Functional Integral Representations for Quantum Many-Particle Systems

by

Cindy Marie Blois

B.Sc. Honours in Physics, The University of Alberta, 2006

B.Sc. Honours in Mathematics, The University of Alberta, 2007

M.Sc., The University of Toronto, 2008

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Abstract

Formal functional integrals are commonly used as theoretical tools and as sources of intuition for predicting phase transitions of many-body systems in Condensed Matter Physics. In this thesis, we derive rigorous versions of these functional integrals for two types of quantum many-particle systems.

We begin with a brief review of quantum statistical mechanics in Chapter 2 and the formalism of coherent states in Chapter 3, which form the basis for our analysis in Chapters 4 and 5. In Chapter 4, we study a mixed gas of bosons and/or fermions interacting on a finite lattice, with a general Hamiltonian that preserves the total number of particles in each species. We rigorously derive a functional integral representation for the partition function, which employs a large-field cutoff for the boson fields. We then expand the resulting "action" in powers of the fields and find a recursion relation for the coefficients. In the case of a two-body interaction (such as the Coulomb interaction), we also find bounds on the coefficients, which give a domain of analyticity for the action. This domain is large enough for use of the action in the functional integral, provided that the large-field cutoffs are taken to grow not too quickly. In Chapter 5, we study a system of electrons and phonons interacting in a finite lattice, using the Holstein Hamiltonian. Again, we rigorously derive a coherent-state functional integral representation for the partition function of this system and then prove that the "action" in the functional integral is an entire-analytic function of the fields. However, since the Holstein Hamiltonian does not preserve the total number of bosons, the approach from Chapter 4 requires some modification. In particular, we repeatedly use Duhamel expansions in powers of the interaction, rather than sums over particle numbers.

Preface

Chapters 1, 2 and 3 consist of introductory material to provide background and motivation. No original results are presented in these chapters.

All research presented in Chapters 4 and 5 was designed, carried out, and analyzed by the author.

Table of Contents

List of Figures

List of Symbols

The table below is serves as a supplemental reference to keep track of the symbols throughout the thesis. To see each definition or notation in context, please click the corresponding hyperlinked page number below.

Symbol	Page	Meaning
$\sigma(x_1,\ldots,x_n)$	15	$(x_{\sigma(1)},\ldots,x_{\sigma(n)})$
$(\sigma f)(x_1,\ldots,x_n)$	15	$f(x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(n)})$
$\mathcal{B}_n(X)$	16	$\{f \in L^2(X^n) : (\forall\sigma \in S_n)\sigma f = f\}$
$\mathcal{F}_n(X)$	16	$\{f \in L^2(X^n) : (\forall\sigma \in S_n)\sigma f = (-1)^\sigma f\}$
$S_b^{(n)}(X)$	16	$\frac{1}{n!}\sum_{\sigma\in S_n}\sigma$
$S_f^{(n)}(X)$	16	$\frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\sigma$
$\mathbb{N}_0$	17	$\{0,1,2,\ldots\}$
$u\otimes_s v$	18	$\frac{1}{\sqrt{m!(m+n)!n!}}\sum_{\sigma\in S_{m+n}}\sigma(u\otimes v)$
$u\wedge v$	18	$\frac{1}{\sqrt{m!(m+n)!n!}}\sum_{\pi\in S_{n+m}}(-1)^\pi\pi(u\otimes v)$
δ_Y	20	$c_Y\ \delta_{y_1}\otimes_s\cdots\otimes_s\delta_{y_n}$ (for bosons)
c_Y	20	$\left(\prod_{x\in X}\mu_Y(x)!\right)^{-\frac{1}{2}}$
$\mu_Y(x)$	20	the number of times x appears in Y
δ_Y	20	$\delta_{y_1}\wedge\cdots\wedge\delta_{y_n}$ (for fermions)
$\mathcal{D}_n(X)$	21	$L^2(X^n)$

$\mathcal{D}(X)$	21	$\bigoplus_{n=1}^{\infty} \mathcal{D}_n(X)$
$\mathcal{B}(X)$	21	$\bigoplus_{n=1}^{\infty} \mathcal{B}_n(X)$
$\mathcal{F}(X)$	21	$\bigoplus_{n=1}^{\lvert X\rvert} \mathcal{F}_n(X)$
$S_b(X)$	22	$\bigoplus_{n=1}^{\infty} S_b^{(n)}(X)$
$S_f(X)$	22	$\bigoplus_{n=1}^{\infty} S_f^{(n)}(X)$
$P^{(n)}$	22	projection onto the n-particle subspace
$\mathcal{D}^{(finite)}(X)$	23	$\{f \in \mathcal{D}(X) : \{n : P^{(n)} f \neq 0\}$ is finite$\}$
$\lvert\phi\rangle$	34, 41	$e^{\int_X dy\ \ \phi(y) a_\gamma^\dagger(y)} \lvert 0\rangle$ (where $\gamma = b$ or f)
$\mathcal{H}$	44	$\mathcal{B} \otimes \mathcal{F}$
$\mathcal{B}$	44	$\mathcal{B}(X_1) \otimes \cdots \otimes \mathcal{B}(X_{P_b})$
$\mathcal{F}$	44	$\mathcal{F}(X_{P_b+1}) \otimes \cdots \otimes \mathcal{F}(X_P)$
$\mathcal{B}_{\mathbf{n}}$	44	$\mathcal{B}_{n_1}(X_1) \otimes \mathcal{B}_{n_2}(X_2) \otimes \cdots \otimes \mathcal{B}_{n_{P_b}}(X_{P_b})$
$\mathcal{B}_{\leq \mathbf{n}}$	44	$(\bigoplus_{i=0}^{n_1} \mathcal{B}_i(X_1)) \otimes \cdots \otimes \left(\bigoplus_{i=0}^{n_{P_b}} \mathcal{B}_i(X_{P_b})\right)$
$\mathbf{m} \leq \mathbf{n}$	44	$(\forall p) m_p \leq n_p$
$\mathcal{H}_{\leq \mathbf{n}}$	44	$\mathcal{B}_{\leq \mathbf{n}} \otimes \mathcal{F}$
$\mathcal{H}_{\mathbf{n}}$	44	$\mathcal{B}_{\mathbf{n}} \otimes \mathcal{F}$
$P_{\mathbf{n}}$	44	the projection from $\mathcal{H}$ onto $\mathcal{H}_{\leq \mathbf{n}}$
$P^{(\mathbf{n})}$	44	the projection from $\mathcal{H}$ onto $\mathcal{H}_{\mathbf{n}}$
$\boldsymbol{\phi}$	45	$(\phi_1, \ldots, \phi_{P_b}, \phi_{P_b+1}, \ldots, \phi_P)$
$\lvert\boldsymbol{\phi}\rangle$	45	$\lvert\phi_1\rangle \otimes \lvert\phi_2\rangle \otimes \cdots \otimes \lvert\phi_P\rangle$
I_r	47	$\int_{L^2(X)} d\mu_r(\phi^*, \phi) e^{-\lVert\phi\rVert^2} \lvert\phi\rangle\langle\phi\rvert$
$\mathrm{I}_{\underline{\mathbf{r}}}$	51	$\int d\mu_{\underline{\mathbf{r}}}(\boldsymbol{\phi}^*, \boldsymbol{\phi}) e^{-\int d\mathbf{x}\ \ \boldsymbol{\phi}^*(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x})} \lvert\boldsymbol{\phi}\rangle\langle\boldsymbol{\phi}\rvert$
$X^{\mathbf{n}}/S_{\mathbf{n}}$	55	$X_1^{n_1}/S_{n_1} \times X_2^{n_2}/S_{n_2} \times \cdots \times X_{P_b}^{n_{P_b}}/S_{n_{P_b}}$
$\mathbf{x} \circ \mathbf{y}$	59	$(\vec{x}_1 \circ \vec{y}_1, \vec{x}_2 \circ \vec{y}_2, \ldots, \vec{x}_P \circ \vec{y}_P)$
$\vec{x}_p \circ \vec{y}_p$	59	$(x_{p_1}, x_{p_2}, \ldots, x_{p_{n_p}}, y_{p_1}, y_{p_2}, \ldots, y_{p_{m_p}})$

$X^{\mathbf{n}}$	59	$X_2^{n_1} \times \cdots \times X_P^{n_P}$
$\mathbf{X}$	59	$\bigcup_{\mathbf{n}\in\mathbb{N}_0^P} X^{\mathbf{n}}$
$\lvert\mathbf{n}\rvert$	59	$\sum_{p=1}^{P} n_p$
$\mathbb{I}_{\mathbf{x}}$	60	the set of all indices of components of $\mathbf{x}$
$\vec{x}_{[\le m]}$	60	$(x_1, \ldots, x_m)$
$\mathbf{x}_{[\le \mathbf{m}]}$	61	$(\vec{x}_{1_{[\le m_1]}}, \ldots, \vec{x}_{P_{[\le m_P]}})$
$\mathbf{x}_{[> \mathbf{m}]}$	61	$(\vec{x}_{1_{[> m_1]}}, \ldots, \vec{x}_{P_{[> m_P]}})$
$\alpha^*(\mathbf{x})$	61	$\alpha_1^*(\vec{x}_1) \cdots \alpha_P^*(\vec{x}_P)$
$\alpha_p^*(\vec{x}_p)$	61	$\alpha_p^*(x_{p_1})\alpha_p^*(x_{p_2}) \cdots \alpha_p^*(x_{p_{n_p}})$
$\frac{\partial}{\partial\alpha^*(\mathbf{x})}$	61	$\frac{\partial}{\partial\alpha_P^*(\vec{x}_P)} \cdots \frac{\partial}{\partial\alpha_1^*(\vec{x}_1)}$
$\frac{\partial}{\partial\alpha_p^*(\vec{x}_p)}$	62	$\frac{\partial}{\partial\alpha_p^*(x_{p,n_p})} \frac{\partial}{\partial\alpha_p^*(x_{p,n_p-1})} \cdots \frac{\partial}{\partial\alpha_p^*(x_{p,1})}$
$\mathbf{n}!$	62	$n_1! n_2! \cdots n_P!$
$S_{\mathbf{n}}$	62	$S_{n_1} \times S_{n_2} \times \cdots \times S_{n_P}$
$(-1)^{\boldsymbol{\sigma}}$	62	$\prod_{p=P_b+1}^{P} (-1)^{\sigma_p}$
$S^{(\mathbf{n})}$	62	$\frac{1}{\mathbf{n}!} \sum_{\boldsymbol{\sigma}\in S_{\mathbf{n}}} (-1)^{\boldsymbol{\sigma}} \boldsymbol{\sigma}$
$a_p^\dagger(\vec{x})$	63	$a_p^\dagger(x_1) a_p^\dagger(x_2) \cdots a_p^\dagger(x_n)$
$a_p(\vec{x})$	63	$a_p(x_n) a_p(x_{n-1}) \cdots a_p(x_1)$
$a^\dagger(\mathbf{x})$	63	$a_1^\dagger(\vec{x}_1) \otimes \cdots \otimes a_P^\dagger(\vec{x}_P)$
$a(\mathbf{x})$	63	$a_1(\vec{x}_1) \otimes \cdots \otimes a_P(\vec{x}_P)$
N_p	63	$\int_{X_p} dx\, a_p^\dagger(x) a_p(x)$
M	64	the "order" of the interaction
k_L	68	$\frac{-P_b B^2}{4A}$, a lower bound for K
$\mathbf{e}_p$	78	$(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in p^{th} coordinate
λ_v	90	$2^{P+2} \left(\sum_{p,q} \lVert v_{p,q} \rVert_{1,\infty}\right) \sum_{0\ne\mathbf{m}\in\mathbb{N}_0^P} \frac{1}{\lvert\mathbf{m}\rvert^{P+1}}$

C_v	99	$2\lvert X_b\rvert\lvert X_f\rvert\lVert v\rVert_\infty$
κ_L	103	$\frac{\lambda_b-\mu_b}{2}+\frac{\left(2\lvert X_b\rvert\lvert X_f\rvert\lVert v\rVert_\infty\right)^2}{\gamma}$, a lower bound for K
$\Omega_n(t)$	105	$\{(s_1,\ldots,s_n)\in[0,t]^n:\sum_{k=1}^n s_k\leq t\}$
C_b	107	$\min\{\sqrt{\lambda_b-\mu_b},1\}$
$C_2(t)$	107	$4\frac{C_b}{C_v}e^{\frac{1}{4}t(\lambda_b-\mu_b)}\max\{t,\sqrt{t}\}$
c_2	125	$\frac{4C_v}{C_b}e^{\frac{1}{4}(\lambda_b-\mu_b)}$

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Dedication

To Mom and Dad.

Chapter 1

Introduction

Functional integrals have become a ubiquitous tool in theoretical and mathematical physics, providing new intuition, connections between different problems, and stepping stones for further analysis. This thesis is dedicated to rigorously constructing such integrals for two classes of quantum many-particle systems. Before outlining our goals and strategies, we will introduce functional integrals through three main examples: the Feynman path integral, the Feynman-Kac formula and the coherent-state formal functional integral for the partition function of a boson gas.

Consider a single particle of mass m allowed to move in space $\mathbb{R}^3$ under the influence of a force corresponding to the potential energy $V : \mathbb{R}^3 \to \mathbb{R}$. In quantum mechanics, all information about the motion of this particle is contained in its *wave function*, ψ_t, which is an element of the Hilbert space $L^2(\mathbb{R}^3)$, for each time $t \geq 0$. For example, the probability of finding the particle within a region $\Lambda \subset \mathbb{R}^3$ at time t is $\int_\Lambda |\psi_t(x)|^2 \, dx$. The time evolution of the wave function ψ_t is governed by the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \psi_t = H \psi_t, \tag{1.1}$$

for all $t > 0$. The *Hamiltonian* operator H represents the energy of system and is defined by

$$H := -\frac{\hbar^2}{2m} \Delta + V.$$

Here, Δ is the Laplacian on $\mathbb{R}^3$ and V is the multiplication operator corresponding to the potential energy function of the same name. (Note that the domain of H surely does not contain all of $L^2(\mathbb{R}^3)$. However, it is not necessary to precisely define it for introductory purposes.) For convenience, we will let $\hbar = 1$ and $H_0 := -\frac{1}{2m}\Delta$, so that $H = H_0 + V$. The solutions of (1.1) must satisfy

$$\psi_t = e^{-itH}\psi_0.$$

This formula is difficult to work with in practice, because it involves the unbounded operator H. We would like to simplify it into an integral equation of the form,

$$\psi_t(x) = \int_{\mathbb{R}^3} dy\ I(x, y; t)\psi_0(y),$$

where $I(x, y; t)$ is the integral kernel of the operator e^{-itH}. Although we do not have a formula for $I(x, y; t)$ offhand, we can use Fourier transforms [1] to prove that the integral kernel of the free time-evolution operator e^{-itH_0} is

$$I_0(x, y; t) = \left(\frac{m}{2\pi it}\right)^{3/2} e^{i\frac{m}{2t}|x-y|^2}. \tag{1.2}$$

Unfortunately, because H_0 and V do not commute, we cannot simply separate the exponential of their sum into a product of their exponentials. That is,

$$e^{-it(H_0+V)} \neq e^{-itH_0}e^{-itV}.$$

However, this formula can be modified to a correct form,

$$e^{-it(H_0+V)} = \lim_{n\to\infty}\left(e^{-\frac{it}{n}H_0}e^{-\frac{it}{n}V}\right)^n,$$

which is an instance of the Trotter product formula [2]. The limit is taken in the strong operator topology. Hence,

$$\psi_t = e^{-itH}\psi_0 = \lim_{n\to\infty}\left(e^{-\frac{it}{n}H_0}e^{-\frac{it}{n}V}\right)^n \psi_0.$$

We can now use (1.2) to replace each $e^{-\frac{it}{n}H_0}$ with an integral, so that for any x_0,

$$\psi_t(x_0) = \lim_{n\to\infty}\left(\frac{m}{2\pi i\frac{t}{n}}\right)^{\frac{3}{2}n} \int_{\mathbb{R}^{3n}} dx_1\cdots dx_n\ e^{\frac{im}{2(t/n)}|x_0-x_1|^2}e^{-\frac{it}{n}V(x_1)}\cdots$$

$$\cdots e^{\frac{im}{2(t/n)}|x_{n-1}-x_n|^2} e^{-\frac{it}{n}V(x_n)}\psi_0(x_n)$$

$$= \lim_{n\to\infty} \left(\frac{mn}{2\pi it}\right)^{\frac{3}{2}n} \int_{\mathbb{R}^{3n}} dx_1 \cdots dx_n \; e^{iS_n(x_0,\cdots,x_n;t)}\psi_0(x_n), \tag{1.3}$$

where

$$S_n(x_0, \cdots, x_n; t) := \sum_{j=1}^{n} \frac{t}{n} \left(\frac{m}{2} \left(\frac{|x_{j-1} - x_j|}{t/n} \right)^2 - V(x_j) \right).$$

Everything up to this point can be proven rigorously. Now, formally taking the limit as $n \to \infty$, one could imagine that the sequences $\{x_0, \cdots, x_n\}$ would become continuous paths $X : [0,t] \to \mathbb{R}^3$ with $X(0) = x_0$ and that $S_n(x_0, \cdots, x_n; t)$ would become the *action* over each path,

$$S(X;t) := \int_0^t d\tau \; \left(\frac{m}{2} X'(\tau)^2 - V(X(\tau)) \right).$$

Assuming that the (complex and divergent) factor $\left(\frac{nm}{2\pi it}\right)^{\frac{3}{2}n}$ could be absorbed into the "measure", which we will call dX, we would find

$$\psi_t(x_0) = \int_{\Omega_{x_0}} dX \; e^{iS(X;t)}\psi_0(X(t)). \tag{1.4}$$

"Path integral" formulas such as this were first invented in the context of quantum mechanics by Richard Feynman in his 1942 PhD thesis. Physicists now routinely use this formula and others like it as a symbolic tool in order to gain intuition and do formal calculations by expanding about classical paths, which pleasantly arise as stationary points for the quantum action S. However, unfortunately, as $n \to \infty$, $\left(\frac{mn}{2\pi it}\right)^{\frac{3}{2}n} dx_1 \cdots dx_n$ does not converge to a measure, both because there is no infinite-dimensional analogue of Lebesgue measure and because the factor in front, $\left(\frac{mn}{2\pi it}\right)^{\frac{3}{2}n}$ blows up as $n \to \infty$. Furthermore, without restrictions on the paths X, the integrand fluctuates wildly, if it is defined at all. Hence, "dX" in (1.4) is not actually a measure and the integral is not well-defined. There have been some efforts to create a new definition that incorporates (1.4) and rigorously allows for expansion about classical paths, for example in [3].

If we consider the above analysis in "imaginary time", i.e. for the operator

e^{-tH} rather than e^{-itH}, the same work carries through to obtain the following analogue of (1.3),

$$(e^{-tH}\psi_0)(x_0) = \lim_{n\to\infty} \left(\frac{mn}{2\pi t}\right)^{\frac{3}{2}n} \int_{\mathbb{R}^{3n}} dx_1 \cdots dx_n \; e^{-\tilde{S}_n(x_0,\cdots,x_n;t)}\psi_0(x_n), \quad (1.5)$$

where

$$\tilde{S}_n(x_0,\cdots,x_n;t) = \sum_{j=1}^{n} \frac{t}{n}\left(\frac{m}{2}\left(\frac{|x_{j-1}-x_j|}{t/n}\right)^2 + V(x_j)\right).$$

The critical insight here is that the integrand now contains a product of Gaussians (rather than the wildly oscillatory factors of $e^{i\frac{m}{2t}|x-y|^2}$ above) due to the contribution of the imaginary-time free propagators. These can be combined with the Lebesgue measure on $\mathbb{R}^{3n}$ to form the $3n$-dimensional Gaussian measure,

$$d\mu_n(x_1,\ldots,x_n) = \prod_{j=1}^{n}\left[\left(\frac{mn}{2\pi t}\right)^{\frac{3}{2}} e^{-\frac{m}{2}\frac{|x_{j-1}-x_j|^2}{t/n}} dx_j\right].$$

In the limit $n \to \infty$, this measure converges to a probability measure known as the *Wiener measure* μ_{x_0} on the space Ω of continuous paths $X : [0,t] \to \mathbb{R}^3$ [1]. (This measure had already been invented by Wiener in the early 1920s to describe Brownian motion [4].) Assuming that V is sufficiently "nice" (e.g. in $L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ [1]), (1.5) converges to

$$(e^{-tH}\psi_0)(x_0) = \int_\Omega d\mu_{x_0}(\omega)\; e^{-\int_0^t d\tau\; V((\omega(\tau))}\psi_0(\omega(t)).$$

This functional integral is known as the *Feynman-Kac* formula. It has applications in many different fields, including quantum physics, PDEs, probability, and finance, thereby providing connections between these fields [5]. For example, the Feynman-Kac formula is a solution to the PDE,

$$\frac{\partial\psi_t}{\partial t} - \frac{1}{2m}\Delta\psi_t + V\psi_t = 0.$$

This is a heat equation, where V represents a source of external cooling. On the other hand, the Feynman-Kac formula may be viewed as an expectation value with respect to the Wiener measure, which describes Brownian motion.

This link makes it possible to use properties of the PDE to deduce properties of Brownian motion, and vice versa [8]. Particularly relevant to this thesis (although we do not use them here) are applications of Feynman-Kac formulae to many-particle systems. For example, Ginibre [6] and Brydges and Federbush [7] constructed Feynman-Kac formulae for N particles and then summed over all N to obtain the partition function and correlation functions for the full Fock space, in the case where the chemical potential is negative and sufficiently large in magnitude.

There is another approach to constructing functional integrals for quantum many-particle systems, wherein fields are introduced through "coherent states". Coherent-state functional integrals appear to be the most promising tool for rigorously constructing the many-particle models, because they (like the Feynman path integrals described above) naturally contain an action which provides intuition about the low-energy behaviour of the system and suggests phase transitions. To see this roughly and quickly, consider a system of interacting bosons in $\mathbb{R}^3$. Through formal manipulations [9], the coherent-state approach yields a formal functional integral for the partition function of the following form [10]:

$$Z = \int \cdots \int \prod_{\substack{x \in \mathbb{R}^3 \\ 0 \le \tau \le \beta}} \frac{d\phi_\tau^*(x) d\phi_\tau(x)}{2\pi i} \; e^{\mathcal{A}(\phi^*, \phi)}, \tag{1.6}$$

where the domain is over all fields $\phi_\tau : \mathbb{R}^3 \to \mathbb{C}$, as functions of the (imaginary) time-like variable $\tau \in [0, \beta]$, with the periodic boundary condition, $\phi_0 = \phi_\beta$. For each $\tau \in [0, \beta]$ and $x \in \mathbb{R}^3$,

$$\int \frac{d\phi_\tau^*(x) d\phi_\tau(x)}{2i}$$

represents an integral over $\mathbb{C} \cong \mathbb{R}^2$ with Lebesgue measure. That is, for $z = u + iv$, $\frac{dz^* dz}{2i} = du\, dv$.

As in the Feynman path integral (1.4), the exponent $\mathcal{A}(\phi^*, \phi)$ in Equation (1.6) is called the "action" and it depends on all $\phi_\tau(x), \phi_\tau^*(x)$. In particular,

$$\mathcal{A}(\alpha^*, \phi) := \int_0^\beta d\tau \int_{\mathbb{R}^3} dx \; \alpha_\tau^*(x) \frac{\partial}{\partial \tau} \phi_\tau(x) - \int_0^\beta d\tau \; K(\alpha_\tau^*, \phi_\tau),$$

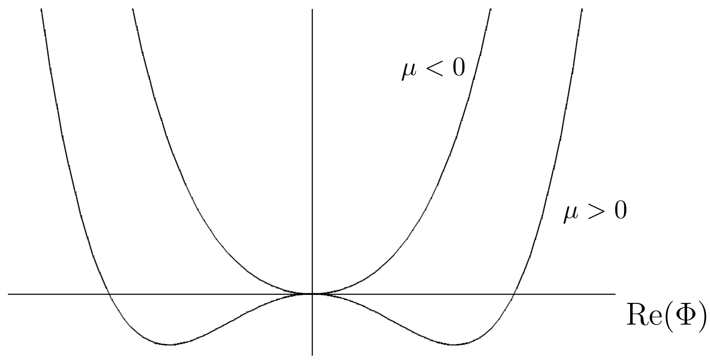

Figure 1.1: The naïve effective potential, for positive and negative chemical potentials μ.

where

$$K(\alpha^*, \phi) = \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\ \alpha^*(x) h(x,y) \phi(y) - \mu \int_{\mathbb{R}^3} dx\ \alpha^*(x)\phi(x) \qquad (1.7)$$
$$+ \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy\ \alpha^*(x)\alpha^*(y) v(x,y)\phi(x)\phi(y).$$

Here, h is a kernel for the single-particle kinetic energy operator, μ is the chemical potential, and $v(x, y)$ represents the potential energy due to a particle at position x interacting with a particle at position y.

The formal functional integral (1.6) provides intuition about phase transitions of the system. To see this briefly, assume for simplicity that the field ϕ is a constant, that is $\phi_\tau(x) = \Phi \in \mathbb{C}$ for all τ and x. Then the action simplifies to a function of the form,

$$\mathcal{A}(\Phi^*, \Phi) = -\int_0^\beta d\tau \int_{\mathbb{R}^3} dx\ \left(v_0 |\Phi|^4 - \mu |\Phi|^2 \right),$$

where $v_0 = \int_{\mathbb{R}^3} dy\ v(x, y)$. So $\mathcal{A}(\Phi^*, \Phi)$ is minus the volume of "space-time" (which is infinite, but we'll ignore that) times the "naïve effective potential", $v_0|\Phi|^4 - \mu|\Phi|^2$. When we look for its stationary points, we see two cases: when $\mu \leq 0$, the only minimum occurs at $\Phi = 0$; otherwise, if $\mu > 0$, there is a degenerate minimum on the circle $|\Phi| = \sqrt{\frac{\mu}{2v_0}}$. This is depicted in Figure 1.1. Hence at low temperatures, we expect the fields to be near zero when $\mu < 0$ and nonzero when $\mu > 0$. This signals a phase transition.

Unfortunately, the formal functional integral in (1.6) is not well-defined. Since the domain is (uncountably) infinite-dimensional, there is no clear extension of the Lebesgue measure to it. In the spirit of the Wiener measure, one could hope to absorb part of the exponential, $e^{\mathcal{A}(\phi^*,\phi)}$, to construct a proper infinite-dimensional Gaussian measure, however, the purely imaginary term $\int_0^\beta d\tau \int_{\mathbb{R}^3} dx\ \alpha_\tau^*(x)\frac{\partial}{\partial\tau}\phi_\tau(x)$ oscillates so wildly that this is not possible, as shown in Appendix A of [10].

In this thesis, our goal is to rigorously construct functional integrals of the form in Equation (1.6). Our main strategy is to:

1. approximate the physical space $\mathbb{R}^3$ by a finite set X,
2. approximate the time-like interval $[0, \beta)$ by a finite set $\mathcal{T}_q = \{\epsilon, 2\epsilon, \ldots, q\epsilon = \beta\}$,
3. use a large-field cutoff in the integral, so that rather than integrating over the whole complex plane for each τ and x, we only integrate over a ball of finite radius, $|\phi_\tau(x)| < R$.

To recover (1.6), we would like to take the limits as the size of X grows to infinity, as $q \to \infty$, and as $R \to \infty$. In this work, we take the limit as both q and R grow to infinity at the same time, considering R as a function of q. However, we leave the limit $|X| \to \infty$ for future work.

We consider two classes of many-body systems, both of which consist of mixed gases of bosons and fermions. First, we study a system of any finite number of particle species (bosons and/or fermions), interacting on a finite lattice with a general Hamiltonian that preserves the total number of particles in each species. This is a generalization of the work for a purely bosonic gas in [10]. Gases of this type have been predicted and observed to have remarkably rich phase diagrams, including such features as superfluidity, supersolidity, and phase separation [11–13]. Next, we consider a boson-fermion gas with an interaction that does not preserve the number of bosons. The Hamiltonian is called the "Holstein Hamiltonian" [14] and has been used extensively to model systems of electrons and phonons [15]. In these systems, the phenomena of "polarons" (quasiparticles of electrons surrounded by a cloud of phonons) and phonon-mediated superconductivity have garnered much attention and research. Furthermore, this type of Hamiltonian arises when a Hubbard-Stratonovich transformation [16] is performed on the partition function for a purely fermionic gas with a two-body interaction.

We begin in Chapter 2 with a brief overview of background material, including non-relativistic quantum mechanics, Fock spaces, and quantum statistical mechanics. In Chapter 3, we define the coherent states and develop a theoretical toolkit, which includes a resolution of the identity and a trace formula. We follow the approaches of [9] and [10] for the coherent states of single species of bosons and fermions, and then extend to gases with multiple species of bosons and/or fermions. Our main results lie in Chapters 4 and 5. Theorem 4.2.1 gives a functional integral representation for the partition function of any quantum system with a fixed and finite number of particle species (bosons, fermions, or both) interacting on a finite lattice, with a Hamiltonian that preserves the total number of particles for each species. This is of the form,

$$\mathrm{Tr}\ e^{-\beta K} = \lim_{q\to\infty} \int \prod_{\tau\in\mathcal{T}_q} \left[d\mu_{\underline{\mathbf{R}}(q)}(\boldsymbol{\phi}_\tau^*, \boldsymbol{\phi}_\tau) e^{-\int dy\ \boldsymbol{\phi}_\tau^*(y)\cdot\boldsymbol{\phi}_\tau(y)} \right] \prod_{\tau\in\mathcal{T}_q} \langle \boldsymbol{\phi}_{\tau-\epsilon} | e^{-\epsilon K} | \boldsymbol{\phi}_\tau \rangle, \tag{1.8}$$

where β is the inverse temperature, $K = H - \boldsymbol{\mu}\cdot\mathbf{N}$, H is the Hamiltonian, $\boldsymbol{\mu}$ is the vector of chemical potentials, $\mathbf{N}$ is the vector of particle number operators, and $\underline{\mathbf{R}}(q)$ represents a large-boson-field cutoff that grows sufficiently quickly as $q \to \infty$. Each $|\boldsymbol{\phi}_\tau\rangle$ is a coherent state (which is defined in Chapter 3). In Lemma 4.3.3, we expand the "action", i.e. the logarithm of $\langle\boldsymbol{\alpha}|e^{-\epsilon K}|\boldsymbol{\phi}\rangle$, in powers of the fields,

$$\log\langle\boldsymbol{\alpha}|e^{-\epsilon K}|\boldsymbol{\phi}\rangle = \sum_{\mathbf{n}\in\mathbb{N}_0^P} \int_{X^{\mathbf{n}}} d\mathbf{x} \int_{X^{\mathbf{n}}} d\mathbf{y}\ \alpha^*(\mathbf{x}) F_{\mathbf{n}}(\epsilon,\mathbf{x},\mathbf{y}) \phi(\mathbf{y}),$$

and derive a recursion relation for the coefficients $F_{\mathbf{n}}(\epsilon,\mathbf{x},\mathbf{y})$ in Theorem 4.3.3. For two-body interactions (such as the Coulomb interaction), we have shown in Theorem 4.3.5 that the coefficients $F_{\mathbf{n}}(\epsilon,\mathbf{x},\mathbf{y})$ decay as $\mathbf{n}$ grows. In Proposition 4.3.6, these bounds are then used to find a domain of analyticity for the action. This domain is large enough such that an exponential of the action may be substituted for each inner product in (1.8) above, provided that the large-field cutoffs are taken to be not too large.

In Chapter 5, we study the electron-phonon model. Because the Hamiltonian does not preserve the number of bosons, we require extra control over expansions in boson number. For this, we generally use an expansion of $e^{-\epsilon K}$ in powers of the interaction V. This is called a Duhamel expansion and is shown in Theorem 5.3.1. In Theorem 5.3.8, we find a functional integral representation for the partition of this system, which is also of the form in (1.8) above. Finally, we prove in Section 5.4 that the action exists as an entire-analytic function of the fields.

Chapter 2

Quantum Many-Particle Systems

This chapter gives a rapid overview of the quantum theory that underpins this thesis. Readers who have already been introduced to annihilation and creation operators may skip to Section 2.3, and those who are already familiar with the grand canonical ensemble may skip straight to Chapter 3.

2.1 Introduction to Quantum Mechanics

Every *quantum system* is associated with a Hilbert space, $\mathcal{H}$. Each *ray* of this Hilbert space represents a *state* of the system. A ray is defined to be an equivalence class of the relation '$\sim$', where, for any $\psi, \phi \in \mathcal{H}$,

$$\psi \sim \phi :\Longleftrightarrow (\exists \lambda \in \mathbb{C}\backslash\{0\})\ \psi = \lambda\phi.$$

The set of all rays is the *projective Hilbert space*,

$$\mathbb{P}\mathcal{H} := \mathcal{H}/\sim .$$

On the other hand, every *pure state* of the quantum system may be represented by an element of $\mathbb{P}\mathcal{H}$. For convenience, we will often denote elements of the projective space (and hence, pure states) by a single representative element $\psi \in \mathcal{H}$ with norm $\|\psi\| = 1$. (The meaning of the word "pure" in this context will be explained in section 2.3.1.)

Physically, an *observable* is any quantity of the system which may be measured. Examples include position, momentum, energy, and angular momentum. Mathematically, each observable is represented by a self-adjoint operator on $\mathcal{H}$, or a dense subspace thereof. *Measurement* of the observable yields a real number that lies in the spectrum of this operator.

When the system is in state $\psi \in \mathcal{H}$ (with $\|\psi\| = 1$), the *expectation value* of the observable corresponding to the self-adjoint operator A is

$$\langle \psi, A\psi \rangle.$$

By the spectral theorem [2], each self-adjoint operator A is associated with a projection-valued measure,

$$P_S = \chi_S(A),$$

defined for Borel sets $S \subset \mathbb{R}$. Here, χ_S is the characteristic function,

$$\chi_S(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The probability of measuring the corresponding observable to lie within a Borel set $S \subset \mathbb{R}$ while the system is in state ψ is

$$\langle \psi, \chi_S(A)\, \psi \rangle = \langle \psi, P_S\, \psi \rangle = \int_S d\langle \psi, P_\lambda\, \psi \rangle.$$

A self-adjoint operator of particular interest is the one corresponding to the total energy of the system, known as the *Hamiltonian*, because it generates time evolution. That is, the state ψ of a quantum system evolves in time t according to the Schrödinger equation,

$$i\hbar \frac{d\psi}{dt} = H\psi,$$

where $\hbar$ is a constant with dimensions of [energy] $\times$ [time] and H is the Hamiltonian. This equation can be derived from the principles of causality, superposition, and correspondence with classical mechanics in the classical limit [17].

The vocabulary thus far can be summarized as follows:

$$\begin{aligned} \textit{quantum system} &\longleftrightarrow \textit{Hilbert space } \mathcal{H}, \\ \textit{pure state} &\longleftrightarrow \textit{element of } \mathbb{P}\mathcal{H}, \\ \textit{observable} &\longleftrightarrow \textit{self-adjoint operator on } \mathcal{H}. \end{aligned}$$

Since we are working from the mathematical side in this thesis, we may consider the words on the left as vocabulary for the mathematical objects on the right, even though they also correspond with tangible entities of the physical world. Without attempting the philosophical task of clarifying this correspondence, we refer to an example to help the reader gain intuition.

Example 2.1.1 (*A free particle in 1 dimension*)**.** Consider a quantum system consisting of a single particle allowed to move freely within some interval of a 1-dimensional space, $\Omega \subset \mathbb{R}$. The Hilbert space for this system is $L^2(\Omega)$, the set of all square-integrable functions on Ω with Lebesgue measure. The operator corresponding to the position of the particle is multiplication by the identity function,

$$(\hat{x}\psi)(x) := x\psi(x),$$

for any $x \in \Omega$ and $\psi \in L^2(\Omega)$ such that the function on the right side is also an element of $L^2(\Omega)$. For example, if Ω is compact, $\hat{x}$ is well-defined as a bounded operator on all of $L^2(\Omega)$. If the system is in state $\psi \in L^2(\Omega)$ with $\|\psi\| = 1$, then the expected value of the position is

$$\langle \psi, \hat{x}\psi \rangle = \int_\Omega x|\psi(x)|^2 \, dx.$$

This means that if we were to prepare many independent isolated copies of this quantum system, all in state ψ, and we measured the position of the particle in each, we would expect the mean value of all such measurements to be $\int_\Omega x|\psi(x)|^2 \, dx$. The probability of finding the particle within the (Borel) set $\Gamma \subset \Omega$ is

$$\langle \psi, \chi_\Gamma(\hat{x})\psi \rangle = \int_\Omega \psi^*(x)\chi_\Gamma(x)\psi(x)\, dx = \int_\Gamma |\psi(x)|^2 \, dx.$$

In this sense, the state ψ gives rise to a probability distribution $|\psi(x)|^2$ for the position of the particle.

If $\Omega = \mathbb{R}$, the Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2},$$

defined on the domain $C^2(\Omega)$, which is dense in $L^2(\Omega)$. If the derivatives are taken to be weak derivatives, the domain may also be extended to the Sobolev space $H^2(\Omega)$ [17].

Suppose that the system begins in state $\psi_0 \in L^2(\mathbb{R})$ and evolves in time. The state at time t is denoted by $\psi_t \in L^2(\mathbb{R})$. For any $x \in \mathbb{R}$, we use the notation $\psi(x,t) := \psi(t)(x)$. The time evolution is given by the Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t}(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}(x,t), \tag{2.1}$$

with the initial condition,

$$\psi(x,0) = \psi_0(x)$$

for all $x \in \mathbb{R}$. Since Equation (2.1) is a type of wave equation, it is common to call the solution ψ a *wave function*.

2.1.1 n-Particle Systems

For a quantum system consisting of a single particle in a region Ω of space, the corresponding Hilbert space is $L^2(\Omega)$. What is the Hilbert space for two or more particles in Ω? More generally, if two distinguishable quantum systems with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are combined and considered altogether as a single system, what is the Hilbert space of the composite system? The Hilbert space $\mathcal{H}$ of the composite system is the tensor product of its constituent spaces,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$

This is a tensor product of Hilbert spaces, which is similar to a vector-space tensor product, but completed under the topology induced by the inner product,

$$\langle \phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2 \rangle = \langle \phi_1, \psi_1 \rangle \langle \phi_2, \psi_2 \rangle,$$

for all $\phi_1, \psi_1 \in \mathcal{H}_1$ and $\phi_2, \psi_2 \in \mathcal{H}_2$. (A full definition of the Hilbert space tensor product may be found in standard textbooks such as [2].)

Successively combining quantum systems with Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, $\ldots$, $\mathcal{H}_n$, the resulting Hilbert space for the n-component composite system is

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n.$$

Back to our original question of this section, consider a quantum system consisting of n distinguishable particles in a region Ω. The corresponding Hilbert space is then

$$\mathcal{D}_n(\Omega) := \underbrace{L^2(\Omega) \otimes L^2(\Omega) \otimes \cdots \otimes L^2(\Omega)}_{n \text{ times}} \cong L^2(\Omega^n).$$

In this thesis, we are also interested in quantum systems of *indistinguishable* particles. Aiming to define the corresponding Hilbert spaces, let's intuitively start by considering a state $\psi \in L^2(\Omega^2)$ for a quantum system of two distinguishable particles in Ω and then ask what we should demand of ψ if the particles were to become indistinguishable.

First, indistinguishability should imply that no measurement is affected by the permutation of particle coordinates. For example, let ψ_s be the function obtained by interchanging the two particle coordinates,

$$\psi_s(x_1, x_2) = \psi(x_2, x_1).$$

We demand that the expectation value of any observable A be unaffected by this interchange,

$$\langle \psi, A\psi \rangle = \langle \psi_s, A\psi_s \rangle.$$

Since this is true for any observable A, it follows that

$$|\psi(x_1, x_2)|^2 = |\psi_s(x_1, x_2)|^2$$

for all $x_1, x_2 \in \Omega$. Hence, the two functions agree up to a phase shift,

$$\psi(x_2, x_1) = e^{i\theta(x_1,x_2)}\psi(x_1, x_2).$$

for some $\theta(x_1, x_2) \in \mathbb{R}$. In three space-dimensions, θ is a constant, equal to 0 or π. That is,

$$\psi_s = \pm\psi,$$

which means that the wave functions are either symmetric or antisymmetric. Particles with symmetric wave functions are said to satisfy Bose-Einstein statistics and are called *bosons*. Those with antisymmetric wave functions are said to satisfy Fermi-Dirac statistics and are called *fermions*. Furthermore, it can be proven that there is a direct link between the spin of a particle and its statistics. This remarkable result, called the Spin-Statistics Theorem, verifies the experimental observation that particles of integer spin are bosons, whereas particles with half-integer spin are fermions [18].

We can extend this intuition to any number n of indistinguishable particles in three dimensions by demanding that their wave functions $\psi \in L^2(\mathbb{R}^{3n})$ satisfy

$$\psi(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (\pm 1)^\sigma \psi(x_1, \ldots, x_n),$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^{3n}$ and every permutation $\sigma \in S_n$. Again, the $\pm$ sign on the right side may be read as "+" for bosons and "−" for fermions, where $(+1)^\sigma \equiv 1$ and $(-1)^\sigma$ is the sign of the permutation σ.

We will use the following notation for the action of permutations on n-particle coordinates and wave functions.

Definition 1. Let X be any set.

(i) For any $n \in \mathbb{N} \cup \{0\}$ and $\sigma \in S_n$, let

$$\begin{aligned} \sigma : X^n &\to X^n, \\ (x_1, \ldots, x_n) &\mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \end{aligned} \tag{2.2}$$

(ii) For any $n \in \mathbb{N} \cup \{0\}$ and $\sigma \in S_n$, let

$$\begin{aligned} \sigma : L^2(X^n) &\to L^2(X^n), \\ f &\mapsto f \circ \sigma^{-1}. \end{aligned}$$

That is,

$$(\sigma f)(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}), \tag{2.3}$$

for any $(x_1, \ldots, x_n) \in X^n$.

We are now prepared to define the Hilbert spaces for n indistinguishable particles.

Definition 2. Let (X, μ) be a measure space.

(i) For any $n \in \mathbb{N}$, the *Hilbert space for n identical bosons* in X is

$$\mathcal{B}_n(X) := \{f \in L^2(X^n) : (\forall \sigma \in S_n)\sigma f = f\}, \tag{2.4}$$

the set of fixed points for the action of S_n on $L^2(X^n)$, otherwise known as the symmetric functions from X^n to $\mathbb{C}$.

(ii) For any $n \in \mathbb{N}$, the *Hilbert space for n identical fermions* in X is

$$\mathcal{F}_n(X) := \{f \in L^2(X^n) : (\forall \sigma \in S_n)\sigma f = (-1)^\sigma f\}, \tag{2.5}$$

the set of antisymmetric functions from X^n to $\mathbb{C}$.

(iii) The *vacuum* is the normalized state with zero particles,

$$|0\rangle := 1 \in \mathbb{C}.$$

We also define the *zero-boson* and *zero-fermion* spaces to be

$$\mathcal{B}_0(X) = \mathcal{F}_0(X) := \mathbb{C}.$$

For all n, $\mathcal{B}_n(X)$ and $\mathcal{F}_n(X)$ are subspaces of $L^2(X^n)$, closed under the inner product

$$\langle f, g\rangle := \int_{X^n} f^* g \; d\mu,$$

where f^* is the complex conjugate of f. The orthogonal projection onto $\mathcal{B}_n(X)$ is the *symmetrization operator*,

$$S_b^{(n)}(X) := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma. \tag{2.6}$$

Likewise, the *anti-symmetrization operator*,

$$S_f^{(n)}(X) := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \sigma \tag{2.7}$$

is the orthogonal projection onto $\mathcal{F}_n(X)$.

Just as we combine Hilbert spaces corresponding to distinguishable quantum systems using a tensor product, indistinguishable systems can be combined by taking a tensor product of their Hilbert spaces and then symmetrizing or anti-symmetrizing. That is, for any $m, n \in \mathbb{N}_0$ (where $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, the set of whole numbers),

$$\mathcal{B}_{n+m}(X) = S_b^{(n+m)}(X)\ \mathcal{B}_n(X) \otimes \mathcal{B}_m(X)$$

and

$$\mathcal{F}_{n+m}(X) = S_f^{(n+m)}(X)\ \mathcal{F}_n(X) \otimes \mathcal{F}_m(X).$$

We also define corresponding symmetric and antisymmetric tensor products for elements of these spaces. Suppose that $u \in L^2(X^n)$ and $v \in L^2(X^m)$ for some $n, m \in \mathbb{N}_0$. Their tensor product $u \otimes v$ in $L^2(X^{n+m})$ is given by

$$u \otimes v(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) = u(x_1, \dots, x_n)v(x_{n+1}, \dots, x_{n+m}),$$

for all $(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) \in X^{n+m}$. Aiming to define symmetric and anti-symmetric versions of this tensor product that lie in the bosonic and fermionic subspaces respectively, our first thought might be to simply take the tensor product above and then symmetrize or antisymmetrize. Clearly,

$$S_b^{(n+m)}(X)(u \otimes v) \in \mathcal{B}_{n+m}(X) \quad \text{and} \quad S_f^{(n+m)}(u \otimes v) \in \mathcal{F}_{n+m}(X).$$

We could simply define these as the symmetric and antisymmetric tensor products of elements u and v (and some authors do); however, in order to ensure associativity of these products, we will include a combinatorial factor (depending on m and n) in the definitions.

Definition 3. For any $m, n \in \mathbb{N}$, let $u \in L^2(X^n)$ and $v \in L^2(X^m)$.

(i) The *tensor product* of u and v, $u \otimes v \in L^2(X^{m+n})$, is defined by

$$u \otimes v(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) := u(x_1, \dots, x_n)v(x_{n+1}, \dots, x_{n+m}), \tag{2.8}$$

for any $(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) \in X^{n+m}$.

(ii) The *symmetric tensor product* of u and v is

$$\begin{aligned} u \otimes_s v &:= \sqrt{\frac{(m+n)!}{m!n!}}\; S_b^{(m+n)}(X)(u \otimes v) \\ &= \frac{1}{\sqrt{m!(m+n)!n!}} \sum_{\sigma \in S_{m+n}} \sigma(u \otimes v). \end{aligned} \tag{2.9}$$

(iii) The *antisymmetric tensor product* or *wedge product* of u and v is

$$\begin{aligned} u \wedge v &:= \sqrt{\frac{(m+n)!}{m!n!}}\; S_f^{(m+n)}(u \otimes v) \\ &= \frac{1}{\sqrt{m!(m+n)!n!}} \sum_{\pi \in S_{n+m}} (-1)^{\pi} \pi(u \otimes v). \end{aligned} \tag{2.10}$$

It is easy to verify that these products are indeed associative. Furthermore, the *annihilation operators*, which we will define in Section 2.2.2, satisfy Leibniz-like product rules with respect to these products. The symmetric tensor product is clearly commutative by definition, whereas the wedge product satisfies

$$u \wedge v = (-1)^{nm} v \wedge u.$$

for all $u \in \mathcal{F}_n(X)$ and $v \in \mathcal{F}_m(X)$.

