frac{(y_0 + f)b(x) + c(x)}{a(x)} + (y_0 + f)^2 \tag{3.15}$$

and

$$u^2 = |Y(x) - (y_0 - f)|^2 = \frac{(y_0 - f)b(x) + c(x)}{a(x)} + (y_0 - f)^2. \tag{3.16}$$

At this point, we can simply substitute equations (3.15) and (3.16) into equation (3.14), and have a usable test for computationally determining whether a given contour is an ellipse. In the interest of having a final result of a similar form to (3.9), we can simplify the result of said substitution to

$$\begin{aligned} &\frac{-h^2 a(x)b(x)y_0 + 4\,f^2 a(x)b(x)y_0 - h^2 a(x)c(x) + f^2 b^2(x)}{a^2(x)} \\ &= -1/4\,\left(h^2 - 4\,{y_0}^2\right)\left(h^2 - 4\,f^2\right). \end{aligned} \tag{3.17}$$

This final formula allows for easy checking whether this case of the ellipse holds, as all the contents of the equation can be quickly determined using the definitions in equation (3.11).

3.2.3 The Case for an Ellipse with a Minor Axis on the Real Axis

For this case, we proceed mostly as in the other, but without the luxury of having real foci. Instead of being able to reuse our previous work directly, we have, for a general case of a complex z,

$$\begin{aligned} |Y^{\pm}(x) - z|^2 &= (Y^{\pm}(x) - z)(\overline{Y^{\pm}(x)} - \overline{z}) \\ &= Y^{\pm}(x)\overline{Y^{\pm}(x)} - \overline{Y^{\pm}(x)}z - Y^{\pm}(x)\overline{z} + z\overline{z}. \end{aligned} \tag{3.18}$$

What equation (3.18) means for this problem, is that there are two cases that need to be treated, depending on the choice of branch we use, as the choice of branch determines whether we are on the top half of the ellipse or the bottom. This division into cases makes sense, as the foci are not on the real line, so for each individual focus, the modulus may vary based on choice of branch, but as we have two conjugate foci in our equation and two conjugate branches, this division into cases should cancel out in the end.

The necessary arithmetic in these cases is analogous to the derivation of equation (3.8). We obtain, for $z = w + iv, (w, v \in \mathbb{R},)$

$$
\begin{aligned}
|Y^{+}(x) - (w + iv)|^2 &= -\frac{v\sqrt{4\,a(x)c(x) - b^2(x)}}{a(x)} + w^2 + \frac{wb(x)}{a(x)} + v^2 + \frac{c(x)}{a(x)}, \\
|Y^{-}(x) - (w + iv)|^2 &= \frac{v\sqrt{4\,a(x)c(x) - b^2(x)}}{a(x)} + w^2 + \frac{wb(x)}{a(x)} + v^2 + \frac{c(x)}{a(x)}.
\end{aligned}
\tag{3.19}
$$

Finally, we can substitute $w = y_0, iv = f$ into the above equations (3.19) and substitute each one in turn into equation (3.14), noting that the branch-dependent terms containing $\dfrac{v\sqrt{4\,a(x)c(x) - b^2(x)}}{a(x)}$ cancel as expected. Finally, we isolate constants as in section 3.2.2 to get our analogous result to equations (3.9) and (3.17):

$$
\begin{aligned}
&\frac{h^2 y_0 a(x)b(x) + h^2 a(x)c(x) + 4f^2 a(x)c(x) - f^2 b^2(x)}{a^2(x)} \\
&= \frac{h^2}{4}\left(h^2 + 4\,f^2 - 4{y_0}^2\right).
\end{aligned}
\tag{3.20}
$$

Combined with our work in the previous section, the two formulae (3.17) and (3.20) exhaust all possible cases in our application for an elliptical contour.

Important Note

All of the work in this chapter so far has been operating from the path starting from the factorization $Q(x, y) = a(x)y^2 + b(x)y + c(x)$, and all of the tests give information on the contour L. If we want information on the contour M associated with the alternate factorization, we simply replace all of the terms that appear here with their counterparts that appear in table 2.4.2.

