Some new results on the $SU(3)$ Toda system and Lin-Ni problem

by

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Abstract

In this thesis, we mainly consider two problems. First, we study the $SU(3)$ Toda system. Let (M, g) be a compact Riemann surface with volume 1, h_1 and h_2 be a C^1 positive function on M and $\rho_1, \rho_2 \in \mathbb{R}^+$. The $SU(3)$ Toda system is the following one on the compact surface M

$$\begin{cases} \Delta u_1 + 2\rho_1(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1) - \rho_2(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1) = 4\pi \sum_{q \in S_1} \alpha_q(\delta_q - 1), \\ \Delta u_2 - \rho_1(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1) + 2\rho_2(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1) = 4\pi \sum_{q \in S_2} \beta_q(\delta_q - 1), \end{cases}$$

where Δ is the Beltrami-Laplace operator, $\alpha_q \geq 0$ for every $q \in S_1$, $S_1 \subset M$, $\beta_q \geq 0$ for every $q \in S_2$, $S_2 \subset M$ and δ_q is the Dirac measure at $q \in M$. We initiate the program for computing the Leray-Schauder topological degree of $SU(3)$ Toda system and succeed in obtaining the degree formula for $\rho_1 \in (0, 4\pi) \cup (4\pi, 8\pi)$, $\rho_2 \notin 4\pi\mathbb{N}$ when $S_1 = S_2 = \emptyset$.

Second, we consider the following nonlinear elliptic Neumann problem

$$\begin{cases} \Delta u - \mu u + u^q = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

where $q = \frac{n+2}{n-2}$, $\mu > 0$ and Ω is a smooth and bounded domain in $\mathbb{R}^n$. Lin and Ni (1986) conjectured that for μ small, all solutions are constants. In the second part of this thesis, we will show that this conjecture is false for a general domain in $n = 4, 6$ by constructing a nonconstant solution.

Preface

This dissertation is ultimately based on the original intellectual product of the author, Wen Yang under the guidance of his supervisor Prof. Jun-cheng Wei.

A version of Chapter 2 has been submitted and put on the arXiv: http://arxiv.org/abs/1408.5802.

A version of Chapter 3 is contained in a preprint paper joint with Prof. Jun-cheng Wei and Prof. Bin Xu.

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Last but not least, I want to show my gratitude to my parents for their support.

Dedication

To my parents.

Chapter 1

Introduction

This thesis mainly concerns two problems. First, we consider the $SU(3)$ Toda system. Let (M,g) be a compact Riemann surface with volume 1, h_1 and h_2 be C^1 positive functions on M and $\rho_1, \rho_2 \in \mathbb{R}^+$. The $SU(3)$ Toda system is the following one on the compact surface M,

$$\begin{cases} \Delta u_1 + 2\rho_1(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1) - \rho_2(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1) = 4\pi \sum_{q \in S_1} \alpha_q(\delta_q - 1), \\ \Delta u_2 - \rho_1(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1) + 2\rho_2(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1) = 4\pi \sum_{q \in S_2} \beta_q(\delta_q - 1), \end{cases} \tag{1.0.1}$$

where Δ is the Beltrami-Laplace operator, $\alpha_q \geq 0$ for every $q \in S_1,\ S_1 \subset M$, $\beta_q \geq 0$ for every $q \in S_2,\ S_2 \subset M$ and δ_q is the Dirac measure at $q \in M$. A partial result for the degree counting formula of (1.0.1) is obtained.

The second part of this thesis concerns the following nonlinear elliptic Neumann problem

$$\begin{cases} \Delta u - \mu u + u^q = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.0.2}$$

where $1 < q < +\infty,\ \mu > 0$ and Ω is a smooth and bounded domain in $\mathbb{R}^n$. In 1986, Lin and Ni proposed the following conjecture

Lin-Ni's Conjecture [41]. For μ small and $q = \frac{n+2}{n-2}$, problem (1.0.2) admits only the constant solution.

When $n = 4$ and 6, we prove the existence of nonconstant solution to (1.0.2) provided μ is sufficiently small. This gives a counterexample of the conjecture in dimensions $n = 4$ and 6.

1.1 The Degree Counting Formula For $SU(3)$ Toda System

1.1.1 Background And Main Results

Let (M, g) be a compact Riemann surface with volume 1, h_1 and h_2 be C^1 positive functions on M and $\rho_1, \rho_2 \in \mathbb{R}^+$. The $SU(3)$ Toda system on the compact surface M is the following

$$\begin{cases} \Delta u_1 + 2\rho_1\Big(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\Big) - \rho_2\Big(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\Big) = 4\pi \sum_{q\in S_1} \alpha_q(\delta_q - 1), \\ \Delta u_2 - \rho_1\Big(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\Big) + 2\rho_2\Big(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\Big) = 4\pi \sum_{q\in S_2} \beta_q(\delta_q - 1), \end{cases} \tag{1.1.1}$$

where Δ is the Beltrami-Laplace operator, $\alpha_q \geq 0$ for every $q \in S_1,\ S_1 \subset M$, $\beta_q \geq 0$ for every $q \in S_2,\ S_2 \subset M$ and δ_q is the Dirac measure at $q \in M$.

When the two equations in (1.1.1) are identical, i.e., $S_1 = S_2,\ \alpha_q = \beta_q$, $u_1 = u_2 = u$, $h_1 = h_2 = h$ and $\rho_1 = \rho_2 = \rho$, system (1.1.1) is reduced to the following mean field equation

$$\Delta u + \rho\Big(\frac{he^u}{\int_M he^u} - 1\Big) = 4\pi \sum_{q\in S_1} \alpha_q(\delta_q - 1). \tag{1.1.2}$$

Equations (1.1.1) and (1.1.2) arise in many physical and geometric problems. In physics, (1.1.2) or (1.1.1) is one of the limiting equations of the abelian gauge field theory or non-abelian Chern-Simons gauge field theory, one can see [24, 25, 48, 59, 60, 73] and references therein. In conformal geometry, equation (1.1.2) without singular sources corresponds to the Nirenberg problem of prescribing Gaussian curvature. In general, equation (1.1.2) is related to the existence of positive constant curvature metric with conic singularities. As for the Toda system (1.1.1), it is closely related to the classical Plücker formula for a holomorphic curve from M to $\mathbb{CP}^2$, the vortex points and $\alpha_q,\ \beta_q$ are exactly the branch points and its ramification index of this holomorphic curve. See [43] for more precise formulation and also [8, 9, 13, 18, 26, 35] for connection with different aspects of geometry. For the past decade, there are many studies for the $SU(3)$ Toda system, or more generally, system of equations with exponential nonlinearity. We refer the readers to [7, 29, 30, 36, 40, 44–47, 49–51, 54–56, 60, 71, 72] and references therein.

When $S_1 = \emptyset$, equation (1.1.1) becomes the following nonlinear elliptic equation

$$\Delta u + \rho\Big(\frac{he^u}{\int_M he^u} - 1\Big) = 0. \tag{1.1.3}$$

Clearly, (1.1.3) is the Euler-Lagrange equation of the nonlinear functional J_ρ

$$J_\rho(\phi) = \frac{1}{2}\int_M |\nabla\phi|^2 - \rho \log\Big(\int_M he^\phi\Big)$$

for $\phi \in \{f \in H^1(M) \mid \int_M \phi = 0\}$, where $H^1(M)$ denotes the Sobolev space of L^2 functions with L^2-integrable first derivatives. For $\rho < 8\pi$, $J_\rho(\phi)$ is bounded from below and the infinimum of $J_\rho(\phi)$ can be achieved by the well-known inequality due to Moser and Trudinger. For $\rho \geq 8\pi$, the existence of (1.1.3) is more difficult. Struwe and Tarantello [65] were able to obtain nontrivial solutions of (1.1.3) for $8\pi < \rho < 4\pi^2$ when $h^* \equiv 1$ and M is the flat torus with fundamental domain $[0,1]\times[0,1]$. Also, by using a similar approach, Ding, Jost, Li, and Wang [22] proved the existence of solutions to (1.1.3) for $8\pi < \rho < 16\pi$ when M is a compact Riemann surface with genus $g \geq 1$. For the case $M = \mathbb{S}^2$ and $8\pi < \rho < 16\pi$, Lin [39] proved the nonvanishing of the Leray-Schauder degree to equation (1.1.3), and consequently, the existence of solutions follows for the case of genus 0. For the convenience of the reader, we provide a short introduction of the Leray-Schauder degree in Section 7 of Chapter 2. When the value of the parameter satisfies $\rho > 16\pi$, the existence results for (1.1.3) can be deduced by the degree formula obtained by Chen-Lin in [14, 15] (Malchiodi uses a different approach to get the same degree counting formula in [53]). More precisely, Li [37] proposed the problem of studying the existence of solutions to (1.1.3) by the Leray-Schauder topological degree. Obviously, equation (1.1.3) is invariant under adding a constant. Hence, we can seek solutions in the class of functions that are normalized by $\int_M u = 0$. By the results of

Brezis and Merle [11] and Li and Shafrir [38], it follows that for any integer $m > 0$ and for any compact set I in $(8\pi m, 8\pi(m+1))$, the normalized solutions of (1.1.3) are uniformly bounded for any positive C^1 function h and $\rho \in I$. Thus, the Leray-Schauder degree d_ρ at zero of the Fredholm map $I + T(\rho)$ with

$$T(\rho) = \rho\Delta^{-1}\Big(\frac{he^u}{\int_M he^u} - 1\Big)$$

is well defined. Moreover, d_ρ is independent of both the function $h(x)$ and the parameter ρ whenever $\rho \in (8\pi m, 8\pi(m+1))$. Subsequently, Chen-Lin in [14, 15] were able to complete Li's analysis and they arrived at the following formula:

$$d_\rho = \begin{cases} 1, & \text{if } \rho \in (0, 8\pi), \\ \frac{(m-\chi(M))\cdots(1-\chi(M))}{m!}, & \text{if } \rho \in (8m\pi, 8(m+1)\pi),\ m > 0, \end{cases} \tag{1.1.4}$$

where $\chi(M) = 2(1-g)$ is the Euler characteristic of M with genus g.

Since the degree counting formula of (1.1.3) is obtained, it is natural to consider the same problem for equation (1.1.2). For equation (1.1.2), we let the set Σ of the critical parameters be defined by

$$\begin{aligned}\Sigma :&= \big\{8\pi N + \Sigma_{q\in A} 8\pi(1+\alpha_q) \mid A \subseteq S_1,\ N \in \mathbb{N}\cup\{0\}\big\} \setminus \{0\} \\ &= \{8\pi\mathfrak{a}_k \mid k = 1, 2, 3, \cdots.\},\end{aligned}$$

where $\mathfrak{a}_k$ will be defined in (1.1.5). We note that if $S_1 = \emptyset$, $\Sigma = 8m\pi$, which is indeed the set of the critical parameters for (1.1.3). It was proved that if $\rho \notin \Sigma$, then the a-priori estimate for any solution of (1.1.2) holds in $C^2_{\text{loc}}(M \setminus S_1)$. This a-priori bound was obtained by Li and Shafrir [38] for the case without singular sources, i.e., $S_1 = \emptyset$, and by Bartolucci and Tarantello [6] for the general case with singular sources. After establishing the a-priori bound for a non-critical parameter ρ, the Leray Schauder degree for the equation (1.1.2) in the general case is well-defined for $\rho \in \mathbb{R}^+ \setminus \Sigma$. Following the same idea in [14, 15], Chen and Lin in [16, 17] have derived the topological degree counting formula for (1.1.2) as described below.

We denote the topological degree of (1.1.2) for $\rho \notin \Sigma$ by d_ρ. By the homotopic invariant of the topological degree, d_ρ is a constant for $8\pi\mathfrak{a}_k < \rho < 8\pi\mathfrak{a}_{k+1},\ k = 0, 1, 2, \cdots$, where $\mathfrak{a}_0 = 0$. Set $d_m = d_\rho$ for $8\pi\mathfrak{a}_m < \rho < 8\pi\mathfrak{a}_{m+1}$. To state the result, we introduce the following generating function Ξ_0 :

$$\begin{aligned}\Xi_0(x) =&(1 + x + x^2 + x^3 + \cdots)^{-\chi(M)+|S_1|}\Pi_{q\in S_1}(1 - x^{1+\alpha_q})\\ =&1 + \mathfrak{c}_1 x^{\mathfrak{a}_1} + \mathfrak{c}_2 x^{\mathfrak{a}_2} + \cdots + \mathfrak{c}_k x^{\mathfrak{a}_k} + \cdots .\end{aligned} \tag{1.1.5}$$

The degree d_m can be written in terms of $\mathfrak{c}_j$, as shown in the following theorem.

Theorem A. ([17]) *Let d_ρ be the Leray-Schauder degree for (1.1.2). Suppose $8\mathfrak{a}_m\pi < \rho < 8\mathfrak{a}_{m+1}\pi$. Then*

$$d_\rho = \sum_{j=0}^{m} \mathfrak{c}_j,$$

where $d_0 = 1$.

For the application, it often requires that $\alpha_q \in \mathbb{N}$ for all $q \in S_1$. In this case, $\Sigma = \{8\pi m \mid m \in \mathbb{N}\}$ and let $d_m = d_\rho$ for $\rho \in (8\pi m, 8\pi(m + 1))$. Then the generating function

$$\begin{aligned}\Xi_1(x) =&\sum_{k=0}^{\infty} d_k x^k = (1 + x + x^2 + \cdots)^{-\chi(M)+1+|S_1|}\Pi_{q\in S_1}(1 - x^{\alpha_q+1})\\ =&(1 + x + x^2 + \cdots)^{-\chi(M)+1}\Pi_{q\in S_1}(1 + x + x^2 + \cdots + x^{\alpha_q}).\end{aligned} \tag{1.1.6}$$

Clearly, we have $d_m \geq 1,\ \forall m$ provided $\chi(M) \leq 0$. Hence we can obtain the existence of the solution to (1.1.2) when the genus of M is nonzero.

In the first part of this thesis, we want to initiate the program for computing the Leray-Schauder degree formula for the system (1.1.1). However, it seems still a very challenging problem in full generality. Hence we shall consider the simplest (but nontrivial) case, described below. We assume

(i) $S_1, S_2 = \emptyset$,

(ii) $\rho_1 \in (0, 4\pi) \cup (4\pi, 8\pi)$ and $\rho_2 \notin \Sigma_1 = \{4\pi N \mid N \in \mathbb{N}\}$.

In order to state our result on the degree formula for $SU(3)$ Toda system (1.1.1), we first introduce the following generating function

$$\Xi_1(x) = (1 + x + x^2 + x^3 \cdots)^{-\chi(M)+1} = b_0 + b_1 x^1 + b_2 x^2 + \cdots + b_m x^m + \cdots ,$$

which is (1.1.6) provided $\alpha_q = 0,\ \forall q \in S_1$. It is easy to see that

$$b_m = \begin{pmatrix} m - \chi(M) \\ m \end{pmatrix}, \tag{1.1.7}$$

where

$$\begin{pmatrix} m - \chi(M) \\ m \end{pmatrix} = \begin{cases} \frac{(m-\chi(M))\cdots(1-\chi(M))}{m!}, & \text{if } m \geq 1, \\ 1, & \text{if } m = 0. \end{cases}$$

Under the assumption $(i) - (ii)$, our result on computing the Leray-Schauder degree for system (1.1.1) is as follows.

Theorem 1.1.1. *Suppose $S_1 = S_2 = \emptyset$. Let $d^{(2)}_{\rho_1,\rho_2}$ denote the topological degree for (1.1.1) when $\rho_2 \in (4\pi m, 4\pi(m+1))$. Then*

$$d^{(2)}_{\rho_1,\rho_2} = \begin{cases} b_m, & \rho_1 \in (0, 4\pi), \\ b_m - \chi(M)(b_m + b_{m-1}), & \rho_1 \in (4\pi, 8\pi). \end{cases}$$

1.1.2 Sketch Of The Proof Of Theorem 1.1.1

In the following, we shall sketch the proof for the Theorem 1.1.1.

Step 1. Find the critical parameters of (1.1.1).

In order to compute the Leray-Schauder degree of the system, we need to develop a complete understanding of the blow-up phenomena for (1.1.1). The first main issue for the system is to determine the set of critical parameters, i.e., those $\rho = (\rho_1, \rho_2)$ such that the a-priori bounds for solutions of (1.1.1) fail.

Based on the assumption $S_1 = S_2 = \emptyset$, we can write (1.1.1) as

$$\begin{cases} \Delta u_1 + 2\rho_1\Big(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\Big) - \rho_2\Big(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\Big) = 0, \\ \Delta u_2 - \rho_1\Big(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\Big) + 2\rho_2\Big(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\Big) = 0. \end{cases} \tag{1.1.8}$$

For equation (1.1.8), if $u_1 = u_2 = u$, $h_1 = h_2 = h$, and $\rho_1 = \rho_2 = \rho$, equation (1.1.8) turns to be

$$\Delta u + \rho\Big(\frac{he^u}{\int_M he^u} - 1\Big) = 0. \tag{1.1.9}$$

It is known that for mean field equation (1.1.9), the blow up phenomena is closely related to the concentration phenomena, i.e., $\frac{he^{u_k}}{\int_M he^{u_k}}$ tends to a sum of Dirac measures, where u_k is a sequence of blow up solutions to (1.1.9). More precisely, let u_k be a sequence of blow-up solutions to (1.1.9). Then,

$$\rho\frac{he^{u_k}}{\int_M he^{u_k}} \to 8\pi\sum_{p\in B}\delta_p,$$

where B is the set containing all the blow up points of u_k.

As a consequence, once the blow-up phenomena happens, $\rho = \rho\frac{he^{u_k}}{\int_M he^{u_k}} \to 8\mathbb{N}\pi$. Therefore, if $\rho \neq 8\mathbb{N}\pi$, we can get a-priori bound on the solutions of (1.1.9). While for system (1.1.8), we can not find the counterpart result in system. In fact, recently D'Aprile, Pistoia and Ruiz [19] have constructed a sequence of bubbling solutions (u_{1k}, u_{2k}) to (1.1.8) with one of them failing to have the concentration property. So, finding the critical parameter for system (1.1.8) is more difficult than for the single equation, especially for the general system (1.1.1). However, for the case without singular source term, i.e., the equation (1.1.8), we are able to determine all the critical parameters and the result is stated as follows,

Proposition 1.1.1. *Suppose h_i in (1.1.8) are positive functions and $\rho_i \neq 4\pi\mathbb{N}$, $i = 1, 2$. Then, there exists a positive constant c such that for any* *solution of equation (1.1.8), there holds:*

$$|u_i(x)| \leq c, \ \forall x \in M, \ i = 1, 2.$$

Step 2. Find out all the blow up solutions of (1.1.8).

By Proposition 1.1.1, the Leray-Schauder degree $d^{(2)}_{\rho_1,\rho_2}$ for (1.1.1), or equivalently (1.1.8), is well-defined for $\rho_1 \in (0, 4\pi) \cup (4\pi, 8\pi)$ and $\rho_2 \notin 4\pi\mathbb{N}$. Clearly, $d^{(2)}_{\rho_1,\rho_2} = d^{(1)}_{\rho_2}$ if $0 < \rho_1 < 4\pi$, and $\rho_2 \notin 4\pi\mathbb{N}$. Hence, the main result of the first part of the thesis is to compute the degree $d^{(2)}_{\rho_1,\rho_2}$ for $4\pi < \rho_1 < 8\pi$. By the homotopic invariant, for any fixed $\rho_2 \notin 4\pi\mathbb{N}$, $d^{(2)}_{\rho_1,\rho_2}$ is a constant for $\rho_1 \in (0, 4\pi)$, and the same holds true for $\rho_1 \in (4\pi, 8\pi)$. For simplicity, we might let $d^{(2)}_-$ and $d^{(2)}_+$ denote $d^{(2)}_{\rho_1,\rho_2}$ for $\rho_1 \in (0, 4\pi)$ and $\rho_1 \in (4\pi, 8\pi)$. Since $d^{(2)}_-$ is known by Theorem A, computing $d^{(2)}_+$ is equivalent to computing the difference of $d^{(2)}_+ - d^{(2)}_-$, which might be not zero due to the bubbling phenomena of (1.1.8) at $(4\pi, \rho_2)$. To calculate $d^{(2)}_+ - d^{(2)}_-$, we need to compute the topological degree of the bubbling solution of (1.1.8) when ρ_1 crosses 4π, $\rho_2 \notin 4\pi\mathbb{N}$. For convenience, we rewrite (1.1.8) as

$$\begin{cases} \Delta v_1 + \rho_1\left(\frac{h_1 e^{2v_1 - v_2}}{\int_M h_1 e^{2v_1 - v_2}} - 1\right) = 0, \\ \Delta v_2 + \rho_2\left(\frac{h_2 e^{2v_2 - v_1}}{\int_M h_2 e^{2v_2 - v_1}} - 1\right) = 0, \end{cases} \tag{1.1.10}$$

where $v_1 = \frac{1}{3}(2u_1 + u_2)$, $v_2 = \frac{1}{3}(u_1 + 2u_2)$. It is known that the Leray-Schauder degree for (1.1.8) and (1.1.10) are the same. So, our aim is to compute the degree contribution of the bubbling solution of (1.1.10) when ρ_1 crosses 4π, $\rho_2 \notin 4\pi\mathbb{N}$. We consider (v_{1k}, v_{2k}) to be a sequence of solutions of (1.1.10) with $(\rho_{1k}, \rho_{2k}) \to (4\pi, \rho_2)$, and assume $\max_M(v_{1k}, v_{2k}) \to \infty$. Then we have the following theorem

Proposition 1.1.2. *Let (v_{1k}, v_{2k}) be described as above. Then, the followings hold:*

(i)

$$\rho_{1k}\frac{h_1 e^{2v_{1k}-v_{2k}}}{\int_M h_1 e^{2v_{1k}-v_{2k}}} \to 4\pi\delta_p \text{ for some } p\in M, \tag{1.1.11}$$

(ii) $v_{2k}\to \frac{1}{2}w$ in $C^{2,\alpha}(M)$, *where* (p,w) *satisfies*

$$\nabla\big(\log(h_1 e^{-\frac{1}{2}w})(x)+4\pi R(x,x)\big)\,|_{x=p}=0, \tag{1.1.12}$$

and

$$\Delta w+2\rho_2\Big(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}}-1\Big)=0. \tag{1.1.13}$$

Here $R(x,p)$ *refers to the regular part of the Green function* $G(x,p)$.