Example 1. If a single particle in state u is combined with with n particles in state v, where all of these particles (including the single) are identical, what is the resulting $(n+1)$-particle state? In the case of bosons, if $u \in \mathcal{B}_1(X)$ and $v \in \mathcal{B}_n(X)$ for some $n \in \mathbb{N}$,

$$u \otimes_s v(x_1, \ldots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} u(x_j) v(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}).$$

Similarly, in the case of fermions, if $u \in \mathcal{F}_1(X)$ and $v \in \mathcal{F}_n(X)$ for some $n \in \mathbb{N}$,

$$u \wedge v(x_1, \ldots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} (-1)^{j-1} u(x_j) v(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}).$$

Example 2. If n identical particles in states $u_1, \ldots, u_n \in \mathcal{B}_1(X) = \mathcal{F}_1(X) = L^2(X)$ are combined, the resulting n-particle state will be as follows: for bosons,

$$u_1 \otimes_s \cdots \otimes_s u_n(x_1, \ldots, x_n) = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} u_1(x_{\sigma^{-1}(1)}) \cdots u_n(x_{\sigma^{-1}(n)}),$$

and for fermions,

$$u_1 \wedge \cdots \wedge u_n(x_1, \ldots, x_n) = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} (-1)^\sigma u_1(x_{\sigma^{-1}(1)}) \cdots u_n(x_{\sigma^{-1}(n)}).$$

These results follow from Example 1 by induction.

Given an orthonormal basis for the single-particle space $L^2(X)$, we can combine its elements using the symmetric and anti-symmetric tensor products as in Example 2 in order to construct orthonormal bases for $\mathcal{B}_n(X)$ and $\mathcal{F}_n(X)$ respectively.

Proposition 2.1.2. Let X be a finite set. Suppose that $\{\varphi_i : i = 1, \ldots, |X|\}$ is an orthonormal basis for the single-particle Hilbert space, $L^2(X)$.

(i) The set

$$\left\{ \frac{1}{\sqrt{\alpha!}} \varphi^{\otimes_s^\alpha} : \alpha \in \mathbb{N}_0^{|X|}, |\alpha| = n \right\}$$

is an orthonormal basis for $\mathcal{B}_n(X)$. For the "multi-index" α, we use the following notation:

$$\alpha! := \alpha_1! \alpha_2! \ldots \alpha_{|X|}!,$$

$$|\alpha| := \sum_{j=1}^{|X|} \alpha_j,$$

and

$$\varphi^{\otimes_s^\alpha} := \varphi_1^{\otimes_s^{\alpha_1}} \otimes_s \varphi_2^{\otimes_s^{\alpha_2}} \otimes_s \cdots \otimes_s \varphi_{|X|}^{\otimes_s^{\alpha_{|X|}}}.$$

(ii) The set

$$\left\{ \varphi_{i_1} \wedge \varphi_{i_2} \wedge \cdots \wedge \varphi_{i_n} : \{i_j\}_{j=1}^n \subset \{1, \ldots, |X|\} \text{ with } i_1 < i_2 < \cdots < i_n \right\}$$

is an orthonormal basis for $\mathcal{F}_n(X)$.

One particularly useful orthonormal basis for $L^2(X)$ is the set of *Kronecker delta functions*. For any $y \in X$, the Kronecker delta function δ_y is given by

$$\delta_y(x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$. Applying Proposition 2.1.2, we see that

$$\{\delta_Y : Y \in X^n/S_n\}$$

is an orthonormal basis for $\mathcal{B}_n(X)$, where if $(y_1, \ldots, y_n)$ is any representative for $Y \in X^n/S_n$, then

$$\delta_Y := c_Y \ \delta_{y_1} \otimes_s \cdots \otimes_s \delta_{y_n}, \tag{2.11}$$

with

$$c_Y := \frac{1}{\sqrt{\prod\limits_{x \in X} \mu_Y(x)!}}, \tag{2.12}$$

and $\mu_Y(x) := |\{j : x = y_j\}|$, the number of times x appears in Y. Similarly, if we introduce a total ordering (arbitrarily) on X,

$$\{\delta_{y_1} \wedge \delta_{y_2} \wedge \cdots \wedge \delta_{y_n} : y_1, \ldots, y_n \in X, y_i < y_j \text{ for all } \ i < j\} \tag{2.13}$$

is a basis for $\mathcal{F}_n(X)$. If $Y = \{y_1, \ldots, y_n\} \subset X^n$ with ordered elements $y_1 < \cdots < y_n$, then

$$\delta_Y := \delta_{y_1} \wedge \cdots \wedge \delta_{y_n}. \tag{2.14}$$

Hence, the basis in Equation (2.13) can also be written as $\{\delta_Y : Y \subset X\}$.

By counting the elements of these bases, we can see that the dimension of $\mathcal{B}_n(X)$ is $\binom{n+|X|-1}{n}$ and the dimension of $\mathcal{F}_n(X)$ is, when $n \leq |X|$, $\binom{|X|}{n}$.

2.2 The Fock Spaces

We are interested in quantum systems where particles may be created and destroyed, or allowed to move in and out of the system. In this section, we define the corresponding Hilbert spaces, which must allow for arbitrary and varying particle numbers.

Throughout this section, let X be a finite set with the counting measure. That is, for any function $f : X \to \mathbb{C}$,

$$\int_X f(x)\ dx := \sum_{x \in X} f(x).$$

Recall that the Hilbert space for a fixed number, n, of distinguishable particles in the region X is

$$\mathcal{D}_n(X) = \underbrace{L^2(X) \otimes \cdots \otimes L^2(X)}_{n \text{ times}} = L^2(X)^{\otimes n} \cong L^2(X^n) \cong \mathbb{C}^{|X|^n}. \tag{2.15}$$

Taking the direct sum of all n-particle Hilbert spaces, we obtain

$$\mathcal{D}(X) := \bigoplus_{n=0}^{\infty} \mathcal{D}_n(X), \tag{2.16}$$

which contains states with arbitrary particle numbers and superpositions thereof. This Hilbert space is called the *Fock space* for distinguishable particles.

Similarly, the Fock space for indistinguishable bosons is defined to be

$$\mathcal{B}(X) := \bigoplus_{n=1}^{\infty} \mathcal{B}_n(X), \tag{2.17}$$

and for indistinguishable fermions,

$$\mathcal{F}(X) := \bigoplus_{n=1}^{\infty} \mathcal{F}_n(X). \tag{2.18}$$

However, since X is finite, there are no nontrivial antisymmetric functions on X with $|X| + 1$ arguments or more. Hence, $\mathcal{F}_n(X) = \{0\}$ for all $n > |X|$. It follows that $\mathcal{F}(X)$ is finite-dimensional and

$$\mathcal{F}(X) \cong \bigoplus_{n=1}^{|X|} \mathcal{F}_n(X). \tag{2.19}$$

Each of $\mathcal{B}(X)$ and $\mathcal{F}(X)$ are closed subspaces of $\mathcal{D}(X)$; the orthogonal projections onto these spaces are the direct sums of the n-particle symmetrization and anti-symmetrization operators respectively. That is, the projection from $\mathcal{D}(X)$ to $\mathcal{B}(X)$ is the operator

$$S_b(X) := \bigoplus_{n=1}^{\infty} S_b^{(n)}(X), \tag{2.20}$$

where the right-hand side means that the restriction of $S_b(X)$ to each $\mathcal{D}_n(X)$ is $S_b^{(n)}(X)$. The projection from $\mathcal{D}(X)$ to $\mathcal{F}(X)$ is the operator

$$S_f(X) := \bigoplus_{n=1}^{\infty} S_f^{(n)}(X). \tag{2.21}$$

(Again, the latter sum can be truncated at $n = |X|$.)

We will denote by $P^{(n)}$ the projection from the Fock space $\mathcal{D}(X)$ to the n-particle subspace, $\mathcal{D}_n(X)$. Note that $P^{(n)}$ preserves symmetry and anti-symmetry and therefore can also be seen as the projection from $\mathcal{B}(X)$ to $\mathcal{B}_n(X)$ and from $\mathcal{F}(X)$ to $\mathcal{F}_n(X)$.

Recall that self-adjoint operators in the Hilbert space correspond to "observables": quantities of the system that may be measured, such as energy, position, and momentum. We will discuss such operators on Fock spaces, as well as the *creation* and *annihilation operators*, which are not self-adjoint, but will provide for a useful representation of the self-adjoint operators of interest.

2.2.1 Self-Adjoint Operators on Fock Space

Consider a single particle in the space X (which we are still assuming to be a finite set). Let h be a self-adjoint operator on $L^2(X)$ that represents the kinetic energy of the particle. A typical example is

$$h = -\frac{1}{2m}\Delta,$$

where m is the mass the particle and Δ is the discrete Laplacian on X. For example, if X is a one-dimensional lattice with spacing δ between lattice sites, then for any $f : X \to \mathbb{C}$,

$$hf(x) = -\frac{1}{2m}\Delta f(x) = -\frac{1}{2m}\frac{f(x+\delta) - 2f(x) + f(x-\delta)}{\delta^2}.$$

For n distinguishable particles in X, the total kinetic energy is intuitively the sum of the kinetic energies for each particle. This is defined by

$$\begin{aligned} H_n\ \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n :=& (h\psi_1) \otimes \psi_2 \otimes \cdots \otimes \psi_n + \psi_1 \otimes (h\psi_2) \otimes \cdots \otimes \psi_n + \\ & \cdots + \psi_1 \otimes \psi_2 \otimes \cdots \otimes (h\psi_n) \\ =& \sum_{j=1}^{n} \psi_1 \otimes \cdots \otimes (h\psi_j) \otimes \cdots \otimes \psi_n, \end{aligned}$$

for any state of the form $\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n$ where each $\psi_j \in L^2(X)$. We can extend by linearity to all of $\mathcal{D}_n(X)$. The extension of h to the full Fock space,

$$d\Gamma(h) := \bigoplus_{n=1}^{\infty} H_n, \tag{2.22}$$

is known as the *second quantization of h*. This operator is essentially self-adjoint on the domain

$$\mathcal{D}^{(finite)}(X) := \{f \in \mathcal{D}(X) : \{n : P^{(n)} f \neq 0\} \text{ is finite}\} \tag{2.23}$$

and self-adjoint on the domain,

$$D(d\Gamma(h)) := \left\{ \psi \in \mathcal{D}(X) : \sum_{n=1}^{\infty} \left\| H_n P^{(n)} \psi \right\|^2 < \infty \right\}.$$

Example 3. The second-quantization of the identity operator,

$$N := d\Gamma(\mathbb{1}) \tag{2.24}$$

is known as the *particle-number operator*. For example, if $\psi \in \mathcal{D}_n(X)$, then $N\psi = n\psi$.

In order to account for n-body interactions, we will also need to extend the corresponding operators on $\mathcal{D}_n(X)$, where $n \geq 2$, to the Fock space. Most interactions in nature are accurately modelled by two-body potentials.

Suppose that the operator V on $L^2(X^2)$ represents the energy corresponding to a two-body interaction. We may extend V to the n-particle space (where $n \geq 2$) by summing over particle pairs. For example, suppose that V is a multiplication operator by the function $v : X^2 \to \mathbb{C}$, such that for all $\phi \in L^2(X^2)$,

$$V\phi(x_1, x_2) = v(x_1, x_2)\phi(x_1, x_2)$$

A physically important case is the Coulomb interaction, in which v has the form

$$v(x_1, x_2) = \frac{kq_1q_2}{|x_1 - x_2|},$$

where q_1 and q_2 are the charges of the two particles and $|x_1 - x_2|$ is the distance between them.

For each pair i, j with $1 \leq i < j \leq n$, let

$$V^{(ij)}\psi(x_1, \ldots, x_n) := v(x_i, x_j)\psi(x_1, \ldots, x_n),$$

for all $\psi \in L^2(X^n)$. Then if

$$V_n := \sum_{1 \leq i < j \leq n} V^{(ij)},$$

for all $n \geq 2$, the second quantization of V is

$$d\Gamma(V) := \bigoplus_{n=2}^{\infty} V_n. \tag{2.25}$$

Again, this operator is essentially self-adjoint on the domain $\mathcal{D}^{(finite)}(X)$ and self-adjoint on the domain,

$$D(d\Gamma(V)) := \left\{ \psi \in \mathcal{D}(X) : \sum_{n=2}^{\infty} \left\| V_n P^{(n)} \psi \right\|^2 < \infty \right\}.$$

Unless otherwise stated, if A is any operator from $\mathcal{U}_n(X)$ to $\mathcal{U}_n(X)$ for all $n \in \mathbb{N}_0$, where $\mathcal{U} = \mathcal{D}, \mathcal{B}$, or $\mathcal{F}$, we will assume the domain of A to be the set

$$D(A) = \left\{ \psi \in \mathcal{U}(X) : \sum_{n=0}^{\infty} \left\| AP^{(n)} \psi \right\|^2 < \infty \right\}. \tag{2.26}$$

2.2.2 Creation and Annihilation Operators

Let $\psi \in \mathcal{D}_n(X)$ be a n-distinguishable-particle state. Roughly speaking, the *creation operator* at $x \in X$, $a^\dagger(x)$, adds a new particle at the point x. More precisely,

$$a^\dagger(x)\psi := \sqrt{n+1}\ \delta_x \otimes \psi \in \mathcal{D}_{n+1}(X).$$

On the other hand, the *annihilation operator* at $x \in X$, $a(x)$, destroys a particle at the point x. That is, $a(x)\psi \in \mathcal{D}_{n-1}(X)$ is defined by

$$[a(x)\psi](x_1, \ldots, x_{n-1}) = \sqrt{n}\ \psi(x, x_1, \cdots, x_{n-1}).$$

Each of these operators can be linearly extended to $\mathcal{D}^{(finite)}(X)$.

The creation and annihilation operators for bosons and fermions may be obtained from their distinguishable counterparts by symmetrization and anti-symmetrization respectively.

Definition 4. (a) The *boson creation and annihilation operators* are defined by

$$a_b^\dagger(x) := S_b(X)a^\dagger(x)S_b(X) \quad \text{and} \quad a_b(x) := S_b(X)a(x)S_b(X),$$

respectively.

(b) The *fermion creation and annihilation operators* are defined by

$$a_f^\dagger(x) := S_f(X)a^\dagger(x)S_f(X) \qquad \text{and} \qquad a_f(x) := S_f(X)a(x)S_f(X),$$

respectively.

Since $S_f(X)$ projects $\mathcal{D}(X)$ onto a finite-dimensional subspace $\mathcal{F}(X)$, $a_f(x)$ and $a_f^{(\dagger)}(x)$ are bounded operators defined on all of $\mathcal{D}(X)$. The boson creation and annihilation operators are defined on $D^{(finite)}(X)$ and may be extended to the following respective domains: for annihilation operators,

$$D(a_b(x)) := \left\{ u \in \mathcal{D}(X) : \sum_{n=1}^{\infty} \|a_b(x)P^{(n)}u\|^2_{\mathcal{B}_{n-1}(X)} < \infty \right\},$$

and for creation operators,

$$D(a_b^\dagger(x)) := \left\{ u \in \mathcal{D}(X) : \sum_{n=0}^{\infty} \|a_b^\dagger(x) P^{(n)} u\|^2_{\mathcal{B}_{n+1}(X)} < \infty \right\}.$$

Proposition 2.2.1. The creation operators satisfy

$$a_b^\dagger(x)u = \delta_x \otimes_s (S_b(X)u) \quad \text{and} \quad a_f^\dagger(x)u = \delta_x \wedge (S_f(X)u),$$

for all $u \in D^{(finite)}(X)$ and $x \in X$.

Proposition 2.2.2. The creation operators are the adjoints of their respective annihilation operators. That is, for all $x \in X$, $a_f^\dagger(x)$ is the adjoint of $a_f(x)$ and $a_b^\dagger(x)$ is the adjoint of $a_b(x)$, provided that the domains of the boson creation annihilation operators are taken to be $D(a_b^\dagger(x))$ and $D(a_b(x))$ respectively, as defined above.

The creation and annihilation operators often provide a convenient and intuitive way of expressing other many-body operators on the Fock space. The second-quantization of the two-particle operator V, $d\Gamma(V)$ from Equation (2.25), may be expressed as

$$d\Gamma(V) = \int_{X^2} dx\, dy\ a^\dagger(x) a^\dagger(y) v(x,y) a(x) a(y). \tag{2.27}$$

Similarly, the second quantization of the single-particle operator $h : L^2(X) \to L^2(X)$, $d\Gamma(h)$ from Equation (2.22), satisfies

$$d\Gamma(h) = \int_{X^2} dx\, dy\ a^\dagger(x) \tilde{h}(x,y) a(y), \tag{2.28}$$

where $\tilde{h}$ is the integral kernel of h. In particular, for the particle-number operator, Equation (2.28) reads,

$$N = d\Gamma(\mathbb{1}) = \int_X dx\ a^\dagger(x) a(x).$$

For each x, let

$$n(x) := a^\dagger(x) a(x).$$

This is called the *particle-density operator* and represents the number of particles present at the point $x \in X$.

Similarly, the *boson-density operator at $x \in X$* is

$$n_b(x) := S_b(X)\, n(x)\, S_b(X) = a_b^\dagger(x) a_b(x)$$

and the *fermion-density operator at $x \in X$* is

$$n_f(x) := S_f(X)\, n(x)\, S_f(X) = a_f^\dagger(x) a_f(x).$$

Intuitively, the total-particle-number operator for distinguishable particles satisfies $N = \sum_{x \in X} n(x)$. Similarly, for bosons and fermions, we let

$$N_b := S_b(X) N S_b(X) = \sum_{x \in X} n_b(x)$$

and

$$N_f := S_f(X) N S_f(X) = \sum_{x \in X} n_f(x).$$

Proposition 2.2.3 (Canonical Commutation and Anti-Commutation Relations)**.** The boson creation and annihilation operators satisfy the *commutation relations*,

$$\begin{aligned}
[a_b(x), a_b^\dagger(y)]_- &= \delta_x(y), \\
[a_b(x), a_b(y)]_- &= 0, \\
[a_b^\dagger(x), a_b^\dagger(y)]_- &= 0,
\end{aligned}$$

where $[A, B]_-$ represents the *commutator*,

$$[A, B]_- = AB - BA,$$

for all operators A and B for which the right side is well-defined.

The fermion creation and annihilation operators satisfy the *anti*-commutation relations,

$$\begin{aligned}
[a_f(x), a_f^\dagger(y)]_+ &= \delta_x(y), \\
[a_f(x), a_f(y)]_+ &= 0, \\
[a_f^\dagger(x), a_f^\dagger(y)]_+ &= 0.
\end{aligned}$$

The *anti-commutator* $[A, B]_+$ is given by

$$[A, B]_+ = AB + BA,$$

for all operators A and B for which the right side is well-defined.

Considered as operators on the composite Hilbert space, $\mathcal{B} \otimes \mathcal{F}$, the boson and fermion creation and annihilation operators satisfy

$$[a_b^{(\dagger)}(x), a_f^{(\dagger)}(y)]_- = 0$$

for all $x, y \in X$, where each "$\dagger$" may be included or not.

Proposition 2.2.4. For any $n \in \mathbb{N}$, the norms of the creation and annihilation operators, $a^\dagger(x)$ and $a(x)$ for any $x \in X$, restricted to $\mathcal{D}_n(X)$, are

$$\|a^\dagger(x)|_{\mathcal{D}_n(X)}\| = \sqrt{n+1} \ \text{ and } \ \|a(x)|_{\mathcal{D}_n(X)}\| = \sqrt{n}.$$

In particular, for bosons,

$$\|a_b^\dagger(x)|_{\mathcal{B}_n(X)}\| = \sqrt{n+1} \ \text{ and } \ \|a_b(x)|_{\mathcal{B}_n(X)}\| = \sqrt{n}.$$

Lemma 2.2.5. For any $n \in \mathbb{N}$ and $Y \in X^n/S_n$, the results of acting on the basis element δ_Y with the boson creation, annihilation, and density operators at $x \in X$ are:

$$\begin{aligned} a_b(x)\delta_Y &= \sqrt{\mu_Y(x)}\ \delta_{Y\setminus\{x\}}, \\ a_b^\dagger(x)\delta_Y &= \sqrt{\mu_Y(x)+1}\ \delta_{Y\sqcup\{x\}}, \text{ and} \\ n(x)\delta_Y &= \mu_Y(x)\delta_Y. \end{aligned}$$

Here, if $(y_1, \ldots, y_n)$ is a representative for Y, then $Y \sqcup \{x\}$ is the element of X^{n+1}/S_{n+1} with representative $(y_1, \ldots, y_n, x)$. If $x \in Y$, for example, $x = y_1$, then $Y\setminus\{x\}$ is the element of X^{n-1}/S_{n-1} with representative $(y_2, \ldots, y_n)$. If $x \notin Y$, then $Y\setminus\{x\} := Y$. (However, in the latter case, $\mu_Y(x) = 0$, so the right side of the first equation is equal to zero anyway. Note that $\mu_Y(x)$ was defined below Equation (2.12).)

Similarly, for any $Y \subset X$, the fermion annihilation operator $a_f(x)$ acts on

δ_Y as

$$a_f(x)\delta_Y = \begin{cases} (-1)^{k-1}\delta_{Y\setminus\{x\}} & \text{if } y_k = x \text{ for some } k, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \sum_{j=1}^{n} \delta_x(y_j)(-1)^{j-1}\delta_{Y\setminus\{x\}}.$$

The fermion creation operator at x, $a_f^\dagger(x)$ acts on δ_Y as

$$a_f^\dagger(x)\delta_Y = \begin{cases} (-1)^{k-1}\delta_{Y\sqcup\{x\}} & \text{if } x \notin Y \text{ and } y_{k-1} < x < y_k \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.2.6. For any $Y \in X^n/S_n$ with representative $(y_1, \ldots, y_n) \in X^n$, the bosonic basis element δ_Y can be obtained by repeated applications of boson creation operators to the vacuum state:

$$\delta_Y = c_Y\, a_b^\dagger(y_1) \cdots a_b^\dagger(y_n)|0\rangle.$$

For fermions, if $Z = \{z_1, \ldots, z_n\} \subset X$ with $z_1 < \cdots < z_n$,

$$\delta_Z = \delta_{z_1} \wedge \cdots \wedge \delta_{z_n} = a_f^\dagger(z_1) \cdots a_f^\dagger(z_n)|0\rangle.$$

The bosonic case follows from Lemma 2.2.5 by induction.

2.3 Quantum Statistical Mechanics

We will study quantum systems in thermodynamic equilibrium that are free to exchange energy and particles with a heat bath. The set of possible states is known as the *grand canonical ensemble*. The grand canonical ensemble is represented by a *density matrix*, defined in the following section.

2.3.1 Density Matrices

Consider a quantum system with a 2-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$, such as a spin-half particle. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ and suppose that $p_1, p_2 \geq 0$ with $p_1 + p_2 = 1$. Suppose that the expected value of any observable A on $\mathcal{H}$ is

$$\begin{aligned}\langle A \rangle &= p_1 \langle e_1, Ae_1 \rangle + p_2 \langle e_2, Ae_2 \rangle \\ &= \frac{\text{Tr}(\rho A)}{\text{Tr}\rho},\end{aligned}$$

where

$$\rho = p_1 P_{e_1} + p_2 P_{e_2} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix},$$

and P_{e_1} and P_{e_2} are the projections onto e_1 and e_2 respectively. Then we say that system is in a *mixed state*, or more precisely, the mixture of the pure state e_1 with probability p_1 and the pure state e_2 with probability p_2.

In general, any positive, trace-class operator ρ on a Hilbert space $\mathcal{H}$, may be called a *density operator* and represents a statistical ensemble of pure quantum states, also called a *mixed state*. In this context, an element of the Hilbert space (or the projective Hilbert space) is called a *pure state*. As above, if the system is in the mixed state ρ, the expectation value of an operator A is

$$\langle A \rangle = \frac{\text{Tr}(\rho A)}{\text{Tr}\rho}.$$

2.3.2 Grand Canonical Partition Function

For a quantum system of many indistinguishable particles in thermodynamic equilibrium, the grand-canonical ensemble is represented by the density matrix,

$$\rho = e^{-\beta(H-\mu N)},$$

provided that this operator is trace-class. Here, N is the particle-number operator as defined in Section 2.2.1; H is a self-adjoint operator on the Fock space $\mathcal{H}$ called the *Hamiltonian*, which represents the total energy of the system. A typical example of such a Hamiltonian is given by a sum of the operators $d\Gamma(h)$ and $d\Gamma(V)$ from Section 2.2.1. The constants β and μ represent the inverse temperature and chemical potential respectively.

The *grand-canonical partition function* is the trace of the density matrix,

$$Z(\beta, \mu) = \mathrm{Tr}(e^{-\beta(H-\mu N)}). \tag{2.29}$$

Aside from serving as a normalization factor for ρ, the partition function may be modified and analyzed to extract a wealth of information about the quantum system, including the one and two-point correlation functions, whose behaviour changes at the phase transitions described in the Introduction.

Our approach to analyzing the partition function starts with expressing the operator of the form $H - \mu N$ in terms of creation and annihilation operators as outlined in Section 2.2.2, Equations (2.27) and (2.28). (This will be done in more detail for each system of interest in Chapters 4 and 5.) We then aim to expand the trace in the set of "eigenstates" of the annihilation operators. It will be through these *coherent states* that the fields and functional integrals over these fields will arise. In the next chapter, we will define coherent states and explore their properties.

Chapter 3

Coherent States

The goal of this chapter is to find eigenstates of the annihilation operators. The states that will arise are called *coherent states* and will be a critical tool in our construction of functional integral representations for quantum many-particle partition functions.

Our review of coherent states here is similar to Negele and Orland [9], as well as Feldman et. al. [10] in the case of bosons.

As in the previous chapter, let X be a finite set with the counting measure. That is, for any function $f : X \to \mathbb{C}$,

$$\int_X dx\, f(x) := \sum_{x \in X} f(x).$$

3.1 Bosonic Coherent States

We set out to find an element of $\mathcal{B}(X)$ which is simultaneously an eigenvector for all $a_b(x)$ with $x \in X$. Consider first the simple case where $|X| = 1$. In this case, $\mathcal{B}_n(X) \cong \mathbb{C}$ for each $n \in \mathbb{N}_0$. Let e_n be the basis element corresponding to the n-boson space; that is, $\mathcal{B}_n(X) = \mathbb{C}e_n$. For arbitrary $\phi \in \mathcal{B}(X)$, there is a sequence of complex numbers, $\{\varphi_n\}_{n=0}^{\infty}$, such that

$$\sum_{n=0}^{\infty} |\varphi_n|^2 < \infty$$

and

$$\phi = \sum_{n=0}^{\infty} \varphi_n e_n.$$

Acting with the annihilation operator a_b on the n^{th} component,

$$a_b \varphi_n e_n = \varphi_n \sqrt{n} e_{n-1}.$$

Supposing that ϕ is an eigenvector of a_b, we choose the coefficients so that $\varphi_n \sqrt{n} = \alpha \varphi_{n-1}$ for some $\alpha \in \mathbb{C}$. If we choose $\varphi_0 = 1$, it follows by induction (using $\varphi_n \sqrt{n} = \alpha \varphi_{n-1}$) that

$$\varphi_n = \frac{\alpha^n}{\sqrt{n!}}.$$

Clearly, $\phi \in \mathcal{B}(X)$. To see that $\phi \in D(a_b)$,

$$\sum_{n=0}^{\infty} \|a_b \varphi_n e_n\|^2 = \sum_{n=1}^{\infty} \|\alpha \varphi_{n-1} e_{n-1}\|^2 = |\alpha|^2 \|\phi\|^2.$$

Hence,

$$a_b \phi = \sum_{n=0}^{\infty} a_b \varphi_n e_n = \sum_{n=1}^{\infty} \varphi_n \sqrt{n} e_{n-1} = \sum_{n=1}^{\infty} \alpha \varphi_{n-1} e_{n-1} = \alpha \phi.$$

We thus define the *coherent state* $|\alpha\rangle$ to be this eigenvector of the annihilation operator with eigenvalue α,

$$|\alpha\rangle := \bigoplus_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e_n.$$

For the remainder of this chapter, let X be any finite set with the counting measure. We generalize this definition of a coherent state to $\mathcal{B}(X)$, as follows.

Definition 5 (Bosonic Coherent States)**.** Consider a system of bosons with Fock space $\mathcal{B}(X)$.

(i) For any $x \in X$ and $\alpha \in \mathbb{C}$,

$$|\alpha \delta_x\rangle := \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \delta_x^{\otimes_s^n} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} a_b^\dagger(x)^n |0\rangle = e^{\alpha a_b^\dagger(x)} |0\rangle,$$

where $|0\rangle$ is the vacuum state, $|0\rangle = 1 \in \mathbb{C} = \mathcal{B}_0$ and

$$\delta_x^{\otimes_s^n} = \underbrace{\delta_x \otimes \cdots \otimes \delta_x}_{n\ times}.$$

(ii) For any $\phi \in L^2(X)$, the coherent state $|\phi\rangle \in \mathcal{B}(X)$ is

$$|\phi\rangle := \bigotimes_{y \in X}{}_s\, |\phi(y)\delta_y\rangle = e^{\int_X dy\ \phi(y) a_b^\dagger(y)}|0\rangle. \tag{3.1}$$

In the following proposition, we note that after extending this definition of coherent states to the case $|X| > 1$, they continue to serve as eigenstates of the annihilation operators. Also, the creation operators act on the coherent states $|\phi\rangle$ as derivatives with respect to ϕ.

Proposition 3.1.1. For any $x \in X$ and $\phi \in L^2(X)$,

$$a_b(x)|\phi\rangle = \phi(x)|\phi\rangle \quad \text{and} \quad a_b^\dagger(x)|\phi\rangle = \frac{\partial}{\partial \phi(x)}|\phi\rangle. \tag{3.2}$$

Lemma 3.1.2. The coherent states may be expanded in the basis $\{\delta_Y : n \in \mathbb{N}_0, Y \in X^n/S_n\}$ from Part (i) of Proposition 2.1.2, as follows. For any $\phi \in L^2(X)$,

$$|\phi\rangle = \sum_{n=0}^{\infty} \sum_{Y \in X^n/S_n} \phi(Y) c_Y \delta_Y,$$

where, if we think of $Y \in X^n/S_n$ as an unordered list of elements y from X, $\phi(Y)$ is the product of all those $\phi(y)$. To be more precise,

$$\phi(Y) := \prod_{y \in X} \phi(y)^{\mu_Y(y)},$$

where $\mu_Y(y)$ is the number of appearances of y in Y. Also,

$$c_Y = \frac{1}{\sqrt{\prod_{y \in X} \mu_Y(y)!}},$$

as defined in Proposition 2.1.2.

Proposition 3.1.3. For all $\alpha, \phi \in L^2(X)$, the inner product of the coherent states $|\alpha\rangle$ and $|\phi\rangle$ is

$$\langle\alpha|\phi\rangle = e^{\int_X dy\, \alpha(y)^* \phi(y)}.$$

Furthermore, the projection of $|\alpha\rangle$ onto the n-boson subspace has norm

$$\|P^{(n)}|\alpha\rangle\| = \frac{\|\alpha\|^n}{\sqrt{n!}},$$

where the norm on the right side is the norm of α in $L^2(X)$,

$$\|\alpha\| = \left(\int_X dy\, |\alpha(y)|^2\right)^{1/2}.$$

We would also like to construct coherent states for fermions; however, the canonical anti-commutation relations for fermion annihilation operators force any complex eigenvalues to be zero. To see this, suppose that $\lambda \in \mathbb{C}$ is an eigenvalue for $a_f(x)$ for some $x \in X$, with corresponding eigenvector $\phi \in \mathcal{F}(X)$. Then

$$a_f(x)^2\phi = \lambda^2\phi.$$

The left side is equal to $\frac{1}{2}[a_f(x), a_f(x)]_+$, which is zero from Proposition 2.2.3. Therefore, $\lambda = 0$.

However, if we work with a new algebra that includes anti-commuting elements, rather than the complex plane, we will see that it is possible to construct fermionic coherent states that will have similar properties to their bosonic counterparts. We will define this algebra in the following section.

3.2 The Grassmann Algebra

Definition 6. The Grassmann algebra $\wedge\mathcal{V}$ generated by the vector space $\mathcal{V}$, is defined to be the direct sum of all n-fold antisymmetric tensor products of $\mathcal{V}$:

$$\wedge\mathcal{V} := \bigoplus_{n=0}^{\dim\mathcal{V}} \wedge^n\mathcal{V},$$

Note that, for $n > \dim\mathcal{V}$, $\wedge^n\mathcal{V} = \{0\}$.

We will take $\mathcal{V}$ to be the $2|X|$-dimensional vector space freely generated by

$$B_{\mathcal{V}} := \{\phi_f(x) : x \in X\} \sqcup \{\phi_f^*(x) : x \in X\}$$

and let $\mathcal{G}$ be the Grassmann algebra generated by $\mathcal{V}$. Note that $B_{\mathcal{V}}$ is a set of independent basis vectors indexed by $x \in X$ and the presence or absence of a star. To be clear, these are not complex variables and the star index is not related to complex conjugation.

In this section, it will be useful to enumerate all elements of $B_{\mathcal{V}}$ so that $B_{\mathcal{V}} = \{\xi_1, \ldots, \xi_D\}$, where $D := \dim \mathcal{V} = 2|X|$. Note that D is even. Then each element of $\mathcal{G}$ can be expressed in the form,

$$a(\xi) := \sum_{n=0}^{D} \sum_{\substack{\vec{i} \in \{1,\ldots,D\}^n \\ 1 \le i_1 < i_2 < \cdots < i_n \le D}} a_{\vec{i}}\, \xi_{i_1} \xi_{i_2} \cdots \xi_{i_n}, \tag{3.3}$$

where each $a_{\vec{i}} \in \mathbb{C}$. The coefficients for the sum of any two elements $a(\xi)$ and $b(\xi)$ are given by

$$(a+b)_{\vec{i}} = a_{\vec{i}} + b_{\vec{i}}.$$

The product of two elements of the Grassmann algebra (sometimes referred to as *wedge product*) is found by supplementing the usual distributive law with the commutation rule $\xi_i \xi_j = -\xi_j \xi_i$ for all i, j.

It will be useful to define an involution or adjoint operation on the Grassmann algebra. For any element $a(\xi)$ with the form above (3.3),

$$a(\xi)^* := \sum_{n=0}^{D} \sum_{\substack{\vec{i} \in \{1,\ldots,D\}^n \\ 1 \le i_1 < i_2 < \cdots < i_n \le D}} a_{\vec{i}}^*\ \xi_{i_n}^* \xi_{i_{n-1}}^* \cdots \xi_{i_1}^*,$$

where, if $\xi_j = \phi_f(x)$ for some $x \in X$, then $\xi_j^* := \phi_f^*(x)$. On the other hand, if $\xi_j = \phi_f^*(x)$ for some $x \in X$, then $\xi_j^* := \phi_f(x)$. For the complex coefficient, $a_{\vec{i}}^*$ is the complex conjugate of $a_{\vec{i}}$. (Note the reversed order of the subscripts on the right-hand side.)

Note that, for any $a(\xi), b(\xi) \in \mathcal{G}$, $a(\xi)^{**} = a(\xi)$, $(a(\xi)b(\xi))^* = b(\xi)^* a(\xi)^*$, and $(\alpha a(\xi) + \beta b(\xi))^* = \alpha^* a(\xi)^* + \beta^* b(\xi)^*$.

For convenience in notation, we'll also let

$$\begin{aligned} a(-\xi) &:= \sum_{n=0}^{D} \sum_{\substack{\vec{i}\in\{1,\dots,D\}^n \\ 1\le i_1<i_2<\dots<i_n\le D}} a_{\vec{i}}\,(-\xi_{i_1})(-\xi_{i_2})\cdots(-\xi_{i_n}) \\ &= \sum_{n=0}^{D} \sum_{\substack{\vec{i}\in\{1,\dots,D\}^n \\ 1\le i_1<i_2<\dots<i_n\le D}} (-1)^n a_{\vec{i}}\,\xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}. \end{aligned}$$

An element of the Grassmann algebra $a(\xi)$ is said to be *even* if it satisfies $a(-\xi) = a(\xi)$, and *odd* if it satisfies $a(-\xi) = -a(\xi)$. We denote the set of all even elements of $\mathcal{G}$ by $\mathcal{G}_+$ and the set of odd elements by $\mathcal{G}_-$. Then $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$ and we denote by $P_\pm$ the projection onto each subspace, $\mathcal{G}_\pm$.

Even elements commute with all other elements of the Grassmann algebra. That is, for any $a(\xi) \in \mathcal{G}_+$ and any $b(\xi) \in \mathcal{G}$,

$$a(\xi)b(\xi) = b(\xi)a(\xi).$$

On the other hand, odd elements $a(\xi) \in \mathcal{G}_-$ satisfy

$$a(\xi)b(\xi) = b(-\xi)a(\xi).$$

for all $b(\xi) \in \mathcal{G}$. In general, for any $a(\xi), b(\xi) \in \mathcal{G}$,

$$a(\xi)b(\xi) = b(\xi)a_+(\xi) + b(-\xi)a_-(\xi)$$

where $a_\pm(\xi) := P_\pm a(\xi)$.

We now define two linear maps on $\mathcal{G}$ that behave similarly to differentiation and integration in calculus.

Definition 7 (Grassmann Differentiation and Integration)**.**

(i) For any $i \in \{1, \dots, D\}$, the *Grassmann derivative*,

$$\frac{\partial}{\partial \xi_i} : \mathcal{G} \to \mathcal{G}$$

is the unique linear map which satisfies, for any element of the form $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}$ where $1 \le i_1 < i_2 < \cdots < i_n \le D$,

$$\frac{\partial}{\partial \xi}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_n} = \begin{cases} (-1)^{j-1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{j-1}}\xi_{i_{j+1}}\cdots\xi_{i_n} & \text{if } (\exists j)\xi_{i_j} = \xi, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) The *Grassmann integral*,

$$\int d\xi_D \cdots d\xi_1 : \mathcal{G} \to \mathbb{C}$$

is the unique linear map which is equal to zero on $\oplus_{k=0}^{D-1} \wedge^k \mathcal{V}$ and satisfies the property

$$\int d\xi_D \cdots d\xi_1 \; \xi_1 \cdots \xi_D = 1.$$

Note that the Grassmann derivative is not a rate of change and the Grassmann integral makes no reference to a measure. However, these maps satisfy properties that mimic those of their namesakes, including a product rule and an integration-by-parts formula. For more details, see [19].

Lemma 3.2.1. For any elements $a(\xi), b(\xi) \in \mathcal{G}$,

$$\int d\xi_D \ldots d\xi_1 \; a(\xi)b(\xi) = \int d\xi_D \ldots d\xi_1 \; b(-\xi)a(\xi) = \int d\xi_D \ldots d\xi_1 \; b(\xi)a(-\xi).$$

Proof. From the definition of the Grassmann integral, only the terms of the integrand of order D will contribute. Since $D = 2|X|$ is even in this case, we may project the integrand onto the even subspace $\mathcal{G}_+$.

$$\begin{aligned} \int d\xi_D \ldots d\xi_1 \; a(\xi)b(\xi) &= \int d\xi_D \ldots d\xi_1 \; P_+ a(\xi)b(\xi) \\ &= \int d\xi_D \ldots d\xi_1 \; [(P_+a(\xi))(P_+b(\xi)) + (P_-a(\xi))(P_-b(\xi))] \\ &= \int d\xi_D \ldots d\xi_1 \; [(P_+b(-\xi))(P_+a(\xi)) + (P_-b(-\xi))(P_-a(\xi))] \\ &= \int d\xi_D \ldots d\xi_1 \; P_+(b(-\xi)a(\xi)) \\ &= \int d\xi_D \ldots d\xi_1 \; b(-\xi)a(\xi). \end{aligned}$$

In the third line, it is also true that $(P_- a(\xi))(P_- b(\xi)) = (P_- b(\xi))(P_- a(-\xi))$. Following the same steps, we obtain the second equality of the lemma. $\square$

We now allow elements of the Grassmann algebra $\mathcal{G}$ to act as coefficients on the fermionic Fock space $\mathcal{F}(X)$ by forming the tensor product between the two spaces. Then we will define a multiplication between elements of $\mathcal{G}$ and the operators on $\mathcal{F}(X)$, which will be similar to the multiplication of two superalgebras [19]. This definition relies on splitting the space $\mathscr{B}(\mathcal{F}(X))$ of linear operators on $\mathcal{F}(X)$ into even and odd parts. In order to do this, we will require the following lemma.

Lemma 3.2.2. Each linear operator on $\mathcal{F}(X)$ can be expressed as a linear combination of products of creation and annihilation operators.

Proof. Using the basis for $\mathcal{F}(X)$ from Equation (2.13), it suffices to show that the claim holds for any operator of the form $f \mapsto \delta_{y_1} \wedge \cdots \wedge \delta_{y_n} \langle \delta_{x_1} \wedge \cdots \wedge \delta_{x_m}, f\rangle$.

$$\delta_{y_1} \wedge \cdots \wedge \delta_{y_n} \langle \delta_{x_1} \wedge \cdots \wedge \delta_{x_m}, f\rangle = a_f^\dagger(y_1) \cdots a_f^\dagger(y_n) |0\rangle\langle 0| a_f(x_m) \cdots a_f(x_1) f$$

and since

$$|0\rangle\langle 0| = \prod_{y \in X_b} a_f(y) a_f^\dagger(y),$$

the claim follows. $\square$

Let $\mathscr{B}_+(\mathcal{F}(X))$ and $\mathscr{B}_-(\mathcal{F}(X))$ be the subspaces of $\mathscr{B}(\mathcal{F}(X))$ generated by products of even and odd numbers of creation/annihilation operators (respectively). Then $\mathscr{B}(\mathcal{F}(X)) = \mathscr{B}_+(\mathcal{F}(X)) \oplus \mathscr{B}_-(\mathcal{F}(X))$.

We define a multiplication operation on $\mathcal{G} \otimes \mathscr{B}(\mathcal{F}(X))$ which is associative, distributive and satisfies the following property. For any $A, B \in \mathscr{B}(\mathcal{F}(X))$ with $A = A_+ + A_-$, where $A_\pm \in \mathscr{B}_\pm(\mathcal{F}(X))$, and $\psi, \sigma \in \mathcal{G}$ with $\sigma = \sigma_+ + \sigma_-$, where $\sigma_\pm \in \mathcal{G}_\pm$,

$$\begin{aligned}&[\psi \otimes (A_+ + A_-)][(\sigma_+ + \sigma_-) \otimes B] \\ &\quad = \psi\sigma_+ \otimes A_+ B \;+\; \psi\sigma_+ \otimes A_- B \;+\; \psi\sigma_- \otimes A_+ B \;-\; \psi\sigma_- \otimes A_- B.\end{aligned}$$

The odd elements of the two algebras anticommute, while the even parts commute. This is the same as multiplication in the tensor product of two superalgebras [19], however, $\mathscr{B}(\mathcal{F}(X))$ is not a superalgebra since $[a_f(x), a_f^\dagger(y)]_+ = \delta_x(y)$ rather than 0.

In particular, note that for any $x, y \in X$,

$$\phi_f^{(*)}(x) \otimes a_f^{(\dagger)}(y) = -[1 \otimes a_f^{(\dagger)}(y)][\phi_f^{(*)}(x) \otimes \mathbb{1}].$$

This equation is intended to be read either by including or omitting all of the symbols in brackets. Put more casually,

$$\phi_f^{(*)}(x) a_f^{(\dagger)}(y) = -a_f^{(\dagger)}(y) \phi_f^{(*)}(x).$$

We will continue to use this casual notation, identifying an element $\sigma \in \mathcal{G}$ with $\sigma \otimes \mathbb{1} \in \mathcal{G} \otimes \mathscr{B}(\mathcal{F}(X))$, and $A \in \mathscr{B}(\mathcal{F}(X))$ with $1 \otimes A \in \mathcal{G} \otimes \mathscr{B}(\mathcal{F}(X))$. Likewise, we identify $f \in \mathcal{F}(X)$ with $1 \otimes f \in \mathcal{G} \otimes \mathcal{F}(X)$.

We can now regard $\mathcal{G} \otimes \mathscr{B}(\mathcal{F}(X))$ as the space of linear operators from $\mathcal{G} \otimes \mathcal{F}(X)$ to $\mathcal{G} \otimes \mathcal{F}(X)$. Furthermore, any linear operator A on $\mathcal{G}$ (such as Grassmann integration and differentiation, for example) can be considered as an operator on $\mathcal{G} \otimes \mathcal{F}(X)$ via $A(a(\xi) \otimes f) := (Aa(\xi)) \otimes f$ for any $a(\xi) \in \mathcal{G}$ and $f \in \mathcal{F}(X)$.
For any $a(\xi), b(\xi) \in \mathcal{G}$ and $f, g \in \mathcal{F}(X)$, we define the "inner product",

$$\langle a(\xi) f, b(\xi) g \rangle := a(\xi)^* b(\xi) \langle f, g \rangle.$$

This can be extended to all of $\mathcal{G} \otimes \mathcal{F}(X)$,

$$\left\langle \sum_i a_i(\xi) f_i, \sum_j b_j(\xi) g_j \right\rangle := \sum_{i,j} a_i(\xi)^* b_j(\xi) \langle f_i, g_j \rangle$$

for all $a_i(\xi), b_j(\xi) \in \mathcal{G}$ and $f_i, g_j \in \mathcal{F}(X)$.

3.3 Fermionic Coherent States

As in the beginning of the previous section (3.2), let $\mathcal{G}$ be the Grassmann algebra over $\mathcal{V}$, the vector space that is freely generated by

$$B_{\mathcal{V}} = \{\phi_f(x) : x \in X\} \sqcup \{\phi_f^*(x) : x \in X\}.$$

Definition 8. The *fermionic coherent state* is $|\phi_f\rangle$ is defined to be

$$|\phi_f\rangle := e^{-\int_X dx \ \phi_f(x) a_f^\dagger(x)} |0\rangle. \tag{3.4}$$

Here, the exponential is defined on $\mathcal{G} \otimes \mathscr{B}(\mathcal{F}(X))$ by its series expansion,

$$e^u = \sum_{n=0}^{\infty} \frac{1}{n!} u^n,$$

for any $u \in \mathcal{G} \otimes \mathscr{B}(\mathcal{F}(X))$. (Note that this sum must be finite, terminating at some $n \leq |X|$.)