3.3 A Fast Informal Test for Circular Contours

The degenerate cases corresponding to our work in section 3.1, where the contour associated with the BVP approach to a random walk is a circle, have an interesting characteristic for the jump probabilities for that random walk. The arrow representation for the jump probabilities in a random walk in the quarter plane (similar to figure 2.2, but with arrows corresponding to jump probabilities of zero not drawn,) is often used in discussion and informal work on the random walk. In this section we will present two weak, fast, informal tests that can be used on these diagrams.

Observing closely the definitions for $a(x), b(x)$, and $c(x)$,

$$\begin{aligned} Q(x,y) &= xy\sum_{i,j}(p_{ij}x^i y^j) - 1 \\ &= p_{-1,-1} + xp_{0,-1} + yp_{-1,0} + xy(p_{0,0}-1) + x^2 p_{1,-1} \\ &+ y^2 p_{-1,1} + x^2 y p_{1,0} + xy^2 p_{0,1} + x^2y^2 p_{1,1}, \end{aligned}$$

which is factorable to be

$$\begin{aligned} &= (x^2 p_{1,1} + xp_{0,1} + p_{-1,1})y^2 + (x^2 p_{1,0} + x(p_{0,0}-1) + p_{-1,0})y \\ &+ (x^2 p_{1,-1} + xp_{0,-1} + p_{-1,-1}), \end{aligned}$$

wherein we name the coefficients

$$= a(x)y^2 + b(x)y + c(x), \tag{3.21}$$

we may note that the condition given in equation (3.3) is equivalent, by matching like terms, to the vector equation, for $k = r^2$:

$$k\begin{bmatrix} p_{1,1} \\ p_{0,1} \\ p_{-1,1} \end{bmatrix} = \begin{bmatrix} p_{1,-1} \\ p_{0,-1} \\ p_{-1,-1} \end{bmatrix}. \tag{3.22}$$

Equation (3.22) means that we can automatically disqualify a subset of random walks from having a circular contour centered at the origin. The value of k is unimportant for this weak informal test, the only relevant information is what is available from the diagram, that is, whether the jump probabilities are zero or nonzero. The test is described in figure 3.2.

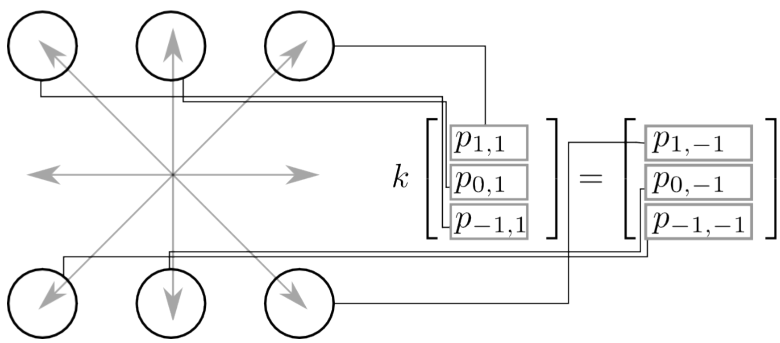

Figure 3.2: The test for a circle with a center at the origin.
If for each $j, p_{1,j} = 0$ and $p_{-1,j} = 0$, or $p_{1,j} \neq 0$ and $p_{-1,j} \neq 0$, then it is possible that the contour is a circle with a center at the origin. Otherwise it is impossible.
What this means for usage is that arrows must come in pairs reflected across the horizontal axis for L to be a circle. A similar result holds for the vertical axis and M. If one test fails, then actually computing that vector equation is unnecessary.

Equation (3.22) and the related test in figure 3.2 have results that coincide with the determinant characterization in equation (3.1), as matrices with linearly dependent rows or columns have a determinant of 0. This trick can be extended somewhat to the circles with moved centers case. We have

$$k \begin{bmatrix} p_{1,1} & + & y_0 p_{1,0} \\ p_{0,1} & + & y_0(p_{0,0} - 1) \\ p_{-1,1} & + & y_0 p_{-1,0} \end{bmatrix} = \begin{bmatrix} p_{1,-1} \\ p_{0,-1} \\ p_{-1,-1} \end{bmatrix}, \tag{3.23}$$

which has the equivalent test depicted in figure 3.3.

