Operators on compositions and noncommutative Schur functions

by

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Abstract

In this thesis, we study a natural noncommutative lift of the ubiquitous Schur functions, called noncommutative Schur functions. These functions were introduced by Bessenrodt, Luoto and van Willigenburg and resemble Schur functions in many regards. We prove some new results for noncommutative Schur functions that are analogues of classical results, and demonstrate that the resulting combinatorics in this setting is equally rich.

First we prove a Murnaghan-Nakayama rule for noncommutative Schur functions. In other words, we give an explicit combinatorial formula for expanding the product of a noncommutative power sum symmetric function and a noncommutative Schur function in terms of noncommutative Schur functions. In direct analogy to the classical Murnaghan-Nakayama rule, the summands are computed using a noncommutative analogue of border strips, and have coefficients ± 1 determined by the height of these border strips. The rule is proved by interpreting the noncommutative Pieri rules for noncommutative Schur functions in terms of box adding operators on compositions.

We proceed to give a backward jeu de taquin slide analogue on semistandard reverse composition tableaux. These tableaux were first studied by Haglund, Luoto, Mason and van Willigenburg when defining quasisymmetric Schur functions. Our algorithm for performing backward jeu de taquin slides on semistandard reverse composition tableaux results in a natural operator on compositions that we call the jdt operator. This operator in turn gives rise to a new poset structure on compositions whose maximal chains we enumerate. As an application, we also give new right Pieri rules for noncommutative Schur functions that use the jdt operators, in contrast to the left Pieri rules given by Bessenrodt, Luoto and van Willigenburg.

Preface

This thesis contains the results from two manuscripts written by the author: [44] and [45]. Chapter 3 is based on the manuscript [44] while Chapter 4 is based on the manuscript [45].

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List of Symbols and Notations

α	composition
$\widetilde{\alpha}$	underlying partition of α
$\beta /\!\!/ \alpha$	skew reverse composition shape
χ	forgetful map
comp	composition associated to a set, or to the descent set of a tableau
$CRHW_n$	set of connected reverse hookwords
$\delta_{i,j}$	1 if $i = j$ and 0 otherwise
Des	descent set of a tableau
$\mathfrak{d}$	box removing operator
$e(T)$	evacuation of a tableau
F_α	fundamental quasisymmetric function
h_λ	complete homogeneous symmetric function
$\mathbf{h}_\alpha$	complete homogeneous noncommutative symmetric function
ht	height statistic associated to a border strip, or an nc border strip
jdt_i	backward jeu de taquin slide on SSRTs from column i
λ	partition
λ/μ	skew shape
$\mathcal{L}_c$	reverse composition poset

μ_i	backward jeu de taquin slide on SSRCTs from column i
NSym	Hopf algebra of noncommutative symmetric functions
p_λ	power sum symmetric function
Ψ_α	noncommutative power sum symmetric function
QSym	Hopf algebra of quasisymmetric functions
$\mathrm{rect}(T)$	rectification of a tableau
ρ_α	bijection between SSRCT and SSRT
RHW_n	set of reverse hookwords
$\mathbf{r}_\alpha$	noncommutative ribbon Schur function
set	set associated to a composition
sh	shape of a tableau
s_λ	Schur function
$\mathcal{S}_\alpha$	quasisymmetric Schur function
$\mathbf{s}_\alpha$	noncommutative Schur function
SRT	standard reverse tableau
SSRT	semistandard reverse tableau
SRCT	standard reverse composition tableau
SSRCT	semistandard reverse composition tableau
$\mathrm{supp}(\beta /\!\!/ \alpha)$	support of a skew reverse composition shape
Sym	Hopf algebra of symmetric functions
$\mathfrak{S}_n$	symmetric group
T	semistandard reverse tableau
τ	semistandard reverse composition tableau
τ_α	canonical standard reverse composition tableau
$\mathfrak{t}$	box adding operator
$\mathfrak{u}$	jeu de taquin operator
w_T	column reading word of a tableau

x^T	monomial associated to a tableau
$\mathcal{Y}$	Young's lattice
$\varnothing$	empty composition or partition
$\vdash$	is a partition of
$\models$	is a composition of
$\lvert\ \rvert$	size of a composition or partition, or size of a skew shape
$\lessdot_c$	cover relation in $\mathcal{L}_c$
$\lessdot_r$	cover relation in $\mathcal{R}_c$
$\prec$	cover relation in $\mathcal{Y}$
$\preccurlyeq$	refines
$\subseteq$	subset of

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Dedication

I dedicate this thesis to my sister Stuti.

Chapter 1

Introduction

The Hopf algebra of symmetric functions, denoted by **Sym**, is a central object in algebraic combinatorics and its most important linear basis is that of Schur functions s_λ for all partitions λ. The ubiquity and importance of Schur functions can be gauged from the numerous ways of defining them. For example, when interpreting symmetric functions in the representation theory of symmetric groups, Schur functions correspond to irreducible representations [39]. Another link to representation theory is that Cauchy's quotient of alternants formula for Schur functions is a specialization of Weyl's character formula. In algebraic geometry, the cohomology rings of Grassmanians can be interpreted as quotients of polynomial rings so that Schubert classes correspond to Schur functions, and in this setting, the famed Littlewood-Richardson coefficients arise as structure constants expressing cup products [12]. Finally, the combinatorial way of defining Schur functions, attributed to Littlewood [28], is to describe s_λ as a generating function for semistandard reverse tableaux.

Another key basis of **Sym** is the basis of power sum symmetric functions p_λ for all partitions λ. The transition matrix between the two bases mentioned gives us the character table of the symmetric group, which can also be computed combinatorially using the Murnaghan-Nakayama rule [42, Section 7.17]. This rule additionally gives a combinatorial formula for computing the product of a power sum symmetric function and a Schur function in terms of Schur functions.

Sym is a Hopf subalgebra of **QSym**, the Hopf algebra of quasisymmetric functions. Quasisymmetric functions are refinements of symmetric functions and were introduced by Gessel [14] in his study of P-partitions. They can also be realized as invariants under a quasisymmetrizing action of the symmetric group [21]. Since its introduction, there has been much activity surrounding **QSym**, and quasisymmetric functions have found applications in areas such as combinatorial Hopf algebras [1, 9], card shuffling [11], permutation enumeration [15], discrete geometry [5], and representation theory [24]. Given the intimate connection between **Sym** and **QSym**, a key line of inquiry is the existence of a Schur-like basis for **QSym**, that is, a basis of **QSym** that refines the algebraic and combinatorial properties of the basis of Schur functions.

A prominent candidate, introduced by Haglund, Luoto, Mason and van Willigenburg [17], is the basis of quasisymmetric Schur functions $\mathcal{S}_\alpha$ where α runs over all compositions. The quasisymmetric Schur function $\mathcal{S}_\alpha$, much like a Schur function, is obtained as a generating function for semistandard reverse composition tableaux, and its original definition was motivated by the combinatorics of Macdonald polynomials [16]. Furthermore, these functions naturally refine Schur functions and many of their combinatorial properties [29].

Equally interesting phenomena happen when we consider the Hopf algebra dual to **QSym**. This is the Hopf algebra of noncommutative symmetric functions, denoted by **NSym**, introduced in the seminal paper [13]. In addition to being the Hopf dual to **QSym**, **NSym** affords **Sym** as a quotient. Thus, it is intimately connected to the two Hopf algebras introduced earlier and provides an alternative route to establishing new results in **Sym** and **QSym**, with the noncommutativity offering a unique perspective on existing results. Noncommutative symmetric functions and their relations with a host of generalizations of symmetric functions have been a subject of intense study, as is evident from [6, 7, 8, 23, 24, 25].

Taking the dual basis corresponding quasisymmetric Schur functions in **NSym**, we obtain the basis of noncommutative Schur functions introduced in [4]. As the name suggests, these functions are noncommutative lifts of classical Schur functions, and resemble them by possessing properties that are noncommutative analogues of Pieri rules, Kostka numbers, and the Littlewood-Richardson rule [4]. They also have a representation-theoretic significance as they can be identified as noncommutative irreducible characters of the symmetric group [48].

In this thesis, we strengthen the connection between noncommutative Schur functions and classical Schur functions by supplementing already existing results with analogues of the Murnaghan-Nakayama rule, jeu de taquin, and classical Pieri rules. We briefly outline our results next and the full statements can be found in Section 2.6. In Chapter 3 of this thesis, we consider the problem of finding an analogue of the classical Murnaghan-Nakayama rule. As mentioned earlier, the classical rule [32, 33] is a combinatorial procedure to compute the character table of the symmetric group. The combinatorial objects involved, which aid the said computation, are called border strips. An alternate description of the Murnaghan-

Nakayama rule is that it gives an explicit combinatorial description of the expansion of the product of a power sum symmetric function p_k, where k is a positive integer, and a Schur function s_μ as a sum of Schur functions

$$p_k \cdot s_\mu = \sum_\lambda (-1)^{ht(\lambda/\mu)} s_\lambda, \tag{1.1}$$

where the sum is over all partitions λ such that the skew shape λ/μ is a border strip of size k, and $ht(\lambda/\mu)$ is a certain statistic associated with the border strip λ/μ.

To obtain our noncommutative analogue to the result above, we expand the product $\Psi_r \cdot \mathbf{s}_\alpha$ in the basis of noncommutative Schur functions. Here Ψ_r denotes the noncommutative power sum symmetric function of the first kind, indexed by a positive integer r, while $\mathbf{s}_\alpha$ denotes the noncommutative Schur function indexed by a composition α. The main result we prove is that, much like (1.1), we have an expansion

$$\Psi_r \cdot \mathbf{s}_\alpha = \sum_\beta (-1)^{ht(\beta /\!\!/ \alpha)} \mathbf{s}_\beta, \tag{1.2}$$

where the sum is over certain compositions β such that the skew reverse composition shape $\beta /\!\!/ \alpha$ gives rise to a noncommutative analogue of a border strip of size r, and $ht(\beta /\!\!/ \alpha)$ is a certain statistic similar to $ht(\lambda/\mu)$ in (1.1). The box adding operators on compositions that we introduce in Chapter 2, and that are reminiscent of those in [2, 10], will be fundamental to deriving (1.2).

In Chapter 4 of this thesis, we introduce an analogue of a combinatorial procedure, called jeu de taquin, for semistandard reverse composition tableaux and study the implications in the setting of **NSym**. Classically, jeu de taquin is a set of sliding rules defined on semistandard Young tableaux (or semistandard reverse tableaux) that preserves the property of being semistandard while changing the underlying shape. This procedure is of utmost importance in algebraic combinatorics, representation theory and algebraic geometry. It was introduced by Schützenberger [40], and its many remarkable properties have been studied in depth since [12, 38, 42]. The procedure plays a crucial role in the combinatorics of permutations and Young tableaux, with deep links to the Robinson-Schensted algorithm and the plactic monoid [26]. One of the most important applications of Schützenberger's jeu de taquin is in giving a combinatorial rule for computing the induced tensor product of two representations of the symmetric group, commonly called the Littlewood-Richardson rule.

Analogues of jeu de taquin have been found for shifted tableaux [19, 37], Littelmann's crystal paths [27], increasing tableaux [46], edge labeled Young tableaux [47] and d-complete posets [35, 36]. Procedures such as promotion and evacuation, which are consequences of jeu de taquin and important in their own right, have also been studied [34, 43]. Recently, an infinite version of jeu de taquin has been studied for infinite Young tableaux [41].

Our focus here is on semistandard reverse composition tableaux and we give an algorithm for performing backward jeu de taquin slides on them. Subsequently, we use this algorithm to give *right Pieri rules*. To give the aforementioned results context, we will now supply some background. A special case of the noncommutative Littlewood-Richardson rule proved in [4] is a noncommutative analogue of the classical Pieri rules, which we will call the *left Pieri rules*. It corresponds to expanding the product of noncommutative Schur functions $\mathbf{s}_{(n)} \cdot \mathbf{s}_\alpha$ and $\mathbf{s}_{(1,\dots,1)} \cdot \mathbf{s}_\alpha$ in the basis of noncommutative Schur functions, where n denotes a positive integer and α denotes a composition. Given that we are computing products in a noncommutative Hopf algebra, it is natural to consider the products $\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)}$ and $\mathbf{s}_\alpha \cdot \mathbf{s}_{(1,\dots,1)}$. The statement of the noncommutative Littlewood-Richardson rule in [4], which allows us to compute the aforementioned products, requires the rectification of standard reverse tableaux and a map defined by Mason in [31], denoted by ρ, which links semistandard reverse tableaux and semistandard reverse composition tableaux. One could ask whether it is possible to eliminate the use of Mason's map and introduce an analogue of rectification for semistandard reverse composition tableaux. This was achieved by Bechard in [3]. For what we aspire to compute, it is preferable to consider a procedure that is the inverse of Bechard's rectification. Our algorithm for performing backward jeu de taquin slides on SSRCTs is precisely such a procedure.

The organization of this thesis is as follows. Chapter 2 introduces the notation and definitions necessary for stating our main results: Section 2.1 defines combinatorial objects such as partitions, compositions, diagrams, and tableaux. In Section 2.2, we define **Sym** and give a combinatorial description for the distinguished basis of Schur functions using semistandard reverse tableaux. We also discuss combinatorial procedures such as jeu de taquin and rectification. Section 2.3 concerns itself with **QSym** and a natural analogue of Schur functions therein, the quasisymmetric Schur functions. Next we consider the Hopf dual to **QSym**, that is, **NSym** and give an implicit description for the basis of noncommutative Schur functions. In Section 2.5, we define certain operators on compositions. These operators are crucial in helping us derive our noncommutative Murnaghan-Nakayama rule and the right Pieri rules. Finally, in Section 2.6, we state our two main results along with comprehensive examples: Theorem 2.6.3 (noncommutative Murnaghan-Nakayama rule), and

Algorithm 2.6.6 (backward jeu de taquin slides for SSRCTs).

In Chapter 3 we give a proof for Theorem 2.6.3. In Section 3.1, we connect sequences of box adding operators with standard reverse composition tableaux, and then describe certain distinguished sequences of box adding operators called reverse hookwords. The goal of Section 3.2 is to give a rudimentary version of the noncommutative Murnaghan-Nakayama rule in terms of box adding operators in Equation (3.6). Finally, after introducing our analogue of classical border strips, called nc border strips, in Section 3.3, we arrive at the noncommutative Murnaghan-Nakayama rule that mirrors the classical version in Theorem 3.3.20.

In Chapter 4, we proceed to give the proofs of validity of the algorithms described earlier in Subsection 2.6.3 and the connection is made precise in Theorem 4.1.21. Next, in Subsection 4.1.1, we study the effect of slides on underlying composition shapes using jdt operators. In Section 4.2, we describe a procedure to perform backward jeu de taquin slides on skew semistandard reverse composition tableaux. Using the jdt operators, in Section 4.3 we endow the set of compositions with a new poset structure $\mathcal{R}_c$. Finally, in Section 4.4, we prove right Pieri rules in Theorem 4.4.3 and enumerate the maximal chains in $\mathcal{R}_c$ in Theorem 4.4.4.

We conclude this thesis by describing some new questions and avenues in Chapter 5.

Chapter 2

Sym, QSym and NSym

We start by defining some of the combinatorial structures that we will encounter. All the classical notions introduced in this chapter are covered in more detail in [30, 38, 42].

2.1 Combinatorial preliminaries

2.1.1 Partitions

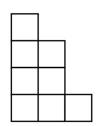
A *partition* λ is a finite list of positive integers $(\lambda_1, \ldots, \lambda_k)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. The integers appearing in the list are called the *parts* of the partition. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, the *size* $|\lambda|$ is defined to be $\sum_{i=1}^{k} \lambda_i$. The number of parts of λ is called its *length*, and is denoted by $l(\lambda)$. If λ is a partition satisfying $|\lambda| = n$, then we denote this by $\lambda \vdash n$. By convention, there is a unique partition of size and length equalling 0, and we denote it by $\varnothing$.

We will depict a partition using its *Young diagram*. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, the Young diagram of λ, also denoted by λ, is the left-justified array of n boxes, with λ_i boxes in the i-th row. We will be using the French convention where the rows are numbered from bottom to top and the columns from left to right, and will refer to the box in the i-th row and j-th column by the ordered pair (i, j). If λ and μ are partitions such that $\mu \subseteq \lambda$, that is, $l(\mu) \leq l(\lambda)$ and $\mu_i \leq \lambda_i$ for all $i = 1, 2, \ldots, l(\mu)$, then the *skew shape* λ/μ is obtained by removing the first μ_i boxes from the i-th row of the Young diagram of λ for $1 \leq i \leq l(\mu)$. Given a skew shape λ/μ, we refer to λ as the *outer shape* and to μ as the *inner shape*. If the inner shape is $\varnothing$, instead of writing $\lambda/\varnothing$, we just write λ and call λ a *straight shape*. The *size* of a skew shape λ/μ, denoted by $|\lambda/\mu|$, is the number of boxes in the skew shape, which is $|\lambda| - |\mu|$. If λ/μ does not have two boxes belonging to the same column, then it is called a *horizontal strip*, while if it does not have two boxes belonging to the same row it is called a *vertical strip*.

Next, we define addable nodes and removable nodes of a partition. An *addable node* of a partition λ is a position (i,j) where a box can be appended to a part of λ so that the result is still a partition. Note that the addable nodes of a partition do not belong to the partition. On the other hand, a *removable node* of λ is a position (i,j) where a box can be removed from a part of λ so that the resulting shape is still a partition. Unlike addable nodes, removable nodes of a partition belong to the partition. Furthermore, given a partition, both addable nodes and removable nodes are uniquely determined by the column to which they belong. In the definitions just given, we have identified the partition λ with its Young diagram.

We will now use the aforementioned notions to define a lattice structure on the set of partitions, called *Young's lattice*. If μ is a partition obtained by adding a box at an addable node of the partition λ, then we say that μ covers λ and we denote it by $\lambda \prec \mu$. The partial order obtained by taking the transitive closure of $\prec$ endows the set of partitions with the structure of a lattice that we will denote by $\mathcal{Y}$ [38, Definition 5.1.2].

Example 2.1.1. *The partition* $\lambda = (3, 2, 2, 1)$ *is represented by the following Young diagram.*



The box in the lower left corner is indexed by $(1,1)$*. The addable nodes of* λ *are* $(1,4)$*,* $(2,3)$*,* $(4,2)$ *and* $(5,1)$ *while the removable nodes are* $(1,3)$*,* $(3,2)$ *and* $(4,1)$*. Also, the partition* $\mu = (3,3,2,1)$ *covers* λ *in* $\mathcal{Y}$*, that is,* $(3,2,2,1) \prec (3,3,2,1)$*.*

2.1.2 Semistandard reverse tableaux

In this subsection, we will introduce certain classical objects that play a central role in the theory of symmetric functions.

Definition 2.1.2. *Given a skew shape* λ/μ*, a* semistandard reverse tableau (SSRT) T *of* shape λ/μ *is a filling of the boxes of* λ/μ *with positive integers satisfying the condition that the entries in* T *are weakly decreasing along each row read from left to right and strictly decreasing along each column read from bottom to top.*

We will denote the set of all SSRTs of shape λ/μ by $\mathrm{SSRT}(\lambda/\mu)$. Given an SSRT T, the entry in box (i,j) is denoted by $T_{(i,j)}$. The shape underlying T is denoted by $\mathrm{sh}(T)$.

A *standard reverse tableau* (SRT) T of shape λ/μ is an SSRT that contains every positive integer in $[|\lambda/\mu|] = \{1, 2, \ldots, |\lambda/\mu|\}$ exactly once.

Given an SSRT, we associate to it a distinguished word called the *column reading word*. This word plays an important role in tableaux combinatorics, and we will encounter it when discussing jeu de taquin and rectification. The *column reading word* of an SSRT T is the word obtained by reading the entries of every column in increasing order from left to right, and we denote it by w_T.

Let $\{x_1, x_2, \ldots\}$ be an alphabet comprising of countably many commuting indeterminates. Given any SSRT T of shape $\lambda \vdash n$, we will associate a monomial x^T with it as follows.

$$x^T = \prod_{(i,j)\in\lambda} x_{T_{(i,j)}}$$

Example 2.1.3. *Shown below are an* SSRT T *of shape* $(4, 3, 3, 1)$ *and its associated monomial.*

$$T = \begin{array}{|c|c|c|c|}\cline{1-1} 1 \\ \cline{1-3} 3 & 3 & 2 \\ \cline{1-3} 6 & 5 & 4 \\ \hline 7 & 7 & 6 & 3 \\ \hline\end{array} \qquad x^T = x_7^2 x_6^2 x_5 x_4 x_3^3 x_2 x_1$$

The column reading word of T *is* 1367 357 246 3.

2.1.3 Compositions

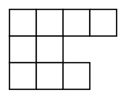
A *composition* α is a finite ordered list of positive integers. The integers appearing in the list are called the *parts* of the composition. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, its *size* $|\alpha|$ is defined to be $\sum_{i=1}^k \alpha_i$. The number of parts of α is called the *length*, and is denoted by $l(\alpha)$. If α is a composition satisfying $|\alpha| = n$, then we write it as $\alpha \vDash n$. By convention, there is a unique composition of size and length 0, and we denote it by $\varnothing$. Finally, given a composition α, we denote by $\widetilde{\alpha}$ the partition obtained by sorting the parts of α in weakly decreasing order.

Just as we used Young diagrams for partitions, we will depict a composition using its reverse composition diagram as follows. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$, the *reverse composition diagram* of α, also denoted by α, is the left-justified array of n boxes with α_i boxes in the i-th row. Here we follow the English convention, that is, the rows are numbered from top to bottom, and the columns from left to right. We will refer to the box in the i-th row and j-th column by the ordered pair (i, j). We adopt a different convention from Young diagrams in order to facilitate easier proofs later.

Recall now the bijection between compositions of n and subsets of $[n-1]$. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$, we can associate a subset of $[n-1]$, denoted by $\mathrm{set}(\alpha)$, by defining it to be $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$. In the opposite direction, given a set $S = \{i_1 < \cdots < i_j\} \subseteq [n-1]$, we can associate a composition of n, denoted by $\mathrm{comp}(S)$, by defining it to be $(i_1, i_2 - i_1, \ldots, i_j - i_{j-1}, n - i_j)$.

Finally, we define the refinement order on compositions. Given compositions α and β, we say that $\alpha \succcurlyeq \beta$ if one obtains parts of α in order by adding together adjacent parts of β in order. The composition β is said to be a *refinement* of α.

Example 2.1.4. *Let* $\alpha = (4, 2, 3) \vDash 9$. *Then* $\mathrm{set}(\alpha) = \{4, 6\} \subseteq [8]$. *Shown below is the reverse composition diagram of* α*.*



Note also that $(4, 2, 3) \succcurlyeq (3, 1, 2, 1, 2)$.

2.1.4 Semistandard reverse composition tableaux

Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ and β be compositions. Define a cover relation $\lessdot_c$ on compositions as follows.

$$\alpha \lessdot_c \beta \text{ iff } \begin{cases} \beta = (1, \alpha_1, \ldots, \alpha_l) & \text{or} \\ \beta = (\alpha_1, \ldots, \alpha_k + 1, \ldots, \alpha_l) & \text{and } \alpha_i \neq \alpha_k \text{ for all } i < k. \end{cases}$$

The *reverse composition poset* $\mathcal{L}_c$ is the poset on compositions where the partial order $<_c$ is obtained by taking the transitive closure of the cover relations above. If $\alpha <_c \beta$, the *skew reverse composition shape* $\beta /\!\!/ \alpha$ is defined to be the array of boxes

$$\beta /\!\!/ \alpha = \{(i, j) \mid (i, j) \in \beta, (i, j) \notin \alpha\}$$

where α is drawn in the bottom left corner of β. We refer to β as the *outer shape* and to α as the *inner shape*. If the inner shape is $\varnothing$, instead of writing $\beta /\!\!/ \varnothing$, we simply write β and refer to β as a *straight shape*. The *size* of the skew reverse composition shape $\beta /\!\!/ \alpha$, denoted by $|\beta /\!\!/ \alpha|$, is the number of boxes in the skew reverse composition shape, that is, $|\beta| - |\alpha|$. If $\beta /\!\!/ \alpha$ does not have two boxes belonging to the same column, then it is called a *horizontal strip*, while if it does not have two boxes lying in the same row it is called a *vertical strip*.

Recall that we had the notion of horizontal and vertical strips for skew shapes as well, and it will be clear from the context which notion we are referring to.

Definition 2.1.5. *A* semistandard reverse composition tableau *(*SSRCT*)* τ *of* shape $\beta /\!\!/ \alpha$ *is a filling*

$$\tau : \beta /\!\!/ \alpha \longrightarrow \mathbb{Z}^+$$

that satisfies the following conditions

1. *the rows are weakly decreasing from left to right,*

2. *the entries in the first column are strictly increasing from top to bottom,*

3. *if* $i < j$ *and* $(j, k+1) \in \beta /\!\!/ \alpha$ *and either* $(i,k) \in \alpha$ *or* $\tau(i,k) \geq \tau(j, k+1)$ *then either* $(i, k+1) \in \alpha$ *or both* $(i, k+1) \in \beta /\!\!/ \alpha$ *and* $\tau(i, k+1) > \tau(j, k+1)$*.*

The shape underlying an SSRCT τ will be denoted by $\mathrm{sh}(\tau)$. A *standard reverse composition tableau* (SRCT) is an SSRCT in which the filling is a bijection $\tau : \alpha /\!\!/ \beta \longrightarrow [|\alpha /\!\!/ \beta|]$. That is, in an SRCT each number from the set $\{1, 2, \ldots, |\alpha /\!\!/ \beta|\}$ appears exactly once. We will denote the set of SSRCTs (SRCTs) of shape $\beta /\!\!/ \alpha$ by $\mathrm{SSRCT}(\beta /\!\!/ \alpha)$ ($\mathrm{SRCT}(\beta /\!\!/ \alpha)$ respectively).

Throughout this thesis, when considering an SSRCT, the boxes of the outer shape will often be shaded. Suppose now that we fill the boxes belonging to the inner shape with bullets and assume further that the ordered pairs of positive integers (i,j) such that $(i,j) \notin \alpha$ and $(i,j) \notin \beta$ are filled with 0s. This given, the third condition in the definition above is equivalent to the non-existence of the following configurations in the filling τ.

$$\begin{array}{cc} \boxed{\bullet} & \boxed{y} \\ & \vdots \\ & \boxed{z} \\ \multicolumn{2}{c}{z \geq y > 0} \end{array} \qquad \begin{array}{cc} \boxed{\bullet} & \\ & \vdots \\ & \boxed{z} \\ \multicolumn{2}{c}{z > 0} \end{array} \qquad \begin{array}{cc} \boxed{x} & \boxed{y} \\ & \vdots \\ & \boxed{z} \\ \multicolumn{2}{c}{x \geq z \geq y > 0} \end{array} \qquad \begin{array}{cc} \boxed{x} & \\ & \vdots \\ & \boxed{z} \\ \multicolumn{2}{c}{x \geq z > 0} \end{array}$$

The existence of a configuration of the above types in a filling will be termed a *triple rule violation*. From this point on, we will refer to the entry in the box in position (i,j) of an SSRCT τ by $\tau_{(i,j)}$.

The *column reading word* of an SSRCT τ is the word obtained by reading the entries in each column in increasing order, where we traverse the columns from left to right. We denote it by w_τ.

Just as we did with SSRTs, we will associate a monomial to an SSRCT. Given any SSRCT τ of shape $\alpha \vDash n$, we will associate a monomial x^{τ} with it as follows.

$$x^{\tau} = \prod_{(i,j)\in\alpha} x_{\tau(i,j)}$$

Example 2.1.6. *An* SSRCT *of shape* $(2,5,4,2)/\!\!/(2,1,2)$ *(left) and an* SRCT *of shape* $(3,4,2,3)$.

$$\begin{array}{|c|c|c|c|c|}
\hline
4 & 3 \\
\cline{1-5}
\bullet & \bullet & 7 & 5 & 1 \\
\cline{1-5}
\bullet & 7 & 6 & 2 \\
\cline{1-4}
\bullet & \bullet \\
\cline{1-2}
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c|}
\hline
6 & 4 & 1 \\
\hline
8 & 7 & 5 & 2 \\
\hline
10 & 3 \\
\cline{1-3}
12 & 11 & 9 \\
\cline{1-3}
\end{array}$$

The column reading word of the SSRCT τ *on the left is* $w_{\tau} = 4\ \ 37\ \ 67\ \ 25\ \ 1$.

Also associated with an SRCT is its *descent set*. Given an SRCT τ of shape $\alpha \vDash n$, its descent set, denoted by $\mathrm{Des}(\tau)$, is defined to be the set of all integers i such that $i+1$ lies weakly to the right of i in τ. Note that $\mathrm{Des}(\tau)$ is a subset of $[n-1]$. The *descent composition* of τ, denoted by $\mathrm{comp}(\tau)$, is the composition of n associated with $\mathrm{Des}(\tau)$. For instance, the descent set of the SRCT in Example 2.1.6 is $\{1,3,4,6,8,10\} \subseteq [11]$. Hence the associated descent composition is $(1,2,1,2,2,2,2) \vDash 12$.