Since the terms $\phi_f(x) a_f^\dagger(x)$ in the exponent of the fermionic coherent state are "even" in $\mathcal{G} \otimes \mathscr{B}(\mathcal{F}(X))$, they must commute with each other. Hence,

$$e^{-\sum_{x \in X} \phi_f(x) a_f^\dagger(x)} = \prod_{x \in X} e^{-\phi_f(x) a_f^\dagger(x)}.$$

For each $x \in X$ and $k > 1$, $(-\phi_f(x) a_f^\dagger(x))^k = 0$. Hence, the series expansion of the exponential $e^{-\phi_f(x) a_f^\dagger(x)}$ terminates after the linear term. That is,

$$e^{-\phi_f(x) a_f^\dagger(x)} = 1 - \phi_f(x) a_f^\dagger(x).$$

Therefore,

$$|\phi_f\rangle = \prod_{x \in X} (1 - \phi_f(x) a_f^\dagger(x)) |0\rangle.$$

Again, since each factor $(1 - \phi_f(x) a_f^\dagger(x))$ is even in $\mathcal{G} \otimes \mathscr{B}(\mathcal{F}(X))$, this product may be taken in any order.

On the other hand, we define

$$\langle \phi_f| := \langle 0|e^{-\sum_{x\in X} a_f(x)\phi_f^*(x)} = \langle 0| \prod_{x\in X}(1 - a_f(x)\phi_f^*(x)).$$

It is easy to see that, for any $u \in \mathcal{G} \otimes \mathcal{F}(X)$, $\langle \phi_f|u\rangle$ is equal to the inner product of $|\phi_f\rangle$ with u, as defined above.

We are now prepared to achieve our goal for this section – to prove that, over the Grassmann algebra, the fermionic coherent states behave as eigenvectors with respect to the annihilation operators. Similarly, the fermion creation operators act as Grassmann derivatives on the coherent states, much like the boson creation operators act as derivatives (with respect to the boson fields) on the boson coherent states.

Proposition 3.3.1. For all $x \in X$,

$$a_f(x)|\phi_f\rangle = \phi_f(x)|\phi_f\rangle \quad \text{and} \quad a_f^\dagger(x)|\phi_f\rangle = -\frac{\partial}{\partial \phi_f(x)}|\phi_f\rangle.$$

Likewise,

$$\langle \phi_f|a_f(x) = \frac{\partial}{\partial \phi_f^*(x)}\langle \phi_f| \quad \text{and} \quad \langle \phi_f|a_f^\dagger(x) = \phi_f^*(x)\langle \phi_f|.$$

Proof.

$$\begin{aligned}
a_f(x)|\phi_f\rangle &= a_f(x) \prod_{y\in X}\left(1 - \phi_f(y)a_f^\dagger(y)\right)|0\rangle \\
&= \left[\prod_{y\neq x}\left(1 - \phi_f(y)a_f^\dagger(y)\right)\right] a_f(x)\left(1 - \phi_f(x)a_f^\dagger(x)\right)|0\rangle \\
&= \left[\prod_{y\neq x}\left(1 - \phi_f(y)a_f^\dagger(y)\right)\right] \phi_f(x)|0\rangle \\
&= \left[\prod_{y\neq x}\left(1 - \phi_f(y)a_f^\dagger(y)\right)\right] \phi_f(x)\left(1 - \phi_f(x)a_f^\dagger(x)\right)|0\rangle \\
&= \phi_f(x) \prod_{y\in X}\left(1 - \phi_f(y)a_f^\dagger(y)\right)|0\rangle \\
&= \phi_f(x)|\phi_f\rangle.
\end{aligned}$$

Similarly,

$$\begin{aligned}
a^\dagger(x)|\phi_f\rangle &= a_f^\dagger(x)\prod_y(1-\phi_f(y)a_f^\dagger(y))|0\rangle \\
&= \left[\prod_{y\neq x}(1-\phi_f(y)a_f^\dagger(y))\right]a_f^\dagger(x)(1-\phi_f(x)a_f^\dagger(x))|0\rangle \\
&= \left[\prod_{y\neq x}(1-\phi_f(y)a_f^\dagger(y))\right]a_f^\dagger(x)|0\rangle \\
&= a_f^\dagger(x)\prod_{y\neq x}(1-\phi_f(y)a_f^\dagger(y))|0\rangle \\
&= -\frac{\partial}{\partial\phi_f(x)}(1-\phi_f(x)a_f^\dagger(x))\prod_{y\neq x}(1-\phi_f(y)a_f^\dagger(y))|0\rangle \\
&= -\frac{\partial}{\partial\phi_f(x)}|\phi_f\rangle.
\end{aligned}$$

The proof is very similar for the dual-type equations in the second part of the proposition.

□

3.4 Coherent States for Mixed Systems

For a mixed system of P species of particles, bosons and/or fermions, a coherent state is simply a tensor product of coherent states for the component species. In this section, we define notation for the Hilbert spaces of these mixed systems and their coherent states.

Let $\mathcal{H}_p$ be the Fock space for the p^{th} particle species. We choose the convention of listing bosons first, so that the first P_b species are bosonic and the last P_f species are fermionic, where $P_b + P_f = P$. We also assume that for each p, there is a finite set X_p, which we call the "configuration space" for the p^{th} species, such that if $p \leq P_b$, $\mathcal{H}_p = \mathcal{B}(X_p)$ and if $p > P_b$, $\mathcal{H}_p = \mathcal{F}(X_p)$. For example, if the p^{th} species consists of spin-$\frac{1}{2}$ fermions on a lattice X, its configuration space is $X_p = X \times S_p$, where $S_p = \{\uparrow, \downarrow\}$.

The Hilbert space for the full combined system is then

$$\begin{aligned}\mathcal{H} &:= \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_P \\ &= \mathcal{B}(X_1) \otimes \cdots \otimes \mathcal{B}(X_{P_b}) \otimes \mathcal{F}(X_{P_b+1}) \otimes \cdots \otimes \mathcal{F}(X_P).\end{aligned} \tag{3.5}$$

The *bosonic space* is

$$\mathcal{B} := \mathcal{B}(X_1) \otimes \cdots \otimes \mathcal{B}(X_{P_b}) \tag{3.6}$$

and the *fermionic space* is

$$\mathcal{F} := \mathcal{F}(X_{P_b+1}) \otimes \cdots \otimes \mathcal{F}(X_P). \tag{3.7}$$

It will be convenient to define, for any $\mathbf{n} = (n_1, n_2, \ldots, n_{P_b}) \in \mathbb{N}_0^{P_b}$, the "**n**-boson subspace" $\mathcal{B}_{\mathbf{n}}$ to be the subspace of $\mathcal{B}$ with n_p particles for each p^{th} boson species,

$$\mathcal{B}_{\mathbf{n}} := \mathcal{B}_{n_1}(X_1) \otimes \mathcal{B}_{n_2}(X_2) \otimes \cdots \otimes \mathcal{B}_{n_{P_b}}(X_{P_b}). \tag{3.8}$$

Taking the following direct sums, we obtain the subspace of $\mathcal{B}$ with at most n_p particles for each p^{th} boson species,

$$\mathcal{B}_{\leq \mathbf{n}} = \left(\bigoplus_{i=0}^{n_1} \mathcal{B}_i(X_1) \right) \otimes \left(\bigoplus_{i=0}^{n_2} \mathcal{B}_i(X_2) \right) \otimes \cdots \otimes \left(\bigoplus_{i=0}^{n_{P_b}} \mathcal{B}_i(X_{P_b}) \right). \tag{3.9}$$

For any $\mathbf{m} = (m_1, \ldots, m_{P_b})$ and $\mathbf{n} = (n_1, \ldots, n_{P_b})$ in $\mathbb{N}_0^{P_b}$, if $m_p \leq n_p$ for all p, we write "$\mathbf{m} \leq \mathbf{n}$". With this notation,

$$\mathcal{B}_{\leq \mathbf{n}} = \bigoplus_{\mathbf{m} \leq \mathbf{n}} \mathcal{B}_{\mathbf{m}}.$$

The corresponding subspaces of the full Hilbert space $\mathcal{H}$ are

$$\mathcal{H}_{\mathbf{n}} := \mathcal{B}_{\mathbf{n}} \otimes \mathcal{F} \qquad \text{and} \qquad \mathcal{H}_{\leq \mathbf{n}} := \mathcal{B}_{\leq \mathbf{n}} \otimes \mathcal{F}. \tag{3.10}$$

We denote the projections onto these subspaces by $P^{(\mathbf{n})}$ and $P_{\mathbf{n}}$ respectively,

$$P^{(\mathbf{n})} = P_{\mathcal{H}_{\mathbf{n}}} \qquad \text{and} \qquad P_{\mathbf{n}} = P_{\mathcal{H}_{\leq \mathbf{n}}}. \tag{3.11}$$

For $p \leq P_b$, let $\phi_p \in L^2(X_p)$. For $p > P_b$, let $\mathcal{G}_p$ be the Grassmann Algebra over $\mathcal{V}_p$, the vector space freely generated by

$$\{\phi_p(x) : x \in X_p\} \sqcup \{\phi_p^*(x) : x \in X_p\}.$$

The fields for the mixed systems are vector-valued with P components, one for each species. We will represent these by bold Greek letters,

$$\boldsymbol{\phi} := (\phi_1, \ldots, \phi_{P_b}, \phi_{P_b+1}, \ldots, \phi_P).$$

The first P_b components are complex fields while the last P_f components are *Grassmann fields*. The multi-species coherent state $|\boldsymbol{\phi}\rangle$ is then defined to be the tensor product,

$$|\boldsymbol{\phi}\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_P\rangle.$$

For such $\boldsymbol{\phi}$, we also define $\boldsymbol{\phi}^* := (\phi_1^*, \phi_2^*, \ldots, \phi_P^*)$. (For $p \leq P_b$, ϕ_p^* is the complex conjugate of $\phi_p \in L^2(X_p)$, whereas for $p > P_b$, ϕ_p^* is the $*$-*Grassmann field*, representing the set of starred basis elements, $\phi_p^*(x)$ for $x \in X_p$.)

It will be useful to sometimes split up the bosonic and fermionic parts of the coherent state and write

$$|\boldsymbol{\phi}\rangle = |\boldsymbol{\phi}_B\rangle \otimes |\boldsymbol{\phi}_F\rangle,$$

where $|\boldsymbol{\phi}_B\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_{P_b}\rangle$ and $|\boldsymbol{\phi}_F\rangle = |\phi_{P_b+1}\rangle \otimes \cdots \otimes |\phi_P\rangle$.

For the dual map, we will often write,

$$\langle\boldsymbol{\phi}| = \langle\phi_P| \otimes \langle\phi_{P-1}| \otimes \cdots \otimes \langle\phi_1|.$$

However, since the creation and annihilation operators corresponding to different particle species commute, as do the corresponding Grassmann variables, the order on the right side does not matter.

The inner product between two coherent states for the mixed system is then

$$\langle\boldsymbol{\alpha}|\boldsymbol{\phi}\rangle = \prod_{p=1}^{P} \langle\alpha_p|\phi_p\rangle = \prod_{p=1}^{P} e^{\int_{X_p} dx \;\; \alpha_p(x)^* \phi_p(x)}.$$

We will sometimes use the following shorthand notation for the right-hand side,

$$e^{\int d\mathbf{x}\ \boldsymbol{\alpha}^*(\mathbf{x})\cdot\boldsymbol{\phi}(\mathbf{x})} := \prod_{p=1}^{P} e^{\int_{X_p} dx\ \alpha_p(x)^*\phi_p(x)}.$$

3.5 Approximate Resolutions of the Identity

Our goal is to expand the partition function in the set of coherent states. Since

$$\langle\phi|\psi\rangle = e^{\int_X dx\ \phi^*(x)\psi(x)} \neq 0$$

for all $\phi, \psi \in L^2(X)$, the bosonic coherent states do not form an orthonormal basis. However, they do span the whole bosonic Fock space $\mathcal{B}(X)$.

In this section, we aim prove that the coherent states satisfy a type of "closure relation" of the form,

$$\mathbb{1} = \int \prod_{x\in X}\left[\frac{d\phi^*(x)d\phi(x)}{2\pi i}\right] e^{-\int dy\, \phi^*(y)\phi(y)}|\phi\rangle\langle\phi|, \tag{3.12}$$

where the measure

$$\frac{d\phi^*(x)d\phi(x)}{2i} = d\,\mathrm{Re}(\phi(x))d\,\mathrm{Im}(\phi(x))$$

represents the Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$. The right-hand side is often called a *resolution of the identity*.

3.5.1 For Bosons

From Proposition 3.1.3, for any $\phi \in L^2(X)$, the norm of the operator

$$e^{-\int dy\,|\phi(y)|^2}|\phi\rangle\langle\phi|$$

is equal to 1. Since the integral in (3.12) runs over a domain of infinite measure, $L^2(X) \cong \mathbb{C}^{|X|}$, its convergence is unclear at first glance. We will make a "large-field cutoff", replacing the domain by a ball of radius R, and then take the limit as $R \to \infty$.

Definition 9. For any $R > 0$, the measure μ_R on $L^2(X)$ is defined by

$$d\mu_R(\phi^*, \phi) := \prod_{x\in X} \left[\frac{d\phi^*(x) d\phi(x)}{2\pi i} \chi_R(|\phi(x)|) \right],$$

where χ_R is the characteristic function of the interval $[0, R]$; that is,

$$\chi_R(x) = \begin{cases} 1 & \text{if } 0 \le x \le R, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in \mathbb{R}$.

Definition 10. For all $r > 0$, the *bosonic approximate resolution of the identity* I_r is defined on $\mathcal{B}(X)$ by

$$I_r f := \int_{L^2(X)} d\mu_r(\phi^*, \phi) e^{-\|\phi\|^2} |\phi\rangle\langle\phi|f\rangle \tag{3.13}$$

for all $f \in \mathcal{B}(X)$.

In the following theorem, we outline some important properties of I_r. In particular, I_r indeed converges to the identity operator (in the strong operator topology) as $r \to \infty$. (This is essentially a copy of Theorem 2.26 from [10].)

Theorem 3.5.1. For all $r > 0$, the bosonic approximate resolution of the identity I_r satisfies the following properties.

(i) Each element of the basis $\{\delta_Y : Y \in X^n/S_n\}$ is an eigenvector of I_r with eigenvalue

$$\lambda_r(Y) = \prod_{x\in X} \frac{\Gamma_r(\mu_Y(x))}{\mu_Y(x)!}, \tag{3.14}$$

where

$$\Gamma_r(s) := \int_0^{r^2} dt\ e^{-t} t^s$$

for all $s > -1$.

(ii) The boson number operator N_b commutes with I_r.

(iii) The operator norm of I_r satisfies the bound

$$\|I_r\| \leq 1.$$

(iv) In the limit as $r \to \infty$, I_r converges strongly to the identity operator. That is, for all $f \in \mathcal{B}(X)$,

$$\lim_{r\to\infty} \|I_r f - f\| = 0.$$

(v) The difference between I_r and the identity operator $\mathbb{1}$ satisfies the following bound:

$$\|(\mathbb{1} - I_r)P_n\| \leq |X| 2^{n+1} e^{-\frac{1}{2}r^2}.$$

Here, P_n is the orthogonal projection onto the subspace of at most n bosons, $\bigoplus_{j=0}^n \mathcal{B}_n(X)$.

3.5.2 For Fermions

In the section, we use the notation from Section 3.3 and prove that fermionic analogue of Equation (3.12) is exact.

Lemma 3.5.2. The operator

$$\int \left(\prod_{x\in X} d\phi_f^*(x) d\phi_f(x) \right) e^{-\int_X dx\ \ \phi_f^*(x)\phi_f(x)} |\phi_f\rangle\langle\phi_f|$$

is the identity operator on $\mathcal{G} \otimes \mathcal{F}(X)$. We call this operator a *fermionic resolution of the identity.* Here, $\int_X dx\ \phi_f^*(x)\phi_f(x) = \sum_{x\in X} \phi_f^*(x)\phi_f(x)$.

Proof. For any $Y \subset X$ with ordered elements $y_1 < \cdots < y_n$, note that

$$\langle\phi_f|\delta_Y\rangle = \phi^*(y_1)\dots\phi^*(y_n)$$

and

$$\langle\delta_Y|\phi_f\rangle = \phi_f(y_n)\cdots\phi_f(y_1).$$

Let A be the proposed resolution of the identity,

$$A := \int \left(\prod_{x \in X} d\phi_f^*(x) d\phi_f(x) \right) e^{-\int_X dx \ \phi_f^*(x)\phi_f(x)} |\phi_f\rangle\langle\phi_f|.$$

Then, for any $Y \subset X$ and $Z \subset X$ with ordered elements $y_1 < \cdots < y_n$ and $z_1 < \cdots < z_m$ respectively, the matrix element $\langle \delta_Z, A\delta_Y \rangle$ is given by

$$\begin{aligned}\langle \delta_Z, A\delta_Y \rangle &= \int \left(\prod_{x \in X} d\phi_f^*(x) d\phi_f(x) \right) e^{-\int_X dx \ \phi_f^*(x)\phi_f(x)} \langle \delta_Z | \phi_f \rangle \langle \phi_f | \delta_Y \rangle \\ &= \int \left(\prod_{x \in X} d\phi_f^*(x) d\phi_f(x) \right) \left(\prod_{x \in X} (1 - \phi_f^*(x)\phi_f(x)) \right) \\ &\qquad \phi_f(z_m) \cdots \phi_f(z_1) \phi^*(y_1) \ldots \phi^*(y_n).\end{aligned}$$

If there is some i such that $z_i \notin Y$, then the integrand is of the form

$$\begin{aligned}&\left(\prod_{x \neq z_i} f_x(\phi_f(x), \phi_f^*(x)) \right) (1 - \phi_f^*(z_i)\phi_f(z_i))\phi_f(z_i) \\ &\qquad = \left(\prod_{x \neq z_i} f_x(\phi_f(x), \phi_f^*(x)) \right) \phi_f(z_i)\end{aligned}$$

where $f_x(\phi_f(x), \phi_f^*(x))$ is a polynomial in $\phi_f(x)$ and $\phi_f^*(x)$. This expression contains no $\phi_f^*(z_i)$ and hence, the Grassmann integral $\int \left(\prod_{x \in X} d\phi_f^*(x) d\phi_f(x) \right)$ will map it to 0. Hence, if the integral is non-zero, $Z \subset Y$. We can use the same reasoning to deduce that $Y \subset Z$ if the integral is non-zero. Therefore, $\langle \delta_Z, A\delta_Y \rangle = 0$ unless $Y = Z$. In the case where $Y = Z$,

$$\begin{aligned}\langle \delta_Y, A\delta_Y \rangle &= \int \left(\prod_{x \in X} d\phi_f^*(x) d\phi_f(x) \right) \left(\prod_{x \in X} (1 - \phi_f^*(x)\phi_f(x)) \right) \\ &\qquad \phi_f(y_n) \cdots \phi_f(y_1) \phi^*(y_1) \ldots \phi^*(y_n) \\ &= \int \left(\prod_{x \in X} d\phi_f^*(x) d\phi_f(x) \right) \left(\prod_{x \notin Y} (1 - \phi_f^*(x)\phi_f(x)) \right) \\ &\qquad \prod_{i=1}^{n} (1 - \phi_f^*(y_i)\phi_f(y_i)) \phi_f(y_i) \phi_f^*(y_i)\end{aligned}$$

$$= \int \left(\prod_{x \in X} d\phi_f^*(x) d\phi_f(x) \right) \left(\prod_{x \notin Y} \phi_f(x) \phi_f^*(x) \right) \left(\prod_{i=1}^{n} \phi_f(y_i) \phi_f^*(y_i) \right)$$
$$= 1$$

This completes the proof. □

3.5.3 For Mixed Systems

As in Section 3.4, we consider a mixed system of P species of particles. Recall that the Hilbert space is

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_P \\ &= \mathcal{B}(X_1) \otimes \cdots \otimes \mathcal{B}(X_{P_b}) \otimes \mathcal{F}(X_{P_b+1}) \otimes \cdots \otimes \mathcal{F}(X_P), \end{aligned}$$

where each X_p is a finite set. Here, P_b is the number of bosonic species, P_f is the number of fermionic species, and $P = P_b + P_f$.

To each fermionic species (with $p > P_b$), we associate a Grassmann Algebra $\mathcal{G}_p$ over the vector space that is freely generated by

$$\{\phi_p(x) : x \in X_p\} \sqcup \{\phi_p^*(x) : x \in X_p\}.$$

Definition 11. For any $\underline{\mathbf{r}} = (r_1, \ldots, r_{P_b}) \in (0, \infty)^{P_b}$, let $\mu_{\underline{\mathbf{r}}}^b$ be the product measure,

$$\mu_{\underline{\mathbf{r}}}^b = \mu_{r_1} \times \mu_{r_2} \times \cdots \times \mu_{r_{P_b}}.$$

That is, $\mu_{\underline{\mathbf{r}}}^b$ is the measure on the product space $L^2(X_1) \times \cdots \times L^2(X_{P_b})$ given by

$$d\mu_{\underline{\mathbf{r}}}^b(\boldsymbol{\phi}_B^*, \boldsymbol{\phi}_B) = \prod_{p=1}^{P_b} d\mu_{r_p}(\phi_p^*, \phi_p) = \prod_{p=1}^{P} \prod_{x \in X_p} \left[\frac{d\phi_p^*(x) d\phi_p(x)}{2\pi i} \chi_{r_p}(|\phi_p(x)|) \right]$$

where $\boldsymbol{\phi}_B := (\phi_1, \ldots, \phi_{P_b})$.

We use a similar notation for the products of Grassmann integrals corresponding to the fermionic particle species. That is, $\int d\mu^f(\boldsymbol{\phi}_F^*, \boldsymbol{\phi}_F)$ is the map

$$\int d\mu^f(\boldsymbol{\phi}_F^*, \boldsymbol{\phi}_F) = \prod_{p=P_b+1}^{P} \int \prod_{x\in X_p} d\phi_p^*(x) d\phi_p(x)$$

where $\boldsymbol{\phi}_F = (\phi_{P_b+1}, \dots, \phi_P)$.

Combining the bosonic and fermionic parts, we let

$$\int d\mu_{\underline{\mathbf{r}}}(\boldsymbol{\phi}^*, \boldsymbol{\phi}) = \int d\mu^b_{\underline{\mathbf{r}}}(\boldsymbol{\phi}_B^*, \boldsymbol{\phi}_B) \int d\mu^f(\boldsymbol{\phi}_F^*, \boldsymbol{\phi}_F).$$

Definition 12. For any $\underline{\mathbf{r}} \in (0,\infty)^{P_b}$, the *approximate resolution of the identity* $\mathrm{I}_{\underline{\mathbf{r}}}$ is the operator on $\mathcal{H}$ given by

$$\mathrm{I}_{\underline{\mathbf{r}}} := \int d\mu_{\underline{\mathbf{r}}}(\boldsymbol{\phi}^*, \boldsymbol{\phi}) e^{-\int d\mathbf{x}\ \boldsymbol{\phi}^*(\mathbf{x})\cdot\boldsymbol{\phi}(\mathbf{x})} |\boldsymbol{\phi}\rangle\langle\boldsymbol{\phi}|, \tag{3.15}$$

where

$$\int d\mathbf{x}\ \boldsymbol{\phi}^*(\mathbf{x})\cdot\boldsymbol{\phi}(\mathbf{x}) := \sum_{p=1}^{P} \int_{X_p} dx\ \phi_p^*(x)\phi_p(x).$$

Note that

$$\mathrm{I}_{\underline{\mathbf{r}}} = \left(\bigotimes_{p=1}^{P_b} I_{p,r_p} \right) \otimes \mathbb{1}_{\mathcal{F}},$$

where I_{p,r_p} is the approximate resolution of the identity on $\mathcal{B}(X_p)$,

$$\mathrm{I}_{p,r_p} = \int d\mu_{r_p}(\phi^*, \phi) e^{-\int_{X_p} \phi^*(x)\phi(x)dx} |\phi\rangle\langle\phi|.$$

Theorem 3.5.3. The approximate resolution of the identity $\mathrm{I}_{\underline{\mathbf{r}}}$ satisfies the following properties.

(i) For all $\mathbf{n} = (n_1, \dots, n_{P_b}) \in \mathbb{N}_0^{P_b}$ and all $\mathbf{Y} = (Y_1, \dots, Y_{P_b}) \in X_1^{n_1}/S_{n_1} \times \cdots \times X_P^{n_{P_b}}/S_{n_{P_b}}$,

$$\delta_{\mathbf{Y}} := \delta_{Y_1} \otimes \cdots \otimes \delta_{Y_{P_b}}$$

is an eigenvector of $\mathrm{I}_{\underline{\mathbf{r}}}$. That is,

$$\mathrm{I}_{\underline{\mathbf{r}}}\delta_{\mathbf{Y}} = \lambda_{\underline{\mathbf{r}}}(\mathbf{Y})\delta_{\mathbf{Y}},$$

with eigenvalue

$$\lambda_{\underline{\mathbf{r}}}(\mathbf{Y}) := \prod_{p=1}^{P_b} \lambda_{r_p}(Y_p),$$

where $\lambda_{r_p}(Y_p)$ is the eigenvalue of I_{p,r_p} corresponding to δ_{Y_p}, as defined in (3.14).

(ii) For each p, $\mathrm{I}_{\underline{\mathbf{r}}}$ commutes with the p^{th}-species particle number operator,

$$N_p := \int_{X_p} dx \; a_p^\dagger(x) a_p(x),$$

where, for all $x \in X_p$, $a_p^\dagger(x)$ is the creation operator and $a_p(x)$ is the annihilation operator for the p^{th} particle species. That is,

$$a_p^{(\dagger)}(x) = \begin{cases} a_b^{(\dagger)}(x) & \text{if } x \leq P_b, \\ a_f^{(\dagger)}(x) & \text{if } x > P_b, \end{cases}$$

where instances of "(†)" may be all included (and read as "†") or all excluded.

(iii) For all $\underline{\mathbf{r}} \in (0,\infty)^{P_b}$, $\|\mathrm{I}_{\underline{\mathbf{r}}}\| \leq 1$, and $\mathrm{I}_{\underline{\mathbf{r}}}$ converges strongly to the identity operator on $\mathcal{H}$ as $\min_{1 \leq p \leq P_b} r_p \to \infty$.

(iv) For all $\mathbf{n} \in \mathbb{N}_0^{P_b}$ and $\underline{\mathbf{r}} \in (0,\infty)^{P_b}$,

$$\|(\mathbb{1} - \mathrm{I}_{\underline{\mathbf{r}}}) P_{\mathbf{n}}\| \leq \sum_{p=1}^{P_b} |X_p| 2^{n_p+1} e^{-r_p^2/2},$$

where $P_{\mathbf{n}}$ is the projection onto $\mathcal{H}_{\leq \mathbf{n}}$ as defined in Equation (3.11).

Theorem 3.5.3 is an extension of its single-species boson analogue, Theorem 2.26 in [10]. Parts (ii) and (iii) follow immediately from the definition of $\mathrm{I}_{\underline{\mathbf{r}}}$ and Theorem 2.26 (b and c) in [10]. Part (iv) may be obtained by proving the inequality for simple tensors and then extending linearly to $\mathcal{H}_{\leq \mathbf{n}}$. The difference between $f_1 \otimes \cdots \otimes f_{P_b}$ and $I_{1,r_1} f_1 \otimes \cdots \otimes I_{P_b, r_{P_b}} f_{P_b}$ may be written as a telescoping sum, so that we can apply the bound from Theorem 2.26 (d) in [10] for each species of bosons.

3.6 Trace Formula

When attempting to expand the partition function,

$$Z(\mu, \beta) = \mathrm{Tr}(e^{-\beta(H-\mu N)}),$$

using coherent states, we might intuitively start by taking inner products of the form

$$\langle \phi | e^{-\beta(H-\mu N)} | \phi \rangle,$$

then dividing by $\langle \phi | \phi \rangle$ and integrating over all ϕ. Roughly speaking, we might expect a formula that looks like

$$\mathrm{Tr}(e^{-\beta(H-\mu N)}) = \int d\phi^* d\phi \ e^{-\int_X dx \ \ \phi^*(x)\phi(x)} \langle \phi | e^{-\beta(H-\mu N)} | \phi \rangle.$$

In this section, we will show that this approach essentially works, provided that we take an appropriate large-field cutoff for bosons.

3.6.1 For Bosons

For bosons, we quote Proposition 2.28 (a) from [10], except that we do not require the operator B to commute with the boson number operator. (The proof given in [10] does not actually require this condition.)

Proposition 3.6.1. Let X be a finite set and let B be a bounded operator on $\mathcal{B}(X)$. For all $r > 0$, BI_r is trace class and

$$\mathrm{Tr} BI_r = \int d\mu_r(\phi^*, \phi) \ e^{-\int_X dy \ \ |\phi(y)|^2} \langle \phi | B | \phi \rangle.$$

3.6.2 For Fermions

For fermions, the intuitive formula is exactly correct, except for a minus sign. The integral in this case is a Grassmann integral.

Proposition 3.6.2. Let X be a finite set. For any linear operator B on the fermionic Fock space $\mathcal{F}(X)$,

$$\mathrm{Tr} \ B = \int \prod_{x \in X} d\phi_f^*(x) d\phi_f(x) \ e^{-\int_X dx \ \ \phi_f^*(x)\phi_f(x)} \langle -\phi_f | B | \phi_f \rangle,$$

where

$$\langle -\phi_f| = \langle 0| e^{\sum_{x\in X} a_f(x)\phi_f^*(x)} = \langle 0| \prod_{x\in X} (1 + a_f(x)\phi_f^*(x)).$$

Proof. Let $\{u_j\}_{j=1}^{2^{|X|}}$ be an orthonormal basis for $\mathcal{F}(X)$. Expanding the trace of B in this basis,

$$\text{Tr } B = \sum_{j=1}^{2^{|X|}} \langle u_j, Bu_j\rangle.$$

Then inserting the fermionic resolution of the identity from Lemma 3.5.2 into each inner product,

$$\begin{aligned}
\text{Tr } B &= \sum_{j=1}^{2^{|X|}} \langle u_j, B \int \left(\prod_{x\in X} d\phi_f^*(x) d\phi_f(x)\right) e^{-\int_X dx\ \phi_f^*(x)\phi_f(x)} |\phi_f\rangle\langle\phi_f|u_j\rangle \\
&= \int \left(\prod_{x\in X} d\phi_f^*(x) d\phi_f(x)\right) e^{-\int_X dx\ \phi_f^*(x)\phi_f(x)} \sum_{j=1}^{2^{|X|}} \langle u_j, B|\phi_f\rangle\langle\phi_f|u_j\rangle \\
&= \int \left(\prod_{x\in X} d\phi_f^*(x) d\phi_f(x)\right) e^{-\int_X dx\ \phi_f^*(x)\phi_f(x)} \sum_{j=1}^{2^{|X|}} \langle -\phi_f|u_j\rangle\langle u_j, B|\phi_f\rangle \\
&= \int \left(\prod_{x\in X} d\phi_f^*(x) d\phi_f(x)\right) e^{-\int_X dx\ \phi_f^*(x)\phi_f(x)} \langle -\phi_f|B|\phi_f\rangle.
\end{aligned}$$

In the third line, we've applied Lemma 3.2.1. □

3.6.3 For Mixed Systems

For mixtures of P species of bosons and/or fermions, we use the notation from Sections 3.4 and 3.5.3.

Proposition 3.6.3. Suppose that B is a bounded operator on $\mathcal{H}$. Then for all $\underline{\mathbf{r}} \in (0,\infty)^{P_b}$, $B\mathrm{I}_{\underline{\mathbf{r}}}$ is trace class and

$$\text{Tr } B\mathrm{I}_{\underline{\mathbf{r}}} = \int d\mu_{\underline{\mathbf{r}}}(\boldsymbol{\phi}^*, \boldsymbol{\phi})\ e^{-\int d\mathbf{y}\ \boldsymbol{\phi}^*(\mathbf{y})\cdot\boldsymbol{\phi}(\mathbf{y})} \langle\tilde{\boldsymbol{\phi}}|B|\boldsymbol{\phi}\rangle,$$

where $|\tilde{\boldsymbol{\phi}}\rangle = |\boldsymbol{\phi}_B\rangle \otimes |-\boldsymbol{\phi}_F\rangle$ and

$$\int d\mathbf{y}\ \boldsymbol{\phi}^*(\mathbf{y}) \cdot \boldsymbol{\phi}(\mathbf{y}) := \sum_{p=1}^{P} \int_{X_p} dy\ \phi_p^*(y)\phi_p(y).$$

Proof. Since B is bounded and $\mathrm{I}_{\underline{\mathbf{r}}}$ is trace-class, $B\mathrm{I}_{\underline{\mathbf{r}}}$ is trace class.

For each $\mathbf{n} = (n_1, n_2, \ldots, n_{P_b}) \in \mathbb{N}_0^{P_b}$, let

$$X^{\mathbf{n}}/S_{\mathbf{n}} := X_1^{n_1}/S_{n_1} \times X_2^{n_2}/S_{n_2} \times \cdots \times X_{P_b}^{n_{P_b}}/S_{n_{P_b}}. \tag{3.16}$$

Furthermore, for each $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_{P_b}) \in X^{\mathbf{n}}/S_{\mathbf{n}}$, let

$$\delta_{\mathbf{Y}} := \delta_{Y_1} \otimes \delta_{Y_2} \otimes \cdots \otimes \delta_{Y_{P_b}}.$$

Then $\{\delta_{\mathbf{Y}} : \mathbf{Y} \in X^{\mathbf{m}}/S_{\mathbf{m}}, \mathbf{m} \leq \mathbf{n}\}$ is an orthonormal basis for $\mathcal{B}_{\leq \mathbf{n}}$.

Let $\{f_k : k = 1, \ldots, \dim \mathcal{F}\}$ be any orthonormal basis for $\mathcal{F}$. (Note that $\dim \mathcal{F} = 2^{\sum_{p=P_b+1}^{P} |X_p|}$.)

Expanding the trace of $B\mathrm{I}_{\underline{\mathbf{r}}}$ in the basis $\{\delta_{\mathbf{Y}} \otimes f_k\}$,

$$\begin{aligned} \mathrm{Tr}(B\mathrm{I}_{\underline{\mathbf{r}}}) &= \sum_{k=1}^{\dim \mathcal{F}} \sum_{\mathbf{n} \in \mathbb{N}_0^{P_b}} \sum_{\mathbf{Y} \in X^{\mathbf{n}}/S_{\mathbf{n}}} \langle \delta_{\mathbf{Y}} \otimes f_k, B\mathrm{I}_{\underline{\mathbf{r}}}\ \delta_{\mathbf{Y}} \otimes f_k \rangle & (3.17) \\ &= \sum_{k=1}^{\dim \mathcal{F}} \sum_{\mathbf{n} \in \mathbb{N}_0^{P_b}} \sum_{\mathbf{Y} \in X^{\mathbf{n}}/S_{\mathbf{n}}} \lambda_{\underline{\mathbf{r}}}(\mathbf{Y}) \langle \delta_{\mathbf{Y}} \otimes f_k, B\ \delta_{\mathbf{Y}} \otimes f_k \rangle. & (3.18) \end{aligned}$$

Since $\lambda_{\underline{\mathbf{r}}}(\mathbf{Y})$ satisfies the bound,

$$\max_{\mathbf{Y} \in X^{\mathbf{n}}/S_{\mathbf{n}}} \left| \lambda_{\underline{\mathbf{r}}}(\mathbf{Y}) \right| \leq \prod_{p=1}^{P_b} \max_{Y_p \in X_p^{n_p}/S_{n_p}} |\lambda_{r_p}(Y_p)| \leq \prod_{p=1}^{P_b} \frac{(r_p^2 |X_p|)^{n_p}}{n_p!},$$

and $\dim \mathcal{H}_{\mathbf{n}} \leq 2^{\sum_{p=P_b+1}^{P} |X_p|} \prod_{p=1}^{P_b} |X_p|^{n_p}$, the sum in (3.18) is absolutely convergent; that is,

$$\sum_{k=1}^{\dim \mathcal{F}} \sum_{\mathbf{n}\in\mathbb{N}_0^{P_b}} \sum_{\mathbf{Y}\in X^{\mathbf{n}}/S_{\mathbf{n}}} \left|\lambda_{\underline{\mathbf{r}}}(\mathbf{Y})\langle\delta_{\mathbf{Y}}\otimes f_k, B\ \delta_{\mathbf{Y}}\otimes f_k\rangle\right| \\ \leq 2^{\sum_{p=P_b+1}^{P}|X_p|}\|B\| \prod_{p=1}^{P_b}\left[\sum_{n_p=0}^{\infty}\frac{(r_p|X_p|)^{2n_p}}{n_p!}\right] < \infty.$$

Hence, $\mathrm{Tr}(B\mathrm{I}_{\underline{\mathbf{r}}}P_{\mathbf{n}})$ converges to $\mathrm{Tr}(B\mathrm{I}_{\underline{\mathbf{r}}})$ as each $n_p \to \infty$.

By definition of $\mathrm{I}_{\underline{\mathbf{r}}}$,

$$\begin{aligned}\mathrm{Tr}\left(B\mathrm{I}_{\underline{\mathbf{r}}}P_{\mathbf{n}}\right) &= \sum_{\mathbf{m}\leq\mathbf{n}} \sum_{\mathbf{Y}\in X^{\mathbf{m}}/S_{\mathbf{m}}} \int d\mu_{\underline{\mathbf{r}}}(\boldsymbol{\phi}^*,\boldsymbol{\phi})\ e^{-\int d\mathbf{y}\ \boldsymbol{\phi}^*(\mathbf{y})\cdot\boldsymbol{\phi}(\mathbf{y})}\langle\delta_{\mathbf{Y}}, B|\boldsymbol{\phi}\rangle\langle\boldsymbol{\phi}|\delta_{\mathbf{Y}}\rangle \\ &= \sum_{\mathbf{m}\leq\mathbf{n}} \sum_{\mathbf{Y}\in X^{\mathbf{m}}/S_{\mathbf{m}}} \int d\mu_{\underline{\mathbf{r}}}(\boldsymbol{\phi}^*,\boldsymbol{\phi})\ e^{-\int d\mathbf{y}\ \boldsymbol{\phi}^*(\mathbf{y})\cdot\boldsymbol{\phi}(\mathbf{y})}\langle\tilde{\boldsymbol{\phi}}|\delta_{\mathbf{Y}}\rangle\langle\delta_{\mathbf{Y}}, B|\boldsymbol{\phi}\rangle \\ &= \int d\mu_{\underline{\mathbf{r}}}(\boldsymbol{\phi}^*,\boldsymbol{\phi})\ e^{-\int d\mathbf{y}\ \boldsymbol{\phi}^*(\mathbf{y})\cdot\boldsymbol{\phi}(\mathbf{y})}\langle\tilde{\boldsymbol{\phi}}|P_{\mathbf{n}}B|\boldsymbol{\phi}\rangle.\end{aligned}$$

The integrand is bounded by $\|B\|$ for all $\mathbf{n}$. By the dominated convergence theorem, the right-hand side converges to

$$\int d\mu_{\underline{\mathbf{r}}}(\boldsymbol{\phi}^*,\boldsymbol{\phi})\ e^{-\int d\mathbf{y}\ \boldsymbol{\phi}^*(\mathbf{y})\cdot\boldsymbol{\phi}(\mathbf{y})}\langle\tilde{\boldsymbol{\phi}}|B|\boldsymbol{\phi}\rangle.$$

as $\min_{1\leq p\leq P_b} n_p \to \infty$.

□

Chapter 4

Particle-Number-Preserving Interactions

Our goal is to construct a functional integral representation for the partition function of a gas consisting of an arbitrary finite number of particle species (bosons and/or fermions) interacting on a finite lattice, with a Hamiltonian that preserves the total number of particles in each species.

4.1 The Physical Setting and Notation

The first step of this project is to create notation that is compact, intuitive, and general enough to allow for the treatment of any finite number of bosonic and/or fermionic particle species. We review and build upon some of the notation that has already been defined in Chapter 3.

4.1.1 The Hilbert Space

Recall from Section 3.4 that the Hilbert space for a mixed gas with P species of particles is

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_P,$$

where $\mathcal{H}_p$ is the Fock space for the p^{th} species. Listing the bosons before fermions,

$$\mathcal{H}_p = \begin{cases} \mathcal{B}(X_p) & \text{if } p \le P_b \\ \mathcal{F}(X_p) & \text{if } p > P_b, \end{cases}$$

where X_p is the configuration space for the p^{th} species of particles and P_b is the number of bosonic species. We also let P_f be the number of fermionic species so that $P = P_b + P_f$.

The bosonic and fermionic spaces are

$$\mathcal{B} = \mathcal{B}(X_1) \otimes \cdots \otimes \mathcal{B}(X_{P_b}) \text{ and } \mathcal{F} = \mathcal{F}(X_{P_b+1}) \otimes \cdots \otimes \mathcal{F}(X_P)$$

respectively.

For each $\mathbf{n} = (n_1, n_2, \ldots, n_{P_b}) \in \mathbb{N}_0^{P_b}$,

$$\mathcal{B}_{\mathbf{n}} = \mathcal{B}_{n_1}(X_1) \otimes \mathcal{B}_{n_2}(X_2) \otimes \cdots \otimes \mathcal{B}_{n_{P_b}}(X_{P_b}).$$

and

$$\mathcal{B}_{\le \mathbf{n}} = \left(\bigoplus_{i=0}^{n_1} \mathcal{B}_i(X_1) \right) \otimes \left(\bigoplus_{i=0}^{n_2} \mathcal{B}_i(X_2) \right) \otimes \cdots \otimes \left(\bigoplus_{i=0}^{n_{P_b}} \mathcal{B}_i(X_{P_b}) \right).$$

The corresponding subspaces of $\mathcal{H}$ are

$$\mathcal{H}_{\mathbf{n}} = \mathcal{B}_{\mathbf{n}} \otimes \mathcal{F} \qquad \text{and} \qquad \mathcal{H}_{\le \mathbf{n}} = \mathcal{B}_{\le \mathbf{n}} \otimes \mathcal{F}.$$

We denote the projections onto these subspaces by $P^{(\mathbf{n})}$ and $P_{\mathbf{n}}$ respectively,

$$P^{(\mathbf{n})} = P_{\mathcal{H}_{\mathbf{n}}} \qquad \text{and} \qquad P_{\mathbf{n}} = P_{\mathcal{H}_{\le \mathbf{n}}}.$$

4.1.2 Notation for Particle Coordinates and Fields

For each p, we will typically use the lower-case letters x and y to denote elements of X_p. For any $n_p \in \mathbb{N}$, a vector of n_p elements of X_p will be indicated with an arrow,

$$\vec{x}_p = (x_{p_1}, x_{p_2}, \ldots, x_{p_{n_p}}) \in X_p^{n_p}.$$

Furthermore, these elements of $X_p^{n_p}$ will be collected to form the (nested) vector,

$$\mathbf{x} := (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_P) \in X_1^{n_1} \times X_2^{n_2} \times \cdots \times X_P^{n_P}.$$

To condense the notation on the right-hand side, if $\mathbf{n} := (n_1, \ldots, n_P)$, let

$$X^{\mathbf{n}} := X_1^{n_1} \times \cdots \times X_P^{n_P}. \tag{4.1}$$

The union of all such elements is

$$\mathbf{X} := \bigcup_{\mathbf{n} \in \mathbb{N}_0^P} X^{\mathbf{n}}. \tag{4.2}$$

For any element $\mathbf{x}$ of $\mathbf{X}$, let $\mathbf{n}(\mathbf{x}) = (n_1(\mathbf{x}), \ldots, n_P(\mathbf{x}))$ be the element $\mathbf{n}$ of $\mathbb{N}_0^P$ such that $\mathbf{x} \in X^{\mathbf{n}}$.

We will denote the ℓ^1 norm of an element $\mathbf{n} \in \mathbb{N}_0^P$ by $|\mathbf{n}|$. That is,

$$|\mathbf{n}| = \sum_{p=1}^{P} n_p. \tag{4.3}$$

Hence, for any $\mathbf{x} \in \mathbf{X}$, then $|\mathbf{n}(\mathbf{x})| = \sum_{p=1}^{P} n_p(\mathbf{x})$ is the total number of particles whose positions are represented by $\mathbf{x}$.

The *concatenation* of two elements $\mathbf{x} = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_P) \in X^{\mathbf{n}}$ and $\mathbf{y} = (\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_P) \in X^{\mathbf{m}}$ is defined to be

$$\mathbf{x} \circ \mathbf{y} = (\vec{x}_1 \circ \vec{y}_1, \vec{x}_2 \circ \vec{y}_2, \ldots, \vec{x}_P \circ \vec{y}_P) \in X^{\mathbf{n}+\mathbf{m}}, \tag{4.4}$$

where, for each p,

$$\vec{x}_p \circ \vec{y}_p = (x_{p_1}, x_{p_2}, \ldots, x_{p_{n_p}}, y_{p_1}, y_{p_2}, \ldots, y_{p_{m_p}}). \tag{4.5}$$

Given an element $\mathbf{x} \in \mathbf{X}$ with $\mathbf{n}(\mathbf{x}) = \mathbf{n}$, we can write

$$\mathbf{x} = (\vec{x}_1, \cdots, \vec{x}_P) = (x_{1_1}, \ldots, x_{1_{n_1}}, \cdots, x_{P_1}, \ldots, x_{P_{n_P}})$$

Consider the subscripts of each x as an ordered pair (i, j) so that

$$x_{(i,j)} := x_{i_j}.$$

Let $\mathbb{I}_{\mathbf{x}}$ be the set of all indices for components of $\mathbf{x}$,

$$\mathbb{I}_{\mathbf{x}} := \{(1,1), \dots, (1, n_1), (2,1) \dots, (2, n_2), \dots, (P, 1), \dots, (P, n_P)\}. \tag{4.6}$$

There are $|\mathbf{n}(\mathbf{x})|$ distinct elements in this set. We define an order on these elements, just as they are listed above in (4.6). That is,

$$(p, n) < (q, m)$$

whenever $p < q$, or in the case $p = q$, whenever $n < m$.

For any subset U of $\mathbb{I}_{\mathbf{x}}$, $\mathbf{x}_U$ is defined to be the vector of components of $\mathbf{x}$ whose indices lie in U, with the order of their indices preserved. In other words, the components of $\mathbf{x}_U$ form the set $\{x_{(p,n)} : (p, n) \in U\}$, and they are ordered by increasing (p, n).

For $U \subset \mathbb{I}_{\mathbf{x}}$, we also let $n_p(U)$ be the number of coordinates in U representing particles of the p^{th} species. That is,

$$n_p(U) := |\{m \in \mathbb{N} : (p, m) \in U\}|. \tag{4.7}$$

Since we will often want to count only the coordinates corresponding to fermions, we also let

$$n_p^{(f)}(U) := n_p(U)\chi_{[P_b+1,P]}(p), \tag{4.8}$$

where $\chi_{[P_b+1,P]}$ is the characteristic function of the interval $[P_b + 1, P]$. That is, $\chi_{[P_b+1,P]}(p)$ is equal to 1 when the p^{th} species is fermionic, and 0 otherwise.