The tests presented in figures 3.2 and 3.3 are only fast visual checks one can make to quickly obtain negative results, that is to say, to quickly

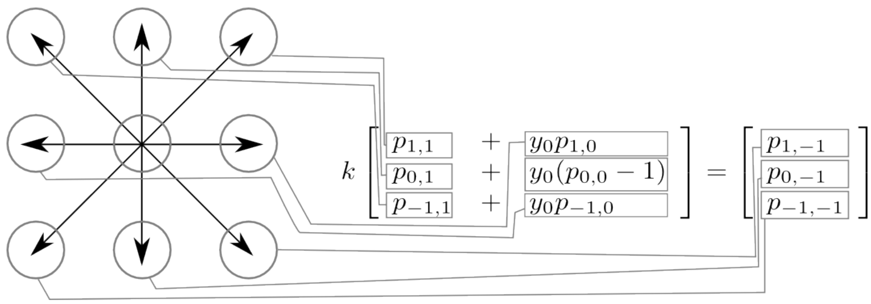

Figure 3.3: The test for a general circular contour.

Formally, the test states $\forall j \in \{1, -1\}, (p_{1,j} = 0 \wedge p_{0,j} = 0 \wedge p_{-1,j} \neq 0) \implies$ the contour is not a circle. A similar implication is derived by dividing through by k.

What this means for usage is to look for adjacent pairs of arrows not drawn (probability zero) along the edge. If such a pair exists without the third one in the line also not being drawn, then one of the contours cannot be a circle, so actual computation of the related vector equation is unnecessary.

determine that a given random walk can not have a circular contour when one approaches it using BVP methods.

Please also note that the reflection of these tests along the diagonal also gives a valid test: the tests pictured in figures 3.2 and 3.3 are based on the equations (3.3) and (3.9), which have counterparts $r = \sqrt{\frac{\tilde{c}(y)}{\tilde{a}(y)}}$ and $r^2 - {x_0}^2 = \frac{\tilde{c}(y)+x_0\tilde{b}(y)}{\tilde{a}(y)}$, which correspond to vector equations that correspond to the transposed tests.

The cases where one wishes to determine if a contour is an ellipse are complicated enough that one is better off seeing if formulae (3.17) and (3.20) hold than attempting to use a test analogous to those that appear in this section.

Chapter 4

Homogeneous Queueing BVPs

In this chapter, we extend a known approach for obtaining the generating function for the stationary probability to be compatible with a broad class of F.E.s known as the homogeneous case, which is described shortly. The approach we take follows Avrachenkov's in [1], wherein we attack the problem with a Riemann-Hilbert BVP. Our extension to the mathematics of the existing literature consists of an algorithmic method for determining the special branches of Y and a proof extending a required result in [1] to cover the arbitrary contours that may appear in the homogeneous case. We also hope to make it clear and explicit exactly what has to be done to fit this approach to a given fundamental equation.

The class of problems that this approach is designed to address is the class of random walks in the quarter plane which satisfies the conditions of section 2.2. From this class, we will take a representative with a F.E. constructed arbitrarily. Additionally, this approach also requires that in equation (2.23), $q_0(x, y) \equiv 0$; This generating function being zero is why we call this class of F.E.s the homogeneous case.

Important note

The construction of section 2.3 is effectively incompatible with the homogeneous case that appears here. Other constructions, (like [1],) can result in a homogeneous F.E., but observing closely the definitions applied in section 2.3, setting $q_0(x, y) \equiv 0$ in the setup above implies that the Markov chain can never leave the empty state, which, as one would imagine, does not require much labor for one to determine a stationary distribution.

We will proceed under the assumption that the starting point of our analysis is the homogeneous F.E. itself, composed of generating functions analytic over the unit disc, that fits the form of equation (2.23), and proceed to make definitions as appear as in section 2.4.