Given a composition α, there exists a unique SRCT τ_{α} so that the underlying shape of τ_{α} is α and $\mathrm{comp}(\tau_{\alpha}) = \alpha$. This tableau is called the *canonical tableau* of shape α. To construct τ_{α} given $\alpha = (\alpha_1, \ldots, \alpha_k)$, place consecutive integers from $1 + \sum_{i=1}^{r-1} \alpha_i$ to $\sum_{i=1}^{r} \alpha_i$ in that order from right to left in the r-th row from the top in the reverse composition diagram of α for $1 \leq r \leq k$.

Example 2.1.7. *Let* $\alpha = (3,4,2,3)$. *Then* τ_{α} *is as shown below.*

$$\begin{array}{|c|c|c|c|}
\hline
3 & 2 & 1 \\
\hline
7 & 6 & 5 & 4 \\
\hline
9 & 8 \\
\cline{1-3}
12 & 11 & 10 \\
\cline{1-3}
\end{array}$$

Next, we will show that SSRCTs and SSRTs are intimately related via a generalization of the bijection defined by Mason in [31] (the generalization itself is also bijective and is presented in [29]). Let $\mathrm{SSRCT}(-/\!\!/\alpha)$ denote the set of all SSRCTs with inner shape α, and let $\mathrm{SSRT}(-/\widetilde{\alpha})$ denote the set of SSRTs with inner shape $\widetilde{\alpha}$ (which might be empty). Then the *generalized ρ map*

$$\rho_\alpha : \mathrm{SSRCT}(-/\!\!/\alpha) \longrightarrow \mathrm{SSRT}(-/\widetilde{\alpha})$$

is defined as follows. Given an SSRCT τ, write the entries in each column in decreasing order from bottom to top, and bottom justify the new columns thus obtained on the inner shape $\widetilde{\alpha}$. This yields $\rho_\alpha(\tau)$.

We give the description of the inverse map ρ_α^{-1} next. Let T be an SSRT.

1. Take the first column of the SSRT, and place the entries of this column above the first column of the inner shape α in increasing order from top to bottom.

2. Now take the set of entries in the second column in decreasing order and place them in the row with smallest index so that either

   - the box to the immediate left of the number being placed is filled and the entry therein is weakly greater than the number being placed
   - the box to the immediate left of the number being placed belongs to the inner shape α.

As a matter of convention, instead of writing $\rho_\varnothing$ or $\rho_\varnothing^{-1}$, we will use ρ or ρ^{-1} respectively.

Example 2.1.8. *An* SSRCT *of shape* $(2,5,4,2)/\!\!/(2,1,2)$ *(left), and the corresponding* SSRT *of shape* $(5,4,2,2)/(2,2,1)$ *(right).*

$$\begin{array}{|c|c|c|c|c|}\hline 4 & 3 \\ \hline \bullet & \bullet & 7 & 5 & 1 \\ \hline \bullet & 7 & 6 & 2 \\ \hline \bullet & \bullet \\ \hline \end{array} \quad \underset{\rho^{-1}_{(2,1,2)}}{\overset{\rho_{(2,1,2)}}{\rightleftarrows}} \quad \begin{array}{|c|c|c|c|c|}\hline 4 & 3 \\ \hline \bullet & 7 \\ \hline \bullet & \bullet & 6 & 2 \\ \hline \bullet & \bullet & 7 & 5 & 1 \\ \hline \end{array}$$

2.2 Symmetric functions and Schur functions

The algebra of symmetric functions, denoted by **Sym**, is the algebra freely generated over $\mathbb{Q}$ by countably many commuting variables $\{h_1, h_2, \ldots\}$. Assigning degree i to h_i (and then extending this multiplicatively) allows us to endow **Sym** with a structure of a graded algebra. A basis for the degree n component of **Sym**, denoted by $\mathbf{Sym}^n$, is given by the *complete homogeneous symmetric functions* of degree n,

$$\{h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k} : \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n\}.$$

A concrete realization of **Sym** is obtained by embedding **Sym** in $\mathbb{Q}[[x_1, x_2, \ldots]]$, that is, the ring of formal power series in countably many commuting indeterminates $x_1, x_2, \ldots$, under the identification (extended multiplicatively)

$$h_i \longmapsto \text{ sum of all distinct monomials in } x_1, x_2, \ldots \text{ of degree } i.$$

This viewpoint allows us to think of symmetric functions as being certain formal power series f in the x variables with the property that $f(x_{\pi(1)}, x_{\pi(2)}, \ldots) = f(x_1, x_2, \ldots)$ for every permutation π of the positive integers $\mathbb{N}$.

Definition 2.2.1. *The* Schur function *indexed by the partition λ, denoted by s_λ, is defined combinatorially as*

$$s_\lambda = \sum_{T \in \mathrm{SSRT}(\lambda)} x^T.$$

Moreover, we define $s_\varnothing = 1$.

Though not evident from the definition above, s_λ is a symmetric function [38, Proposition 4.4.2]. Furthermore, the elements of the set $\{s_\lambda : \lambda \vdash n\}$ form a basis of $\mathbf{Sym}^n$ for any positive integer n.

Example 2.2.2. *We obtain the expansion*

$$s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + 2x_1x_2x_3 + \cdots$$

from the tableaux below.

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array}\quad \begin{array}{|c|}\hline 1 \\ \hline \end{array}\quad \begin{array}{|c|}\hline 1 \\ \hline \end{array}\quad \begin{array}{|c|}\hline 2 \\ \hline \end{array}$$
$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline \end{array}\quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline \end{array}\quad \cdots$$

Now we will give the reader a hint of the importance of the Schur basis in the representation theory of symmetric groups.

2.2.1 The Murnaghan-Nakayama rule

Another important class of symmetric functions is given by the *power sum symmetric functions.* The power sum symmetric function p_k for $k \geq 1$ is defined as

$$p_k = \sum_{i \geq 1} x_i^k.$$

The set $\{p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k} \mid \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n\}$ also forms a basis for $\mathbf{Sym}^n$ for any positive integer n. The classical Murnaghan-Nakayama rule gives an algorithm to compute the product $p_k \cdot s_\mu$ in terms of Schur functions. To state the rule, we need the notion of a border strip. A skew shape λ/μ is called a *border strip* if it is connected and contains no 2×2 array of boxes. The *height* of a border strip λ/μ, denoted by $ht(\lambda/\mu)$, is defined to be one less than the number of rows occupied by the border strip.

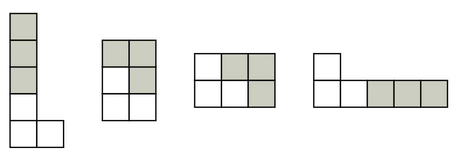
Theorem 2.2.3 (Murnaghan-Nakayama rule)**.** *[42, Theorem 7.17.1] Given a positive integer k and $\mu \vdash n$, we have*

$$p_k \cdot s_\mu = \sum_{\lambda \vdash |\mu|+k} (-1)^{ht(\lambda/\mu)} s_\lambda,$$

where the sum is over all partitions λ such that λ/μ is a border strip of size k.

As an aid to understanding the theorem above, we will do an example.

Example 2.2.4. *Consider the computation of $p_3 \cdot s_{(2,1)}$. All partitions λ such that $\lambda/(2,1)$ is a border strip of size 3 are listed below.*



The statistic $ht(\lambda/(2,1))$ for the partitions above from left to right is $2, 1, 1, 0$ respectively. Hence, the Murnaghan-Nakayama rule yields that

$$p_3 \cdot s_{(2,1)} = s_{(2,1,1,1,1)} - s_{(2,2,2)} - s_{(3,3)} + s_{(5,1)}.$$

We can use Theorem 2.2.3 iteratively to expand a power sum symmetric function indexed by a partition in the basis of Schur functions, and combinatorially compute the structure coefficients that arise in doing so. These structure coefficients are in fact the character values of the symmetric group [42, Corollary 7.17.4].

A noncommutative analogue of Theorem 2.2.3 will be the main result in Chapter 3. Next we discuss an important combinatorial procedure whose noncommutative analogue will be our focus in Chapter 4.

2.2.2 Jeu de taquin slides on SSRTs

In this subsection, we will describe backward jeu de taquin slides in the setting of SSRTs. The exposition here follows that in [38] analogously.

Let $\lambda \vdash n$ and $T \in \mathrm{SSRT}(\lambda)$. Given $i_0, j_0 \geq 1$ such that λ has an addable node in position $c = (i_0, j_0)$, a *backward jeu de taquin slide* on T starting from c gives an SSRT $\mathrm{jdt}_{j_0}(T)$ in the manner outlined below.

1. Let $c = (i_0, j_0)$.

2. **While** c is not equal to $(1, 1)$ **do**

   - If $c = (i, j)$, then let c' be the box of $\min(T_{(i-1,j)}, T_{(i,j-1)})$. If only one of $T_{(i-1,j)}$ and $T_{(i,j-1)}$ exists then the minimum is taken to be that single value. Furthermore, if $T_{(i-1,j)} = T_{(i,j-1)}$, then $c' = (i-1, j)$.
   - Slide $T_{c'}$ into position c. Let $c := c'$.

3. The final skew SSRT obtained is $\mathrm{jdt}_{j_0}(T)$.

We give an example demonstrating the above algorithm.

Example 2.2.5. *Let $\lambda = (3, 2, 2, 1)$ and let $T \in \mathrm{SSRT}(\lambda)$ be*

$$\begin{ytableau} 2 \\ 5 & 4 \\ 6 & 5 \\ 8 & 7 & 1 \end{ytableau}\ .$$

If we start a backward jeu de taquin slide from the addable node given by $c = (2, 3)$, then the algorithm executes in the following manner.

$$\begin{ytableau} 2 \\ 5 & 4 \\ 6 & 5 & \bullet \\ 8 & 7 & 1 \end{ytableau} \to \begin{ytableau} 2 \\ 5 & 4 \\ 6 & 5 & 1 \\ 8 & 7 & \bullet \end{ytableau} \to \begin{ytableau} 2 \\ 5 & 4 \\ 6 & 5 & 1 \\ 8 & \bullet & 7 \end{ytableau} \to \begin{ytableau} 2 \\ 5 & 4 \\ 6 & 5 & 1 \\ \bullet & 8 & 7 \end{ytableau}$$

Thus, we have that

$$\mathrm{jdt}_3(T) = \begin{ytableau} 2 \\ 5 & 4 \\ 6 & 5 & 1 \\ \none & 8 & 7 \end{ytableau}.$$

The above algorithm can be generalized naturally to SSRT of skew shape. Now let T denote an SSRT of skew shape λ/μ. Suppose $i_0, j_0 \geq 1$ are such that the position $c = (i_0, j_0)$ is an addable node of λ. Then $\mathrm{jdt}_{j_0}(T)$ is obtained as outlined below.

1. Let $c = (i_0, j_0)$.
2. **While** c is not an addable node of μ **do**
   - If $c = (i, j)$, then let c' be the box of $\min(T_{(i-1,j)}, T_{(i,j-1)})$. If only one of $T_{(i-1,j)}$ and $T_{(i,j-1)}$ exists then the minimum is taken to be that single value. Furthermore, if $T_{(i-1,j)} = T_{(i,j-1)}$, then $c' = (i-1, j)$.
   - Slide $T_{c'}$ into position c. Let $c := c'$.
3. The final skew SSRT obtained is $\mathrm{jdt}_{j_0}(T)$.

Example 2.2.6. *Consider the* SSRT T *of shape* $(5, 4, 3, 1)/(3, 2)$ *as shown below.*

$$\begin{array}{ccccc} 1 & & & & \\ 3 & 1 & 1 & & \\ \bullet & \bullet & 2 & 1 & \\ \bullet & \bullet & \bullet & 4 & 3 \end{array}$$

Suppose we want to compute $\mathrm{jdt}_4(T)$. *Then the following sequence of sliding moves occurs, where the shaded box denotes the box into which entries slide.*

$$\begin{array}{ccccc} 1 & & & & \\ 3 & 1 & 1 & \bullet & \\ \bullet & \bullet & 2 & 1 & \\ \bullet & \bullet & \bullet & 4 & 3 \end{array} \longrightarrow \begin{array}{ccccc} 1 & & & & \\ 3 & 1 & 1 & 1 & \\ \bullet & \bullet & 2 & \bullet & \\ \bullet & \bullet & \bullet & 4 & 3 \end{array} \longrightarrow \begin{array}{ccccc} 1 & & & & \\ 3 & 1 & 1 & 1 & \\ \bullet & \bullet & \bullet & 2 & \\ \bullet & \bullet & \bullet & 4 & 3 \end{array}$$

Thus, we have

$$\mathrm{jdt}_4(T) = \begin{array}{ccccc} 1 & & & & \\ 3 & 1 & 1 & 1 & \\ & & & 2 & \\ & & & 4 & 3 \end{array} .$$

For completeness, we will discuss the *forward jeu de taquin* slide next. It is the inverse of the backward jeu de taquin slide discussed above. Let T denote an SSRT of skew shape λ/μ. Suppose $i_0, j_0 \geq 1$ are such that the position $c = (i_0, j_0)$ is a removable node of μ. Then the SSRT obtained after a forward jeu de taquin slide is initiated from position c, which we denote by $\mathrm{jdt}^f_{j_0}(T)$, is obtained as outlined below.

1. Let $c = (i_0, j_0)$.

2. **While** c is not an addable node of λ **do**

   - If $c = (i, j)$, then let c' be the box of $\max(T_{(i+1,j)}, T_{(i,j+1)})$. If only one of $T_{(i+1,j)}$ and $T_{(i,j+1)}$ exists then the maximum is taken to be that single value. Furthermore, if $T_{(i+1,j)} = T_{(i,j+1)}$, then $c' = (i + 1, j)$.
   - Slide $T_{c'}$ into position c. Let $c := c'$.

3. The final SSRT obtained is $\mathrm{jdt}^f_{j_0}(T)$.

Example 2.2.7. *Let T denote the* SSRT *below.*

$$\begin{array}{|c|c|c|c|c|} \hline 1 \\ \hline 3 & 1 & 1 & 1 \\ \hline \bullet & \bullet & \bullet & 2 \\ \hline \bullet & \bullet & \bullet & 4 & 3 \\ \hline \end{array}$$

Then $\mathrm{jdt}^f_3(T)$ is obtained by reversing the steps in Example 2.2.6 and we get that

$$\mathrm{jdt}^f_3(T) = \begin{array}{|c|c|c|c|c|} \hline 1 \\ \hline 3 & 1 & 1 \\ \hline \bullet & \bullet & 2 & 1 \\ \hline \bullet & \bullet & \bullet & 4 & 3 \\ \hline \end{array}\ .$$

2.2.3 Rectification of SRTs

A procedure that is closely tied to the jeu de taquin slide is rectification. To describe it in a way that is convenient for us, we need to introduce a slight variant of the Robinson-Schensted algorithm.

A discussion of the original algorithm is present in Section 4.5. Our variant establishes a bijection between permutations in $\mathfrak{S}_n$ and pairs of SRTs of the same shape. We will need the notion of a *partial tableau* that we define to be an SSRT with distinct entries. Furthemore, we will abuse notation and use $\varnothing$ to denote the empty tableau.

We will outline the algorithm next. Our input is a permutation $\sigma \in \mathfrak{S}_n$, and the output is a pair of SRTs of the same shape and size n.

Algorithm 2.2.8 (Variant of Robinson-Schensted insertion algorithm)**.**

1. Let $P_0 = Q_0 = \varnothing$.

2. ***For*** *$i = 1$ to n, let $y := \sigma(i)$ and* ***do***

   (a) *Set $r :=$ first row of P_{i-1}.*

   (b) ***While*** *y is greater than some number in row r,* ***do***

   - *Let x be the greatest number smaller than y and replace x by y in P_{i-1}.*
   - *Set $y := x$ and $r :=$ the next row.*

   (c) *Now that y is smaller than every number in row r, place y at the end of row r and call the resulting partial tableau P_i. If (a, b) is where y was placed, the partial tableau Q_i is obtained by placing $n - i + 1$ in position (a, b) in Q_{i-1}.*

3. *Finally, set $P := P_n$ and $Q := Q_n$.*

The SRTs P and Q thus obtained are called the *insertion tableau* and *recording tableau* respectively. We present an example next.

Example 2.2.9. *Consider the permutation $\sigma = 3157624$ in one line notation. Then the insertion algorithm executes as shown below, where at each stage we note the pair (P_i, Q_i).*

$$(\varnothing, \varnothing) \mapsto \left(\begin{array}{|c|}\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 7\\\hline\end{array}\right) \mapsto \left(\begin{array}{|c|c|}\hline 3&1\\\hline\end{array}, \begin{array}{|c|c|}\hline 7&6\\\hline\end{array}\right) \mapsto \left(\begin{array}{cc} 3 & \\ 5 & 1 \end{array}, \begin{array}{cc} 5 & \\ 7 & 6 \end{array}\right) \mapsto \left(\begin{array}{cc} 3 & \\ 5 & \\ 7 & 1 \end{array}, \begin{array}{cc} 4 & \\ 5 & \\ 7 & 6 \end{array}\right) \mapsto \left(\begin{array}{cc} 3 & \\ 5 & 1 \\ 7 & 6 \end{array}, \begin{array}{cc} 4 & \\ 5 & 3 \\ 7 & 6 \end{array}\right)$$

$$\mapsto \left(\begin{array}{ccc} 3 & & \\ 5 & 1 & \\ 7 & 6 & 2 \end{array}, \begin{array}{ccc} 4 & & \\ 5 & 3 & \\ 7 & 6 & 2 \end{array}\right) \mapsto \left(\begin{array}{ccc} 3 & 1 & \\ 5 & 2 & \\ 7 & 6 & 4 \end{array}, \begin{array}{ccc} 4 & 1 & \\ 5 & 3 & \\ 7 & 6 & 2 \end{array}\right)$$

Recall that given an SRT T, its column reading word is w_T. Furthermore, w_T is a permutation written in single line notation. The *rectification* of T, denoted $\mathrm{rect}(T)$ is defined to be $P(w_T)$, that is, the insertion tableau obtained on performing the above variant of Robinson-Schensted algorithm on w_T.

Example 2.2.10. *Consider the* SRT *T of skew shape given below.*

$$\begin{array}{cccc} 3 & 1 & & \\ & 5 & & \\ & 7 & 6 & 2 \\ & & & 4 \end{array}$$

Then the column reading word is 3 157 6 24*. The insertion tableau for this reading word has* *already been computed in Example 2.2.9, and thus* $\operatorname{rect}(T)$ *is as follows.*

$$\begin{ytableau} 3 & 1 \\ 5 & 2 \\ 7 & 6 & 4 \end{ytableau}$$

An important property of rectification that will prove crucial for us is the fact that it behaves nicely with respect to jeu de taquin slides. More precisely, if T' is some SRT obtained by performing a backward jeu de taquin slide on some SRT T, then $\operatorname{rect}(T) = \operatorname{rect}(T')$ [38, Theorem 3.7.7]. Rectification also allows us to describe another important combinatorial procedure called evacuation which we describe next.

Given an SRT T, the *evacuation* action on T, denoted by $e(T)$, is obtained using the algorithm below.

Algorithm 2.2.11 (Evacuation)**.**

1. *Let U be a Young diagram of shape λ, where $\lambda = sh(T) \vdash n$.*

2. *Let the entry in the box in position $(1,1)$ in T be a. Remove this entry, and compute the rectification of the remaining* SRT *of skew shape. Equivalently, one can remove the entry in the box in position $(1,1)$ in T and let T' be the resulting skew* SRT*, and compute* $\operatorname{jdt}_1^f(T')$. *This viewpoint makes it clear that the underlying shape of the rectified tableau is obtained by removing a removable node from λ. Let this node be in position (i,j).*

3. *To the box given by (i,j) in U, assign the entry $n - a + 1$.*

4. *Repeat the previous two steps until all the boxes in U are filled. Finally, $e(T) = U$.*

Example 2.2.12. *Consider the insertion tableau from Example 2.2.9. We demonstrate Algorithm 2.2.11 by drawing the tableau T and the partially filled U at each step.*

$$\begin{ytableau} 3 & 1 \\ 5 & 2 \\ 7 & 6 & 4 \end{ytableau}, \begin{ytableau} \; & \; \\ \; & \; \\ \; & \; & \; \end{ytableau} \mapsto \begin{ytableau} 3 & 1 \\ 5 & 2 \\ 6 & 4 \end{ytableau}, \begin{ytableau} \; & \; \\ \; & \; \\ \; & \; & 1 \end{ytableau} \mapsto \begin{ytableau} 1 \\ 3 & 2 \\ 5 & 4 \end{ytableau}, \begin{ytableau} \; & 2 \\ \; & \; \\ \; & \; & 1 \end{ytableau} \mapsto \begin{ytableau} 1 \\ 3 \\ 4 & 2 \end{ytableau}, \begin{ytableau} \; & 2 \\ \; & 3 \\ \; & \; & 1 \end{ytableau} \mapsto \begin{ytableau} 1 \\ 3 & 2 \end{ytableau}, \begin{ytableau} 4 & 2 \\ \; & 3 \\ \; & \; & 1 \end{ytableau}$$

$$\mapsto \begin{ytableau} 1 \\ 2 \end{ytableau}, \begin{ytableau} 4 & 2 \\ \; & 3 \\ \; & 5 & 1 \end{ytableau} \mapsto \begin{ytableau} 1 \end{ytableau}, \begin{ytableau} 4 & 2 \\ 6 & 3 \\ \; & 5 & 1 \end{ytableau} \mapsto \varnothing, \begin{ytableau} 4 & 2 \\ 6 & 3 \\ 7 & 5 & 1 \end{ytableau}$$

Thus, $e(T)$ is the SRT *obtained in the last step above.*

Next, we will note down a property of our variant of the Robinson-Schensted algorithm. The proof is presented in Section 4.5.

Lemma 2.2.13. *Let $w \in \mathfrak{S}_n$ and suppose that $w \mapsto (P, Q)$ using the variant of Robinson-Schensted algorithm outlined in Algorithm 2.2.8. Then*

$$w^{-1} \mapsto (e(Q), e(P)).$$

2.2.4 The classical Pieri rules

As an application of the concepts in the previous subsection we obtain the classical Pieri rules. These rules give an explicit description for the products $s_{(n)} \cdot s_\mu$ and $s_{(1,\dots,1)} \cdot s_\mu$ in terms of Schur functions. They are a special case of the Littlewood-Richardson rule which is a combinatorial rule for expressing the product of two Schur functions as a sum of Schur functions,

Theorem 2.2.14 (Pieri rules)**.** *[42, Theorem 7.15.7] Let μ be a partition and n be a positive integer. Let (1^n) denote the partition with n parts all equalling 1. Then we have the following expansions.*

$$s_{(n)} \cdot s_\mu = \sum_{\substack{\lambda/\mu \text{ a horizontal strip} \\ |\lambda/\mu| = n}} s_\lambda$$

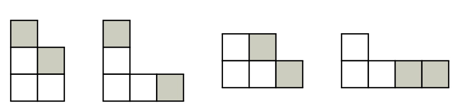
$$s_{(1^n)} \cdot s_\mu = \sum_{\substack{\lambda/\mu \text{ a vertical strip} \\ |\lambda/\mu| = n}} s_\lambda$$

We consider an example next.

Example 2.2.15. *Let $\lambda = (2, 1)$ and $n = 2$. Then we obtain the expansion*

$$s_{(2)} \cdot s_{(2,1)} = s_{(2,2,1)} + s_{(3,1,1)} + s_{(3,2)} + s_{(4,1)}$$

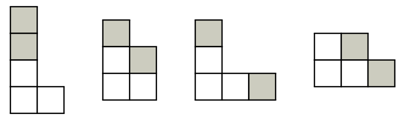
from the following horizontal strips (shown shaded).



Similarly, we obtain the expansion

$$s_{(1,1)} \cdot s_{(2,1)} = s_{(2,1,1,1)} + s_{(2,2,1)} + s_{(3,1,1)} + s_{(3,2)}$$

from the vertical strips below (shown shaded).



In Chapter 4, we will obtain a noncommutative analogue of Theorem 2.2.14 above.

2.3 Quasisymmetric functions and quasisymmetric Schur functions

We will now introduce an important Hopf algebra of which **Sym** is a Hopf subalgebra. This is the algebra of quasisymmetric functions, referred to as **QSym**, and was originally defined by Gessel [14].

A formal power series $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ is said to be a *quasisymmetric function* if it satisfies the following two conditions.

1. The degrees of the monomials occurring in f are bounded.
2. For all compositions $(\alpha_1, \ldots, \alpha_k)$, all monomials $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ in f with $i_1 < i_2 < \cdots < i_k$ have the same coefficient.

It can be easily verified that the product of two quasisymmetric functions is quasisymmetric. Hence the set of quasisymmetric functions can be endowed with an algebra structure. It is this algebra that we refer to as **QSym**. Just as in the case of **Sym**, we have that **QSym** is a graded algebra, where the degree gives the grading. We denote the degree n component by $\mathbf{QSym}^n$.

QSym has a host of interesting bases. We will first introduce the basis of fundamental quasisymmetric functions and then use it to define the basis of quasisymmetric Schur functions.

Let $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$. Then the *fundamental quasisymmetric function* F_α is defined by

$$F_\alpha = \sum x_{i_1} \cdots x_{i_n}$$

where the sum is over all $1 \leq i_1 \leq \cdots \leq i_n$ and $i_j < i_{j+1}$ if $j \in \mathrm{set}(\alpha)$. Furthermore, we define $F_\varnothing$ to be 1. Additionally we have that $\mathbf{QSym}^n = \mathrm{span}\{F_\alpha \mid \alpha \vDash n\}$, which immediately yields that $\dim(\mathbf{QSym}^n) = 2^{n-1}$.

Example 2.3.1. *Let* $\alpha = (1, 2)$*. Then* $\mathrm{set}(\alpha) = \{1\}$*. We obtain the following expansion for* $F_{(1,2)}$*.*

$$F_{(1,2)} = x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + \cdots + x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 + \cdots$$

Now we are set to define quasisymmetric Schur functions. We will give two equivalent definitions: one involving sum of monomials associated with SSRCTs (akin to our combinatorial definition of Schur functions earlier), and the other an expansion in the basis of fundamental quasisymmetric functions.

Definition 2.3.2. *Given a composition* α*, the* quasisymmetric Schur function *indexed by* α*, denoted by* $\mathcal{S}_\alpha$ *is defined by*

$$\mathcal{S}_\alpha = \sum_{\tau \in \mathrm{SSRCT}(\alpha)} x^\tau = \sum_{\tau \in \mathrm{SRCT}(\alpha)} d_{\alpha\beta} F_\beta$$

where $d_{\alpha\beta}$ *denotes the number of* SRCT*s of shape* α *and descent composition* β*. Furthermore, we define* $\mathcal{S}_\varnothing$ *to be* 1*.*

Example 2.3.3. *Let* $\alpha = (1, 2)$*. Then we obtain the following expansion*

$$\begin{aligned}\mathcal{S}_{(1,2)} =& x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + x_1x_2x_3 + x_1x_2x_4 \cdots \\ =& F_{(1,2)}.\end{aligned}$$

from the following list of SSRCT*s.*

$$\begin{array}{|c|c} \hline 1 & \\ \hline 2 & \multicolumn{1}{|c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 & \\ \hline 3 & \multicolumn{1}{|c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 2 & \\ \hline 3 & \multicolumn{1}{|c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 & \\ \hline 3 & \multicolumn{1}{|c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 1 & \\ \hline 4 & \multicolumn{1}{|c|}{2} \\ \hline \end{array} \cdots$$

The expansion in fundamental quasisymmetric functions is obtained from the only SRCT *of shape* $(1, 2)$ *(the one with shaded boxes).*

As an additional example, consider $\alpha = (3, 2)$*. Then we obtain the following expansion*

$$\mathcal{S}_{(3,2)} = F_{(3,2)} + F_{(1,3,1)} + F_{(2,2,1)}$$

from the following list of SRCT*s.*

$$\begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 4 & 3 & 2 \\ \hline 5 & 1 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 5 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array}$$

In addition to Definition 2.3.2, some of the fundamental properties of $\mathcal{S}_\alpha$ that were proved in [17] were the following.

1. $\mathcal{S}_\alpha \in \mathbf{QSym}$.

2. $\mathbf{QSym}^n = \operatorname{span}\{\mathcal{S}_\alpha \mid \alpha \vDash n\}$.

3. $s_\lambda = \sum_{\widetilde{\alpha}=\lambda} \mathcal{S}_\alpha$.

The third point above explains the reasoning behind calling the functions quasisymmetric Schur functions. Analogues of the classical Littlewood-Richardson rule and Pieri rules have also been found (see [4, 17, 18]), casting new light on classical results and giving rise to new avenues. The Hopf algebra dual to **QSym** is equally interesting, and we will introduce it next.

2.4 Noncommutative symmetric functions and noncommutative Schur functions

An algebra closely related to **Sym** and **QSym** is the algebra of *noncommutative symmetric functions* **NSym**, introduced in [13]. The algebra **NSym** is the free associative algebra $\mathbb{Q}\langle \mathbf{h}_1, \mathbf{h}_2, \ldots \rangle$ generated by a countably infinite number of indeterminates $\mathbf{h}_k$ for $k \geq 1$. Assigning degree k to $\mathbf{h}_k$, and extending this multiplicatively allows us to endow **NSym** with a structure of a graded algebra. A natural basis for the degree n graded component of **NSym**, denoted by $\mathbf{NSym}^n$, is given by the *noncommutative complete homogeneous symmetric functions*, $\{\mathbf{h}_\alpha = \mathbf{h}_{\alpha_1} \cdots \mathbf{h}_{\alpha_k} : \alpha = (\alpha_1, \ldots, \alpha_k) \vDash n\}$. The link between **Sym** and **NSym** is made manifest through the *forgetful* map, $\chi : \mathbf{NSym} \to \mathbf{Sym}$, defined by mapping $\mathbf{h}_i$ to h_i and extending multiplicatively. Thus, the images of elements of **NSym** under χ are elements of **Sym**, imparting credibility to the term noncommutative symmetric function.