For any $p \in \{1, \dots, P\}$, $m, n \in \mathbb{N}$ with $m \le n$, and $\vec{x} \in X_p^n$, let

$$\vec{x}_{[\le m]} := (x_1, \dots, x_m) \tag{4.9}$$

and

$$\vec{x}_{[>m]} := (x_{m+1}, \dots, x_n).$$

Similarly, for any $\mathbf{m}, \mathbf{n} \in \mathbb{N}_0^P$ with $\mathbf{m} \leq \mathbf{n}$, and any $\mathbf{x} \in X^{\mathbf{n}}$,

$$\mathbf{x}_{[\leq \mathbf{m}]} = (\vec{x}_{1_{[\leq m_1]}}, \ldots, \vec{x}_{P_{[\leq m_P]}}) \tag{4.10}$$

and

$$\mathbf{x}_{[> \mathbf{m}]} = (\vec{x}_{1_{[> m_1]}}, \ldots, \vec{x}_{P_{[> m_P]}}). \tag{4.11}$$

It will be useful to consider partitions of the set $\mathbb{I}_{\mathbf{x}}$. A partition $\mathcal{P}$ of $\mathbb{I}_{\mathbf{x}}$ is a collection of nonempty disjoint subsets U of $\mathbb{I}_{\mathbf{x}}$ whose union is equal to $\mathbb{I}_{\mathbf{x}}$. For each $U \in \mathcal{P}$, let $\vec{U}$ be the vector formed by the components of U in increasing order. We then order the elements of $\mathcal{P}$ by increasing largest element, so that

$$\mathbb{I}_{\mathbf{x}} = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_{|\mathcal{P}|-1} \sqcup U_{|\mathcal{P}|},$$

where $U_{|\mathcal{P}|}$ contains the largest element (P, n_P) of $\mathbb{I}_{\mathbf{x}}$, $U_{|\mathcal{P}|-1}$ contains the largest element of $\mathbb{I}_{\mathbf{x}} \backslash U_{|\mathcal{P}|}$, and so on. Then the *sign* of the partition $\mathcal{P}$, $(-1)^{\mathcal{P}}$, is defined to be the sign of the permutation required to bring the coordinates of each fermionic species from the concatenated list

$$\vec{U_1} \circ \cdots \circ \vec{U_{|\mathcal{P}|}}$$

back to their order in $\mathbb{I}_{\mathbf{x}}$.

Recall that the fields for the mixed gas are of the form

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{P_b}, \alpha_{P_b+1}, \ldots, \alpha_P)$$

where the first P_b components are complex fields and the last P_f components are Grassmann fields. For a given $\mathbf{x} \in X_1 \times \cdots \times X_P$, $\alpha^*(\mathbf{x})$ is defined to be the product

$$\alpha^*(\mathbf{x}) := \alpha_1^*(\vec{x}_1) \cdots \alpha_P^*(\vec{x}_P), \tag{4.12}$$

where for each p,

$$\alpha_p^*(\vec{x}_p) := \alpha_p^*(x_{p_1}) \alpha_p^*(x_{p_2}) \cdots \alpha_p^*(x_{p_{n_p}}). \tag{4.13}$$

We similarly define a differential operator,

$$\frac{\partial}{\partial \alpha^*(\mathbf{x})} := \frac{\partial}{\partial \alpha_P^*(\vec{x}_P)} \cdots \frac{\partial}{\partial \alpha_1^*(\vec{x}_1)}, \tag{4.14}$$

where, for each p,

$$\frac{\partial}{\partial \alpha_p^*(\vec{x}_p)} := \frac{\partial}{\partial \alpha_p^*(x_{p,n_p})} \frac{\partial}{\partial \alpha_p^*(x_{p,n_p-1})} \cdots \frac{\partial}{\partial \alpha_p^*(x_{p,1})}. \tag{4.15}$$

Note that the order of the position coordinates are decreasing in this product, whereas they are increasing in Equation (4.13).

For any $\mathbf{n} = (n_1, n_2, \ldots, n_P) \in \mathbb{N}_0^P$, we use the multi-index notation,

$$\mathbf{n}! := n_1! n_2! \cdots n_P!. \tag{4.16}$$

We also define

$$S_{\mathbf{n}} := S_{n_1} \times S_{n_2} \times \cdots \times S_{n_P}. \tag{4.17}$$

The *sign* of $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_P) \in S_{\mathbf{n}}$ is

$$(-1)^{\boldsymbol{\sigma}} := \prod_{p=P_b+1}^{P} (-1)^{\sigma_p}. \tag{4.18}$$

We may let any vector of permutations $\boldsymbol{\sigma}$ of $S_{\mathbf{n}}$ act on $X^{\mathbf{n}}$ via

$$\begin{aligned} \boldsymbol{\sigma} : X^{\mathbf{n}} &\to X^{\mathbf{n}} \\ (\vec{x}_1, \ldots, \vec{x}_P) &\mapsto (\sigma_1(\vec{x}_1), \ldots, \sigma_P(\vec{x}_P)). \end{aligned} \tag{4.19}$$

(To be clear, each $\vec{x}_p \in X_p^{n_p}$ in the second line.) Likewise, we can allow $\boldsymbol{\sigma}$ to act on functions f with domain $X^{\mathbf{n}}$ as

$$(\boldsymbol{\sigma} f)(\mathbf{x}) = f \circ \boldsymbol{\sigma}^{-1}(\mathbf{x}) = f(\boldsymbol{\sigma}^{-1}(\mathbf{x})),$$

where $\boldsymbol{\sigma}^{-1} = (\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_P^{-1})$.

The *symmetrization operator* $S^{(\mathbf{n})}$ is defined as

$$S^{(\mathbf{n})} := \frac{1}{\mathbf{n}!} \sum_{\boldsymbol{\sigma} \in S_{\mathbf{n}}} (-1)^{\boldsymbol{\sigma}} \boldsymbol{\sigma}. \tag{4.20}$$

Note that this is also equal to

$$S^{(\mathbf{n})} = \bigotimes_{p=1}^{P_b} S_b^{(n_p)}(X_p) \otimes \bigotimes_{p=P_b+1}^{P} S_f^{(n_p)}(X_p).$$

In instances where a function has several arguments in $X^{\mathbf{n}}$, we will use a subscript to indicate which argument is being symmetrized, as in

$$S_{\mathbf{x}}^{(\mathbf{n})} f(\mathbf{x}, \mathbf{y}) = \frac{1}{\mathbf{n}!} \sum_{\boldsymbol{\sigma} \in S_{\mathbf{n}}} (-1)^{\boldsymbol{\sigma}} f(\boldsymbol{\sigma}^{-1}(\mathbf{x}), \mathbf{y}).$$

4.1.3 Creation and Annihilation Operators

For each p^{th} species, creation and annihilation operators at $x \in X_p$ are denoted by $a_p^\dagger(x)$ and $a_p(x)$, respectively.

For any p and $\vec{x} = (x_1, x_2, \ldots, x_n) \in X_p^n$, let

$$a_p^\dagger(\vec{x}) := a_p^\dagger(x_1) a_p^\dagger(x_2) \cdots a_p^\dagger(x_n) \text{ and } a_p(\vec{x}) := a_p(x_n) a_p(x_{n-1}) \cdots a_p(x_1).$$

(Note that the order of the annihilation operators is reversed in the second equation above.)

Furthermore, for any $\mathbf{x} = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_P) \in \mathbf{X}$, we define

$$a^\dagger(\mathbf{x}) := a_1^\dagger(\vec{x}_1) \otimes \cdots \otimes a_P^\dagger(\vec{x}_P) \text{ and } a(\mathbf{x}) := a_1(\vec{x}_1) \otimes \cdots \otimes a_P(\vec{x}_P).$$

The number operator (densely defined for bosons) on $\mathcal{H}_p$ for the p^{th} species is

$$N_p = \int_{X_p} dx \; a_p^\dagger(x) a_p(x).$$

4.1.4 The Hamiltonian

The Hamiltonian H is a linear operator on the Hilbert space $\mathcal{H}$ that represents the total energy of the system. Generalizing Proposition 2.14 of [10], we write H in the form,

$$H = \int_{\mathbf{X}^2} d\mathbf{x}\, d\mathbf{y}\; a^\dagger(\mathbf{x})\mathfrak{h}(\mathbf{x},\mathbf{y})a(\mathbf{y}),$$

where $\mathfrak{h} : \mathbf{X}^2 \to \mathbb{C}$ is the kernel of H. For example, if H includes the kinetic energy of each particle as well as a two-body interaction, $\mathfrak{h}$ would be of the form

$$\mathfrak{h}(\mathbf{x},\mathbf{y}) = \begin{cases} h_p(x,y) & \text{if } \mathbf{x} = x \in X_p \text{ and } \mathbf{y} = y \in X_p, \\ v(x_1,x_2)\delta_{x_1}(y_1)\delta_{x_2}(y_2) & \text{if } \mathbf{n}(\mathbf{x}) = \mathbf{n}(\mathbf{y}) \text{ and } |\mathbf{n}(\mathbf{x})| = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.21}$$

Note that $\mathfrak{h}(\mathbf{x},\mathbf{y})$ depends on the sizes of its arguments, $\mathbf{n}(\mathbf{x})$ and $\mathbf{n}(\mathbf{y})$, as well as their values.

We make four main assumptions about the Hamiltonian H in this chapter:

1. The kernel satisfies $\mathfrak{h}(\mathbf{y},\mathbf{x}) = \mathfrak{h}^*(\mathbf{x},\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in \mathbf{X}$;

2. H preserves the number of particles in each species;

3. The *order* M of H is finite and strictly greater than 1, where

$$M := \max\{\max(|\mathbf{n}(\mathbf{x})|, |\mathbf{n}(\mathbf{y})|) : \mathbf{x},\mathbf{y} \in \mathbf{X} \text{ and } \mathfrak{h}(\mathbf{x},\mathbf{y}) \neq 0\}.$$

4. H satisfies the bounds

$$\sum_{p=1}^{P_b} \left(AN_p^2 - \tilde{B}N_p\right) \leq H \leq \tilde{C} \prod_{p=1}^{P_b} N_p^M, \tag{4.22}$$

where $A > 0$, $\tilde{B}$, and $\tilde{C}$ are real constants that may depend on M and each $|X_p|$.

Note that the second assumption implies that $\mathfrak{h}(\mathbf{x},\mathbf{y})$ is only nonzero when $\mathbf{n}(\mathbf{x}) = \mathbf{n}(\mathbf{y})$.

In [10], the bounds of Equation (4.22) for one species of bosons with a $M = 2$-body interaction were not assumed directly, but rather deduced from properties assumed of the kernel $\mathfrak{h}$. In particular, the lower bound follows from the assumption that the second-order interaction is *repulsive*, just as particles of like charge repel under the Coulomb interaction. In particular, repulsive 2-body interactions lead to lower bounds on H that are quadratic in the particle number, as in (4.22); see Definition 2.16 and Proposition 2.17 of [10]. However, if we allow higher-order interactions, we could also deduce this bound by assuming that the term of highest order in each boson species is repulsive, in some sense. For example, for any $M \in \mathbb{N}$, consider one species of bosons with an M^{th}-order interaction of the form

$$V = \int_{X^M} dx_1 \cdots dx_M \; a^\dagger(x_1) \cdots a^\dagger(x_M) v(x_1, \cdots, x_M) a(x_1) \cdots a(x_M),$$

where $v : X^M \to \mathbb{R}$. We assume that V is repulsive in the sense that

$$v_0 := \inf \left\{ \int_{X^M} dx_1 \cdots dx_M \; v(x_1, \cdots, x_M) \rho(x_1) \cdots \rho(x_M) \middle| \int dx \; \rho(x)^M = 1, \rho \geq 0 \right\}$$

is (strictly) positive. Then for any $n \in \mathbb{N}$ and $Y \in X^n/S_n$,

$$\begin{aligned}
\langle \delta_Y, V \delta_Y \rangle &= \int_{X^M} dx_1 \cdots dx_M \; v(x_1, \cdots, x_M) \langle \delta_Y, n(x_1) \cdots n(x_M) \delta_Y \rangle \\
&= \int_{X^M} dx_1 \cdots dx_M \; v(x_1, \cdots, x_M) \mu_Y(x_1) \cdots \mu_Y(x_M) \\
&\geq v_0 \int_X dx \; \mu_Y(x)^M \\
&\geq v_0 \max_{x \in X} \mu_Y(x)^M \\
&\geq v_0 \left(\frac{\int_X dx \; \mu_Y(x)}{|X|} \right)^M \\
&= \frac{v_0}{|X|^M} n^M.
\end{aligned}$$

Since the set $\{\delta_Y : Y \in X^n/S_n\}$ forms an orthonormal basis of eigenvectors of V, it follows that

$$V \geq \frac{v_0}{|X|^M} N^M,$$

where N is the boson number operator. Of course, $N^M \geq N^2$; hence, the lower bound of the form (4.22) indeed applies to V. We could perform a similar analysis for other higher-order interactions, both for the lower and upper bounds. However, we will simply assume the bounds in Equation (4.22), rather than assuming conditions on $\mathfrak{h}$.

Since the creation and annihilation operators satisfy (anti)commutation relations, the kernel $\mathfrak{h}$ must have symmetry properties as well. That is, for any $\boldsymbol{\sigma}, \boldsymbol{\pi} \in S_{\mathbf{n}}$ and $\mathbf{x}, \mathbf{y} \in X^{\mathbf{n}}$,

$$\mathfrak{h}(\boldsymbol{\sigma}(\mathbf{x}), \boldsymbol{\pi}(\mathbf{y})) = (-1)^{\boldsymbol{\sigma}}(-1)^{\boldsymbol{\pi}}\mathfrak{h}(\mathbf{x}, \mathbf{y}). \tag{4.23}$$

(To review the notation above, please see page 62.)

Under these assumptions, it is easy to see that H is essentially self-adjoint on the subspace of $\mathcal{H}$ with a finite number of bosons,

$$\mathcal{H}^{(finite)} = \mathcal{B}^{(finite)}(X_1) \otimes \mathcal{B}^{(finite)}(X_2) \otimes \cdots \otimes \mathcal{B}^{(finite)}(X_{P_b}) \otimes \mathcal{F},$$

where

$$\mathcal{B}^{(finite)}(X) := \{f \in \mathcal{B}(X) : (\exists m \in \mathbb{N})(\forall n > m) P^{(n)} f = 0\}.$$

(Note that m depends on f.) Alternately, $\mathcal{B}^{(finite)}(X) = S_b\, \mathcal{D}^{(finite)}(X)$, as defined in Chapter 2.

Recall from Equation (2.29) of Section 2.3.2 that the partition function for a single species of indistinguishable particles is

$$\mathrm{Tr}(e^{-\beta(H-\mu N)}),$$

where β is the inverse temperature, H is the Hamiltonian, μ is the chemical potential, and N is the particle number operator.

With p different species of particles, we replace μN with the sum $\sum_{p=1}^{P} \mu_p N_p$, where μ_p is the chemical potential for the p^{th} particle species. For convenience, we form the vector of all P chemical potentials,

$$\boldsymbol{\mu} := (\mu_1, \ldots, \mu_P),$$

as well as the vector of all P particle number operators,

$$\mathbf{N} := (N_1, \ldots, N_P).$$

Then the partition function may be written as

$$Z(\beta, \boldsymbol{\mu}) = \operatorname{Tr} e^{-\beta(H - \boldsymbol{\mu}\cdot\mathbf{N})}.$$

Since our main goal is to find a functional integral representation for the partition function, we will be more commonly working with the operator in the exponent rather than the Hamiltonian H. We will denote this self-adjoint operator by K,

$$K := H - \boldsymbol{\mu} \cdot \mathbf{N}.$$

Since H and $\boldsymbol{\mu} \cdot \mathbf{N}$ each preserve the number of particles in each species, so too must K. Provided that the order of H is nonzero, H and K also have the same order.

Let $k : \mathbf{X}^2 \to \mathbb{C}$ be the kernel of K, such that

$$K = \int_{\mathbf{X}^2} d\mathbf{x}\, d\mathbf{y}\ a^\dagger(\mathbf{x}) k(\mathbf{x}, \mathbf{y}) a(\mathbf{y}).$$

Assumptions 1-4 above on H also carry over for K. That is,

1. The kernel satisfies $k(\mathbf{y}, \mathbf{x}) = k^*(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$;

2. K preserves the number of particles in each species;

3. The *order* M of K is finite, where

$$M := \max\{\max(|\mathbf{n}(\mathbf{x})|, |\mathbf{n}(\mathbf{y})|) : \mathbf{x}, \mathbf{y} \in \mathbf{X} \text{ and } k(\mathbf{x}, \mathbf{y}) \neq 0\}.$$

4. K is *stable* in the sense that it is bounded from below. In particular,

$$\sum_{p=1}^{P_b} \left(A N_p^2 - B N_p \right) \leq K \leq C \prod_{p=1}^{P_b} N_p^M \tag{4.24}$$

where $A > 0$ and B are constants and N_p is the total boson number operator for the p^{th} species.

In particular, note that Equation (4.24) implies that K is bounded below by a constant k_L,

$$K \geq k_L, \text{ where } k_L := \frac{-P_b B^2}{4A}. \tag{4.25}$$

4.2 A Functional Integral Representation for the Partition Function

The main goal of this section is to prove the coherent-state functional integral representation for the partition function in the following theorem.

Theorem 4.2.1. Let $\underline{\mathbf{R}}$ be a map from $\mathbb{N}$ to $(0, \infty)^{P_b}$ satisfying

$$\lim_{q\to\infty} q e^{-R_p(q)^2/2} = 0$$

for each $p \in \{1, \ldots, P_b\}$. Then

$$\text{Tr } e^{-\beta K} = \lim_{q\to\infty} \int \prod_{\tau\in\mathcal{T}_q} \left[d\mu_{\underline{\mathbf{R}}(q)}(\boldsymbol{\phi}^*_\tau, \boldsymbol{\phi}_\tau) e^{-\int d\mathbf{y} \; \boldsymbol{\phi}^*_\tau(\mathbf{y})\cdot\boldsymbol{\phi}_\tau(\mathbf{y})} \right] \prod_{\tau\in\mathcal{T}_q} \langle \boldsymbol{\phi}_{\tau-\epsilon} | e^{-\epsilon K} | \boldsymbol{\phi}_\tau \rangle \tag{4.26}$$

where $\mathcal{T}_q := \{\epsilon, 2\epsilon, \ldots, q\epsilon\}$, $\epsilon := \frac{\beta}{q}$,

$$\boldsymbol{\phi}_0 := \tilde{\boldsymbol{\phi}}_\beta = (\phi_{0_1}, \ldots, \phi_{0_{P_b}}, -\phi_{0_{P_b+1}}, \ldots, -\phi_{0_P}),$$

and

$$\int d\mathbf{y} \; \boldsymbol{\phi}^*_\tau(\mathbf{y}) \cdot \boldsymbol{\phi}_\tau(\mathbf{y}) := \sum_{p=1}^{P} \int_{X_p} dy \; \phi^*_{\tau_p}(y) \phi_{\tau_p}(y).$$

The lemma below will be required to prove this theorem.

Lemma 4.2.2. Let $\underline{\mathbf{R}}$ be a map from $\mathbb{N}$ to $(0, \infty)^{P_b}$ satisfying

$$\lim_{q\to\infty} q e^{-R_p(q)^2/2} = 0$$

for each $p \in \{1, \ldots, P_b\}$. Then,

$$\text{Tr } e^{-\beta K} = \lim_{q\to\infty} \text{Tr} \left(e^{-\frac{\beta}{q}K} \mathrm{I}_{\underline{\mathbf{R}}(q)} \right)^{q-1} e^{-\frac{\beta}{q}K}$$

and for any $\boldsymbol{\phi}, \boldsymbol{\phi}'$,

$$\langle \boldsymbol{\phi} | e^{-\beta K} | \boldsymbol{\phi}' \rangle = \lim_{q\to\infty} \langle \boldsymbol{\phi} | \left(e^{-\frac{\beta}{q}K} \mathrm{I}_{\underline{\mathbf{R}}(q)} \right)^{q-1} e^{-\frac{\beta}{q}K} | \boldsymbol{\phi}' \rangle.$$

Proof. This proof is very similar to that of Lemma 3.4 in [10]. As in [10], we introduce the notation

$$A_i := \begin{cases} e^{-\frac{\beta}{q}K} & \text{if } i \text{ is odd} \\ \mathrm{I}_{\underline{\mathbf{R}}(q)} & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad B_i := \begin{cases} e^{-\frac{\beta}{q}K} & \text{if } i \text{ is odd} \\ \mathbb{1} & \text{if } i \text{ is even} \end{cases}$$

so that $\left(e^{-\frac{\beta}{q}K} \mathrm{I}_{\underline{\mathbf{R}}(q)} \right)^{q-1} e^{-\frac{\beta}{q}K} = \prod_{i=1}^{2q-1} A_i$ and $e^{-\beta K} = \prod_{i=1}^{2q-1} B_i$.

Let $\mathbf{n} := (n, n, \ldots, n) \in \mathbb{N}^{P_b}$ for $n \in \mathbb{N}$. Then $P_{\mathbf{n}}$ is the projection onto

$$\mathcal{H}_{\leq \mathbf{n}} = \mathcal{B}_{\leq \mathbf{n}} \otimes \mathcal{F} = \left(\bigoplus_{i=0}^{n} \mathcal{B}_i(X_1) \right) \otimes \left(\bigoplus_{i=0}^{n} \mathcal{B}_i(X_2) \right) \otimes \cdots \otimes \left(\bigoplus_{i=0}^{n} \mathcal{B}_i(X_{P_b}) \right) \otimes \mathcal{F}.$$

By the triangle inequality for the trace norm $\|\cdot\|_1 = \mathrm{Tr}\,|\cdot|$,

$$\begin{aligned} \mathrm{Tr} \left| \prod_i A_i - \prod_i B_i \right| \leq \mathrm{Tr} \left| \left(\prod_i A_i - \prod_i B_i \right) \mathrm{P}_{\mathbf{n}} \right| + \mathrm{Tr} \left| \prod_i A_i (\mathbb{1} - \mathrm{P}_{\mathbf{n}}) \right| \\ + \mathrm{Tr} \left| \prod_i B_i (\mathbb{1} - \mathrm{P}_{\mathbf{n}}) \right|. \end{aligned} \tag{4.27}$$

We start by bounding the first term, which can be written as a telescoping sum,

$$\begin{aligned} \mathrm{Tr} \left| \left(\prod_i A_i - \prod_i B_i \right) \mathrm{P}_{\mathbf{n}} \right| &= \mathrm{Tr} \left| \sum_{j=1}^{2q-1} \left(\prod_{i=1}^{j-1} A_i \; (A_j - B_j) \prod_{i=j+1}^{2q-1} B_i \right) \mathrm{P}_{\mathbf{n}} \right| \\ &\leq \mathrm{Tr}(\mathrm{P}_{\mathbf{n}}) \sum_{j=1}^{2q-1} \left\| \prod_{i=1}^{j-1} A_i \; (A_j - B_j) \prod_{i=j+1}^{2q-1} B_i \; \mathrm{P}_{\mathbf{n}} \right\|. \end{aligned}$$

Note that

$$\|A_i\|, \|B_i\| \leq \begin{cases} e^{\frac{\beta}{q} k_L} & \text{if } i \text{ is odd}, \\ 1 & \text{if } i \text{ is even}, \end{cases}$$

where $k_L \in \mathbb{R}$ is the lower bound for K from Equation (4.25).

Since $A_i - B_i = I_{\underline{\mathbf{R}}(q)} - \mathbb{1}$ when i is even and is zero otherwise,

$$\begin{aligned} \operatorname{Tr}\left|\left(\prod A_i - \prod B_i\right) \mathrm{P}_{\mathbf{n}}\right| &\leq \operatorname{Tr}(\mathrm{P}_{\mathbf{n}})(q-1)e^{\beta k_L} \|(I_{\underline{\mathbf{R}}(q)} - \mathbb{1})\mathrm{P}_{\mathbf{n}}\| \\ &\leq \operatorname{Tr}(\mathrm{P}_{\mathbf{n}})(q-1)e^{\beta k_L} \sum_{p=1}^{P_b} |X_p| 2^{n_p+1} e^{-R_p(q)^2/2}, \end{aligned} \tag{4.28}$$

where the bound on $\|(I_{\underline{\mathbf{R}}(q)} - \mathbb{1})\mathrm{P}_{\mathbf{n}}\|$ in the second line is from Theorem 3.5.3. The large-field cutoffs $R_p(q)$ have been chosen to grow quickly enough so that the bound in (4.28) shrinks to 0 as $q \to \infty$, for any fixed $n \in \mathbb{N}$.

Using the bound from Equation (4.24), the other two terms in (4.27) satisfy

$$\operatorname{Tr}\left|\prod A_i(\mathbb{1} - \mathrm{P}_{\mathbf{n}})\right|, \operatorname{Tr}\left|\prod B_i(\mathbb{1} - \mathrm{P}_{\mathbf{n}})\right| \leq \sum_{\substack{\mathbf{m} \in \mathbb{N}^{P_b} \\ (\exists p) m_p > n}} e^{-\beta(A\mathbf{m}^2 - B|\mathbf{m}|)} \operatorname{Tr}(\mathrm{P}^{(\mathbf{m})}), \tag{4.29}$$

where $\mathbf{m}^2 = \sum_{p=1}^{P_b} m_p^2$ and $|m| = \sum_{p=1}^{P_b} m_p$. The trace of $\mathrm{P}^{(\mathbf{m})}$ is equal to the dimension of $\mathcal{H}_{\mathbf{m}} = \mathcal{B}_{\mathbf{m}} \otimes \mathcal{F}$,

$$\dim(\mathcal{H}_{\mathbf{m}}) = \prod_{p=1}^{P_b} \dim(\mathcal{B}_{m_p}(X_p)) \prod_{p=P_b+1}^{P} 2^{|X_p|}.$$

The dimension of the m_p-boson space, $\mathcal{B}_{m_p}(X_p)$ is

$$\begin{aligned} \dim(\mathcal{B}_{m_p}(X_p)) &= \frac{(m_p + |X_p| - 1)!}{m_p!(|X_p| - 1)!} \\ &\leq \frac{(m_p + |X_p| - 1)^{|X_p|-1}}{(|X_p| - 1)!} \end{aligned}$$

$$\leq \frac{|X_p|^{|X_p|-1}}{(|X_p|-1)!} m_p^{|X_p|-1}.$$

Hence,

$$\mathrm{Tr}(\mathrm{P}_{\mathbf{m}}) \leq C \prod_{p=1}^{P_b} m_p^{|X_p|-1}$$

where $C = \prod_{p=1}^{P_b} \frac{|X_p|^{|X_p|-1}}{(|X_p|-1)!} \prod_{p=P_b+1}^{P} 2^{|X_p|}$ is a constant that depends on the size of each X_p (but not on q or m_p).

Applying this to Equation (4.29),

$$\begin{aligned}
\mathrm{Tr}\left|\prod A_i(\mathbb{1}-\mathrm{P}_{\mathbf{n}})\right|, &\mathrm{Tr}\left|\prod B_i(\mathbb{1}-\mathrm{P}_{\mathbf{n}})\right| \\
&\leq \sum_{\substack{\mathbf{m}\in\mathbb{N}^{P_b} \\ (\exists p) m_p > n}} e^{-\beta(A\mathbf{m}^2 - B|\mathbf{m}|)} C \prod_{p=1}^{P_b} m_p^{|X_p|-1} \\
&\leq C \sum_{\substack{\mathbf{m}\in\mathbb{N}^{P_b} \\ (\exists p) m_p > n}} \prod_{p=1}^{P_b} e^{-\beta(Am_p^2 - Bm_p)} m_p^{|X_p|-1} \\
&\leq C \left(\sum_{m=0}^{\infty} e^{-\beta(Am^2 - Bm)} m^{M_X} \right)^{P_b - 1} \sum_{m=n}^{\infty} e^{-\beta(Am^2 - Bm)} m^{M_X} \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}$$

where $M_X := \max_{1\leq p \leq P_b} |X_p| - 1$. Hence, the bound in (4.29) can be made arbitrarily small, as long as n is chosen to be sufficiently large (independently of q). It follows that $\prod A_i - \prod B_i \to 0$ as $q \to \infty$ in the trace norm, and hence also in operator norm, as well as weakly. The claim follows. □

Proof of Theorem 4.2.1. We start by expressing the partition function as a sum over all **n**-particle spaces,

$$\mathrm{Tr}\, e^{-\beta K} = \sum_{\mathbf{n}\in\mathbb{N}_0^{P_b}} \mathrm{Tr}_{\mathcal{H}_{\mathbf{n}}} \left(e^{-\beta K} P^{(\mathbf{n})} \right).$$

Since each $\mathcal{H}_{\mathbf{n}}$ is finite-dimensional, $\mathrm{I}_{\underline{\mathbf{R}}(q)}$ maps each $\mathcal{H}_{\mathbf{n}}$ to itself, and $\mathrm{I}_{\underline{\mathbf{R}}(q)}$

converges strongly to $\mathbb{1}$ as $q \to \infty$ (from Theorem 3.5.3), it follows that

$$\mathrm{Tr}\ e^{-\beta K} = \sum_{\mathbf{n}\in\mathbb{N}_0^{P_b}} \lim_{q\to\infty} \mathrm{Tr}_{\mathcal{H}_\mathbf{n}}\left(e^{-\beta K} P^{(\mathbf{n})} \mathrm{I}_{\underline{\mathbf{R}}(q)}\right).$$

Note that

$$\left|\mathrm{Tr}_{\mathcal{H}_\mathbf{n}}\left(e^{-\beta K}\mathrm{I}_{\underline{\mathbf{R}}(q)}\right)\right| \le \dim(\mathcal{H}_\mathbf{n})\|e^{-\beta K}\mathrm{P}^{(\mathbf{n})}\| \le \dim(\mathcal{H}_\mathbf{n})e^{-\beta(A\mathbf{n}^2 - B|\mathbf{n}|)},$$

which is summable over $\mathbf{n}$, as we've just seen in the proof of the Lemma 5.3.9. Hence, by the dominated convergence theorem,

$$\mathrm{Tr}\ e^{-\beta K} = \lim_{q\to\infty} \mathrm{Tr}\left(e^{-\beta K}\mathrm{I}_{\underline{\mathbf{R}}(q)}\right).$$

From Proposition 3.6.3, it follows that

$$\mathrm{Tr}\ e^{-\beta K} = \lim_{q\to\infty} \int d\mu_{\underline{\mathbf{R}}(q)}(\boldsymbol{\phi}_0^*, \boldsymbol{\phi}_0) e^{-\int dy\ \ \boldsymbol{\phi}_0^*(y)\cdot\boldsymbol{\phi}_0(y)} \langle\tilde{\boldsymbol{\phi}}_0|e^{-\beta K}|\boldsymbol{\phi}_0\rangle$$

Comparing this with the right-hand side of Equation (4.26),

$$\begin{aligned}
&\int d\mu_{\underline{\mathbf{R}}(q)}(\boldsymbol{\phi}_0^*, \boldsymbol{\phi}_0) e^{-\int dy\ \ \boldsymbol{\phi}_0^*(y)\cdot\boldsymbol{\phi}_0(y)} \langle\tilde{\boldsymbol{\phi}}_0|e^{-\beta K}|\boldsymbol{\phi}_0\rangle - \\
&\qquad\qquad \int \prod_{\tau\in\mathcal{T}_s}\left[d\mu_{\underline{\mathbf{R}}(q)}(\boldsymbol{\phi}_\tau^*, \boldsymbol{\phi}_\tau) e^{-\int dy\ \ \boldsymbol{\phi}_\tau^*(y)\cdot\boldsymbol{\phi}_\tau(y)}\right] \prod_{\tau\in\mathcal{T}_s} \langle\tilde{\boldsymbol{\phi}}_{\tau-\epsilon}|e^{-\epsilon K}|\boldsymbol{\phi}_\tau\rangle \\
&= \int d\mu_{\underline{\mathbf{R}}(q)}(\boldsymbol{\phi}_0^*, \boldsymbol{\phi}_0) e^{-\int dy\ \ \boldsymbol{\phi}_0^*(y)\cdot\boldsymbol{\phi}_0(y)} \langle\tilde{\boldsymbol{\phi}}_0|e^{-\beta K}|\boldsymbol{\phi}_0\rangle \\
&\qquad - \int d\mu_{\underline{\mathbf{R}}(q)}(\boldsymbol{\phi}_0^*, \boldsymbol{\phi}_0) e^{-\int dy\ \ \boldsymbol{\phi}_0^*(y)\cdot\boldsymbol{\phi}_0(y)} \langle\tilde{\boldsymbol{\phi}}_0|\left(e^{-\frac{\beta}{q}K}\mathrm{I}_{\underline{\mathbf{R}}(q)}\right)^{q-1} e^{-\frac{\beta}{q}K}|\boldsymbol{\phi}_0\rangle \\
&= \int d\mu_{\underline{\mathbf{R}}(q)}(\boldsymbol{\phi}_0^*, \boldsymbol{\phi}_0) e^{-\int dy\ \ \boldsymbol{\phi}_0^*(y)\cdot\boldsymbol{\phi}_0(y)} \langle\tilde{\boldsymbol{\phi}}_0|e^{-\beta K} - \left(e^{-\frac{\beta}{q}K}\mathrm{I}_{\underline{\mathbf{R}}(q)}\right)^{q-1} e^{-\frac{\beta}{q}K}|\boldsymbol{\phi}_0\rangle \\
&= \int \prod_{p=1}^{P_b}\prod_{x\in X_p}\left[\frac{d\phi_p^*(x)d\phi_p(x)}{2\pi i}\chi_{R_p(q)}(|\phi_p(x)|)\right] e^{-\int dy\ \ \boldsymbol{\phi}_{0_B}^*(y)\cdot\boldsymbol{\phi}_{0_B}(y)} \\
&\qquad\qquad \langle\boldsymbol{\phi}_{0_B}|\mathrm{Tr}_\mathcal{F}\left[e^{-\beta K} - \left(e^{-\frac{\beta}{q}K}\mathrm{I}_{\underline{\mathbf{R}}(q)}\right)^{q-1} e^{-\frac{\beta}{q}K}\right]|\boldsymbol{\phi}_{0_B}\rangle,
\end{aligned} \tag{4.30}$$

where $\boldsymbol{\phi}_{0_B}$ is the bosonic component of $\boldsymbol{\phi}_0$,

$$\boldsymbol{\phi}_{0_B} := (\phi_{0_1}, \ldots, \phi_{0_{P_b}}).$$

From Lemma 4.2.2, we know that the integrand converges to 0 as $q \to \infty$. In order to apply the dominated convergence theorem again, note that for any q,

$$\begin{aligned}\left|\langle \boldsymbol{\phi}_{0_B}|\mathrm{Tr}_{\mathcal{F}}\left[e^{-\beta K}\right]|\boldsymbol{\phi}_{0_B}\rangle\right| &\leq \dim \mathcal{F} \sum_{\mathbf{n}\in\mathbb{N}_0^{P_b}} e^{-\beta(A\mathbf{n}^2 - B|\mathbf{n}|)} \|P^{(\mathbf{n})}|\boldsymbol{\phi}_{0_B}\rangle\| \\ &\leq 2^{\sum_{p=P_b+1}^{P}|X_p|} \sum_{\mathbf{n}\in\mathbb{N}_0^{P_b}} \prod_{p=1}^{P_b} e^{-\beta(An_p^2 - Bn_p)} \frac{\|\phi_p\|^{2n_p}}{n_p!}.\end{aligned} \tag{4.31}$$

In the exponent, $An_p^2 - Bn_p$ can be bounded from below by a linear function of n_p of arbitrary slope. That is, for any (fixed) $\gamma > 0$,

$$An_p^2 - Bn_p \geq \gamma n_p - \frac{(B+\gamma)^2}{2A}. \tag{4.32}$$

Hence,

$$\begin{aligned}\left|\langle \boldsymbol{\phi}_{0_B}|\mathrm{Tr}_{\mathcal{F}}\left[e^{-\beta K}\right]|\boldsymbol{\phi}_{0_B}\rangle\right| &\leq 2^{\sum_{p=P_b+1}^{P}|X_p|} \sum_{\mathbf{n}\in\mathbb{N}_0^{P_b}} \prod_{p=1}^{P_b} e^{-\beta\left(\gamma n_p - \frac{(B+\gamma)^2}{4A}\right)} \frac{\|\phi_p\|^{2n_p}}{n_p!} \\ &\leq 2^{\sum_{p=P_b+1}^{P}|X_p|} e^{\frac{\beta P_b (B+\gamma)^2}{4A}} \prod_{p=1}^{P_b} \left(\sum_{n_p=0}^{\infty} \frac{(e^{-\beta\gamma}\|\phi_p\|^2)^{n_p}}{n_p!} \right) \\ &\leq 2^{\sum_{p=P_b+1}^{P}|X_p|} e^{\frac{\beta P_b (B+\gamma)^2}{4A}} \prod_{p=1}^{P_b} e^{e^{-\beta\gamma}\|\phi_p\|^2} \\ &\leq 2^{\sum_{p=P_b+1}^{P}|X_p|} e^{\frac{\beta P_b (B+\gamma)^2}{4A}} e^{\left[e^{-\beta\gamma}\int d\mathbf{y}\ \boldsymbol{\phi}_{0_B}^*(\mathbf{y})\cdot\boldsymbol{\phi}_{0_B}(\mathbf{y})\right]}.\end{aligned} \tag{4.33}$$

The same bound holds for

$$\left|\langle \boldsymbol{\phi}_{0_B}|\mathrm{Tr}_{\mathcal{F}}\left[\left(e^{-\frac{\beta}{q}K}\mathrm{I}_{\underline{\mathbf{R}}(q)}\right)^{q-1} e^{-\frac{\beta}{q}K}\right]|\boldsymbol{\phi}_{0_B}\rangle\right|.$$

Hence, the integrand in (4.30) is bounded by

$$
\begin{aligned}
&2 \cdot 2^{\sum_{p=P_b+1}^{P} |X_p|} e^{\frac{\beta P_b (B+\gamma)^2}{4A}} \left(\prod_{p=1}^{P_b} \chi_{R_p(q)}(|\phi_p(x)|) \right) e^{-(1-e^{-\beta\gamma}) \int \boldsymbol{\phi}_{0_B}^*(y) \cdot \boldsymbol{\phi}_{0_B}(y) dy} \\
\leq & 2 \cdot 2^{\sum_{p=P_b+1}^{P} |X_p|} e^{\frac{\beta P_b (B+\gamma)^2}{4A}} e^{-(1-e^{-\beta\gamma}) \int \boldsymbol{\phi}_{0_B}^*(y) \cdot \boldsymbol{\phi}_{0_B}(y) dy},
\end{aligned}
$$

which is independent of q and integrable with respect to the measure

$$
\prod_{p=1}^{P_b} \prod_{x \in X_p} d\phi_p^*(x) d\phi_p(x),
$$

since $e^{-\beta\gamma} < 1$. An application of the dominated convergence theorem completes the proof. □

4.3 The Action

In this section, we analyze the *action*,

$$
F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) := \log \langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle,
$$

which is the logarithm of the inner product that appears in the functional integral representation of Theorem 4.2.1.

Lemma 4.3.1. For each $\epsilon > 0$, there is a function $F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ that is Grassmann-valued, analytic in the boson fields $\alpha_1^*, \ldots, \alpha_{P_b}^*, \phi_1, \ldots, \phi_{P_b}$ in a neighbourhood of the origin and even in the Grassmann fields such that

$$
e^{F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})} = \langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle. \tag{4.34}
$$

F satisfies the differential equation,

$$
\frac{\partial F}{\partial \epsilon} = -\int_{\mathbf{X}^2} d\mathbf{x} d\mathbf{y} \; \alpha^*(\mathbf{x}) k(\mathbf{x}, \mathbf{y}) \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{U \in \mathcal{P}} \frac{\partial^{|U|} F}{\partial \alpha^*(\mathbf{y}_U)}, \tag{4.35}
$$

where the sum is over partitions $\mathcal{P}$ of $\mathbb{I}_{\mathbf{y}}$. The product over $U \in \mathcal{P}$ is ordered from left to right by decreasing highest element.

The initial condition is

$$F(0, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) = \int d\mathbf{x}\ \boldsymbol{\alpha}^*(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x}) \equiv \sum_{p=1}^{P} \int_{X_p} dx\ \alpha_p^*(x)\phi_p(x). \tag{4.36}$$

Proof. Since $\langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle$ is an entire-analytic function of the boson fields $\alpha_1^*, \ldots, \alpha_{P_b}^*$ and $\phi_1, \ldots, \phi_{P_b}$, $F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ is also analytic in any region where $\langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle$ is nonzero. When $\epsilon = 0$,

$$\langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle = \langle \boldsymbol{\alpha} | \boldsymbol{\phi} \rangle = e^{\int d\mathbf{x}\ \boldsymbol{\alpha}^*(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x})} \neq 0.$$

By bounding $\frac{\partial}{\partial \epsilon} \langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle$, it is easy to explicitly find a small neighbourhood around the origin in which $\langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle$ is nonzero.

In order to obtain the differential equation (4.35), we differentiate both sides of the equation $e^{F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})} = \langle \boldsymbol{\alpha} | e^{\epsilon K} | \boldsymbol{\phi} \rangle$ with respect to ϵ.

$$\begin{aligned} e^{F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})} \frac{\partial F}{\partial \epsilon}(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) &= -\langle \boldsymbol{\alpha} | K e^{-\epsilon K} | \boldsymbol{\phi} \rangle \\ &= -\langle \boldsymbol{\alpha} | \int_{\mathbf{X}^2} d\mathbf{x} d\mathbf{y}\ a^\dagger(\mathbf{x}) k(\mathbf{x}, \mathbf{y}) a(\mathbf{y})\ e^{-\epsilon K} | \boldsymbol{\phi} \rangle \\ &= -\int_{\mathbf{X}^2} d\mathbf{x} d\mathbf{y}\ \alpha^*(\mathbf{x}) k(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial \alpha^*(\mathbf{y})} \langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle \\ &= -\int_{\mathbf{X}^2} d\mathbf{x} d\mathbf{y}\ \alpha^*(\mathbf{x}) k(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial \alpha^*(\mathbf{y})} e^{F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})}. \end{aligned}$$

We now claim that

$$\frac{\partial}{\partial \alpha^*(\mathbf{y})} e^F = \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{U \in \mathcal{P}} \frac{\partial^{|U|} F}{\partial \alpha^*(\mathbf{y}_U)} e^F,$$

where the sum is over partitions $\mathcal{P}$ of $\mathbb{I}_{\mathbf{y}}$ and the product is ordered by decreasing largest element in $U \in \mathcal{P}$. The claim is trivial in the case $|\mathbf{n}(\mathbf{y})| = 1$. Suppose that it also holds for $|\mathbf{n}(\mathbf{y})| = n - 1$ and consider any $\mathbf{y} \in X_1^{n_1} \times \cdots \times X_P^{n_P}$, with $\sum_p n_p = n$. Without loss of generality, assume that $n_P \geq 1$. Let $\mathbf{y}'$ be the projection of $\mathbf{y}$ onto $X_1^{n_1} \times \cdots \times X_P^{n_P - 1}$. (That is, $\mathbf{y}'$

is obtained by removing the last particle coordinate from $\mathbf{y}$.) Then,

$$\begin{aligned}
\frac{\partial}{\partial\alpha^*(\mathbf{y})}e^F &= \frac{\partial}{\partial\alpha_P^*(y_{(P,n_P)})}\frac{\partial}{\partial\alpha^*(\mathbf{y}')}e^F \\
&= \frac{\partial}{\partial\alpha_P^*(y_{(P,n_P)})}\sum_{\mathcal{P}'}(-1)^{\mathcal{P}'}\prod_{U\in\mathcal{P}'}\frac{\partial^{|U|}F}{\partial\alpha^*(\mathbf{y}'_U)}e^F \\
&= \sum_{\mathcal{P}'}(-1)^{\mathcal{P}'}\frac{\partial}{\partial\alpha_P^*(y_{(P,n_P)})}\prod_{j=1}^{|\mathcal{P}'|}\frac{\partial^{|U_j|}F}{\partial\alpha^*(\mathbf{y}'_{U_j})}e^F
\end{aligned}$$

where the sum is over partitions $\mathcal{P}'$ of $\mathbb{I}_{\mathbf{y}'}$. In the third line, we've enumerated the sets U according to the order of their greatest element. In particular, U_1 contains the greatest element of all the sets $U \in \mathcal{P}'$, which is $(P, n_P - 1)$ if $n_P > 1$, or $(P-1, n_{P-1})$ otherwise. The product is ordered by increasing j, which corresponds to the same order in the line above. We now apply the product rule, taking care to keep track of any minus signs that arise from the anticommutation relations which play a role when $\alpha_P^*(y_{(P,n_P)})$ is a Grassmann variable. Recall from Equations (4.7) and (4.8),

$$n_p^{(f)}(U) := |\{m \in \mathbb{N} : (p,m) \in U\}|\, \delta_{[P_b+1,P]}(p).$$

Then

$$\begin{aligned}
\frac{\partial}{\partial\alpha^*(\mathbf{y})}e^F &= \sum_{\mathcal{P}'}(-1)^{\mathcal{P}'}\Bigg[\sum_{k=1}^{|\mathcal{P}'|}(-1)^{\sum_{j=1}^{k-1}n_p^{(f)}(U_j)}\times \\
&\qquad \left(\prod_{j=1}^{k-1}\frac{\partial^{|U_j|}F}{\partial\alpha^*(\mathbf{y}_{U_j})}\right)\frac{\partial^{|U_k|+1}F}{\partial\alpha^*(\mathbf{y}_{\{(P,n_P)\}\cup U_k})}\left(\prod_{j=k+1}^{|\mathcal{P}'|}\frac{\partial^{|U_j|}F}{\partial\alpha^*(\mathbf{y}_{U_j})}\right) \\
&\qquad + (-1)^{n_P(\mathbf{y}')}\left(\prod_{j=1}^{|\mathcal{P}'|}\frac{\partial^{|U_j|}F}{\partial\alpha^*(\mathbf{y}'_{U_j})}\right)\frac{\partial F}{\partial\alpha_P^*(y_{(P,n_P)})}\Bigg]e^F \\
&= \sum_{\mathcal{P}'}(-1)^{\mathcal{P}'}\sum_{k=1}^{|\mathcal{P}'|+1}(-1)^{\sum_{p=1}^{P}\sum_{j=1}^{k-1}n_p^{(f)}(U_j)n_p^{(f)}(U_k)}\times \\
&\qquad \frac{\partial^{|U_k|+1}F}{\partial\alpha^*(\mathbf{y}_{\{(P,n_P)\}\cup U_k})}\left(\prod_{j=0}^{k-1}\frac{\partial^{|U_j|}F}{\partial\alpha^*(\mathbf{y}_{U_j})}\right)\left(\prod_{j=k+1}^{|\mathcal{P}'|}\frac{\partial^{|U_j|}F}{\partial\alpha^*(\mathbf{y}_{U_j})}\right)e^F \\
&= \sum_{\mathcal{P}}(-1)^{\mathcal{P}}\prod_{U\in\mathcal{P}}\frac{\partial^{|U|}F}{\partial\alpha^*(\mathbf{y}_U)}e^F,
\end{aligned}$$

where $U_{|\mathcal{P}'|+1}$ (in the second line above) is the empty set. In the last line, the sum is over partitions $\mathcal{P}$ of $\mathbb{I}_{\mathbf{y}}$. Each $\mathcal{P}$ is equal to

$$\{U_1, \ldots, U_k \cup \{(P, n_P)\}, \ldots, U_{|\mathcal{P}'|}\}$$

for some partition $\mathcal{P}' = \{U_1, \cdots, U_{|\mathcal{P}'|}\}$ of $\mathbb{I}_{\mathbf{y}'}$ and some k. The sign of such $\mathcal{P}$, $(-1)^{\mathcal{P}}$, is precisely equal to $(-1)^{\sum_{p=1}^{P}\sum_{j=1}^{k-1} n_p^{(f)}(U_j) n_p^{(f)}(U_k)}(-1)^{\mathcal{P}'}$. This completes the proof. □

Lemma 4.3.2. The differential equation (4.35) satisfied by F can also be expressed as

$$\frac{\partial F}{\partial \epsilon} = -\sum_{\ell \geq 1} \sum_{\substack{\mathbf{m}_1, \ldots, \mathbf{m}_\ell \neq 0 \\ \mathbf{m}_j \in \mathbb{N}_0^P}} \frac{(\sum_{j=1}^{\ell} \mathbf{m}_j)!}{\ell!} \int_{\mathbf{X}} d\mathbf{x} \prod_{j=1}^{\ell} \left(\int_{\mathbf{X}^{\mathbf{m}_j}} d\mathbf{y}_j \right)$$
$$\alpha^*(\mathbf{x}) k(\mathbf{x}, \mathbf{y}_1 \circ \cdots \circ \mathbf{y}_\ell) \prod_{j=1}^{\ell} \frac{1}{\mathbf{m}_j!} \frac{\partial^{|\mathbf{m}_j|} F}{\partial \alpha^*(\mathbf{y}_j)},$$

where for any $\mathbf{m} = (m_1, \ldots, m_P) \in \mathbb{N}_0^P$,

$$\mathbf{m}! := \prod_{p=1}^{P} m_p!.$$

Recall also that $\mathbf{y}_1 \circ \cdots \circ \mathbf{y}_\ell$ is the vector formed by concatenating $\mathbf{y}_1, \ldots, \mathbf{y}_\ell$, as defined in Equations (4.4) and (4.5).