Chapter overview

Our goal is to manipulate the F.E. into a form amenable to treatment by a Riemann-Hilbert BVP. We take measures to satisfy the requirements on the boundary condition, but this requires some setup. First, we will detail the construction and provide an example construction of additional branches of $Y(x)$ and $X(y)$ labeled $X_0(y)$ and $Y_0(x)$. These are created first to allow easy manipulation of the F.E., and second to have useful properties that allow us to make analytic continuations up to the boundary, as required by the BVP setup. The details of these necessary properties are treated immediately after the construction of the branches. Second, we do the actual manipulation into the form of a Riemann-Hilbert BVP. After the aforementioned analytic continuation, and modification to create the boundary condition, we eliminate one of the unknown π_is and and coerce the F.E. into the BVP form in one step. Third, we meet the rest of the BVP requirements by verifying analyticity, and moving around roots, and use the BVP as intended. Finally, we recover π_1 and π_2, then make a brief argument allowing us to recover $\pi(x, y)$, and we are done.

4.1 General Theory for Homogeneous Fundamental Equations

4.1.1 Construction of New Branches

In table 2.4.2, we documented how the existence of a different factorization at the beginning induced counterpart definitions, mirroring our work. These bodies of work are more than similarly constructed, in fact they are closely linked through a relationship between the branches of $X(y)$ and $Y(x)$.

According to [4], there exists a choice of branches of these functions, call them $X_0(y)$ and $Y_0(x)$ such that $X_0 \circ Y_0$ is the identity map on $G_M \backslash x_1 x_2$, and $Y_0 \circ X_0$ is the identity map on $G_L \backslash y_1 y_2$. These functions are also analytic everywhere except the cut between the first and second branch points, and the cut between the third and fourth branch points. Additionally, recall equation (2.49), which says we can construct L and M using these branches. Constructing these new branches allows us to easily move from the x-plane to the y-plane and vice versa, as they also have several useful properties that will make our analysis easier later on.

Constructing these branches explicitly has mainly been glossed over in the literature, as their existence is well proven, but we will discuss it for the sake of completeness here. We are primarily interested in the curve $\mathfrak{Im}(D(x)) = 0$, as this is where the choice of the branch of the square root matters. As long as we make changes of branch only here, we do not risk breaking analyticity anywhere else, changes of branch here cannot worsen things.

We consider the partitions generated by looking at the signs of $\mathfrak{Re}(D(x))$ and $\mathfrak{Im}(D(x))$ to construct possible branches for $Y_0(x)$. Once all of these cases are enumerated, we can thin them out by looking at which ones are analytic on the real line minus the cuts, e.g. $(-\infty, y_1) \cup (y_2, y_3) \cup (y_4, \infty)$. We do the same with their image under $\tilde{D}(y)$, and test to see which pairing results in a pair of inverse functions, which will result in our choice of branch for X_0 and Y_0. The 'opposite' choice of branch on each partition in each case will be meromorphic except on the cuts as well, and will be declared as X_1 and Y_1 respectively. (See [4].)

Example Construction of Analytic Branches

The explicit formulation of these branches comes directly from [8]. The authors also state that all of these branches are analytic in the cut plane, except X_1, which has a pole at 0, and is thus meromorphic over the cut plane. These branches off of the branches Y^+, Y^-, X^+, and X^- as follows:

$$X_0(y) = \begin{cases} X^+(y) & \text{if } y \in \left\{ z \;\middle|\; \begin{array}{l} \mathfrak{Re}(z) \le y_2, \\ \mathfrak{Im}(\tilde{D}(\mathfrak{Re}(z) + i|\mathfrak{Im}(z)|)) < 0 \end{array} \right\}, \\ X^+(y) & \text{if } y \in (-\infty, y_1), \\ X^-(y) & \text{otherwise.} \end{cases} \tag{4.1}$$

$$X_1(y) = \begin{cases} X^-(y) & \text{if } y \in \left\{ z \;\middle|\; \begin{array}{l} \mathfrak{Re}(z) \le y_2, \\ \mathfrak{Im}(\tilde{D}(\mathfrak{Re}(z) + i|\mathfrak{Im}(z)|)) < 0 \end{array} \right\}, \\ X^-(y) & \text{if } y \in (-\infty, y_1), \\ X^+(y) & \text{otherwise.} \end{cases} \tag{4.2}$$