NSym has another important basis called the noncommutative ribbon Schur basis. We will denote the *noncommutative ribbon Schur function* indexed by a composition β by $\mathbf{r}_\beta$. The following [13, Proposition 4.13] can be taken as the definition of the noncommutative ribbon Schur functions.

$$\mathbf{r}_\beta \;=\; \sum_{\alpha \succcurlyeq \beta} (-1)^{l(\beta)-l(\alpha)} \mathbf{h}_\alpha$$

A multiplication rule for noncommutative ribbon Schur functions was proved in [13], and we will be needing it later.

Theorem 2.4.1 (Proposition 3.13, [13]). *Let $\alpha = (\alpha_1, \ldots, \alpha_{k_1})$ and $\beta = (\beta_1, \ldots, \beta_{k_2})$ be compositions. Define two new compositions γ and μ as follows*

$$\gamma = (\alpha_1, \ldots, \alpha_{k_1}, \beta_1, \ldots, \beta_{k_2}), \;\; \delta = (\alpha_1, \ldots, \alpha_{k_1} + \beta_1, \beta_2, \ldots, \beta_{k_2}).$$

Then

$$\mathbf{r}_\alpha \cdot \mathbf{r}_\beta = \mathbf{r}_\gamma + \mathbf{r}_\delta.$$

We will also need a noncommutative analogue of the power sum symmetric functions. In [13] they have defined two such analogues, the Φ basis and the Ψ basis. Our interest is in the Ψ basis. For a positive integer n, define

$$\Psi_n = \sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k)}. \tag{2.1}$$

This given, for a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, we define Ψ_α multiplicatively as $\Psi_{\alpha_1} \cdots \Psi_{\alpha_k}$. It can be shown that $\chi(\Psi_n) = p_n$.

While the transition matrix between the Ψ basis and $\mathbf{r}$ basis is discussed in [13, Section 4.8], no explicit rule for $\Psi_n \cdot \mathbf{r}_\alpha$ is given. Since $\mathbf{r}_\alpha$ has a representation theoretic meaning [24], this product may also be considered a type of Murnaghan-Nakayama rule and hence we will give a rule to compute $\Psi_n \cdot \mathbf{r}_\alpha$ where n is a positive integer and α is a composition.

Lemma 2.4.2. *Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a composition and let n be a positive integer. Then*

$$\Psi_n \cdot \mathbf{r}_\alpha = \sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k, \alpha_1, \ldots, \alpha_m)} + \sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k+\alpha_1, \alpha_2, \ldots, \alpha_m)}.$$

Furthermore, the summands on the right are all distinct, that is, there is no cancellation in the sum on the right.

Proof. From (2.1) and the multiplication rule in Theorem 2.4.1, it follows that

$$\Psi_n \cdot \mathbf{r}_\alpha = \sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k, \alpha_1, \ldots, \alpha_m)} + \sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k+\alpha_1, \alpha_2, \ldots, \alpha_m)}.$$

Hence we only need to establish that the summands above are all distinct. Consider the following multisets of compositions.

$$\begin{aligned} X_1 =& \{(1^k, n-k, \alpha_1, \ldots, \alpha_m) \mid 0 \leq k \leq n-1\} \\ X_2 =& \{(1^k, n-k+\alpha_1, \alpha_2, \ldots, \alpha_m) \mid 0 \leq k \leq n-1\} \end{aligned}$$

By considering the lengths of the compositions therein, it follows that the elements in X_1 (and X_2) are all distinct. Hence, to establish that there is no cancellation, we only need to show that the set $X_1 \cap X_2$ is empty. Assume otherwise. Then there exist positive integers $0 \leq a, b \leq n-1$ such that

$$(1^a, n-a, \alpha_1, \ldots, \alpha_m) = (1^b, n-b+\alpha_1, \alpha_2, \ldots, \alpha_m).$$

That the lengths of the two compositions above have to be equal implies that $a+1=b$. But then the equality of compositions above implies that $n-a=1$, which in turn yields that $b=n$. But we know that $0 \leq b \leq n-1$. Hence we have a contradiction, and the claim follows. $\square$

Example 2.4.3. *Lemma 2.4.2 gives the following expansion for* $\Psi_{(3)} \cdot \mathbf{r}_{(1,3,2)}$.

$$\Psi_{(3)} \cdot \mathbf{r}_{(1,3,2)} = (\mathbf{r}_{(3,1,3,2)}) - \mathbf{r}_{(1,2,1,3,2)} + \mathbf{r}_{(1,1,1,1,3,2)}) + (\mathbf{r}_{(4,3,2)} - \mathbf{r}_{(1,3,3,2)} + \mathbf{r}_{(1,1,2,3,2)})$$

Now that we have defined a noncommutative analogue of power sum symmetric functions, to state a noncommutative Murnaghan-Nakayama rule, we need to define our analogue of the Schur functions in **NSym**. The next subsection achieves that aim.

2.4.1 Noncommutative Schur functions

We will now describe a distinguished basis for **NSym**, introduced in [4], called the basis of *noncommutative Schur functions*. They are naturally indexed by compositions, and the noncommutative Schur function indexed by a composition α will be denoted by $\mathbf{s}_\alpha$.

Definition 2.4.4. *The noncommutative Schur functions* $\mathbf{s}_\alpha$ *for* $\alpha \vDash n$ *are defined implicitly using the relation*

$$\mathbf{r}_\beta = \sum_{\alpha \vDash |\beta|} d_{\alpha\beta} \mathbf{s}_\alpha \tag{2.2}$$

where $d_{\alpha\beta}$ *is the number of* SRCT*s of shape* α *and descent composition* β.

Remark 2.4.5. *The original definition of noncommutative Schur functions in [4, Section 2.4] utilized the duality between the Hopf algebras* **NSym** *and* **QSym***. The basis of noncommutative Schur functions is defined to be the dual basis corresponding to the basis of quasisymmetric Schur functions. The relation* (2.2) *is obtained by considering the dual version of the expansion of a quasisymmetric Schur function in terms of the fundamental quasisymmetric functions in Definition 2.3.2.*

The noncommutative Schur function $\mathbf{s}_\alpha$ satisfies the following important property [4, Equation 2.12].

$$\chi(\mathbf{s}_\alpha) = s_{\widetilde{\alpha}}$$

Thus, the noncommutative Schur functions are lifts of the Schur functions in **NSym**. They share many properties with the Schur functions. The interested reader should refer to [4, 29] for an in-depth study of these functions. For our purposes, we require the noncommutative Littlewood-Richardson rule proved in [4].

Theorem 2.4.6. *[4, Theorem 3.5] Given compositions α and β, the product $\mathbf{s}_\alpha \cdot \mathbf{s}_\beta$ can be expanded in the basis of noncommutative Schur functions as*

$$\mathbf{s}_\alpha \cdot \mathbf{s}_\beta = \sum_\gamma C^\gamma_{\alpha\beta} \mathbf{s}_\gamma$$

where $C^\gamma_{\alpha\beta}$ counts the number of SRCT*s of shape $\gamma /\!\!/ \beta$ that rectify to the canonical tableau of shape α.*

We should clarify what it means to rectify for SRCTs. We say that an SRCT τ_1 of shape $\gamma /\!\!/ \beta$ rectifies to a tableau τ_2 of straight shape α if $\rho^{-1}(P(w_{\tau_1})) = \tau_2$.

As a corollary of the above result, we obtain the noncommutative Pieri rules for noncommutative Schur functions proved in [4], which we state below.

Theorem 2.4.7 (Left Pieri rules)**.** *[4, Corollary 3.8] Given a composition β, we have*

$$\mathbf{s}_{(n)} \cdot \mathbf{s}_\alpha = \sum_\gamma \mathbf{s}_\gamma$$

where the sum on the right runs over all compositions $\gamma >_c \alpha$ such that $|\gamma /\!\!/ \alpha| = n$ and $\gamma /\!\!/ \alpha$ is a horizontal strip where the boxes have been added in columns from left to right. Similarly,

$$\mathbf{s}_{(1^n)} \cdot \mathbf{s}_\alpha = \sum_\gamma \mathbf{s}_\gamma$$

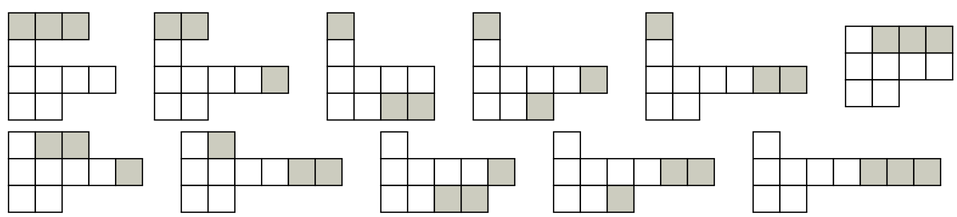
where the sum on the right runs over all compositions $\gamma >_c \alpha$ such that $|\gamma /\!\!/ \alpha| = n$ and $\gamma /\!\!/ \alpha$ is a vertical strip where the boxes have been added in columns from right to left.

The following example will illustrate the theorem above.

Example 2.4.8. *Let α be $(1,4,2)$ and $n = 3$. We compute the following expansion*

$$\begin{aligned}\mathbf{s}_{(3)} \cdot \mathbf{s}_{(1,4,2)} =& \mathbf{s}_{(3,1,4,2)} + \mathbf{s}_{(2,1,5,2)} + \mathbf{s}_{(1,1,4,4)} + \mathbf{s}_{(1,1,5,3)} + \mathbf{s}_{(1,1,6,2)} \\ &+ \mathbf{s}_{(4,4,2)} + \mathbf{s}_{(3,5,2)} + \mathbf{s}_{(2,6,2)} + \mathbf{s}_{(1,5,4)} + \mathbf{s}_{(1,6,3)} + \mathbf{s}_{(1,7,2)}\end{aligned}$$

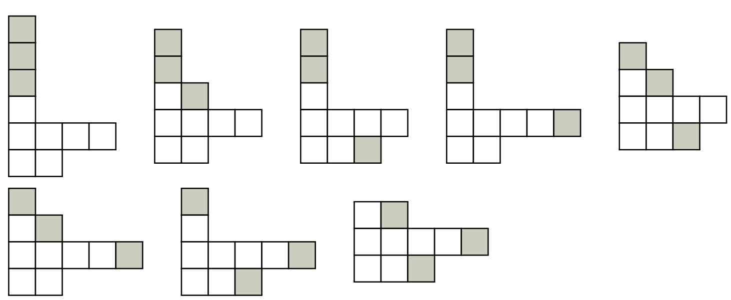
from the following horizontal strips (shown shaded).



Similarly, we compute the expansion

$$\begin{aligned}\mathbf{s}_{(1,1,1)} \cdot \mathbf{s}_{(1,4,2)} =& \mathbf{s}_{(1,1,1,1,4,2)} + \mathbf{s}_{(1,1,2,4,2)} + \mathbf{s}_{(1,1,1,4,3)} + \mathbf{s}_{(1,1,1,5,2)} \\ &+ \mathbf{s}_{(1,2,4,3)} + \mathbf{s}_{(1,2,5,2)} + \mathbf{s}_{(1,1,5,3)} + \mathbf{s}_{(2,5,3)}\end{aligned}$$

from the vertical strips below (shown shaded).



Next, we will state an easy equality that will prove useful in Chapter 3.

Lemma 2.4.9. *Let $n \geq 0$. Then we have that $\mathbf{s}_{(n)} = \mathbf{r}_{(n)}$ and $\mathbf{s}_{(1^n)} = \mathbf{r}_{(1^n)}$.*

Proof. By (2.2) we have that

$$\mathbf{r}_\beta = \sum_{\alpha \vDash n} d_{\alpha\beta} \mathbf{s}_\alpha.$$

where $d_{\alpha\beta}$ is the number of SRCTs of shape α and descent composition β. Now our claim follows once we note that if $\beta = (n)$ or $\beta = (1^n)$ then $d_{\alpha\beta} = \delta_{\alpha\beta}$. Here $\delta_{\alpha\beta}$ denotes the Kronecker delta function that equals 1 if $\alpha = \beta$ and 0 otherwise. $\square$

2.5 Operators on compositions

In this section, we will define certain operators on compositions that play a key role in this thesis. We begin by defining the box adding operators that act on compositions and add a box to a composition in accordance with the Pieri rules stated in Theorem 2.4.7.

2.5.1 Box adding operators

Given a positive integer i and a composition α, we define $\mathfrak{t}_i(\alpha)$ in the following manner.

$$\mathfrak{t}_i(\alpha) = \begin{cases} \beta & \text{if } \alpha \lessdot_c \beta \text{ and } \beta /\!\!/ \alpha \text{ is a box in the } i\text{-th column} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear from the above definition that $\mathfrak{t}_i(\alpha)$ is nonzero if and only if α has a part equal to $i-1$.

Example 2.5.1. *Let $\alpha = (2,1,3,1)$. Then $\mathfrak{t}_1(\alpha) = (1,2,1,3,1)$, $\mathfrak{t}_2(\alpha) = (2,2,3,1)$ and $\mathfrak{t}_5(\alpha) = 0$.*

Next we establish a useful relation satisfied by box adding operators.

Lemma 2.5.2. *Given positive integers i, j such that $|i-j| \geq 2$, we have the following equality as operators on compositions.*

$$\mathfrak{t}_i\mathfrak{t}_j = \mathfrak{t}_j\mathfrak{t}_i$$

Proof. We will assume, without loss of generality that $i < j$. Consider first the case where $i = 1$ and $j \geq 3$. If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition, and no part of α equals $j-1$, then we have $\mathfrak{t}_1\mathfrak{t}_j(\alpha) = \mathfrak{t}_j\mathfrak{t}_1(\alpha) = 0$. If, on the other hand, there is a part of α that equals $j-1$, pick the leftmost such part. Let this part be α_r. Then $\mathfrak{t}_1\mathfrak{t}_j(\alpha) = (1, \alpha_1, \ldots, \alpha_r+1, \ldots, \alpha_k) = \mathfrak{t}_j\mathfrak{t}_1(\alpha)$.

Now consider the case where $i \geq 2$. Clearly, if either $i-1$ or $j-1$ is not a part of α, we have $\mathfrak{t}_i\mathfrak{t}_j(\alpha) = \mathfrak{t}_j\mathfrak{t}_i(\alpha) = 0$. If both $i-1$ and $j-1$ are parts of α, let α_r and α_s be their leftmost occurrences in α respectively. It is easily seen that the application of either $\mathfrak{t}_i$ first or $\mathfrak{t}_j$ first does not change the position of the leftmost instance of a part equalling $j-1$ or $i-1$ in α respectively. Hence $\mathfrak{t}_i\mathfrak{t}_j = \mathfrak{t}_j\mathfrak{t}_i$ for $|i-j| \geq 2$. $\square$

2.5.2 Box removing operators

The *box removing operators* on compositions, denoted by $\mathfrak{d}_i$ for $i \geq 1$, have the following description: $\mathfrak{d}_i(\alpha) = \alpha'$ where α' is the composition obtained by subtracting 1 from the rightmost part of length i in α and omitting any 0s that might result in so doing. If there is no such part, then $\mathfrak{d}_i(\alpha) = 0$.

Example 2.5.3. *Let* $\alpha = (2, 1, 2)$*. Then* $\mathfrak{d}_1(\alpha) = (2, 2)$*,* $\mathfrak{d}_2(\alpha) = (2, 1, 1)$*, and* $\mathfrak{d}_3(\alpha) = 0$*.*

Given a nonempty set of positive integers $S = \{b_1 < b_2 < \cdots < b_r\}$, we will also be needing the following product of box removing operators.

$$\mathfrak{v}_S = \mathfrak{d}_{b_1}\mathfrak{d}_{b_2}\cdots\mathfrak{d}_{b_r}$$

Note that the product above is not commutative, and we will consider $\mathfrak{v}_S$ as operators $\mathfrak{d}_{b_r}$, $\mathfrak{d}_{b_{r-1}}$, …, $\mathfrak{d}_{b_1}$ applied in succession to a composition, in that order. As a matter of convention, $\mathfrak{v}_\varnothing$ is to be interpreted as the identity map. Of particular interest to us is the operator $\mathfrak{v}_{[i]}$ for $i \geq 1$, where $[i] = \{1, \ldots, i\}$. We define $[0]$ to be the empty set $\varnothing$. Then $\mathfrak{v}_{[0]}$ is the identity map.

2.5.3 Jeu de taquin operators

Next, define an operator a_i that appends a part of length i to a composition α at the end, for $i \geq 1$. For example, $a_2((2, 1, 3)) = (2, 1, 3, 2)$. Notice also that $a_j\mathfrak{d}_2((3, 5, 1)) = 0$ for any positive integer j, as $\mathfrak{d}_2((3, 5, 1)) = 0$.

With the definitions of a_i and $\mathfrak{v}_{[i]}$, we can define the *jeu de taquin operators* (henceforth abbreviated to *jdt operators*), $\mathfrak{u}_i$ for $i \geq 1$, as follows.

$$\mathfrak{u}_i = a_i\mathfrak{v}_{[i-1]}$$

We will be using the jdt operators for studying the change in underlying shape when applying backward jeu de taquin slides on SSRCTs. Additionally, the role played by the box adding operators in the left Pieri rule stated in the previous section, is played by these operators in the right Pieri rules in Chapter 4.

Example 2.5.4. *We will compute* $\mathfrak{u}_4(\alpha)$ *where* $\alpha = (6,3,2,3,1,5,4)$*. Firstly,* $\mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_3(\alpha) = (6,3,2,1,5,4)$*, and thus* $a_4\mathfrak{v}_{[3]}(\alpha) = (6,3,2,1,5,4,4)$*.*

Remark 2.5.5. *Let* α *be a composition and* $\lambda = \widetilde{\alpha}$*. Let* $\beta = \mathfrak{u}_i(\alpha)$ *for some positive integer* i*. Then* $\widetilde{\beta}$ *is the partition obtained by adding a box at the addable node to* λ *in column* i*.*

2.6 Statement of main results

After setting up the necessary notation, we will state our main results in the next two subsections.

2.6.1 A noncommutative Murnaghan-Nakayama rule

Let α and β be compositions such that $\alpha <_c \beta$. Define the *support*, $\mathrm{supp}(\beta /\!\!/ \alpha)$, as follows.

$$\mathrm{supp}(\beta /\!\!/ \alpha) = \{j : (i,j) \in \beta /\!\!/ \alpha\}$$

The skew reverse composition shape $\beta /\!\!/ \alpha$ whose support is an interval in $\mathbb{Z}_{\geq 1}$ will be called an *interval shape*. Finally, an interval shape $\beta /\!\!/ \alpha$ will be called an *nc border strip* if it satisfies the following conditions.

1. If there exist boxes in positions $(i,1)$ and $(i,2)$ in $\beta /\!\!/ \alpha$, then the box in position $(i,1)$ is the bottommost box in column 1 of $\beta /\!\!/ \alpha$.

2. If there exist boxes in positions (i,j) and $(i,j+1)$ in $\beta /\!\!/ \alpha$ for $j \geq 2$, then the box in position (i,j) is the topmost box in column j of $\beta /\!\!/ \alpha$.

Given an nc border strip $\beta /\!\!/ \alpha$, its *height* $ht(\beta /\!\!/ \alpha)$ is defined to be one less than the number of rows it occupies, that is, one less than the number of rows that contain a non-zero number of boxes. An nc border strip can be seen in Example 2.6.1.

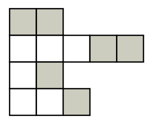
Suppose we are given an interval shape $\beta /\!\!/ \alpha$. We discuss three notions that have to do with the relative positions of boxes in consecutive columns of $\beta /\!\!/ \alpha$. We say that column $j+1$ is *south-east* of column j in $\beta /\!\!/ \alpha$ if there is a box in column $j+1$ that is strictly south-east of a box in column j. We say that column $j+1$ is *north-east* of column j, if for any pair of boxes $(i_1, j+1)$ and (i_2, j) in $\beta /\!\!/ \alpha$, we have $i_1 < i_2$. Here, we assume that there exists at least one box in the $j+1$-th column. Finally, we say that column $j+1$ is *east* of column j if there are boxes in positions (i,j) and $(i,j+1)$ for some i.

This given, we associate three statistics with an interval shape $\beta /\!\!/ \alpha$ as follows.

$$\begin{aligned} E(\beta /\!\!/ \alpha) &= \{j : \text{column } j+1 \text{ is east of column } j\} \\ SE(\beta /\!\!/ \alpha) &= \{j : \text{column } j+1 \text{ is south-east of column } j \text{ and } j \notin E(\beta /\!\!/ \alpha)\ \} \\ NE(\beta /\!\!/ \alpha) &= \{j : \text{column } j+1 \text{ is north-east of column } j\} \end{aligned}$$

The statistic $NE(\beta /\!\!/ \alpha)$ will play a crucial role in our noncommutative Murnaghan-Nakayama rule.

Example 2.6.1. *Consider the following interval shape $\beta /\!\!/ \alpha$, where the shaded boxes belong to the outer shape.*



In this case we have $E(\beta /\!\!/ \alpha) = \{1, 4\}, SE(\beta /\!\!/ \alpha) = \{2\}, NE(\beta /\!\!/ \alpha) = \{3\}$. Notice that the above interval shape is an nc border strip whose height is 3.

Remark 2.6.2. *Given an interval shape $\beta /\!\!/ \alpha$, note that $E(\beta /\!\!/ \alpha)$, $SE(\beta /\!\!/ \alpha)$, $NE(\beta /\!\!/ \alpha)$ are always pairwise disjoint. Furthermore, if p is the maximum element of* $\operatorname{supp}(\beta /\!\!/ \alpha)$*, then we have*

$$\operatorname{supp}(\beta /\!\!/ \alpha) \quad = \quad \{p\} \uplus E(\beta /\!\!/ \alpha) \uplus SE(\beta /\!\!/ \alpha) \uplus NE(\beta /\!\!/ \alpha).$$

Now, given a positive integer n, consider the following sets of compositions.

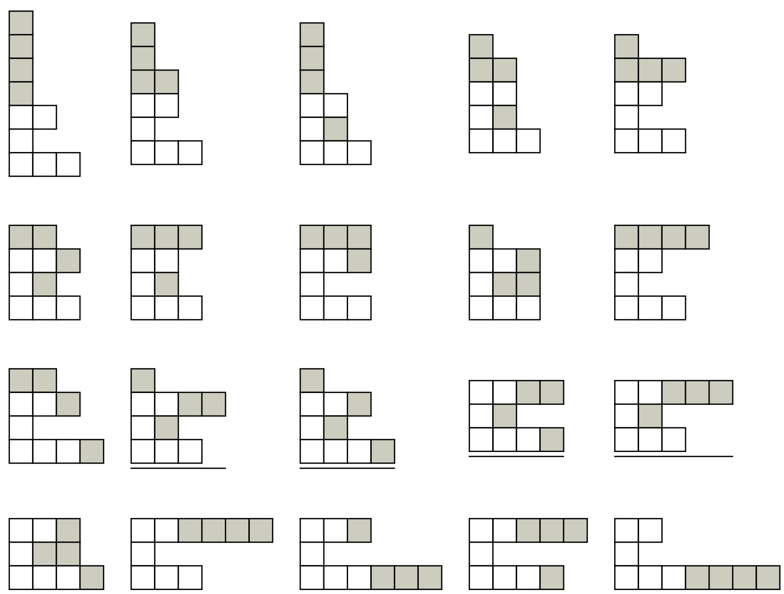
$$\begin{aligned} &B_{\alpha,n} = \{\beta : \alpha <_c \beta \text{ and } \beta /\!\!/ \alpha \text{ is an nc border strip of size } n\} \\ &P_{\alpha,n} = \{\beta : \beta \in B_{\alpha,n}, |NE(\beta /\!\!/ \alpha)| = 0\} \end{aligned}$$

Thus, $P_{\alpha,n}$ consists of all compositions β such that $\beta /\!\!/ \alpha$ is an nc border strip of size n satisfying $|NE(\beta /\!\!/ \alpha)| = 0$. This given, we can write down our analogue of the Murnaghan-Nakayama rule for noncommutative Schur functions as follows.

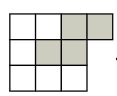
Theorem 2.6.3 (Noncommutative Murnaghan-Nakayama rule)**.** *Given a composition α and a positive integer n, we have the following expansion.*

$$\Psi_n \cdot \mathbf{s}_\alpha = \sum_{\beta \in P_{\alpha,n}} (-1)^{ht(\beta /\!\!/ \alpha)} \mathbf{s}_\beta$$

Example 2.6.4. *Consider the computation of $\Psi_4 \cdot \mathbf{s}_{(2,1,3)}$. We need to find all compositions $\beta \in B_{(2,1,3),4}$ satisfying $|NE(\beta /\!\!/ (2,1,3))| = 0$. We will start by listing all the elements of $B_{(2,1,3),4}$ where the boxes of $\beta /\!\!/ \alpha$ will be shaded. Furthermore, if β is underlined, then $|NE(\beta /\!\!/ (2,1,3)| \geq 1$.*



To illustrate the parity condition, we will compute the coefficient of



.

Since $ht((4,3,3) /\!\!/ (2,1,3)) = 1$, the coefficient of $\mathbf{s}_{(4,3,3)}$ in $\Psi_4 \cdot \mathbf{s}_{(2,1,3)}$ is -1. The complete expansion is as follows.

$$\begin{aligned} \Psi_4 \cdot \mathbf{s}_{(2,1,3)} = \quad & -\mathbf{s}_{(1,1,1,1,2,1,3)} + \mathbf{s}_{(1,1,2,2,1,3)} - \mathbf{s}_{(1,1,1,2,2,3)} + \mathbf{s}_{(1,2,2,2,3)} - \mathbf{s}_{(1,3,2,1,3)} + \mathbf{s}_{(1,2,3,1,3)} \\ & +\mathbf{s}_{(2,3,2,3)} - \mathbf{s}_{(3,2,2,3)} - \mathbf{s}_{(3,3,1,3)} + \mathbf{s}_{(1,3,3,3)} + \mathbf{s}_{(4,2,1,3)} - \mathbf{s}_{(3,2,1,4)} - \mathbf{s}_{(2,4,1,3)} \\ & +\mathbf{s}_{(2,3,1,4)} - \mathbf{s}_{(4,3,3)} + \mathbf{s}_{(3,3,4)} + \mathbf{s}_{(6,1,3)} - \mathbf{s}_{(3,1,6)} - \mathbf{s}_{(5,1,4)} + \mathbf{s}_{(2,1,7)} \end{aligned}$$

2.6.2 Jeu de taquin and right Pieri rules

We will now state our algorithm for our analogue of the classical backward jeu de taquin slide. We will work under the following assumptions in this subsection and for all the proofs in the Chapter 4, unless otherwise stated.

- i will denote a positive integer ≥ 1.

- T will refer to a fixed SSRT of shape λ.

- The SSRCT $\rho^{-1}(T)$ will be denoted by τ and we will assume that it has shape α where α clearly satisfies $\widetilde{\alpha} = \lambda$.

- There exists an addable node in column $i+1$ of λ.

2.6.3 Backward jeu de taquin slides for SSRCTs

With the notation from above, define a positive integer r_1 to be such that such that α_{r_1} is the bottommost part of length i in α. This given define positive integers r_j for $j \geq 2$ recursively subject to the following conditions.

I. $r_j < r_{j-1}$.

II. r_j is the greatest positive integer such that $\tau(r_j, i) > \tau(r_{j-1}, i) \geq \tau(r_j, i+1)$. If the box in position $(r_j, i+1)$ does not belong to τ, we let $\tau(r_j, i+1)$ equal 0.

Clearly the r_js defined above are finitely many. Let r_k denote the minimum element and $S = \{r_1 > \cdots > r_k\}$. Next we give an algorithm ϕ_{i+1} that takes τ as input and outputs an SSRCT $\phi_{i+1}(\tau)$ using the set S above.

Algorithm 2.6.5.

(i) For j=k down to 1 *in that order,*

- *Replace the entry in box* (r_j, i) *by the entry in box* (r_{j-1}, i)*. The entry* $\tau(r_k, i)$ exits *the tableau in every iteration.*

- *Remove the box in position* (r_1, i)*.*

(ii) Denote the resulting filling by $\phi_{i+1}(\tau)$*.*

We will use the above algorithm and give a procedure on SSRCTs of straight shape, referred to as μ_i, and this is analogous to the procedure jdt_i on SSRTs of straight shape. Our input is again τ and the output is an SSRCT $\mu_i(\tau)$.

Algorithm 2.6.6 (Backward jeu de taquin slides for SSRCTs)**.**

(i) If $i \geq 2$, compute first the SSRCT $\phi_2 \circ \cdots \circ \phi_i(\tau)$*. Denote this by τ'. If $i = 1$, define τ' to be τ. Furthermore, store the entries that exit in an array H, in the case $i \geq 2$.*

(ii) Let $\mathrm{sh}(\tau') = \gamma = (\gamma_1, \ldots, \gamma_k)$*. Let $\delta = (\gamma_1, \ldots, \gamma_k, i)$.*

(iii) Consider a filling of the skew reverse composition shape $\delta /\!\!/ (1)$ where the top k rows are filled according to τ', and the last row is filled by the entries in H in decreasing order from left to right.