Proof. Recall that each partition $\mathcal{P}$ of $\mathbb{I}_{\mathbf{y}}$ is a set of disjoint subsets U of $\mathbb{I}_{\mathbf{y}}$ such that $\sqcup_{U \in \mathcal{P}} U = \mathbb{I}_{\mathbf{y}}$. If we arbitrarily order the elements of $\mathcal{P}$ as $U_1, U_2, \cdots, U_{|\mathcal{P}|}$, we will denote the vector $(U_1, U_2, \cdots, U_{|\mathcal{P}|})$ by $\vec{\mathcal{P}}$ and call it an *ordered partition*. For a given partition $\mathcal{P}$, there are $|\mathcal{P}|!$ corresponding ordered partitions $\vec{\mathcal{P}}$. We will also let $|\vec{\mathcal{P}}| := |\mathcal{P}|$. The sign $(-1)^{\vec{\mathcal{P}}}$ of the ordered partition $\vec{\mathcal{P}}$ is the sign of the permutation required to bring the coordinates of each fermionic species from the concatenated list $\vec{U_1} \circ \cdots \circ \vec{U_{|\mathcal{P}|}}$ back to their order in $\mathbb{I}_{\mathbf{x}}$, where the elements of each $\vec{U_j}$ is arranged in increasing order.

If we sum over ordered partitions in Equation (4.35) rather than partitions, we obtain,

$$\frac{\partial F}{\partial \epsilon} = -\int_{\mathbf{X}^2} d\mathbf{x}d\mathbf{y}\ \alpha^*(\mathbf{x})k(\mathbf{x},\mathbf{y}) \sum_{\ell=1}^{|\mathbf{n}(\mathbf{y})|} \frac{1}{\ell!} \sum_{\substack{\vec{\mathcal{P}} \\ |\vec{\mathcal{P}}|=\ell}} (-1)^{\vec{\mathcal{P}}} \prod_{j=1}^{\ell} \frac{\partial^{|U_j|} F}{\partial \alpha^*(\mathbf{y}_{U_j})}.$$

Since the kernel is symmetric in the boson coordinates and antisymmetric in the fermion coordinates, rearranging the arguments of $\mathbf{y}$ to the order $\mathbf{y}_{U_1} \circ \cdots \circ \mathbf{y}_{U_\ell}$ results in a sign of $(-1)^{\vec{\mathcal{P}}}$. Hence,

$$\begin{aligned}
\frac{\partial F}{\partial \epsilon} &= -\sum_{\ell \geq 1} \frac{1}{\ell!} \sum_{|\vec{\mathcal{P}}|=\ell} \int_{\mathbf{X}^2} d\mathbf{x}d\mathbf{y}\ \alpha^*(\mathbf{x})k(\mathbf{x},\mathbf{y}_{U_1} \circ \cdots \circ \mathbf{y}_{U_\ell}) \prod_{j=1}^{\ell} \frac{\partial^{|U_j|} F}{\partial \alpha^*(\mathbf{y}_{U_j})} \\
&= -\sum_{\ell \geq 1} \frac{1}{\ell!} \sum_{\mathbf{m}_1,\ldots,\mathbf{m}_\ell \neq 0} \frac{(\mathbf{m}_1 + \cdots + \mathbf{m}_\ell)!}{\mathbf{m}_1! \cdots \mathbf{m}_\ell!} \int_{\mathbf{X}} d\mathbf{x} \int_{X^{\mathbf{m}_1}} d\mathbf{y}_1 \cdots \int_{X^{\mathbf{m}_\ell}} d\mathbf{y}_\ell \\
&\qquad \alpha^*(\mathbf{x})k(\mathbf{x},\mathbf{y}_1 \circ \cdots \circ \mathbf{y}_\ell) \prod_{j=1}^{\ell} \frac{\partial^{|\mathbf{m}_j|} F}{\partial \alpha^*(\mathbf{y}_j)}.
\end{aligned}$$

In the second line, we essentially replaced each set U_j with its corresponding vector $\mathbf{m}_j := \mathbf{n}(U_j) := (n_1(U_j), \ldots, n_P(U_j))$ of particle numbers. Each $\mathbf{y}_j$ is an independent variable with $\mathbf{m}_j$ coordinates. The number of ordered partitions corresponding to each sequence $\{\mathbf{m}_j\}$ is equal to the multinomial coefficient $\frac{(\mathbf{m}_1+\cdots+\mathbf{m}_\ell)!}{\mathbf{m}_1!\cdots\mathbf{m}_\ell!}$. This completes the proof. □

In the lemma below, for each $p \in \{1, \ldots, P\}$, let $\mathbf{e}_p$ be the unit vector,

$$\mathbf{e}_p := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}_0^P, \tag{4.37}$$

where the 1 lies in the p^{th} argument.

Theorem 4.3.3. *F has an expansion*

$$F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} \int_{X^{\mathbf{n}}} d\mathbf{x} \int_{X^{\mathbf{n}}} d\mathbf{y}\ \alpha^*(\mathbf{x}) F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y}) \phi(\mathbf{y}), \tag{4.38}$$

where the coefficients $F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y}) \in \mathbb{C}$ are given by the following recursion relations. For each $p \in \{1, \dots, P\}$,

$$F_{\mathbf{e}_p}(\epsilon, x, y) = (e^{-\epsilon h_p})(x, y), \tag{4.39}$$

where $h_p = P^{(\mathbf{e}_p)} K P^{(\mathbf{e}_p)}$ is the single-particle operator for species p (which would include both the kinetic energy and the chemical potential). In other words, the kernel of h_p satisfies

$$h_p(x, y) = k(x\mathbf{e}_p, y\mathbf{e}_p).$$

For all $\mathbf{n}$ with $|\mathbf{n}| > 1$ and $\mathbf{x}, \mathbf{x}' \in X^{\mathbf{n}}$, let

$$H_{\mathbf{n}}(\mathbf{x}, \mathbf{x}') := S^{(\mathbf{n})}_{\mathbf{x}} \sum_{0 \neq \mathbf{m} \leq \mathbf{n}} \frac{\mathbf{n}!}{(\mathbf{n} - \mathbf{m})!} k(\mathbf{x}_{[\leq \mathbf{m}]}, \mathbf{x}'_{[\leq \mathbf{m}]}) \delta_{\mathbf{x}_{[>\mathbf{m}]}}(\mathbf{x}'_{[>\mathbf{m}]}). \tag{4.40}$$

Given $\ell \in \mathbb{N}$, let $\underline{\mathbf{n}} := (\mathbf{n}_1, \dots, \mathbf{n}_\ell)$ where each $\mathbf{n}_j \in \mathbb{N}_0^P$ and $\sum_{j=1}^{\ell} \mathbf{n}_j = \mathbf{n}$. Then for all $\mathbf{x} \in X^{\mathbf{n}}$ and $\mathbf{x}'_j \in X^{\mathbf{n}_j}$ where $j \in \{1, \dots, \ell\}$, let

$$\begin{aligned} K_{\underline{\mathbf{n}}}(\mathbf{x}, \mathbf{x}'_1, \dots, \mathbf{x}'_\ell) =& S^{(\mathbf{n})}_{\mathbf{x}} \sum_{\substack{(\mathbf{m}_j)_{j=1}^{\ell} \\ 0 \neq \mathbf{m}_j \leq \mathbf{n}_j}} \frac{\mathbf{m}!}{\ell!} \frac{\underline{\mathbf{n}}!}{\underline{\mathbf{m}}!(\underline{\mathbf{n}} - \underline{\mathbf{m}})!} (-1)^{\sum\limits_{p=P_b+1}^{P} \sum\limits_{j=2}^{\ell} (n_{jp} - m_{jp}) \sum\limits_{k=1}^{j-1} n_{kp}} \\ & k(\mathbf{x}_{[\leq \mathbf{m}]}, \mathbf{x}'_{1[\leq \mathbf{m}_1]} \circ \cdots \circ \mathbf{x}'_{\ell[\leq \mathbf{m}_\ell]}) \delta_{\tilde{\mathbf{x}}_1}(\mathbf{x}'_{1[>\mathbf{m}_1]}) \dots \delta_{\tilde{\mathbf{x}}_\ell}(\mathbf{x}'_{\ell[>\mathbf{m}_\ell]}) \end{aligned} \tag{4.41}$$

where $\mathbf{m} := \sum_{j=1}^{\ell} \mathbf{m}_j$, $\underline{\mathbf{n}}! := \prod_{j=1}^{\ell} \mathbf{n}_j!$ and $\underline{\mathbf{m}}! := \prod_{j=1}^{\ell} \mathbf{m}_j!$. Each $\tilde{\mathbf{x}}_j \in X^{\mathbf{n}_j - \mathbf{m}_j}$ is defined by $\mathbf{x}_{[>\mathbf{m}]} = \tilde{\mathbf{x}}_1 \circ \cdots \circ \tilde{\mathbf{x}}_\ell$.

Then, for all $\mathbf{n}$ with $|\mathbf{n}| > 1$,

$$\begin{aligned} F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y}) = - \int_0^{\epsilon} d\tau \int_{X^{\mathbf{n}}} d\mathbf{x}' \; e^{-(\epsilon - \tau) H_{\mathbf{n}}}(\mathbf{x}, \mathbf{x}') \sum_{\ell > 1} \sum_{\substack{\underline{\mathbf{n}} = (\mathbf{n}_j)_{j=1}^{\ell} \\ \sum \mathbf{n}_j = \mathbf{n} \\ \mathbf{n}_j \neq 0}} \\ \int_{X^{\mathbf{n}_\ell}} d\mathbf{x}''_\ell \cdots \int_{X^{\mathbf{n}_1}} d\mathbf{x}''_1 \; K_{\underline{\mathbf{n}}}(\mathbf{x}', \mathbf{x}''_1, \dots, \mathbf{x}''_\ell) \prod_{j=1}^{\ell} F_{\mathbf{n}_j}(\tau, \mathbf{x}''_j, \mathbf{y}_j), \end{aligned} \tag{4.42}$$

where each $\mathbf{y}_j \in X^{\mathbf{n}_j}$ is defined by

$$\mathbf{y} = \mathbf{y}_1 \circ \mathbf{y}_2 \circ \cdots \circ \mathbf{y}_\ell.$$

Proof. Since $F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ is analytic in $\boldsymbol{\alpha}^*$, we can make the expansion,

$$F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) = \sum_{\mathbf{n} \in \mathbb{N}_0^P} \int_{X^{\mathbf{n}}} d\mathbf{x} \ \alpha^*(\mathbf{x}) F_{\mathbf{n}}(\epsilon, \mathbf{x}, \boldsymbol{\phi}),$$

so that

$$F_{\mathbf{n}}(\epsilon, \mathbf{x}, \boldsymbol{\phi}) = \frac{1}{\mathbf{n}!} \frac{\partial}{\partial \alpha^*(\mathbf{x})} F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) \bigg|_{\boldsymbol{\alpha}^*=0}. \tag{4.43}$$

For a Grassmann variable α_p^* with $p > P_b$, the evaluation at $\alpha_p^* = 0$ is equal to the term of zeroth order in α_p^*.

We now find a differential equation for each coefficient $F_{\mathbf{n}}$ by differentiating the above equation (4.43) with respect to ϵ and applying Lemma 4.3.2 on the right-hand side. Throughout the calculation below, let $\mathbf{m} := \sum_{j=1}^{\ell} \mathbf{m}_j$.

$$\begin{aligned}
\frac{\partial F_{\mathbf{n}}}{\partial \epsilon}(\epsilon, \mathbf{x}, \boldsymbol{\phi}) = & -\frac{1}{\mathbf{n}!} \frac{\partial}{\partial \alpha^*(\mathbf{x})} \sum_{\ell \geq 1} \sum_{\substack{\mathbf{m}_1, \ldots, \mathbf{m}_\ell \\ \neq 0}} \frac{\mathbf{m}!}{\ell!} \int_{\mathbf{X}} d\mathbf{z} \prod_{j=1}^{\ell} \int_{\mathbf{X}^{\mathbf{m}_j}} d\mathbf{y}_j \\
& \qquad \alpha^*(\mathbf{z}) k(\mathbf{z}, \mathbf{y}_1 \circ \cdots \circ \mathbf{y}_\ell) \prod_{j=1}^{\ell} \frac{1}{\mathbf{m}_j!} \frac{\partial^{|\mathbf{m}_j|} F}{\partial \alpha^*(\mathbf{y}_j)} \bigg|_{\boldsymbol{\alpha}^*=0} \\
= & -\frac{1}{\mathbf{n}!} S_{\mathbf{x}}^{(\mathbf{n})} \sum_{\ell \geq 1} \sum_{\substack{\mathbf{m}_1, \ldots, \mathbf{m}_\ell \\ \neq 0}} \frac{\mathbf{m}!}{\ell!} \prod_{j=1}^{\ell} \int_{\mathbf{X}^{\mathbf{m}_j}} d\mathbf{y}_j \ \frac{\mathbf{n}!}{\mathbf{m}!(\mathbf{n}-\mathbf{m})!} \mathbf{m}! \\
& \qquad k(\mathbf{x}_{[\leq \mathbf{m}]}, \mathbf{y}_1 \circ \cdots \circ \mathbf{y}_\ell) \frac{\partial}{\partial \alpha^*(\mathbf{x}_{[>\mathbf{m}]})} \prod_{j=1}^{\ell} \frac{1}{\mathbf{m}_j!} \frac{\partial^{|\mathbf{m}_j|} F}{\partial \alpha^*(\mathbf{y}_j)} \bigg|_{\boldsymbol{\alpha}^*=0} \\
= & - S_{\mathbf{x}}^{(\mathbf{n})} \sum_{\ell \geq 1} \sum_{\substack{\mathbf{m}_1, \ldots, \mathbf{m}_\ell \\ \neq 0}} \frac{\mathbf{m}!}{\ell!(\mathbf{n}-\mathbf{m})!} \prod_{j=1}^{\ell} \int_{\mathbf{X}^{\mathbf{m}_j}} d\mathbf{y}_j \ k(\mathbf{x}_{[\leq \mathbf{m}]}, \mathbf{y}_1 \circ \cdots \circ \mathbf{y}_\ell) \times \\
& \sum_{\substack{\mathbf{w}_1, \ldots, \mathbf{w}_\ell \geq 0 \\ \sum \mathbf{w}_j = \mathbf{n}-\mathbf{m}}} (\mathbf{n}-\mathbf{m})! \prod_{j=1}^{\ell} \frac{(-1)^{\gamma_j}}{\mathbf{w}_j! \mathbf{m}_j!} \frac{\partial^{|\mathbf{m}_j|+|\mathbf{w}_j|} F}{\partial \alpha^*(\tilde{\mathbf{x}}_j) \partial \alpha^*(\mathbf{y}_j)} \bigg|_{\boldsymbol{\alpha}^*=0},
\end{aligned}$$

where

$$\mathbf{x}_{[>\mathbf{m}]} = \tilde{\mathbf{x}}_1 \circ \cdots \circ \tilde{\mathbf{x}}_\ell \text{ with each } \tilde{\mathbf{x}}_j \in X^{\mathbf{w}_j},$$

and

$$\begin{aligned}\gamma_j &:= \sum_{p=P_b+1}^{P} n_p(\tilde{\mathbf{x}}_j) \sum_{k=1}^{j-1} (n_p(\tilde{\mathbf{x}}_k) + n_p(\mathbf{y}_k)) \\ &= \sum_{p=P_b+1}^{P} w_{j_p} \sum_{k=1}^{j-1} \Big(w_{k_p} + m_{k_p}\Big).\end{aligned}$$

Here, $(-1)^{\gamma_j}$ is a sign that arises from the product rule for Grassmann derivatives. Noting that

$$\frac{\partial^{|\mathbf{m}_j|+|\mathbf{w}_j|}F}{\partial\alpha^*(\tilde{\mathbf{x}}_j)\partial\alpha^*(\mathbf{y}_j)} = \frac{\partial^{|\mathbf{m}_j|+|\mathbf{w}_j|}F}{\partial\alpha^*(\mathbf{y}_j \circ \tilde{\mathbf{x}}_j)},$$

we can evaluate at $\boldsymbol{\alpha}^* = 0$ and use the definition of $F_\mathbf{n}$ to conclude that

$$\begin{aligned}\frac{\partial F_\mathbf{n}}{\partial \epsilon}(\epsilon, \mathbf{x}, \boldsymbol{\phi}) = -S_\mathbf{x}^{(\mathbf{n})} \sum_{\ell \geq 1} \sum_{\substack{\mathbf{m}_1,\ldots,\mathbf{m}_\ell \\ \neq 0}} \frac{\mathbf{m}!}{\ell!} \prod_{j=1}^{\ell} \int_{\mathbf{X}^{\mathbf{m}_j}} d\mathbf{y}_j \, k(\mathbf{x}_{[\leq \mathbf{m}]}, \mathbf{y}_1 \circ \cdots \circ \mathbf{y}_\ell) \\ \sum_{\substack{\mathbf{w}_1,\ldots,\mathbf{w}_\ell \geq 0 \\ \sum \mathbf{w}_j = \mathbf{n}-\mathbf{m}}} \prod_{j=1}^{\ell} \frac{(\mathbf{m}_j + \mathbf{w}_j)!}{\mathbf{w}_j! \mathbf{m}_j!} (-1)^{\gamma_j} F_{\mathbf{m}_j + \mathbf{w}_j}(\epsilon, \mathbf{y}_j \circ \tilde{\mathbf{x}}_j, \boldsymbol{\phi}).\end{aligned}$$

For each $\ell \in \mathbb{N}$ and $j \in \{1, \ldots, \ell\}$, set

$$\mathbf{n}_j = \mathbf{m}_j + \mathbf{w}_j$$

and

$$\begin{aligned}K_{\underline{\mathbf{n}}}(\mathbf{x}, \mathbf{x}'_1, \ldots, \mathbf{x}'_\ell) = S_\mathbf{x}^{(\mathbf{n})} \sum_{\substack{(\mathbf{m}_j)_{j=1}^{\ell} \\ 0 \neq \mathbf{m}_j \leq \mathbf{n}_j}} \frac{\mathbf{m}!}{\ell!} \frac{\underline{\mathbf{n}}!}{\underline{\mathbf{m}}!(\underline{\mathbf{n}} - \underline{\mathbf{m}})!} (-1)^{\sum_{p=P_b+1}^{P} \sum_{j=1}^{\ell} (n_{j_p} - m_{j_p}) \sum_{k=1}^{j-1} n_{k_p}} \\ k(\mathbf{x}_{[\leq \mathbf{m}]}, \mathbf{x}'_{1[\leq \mathbf{m}_1]} \circ \cdots \circ \mathbf{x}'_{\ell[\leq \mathbf{m}_\ell]}) \delta_{\tilde{\mathbf{x}}_1}(\mathbf{x}'_{1[>\mathbf{m}_1]}) \ldots \delta_{\tilde{\mathbf{x}}_\ell}(\mathbf{x}'_{\ell[>\mathbf{m}_\ell]}).\end{aligned}$$

where $\underline{\mathbf{n}}! := \prod_{j=1}^{\ell} \mathbf{n}_j!$. Then,

$$\frac{\partial F_{\mathbf{n}}}{\partial \epsilon}(\epsilon, \mathbf{x}, \boldsymbol{\phi}) = -\sum_{\ell \geq 1} \sum_{\substack{\underline{\mathbf{n}} = (\mathbf{n}_j)_{j=1}^{\ell} \\ \sum \mathbf{n}_j = \mathbf{n} \\ \mathbf{n}_j \neq 0}} \int_{X^{\mathbf{n}_\ell}} d\mathbf{x}'_\ell \cdots \int_{X^{\mathbf{n}_1}} d\mathbf{x}'_1 \; K_{\underline{\mathbf{n}}}(\mathbf{x}, \mathbf{x}'_1, \ldots, \mathbf{x}'_\ell) \times \prod_{j=1}^{\ell} F_{\mathbf{n}_j}(\epsilon, \mathbf{x}'_j, \boldsymbol{\phi}).$$

Singling out the term involving $F_{\mathbf{n}}$,

$$\frac{\partial F_{\mathbf{n}}}{\partial \epsilon}(\epsilon, \mathbf{x}, \boldsymbol{\phi}) = -\int_{X^{\mathbf{n}}} d\mathbf{x}' \, H_{\mathbf{n}}(\mathbf{x}, \mathbf{x}') F_{\mathbf{n}}(\epsilon, \mathbf{x}', \boldsymbol{\phi}) - \sum_{\ell > 1} \sum_{\substack{\underline{\mathbf{n}} = (\mathbf{n}_j)_{j=1}^{\ell} \\ \sum \mathbf{n}_j = \mathbf{n} \\ \mathbf{n}_j \neq 0}} \int_{X^{\mathbf{n}_\ell}} d\mathbf{x}'_\ell \cdots \int_{X^{\mathbf{n}_1}} d\mathbf{x}'_1 \; K_{\underline{\mathbf{n}}}(\mathbf{x}, \mathbf{x}'_1, \ldots, \mathbf{x}'_\ell) \prod_{j=1}^{\ell} F_{\mathbf{n}_j}(\epsilon, \mathbf{x}'_j, \boldsymbol{\phi}),$$

where $H_{\mathbf{n}}$ is equal to $K_{\underline{\mathbf{n}}}$ in the case where $\underline{\mathbf{n}}$ is a sequence with only one element, $\underline{\mathbf{n}} = (\mathbf{n})$. For $\mathbf{x}, \mathbf{x}' \in X^{\mathbf{n}}$,

$$H_{\mathbf{n}}(\mathbf{x}, \mathbf{x}') = S_{\mathbf{x}} \sum_{0 \neq \mathbf{m} \leq \mathbf{n}} \frac{\mathbf{n}!}{(\mathbf{n} - \mathbf{m})!} k(\mathbf{x}_{[\leq \mathbf{m}]}, \mathbf{x}'_{[\leq \mathbf{m}]}) \delta_{\mathbf{x}_{[>\mathbf{m}]}}(\mathbf{x}'_{[>\mathbf{m}]}).$$

In the case $\mathbf{n} = \mathbf{e}_p$, the differential equation can be simplified to

$$\frac{\partial F_{\mathbf{e}_p}}{\partial \epsilon}(\epsilon, x, \boldsymbol{\phi}) = -\int_{X_p} dx' \; h_p(x, x') F_{\mathbf{n}}(x')$$

where $h_p(x, x') = k(x\mathbf{e}_p, x'\mathbf{e}_p)$.

The initial conditions are

$$F_{\mathbf{n}}(0, \mathbf{x}, \boldsymbol{\phi}) = \begin{cases} \boldsymbol{\phi}(\mathbf{x}) & \text{if } \mathbf{n} = \mathbf{e}_p \text{ for some } p \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

Using the integrating factor $e^{\epsilon H_{\mathbf{n}}}$, we solve the differential equation for $F_{\mathbf{n}}$ to obtain the recursive integral formula,

$$F_{\mathbf{n}}(\epsilon,\mathbf{x},\boldsymbol{\phi}) = -\int_0^\epsilon d\tau \int_{X^{\mathbf{n}}} d\mathbf{x}' e^{-(\epsilon-\tau)H_{\mathbf{n}}}(\mathbf{x},\mathbf{x}') \sum_{\ell>1} \sum_{\substack{\underline{\mathbf{n}}=(\mathbf{n}_j)_1^\ell \\ \sum \mathbf{n}_j=\mathbf{n} \\ \mathbf{n}_j\neq 0}} \int_{X^{\mathbf{n}_\ell}} d\mathbf{x}''_\ell \cdots \int_{X^{\mathbf{n}_1}} d\mathbf{x}''_1$$

$$K_{\underline{\mathbf{n}}}(\mathbf{x}',\mathbf{x}''_1,\ldots,\mathbf{x}''_\ell) \prod_{j=1}^{\ell} F_{\mathbf{n}_j}(\tau,\mathbf{x}''_j,\boldsymbol{\phi}), \tag{4.44}$$

for all $|\mathbf{n}(\mathbf{x})| > 1$. For $\mathbf{n} = \mathbf{e}_p$,

$$F_{\mathbf{e}_p}(\epsilon, x, \phi) = (e^{-\epsilon h_p}\phi)(x).$$

By induction, $F_{\mathbf{n}}(\epsilon,\mathbf{x},\boldsymbol{\phi})$ must also be of degree $\mathbf{n}$ in $\boldsymbol{\phi}$. Let

$$F_{\mathbf{n}}(\epsilon,\mathbf{x},\boldsymbol{\phi}) = \int_{X^{\mathbf{n}}} d\mathbf{y}\; F_{\mathbf{n}}(\epsilon,\mathbf{x},\mathbf{y})\phi(\mathbf{y}). \tag{4.45}$$

Then

$$\prod_{j=1}^{\ell} F_{\mathbf{n}_j}(\tau,\mathbf{x}''_j,\boldsymbol{\phi}) = \prod_{j=1}^{\ell} \int_{X^{\mathbf{n}_j}} d\mathbf{y}_j\; F_{\mathbf{n}_j}(\tau,\mathbf{x}''_j,\mathbf{y}_j)\phi(\mathbf{y}_j).$$

The recursion relations for $F_{\mathbf{n}}(\epsilon,\mathbf{x},\mathbf{y})$ become

$$F_{\mathbf{e}_p}(\epsilon, x, y) = (e^{-\epsilon h_p})(x, y)$$

and, for $|\mathbf{n}| > 1$,

$$F_{\mathbf{n}}(\epsilon,\mathbf{x},\mathbf{y}) = -\int_0^\epsilon d\tau \int_{X^{\mathbf{n}}} d\mathbf{x}' e^{-(\epsilon-\tau)H_{\mathbf{n}}}(\mathbf{x},\mathbf{x}') \sum_{\ell>1} \sum_{\substack{\underline{\mathbf{n}}=(\mathbf{n}_j)_{j=1}^\ell \\ \sum \mathbf{n}_j=\mathbf{n} \\ \mathbf{n}_j\neq 0}}$$

$$\int_{X^{\mathbf{n}_\ell}} d\mathbf{x}''_\ell \cdots \int_{X^{\mathbf{n}_1}} d\mathbf{x}''_1\, K_{\underline{\mathbf{n}}}(\mathbf{x}',\mathbf{x}''_1,\ldots,\mathbf{x}''_\ell) \prod_{j=1}^{\ell} F_{\mathbf{n}_j}(\tau,\mathbf{x}''_j,\mathbf{y}_j).$$

where, for each ℓ, $\mathbf{y} = \mathbf{y}_1 \circ \cdots \circ \mathbf{y}_\ell$ with each $\mathbf{y}_j \in \mathbf{X}^{\mathbf{n}_j}$. This can be seen by inserting (4.45) into the recursion relation (4.44) and letting $\mathbf{y} := \mathbf{y}_1 \circ \cdots \circ \mathbf{y}_\ell$.

That is,

$$F_{\mathbf{n}}(\epsilon, \mathbf{x}, \boldsymbol{\phi}) = -\int_0^{\epsilon} d\tau \int_{X^{\mathbf{n}}} d\mathbf{x}' e^{-(\epsilon-\tau)H_{\mathbf{n}}}(\mathbf{x}, \mathbf{x}') \sum_{\ell>1} \sum_{\substack{\underline{\mathbf{n}}=(\mathbf{n}_j)_1^{\ell} \\ \sum \mathbf{n}_j = \mathbf{n} \\ \mathbf{n}_j \neq 0}} \int_{X^{\mathbf{n}_\ell}} d\mathbf{x}''_\ell \cdots \int_{X^{\mathbf{n}_1}} d\mathbf{x}''_1$$

$$K_{\underline{\mathbf{n}}}(\mathbf{x}', \mathbf{x}''_1, \ldots, \mathbf{x}''_\ell) \prod_{j=1}^{\ell} \int_{\mathbf{X}^{\mathbf{n}_j}} d\mathbf{y}_j \; F_{\mathbf{n}_j}(\tau, \mathbf{x}''_j, \mathbf{y}_j) \phi(\mathbf{y}_j)$$

$$= \int_{\mathbf{X}} d\mathbf{y} \Bigg[-\int_0^{\epsilon} d\tau \int_{X^{\mathbf{n}}} d\mathbf{x}' e^{-(\epsilon-\tau)H_{\mathbf{n}}}(\mathbf{x}, \mathbf{x}') \sum_{\ell>1} \sum_{\substack{\underline{\mathbf{n}}=(\mathbf{n}_j)_{j=1}^{\ell} \\ \sum \mathbf{n}_j = \mathbf{n} \\ \mathbf{n}_j \neq 0}}$$

$$\int_{X^{\mathbf{n}_\ell}} d\mathbf{x}''_\ell \cdots \int_{X^{\mathbf{n}_1}} d\mathbf{x}''_1 \;\; K_{\underline{\mathbf{n}}}(\mathbf{x}', \mathbf{x}''_1, \ldots, \mathbf{x}''_\ell) \prod_{j=1}^{\ell} \; F_{\mathbf{n}_j}(\tau, \mathbf{x}''_j, \mathbf{y}_j) \Bigg] \phi(\mathbf{y}).$$

This completes the proof.

□

4.3.1 Bounds on $F_{\mathbf{n}}$ for 2-Body Interactions

We now specialize to the case where K has order $M = 2$. In particular,

$$K = \sum_{p=1}^{P} \int_{X_p^2} dx\, dy\ a_p(x)^\dagger h_p(x,y) a_p(y) \\ + \sum_{p=1}^{P}\sum_{q=1}^{P} \int_{X_p} dx \int_{X_q} dy\ a_p^\dagger(x) a_q^\dagger(y) v_{pq}(x,y) a_p(x) a_q(y), \qquad (4.46)$$

where for all $p, q \in \{1, \ldots, P\}$, $h_p : X_p^2 \to \mathbb{R}$, $v_{pq} : X_p \times X_q \to \mathbb{R}$, and for all $x \in X_p$ and $y \in X_q$,

$$v_{pq}(x,y) = v_{qp}(y,x).$$

In the notation used thus far in this chapter where

$$K = \int_{\mathbf{X}^2} d\mathbf{x}\, d\mathbf{y}\ a^\dagger(\mathbf{x}) k(\mathbf{x},\mathbf{y}) a(\mathbf{y}),$$

the kernel k of K in this case satisfies

$$k(\mathbf{x},\mathbf{y}) = \begin{cases} h_p(x,y) & \text{if } \mathbf{x} = (x)\,\mathbf{e}_p \text{ and } \mathbf{y} = (y)\,\mathbf{e}_p, \\ v_{pq}(x_p, x_q)\delta_{x_p}(y_p)\delta_{x_q}(y_q) & \text{if } p \neq q,\ \mathbf{x} = (x_p)\mathbf{e}_p + (x_q)\mathbf{e}_q \\ & \text{and } \mathbf{y} = (y_p)\mathbf{e}_p + (y_q)\mathbf{e}_q, \\ v_{pp}(x_{p1}, x_{p2})\delta_{x_{p1}}(y_{p1})\delta_{x_{p2}}(y_{p2}) & \text{if } \mathbf{x} = (x_{p1}, x_{p2})\mathbf{e}_p \text{ and} \\ & \mathbf{y} = (y_{p1}, y_{p2})\mathbf{e}_p, \\ 0 & \text{otherwise.} \end{cases}$$

We assume that K is *repulsive* in the sense that

$$\lambda_0(v) := \inf \left\{ \sum_{p,q=1}^{P} \int_{X_p} dx \int_{X_q} dy\ \rho_p(x) v_{pq}(x,y) \rho_q(y) \right. \\ \left. \middle| \sum_{p=1}^{P} \int_{X_p} dx\ \rho_p(x)^2 = 1, (\forall p)(\forall x \in X_p) \rho_q(x) \geq 0 \right\} \qquad (4.47)$$

is (strictly) positive for all $p \in \{1, \ldots, P\}$.

The function $H_{\mathbf{n}}(\mathbf{x}, \mathbf{x}')$ defined in Equation (4.40) as

$$H_{\mathbf{n}}(\mathbf{x}, \mathbf{x}') := S_{\mathbf{x}}^{(\mathbf{n})} \sum_{0 \neq \mathbf{m} \leq \mathbf{n}} \frac{\mathbf{n}!}{(\mathbf{n} - \mathbf{m})!} k(\mathbf{x}_{[\leq \mathbf{m}]}, \mathbf{x}'_{[\leq \mathbf{m}]}) \delta_{\mathbf{x}_{[>\mathbf{m}]}}(\mathbf{x}'_{[>\mathbf{m}]})$$

may be simplified in this case to

$$\begin{aligned} H_{\mathbf{n}}(\mathbf{x}, \mathbf{x}') = &\sum_{p=1}^{P} \sum_{j=1}^{n_p} h_p(x_{p_j}, x'_{p_j}) \prod_{(q,i) \neq (p,j)} \delta_{x_{q_i}}(x'_{q_i}) \\ &+ \sum_{p=1}^{P} \sum_{i=1}^{n_p} \left(\sum_{q=1}^{P} \sum_{j=1}^{n_q} v_{pq}(x_{p_i}, x_{q_j}) - v_{pp}(x_{p_i}, x_{p_i}) \right) \delta_{\mathbf{x}}(\mathbf{x}'). \end{aligned}$$

We define

$$\mathrm{h}_{\mathbf{n}}(\mathbf{x}, \mathbf{x}') := \sum_{p=1}^{P} \sum_{j=1}^{n_p} h_p(x_{p_j}, x'_{p_j}) \prod_{(q,m) \neq (p,j)} \delta_{x_{q_m}}(x'_{q_m})$$

and

$$V_{\mathbf{n}}(\mathbf{x}, \mathbf{x}') := \sum_{p=1}^{P} \sum_{i=1}^{n_p} \left(\sum_{q=1}^{P} \sum_{j=1}^{n_q} v_{pq}(x_{p_i}, x_{q_j}) - v_{pp}(x_{p_i}, x_{p_i}) \right) \delta_{\mathbf{x}}(\mathbf{x}')$$

so that $H_{\mathbf{n}}(\mathbf{x}, \mathbf{x}') = \mathrm{h}_{\mathbf{n}}(\mathbf{x}, \mathbf{x}') + V_{\mathbf{n}}(\mathbf{x}, \mathbf{x}')$.

Recall from Equation (4.41), for $\ell > 1$,

$$\begin{aligned} K_{\underline{\mathbf{n}}}(\mathbf{x}, \mathbf{x}'_1, \dots, \mathbf{x}'_\ell) = &S_{\mathbf{x}}^{(\mathbf{n})} \sum_{\substack{(\mathbf{m}_j)_{j=1}^{\ell} \\ (\forall j) 0 \neq \mathbf{m}_j \leq \mathbf{n}_j}} \frac{\mathbf{m}!}{\ell!} \frac{\underline{\mathbf{n}}!}{\underline{\mathbf{m}}!(\underline{\mathbf{n}} - \underline{\mathbf{m}})!} (-1)^{\sum_{p=P_b+1}^{P} \sum_{j=1}^{\ell} (n_{jp} - m_{jp}) \sum_{k=1}^{j-1} n_{kp}} \\ &k(\mathbf{x}_{[\leq \mathbf{m}]}, \mathbf{x}'_{1[\leq \mathbf{m}_1]} \circ \dots \circ \mathbf{x}'_{\ell[\leq \mathbf{m}_\ell]}) \delta_{\tilde{\mathbf{x}}_1}(\mathbf{x}'_{1[>\mathbf{m}_1]}) \dots \delta_{\tilde{\mathbf{x}}_\ell}(\mathbf{x}'_{\ell[>\mathbf{m}_\ell]}), \end{aligned}$$

where each $\tilde{\mathbf{x}}_j \in X^{\mathbf{n}_j - \mathbf{m}_j}$ is defined by $\mathbf{x}_{[>\mathbf{m}]} = \tilde{\mathbf{x}}_1 \circ \dots \circ \tilde{\mathbf{x}}_\ell$.

Since each $\mathbf{m}_j \neq 0$ and $k(\mathbf{x}_{[\leq \mathbf{m}]}, \mathbf{x}'_{1[\leq \mathbf{m}_1]} \circ \dots \circ \mathbf{x}'_{\ell[\leq \mathbf{m}_\ell]}) = 0$ unless $|\mathbf{m}| \leq 2$, we must conclude that the only nonzero terms in the above equation have $\ell = 2$

and $|\mathbf{m}_1| = |\mathbf{m}_2| = 1$.

In this case, we can simplify $K_{\underline{\mathbf{n}}}$ to the form,

$$K_{(\mathbf{n}_1,\mathbf{n}_2)}(\mathbf{x}, \mathbf{x}'_1, \mathbf{x}'_2) = S^{(\mathbf{n})}_{\mathbf{x}} \sum_{p=1}^{P} \sum_{q=p}^{P} n_{1_p} n_{2_q} (-1)^{s_q(\underline{\mathbf{n}})} v_{pq}(x_{p_1}, x_{q_{n_{1_q}+1}}) \delta_{\mathbf{x}}(\mathbf{x}'_1 \circ \mathbf{x}'_2). \tag{4.48}$$

where, for each $p \in \{1, \ldots, P\}$,

$$s_p(\underline{\mathbf{n}}) := \sum_{p \neq p' > P_b} n_{2_{p'}} n_{1_{p'}} + \chi_{[>P_b]}(p)(n_{2_p} - 1) n_{1_p}$$

and

$$\chi_{[>P_b]}(p) = \begin{cases} 1 & \text{if } p > P_b \\ 0 & \text{otherwise.} \end{cases}$$

With $\ell = 2$, the recursion relation (4.42) from Lemma 4.3.3 becomes

$$F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y}) = -\int_0^{\epsilon} d\tau \int_{X^{\mathbf{n}}} d\mathbf{x}' e^{-(\epsilon-\tau)H_{\mathbf{n}}}(\mathbf{x}, \mathbf{x}') \sum_{\substack{\mathbf{n}_1,\mathbf{n}_2 \neq 0 \\ \mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}} \int_{X^{\mathbf{n}_2}} d\mathbf{x}''_2 \int_{X^{\mathbf{n}_1}} d\mathbf{x}''_1$$
$$K_{(\mathbf{n}_1,\mathbf{n}_2)}(\mathbf{x}', \mathbf{x}''_1, \mathbf{x}''_2) F_{\mathbf{n}_1}(\tau, \mathbf{x}''_1, \mathbf{y}_{[\leq \mathbf{n}_1]}) F_{\mathbf{n}_2}(\tau, \mathbf{x}''_2, \mathbf{y}_{[>\mathbf{n}_1]}). \tag{4.49}$$

By substituting $K_{(\mathbf{n}_1,\mathbf{n}_2)}(\mathbf{x}', \mathbf{x}''_1, \mathbf{x}''_2)$ with the expression from Equation (4.48),

$$\begin{aligned} F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y}) = &-\int_0^{\epsilon} d\tau \int_{X^{\mathbf{n}}} d\mathbf{x}' e^{-(\epsilon-\tau)H_{\mathbf{n}}}(\mathbf{x}, \mathbf{x}') \sum_{\substack{\mathbf{n}_1,\mathbf{n}_2 \neq 0 \\ \mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}} \int_{X^{\mathbf{n}_2}} d\mathbf{x}''_2 \int_{X^{\mathbf{n}_1}} d\mathbf{x}''_1 \\ &S^{(\mathbf{n})}_{\mathbf{x}'} \sum_{p=1}^{P} \sum_{q=p}^{P} n_{1_p} n_{2_q} (-1)^{s_q(\underline{\mathbf{n}})} v_{pq}(x'_{p_1}, x'_{q_{n_{1_q}+1}}) \delta_{\mathbf{x}'}(\mathbf{x}''_1 \circ \mathbf{x}''_2) \\ &F_{\mathbf{n}_1}(\tau, \mathbf{x}''_1, \mathbf{y}_{[\leq \mathbf{n}_1]}) F_{\mathbf{n}_2}(\tau, \mathbf{x}''_2, \mathbf{y}_{[>\mathbf{n}_1]}) \\ = &-\int_0^{\epsilon} d\tau \int_{X^{\mathbf{n}}} d\mathbf{x}' e^{-(\epsilon-\tau)H_{\mathbf{n}}}(\mathbf{x}, \mathbf{x}') \sum_{\substack{\mathbf{n}_1,\mathbf{n}_2 \neq 0 \\ \mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}} \\ &S^{(\mathbf{n})}_{\mathbf{x}'} \sum_{p=1}^{P} \sum_{q=p}^{P} n_{1_p} n_{2_q} (-1)^{s_q(\underline{\mathbf{n}})} v_{pq}(x'_{p_1}, x'_{q_{n_{1_q}+1}}) \cdot \end{aligned}$$

$$
\cdot F_{\mathbf{n}_1}(\tau, \mathbf{x}'_{[\leq \mathbf{n}_1]}, \mathbf{y}_{[\leq \mathbf{n}_1]}) F_{\mathbf{n}_2}(\tau, \mathbf{x}'_{[>\mathbf{n}_1]}, \mathbf{y}_{[>\mathbf{n}_1]})
$$

$$
= -\int_0^\epsilon d\tau \int_{X^\mathbf{n}} d\mathbf{x}' e^{-(\epsilon-\tau)H_\mathbf{n}}(\mathbf{x}, \mathbf{x}') \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \neq 0 \\ \mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}} S^{(\mathbf{n})}_{\mathbf{x}'} \sum_{p=1}^{P} \sum_{q=p}^{P} \sum_{i=1}^{n_{1_p}} \sum_{j=n_{1_q}+1}^{n_p}
$$

$$
(-1)^{s_q(\underline{\mathbf{n}})} v_{pq}(x'_{p_i}, x'_{q_j}) F_{\mathbf{n}_1}(\tau, \mathbf{x}'_{[\leq \mathbf{n}_1]}, \mathbf{y}_{[\leq \mathbf{n}_1]}) F_{\mathbf{n}_2}(\tau, \mathbf{x}'_{[>\mathbf{n}_1]}, \mathbf{y}_{[>\mathbf{n}_1]}).
$$

Definition 13. For any $\mathbf{n} \in \mathbb{N}_0^P$ and $F : X^\mathbf{n} \to \mathbb{C}$, we define

$$
\|F\|_{1,\infty} = \max_{p \in \{1,\ldots,P\}} \max_{1 \leq j \leq n_p} \max_{x_j \in X_p} \int_{X^{\mathbf{n}-\mathbf{e}_p}} \prod_{(q,i) \neq (p,j)} dx_{q_i} \; |F(\mathbf{x})|.
$$

Lemma 4.3.4. For any $\mathbf{n} \in \mathbb{N}_0^P$ and $F : X^\mathbf{n} \times X^\mathbf{n} \to \mathbb{C}$,

(a) $\|e^{-\epsilon \mathrm{h}_\mathbf{n}} F\|_{1,\infty} \leq e^{|\mathbf{n}|\epsilon h_0} \|F\|_{1,\infty}$, where $h_0 := \max_p \|h_p\|_{1,\infty}$,

(b) $\|e^{-\epsilon V_\mathbf{n}} F\|_{1,\infty} \leq e^{|\mathbf{n}|\epsilon v_0} \|F\|_{1,\infty}$, where $v_0 := \max_{p, x \in X_p} v_{pp}(x,x)$, and

(c) $\|e^{-\epsilon H_\mathbf{n}} F\|_{1,\infty} \leq e^{(h_0+v_0)|\mathbf{n}|\epsilon} \|F\|_{1,\infty}$.

Proof. The proof is very similar to that of Lemma (3.10) in [10].

(a) For each p, let

$$
\mathrm{h}^{(p)}_\mathbf{n}(\mathbf{x}, \mathbf{x}') := \sum_{j=1}^{n_p} h_p(x_{pj}, x'_{pj}) \prod_{(q,m) \neq (p,j)} \delta_{x_{qm}}(x'_{qm}),
$$

so that $\mathrm{h}_\mathbf{n} = \sum_{p=1}^P \mathrm{h}^{(p)}_\mathbf{n}$. From [10], we know that for each p,

$$
\|e^{-\epsilon h_p}\|_{1,\infty} \leq e^{\epsilon \|h_p\|_{1,\infty}}
$$

and

$$
\|e^{-\epsilon \mathrm{h}^{(p)}_\mathbf{n}} F\|_{1,\infty} \leq e^{n_p \epsilon \|h_p\|_{1,\infty}} \|F\|_{1,\infty}.
$$

(The proof is the same, regardless of whether the p^{th} species is bosonic or fermionic.) Since $\mathrm{h}_{\mathbf{n}}^{(p)}$ and $\mathrm{h}_{\mathbf{n}}^{(q)}$ commute,

$$\begin{aligned}
\|e^{-\epsilon \mathrm{h}_{\mathbf{n}}} F\|_{1,\infty} &= \left\| \prod_{p=1}^{P} e^{-\epsilon \mathrm{h}_{\mathbf{n}}^{(p)}} \; F \right\|_{1,\infty} \\
&\leq \prod_{p=1}^{P} e^{n_p \epsilon \|h_p\|_{1,\infty}} \|F\|_{1,\infty} \\
&= e^{|\mathbf{n}| \epsilon h_0} \|F\|_{1,\infty},
\end{aligned}$$

where $h_0 := \max_{1 \leq p \leq P} \|h_p\|_{1,\infty}$.