$$Y_0(x) = \begin{cases} Y^+(x) & \text{if } x \in \left\{ z \;\middle|\; \begin{array}{l} \mathfrak{Re}(z) \le x_2 \\ \mathfrak{Im}(D(\mathfrak{Re}(z) + i|\mathfrak{Im}(z)|)) < 0 \end{array} \right\}, \\ Y^+(x) & \text{if } x \in \left\{ z \;\middle|\; \begin{array}{l} \mathfrak{Re}(z) \ge x_3, \\ \mathfrak{Im}(D(\mathfrak{Re}(z) + i|\mathfrak{Im}(z)|)) > 0 \end{array} \right\}, \\ Y^+(x) & \text{if } x \in (-\infty, x_1) \cup (x_4, \infty), \\ Y^-(x) & \text{otherwise.} \end{cases} \tag{4.3}$$

$$Y_1(x) = \begin{cases} Y^-(x) & \text{if } x \in \left\{ z \;\middle|\; \begin{array}{l} \mathfrak{Re}(z) \le x_2 \\ \mathfrak{Im}(D(\mathfrak{Re}(z) + i|\mathfrak{Im}(z)|)) < 0 \end{array} \right\}, \\ Y^-(x) & \text{if } x \in \left\{ z \;\middle|\; \begin{array}{l} \mathfrak{Re}(z) \ge x_3, \\ \mathfrak{Im}(D(\mathfrak{Re}(z) + i|\mathfrak{Im}(z)|)) > 0 \end{array} \right\}, \\ Y^-(x) & \text{if } x \in (-\infty, x_1) \cup (x_4, \infty), \\ Y^+(x) & \text{otherwise.} \end{cases} \tag{4.4}$$

4.1.2 Bounds for $X_0(y)$ and $Y_0(x)$

We will also require several facts about the behavior of these branches for our future use, bounds to help prove analyticity when the functions are composed. Here we introduce the notation Γ to represent the unit circle, and hence G_Γ to represent the unit disc, abusing notation so that Γ and G_Γ

are part of the x-plane or y-plane as needed.

We state our bounds-related results here:

1. $|X_0(y)| \leq 1$ if $|y| = 1$, with equality only possible at 1.
2. $X_0(y) \in G_M \cup G_{M^{ext}}, \forall y \in \mathbb{C} \backslash y_1 y_2$.
3. $G_L \backslash x_1 x_2$ and $G_M \backslash y_1 y_2$ are homotopically equivalent via X_0 and Y_0.
4. $|X_0(y)| \leq 1$ for $y \in G_L \backslash G_\Gamma$.

Although the first three points here have backing in the literature[3] [4, pp 109, 118, 119], the general case of point number four that we need today has only been treated in specific examples [1, 3]. We present a proof of point number four for our general case here.

Proof. We want to show that $|X_0(y)| \leq 1$ for $y \in G_L \backslash G_\Gamma$. The overarching proof structure is as follows: We show that the desired property holds over the image $Y_0(G_\Gamma \cap G_M \backslash x_1 x_2)$, and then demonstrate that that image contains $G_L \backslash G_\Gamma$. Effectively, we are illustrating how X_0 and Y_0 map some regions to others.

– *Lemma 1*

  We demonstrate that $|X_0(s)| < 1, \forall s \in Y_0(G_\Gamma \cap G_M \backslash x_1 x_2)$. First, we know that $X_0 \circ Y_0(t) = t, \forall t \in G_M \backslash x_1 x_2$. Second, we take an element $e \in G_\Gamma \cap G_M \backslash x_1 x_2$. Then $X_0(Y_0(e)) = e \in G_\Gamma \cap G_M \backslash x_1 x_2$. Note that since $e \in G_\Gamma$, $|X_0(Y_0(e))| \leq 1$. Finally, we identify $s = Y_0(e) \in Y_0(G_\Gamma \cap G_M \backslash x_1 x_2)$. This results the range of possible s being identically $Y_0(G_\Gamma \cap G_M \backslash x_1 x_2)$ and so the lemma holds.