(iv) The filling of the skew reverse composition shape $\delta /\!\!/ (1)$ obtained above is defined to be $\mu_i(\tau)$.

Now we will state a theorem that gives credence to our claim that μ_i is analogous to jdt_i.

Theorem 2.6.7. *With the notation as before, the following diagram commutes.*

$$\begin{tikzcd}
\text{SSRTs} \arrow[r, "\rho^{-1}"] \arrow[d, "\mathrm{jdt}_i"'] & \text{SSRCTs} \arrow[d, "\mu_i"] \\
\text{SSRTs} \arrow[r, "\rho^{-1}_{(1)}"'] & \text{SSRCTs}
\end{tikzcd}$$

Example 2.6.8. *Given below is an* SSRT T *(left) and the* SSRCT $\tau = \rho^{-1}(T)$ *(right).*

$$T = \begin{array}{ccccccccc}
8 \\
19 & 9 & 6 \\
24 & 17 & 7 & 3 \\
26 & 23 & 18 & 10 \\
37 & 28 & 25 & 20 \\
43 & 35 & 27 & 22 & 16 & 12 & 1 \\
44 & 38 & 36 & 29 & 21 & 15 & 4 & 2 \\
47 & 42 & 41 & 33 & 32 & 31 & 13 & 5 \\
48 & 46 & 45 & 40 & 39 & 34 & 30 & 14 & 11
\end{array}
\qquad
\tau = \begin{array}{ccccccccc}
8 \\
19 & 17 & 7 & 3 \\
24 & 23 & 18 & 10 \\
26 & 9 & 6 \\
37 & 35 & 27 & 22 & 21 & 15 & 13 & 5 \\
43 & 42 & 41 & 40 & 39 & 34 & 30 & 14 & 11 \\
44 & 38 & 36 & 33 & 32 & 31 & 4 & 2 \\
47 & 46 & 45 & 29 & 16 & 12 & 1 \\
48 & 28 & 25 & 20
\end{array}$$

We will compute $\mu_9(\tau)$. We start by computing $\phi_2 \circ \cdots \circ \phi_9(\tau)$. To comprehensively illustrate the theorem, this is shown step by step below. The entries in rows r_j for $1 \leq j \leq k$ that we need in applying Algorithm 2.6.5 are depicted in green, except the entry that exits, *which is depicted in red.*

8								
19	17	7	3					
24	23	18	10					
26	9	6						
37	35	27	22	21	15	13	5	
43	42	41	40	39	34	30	14	11
44	38	36	33	32	31	4	2	
47	46	45	29	16	12	1		
48	28	25	20					

$\phi_9(\tau)$

8								
19	17	7	3					
24	23	18	10					
26	9	6						
37	35	27	22	21	15	13	2	
43	42	41	40	39	34	30	14	11
44	38	36	33	32	31	4		
47	46	45	29	16	12	1		
48	28	25	20					

$\phi_8 \circ \phi_9(\tau)$

8								
19	17	7	3					
24	23	18	10					
26	9	6						
37	35	27	22	21	15	4	2	
43	42	41	40	39	34	30	14	11
44	38	36	33	32	31	1		
47	46	45	29	16	12			
48	28	25	20					

$\phi_7 \circ \phi_8 \circ \phi_9(\tau)$

8								
19	17	7	3					
24	23	18	10					
26	9	6						
37	35	27	22	21	15	4	2	
43	42	41	40	39	31	30	14	11
44	38	36	33	32	12	1		
47	46	45	29	16				
48	28	25	20					

$\phi_6 \circ \cdots \circ \phi_9(\tau)$

8								
19	17	7	3					
24	23	18	10					
26	9	6						
37	35	27	22	21	15	4	2	
43	42	41	40	32	31	30	14	11
44	38	36	33	16	12	1		
47	46	45	29					
48	28	25	20					

$\phi_5 \circ \cdots \circ \phi_9(\tau)$

8								
19	17	7	3					
24	23	18	10					
26	9	6						
37	35	27	22	21	15	4	2	
43	42	41	33	32	31	30	14	11
44	38	36	29	16	12	1		
47	46	45	20					
48	28	25						

$\phi_4 \circ \cdots \circ \phi_9(\tau)$

8								
19	17	7	3					
24	23	18	10					
26	9	6						
37	35	27	22	21	15	4	2	
43	42	41	33	32	31	30	14	11
44	38	36	29	16	12	1		
47	46	25	20					
48	28							

$\phi_3 \circ \cdots \circ \phi_9(\tau)$

8								
19	17	7	3					
24	23	18	10					
26	9	6						
37	35	27	22	21	15	4	2	
43	42	41	33	32	31	30	14	11
44	38	36	29	16	12	1		
47	28	25	20					
48								

$\phi_2 \circ \cdots \circ \phi_9(\tau)$

Thus, finally we have the following.

8								
19	17	7	3					
24	23	18	10					
26	9	6						
37	35	27	22	21	15	4	2	
43	42	41	33	32	31	30	14	11
44	38	36	29	16	12	1		
47	28	25	20					
	48	46	45	40	39	34	13	5

$\mu_9(\tau)$

8								
19	9	6						
24	17	7	3					
26	23	18	10					
37	28	25	20					
43	35	27	22	16	12	1		
44	38	36	29	21	15	4	2	
47	42	41	33	32	31	30	13	5
	48	46	45	40	39	34	14	11

$\mathrm{jdt}_9(T)$

One can see that $\rho_{(1)}(\mu_9(\tau)) = \mathrm{jdt}_9(T)$.

2.6.4 Right Pieri rules

Next we will state our right Pieri rules using jdt operators and then give an example.

Theorem 2.6.9 (Right Pieri rules)**.** *Let* α *be a composition and* n *a positive integer. Then*

$$\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)} = \sum_{\substack{\beta \vDash |\alpha|+n, \mathfrak{u}_{i_n} \cdots \mathfrak{u}_{i_1}(\alpha)=\beta \\ i_n > \cdots > i_1}} \mathbf{s}_\beta,$$

$$\mathbf{s}_\alpha \cdot \mathbf{s}_{(1^n)} = \sum_{\substack{\beta \vDash |\alpha|+n, \mathfrak{u}_{i_n} \cdots \mathfrak{u}_{i_1}(\alpha)=\beta \\ i_n \leq \cdots \leq i_1}} \mathbf{s}_\beta.$$

In the statement above, (1^n) denotes the composition with n parts equalling 1 each.

Example 2.6.10. *Let* $\alpha = (1,4,2)$. *We will compute* $\mathbf{s}_\alpha \cdot \mathbf{s}_{(3)}$ *first. Thus, we are interested in finding positive integers* $i_1 < i_2 < i_3$ *such that* $\mathfrak{u}_{i_3}\mathfrak{u}_{i_2}\mathfrak{u}_{i_1}(\alpha)$ *is a composition of size* 10. *One can see that we have the following choices.*

$$\begin{array}{lll}
\mathfrak{u}_3\mathfrak{u}_2\mathfrak{u}_1(\alpha) = (1,4,2,3) & \mathfrak{u}_5\mathfrak{u}_2\mathfrak{u}_1(\alpha) = (1,2,2,5) & \mathfrak{u}_4\mathfrak{u}_3\mathfrak{u}_1(\alpha) = (1,4,1,4) \\
\mathfrak{u}_5\mathfrak{u}_3\mathfrak{u}_1(\alpha) = (1,3,1,5) & \mathfrak{u}_6\mathfrak{u}_5\mathfrak{u}_1(\alpha) = (1,2,1,6) & \mathfrak{u}_4\mathfrak{u}_3\mathfrak{u}_2(\alpha) = (4,2,4) \\
\mathfrak{u}_5\mathfrak{u}_3\mathfrak{u}_2(\alpha) = (3,2,5) & \mathfrak{u}_6\mathfrak{u}_5\mathfrak{u}_2(\alpha) = (2,2,6) & \mathfrak{u}_5\mathfrak{u}_4\mathfrak{u}_3(\alpha) = (1,4,5) \\
\mathfrak{u}_6\mathfrak{u}_5\mathfrak{u}_3(\alpha) = (1,3,6) & \mathfrak{u}_7\mathfrak{u}_6\mathfrak{u}_5(\alpha) = (1,2,7) &
\end{array}$$

This gives us the following expansion for $\mathbf{s}_{(1,4,2)} \cdot \mathbf{s}_{(3)}$.

$$\begin{aligned}\mathbf{s}_{(1,4,2)} \cdot \mathbf{s}_{(3)} = \quad & \mathbf{s}_{(1,4,2,3)} + \mathbf{s}_{(1,2,2,5)} + \mathbf{s}_{(1,4,1,4)} + \mathbf{s}_{(1,3,1,5)} + \mathbf{s}_{(1,2,1,6)} + \mathbf{s}_{(4,2,4)} \\ & + \mathbf{s}_{(3,2,5)} + \mathbf{s}_{(2,2,6)} + \mathbf{s}_{(1,4,5)} + \mathbf{s}_{(1,3,6)} + \mathbf{s}_{(1,2,7)}\end{aligned}$$

Next, we compute $\mathbf{s}_\alpha \cdot \mathbf{s}_{(1,1,1)}$. *Then we are interested in finding positive integers* $i_1 \geq i_2 \geq i_3$ *such that* $\mathfrak{u}_{i_3}\mathfrak{u}_{i_2}\mathfrak{u}_{i_1}(\alpha)$ *is a composition of size* 10. *Thus, we have the following choices.*

$$\begin{array}{lll}\mathfrak{u}_1\mathfrak{u}_1\mathfrak{u}_1(\alpha) = (1,4,2,1,1,1) & \mathfrak{u}_1\mathfrak{u}_1\mathfrak{u}_2(\alpha) = (4,2,2,1,1) & \mathfrak{u}_1\mathfrak{u}_1\mathfrak{u}_3(\alpha) = (1,4,3,1,1) \\ \mathfrak{u}_1\mathfrak{u}_1\mathfrak{u}_5(\alpha) = (1,2,5,1,1) & \mathfrak{u}_1\mathfrak{u}_2\mathfrak{u}_3(\alpha) = (4,3,2,1) & \mathfrak{u}_1\mathfrak{u}_2\mathfrak{u}_5(\alpha) = (2,5,2,1) \\ \mathfrak{u}_1\mathfrak{u}_3\mathfrak{u}_5(\alpha) = (1,5,3,1) & \mathfrak{u}_2\mathfrak{u}_3\mathfrak{u}_5(\alpha) = (5,3,2) & \end{array}$$

This gives us the following expansion for $\mathbf{s}_{(1,4,2)} \cdot \mathbf{s}_{(1,1,1)}$.

$$\begin{aligned}\mathbf{s}_{(1,4,2)} \cdot \mathbf{s}_{(1,1,1)} = \quad & \mathbf{s}_{(1,4,2,1,1,1)} + \mathbf{s}_{(4,2,2,1,1)} + \mathbf{s}_{(1,4,3,1,1)} + \mathbf{s}_{(1,2,5,1,1)} + \mathbf{s}_{(4,3,2,1)} \\ & + \mathbf{s}_{(2,5,2,1)} + \mathbf{s}_{(1,5,3,1)} + \mathbf{s}_{(5,3,2)}\end{aligned}$$

Chapter 3

A Murnaghan-Nakayama rule for noncommutative Schur functions

In this chapter, we will give a proof for Theorem 2.6.3. We will start by discussing some combinatorics of box adding operators. More specifically, we describe how sequences of box adding operators correspond to SRCTs, and then we recast the left Pieri rules (Theorem 2.4.7) in terms of box adding operators.

3.1 Box adding operators, Pieri rules and reverse hookwords

3.1.1 SRCTs associated with sequence of box adding operators

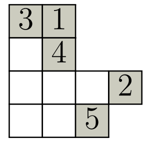
Let α be a composition and suppose $w = \mathfrak{t}_{i_1} \cdots \mathfrak{t}_{i_n}$ is such that $w(\alpha) = \beta$, where β is not 0. Then, by [4, Proposition 2.11], we see that there is a unique SRCT of shape $\beta /\!\!/ \alpha$ associated with the word w, as we can associate the following maximal chain in $\mathcal{L}_c$ with w.

$$\alpha \lessdot_c \mathfrak{t}_{i_n}(\alpha) \lessdot_c \cdots \lessdot_c \mathfrak{t}_{i_2} \cdots \mathfrak{t}_{i_n}(\alpha) \lessdot_c \mathfrak{t}_{i_1} \cdots \mathfrak{t}_{i_n}(\alpha) = \beta$$

This maximal chain in turn gives rise to a unique SRCT of size n and shape $\beta /\!\!/ \alpha$, where $n - i + 1$ is placed in the i-th box added.

Example 3.1.1. *Let* $\alpha = (1, 3, 2)$ *and* $\beta = \mathfrak{t}_2\mathfrak{t}_4\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3(\alpha) = (2, 2, 4, 3)$*. Then* $w = \mathfrak{t}_2\mathfrak{t}_4\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3$

uniquely corresponds to the following SRCT *of shape* $\beta /\!\!/ \alpha$.



3.1.2 Pieri rules using box adding operators

Let $\mathbb{C}Comp$ denote the vector space consisting of formal sums of compositions. Consider the map $\Phi : \mathbf{NSym} \longrightarrow \mathbb{C}Comp$ defined by sending $\mathbf{s}_\alpha \mapsto \alpha$ and extending linearly. We give $\mathbb{C}Comp$ an algebra structure by defining the product between two composition $\alpha \cdot \beta$ as follows.

$$\alpha \cdot \beta \;=\; \Phi(\Phi^{-1}(\alpha) \cdot \Phi^{-1}(\beta))$$

It is easily seen that with the above definition of Φ, we have an $\mathbb{C}$-algebra isomorphism between $\mathbf{NSym}$ and $\mathbb{C}Comp$. We will freely use the notation $\alpha \cdot \beta$ to denote the product $\mathbf{s}_\alpha \cdot \mathbf{s}_\beta$, and for the sake of convenience, we will not distinguish between $\mathbf{s}_\alpha \cdot \mathbf{s}_\beta$ and $\Phi(\mathbf{s}_\alpha \cdot \mathbf{s}_\beta)$.

With the above setup and armed with the definition of box adding operators, we restate Theorem 2.4.7 in the language of box adding operators.

Proposition 3.1.2. *Let α be a composition and n be a positive integer. Then*

$$(n) \cdot \alpha \;=\; \left(\sum_{i_n > \cdots > i_1} \mathfrak{t}_{i_n} \cdots \mathfrak{t}_{i_1} \right)(\alpha),$$
$$(1^n) \cdot \alpha \;=\; \left(\sum_{i_n \leq \cdots \leq i_1} \mathfrak{t}_{i_n} \cdots \mathfrak{t}_{i_1} \right)(\alpha).$$

Recall now Equation (2.1) which gives the following expansion.

$$\Psi_n = \sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k)}. \tag{3.1}$$

Thus, in order to to obtain a noncommutative analogue of the Murnaghan-Nakayama rule, we would like to compute $\mathbf{r}_{(1^k,n-k)} \cdot \mathbf{s}_\alpha$ using box adding operators. Note that we already know the answer in the special cases $k = 0$ or $k = n-1$, as these are the cases encompassed by Proposition 3.1.2. To this end, we will need some more notation, which is the aim of the next subsection.

3.1.3 Reverse hookwords and multiplication by $\mathbf{r}_{(1^k,n-k)}$

Given positive integers n and k such that $0 \leq k \leq n-1$, define $w = \mathfrak{t}_{i_1} \dots \mathfrak{t}_{i_n}$ to be a *reverse k-hookword* if $i_1 \leq i_2 \leq \cdots \leq i_{k+1} > i_{k+2} > \cdots > i_n$. If w is a reverse k-hookword for some k, then we will call w a *reverse hookword*. Denote by $\mathrm{supp}(w)$ the set formed by the indices $i_1, \dots, i_n$ (by discarding duplicates). Call w *connected* if $\mathrm{supp}(w)$ is an interval in $\mathbb{Z}_{\geq 1}$, otherwise call it *disconnected*. Put differently, w is connected if the set of indices, when considered in ascending (or descending) order, form a contiguous block of positive integers. Let

$$\begin{aligned} RHW_n &= \text{set of reverse hookwords of length } n, \\ CRHW_n &= \text{set of connected reverse hookwords of length } n. \end{aligned}$$

Moreover, let $RHW_{n,k}$ denote the subset of RHW_n consisting of reverse k-hookwords.

For the same w as before, we define the *content vector* of w to be a finite ordered list of nonnegative integers $(c_1, c_2, \dots)$ where c_i denotes the number of times the operator $\mathfrak{t}_i$ appears in w. Also associated with w are the notions of $\mathrm{arm}(w)$ and $\mathrm{leg}(w)$ defined as follows.

$$\begin{aligned} \mathrm{arm}(w) &= \{i_j : 1 \leq j \leq k+1\} \\ \mathrm{leg}(w) &= \{i_j : k+1 \leq j \leq n\} \end{aligned}$$

We note that $\mathrm{arm}(w)$ is a multiset, while $\mathrm{leg}(w)$ is a set. Finally, let

$$\mathrm{asc}(w) = |\mathrm{arm}(w)| - 1 = n - |\mathrm{leg}(w)|.$$

Example 3.1.3. *Let $w = \mathfrak{t}_2\mathfrak{t}_5\mathfrak{t}_6\mathfrak{t}_7\mathfrak{t}_8\mathfrak{t}_8\mathfrak{t}_9\mathfrak{t}_9\mathfrak{t}_7\mathfrak{t}_6\mathfrak{t}_4\mathfrak{t}_3$. Then w is a connected reverse 7-hookword with content vector given by $(0, 1, 1, 1, 1, 2, 2, 2, 2)$. Furthermore, we also have that*

$$\begin{aligned} \mathrm{arm}(w) &= \{9, 9, 8, 8, 7, 6, 5, 2\}, \\ \mathrm{leg}(w) &= \{3, 4, 6, 7, 9\}. \end{aligned}$$

Now, using reverse hookwords, we can express the multiplication of noncommutative Schur functions by noncommutative ribbon Schur functions in terms of box adding operators.

Lemma 3.1.4. *Let $0 \leq k \leq n-1$, and let α be a composition. Then*

$$\mathbf{r}_{(1^k,n-k)} \cdot \mathbf{s}_\alpha = \left(\sum_{w \in RHW_{n,k}} w \right) (\alpha).$$

Proof. Note first that in the case $k = 0$, the claim is true as it is equivalent to the first identity of Proposition 3.1.2. We will establish the claim by induction on k. Assume that the claims holds for all integers $\leq k$ for some $k \geq 1$. We will compute $(\mathbf{r}_{(1^{k+1})} \cdot \mathbf{r}_{(n-k-1)}) \cdot \mathbf{s}_\alpha$ in two ways. First notice that by using Theorem 2.4.1, we get

$$(\mathbf{r}_{(1^{k+1})} \cdot \mathbf{r}_{(n-k-1)}) \cdot \mathbf{s}_\alpha \quad = \quad \mathbf{r}_{(1^{k+1},n-k-1)} \cdot \mathbf{s}_\alpha + \mathbf{r}_{(1^k,n-k)} \cdot \mathbf{s}_\alpha. \tag{3.2}$$

Now, using Proposition 3.1.2, we have that

$$\begin{aligned} (\mathbf{r}_{(1^{k+1})} \cdot \mathbf{r}_{(n-k-1)}) \cdot \mathbf{s}_\alpha &= \left(\sum_{i_1 \leq \cdots \leq i_{k+1}} \mathfrak{t}_{i_1} \cdots \mathfrak{t}_{i_{k+1}} \right) \left(\sum_{i_{k+2} > \cdots > i_n} \mathfrak{t}_{i_{k+2}} \cdots \mathfrak{t}_{i_n} \right) (\alpha) \\ &= \left(\sum_{w \in RHW_{n,k}} w \right) (\alpha) + \left(\sum_{w' \in RHW_{n,k+1}} w' \right) (\alpha). \end{aligned} \tag{3.3}$$

By the induction hypothesis we know that

$$\mathbf{r}_{(1^k,n-k)} \cdot \mathbf{s}_\alpha \quad = \quad \left(\sum_{w \in RHW_{n,k}} w \right) (\alpha). \tag{3.4}$$

Now, using (3.2), (3.3) and (3.4), the claim follows. $\square$

3.2 Multiplication by noncommutative power sums in terms of box adding operators

On using the definition of the noncommutative power sum function Ψ_n in terms of the noncommutative ribbon Schur functions, and Lemma 3.1.4 subsequently, we get that

$$\begin{aligned}
\Psi_n \cdot \mathbf{s}_\alpha &= \left(\sum_{k=0}^{n-1}(-1)^k \mathbf{r}_{(1^k,n-k)}\right)\cdot \mathbf{s}_\alpha \\
&= \sum_{k=0}^{n-1}(-1)^k \sum_{w\in RHW_{n,k}} w(\alpha) \\
&= \sum_{w\in RHW_n}(-1)^{\mathrm{asc}(w)} w(\alpha). \qquad (3.5)
\end{aligned}$$

Using the involution described in Section 5 of [10], we can simplify (3.5) so that the sum runs over the set of connected reverse hookwords. For completeness, we give a summary of this involution next. Let w be a disconnected reverse hookword and let its content vector be β. Let j be the smallest integer such that $j+1 \notin \mathrm{supp}(w)$, and j is not the maximum element in $\mathrm{supp}(w)$. Such a j exists since $\mathrm{supp}(w)$ is not an interval. Then, if $j \in \mathrm{leg}(w)$, define $\Theta(w)$ to be the unique reverse hookword with content vector β and $\mathrm{leg}(\Theta(w)) = \mathrm{leg}(w)\setminus\{j\}$. Similarly, if $j \in \mathrm{arm}(w)$ but $j \notin \mathrm{leg}(w)$, then define $\Theta(w)$ to be the unique reverse hookword with content vector β and $\mathrm{leg}(\Theta(w)) = \mathrm{leg}(w)\cup\{j\}$. It is easy to see that Θ is an involution on the set of disconnected reverse hookwords, but more importantly, from Lemma 2.5.2 it follows that $\Theta(w)(\alpha) = w(\alpha)$. Furthermore, $\mathrm{asc}(w)$ and $\mathrm{asc}(\Theta(w))$ are of opposite parity. The preceding arguments allow us to rewrite (3.5) as follows.

$$\Psi_n \cdot \mathbf{s}_\alpha = \sum_{w\in CRHW_n}(-1)^{\mathrm{asc}(w)} w(\alpha) \qquad (3.6)$$

While the equation above is a legitimate way to compute $\Psi_n \cdot \mathbf{s}_\alpha$, it is not cancellation-free. Giving a cancellation-free expansion for $\Psi_n \cdot \mathbf{s}_\alpha$ will be our aim in the next section.

Example 3.2.1. *We will compute $\Psi_2 \cdot \mathbf{s}_\alpha$ where $\alpha = (1,3,2)$. The possible elements of*

$CRHW_2$ are $\mathfrak{t}_i\mathfrak{t}_i$, $\mathfrak{t}_i\mathfrak{t}_{i+1}$ and $\mathfrak{t}_{i+1}\mathfrak{t}_i$ for $i \geq 1$. Thus, we have

$$\begin{aligned}
\Psi_2 \cdot \mathbf{s}_{(1,3,2)} &= \sum_{i\geq 1}(\mathfrak{t}_{i+1}\mathfrak{t}_i - \mathfrak{t}_i\mathfrak{t}_{i+1} - \mathfrak{t}_i^2)((1,3,2)) \\
&= (\mathfrak{t}_2\mathfrak{t}_1 + \mathfrak{t}_3\mathfrak{t}_2 + \mathfrak{t}_4\mathfrak{t}_3 + \mathfrak{t}_5\mathfrak{t}_4 - \mathfrak{t}_1\mathfrak{t}_2 - \mathfrak{t}_2\mathfrak{t}_3 - \mathfrak{t}_3\mathfrak{t}_4 - \mathfrak{t}_1\mathfrak{t}_1)((1,3,2)) \\
&= (2,1,3,2)+(3,3,2)+(1,4,3)+(1,5,2)-(1,2,3,2) \\
&\quad -(2,3,3)-(1,4,3)-(1,1,1,3,2) \\
&= (2,1,3,2)+(3,3,2)+(1,5,2)-(1,2,3,2)-(2,3,3)-(1,1,1,3,2).
\end{aligned}$$

Note that this example reaffirms our claim that (3.6) is not cancellation-free. The composition (1,4,3) appeared once with coefficient 1, and once with coefficient -1. Hence it does not appear in the final result.

3.3 The Murnaghan-Nakayama rule for noncommutative Schur functions

By (3.6) we know that

$$\Psi_n \cdot \mathbf{s}_\alpha = \sum_\beta k_\beta \mathbf{s}_\beta \tag{3.7}$$

where

$$k_\beta = \sum_{\substack{w \in CRHW_n \\ w(\alpha)=\beta}} (-1)^{\mathrm{asc}(w)}.$$

Thus, our aim is to compute k_β given any composition β. Considering the equation above, it would be helpful to be able to tell, given the reverse composition diagrams of α and β, whether there exists a connected reverse hookword w such that $w(\alpha) = \beta$. To this end, we will recall the notion of an nc border strip introduced in Section 2.6.

Let α and β be compositions such that $\alpha <_c \beta$. Define the *support*, $\mathrm{supp}(\beta /\!\!/ \alpha)$, as follows.

$$\mathrm{supp}(\beta /\!\!/ \alpha) = \{j \mid (i,j) \in \beta /\!\!/ \alpha\}$$

The skew reverse composition shape $\beta /\!\!/ \alpha$ whose support is an interval in $\mathbb{Z}_{\geq 1}$ will be called an *interval shape*. Finally, an interval shape $\beta /\!\!/ \alpha$ will be called an *nc border strip* if it satisfies the following conditions.

1. If there exist boxes in positions $(i,1)$ and $(i,2)$ in $\beta/\!\!/\alpha$, then the box in position $(i,1)$ is the bottommost box in column 1 of $\beta/\!\!/\alpha$.

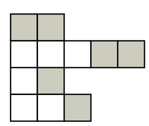
2. If there exist boxes in positions (i,j) and $(i,j+1)$ in $\beta/\!\!/\alpha$ for $j \geq 2$, then the box in position (i,j) is the topmost box in column j of $\beta/\!\!/\alpha$.

Suppose we are given an interval shape $\beta/\!\!/\alpha$. We discuss three notions that have to do with the relative positions of boxes in consecutive columns of $\beta/\!\!/\alpha$. We say that column $j+1$ is *south-east* of column j in $\beta/\!\!/\alpha$ if there is a box in column $j+1$ that is strictly south-east of a box in column j. We say that column $j+1$ is *north-east* of column j, if for any pair of boxes $(i_1, j+1)$ and (i_2, j) in $\beta/\!\!/\alpha$, we have $i_1 < i_2$. Here, we assume that there exists at least one box in the $j+1$-th column. Finally, we say that column $j+1$ is *east* of column j if there are boxes in positions (i,j) and $(i,j+1)$ for some i.

This given, we associate three statistics with an interval shape $\beta/\!\!/\alpha$ as follows.

$$\begin{aligned}
E(\beta/\!\!/\alpha) &= \{j : \text{column } j+1 \text{ is east of column } j\}\\
SE(\beta/\!\!/\alpha) &= \{j : \text{column } j+1 \text{ is south-east of column } j \text{ and } j \notin E(\beta/\!\!/\alpha)\ \}\\
NE(\beta/\!\!/\alpha) &= \{j : \text{column } j+1 \text{ is north-east of column } j\}
\end{aligned}$$

Example 3.3.1. *Consider the following interval shape $\beta/\!\!/\alpha$, where the shaded boxes belong to the outer shape.*



In this case we have $E(\beta/\!\!/\alpha) = \{1,4\}, SE(\beta/\!\!/\alpha) = \{2\}, NE(\beta/\!\!/\alpha) = \{3\}$. Notice that the above interval shape is an nc border strip whose height is 3.

Remark 3.3.2. *Given an interval shape $\beta/\!\!/\alpha$, note that $E(\beta/\!\!/\alpha)$, $SE(\beta/\!\!/\alpha)$, $NE(\beta/\!\!/\alpha)$ are always pairwise disjoint. Furthermore, if p is the maximum element of* $\mathrm{supp}(\beta/\!\!/\alpha)$, *then we have*

$$\mathrm{supp}(\beta/\!\!/\alpha) = \{p\} \uplus E(\beta/\!\!/\alpha) \uplus SE(\beta/\!\!/\alpha) \uplus NE(\beta/\!\!/\alpha).$$

Now, consider the following two sets

$$\begin{aligned} A_{\alpha,n} &= \{\beta : \beta = w(\alpha) \text{ for some } w \in CRHW_n\}, \\ B_{\alpha,n} &= \{\beta : \alpha <_c \beta \text{ and } \beta /\!\!/ \alpha \text{ is an nc border strip of size } n\}. \end{aligned}$$

Our next aim is to establish that $A_{\alpha,n} = B_{\alpha,n}$. This means that we will be able to replace checking whether there exists $w \in CRHW_n$ such that $w(\alpha) = \beta$ with checking whether $\beta /\!\!/ \alpha$ is an nc border strip.