Step 3. Computing the degree contributed by the blow up solution of (1.1.10).

We write (1.1.12) and (1.1.13) as

$$\begin{cases} \Delta w+2\rho_2\big(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}}-1\big)=0,\\ \nabla\big(\log(h_1 e^{-\frac{1}{2}w})(x)+4\pi R(x,x)\big)\,|_{x=p}=0. \end{cases} \tag{1.1.14}$$

The system (1.1.14) is called the *shadow system* of (1.1.10). We say (p,w) is called a non-degenerate solution of (1.1.14) if the linearized equation. i.e., for (ϕ,ν), where $\nu\in\mathbb{R}^2$

$$\begin{cases} \Delta\phi+2\rho_2\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}}\phi \\ \qquad -2\rho_2\frac{h_2 e^{w-4\pi G(x,p)}}{\big(\int_M h_2 e^{w-4\pi G(x,p)}\big)^2}\int_M\big(h_2 e^{w-4\pi G(x,p)}\phi\big) \\ \qquad -8\pi\rho_2\frac{h_2 e^{w-4\pi G(x,p)}(\nabla G(x,p)\nu)}{\int_M h_2 e^{w-4\pi G(x,p)}} \\ \qquad +8\pi\rho_2\frac{h_2 e^{w-4\pi G(x,p)}\int_M\big(h_2 e^{w-4\pi G(x,p)}(\nabla G(x,p)\nu)\big)}{\big(\int_M h_2 e^{w-4\pi G(x,p)}\big)^2}=0, \\ \nabla^2\big(\log(h_1 e^{-\frac{1}{2}w})(x)+4\pi R(x,x)\big)\,|_{x=p}\,\nu-\frac{1}{2}\nabla\phi(p)=0,\ \ \int_M\phi=0, \end{cases}$$

admits only trivial solution, i.e., $(\phi,\nu)=(0,0)$. For a given solution (p,w)

of equation (1.1.14), we say λ is an eigenvalue of the linearized equation (1.1.14) if there exists an nontrivial pair (ϕ, ν) such that

$$\begin{cases} \Delta\phi + 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}}\phi \\ \qquad -2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\left(\int_M h_2 e^{w-4\pi G(x,p)}\right)^2} \int_M \left(h_2 e^{w-4\pi G(x,p)}\phi\right) \\ \qquad -8\pi\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}(\nabla G(x,p)\nu)}{\int_M h_2 e^{w-4\pi G(x,p)}} \\ \qquad +8\pi\rho_2 \frac{h_2 e^{w-4\pi G(x,p)} \int_M \left(h_2 e^{w-4\pi G(x,p)}(\nabla G(x,p)\nu)\right)}{\left(\int_M h_2 e^{w-4\pi G(x,p)}\right)^2} + \lambda\phi = 0, \\ \nabla^2\big(\log(h_1 e^{-\frac{1}{2}w})(x) + 4\pi R(x,x)\big)\,|_{x=p}\,\nu - \frac{1}{2}\nabla\phi(p) + \lambda\nu = 0, \;\; \int_M \phi = 0. \end{cases}$$

The Morse index of solution (p, w) to (1.1.14) is the total number (counting multiplicity) of the negative eigenvalues of the linearized system, and the Leray-Schauder topological degree contributed by (p, w) is given by $(-1)^N$, where N is the Morse index. From Proposition 1.1.2, it is known that any blow up solution of (1.1.10) is closely related to (1.1.14). Furthermore, we shall prove that the topological degree contributed by all the blow-up solutions equals the topological degree of the shadow system (1.1.14) up to some factor, i.e., we have the following result.

Proposition 1.1.3. *Let d_T denote the topological degree contributed by all the blow up solutions of (1.1.10) and d_S denote the topological degree contributed by (1.1.14). Then*

$$d_T = -d_S.$$

Step 4. Computing the degree contributed by the shadow system (1.1.14).

The equation (1.1.14) is a coupled system, and we can not directly compute the topological degree of (1.1.14). In this step, we introduce a deformation to decouple the system (1.1.14)

$$(S_t)\begin{cases} \Delta w + 2\rho_2\big(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1\big) = 0, \\ \nabla\big(\log(h_1 e^{-\frac{1}{2}w\cdot(1-t)}) + 4\pi R(x,x)\big)\,|_{x=p} = 0. \end{cases} \tag{1.1.15}$$

During the deformation from (S_1) to (S_0), we have the following lemma.

Lemma 1.1.1. *Let $\rho_2 \notin 4\pi\mathbb{N}$. Then there is uniform constant C_{ρ_2} such that for all solutions to (1.1.15), we have $|w|_{L^\infty(M)} < C_{\rho_2}$.*

By using Lemma 1.1.1, computing the topological degree for the coupled system (S_0) is equivalent to computing the topological degree for the decoupled system (S_1). For system (S_1), we are able to compute its topological degree. Therefore, we can get the degree formula for (1.1.14) and the degree contributed by all the blow-up solutions. Then, we are able to complete the proof of Theorem 1.1.1.

Remark: Based on Theorem 1.1.1, a natural question is: what happens when ρ_1 crosses 8π? As the case ρ_1 crosses 8π, we can also get a corresponding shadow system and the equation is as follows,

$$\begin{cases} \Delta w + 2\rho_2\big(\frac{h_2 e^{w-4\pi G(x,p_1)-4\pi G(x,p_2)}}{\int_M h_2 e^{w-4\pi G(x,p_1)-4\pi G(x,p_2)}} - 1\big) = 0, \\ \nabla\big(\log(h_1 e^{-\frac{1}{2}w})(x) + 4\pi R(x,x) + 8\pi G(x,p_2)\big)\,|_{x=p_1} = 0, \\ \nabla\big(\log(h_1 e^{-\frac{1}{2}w})(x) + 4\pi R(x,x) + 8\pi G(x,p_1)\big)\,|_{x=p_2} = 0. \end{cases} \tag{1.1.16}$$

For the equation (1.1.16), if we treat it by a similar way as we did for (1.1.14), i.e., replace $-\frac{1}{2}w$ by $-\frac{1}{2}w(1-t)$ for the second and third equation of (1.1.16), we can get a decoupled system by letting $t = 1$. However, during the deformation, we can not provide a uniform estimate for the solution w until now. Indeed, when t changes form 0 to 1, the points p_2 and p_1 may go to a same point, which may cause blow up phenomena for the solution w and the degree may change after the deformation, which is the most difficult point when we consider ρ_1 crossing $4m\pi$ for $m \geq 2, m \in \mathbb{N}$.

1.2 Lin-Ni Problem

1.2.1 Background

In the second part of my thesis, I consider the following nonlinear Neumann elliptic problem

$$\begin{cases} \Delta u - \mu u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.1}$$

where $1 < p < +\infty$, $\mu > 0$, ν denotes the outward unit normal vector of $\partial\Omega$, and Ω is a smooth and bounded domain in $\mathbb{R}^n$.

Problem (1.2.1) arises in many applied models concerning biological pattern formations. For example, it gives rise to steady states in the Keller-Segel model of the chemotactic aggregation of the cellular slime molds. Chemotaxis is the oriented movement of cells in response to chemicals in their environment. Cellular slime molds (amoebae) release a certain chemical, move toward places of its higher concentration, and eventually form aggregates. Keller and Segel [33] proposed a model to describe the chemotactic aggregation stage of cellular slime molds. Let $u(x,t)$ denote the population of amoebae at place x and at time t and $v(x,t)$ be the concentration of the chemical. Then the simplified Keller-Segel system is written as

$$\begin{cases} (KS1) & u_t = D_1 \Delta u - \chi \nabla \cdot (u \nabla \phi(v)) & \text{in } \Omega \times (0, +\infty), \\ (KS2) & v_t = D_2 \Delta v + k(u,v) & \text{in } \Omega \times (0, +\infty), \\ (IC) & u(x,0) = u_0 > 0,\ v(x,0) = v_0 > 0 & \text{in } \Omega, \\ (BC) & \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} & \text{on } \partial\Omega, \end{cases}$$

where D_1, D_2 and χ are positive constants; ϕ is so-called sensitivity function which is a smooth function such that $\phi'(r) > 0$ for $r > 0$; k is a smooth function with $k_u \geq 0$ and $k_v \leq 0$.

We are concerned with stationary solutions to Keller-Segel system in the case of logarithmic sensitivity $\phi(v) = \ln v$ and $k(u,v) = -av + bu$, where a and b are positive constants. Since $\int_\Omega u(x,t)dx = \int_\Omega u_0(x,t)dx$ for all $t > 0$ by virtue of $(KS1)$ and (BC), we consider the following problem for positive functions u and v :

$$\begin{cases} D_1\Delta u - \chi\nabla\cdot(u\nabla\ln(v)) = 0 & \text{in } \Omega, \\ D_2\Delta v - av + bu = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial\nu} = 0 = \frac{\partial v}{\partial\nu} & \text{on } \partial\Omega, \\ |\Omega|^{-1}\int_\Omega u(x)dx = \overline{u}, \end{cases} \tag{1.2.2}$$

where $|\Omega|$ denotes the volume of Ω, and $\overline{u} > 0$ is a given constant. Obviously, $(u, v) = (\overline{u}, \overline{v})$ with $\overline{v} = a^{-1}b\overline{u}$ is a solution.

It is not difficult to reduce system (1.2.2) to a single equation. Indeed, we write the first equation as

$$\nabla\cdot\{D_1u\nabla[\ln u - \chi D_1^{-1}\ln v]\} = 0$$

and using boundary condition we see that $u = \lambda v^{\frac{\chi}{D_1}}$ for some positive constant λ. Thus (1.2.2) is equivalent to the following system for (v, λ):

$$\begin{cases} D_2\Delta v - av + b\lambda v^{\frac{\chi}{D_1}} = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial\nu} = 0 & \text{on } \partial\Omega, \\ |\Omega|^{-1}\int_\Omega v(x)dx = \overline{v}. \end{cases} \tag{1.2.3}$$

Now we set $p = \frac{\chi}{D_1}$, $d = \frac{D_2}{a}$, $\theta = (a^{-1}b\lambda)^{\frac{1}{q-1}}$, and $w(x) = \theta v(x)$, we have

$$d\Delta w - w + w^p = 0 \text{ in } \Omega, \quad \frac{\partial w}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{1.2.4}$$

By abuse of notation, let $w(x) = d^{\frac{1}{p-1}}u(x)$ and $\mu = \frac{1}{d}$, then (1.2.4) turns to be (1.2.1).

Problem (1.2.1) can be also viewed as the steady-state equation for the Gierer-Meinhardt system, which is a system of reaction-diffusion equation of the form

$$\begin{cases} a_t = d\Delta a - a + \frac{a^p}{h^q} & \text{in } \Omega\times(0,+\infty), \\ \tau h_t = D\Delta h - h + \frac{a^r}{h^s} & \text{in } \Omega\times(0,\infty), \\ \frac{\partial a}{\partial\nu} = \frac{\partial h}{\partial\nu} = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \tag{1.2.5}$$

where d, D, p, q, r, s are all positive constants, $s \geq 0$, and

$$0 < \frac{p-1}{q} < \frac{r}{s+1}.$$

This system was motivated by biological experiments on *hydra* in morphogenesis. *Hydra*, an animal of a few millimeters in length, is made up of approximately $100,000$ cells of about fifteen different types. It consists of a "head" region located at one end along its length. Typical experiments on *hydra* involve removing part of the "head" region and transplanting it to other parts of the body column. Then, a new "head" will form if and only if the transplanted area is sufficiently far from the (old) head. These observations have led to the assumption of the existence of two chemical substances: a slowly diffusing (short-range) activator and a rapidly diffusing (long-range) inhibitor. Here $a(x,t), h(x,t)$ represent the density of the activator and inhibitor respectively.

For the full Gierer-Meinhardt system (1.2.5), there are a lot of works concerning the existence and the stability of the solutions with specific configuration. For example, D. Iron, M. Ward, and J. Wei [28] studied the stability of the symmetric k-peaked solutions by using matched asymptotic analysis. For more results in this direction, one can see [28], [64], [69] and references therein.

In general, the full (GM) system (1.2.5) is still very difficult to study. A very useful idea, which goes back to Keener [32] and Nishiura [58], is to consider the so-called *shadow system*. Namely, we let $D \to +\infty$ first. Suppose that the quantity $-h + \frac{a^p}{h^q}$ remains bounded, then we obtain

$$\Delta h \to 0, \ \frac{\partial h}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{1.2.6}$$

Thus $h(x,t) \to \xi(t)$, a constant. To derive the equation for $\xi(t)$, we integrate both sides of the equation for h over Ω and then we obtain the following so-called shadow system

$$\begin{cases} a_t = d\Delta a - a + \frac{a^p}{\xi^q} & \text{in } \Omega, \\ \tau\xi_t = -\xi + \frac{1}{|\Omega|}\int_\Omega \frac{a^r}{\xi^s}, \\ a > 0 \text{ in } \Omega \text{ and } \frac{\partial a}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2.7}$$

The advantage of shadow system is that by a simple scaling

$$a = \xi^{\frac{q}{p-1}} w, \quad \xi = \left(\frac{1}{|\Omega|}\int_\Omega w^r\right)^{\frac{p-1}{(p-1)(s+1)-qr}}, \tag{1.2.8}$$

the stationary shadow system can be reduced to a single equation

$$d\Delta w - w + w^p = 0 \text{ in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

which is exactly the same form as (1.2.4). By a same transformation as we did for (1.2.4), we can obtain (1.2.1).

Equation (1.2.1) enjoys at least one solution, namely the constant solution $u = \mu^{\frac{1}{p-1}}$. In a series of seminal works, Lin, Ni and Takagi [42] and Ni and Takagi [57] initiated quantitative analysis of non-constant solution to equation (1.2.1). In particular, it is proved in [42],[57] that for μ large, the least energy solution concentrates at the boundary point of maximum mean curvature. On the other hand, in the subcritical case $1 < p < 2^* - 1$, blow up arguments and the compactness of embedding imply that for small positive μ, the constant solution is the only solution. This uniqueness result motivated Lin and Ni to raise the following conjecture, the extension of this result to the critical case $p = 2^* - 1$.

Lin-Ni's Conjecture [41]. For μ small and $p = \frac{n+2}{n-2}$, problem (1.2.1) admits only the constant solution.

1.2.2 Previous Results On Lin-Ni Problem

In the following, we recall the the main results towards proving or disproving Lin-Ni's conjecture. Adimurthi-Yadava [2]-[3] and Budd-Knapp-Peletier [12] first considered the following problem

$$\begin{cases} \Delta u - \mu u + u^{\frac{n+2}{n-2}} = 0 & \text{in } B_R(0), \\ u > 0 & \text{in } B_R(0), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_R(0). \end{cases} \tag{1.2.9}$$

They proved the following result:

Theorem B. ([2–4, 12]) *For μ sufficiently small*

(1) *if $n = 3$ or $n \geq 7$, problem (1.2.9) admits only the constant solution,*

(2) *if $n = 4, 5, 6$, problem (1.2.9) admits a nonconstant solution.*

The proof of Theorem B relies on the radial symmetry of the domain. In the asymmetric case, the complete answer is not known yet, but there are a few results. In the general three-dimension domain case, Zhu [74] proved

Theorem C. ([70, 74]) *The conjecture is true if $n = 3$ $(p = 5)$ and Ω is convex.*

Zhu's proof relies on a priori estimate. Later, Wei and Xu [70] gave a direct proof of Theorem C by using a method based only on integration by parts only. In comparison with the strong convexity condition assumed on the domain, under the assumption on the bound of the energy and a weaker convexity condition (mean convex domains), Druet, Robert and Wei [23] showed the following result:

Theorem D. ([23]) *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n = 3$ or $n \geq 7$. Assume that $H(x) > 0$ for all $x \in \partial\Omega$, where $H(x)$ is the mean curvature of $\partial\Omega$ at $x \in \partial\Omega$. Then for all $\mu > 0$, there exists $\mu_0(\Omega, \Lambda) > 0$ such that for all $\mu \in (0, \mu_0(\Omega, \Lambda))$ and for any $u \in C^2(\overline{\Omega})$, we have that*

$$\left\{ \begin{array}{ll} -\Delta u + \mu u = n(n-2)u^{2^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \\ \int_\Omega u^{2^*} dx \leq \Lambda & \end{array} \right\} \Rightarrow u \equiv \left(\frac{\mu}{n(n-2)}\right)^{\frac{n-2}{4}}.$$

It should be mentioned that the assumption of the bounded energy is necessary in obtaining Theorem D. Without this technical assumption, it was proved that the solutions to (1.2.9) may accumulate with infinite energy when the mean curvature is negative somewhere (see Wang-Wei-Yan [66]). More precisely, Wang, Wei and Yan gave a negative answer to Lin-Ni's conjecture in all dimensions ($n \geq 3$) for non-convex domain by assuming that Ω is a smooth and bounded domain satisfying the following conditions:

(H_1) $y \in \Omega$ if and only if $(y_1, y_2, y_3, \cdots, -y_i, \cdots, y_n) \in \Omega,\ \forall i = 3, \cdots, n$.
(H_2) If $(r, 0, y'') \in \Omega$, then $(r\cos\theta, r\sin\theta, y'') \in \Omega,\ \forall \theta \in (0, 2\pi)$, where $y'' = (y_3, \cdots, y_n)$.
(H_3) Let $T := \partial\Omega \cap \{y_3 = \cdots = y_n = 0\}$, there exists a connected component Γ of T such that $H(x) \equiv \gamma < 0,\ \forall x \in \Gamma$.

Theorem E. ([66]) *Suppose* $n \geq 3,\ q = \frac{n+2}{n-2}$ *and* Ω *is a bounded smooth domain satisfying* (H_1)-(H_3). *Let* μ *be any fixed positive number. Then problem (1.2.9) has infinitely many positive solutions, whose energy can be made arbitrarily large.*

Wang, Wei and Yan [67] also gave a negative answer to Lin-Ni's conjecture in some convex domain including the balls for $n \geq 4$.

Theorem F. ([67]) *Suppose* $n \geq 4,\ q = \frac{n+2}{n-2}$ *and* Ω *satisfies* (H_1)-(H_2). *Let* μ *be a any fixed positive number. Then problem (1.2.9) has infinitely many positive solutions, whose energy can be made arbitrarily large.*

Theorem B-F reveal that Lin-Ni's conjecture depends very sensitively not only on the dimensions, but also on the shape of the domain. A natural question is: what about a general domain? Inspired by the result of Theorem B, we expect to give a negative answer to the case $n = 4, 5, 6$. The only approach in this direction is given by Rey and Wei [63]. They disproved the conjecture in the five-dimensional case by establishing a nontrivial solution which blows up at K interior points in Ω provided μ is sufficiently small. The second part of my thesis is to establish a result similar to (2) of Theorem B in general four, and six-dimensional domains by establishing a nontrivial solution which blows up at a single point in Ω provided μ is sufficiently small. Namely, we consider the problem

$$\Delta u - \mu u + u^{\frac{n+2}{n-2}} = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{1.2.10}$$

where $n = 4, 6$ and Ω is a smooth bounded domain in $\mathbb{R}^n$ and $\mu > 0$ very small. Our main result is stated as follows

Theorem 1.2.1. *For problem (1.2.10) in $n = 4, 6$, there exists $\mu_0 > 0$ such that for all $0 < \mu < \mu_0$, equation (1.2.10) possesses a nontrivial solution.*

1.2.3 Sketch Of The Proof Of Theorem 1.2.1

Our proof use the *localized energy method*, which was introduced in [27] and [52] in dealing with spikes. This method usually consists of five steps.

Step 1. Find out good approximate solutions.

We set

$$\Omega_\varepsilon := \Omega/\varepsilon = \{z | \varepsilon z \in \Omega\},$$

and

$$\mu = \begin{cases} \left(\frac{c_1}{-\ln \varepsilon}\right)^{\frac{1}{2}}, & n = 4, \\ \varepsilon\ , & n = 6, \end{cases} \tag{1.2.11}$$

where c_1 is some constant that depends on the domain only, to be determined later.