– *Lemma 2*

  We show that the image $Y_0(G_\Gamma \cap G_M)$ contains $G_L \backslash G_\Gamma$. First some background facts: Since we know that $|Y_0(x)| \leq 1$ if $|x| = 1$, with equality only possible at 1, the contour $Y_0(\Gamma) \subset G_\Gamma$. Also, according

[3] Point three is not explicitly declared as it appears here, but it meets the definition.

to the definition of L in equation (2.49), it is the contour coming from the limit of contours approaching the line segment x_1x_2 under Y_0.

Now we proceed by a topological argument. Our definitions for homology groups and homotopic equivalences come from [14]. Denote $A = G_\Gamma \cap G_M \backslash y_1y_2$, and $B = G_L \backslash \overline{G_{Y_0(\text{bdy } G_\Gamma \cap G_M)}}$. Our immediately preceding points indicate that B contains $G_L \backslash G_\Gamma$.

The boundaries of A map to the boundaries of B, we show that the image of A under Y_0 must be exactly B:

We need to work with closed sets for the time being, so we create either an open annulus or the open unit disk to be homotopically equivalent to A, and take the closure of whichever was needed to be $\mathcal{A}$. We also denote the closures of $Y_0(A)$ and B as $\mathcal{A}'$ and $\mathcal{B}$ respectively. Finally, we extend the homotopic equivalence of A and $Y_0(A)$ through Y_0 to these new sets, denoting the extended map $\mathcal{Y}$, which maps boundaries to boundaries by taking limit points to limit points. A commutative diagram appears below depicting this construction, with the arrows between $Y_0(A)$ and $\mathcal{A}'$, and also between B and $\mathcal{B}$ denoting inclusion. Our current goal is to show that $\mathcal{A}' = \mathcal{B}$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left, "\mathcal{Y}"] & \mathcal{A}' \arrow[l, shift left, "\mathcal{Y}^{-1}"] \arrow[r, equal, "?"] & \mathcal{B} \\
A \arrow[u] \arrow[r, shift left, "Y_0"] & Y_0(A) \arrow[u] \arrow[l, shift left, "X_0"] & B \arrow[u]
\end{tikzcd}$$

For the case where $\mathcal{A}$ is an annulus, consider a path p from one boundary of $\mathcal{A}$ to the other entirely within the interior of $\mathcal{A}$ except at the endpoints. $\mathcal{Y}(p)$ leaving and returning to $\mathcal{A}'$ would imply that the image of a non-boundary point was the same as the image of a point on the boundary, contradicting the fact that $\mathcal{Y}$ is injective. The case where $\mathcal{A}$ is a disc admits a similar argument and result for images of paths inside the disc mapping to paths entering and leaving the disc. So $\mathcal{B} \supseteq \mathcal{A}'$.

The homology groups of $\mathcal{A}$ are the same as those of $\mathcal{A}'$ due to the homotopic equivalence. Further, the boundaries $\mathcal{B}$ are part of $\mathcal{A}'$. Therefore, it is impossible for any point in $\mathcal{B}$ to not be part of $\mathcal{A}'$ as this would put in holes different from the ones that come from the boundaries of $\mathcal{A}$ and change the first homology group. So $\mathcal{B} \supseteq \mathcal{A}' \supseteq \mathcal{B}$.

Hence, int $\mathcal{A}' =$ int $\mathcal{B}$, and therefore $Y_0(A) = B \supseteq G_L \backslash G_\Gamma$.

The combination of these two lemmas gives the proof as desired. □

4.1.3 Solution via BVP

Boundary Value Problem Setup

Our goal at this point is to manipulate our F.E. into a BVP, using $Q(x, y) = 0$ (equivalently, $y = Y(x)$, either branch,) to reduce the complexity and set the values of y on L simultaneously. This gives us our starting point, with restrictions based on the intersections of the regions of analyticity of the functions:

$$q_1(x, Y_0(x))\pi_1(x) + q_2(x, Y_0(x))\pi_2(Y_0(x)) = 0, \; x \in G_\Gamma \backslash x_1 x_2. \tag{4.5}$$