(b) Recall,

$$V_{\mathbf{n}}(\mathbf{x}, \mathbf{x}') := \sum_{p=1}^{P} \sum_{i=1}^{n_p} \left(\sum_{q=1}^{P} \sum_{j=1}^{n_q} v_{pq}(x_{p_i}, x_{q_j}) - v_{pp}(x_{p_i}, x_{p_i}) \right) \delta_{\mathbf{x}}(\mathbf{x}').$$

Since V is repulsive, we can let $\rho_p(x) = \frac{1}{\sqrt{P n_p}} \sum_{i=1}^{n_p} \delta_{x_{p_i}}(x)$ in Equation (4.47) to show that

$$\sum_{p,q=1}^{P} \sum_{i=1}^{n_p} \sum_{j=1}^{n_q} v_{pq}(x_{p_i}, x_{q_j}) \geq \lambda_0(v) > 0.$$

Hence,

$$\begin{aligned}
\sum_{p=1}^{P} \sum_{i=1}^{n_p} \left(\sum_{q=1}^{P} \sum_{j=1}^{n_q} v_{pq}(x_{p_i}, x_{q_j}) - v_{pp}(x_{p_i}, x_{p_i}) \right) &\geq - \sum_{p=1}^{P} \sum_{i=1}^{n_p} v_{pp}(x_{p_i}, x_{p_i}) \\
&\geq -|\mathbf{n}| v_0,
\end{aligned}$$

where

$$v_0 := \max_{1 \leq p \leq P} \max_{x \in X_p} v_{pp}(x, x).$$

Therefore,

$$|e^{-\epsilon V_{\mathbf{n}}} F| \leq e^{|\mathbf{n}| \epsilon v_0} |F|.$$

The claim follows.

(c) Part (c) follows from the Trotter product formula and Parts (a) and (b), as in [10].

□

Theorem 4.3.5. For a system with P species of bosons and/or fermions interacting with a repulsive 2-body potential as defined in Equations (4.46) and (4.47), the coefficients $F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y})$ in the expansion (4.38) for the action $F(\epsilon, \mathbf{x}, \mathbf{y})$ satisfy the bound

$$\|F_{\mathbf{n}}(\epsilon, \cdot, \cdot)\|_{1,\infty} \leq (\lambda_v \epsilon)^{|\mathbf{n}|-1} e^{(h_0+v_0)|\mathbf{n}|\epsilon} \frac{1}{|\mathbf{n}|^{P+2}},$$

where $|\mathbf{n}| := \sum_p n_p$ and

$$\lambda_v := 2^{P+2} \left(\sum_{p,q=1}^{P} \|v_{p,q}\|_{1,\infty} \right) \sum_{0 \neq \mathbf{m} \in \mathbb{N}_0^P} \frac{1}{|\mathbf{m}|^{P+1}}. \tag{4.50}$$

Proof. For convenience, within this proof, let $C_1 := h_0 + v_0$. In the case $|\mathbf{n}| = 1$, $\mathbf{n} = \mathbf{e}_p$ for some p, and from Equation (4.39), $F_{\mathbf{e}_p}(\epsilon, x, y) = (e^{-\epsilon h_p})(x, y)$. Hence,

$$\|F_{\mathbf{n}}(\epsilon, \cdot, \cdot)\|_{1,\infty} = \|e^{-\epsilon h_p}\|_{1,\infty} \leq e^{\epsilon \|h_p\|_{1,\infty}}.$$

For any $n \geq 2$, assume the bound for all $F_{\mathbf{m}}$ with $|\mathbf{m}| = \sum_p m_p < n$. Then for any $\mathbf{n}$ with $|\mathbf{n}| = n$,

$$\|F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y})\|_{1,\infty} = \Bigg\| \int_0^\epsilon d\tau \int_{X^{\mathbf{n}}} d\mathbf{x}' e^{-(\epsilon-\tau)H_{\mathbf{n}}}(\mathbf{x}, \mathbf{x}') \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \neq 0 \\ \mathbf{n}_1 + \mathbf{n}_2 = \mathbf{n}}} S_{\mathbf{x}'}^{(\mathbf{n})} \sum_{p=1}^{P} \sum_{q=p}^{P}$$
$$\sum_{i=1}^{n_{1_p}} \sum_{j=n_{1_q}+1}^{n_p} (-1)^{s_q(\underline{\mathbf{n}})} v_{pq}(x'_{p_i}, x'_{q_j}) F_{\mathbf{n}_1}(\tau, \mathbf{x}'_{[\leq \mathbf{n}_1]}, \mathbf{y}_1) F_{\mathbf{n}_2}(\tau, \mathbf{x}'_{[>\mathbf{n}_1]}, \mathbf{y}_2) \Bigg\|_{1,\infty}$$
$$\leq \int_0^\epsilon d\tau \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \neq 0 \\ \mathbf{n}_1 + \mathbf{n}_2 = \mathbf{n}}} \sum_{p=1}^{P} \sum_{q=p}^{P} \sum_{i=1}^{n_{1_p}} \sum_{j=n_{1_q}+1}^{n_p} \Bigg\| \int_{X^{\mathbf{n}}} d\mathbf{x}' e^{-(\epsilon-\tau)H_{\mathbf{n}}}(\mathbf{x}, \mathbf{x}') S_{\mathbf{x}'}^{(\mathbf{n})}$$
$$v_{pq}(x'_{p_i}, x'_{q_j}) F_{\mathbf{n}_1}(\tau, \mathbf{x}'_{[\leq \mathbf{n}_1]}, \mathbf{y}_1) F_{\mathbf{n}_2}(\tau, \mathbf{x}'_{[>\mathbf{n}_1]}, \mathbf{y}_2) \Bigg\|_{1,\infty}$$
$$\leq \int_0^\epsilon d\tau \sum_{\substack{\mathbf{n}_1, \mathbf{n}_2 \neq 0 \\ \mathbf{n}_1 + \mathbf{n}_2 = \mathbf{n}}} \sum_{\substack{p,q \\ p \leq q}} n_{1_p} n_{2_q} e^{(h_0+v_0)|\mathbf{n}|(\epsilon-\tau)} \|v_{pq}\|_{1,\infty}$$

$$
\begin{aligned}
&\qquad\qquad \cdot \|F_{\mathbf{n}_1}(\tau,\cdot,\cdot)\|_{1,\infty}\|F_{\mathbf{n}_2}(\tau,\cdot,\cdot)\|_{1,\infty} \\
&\le \int_0^\epsilon d\tau \sum_{\substack{\mathbf{n}_1,\mathbf{n}_2\neq 0\\ \mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}} \sum_{\substack{p,q\\ p\le q}} n_{1_p} n_{2_q} e^{C_1|\mathbf{n}|(\epsilon-\tau)}\|v_{pq}\|_{1,\infty} \\
&\qquad (\lambda_v\tau)^{|\mathbf{n}_1|-1} e^{C_1|\mathbf{n}_1|\tau}\frac{1}{|\mathbf{n}_1|^{P+2}}(\lambda_v\tau)^{|\mathbf{n}_2|-1}e^{\lambda_v|\mathbf{n}_1|\tau}\frac{1}{|\mathbf{n}_2|^{P+2}} \\
&\le \int_0^\epsilon d\tau \sum_{\substack{\mathbf{n}_1,\mathbf{n}_2\neq 0\\ \mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}} \sum_{\substack{p,q\\ p\le q}} n_{1_p} n_{2_q} e^{C_1|\mathbf{n}|(\epsilon-\tau)}\|v_{pq}\|_{1,\infty}(\lambda_v\tau)^{n-2}e^{C_1 n\tau}\frac{1}{|\mathbf{n}_1|^{P+2}|\mathbf{n}_2|^{P+2}} \\
&\le e^{C_1|\mathbf{n}|\epsilon}\left(\int_0^\epsilon d\tau(\lambda_v\tau)^{n-2}\right) \sum_{\substack{\mathbf{n}_1,\mathbf{n}_2\neq 0\\ \mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}} \sum_{\substack{p,q\\ p\le q}} n_{1_p} n_{2_q}\|v_{pq}\|_{1,\infty}\frac{1}{|\mathbf{n}_1|^{P+2}|\mathbf{n}_2|^{P+2}} \\
&\le e^{C_1|\mathbf{n}|\epsilon}\frac{(\lambda_v\epsilon)^{n-1}}{n-1}\frac{1}{\lambda_v} \sum_{\substack{\mathbf{n}_1,\mathbf{n}_2\neq 0\\ \mathbf{n}_1+\mathbf{n}_2=\mathbf{n}}} \sum_{\substack{p,q\\ p\le q}} n_{1_p} n_{2_q}\|v_{pq}\|_{1,\infty}\frac{1}{|\mathbf{n}_1|^{P+2}|\mathbf{n}_2|^{P+2}} \\
&\le e^{C_1|\mathbf{n}|\epsilon}(\lambda_v\epsilon)^{n-1}\frac{1}{\lambda_v}\left(\sum_{\substack{p,q\\ p\le q}}\|v_{pq}\|_{1,\infty}\right)\frac{1}{n-1}\sum_{\substack{\mathbf{m}\le\mathbf{n}\\ 0\neq\mathbf{m}\neq\mathbf{n}}}\frac{1}{|\mathbf{m}|^{P+1}|\mathbf{n}-\mathbf{m}|^{P+1}}.
\end{aligned}
$$

The sum over $\mathbf{m}$ has a symmetry about $\mathbf{m} = \frac{1}{2}\mathbf{n}$. Hence,

$$
\begin{aligned}
\sum_{\substack{\mathbf{m}\le\mathbf{n}\\ 0\neq\mathbf{m}\neq\mathbf{n}}}\frac{1}{|\mathbf{m}|^{P+1}|\mathbf{n}-\mathbf{m}|^{P+1}} &\le 2\sum_{\substack{\mathbf{m}\le\frac{1}{2}\mathbf{n}\\ 0\neq\mathbf{m}\neq\mathbf{n}}}\frac{1}{|\mathbf{m}|^{P+1}|\mathbf{n}-\mathbf{m}|^{P+1}} \\
&\le 2\frac{1}{\left(\frac{1}{2}|\mathbf{n}|\right)^{P+1}}\sum_{\substack{\mathbf{m}\le\frac{1}{2}\mathbf{n}\\ 0\neq\mathbf{m}\neq\mathbf{n}}}\frac{1}{|\mathbf{m}|^{P+1}} \\
&\le \frac{2^{P+2}}{\mathbf{n}^{P+1}}\sum_{\substack{\mathbf{m}\in\mathbb{N}_0^P\\ \mathbf{m}\neq 0}}\frac{1}{|\mathbf{m}|^{P+1}}.
\end{aligned}
$$

This is finite, since

$$
\begin{aligned}
\sum_{\substack{\mathbf{m}\in\mathbb{N}_0^P\\ \mathbf{m}\neq 0}}\frac{1}{|\mathbf{m}|^{P+1}} &\le C\int_{\substack{\mathbb{R}^P\\ \|\mathbf{x}\|>1}} d\mathbf{x}\frac{1}{\|\mathbf{x}\|^{P+1}} \\
&\le C'\int_1^\infty \frac{1}{r^{P+1}}r^{P-1}dr
\end{aligned}
$$

$$= C' \int_1^\infty \frac{1}{r^2} dr < \infty,$$

for some constants C and C'. Hence, if we let

$$\lambda_v := 2^{P+2} \left(\sum_{p,q=1}^{P} \|v_{pq}\|_{1,\infty} \right) \sum_{\substack{\mathbf{m} \in \mathbb{N}_0^P \\ \mathbf{m} \neq 0}} \frac{1}{|\mathbf{m}|^{P+1}},$$

then

$$\|F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y})\|_{1,\infty} \leq (\lambda_v \epsilon)^{|\mathbf{n}|-1} e^{C_1 |\mathbf{n}| \epsilon} \frac{1}{|\mathbf{n}|^{P+2}},$$

as claimed. This completes the proof. □

Proposition 4.3.6. For each $\epsilon > 0$, the action, $F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ is an analytic function of the fields $\boldsymbol{\alpha}^*$ and $\boldsymbol{\phi}$ on the domain defined by

$$|\alpha_p^*|_{X_p}, |\phi_p|_{X_p} < \frac{e^{-\frac{1}{2}(h_0+v_0)\epsilon}}{\sqrt{\lambda_v \epsilon}},$$

for all $p \leq P_b$, where $|\cdot|_{X_p}$ is the supremum norm on X_p,

$$|\phi_p|_{X_p} := \max_{x \in X_p} |\phi_p(x)|.$$

As defined in Theorem 4.3.5, λ_v is the constant,

$$\lambda_v = 2^{P+2} \left(\sum_{p,q=1}^{P} \|v_{p,q}\|_{1,\infty} \right) \sum_{0 \neq \mathbf{m} \in \mathbb{N}_0^P} \frac{1}{|\mathbf{m}|^{P+1}}.$$

Proof. For each $\mathbf{n} = (n_1, \ldots, n_P) \in \mathbb{N}_0^P$, let $\mathbf{n}_b$ represent the vector of only the boson numbers,

$$\mathbf{n}_b := (n_1, \ldots, n_{P_b})$$

and let $\mathbf{n}_f$ represent the vector of only the fermion numbers,

$$\mathbf{n}_f := (n_{P_b+1}, \ldots, n_P).$$

Then $\mathbf{n} = \mathbf{n}_b \circ \mathbf{n}_f$. Similarly, for each $\mathbf{x} = (\vec{x}_1, \ldots, \vec{x}_P) \in X^{\mathbf{n}}$, let

$$\mathbf{x}_b := (\vec{x}_1, \ldots, \vec{x}_{P_b}) \in X^{\mathbf{n}_b}$$

and

$$\mathbf{x}_f := (\vec{x}_{P_b+1}, \ldots, \vec{x}_P) \in X^{\mathbf{n}_f},$$

so that $\mathbf{x} = \mathbf{x}_b \circ \mathbf{x}_f$.

With this notation,

$$\begin{aligned}
F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) &= \sum_{\mathbf{n} \in \mathbb{N}_0^P} \int_{X^{\mathbf{n}}} d\mathbf{x} \int_{X^{\mathbf{n}}} d\mathbf{y}\ \alpha^*(\mathbf{x}) \phi(\mathbf{y}) F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y}) \\
&= \sum_{\mathbf{n}_f \in \{1, \ldots, |X_f|\}^{P_f}} \int_{X^{\mathbf{n}_f}} d\mathbf{x}_f \int_{X^{\mathbf{n}_f}} d\mathbf{y}_f\ \alpha^*(\mathbf{x}_f) \phi(\mathbf{y}_f) \\
&\quad \cdot \sum_{\mathbf{n}_b \in \mathbb{N}_0^{P_b}} \int_{X^{\mathbf{n}_b}} d\mathbf{x}_b \int_{X^{\mathbf{n}_b}} d\mathbf{y}_b\ \alpha^*(\mathbf{x}_b) \phi(\mathbf{y}_b) F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y})
\end{aligned}$$

Since the outer sum over fermion numbers is finite, we can focus on the inner sum over boson numbers. Using the bound from Theorem 4.3.5,

$$\begin{aligned}
&\sum_{\mathbf{n}_b \in \mathbb{N}_0^{P_b}} \int_{X^{\mathbf{n}_b}} d\mathbf{x}_b \int_{X^{\mathbf{n}_b}} d\mathbf{y}_b\ \ |\alpha^*(\mathbf{x}_b) \phi(\mathbf{y}_b) F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y})| \\
&\quad \leq \sum_{\mathbf{n}_b \in \mathbb{N}_0^{P_b}} \left(\prod_{p=1}^{P_b} |\alpha_p|_{X_p} |\phi_p|_{X_p} \right) \int_{X^{\mathbf{n}_b}} d\mathbf{x}_b \int_{X^{\mathbf{n}_b}} d\mathbf{y}_b\ \ |F_{\mathbf{n}}(\epsilon, \mathbf{x}, \mathbf{y})| \\
&\quad \leq \left(\max_p |X_p| \right) \sum_{\mathbf{n}_b \in \mathbb{N}_0^{P_b}} \left(\prod_{p=1}^{P_b} |\alpha_p|_{X_p}^{n_p} |\phi_p|_{X_p}^{n_p} \right) \|F_{\mathbf{n}}(\epsilon, \cdot, \cdot)\|_{1,\infty} \\
&\quad \leq \left(\max_p |X_p| \right) \sum_{\mathbf{n}_b \in \mathbb{N}_0^{P_b}} \prod_{p=1}^{P_b} \Big(|\alpha_p|_{X_p} |\phi_p|_{X_p} \Big)^{n_p} (\lambda_v \epsilon)^{|\mathbf{n}|-1} e^{(h_0+v_0)|\mathbf{n}|\epsilon} \frac{1}{|\mathbf{n}|^{P+2}} \\
&\quad \leq \left(\max_p |X_p| \right) (\lambda_v \epsilon)^{|\mathbf{n}_f|-1} e^{(h_0+v_0)|\mathbf{n}_f|\epsilon} \prod_{p=1}^{P_b} \sum_{n_p=0}^{\infty} \Big(|\alpha_p|_{X_p} |\phi_p|_{X_p} e^{(h_0+v_0)\epsilon} \lambda_v \epsilon \Big)^{n_p}.
\end{aligned}$$

This is absolutely convergent as long as, for each $p \leq P_b$,

$$|\alpha_p|_{X_p} |\phi_p|_{X_p} < \frac{1}{e^{(h_0+v_0)\epsilon} \lambda_v \epsilon},$$

which is certainly true in the domain with

$$|\alpha_p|_{X_p}, |\phi_p|_{X_p} < \frac{e^{-\frac{1}{2}(h_0+v_0)\epsilon}}{\sqrt{\lambda_v \epsilon}}.$$

This completes the proof. □

Note that it is easy to choose the large-field cutoffs $R_p(q)$ in Theorem 4.2.1 so that they satisfy both the analyticity condition

$$R_p(q) < \frac{e^{-\frac{1}{2}(h_0+v_0)\frac{\beta}{q}}}{\sqrt{\lambda_v \frac{\beta}{q}}} \tag{4.51}$$

and the convergence condition

$$\lim_{q\to\infty} q e^{-\frac{1}{2}R_p(q)^2} = 0. \tag{4.52}$$

For example,

$$R_p(q) = C\sqrt{q}, \quad \text{with} \quad C = \frac{1}{\sqrt{\lambda_v \beta}} e^{-\frac{1}{2}(h_0+v_0)\beta},$$

clearly satisfies both conditions (4.51) and (4.52) above.

Hence, we can replace each inner product $\langle \mathbf{\phi}_{\tau-\epsilon} | e^{-\epsilon K} | \mathbf{\phi}_\tau \rangle$ in the functional integral (4.26) with an exponential of the action $F(\epsilon, \mathbf{\phi}^*_{\tau-\epsilon}, \mathbf{\phi}_\tau)$ to obtain

$$\mathrm{Tr}\ e^{-\beta K} = \lim_{q\to\infty} \int \prod_{\tau\in\mathcal{T}_q} \left[d\mu_{\underline{\mathbf{R}}(q)}(\mathbf{\phi}^*_\tau, \mathbf{\phi}_\tau) e^{-\int d\mathbf{y}\ \mathbf{\phi}^*_\tau(\mathbf{y})\cdot\mathbf{\phi}_\tau(\mathbf{y})} \right] \prod_{\tau\in\mathcal{T}_q} e^{F(\epsilon, \mathbf{\phi}^*_{\tau-\epsilon}, \mathbf{\phi}_\tau)}.$$

We have assumed here that the interactions are of order at most $M = 2$ and that, for each $p \in \{1, \ldots, P\}$, $R_p(q)$ satisfies both conditions (4.51) and (4.52) above.

Chapter 5

Systems of Electrons and Phonons

We now consider a system of bosons and fermions, with a particular Hamiltonian, known as the Holstein Hamiltonian, that does not preserve the number of bosons. If $a_b^\dagger(x)$, $a_b(x)$ are the creation and annihilation operators at a point x of the boson configuration space X_b and $a_f^\dagger(x)$, $a_f(x)$ are the creation and annihilation operators at a point x of the fermion configuration space X_f, the operator representing the interaction between bosons and fermions has the form

$$V = \int_{X_b} dx \int_{X_f} dy \; v(x,y) a_f^\dagger(y) a_f(y) (a_b^\dagger(x) + a_b(x)),$$

where $v : X_b \times X_f \to \mathbb{R}$. This type of interaction is often used for modelling systems of electrons and phonons [14, 15, 20].

In Section 5.1, we'll provide a brief review of the notation defined in Chapters 2-4, as applied to a system of one species of bosons and one species of fermions. In Section 5.2, we'll define the Hamiltonian and prove bounds on it which will be used repeatedly throughout the remainder of the Chapter. In Section 5.3, we'll derive the functional integral representation, which takes the same form as the functional integral in (4.26) for particle-preserving interactions. However, the non-preservation of boson number will demand new techniques in the proof. In particular, we'll use a Duhamel expansion in powers of V. Finally, in Section 5.4, we'll show that the "action" is an entire-analytic non-zero function of the fields.

5.1 Notation

We denote the configuration space for the bosons by X_b and the configuration space for fermions by X_f. Each of these incorporate the position space as well as any internal states, such as spin.

The Hilbert space for a system of indistinguishable bosons on the configuration space X_b is

$$\mathcal{B}(X_b) = \bigoplus_{n=0}^{\infty} \mathcal{B}_n(X_b)$$

where $\mathcal{B}_n(X_b)$ is the space of symmetric functions of n arguments on X_b,

$$\mathcal{B}_n(X_b) := \{f \in L^2(X_b^n) : (\forall \sigma \in S_n)\sigma f = f\}.$$

Similarly, the Hilbert space for a system of indistinguishable fermions on the configuration space X_f is

$$\mathcal{F}(X_f) = \bigoplus_{n=0}^{|X_f|} \mathcal{F}_n(X_f),$$

where

$$\mathcal{F}_n(X_f) := \{f \in L^2(X_f^n) : (\forall \sigma \in S_n)\sigma f = (-1)^{\sigma} f\}.$$

The Hilbert space for the combined system of bosons and fermions is

$$\mathcal{H} := \mathcal{B}(X_b) \otimes \mathcal{F}(X_f).$$

Throughout this chapter, we will sometimes suppress the configuration spaces in the notation above and denote $\mathcal{B}(X_b)$ and $\mathcal{F}(X_f)$ by $\mathcal{B}$ and $\mathcal{F}$ respectively. Similarly, the n-boson and n-fermion subspaces of $\mathcal{B}$ and $\mathcal{F}$ will be denoted by $\mathcal{B}_n$ and $\mathcal{F}_n$ respectively.

We denote the n-boson subspace of $\mathcal{H}$ by

$$\mathcal{H}_n := \mathcal{B}_n \otimes \mathcal{F},$$

while the projection onto $\mathcal{H}^{(n)}$ is denoted by $P^{(n)}$. The subspace of $\mathcal{H}$ with at most n bosons is

$$\mathcal{H}_{[\leq n]} := \sum_{m=1}^{n} \mathcal{H}_m,$$

and P_n is the projection onto this space.

Recall from Section 2.1.1 that for each $n \in \mathbb{N}_0$, $\{\delta_Y : Y \in X_b^n/S_n\}$ is a basis for $\mathcal{B}_n(X_b)$ and $\{\delta_Y : Y \subset X_f\}$ is a basis for $\mathcal{F}(X_f)$.

The *power set* of X_f is the set of all subsets of X_f and is denoted by $\mathcal{P}(X_f)$. We also denote the set of all subsets with exactly n elements by $\mathcal{P}_n(X_f)$,

$$\mathcal{P}_n(X_f) := \{Y \subset X_f : |Y| = n\}.$$

For any $Y = (Y_b, Y_f) \in X^{n_b}/S_{n_b} \times \mathcal{P}(X_f)$, we let $\delta_Y := \delta_{Y_b} \otimes \delta_{Y_f}$. Then

$$\{\delta_Y : Y \in X^{n_b}/S_{n_b} \times \mathcal{P}(X_f)\}$$

is an orthonormal basis for $\mathcal{H}_{n_b}$. The set

$$B(\mathcal{H}) := \left\{\delta_Y : Y \in \left(\bigsqcup_{n=0}^{\infty} X_b^n/S_n\right) \times \mathcal{P}(X_f)\right\}$$

is an orthonormal basis for $\mathcal{H}$.

For any $\mathbf{n} = (n_b, n_f) \in \mathbb{N}_0^2$, let

$$X^{\mathbf{n}} := X_b^{n_b} \times X_f^{n_f}.$$

Then let

$$\mathbf{X} := \bigsqcup_{\mathbf{n} \in \mathbb{N}_0^2} X^{\mathbf{n}}.$$

As with all other finite sets in this work, integrals over $\mathbf{X}$ and any subset thereof will be with respect to the counting measure. For example, if $f : \mathbf{X} \to \mathbb{C}$,

$$\int_{\mathbf{X}} d\mathbf{x}\ f(\mathbf{x}) = \sum_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}).$$

For each $\mathbf{x} \in \mathbf{X}$, we let $\mathbf{n}(\mathbf{x})$ be the element $\mathbf{n} = (n_b, n_f)$ of $\mathbb{N}_0^2$ such that

$\mathbf{x} \in X^{\mathbf{n}}$. Then there is a vector $\vec{x}_b$ of boson coordinates,

$$\vec{x}_b = (x_{b1}, \ldots, x_{bn_b}),$$

and a vector of fermion coordinates,

$$\vec{x}_f = (x_{f1}, \ldots, x_{fn_f}),$$

such that $\mathbf{x} = (\vec{x}_f, \vec{x}_b)$. In this case, the creation and annihilation operators at $\mathbf{x}$ are

$$a^\dagger(\mathbf{x}) := a_b^\dagger(x_{b1}) \ldots a_b^\dagger(x_{bn_b}) a_f^\dagger(x_{f1}) \ldots a_f^\dagger(x_{fn_f})$$

and

$$a(\mathbf{x}) := a_f(x_{fn_f}) \ldots a_f(x_{f1}) a_b(x_{bn_b}) \ldots a_b(x_{b1}).$$

respectively.

5.2 The Hamiltonian

The Hamiltonian H will be defined as the sum of two operators, H_0 and V. The *kinetic energy operator* H_0 is defined by

$$H_0 := \int_{X_b^2} dx\, dy\ a_b^\dagger(x) h_b(x,y) a_b(y) \ + \ \int_{X_f^2} dx\, dy\ a_f^\dagger(x) h_f(x,y) a_f(y),$$

where h_b is a complex-valued function on X_b^2 that satisfies $h_b(x,y) = h_b^*(y,x)$ for all $x, y \in X_b$. Likewise, $h_f : X_f^2 \to \mathbb{C}$ with $h_f(x,y) = h_f^*(y,x)$ for all $x, y \in X_f$. H_0 is well-defined as a map from the finite-dimensional space $\mathcal{H}_n$ to $\mathcal{H}_n$, for each $n \in \mathbb{N}$, and may be extended linearly to the domain,

$$\mathcal{H}^{(finite)} = \{\psi \in \mathcal{H} : (\exists N \in \mathbb{N})(\forall n \geq N) P^{(n)}\psi = 0\}.$$

Lemma 5.2.1. The kinetic energy operator H_0 is essentially self-adjoint on $\mathcal{H}^{(finite)}$.

Proof. Due to the symmetry conditions on the functions h_b and h_f, H_0 is symmetric. As a symmetric operator on a finite-dimensional space, each $H_0|_{\mathcal{H}_n}$ satisfies $\mathrm{Ran}(H_0|_{\mathcal{H}_n} \pm i) = \mathcal{H}_n$. Hence, with domain $\mathcal{H}^{(finite)}$, the operator $H_0 \pm i$ has range $\mathcal{H}^{(finite)}$ as well, which is dense in $\mathcal{H}$. It follows that H_0 is essentially self-adjoint on $\mathcal{H}^{(finite)}$.

□

Note that H_0 satisfies the bounds,

$$\boldsymbol{\lambda} \cdot \mathbf{N} \leq H_0 \leq \boldsymbol{\Lambda} \cdot \mathbf{N}, \tag{5.1}$$

where $\mathbf{N} := (N_b, N_f)$ the vector of particle number operators. On the left side, $\boldsymbol{\lambda} = (\lambda_b, \lambda_f)$ and λ_b, λ_f are the lowest eigenvalues of h_b and h_f respectively. On the right side, $\boldsymbol{\Lambda} = (\Lambda_b, \Lambda_f)$, where Λ_b, Λ_f are the largest eigenvalues of h_b and h_f respectively. We assume that the lowest eigenvalue of h_b, λ_b is strictly positive.

The energy due to interaction between the bosons and fermions is represented by the operator,

$$V := \int_{X_b} dx \int_{X_f} dy \; v(x,y) a_f^\dagger(y) a_f(y) \left(a_b(x) + a_b^\dagger(x) \right),$$

where v is a real-valued function on $X_b \times X_f$. This operator is well-defined operator on each $\mathcal{H}_n$. Although it preserves fermion number, V does not preserve boson number. When applied to an n-boson state, this operator will return a sum of $(n-1)$-boson and $(n+1)$-boson states. Hence, V maps each $\mathcal{H}_n$ to $\mathcal{H}_{n-1} \oplus \mathcal{H}_{n+1}$, and may be extended linearly to $\mathcal{H}^{(finite)}$. It is easy to see that V is a symmetric operator on $\mathcal{H}^{(finite)}$.

Lemma 5.2.2. The interaction operator V satisfies the following bounds.

(a) For all $m, n \in \mathbb{N}_0$,

$$\|P^{(n)} V P^{(m)}\| \leq \begin{cases} \frac{1}{2} C_v \sqrt{n+1} & \text{if } m = n+1, \\ \frac{1}{2} C_v \sqrt{n} & \text{if } m = n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

where

$$C_v := 2|X_b||X_f|\|v\|_\infty. \tag{5.3}$$

and

$$\|v\|_\infty := \max_{x,y\in X_b\times X_f} |v(x,y)|.$$

(b) On the domain $\mathcal{H}^{(finite)}$,

$$-C_v\sqrt{N_b+1} \le V \le C_v\sqrt{N_b+1}. \tag{5.4}$$

Proof. (a) Recall from Proposition 2.2.4 that the boson creation and annihilation operators satisfy the bounds,

$$\|a_b(x)P^{(n)}\| \le \sqrt{n} \quad \text{and} \quad \|a_b^\dagger(x)P^{(n)}\| \le \sqrt{n+1},$$

for any $x \in X_b$. Also, the fermion number operator at $y \in X_f$ satisfies $\|n_f(y)\| = \|a_f^\dagger(y)a_f(y)\| \le 1$. Hence,

$$\begin{aligned}
&\|P^{(m)}VP^{(n)}\| \\
&\quad = \left\| P^{(m)} \int_{X_b} dx \int_{X_f} dy\; v(x,y)a_f^\dagger(y)a_f(y)\left(a_b(x)+a_b^\dagger(x)\right)P^{(n)}\right\| \\
&\quad \le \|v\|_\infty |X_f| \int_{X_b} dx \left\| P^{(m)}\left(a_b(x)+a_b^\dagger(x)\right)P^{(n)}\right\| \\
&\quad \le \begin{cases} \|v\|_\infty |X_f||X_b|\sqrt{n+1} & \text{if } m=n+1, \\ \|v\|_\infty |X_f||X_b|\sqrt{n} & \text{if } m=n-1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

(b) Let ψ be an arbitrary element of $\mathcal{H}^{(finite)}$. Denote the n-boson projection, $P^{(n)}\psi$, by ψ_n for all $n \in \mathbb{N}$. For convenience, let $\phi_{-1} := 0$. Since $\psi \in \mathcal{H}^{(finite)}$, we can choose an integer N such that $\psi_n = 0$ for all $n > N$. From Part a,

$$\begin{aligned}
\langle \psi, V\psi\rangle &= \sum_{n=0}^{\infty} \left(\langle \psi_n, V, \psi_{n+1}\rangle + \langle \psi_n, V, \psi_{n-1}\rangle\right) \\
&\le |X_b||X_f|\|v\|_\infty \sum_{n=0}^{\infty}\left(\sqrt{n+1}\;\|\psi_n\|\|\psi_{n+1}\| + \sqrt{n}\;\|\psi_n\|\|\psi_{n-1}\|\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2|X_b||X_f|\|v\|_\infty \sum_{n=0}^{\infty} \sqrt{n+1}\ \|\psi_n\|\|\psi_{n+1}\| \\
&\leq |X_b||X_f|\|v\|_\infty \sum_{n=0}^{\infty} \sqrt{n+1}\ \left(\|\psi_n\|^2 + \|\psi_{n+1}\|^2\right) \\
&\leq 2|X_b||X_f|\|v\|_\infty \sum_{n=0}^{\infty} \sqrt{n+1}\ \|\psi_n\|^2 \\
&= 2|X_b||X_f|\|v\|_\infty \langle \psi, \sqrt{N_b+1}\ \psi\rangle.
\end{aligned}
$$

By the same reasoning,

$$
\langle \psi, V\psi\rangle \geq -2|X_b||X_f|\|v\|_\infty \langle \psi, \sqrt{N_b+1}\ \psi\rangle.
$$

Hence,

$$
-C_v\sqrt{N_b+1} \leq V \leq C_v\sqrt{N_b+1},
$$

as claimed.

□

Proposition 5.2.3. The *Hamiltonian*,

$$
\begin{aligned}
H &:= H_0 + V \\
&= \int_{X_b^2} dx\, dy\ a_b^\dagger(x) h_b(x,y) a_b(y)\ +\ \int_{X_f^2} dx\, dy\ a_f^\dagger(x) h_f(x,y) a_f(y) \\
&\quad + \int_{X_b} dx \int_{X_f} dy\ v(x,y) a_f^\dagger(y) a_f(y) \left(a_b(x) + a_b^\dagger(x)\right)
\end{aligned}
$$

is essentially self-adjoint on the domain $\mathcal{H}^{(finite)}$.

Proof. From Lemma 5.2.1, H_0 is essentially self-adjoint on $\mathcal{H}^{(finite)}$. By the Kato-Rellich Theorem, Theorem X.12 of [1], if we can find real constants a and b, with $a < 1$, such that

$$
\|V\phi\| \leq a\|H_0\phi\| + b\|\phi\|, \tag{5.5}
$$

for all $\phi \in \mathcal{H}^{(finite)}$, the proof will be complete.

Let ϕ be an arbitrary element of $\mathcal{H}^{(finite)}$ and let $\phi_n := P^{(n)}\phi$ for all $n \in \mathbb{N}$. For convenience, let $\phi_{-1} := 0$. Since $\phi \in \mathcal{H}^{(finite)}$, we may choose $N \in \mathbb{N}$

such that $\phi_n = 0$ for all $n > N$. Then, $\phi = \sum_{n=0}^{N} \phi_n$ and

$$\begin{aligned}
\|V\phi\|^2 &= \sum_{n=0}^{N+1} \|P^{(n)} V\phi\|^2 \\
&= \sum_{n=0}^{N+1} \|P^{(n)} V(\phi_{n-1} + \phi_{n+1})\|^2 \\
&\leq (\|v\|_\infty |X_f||X_b|)^2 \sum_{n=0}^{N+1} \left(\sqrt{n}\|\phi_{n-1}\| + \sqrt{n+1}\|\phi_{n+1}\|\right)^2 \\
&\leq \frac{1}{2} C_v^2 \sum_{n=0}^{N} (2n+1)\|\phi_n\|^2,
\end{aligned}$$

where $C_v = 2|X_b||X_f|\|v\|_\infty$, as defined in Lemma 5.2.2. Noting that $n \leq \frac{n^2}{M} + M$ for any $M > 0$, let $M = \frac{C_v^2}{\epsilon^2 \lambda_b^2}$ for any $\epsilon > 0$, where λ_b is the smallest eigenvalue of h_b. Then,

$$\begin{aligned}
\|V\phi\|^2 &\leq \sum_{n=0}^{N} \left(\epsilon^2 \lambda_b^2 n^2 + \frac{C_v^4}{\epsilon^2 \lambda_b^2} + \frac{C_v^2}{2}\right) \|\phi_n\|^2 \\
&\leq \epsilon^2 \lambda_b^2 \langle \phi, N_b^2 \phi \rangle + \left(\frac{C_v^4}{\epsilon^2 \lambda_b^2} + \frac{C_v^2}{2}\right) \|\phi\|^2 \\
&\leq \epsilon^2 \langle \phi, H_0^2 \phi \rangle + \left(\frac{C_v^4}{\epsilon^2 \lambda_b^2} + \frac{C_v^2}{2}\right) \|\phi\|^2 \\
&= \epsilon^2 \|H_0 \phi\|^2 + \left(\frac{C_v^4}{\epsilon^2 \lambda_b^2} + \frac{C_v^2}{2}\right) \|\phi\|^2.
\end{aligned}$$

With $a = \epsilon$ and $b = \sqrt{\frac{C_v^4}{\epsilon^2 \lambda_b^2} + \frac{C_v^2}{2}}$, the inequality (5.5) holds. This completes the proof.

$\square$

Since our goal is to find a functional integral representation for the partition function,

$$Z = \text{Tr } e^{-\beta(H - \boldsymbol{\mu} \cdot \mathbf{N})}$$

where $\boldsymbol{\mu} = (\mu_b, \mu_f)$ is the vector of chemical potentials, are also interested in the operator

$$K := H - \boldsymbol{\mu} \cdot \mathbf{N} = H_0 + V - \boldsymbol{\mu} \cdot \mathbf{N}. \tag{5.6}$$

We can use the same argument as in the proof of Proposition 5.2.3 to show that K is essentially self-adjoint on $\mathcal{H}^{(finite)}$, provided that $\lambda_b - \mu_b > 0$. Indeed, we assume that both

$$\lambda_b > \mu_b \quad \text{and} \quad \lambda_f > \mu_f. \tag{5.7}$$

We use these assumptions along with the bounds on H_0 and V to find bounds on the operator K.

5.2.1 Bounds on K

From Equations (5.1), (5.4), and (5.6),

$$(\boldsymbol{\lambda} - \boldsymbol{\mu}) \cdot \mathbf{N} - C_v\sqrt{N_b + 1} \leq K \leq (\boldsymbol{\Lambda} - \boldsymbol{\mu}) \cdot \mathbf{N} + C_v\sqrt{N_b + 1}. \tag{5.8}$$

For any $\gamma > 0$,

$$C_v\sqrt{N_b + 1} \leq \gamma(N_b + 1) + \frac{C_v^2}{\gamma}.$$

Hence,

$$(\boldsymbol{\lambda} - \boldsymbol{\mu}) \cdot \mathbf{N} - C_v\sqrt{N_b + 1} \geq (\lambda_b - \mu_b - \gamma)\, N_b - \gamma - \frac{C_v^2}{\gamma}$$

If we take $\gamma = \frac{\lambda_b - \mu_b}{2}$, then

$$(\boldsymbol{\lambda} - \boldsymbol{\mu}) \cdot \mathbf{N} - C_v\sqrt{N_b + 1} \geq \frac{\lambda_b - \mu_b}{2} N_b - \kappa_L$$

where

$$\kappa_L := \frac{\lambda_b - \mu_b}{2} + \frac{C_v^2}{\gamma}. \tag{5.9}$$

Thus, we have a lower bound for K that is purely linear in boson number,

$$K \geq \frac{\lambda_b - \mu_b}{2} N_b - \kappa_L, \tag{5.10}$$

as well as the constant lower bound,

$$K \geq -\kappa_L. \tag{5.11}$$

5.3 A Functional Integral Representation for the Partition Function

Definition 14. The *partition function* is

$$Z = \text{Tr } e^{-\beta K}$$

where $\beta > 0$ represents the inverse temperature.

The goal in this section is to derive a functional representation for the partition function in the same style as (4.26):

$$\text{Tr } e^{-\beta K} = \lim_{q\to\infty} \int \prod_{\tau\in\mathcal{T}_q} \left[d\mu_{R(q)}(\boldsymbol{\phi}^*_\tau, \boldsymbol{\phi}_\tau) e^{-\int d\mathbf{y}\ \boldsymbol{\phi}^*_\tau(\mathbf{y})\cdot\boldsymbol{\phi}_\tau(\mathbf{y})}\right] \prod_{\tau\in\mathcal{T}_q} \langle \boldsymbol{\phi}_{\tau-\epsilon}| e^{-\epsilon K} |\boldsymbol{\phi}_\tau\rangle,$$

where $\epsilon := \frac{\beta}{q}$, $\mathcal{T}_q := \{\epsilon, 2\epsilon, \ldots, q\epsilon\}$ and

$$\int d\mathbf{y}\ \boldsymbol{\phi}^*_\tau(\mathbf{y})\cdot\boldsymbol{\phi}_\tau(\mathbf{y}) := \int_{X_b} dy\ \phi^*_b(y)\phi_b(y) + \int_{X_f} dy\ \phi^*_f(y)\phi_f(y).$$

The field $\boldsymbol{\phi}_0$ is defined by the boundary condition,

$$\boldsymbol{\phi}_0 = \tilde{\boldsymbol{\phi}}_\beta = (\phi_b, -\phi_f).$$

In order for the right-hand side to converge, the cutoff radius $R(q)$ must grow sufficiently quickly as $q \to \infty$. (We will see this in more detail later on.)

In the case of particle-preserving interactions, we repeatedly used an expansion over fixed-particle number subspaces in order to derive this type of functional integral representation. In this case, the operator K does not preserve bosons; as a result, the density operator $e^{-\beta K}$ may connect states with different boson numbers. That is, $P^{(n)}e^{-\beta K}P^{(m)}$ need not be zero when $n \neq m$. However, if we let

$$K_0 := H_0 - \boldsymbol{\mu}\cdot\mathbf{N},$$

then K_0 does preserve the number of bosons. The interaction V changes the boson number by exactly one and we have bounds on $P^{(n)}VP^{(n\pm1)}$. We will see that we can exploit these properties by expanding in powers of the interaction V and then carrying out our analysis. This is called a *Duhamel* *expansion*; we prove one version of this below.

Theorem 5.3.1. (Duhamel's Formula) For any $\varphi, \psi \in \mathcal{H}^{(finite)}$, $t > 0$ and $N \in \mathbb{N}_0$,

$$\langle \varphi, e^{-tK}\psi \rangle = \sum_{n=0}^{N} A_n(t,\varphi,\psi) + \epsilon_{N+1}(t,\varphi,\psi)$$

where for all $n \in \mathbb{N}$,

$$A_n(t,\varphi,\psi) := (-1)^n \int_{\Omega_n(t)} ds_1 \dots ds_n \ \left\langle \varphi, e^{-(t-\sum_{k=1}^n s_k)K_0} \left[\prod_{k=1}^{n} V e^{-s_k K_0} \right] \psi \right\rangle. \tag{5.12}$$

The domain of integration here is

$$\Omega_n(t) := \{(s_1, \dots, s_n) \in [0,t]^n : \sum_{k=1}^{n} s_k \le t\}. \tag{5.13}$$

In the case $n = 0$, $A_0(t,\varphi,\psi) = \langle \varphi, e^{-tK_0}\psi \rangle$. The "error term" is defined by

$$\epsilon_n(t,\varphi,\psi) = (-1)^n \int_{\Omega_n(t)} ds_1 \dots ds_n \left\langle \varphi, e^{-(t-\sum_{k=1}^n s_k)K} \left[\prod_{k=1}^{n} V e^{-s_k K_0} \right] \psi \right\rangle. \tag{5.14}$$

Proof. Fix φ and ψ in $\mathcal{H}^{(finite)}$ and let

$$J_t(s) := \left\langle \varphi, e^{-(t-s)(K_0+V)} e^{-sK_0} \psi \right\rangle,$$

for all s, t with $0 \le s \le t$. Then $J_t(0) = \langle \varphi, e^{-tK}\psi \rangle$, $J_t(t) = \langle \varphi, e^{-tK_0}\psi \rangle$, and

$$\begin{aligned} J_t'(s) &= \left\langle \varphi, \left(e^{-(t-s)(K_0+V)}(K_0+V)e^{-sK_0} - e^{-(t-s)(K_0+V)}e^{-sK_0}K_0 \right) \psi \right\rangle \\ &= \left\langle \varphi, e^{-(t-s)(K_0+V)} V e^{-sK_0} \psi \right\rangle. \end{aligned}$$

Hence,

$$\begin{aligned} \left\langle \varphi, e^{-t(K_0+V)}\psi \right\rangle &= J_t(0) = J_t(t) - \int_0^t ds \ J_t'(s) \\ &= \left\langle \varphi, e^{-tK_0}\psi \right\rangle - \int_0^t ds \ \left\langle \varphi, e^{-(t-s)(K_0+V)} V e^{-sK_0} \psi \right\rangle. \end{aligned} \tag{5.15}$$

This is the $N = 0$ case of the claim. We proceed by induction. It suffices to show that $\epsilon_n(t,\varphi,\psi) = A_n(t,\varphi,\psi) + \epsilon_{n+1}(t,\varphi,\psi)$ for all $n \ge 1$. We use the same idea from the base case, applying it to the exponential function of K that appears in the definition of $\epsilon_n(t,\varphi,\psi)$. That is,

$$\epsilon_n(t,\varphi,\psi) = (-1)^n \int_{\Omega_n(t)} ds_1 \dots ds_n \left\langle \varphi, e^{-(t-\sum_{k=1}^n s_k)K} \left[\prod_{k=1}^n V e^{-s_k K_0}\right] \psi \right\rangle$$

$$= (-1)^n \int_{\Omega_n(t)} ds_1 \dots ds_n \left\langle \varphi, e^{-(t-\sum\limits_{k=1}^{n+1} s_k)K} e^{-s_{n+1}K_0} \left[\prod_{k=1}^n V e^{-s_k K_0}\right] \psi \right\rangle\Bigg|_{s_{n+1}=0}$$

Let's consider the right-hand side as a function f of s_{n+1} evaluated at $s_{n+1} = 0$; that is, let

$$f(s) := (-1)^n \int_{\Omega_n(t)} ds_1 \dots ds_n \left\langle \varphi, e^{-(t-s-\sum_{k=1}^n s_k)K} e^{-sK_0} \left[\prod_{k=1}^n V e^{-s_k K_0}\right] \psi \right\rangle.$$

Then

$$\epsilon_n(t,\varphi,\psi) = f(0) = f\left(t - \sum_{k=1}^n s_k\right) - \int_0^{t-\sum\limits_{k=1}^n s_k} ds_{n+1}\ f'(s_{n+1}).$$

Here,

$$f\left(t - \sum_{k=1}^n s_k\right) = (-1)^n \int_{\Omega_n(t)} ds_1 \dots ds_n \left\langle \varphi, e^{-\left(t-\sum\limits_{k=1}^n s_k\right)K_0} \left[\prod_{k=1}^n V e^{-s_k K_0}\right] \psi \right\rangle$$
$$= A_n(t,\varphi,\psi).$$

The second term is

$$- \int_0^{t-\sum\limits_{k=1}^n s_k} ds_{n+1}\ f'(s_{n+1})$$
$$= - \int_0^{t-\sum\limits_{k=1}^n s_k} ds_{n+1} \frac{\partial}{\partial s_{n+1}} (-1)^n \int_{\Omega_n(t)} ds_1 \dots ds_n \left\langle \varphi, e^{-(t-\sum_{k=1}^{n+1} s_k)K} e^{-s_{n+1}K_0} \right.$$
$$\left. \cdot \left[\prod_{k=1}^n V e^{-s_k K_0}\right] \psi \right\rangle$$
$$= (-1)^{n+1} \int_{\Omega_{n+1}(t)} ds_1 \dots ds_{n+1} \left\langle \varphi, e^{-(t-\sum_{k=1}^{n+1} s_k)K} (K - K_0) e^{-s_{n+1}K_0} \right.$$

$$\cdot \left[\prod_{k=1}^{n} V e^{-s_k K_0}\right] \psi \Big\rangle$$

$$= (-1)^{n+1} \int_{\Omega_{n+1}(t)} ds_1 \dots ds_{n+1} \left\langle \varphi, e^{-(t-\sum_{k=1}^{n+1} s_k)K} \left[\prod_{k=1}^{n+1} V e^{-s_k K_0}\right] \psi \right\rangle$$

$$= \epsilon_{n+1}(t, \varphi, \psi).$$

Hence,

$$\epsilon_n(t, \varphi, \psi) = A_n(t, \varphi, \psi) + \epsilon_{n+1}(t, \varphi, \psi).$$

This completes the proof.