For the reason of normalization, we consider the following equation

$$\Delta u - \mu u + n(n-2)u^{\frac{n+2}{n-2}} = 0, \ u > 0 \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{1.2.12}$$

Through the transformation $u(x) \longmapsto \varepsilon^{-\frac{n-2}{2}} u(x/\varepsilon)$, (1.2.12) yields the rescaled problem written as

$$\Delta u - \mu\varepsilon^2 u + n(n-2)u^{\frac{n+2}{n-2}} = 0,\ u > 0 \text{ in } \Omega_\varepsilon,\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Omega_\varepsilon. \qquad (1.2.13)$$

For any $Q \in \Omega$, we use $U_{\Lambda, \frac{Q}{\varepsilon}}$ as an approximate solution of (1.2.13), where

$$U_{\Lambda,Q} = \Big(\frac{\Lambda}{\Lambda^2 + |x-Q|^2}\Big)^{\frac{n-2}{2}},\ \Lambda > 0,\ Q \in \mathbb{R}^n.$$

Because of the appearance of the additional term $\mu\varepsilon^2 u$, we need to add an extra term to get a better approximation. To this end, for $n = 4$, we consider the following equation

$$\Delta\bar{\Psi} + U_{1,0} = 0 \quad \text{in } \mathbb{R}^4, \quad \bar{\Psi}(0) = 1.$$

Let

$$\Psi_{\Lambda,Q} = \frac{\Lambda}{2}\ln\frac{1}{\Lambda\varepsilon} + \Lambda\bar{\Psi}(\frac{y-Q}{\Lambda}).$$

Then

$$\Delta\Psi_{\Lambda,Q} + U_{\Lambda,Q} = 0.$$

For $n = 6$, we denote $\Psi(|y|)$ as the radial solution of

$$\Delta\Psi + U_{1,0} = 0 \text{ in } \mathbb{R}^6, \quad \Psi \to 0 \text{ as } |\mathrm{y}| \to +\infty.$$

We set $\Psi(\Lambda, Q)(y) = \Psi(\frac{y-Q}{\Lambda})$, then

$$\Delta\Psi_{\Lambda,Q}(y) + U_{\Lambda,Q} = 0 \text{ in } \mathbb{R}^6.$$

In order to match the boundary condition, we need an extra correction term. For this purpose, we first introduce the Green's function

$$\Delta_x G(x,Q) + \delta_Q - \frac{1}{|\Omega|} = 0 \text{ in } \Omega,\ \frac{\partial G}{\partial \nu} = 0 \text{ on } \partial\Omega,\ \int_\Omega G(x,Q) = 0.$$

We decompose

$$G(x,Q) = K(|x-Q|) - H(x,Q),\ K(r) = \frac{1}{c_n r^{n-2}},\ c_n = (n-2)|\mathbb{S}^n|.$$

We define

$$\hat{U}_{\Lambda,\frac{Q}{\varepsilon}}(z) = -\Psi_{\Lambda,\frac{Q}{\varepsilon}}(z) - c_n\mu^{-1}\varepsilon^{n-4}\Lambda^{\frac{n-2}{2}}H(\varepsilon z,Q) + R_{\varepsilon,\Lambda,Q}(z)\chi(\varepsilon z),$$

where $R_{\varepsilon,\Lambda,Q}$ is defined by $\Delta R_{\varepsilon,\Lambda,Q} - \varepsilon^2 R_{\varepsilon,\Lambda,Q} = 0$ in Ω_ε and

$$\mu\varepsilon^2\frac{\partial R_{\varepsilon,\Lambda,Q}}{\partial\nu} = -\frac{\partial}{\partial\nu}\Big[U_{\frac{Q}{\varepsilon}} - \mu\varepsilon^2\Psi_{\frac{Q}{\varepsilon}} - c_n\varepsilon^{n-2}\Lambda^{\frac{n-2}{2}}H(\varepsilon z,Q)\Big] \text{ on } \partial\Omega_\varepsilon.$$

Here $\chi(x)$ is a smooth cut-off function in Ω such that $\chi(x) = 1$ for $d(x,\partial\Omega) < \delta/4$ and $\chi(x) = 0$ for $d(x,\partial\Omega) > \delta/2$.

Since in $n = 4$ and 6, the constructions of the approximate solutions are different, we shall treat them differently in the following. For $n = 4$, let

$$\Lambda_{4,1} \le \Lambda \le \Lambda_{4,2},\ Q \in \mathcal{M}_{\delta_4} := \{x \in \Omega |\ d(x,\partial\Omega) > \delta_4\}, \tag{1.2.14}$$

where $\Lambda_{4,1}$ and $\Lambda_{4,2}$ are constants depend on the domain and δ_4 is a small constant, to be determined in Chapter 3. We write

$$\bar{Q} = \frac{1}{\varepsilon}Q,$$

and define our approximate solutions as

$$W_{\varepsilon,\Lambda,Q} = U_{\Lambda,\bar{Q}} + \mu\varepsilon^2\hat{U}_{\Lambda,\bar{Q}} + \frac{c_4\Lambda}{|\Omega|}\mu^{-1}\varepsilon^2. \tag{1.2.15}$$

For $n = 6$, let

$$\sqrt{\frac{|\Omega|}{c_6}(\frac{1}{96} - \Lambda_6 \varepsilon^{\frac{2}{3}})} \leq \Lambda \leq \sqrt{\frac{|\Omega|}{c_6}(\frac{1}{96} + \Lambda_6 \varepsilon^{\frac{2}{3}})},$$
$$Q \in \mathcal{M}_{\delta_6} := \{x \in \Omega | \ d(x, \partial\Omega) > \delta_6\},$$
$$\frac{1}{48} - \eta_6 \varepsilon^{\frac{1}{3}} \leq \eta \leq \frac{1}{48} + \eta_6 \varepsilon^{\frac{1}{3}}, \tag{1.2.16}$$

where Λ_6 and η_6 are some constants that may depend on the domain, δ_6 is a small constant, which are also given in Chapter 3. Our approximate solution for $n = 6$ is the following

$$W_{\varepsilon,\Lambda,Q,\eta} = U_{\Lambda,\bar{Q}} + \mu\varepsilon^2 \hat{U}_{\Lambda,\bar{Q}} + \eta\mu^{-1}\varepsilon^4. \tag{1.2.17}$$

Step 2. A-priori estimate for a linear problem.

This is the most important step in reducing an infinite-dimensional problem to finite dimensional one. The key result we need is the non-degeneracy of the following solution u:

$$\Delta u + u^{\frac{n+2}{n-2}} = 0, \ u > 0 \text{ in } \mathbb{R}^n, \ u(y) \to 0 \text{ as } |y| \to \infty.$$

Using this non-degeneracy condition, we can show the solvability of the following linearized problem in suitable function space for $n = 4$,

$$\begin{cases} -\Delta\phi + \mu\varepsilon^2\phi - 24W^2\phi = h + \Sigma_{i=0}^4 c_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi\rangle = 0, & 0 \leq i \leq 4, \end{cases}$$

where

$$\begin{cases} Z_0 = -\Delta\frac{\partial W}{\partial\Lambda} + \mu\varepsilon^2\frac{\partial W}{\partial\Lambda}, \\ Z_i = -\Delta\frac{\partial W}{\partial\bar{Q}_i} + \mu\varepsilon^2\frac{\partial W}{\partial\bar{Q}_i}, \ 1 \leq i \leq 4. \end{cases}$$

and for $n = 6$,

$$\begin{cases} -\Delta\phi + \mu\varepsilon^2\phi - 48W\phi = h + \Sigma_{i=0}^{7} d_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi\rangle = 0, & 0 \le i \le 7, \end{cases}$$

where

$$\begin{cases} Z_0 = -\Delta\frac{\partial W}{\partial\Lambda} + \mu\varepsilon^2\frac{\partial W}{\partial\Lambda}, \\ Z_i = -\Delta\frac{\partial W}{\partial Q_i} + \mu\varepsilon^2\frac{\partial W}{\partial Q_i}, \ 1 \le i \le 6, \\ Z_7 = -\Delta\frac{\partial W}{\partial\eta} + \mu\varepsilon^2\frac{\partial W}{\partial\eta}. \end{cases}$$

Step 3. The solvability of the nonlinear problem.

Using $W_{\varepsilon,\Lambda,Q}$ as the approximate solution for $n = 4$, $W_{\varepsilon,\Lambda,Q,\eta}$ as the approximate solution for $n = 6$ and the result of Step 2, we can find that there exists a constant ε_0 such that for all $\varepsilon \le \varepsilon_0$, there is ϕ such that

$$\begin{cases} -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - 8(W+\phi)^3 = \sum_i c_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi\rangle = 0, & 0 \le i \le 4 \end{cases}$$

for $n = 4$, and

$$\begin{cases} -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - 24(W+\phi)^2 = \sum_i d_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi\rangle = 0, & 0 \le i \le 7 \end{cases}$$

for $n = 6$.

Step 4. The reduction lemma.

We define

$$I_\varepsilon(\Lambda, \bar{Q}) \equiv J_\varepsilon[W_{\varepsilon,\Lambda,Q} + \phi_{\varepsilon,\Lambda,Q}]$$

for $n = 4$ and

$$I_\varepsilon(\Lambda, \eta, \bar{Q}) \equiv J_\varepsilon[W_{\varepsilon,\Lambda,Q,\eta} + \phi_{\varepsilon,\Lambda,Q,\eta}]$$

for $n = 6$.

The reduction lemma says that if $(\Lambda, \bar{Q})$ and $(\Lambda, \eta, \bar{Q})$ are the critical points of I_ε for $n = 4$ and $n = 6$ respectively, then $u = W_{\varepsilon,\Lambda,Q} + \phi_{\varepsilon,\Lambda,\bar{Q}}$ and $u = W_{\varepsilon,\Lambda,Q,\eta} + \phi_{\varepsilon,\Lambda,\eta,\bar{Q}}$ are solution to problem (1.2.13) for $n = 4$ and $n = 6$ respectively.

Step 5. Finding the critical points of I_ε for $n = 4$ and $n = 6$ respectively.

For $n = 4$, we find the maximal value of I_ε in (1.2.14), and for $n = 6$, we find the min-max value of I_ε in (1.2.16).

Once we find the critical points of I_ε for $n = 4$ and $n = 6$. Using Step 4, we get a solution to (1.2.13). Furthermore, it is obvious that the solution we construct is nontrivial. Hence, we get Theorem 1.2.1, thereby disproving the Lin-Ni's conjecture in $n = 4$ and $n = 6$ for a general domain.

1.3 Organization Of The Thesis

In Chapter 2, we prove Theorem 1.1.1. The proof of Proposition 1.1.1 , Proposition 1.1.2 and the existence of smooth positive function h_1 and h_2 such that any solution of the shadow system (1.1.14) is non-degenerate are given in Section 2.1. In Section 2.2, we get the a-priori estimate for solutions of (1.1.10) when $\rho_1 \to 4\pi$ and $\rho_2 \notin 4\pi\mathbb{N}$. In Section 2.3 and Section 2.4, we use the solutions of the shadow system (1.1.14) to get a good approximation of some bubbling solutions of (1.1.10) and thereby prove Proposition 1.1.3 except for some important estimates which are shown in Section 2.6. In Section 2.5, we derive the degree formula for the shadow system (1.1.14) and prove Theorem 1.1.1.

Chapter 3 is devoted to the proof of Theorem 1.2.1. In Section 3.1, we construct suitable approximated bubble solution W, and list their properties and some important estimates with the proof given in Section 3.6. In Section 3.2, we solve the linearized problem at W in a finite-codimensional space. Then, in Section 3.3, we are able to solve the nonlinear problem in that space. In Section 3.4, we study the remaining finite-dimensional problem and solve it in Section 3.5 by finding critical points of the reduced energy functional.

Chapter 2

The $SU(3)$ Toda System

2.1 Proof Of Proposition 1.1.1, Proposition 1.1.2 And Shadow System

We shall prove Proposition 1.1.1 and Proposition 1.1.2 in this section. For a sequence of bubbling solutions (u_{1k}, u_{2k}) of (1.1.8), we set

$$\tilde{u}_{ik} = u_{ik} - \int_M h_i e^{u_{ik}}, \ i = 1, 2.$$

Then $\tilde{u}_{ik}$ satisfy

$$\begin{cases} \Delta \tilde{u}_{1k} + 2\rho_1 (h_1 e^{\tilde{u}_{1k}} - 1) - \rho_2 (h_2 e^{\tilde{u}_{2k}} - 1) = 0, \\ \Delta \tilde{u}_{2k} - \rho_1 (h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2 (h_2 e^{\tilde{u}_{2k}} - 1) = 0. \end{cases} \tag{2.1.1}$$

We define the blow up set for $\tilde{u}_{ik}$ as follows

$$\mathfrak{S}_i = \{ p \in M \mid \exists \{x_k\}, \ x_k \to p, \ \tilde{u}_{ik}(x_k) \to +\infty \} \tag{2.1.2}$$

and $\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2$. We note that

$$u_{ik} = \tilde{u}_{ik} + \int_M h_i e^{u_{ik}} \geq \tilde{u}_{ik} + C e^{\int_M u_{ik}} \geq \tilde{u}_{ik} + C,$$

where we used the Jensen's inequality and h_i (here $h_i = h_i^*$) is a positive function in M. So, if p is a blow up point of $\tilde{u}_{ik}$, then p is also a blow up point of u_{ik}. For any $p \in \mathfrak{S}$, we define the local mass by

$$\sigma_{ip} = \lim_{\delta \to 0} \lim_{k \to +\infty} \frac{1}{2\pi} \int_{B_\delta(p)} \rho_i h_i e^{\tilde{u}_{ik}}. \tag{2.1.3}$$

Lemma 2.1.1. *If* $\sigma_{1p}, \sigma_{2p} < \frac{1}{3}$, *we have* $p \notin \mathfrak{S}$.

Proof. Since $\sigma_{ip} < \frac{1}{3}$, we can choose small r_0, such that in $B_{r_0}(p)$, the following holds

$$\int_{B_{r_0}(p)} \rho_i h_i e^{\tilde{u}_{ik}} < \pi, \tag{2.1.4}$$

which implies $\int_{B_{r_0}(p)} \tilde{u}_{ik}^{+} \leq C$, where C is some constant independent of k. In the following, C always denotes some generic constant independent of k, and may depend on the domain $B_{r_0}(p)$. For the first equation in (2.1.1), we decompose $\tilde{u}_{1k} = \sum_{j=1}^{3} \tilde{u}_{1k,j}$, where $\tilde{u}_{1k,j}$ satisfy the following equation

$$\begin{cases} -\Delta \tilde{u}_{1k,1} = 2\rho_1 h_1 e^{\tilde{u}_{1k}} - \rho_2 h_2 e^{\tilde{u}_{2k}} & \text{in } B_{r_0}(p), \quad \tilde{u}_{1k,1} = 0 \quad \text{on } \partial B_{r_0}(p), \\ -\Delta \tilde{u}_{1k,2} = -2\rho_1 + \rho_2 & \text{in } B_{r_0}(p), \quad \tilde{u}_{1k,2} = 0 \quad \text{on } \partial B_{r_0}(p), \\ -\Delta \tilde{u}_{1k,3} = 0 & \text{in } B_{r_0}(p), \quad \tilde{u}_{1k,3} = \tilde{u}_{1k} \quad \text{on } \partial B_{r_0}(p). \end{cases} \tag{2.1.5}$$

For the first equation in (2.1.5), since

$$\int_{B_{r_0}(p)} \left| 2\rho_1 h_1 e^{\tilde{u}_{1k}} - \rho_2 h_2 e^{\tilde{u}_{2k}} \right| < 3\pi,$$

by [11, Theorem 1], we have

$$\int_{B_{r_0}(p)} \exp((1+\delta)|\tilde{u}_{1k,1}|)dx \leq C, \tag{2.1.6}$$

where $\delta \in (0, \frac{1}{3})$. Therefore, we have

$$\int_{B_{r_0}(p)} |\tilde{u}_{1k,1}| \leq C. \tag{2.1.7}$$

For the second equation in (2.1.5), we can easily get

$$\int_{B_{r_0}(p)} |\tilde{u}_{1k,2}| \leq C, \text{ and } |\tilde{u}_{1k,2}| \leq C. \tag{2.1.8}$$

For the third equation in (2.1.5). By the mean value theorem for harmonic function we have

$$\begin{aligned}\|\tilde{u}^{+}_{1k,3}\|_{L^\infty(B_{\frac{r_0}{2}}(p))} &\le C\|\tilde{u}^{+}_{1k,3}\|_{L^1(B_{r_0}(p))}\\ &\le C\Big[\|\tilde{u}^{+}_{1k}\|_{L^1(B_{r_0}(p))}+\|\tilde{u}_{1k,1}\|_{L^1(B_{r_0}(p))}+\|\tilde{u}_{1k,2}\|_{L^1(B_{r_0}(p))}\Big]\\ &\le C. \end{aligned} \tag{2.1.9}$$

From (2.1.8)-(2.1.9), we have

$$2\rho_1 h_1 e^{\tilde{u}_{1k,2}+\tilde{u}_{1k,3}} \le C \text{ in } B_{\frac{r_0}{2}}(p). \tag{2.1.10}$$

By (2.1.6), (2.1.10) and Hölder inequality, we obtain

$$e^{\tilde{u}_{1k}} \in L^{1+\delta_1}(B_{r_0}(p))$$

with $\delta_1 > 0$ independent of k. Similarly, we have

$$e^{\tilde{u}_{2k}} \in L^{1+\delta_2}(B_{r_0}(p))$$

with $\delta_2 > 0$ independent of k. By using the standard elliptic estimate for the first equation in (2.1.5), we get $\|\tilde{u}_{1k,1}\|_{L^\infty}(B_{r_0/2}(p))$ is uniformly bounded. Combined with (2.1.8) and (2.1.9), we have $\tilde{u}_{1k}$ is uniformly bounded above in $B_{\frac{r_0}{2}}(p)$. Following a same process, we can also obtain $\tilde{u}_{2k}$ is uniformly bounded above in $B_{\frac{r_0}{2}}(p)$. Hence, we finish the proof of the lemma. □

From Lemma 2.1.1, we get if $p \in \mathfrak{S}$, either $\sigma_{1p} \ge \frac{1}{3}$ or $\sigma_{2p} \ge \frac{1}{3}$, which implies $|\mathfrak{S}| < \infty$ and $\mathfrak{S}$ is discrete in M. In fact, in next lemma, we shall prove that if $p \in \mathfrak{S}_i$, σ_{ip} must be positive.

Lemma 2.1.2. *If $p \in \mathfrak{S}_i$, $\sigma_{ip} > 0$.*

Proof. We prove it by contradiction. Without loss of generality, we may assume $p \in \mathfrak{S}_2$ and $\sigma_{2p} = 0$. First, we claim that there is a constant $C_K > 0$ that depends on the compact set K such that

$$|u_{ik}(x)| \le C_K,\ \forall x \in K \subset\subset M \setminus \mathfrak{S},\ i = 1, 2. \tag{2.1.11}$$

We only prove for $i=1$, the other one can be obtained similarly

$$
\begin{aligned}
u_{1k}(x) &= \int_M G(x,z)\Big(2\rho_1(h_1 e^{\tilde{u}_{1k}}-1)-\rho_2(h_2 e^{\tilde{u}_{2k}}-1)\Big)\\
&= \int_{M_1} G(x,z)\Big(2\rho_1(h_1 e^{\tilde{u}_{1k}}-1)-\rho_2(h_2 e^{\tilde{u}_{2k}}-1)\Big)\\
&\quad + \int_{M\setminus M_1} G(x,z)\Big(2\rho_1(h_1 e^{\tilde{u}_{1k}}-1)-\rho_2(h_2 e^{\tilde{u}_{2k}}-1)\Big),
\end{aligned}
$$

where $M_1 = \cup_{p\in\mathfrak{S}} B_{r_0}(p)$ and r_0 is small enough to make $K \subset\subset M\setminus M_1$. It is easy to see that

$$
\int_{M_1} G(x,z)\Big(2\rho_1(h_1 e^{\tilde{u}_{1k}}-1)-\rho_2(h_2 e^{\tilde{u}_{2k}}-1)\Big) = O(1),
$$

because $G(x,z)$ is bounded due to the distance $d(x,z)\geq \delta_0 > 0$ for $z\in M_1$, and $x\in K$. In $M\setminus M_1$, we can see that $\tilde{u}_{ik}$ are bounded above by some constant depends on r_0, then it is not difficult to obtain that

$$
\int_{M\setminus M_1} G(x,z)\Big(2\rho_1(h_1 e^{\tilde{u}_{1k}}-1)-\rho_2(h_2 e^{\tilde{u}_{2k}}-1)\Big) = O(1).
$$

Therefore, we prove the claim.