Lemma 3.3.3.

$$B_{\alpha,n} \subseteq A_{\alpha,n}.$$

Proof. To establish the claim, we need to show that given $\beta \in B_{\alpha,n}$ (that is, $\beta /\!\!/ \alpha$ is an nc border strip), there exists a $w \in CRHW_n$ such that $w(\alpha) = \beta$. We will construct an SRCT of shape $\beta /\!\!/ \alpha$ that will be associated (in the sense of Section 3.1.1) with an element of $CRHW_n$.

Let $E(\beta /\!\!/ \alpha) = \{x_1 < \cdots < x_s\}$. For every x_i, place the integer $n - i + 1$ in the topmost box of the x_i-th column of $\beta /\!\!/ \alpha$ if $x_i > 1$ and in the bottommost box if $x_i = 1$. After this step, traverse the remaining $n - s$ unfilled boxes in $\beta /\!\!/ \alpha$ in the manner described below and place the integers $n - s$ down to 1 in that order.

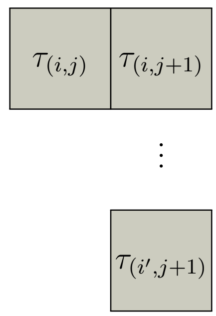
- Start from the rightmost column that contains an empty box.
- Visit every empty box from top to bottom if column under consideration is not the first column, otherwise visit boxes from bottom to top.
- Repeat the above steps for the next rightmost column in $\beta /\!\!/ \alpha$ that contains an empty box.

We claim that the resulting filling, which we call τ, is an SRCT. By construction, if $1 \in \mathrm{supp}(\beta /\!\!/ \alpha)$, then the entries in the boxes in column 1 strictly increase from top to bottom. In all other columns, the entries decrease from top to bottom.

To show that entries decrease from left to right along rows, assume first the existence of boxes in position (i, j) and $(i, j+1)$ in $\beta /\!\!/ \alpha$. Clearly $j \in E(\beta /\!\!/ \alpha)$. Since $\beta /\!\!/ \alpha$ is an nc border strip, we know that the box in position (i, j) is the topmost box in the j-th column of $\beta /\!\!/ \alpha$ if $j \geq 2$, and is the bottommost box if $j = 1$. By our construction of the filling, if $j \in E(\beta /\!\!/ \alpha)$, every box of $\beta /\!\!/ \alpha$ in column $j + 1$ contains a number strictly less than $\tau_{(i,j)}$.

Hence in particular, $\tau_{(i,j)} > \tau_{(i,j+1)}$ and thus, the entries decrease from left to right along rows.

Now we have to show that there are no triple rule violations in τ. Since, in every column k where $k \geq 2$, the entries decrease from top to bottom, we are guaranteed there are no triple rule violations of the form



where $j \geq 1$, and $(i', j+1)$, $(i, j+1) \in \beta /\!\!/ \alpha$ with $i' > i$.

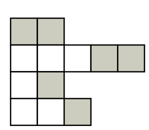
Now assume that $(i, j+1) \notin \beta /\!\!/ \alpha$. We will show that $\tau_{(i,j)} < \tau_{(i',j+1)}$ for any $i' > i$ where $(i', j+1) \in \beta /\!\!/ \alpha$. Assume first that $j \in E(\beta /\!\!/ \alpha)$. If $j = 1$, then we know that (i, j) is not the bottommost box in column j of $\beta /\!\!/ \alpha$, as $(i, j+1) \notin \beta /\!\!/ \alpha$. Thus, by construction, $\tau_{(i,j)}$ is strictly less than every entry in column 2. In particular, $\tau_{(i',j+1)} > \tau_{(i,j)}$, and we do not have triple rule violations. If $j \geq 2$, then we know that the box in position (i, j) is not the topmost box in column j. Again, we have that $\tau_{(i,j)}$ is strictly less than every entry in column $j+1$, and as before, we have no triple rule violations.

Now suppose that $(i, j+1) \notin \beta /\!\!/ \alpha$, and that $j \notin E(\beta /\!\!/ \alpha)$. Then, the way the filling τ has been constructed implies that the greatest entry in column j is less than the smallest entry in column $j+1$. This guarantees that $\tau_{(i',j+1)} > \tau_{(i,j)}$, and we have no triple rule violations.

Hence, τ is an SRCT. Clearly, the word associated with τ is an element of RHW_n. The fact that it is an element of $CRHW_n$ follows from the fact that $\beta /\!\!/ \alpha$ is an interval shape. Hence there exists a $w \in CRHW_n$ such that $w(\alpha) = \beta$, implying $\beta \in A_{\alpha,n}$. $\square$

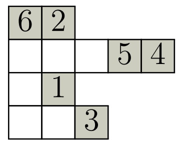
Next, we give an example of the construction presented in the algorithm above.

Example 3.3.4. *Consider the shape $\beta /\!\!/ \alpha$ presented in Example 3.3.1.*



Here $|\beta /\!\!/ \alpha| = 6$. Hence we will be placing the numbers 1 to 6 in the shaded boxes above.

Since $E(\beta/\!\!/\alpha) = \{1, 4\}$, the bottommost box in column 1 of $\beta/\!\!/\alpha$ will have 6 placed in it, while the topmost box in column 4 will have 5 placed in it. Now the columns with empty boxes, considered from right to left, are 5, 3 and 2. We fill in these boxes with the numbers $4, 3, 2, 1$ in that order from top to bottom in each column to obtain the following filling.



Lemma 3.3.5. *Suppose $w \in CRHW_n$ is such that $w(\alpha) = \beta$. Let τ be the* SRCT *of shape $\beta/\!\!/\alpha$ corresponding to the word w. Then, the entries in τ in column j strictly decrease from top to bottom for all $j \geq 2$.*

Proof. Suppose w has c occurrences of $\mathfrak{t}_j$ where $j \geq 2$. We can then factorize w as follows

$$w = w_3 \mathfrak{t}_j^{c-1} w_2 \mathfrak{t}_j w_1, \tag{3.8}$$

where w_1, w_2 could be empty words. Notice that since w is a reverse hookword, all the operators that constitute w_2 are of the form $\mathfrak{t}_k$ where $k > j$. Hence they only add boxes to parts of length $\geq j$ in the composition $\mathfrak{t}_j w_1(\alpha)$. Now $\mathfrak{t}_j^{c-1}$ adds boxes in the j-th column. Thus, we have added c boxes in the j-th column and the definition of the box adding operators implies that repeated applications of $\mathfrak{t}_j$ add boxes from top to bottom, when $j \geq 2$. Hence the entries in these c boxes strictly decrease from top to bottom. $\square$

Next, we note down a couple of lemmas that are immediate consequences of the bijection mentioned in Subsection 3.1.1. For Lemmas 3.3.6 and 3.3.7, and Corollary 3.3.8, we will work under the assumption that $w \in CRHW_n$ is such that $w(\alpha) = \beta$, and that τ is the SRCT of shape $\beta/\!\!/\alpha$ corresponding to w. We will assume further that j belongs to $\mathrm{supp}(\beta/\!\!/\alpha)$ (or $\mathrm{supp}(w)$), but is not the greatest element therein.

Lemma 3.3.6. *Suppose $j \notin \mathrm{leg}(w)$ for some $j \geq 1$. Then the greatest entry in column j of τ is strictly less than the smallest entry in column $j+1$ in τ.*

Lemma 3.3.7. *Suppose $j \in \mathrm{leg}(w)$ for some $j \geq 1$. Then*

- *the greatest entry in column j of τ is strictly greater than the greatest entry in column $j+1$ in τ, and*

- *all other entries in column j of τ are strictly smaller than the smallest entry in column $j+1$ in τ.*

We will note down an important corollary of the two lemmas above.

Corollary 3.3.8. *For all $j \geq 1$, every entry in column j except the greatest one, is strictly smaller than the smallest entry in column $j+1$.*

Lemma 3.3.9.

$$A_{\alpha,n} = B_{\alpha,n}.$$

Proof. We have already shown in Lemma 3.3.3 that every element of $B_{\alpha,n}$ is an element of $A_{\alpha,n}$. Thus, now we need to show that $A_{\alpha,n} \subseteq B_{\alpha,n}$.

Assume $\beta \in A_{\alpha,n}$. We have to show that $\beta \in B_{\alpha,n}$ as well. It is clear that if $\beta = w(\alpha)$ for some $w \in CRHW_n$, then $\alpha <_c \beta$. The fact that $\beta /\!\!/ \alpha$ is an interval shape of size n follows from the fact that w is connected and has length n. Let τ denote the SRCT of shape $\beta /\!\!/ \alpha$ corresponding to w.

Next, suppose there is a configuration of the following type.

$$\begin{array}{|c|c|}\hline \tau_{(i,j)} & \tau_{(i,j+1)} \\ \hline \end{array}$$

If $\tau_{(i,j)}$ is not the greatest entry in column j, then Corollary 3.3.8 implies that $\tau_{(i,j)} < \tau_{(i,j+1)}$, which contradicts the fact that τ is an SRCT. Hence $\tau_{(i,j)}$ is the greatest entry in column j. Hence the box in position (i,j) is the bottommost box in column j if $j = 1$, or the topmost box in column j otherwise, by invoking Lemma 3.3.5.

This finishes the proof that $\beta /\!\!/ \alpha$ is an nc border strip if $\beta \in A_{\alpha,n}$. Thus $A_{\alpha,n} \subseteq B_{\alpha,n}$, and we are done. ☐

Remark 3.3.10. *A skew shape λ/μ is called a* disconnected border strip *if it is a disjoint union of border strips. Notice that if α is a composition such that $\widetilde{\alpha} = \mu$, and $w \in CRHW_n$ is such that $\widetilde{w(\alpha)} = \lambda$, then λ/μ is a disconnected border strip of size n whose support is an interval. Equivalently if $\beta /\!\!/ \alpha$ is an nc border strip, then $\widetilde{\beta}/\widetilde{\alpha}$ is a disconnected border strip whose support is an interval. This explains our reasoning for calling $\beta /\!\!/ \alpha$ an nc border strip.*

We will outline our strategy for the remainder of this section. Now that we have established that $A_{\alpha,n} = B_{\alpha,n}$, our next aim is to enumerate all hookwords w such that $w(\alpha) = \beta$

where $\beta \in B_{\alpha,n}$, that is, $\beta /\!\!/ \alpha$ is an nc border strip of size n. This will be achieved in Lemma 3.3.16. Once this is done, we will be able to compute the coefficient

$$k_\beta = \sum_{\substack{w \in CRHW_n \\ w(\alpha)=\beta}} (-1)^{\mathrm{asc}(w)}.$$

Recall that k_β was defined to be the coefficient of $\mathbf{s}_\beta$ in the expansion $\Psi_n \cdot \mathbf{s}_\alpha$. This will give us a noncommutative analogue of the Murnaghan-Nakayama rule that we will state in Theorem 3.3.20.

Lemma 3.3.11. *Given $\beta \in B_{\alpha,n}$, if for any $w \in CRHW_n$ such that $w(\alpha) = \beta$ we have that column $j \in E(\beta /\!\!/ \alpha)$, then $j \in \mathrm{leg}(w)$.*

Proof. Let τ be the SRCT of shape $\beta /\!\!/ \alpha$ corresponding to the word $w = \mathfrak{t}_{i_1} \cdots \mathfrak{t}_{i_n}$. If $j \in E(\beta /\!\!/ \alpha)$, then there exists a configuration in $\beta /\!\!/ \alpha$ of the following form.

$$\begin{array}{|c|c|}\hline \tau_{(i,j)} & \tau_{(i,j+1)} \\ \hline \end{array}$$

We know that $\tau_{(i,j)} > \tau_{(i,j+1)}$, and this implies that there exists $1 \le p < q \le n$ such that $\mathfrak{t}_{i_p} = \mathfrak{t}_{j+1}$ and $\mathfrak{t}_{i_q} = \mathfrak{t}_j$. But since w is a reverse hookword, this immediately implies that $j \in \mathrm{leg}(w)$. □

Lemma 3.3.12. *Given $\beta \in B_{\alpha,n}$, if for any $w \in CRHW_n$ such that $w(\alpha) = \beta$ we have that column $j \in SE(\beta /\!\!/ \alpha)$, then $j \notin \mathrm{leg}(w)$.*

Proof. Let τ be the SRCT of shape $\beta /\!\!/ \alpha$ corresponding to the word w. Assume first that $j = 1$. Let the bottommost box in the first column of $\beta /\!\!/ \alpha$ be in position $(i, 1)$. Since $1 \in SE(\beta /\!\!/ \alpha)$, we know that all boxes in column 2 that belong to $\beta /\!\!/ \alpha$ lie strictly southeast of the box in position $(i, 1)$. Notice that $\tau_{(i,1)}$ is the greatest amongst all the numbers in column 1 of τ. We have the following configuration in τ.

$$\begin{array}{cc} \begin{array}{|c|}\hline \tau_{(i,1)} \\ \hline \end{array} & \\ \vdots & \\ \begin{array}{|c|}\hline \phantom{\tau_{(i,1)}} \\ \hline \end{array} & \!\!\!\!\begin{array}{|c|}\hline \tau_{(i',2)} \\ \hline \end{array} \end{array}$$

Since τ is an SRCT, we know that $\tau_{(i',2)} > \tau_{(i,1)}$. Thus in particular, the smallest entry in the second column of τ is strictly greater than $\tau_{(i,1)}$. Thus, the first time a box is added in the first column happens after all the boxes that were to be added in the second column by w have been added. Hence $1 \notin \mathrm{leg}(w)$.

Now assume $j \geq 2$. Let $\tau_{(i,j)}$ be the entry in the topmost box of the j-th column of τ. This entry is greater than every other entry in the j-th column of τ by Lemma 3.3.5. Since $j \in SE(\beta /\!\!/ \alpha)$, we are guaranteed the existence of a box in position $(i', j+1)$ where $i' > i$. Arguing like before, we must have $\tau_{(i',j+1)} > \tau_{(i,j)}$. Hence there is at least one entry in the $j+1$-th column of τ that is greater than every entry in the j-th column of τ. Given that w is a reverse hookword, this implies that $j \notin \mathrm{leg}(w)$. $\square$

Lemma 3.3.13. *Let $\beta \in B_{\alpha,n}$. If $j \in NE(\beta /\!\!/ \alpha)$, then $j \geq 2$.*

Proof. Let $w \in CRHW_n$ be such that $w(\alpha) = \beta$. Assume that the claim is not true. Hence $1 \in NE(\beta /\!\!/ \alpha)$. Thus, we must have that $\{1, 2\} \subseteq \mathrm{supp}(\beta /\!\!/ \alpha)$. If $1 \in \mathrm{leg}(w)$, then the way the box adding operators act implies that $1 \in E(\beta /\!\!/ \alpha)$, as $\mathfrak{t}_2$ will add a box adjacent to a newly added box resulting from the operator $\mathfrak{t}_1$ applied before it.

Now, assume $1 \notin \mathrm{leg}(w)$. Then w factors uniquely as $\mathfrak{t}_1^k w_1$, where w_1 has no instances of $\mathfrak{t}_1$. Then we know that in computing $\mathfrak{t}_1^k(w_1(\alpha))$, the boxes added in column 1 will be strictly northwest of the boxes added in column 2 while computing $w_1(\alpha)$. This also follows from how $\mathfrak{t}_1$ acts. But then $1 \in SE(\beta /\!\!/ \alpha)$. Again, this is a contradiction. Hence we must have that $j \geq 2$. $\square$

Lemma 3.3.14. *Let $j \geq 2$ be a positive integer and μ be a composition. Suppose that μ has parts equalling j and $j-1$ and that the number of parts that equal j and lie to the left of the leftmost instance of a part equalling $j-1$ is m. Then for all $0 \leq k \leq m$, we have*

$$\mathfrak{t}_j \mathfrak{t}_{j+1}^k(\mu) = \mathfrak{t}_{j+1}^k \mathfrak{t}_j(\mu).$$

Proof. The case where $m = 0$ is trivial. Assume that $m \geq 1$.

The operator $\mathfrak{t}_{j+1}^k$ adds a box to each of the k leftmost parts in μ that equal j. We know that there are m parts in μ that equal j and lie to the left of the leftmost instance of a part that equals $j-1$. Since $k \leq m$, we are guaranteed that $\mathfrak{t}_j \mathfrak{t}_{j+1}^k(\mu) = \mathfrak{t}_{j+1}^k \mathfrak{t}_j(\mu)$. $\square$

Lemma 3.3.15. *Suppose $\beta \in B_{\alpha,n}$ and that $j \in NE(\beta /\!\!/ \alpha)$. Let $w \in CRHW_n$ such that $w(\alpha) = \beta$. Define an element $w' \in CRHW_n$ as follows.*

- *If $j \in \mathrm{leg}(w)$, then let w' be the unique reverse hookword with the same content as w obtained by setting $\mathrm{leg}(w') = \mathrm{leg}(w) \setminus \{j\}$.*

- *If $j \notin \operatorname{leg}(w)$, then let w' be the unique reverse hookword with the same content as w obtained by setting $\operatorname{leg}(w') = \operatorname{leg}(w) \cup \{j\}$.*

Then $w'(\alpha) = \beta$.

Proof. Since $j \in NE(\beta /\!\!/ \alpha)$, we know that j is not the maximum element of $\operatorname{supp}(\beta /\!\!/ \alpha)$. Hence, either $j \in \operatorname{leg}(w)$ or $j \notin \operatorname{leg}(w)$. Thus, the word w' is well-defined. Furthermore, by Lemma 3.3.13 we know that $j \geq 2$.

Let the number of $\mathfrak{t}_{j+1}$ in w equal c where $c \geq 1$. Let the largest common suffix of w and w' be w_1. All operators $\mathfrak{t}_k$ that appear in w_1 satisfy $k \leq j-1$. Let $\mu = w_1(\alpha)$. Since $j \in NE(\beta /\!\!/ \alpha)$, we know that there are at least c instances of a part of length j to the left of the leftmost part of length $j-1$ in μ. Thus $\mathfrak{t}_j \mathfrak{t}_{j+1}^m(\mu) = \mathfrak{t}_{j+1}^m \mathfrak{t}_j(\mu)$ for $0 \leq m \leq c$ by Lemma 3.3.14. This combined with the fact that $\mathfrak{t}_j \mathfrak{t}_i = \mathfrak{t}_i \mathfrak{t}_j$ for $i - j \geq 2$ implies that $w(\alpha) = w'(\alpha) = \beta$, as required. □

Lemma 3.3.16. *If $\beta \in B_{\alpha,n}$, then the number of words $w \in CRHW_n$ such that $w(\alpha) = \beta$ equals $2^{|NE(\beta /\!\!/ \alpha)|}$.*

Proof. Let p be the maximum element of $\operatorname{supp}(\beta /\!\!/ \alpha)$. Then, by Remark 3.3.2, we know that

$$\operatorname{supp}(\beta /\!\!/ \alpha) = \{p\} \uplus E(\beta /\!\!/ \alpha) \uplus SE(\beta /\!\!/ \alpha) \uplus NE(\beta /\!\!/ \alpha). \tag{3.9}$$

Let w be an element of $CRHW_n$ such that $w(\alpha) = \beta$. Since we know that the number of instances of $\mathfrak{t}_k$ in w is equal to the number of boxes in column k of $\beta /\!\!/ \alpha$, w is completely determined by $\operatorname{leg}(w)$.

Now by Lemma 3.3.11, we have that every element of $E(\beta /\!\!/ \alpha)$ belongs to $\operatorname{leg}(w)$. Lemma 3.3.12 implies that every element of $SE(\beta /\!\!/ \alpha)$ does not belong to $\operatorname{leg}(w)$. As far as elements of $NE(\beta /\!\!/ \alpha)$ are concerned, we know by Lemma 3.3.15, that it does not matter whether they belong to $\operatorname{leg}(w)$ or not, as the final shape β is going to be the same. Thus, for every subset $X \subseteq NE(\beta /\!\!/ \alpha)$, the word $w \in CRHW_n$ satisfying

$$\operatorname{leg}(w) = E(\beta /\!\!/ \alpha) \uplus X \uplus \{p\} \tag{3.10}$$

has the property that $w(\alpha) = \beta$. Hence the number of such words is $2^{|NE(\beta /\!\!/ \alpha)|}$ as claimed. □

Lemma 3.3.17. *Let $\beta \in B_{\alpha,n}$. Then*

$$\sum_{\substack{w \in CRHW_n \\ w(\alpha)=\beta}} (-1)^{\operatorname{asc}(w)} = (-1)^{n-1-|E(\beta /\!\!/ \alpha)|} \delta_{0,|NE(\beta /\!\!/ \alpha)|}$$

where δ denotes the Kronecker delta function.

Proof. Firstly, recall that $\operatorname{asc}(w) = |\operatorname{arm}(w)| - 1$. This in turn implies that $\operatorname{asc}(w) = n - |\operatorname{leg}(w)|$. By Lemma 3.3.11, we know that $E(\beta /\!\!/ \alpha) \subseteq \operatorname{leg}(w)$. Further, if p is the maximum element of $\operatorname{supp}(w)$, then $p \in \operatorname{leg}(w)$ as well. Lemma 3.3.16 implies that any subset of $NE(\beta /\!\!/ \alpha)$ can belong to $\operatorname{leg}(w)$, and Lemma 3.3.12 gives that no element of $SE(\beta /\!\!/ \alpha)$ belongs to $\operatorname{leg}(w)$.

Thus, we have the following sequence of equalities.

$$\begin{aligned}
\sum_{\substack{w \in CRHW_n \\ w(\alpha)=\beta}} (-1)^{\operatorname{asc}(w)} &= \sum_{\substack{w \in CRHW_n \\ w(\alpha)=\beta}} (-1)^{n-|\operatorname{leg}(w)|} \\
&= \sum_{X \subseteq NE(\beta /\!\!/ \alpha)} (-1)^{n-1-|E(\beta /\!\!/ \alpha)|-|X|} \\
&= (-1)^{n-1-|E(\beta /\!\!/ \alpha)|} \sum_{X \subseteq NE(\beta /\!\!/ \alpha)} (-1)^{|X|} \\
&= (-1)^{n-1-|E(\beta /\!\!/ \alpha)|} \delta_{0,|NE(\beta /\!\!/ \alpha)|} \qquad (3.11)
\end{aligned}$$

□

What follows is essentially a restatement of the above lemma.

Corollary 3.3.18. *If $\beta \in B_{\alpha,n}$ then*

$$k_\beta = \begin{cases} 0 & |NE(\beta /\!\!/ \alpha)| \geq 1 \\ (-1)^{n-1-|E(\beta /\!\!/ \alpha)|} & |NE(\beta /\!\!/ \alpha)| = 0. \end{cases}$$

Now, consider the following set of compositions.

$$P_{\alpha,n} = \{\beta : \beta \in B_{\alpha,n}, |NE(\beta /\!\!/ \alpha)| = 0\}$$

Before we state a Murnaghan-Nakayama rule in its final definitive form, we will need another short lemma.

Lemma 3.3.19. *If $\beta /\!\!/ \alpha$ is an nc border strip of size n, then $n - 1 - |E(\beta /\!\!/ \alpha)| = ht(\beta /\!\!/ \alpha)$.*

Proof. Suppose there are $k > 0$ boxes in some row of $\beta /\!\!/ \alpha$, then this row contributes $k - 1$ to $|E(\beta /\!\!/ \alpha)|$. Let the number of boxes in rows that contain at least one box be $k_1, k_2, \ldots, k_r$

for some positive integer r. Notice that $\sum_{j=1}^{r}(k_j - 1)$ equals $n - r$. Thus we have that

$$\begin{aligned} n - 1 - |E(\beta /\!\!/ \alpha)| &= n - 1 - \sum_{j=1}^{r}(k_j - 1) \\ &= r - 1. \end{aligned} \tag{3.12}$$

Now notice that $r - 1$ is precisely $ht(\beta /\!\!/ \alpha)$. $\square$

This given, we can write down our analogue of the Murnaghan-Nakayama rule for noncommutative Schur functions as follows.

Theorem 3.3.20. *Given a composition α and a positive integer n, we have the following expansion.*

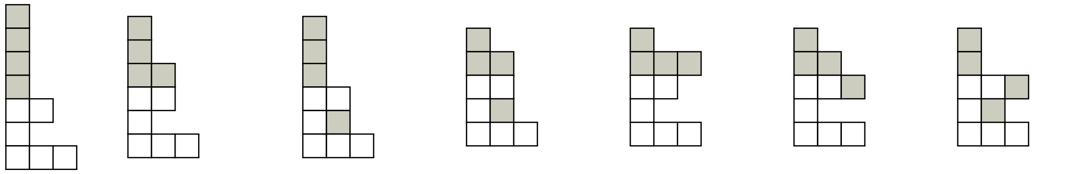
$$\Psi_n \cdot \mathbf{s}_\alpha = \sum_{\beta \in P_{\alpha,n}} (-1)^{ht(\beta /\!\!/ \alpha)} \mathbf{s}_\beta$$

While we do not attempt it here, the above theorem can be used by compute the expansion of $\Psi_{(\gamma_1,\ldots,\gamma_k)}$ in the basis of noncommutative Schur functions. To see this, consider the equality below.

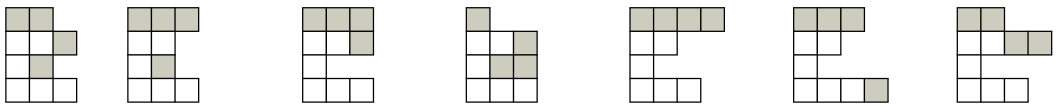
$$\Psi_{(\gamma_1,\ldots,\gamma_k)} = \Psi_{\gamma_1} \cdots \Psi_{\gamma_k} \cdot \mathbf{s}_\varnothing$$

Now applying Theorem 3.3.20 iteratively computes the desired expansion. Thus, in theory, we can compute the transition matrix between the basis of noncommutative power sum symmetric functions and the noncommutative Schur functions using Theorem 3.3.20.

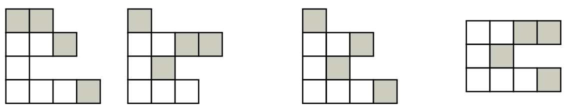
Example 3.3.21. *Consider the computation of $\Psi_4 \cdot \mathbf{s}_{(2,1,3)}$. We need to find all compositions $\beta \in B_{(2,1,3),4}$ satisfying $|NE(\beta /\!\!/ (2,1,3))| = 0$. We will start by listing all the elements of $B_{(2,1,3),4}$ where the boxes of $\beta /\!\!/ \alpha$ will be shaded. In the list below, beneath every composition we have noted those reverse hookwords that act on $(2,1,3)$ to give the composition under consideration. For the sake of clarity, the occurrences of $\mathfrak{t}$ in the reverse hookwords have been suppressed. Thus, for example, 1121 denotes $\mathfrak{t}_1\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_1$. Furthermore, if the words below* *a composition β are underlined, then $|NE(\beta /\!\!/ (2,1,3)| \geq 1$.*



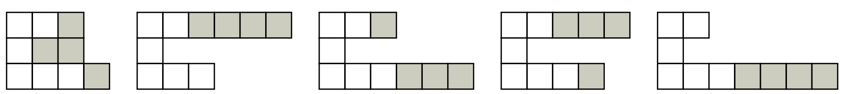
1111 1121 1112 1221 1321 1231 1132,1123



2231 2321 3321 1332 4321 3421 2431

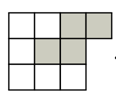


2341 1432, 1243 1342,1234 4432,2443 5432,2543 3542,2354 3432



3342 6543 3654 4543 7654

To illustrate the parity condition, we will compute the coefficient of



.

Since $ht((4,3,3) /\!\!/ (2,1,3)) = 1$, the coefficient of $\mathbf{s}_{(4,3,3)}$ in $\Psi_4 \cdot \mathbf{s}_{(2,1,3)}$ is -1. The complete expansion is as follows.

$$\begin{aligned}\Psi_4 \cdot \mathbf{s}_{(2,1,3)} = \quad & -\mathbf{s}_{(1,1,1,1,2,1,3)} + \mathbf{s}_{(1,1,2,2,1,3)} - \mathbf{s}_{(1,1,1,2,2,3)} + \mathbf{s}_{(1,2,2,2,3)} - \mathbf{s}_{(1,3,2,1,3)} + \mathbf{s}_{(1,2,3,1,3)} \\ & +\mathbf{s}_{(2,3,2,3)} - \mathbf{s}_{(3,2,2,3)} - \mathbf{s}_{(3,3,1,3)} + \mathbf{s}_{(1,3,3,3)} + \mathbf{s}_{(4,2,1,3)} - \mathbf{s}_{(3,2,1,4)} - \mathbf{s}_{(2,4,1,3)} \\ & +\mathbf{s}_{(2,3,1,4)}) - \mathbf{s}_{(4,3,3)} + \mathbf{s}_{(3,3,4)} + \mathbf{s}_{(6,1,3)} - \mathbf{s}_{(3,1,6)} - \mathbf{s}_{(5,1,4)} + \mathbf{s}_{(2,1,7)}\end{aligned}$$

Chapter 4

Jeu de taquin and right Pieri rules

In this chapter, we will give proofs for the results in Subsection 2.6.3.

4.1 Proof of validity of algorithm μ_i

We will work under the following assumptions in this section.