□

Our next step is to take the limit as $N \to \infty$ in the Duhamel expansion of Theorem 5.3.1 and prove that the resulting sum is absolutely convergent. To this end, we'll need bounds on each term in the expansion, $A_k(t, \phi, \psi)$, as well as the error term, $\epsilon_k(t, \phi, \psi)$, for large k.

Lemma 5.3.2. Let $t > 0$ and $n, m \in \mathbb{N}_0$ be arbitrary.

(a) For any $M \in \mathbb{N}$, $\psi \in \mathcal{H}$ and $\phi \in \mathcal{H}_n$ with $\|\psi\| = \|\phi\| = 1$,

$$|\epsilon_M(t, \psi, \phi)| \leq \frac{(4\frac{C_v}{C_b} \max\{t, \sqrt{t}\})^M}{\sqrt{M!}} e^{t\kappa_L}, \tag{5.16}$$

where $C_b := \min\{\sqrt{\lambda_b - \mu_b}, 1\}$.

(b) For any $\psi \in \mathcal{H}_m, \phi \in \mathcal{H}_n$ with $\|\psi\| = \|\phi\| = 1$,

$$|A_k(t, \psi, \phi)| \leq e^{-\frac{1}{8}t(\lambda_b - \mu_b)m} \frac{C_2(t)^k}{\sqrt{k!}}, \tag{5.17}$$

where

$$\begin{aligned} C_2(t) &:= 4\frac{C_v}{C_b} \max\{t, \sqrt{t}\} e^{\frac{1}{4}t(\lambda_b - \mu_b)} \\ &= \frac{8|X_b||X_f|\|v\|_\infty}{\min\{\sqrt{\lambda_b - \mu_b}, 1\}} \max\{t, \sqrt{t}\} e^{\frac{1}{4}t(\lambda_b - \mu_b)}. \end{aligned} \tag{5.18}$$

(c) For any $\psi \in \mathcal{H}_m, \phi \in \mathcal{H}_n$ with $\|\psi\| = \|\phi\| = 1$,

$$|A_k(t,\psi,\phi)| \leq \frac{C_2(t)^k}{(k!|m-n|!)^{1/4}} e^{-\frac{1}{8}t(\lambda_b-\mu_b)m}. \tag{5.19}$$

Proof. (a) For any $M \in \mathbb{N}$, $\psi \in \mathcal{H}$ and $\phi \in \mathcal{H}_n$ with $\|\psi\| = \|\phi\| = 1$,

$$\begin{aligned}|\epsilon_M(t,\psi,\phi)| &= \left|\int_{\Omega_M(t)} ds_1 \dots ds_M \ \left\langle \psi, e^{-(t-\sum_{k=1}^M s_k)K} \left[\prod_{k=1}^M V e^{-s_k K_0}\right] \phi \right\rangle\right| \\ &\leq \int_{\Omega_M(t)} ds_1 \dots ds_M \ e^{(t-\sum_{k=1}^M s_k)\kappa_L} \left\| \left[\prod_{k=1}^M V e^{-s_k K_0}\right] P^{(n)} \right\|\end{aligned}$$

where $-\kappa_L$ is the lower bound for K from Equation (5.11). Note that K_0 preserves boson number and satisfies the bounds

$$(\boldsymbol{\lambda} - \boldsymbol{\mu}) \cdot \mathbf{N} \leq K_0 \leq (\boldsymbol{\Lambda} - \boldsymbol{\mu}) \cdot \mathbf{N}. \tag{5.20}$$

Also, V changes the boson number by one. Hence,

$$\begin{aligned}|\epsilon_M(t,\psi,\phi)| \leq \int_{\Omega_M(t)} ds_1 \dots ds_M \ e^{(t-\sum_{k=1}^M s_k)\kappa_L} \\ \cdot \sum_{\substack{\{n_k\}_{k=0}^M \\ n_M = n \\ (\forall k)|n_k - n_{k-1}|=1}} \prod_{k=1}^M \left\| P^{(n_{k-1})} V P^{(n_k)} \right\| e^{-s_k(\lambda_b-\mu_b)n_k}.\end{aligned}$$

From now on, we'll understand that the sum is only over sequences $\{n_k\}_{k=0}^M$ with $n_M = n$ and $|n_k - n_{k-1}| = 1$ for all $k \geq 1$, without writing all conditions explicitly. Using the bounds on $\left\|P^{(n)} V P^{(m)}\right\|$ from Equation (5.2) and bounding $\sum_{k=1}^M s_k$ from below by zero,

$$\begin{aligned}|\epsilon_M(t,\psi,\phi)| \leq \left(\frac{C_v}{2}\right)^M \int_{\Omega_M(t)} ds_1 \dots ds_M \ e^{t\kappa_L} \\ \cdot \sum_{\{n_k\}} \prod_{k=1}^M \left[\sqrt{n_k+1} \ e^{-s_k(\lambda_b-\mu_b)n_k}\right].\end{aligned}$$

We'll use the exponential inside the product over k to cancel the factor of $\sqrt{n_k+1}$. To this end, note that for any $\alpha > 0$ and $n \geq 1$,

$$\begin{aligned} \alpha(n+1) \leq 2\alpha n &\leq e^{2\alpha n}. \\ \implies \sqrt{\alpha(n+1)} &\leq e^{\alpha n} \\ \implies e^{-\alpha n} &\leq \frac{1}{\sqrt{\alpha(n+1)}}. \end{aligned}$$

Hence, if $n_k \geq 1$,

$$\sqrt{n_k+1}\; e^{-s_k(\lambda_b-\mu_b)n_k} \leq \frac{1}{\sqrt{s_k(\lambda_b-\mu_b)}}. \tag{5.21}$$

Otherwise $n_k = 0$ and $\sqrt{n_k+1}\; e^{-s_k(\lambda_b-\mu_b)n_k} = 1$. To account for both cases, we simply let the product run only over k such that $n_k > 0$. That is,

$$|\epsilon_M(t,\psi,\phi)| \leq \left(\frac{C_v}{2C_b}\right)^M e^{t\kappa_L} \int_{\Omega_M(t)} ds_1 \cdots ds_M \sum_{\{n_k\}} \prod_{\substack{k=1 \\ n_k>0}}^{M} \frac{1}{\sqrt{s_k}}.$$

where $C_b := \min\{\sqrt{\lambda_b-\mu_b}, 1\}$.

We will show in the two lemmas after this proof that

$$\int_{\Omega_M(t)} ds_1 \ldots ds_M \prod_{\substack{k=1 \\ n_k>0}}^{M} \frac{1}{\sqrt{s_k}} \leq \frac{(4\max\{t,\sqrt{t}\})^M}{\sqrt{M!}},$$

a bound that it is independent of the sequence $\{n_k\}$. Since all sequences $\{n_k\}_{k=1}^M$ must have $n_M = n$, they can each be identified by their sequence of differences $\Delta_k = n_k - n_{k-1} = \pm 1$. Therefore, the number of allowed sequences $\{n_k\}_{k=0}^M$ is equal to 2^M. Hence,

$$|\epsilon_M(t,\psi,\phi)| \leq \frac{(4\frac{C_v}{C_b}\max\{t,\sqrt{t}\})^M}{\sqrt{M!}} e^{t\kappa_L}.$$

(b) Let $\psi \in \mathcal{H}_m, \phi \in \mathcal{H}_n$ with $\|\psi\| = \|\phi\| = 1$.

Note that $A_k(t,\psi,\phi)$ and $\epsilon_k(t,\psi,\phi)$ differ only in the appearance of K_0 or K in the leftmost exponent. Since K_0 is also bounded below (by zero), we can apply exactly the same analysis as above to obtain the bound

$$|A_k(t,\psi,\phi)| \leq \frac{(4\frac{C_v}{C_b}\max\{t,\sqrt{t}\})^k}{\sqrt{k!}}. \tag{5.22}$$

The right-hand side is the same as in (5.16), except with κ_L replaced by 0.

However, we'll also require a bound that diminishes as $m,n \to \infty$. Many of the steps in the following calculation are the same as in the proof of Part a, except that we use the assumption that $\psi \in \mathcal{H}_m$ (to project onto $\mathcal{H}_m$ in the second line below) and we'll be more careful keep bounds that diminish as $n_k \to 0$ at each step.

$$\begin{aligned}
|A_k(t,\psi,\phi)| &= \left|\int_{\Omega_k(t)} ds_1 \dots ds_k \left\langle \psi, e^{-(t-\sum_{j=1}^k s_j)K_0} \left[\prod_{j=1}^k V e^{-s_j K_0}\right]\phi\right\rangle\right| \\
&\leq \int_{\Omega_k(t)} ds_1 \dots ds_k\ e^{-(t-\sum_{j=1}^k s_j)(\lambda_b-\mu_b)m} \left\| P^{(m)} \left[\prod_{j=1}^k V e^{-s_j K_0}\right] P^{(n)} \right\| \\
&\leq \int_{\Omega_k(t)} ds_1 \dots ds_k\ e^{-(t-\sum_{j=1}^k s_j)(\lambda_b-\mu_b)m} \\
&\qquad \cdot \sum_{\substack{\{n_j\}_{j=0}^k \\ n_0=m, n_k=n \\ (\forall j)|n_j-n_{j-1}|=1}} \prod_{j=1}^k \left(\left\|P^{(n_{j-1})} V P^{(n_j)}\right\| \left\|P^{(n_j)} e^{-s_j K_0} P^{(n_j)}\right\| \right) \\
&\leq \int_{\Omega_k(t)} ds_1 \dots ds_k\ e^{-(t-\sum_{j=1}^k s_j)(\lambda_b-\mu_b)m} \sum_{\{n_j\}} \prod_{j=1}^k \left(\frac{C_v}{2}\sqrt{n_j+1} e^{-s_j(\lambda_b-\mu_b)n_j} \right)
\end{aligned} \tag{5.23}$$

In the last line, it is understood that the sequences $\{n_j\}_{j=0}^k$ in the domain of the sum still must satisfy the conditions $n_0 = m$, $n_k = n$, and $|n_j - n_{j-1}| = 1$ for all $j \geq 1$. Note that, in order for there to be any such sequences, $m+n-k$ must be even, in which case these sequences must satisfy

$$n_j \geq \frac{m+n-k}{2},$$

for all $j \in \{1, \ldots, k\}$. Consider the case

$$k < \frac{m+n}{2}. \tag{5.24}$$

In this case, $n_j \geq \frac{m+n}{4}$. Then

$$\begin{aligned}\prod_{j=1}^{k} e^{-s_j(\lambda_b-\mu_b)n_j} &\leq \prod_{j=1}^{k}\left(e^{-\frac{1}{2}s_j(\lambda_b-\mu_b)n_j} e^{-\frac{1}{2}s_j(\lambda_b-\mu_b)\frac{m+n}{4}}\right) \\ &\leq e^{-\frac{1}{8}\sum_{j=1}^{k} s_j(\lambda_b-\mu_b)(m+n)} \prod_{j=1}^{k} e^{-\frac{1}{2}s_j(\lambda_b-\mu_b)n_j}\end{aligned}$$

Hence,

$$\begin{aligned}|A_k(t,\psi,\phi)| \leq \int_{\Omega_k(t)} & ds_1 \ldots ds_k \; e^{-(t-\frac{7}{8}\sum_{j=1}^{k} s_j)(\lambda_b-\mu_b)m} \\ & \cdot \sum_{\{n_j\}} \prod_{j=1}^{k}\left(\frac{C_v}{2}\sqrt{n_j+1}e^{-\frac{1}{2}s_j(\lambda_b-\mu_b)n_j}\right) \\ \leq \left(\frac{C_v}{2}\right)^k e^{-\frac{1}{8}t(\lambda_b-\mu_b)m} & \int_{\Omega_k(t)} ds_1 \ldots ds_k \; \sum_{\{n_j\}} \prod_{j=1}^{k}\left(\sqrt{n_j+1}e^{-\frac{1}{2}s_j(\lambda_b-\mu_b)n_j}\right)\end{aligned}$$

As in (5.21), when $n_k > 0$,

$$\sqrt{n_k+1}\; e^{-\frac{1}{2}s_k(\lambda_b-\mu_b)n_k} \leq \frac{1}{\sqrt{\frac{1}{2}s_k(\lambda_b-\mu_b)}}.$$

Hence,

$$\begin{aligned}|A_k(t,\psi,\phi)| &\leq \left(\frac{C_v}{2}\right)^k e^{-\frac{1}{8}t(\lambda_b-\mu_b)m} \int_{\Omega_k(t)} ds_1 \ldots ds_k \sum_{\{n_j\}} \prod_{\substack{j=1 \\ n_j>0}}^{k} \sqrt{\frac{2}{s_k(\lambda_b-\mu_b)}} \\ &\leq \left(\frac{C_v}{\sqrt{2}C_b}\right)^k e^{-\frac{1}{8}t(\lambda_b-\mu_b)m} \int_{\Omega_k(t)} ds_1 \ldots ds_k \;\; \sum_{\{n_j\}} \prod_{\substack{j=1 \\ n_j>0}}^{k} \frac{1}{\sqrt{s_k}}\end{aligned}$$

$$\leq \left(\frac{C_v}{\sqrt{2}C_b}\right)^k e^{-\frac{1}{8}t(\lambda_b-\mu_b)m} \frac{(4\max\{t,\sqrt{t}\})^k}{\sqrt{k!}}$$
$$\leq \frac{\left(4\frac{C_v}{C_b}\max\{t,\sqrt{t}\}\right)^k}{\sqrt{k!}} e^{-\frac{1}{8}t(\lambda_b-\mu_b)m}.$$

Thus, in the case $k < \frac{m+n}{2}$,

$$|A_k(t,\psi,\phi)| \leq \frac{\left(4\frac{C_v}{C_b}\max\{t,\sqrt{t}\}\right)^k}{\sqrt{k!}} e^{-\frac{1}{8}t(\lambda_b-\mu_b)m}. \tag{5.25}$$

Let's now consider the case $k \geq \frac{m+n}{2}$. Here, we can use the bound from (5.22),

$$|A_k(t,\psi,\phi)| \leq \frac{(4\frac{C_v}{C_b}\max\{t,\sqrt{t}\})^k}{\sqrt{k!}}.$$

In the calculation below, let $C_1(t) := 4\frac{C_v}{C_b}\max\{t,\sqrt{t}\}$. Since $k \geq \frac{m}{2}$,

$$\begin{aligned}
|A_k(t,\psi,\phi)| &\leq \frac{C_1(t)^k}{\sqrt{k!}} \\
&\leq \frac{C_1(t)^k}{\sqrt{k!}} \frac{e^{-\frac{1}{8}t(\lambda_b-\mu_b)m}}{e^{-\frac{1}{8}t(\lambda_b-\mu_b)(2k)}} \\
&\leq \frac{\left(C_1(t)e^{\frac{1}{4}t(\lambda_b-\mu_b)}\right)^k}{\sqrt{k!}} e^{-\frac{1}{8}t(\lambda_b-\mu_b)m} \\
&\leq \frac{C_2(t)^k}{\sqrt{k!}} e^{-\frac{1}{8}t(\lambda_b-\mu_b)m},
\end{aligned} \tag{5.26}$$

where

$$C_2(t) := C_1(t)e^{\frac{1}{4}t(\lambda_b-\mu_b)} \geq C_1(t).$$

From (5.25) for $k \leq \frac{m+n}{2}$ and (5.26) for $k > \frac{m+n}{2}$, we can conclude that, for all $k \in \mathbb{N}$,

$$|A_k(t,\psi,\phi)| \leq \frac{C_2(t)^k}{\sqrt{k!}} e^{-\frac{1}{8}t(\lambda_b-\mu_b)m}.$$

(c) In part (b), note that $|m-n| \leq k$ in order for $A_k(t, \psi, \phi)$ to be nonzero. Hence,

$$|A_k(t, \psi, \phi)| \leq \frac{C_2(t)^k}{(k!|m-n|!)^{1/4}} e^{-\frac{1}{8}t(\lambda_b - \mu_b)m}.$$

□

Lemma 5.3.3. For all $n \in \mathbb{N}$,

$$I_n(t) := \int_{\Omega_n(t)} \prod_{k=1}^{n} \frac{1}{\sqrt{s_k}} \; ds_1 \dots ds_n \;\leq \frac{(4\sqrt{t})^n}{\sqrt{n!}},$$

where, as above, $\Omega_n(t) := \{(s_1, \dots, s_n) \in [0, t]^n : \sum_{k=1}^{n} s_k \leq t\}$.

Proof. Make the substitution $u_j := s_j/t$ for each j. Then,

$$I_n(t) = \int_{\Omega_n(1)} \frac{1}{\prod_{k=1}^{n} \sqrt{tu_k}} \; t^n \; du_1 \dots du_n = t^{n/2} I_n(1).$$

In the case $n = 1$,

$$I_1(1) = \int_0^1 \frac{1}{\sqrt{s}} ds = 2\sqrt{s}\Big|_0^1 = 2.$$

For any $n \in \mathbb{N}$,

$$\begin{aligned} I_{n+1}(1) &= \int_{\Omega_{n+1}(1)} \frac{1}{\prod_{k=1}^{n+1} \sqrt{s_k}} \; ds_1 \dots ds_n ds_{n+1} \\ &= \int_0^1 ds_{n+1} \frac{1}{\sqrt{s_{n+1}}} \int_{\Omega_n(1-s_{n+1})} ds_1 \dots ds_n \frac{1}{\prod_{k=1}^{n} \sqrt{s_k}} \\ &= \int_0^1 ds \frac{1}{\sqrt{s}} I_n(1-s) \\ &= \int_0^1 ds \frac{1}{\sqrt{s}} (1-s)^{n/2} I_n(1). \end{aligned}$$

Now, for any $\epsilon \in (0,1)$,

$$
\begin{aligned}
\int_0^1 ds \frac{(1-s)^{n/2}}{\sqrt{s}} &= \int_0^\epsilon ds \frac{(1-s)^{n/2}}{\sqrt{s}} + \int_\epsilon^1 ds \frac{(1-s)^{n/2}}{\sqrt{s}} \\
&\leq \int_0^\epsilon ds \frac{1}{\sqrt{s}} + \int_0^1 ds \frac{(1-s)^{n/2}}{\sqrt{\epsilon}} \\
&= 2\sqrt{s}\Big|_0^\epsilon + \frac{1}{\sqrt{\epsilon}} \left[-\frac{(1-s)^{n/2+1}}{n/2+1} \right]_0^1 \\
&= 2\sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}} \cdot \frac{1}{n/2+1}.
\end{aligned}
$$

If we take $\epsilon = \frac{1}{n+1}$, then we find

$$
\int_0^1 ds \frac{(1-s)^{n/2}}{\sqrt{s}} \leq \frac{2}{\sqrt{n+1}} + \frac{\sqrt{n+1}}{n/2+1} \leq \frac{4}{\sqrt{n+1}}.
$$

Hence,

$$
I_{n+1}(1) \leq \frac{4}{\sqrt{n+1}} I_n(1).
$$

By induction, it follows that

$$
I_n(1) \leq \frac{4^{n-1}}{\sqrt{n!}} I_1(1) < \frac{4^n}{\sqrt{n!}}.
$$

Hence,

$$
I_n(t) \leq \frac{(4\sqrt{t})^n}{\sqrt{n!}},
$$

as claimed. □

Lemma 5.3.4. For any $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $0 \leq k \leq n$,

$$
J_{n,k}(t) := \int_{\Omega_n(t)} ds_1 \dots ds_n \prod_{j=k+1}^{n} \frac{1}{\sqrt{s_j}} \leq \frac{4^n t^{\frac{n+k}{2}}}{\sqrt{(n+k)!}}.
$$

Furthermore,

$$
J_{n,k}(t) \leq \frac{(4\max\{t, \sqrt{t}\})^n}{\sqrt{n!}}.
$$

This second bound does not depend on k.

Proof. The proof is by induction on k, with arbitrary $n \in \mathbb{N}$ satisfying $n \geq k$. Lemma 5.3.3 is the case $k = 0$. Assuming that the claim holds for some k and all $n \geq k$, then for any $n \geq k+1$ with $n > 1$,

$$\begin{aligned}
J_{n,k+1}(t) &= \int_{\Omega_n(t)} ds_1 \dots ds_n \prod_{j=k+2}^{n} \frac{1}{\sqrt{s_j}} \\
&= \int_0^t ds_1 \int_{\Omega_{n-1}(t-s_1)} ds_2 \dots ds_n \prod_{j=k+2}^{n} \frac{1}{\sqrt{s_j}} \\
&= \int_0^t ds\ J_{n-1,k}(t-s) \\
&\leq \int_0^t ds\ \frac{4^{n-1}(t-s)^{\frac{n-1+k}{2}}}{\sqrt{(n-1+k)!}} \\
&= \frac{4^{n-1}}{\sqrt{(n+k-1)!}} \left[\frac{-(t-s)^{\frac{n+k+1}{2}}}{\frac{n+k+1}{2}} \right]_{s=0}^{t} \\
&= \frac{2 \cdot 4^{n-1}}{\sqrt{(n+k-1)!}} \frac{t^{\frac{n+k+1}{2}}}{n+k+1} \\
&\leq \frac{4^n t^{\frac{n+k+1}{2}}}{\sqrt{(n+k+1)!}},
\end{aligned}$$

as claimed. In the case $n = 1$, k must be equal to 0. Then

$$J_{n,k+1}(t) = J_{1,1}(t) = \int_0^t ds\ 1 = t,$$

which also satisfies the bound above.

□

Lemma 5.3.5. For all $m, n \in \mathbb{N}_0$,

$$\|P^{(m)} e^{-tK} P^{(n)}\| \leq \sqrt{2} e^{C_2(t)^2} e^{-\frac{1}{8}t(\lambda_b - \mu_b)m}. \tag{5.27}$$

Furthermore, for any $r > 1$,

$$\|P^{(m)}e^{-tK}P^{(n)}\| \leq \frac{e^{-\frac{1}{8}t(\lambda_b-\mu_b)m}}{|m-n|!^{1/4}}\left(\frac{1}{1-\frac{1}{r^{4/3}}}\right)^{3/4} e^{\frac{1}{4}(rC_2(t))^4}. \tag{5.28}$$

Proof. For any $m, n \in \mathbb{N}_0$,

$$\|P^{(m)}e^{-tK}P^{(n)}\| = \sup_{\substack{\psi\in\mathcal{H}_m,\,\phi\in\mathcal{H}_n\\ \|\psi\|=\|\phi\|=1}} \left|\langle\psi, e^{-tK}\phi\rangle\right|.$$

From Lemma 5.3.2a, for any $\psi \in \mathcal{H}_m$, $\phi \in \mathcal{H}_n$,

$$\lim_{M\to\infty}|\epsilon_M(t,\psi,\phi)| = 0.$$

Hence,

$$\begin{aligned}\left|\langle\psi, e^{-tK}\phi\rangle\right| &= \left|\sum_{k=0}^{\infty} A_k(t,\psi,\phi)\right| \\ &\leq \sum_{k=0}^{\infty}|A_k(t,\psi,\phi)| \\ &\leq \sum_{k=0}^{\infty}\frac{C_2(t)^k}{\sqrt{k!}}e^{-\frac{1}{8}t(\lambda_b-\mu_b)m},\end{aligned}$$

where the last line is from Lemma 5.3.2b.

Note that

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{C_2(t)^k}{\sqrt{k!}} &= \sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k\frac{(\sqrt{2}C_2(t))^k}{\sqrt{k!}} \\ &\leq \left(\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k\right)^{1/2}\left(\sum_{k=0}^{\infty}\frac{(2C_2(t)^2)^k}{k!}\right)^{1/2} \\ &\leq \sqrt{2}e^{C_2(t)^2}.\end{aligned}$$

Hence,

$$\|P^{(m)}e^{-tK}P^{(n)}\| \leq \sqrt{2}e^{C_2(t)^2}e^{-\frac{1}{8}t(\lambda_b-\mu_b)m},$$

as claimed in (5.27).

To derive the tighter bound in (5.28), we use Lemma 5.3.2c.

$$\begin{aligned}
\|P^{(m)}e^{-tK}P^{(n)}\| &= \sup_{\substack{\psi\in\mathcal{H}_m,\phi\in\mathcal{H}_n\\ \|\psi\|=\|\phi\|=1}} \left|\langle\psi, e^{-tK}\phi\rangle\right| \\
&\leq \sup_{\substack{\psi\in\mathcal{H}_m,\phi\in\mathcal{H}_n\\ \|\psi\|=\|\phi\|=1}} \sum_{k=0}^{\infty}|A_k(t,\psi,\phi)| \\
&\leq \sum_{k=0}^{\infty}\frac{C_2(t)^k}{(k!|m-n|!)^{1/4}}e^{-\frac{1}{8}t(\lambda_b-\mu_b)m} \\
&\leq \frac{e^{-\frac{1}{8}t(\lambda_b-\mu_b)m}}{|m-n|!^{1/4}}\sum_{k=0}^{\infty}\frac{1}{r^k}\frac{(rC_2(t))^k}{(k!)^{1/4}} \\
&\leq \frac{e^{-\frac{1}{8}t(\lambda_b-\mu_b)m}}{|m-n|!^{1/4}}\left(\sum_{k=0}^{\infty}\frac{1}{r^{\frac{4}{3}k}}\right)^{3/4}\left(\sum_{k=0}^{\infty}\frac{(rC_2(t))^{4k}}{k!}\right)^{1/4} \\
&\leq \frac{e^{-\frac{1}{8}t(\lambda_b-\mu_b)m}}{|m-n|!^{1/4}}\left(\frac{1}{1-\frac{1}{r^{4/3}}}\right)^{3/4} e^{\frac{1}{4}(rC_2(t))^4}.
\end{aligned}$$

where $r > 1$ is an arbitrary constant.

□

Theorem 5.3.6. For any $t > 0$, the partition function can be expanded as

$$\mathrm{Tr}(e^{-tK}) = \sum_{n=0}^{\infty}\sum_{\substack{Y=(Y_b,Y_f)\\ Y_b\in X_b^n/S_n\\ Y_f\subset X_f}}\sum_{k=0}^{\infty}A_k(t,\delta_Y,\delta_Y), \tag{5.29}$$

where $\delta_Y := \delta_{Y_b}\otimes\delta_{Y_f}$ for each $Y = (Y_b, Y_f)$. Furthermore, the series above is absolutely convergent.

Proof. First, we prove that the sum on the right-hand side of Equation (5.29) is absolutely convergent. Noting that there are $2^{|X_f|}$ subsets of X_f

and $\binom{|X_b|+n-1}{n}$ elements of X_b^n/S_n and applying Lemma 5.3.2b,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{\substack{Y=(Y_b,Y_f)\\ Y_b\in X_b^n/S_n\\ Y_f\subset X_f}} \sum_{k=0}^{\infty} |A_k(t,\delta_Y,\delta_Y)| \\
&\qquad \le \sum_{n=0}^{\infty} 2^{|X_f|} \binom{|X_b|+n-1}{n} \sum_{k=0}^{\infty} e^{-\frac{1}{8}t(\lambda_b-\mu_b)n} \frac{C_2(t)^k}{\sqrt{k!}} \\
&\qquad \le 2^{|X_f|} \left(\sum_{n=0}^{\infty} \frac{(|X_b|+n-1)^{|X_b|-1}}{(|X_b|-1)!} e^{-\frac{1}{8}t(\lambda_b-\mu_b)n} \right) \sum_{k=0}^{\infty} \frac{C_2(t)^k}{\sqrt{k!}} \qquad (5.30) \\
&\qquad < \infty.
\end{aligned}$$

Indeed, the sum converges absolutely.

By Theorem 5.3.1, for any $n, M \in \mathbb{N}$ and $Y = (Y_b, Y_f)$ with $Y_b \in X_b^n/S_n$ and $Y_f \subset X_f$,

$$\langle \delta_Y, e^{-tK}\delta_Y \rangle = \sum_{k=0}^{M} A_k(t,\delta_Y,\delta_Y) + \epsilon_{M+1}(t,\delta_Y,\delta_Y).$$

In the limit $M \to \infty$, $\epsilon_{M+1}(t,\delta_Y,\delta_Y) \to 0$, by Lemma 5.3.2a. Hence,

$$\langle \delta_Y, e^{-tK}\delta_Y \rangle = \sum_{k=0}^{\infty} A_k(t,\delta_Y,\delta_Y).$$

Therefore,

$$\begin{aligned}
\mathrm{Tr}(e^{-tK}) &= \sum_{n=0}^{\infty} \sum_{\substack{Y=(Y_b,Y_f)\\ Y_b\in X_b^n/S_n\\ Y_f\subset X_f}} \langle \delta_Y, e^{-tK}\delta_Y \rangle \\
&= \sum_{n=0}^{\infty} \sum_{\substack{Y=(Y_b,Y_f)\\ Y_b\in X_b^n/S_n\\ Y_f\subset X_f}} \sum_{k=0}^{\infty} A_k(t,\delta_Y,\delta_Y),
\end{aligned}$$

as claimed.

□

Corollary 5.3.7. For any $z \in \mathbb{C}$, let $K(z) := K_0 + zV$. As a function of z, the partition function,

$$Z(z) := \mathrm{Tr}\ e^{-tK(z)},$$

is entire-analytic.

Proof. Replacing all instances of V with zV in Theorem 5.3.6, we see that the right side of Equation (5.29) is a power series in z for $Z(z)$,

$$Z(z) = \sum_{n=0}^{\infty} \sum_{\substack{Y=(Y_b,Y_f) \\ Y_b \in X_b^n/S_n \\ Y_f \subset X_f}} \sum_{k=0}^{\infty} A_k(t, \delta_Y, \delta_Y) z^k.$$

Multiplying each term by z^k in (5.30) does not influence the absolute convergence of the series, for any value of $z \in \mathbb{C}$. We conclude that the series is absolutely convergent, uniformly convergent on compact subsets of $\mathbb{C}$, and therefore analytic on $\mathbb{C}$. □

In the proof of the following theorem, we will use *partial traces*, which are defined (in this context) as follows. Suppose that A is a trace-class operator on $\mathcal{H} = \mathcal{B} \otimes \mathcal{F}$. If $\{\phi_n\}_{n=1}^{\infty}$ is any orthonormal basis for $\mathcal{B}$, then the partial trace of A over $\mathcal{B}$ is

$$\mathrm{Tr}_{\mathcal{B}}(A) := \sum_{n=1}^{\infty} \langle \phi_n, A\phi_n \rangle.$$

Note that $\mathrm{Tr}_{\mathcal{B}}(A)$ is an operator on $\mathcal{F}$. Similarly, if $\{\psi_n\}_{n=1}^{|X_f|}$ is any basis for $\mathcal{F}$, the partial trace of A over $\mathcal{F}$ is

$$\mathrm{Tr}_{\mathcal{F}}(A) := \sum_{n=1}^{|X_f|} \langle \psi_n, A\psi_n \rangle,$$

an operator on $\mathcal{B}$. Since $\{\phi_i \otimes \psi_j\}_{i,j \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{B} \otimes \mathcal{F}$,

$$\begin{aligned}
\mathrm{Tr}\, A &= \sum_{i,j=1}^{\infty} \langle \phi_i \otimes \psi_j, A\phi_i \otimes \psi_j \rangle \\
&= \sum_{i=1}^{\infty} \langle \phi_i, \left(\sum_{j=1}^{\infty} \langle \psi_j, A\psi_j \rangle \right) \phi_i \rangle \\
&= \mathrm{Tr}_{\mathcal{B}}(\mathrm{Tr}_{\mathcal{F}}(A)).
\end{aligned}$$

Likewise,

$$\operatorname{Tr} A = \operatorname{Tr}_{\mathcal{F}}(\operatorname{Tr}_{\mathcal{B}}(A)).$$

Theorem 5.3.8. Let R be a map from $\mathbb{N}$ to $(0, \infty)$ satisfying

$$\lim_{p\to\infty} \frac{R(p)}{p\sqrt{\log p}} = \infty.$$

(For example, $R(p) = p^2$ would work, as would any power of p that is strictly bigger than 1.) The partition function may be represented as the following functional integral:

$$\operatorname{Tr}\, e^{-\beta K} = \lim_{p\to\infty} \int \prod_{\tau\in\mathcal{T}_p} \left[d\mu_{R(p)}(\boldsymbol{\phi}_\tau^*, \boldsymbol{\phi}_\tau) e^{-\int d\mathbf{x}\, \boldsymbol{\phi}_\tau^*(\mathbf{x})\cdot\boldsymbol{\phi}_\tau(\mathbf{x})} \right] \prod_{\tau\in\mathcal{T}_p} \langle \boldsymbol{\phi}_{\tau-\epsilon} | e^{-\epsilon K} | \boldsymbol{\phi}_\tau \rangle, \tag{5.31}$$

where $\epsilon := \frac{\beta}{p}$, $\mathcal{T}_p := \{\epsilon, 2\epsilon, \ldots, p\epsilon\}$, and $\boldsymbol{\phi}_0 := \tilde{\boldsymbol{\phi}}_\beta = (\phi_{\beta_b}, -\phi_{\beta_f})$. In the exponent,

$$\int d\mathbf{x}\ \boldsymbol{\phi}_\tau^*(\mathbf{x}) \cdot \boldsymbol{\phi}_\tau(\mathbf{x}) := \int_{X_b} dy\, \phi_b^*(y)\phi_b(y) + \int_{X_f} dy\, \phi_f^*(y)\phi_f(y).$$

Proof. We start by taking the partial trace of $e^{-\beta K}$ over the boson Fock space $\mathcal{B}$. That is,

$$\operatorname{Tr}_{\mathcal{B}}\, e^{-\beta K} = \sum_{n=0}^{\infty} \sum_{Y\in X_b^n/S_n} \langle \delta_Y, e^{-\beta K} \delta_Y \rangle.$$

By the strong convergence of the bosonic resolution of the identity $I_{R(p)}$ to the identity operator on $\mathcal{B}$,

$$\operatorname{Tr}_{\mathcal{B}}\, e^{-\beta K} = \sum_{n=0}^{\infty} \sum_{Y\in X_b^n/S_n} \lim_{p\to\infty} \langle \delta_Y, e^{-\beta K} I_{R(p)} \delta_Y \rangle.$$

For all $p > 0$ and any $n \in \mathbb{N}_0$ and $Y \in X_b^n/S_n$,

$$\begin{aligned} \left\| \langle \delta_Y, e^{-\beta K} I_{R(p)} \delta_Y \rangle \right\| &\leq \left\| P^{(n)} e^{-\beta K} P^{(n)} \right\| \\ &\leq \sqrt{2} e^{-\frac{1}{8}\beta(\lambda_b - \mu_b)n}, \end{aligned}$$

from the bound (5.27) in Lemma 5.3.5. Since there are

$$\binom{|X_b|+n-1}{n} \leq \frac{(|X_b|+n-1)^{|X_b-1|}}{(|X_b|-1)!}$$

elements of X_b^n/S_n,

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{Y\in X_b^n/S_n} \left\| \langle \delta_Y, e^{-\beta K} I_{R(p)} \delta_Y \rangle \right\| &\leq 1 + \sum_{n=1}^{\infty} \frac{(cn)^{|X_b-1|}}{(|X_b|-1)!} \sqrt{2} e^{C_2(\beta)^2} e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)n} \\
&\leq 1 + \frac{\sqrt{2}e^{C_2(\beta)^2}}{(|X_b|-1)!} \sum_{n=1}^{\infty} (cn)^{|X_b-1|} e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)n} \\
&< \infty,
\end{aligned}$$

where c is a constant that depends on $|X_b|$. By the dominated convergence theorem, it follows that

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{B}}\ e^{-\beta K} &= \lim_{p\to\infty} \sum_{n=0}^{\infty} \sum_{Y\in X_b^n/S_n} \langle \delta_Y, e^{-\beta K} I_{R(p)} \delta_Y \rangle \\
&= \lim_{p\to\infty} \mathrm{Tr}_{\mathcal{B}}(e^{-\beta K} I_{R(p)}).
\end{aligned}$$

By the approximate trace theorem for bosons (Proposition 3.6.1),

$$\mathrm{Tr}_{\mathcal{B}}(e^{-\beta K} I_{R(p)}) = \int d\mu_{R(p)}(\phi^*, \phi)\ e^{-\int_{X_b} dy\ |\phi(y)|^2} \langle \phi | e^{-\beta K} | \phi \rangle.$$

We proceed by proving an intermediate version of Equation (5.31) that involves only the bosonic fields $\phi_\tau \in L^2(X_b)$,

$$\mathrm{Tr}_{\mathcal{B}}(e^{-\beta K}) = \lim_{p\to\infty} \int \prod_{\tau\in\mathcal{T}_p} \left[d\mu_{R(p)}(\phi_\tau^*, \phi_\tau) e^{-\|\phi_\tau\|^2} \right] \prod_{\tau\in\mathcal{T}_p} \left\langle \phi_{\tau-\epsilon} | e^{-\epsilon K} | \phi_\tau \right\rangle.$$

To this end, we will show that the following difference converges to zero as $p \to \infty$:

$$\begin{aligned}
&\left\| \mathrm{Tr}_{\mathcal{B}}(e^{-\beta K} \mathrm{I}_{R(p)}) - \int \prod_{\tau\in\mathcal{T}_p} \left[d\mu_{R(p)}(\phi_\tau^*, \phi_\tau) e^{-\|\phi_\tau\|^2} \right] \prod_{\tau\in\mathcal{T}_p} \left\langle \phi_{\tau-\epsilon} | e^{-\epsilon K} | \phi_\tau \right\rangle \right\| \\
&\qquad \leq \int d\mu_{R(p)}(\phi_0^*, \phi_0) e^{-\|\phi_0\|^2} \left\| \langle \phi_0 | e^{-\beta K} - \left(e^{-\epsilon K} I_{R(p)} \right)^{p-1} e^{-\epsilon K} | \phi_0 \rangle \right\|
\end{aligned}$$

$$= \int \prod_{x\in X_b} \frac{d\phi_0^*(x) d\phi_0(x)}{2\pi i} \chi_{R(p)}(|\phi_0(x)|) e^{-\|\phi_0\|^2} \cdot \left\| \langle \phi_0 | e^{-\beta K} - \left(e^{-\epsilon K} I_{R(p)} \right)^{p-1} e^{-\epsilon K} | \phi_0 \rangle \right\|. \tag{5.32}$$

We will prove in Lemma 5.3.9 that, as long as $\lim_{p\to\infty} \frac{R(p)}{p\sqrt{\log p}} = \infty$,

$$\left\| \langle \phi_0 | e^{-\beta K} - \left(e^{-\frac{\beta}{p} K} I_{R(p)} \right)^{p-1} e^{-\frac{\beta}{p} K} | \phi_0 \rangle \right\| \leq 2|X_b| \left(\frac{16}{\beta(\lambda_b - \mu_b)} \right)^2 e^{\frac{1}{4}\left(e^{-\frac{1}{16}\beta(\lambda_b - \mu_b)} + 3 \right) \|\phi_0\|^2},$$

for sufficiently large p.

Therefore, the integrand of (5.32) satisfies the bound,

$$\chi_r(|\phi_0(x)|) e^{-\|\phi_0\|^2} \left\| \langle \phi_0 | e^{-\beta K} - \left(e^{-\epsilon K} I_{R(p)} \right)^{p-1} e^{-\epsilon K} | \phi_0 \rangle \right\| \leq 4|X_b| \left(\frac{16}{\beta(\lambda_b - \mu_b)} \right)^2 e^{-\frac{1}{4}\left(1 - e^{-\frac{1}{16}\beta(\lambda_b - \mu_b)} \right) \|\phi_0\|^2},$$

which is independent of p and integrable with respect to the measure

$$\prod_{x\in X_b} \frac{d\phi_0^*(x) d\phi_0(x)}{2\pi i}.$$

It follows by the dominated convergence theorem that

$$\mathrm{Tr}_{\mathcal{B}}(e^{-\beta K}) = \lim_{p\to\infty} \int \prod_{\tau\in\mathcal{T}_p} \left[d\mu_{R(p)}(\phi_\tau^*, \phi_\tau) e^{-\|\phi_\tau\|^2} \right] \prod_{\tau\in\mathcal{T}_p} \left\langle \phi_{\tau-\epsilon} | e^{-\epsilon K} | \phi_\tau \right\rangle$$

in operator norm.

We now trace out the fermion space $\mathcal{F}$. For clarity, we replace each "ϕ" in the above expression with "ϕ_b". Since $\mathcal{F}$ is finite-dimensional,

$$\mathrm{Tr}\, e^{-\beta K} = \lim_{p\to\infty} \mathrm{Tr}_{\mathcal{F}} \int \prod_{\tau\in\mathcal{T}_p} \left[d\mu_{R(p)}(\phi_{b,\tau}^*, \phi_{b,\tau}) e^{-\|\phi_{b,\tau}\|^2} \right] \prod_{\tau\in\mathcal{T}_p} \left\langle \phi_{b,\tau-\epsilon} | e^{-\epsilon K} | \phi_{b,\tau} \right\rangle.$$

By the fermionic trace formula in Proposition 3.6.2,

$$\mathrm{Tr}\, e^{-\beta K} = \lim_{p\to\infty} \int d\mu(\phi^*_{0,f}, \phi_{0,f}) e^{-\int_{X_f} \phi^*_{0,f}(x)\phi_{f,0}(x)} \langle -\phi_{f,0}|U|\phi_{f,0}\rangle.$$

where

$$U := \int \prod_{\tau\in\mathcal{T}_p} \left[d\mu_{R(p)}(\phi^*_{b,\tau}, \phi_{b,\tau}) e^{-\|\phi_{b,\tau}\|^2}\right] \prod_{\tau\in\mathcal{T}_p} \left\langle \phi_{b,\tau-\epsilon}|e^{-\epsilon K}|\phi_{b,\tau}\right\rangle$$

and

$$d\mu(\phi^*_f, \phi_f) := \prod_{x\in X} d\phi^*_f(x) d\phi_f(x).$$

We then apply the trace formula for fermions and insert the fermionic resolution of the identity p times, each alongside an insertion of the bosonic approximate resolution of the identity $I_{R(p)}$. Recall that the fermionic resolution of the identity is the Grassmann integral,

$$\mathbb{1}_{\mathcal{F}} = \int d\mu(\phi^*_f, \phi_f) e^{-\int_{X_f} dy \;\; \phi^*_f(y)\phi_f(y)} |\phi_f\rangle\langle\phi_f|,$$

where

$$d\mu(\phi^*_f, \phi_f) := \prod_{x\in X_f} d\phi_f(x) d\phi^*_f(x).$$

Hence,

$$\begin{aligned}
\mathrm{Tr}(e^{-\beta K}) &= \lim_{p\to\infty} \int d\mu(\phi^*_{f,\tau}, \phi_{f,\tau}) e^{-\int_{X_f} dy \;\; \phi^*_{f,\tau}(y)\phi_{f,\tau}(y)} \\
&\quad \cdot \langle -\phi_{f,\tau}| \int \prod_{\tau\in\mathcal{T}_p} \left[d\mu_{R(p)}(\phi^*_{b,\tau}, \phi_{b,\tau}) e^{-\|\phi_{b,\tau}\|^2}\right] \prod_{\tau\in\mathcal{T}_p} \left\langle \phi_{b,\tau-\epsilon}|e^{-\epsilon K}|\phi_{b,\tau}\right\rangle |\phi_{f,\tau}\rangle \\
&= \lim_{p\to\infty} \int \prod_{\tau\in\mathcal{T}_p} \left[d\mu_{R(p)}(\phi^*_{b,\tau}, \phi_{b,\tau}) e^{-\|\phi_{b,\tau}\|^2} d\mu(\phi^*_{f,\tau}, \phi_{f,\tau}) e^{-\int_{X_f} dy \;\; \phi^*_{f,\tau}(y)\phi_{f,\tau}(y)}\right] \\
&\qquad \prod_{\tau\in\mathcal{T}_p} \langle\phi_{f,\tau-\epsilon}|\langle\phi_{b,\tau-\epsilon}|e^{-\epsilon K}|\phi_{b,\tau}\rangle|\phi_{f,\tau}\rangle \\
&= \lim_{p\to\infty} \int \prod_{\tau\in\mathcal{T}_q} \left[d\mu_{R(p)}(\boldsymbol{\Phi}^*_\tau, \boldsymbol{\Phi}_\tau) e^{-\int d\mathbf{x}\, \boldsymbol{\Phi}^*_\tau(\mathbf{x})\cdot\boldsymbol{\Phi}_\tau(\mathbf{x})}\right] \prod_{\tau\in\mathcal{T}_q} \langle \boldsymbol{\Phi}_{\tau-\epsilon}|e^{-\epsilon K}|\boldsymbol{\Phi}_\tau\rangle,
\end{aligned}$$

where $\phi_{f,0} := -\phi_{f,\tau}$. This completes the proof. □

The following lemma was used in the above proof.

Lemma 5.3.9. Let R be a map from $\mathbb{N}$ to $(0,\infty)$ satisfying

$$\lim_{q\to\infty} \frac{R(q)}{q\sqrt{\log q}} = \infty \tag{5.33}$$

For example, $R(q) \cong q^s$ would satisfy this condition for any $s > 1$.