Since $\sigma_{2p}=0$, we can find some r_0, such that

$$
\int_{B_{r_0}(p)} \rho_2 h_2 e^{\tilde{u}_{2k}} \leq \pi \tag{2.1.12}
$$

for all k (passing to a subsequence if necessary) and $r_0 \leq \frac{1}{2} d(p, \mathfrak{S}\setminus\{p\})$. On $\partial B_{r_0}(p)$, by (2.1.11)

$$
|u_{1k}|,\ |u_{2k}| \leq C \text{ on } \partial B_{r_0}(p). \tag{2.1.13}
$$

Let w_k satisfy the following equation

$$
\begin{cases}
\Delta w_k = \rho_1\Big(\frac{h_1 e^{u_{1k}}}{\int_M h_1 e^{u_{1k}}}-1\Big) & \text{in } B_{r_0}(p),\\
w_k = u_{1k} & \text{on } \partial B_{r_0}(p).
\end{cases} \tag{2.1.14}
$$

We set $w_k = w_{k1} + w_{k2}$ where w_{k1}, w_{k2} satisfy

$$\begin{cases} \Delta w_{k1} = \rho_1 \frac{h_1 e^{u_{1k}}}{\int_M h_1 e^{u_{1k}}} & \text{in } B_{r_0}(p), \quad w_{k1} = u_{1k} \quad \text{on } \partial B_{r_0}(p), \\ \Delta w_{k2} = -\rho_1 & \text{in } B_{r_0}(p), \quad w_{k1} = 0 \quad \text{on } \partial B_{r_0}(p). \end{cases} \tag{2.1.15}$$

By maximum principle and (2.1.13), we have $w_{k1} \leq \max_{\partial B_{r_0}(p)} u_{1k} \leq C$ for $x \in B_{r_0}(p)$. By elliptic estimate, we can easily get $|w_{k2}| \leq C$. Therefore,

$$w_k \leq C, \ \forall x \in B_{r_0}(p). \tag{2.1.16}$$

We set $u_{2k} = f_{k1} + f_{k2} + w_k$, where f_{k1} and f_{k2} satisfy

$$\begin{cases} \Delta f_{k1} = -2\rho_2 \frac{h_2 e^{u_{2k}}}{\int_M h_2 e^{u_{2k}}} & \text{in } B_{r_0}(p), \quad f_{k1} = 0 \quad \text{on } \partial B_{r_0}(p), \\ \Delta f_{k2} = 2\rho_2 & \text{in } B_{r_0}(p), \quad f_{k2} = u_{2k} - w_k \quad \text{on } \partial B_{r_0}(p). \end{cases} \tag{2.1.17}$$

For the second equation in (2.1.17), we have

$$|f_{k2}| \leq |u_{2k}| + |w_k| = |u_{2k}| + |u_{1k}| \leq C \text{ on } \partial B_{r_0}(p).$$

Thus $|f_{k2}| \leq C$ in $B_{r_0}(p)$. We denote $g_k = e^{f_{k2} + w_k}$, and the first equation in (2.1.17) can be written as

$$\Delta f_{k1} + 2\rho_2 \frac{h_2 e^{g_k}}{\int_M h_2 e^{u_{2k}}} e^{f_{k1}} = 0 \text{ in } B_{r_0}(p), \quad f_{k1} = 0 \text{ on } \partial B_{r_0}(p). \tag{2.1.18}$$

By using the Jensen's inequality, we have $\int_M h_2 e^{u_{2k}} \geq C e^{\int_M u_{2k}} \geq C > 0$. We set $V_k = 2\rho_2 \frac{h_2 e^{g_k}}{\int_M h_2 e^{u_{2k}}}$, and have $V_k \leq C$, where C depends on r_0. Using (2.1.12), we get $\int_{B_{r_0}(p)} V_k e^{f_{k1}} \leq 2\pi$. By [11, Corollary 3], we have $|f_{k1}| \leq C$ and

$$u_{2k} \leq f_{k1} + f_{k2} + w_k \leq C.$$

This leads to $\tilde{u}_{2k} = u_{2k} - \int_M h_2 e^{u_{2k}} \leq C$, which contradicts to the assumption $\tilde{u}_{2k}$ blows up at p. Thus we finish the proof of this lemma. $\square$

By these two lemmas, we now begin to prove Proposition 1.1.1.

Proof of Proposition 1.1.1. We note that it is enough for us to prove $\tilde{u}_{ik}$ is uniformly bounded above. We shall prove it by contradiction.

First, we claim $\mathfrak{S}_1 \neq \emptyset$. If not, $\tilde{u}_{1k}$ is uniformly bounded above and $\tilde{u}_{2k}$ blows up. We decompose $u_{2k} = u_{2k,1} + u_{2k,2}$, where $u_{2k,1}$ and $u_{2k,2}$ satisfy the following

$$\begin{cases} \Delta u_{2k,1} - \rho_1(h_1\tilde{u}_{1k} - 1) = 0, & \int_M u_{2k,1} = 0, \\ \Delta u_{2k,2} + 2\rho_2\big(\frac{\tilde{h}_{2k}e^{u_{2k,2}}}{\int_M \tilde{h}_{2k}e^{u_{2k,2}}} - 1\big) = 0, & \int_M u_{2k,2} = 0, \end{cases}$$

where $\tilde{h}_{2k} = h_2 e^{u_{2k,1}}$. By the L^p estimate, $u_{2k,1}$ is bounded in $W^{2,p}$ for any $p > 1$. Thus $u_{2k,1}$ is bounded in $C^{1,\alpha}$ for any $\alpha \in (0,1)$, after passing to a subsequence if necessary, we gain $u_{2k,1}$ converges to u_0 in $C^{1,\alpha}$. As a consequence, $\tilde{h}_{2k} \to h_2 e^{u_0}$ in $C^{1,\alpha}$. Since $\tilde{u}_{2k}$ blows up, u_{2k} and $u_{2k,2}$ both blow up. Then applying the result of Li and Shafrir in [38], we have $\rho_2 \in 4\pi\mathbb{N}$, which contradicts to our assumption. Thus $\mathfrak{S}_1 \neq \emptyset$. Similarly, we can prove that $\mathfrak{S}_2 \neq \emptyset$.

We note that our argument above can be applied to the local case, which yields $\mathfrak{S}_1 \cap \mathfrak{S}_2 \neq \emptyset$. Suppose $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \emptyset$. For any point $p \in \mathfrak{S}_2$, we consider the behavior of u_{1k} and u_{2k} in $B_{r_0}(p)$, where r_0 is small enough such that $B_{r_0}(p) \cap (\mathfrak{S} \setminus \{p\}) = \emptyset$. We decompose $\tilde{u}_{2k} = u_{2k,3} + u_{2k,4}$, where $u_{2k,3}$ and $u_{2k,4}$ satisfy

$$\begin{cases} \Delta u_{2k,3} - \rho_1(h_1 e^{\tilde{u}_{1k}} - 1) = 0 & \text{in } B_{r_0}(p), \quad u_{2k,3} = 0 \quad \text{on } \partial B_{r_0}(p), \\ \Delta u_{2k,4} + 2\rho_2(\tilde{h}_{2,k}e^{u_{2k,4}} - 1) = 0 & \text{in } B_{r_0}(p), \quad u_{2k,4} = \tilde{u}_{2k} \quad \text{on } \partial B_{r_0}(p), \end{cases} \tag{2.1.19}$$

where $\tilde{h}_{2,k} = h_2 e^{u_{2k,3}}$. By using $\tilde{u}_{1k}$ uniformly bounded from above in $B_{r_0}(p)$, we have $\tilde{h}_{2,k}$ converges in $C^{1,\alpha}(B_{r_0}(p))$. Since $u_{2k,4}$ blows up simply at p, we have

$$\Big| u_{2k,4} - \log\Big(\frac{e^{u_{2k,4}(p^{(k)})}}{(1 + \frac{\rho_2 \tilde{h}_{2,k}(p^{(k)})e^{u_{2k,4}(p^{(k)})}}{4}|x - p^{(k)}|^2)^2}\Big)\Big| \le C, \tag{2.1.20}$$

where $u_{2k,4}(p^{(k)}) = \max_{B_{r_0}(p)} u_{2k,4}$. (2.1.20) is proved in [5] and [37]. From

(2.1.20), we have

$$u_{2k,4} \to -\infty \text{ in } B_{r_0(p)} \setminus \{p\} \quad \text{and} \quad \rho_2 h_2 e^{\tilde{u}_{2k}} \to 4\pi\delta_p \text{ in } B_{r_0}(p), \tag{2.1.21}$$

which implies

$$\rho_2 = \lim_{k\to\infty} \int_M \rho_2 h_2 e^{\tilde{u}_{2k}} = 4\pi|\mathfrak{S}_2|, \tag{2.1.22}$$

a contradiction to our assumption $\rho_2 \notin 4\pi\mathbb{N}$, so $\mathfrak{S}_1 \cap \mathfrak{S}_2 \neq \emptyset$.

Let $p \in \mathfrak{S}_1 \cap \mathfrak{S}_2$, and $\sigma_{ip}, i = 1, 2$ be the local masses of them at p. Applying the result of Jost-Lin-Wang (Proposition 2.4 in [29]), we have $(\sigma_{1p}, \sigma_{2p})$ is one of $(2, 4)$, $(4, 2)$ and $(4, 4)$.

In the following, we claim if $\sigma_{ip} = 4$, then $\tilde{u}_{ik}$ concentrate, i.e., $\tilde{u}_{ik} \to -\infty$ uniformly in any compact set of $B_{r_0}(p) \setminus \{p\}$. This implies $\int_M h_i e^{u_{ik}} \to +\infty$ and $\tilde{u}_{ik} \to -\infty$ uniformly in any compact set of $M \setminus \mathfrak{S}_i$. Then,

$$\rho_i h_i e^{\tilde{u}_{ik}} \to \alpha_q \sum_{q\in\mathfrak{S}_i\setminus\{p\}} \delta_q + 8\pi\delta_p \text{ with } \alpha_q = 4\pi \text{ or } 8\pi, \tag{2.1.23}$$

which implies $\rho_i = 4\pi\mathbb{N}$ and again yields a contradiction. This completes the proof of Proposition 1.1.1. The proof of the claim is given in Lemma 2.1.3 below. □

Lemma 2.1.3. *Suppose $\tilde{u}_{ik}, i = 1, 2$ both blow up at p and let 4 be the local mass of $\tilde{u}_{ik}$ as before. Then $\tilde{u}_{ik} \to -\infty$ in $B_{r_0}(p) \setminus \{p\}$.*

Proof. If the claim is not true, we have $\tilde{u}_{ik}$ is bounded by some constant C in $L^\infty(\partial B_{r_0}(p))$. Without loss of generality, we assume $i = 2$. Then $\sigma_{1p} = 0,\ 2$ or 4. In fact, the proof of these three cases are the same, so, we only give the proof of the case $\sigma_{1p} = 4$. Let $f_{1k} = -\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1)$ and z_k be the solution of

$$\begin{cases} -\Delta z_k = f_{1k} & \text{in } B_{r_0}(p), \\ z_k = -C & \text{on } \partial B_{r_0}(p). \end{cases} \tag{2.1.24}$$

Note that $f_{1k} \to f_1$ uniformly in any compact set of $B_{r_0}(p) \setminus \{p\}$ and the integration of the right hand side of (2.1.24) over $B_{r_0}(p)$ is $8\pi + o(1)$ as

$r_0 \to 0$. By maximum principle, $\tilde{u}_{2k} \geq z_k$ in $B_{r_0}(p)$. In particular

$$\int_{B_{r_0}(p)} e^{z_k} \leq \int_{B_{r_0}(p)} e^{\tilde{u}_{2k}} < \infty.$$

On the other hand, using Green representation formula for z_k, we have

$$z_k(x) = -\int_{B_{r_0}(p)} \frac{1}{2\pi} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1)\big) + O(1), \tag{2.1.25}$$

where we used the regular part of the Green function is bounded. For any $x \in B_{r_0}(p) \setminus \{p\}$, we denote the distance between x and p by $2r$. From (2.1.25), we have

$$\begin{aligned} z_k(x) &= -\int_{B_{r_0}(p)} \frac{1}{2\pi} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1)\big) + O(1) \\ &= -\int_{B_{r_0}(p)\cap B_r(x)} \frac{1}{2\pi} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1)\big) \\ &\quad -\int_{B_{r_0}(p)\setminus B_r(x)} \frac{1}{2\pi} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1)\big) + O(1). \end{aligned}$$

It is easy to see

$$\Big| \int_{B_{r_0}(p)\cap B_r(x)} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1)\big) \Big| \leq C,$$

where we used the fact that $\tilde{u}_{ik}$ is uniformly bounded above in $B_r(x),\ i = 1, 2$ and C depends only on x. For $y \in B_{r_0}(p) \setminus B_r(x)$, we have $|x - y| \geq r$ and

$$\begin{aligned} &\int_{B_{r_0}(p)\setminus B_r(x)} \ln|x-y| \big(-\rho_1(h_1 e^{\tilde{u}_{1k}} - 1) + 2\rho_2(h_2 e^{\tilde{u}_{2k}} - 1)\big) \\ &= (8\pi + o(1)) \ln|x-p| + O(1). \end{aligned}$$

Therefore, we get $z_k(x)$ is uniformly bounded below by some constant that depends on x only. Thus, we have $z_k \to z$ in $C^2_{loc}(B_{r_0}(p) \setminus \{p\})$, where z

satisfies

$$\begin{cases} -\Delta z = f_1 & \text{in } B_{r_0}(p) \setminus \{p\}, \\ z = -C & \text{on } \partial B_{r_0}(P). \end{cases}$$

For $\varphi \in C_0^\infty(B_{r_0}(p))$,

$$\begin{aligned} -\lim_{k\to+\infty} \int_{B_{r_0}(p)} \varphi \Delta z_k &= -\int_{B_{r_0}(p)} (\varphi(x) - \varphi(p))\Delta z_k + \varphi(p)(\int_{B_{r_0}(p)} f_1 + 8\pi) \\ &= \int_{B_{r_0}(p)} \varphi(x) f_1 + 8\pi\varphi(p). \end{aligned}$$

Thus $-\Delta z = f_1 + 8\pi\delta_p$. Therefore, we have $z(x) \geq 4\log\frac{1}{|x-p|} + O(1)$ as $x \to p$, which implies $\int_{B_{r_0}(p)} e^z = \infty$, a contradiction. Hence

$$\tilde{u}_{2k} \to -\infty \text{ in } B_{r_0}(p) \setminus \{p\}. \tag{2.1.26}$$

Thus, Lemma 2.1.3 holds. □

Next, we prove Proposition 1.1.2 and derive the shadow system (1.1.14).

Proof of Proposition 1.1.2. As $\rho_{1k} \to 4\pi, \rho_{2k} \to \rho_2$ and $\rho_2 \notin 4\pi\mathbb{N}$, we consider a sequence of solutions (v_{1k}, v_{2k}) to (1.1.10) such that $\max_M(v_{1k}, v_{2k}) \to +\infty$. We claim $\max_M(\tilde{u}_{1k}, \tilde{u}_{2k}) \to +\infty$. Otherwise, $\tilde{u}_{1k}, \tilde{u}_{2k}$ are uniformly bounded above. From Green representation theorem and L^p estimate, we can get u_{1k}, u_{2k} are uniformly bounded. This implies v_{1k}, v_{2k} are uniformly bounded, which contradicts to our assumption. Let $\mathfrak{S}_i$ denote the blow up point of $\tilde{u}_{ik},\ i = 1, 2$ as before.

We claim $\mathfrak{S}_2 = \emptyset$ and $\mathfrak{S}_1$ consists of one point only. Suppose first $\mathfrak{S}_2 \neq \emptyset$. From the proof of Proposition 1.1.1, if $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \emptyset$, then $\tilde{u}_{2k}$ would concentrate, i.e., $\tilde{u}_{2k} \to -\infty,\ \forall x \in M \setminus \mathfrak{S}_2$, which implies $\rho_2 = \lim_{k\to+\infty} \int_M h_2 e^{\tilde{u}_{2k}} \in 4\pi\mathbb{N}$, a contradiction. Thus $\mathfrak{S}_1 \cap \mathfrak{S}_2 \neq \emptyset$. Suppose $q \in \mathfrak{S}_1 \cap \mathfrak{S}_2$, from Proposition 2.4 in [29] and the condition $\rho_{1k} < 8\pi$, we conclude $\sigma_{1q} = 2$ and $\sigma_{2q} = 4$. By Lemma 2.1.3, we have $\tilde{u}_{2k}$ concentrate, which implies $\rho_2 \in 4\pi\mathbb{N}$, a contradiction again. Hence $\mathfrak{S}_2 = \emptyset$. By Lemma

2.1.2, $\tilde{u}_{2k}$ is uniformly bounded from above in M. Since $\max_M(\tilde{u}_{1k}, \tilde{u}_{2k}) \to +\infty$, we get $\mathfrak{S}_1 \neq \emptyset$. By the fact $\rho_{1k} \to 4\pi$, we have $\mathfrak{S}_1$ contains only one point.

We write the equation for $v_{ik},\ i = 1, 2$ as

$$\begin{cases} \Delta v_{1k} + \rho_{1k}\Big(\frac{h_1 e^{2v_{1k}-v_{2k}}}{\int_M h_1 e^{2v_{1k}-v_{2k}}} - 1\Big) = 0, \\ \Delta v_{2k} + \rho_{2k}\Big(\frac{h_2 e^{2v_{2k}-v_{1k}}}{\int_M h_2 e^{2v_{2k}-v_{1k}}} - 1\Big) = 0. \end{cases} \tag{2.1.27}$$

Since $\tilde{u}_{2k}$ is uniformly bounded above, the second equation of (2.1.27) implies that v_{2k} is uniformly bounded in M and converges to some function $\frac{1}{2}w$ in $C^{1,\alpha}(M)$. From the first equation of (2.1.27) and $\rho_{1k} \to 4\pi$, v_{1k} blows up at only one point, say $p \in M$.

We write the first equation in (2.1.27) as

$$\Delta v_{1k} + \rho_{1k}\Big(\frac{\tilde{h}_k e^{2v_{1k}}}{\int_M \tilde{h}_k e^{2v_{1k}}} - 1\Big) = 0, \tag{2.1.28}$$

where $\tilde{h}_k = h_1 e^{-v_{2k}}$. We define $\tilde{v}_{1k} = v_{1k} - \frac{1}{2}\log\int_M \tilde{h}_k e^{2v_{1k}}$. Due to the $C^{1,\alpha}$ convergence of $\tilde{h}_k$, $\tilde{v}_{1k}$ simply blows up at p by a result of Li [37] (one can also see [5]), i.e., the following inequality holds:

$$\left|2\tilde{v}_{1k} - \log\frac{e^{\lambda_k}}{\big(1 + \frac{\rho_{1k}\tilde{h}_k(p^{(k)})e^{\lambda_k}}{4}|x - p^{(k)}|^2\big)^2}\right| < c \text{ for } |x - p^{(k)}| < r_0, \tag{2.1.29}$$

where $\lambda_k = 2\tilde{v}_{1k}(p^{(k)}) = \max_{x \in B_{r_0}(p)} 2\tilde{v}_{1k}$. By using this sharp estimate, we get

$$\tilde{v}_{1k} \to -\infty \text{ in } M \setminus \{p\}, \quad \rho_{1k}\frac{h_1 e^{2v_{1k}-v_{2k}}}{\int_M h_1 e^{2v_{1k}-v_{2k}}} \to 4\pi\delta_p, \tag{2.1.30}$$

and

$$\nabla\Big(\log(h_1 e^{-\frac{1}{2}w}) + 4\pi R(x,x)\Big)\,|_{x=p} = 0, \tag{2.1.31}$$

which proves (1.1.11) and (1.1.12).

In the following, we claim $v_{2k} \to \frac{1}{2}w$ in $C^{2,\alpha}(M)$. From this claim and (2.1.28), it is easy to get

$$v_{1k} \to 8\pi G(x,p) \text{ in } C^{2,\alpha}(M \setminus \{p\}).$$

Combined with $v_{2k} \to \frac{1}{2}w$ in $C^{2,\alpha}(M)$, we have w satisfies the following equation

$$\Delta w + 2\rho_2\Big(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1\Big) = 0. \tag{2.1.32}$$

This proves (1.1.13). Therefore, we finish the proof of Proposition 1.1.2. The proof of the claim is given in the following Lemma 2.1.4. □

Lemma 2.1.4. *Let v_{1k}, v_{2k} be a sequence of blow up solutions of (2.1.27), which v_{1k} blows at p and $v_{2k} \to \frac{1}{2}w$ in $C^{1,\alpha}(M)$. Then $v_{2k} \to \frac{1}{2}w$ in $C^{2,\alpha}(M)$.*

Proof. By (2.1.29), we have

$$|\lambda_k - \log \int_M \tilde{h}_k e^{2v_{1k}}| < c. \tag{2.1.33}$$

To prove $v_{2k} \to \frac{1}{2}w$ in $C^{2,\alpha}$, we need the following estimate

$$\left|2\nabla\tilde{v}_{1k} - \nabla\Big(\log \frac{e^{\lambda_k}}{\big(1+\frac{\rho_{1k}h(p)e^{\lambda_k}}{4}|x-p|^2\big)^2}\Big)\right| < c \text{ for } |x-p| < r_0, \tag{2.1.34}$$

where (2.1.34) comes from the error estimate of [14, Lemma 4.1]. We write

$$h_2 e^{2v_{2k}-v_{1k}} = h_2 e^{-v_{1k}} e^{2v_{2k}}.$$

By (2.1.29) and (2.1.34), it is not difficult to show

$$\nabla\big(h_2 e^{-v_{1k}}\big) \in L^\infty(M).$$

Therefore, by classical elliptic regularity theory and Sobolev inequality, we can show that

$$v_{2k} \to \frac{1}{2}w \text{ in } C^{2,\alpha} \text{ for any } \alpha \in (0,1). \tag{2.1.35}$$

Then we finished the proof of this lemma. $\square$

After deriving the shadow system (1.1.14), we show the non-degeneracy of (1.1.14) by applying the well-known transversality theorem, which can be found in [1], [61] and references therein. First, we recall that

Theorem 2.1.1. *Let $F : \mathcal{H} \times \mathcal{B} \to \mathcal{E}$ be a C^k map. $\mathcal{H}$, $\mathcal{B}$ and $\mathcal{E}$ Banach manifolds with $\mathcal{H}$ and $\mathcal{E}$ separable. If 0 is a regular value of F and $F_b = F(\cdot, b)$ is a Fredholm map of index $< k$, then the set $\{b \in \mathcal{B} : 0 \text{ is a regular value of } F_b\}$ is residual in $\mathcal{B}$.*

We say $y \in \mathcal{E}$ is a regular value if every point $x \in F^{-1}(y)$ is a regular point, where $x \in \mathcal{H} \times \mathcal{B}$ is a regular point of F if $D_xF : T_x(\mathcal{H} \times \mathcal{B}) \to T_{F(x)}\mathcal{E}$ is onto. We say a set A is a residual set if A is a countable intersection of open dense sets in B, see [1], which implies A is dense in B (B is a Banach space), see [31].