- i will denote a positive integer ≥ 1.
- T will refer to a fixed SSRT of shape λ.
- The SSRCT $\rho^{-1}(T)$ will be denoted by τ and we will assume that it has shape α where α clearly satisfies $\widetilde{\alpha} = \lambda$.
- There exists an addable node in column $i+1$ of λ.

A particular entry of T that plays a significant role in what follows is the first entry that moves horizontally while computing $\mathrm{jdt}_{i+1}(T)$. This entry is $T_{j,i}$ where j is the largest integer such that $T_{j,i} < T_{j-1,i+1}$. (We assume $T_{0,s} = \infty$ for all positive integers s.) We shall denote this entry by $f_{i+1}(T)$ (the reason behind the name being that it is the $f$irst entry that moves horizontally). We will use this in the next couple of lemmas, whose proofs are straightforward.

Lemma 4.1.1. *Let $f_{i+1}(T) = T_{r,i}$. Let T' be the tableau defined below.*

$$T'_{p,q} = T_{p,q} \text{ if } q \neq i$$
$$T'_{p,i} = T_{p,i} \text{ if } p < r$$
$$T'_{p,i} = T_{p+1,i} \text{ if } p \geq r.$$

Then T' is an SSRT.

Remark 4.1.2. *In words, T' is obtained from T by first removing $T_{r,i}$ and sliding all entries above it in the i-th column down by one row. Note that the rest of the tableau is left unchanged.*

Proof. The only thing we need to check is that the entries in each row of T' are weakly decreasing when read from left to right. More specifically, it follows from the preceding remark that we only need to check if $T'_{j,i} \geq T'_{j,i+1}$ and $T'_{j,i-1} \geq T'_{j,i}$ (assuming $i \geq 2$ for this case) for $j \geq r$.

The former inequality is clear since $T'_{j,i} = T_{j+1,i}$ and $T'_{j,i+1} = T_{j,i+1}$ for $j \geq r$ and by our hypothesis $T_{r,i}$ is the first entry that moves horizontally when a backward jeu de taquin slide is initiated from the $i+1$-th column.

Now we will show that $T'_{j,i-1} \geq T'_{j,i}$ for $j \geq r$. We have $T'_{j,i} = T_{j+1,i}$ and $T'_{j,i-1} = T_{j,i-1}$. But it is clear that $T_{j,i-1} > T_{j+1,i}$ in any SSRT T. Hence the claim follows. $\square$

We will denote the SSRT T' above by $\mathrm{tjdt}_{i+1}(T)$ (the reason behind the name tjdt is that it stands for a 'truncated' jeu de taquin slide). Furthermore, we define $\mathrm{tjdt}_1(T) = T$, as no entries move horizontally when a backward jeu de taquin slide is initiated from the first column.

Example 4.1.3. *Let*

$$T = \begin{array}{|c|c|c|c|}\cline{1-1} 6 \\ \cline{1-2} 7 & 6 \\ \cline{1-3} 8 & 7 & 4 \\ \cline{1-3} 10 & 8 & 5 \\ \hline 11 & 9 & 9 & 8 \\ \hline \end{array} .$$

If we consider a backward jeu de taquin slide from the addable node in column 3, then we have $f_3(T) = T_{2,2} = 8$. Thus

$$\mathrm{tjdt}_3(T) = \begin{array}{|c|c|c|c|}\cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-3} 8 & 6 & 4 \\ \cline{1-3} 10 & 7 & 5 \\ \hline 11 & 9 & 9 & 8 \\ \hline \end{array} .$$

In the computation of $\mathrm{jdt}_{i+1}(T)$, clearly i entries in T move horizontally. It is crucial for us to know these entries. Define

$$H_{i+1}(T) \quad = \quad \text{multiset of entries in } T \text{ that move horizontally under } \mathrm{jdt}_{i+1}(T).$$

We define $H_1(T)$ to be the empty set. With this definition at hand, we state a second lemma.

Lemma 4.1.4.

$$H_i(\mathrm{tjdt}_{i+1}(T)) \cup \{f_{i+1}(T)\} \quad = \quad H_{i+1}(T).$$

In the equality above, we are considering union of multisets.

Proof. Let $T' = \mathrm{tjdt}_{i+1}(T)$ and $f_{i+1}(T) = T_{r,i}$. Our construction of T' implies that there is an addable node in column i in the partition shape underlying T'. Hence, we can initiate a backward jeu de taquin slide from the i-th column of T' and $H_i(T')$ is well-defined.

If $i = 1$, then we have nothing to prove. Hence, we can assume that $i \geq 2$. We will show that no entry of the form $T'_{j,i-1}$ with $j > r$ moves horizontally when a backward jeu de taquin slide is initiated from column i in T', as this clearly suffices to establish the claim.

Suppose this was not the case. Then there exists $j > r$ such that

$$T'_{j,i-1} < T'_{j-1,i}.$$

Since $T'_{j,i-1} = T_{j,i-1}$ and $T'_{j-1,i} = T_{j,i}$ for $j > r$, the inequality above implies

$$T_{j,i-1} < T_{j,i}.$$

But this contradicts the fact that T is an SSRT. $\square$

Example 4.1.5. *Let*

$$T = \begin{array}{|c|c|c|c|} \hline 6 \\ \cline{1-1} 7 & 6 \\ \cline{1-2} 8 & 7 & 2 \\ \hline 10 & 8 & 5 & 3 \\ \hline 11 & 9 & 9 & 8 \\ \hline \end{array}\;.$$

If we consider a backward jeu de taquin slide from the addable node in column 4, then the *entries in the shaded boxes move horizontally. Thus, we have*

$$H_4(T) = \{2, 8, 11\} \text{ and } f_4(T) = 2.$$

We also have that

$$\mathrm{tjdt}_4(T) = \begin{array}{|c|c|c|c|} \cline{1-1} 6 & \multicolumn{3}{c}{} \\ \cline{1-2} 7 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 8 & 7 & \multicolumn{2}{c}{} \\ \hline 10 & 8 & 5 & 3 \\ \hline 11 & 9 & 9 & 8 \\ \hline \end{array}.$$

Now if one does a backward jeu de taquin slide on the tableau above from the third column, then the entries in the shaded boxes move horizontally, therefore $H_3(\mathrm{tjdt}_4(T)) = \{8, 11\}$. *Thus, we can verify in this case that*

$$H_4(T) = H_3(\mathrm{tjdt}_4(T)) \cup f_4(T).$$

The lemma above allows us to recursively compute all the entries that move horizontally when a backward jeu de taquin slide is performed from an addable node in column $i + 1$. We already know how to obtain $\mathrm{tjdt}_{i+1}(T)$ given an SSRT T. Before we define a backward jeu de taquin slide for SSRCTs, let us consider the relatively easier question of computing $\rho^{-1}(\mathrm{tjdt}_{i+1}(T))$ given τ. This is precisely what the algorithm ϕ_{i+1} described in Subsection 2.6.3 achieves. We will establish that ϕ_{i+1} maps SSRCTs to SSRCTs such that the following diagram commutes.

$$\begin{tikzcd} \text{SSRTs} \arrow[r, "\rho^{-1}"] \arrow[d, "\mathrm{tjdt}_{i+1}"'] & \text{SSRCTs} \arrow[d, "\phi_{i+1}"] \\ \text{SSRTs} \arrow[r, "\rho^{-1}"] & \text{SSRCTs} \end{tikzcd}$$

To allow for cleaner proofs, we will now describe how we will think of T and τ diagrammatically. We will think of T as aligned to the left in a rectangular Ferrers diagram where

$$\begin{aligned} \text{number of rows, } n &= l(\lambda), \\ \text{number of columns} &\geq \lambda_1 + 1. \end{aligned}$$

The empty boxes of this Ferrers diagram are assumed to be filled with 0s. We will call this diagram $T_\square$. Note that the 0s do not play any role as far as doing the backward jeu de taquin slides are concerned.

Similarly, we will think of τ as being placed left aligned in a rectangular Ferrers diagram where

$$\begin{aligned} \text{number of rows, } n &= l(\alpha), \\ \text{number of columns } &\geq \lambda_1 + 1. \end{aligned}$$

The empty boxes of this Ferrers diagram are assumed to be filled with 0s. We will assume that the dimensions of the rectangular Ferrers diagram are the same for both T and τ. The exact dimensions will not matter as long as they satisfy the constraints mentioned above. We will call the rectangular filling $\tau_\square$.

For computing $\mathrm{tjdt}_{i+1}(T)$, we only need to know what the entries in columns i and $i+1$ are. Since we will work with $T_\square$ we will take the 0s into account when considering the i-th and $i+1$-th columns. Let the i-th and $i+1$-th columns of $T_\square$ be as shown below.

$$\begin{array}{|c|c|} \hline a_1 & b_1 \\ \hline a_2 & b_2 \\ \hline \vdots & \vdots \\ \hline a_n & b_n \\ \hline \end{array}$$

Furthermore, let $a_{n+1} = b_{n+1} = \infty$. Since we have assumed that there is an addable node in the $i+1$-th column of λ, we know that

$$|\{b_r : b_r = 0,\ 1 \leq r \leq n\}| > |\{a_r : a_r = 0,\ 1 \leq r \leq n\}|. \tag{4.1}$$

Let

$$min = \text{smallest positive integer such that } a_{min} < b_{min+1}.$$

Then, we have that

$$a_{min} = f_{i+1}(T).$$

Clearly, $a_{min} > 0$. Another trivial observation, which we will use without mentioning is that $a_p = a_q$ for some $1 \leq p, q \leq n$ if and only if $a_p = a_q = 0$.

Corresponding to $T_\square$ above, the i-th and $i+1$-th columns of $\tau_\square$ are as shown below.

$$\begin{array}{|c|c|}\hline c_1 & d_1 \\ \hline c_2 & d_2 \\ \hline \vdots & \vdots \\ \hline c_n & d_n \\ \hline \end{array}$$

Clearly the sequence $c_1, c_2, \ldots, c_n$ is a rearrangement of the sequence $a_1, a_2, \ldots, a_n$. A similar statement holds for the sequences $d_1, d_2, \ldots, d_n$ and $b_1, b_2, \ldots, b_n$.

Now let us consider what we require from the procedure ϕ_{i+1}. We would like it to locate $f_{i+1}(T)$ in τ and then remove it from τ and rearrange the remaining entries so that what we have is still an SSRCT. The algorithm to accomplish this will be presented after some intermediate lemmas. For all the lemmas that follow, we will assume that $\mathbf{s_1}$ **is the positive integer such that**

$$c_{s_1} = a_{min}.$$

Our first lemma implies that a_{min} does not equal any other entry in the i-th and $i+1$-th columns of $T_\square$. We know that $a_{min} > 0$. Hence it clearly can not equal any other entry in the i-th column since T is an SSRT. To see that a_{min} does not equal any entry in the $i+1$-th column, the following suffices.

Lemma 4.1.6. $a_{min} \neq b_{min}$.

Proof. Assume, contrary to what is to be established, that $a_{min} = b_{min}$. Since $a_{min} > 0$, we get that $min > 1$, as b_1 is definitely 0 given the inequality in (4.1). Now, since $a_{min} > a_{min-1}$, our assumption gives $a_{min-1} < b_{min}$. But this contradicts the definition of min. $\square$

Our next lemma deals with entries greater than $c_{s_1} = a_{min}$ in $\tau_\square$.

Lemma 4.1.7. $c_t > c_{s_1} \iff d_t \geq b_{min+1}$.

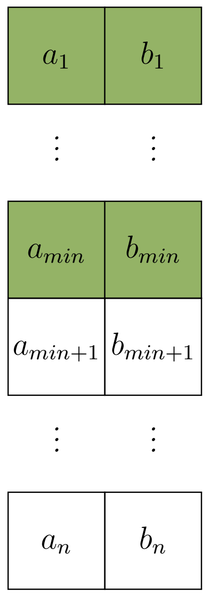
Proof. Since $c_{s_1} = a_{min}$, we know that $b_{min+1} > c_{s_1}$. Thus, $b_{min+1} > c_j$ for all $1 \leq j \leq n$ satisfying $c_j \leq c_{s_1}$. Thus any entry in the $i+1$-th column of $T_\square$ that is $\geq b_{min+1}$ can not be the neighbour of any entry that is $\leq c_{s_1}$ in the i-th column of $\tau_\square$. Consider the following sets.

$$\begin{aligned} X &= \{c_j : c_j > c_{s_1}\} \\ Y &= \{b_j : b_j \geq b_{min+1}\} \end{aligned}$$

The fact that X and Y are sets, and not multisets, follows from the fact that $b_{min+1} > c_{s_1} = a_{min} > 0$. Now, we know from our argument at the beginning of this proof that an element of Y can only neighbour an element of X in $\tau_\square$. But $|X| = |Y|$, as X is the same as $\{a_j : a_j > a_{min}\}$. Hence each element of Y is the neighbour to a unique element of X, and each element of X has a neighbour that is an element of Y. Hence the claim follows. $\square$

We note down an important consequence of the above lemma in the following remark.

Remark 4.1.8. *Consider the i-th and $i+1$-th columns of $T_\square$ again.*



The boxes in the i-th column that contain entries $\leq a_{min}$ and the boxes in the $i+1$-th column that contain entries $\leq b_{min}$ are shaded. All other boxes, that is, the boxes in the i-th column that contain entries $> a_{min}$ and the boxes in the $i+1$-th column that contain entries $> b_{min}$ are unshaded. Notice that the sets X and Y defined in the proof of Lemma 4.1.7 correspond to the unshaded entries in the i-th column and $i+1$-th column respectively. Then, Lemma 4.1.7 implies that in $\tau_\square$, the values belonging to the shaded boxes in $T_\square$ get neighbours from the shaded boxes only, while the values that belong to the unshaded boxes in $T_\square$ get neighbours from the unshaded boxes only.

Recall now that s_1 is the positive integer such that $c_{s_1} = a_{min}$. Define recursively integers s_j satisfying $1 \leq s_j \leq n$ for $j \geq 2$ using the following criteria:

I. $s_j > s_{j-1}$,

II. c_{s_j} is the greatest positive integer that satisfies

$$c_{s_{j-1}} > c_{s_j} \geq d_{s_{j-1}}.$$

Clearly, only finitely many s_j exist. Let the maximum be given by s_k and let

$$S = \{s_1, \ldots, s_k\}.$$

With this setup, we have the following lemma.

Lemma 4.1.9. *If $t > s_j$ for some $1 \leq j \leq k$ such that $c_t > c_{s_j}$, then $c_t > c_{s_1}$.*

Proof. If $j = 1$, the claim is clearly true. Hence assume $j \geq 2$. Shown below is the configuration in $\tau_\square$ that we will focus on.

$$\begin{array}{|c|c|}
\hline
c_{s_{j-1}} & d_{s_{j-1}} \\
\hline
\multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} \\
\hline
c_{s_j} & d_{s_j} \\
\hline
\multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{} \\
\cline{1-1}
c_t & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}$$

Firstly, note that since $c_{s_1} > c_{s_2} > \cdots > c_{s_k}$, we can not have t being equal to any s_i for some $1 \leq i \leq k$. For if $t = s_i$ for some i, then we know that $c_t < c_{s_j}$ for all $s_j < t$. This contradicts our assumption on t.

We will now show that the hypothesis implies $c_t > c_{s_{j-1}}$, as this suffices to establish the claim. We know that

$$c_{s_{j-1}} > c_{s_j} \geq d_{s_{j-1}}. \tag{4.2}$$

We also know, by the definition of s_j, that $s_j > s_{j-1}$ is such that c_{s_j} is the greatest positive integer satisfying the inequalities in (4.2). Since $t > s_j$, we know that c_t does not satisfy

$$c_{s_{j-1}} > c_t \geq d_{s_{j-1}}. \tag{4.3}$$

But clearly c_t satisfies $c_t > d_{s_{j-1}}$, by using the hypothesis that $c_t > c_{s_j}$ and (4.2). Thus, we must have that $c_t \geq c_{s_{j-1}}$, if c_t is not to satisfy (4.3). Now, since $c_{s_{j-1}}$ is positive by definition, we get that so is c_t. Thus, in fact, $c_t > c_{s_{j-1}}$. □

Lemma 4.1.10. *Let t be a positive integer such that $s_j < t \neq s_{j+1}$ for some $1 \leq j \leq k-1$. Then exactly one of the following holds.*

1. $c_t > c_{s_1}$.

2. $c_t < c_{s_{j+1}}$.

Proof. Notice that if $c_t > c_{s_j}$, then Lemma 4.1.9 implies that $c_t > c_{s_1}$, which is precisely the first condition in the claim.

Now, assume that $c_{s_j} > c_t > c_{s_{j+1}}$. The definition of s_{j+1} implies that $c_{s_j} > c_{s_{j+1}} \geq d_{s_j}$. Thus we get that

$$c_{s_j} > c_t \geq d_{s_j}. \tag{4.4}$$

But since $c_t > c_{s_{j+1}}$ and $t > s_j$, (4.4) and the recursive definition of the elements of S would imply that s_{j+1} equals t. This is clearly not the case. Therefore $c_t \leq c_{s_{j+1}}$. Again using the fact that $c_{s_{j+1}}$ is positive, we get that $c_t < c_{s_{j+1}}$. □

This brings us to the following important lemma which states that c_{s_k} occupies the rightmost box in the bottommost row of size i in α, where recall that α is the shape of τ. Here we are identifying α with its reverse composition diagram. Establishing that c_{s_k} occupies the rightmost box in a row of size i is the same as proving that $d_{s_k} = 0$. That it belongs to the bottommost row of length i requires us to prove that an entry c_t that lies strictly south of c_{s_k} is either 0 or is such that its neighbour $d_t > 0$.

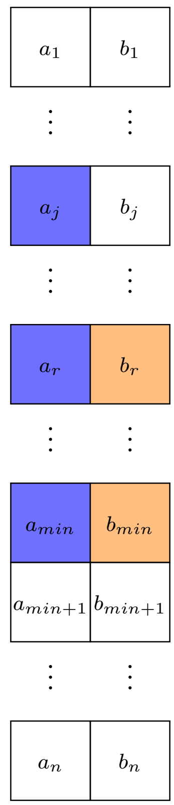
Lemma 4.1.11. *1. $d_{s_k} = 0$.*

2. If $t > s_k$ and $c_t > 0$, then $c_t > c_{s_1}$ and $d_t > 0$.

Proof. Consider first the i-th and $i+1$-th columns of $T_{\square}$. For any positive integer r satisfying $1 \leq r \leq min$, consider the following multisets.

$$\begin{aligned} X_r &= \{a_j : a_{min} \geq a_j \geq b_r\} \\ Y_r &= \{b_j : b_{min} \geq b_j \geq b_r\} \end{aligned}$$

Notice that X_r and Y_r can be multisets only when $b_r = 0$. Notice further that if $b_r = 0$, then the cardinalities of the multisets X_r and Y_r are equal. For a generic r, the following diagram describes the elements of X_r (boxes shaded darker in the i-th column) and Y_r (boxes shaded lighter in the $i+1$-th column).



Assume, contrary to what is to be proved, that $d_{s_k} > 0$. Then, we know that there exists a unique positive integer r such that $1 \leq r \leq n$ and $d_{s_k} = b_r$, and hence $b_r > 0$. Note also that $c_{s_k} \leq c_{s_1} < b_{min+1}$. This follows since $c_{s_1} = a_{min}$ and $a_{min} < b_{min+1}$.

Now, since $c_{s_k} \geq d_{s_k}$, we have that

$$b_r = d_{s_k} < b_{min+1}.$$

Using the above, combined with the fact fact that b_1 is always 0, we get that $1 < r \leq min$. Thus, we can consider the multisets X_r and Y_r. In fact, they are sets consisting of positive integers as $b_r > 0$. We have, by the definition of Y_r, that

$$d_{s_k} = b_r \in Y_r.$$

Furthermore, since

$$a_{min} = c_{s_1} \geq c_{s_k} \geq d_{s_k} = b_r,$$

we have $c_{s_k} \in X_r$. Notice now the important fact that the elements of Y_r can only be neighbours of elements in X_r, when we consider their respective positions in the i-th and $i+1$-th columns of $\tau_\square$. This follows from Lemma 4.1.7, in a manner similar to the remark following it.

We know that $a_{r-1} \geq b_r = d_{s_k}$ as $r \leq min$. Hence, we obtain the following inequality.

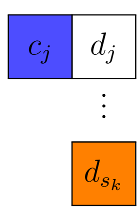
$$|X_r| \geq |Y_r| + 1$$

This in turn implies

$$|X_r \setminus \{c_{s_k}\}| \geq |Y_r \setminus \{d_{s_k}\}| + 1. \tag{4.5}$$

Now consider the i-th and $i+1$-th columns of $\tau_\square$. Each element of Y_r neighbours some element of X_r. Hence each element of $Y_r \setminus \{d_{s_k}\}$ neighbours a unique element of $X_r \setminus \{c_{s_k}\}$. Now, (4.5) implies that there is at least one element of $X_r \setminus \{c_{s_k}\}$ whose neighbour does not belong to $Y_r \setminus \{d_{s_k}\}$. This neighbour has to be strictly smaller than d_{s_k} clearly, as $d_{s_k} = b_r$ is the smallest entry in Y_r. The strictness follows from $d_{s_k} > 0$.

If, in $\tau_\square$, all elements of the set $X_r \setminus \{c_{s_k}\}$ lie strictly north of the box occupied by c_{s_k}, then the argument earlier implies that for some $c_j \in X_r \setminus \{c_{s_k}\}$, we have $d_j < d_{s_k}$. But this corresponds to a triple rule violation because of the existence of the configuration shown below.



satisfying $c_j \geq d_{s_k} > d_j \geq 0$

Hence, there is at least one element $c_j \in X_r \setminus \{c_{s_k}\}$ that lies strictly south of the box occupied by c_{s_k}. Any such element has to be greater than c_{s_k}. If not then it would satisfy the inequality $c_{s_k} > c_j \geq d_{s_k}$. But this would allows us to define s_{k+1}, which we know does not exist. Hence $c_j > c_{s_k}$. But then Lemma 4.1.9 implies that $c_j > c_{s_1}$. As $c_{s_1} = a_{min}$, this contradicts the fact that $c_j \in X_r \setminus \{c_{s_k}\}$. Hence, we conclude that $d_{s_k} = 0$. This establishes the first part of the claim.

As for the second part, note for any positive integer $j > s_k$ such that $c_j > 0$, we must have $c_j > c_{s_k}$ (again by the fact that s_{k+1} does not exist). Lemma 4.1.9 implies that $c_j > c_{s_1}$, while Lemma 4.1.7 implies that $d_j \geq b_{min+1} > 0$. This establishes the second part of the claim. □

Remark 4.1.12. *The lemma above implies that position of c_{s_k} in τ only depends on the shape of τ, and not the filling itself.*

We would like to be able to construct the integers s_j in reverse, since locating c_{s_k} is easier than locating c_{s_1} in $\tau_\square$. Thus, we would like an algorithm that takes as input the integer s_k (and $\tau_\square$) and terminates by giving s_1 as output. Such an algorithm would give us a way of computing $f_{i+1}(T) = a_{min} = c_{s_1}$ given τ. Before we achieve this aim, we need some notation.

Let q_1 be the greatest positive integer such that $d_{q_1} = 0$ and $c_{q_1} > 0$. Define a sequence of positive integers q_j satisfying $1 \leq q_j \leq n$ for $j \geq 2$ recursively according to the following conditions.

1. $q_j < q_{j-1}$,
2. q_j is the greatest positive integer such that

$$c_{q_j} > c_{q_{j-1}} \geq d_{q_j}.$$

By virtue of Lemma 4.1.11, we have that $q_1 = s_k$. In fact, $q_j = s_{k+1-j}$ for $1 \leq j \leq k$ as the next lemma shows.

Lemma 4.1.13. *For $1 \leq j \leq k$, we have $q_j = s_{k+1-j}$.*

Proof. Suppose that for some $1 \leq j \leq k-1$, we know that $q_j = s_{k+1-j}$. We will show that $q_{j+1} = s_{k-j}$. By the definition of q_{j+1}, we have

$$c_{q_{j+1}} > c_{q_j} \geq d_{q_{j+1}}. \tag{4.6}$$

Furthermore, since $q_j = s_{k+1-j}$, we have

$$c_{s_{k-j}} > c_{q_j} \geq d_{s_{k-j}}. \tag{4.7}$$

The recursive definition of q_{j+1} combined with (4.6) and (4.7) implies $q_{j+1} \geq s_{k-j}$. Suppose, contrary to what we seek to prove, that $q_{j+1} > s_{k-j}$. Thus, we have that $q_j > q_{j+1} > s_{k-j}$.

Diagrammatically, this corresponds to the following configuration in $\tau_\square$.

$$\begin{array}{|c|c|}
\hline
c_{s_{k-j}} & d_{s_{k-j}} \\
\hline
\multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} \\
\hline
c_{q_{j+1}} & d_{q_{j+1}} \\
\hline
\multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{} \\
\cline{1-1}
c_{q_j} & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}$$

From (4.6) and (4.7), we have $c_{q_{j+1}} > d_{s_{k-j}}$. Since $q_j = s_{k+1-j}$, we must have $c_{q_{j+1}} > c_{s_{k-j}}$. To see this, notice that had $c_{q_{j+1}} < c_{s_{k-j}}$ been true, the recursive definition of s_{k+1-j} given s_{k-j} would have implied that $s_{k+1-j} \leq q_{j+1}$, which is clearly not the case. Thus, we get that $c_{q_{j+1}} > c_{s_{k-j}}$.

But then, Lemma 4.1.9 implies $c_{q_{j+1}} > c_{s_1}$. By Lemma 4.1.7, this implies that

$$d_{q_{j+1}} \geq b_{min+1}. \tag{4.8}$$

Since $c_{s_1} \geq c_{s_{k+1-j}} = c_{q_j}$, we know that $c_{q_j} < b_{min+1}$, as $c_{s_1} = a_{min} < b_{min+1}$. But then the inequality in (4.8) is in contradiction to the inequality in (4.6). Hence $q_{j+1} = s_{k-j}$. Since $q_1 = s_k$, the claim follows by induction. $\square$

Thus, now we know that $q_k = s_1$. The next lemma will show that q_{k+1} does not exist.

Lemma 4.1.14. *There does not exist a positive integer t such that $t < s_1$ and*

$$c_t > c_{s_1} \geq d_t.$$

Proof. Suppose such a t exists. By Lemma 4.1.7, we have $d_t \geq b_{min+1}$. But we also know that $b_{min+1} > a_{min} = c_{s_1}$. Thus, the inequality in the statement of the lemma can not be satisfied. $\square$

We now know how to compute c_{s_1} given $\tau_\square$. Our next aim is to remove c_{s_1} from the i-th column of $\tau_\square$ and then rearrange entries therein so as to obtain $(\rho^{-1}(\mathrm{tjdt}_{i+1}(T)))_\square$. The short algorithm that accomplishes this is presented below.

Algorithm 4.1.15. *1. For $j = k, k-1, \ldots 2$ in that order, place $c_{q_{j-1}}$ in the box that originally contained c_{q_j}.*

2. *Place a* 0 *in the box that originally contained* c_{q_1}*. Call the resulting tableau* $\beta(\tau_\square)$*.*

3. *If there is a row in* $\beta(\tau_\square)$ *comprising entirely of* 0*s, we remove that row. The resulting rectangular filling is* $(\rho^{-1}(\mathrm{tjdt}_{i+1}(T)))_\square$*. If we omit* 0*s altogether, we obtain what we will call* $\phi_{i+1}(\tau)$*.*

We will demonstrate the above algorithm with an example. The proof of validity of the algorithm will be presented subsequently.

Example 4.1.16. *Suppose that*

$$T_\square = \begin{array}{|c|c|c|c|c|}\hline 7 & 3 & 0 & 0 & 0 \\ \hline 8 & 5 & 0 & 0 & 0 \\ \hline 9 & 6 & 2 & 1 & 0 \\ \hline \end{array}.$$

Say we want to compute $\mathrm{tjdt}_3(T)$*. It is clear that if we start a backward jeu de taquin slide from the addable node in the third column, then* 6 *is the first entry that moves horizontally. Thus* $\mathrm{tjdt}_3(T)$ *would be obtained by removing* 6 *from the second column. Then, we have*

$$(\mathrm{tjdt}_3(T))_\square = \begin{array}{|c|c|c|c|c|}\hline 7 & 0 & 0 & 0 & 0 \\ \hline 8 & 3 & 0 & 0 & 0 \\ \hline 9 & 5 & 2 & 1 & 0 \\ \hline \end{array}.$$

Now we try and repeat the above with the SSRCT $\tau = \rho^{-1}(T)$ *using Algorithm 4.1.15. We have*

$$\tau_\square = \begin{array}{|c|c|c|c|c|}\hline 7 & 6 & 2 & 1 & 0 \\ \hline 8 & 5 & 0 & 0 & 0 \\ \hline 9 & 3 & 0 & 0 & 0 \\ \hline \end{array}.$$

Then we have $q_1 = 3$ *as* c_3 *is the lowermost non-zero entry in the second column that has a neighbour equalling* 0*. Since* $c_2 > c_3 \geq d_2$ *we have* $q_2 = 2$*. Similarly, we have* $q_3 = 1$*. Thus,*

$$\beta(\tau_\square) = \begin{array}{|c|c|c|c|c|}\hline 7 & 5 & 2 & 1 & 0 \\ \hline 8 & 3 & 0 & 0 & 0 \\ \hline 9 & 0 & 0 & 0 & 0 \\ \hline \end{array}.$$

As there is no row comprising entirely of 0*s, by the third step in the algorithm, we have that* $(\rho^{-1}(\mathrm{tjdt}_3(T)))_\square$ *is the tableau above. We can remove the* 0*s to get a better picture. Then*

Algorithm 4.1.15 implies that

$$\phi_3(\tau) = \begin{array}{|c|c|c|c|} \hline 7 & 5 & 2 & 1 \\ \hline 8 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 9 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} .$$

We also have

$$\mathrm{tjdt}_3(T) = \begin{array}{|c|c|c|c|} \cline{1-1} 7 & \multicolumn{3}{c}{} \\ \cline{1-2} 8 & 3 & \multicolumn{2}{c}{} \\ \hline 9 & 5 & 2 & 1 \\ \hline \end{array} .$$

It is easily checked that $\phi_3(\tau) = \rho^{-1}(\mathrm{tjdt}_3(T))$.