If we rewrite equation (4.5) as

$$q_1(x, Y_0(x))\pi_1(x) = -q_2(x, Y_0(x))\pi_2(Y_0(x)), \; x \in G_\Gamma \backslash x_1 x_2, \tag{4.6}$$

we find that we can analytically continue the left hand side out to M. The reason that the RHS is analytic is that problems can only occur when the arguments of the functions π_i leave the unit circle, as we defined them to be generating functions (2.31), which only have guaranteed convergence on the unit circle. To extend this, we note that Y_0's range is the unit disc given that we take arguments in $G_M \backslash G_\Gamma$, by our work in section (4.1.1). The term that appears on the RHS is the composition $\pi_2 \circ Y_0$, and so π_1 can be analytically continued out to M, leaving it with a region of analyticity of G_M.

The interval $[x_1, x_2]$, and its relevance to the definition of L is all that is of interest for our BVP setup, so we ignore most of the region of analyticity in favor of the following:

$$q_1(x, Y_0(x))\pi_1(x) = -q_2(x, Y_0(x))\pi_2(Y_0(x)), \text{ as } x \text{ approaches } [x_1, x_2]. \tag{4.7}$$

We now apply our next bit of information from (4.1.1), that $Y_0 \circ X_0$ is the identity function. This allows us to switch to the other way of tracking that $Q(x, y) = 0$, by setting $x = X_0(y)$. This change of variable applied to equation (4.7) results in a condition over L:

$$q_1(X_0(y), y)\pi_1(X_0(y)) = -q_2(X_0(y), y)\pi_2(y), \; y \in L. \tag{4.8}$$

The next thing we do is observe the zeroes of $q_1(X_0(y), y)$. We want to isolate $\pi_1(X_0(y))$ via dividing through by the coefficient. $\pi_1(X_0(y))$ doesn't have any zeroes on L, so we are free to divide as we like, resulting in our next step:

$$\frac{q_2(X_0(y), y)}{q_1(X_0(y), y)}\pi_2(y) = -\pi_1(X_0(y)), \; y \in L. \tag{4.9}$$

When $y \in L, X_0(y) \in [x_1, x_2]$, so $-\pi_1(X_0(y)) \in \mathbb{R}$. π_1 here is a generating function for a real data set, so putting real values into π_1 returns real values. We may now proceed to our next step by exploiting this fact, eliminating $\pi_1(X_0(y))$:

$$\begin{aligned}\mathfrak{Im}&\left(\frac{q_2(X_0(y), y)}{q_1(X_0(y), y)}\pi_2(y)\right)\\ &= \frac{1}{2}\left[\frac{q_2(X_0(y), y)}{q_1(X_0(y), y)}\pi_2(y) - \overline{\frac{q_2(X_0(y), y)}{q_1(X_0(y), y)}\pi_2(y)}\right] = 0, \; y \in L.\end{aligned} \tag{4.10}$$

A small amount of algebraic manipulation of (4.10) results in

$$-\frac{\frac{q_2(X_0(y),y)}{q_1(X_0(y),y)}}{\overline{\left(\frac{q_2(X_0(y),y)}{q_1(X_0(y),y)}\right)}}\pi_2(y) = \overline{\pi_2(y)}, \; y \in L. \tag{4.11}$$

Now, equation (4.11) would fit a BVP perfectly, if the coefficient of $\pi_2(y)$

were nonvanishing and π_2 itself were analytic inside, and continuous up to the boundary of G_L. We will show that this is in fact the case.

We handle the analyticity first. We make the argument via analytic continuation on equation (4.9), specifically the right hand side. We know that π_1 is analytic on the unit disc, and that X_0 is analytic inside $G_L \backslash x_1 x_2$, and continuous up to the boundary, and maps the same to the unit disc. Hence the composition is analytic inside $G_L \backslash x_1 x_2$, as desired.