Following the notations in Theorem 2.1.1, we denote

$$\mathcal{H} = M \times \mathring{W}^{2,\mathfrak{p}}(M),\ \mathcal{B} = C^{2,\alpha}(M) \times C^{2,\alpha}(M),\ \mathcal{E} = \mathbb{R}^2 \times \mathring{W}^{0,\mathfrak{p}}(M),$$

where

$$\mathring{W}^{2,\mathfrak{p}}(M) := \{f \in W^{2,\mathfrak{p}} \mid \int_M f = 0\},\ \mathring{W}^{0,\mathfrak{p}}(M) := \{f \in L^{\mathfrak{p}} \mid \int_M f = 0\},$$

and

$$C^{2,\alpha}(M) = \{f \in C^{2,\alpha}(M)\}.$$

We consider the map

$$T(w,p,h_1,h_2) = \begin{bmatrix} \Delta w + 2\rho_2\left(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1\right) \\ \nabla \log\left(h_1 e^{-\frac{1}{2}w} + 4\pi R(x,x)\right)(p) \end{bmatrix}. \tag{2.1.36}$$

Clearly, T is C^1. Next, we claim

(i) $T(\cdot,\cdot,h_1,h_2)$ is a Fredholm map of index 0,

(ii) 0 is a regular value of T.

For the first claim, after computation, we get

$$T'_{w,p}(w,p,h_1,h_2)[\phi,\nu]=\begin{bmatrix} T_0(w,p,h_1,h_2)[\phi,\nu] \\ T_1(w,p,h_1,h_2)[\phi,\nu] \end{bmatrix}, \tag{2.1.37}$$

where

$$\begin{aligned} T_0(w,p,h_1,h_2)[\phi,\nu] =&\Delta\phi+2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\phi \\ &-2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{(\int_M h_2e^{w-4\pi G(x,p)})^2}\int_M h_2e^{w-4\pi G(x,p)}\phi \\ &-8\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\nabla G(x,p)\cdot\nu \\ &+8\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{(\int_M h_2e^{w-4\pi G(x,p)})^2}\int_M h_2e^{w-4\pi G(x,p)}\nabla G(x,p)\cdot\nu, \end{aligned}$$

$$T_1(w,p,h_1,h_2)[\phi,\nu]=\nabla_x^2\big(\log h_1e^{-\frac{1}{2}w}+4\pi R(x,x)\big)\mid_{x=p}\cdot\nu-\frac{1}{2}\nabla\phi(p).$$

We decompose

$$T'_{w,p}[\phi,\nu]=\begin{bmatrix} T_{01} \\ T_{11} \end{bmatrix}[\phi,\nu]+\begin{bmatrix} T_{02} \\ T_{12} \end{bmatrix}[\phi,\nu], \tag{2.1.38}$$

where

$$T_{11}=0,\ T_{12}=T_1,$$

$$\begin{aligned} T_{01}(w,p,h_1,h_2)[\phi,\nu] =&\Delta\phi+2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\phi \\ &-2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{(\int_M h_2e^{w-4\pi G(x,p)})^2}\int_M h_2e^{w-4\pi G(x,p)}\phi, \end{aligned}$$

and

$$\begin{aligned} T_{02}(w,p,h_1,h_2)[\phi,\nu] =& -8\pi\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} \nabla G(x,p)\nu \\ & + 8\pi\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{(\int_M h_2 e^{w-4\pi G(x,p)})^2} \int_M h_2 e^{w-4\pi G(x,p)} \nabla G(x,p)\nu. \end{aligned}$$

We define $\mathfrak{T}_1 = \begin{bmatrix} T_{01} \\ T_{11} \end{bmatrix}$ and $\mathfrak{T}_2 = \begin{bmatrix} T_{02} \\ T_{12} \end{bmatrix}$. We can easily see that $\mathfrak{T}_1$ is symmetric, it follows from the basic theory of elliptic operators that $\mathfrak{T}_1$ is a Fredholm operator of index 0. Combining the Sobolev inequality and $\mathbb{R}^2$ is a finite Euclidean space, we can show that $\mathfrak{T}_2$ is a compact operator. Therefore, by the standard linear operator theory [31], we get $\mathfrak{T}_1 + \mathfrak{T}_2$ is also a Fredholm linear operator with index 0. Hence, we prove the first claim that T is a Fredholm map with index 0.

It remains to show that 0 is a regular value. We derive the differentiation of the operator T with respect to h_1 and h_2,

$$T'_{h_1}(w,p,h_1,h_2)[H_1] = \begin{bmatrix} 0 \\ \frac{\nabla H_1}{h_1}(p) - \frac{\nabla h_1}{(h_1)^2} H_1(p) \end{bmatrix},$$

and

$$\begin{aligned} & T'_{h_2}(w,p,h_1,h_2)[H_2] \\ =& \begin{bmatrix} 2\rho_2 \frac{H_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{(\int_M h_2 e^{w-4\pi G(x,p)})^2} \int_M H_2 e^{w-4\pi G(x,p)} \\ 0 \end{bmatrix}. \end{aligned}$$

By choosing $\nu = 0$, and H_1 such that $\frac{\nabla H_1}{h_1} - \frac{\nabla h_1}{(h_1)^2} H_1 = \frac{1}{2}\nabla\phi$ at p. We get

$$\begin{aligned} & T'_{w,p}(w,p,h_1,h_2)[\phi,\nu] + T'_{h_1}(w,p,h_1,h_2)[H_1] \\ =& \begin{bmatrix} \Delta\phi + 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}}\phi - 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{(\int_M h_2 e^{w-4\pi G(x,p)})^2} \int_M h_2 e^{w-4\pi G(x,p)}\phi \\ 0 \end{bmatrix}. \end{aligned}$$

Next, we claim that the vector space spanned by $T'_{w,p}(w,p,h_1,h_2)[\phi,\nu]$,

$T'_{h_1}(w,p,h_1,h_2)[H_1]$ and $T'_{h_2}(w,p,h_1,h_2)[H_2]$ contains $\begin{bmatrix} f \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ for all $f \in \mathring{W}^{0,p}$. It is enough for us to prove that only $\phi = 0$ can satisfy

$$\phi \in \mathrm{Ker}\Big\{\Delta \cdot + 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} \cdot - 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{(\int_M h_2 e^{w-4\pi G(x,p)})^2} \int_M h_2 e^{w-4\pi G(x,p)} \cdot \Big\}$$

and

$$\Big\langle \phi,\ 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} \frac{H_2}{h_2} - 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{(\int_M h_2 e^{w-4\pi G(x,p)})^2} \int_M h_2 e^{w-4\pi G(x,p)} \frac{H_2}{h_2} \Big\rangle = 0,$$

for all $H_2 \in C^{2,\alpha}(M)$. We set

$$L = \Delta \cdot + 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} \cdot - 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{(\int_M h_2 e^{w-4\pi G(x,p)})^2} \int_M h_2 e^{w-4\pi G(x,p)} \cdot .$$

Using $\phi \in \mathrm{Ker}(L)$, we obtain that for any $\overline{H}_2 \in W^{0,\mathfrak{p}}(M)$,

$$\int_M L(\phi) \cdot \overline{H}_2 = 0. \tag{2.1.39}$$

Since $C^{2,\alpha}(M)$ is dense in $W^{0,\mathfrak{p}}(M)$ and

$$\Big\langle \phi, 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} \overline{H}_2 - 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{(\int_M h_2 e^{w-4\pi G(x,p)})^2} \int_M h_2 e^{w-4\pi G(x,p)} \overline{H}_2 \Big\rangle = 0,$$

we deduce

$$\int_M \Delta\phi \cdot \overline{H}_2 = 0, \quad \forall\, \overline{H}_2 \in W^{0,\mathfrak{p}}(M). \tag{2.1.40}$$

Thus

$$\Delta\phi = 0 \text{ in } M, \qquad \int_M \phi = 0. \tag{2.1.41}$$

So $\phi \equiv 0$. Therefore the claim is proved.

On the other hand, we choose two functions, $H_{1,1}$ and $H_{1,2}$ such that

$$\frac{\nabla H_{1,1}}{h_1}(p) - \frac{\nabla h_1}{(h_1)^2}H_{1,1}(p) = (1,0),$$

and

$$\frac{\nabla H_{1,2}}{h_1}(p) - \frac{\nabla h_1}{(h_1)^2}H_{1,2}(p) = (0,1).$$

Then it is not difficult to see that (by setting $\phi = 0, \nu = 0$)

$$\begin{bmatrix} 0 \\ c \end{bmatrix} \subset DT(w,p,h_1,h_2)[\phi,\nu]$$

for all $c \in \mathbb{R}^2$. Therefore, we have proved that the differential map is onto. As a consequence, 0 is a regular point of T. By Theorem 2.1.1,

$$\Big\{(h_1,h_2) \in \mathcal{B} : 0 \text{ is a regular value of } T(\cdot,\cdot,h_1,h_2)\Big\}$$

is residual in $\mathcal{B}$. Since $T(w,p,h_1,h_2)$ is a Fredholm map of index 0 for fixed h_1, h_2, we have

$$\Big\{(h_1,h_2) \in \mathcal{B} : \text{ the solution } (w,p) \text{ of } T(\cdot,\cdot,h_1,h_2) = 0 \text{ is nondegenerate}\Big\}$$

is residual in $\mathcal{B}$. Thus, we can choose $h_1, h_2 > 0$ such that the solution of (1.1.14) is non-degenerate.

2.2 A-priori Estimate

In this section, we shall prove that all the blow up solutions of (1.1.10) must be contained in the set $S_{\rho_1}(p,w) \times S_{\rho_2}(p,w)$ when $\rho_1 \to 4\pi$, $\rho_2 \notin 4\pi\mathbb{N}$, where the definition of $S_{\rho_i}(p,w), i = 1,2$ are given in (2.2.14) and (2.2.15) of this section.

To simplify our description, we may assume M has a flat metric near a neighborhood of each blow up point. Of course we can modify our arguments without any difficulty for the general case, as in [15].

We start to define the set $S_{\rho_i}(p,w)$. For any given non-degenerate solution (p,w) of (1.1.14), we set (by abuse of the notation)

$$h = h_1 e^{-\frac{1}{2}w}. \tag{2.2.1}$$

By noting that

$$\nabla_x\big(\log h + 4\pi R(x,x)\big)\mid_{x=p} = \nabla_x\big(\log h(x) + 8\pi R(x,p)\big)\mid_{x=p} = 0, \tag{2.2.2}$$

whenever (p,w) is a solution of shadow system (1.1.14). For q such that $|q-p| \ll 1$ and large $\lambda > 0$, we set

$$U(x) = \lambda - 2\log\big(1 + \frac{\rho_1 h(q)}{4} e^{\lambda}|x-q|^2\big), \tag{2.2.3}$$

and $U(x)$ satisfies the following equation

$$\Delta U(x) + 2\rho_1 h(q) e^{U} = 0 \text{ in } \mathbb{R}^2, \quad U(q) = \max_{\mathbb{R}^2} U(x) = \lambda. \tag{2.2.4}$$

Let

$$H(x) = \exp\Big\{\log\frac{h(x)}{h(q)} + 8\pi R(x,q) - 8\pi R(q,q)\Big\} - 1, \tag{2.2.5}$$

and

$$s = \lambda + 2\log\Big(\frac{\rho_1 h(q)}{4}\Big) + 8\pi R(q,q) + \frac{\Delta H(q)}{\rho_1 h(q)}\frac{\lambda^2}{e^{\lambda}}. \tag{2.2.6}$$

Let $\sigma_0(t)$ be a cut-off function:

$$\sigma_0(t) = \begin{cases} 1, & \text{if } |t| < r_0, \\ 0, & \text{if } |t| \geq 2r_0. \end{cases}$$

Set $\sigma(x) = \sigma_0(|x-q|)$ and

$$J(x) = \begin{cases} \big(H(x) - \nabla H(q)\cdot(x-q)\big)\sigma, & x \in B_{2r_0}(q), \\ 0, & x \notin B_{2r_0}(q). \end{cases}$$

Let $\eta(x)$ satisfy

$$\begin{cases} \Delta\eta + 2\rho_1 h(q) e^{U}(\eta + J(x)) = 0 \quad \text{on } \mathbb{R}^2, \\ \eta(q) = 0, \nabla\eta(q) = 0. \end{cases} \tag{2.2.7}$$

The existence of η was proved in [15]. Furthermore, we have the following lemma

Lemma 2.2.1. *Let $R = \sqrt{\frac{\rho_1 h(q)}{4} e^{\lambda}}$. For $h \in C^{2,\alpha}(M)$ and large λ. there exists a solution η satisfying (2.2.7) and the following*

(i) $\eta(x) = -\frac{4\Delta H(q)}{\rho_1 h(q)} e^{-\lambda}[\log(R|x-q|+2)]^2 + O(\lambda e^{-\lambda})$ on $B_{2r_0}(q)$,

(ii) $\eta, \nabla_x\eta, \partial_q\eta, \partial_\lambda\eta, \nabla_x\partial_q\eta, \nabla_x\partial_\lambda\eta = O(\lambda^2 e^{-\lambda})$ on $B_{2r_0}(q)$.

The proof of Lemma 2.2.1 was given in [15].

We set

$$\begin{cases} v_q(x) = \big(U(x) + \eta(x) + 8\pi(R(x,q) - R(q,q)) + s\big)\sigma(x) \\ \qquad\qquad +8\pi G(x,q)\big(1 - \sigma(x)\big), \\ \overline{v}_q = \frac{1}{|M|}\int_M v_q, \\ v_{q,\lambda,a} = a(v_q - \overline{v}_q). \end{cases} \tag{2.2.8}$$

Next, we define $O^{(1)}_{q,\lambda}$ and $O^{(2)}_{q,\lambda}$:

$$O^{(1)}_{q,\lambda} = \Big\{\phi \in \mathring{H}^1(M) \;\Big|\; \int_M \nabla\phi\cdot\nabla v_q = \int_M \nabla\phi\cdot\nabla\partial_q v_q = \int_M \nabla\phi\cdot\nabla\partial_\lambda v_q = 0\Big\}, \tag{2.2.9}$$

and

$$O^{(2)}_{q,\lambda} = \Big\{\psi \in W^{2,\mathfrak{p}}(M) \;\Big|\; \int \psi = 0\Big\}, \quad \mathfrak{p} > 2. \tag{2.2.10}$$

For each (q,λ), we set

$$t = \lambda + 8\pi R(q,q) + 2\log\frac{\rho_1 h(q)}{4} + \frac{\Delta H(q)}{\rho_1 h(q)}\lambda^2 e^{-\lambda} - \overline{v}_q. \tag{2.2.11}$$

For $\rho_1 \neq 4\pi$, we define $\lambda(\rho_1)$ such that

$$\rho_1 - 4\pi = \frac{\Delta \log h(p) + 8\pi - 2K(p)}{h(p)} \lambda(\rho_1) e^{-\lambda(\rho_1)}, \tag{2.2.12}$$

where (p, w) is the non-degenerate solution of (1.1.14) and $K(p)$ denotes the Gaussian curvature of p. By using the equation (1.1.12), we have $e^{-4\pi G(x,p)} \mid_{x=p} = 0$ and $\Delta w(p) = 2\rho_2$. Thus

$$\Delta \log h(p) + 8\pi - 2K(p) = \Delta \log h_1(p) - \rho_2 + 8\pi - 2K(p). \tag{2.2.13}$$

Obviously, $\lambda(\rho_1)$ can be well-defined only if

$$\Delta \log h_1(p) - \rho_2 + 8\pi - 2K(p) \neq 0.$$

Let c_1 be a positive constant, which will be chosen later. By using ρ_1, we set

$$\begin{aligned} S_{\rho_1}(p, w) = \Big\{ v_1 = \frac{1}{2} v_{q,\lambda,a} + \phi \ \Big| \ & |q - p| \leq c_1 \lambda(\rho_1) e^{-\lambda(\rho_1)}, \\ & |\lambda - \lambda(\rho_1)| \leq c_1 \lambda(\rho_1)^{-1}, \ |a - 1| \leq c_1 \lambda(\rho_1)^{-\frac{1}{2}} e^{-\lambda(\rho_1)}, \\ & \phi \in O^{(1)}_{q,\lambda} \text{ and } \|\phi\|_{H^1(M)} \leq c_1 \lambda(\rho_1) e^{-\lambda(\rho_1)} \Big\}, \end{aligned} \tag{2.2.14}$$

and

$$S_{\rho_2}(p, w) = \Big\{ v_2 = \frac{1}{2} w + \psi \ \Big| \ \psi \in O^{(2)}_{q,\lambda} \text{ and } \|\psi\|_* \leq c_1 \lambda(\rho_1) e^{-\lambda(\rho_1)} \Big\}, \tag{2.2.15}$$

where $\|\psi\|_* = \|\psi\|_{W^{2,\mathfrak{p}}(M)}$.

Now suppose (v_{1k}, v_{2k}) is a sequence of bubbling solutions of (1.1.10) such that v_{1k} blows up at p and weakly converges to $4\pi G(x, p)$, while $v_{2k} \to \frac{1}{2} w$ in $C^{2,\alpha}(M)$. Then we want to prove that

$$(v_{1k}, v_{2k}) \in S_{\rho_1}(p, w) \times S_{\rho_2}(p, w).$$

First of all, we prove the following lemma.

Lemma 2.2.2. *Let (v_{1k}, v_{2k}) be a sequence of blow up solutions of (1.1.10), which v_{1k} blows up at p, weakly converges to $4\pi G(x,p)$ and $v_{2k} \to \frac{1}{2}w$ in $C^{2,\alpha}(M)$. Suppose (p,w) is a non-degenerate solution of (1.1.14) and*

$$\Delta \log h_1(p) - \rho_2 + 8\pi - 2K(p) \neq 0. \tag{2.2.16}$$

Then there exist q_k^*, λ_k^*, a_k^*, ϕ_k^*, ψ_k^* such that

$$v_{1k} = \frac{1}{2} v_{q_k^*, \lambda_k^*, a_k^*} + \phi_k^*, \qquad v_{2k} = \frac{1}{2} w + \psi_k^*, \tag{2.2.17}$$

and $(v_{1k}, v_{2k}) \in S_{\rho_1}(p,w) \times S_{\rho_2}(p,w)$.

Remark 1. Because the proof of this lemma is very long, we describe the process briefly. First of all, we obtain a good approximation of v_{1k}. Since v_{2k} converges to $\frac{1}{2}w$ in $C^{2,\alpha}(M)$, this fine estimate can be obtained by the same proof in [14]. Next, we substitute v_{1k} into the second equation of v_{2k}. Then we use the non-degeneracy of (1.1.14) to get the sharp estimates of ψ_k and $|\tilde{q}_k - p|$, where $\psi_k = v_{2k} - \frac{1}{2}w$ and $\tilde{q}_k$ is the point where v_{1k} obtains its maximal value. After that, we get the lemma. In the following proof, we use the same notation as the proof of Proposition 1.1.2.