Now we will proceed to prove the validity of Algorithm 4.1.15 by showing that $\phi_{i+1}(\tau)$ is an SSRCT. Since $\phi_{i+1}(\tau)$ is obtained from $\beta(\tau_\square)$ by removing extraneous 0s, we will work with $\beta(\tau_\square)$ as it is more convenient. We need to show that $\beta(\tau_\square)$ satisfies the conditions below.

- Excepting 0s, the first column of $\beta(\tau_\square)$ is strictly decreasing from bottom to top.
- It has no triple rule violations.
- The rows are weakly decreasing from left to right.

The third condition above is verified by an easy argument: Note that if the position of an entry in the i-th column remains unchanged, then the condition is automatically verified. Otherwise the entry c_{s_j} is replaced by $c_{s_{j+1}}$ for some $1 \leq j \leq k-1$. Since $c_{s_{j+1}} \geq d_{s_j}$ and $c_{s_j} > c_{s_{j+1}}$, we see that the algorithm does not change the fact that the rows were weakly decreasing from left to right. Finally, notice that c_{s_k} is replaced by a 0, but since d_{s_k} was already 0 by Lemma 4.1.11, we still have rows weakly decreasing from left to right.

Now we will verify that the first condition holds as well. Notice that if $i \geq 2$, then no changes are ever made to the first column by Algorithm 4.1.15. Hence, in this scenario $\phi_{i+1}(\tau)$ has the entries in the first column strictly decreasing from bottom to top. Consider now the case $i = 1$. Then, it is easily seen that Algorithm 4.1.15 replaces the bottom most non-zero entry (in the first column) whose neighbour is a 0 with a 0, and no other change is made. Hence, if one omits the 0s, the first column of $\beta(\tau_\square)$ is strictly decreasing from bottom to top.

Thus, we only need to prove that there are no triple rule violations in $\beta(\tau_\square)$. We will do this in cases.

Lemma 4.1.17. *In $\beta(\tau_\Box)$, consider any configuration of the type shown below*

$$\begin{array}{cc} \boxed{x} & \boxed{y} \\ & \vdots \\ & \boxed{z} \end{array},$$

where y and z belong to the $i+1$-th column. If $z > 0$ and $z > y$, then $z > x$.

Proof. Notice that Algorithm 4.1.15 does not alter the position of any entry in the $i+1$-th column. Let x' be the entry in $\tau_\Box$ in the box currently occupied by x in $\beta(\tau_\Box)$. Since there are no triple rule violations in $\tau_\Box$, we know that

$$z > 0 \text{ and } z > y \Longrightarrow z > x' \tag{4.9}$$

We have two possibilities to deal with. Firstly, if $x = x'$, then the claim follows easily from (4.9). Now, assume $x \neq x'$. Algorithm 4.1.15 implies that $x < x'$. Again, (4.9) implies the claim. $\Box$

Lemma 4.1.18. *In $\beta(\tau_\Box)$, consider any configuration of the type shown below*

$$\begin{array}{cc} \boxed{x} & \boxed{y} \\ & \vdots \\ & \boxed{z} \end{array},$$

where y and z belong to the i-th column. If $z > 0$ and $z > y$, then $z > x$.

Proof. Firstly, notice that we only need to worry about this configuration when $i \geq 2$.

Let x', y' and z' be the entries in $\tau_\Box$ in the boxes corresponding to those occupied by x, y and z in $\beta(\tau_\Box)$ respectively. Since Algorithm 4.1.15 does not change the $i-1$-th column, we have $x = x'$. This along with the fact that there are no triple rule violations in $\tau_\Box$ implies

$$z' > 0 \text{ and } z' > y' \Longrightarrow z' > x.$$

If $y = y'$ and $z = z'$, then we have nothing to prove.

Now assume that $z = z'$ but $y \neq y'$. Then we have two cases. In the first case, we have $y' = c_{s_k}$ and thus, $y = 0$. Furthermore, since z occupies a box below c_{s_k}, we know that $z > c_{s_1}$, by Lemma 4.1.11. Now the claim follows readily for this case. In the second case, we have $y' = c_{s_j}$ and thus, $y = c_{s_{j+1}}$ for some $1 \leq j \leq k-1$. The possible relative positions of y, y' and z in $\tau_\Box$ are shown below.

$$\begin{array}{ccc} \boxed{y'} & \quad & \boxed{y'} \\ \vdots & & \vdots \\ \boxed{y} & & \boxed{z} \\ \vdots & & \vdots \\ \boxed{z} & & \boxed{y} \end{array}$$

According to Lemma 4.1.10, we have either $z > c_{s_1}$ (and hence $z > y, y'$) or $z < y$ (and hence $z < y'$). In any case, we see that if $z > 0$ and $z > y$, then $z > x$.

Now assume that $y \neq y'$ and $z \neq z'$. Then we must have $y = c_{s_{i_1}}$ and $z = c_{s_{i_2}}$ where $2 \leq i_1 < i_2 \leq k$. But then the situation where $z > y$ does not arise. $\square$

Thus, we have established that $\phi_{i+1}(\tau)$ is an SSRCT indeed. We will prove another lemma that is of similar flavour to the previous two lemmas, and which we will soon use.

Lemma 4.1.19. *In $\beta(\tau_\Box)$, consider a configuration of the type*

$$\boxed{x}\boxed{y} \qquad \text{where } x \text{ is in } i\text{-th column and } y \text{ in } i+1\text{-th column.}$$

If $c_{s_1} > y \geq 0$ then $c_{s_1} > x$.

Proof. For any configuration of the type in the statement of the claim, we first show that c_{s_1} can neither equal x nor y.

Recall that $c_{s_1} = a_{min}$ is the first entry that moves horizontally when computing $\mathrm{jdt}_{i+1}(T)$. This entry is removed from the i-th column of $\tau_\Box$ when $\beta(\tau_\Box)$ is computed. Hence it can not equal x. Lemma 4.1.6 implies it can not equal y either.

If $c_{s_1} > y \geq 0$, then we know that $y \leq b_{min}$. But then Lemma 4.1.7 yields that $x < c_{s_1}$, which is what we wanted. $\square$

We are now in a position to state our first important theorem.

Theorem 4.1.20. *The following diagram commutes.*

$$\begin{tikzcd} \text{SSRTs} \arrow[r, "\rho^{-1}"] \arrow[d, "\mathrm{tjdt}_{i+1}"'] & \text{SSRCTs} \arrow[d, "\phi_{i+1}"] \\ \text{SSRTs} \arrow[r, "\rho^{-1}"'] & \text{SSRCTs} \end{tikzcd}$$

Proof. We have already established that $\phi_{i+1}(\tau)$ is an SSRCT. This observation and the fact that we have removed the same entry from the i-th column in both T and τ while ensuring that none of the other entries have changed columns, establishes the claim. $\square$

We define another SSRCT using the procedure ϕ_i as follows.

$$\Phi_i(\tau) \quad = \quad \begin{cases} \phi_2 \circ \phi_3 \circ \cdots \circ \phi_i(\tau) & i > 1 \\ \tau & i = 1 \end{cases}$$

Let $\gamma = \mathrm{sh}(\Phi_i(\tau)) = (\gamma_1, \ldots, \gamma_s)$. Let $\gamma' = (\gamma, i)$ be the composition obtained by appending a part of length i to γ. Consider the filling of shape $\gamma' /\!\!/ (1)$ where the first s rows are filled as $\Phi_i(\tau)$. Fill the last row with the entries of $H_i(T)$ in decreasing order from left to right. Call the resulting filling $\mu_i(\tau)$.

Theorem 4.1.21. *With the notation as above, the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{SSRTs} \arrow[r, "\rho^{-1}"] \arrow[d, "\mathrm{jdt}_i"'] & \mathrm{SSRCTs} \arrow[d, "\mu_i"] \\
\mathrm{SSRTs} \arrow[r, "\rho^{-1}_{(1)}"'] & \mathrm{SSRCTs}
\end{tikzcd}$$

Proof. First, we start by showing that $\mu_i(\tau)$ is indeed an SSRCT. Notice that the entries in every column are all distinct, using Lemma 4.1.6. Next, since the rows are weakly decreasing by construction, and the top s rows already form an SSRCT, we just need to check that there is no configuration of the form

$$\begin{array}{|c|c|} \hline x & y \\ \hline \end{array} \atop {\begin{array}{cc} & \vdots \\ & \boxed{z} \end{array}} \qquad \text{satisfying } x \geq z > y \text{ and } z \text{ is in the bottommost row.}$$

But this follows from Lemma 4.1.19. Hence $\mu_i(\tau)$ is an SSRCT. Now the claim follows as the set of entries in each column of $\mathrm{jdt}_i(T)$ and $\mu_i(\tau)$ is the same. $\square$

Given the theorem above, we are justified in saying that μ is an analogue of the backward jeu de taquin slide. From this point on, by a backward jeu de taquin slide on SSRCTs of straight shape we will mean an application of μ.

4.1.1 Skew shape after μ

If we perform backward jeu de taquin slides on SSRTs of straight shape, it is easy to predict the resulting shape once the slide is completed. In the case of SSRCTs, when we apply the procedure μ, it is not immediate what the resulting shape is. The aim of this subsection is to use the operators on compositions defined in Subsections 2.5.2 and 2.5.3 to predict it.

Lemma 4.1.22. *The shape of $\phi_{i+1}(\tau)$ is $\mathfrak{d}_i(\alpha)$ if $i \geq 1$.*

Proof. It is clear from Algorithm 4.1.15 that box that contained c_{s_k} in $\tau_\square$ contains 0 in $\phi_{i+1}(\tau_\square)$. We also know from Lemma 4.1.11 that all parts of α that are below α_{s_k} in the reverse composition diagram corresponding to α have either length strictly less than α_{s_k} or strictly greater than α_{s_k}. Furthermore, the same lemma also implies that α_{s_k} has length i. Thus, the claim follows. $\square$

As an immediate corollary, we obtain the following.

Corollary 4.1.23. *The shape of $\Phi_{i+1}(\tau)$ is $\mathfrak{v}_{[i]}(\alpha)$ if $i \geq 0$.*

Proof. Φ_1 fixes the SSRCT τ. In this case the claim is clear. If we are computing $\Phi_{i+1}(\tau)$ for $i \geq 1$, we are computing $\phi_2 \circ \cdots \circ \phi_{i+1}(\tau)$. Using Lemma 4.1.22 repeatedly, we get that $\Phi_{i+1}(\tau)$ has shape $\mathfrak{d}_1 \cdots \mathfrak{d}_i(\alpha) = \mathfrak{v}_{[i]}(\alpha)$. $\square$

Theorem 4.1.24. *The* SSRCT *$\mu_i(\tau)$ is of shape $\gamma /\!\!/ (1)$ where $\gamma = \mathfrak{u}_i(\alpha)$.*

Proof. This follows from the description of the procedure μ_i and Corollary 4.1.23. $\square$

4.2 Backward jeu de taquin slides for SSRCTs of skew shape

After giving a description for the backward jeu de taquin slide for SSRCTs of straight shape, it is only natural to ask for a generalization, that is, a backward jeu de taquin slide for SSRCTs of other shapes. For the proofs in this section, we will consider SSRCTs whose entries are drawn from the ordered alphabet $X = A \cup \bar{A}$ where

$$\begin{aligned} A &= \{1 < 2 < 3 < \cdots\} \\ \bar{A} &= \{\bar{1} < \bar{2} < \bar{3} < \cdots\}. \end{aligned}$$

We will further assume that $\bar{1}$ is greater than any letter in A. Thus, we have a total order on X.

Consider a skew reverse composition shape $\alpha /\!\!/ \beta$. Let τ_o be an SSRCT of shape $\alpha /\!\!/ \beta$ constructed using the alphabet A, and let $\bar{\tau}$ be an SSRCT of straight shape β constructed using the alphabet $\bar{A}$. Then we will denote by τ the SSRCT of straight shape α formed by filling the inner shape in τ_o according to $\bar{\tau}$.

Let $i \geq 1$. Assume, for the moment, that we are computing $\phi_{i+1}(\tau)$. We will show that the shape of the SSRCT obtained by considering entries that belong to A in $\phi_{i+1}(\tau)$ is independent of the filling $\bar{\tau}$. The example next will clarify what we are aspiring to.

Example 4.2.1. *Consider the two tableaux below.*

$$\tau_o = \begin{array}{cccc} 3 & & & \\ \bullet & \bullet & 2 & 1 \\ \bullet & \bullet & \bullet & 5 \\ \bullet & 4 & & \end{array} \quad \textit{and } \bar{\tau} = \begin{array}{ccc} \bar{3} & \bar{1} & \\ \bar{5} & \bar{4} & \bar{2} \\ \bar{6} & & \end{array}$$

Thus

$$\tau = \begin{array}{cccc} 3 & & & \\ \bar{3} & \bar{1} & 2 & 1 \\ \bar{5} & \bar{4} & \bar{2} & 5 \\ \bar{6} & 4 & & \end{array}.$$

Then $\phi_3(\tau)$ is the SSRCT *below.*

$$\begin{array}{cccc} 3 & & & \\ \bar{3} & 4 & 2 & 1 \\ \bar{5} & \bar{4} & \bar{2} & 5 \\ \bar{6} & & & \end{array}$$

If we replace the entries in the above tableau that belong to $\bar{A}$ by bullets, then we obtain the following SSRCT*.*

$$\begin{array}{cccc} 3 & & & \\ \bullet & 4 & 2 & 1 \\ \bullet & \bullet & \bullet & 5 \\ \bullet & & & \end{array}$$

We would like to demonstrate that the above tableau is obtained independent of our choice of the filling $\bar{\tau}$.

We will use the same notation as the one we established when we described the process ϕ_{i+1}. Thus, our focus is on the i-th and $i+1$-th columns of $T_\square$ and $\tau_\square$ respectively, as shown below. Recall that $\rho(\tau) = T$.

$$\begin{array}{|c|c|}\hline a_1 & b_1 \\ \hline a_2 & b_2 \\ \hline \vdots & \vdots \\ \hline a_n & b_n \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline c_1 & d_1 \\ \hline c_2 & d_2 \\ \hline \vdots & \vdots \\ \hline c_n & d_n \\ \hline \end{array}$$

The only difference from the earlier situation is that the entries are now allowed to be elements of $\bar{A}$. We will recall the notation here for the sake of convenience. We have that a_{min} is the first entry that moves horizontally when a backward jeu de taquin slide is initiated from the $i+1$-th column in T. Hence min is the smallest positive integer such that $a_{min} < b_{min+1}$. Also we assume that, in $\tau_\square$, the entry corresponding to a_{min} is c_{s_1}, that is, s_1 is the positive integer such that $c_{s_1} = a_{min}$. Finally, we define recursively integers s_j satisfying $1 \leq s_j \leq n$ for $j \geq 2$ using the following criteria.

1. $s_j > s_{j-1}$,

2. c_{s_j} is the greatest positive integer that satisfies

$$c_{s_{j-1}} > c_{s_j} \geq d_{s_{j-1}}. \tag{4.10}$$

Clearly, only finitely many s_j exist. Let the maximum be given by s_k and let

$$S = \{s_1, \ldots, s_k\}. \tag{4.11}$$

Lemma 4.2.2. *Assume $c_{s_1} \in \bar{A}$. Let $1 \leq j \leq k$ be the greatest positive integer such that c_{s_j} belongs to $\bar{A}$. If $t > s_j$ is a positive integer such that $c_t > 0$, then one of the following statements holds.*

1. $c_t \in A$.

2. $c_t \in \bar{A}$ and $d_t \in \bar{A}$.

Proof. Firstly, note that $c_{s_1} \in \bar{A}$ implies that $b_{min+1} \in \bar{A}$. Notice also that if $c_{s_j} \in \bar{A}$, then for all $1 \leq r \leq j$, we must have that $c_{s_r} \in \bar{A}$ as $c_{s_1} > \cdots > c_{s_j} > \cdots > c_{s_k}$.

If $c_t \in A$, then the first condition is already satisfied. Hence assume that $c_t \in \bar{A}$. If $j = k$, then Lemma 4.1.11 implies that $c_t > c_{s_1}$. Then, Lemma 4.1.7 implies $d_t \geq b_{min+1}$. Hence $d_t \in \bar{A}$.

Now assume that $j < k$. We have that $c_{s_j} \in \bar{A}$ and $c_{s_{j+1}} \in A$. Since the definition of s_{j+1} implies that

$$c_{s_j} > c_{s_{j+1}} \geq d_{s_j},$$

we get that $d_{s_j} \in A$.

By our assumptions, we know that $t > s_j$ and $c_t \in \bar{A}$. Using this and the fact that j is the largest integer such that $c_{s_j} \in \bar{A}$, we conclude that $t \neq s_{j+1}$. But then, we must have that $c_t > c_{s_j}$. To see this, note that if $c_t < c_{s_j}$, then we have the following inequality using the fact that $d_{s_j} \in A$.

$$c_{s_j} > c_t > d_{s_j}$$

But then, we get that $c_{s_{j+1}} > c_t$ and hence, must belong to $\bar{A}$, which is not the case. Hence $c_t > c_{s_j}$ indeed.

Lemma 4.1.10 implies the inequality $c_t > c_{s_1}$ and then Lemma 4.1.7 implies that $d_t \geq b_{min+1}$. Thus, we get that $d_t \in \bar{A}$. $\square$

Note that the above lemma assumes nothing about the filling $\bar{\tau}$.

Lemma 4.2.3. *Assume $c_{s_1} \in \bar{A}$. Consider the* SSRCT *of straight shape inside $\phi_{i+1}(\tau)$ formed by the entries that belong to $\bar{A}$. The shape of this* SSRCT *is $\mathfrak{d}_i(\beta)$.*

Proof. Let $1 \leq j \leq k$ be the greatest positive integer such that $c_{s_j} \in \bar{A}$. Then from Lemma 4.2.2, it follows that c_{s_j} occupies the rightmost box in the bottommost row of length i in the reverse composition diagram of β. According to Algorithm 4.1.15, this entry gets replaced either by an element of A or 0. In either case, the shape of the SSRCT formed by the entries that belong to $\bar{A}$ is $\mathfrak{d}_i(\beta)$. $\square$

Lemma 4.2.4. *Assume that $c_{s_1} \in A$. Then the* SSRCT *in $\phi_{i+1}(\tau)$ formed by considering the entries that belong to A is independent of the choice of $\bar{\tau}$. Furthermore, this* SSRCT *has shape $\mathfrak{d}_i(\alpha)/\!\!/\beta$.*

Proof. Since c_{s_1} belongs to A and $c_{s_1} > c_{s_j}$ for all $2 \leq j \leq k$, we get that $c_{s_j} \in A$ for $2 \leq j \leq k$. Note by Lemma 4.1.11 that c_{s_k} occupies the rightmost box in the bottommost part of length i in the reverse composition diagram of α. Now Algorithm 4.1.15 immediately implies the claim. $\square$

Lemma 4.2.5. *Assume that $c_{s_1} \in \bar{A}$. Then the* SSRCT *in $\phi_{i+1}(\tau)$ formed by considering the entries that belong to A is independent of the choice of $\bar{\tau}$. Furthermore, this* SSRCT *has shape $\mathfrak{d}_i(\alpha) /\!\!/ \mathfrak{d}_i(\beta)$.*

Proof. Let j be the greatest positive integer satisfying $1 \leq j \leq k$ such that $c_{s_j} \in \bar{A}$. By Lemma 4.2.3, we know that c_{s_j} occupies the rightmost box in the bottommost row of length i in the reverse composition diagram of β. Also, by Lemma 4.1.11 we know that c_{s_k} occupies the rightmost box in the bottommost row of length i in the reverse composition diagram of α (recall also that we draw the reverse composition diagram of β in the bottom left corner of the reverse composition diagram of α when depicting $\alpha /\!\!/ \beta$). Observe now that s_j does not depend on $\bar{\tau}$. Thus, from Algorithm 4.1.15, it follows that the SSRCT in $\phi_{i+1}(\tau)$ formed by the entries that belong to A is independent of $\bar{\tau}$ and that it has shape $\mathfrak{d}_i(\alpha) /\!\!/ \mathfrak{d}_i(\beta)$. $\square$

Now we define $\mu_i(\tau_o)$ as follows.

$$\mu_i(\tau_o) = \text{ the SSRCT formed by considering the entries that belong to } A \text{ in } \mu_i(\tau).$$

It is not clear from the above definition that $\mu_i(\tau_o)$ does not depend on the choice of $\bar{\tau}$. We establish this next.

Let $T_o = \rho_\beta(\tau_o)$ and $\bar{T} = \rho(\bar{\tau})$. Define T to be $\rho(\tau)$. Consider a backward jeu de taquin slide on T starting from an addable node in column i. Let j be the greatest integer satisfying $1 \leq j \leq i-1$ such that entry that moves horizontally from column j to column $j+1$ when computing $\mathrm{jdt}_i(T)$ is an element of $\bar{A}$. If there is no such j, define j to be 0. If $j \geq 1$, then the entries moving horizontally when computing $\mathrm{jdt}_i(T)$ from all columns between 1 and j (inclusive) are all elements of $\bar{A}$, because the entries moving horizontally during a backward jeu de taquin slide increase weakly as we move from column $i-1$ to column 1. Now by using Theorem 4.1.20, we get that if we compute $\phi_2 \circ \cdots \circ \phi_i(\tau)$, all the entries that exit from columns between 1 and j belong to $\bar{A}$.

The preceding discussion along with Lemmas 4.2.4 and 4.2.5 implies the following proposition.

Proposition 4.2.6. *Let γ be the* SSRCT *$\Phi_i(\tau)$ where $i \geq 1$. Then the* SSRCT *formed by considering the entries in γ that belong to $\bar{A}$ has shape $\mathfrak{v}_{[j]}(\beta)$. The* SSRCT *formed by considering the entries in γ that belong to A has shape $\mathfrak{v}_{[i-1]}(\alpha) /\!\!/ \mathfrak{v}_{[j]}(\beta)$. Furthermore, this* SSRCT *is independent of the choice of $\bar{\tau}$.*

This proposition along with our description of the procedure μ prior to Theorem 4.1.21, implies the following theorem, which proves that $\mu_i(\tau_o)$ is well-defined and does not depend on the choice of filling $\bar{\tau}$.

Theorem 4.2.7. *Let γ be the* SSRCT *$\mu_i(\tau)$ where $i \geq 1$. Then the* SSRCT *formed by considering the entries in γ that belong to $\bar{A}$ has shape $\mathfrak{u}_{j+1}(\beta)/\!\!/(1)$. The* SSRCT *formed by considering the entries in γ that belong to A has shape $\mathfrak{u}_i(\alpha)/\!\!/\mathfrak{u}_{j+1}(\beta)$, and is independent of the choice of $\bar{\tau}$.*

From this point on, we will not require the alphabet $\bar{A}$.

4.3 A new poset on compositions

In this section we will introduce a new poset structure on the set of compositions that arises from performing backward jeu de taquin slides on SSRCTs. But before that we need to understand the link between maximal chains in Young's lattice $\mathcal{Y}$ and SRTs.

4.3.1 Growth words and SRTs

With any maximal chain in $\mathcal{Y}$ starting at the empty partition, we can associate a sequence of positive integers as explained next. Let

$$\lambda_0 = \varnothing \prec \lambda_1 \prec \cdots \prec \lambda_n$$

be a maximal chain. Then each partition λ_k for $1 \leq k \leq n$ is obtained by adding a box at an addable node of λ_{k-1}, say, in column i_k. The sequence $i_n \cdots i_1$ is called the *column growth word* corresponding to this maximal chain. Note that the column growth word is always *reverse lattice*, that is, in any suffix of the column growth word the number of js weakly exceeds the number of $j+1$s for all positive integers j. What we have outlined is clearly a bijection between reverse lattice words of length n and maximal chains in $\mathcal{Y}$ starting at $\varnothing$ and ending at a partition of n. We will now associate an SRT of size n given a column growth word $w = i_n \cdots i_1$. This is done by considering the corresponding maximal chain and assigning $n-k+1$ to the box in λ_k that is not in λ_{k-1}. This SRT will be denoted by T_w.

Before we state our next lemma connecting the column reading word of T_w to its associated reverse lattice word, we require the notion of standardization. Given a word $w = i_n \cdots i_1$

where the i_js are positive integers, define the *inversion set* as follows.

$$\mathrm{Inv}(w) = \{(p,q) : i_p < i_q\}$$

There exists a unique permutation in $\sigma \in \mathfrak{S}_n$ such that $\mathrm{Inv}(\sigma) = \mathrm{Inv}(w)$. We call σ the *standardization* of w, and denote it by $\mathrm{std}(w)$.

Lemma 4.3.1. *Let $w = i_n \cdots i_1$ be a reverse lattice word, and let $\sigma = \mathrm{std}(w)$. Then the column reading word of T_w equals σ^{-1}.*

Proof. Notice that all the instances of a positive integer j in w are in positions given by the entries in the j-th column of T_w read in increasing order. This implies the claim. $\square$

Example 4.3.2. *Consider the reverse lattice word $w = 341123121$. Then we have that $\sigma = \mathrm{std}(w) = 791258364$ in single line notation. Notice that T_w is the* SSRT *below.*

$$\begin{array}{|c|ccc}
\cline{1-1} 3 & & & \\
\cline{1-1} 4 & & & \\
\cline{1-3} 7 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{1} & \\
\hline 9 & \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{2} \\
\hline
\end{array}$$

The column reading word of T_w is the following permutation.

$$347958162$$

It can be checked that the permutation above is precisely σ^{-1}.

Now recall that if we perform our variant of the Robinson-Schensted algorithm on the column reading word of a tableau T, we recover T as the insertion tableau. This implies the following corollary to the previous lemma, establishing T_w as an insertion tableau for permutation associated naturally to it.

Corollary 4.3.3. *Let $w = i_n \cdots i_1$ be a reverse lattice word, and let $\sigma = \mathrm{std}(w)$. Then $T_w = P(\sigma^{-1})$.*

This interepretation of T_w will be of importance to us in the next subsection.

4.3.2 A poset on compositions

We will now define a new poset on the set of compositions, denoted by $\mathcal{R}_c$, using its cover relation $\lessdot_r$ defined by setting $\alpha \lessdot_r \beta$ if and only if $\beta = \mathfrak{u}_i(\alpha)$ for some $i \geq 1$. The order relation $<_r$ in $\mathcal{R}_c$ is obtained by taking the transitive closure of the cover relation $\lessdot_r$.

Example 4.3.4. *Let* $\alpha = (3,1,4,2,1)$ *and* $\beta = \mathfrak{u}_4(\alpha) = (2,1,4,1,4)$. *Then* $\alpha \lessdot_r \beta$ *in* $\mathcal{R}_c$.

The next theorem is about counting maximal chains in $\mathcal{R}_c$, and the reader should compare it with the analogous result in the case of Young's lattice $\mathcal{Y}$ [42, Proposition 7.10.3] and, more recently, in the case of $\mathcal{L}_c$ [4, Proposition 2.11].

Theorem 4.3.5. *The number of maximal chains from* $\varnothing$ *to a composition* α *in* $\mathcal{R}_c$ *is equal to the number of* SRCT*s of shape* α.

To prove the above theorem, we will use our variant of the Robinson-Schensted algorithm defined in Subsection 2.2.3 and answer a more general question. We will start by proving a result that allows us to compute the resulting inner shape after a sequence of backward jeu de taquin slides has been applied to an SSRCT of straight shape.

Let τ be an SSRCT of shape α. Consider the computation of $\mu_{i_n} \cdots \mu_{i_1}(\tau)$, where we assume that the sequence of backward jeu de taquin slides we are executing is valid at all steps. This means that for all $1 \leq k \leq n-1$, the outer shape β underlying the SSRCT $\mu_{i_k} \cdots \mu_{i_1}(\tau)$ is such that $\widetilde{\beta}$ has an addable node in column i_{k+1}. Furthermore, we also assume that $\widetilde{\alpha}$ has an addable node in column i_1, so that we can compute $\mu_{i_1}(\tau)$ to start with.

We will deviate a little from our earlier description of backward jeu de taquin slides for an SSRCT of straight shape. Let $max(\tau)$ denote the maximum entry in the SSRCT τ (if $\tau = \varnothing$ then this is 0). We will assume that in computing $\mu_i(\tau)$, instead of getting an SSRCT of skew reverse composition shape, with the vacant lower left corner filled by a bullet $\bullet$, we insert the number $max(\tau)+1$ in the corner left vacant. In this way we ensure that $\mu_i(\tau)$ is also an SSRCT of straight shape. We will do backward jeu de taquin slides (the procedure jdt) on SSRTs of straight shape with a similar rule.