Then,

$$\mathrm{Tr}\ e^{-\beta K} = \lim_{q\to\infty} \mathrm{Tr}\left(e^{-\frac{\beta}{q}K}\mathbf{I}_{R(q)}\right)^{q-1} e^{-\frac{\beta}{q}K}. \tag{5.34}$$

Furthermore, for any $\phi \in L^2(X_b)$, the operator on $\mathcal{F}$,

$$\langle\phi|e^{-\beta K} - \left(e^{-\frac{\beta}{q}K}I_{R(q)}\right)^{q-1} e^{-\frac{\beta}{q}K}|\phi\rangle,$$

satisfies the bound,

$$\begin{aligned}\Bigg\|\langle\phi|e^{-\beta K} &- \left(e^{-\frac{\beta}{q}K}I_{R(q)}\right)^{q-1} e^{-\frac{\beta}{q}K}|\phi\rangle\Bigg\| \\ &\leq 4|X_b|\left(\frac{16}{\beta(\lambda_b-\mu_b)}\right)^2 e^{\frac{1}{4}\left(e^{-\frac{1}{16}\beta(\lambda_b-\mu_b)}+3\right)\|\phi\|^2_{L^2(X_b)}},\end{aligned} \tag{5.35}$$

for sufficiently large q.

Proof. For each $i, q \in \mathbb{N}$, let

$$L_i := \begin{cases} e^{-\frac{\beta}{q}K} & \text{if } i \text{ is odd} \\ \mathrm{I}_{R(q)} & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad M_i := \begin{cases} e^{-\frac{\beta}{q}K} & \text{if } i \text{ is odd} \\ \mathbb{1} & \text{if } i \text{ is even} \end{cases}$$

so that

$$\left(e^{-\frac{\beta}{q}K}\mathrm{I}_{R(q)}\right)^{q-1} e^{-\frac{\beta}{q}K} = \prod_{i=1}^{2q-1} L_i \quad \text{and} \quad e^{-\beta K} = \prod_{i=1}^{2q-1} M_i.$$

Noting that L_i and M_i differ only for even i, in which case, $L_i - M_i = \mathrm{I}_{R(q)} - \mathbb{1}$,

$$\left(e^{-\frac{\beta}{q}K}\mathrm{I}_{R(q)}\right)^{q-1} e^{-\frac{\beta}{q}K} - e^{-\beta K} = \sum_{\ell=1}^{q-1}\left(\prod_{j=1}^{2\ell-1} L_i\right)\left(\mathrm{I}_{R(q)} - \mathbb{1}\right)\left(\prod_{j=2\ell+1}^{2q-1} M_i\right).$$

Hence,

$$
\begin{aligned}
&\left|\mathrm{Tr}\ \left(\left(e^{-\frac{\beta}{q}K}\mathrm{I}_{R(q)}\right)^{q-1}e^{-\frac{\beta}{q}K}-e^{-\beta K}\right)\right| \\
&=\left|\mathrm{Tr}\ \left(\sum_{\ell=1}^{q-1}\left(\prod_{j=1}^{2\ell-1}L_i\right)\left(\mathrm{I}_{R(q)}-\mathbb{1}\right)\left(\prod_{j=2\ell+1}^{2q-1}M_i\right)\right)\right| \\
&\leq\sum_{n\in\mathbb{N}_0}\dim\mathcal{H}_n\left\|P^{(n)}\sum_{\ell=1}^{q-1}\left(\prod_{j=1}^{2\ell-1}L_i\right)\left(\mathrm{I}_{R(q)}-\mathbb{1}\right)\left(\prod_{j=2\ell+1}^{2q-1}M_i\right)P^{(n)}\right\| \\
&\leq\sum_{n\in\mathbb{N}_0}\dim\mathcal{H}_n\sum_{\ell=1}^{q-1}\sum_{\substack{\{n_j\}_{j=0}^{q}\\ n_0=n_q=n}}\left(\prod_{j=1}^{q}\|P^{(n_{j-1})}e^{-\frac{\beta}{q}K}P^{(n_j)}\|\right)\left\|\left(\mathrm{I}_{R(q)}-\mathbb{1}\right)P^{(n_\ell)}\right\| \\
&\leq\sum_{n\in\mathbb{N}_0}\dim\mathcal{H}_n\sum_{\ell=1}^{q-1}\sum_{\substack{\{n_j\}_{j=0}^{q}\\ n_0=n_q=n}}\left(\prod_{j=0}^{q-1}\sqrt{2}e^{C_2(\epsilon)^2}e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n_j}\right)|X_b|e^{-R(q)^2}\sum_{k=0}^{n_\ell}\frac{R(q)^{2k}}{k!} \qquad (5.36)\\
&\leq(q-1)e^{-R(q)^2}|X_b|\left(\sqrt{2}e^{C_2(\epsilon)^2}\right)^q\left(\sum_{n=0}^{\infty}\dim\mathcal{H}_n e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n}\right) \\
&\qquad\cdot\left(\sum_{n_j=0}^{\infty}e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n_j}\right)^{q-2}\left(\sum_{n_\ell=0}^{\infty}e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n_\ell}\sum_{k=0}^{n_\ell}\frac{R(q)^{2k}}{k!}\right). \qquad (5.37)
\end{aligned}
$$

where, in line (5.36) above, we've applied (5.27) from Lemma 5.3.5. As long as $q>\beta$, $\epsilon<1$ and

$$
\begin{aligned}
C_2(\epsilon)&=\frac{8|X_b||X_f|\|v\|_\infty}{\min\{\sqrt{\lambda_b-\mu_b},1\}}\max\{\epsilon,\sqrt{\epsilon}\}e^{\frac{1}{4}\epsilon(\lambda_b-\mu_b)} \\
&\leq\frac{8|X_b||X_f|\|v\|_\infty}{\min\{\sqrt{\lambda_b-\mu_b},1\}}e^{\frac{1}{4}(\lambda_b-\mu_b)}\sqrt{\epsilon} \\
&\leq c_2\sqrt{\epsilon}
\end{aligned}
$$

where

$$
c_2:=\frac{8|X_b||X_f|\|v\|_\infty e^{\frac{1}{4}(\lambda_b-\mu_b)}}{\min\{\sqrt{\lambda_b-\mu_b},1\}}. \qquad (5.38)
$$

The dimension of $\mathcal{H}_n$ satisfies

$$\begin{aligned}\dim \mathcal{H}_n &= \dim \mathcal{F} \cdot \dim \mathcal{B}_n \\ &= 2^{|X_f|} \binom{|X_b| + n - 1}{n} \\ &\leq 2^{|X_f|} \frac{(|X_b| + n - 1)^{|X_b|-1}}{(|X_b| - 1)!}.\end{aligned}$$

If $n \geq 1$,

$$(|X_b| + n - 1)^{|X_b|-1} \leq (|X_b| n)^{|X_b|-1} \leq |X_b|^{|X_b|-1} \frac{(|X_b| - 1)!}{\alpha^{|X_b|-1}} e^{\alpha n},$$

where α is any positive constant. If we take $\alpha = \frac{1}{16}\epsilon(\lambda_b - \mu_b)$,

$$\begin{aligned}\dim \mathcal{H}_n &\leq \frac{2^{|X_f|}}{(|X_b| - 1)!} |X_b|^{|X_b|-1} \frac{(|X_b| - 1)!}{\left(\frac{1}{16}(\lambda_b - \mu_b)\frac{\beta}{q}\right)^{|X_b|-1}} e^{\frac{1}{16}(\lambda_b - \mu_b)\frac{\beta}{q} n} \\ &\leq 2^{|X_f|} |X_b|^{|X_b|-1} \left(\frac{16q}{(\lambda_b - \mu_b)\beta}\right)^{|X_b|-1} e^{\frac{1}{16}\frac{\beta}{q}(\lambda_b - \mu_b) n} \\ &\leq C_3\, q^{|X_b|-1}\, e^{\frac{1}{16}\frac{\beta}{q}(\lambda_b - \mu_b) n},\end{aligned} \tag{5.39}$$

where

$$C_3 := 2^{|X_f|} \left(\frac{16|X_b|}{(\lambda_b - \mu_b)\beta}\right)^{|X_b|-1}.$$

In the case $n = 0$, $\dim \mathcal{H}_n = 2^{|X_f|}$. Hence, the bound in (5.39) applies for all $n \geq 0$ as long as $q > \frac{(\lambda_b - \mu_b)\beta}{16|X_b|}$.

The first sum over $n = n_0 = n_q$ in Equation (5.37) therefore satisfies

$$\begin{aligned}\sum_{n=0}^{\infty} \dim \mathcal{H}_n e^{-\frac{1}{8}\epsilon(\lambda_b - \mu_b) n} &\leq \sum_{n=0}^{\infty} C_3 q^{|X_b|-1} e^{\frac{1}{16}\epsilon(\lambda_b - \mu_b) n} e^{-\frac{1}{8}\epsilon(\lambda_b - \mu_b) n} \\ &= C_3 q^{|X_b|-1} \sum_{n=0}^{\infty} e^{-\frac{1}{16}\epsilon(\lambda_b - \mu_b) n} \\ &= \frac{C_3 q^{|X_b|-1}}{1 - e^{-\frac{1}{16}\epsilon(\lambda_b - \mu_b)}}\end{aligned}$$

$$\leq \frac{C_3 q^{|X_b|-1}}{\frac{1}{32}\epsilon(\lambda_b - \mu_b)}$$

$$= \frac{32C_3}{\beta(\lambda_b - \mu_b)} q^{|X_b|}, \tag{5.40}$$

provided that $q > \frac{\beta(\lambda_b - \mu_b)}{16}$. The sum over n_ℓ is

$$\begin{aligned}
\sum_{n_\ell=0}^{\infty} e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n_\ell} \sum_{k=0}^{n_\ell} \frac{R(q)^{2k}}{k!} &= \sum_{k=0}^{\infty} \frac{R(q)^{2k}}{k!} \sum_{n_\ell=k}^{\infty} e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n_\ell} \\
&= \sum_{k=0}^{\infty} \frac{R(q)^{2k}}{k!} \frac{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)k}}{1-e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)}} \\
&= \frac{e^{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)}R(q)^2}}{1-e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)}} \\
&\leq \frac{e^{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)}R(q)^2}}{\frac{1}{16}\epsilon(\lambda_b-\mu_b)} \\
&\leq \frac{16}{\beta(\lambda_b-\mu_b)} q e^{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)}R(q)^2}.
\end{aligned} \tag{5.41}$$

We've assumed in the fourth line that q is large enough so that $\frac{1}{8}\frac{\beta}{q}(\lambda_b - \mu_b) < 1$, i.e. $q > \frac{1}{8}\beta(\lambda_b - \mu_b)$ The sum over n_j, $0 < j < q$, $j \neq \ell$ is

$$\begin{aligned}
\sum_{n_j=0}^{\infty} e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n_j} &= \frac{1}{1-e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)}} \\
&\leq \frac{1}{\frac{1}{16}\epsilon(\lambda_b-\mu_b)} = \frac{16q}{\beta(\lambda_b-\mu_b)}.
\end{aligned} \tag{5.42}$$

Applying each of the bounds from (5.40), (5.41), and (5.42) to Equation (5.37),

$$\begin{aligned}
&\left| \mathrm{Tr}\left(\left(e^{-\frac{\beta}{q}K} \mathrm{I}_{R(q)} \right)^{q-1} e^{-\frac{\beta}{q}K} - e^{-\beta K} \right) \right| \\
&\leq q e^{-R(q)^2} |X_b| \left(\sqrt{2} e^{C_2(\epsilon)^2} \right)^q \left(\sum_{n=0}^{\infty} \dim \mathcal{H}_n e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n} \right)
\end{aligned}$$

$$\cdot \left(\sum_{n_j=0}^{\infty} e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n_j} \right)^{q-2} \left(\sum_{n_\ell=0}^{\infty} e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n_\ell} \sum_{k=0}^{n_\ell} \frac{R(q)^{2k}}{k!} \right)$$
$$\leq q e^{-R(q)^2} |X_b| 2^{q/2} e^{c_2^2\beta} \frac{32C_3}{\beta(\lambda_b-\mu_b)} q^{|X_b|} \left(\frac{16q}{\beta(\lambda_b-\mu_b)} \right)^{q-2} \frac{16q e^{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)} R(q)^2}}{\beta(\lambda_b-\mu_b)}$$
$$\leq 2C_3 |X_b| q^{|X_b|} e^{c_2^2\beta} \left(\frac{16\sqrt{2}q}{\beta(\lambda_b-\mu_b)} \right)^{q} e^{-(1-e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)})R(q)^2}$$
$$\leq 2C_3 |X_b| e^{c_2^2\beta} e^{-\frac{1}{16}\beta(\lambda_b-\mu_b)\frac{R(q)^2}{q} + q\log(c_3 q) + |X_b|\log q},$$

where $c_3 := \frac{16\sqrt{2}}{\beta(\lambda_b-\mu_b)}$. We've assumed that q is larger than both β and $\frac{1}{8}\beta(\lambda_b-\mu_b)$.

This converges to zero if $R(q) \to \infty$ sufficiently quickly, for example, if $R(q) \sim q^{3/2}$ for large q. More generally, if R satisfies the condition

$$\lim_{q\to\infty} \frac{R(q)}{q\sqrt{\log q}} = \infty$$

as assumed in (5.33), we may conclude that

$$\left| \mathrm{Tr} \left(\left(e^{-\frac{\beta}{q}K} \mathrm{I}_{R(q)} \right)^{q-1} e^{-\frac{\beta}{q}K} - e^{-\beta K} \right) \right| \to 0$$

as $q \to \infty$. Since

$$\mathbf{I}_{R(q)} = \mathrm{I}_{R(q)} \otimes \mathbb{1}_{\mathcal{F}},$$

this completes the proof of the first statement (5.34) in the lemma. We now move on to prove the second statement, (5.35).

For any $\phi \in L^2(X_b)$, $\langle \phi | e^{-\beta K} - \left(e^{-\frac{\beta}{q}K} I_{R(q)} \right)^{q-1} e^{-\frac{\beta}{q}K} | \phi \rangle$ is an operator on $\mathcal{F}$. In order to find a bound on the norm of this operator, we will apply the same ideas from the proof above, except that we will use the bound (5.28) of Lemma 5.3.5, which is tighter than (5.27). The resulting bound on $\left\| \langle \phi | e^{-\beta K} - \left(e^{-\frac{\beta}{q}K} I_{R(q)} \right)^{q-1} e^{-\frac{\beta}{q}K} | \phi \rangle \right\|$ will allow for the application of the dominated convergence theorem in the proof of Theorem 5.3.8.

$$\begin{aligned}
&\left\| \langle\phi| e^{-\beta K} - \left(e^{-\frac{\beta}{q}K} I_{R(q)}\right)^{q-1} e^{-\frac{\beta}{q}K}|\phi\rangle \right\| \\
&\leq \sum_{m,n\in\mathbb{N}_0} \frac{\|\phi\|^{n+m}}{\sqrt{n!m!}} \left\| P^{(n)} \left(e^{-\beta K} - \left(e^{-\frac{\beta}{q}K} I_{R(q)}\right)^{q-1} e^{-\frac{\beta}{q}K}\right) P^{(m)} \right\| \\
&\leq \sum_{m,n\in\mathbb{N}_0} \frac{\|\phi\|^{n+m}}{\sqrt{n!m!}} \left\| P^{(n)} \left(\sum_{\ell=1}^{q-1} \left(\prod_{j=1}^{2\ell-1} L_i\right) \left(\mathrm{I}_{R(q)} - \mathbb{1}\right) \left(\prod_{j=2\ell+1}^{2q-1} M_i\right)\right) P^{(m)} \right\| \\
&\leq \sum_{m,n\in\mathbb{N}_0} \frac{\|\phi\|^{n+m}}{\sqrt{n!m!}} \sum_{\ell=1}^{q-1} \sum_{\substack{\{n_j\}_{j=0}^{q} \\ n_0=n, n_q=m}} \left(\prod_{j=1}^{q} \|P^{(n_{j-1})} e^{-\frac{\beta}{q}K} P^{(n_j)}\|\right) \left\|\left(\mathrm{I}_{R(q)} - \mathbb{1}\right) P^{(n_\ell)}\right\| \\
&\leq \sum_{m,n\in\mathbb{N}_0} \frac{\|\phi\|^{n+m}}{\sqrt{n!m!}} \sum_{\ell=1}^{q-1} \sum_{\substack{\{n_j\}_{j=0}^{q} \\ n_0=n, n_q=m}} \prod_{j=0}^{q-1} \left(\frac{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n_j}}{|n_{j+1}-n_j|!^{1/4}} \left(\frac{1}{1-\frac{1}{r^{4/3}}}\right)^{3/4} e^{\frac{1}{4}(rC_2(\epsilon))^4} \right) \\
&\qquad\qquad |X_b| e^{-R(q)^2} \sum_{k=0}^{n_\ell} \frac{R(q)^{2k}}{k!} \\
&\leq \sum_{m,n\in\mathbb{N}_0} \frac{\|\phi\|^{n+m}}{\sqrt{n!m!}} e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)n} \sum_{\substack{\{n_j\}_{j=0}^{q} \\ n_0=n, n_q=m}} \sum_{\ell=1}^{q-1} \left[|X_b| e^{-R(q)^2} \sum_{k=0}^{n_\ell} \frac{R(q)^{2k}}{k!} \right. \\
&\qquad\qquad \left. \cdot \prod_{j=0}^{q-1} \left(\frac{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)(n_j-n)}}{|n_{j+1}-n_j|!^{1/4}} \left(\frac{1}{1-\frac{1}{r^{4/3}}}\right)^{3/4} e^{\frac{1}{4}(rC_2(\epsilon))^4} \right) \right] \\
&\leq |X_b| e^{-R(q)^2} \left(\frac{1}{1-\frac{1}{r^{4/3}}}\right)^{\frac{3}{4}q} e^{\frac{q}{4}(rC_2(\epsilon))^4} \sum_{m,n\in\mathbb{N}_0} \frac{\|\phi\|^{n+m}}{\sqrt{n!m!}} e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)n} \\
&\qquad\qquad \cdot \sum_{\ell=1}^{q-1} \sum_{\substack{\{n_j\}_{j=0}^{q} \\ n_0=n, n_q=m}} \sum_{k=0}^{n_\ell} \frac{R(q)^{2k}}{k!} \prod_{j=0}^{q-1} \left(\frac{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)(n_j-n)}}{|n_{j+1}-n_j|!^{1/4}} \right) \\
&\leq |X_b| e^{-R(q)^2} \left(\frac{1}{1-\frac{1}{r^{4/3}}}\right)^{\frac{3}{4}q} e^{\frac{q}{4}(rC_2(\epsilon))^4} \sum_{m,n\in\mathbb{N}_0} \frac{\|\phi\|^{n+m}}{\sqrt{n!m!}} e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)n} \sum_{\ell=1}^{q-1} \left(\sum_{n'_\ell=0}^{\infty} \right.
\end{aligned}$$

$$e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)(n'_\ell-n)}\sum_{k=0}^{n'_\ell}\frac{R(q)^{2k}}{k!}\Bigg)\sum_{\substack{\{n_j\}_{j=0}^{q}\\ n_0=n,n_q=m}} e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)(n_\ell-n)}\prod_{j=0}^{q-1}\left(\frac{e^{-\frac{\epsilon}{8}(\lambda_b-\mu_b)(n_j-n)}}{|n_{j+1}-n_j|!^{1/4}}\right) \tag{5.43}$$

where $r > 1$ is an arbitrary constant.

Bounding the sum over n'_ℓ,

$$\begin{aligned}\sum_{n'_\ell=0}^{\infty} e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)(n'_\ell-n)}\sum_{k=0}^{n'_\ell}\frac{R(q)^{2k}}{k!} &= e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)n}\sum_{n'_\ell=0}^{\infty} e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n'_\ell}\sum_{k=0}^{n'_\ell}\frac{R(q)^{2k}}{k!}\\ &= e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)n}\sum_{k=0}^{\infty}\frac{R(q)^{2k}}{k!}\sum_{n'_\ell=k}^{\infty} e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)n'_\ell}\\ &= e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)n}\sum_{k=0}^{\infty}\frac{R(q)^{2k}}{k!}\frac{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)k}}{1-e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)}}\\ &= e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)n}\frac{1}{1-e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)}}e^{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)}R(q)^2}.\end{aligned} \tag{5.44}$$

Next, we bound the sum over the sequences $\{n_j : j \in \{0,\ldots,q\}, n_0 = n, n_q = m\}$ that appears as the rightmost sum in (5.43). For each $j \in \{1,\cdots,q\}$, let $m_j := n_j - n_{j-1}$.

$$\begin{aligned}&\sum_{\substack{\{n_j\}_{j=0}^{q}\\ n_0=n,n_q=m}} e^{+\frac{1}{8}\epsilon(\lambda_b-\mu_b)(n_\ell-n)}\prod_{j=0}^{q-1}\frac{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)(n_j-n)}}{|n_{j+1}-n_j|!^{1/4}}\\ &\qquad\le \sum_{\substack{\{n_j\}_{j=0}^{q}\\ n_0=n,n_q=m}}\prod_{j=0}^{q-1}\frac{e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)|n_j-n|}}{|n_{j+1}-n_j|!^{1/4}}\\ &\qquad\le \sum_{\substack{\{m_j\}_{j=1}^{q}\in\mathbb{Z}^q\\ \sum_{j=1}^{q}m_j=m-n}}\prod_{j=0}^{q-1}\frac{e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)\left|\sum_{i=1}^{j}m_i\right|}}{|m_{j+1}|!^{1/4}}\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\substack{\{m_j\}_{j=1}^q \in \mathbb{Z}^q \\ \sum_j m_j = m-n}} \prod_{j=1}^q \frac{e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)\sum_{i=1}^{j-1}|m_i|}}{|m_j|!^{1/4}} \\
&= \sum_{\substack{\{m_j\}_{j=1}^q \in \mathbb{Z}^q \\ \sum_j m_j = m-n}} \prod_{j=1}^q \frac{e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)(q-j)|m_j|}}{|m_j|!^{1/4}} \\
&\leq 2^q \prod_{j=1}^q \sum_{m_j=0}^\infty \frac{e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)(q-j)m_j}}{m_j!^{1/4}}.
\end{aligned}
$$

Note that m_q is determined by m, n and the other m_j, but we summed over it anyway. By Hölder's inequality,

$$
\begin{aligned}
\sum_{m=0}^\infty \frac{e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)(q-j)m}}{m!^{1/4}} &= \sum_{m\in\mathbb{Z}} \frac{1}{r'^m} \frac{r'^m e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)(q-j)m}}{m!^{1/4}} \\
&\leq \left(\sum_{m=0}^\infty \frac{1}{r'^{\frac{4}{3}m}}\right)^{3/4} \left(\sum_{m=0}^\infty \frac{\left(r'^4 e^{\frac{1}{2}\epsilon(\lambda_b-\mu_b)(q-j)}\right)^m}{m!}\right)^{1/4} \\
&\leq \left(\frac{1}{1-r'^{-\frac{4}{3}}}\right)^{3/4} \left(e^{r'^4 e^{\frac{1}{2}\epsilon(\lambda_b-\mu_b)(q-j)}}\right)^{1/4},
\end{aligned}
$$

where $r' > 1$ is an arbitrary constant. Hence,

$$
\begin{aligned}
&\sum_{\substack{\{n_j\}_{j=0}^q \\ n_0=n, n_q=m}} e^{+\frac{1}{8}\epsilon(\lambda_b-\mu_b)(n_\ell-n)} \prod_{j=0}^{q-1} \frac{e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)(n_j-n)}}{|n_{j+1}-n_j|!^{1/4}} \\
&\qquad \leq \left(\frac{2}{(1-r'^{-\frac{4}{3}})^{\frac{3}{4}}}\right)^q \prod_{j=1}^q e^{\frac{r'^4}{4} e^{\frac{1}{2}\epsilon(\lambda_b-\mu_b)(q-j)}} \\
&\qquad \leq \left(\frac{2}{(1-r'^{-\frac{4}{3}})^{\frac{3}{4}}}\right)^q e^{\frac{r'^4}{4}\sum_{j=1}^q e^{\frac{1}{2}\epsilon(\lambda_b-\mu_b)(q-j)}} \\
&\qquad = \left(\frac{2}{(1-r'^{-\frac{4}{3}})^{\frac{3}{4}}}\right)^q e^{\frac{r'^4}{4}\sum_{j=0}^{q-1} e^{\frac{1}{2}\epsilon(\lambda_b-\mu_b)j}} \\
&\qquad = \left(\frac{2}{(1-r'^{-\frac{4}{3}})^{\frac{3}{4}}}\right)^q \exp\left(\frac{r'^4}{4}\frac{e^{\frac{1}{2}q\epsilon(\lambda_b-\mu_b)}-1}{e^{\frac{1}{2}\epsilon(\lambda_b-\mu_b)}-1}\right). \qquad (5.45)
\end{aligned}
$$

Applying the bounds in (5.44) and (5.45) to (5.43),

$$\begin{aligned}
&\left\| \langle\phi| e^{-\beta K} - \left(e^{-\frac{\beta}{q}K} I_{R(q)}\right)^{q-1} e^{-\frac{\beta}{q}K}|\phi\rangle \right\| \\
&\le |X_b| e^{-R(q)^2} \left(\frac{1}{1-\frac{1}{r^{4/3}}}\right)^{\frac{3}{4}q} e^{\frac{q}{4}(rC_2(\epsilon))^4} \sum_{m,n\in\mathbb{N}_0} \frac{\|\phi\|^{n+m}}{\sqrt{n!m!}} e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)n} q \\
&\qquad \cdot \left(\frac{2}{(1-r'^{-\frac{4}{3}})^{\frac{3}{4}}}\right)^q \exp\left(\frac{r'^4}{4}\frac{e^{\frac{1}{2}q\epsilon(\lambda_b-\mu_b)}-1}{e^{\frac{1}{2}\epsilon(\lambda_b-\mu_b)}-1}\right) e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)n} \frac{1}{1-e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)}} \\
&\qquad\qquad \cdot \exp\left(e^{-\frac{1}{8}\epsilon(\lambda_b-\mu_b)} R(q)^2\right) \qquad (5.46) \\
&\le |X_b| \left(\frac{2}{(1-r^{-\frac{4}{3}})^{\frac{3}{2}}}\right)^q e^{\frac{q}{4}(rC_2(\epsilon))^4} \sum_{m,n\in\mathbb{N}_0} \frac{\|\phi\|^{n+m}}{\sqrt{n!m!}} e^{-\frac{1}{8}(\beta-\epsilon)(\lambda_b-\mu_b)n} q \\
&\qquad\qquad \cdot \exp\left(\frac{r^4}{4}\frac{e^{\frac{1}{2}\beta(\lambda_b-\mu_b)}-1}{\frac{1}{4}\epsilon(\lambda_b-\mu_b)}\right) \frac{1}{\frac{1}{16}\epsilon(\lambda_b-\mu_b)} e^{-\frac{1}{16}\epsilon(\lambda_b-\mu_b)R(q)^2} \\
&\le |X_b| \frac{16}{\beta(\lambda_b-\mu_b)} e^{-\frac{1}{16}\beta(\lambda_b-\mu_b)\frac{R(q)^2}{q} + \frac{1}{4}(rc_2)^4\frac{\beta^2}{q} + \left(r^4\frac{e^{\frac{1}{2}\beta(\lambda_b-\mu_b)}}{\beta(\lambda_b-\mu_b)} + c_3\right)q + 2\log q} \\
&\qquad\qquad \cdot \sum_{m,n\in\mathbb{N}_0} \frac{\|\phi\|^{n+m}}{\sqrt{n!m!}} e^{-\frac{1}{8}(\beta-\epsilon)(\lambda_b-\mu_b)n}, \qquad (5.47)
\end{aligned}$$

where

$$c_3 := \log\left(\frac{2}{(1-r^{-\frac{4}{3}})^{\frac{3}{2}}}\right).$$

In the second inequality, we've set the arbitrary constants r and r' equal to each other. We've also assumed that q is larger than both $\frac{1}{8}\beta(\lambda_b-\mu_b)$ and β, so that

$$1 - e^{\frac{1}{8}\epsilon(\lambda_b-\mu_b)} \ge \frac{1}{16}\epsilon(\lambda_b-\mu_b).$$

and

$$C_2(\epsilon) \le c_2 \max\{\epsilon, \sqrt{\epsilon}\} \le c_2\sqrt{\epsilon} = c_2\sqrt{\frac{\beta}{q}},$$

where c_2 was defined in (5.38) to be

$$c_2 = \frac{8|X_b||X_f|\|v\|_\infty e^{\frac{1}{4}(\lambda_b-\mu_b)}}{\min\{\sqrt{\lambda_b-\mu_b}, 1\}}.$$

Let $s > 1$ be an arbitrary constant.

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{\|\phi\|^m}{\sqrt{m!}} &= \sum_{m=0}^{\infty} \frac{1}{s^{m/2}} \frac{(\sqrt{s}\|\phi\|)^m}{\sqrt{m!}} \\ &\leq \left(\sum_{m=0}^{\infty} \frac{1}{s^m} \right)^{1/2} \left(\sum_{m=0}^{\infty} \frac{s^m \|\phi\|^{2m}}{m!} \right)^{1/2} \\ &\leq \left(\frac{s}{s-1} \right)^{1/2} e^{\frac{1}{2}s\|\phi\|^2}. \end{aligned} \tag{5.48}$$

Similarly,

$$\sum_{n=0}^{\infty} \frac{(\|\phi\| e^{-\frac{1}{8}(\beta-\epsilon)(\lambda_b-\mu_b)})^n}{\sqrt{n!}} \leq \left(\frac{s}{s-1} \right)^{1/2} e^{\frac{1}{2}se^{-\frac{1}{4}(\beta-\epsilon)(\lambda_b-\mu_b)}\|\phi\|^2}. \tag{5.49}$$

Applying these inequalities (5.48) and (5.49) to the main bound in (5.47),

$$\begin{aligned} &\left\| \langle\phi| e^{-\beta K} - \left(e^{-\frac{\beta}{q}K} I_{R(q)} \right)^{q-1} e^{-\frac{\beta}{q}K} |\phi\rangle \right\| \\ &\leq |X_b| \frac{16}{\beta(\lambda_b-\mu_b)} e^{-\frac{1}{16}\beta(\lambda_b-\mu_b)\frac{R(q)^2}{q} + \frac{1}{4}(rc_2)^4 e^{\frac{\beta(\lambda_b-\mu_b)}{q}} \frac{\beta^2}{q} + \left(r^4 \frac{e^{\frac{1}{2}\beta(\lambda_b-\mu_b)}}{\beta(\lambda_b-\mu_b)} + c_3 \right) q + 2\log q} \\ &\qquad \cdot \left(\frac{s}{s-1} \right) e^{\frac{1}{2}s\left(e^{-\frac{1}{4}(\beta-\epsilon)(\lambda_b-\mu_b)} + 1 \right)\|\phi\|^2} \\ &\leq |X_b| \frac{16}{\beta(\lambda_b-\mu_b)} e^{-\frac{1}{16}\beta(\lambda_b-\mu_b)\frac{R(q)^2}{q} + \frac{1}{4}(rc_2)^4 e^{\frac{\beta(\lambda_b-\mu_b)}{q}} \frac{\beta^2}{q} + \left(r^4 \frac{e^{\frac{1}{2}\beta(\lambda_b-\mu_b)}}{\beta(\lambda_b-\mu_b)} + c_3 \right) q + 2\log q} \\ &\qquad \cdot \left(\frac{s}{s-1} \right) e^{\frac{1}{2}s\left(e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)} + 1 \right)\|\phi\|^2}, \end{aligned}$$

as long as $q > 2$.

If we let

$$s = \frac{1}{2}\left(1 + \frac{2}{e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)} + 1} \right),$$

then

$$\frac{1}{2}s\left(e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)} + 1 \right) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)} + 1} \right)\left(e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)} + 1 \right)$$

$$= \frac{1}{4}\left(e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)} + 3\right).$$

Also,

$$\frac{s}{s-1} = \frac{3+e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)}}{1-e^{-\frac{1}{8}\beta(\lambda_b-\mu_b)}} \leq \frac{4\cdot 16}{\beta(\lambda_b-\mu_b)}.$$

Since $\lim_{q\to\infty}\frac{R(q)}{q\sqrt{\log q}} = \infty$,

$$e^{-\frac{1}{16}\beta(\lambda_b-\mu_b)\frac{R(q)^2}{q}+\frac{1}{4}(rc_2)^4 e^{\frac{\beta(\lambda_b-\mu_b)}{q}}\frac{\beta^2}{q}+\left(r^4\frac{e^{\frac{1}{2}\beta(\lambda_b-\mu_b)}}{\beta(\lambda_b-\mu_b)}+c_3\right)q+2\log q} \to 0$$

as $q\to\infty$. If q is sufficiently large, this is less than 1, and

$$\begin{aligned}\left\|\langle\phi|e^{-\beta K} - \left(e^{-\frac{\beta}{q}K}I_{R(q)}\right)^{q-1} e^{-\frac{\beta}{q}K}|\phi\rangle\right\| \\ \leq 4|X_b|\left(\frac{16}{\beta(\lambda_b-\mu_b)}\right)^2 e^{\frac{1}{4}\left(e^{-\frac{1}{16}\beta(\lambda_b-\mu_b)}+3\right)\|\phi\|^2},\end{aligned}$$

as claimed.

□

The following lemma was used in the proof above.

Lemma 5.3.10. For any $r>0$ and $n\in\mathbb{N}$,

$$\|(\mathbb{1}-I_r)P_n\| \leq |X_b|e^{-r^2}\sum_{k=0}^{n}\frac{r^{2k}}{k!}$$

Proof. Recall that, from [10] (pg. 1246), for any $r>1$,

$$\begin{aligned}\|(\mathbb{1}-I_r)P_n\| &\leq |X_b|\max_{|Y|\leq n}\max_{x\in X_b}\frac{1}{\mu_Y(x)!}\int_{r^2}^{\infty}dt\; e^{-t}t^{\mu_Y(x)} \\ &= |X_b|\max_{|Y|\leq n}\max_{x\in X_b}e^{-r^2}\sum_{k=0}^{\mu_Y(x)}\frac{r^{2k}}{k!} \\ &\leq |X_b|e^{-r^2}\sum_{k=0}^{n}\frac{r^{2k}}{k!}.\end{aligned}$$

The second line can easily be proven by induction on $\mu_Y(X)$, as we now show. The base case is the equation $\int_{r^2}^\infty e^{-t}dt = e^{-r^2}$. The induction step is

$$\begin{aligned}\int_{r^2}^\infty e^{-t}\frac{t^n}{n!}dt &= -\left. e^{-t}\frac{t^n}{n!}\right|_{r^2}^\infty + \int_{r^2}^\infty e^{-t}\frac{t^{n-1}}{(n-1)!}dt \\ &= e^{-r^2}\frac{r^{2n}}{n!} + e^{-r^2}\sum_{k=0}^{n-1}\frac{r^{2k}}{k!} \\ &= e^{-r^2}\sum_{k=0}^{n}\frac{r^{2k}}{k!}.\end{aligned}$$

□

This completes the proof of Theorem 5.3.8.

5.4 The Action

We define the *action* $F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ to be logarithm of the inner product which appeared in the functional integral (5.31),

$$F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) := \log\langle\boldsymbol{\alpha}|e^{-\epsilon K}|\boldsymbol{\phi}\rangle,$$

wherever this logarithm exists. We will see in this short section that $F(\epsilon, \cdot, \cdot)$ is an entire-analytic function of the fields $\boldsymbol{\alpha}^* = (\alpha_b^*, \alpha_f^*)$ and $\boldsymbol{\phi} = (\phi_b, \phi_f)$.

Proposition 5.4.1. The inner product,

$$G(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) := \langle\boldsymbol{\alpha}|e^{-\epsilon K}|\boldsymbol{\phi}\rangle$$

is an entire-analytic nonzero function of the fermion fields, $\alpha_f^*(y)$ and $\phi_f(y)$, for all $y \in X_f$.

Proof. For each $n \in \mathbb{N}$, let $P_f^{(n)}$ be the projection onto the n-fermion subspace of $\mathcal{H}$, $\mathcal{B} \otimes \mathcal{F}_n$.

Since K preserves fermion number,

$$\begin{aligned} G(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) &= \langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle \\ &= \sum_{n=0}^{|X_f|} \langle \boldsymbol{\alpha} | P_f^{(n)} e^{-\epsilon K} P_f^{(n)} | \boldsymbol{\phi} \rangle \\ &= \sum_{n=0}^{|X_f|} \langle \alpha_f | P_f^{(n)} \left[\langle \alpha_b | P_f^{(n)} e^{-\epsilon K} P_f^{(n)} | \phi_b \rangle \right] P_f^{(n)} | \phi_f \rangle, \end{aligned} \tag{5.50}$$

noting that each $\langle \alpha_b | P_f^{(n)} e^{-\epsilon K} P_f^{(n)} | \phi_b \rangle$ is an operator on $\mathcal{F}_n$.

For each n,

$$\begin{aligned} P_f^{(n)} | \phi_f \rangle &= \sum_{\substack{Y \subset X_f \\ |Y| = n}} \langle \delta_Y | \phi_f \rangle \delta_Y \\ &= \sum_{\substack{y_1, \ldots, y_n \in X_f \\ y_1 < \cdots < y_n}} \phi_f(y_1) \cdots \phi_f(y_n) \delta_{\{y_1, \ldots, y_n\}}. \end{aligned} \tag{5.51}$$

Similarly,

$$\begin{aligned} \langle \alpha_f | P_f^{(n)} &= \sum_{\substack{Y \subset X_f \\ |Y| = n}} \langle \alpha_f | \delta_Y \rangle \delta_Y^* \\ &= \sum_{\substack{y_1, \ldots, y_n \in X_f \\ y_1 < \cdots < y_n}} \alpha_f^*(y_n) \cdots \alpha_f^*(y_1) \delta^*_{\{y_1, \ldots, y_n\}}. \end{aligned} \tag{5.52}$$

where δ_Y^* is the dual of δ_Y. From Equations (5.50), (5.51), and (5.52), $G(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ is the following polynomial in the fermion fields ϕ_f and α_f^*,

$$\begin{aligned} G(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) = \sum_{n=0}^{|X_f|} \sum_{\substack{y_1 < \cdots < y_n \\ x_1 < \cdots < x_n}} \langle \delta_{\{y_1, \ldots, y_n\}}, \left[\langle \alpha_b | P_f^{(n)} e^{-\epsilon K} P_f^{(n)} | \phi_b \rangle \right] \delta_{\{x_1, \ldots, x_n\}} \rangle \\ \cdot \alpha_f^*(y_n) \cdots \alpha_f^*(y_1) \phi_f(x_1) \cdots \phi_f(x_n). \end{aligned}$$

Hence, $G(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ is an analytic function of the fermion fields, ϕ_f and α_f^*.

To see that $G(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ is nonzero everywhere, it suffices to prove that the zeroth-order term in the fermion fields, $G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) := \langle \alpha_b | P_f^{(0)} e^{-\epsilon K} P_f^{(0)} | \phi_b \rangle$, is nonzero everywhere. This is done in Lemma 5.4.2 below. □

Lemma 5.4.2. In the expansion for $G(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) = \langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle$ in the fermion fields α_f^* and ϕ_f, the zeroth-order term, $G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$, is an entire-analytic function of the boson fields α_b^* and ϕ_b and is non-zero everywhere. Moreover,

$$G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) = \exp\left(\int_{X_b^2} dx\, dy\ \alpha_b^*(x) e^{-\epsilon(h_b - \mu_b)}(x, y) \phi_b(y) \right). \tag{5.53}$$

Proof. First note that

$$\begin{aligned} G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) &= \langle \boldsymbol{\alpha} | P_f^{(0)} e^{-\epsilon K} P_f^{(0)} | \boldsymbol{\phi} \rangle \\ &= \langle \alpha_b | e^{-\epsilon(H_b - \mu_b N_b)} | \phi_b \rangle, \end{aligned}$$

where

$$H_b := \int_{X_b^2} dx\, dy\ a_b^\dagger(x) h_b(x, y) a_b(y).$$

Also,

$$H_b - \mu_b N_b = \int_{X_b^2} dx\, dy\ a_b^\dagger(x) (h_b(x, y) - \mu_b) a_b(y).$$

From Lemma 2.23 of [10],

$$\begin{aligned} \langle \alpha_b | e^{-\epsilon(H_b - \mu_b N_b)} | \phi_b \rangle &= \langle \alpha_b | e^{-\epsilon(h_b - \mu_b)} \phi_b \rangle \\ &= \exp\left(\langle \alpha_b, e^{-\epsilon(h_b - \mu_b)} \phi_b \rangle_{L^2(X_b)} \right) \\ &= \exp\left(\int_{X_b^2} dx\, dy\ \alpha_b^*(x) e^{-\epsilon(h_b - \mu_b)}(x, y) \phi_b(y) \right), \end{aligned}$$

as claimed. Indeed, this is an entire-analytic function of the boson fields, α_b^* and ϕ_b, that is nonzero everywhere. □

Proposition 5.4.3. The action,

$$F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) = \log \langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle,$$

is an entire-analytic function of the fields $\boldsymbol{\alpha}^*$ and $\boldsymbol{\phi}$.

Proof. For each $n \in \{0, \ldots, |X_f|\}$, let $G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ be the component of

$G(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ of order n in the fermion fields,

$$G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) := \langle \boldsymbol{\alpha} | P_f^{(n)} e^{-\epsilon K} P_f^{(n)} | \boldsymbol{\phi} \rangle.$$

Note that each $G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ is an entire-analytic function of the fields $\boldsymbol{\alpha}^*$ and $\boldsymbol{\phi}$. Then,

$$G(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) = \sum_{n=0}^{|X_f|} G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}).$$

From Lemma 5.4.2, $G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ is nonzero everywhere and may be factored out of the sum,

$$G(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) = G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) \left(1 + \sum_{n=1}^{|X_f|} \frac{G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})}{G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})} \right).$$

Taking the logarithm,

$$\begin{aligned} F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) &= \log G(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) \\ &= \log G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) + \log \left(1 + \sum_{n=1}^{|X_f|} \frac{G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})}{G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})} \right), \end{aligned} \tag{5.54}$$

where

$$\log G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) = \int_{X_b^2} dx\, dy \ \alpha_b^*(x) e^{-\epsilon(h_b - \mu_b)}(x, y) \phi_b(y),$$

by Equation (5.53). Since each $G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ is of order at least 1 in the Grassmann algebra, the second term in (5.54) is given by the Taylor polynomial,

$$\begin{aligned} \log \left(1 + \sum_{n=1}^{|X_f|} \frac{G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})}{G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})} \right) &= \sum_{m=1}^{|X_f|} \frac{1}{m} \left(\sum_{n=1}^{|X_f|} \frac{G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})}{G_0(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})} \right)^m \\ &= \sum_{m=1}^{|X_f|} \frac{1}{m} e^{-m \int_{X_b^2} dx\, dy \ \alpha_b^*(x) e^{-\epsilon(h_b - \mu_b)}(x,y) \phi_b(y)} \left(\sum_{n=1}^{|X_f|} G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) \right)^m . \end{aligned}$$

Hence,

$$F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) = \int_{X_b^2} dx\, dy\ \alpha_b^*(x) e^{-\epsilon(h_b - \mu_b)}(x,y)\phi_b(y) + \sum_{m=1}^{|X_f|} \frac{1}{m} e^{-m\int_{X_b^2} dx\, dy\ \alpha_b^*(x) e^{-\epsilon(h_b-\mu_b)}(x,y)\phi_b(y)} \left(\sum_{n=1}^{|X_f|} G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) \right)^m . \tag{5.55}$$

Since each $G_n(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ is an entire-analytic function of the fields $\boldsymbol{\alpha}^*$ and $\boldsymbol{\phi}$, it follows that $F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi})$ is as well. □

Hence, in the functional integral for the partition function (5.31) of Theorem 5.3.8, we can replace each inner product $\langle \boldsymbol{\phi}_{\tau-\epsilon} | e^{-\epsilon K} | \boldsymbol{\phi}_\tau \rangle$ with the exponential of the action $F(\epsilon, \boldsymbol{\phi}^*_{\tau-\epsilon}, \boldsymbol{\phi}_\tau)$ to obtain,

$$\text{Tr } e^{-\beta K} = \lim_{p\to\infty} \int \prod_{\tau \in \mathcal{T}_p} \left[d\mu_{R(p)}(\boldsymbol{\phi}^*_\tau, \boldsymbol{\phi}_\tau) e^{-\int d\mathbf{x}\, \boldsymbol{\phi}^*_\tau(\mathbf{x}) \cdot \boldsymbol{\phi}_\tau(\mathbf{x})} \right] \prod_{\tau \in \mathcal{T}_p} e^{F(\epsilon, \boldsymbol{\phi}^*_{\tau-\epsilon}, \boldsymbol{\phi}_\tau)}. \tag{5.56}$$

Chapter 6

Conclusion

We have rigorously constructed functional integral representations for two types of quantum many-particle systems. The first system consists of a mixed gas with a finite number of particle species, bosons and/or fermions, interacting on a finite lattice with a general Hamiltonian that preserves the total number of particles in each species. Our first main result is the coherent-state functional integral representation for the partition function,

$$\mathrm{Tr}\, e^{-\beta K} = \lim_{q\to\infty} \int \prod_{\tau\in\mathcal{T}_q} \left[d\mu_{\underline{\mathbf{R}}(q)}(\boldsymbol{\phi}_\tau^*, \boldsymbol{\phi}_\tau) e^{-\int d\mathbf{y}\ \boldsymbol{\phi}_\tau^*(\mathbf{y})\cdot\boldsymbol{\phi}_\tau(\mathbf{y})} \right] \prod_{\tau\in\mathcal{T}_q} \langle \boldsymbol{\phi}_{\tau-\epsilon} | e^{-\epsilon K} | \boldsymbol{\phi}_\tau \rangle, \tag{6.1}$$

in Theorem 4.2.1. Here, $\underline{\mathbf{R}}(q)$ is a large-boson-field cutoff which grows sufficiently quickly as $q \to \infty$. In Section 4.3, we expanded the "action" $F(\epsilon, \boldsymbol{\alpha}^*, \boldsymbol{\phi}) := \log \langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle$ in powers of the fields $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ and derived a recursion relation for the coefficients. In the case of a 2-body interaction, such as the Coulomb interaction, we proved bounds on these coefficients and used them to find a domain of analyticity for the action. This domain is large enough that $e^{F(\epsilon,\boldsymbol{\alpha}^*,\boldsymbol{\phi})}$ may be substituted for each inner product of the form $\langle \boldsymbol{\alpha} | e^{-\epsilon K} | \boldsymbol{\phi} \rangle$ in the functional integral (6.1) above, as long as the large-field-cutoffs $\underline{\mathbf{R}}(q)$ are taken to be not too large.

The second system consists of one species of bosons and one species of fermions interacting on a finite lattice, which can be thought of as phonons and electrons respectively. The Hamiltonian takes a specific form, wherein the interaction term does not preserve the number of bosons. In this case, we generally use a Duhamel formula to expand in powers of the interaction, rather than over particle numbers. In Theorem 5.3.8, we find a rigorous coherent-state functional integral representation for the partition function, which is also of the form (6.1) above. Finally, we use properties of the Grassmann algebra in Section 5.4 to show that the "action" is an entire-analytic function of the fields.

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