Proof. Let v_{1k} and v_{2k} be a sequence of blow up solutions of (1.1.10),

$$\begin{cases} \Delta v_{1k} + \rho_{1k}\Big(\frac{h_1 e^{2v_{1k} - v_{2k}}}{\int_M h_1 e^{2v_{1k} - v_{2k}}} - 1\Big) = 0, \\ \Delta v_{2k} + \rho_{2k}\Big(\frac{h_2 e^{2v_{2k} - v_{1k}}}{\int_M h_2 e^{2v_{2k} - v_{1k}}} - 1\Big) = 0. \end{cases} \tag{2.2.18}$$

For convenience, we write the first equation in (2.2.18) as,

$$\Delta v_{1k} + \rho_{1k}\Big(\frac{\tilde{h}_k e^{2v_{1k}}}{\int_\Omega \tilde{h}_k e^{2v_{1k}}} - 1\Big) = 0, \tag{2.2.19}$$

where

$$\tilde{h}_k = h_1 e^{-v_{2k}} = h e^{-\psi_k} \text{ and } \psi_k = v_{2k} - \frac{1}{2}w. \tag{2.2.20}$$

Since $\tilde{h}_k \to h$ in $C^{2,\alpha}(M)$, all the estimates in [14] can be applied to our case here, although in [14] the coefficient $\tilde{h}_k$ is independent of k. In the followings (up to (2.2.28) below), we sketch the estimates in [14, 15] which will be used here. We denote $\tilde{q}_k$ to be the maximal point of $\tilde{v}_{1k}$ near p, where $\tilde{v}_{1k} = v_{1k} - \frac{1}{2}\log\int_M \tilde{h}_k e^{2v_{1k}}$. Let

$$\lambda_k = 2\tilde{v}_{1k}(\tilde{q}_k) - \log\int_M \tilde{h}_k e^{2v_{1k}}.$$

In the local coordinate near $\tilde{q}_k$, we set

$$\tilde{U}_k(x) = \log\frac{e^{\lambda_k}}{(1+\frac{\rho_{1k}\tilde{h}_k(q_k)}{4}e^{\lambda_k}|x-q_k|^2)^2},$$

where q_k is chosen such that

$$\nabla\tilde{U}_k(\tilde{q}_k) = \nabla\log\tilde{h}_k(\tilde{q}_k).$$

Clearly, $|q_k - \tilde{q}_k| = O(e^{-\lambda_k})$. Then the error term inside $B_{r_0}(q_k)$ is set by

$$\tilde{\eta}_k(x) = 2\tilde{v}_{1k} - \tilde{U}_k(y) - (8\pi R(x,q_k) - 8\pi R(q_k,q_k)), \tag{2.2.21}$$

and the error term outside $B_{r_0}(q_k)$ is set by

$$\xi_k(x) = 2v_{1k}(x) - 8\pi G(x,q_k). \tag{2.2.22}$$

By Green's representation for v_{1k}, it is not difficult to obtain

$$\xi_k(x) = O(\lambda_k e^{-\lambda_k}) \text{ for } x \in M\setminus B_{r_0}(q_k). \tag{2.2.23}$$

By a straightforward computation, the error term $\tilde{\eta}_k$ satisfies

$$\Delta\tilde{\eta}_k + 2\rho_{1k}\tilde{h}_k(q_k)e^{\tilde{U}_k}\tilde{H}_k(x,\tilde{\eta}_k) = 0, \tag{2.2.24}$$

where

$$\begin{aligned}\tilde{H}_k(x,t) =&\exp\Big\{\log\frac{\tilde{h}_k(x)}{\tilde{h}_k(q_k)}+8\pi\big(R(x,q_k)-R(q_k,q_k)\big)+t\Big\}-1\\ =&H_k(x)+t+O(|t|^2),\end{aligned}$$

and

$$H_k(x)=\exp\Big\{\log\frac{\tilde{h}_k(x)}{\tilde{h}_k(q_k)}+8\pi R(x,q_k)-8\pi R(q_k,q_k)\Big\}-1.$$

We see that except for the higher-order term $O(|\tilde{\eta}_k|^2)$, equation (2.2.24) is exactly like (2.2.7). By Lemma 2.2.1, we can prove

$$\tilde{\eta}_k(x)=-\frac{4}{\rho_{1k}\tilde{h}_k(q_k)}\Delta H_k(q_k)e^{-\lambda_k}[\log(R_k|x-q_k|+2)]^2+O(\lambda_k e^{-\lambda_k}) \quad (2.2.25)$$

for $x\in B_{2r_0}(q_k)$, where $R_k=\sqrt{\frac{\rho_{1k}\tilde{h}_k(q_k)}{4}e^{\lambda_k}}$.

From [14, Theorem 1.1, Theorem 1.4 and Lemma 5.4], we have

$$\rho_{1k}-4\pi=\frac{\Delta\log\tilde{h}_k(q_k)+8\pi-2K(q_k)}{\tilde{h}_k(q_k)}\lambda_k e^{-\lambda_k}+O(e^{-\lambda_k}), \quad (2.2.26)$$

$$2\bar{\tilde{v}}_{1k}+\lambda_k+2\log\frac{\rho_{1k}\tilde{h}_k(q_k)}{4}+8\pi R(q_k,q_k)+\frac{\Delta H_k(q_k)}{\rho_{1k}\tilde{h}_k(q_k)}\lambda_k^2e^{-\lambda_k}=O(\lambda_k e^{-\lambda_k}), \quad (2.2.27)$$

and

$$|\nabla H_k(q_k)|=O(\lambda_k e^{-\lambda_k}). \quad (2.2.28)$$

Now we let η_k be defined as in (2.2.7), v_{q_k} and v_{q_k,λ_k,a_k} be defined as in (2.2.8) with $q=q_k$, $\lambda=\lambda_k$ and $a=a_k=1$. By Lemma 2.2.1, (2.2.25) and (2.2.28), we have

$$\eta_k(x)=\tilde{\eta}_k+O(\lambda_k e^{-\lambda_k}) \text{ for } x\in B_{2r_0}(q_k). \quad (2.2.29)$$

Note that for $x \in B_{r_0}(q_k)$,

$$\begin{aligned} v_{q_k,\lambda_k,a_k} =& \tilde{U}_k(x) + \eta_k(x) + \big(8\pi R(x,q_k) - 8\pi R(q_k,q_k)\big) + \lambda_k + 2\log\frac{\rho_{1k}\tilde{h}_k(q_k)}{4} \\ &+ 8\pi R(q_k,q_k) + \frac{\Delta H_k(q_k)}{\rho_{1k}\tilde{h}_k(q_k)}\lambda_k^2 e^{-\lambda_k} - \overline{v}_{q_k}, \end{aligned}$$

where $\overline{v}_{q_k}$ denotes the average of v_{q_k}. From [15, Lemma 2.2 and Lemma 2.3], we have

$$v_{q_k} - 8\pi G(x,q_k) = O(\lambda_k e^{-\lambda_k}) \text{ in } M \setminus B_{2r_0}(q_k), \text{ and } \overline{v}_{q_k} = O(\lambda_k e^{-\lambda_k}). \tag{2.2.30}$$

By (2.2.21), (2.2.27), (2.2.29) and (2.2.30), we have

$$\begin{aligned} 2v_{1k} - v_{q_k,\lambda_k,a_k} =& 2\tilde{v}_{1k} + \int_M \tilde{h}_k e^{2v_{1k}} - v_{q_k,\lambda_k,a_k} \\ =& 2\tilde{v}_{1k} - \tilde{U}_k - \big(8\pi R(x,x) - 8\pi R(x,q_k)\big) - \eta_k(x) + O(\lambda_k e^{-\lambda_k}) \\ =& \tilde{\eta}_k(x) - \eta_k(x) + O(\lambda_k e^{-\lambda_k}) = O(\lambda_k e^{-\lambda_k}) \end{aligned} \tag{2.2.31}$$

for $x \in B_{r_0}(q_k)$. For $x \in M \setminus B_{2r_0}(q_k)$, by (2.2.22) and (2.2.30), we get

$$2v_{1k} - v_{q_k,\lambda_k,a_k} = 2v_{1k} - 8\pi G(x,q_k) - (v_{q_k} - 8\pi G(x,q_k)) + \overline{v}_{q_k} = O(\lambda_k e^{-\lambda_k}).$$

For the intermediate domain $B_{2r_0}(q_k) \setminus B_{r_0}(q_k)$, following a similar way, we can obtain that $2v_{1k} - v_{q_k,\lambda_k,a_k} = O(\lambda_k e^{-\lambda_k})$. Thus, we find a good approximation $\frac{1}{2}v_{q_k,\lambda_k,a_k}$ for v_{1k}. For convenience, we write

$$v_{1k} = \frac{1}{2}v_{q_k,\lambda_k,a_k} + \phi_k, \text{ where } \|\phi_k\|_{L^\infty(M)} < \tilde{c}\lambda_k e^{-\lambda_k}, \tag{2.2.32}$$

where $\tilde{c}$ is independent of ψ_k.

Next, we substitute (2.2.32) and $v_{2k} = \frac{1}{2}w + \psi_k$ into the second equation of (2.2.18), after computation, we obtain

$$\mathcal{L}\psi_k = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3, \quad \int_M \psi_k = 0, \tag{2.2.33}$$

where

$$\begin{aligned}\mathcal{L}\psi_k =&\Delta\psi_k + 2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\psi_k\\&- 2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{(\int_M h_2e^{w-4\pi G(x,p)})^2}\int_M(h_2e^{w-4\pi G(x,p)}\psi_k)\\&- 4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\big(\nabla G(x,p)(q_k-p)\big)\\&+ 4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{(\int_M h_2e^{w-4\pi G(x,p)})^2}\int_M\big(h_2e^{w-4\pi G(x,p)}(\nabla G(x,p)(q_k-p))\big),\end{aligned}$$

$$\mathbb{I}_1 = -\rho_2\frac{h_2e^{w+2\psi_k-v_{1k}}}{\int_M h_2e^{w+2\psi_k-v_{1k}}} + \rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,q_k)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,q_k)}},$$

$$\begin{aligned}\mathbb{I}_2 =&\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}} - \rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,p)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,p)}}\\&+ 2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\psi_k\\&- 2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{(\int_M h_2e^{w-4\pi G(x,p)})^2}\int_M(h_2e^{w-4\pi G(x,p)}\psi_k),\end{aligned}$$

and

$$\begin{aligned}\mathbb{I}_3 =& -\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,q_k)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,q_k)}} + \rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,p)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,p)}}\\&- 4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}(\nabla G(x,p)(q_k-p))\\&+ 4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{(\int_M h_2e^{w-4\pi G(x,p)})^2}\int_M\big(h_2e^{w-4\pi G(x,p)}(\nabla G(x,p)(q_k-p))\big).\end{aligned}$$

We shall analyze the right hand side of (2.2.33) term by term in the following.

For $\mathbb{I}_1$, we set

$$\mathfrak{E}_1 = \exp\big(w + 2\psi_k - 4\pi G(x, q_k)\big) - \exp\big(w + 2\psi_k - \frac{1}{2}v_{q_k,\lambda_k,a_k} - \phi_k\big).$$

For $x \in M \setminus B_{r_0}(q_k)$. We see that the difference between $4\pi G(x, q_k)$ and v_{q_k,λ_k,a_k} is of order $\lambda_k e^{-\lambda_k}$. As a consequence, $\mathfrak{E}_1 = O(\lambda_k e^{-\lambda_k})$.

For $x \in B_{r_0}(q_k)$,

$$\begin{aligned}
4\pi G(x, q_k) - \frac{1}{2}v_{q_k,\lambda_k,a_k} =& 4\pi G(x, q_k) - 4\pi R(x, q_k) - \log\Big(\frac{\rho_1 \tilde{h}_k(q_k)}{4}\Big) \\
&+ \log\Big(1 + \frac{\rho_1 \tilde{h}_k(q_k) e^{\lambda_k}}{4}|x - q_k|^2\Big) - \lambda_k \\
&- \frac{1}{2}\Big(\eta_k + \frac{\Delta H_k(q_k)}{\rho_1 \tilde{h}_k(q_k)}\frac{\lambda_k^2}{e^{\lambda_k}}\Big) + O(\lambda_k e^{-\lambda_k}) \\
=& \log\Big(\frac{4}{\rho_1 \tilde{h}_k(q_k) e^{\lambda_k}|x - q_k|^2} + 1\Big) \\
&- \frac{1}{2}\Big(\eta_k + \frac{\Delta H_k(q_k)}{\rho_1 \tilde{h}_k(q_k)}\frac{\lambda_k^2}{e^{\lambda_k}}\Big) + O(\lambda_k e^{-\lambda_k}).
\end{aligned}$$

Since $\phi_k = O(\lambda_k e^{-\lambda_k})$,

$$\exp\big(w + 2\psi_k - \frac{1}{2}v_{q_k,\lambda_k,a_k} - \phi_k\big) = \exp\big(w + 2\psi_k - \frac{1}{2}v_{q_k,\lambda_k,a_k}\big) + O(\lambda_k e^{-\lambda_k}).$$

Then, we have

$$\begin{aligned}
&\exp\big(w + 2\psi_k - 4\pi G(x, q_k)\big) - \exp\big(w + 2\psi_k - \frac{1}{2}v_{q_k,\lambda_k,a_k} - \phi_k\big) \\
&= \exp\big(w + 2\psi_k - 4\pi G(x, q_k)\big) - \exp\big(w + 2\psi_k - \frac{1}{2}v_{q_k,\lambda_k,a_k}\big) + O(\lambda_k e^{-\lambda_k}) \\
&= \exp\big(w + 2\psi_k - 4\pi G(x, q_k)\big)\big(1 - \exp\big(4\pi G(x, q_k) - \frac{1}{2}v_{q_k,\lambda_k,a_k}\big)\big) + O(\lambda_k e^{-\lambda_k}) \\
&= \exp\big(w + 2\psi_k - 4\pi G(x, q_k)\big)\Big(1 - \exp\Big[\log\Big(1 + \frac{4}{\rho_1 \tilde{h}_k e^{\lambda_k}|x - q_k|^2}\Big) \\
&\quad + O\Big(\eta_k + \frac{\Delta H_k(q_k)}{\rho_1 \tilde{h}_k(q_k)}\frac{\lambda_k^2}{e^{\lambda_k}}\Big) + O(\lambda_k e^{-\lambda_k})\Big]\Big) + O(\lambda_k e^{-\lambda_k})
\end{aligned}$$

When $|x-q_k| = O(e^{-\frac{\lambda_k}{2}})$, we have

$$\exp\big(w + 2\psi_k - 4\pi G(x,q_k)\big) = O(|x-q_k|^2),$$

and

$$\log\Big(1+\frac{4}{\rho_1\tilde{h}_k e^{\lambda_k}|x-q_k|^2}\Big) + O\Big(\eta_k + \frac{\Delta H_k(q_k)}{\rho_1\tilde{h}_k(q_k)}\frac{\lambda_k^2}{e^{\lambda_k}}\Big) = O\big(\log(e^{-\lambda_k}|x-q_k|^{-2})\big),$$

hence

$$\mathfrak{E}_1 = O(\lambda_k e^{-\lambda_k}) \text{ for } |x-q_k| = O(e^{-\frac{\lambda_k}{2}}).$$

When $|x-q_k| \gg e^{-\frac{\lambda_k}{2}}$,

$$\begin{aligned} 1-\exp\Big(\log\big(1+\frac{4}{\rho_1\tilde{h}_k e^{\lambda_k}|x-q_k|^2}\big) + O\big(\eta_k + \frac{\Delta H_k(q_k)}{\rho_1\tilde{h}_k(q_k)}\frac{\lambda_k^2}{e^{\lambda_k}}\big)\Big) + O(\lambda_k e^{-\lambda_k}) \\ = O\Big(\frac{4}{\rho_1\tilde{h}_k e^{\lambda_k}|x-q_k|^2} + \lambda_k e^{-\lambda_k}\Big), \end{aligned}$$

which gives

$$\mathfrak{E}_1 = O(\lambda_k e^{-\lambda_k}) \text{ for } r_0 \geq |x-q_k| \gg e^{-\frac{\lambda_k}{2}}.$$

Thus, $\|\mathfrak{E}_1\|_{L^\infty(M)} = O(\lambda_k e^{-\lambda_k})$, which implies $\mathbb{I}_1 = O(\lambda_k e^{-\lambda_k})$.

For the second term, it is easy to see that $\mathbb{I}_2 = O(\|\psi_k\|_*^2)$. It remains to estimate $\mathbb{I}_3$. We divide it into three parts. $\mathbb{I}_3 = \mathbb{I}_{31} + \mathbb{I}_{32} + \mathbb{I}_{33}$, where

$$\begin{aligned} \mathbb{I}_{31} = &- \rho_2\frac{h_2 e^{w+2\psi_k-4\pi G(x,q_k)}}{\int_M h_2 e^{w+2\psi_k-4\pi G(x,q_k)}} + \rho_2\frac{h_2 e^{w+2\psi_k-4\pi G(x,p)}}{\int_M h_2 e^{w+2\psi_k-4\pi G(x,p)}} \\ &- 4\pi\rho_2\frac{h_2 e^{w+2\psi_k-4\pi G(x,p)}}{\int_M h_2 e^{w+2\psi_k-4\pi G(x,p)}}\big(\nabla G(x,p)(q_k-p)\big), \\ &+ 4\pi\rho_2\frac{h_2 e^{w+2\psi_k-4\pi G(x,p)}}{(\int_M h_2 e^{w+2\psi_k-4\pi G(x,p)})^2}\int_M\Big(h_2 e^{w+2\psi_k-4\pi G(x,p)}\big(\nabla G(x,p)(q_k-p)\big)\Big), \end{aligned}$$

$$\begin{aligned}\mathbb{I}_{32} =&4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{(\int_M h_2e^{w-4\pi G(x,p)})^2}\int_M\Big(h_2e^{w-4\pi G(x,p)}(\nabla G(x,p)(q_k-p))\Big)\\&-4\pi\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,p)}}{(\int_M h_2e^{w+2\psi_k-4\pi G(x,p)})^2}\int_M\Big(h_2e^{w+2\psi_k-4\pi G(x,p)}(\nabla G(x,p)(q_k-p))\Big),\end{aligned}$$

and

$$\begin{aligned}\mathbb{I}_{33} =&4\pi\rho_2\frac{h_2e^{w+2\psi_k-4\pi G(x,p)}}{\int_M h_2e^{w+2\psi_k-4\pi G(x,p)}}\big(\nabla G(x,p)(q_k-p)\big)\\&-4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\big(\nabla G(x,p)(q_k-p)\big).\end{aligned}$$

It is not difficult to see

$$\mathbb{I}_{31}=O(|q_k-p|^2),\ \mathbb{I}_{32}=O(1)\|\psi\|_*|q_k-p|,\ \mathbb{I}_{33}=O(1)\|\psi\|_*|q_k-p|.$$

Then (2.2.33) can be written as

$$\mathcal{L}(\psi_k)=o(1)\|\psi_k\|_*+O(\|\psi_k\|_*^2+\lambda_ke^{-\lambda_k})+O(|p-q_k|^2). \tag{2.2.34}$$

By the definition of H_k and (2.2.28), we have

$$\nabla H_k(q_k)=\nabla\log h(q_k)-\nabla\psi_k(q_k)+8\pi\nabla R(q_k,q_k)=O(\lambda_ke^{-\lambda_k}). \tag{2.2.35}$$

By (2.2.2) and (2.2.35), we have

$$\begin{aligned}&\nabla^2\big(\log h(p)+8\pi R(p,p)\big)(q_k-p)-\nabla\psi_k(p)\\=&\nabla\log h(q_k)-\nabla\psi_k(q_k)+8\pi\nabla R(q_k,q_k)\\&-\big(\nabla\log h(p)+8\pi\nabla R(p,p)\big)\\&+\nabla\psi_k(q_k)-\nabla\psi_k(p)\\&+O(|p-q_k|^2)\\=&\nabla H_k(q_k)-\nabla H(p)+O(|p-q_k|^\gamma\|\psi_k\|_*)\\&+O(|p-q_k|^2),\end{aligned} \tag{2.2.36}$$

where γ depends on $\mathfrak{p}$. We note that $\nabla H(p)=0$. From (2.2.34)-(2.2.36) and the non-degeneracy of p, w, we obtain

$$\|\psi_k\|_* + |p - q_k| \le C(\lambda_k e^{-\lambda_k} + o(1)\|\psi_k\|_* + \|\psi_k\|_*^2 + |p - q_k|^2), \quad (2.2.37)$$

where C is a generic constant, independent of k and ψ_k. Therefore, we have

$$\psi_k = O(\lambda_k e^{-\lambda_k}),\ |p - q_k| = O(\lambda_k e^{-\lambda_k}). \quad (2.2.38)$$

As a conclusion of (2.2.12), (2.2.26) and (2.2.38), we have

$$\lambda_k - \lambda(\rho_1) = O(\lambda(\rho_1)^{-1}),\ \tilde{h}_k = h + O(\lambda(\rho_1)e^{-\lambda(\rho_1)}),\ |q_k - p| = O(\lambda(\rho_1)e^{-\lambda(\rho_1)}) \quad (2.2.39)$$

and

$$v_{2k} - \frac{1}{2}w = O(\lambda(\rho_1)e^{-\lambda(\rho_1)}). \quad (2.2.40)$$

We replace $\tilde{h}_k$ by h in the definition of v_q, we denote the new terms by v_q. By (2.2.38), we have

$$v_{q_k} - v_q = O(\lambda(\rho_1)e^{-\lambda(\rho_1)}).$$

We set

$$v_{q,\lambda,a} = v_q - \overline{v}_q. \quad (2.2.41)$$

By (2.2.32) and (2.2.41), we obtain

$$v_{1k} - \frac{1}{2}v_{q,\lambda,a} = O(\lambda(\rho_1)e^{-\lambda(\rho_1)}). \quad (2.2.42)$$

By [15, Lemma 3.2], if we choose c_1 in $S_{\rho_1}(p, w)$ big enough, there exists a triplet $(q_k^*, \lambda_k^*, a_k^*)$ and $\phi^* \in O^{(1)}_{q_k^*,\lambda_k^*}$ such that

$$v_{1k} = \frac{1}{2}v_{q_k^*,\lambda_k^*,a_k^*} + \phi_k^*, \quad (2.2.43)$$

where $q_k^*,\ \lambda_k^*,\ a_k^*$ satisfy the condition in $S_{\rho_1}(p,w)$. Thus, we have proved

$$(v_{1k}, v_{2k}) \in S_{\rho_1}(p,w) \times S_{\rho_2}(p,w).$$

□

In conclusion, we have the following result,

Theorem 2.2.1. *Suppose h_1, h_2 are two positive $C^{2,\alpha}$ function on M such that any solution (p,w) of (1.1.14) is non-degenerate and $\Delta \log h_1(p) - \rho_2 + 8\pi - 2K(p) \neq 0$. Then there exists $\varepsilon_0 > 0$ and $C > 0$ such that for any solution of (1.1.10) with $\rho_1 \in (4\pi - \varepsilon_0, 4\pi + \varepsilon_0), \rho_2 \notin 4\pi\mathbb{N}$, either $|v_1|, |v_2| \leq C, \forall x \in M$ or $(v_1, v_2) \in S_{\rho_1}(p,w) \times S_{\rho_2}(p,w)$ for some solution (p,w) of (1.1.14).*

2.3 Approximate Blow-up Solution

In the following two sections, we shall construct the blow up solutions of (1.1.10) when $\rho_1 \to 4\pi$. The construction of such bubbling solution is based on a non-degenerate solution of (1.1.14). Our aim is to compute the degree of the following nonlinear operator

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (-\Delta)^{-1} \begin{pmatrix} \rho_1\left(\frac{h_1 e^{2v_1 - v_2}}{\int_M h_1 e^{2v_1 - v_2}} - 1\right) \\ \rho_2\left(\frac{h_2 e^{2v_2 - v_1}}{\int_M h_2 e^{2v_2 - v_1}} - 1\right) \end{pmatrix}$$

in the space $S_{\rho_1}(p,w) \times S_{\rho_2}(p,w)$.