The nonvanishing part takes a bit more work: Beginning with equation (4.9), we note that if $q_2(X_0(y), y)$ had any zeroes, say w_i, in G_L, $i \in I, (I = \{1, 2, \dots, n\},)$ wherein multiple roots correspond to multiple w_i, then we could define the following functions to move the roots into π_2.

$$U_*(y) = \frac{q_2(X_0(y), y)}{q_1(X_0(y), y) \prod_{i \in I}(y - w_i)} \quad P(y) = \pi_2(y) \prod_{i \in I}(y - w_i). \tag{4.12}$$

We can now claim that

$$U_*(y)P(y) = \frac{q_2(X_0(y), y)}{q_1(X_0(y), y)} \pi_2(y), \tag{4.13}$$

allowing us to claim that the BVP setup is now complete for the boundary condition

$$U(y)P(y) = \overline{\pi_2(y)}, \;\; y \in L \tag{4.14}$$

where we define

$$U(y) = -\frac{U_*(y)}{\left(\overline{U_*(y)}\right)}. \tag{4.15}$$

Boundary Value Problem Solution

Now that we have a BVP, we can begin to specify a solution. First, however, we need to calculate the index of the BVP. Attentive readers may have wondered why we took all the zeroes in $\overline{G_L}$ rather than just on L, as required for the BVP setup. Doing it the way we did allows us to greatly simplify our index calculation. Moving all of the zeroes elsewhere, and therefore not having any roots in $\overline{G_L}$ ensured that the index is zero, by the

Cauchy argument principle.

Thus the BVP solution, using the Riemann-Hilbert BVP setup, is as follows:

$$P(y) = \exp\left(\frac{1}{2\pi i}\int_L \frac{\log(U(t))dt}{t-y}\right) \tag{4.16}$$

Equation (4.16) is a formal solution, it is rare that this integral can be evaluated analytically.

We may now use this to solve for the generating functions that remain unknown. From the BVP solution in equation (4.16) we can easily extract $\pi_2(y)$, as

$$\pi_2(y) = \frac{1}{\prod_{i\in I}(y-y_i)} \exp\left(\frac{1}{2\pi i}\int_L \frac{\log(U(t))dt}{t-y}\right), \tag{4.17}$$

and by a quick reference to table 2.4.2, we can determine $\pi_1(x)$ by the same approach. Equation (4.17), with a quick reference to equation (2.23), allows us to determine $\pi(x,y)$, as

$$\pi(x,y) = \frac{q_1(x,y)\pi_1(x) + q_2(x,y)\pi_2(y)}{Q(x,y)}. \tag{4.18}$$

At this point the $\pi_{i,j}$ may be extracted by taking derivatives, as is usual for generating functions.

Chapter 5

Conclusion

It is our hope that our work in developing tests for identifying elegantly reparameterizable contours will be useful in working towards analytic solutions of the associated Riemann-Hilbert BVPs for this subset of problems, and that our efforts in creating a formal general solution that extends existing methods for approaching homogeneous F.E.s associated with random walks in the quarter plane will be of use for students and queueing theorists addressing those F.E.s in the future. Non-homogeneous BVPs are still best attacked with the methods found in chapters 4, 5, and 6 of the text by Fayolle et al. on the subject [4].

The bodies of our work in chapters 3 and 4 are developed independently. Worse yet, they are fundamentally incompatible, as chapter 3 relies on a construction that renders the work in chapter 4 trivial. Even if a solution to the incompatibility were found, there would be a poor return on the investment of going through the work in chapter 3 in addition to that of chapter 4, as the results of chapter 4 stop at the formal solution. The aforementioned analytic solutions that we hope to use the results of chapter 3 to find remain a goal rather than a reality at the time being. In any case, we would still be restricting the general class of F.E.s that chapter 4 treats to a limited subset.

There are several avenues that future work may take to extend what appears here. Our approach for identifying special contours could be extended to any locus-characterizable contour, but it is certain that more work needs to be done before our approach is a usable tool for queueing theorists. Integration over these simpler contours should be easier than the natural parameterization resulting from drawing values from $[x_1, x_2]$ in equation (2.48), but a general solution for the integrals that occur in the solution of

Riemann-Hilbert BVPs is yet to be found. It has been suggested that looking at the non-BVP solution found in [5] alongside the formal BVP solution found here could give hints on how to compute this integral analytically, and others like it.

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