Example 4.3.6. *Consider the* SRCT $\tau = \begin{array}{|c|c|c|c|} \hline 5 & 2 \\ \hline 8 & 7 & 4 & 1 \\ \hline 9 & 6 & 3 \\ \hline 11 & 10 \\ \hline \end{array}$ *of shape* $(2,4,3,2)$. *We will compute* $\mu_4\mu_3\mu_4(\tau)$ *under the set of rules outlined prior to the example. Firstly,* $max(\tau) = 11$. *We obtain the following* SSRCT*s sequentially (where the entries that come in the lower left corner are highlighted).*

$$\begin{array}{|c|c|c|c|} \hline 5 & 2 \\ \hline 8 & 7 & 3 & 1 \\ \hline 9 & 6 \\ \hline \boxed{12} & 11 & 10 & 4 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|} \hline 5 & 2 \\ \hline 8 & 6 & 3 & 1 \\ \hline \boxed{12} & 11 & 10 & 4 \\ \hline \boxed{13} & 9 & 7 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|} \hline 5 & 2 \\ \hline 8 & 6 & 3 & 1 \\ \hline \boxed{12} & 9 & 7 & 4 \\ \hline \boxed{14} & \boxed{13} & 11 & 10 \\ \hline \end{array}$$

We will next state a theorem that allows us to compute the final SRCT (under our new setup) obtained when a sequence of slides is applied to an empty SRCT $\varnothing$.

Theorem 4.3.7. *Let $w = i_n \cdots i_1$ be a reverse lattice word, and let $\sigma = \mathrm{std}(w)$. Then*

$$\mu_{i_n} \cdots \mu_{i_1}(\varnothing) = \rho^{-1}(Q(\sigma)).$$

Proof. Let $\mu_{i_n} \cdots \mu_{i_1}(\varnothing) = \tau$. Clearly, τ is an SRCT. Now consider the corresponding sequence of slides starting from the empty SRT, also denoted $\varnothing$, and suppose that the final SRT obtained is T. Thus, $\mathrm{jdt}_{i_n} \cdots \mathrm{jdt}_{i_1}(\varnothing) = T$. From Theorem 4.1.21, it follows that $\rho(\tau) = T$ in this new setting.

Observe that in obtaining T, we encounter a sequence of SRTs for $1 \leq k \leq n$ as follows.

$$T_k = \mathrm{jdt}_{i_k} \cdots \mathrm{jdt}_{i_1}(\varnothing)$$

If we let λ_k denote $\mathrm{sh}(T_k) \vdash k$, we get a maximal chain in $\mathcal{Y}$.

$$\varnothing \prec \lambda_1 \prec \cdots \prec \lambda_n$$

Thus, we have that $w = i_n \cdots i_1$ is the column growth word corresponding to the maximal chain above. This combined with the reversibility of jeu de taquin for SSRTs gives us the key fact that $T_w = e(T)$ (essentially as evacuation undoes backward jeu de taquin slides).

From Corollary 4.3.3 we know that $T_w = P(\sigma^{-1})$, and by Lemma 2.2.13, we know that $P(\sigma^{-1}) = e(Q(\sigma))$. Combining the previous two facts, we obtain the following.

$$e(T) = e(Q(\sigma))$$

Since evacuation is an involution as will shown in Section 4.5, we get that $T = Q(\sigma)$. This implies that the SRCT given by $\mu_{i_n} \cdots \mu_{i_1}(\varnothing) = \tau = \rho^{-1}(T)$, equals $\rho^{-1}(Q(\sigma))$. $\square$

We claim that the above theorem also supplies us with a bijection between the set of maximal chains from $\varnothing$ to α in $\mathcal{R}_c$ and the set of SRCTs of shape α.

Proof. (of Theorem 4.3.5) Consider a maximal chain in $\mathcal{R}_c$ as shown below.

$$\varnothing \lessdot_r \alpha_1 \lessdot_r \cdots \lessdot_r \alpha_n = \alpha$$

Just as in the case of $\mathcal{Y}$, the maximal chain above gives us a reverse lattice word $w = i_n \cdots i_1$ where $\alpha_k = \mathfrak{u}_{i_k}(\alpha_{k-1})$ for $1 \leq k \leq n$. Associate the SRCT $\mu_{i_n} \cdots \mu_{i_1}(\varnothing)$ with this maximal chain. Note that the shape of this SRCT, by Theorem 4.1.24, is $\mathfrak{u}_{i_n} \cdots \mathfrak{u}_{i_1}(\varnothing)$ where $\varnothing$ denotes the empty composition. But this is clearly α. To establish that the map associating the SRCT $\mu_{i_n} \cdots \mu_{i_1}(\varnothing)$ to the reverse lattice word $i_n \cdots i_1$ obtained from the maximal chain is indeed a bijection, we will use the invertibility of jeu de taquin slides for SSRTs.

Consider an SRCT τ of shape α, and let $T = \rho(\tau)$. We will analyze the execution of the evacuation action on T. At every step a forward jeu de taquin slide is initiated from the lower left corner, and this slide finishes in an addable node to the shape underlying the rectified tableau. Noting down the columns of these addable nodes gives us a sequence of n positive integers i_n, i_{n-1},$\ldots$, i_1. This sequence is **uniquely determined by** T (by the invertibility of jeu de taquin slides on SSRTs) and the word $w = i_n \cdots i_1$ is reverse lattice. Now since backward jeu de taquin slides undo the forward jeu de taquin slides, we get that

$$T = \mathrm{jdt}_{i_n} \cdots \mathrm{jdt}_{i_1}(\varnothing),$$

and hence that

$$\tau = \mu_{i_n} \cdots \mu_{i_1}(\varnothing).$$

Since τ had shape α to start with, we get that $\mathfrak{u}_{i_n} \cdots \mathfrak{u}_{i_1}(\varnothing) = \alpha$ as well (here again $\varnothing$ refers to the empty composition). Now we can associate the following (unique) maximal chain from $\varnothing$ to α in $\mathcal{R}_c$ to τ.

$$\varnothing \lessdot_r \mathfrak{u}_{i_1}(\varnothing) \lessdot_r \mathfrak{u}_{i_2}\mathfrak{u}_{i_1}(\varnothing) \lessdot_r \cdots \lessdot_r \mathfrak{u}_{i_n} \cdots \mathfrak{u}_{i_1}(\varnothing) = \alpha$$

This finishes the proof. $\square$

We will give an example illustrating the ideas above.

Example 4.3.8. *To compute* $\tau = \mu_3\mu_4\mu_1\mu_1\mu_2\mu_3\mu_1\mu_2\mu_1(\varnothing)$*, note that the column growth word is* 341123121 *and its standardization (which is a permutation in* $\mathfrak{S}_9$*) in single line notation is* $\sigma = 791258364$*. On applying the Robinson-Schensted variant,* σ *maps to the* *following insertion tableau (left) and recording tableau (right).*

$$P(\sigma) = \begin{array}{|c|c|c|c|}\cline{1-1} 1 \\ \cline{1-1} 2 \\ \cline{1-3} 7 & 5 & 3 \\ \hline 9 & 8 & 6 & 4 \\ \hline \end{array} \qquad Q(\sigma) = \begin{array}{|c|c|c|c|}\cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-3} 8 & 6 & 2 \\ \hline 9 & 7 & 3 & 1 \\ \hline \end{array}$$

Then, by Theorem 4.3.7, we have that $\tau = \rho^{-1}(Q(\sigma))$.

$$\tau = \begin{array}{|c|c|c|c|}\cline{1-1} 4 \\ \cline{1-1} 5 \\ \hline 8 & 7 & 3 & 1 \\ \hline 9 & 6 & 2 \\ \cline{1-3} \end{array}$$

It is straightforward to check that $\mathrm{jdt}_3\mathrm{jdt}_4\mathrm{jdt}_1\mathrm{jdt}_1\mathrm{jdt}_2\mathrm{jdt}_3\mathrm{jdt}_1\mathrm{jdt}_2\mathrm{jdt}_1(\varnothing)$ *gives the following sequence of* SRT*s (the bullets indicating where the next slide begins).*

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array}\ \bullet \ \to\ \begin{array}{cc} \bullet & \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \ \to\ \begin{array}{|c|c|c}\cline{1-1} 2 \\ \cline{1-2} 3 & 1 & \bullet \\ \cline{1-2}\end{array} \ \to\ \begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{\bullet} \\ \hline 4 & 3 & 1 \\ \hline\end{array} \ \to\ \begin{array}{|c|c|c|} \multicolumn{1}{c}{\bullet} \\ \cline{1-2} 4 & 2 \\ \hline 5 & 3 & 1 \\ \hline\end{array} \ \to\ \begin{array}{|c|c|c|} \multicolumn{1}{c}{\bullet} \\ \cline{1-1} 4 \\ \cline{1-2} 5 & 2 \\ \hline 6 & 3 & 1 \\ \hline\end{array} \ \to$$

$$\to\ \begin{array}{|c|c|c|c}\cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-2} 6 & 2 \\ \cline{1-3} 7 & 3 & 1 & \bullet \\ \cline{1-3}\end{array} \ \to\ \begin{array}{|c|c|c|c|}\cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-2} 6 & 2 & \multicolumn{1}{c}{\bullet} \\ \hline 8 & 7 & 3 & 1 \\ \hline\end{array} \ \to\ \begin{array}{|c|c|c|c|}\cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-3} 8 & 6 & 2 \\ \hline 9 & 7 & 3 & 1 \\ \hline\end{array}$$

Note that the final SRT *obtained above is* $\rho(\tau)$.

In the next subsection, we will use Theorem 4.3.7 and the proof strategy for Theorem 4.3.5 to consider the effect of a sequence of slides starting from a nonempty SRCT.

4.3.3 Inner shape after a sequence of slides

Let τ_1 be an SRCT of shape $\alpha \vDash n$, and let $T_1 = \rho(\tau_1)$ be of shape $\lambda \vdash n$. Considering the execution of the evacuation action on T_1 as in the proof of Theorem 4.3.5, we get a sequence of n positive integers i_n, $i_{n-1}, \ldots$, i_1. This sequence is uniquely determined by T_1 and the word $w = i_n \cdots i_1$ is reverse lattice.

Remark 4.3.9. *Note that if we started with the final* SRT *in Example 4.3.8, and performed the procedure discussed prior to this remark, we will be obtain the same sequence of* SRT*s in reverse and the columns to which the bullets belong are precisely the integers* i_n *down to* i_1.

It follows that $T_1 = \mathrm{jdt}_{i_n} \cdots \mathrm{jdt}_{i_1}(\varnothing)$ which in turn implies that $\tau_1 = \mu_{i_n} \cdots \mu_{i_1}(\varnothing)$. Consider now the SRCT τ_2 defined below.

$$\tau_2 = \mu_{k_m} \cdots \mu_{k_1}(\tau_1)$$

Let $\rho(\tau_2)$ be T_2. Clearly, we have that $T_2 = \mathrm{jdt}_{k_m} \cdots \mathrm{jdt}_{k_1}(T_1)$. Denote by u the word $k_m \cdots k_1$.

Let $v = u \cdot w$, where $\cdot$ denotes concatenation of words, and let $\pi = \mathrm{std}(v)$, $\sigma = \mathrm{std}(w)$ and $\omega = \mathrm{std}(u)$. Now, note that Theorem 4.3.7 implies that

$$\tau_2 = \rho^{-1}(Q(\pi)).$$

Observe additionally that the integers $m+n$, $m+n-1, \ldots, n+1$ are the new entries added when computing $\mu_{k_m} \cdots \mu_{k_1}(\tau_1)$, and they form an SSRCT of size m (with all entries distinct) inside τ_2. But this SSRCT is obtained by performing our insertion algorithm on the prefix of length m in π, and then applying ρ^{-1} to the partial Q-tableau obtained at this stage. If one subtracts n from all entries in this partial Q-tableau, one gets an SRT that is precisely the Q-tableau obtained after performing the insertion process on $\omega = \mathrm{std}(u)$.

From this point on, we switch perspectives again, and on performing backward jeu de taquin slides on SSRCTs, instead of filling the vacant box in the lower left corner with larger entries, we will fill them with bullets. The preceding discussion implies the following theorem.

Theorem 4.3.10. *Let τ_1 be an* SRCT *of straight shape α, and τ_2 be the* SRCT $\mu_{j_m} \cdots \mu_{j_1}(\tau_1)$. *Let $w = j_m \cdots j_1$ and $\omega = \mathrm{std}(w) \in \mathfrak{S}_m$. Let $\beta = \mathfrak{u}_{j_m} \cdots \mathfrak{u}_{j_1}(\alpha)$ and $\gamma = \mathrm{sh}(\rho^{-1}(Q(\omega)))$. Then τ_2 is an* SRCT *of shape $\beta /\!\!/ \gamma$.*

The following corollary is obtained from the theorem above.

Corollary 4.3.11. *Let τ_1 be an* SRCT *of straight shape α, and τ_2 be the* SRCT $\mu_{j_m} \cdots \mu_{j_1}(\tau_1)$. *Let $w = j_m \cdots j_1$ and $\omega = \mathrm{std}(w) \in \mathfrak{S}_m$. Let $\beta = \mathfrak{u}_{j_m} \cdots \mathfrak{u}_{j_1}(\alpha)$. Then τ_2 is an* SRCT *of shape $\beta /\!\!/ (n)$ if and only if $j_m > \cdots > j_1$, and it is an* SRCT *of shape $\beta /\!\!/ (1^n)$ if and only if $j_m \leq \cdots \leq j_1$.*

SRT versions of the two results above also exist and they are also implied in the discussion earlier. We will recast the corollary above for SRTs as another corollary.

Corollary 4.3.12. *Let T_1 be an* SRT *of straight shape λ, and T_2 be the* SRT $\mathrm{jdt}_{j_m} \cdots \mathrm{jdt}_{j_1}(T_1)$. *Let $w = j_m \cdots j_1$ and $\omega = \mathrm{std}(w) \in \mathfrak{S}_m$. Let $\beta = \mathfrak{u}_{j_m} \cdots \mathfrak{u}_{j_1}(\alpha)$ and $\delta = \widetilde{\beta}$. Then T_2 is an* SRT *of shape $\delta/(n)$ if and only if $j_m > \cdots > j_1$, and it is an* SRT *of shape $\delta/(1^n)$ if and only if $j_m \leq \cdots \leq j_1$.*

Suppose now that T is an SRT of shape $\delta/(n)$. Then the invertibility of the jeu de taquin slides for SRTs along with the corollary above implies that T has been obtained from an SRT T' by doing a sequence of backward jeu de taquin slides as follows

$$T = \mathrm{jdt}_{i_n} \cdots \mathrm{jdt}_{i_1}(T'),$$

where $i_n > \cdots > i_1$. Similarly, if T is an SRT of shape $\delta/(1^n)$, then it has been obtained from an SRT T' by doing a sequence of backward jeu de taquin slides as follows

$$T = \mathrm{jdt}_{i_n} \cdots \mathrm{jdt}_{i_1}(T'),$$

where $i_n \leq \cdots \leq i_1$. Note now that the maps $\rho_{(n)} : \mathrm{SSRT}(-/(n)) \to \mathrm{SSRCT}(-/\!\!/(n))$, and $\rho_{(1^n)} : \mathrm{SSRT}(-/(1^n)) \to \mathrm{SSRCT}(-/\!\!/(1^n))$ are bijections. Using this and Theorem 4.1.21, it is clear that we can apply the above arguments to SRCTs. This insight is all we need to prove our right Pieri rules for the noncommutative Schur functions stated in Theorem 2.6.9.

4.4 Right Pieri rules for noncommutative Schur functions

To recover Pieri rules from Theorem 2.4.6 that allow us to compute the products $\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)}$ (and $\mathbf{s}_\alpha \cdot \mathbf{s}_{(1^n)}$) we need to find all SRCTs of shape $\gamma/\!\!/(n)$ (respectively $\gamma/\!\!/(1^n)$) that rectify to the canonical tableau of shape α. In light of Corollary 4.3.11 we have the following lemma.

Lemma 4.4.1. *Let $\tau = \mu_{i_n} \cdots \mu_{i_1}(\tau_\alpha)$, where $i_n > \cdots > i_1$. Then τ rectifies to τ_α.*

Proof. Let $T_\alpha = \rho(\tau_\alpha)$ and $T = \rho_{(n)}(\tau)$. Then, by Theorem 4.1.21, we have that $T = \mathrm{jdt}_{i_n} \cdots \mathrm{jdt}_{i_1}(T_\alpha)$, which in turn implies that $\mathrm{rect}(T) = \mathrm{rect}(T_\alpha)$ (as discussed before Algorithm 2.2.11). Therefore, we get that τ rectifies to τ_α. □

Remark 4.4.2. *The arguments after Corollary 4.3.12 imply that if there is an* SRCT *of shape $\gamma/\!\!/(n)$ that rectifies to τ_α, then it has to be obtained from τ_α by doing a sequence of backward jeu de taquin slides from columns $i_1, i_2, \cdots, i_n$ in that order, where $i_n > \cdots > i_1$.*

Similarly, if there is an SRCT *of shape $\gamma /\!\!/ (1^n)$ that rectifies to τ_α, then it has to be obtained from τ_α by doing a sequence of backward jeu de taquin slides from columns i_1, $i_2, \cdots, i_n$ in that order, where $i_n \leq \cdots \leq i_1$.*

We can now prove the right Pieri rules, stated in Theorem 2.6.9.

Theorem 4.4.3. *(Right Pieri rules) Let α be a composition and n a positive integer. Then*

$$\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)} = \sum_{\substack{\beta \models |\alpha|+n, \mathfrak{u}_{i_n} \cdots \mathfrak{u}_{i_1}(\alpha)=\beta \\ i_n > \cdots > i_1}} \mathbf{s}_\beta,$$

$$\mathbf{s}_\alpha \cdot \mathbf{s}_{(1^n)} = \sum_{\substack{\beta \models |\alpha|+n, \mathfrak{u}_{i_n} \cdots \mathfrak{u}_{i_1}(\alpha)=\beta \\ i_n \leq \cdots \leq i_1}} \mathbf{s}_\beta.$$

Proof. We will only give the proof for $\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)}$ as the proof of $\mathbf{s}_\alpha \cdot \mathbf{s}_{(1^n)}$ is similar. Notice first that if we have two distinct sequences $i_n > \cdots > i_1$ and $j_n > \cdots > j_1$ such that

$$\tau_1 = \mu_{i_n} \cdots \mu_{i_1}(\tau_\alpha),$$
$$\tau_2 = \mu_{j_n} \cdots \mu_{j_1}(\tau_\alpha),$$

then τ_1 and τ_2 are distinct SRCTs. This follows from the reversibility of backward jeu de taquin slides for SRTs, once we consider the images of τ_1 and τ_2 under the generalized ρ map of Subsection 2.1.4. By Corollary 4.3.11, we get that

$$\mathrm{sh}(\tau_1) = \gamma_1 /\!\!/ (n) \text{ where } \gamma_1 = \mathfrak{u}_{i_n} \cdots \mathfrak{u}_{i_1}(\alpha),$$
$$\mathrm{sh}(\tau_2) = \gamma_2 /\!\!/ (n) \text{ where } \gamma_2 = \mathfrak{u}_{j_n} \cdots \mathfrak{u}_{j_1}(\alpha).$$

We claim that $\gamma_1 \neq \gamma_2$. But in view of Remark 2.5.5 and given that the sequences $i_n > \cdots > i_1$ and $j_n > \cdots > j_1$ are distinct, this immediately follows. Using Remark 4.4.2, we get that all summands $\mathbf{s}_\gamma$ in $\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)}$ are obtained by picking a sequence $i_n > \cdots > i_1$ such that $\mu_{i_n} \cdots \mu_{i_1}(\tau_\alpha)$ is an SRCT, and then letting γ be the outer shape of this SRCT. This gives us a multiplicity free expansion in the noncommutative Schur basis for the product $\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)}$ as stated in the claim. $\square$

We conclude by proving a generalization of Theorem 4.3.5. Instead of counting maximal chains starting from $\varnothing$ and ending at some composition β, we consider maximal chains starting from a composition α and ending at a composition β where $\alpha <_r \beta$.

Theorem 4.4.4. *Let $f_{\alpha,\beta}$ denote the number of maximal chains from α to β in $\mathcal{R}_c$. Then*

$$f_{\alpha,\beta} = \sum_{\gamma \vDash |\beta|-|\alpha|} C^{\beta}_{\alpha\gamma} f_{\varnothing,\gamma}.$$

Proof. Let $|\alpha| = m$ and $|\beta| = n+m$. If $\alpha <_r \beta$, then there is at least one sequence of positive integers $i_n, \cdots, i_1$ such that $\mathfrak{u}_{i_n} \cdots \mathfrak{u}_{i_1}(\alpha) = \beta$. Let $w = i_n \cdots i_1$ and consider the SRCT $\tau = \mu_{i_n} \cdots \mu_{i_1}(\tau_\alpha)$. Then $\mathrm{sh}(\tau) = \beta /\!/ \gamma$ where $\gamma = \mathrm{sh}(\rho^{-1}(Q(\mathrm{std}(w))))$, by Theorem 4.3.10. Note also that τ rectifies to τ_α, by an argument similar to the proof of Lemma 4.4.1.

As stated in Theorem 2.4.6, we have that

$$C^{\beta}_{\alpha\gamma} = \text{ number of SRCTs of shape } \beta /\!/ \gamma \text{ that rectify to } \tau_\alpha.$$

Now consider any skew SRCT τ of shape $\beta /\!/ \gamma$ that rectifies to τ_α. This can be completed to an SRCT τ_{aug} of straight shape β where the inner shape corresponding to γ has been filled according to an SSRCT of shape γ with distinct entries from the set $B = \{n+m, n+m-1, \ldots, m+1\}$. Now, notice that every choice of the filling of shape γ gives a unique maximal chain from α to β. To see this, perform a forward jeu de taquin slide starting from the lower left corner of $\rho(\tau_{aug})$. As long as the resulting tableau has elements from B, repeat the previous step. Notice that the procedure above stops when we reach $\rho(\tau_\alpha)$. Reversing these slides starting from $\rho(\tau_\alpha)$, and using the generalized ρ map defined in Subsection 2.1.4, at every step gives us the unique maximal chain from α to β.

Using the fact that the number of SRCTs of shape γ is $f_{\varnothing,\gamma}$, as established in Theorem 4.3.5, the claim follows. □

Remark 4.4.5. *It is worth emphasizing that $f_{\alpha,\beta}$ is not the number of* SRCT *of shape $\beta /\!/ \alpha$ in general, although it is true in the case where α is $\varnothing$ by Theorem 4.3.5. The above theorem can be considered as a noncommutative generalization of extracting coefficients in the classical identity* $s_{\lambda/\mu} = \sum_{\nu \vdash |\lambda|-|\mu|} c^{\lambda}_{\mu\nu} s_\nu$.

4.5 Some properties of our Robinson-Schensted variant

We will outline the relation between our variant of the Robinson-Schensted algorithm and the classical Robinson-Schensted algorithm. The latter assigns to a permutation $\sigma \in \mathfrak{S}_n$

a pair of standard Young tableaux of the same shape and size equalling n. We begin by defining standard Young tableaux.

Given a partition λ, a *standard Young tableau (SYT)* T of *shape* λ is a filling of the boxes of λ with distinct positive integers from 1 to $|\lambda|$, satisfying the condition that the entries in T are strictly increasing along each row read from left to right and strictly increasing along each column read from bottom to top.

The classical Robinson-Schensted algorithm and evacuation action for SYTs (which we will denote by evac) are discussed in detail in [38, 42] and we will not present them here. To link the classical algorithms for SYTs with the variant for SRTs outlined in Subsection 2.2.3, we will need the notion of the complement of a permutation, and the complement of an SYT/SRT.

Given a permutation $\sigma = \sigma(1)\cdots\sigma(n)$ in single line notation, the *complement* of σ, denoted by σ^c is obtained by replacing $\sigma(i)$ by $n+1-\sigma(i)$ for $1 \leq i \leq n$. Similarly, given an SYT (SRT) T of shape $\lambda \vdash n$, the *complement* of T is the SRT (SYT), denoted by T^c, obtained by replacing an entry k in T by $n+1-k$. Finally, define T^t to be the transpose of any tableau T.

Observe that the complement and transpose operators on SYTs/SRTs commute. Furthermore, both the evac operator on $SYTs$ and the e operator on SRTs commute with the transpose operator. Finally, note that

$$e(T) = (\mathrm{evac}(T^c))^c \text{ if } T \text{ is an SRT,}$$
$$\mathrm{evac}(T) = (e(T^c))^c \text{ if } T \text{ is an SYT.}$$

Since evac is an involution by [42, Proposition A.1.2.9], we can see that e is also an involution.

Now we are ready to make explicit the relation between the classical algorithm for SYTs and the one for SRTs. Let $\pi \in \mathfrak{S}_n$. We will denote the insertion and recording tableaux obtained by using the classical Robinson-Schensted correspondence by $P(\pi)$ and $Q(\pi)$ respectively, and denote those obtained by using our variant by $P_v(\pi)$ and $Q_v(\pi)$ respectively.

Observe that

$$(P_v(\pi), Q_v(\pi)) = ((P(\pi^c))^c, (Q(\pi^c))^c).$$

Further using [42, Theorem A.1.2.10 and Corollary A.1.2.11] and the fact that evac is an involution, we get that

$$(P(\pi^c), Q(\pi^c)) = (\text{evac}(P(\pi)^t), Q(\pi)^t).$$

The two equalities above together imply that

$$(P_v(\pi), Q_v(\pi)) = ((\text{evac}(P(\pi)^t))^c, (Q(\pi)^t)^c).$$

Using [42, Corollary A.1.2.11] and the fact that $P(\pi^{-1}) = Q(\pi)$ we get that

$$\begin{aligned} P_v(\pi^{-1}) &= (\text{evac}(P(\pi^{-1})^t))^c \\ &= (\text{evac}(Q(\pi)^t))^c. \end{aligned}$$

Since $(Q_v(\pi))^c = Q(\pi)^t$, we get that

$$\begin{aligned} P_v(\pi^{-1}) &= (\text{evac}(Q_v(\pi)^c))^c \\ &= e(Q_v(\pi)). \end{aligned}$$

Again, using [42, Corollary A.1.2.11] and the fact that $Q(\pi^{-1}) = P(\pi)$ we get that

$$\begin{aligned} Q_v(\pi^{-1}) &= (Q(\pi^{-1})^t)^c \\ &= (P(\pi)^t)^c. \end{aligned}$$

Now, since $P(\pi)^t = \text{evac}(P_v(\pi)^c)$, we get that

$$\begin{aligned} Q_v(\pi^{-1}) &= (\text{evac}(P_v(\pi)^c))^c \\ &= e(P_v(\pi)). \end{aligned}$$

This establishes Lemma 2.2.13.

Chapter 5

Conclusion

We now conclude with a discussion of questions borne out of the results and techniques in this thesis.

Just as we utilized the noncommutativity of **NSym** to give companion rules to the left Pieri rules, one possible avenue to consider is to find a combinatorial rule to compute $\mathbf{s}_\alpha \cdot \Psi_n$. This will result in a different lift of the classical Murnaghan-Nakayama rule in **NSym**. Data suggest that the coefficients in the expansion in this case are also ± 1. Given that our proof techniques for computing $\Psi_n \cdot \mathbf{s}_\alpha$ involved the left Pieri rules and box adding operators crucially, our belief is that the jdt operators and the right Pieri rules will be key to computing $\mathbf{s}_\alpha \cdot \Psi_n$ combinatorially. Another avenue is to find a representation-theoretic interpretation for the structure coefficients that arise when we expand a noncommutative power sum symmetric function in terms of noncommutative Schur functions. While we did not address them here, these structure coefficients can indeed be computed by iteratively using our noncommutative Murnaghan-Nakayama rule. In the classical case the structure coefficients are character values of the symmetric group, and it would be interesting to know what they are in the setting of **NSym**. Finally, we must mention the other noncommutative power sum symmetric functions introduced in [13], denoted by Φ_n for n a positive integer. The problem of finding a Murnaghan-Nakayama rule for this basis is still unsolved, and computer experiments suggest that this rule will be substantially more complex than the one considered in this thesis.

It would also be interesting to study the combinatorics of jdt operators arising from our description of the backward jeu de taquin slide for SSRCTs, and implications thereof for the associated poset of compositions $\mathcal{R}_c$. The jdt operators satisfy certain congruence relations that resemble plactic (or Knuth) relations. This hints at a monoid-like structure for jdt operators that resembles the plactic monoid, and raises the question of finding a presentation for the jdt operators. It is known that if we have a monoid with relations compatible with standardization and restriction to intervals, then we have a corresponding combinatorial Hopf algebra. It would be extremely interesting to know if the above result can be used in the setting of jdt operators.

Another area to consider is the computation of noncommutative Littlewood-Richardson (LR) coefficients. Given that jdt operators were introduced to model backward jeu de taquin slides on SSRCTs, which in turn were inspired by the definition of noncommutative LR coefficients involving rectification, it is natural to ask if these operators give a more efficient way to compute noncommutative LR coefficients. We hope to address some of the abovementioned questions in the future.

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