Set

$$T(v_1, v_2) = \begin{pmatrix} T_1(v_1, v_2) \\ T_2(v_1, v_2) \end{pmatrix} = \Delta^{-1} \begin{pmatrix} 2\rho_1\left(\frac{h_1 e^{2v_1 - v_2}}{\int_M h_1 e^{2v_1 - v_2}} - 1\right) \\ 2\rho_2\left(\frac{h_2 e^{2v_2 - v_1}}{\int_M h_2 e^{2v_2 - v_1}} - 1\right) \end{pmatrix}.$$

Since each solution v_1 in $S_{\rho_1}(p,w)$ can be represented by (q, λ, a, ϕ), and v_2

in $S_{\rho_2}(p,w)$ can be represented by w and ψ, therefore the nonlinear operator $2v_1 + T_1(v_1, v_2)$ can be divided according to this representation.

Let $v_1 = \frac{1}{2}v_{q,\lambda,a} + \phi \in S_{\rho_1}(p,w)$. Recalling that $t = s - \overline{v}_q$. For $x \in B_{r_0}(q)$, we have

$$\begin{aligned} v_{q,\lambda,a}(x) + \log\frac{h(x)}{h(q)} =& U + t + H(x) + \eta + (a-1)(U+s) \\ &+ O\big(|a-1|(|y| + |\eta| + |\overline{v}_q|)\big), \end{aligned}$$

where $y = x - q$. Then, we get

$$\begin{aligned} \rho_1 h_1 e^{2v_1 - v_2 - 2\phi + \psi} =& \rho_1 h e^{v_{q,\lambda,a}} = \rho_1 h(q) e^{U+t}\Big[1 + (a-1)(U+s) + \eta \\ &+ H(x) + (a-1)O(|y|) + O(\tilde{\beta}^2)\Big], \end{aligned} \tag{2.3.1}$$

where

$$\tilde{\beta} = \lambda|a-1| + |\eta| + |H(x)| + \overline{v}_q.$$

Therefore in $B_{r_0}(q)$, we have

$$\begin{aligned} \rho_1 h_1 e^{2v_1 - v_2} =& (1+\varphi)\rho_1 h e^{v_{q,\lambda,a}} + (e^{\varphi} - 1 - \varphi)\rho_1 h e^{v_{q,\lambda,a}} \\ =& \rho_1 h(q) e^{U+t}\Big[1 + (a-1)(U+s) + \eta + H(x) \\ &+ (a-1)O(|y|) + \varphi\Big] + \tilde{E}, \end{aligned} \tag{2.3.2}$$

where

$$\tilde{E} = (e^{\varphi} - 1 - \varphi)\rho_1 h e^{v_{q,\lambda,a}} + \rho_1 h(q) e^{U+t} O(\varphi^2 + \tilde{\beta}^2), \tag{2.3.3}$$

and $\varphi = 2\phi - \psi$.

Let $\epsilon_2 > 0$ be small. $\tilde{E}$ can be written into two parts

$$\tilde{E} = \tilde{E}^{+} + \tilde{E}^{-},$$

where

$$\tilde{E}^{+} = \begin{cases} \tilde{E} & \text{if} \quad |\varphi| \geq \epsilon_2 \\ 0 & \text{if} \quad |\varphi| < \epsilon_2, \end{cases} \qquad \tilde{E}^{-} = \begin{cases} 0 & \text{if} \quad |\varphi| \geq \epsilon_2 \\ \tilde{E} & \text{if} \quad |\varphi| < \epsilon_2. \end{cases}$$

Then

$$\tilde{E}^{+} = O(e^{|\varphi|+2\lambda}) \text{ if } |\varphi| \geq \epsilon_2, \tag{2.3.4}$$

and

$$\tilde{E}^{-} = \rho_1 h e^{U+\lambda} O(\varphi^2 + \tilde{\beta}^2). \tag{2.3.5}$$

Using the expression for $\rho_1 h_1 e^{2v_1 - v_2}$ above, we obtain the following estimate for $\int_M \rho_1 h_1 e^{2v_1 - v_2}$.

Lemma 2.3.1. *Let* $v_1 = \frac{1}{2} v_{q,\lambda,a} + \phi \in S_{\rho_1}(p, w)$ and $v_2 = \frac{1}{2} w + \psi \in S_{\rho_2}(p, w)$. *Then as* $\rho_1 \to 4\pi, \rho_2 \notin 4\pi\mathbb{N}$, *we have*

$$\begin{aligned} \int_M \rho_1 h_1 e^{2v_1 - v_2} = \; & 4\pi e^t (1 - \psi(p)) + \frac{4\pi}{\rho_1 h(q)} \Delta H(q) \frac{\lambda}{e^{\lambda}} e^t \\ & + 8\pi\lambda(a-1)e^t + O(|a-1|e^{\lambda} + 1). \end{aligned} \tag{2.3.6}$$

Proof. By (2.3.2)

$$\begin{aligned} \int_M \rho_1 h_1 e^{2v_1 - v_2} = & \int_{B_{r_0}(q)} \{\rho_1 h(q) e^{U+t}[1 + \cdots] + \tilde{E}\} \mathrm{d}y \\ & + \int_{M \setminus B_{r_0}(q)} \rho_1 h_1 e^{2v_1 - v_2}. \end{aligned}$$

By the explicit expression of U, we have

$$\int_{M \setminus B_{r_0}(q)} \rho_1 h_1 e^{2v_1 - v_2} = O(1), \tag{2.3.7}$$

$$\int_{B_{r_0}(q)} \rho_1 h(q) e^{U+t} \mathrm{d}y = 4\pi e^t + O(1), \tag{2.3.8}$$

$$\int_{B_{r_0}(q)} \rho_1 h(q) e^{U+t}(a-1)(U+s)\mathrm{d}y = 8\pi(a-1)\lambda e^t + O(1+|a-1|e^{\lambda}), \tag{2.3.9}$$

where $U + s = 2\lambda - 2\log\left(1 + \frac{\rho_1 h(q)}{4} e^{\lambda}|y|^2\right) + O(1)$ is used. By the equation of η and the fact that $\nabla H(q) \cdot y$ is an odd functions, we have

$$\begin{aligned}\int_{B_{r_0}(q)} \rho_1 h(q) e^{U+t}\big(\eta + H(x)\big)dy &= -\frac{1}{2}e^t \int_{B_{r_0}(0)} \Delta\eta \mathrm{d}y = -\frac{1}{2}e^t \int_{\partial B_{r_0}(0)} \frac{\partial \eta}{\partial \nu} \\ &= \frac{4\pi}{\rho_1 h(q)} \Delta H(q) \lambda e^{t-\lambda} + O(1). \end{aligned}\tag{2.3.10}$$

To estimate the terms involving ϕ and ψ, we use (2.2.8) to obtain

$$\Delta v_q = \Delta U + \Delta\eta + 8\pi \text{ for } x \in B_{r_0}(q),$$

and

$$\Delta v_q = \Delta(v_q - 8\pi G(x,q)) + 8\pi \text{ for } x \notin B_{r_0}(q).$$

This together with Lemma 2.2.1 implies

$$\begin{aligned}\int_{B_{r_0}(q)} 2\rho_1 h(q) e^{U} \phi &= -\int_M \phi \Delta v_q + \int_{B_{r_0}(q)} \phi \Delta\eta + 8\pi \int_M \phi \\ &\quad + \int_{M \setminus B_{r_0}(q)} \phi \Delta(v_q - 8\pi G(x,q)) \\ &= \int_M \nabla\phi \nabla v_q + 8\pi \int_M \phi + \int_{B_{r_0}(q)} \phi \Delta\eta \\ &\quad + \int_{M \setminus B_{r_0}(q)} \phi \Delta(v_q - 8\pi G(x,q)) \\ &\le e^{-\lambda}\lambda \int_M |\phi|, \end{aligned}\tag{2.3.11}$$

where $|\Delta\eta(y)| = O(e^{-\lambda})$ and Lemma 2.2.1 are used. By the Poincaré inequality, we have

$$\Big|\int_{B_{r_0}(q)} \rho h(q) e^U \phi\Big| \le e^{-\lambda}\lambda \int_M |\phi| \le c e^{-\lambda}\lambda \|\phi\|_{H^1}. \tag{2.3.12}$$

While, for the terms involving ψ, we have

$$\begin{aligned}\int_{B_{r_0}(q)} \rho_1 h(q) e^U \psi =& \int_{B_{r_0}(q)} \rho_1 h(q) e^U \psi(q) + \int_{B_{r_0}(q)} \rho_1 h(q) e^U (\psi - \psi(q)) \\ =& 4\pi\psi(q) + O(\lambda e^{-\frac{3}{2}\lambda}).\end{aligned} \tag{2.3.13}$$

For $\tilde{E}^+$, we have

$$\begin{aligned}\int_{B_{r_0}(q)} |\tilde{E}^+| =& O(1) \int_{B_{r_0}(q)\cap\{|\varphi-\overline{\varphi}|\ge\epsilon_2\}} e^{|\varphi|+2\lambda} \\ =& O(1) \int_{B_{r_0}(q)\cap\{|\varphi-\overline{\varphi}|\ge\epsilon_2\}} e^{|\varphi-\overline{\varphi}|+2\lambda},\end{aligned}$$

where $\overline{\varphi} = \frac{\int_{B_{r_0}(q)} \varphi}{\mathrm{vol}(B_{r_0}(q))} = O(\|\varphi\|_{H^1}) = O(\lambda e^{-\lambda})$ for λ is large. We write

$$e^{|\varphi-\overline{\varphi}|} = e^{|\varphi-\overline{\varphi}|(1-\frac{4\pi|\varphi-\overline{\varphi}|}{\|\varphi-\overline{\varphi}\|^2})} e^{\frac{4\pi|\varphi-\overline{\varphi}|^2}{\|\varphi-\overline{\varphi}\|^2}}.$$

Since $\|\varphi-\overline{\varphi}\|^{-2} = \|\varphi-\overline{\varphi}\|_{H^1}^{-2} \gg 2\lambda$, we have

$$e^{|\varphi-\overline{\varphi}|(1-\frac{4\pi|\varphi-\overline{\varphi}|}{\|\varphi-\overline{\varphi}\|^2})} \le e^{\frac{\epsilon_2}{2}(1-\frac{2\pi\epsilon_2}{\|\varphi-\overline{\varphi}\|^2})} \ll e^{-2\lambda} \text{ for } \|\varphi-\overline{\varphi}\| \ge \frac{\epsilon_2}{2}.$$

Hence, by Moser-Trudinger inequality

$$\int_{B_{r_0}(q)\cap\{|\varphi-\overline{\varphi}|\ge\frac{\epsilon_2}{2}\}} e^{|\varphi-\overline{\varphi}|} \le e^{-2\lambda_j} \int_{B_{r_0}(q)} \exp\big(\frac{4\pi|\varphi-\overline{\varphi}|^2}{\|\varphi-\overline{\varphi}\|^2}\big) \le O(1)e^{-2\lambda}, \tag{2.3.14}$$

which implies $\int_{B_{r_0}(q)} |\tilde{E}^+| \le O(1)$.

For $\tilde{E}^-$, (2.3.5) gives

$$\int_{B_{r_0}(q)} |\tilde{E}^-| \le O(1) \int_{B_{r_0}(q)} (|\varphi|^2 + \tilde{\beta}^2)\rho_1 h(q) e^{U+t}. \tag{2.3.15}$$

By (2.6.3) in section 2.6, we can estimate the first term on the right hand side of (2.3.15) by

$$\begin{aligned}\int_{B_{r_0}(q)} \rho_1 h(q) e^{U+t}|\varphi|^2 \le &\ O(1)e^t\Big(\int_M |\nabla\phi|^2 + \|\psi\|_*^2 \int_{B_{r_0}(q)} e^U\Big)\\ = &\ O(1)e^t\lambda^2 e^{-2\lambda} = O(1)\lambda^2 e^{-\lambda}.\end{aligned}$$

For $\tilde{\beta}^2$

$$\begin{aligned}\int_{B_{r_0}(q)} \rho_1 h(q) e^{U+t}\tilde{\beta}^2 =&O(1)e^t\Big(|\overline{v}_q|^2 + (\lambda|a-1|\lambda)^2 + \int_{B_r(0)(q)} (\eta^2 + H(q,\eta))e^U\Big)\\ \le& c.\end{aligned}$$

Therefore, we have

$$\int_{B_{r_0}(q)} |\tilde{E}^-| = O(1). \tag{2.3.16}$$

By (2.3.2) and (2.3.7)-(2.3.16), we obtain (2.3.6). Hence we finish the proof of Lemma 2.3.1. □

Now we want to express $2v_1 + T_1(v_1, v_2)$ in a formula similar to (2.3.2). By Lemma 2.3.1 and the Taylor expansion of the exponential function,

$$\begin{aligned}e^{-t}\int_M \rho_1 h_1 e^{2v_1 - v_2} =&4\pi - 4\pi\psi(q) + \frac{4\pi}{\rho_1 h(q)}\Delta H(q)\lambda e^{-\lambda} + 8\pi\lambda(a-1)\\ &+ O(|a-1|) + O(e^{-\lambda}).\end{aligned} \tag{2.3.17}$$

Hence

$$\begin{aligned}\frac{e^t}{\int_M h_1 e^{2v_1-v_2}} - 1 =&\frac{1}{e^{-t}\int_M \rho_1 h_1 e^{2v_1-v_2}}\big(\rho_1 - \frac{\int_M \rho_1 h_1 e^{2v_1-v_2}}{e^t}\big)\\ =&\theta + O(|a-1|) + O(e^{-\lambda}),\end{aligned} \tag{2.3.18}$$

where θ is defined by

$$\theta = \frac{1}{4\pi}\Big[(\rho_1 - 4\pi) - \frac{4\pi}{\rho_1 h(q)}\Delta H(q)\lambda e^{-\lambda} + 4\pi\psi(q) - 8\pi\lambda(a-1)\Big]. \quad (2.3.19)$$

Let

$$\beta = \Big|\frac{e^t}{\int_M h_1 e^{2v_1 - v_2}} - 1\Big| + \tilde{\beta}, \quad (2.3.20)$$

and

$$E = 2(e^{\varphi} - 1 - \varphi)\frac{\rho_1 h_1 e^{2v_1 - v_2}}{\int_M h_1 e^{2v_1 - v_2}} + 2\rho_1 h(q) e^U \big(O(\varphi^2) + O(\beta^2)\big). \quad (2.3.21)$$

Then in $B_{r_0}(q)$, we have by (2.3.2),

$$\begin{aligned}
\frac{\rho_1 h_1 e^{2v_1 - v_2}}{\int_M h_1 e^{2v_1 - v_2}} =& (1+\varphi)\frac{\rho_1 h e^{v_{q,\lambda,a}}}{\int_M h_1 e^{2v_1 - v_2}} + (e^{\varphi} - 1 - \varphi)\frac{\rho_1 h e^{v_{q,\lambda,a}}}{\int_M h_1 e^{2v_1 - v_2}} \\
=& \rho_1 h(q) e^U \Big[1 + \Big(\frac{e^t}{\int_M h_1 e^{2v_1 - v_2}} - 1\Big) + (a-1)(U+s) \\
& + (a-1)O(|y|) + \eta + H + \varphi + O(\beta^2)\Big] + E. \quad (2.3.22)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\Delta\big(2v_1 + T_1(v_1, v_2)\big) =& 2\Delta v_1 + \frac{2\rho_1 h_1 e^{2v_1 - v_2}}{\int_M h_1 e^{2v_1 - v_2}} - 2\rho_1 \\
=& a(\Delta U + \Delta\eta) + 2\Delta\phi + 8\pi a + \frac{2\rho_1 h_1 e^{2v_1 - v_2}}{\int_M h_1 e^{2v_1 - v_2}} - 2\rho_1 \\
=& -2a\rho_1 h(q) e^U \Big[1 + \eta + H - \nabla_y H \cdot y\Big] + 8\pi - 2\rho_1 \\
& + 8\pi(a-1) + \frac{2\rho_1 h_1 e^{2v_1 - v_2}}{\int_M h_1 e^{2v_1 - v_2}} \\
=& 2\Delta\phi + (8\pi - 2\rho_1) + 8\pi(a-1) \\
& + 2\rho_1 h(q) e^U \Big[(a-1)(U + s - 1) + (a-1)O(|y|)y \cdot \nabla H \\
& + \Big(\frac{e^t}{\int_M h_1 e^{2v_1 - v_2}} - 1\Big) + \varphi\Big] + E. \quad (2.3.23)
\end{aligned}$$

Let $\varepsilon_2 > 0$ be small, which will be chosen later, see in section 6. Write

$$E = E^+ + E^-$$

with

$$E^+ = \begin{cases} E & \text{if } |\varphi| \geq \varepsilon_2 \\ 0 & \text{if } |\varphi| < \varepsilon_2 \end{cases} \quad \text{and} \quad E^- = \begin{cases} 0 & \text{if } |\varphi| \geq \varepsilon_2 \\ E & \text{if } |\varphi| < \varepsilon_2 \end{cases}.$$

As $\lambda \to \infty$, we have

$$E^+ = O(e^{|\varphi|+\lambda})$$

and

$$E^- = \rho_1 h(q) e^U (O(\varphi^2) + O(\beta^2)).$$

In $B_{2r_0}(q) \setminus B_{r_0}(q)$, since $v_q - \overline{v}_q - 8\pi G(x,q)$ is small, see [15, Lemma 2.2], we write $\Delta\big(2v_1 + T_1(v_1, v_2)\big)$ as

$$\begin{aligned} \Delta\big(2v_1 + T_1(v_1, v_2)\big) = \; & 2\Delta\phi + a\Delta(v_q - 8\pi G(x,q)) \\ & + 8\pi - 2\rho_1 + 8\pi(a-1) \\ & + \frac{2\rho_1 h}{\int_M h_1 e^{2v_1 - v_2}} e^{a(v_q - \overline{v}_q - 8\pi G(x,q)) + 8\pi a G(x,q) + \varphi}. \end{aligned} \tag{2.3.24}$$

In $M \setminus B_{2r_0}(q)$, we have

$$\begin{aligned} \Delta\big(2v_1 + T_1(v_1, v_2)\big) = \; & 2\Delta\phi + 8\pi - 2\rho_1 + 8\pi(a-1) \\ & + \frac{2\rho_1 h}{\int_M h_1 e^{2v_1 - v_2}} e^{8\pi a G(x,q) + \varphi - a\overline{v}_q}. \end{aligned} \tag{2.3.25}$$

From (2.3.23)-(2.3.25), we have the following

Lemma 2.3.2. *Let* $v_1 = \frac{1}{2} v_{q,\lambda,a} + \phi \in S_{\rho_1}(p,w)$, $v_2 = \frac{1}{2} w + \phi \in S_{\rho_2}(p,w)$. *Then as* $\rho_1 \to 4\pi$, 1.

$$\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla\phi_1 \rangle = 2\mathfrak{B}(\phi, \phi_1) + O(\lambda e^{-\lambda})\|\phi_1\|_{H_0^1(M)}, \tag{2.3.26}$$

where

$$\mathfrak{B}(\phi, \phi_1) := \int_M \nabla\phi \cdot \nabla\phi_1 - \int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \phi\phi_1,$$

is a positive symmetric, bilinear form satisfying $\mathfrak{B}(\phi, \phi) \geq c_0\|\phi\|^2_{H^1(M)}$ *for some constant* $c_0 > 0$.

2.

$$\begin{aligned}
&\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla\partial_q v_q \rangle \\
=& -8\pi\nabla H(q) + 8\pi\nabla\psi(q) \\
&+ O\Big(\lambda|a-1| + \Big|\frac{e^t}{\int_M h_1 e^{2v_1 - v_2}} - 1 - \psi(q)\Big| + \lambda e^{-\lambda}\Big),
\end{aligned} \tag{2.3.27}$$

3.

$$\begin{aligned}
&\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla\partial_{\lambda_j} v_q \rangle \\
=& -16\pi(a-1)\Big(\lambda - 1 + \log\frac{\rho_1 h(q)}{4} + 4\pi R(q,q)\Big) - 8\pi(\theta - \psi(q)) \\
&+ O(|a-1| + \lambda^2 e^{-\frac{3}{2}\lambda}),
\end{aligned} \tag{2.3.28}$$

4.

$$\begin{aligned}
&\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla v_q \rangle \\
=&\Big(2\lambda - 2 + 8\pi R(q,q) + 2\log\frac{\rho_1 h(q)}{4}\Big) \\
&\times \langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla\partial_\lambda v_q \rangle + 16\pi(a-1)\lambda \\
&+ O(1)\|\phi\|_{H^1(M)} + O(\lambda e^{-\lambda}),
\end{aligned} \tag{2.3.29}$$

We leave the proof of Lemma 2.3.2 in the section 6 because it contains a lot of computations.

2.4 Deformation And Degree Counting Formula

In this section, we want to deform $2v_i + T_i(v_1, v_2)$ into a simple form which can be solvable. Obviously, $v_1 = \frac{1}{2}v_{q,\lambda,a} + \phi,\ v_2 = \frac{1}{2}w + \psi$ is a solution of $2v_1 + T_1(v_1, v_2) = 0$, if and only if the left hand sides of (2.3.26)-(2.3.29) vanish. To solve the system (2.3.26)-(2.3.29) and $2v_2 + T_2(v_1, v_2) = 0$, we recall

$$\mathring{H}^1 = O^{(1)}_{q,\lambda} \bigoplus \text{ the linear subspace spanned by } v_q, \partial_\lambda v_q \text{ and } \partial_q v_q$$

and deform $2v_i + T_i(v_1, v_2)$ to a simpler operator $2v_i + T_i^0(v_1, v_2)$ by defining the operator $2I + T_i^t,\ 0 \le t \le 1,\ i = 1, 2$ through the following relations:

$$\begin{aligned}\langle \nabla(2v_1 + T_1^t(v_1, v_2)), \nabla\phi_1\rangle =& t\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla\phi_1\rangle \\ &+ 2(1-t)\mathfrak{B}(\phi, \phi_1) \text{ for } \phi_1 \in O^{(1)}_{q,\lambda},\end{aligned} \tag{2.4.1}$$

$$\begin{aligned}\langle \nabla(2v_1 + T_1^t(v_1, v_2)), \nabla\partial_q v_q\rangle =& t\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla\partial_q v_q\rangle \\ &+ (1-t)\big(-8\pi\nabla H + 8\pi\nabla\psi(q)\big),\end{aligned} \tag{2.4.2}$$

$$\begin{aligned}\langle \nabla(2v_1 + T_1^t(v_1, v_2)), \nabla\partial_\lambda v_q\rangle =& t\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla\partial_\lambda v_q\rangle \\ &- 8\pi(1-t)\Big[2(a-1)\lambda + (\theta - \psi(q))\Big],\end{aligned} \tag{2.4.3}$$

$$\begin{aligned}\langle\nabla(2v_1+T_1^t(v_1,v_2)),\nabla v_q\rangle =& t\Big[\big(2\lambda+O(1)\big)\langle\nabla(2v_1+T_1^t(v_1,v_2)),\nabla\partial_\lambda v_p\rangle\\ &+O(1)\|\phi\|_{H^1}+O(\lambda e^{-\lambda})\Big]+16\pi(a-1)\lambda,\end{aligned} \tag{2.4.4}$$

$$\begin{aligned}&2v_2+T_2^t(v_1,v_2)=t(2v_2+T_2(v_1,v_2))\\ &\quad+(1-t)\Big(w+2\psi-2\rho_2(-\Delta)^{-1}\big(\frac{h_2e^{w+2\psi-4\pi G(x,q)}}{\int_M h_2e^{w+2\psi-4\pi G(x,q)}}-1\big)\Big),\end{aligned} \tag{2.4.5}$$

where those coefficients $O(1)$ are those terms appeared in (2.4.4) so that $T_1^1(v_1,v_2)=T_1(v_1,v_2)$. From the construction above, we have

$$2v_i+T_i(v_1,v_2)=2v_i+T_i^1(v_1,v_2),\ i=1,2.$$

When $t=0$, the operator T_i^0 is simpler than T_i, $i=1,2$. During the deformation from T_i^1 to T_i^0, $i=1,2$ we have

Lemma 2.4.1. *Assume* $(\rho_1-4\pi)\neq 0$, *and* $\rho_2\notin 4\pi\mathbb{N}$. *Then there is* $\varepsilon_1>0$ *such that* $(2v_1+T_1^t(v_1,v_2),2v_2+T_2^t(v_1,v_2))\neq 0$ *for* $(v_1,v_2)\in\partial\big(S_{\rho_1}(p,w)\times S_{\rho_2}(p,w)\big)$ *and* $0\le t\le 1$ *if* $|\rho_1-4\pi|<\varepsilon_1$ *and* ρ_2 *is fixed.*

Proof. Assume $(v_1,v_2)\in\overline{S}_{\rho_1}(p,w)\times\overline{S}_{\rho_2}(p,w)$, where $\overline{S}_{\rho_i}(p,w)$ denotes the closure of $S_{\rho_i}(p,w)$, and $2v_i+T_i^t(v_1,v_2)=0$, $i=1,2$ for some $0\le t\le 1$. We will show that $(v_1,v_2)\notin\partial\big(S_{\rho_1}(p,w)\times S_{\rho_2}(p,w)\big)$.

From $\langle\nabla(2v_1+T_1^t(v_1,v_2)),\nabla\phi\rangle=0$, we have by Lemma 2.3.2

$$\|\phi\|_{H^1}^2\le O(\lambda e^{-\lambda})\|\phi\|_{H^1}.$$

This implies

$$\|\phi\|_{H^1}=O(\lambda e^{-\lambda})\le c_2\lambda e^{-\lambda}, \tag{2.4.6}$$

for some constant c_2 [1] independent of c_1.

Using $\langle\nabla(2v_1+T_1^t(v_1,v_2)),\nabla\partial_\lambda v_q\rangle=0$ and $\langle\nabla(2v_1+T_1^t(v_1,v_2)),\nabla v_q\rangle=$

[1] Here c_2 is independent of ψ, it can be shown in the proof of Lemma 2.3.2.

0, (2.4.4) and (2.4.6) imply

$$16\pi\lambda(a-1) = O(\lambda e^{-\lambda}), \tag{2.4.7}$$

that is, when ρ_1 is close to 4π,

$$|a-1| = O(e^{-\lambda}) < c_1\lambda_1(\rho)^{\frac{1}{2}}e^{-\lambda_1(\rho)}. \tag{2.4.8}$$

By $\langle\nabla(2v_1+T_1^t(v_1,v_2)),\nabla_\lambda v_q\rangle = 0$, we conclude from (2.3.28) and (2.4.8) that

$$\theta - \psi(q) + 2\lambda(a-1) = O(|a-1| + e^{-\lambda}) = O(e^{-\lambda}), \tag{2.4.9}$$

and

$$e^t\Big(\int_M h_1 e^{2v_1-v_2}\Big)^{-1} - 1 - \theta = O(|a-1| + e^{-\lambda}) = O(e^{-\lambda}). \tag{2.4.10}$$

Together with

$$\langle\nabla(2v_1 + T_1^t(v_1,v_2)),\nabla\partial_q v_q\rangle = 0,$$

(2.4.8), (2.4.10) and part (2) of Lemma 2.3.2, we have

$$\begin{aligned}|\nabla H(q) - \nabla\psi(q)| &= O\Big(\lambda|a-1| + |\frac{e^t}{\int_M h_1 e^{2v_1-v_2}} - 1 - \psi(q)| + \lambda e^{-\lambda}\Big)\\ &\le O(1)\lambda e^{-\lambda},\end{aligned}$$

which implies

$$\begin{aligned}&\Big|\nabla_y^2 H(p)\cdot(q-p) - \nabla\psi(p)\Big|\\ \le\ & O(1)\lambda e^{-\lambda} + O(1)\|\psi\|_*|p-q|^\gamma + O(1)|p-q|^2, \end{aligned}\tag{2.4.11}$$

where we used $\nabla_x H(p) = 0$ and $\mathfrak{p} > 2$.

For the second component, by (2.4.5), we have

$$\begin{aligned} 0 =&(1-t)\Big(\Delta w + 2\Delta\psi + 2\rho_2\Big(\frac{h_2 e^{w+2\psi-4\pi G(x,q)}}{\int_M h_2 e^{w+2\psi-4\pi G(x,q)}} - 1\Big)\Big) \\ &+ t\Big(\Delta w + 2\Delta\psi + 2\rho_2\Big(\frac{h_2 e^{w+2\psi-\frac{1}{2}v_{q,\lambda,a}-\phi}}{\int_M h_2 e^{w+2\psi-\frac{1}{2}v_{q,\lambda,a}-\phi}} - 1\Big)\Big). \end{aligned} \tag{2.4.12}$$

We set $\Theta = 2\rho_2 \frac{h_2 e^{w+2\psi-\frac{1}{2}v_{q,\lambda,a}-\phi}}{\int_M h_2 e^{w+2\psi-\frac{1}{2}v_{q,\lambda,a}-\phi}} - 2\rho_2 \frac{h_2 e^{w+2\psi-4\pi G(x,q)}}{\int_M h_2 e^{w+2\psi-4\pi G(x,q)}}$, and claim

$$\|\Theta\|_{L^{\mathfrak{p}}(M)} \le c_3 \lambda e^{-\lambda}, \tag{2.4.13}$$

where c_3 is a constant that independent of c and $\mathfrak{p}$ is defined in $O^{(2)}_{q,\lambda}$. By (2.4.6), it is not difficult to get

$$\exp\big(w + 2\psi - \frac{1}{2}v_{q,\lambda,a} - \phi\big) = \exp(w + 2\psi - \frac{1}{2}v_{q,\lambda,a}) + \Theta_1,$$

where $\|\Theta_1\|_{\mathfrak{p}} \le c_4\lambda e^{-\lambda}$. By noting (2.4.6), it is enough for us to prove the following one to get (2.4.13)

$$\Big\| \exp(w + 2\psi - \frac{1}{2}v_{q,\lambda,a}) - \exp\big(w + 2\psi - 4\pi a G(x,q)\big)\Big\|_{L^\infty(M)} \le c_5 \lambda e^{-\lambda}. \tag{2.4.14}$$

We leave the proof it in section 6. By (2.4.13), (2.4.12) can be written as

$$\Delta w + 2\Delta\psi + 2\rho_2\Big(\frac{h_2 e^{w+2\psi-4\pi G(x,q)}}{\int_M h_2 e^{w+2\psi-4\pi G(x,q)}} - 1\Big) + t\Theta = 0. \tag{2.4.15}$$

We expand the above equation,

$$\begin{aligned}\mathfrak{R} =&\Delta\psi + 2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\psi \\ &- 2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\Big(\int_M h_2e^{w-4\pi G(x,p)}\Big)^2}\int_M \big(h_2e^{w-4\pi G(x,p)}\psi\big) \\ &- 4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\big(\nabla G(x,p)(q-p)\big) \\ &+ 4\pi\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\Big(\int_M h_2e^{w-4\pi G(x,p)}\Big)^2}\int_M \Big(h_2e^{w-4\pi G(x,p)}\big(\nabla G(x,p)(q-p)\big)\Big),\end{aligned} \tag{2.4.16}$$

where $\mathfrak{R} = t\Theta + o(1)\|\psi\|_* + |q-p|^2$. By the non-degeneracy of (p,w) to (1.1.14), (2.4.11) and (2.4.16), we can get

$$\|\psi\|_* \leq c_6\lambda e^{-\lambda} \text{ and } |q-p| \leq c_7\lambda e^{-\lambda}. \tag{2.4.17}$$

Recall that

$$\theta = \frac{1}{4\pi}\Big((\rho_1 - 4\pi) - \frac{4\pi}{\rho_1 h(q)}\Delta H(q)\lambda e^{-\lambda} + 4\pi\psi(q) - 8\pi\lambda(a-1)\Big).$$

From (2.4.9), we obtain

$$O(1)e^{-\lambda} = \Big(\rho_1 - 4\pi - \frac{4\pi}{\rho_1 h(q)}\Delta H(q)\lambda e^{-\lambda}\Big), \tag{2.4.18}$$

which implies

$$O(e^{-\lambda}) = \frac{1}{4\pi}\Delta H(q)(\frac{\lambda_1(\rho)}{\lambda_1(\rho)} - \frac{\lambda}{e^{\lambda}}).$$

Hence

$$|\lambda - \lambda_1(\rho)| = c_8(\lambda_1(\rho))^{-1} \tag{2.4.19}$$

for some constant c_8 independent of c_1.

Using (2.4.17) and (2.4.19), we have

$$\|\psi\|_* \leq c_9\lambda_1(\rho)e^{-\lambda_1(\rho)} \text{ and } |p-q| \leq c_{10}\lambda_1(\rho)e^{-\lambda_1(\rho)}. \tag{2.4.20}$$

By choosing $c_1 > c_2, c_7, c_8, c_9, c_{10}$. From (2.4.6), (2.4.8), (2.4.19) and (2.4.20), we obtain

$$(v_1, v_2) \notin \partial\Big(S_{\rho_1}(p,w), S_{\rho_2}(p,w)\Big).$$

The proof is completed. □

Then, we want to apply Lemma 2.3.2 and Lemma 2.4.1 to get the degree of the linear operator in $S_{\rho_1}(p,w) \times S_{\rho_2}(p,w)$ when ρ_1 crosses 4π.

To compute the term

$$\deg\Big(\big(2v_1 + T_1(v_1,v_2), 2v_2 + T_2(v_1,v_2)\big); S_{\rho_1}(p,w) \times S_{\rho_2}(p,w), 0\Big),$$

we set

$$S_1^*(p,w) = \Big\{(q,\lambda,a) : \frac{1}{2}v_{q,\lambda,a} + \phi \in S_{\rho_1}(p,w), \phi \in O_{q,\lambda}^{(1)}\Big\}$$

and define the map

$$\Phi_p = (\Phi_{p,1}, \Phi_{p,2}, \Phi_{p,3}, \Phi_{p,4}) :$$

$$\begin{aligned}
\Phi_{p,1} &= \frac{1}{8\pi}\Big(\langle \nabla(2v_1 + T_1^0(v_1,v_2)), \nabla\partial_q v_q\rangle + \langle \nabla 2v_2 + T_2^0(v_1,v_2), 0\rangle\Big),\\
\Phi_{p,2} &= \langle \nabla(2v_1 + T_1^0(v_1,v_2)), \nabla\partial_\lambda v_q\rangle + \langle \nabla(2v_2 + T_2^0(v_1,v_2)), 0\rangle,\\
\Phi_{p,3} &= \langle \nabla(2v_1 + T_1^0(v_1,v_2)), \nabla v_q\rangle + \langle \nabla(2v_2 + T_2^0(v_1,v_2)), 0\rangle,\\
\Phi_{p,4} &= \langle \nabla(2v_1 + T_1^0(v_1,v_2)), 0\rangle + (2v_2 + T_2^0(v_1,v_2)).
\end{aligned}$$

Clearly, by Lemma 2.3.2 and Lemma 2.4.1, we have

$$\begin{aligned}
\deg\Big(\big(2v_1{+}T_1(v_1,v_2), 2v_2 + T_2(v_1,v_2)\big); S_{\rho_1}(p,w) \times S_{\rho_2}(p,w), 0\Big)&\\
= \deg\Big(\Phi_p; S_1^*(p,w) \times S_{\rho_2}(p,w), 0\Big).&
\end{aligned} \tag{2.4.21}$$

Next, we study the right hand side of (2.4.21) and prove Proposition 1.1.3.

Proof of Proposition 1.1.3. To compute the degree, we note that,

$$\Phi_{p,2} = -\Big[2\rho_1 - 8\pi - 8\pi\frac{\Delta H(q)}{\rho_1 h(q)}\lambda e^{-\lambda}\Big]. \tag{2.4.22}$$

Clearly, we have

$$\frac{\partial \Phi_{p,1}}{\partial \lambda} = \frac{\partial \Phi_{p,1}}{\partial a} = \frac{\partial \Phi_{p,2}}{\partial a} = \frac{\partial \Phi_{p,3}}{\partial \psi} = \frac{\partial \Phi_{p,3}}{\partial q} = \frac{\partial \Phi_{p,4}}{\partial a} = \frac{\partial \Phi_{p,4}}{\partial \lambda} = 0, \tag{2.4.23}$$

It is easy to see $\Phi_{p,1} = 0$, $\Phi_{p,3} = 0$ and $\Phi_{p,4} = 0$ if and only if

$$q = p,\ a = 1,\ \psi = 0, \tag{2.4.24}$$

and $\hat{\Phi}_{p,2} = 0$ if and only if

$$\rho_1 - 4\pi = \frac{4\pi}{\rho_1 h(q)}\Delta H(q)\lambda e^{-\lambda}. \tag{2.4.25}$$

It is not difficult to see that if $|\rho_1 - 4\pi|$ is sufficiently small, equation (2.4.25) possesses a unique solution λ Hence $(p, \lambda_1(\rho), a, 0)$ is the solution of Φ_p, where $a = 1$. The degree of Φ_p at $(p, \lambda_1(\rho), a, 0)$ depends on the number of negative eigenvalue for the following matrix

$$\mathcal{M} = \begin{bmatrix} \frac{\partial \Phi_{p,1}}{\partial q}, & \frac{\partial \Phi_{p,1}}{\partial \lambda}, & \frac{\partial \Phi_{p,1}}{\partial a}, & \frac{\partial \Phi_{p,1}}{\partial \psi} \\ \frac{\partial \Phi_{p,2}}{\partial q}, & \frac{\partial \Phi_{Q,2}}{\partial \lambda}, & \frac{\partial \Phi_{p,2}}{\partial a}, & \frac{\partial \Phi_{p,2}}{\partial \psi} \\ \frac{\partial \Phi_{p,3}}{\partial q}, & \frac{\partial \Phi_{Q,3}}{\partial \lambda}, & \frac{\partial \Phi_{p,3}}{\partial a}, & \frac{\partial \Phi_{p,3}}{\partial \psi} \\ \frac{\partial \Phi_{p,4}}{\partial q}, & \frac{\partial \Phi_{Q,4}}{\partial \lambda}, & \frac{\partial \Phi_{p,4}}{\partial a}, & \frac{\partial \Phi_{p,4}}{\partial \psi} \end{bmatrix}$$

Here we say $\mu_{\mathcal{M}}$ is an eigenvalue of $\mathcal{M}$, if there exists $\nu \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$, $a \in \mathbb{R}$, and Ψ such that

$$\mathcal{M}\begin{bmatrix}\nu\\ a\\ \lambda\\ \Psi\end{bmatrix}=\mu_{\mathcal{M}}\begin{bmatrix}\nu\\ a\\ \lambda_1\\ (-\Delta)^{-1}\Psi\end{bmatrix},$$

where $\frac{\partial\Phi_{p,1}}{\partial\psi}[\Psi]=\nabla\Psi(p)$, and

$$\begin{aligned}\frac{\partial\Phi_{p,4}}{\partial\psi}[\Psi]=&\Psi-(-\Delta)^{-1}\Big(2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\int_M h_2e^{w-4\pi G(x,p)}}\Big)\Psi\\ &-2\rho_2\frac{h_2e^{w-4\pi G(x,p)}}{\big(\int_M h_2e^{w-4\pi G(x,p)}\big)^2}\int_M\big(h_2e^{w-4\pi G(x,p)}\Psi\big)\Big).\end{aligned}$$

We set $N(T)$ as the number of the negative eigenvalue of matrix T,

$$\mathcal{M}_1=\begin{bmatrix}\frac{\partial\Phi_{p,1}}{\partial q}, & \frac{\partial\Phi_{p,1}}{\partial\psi}\\ \frac{\partial\Phi_{p,4}}{\partial q}, & \frac{\partial\Phi_{p,4}}{\partial\psi}\end{bmatrix}\text{ and }\mathcal{M}_2=\begin{bmatrix}\frac{\partial\Phi_{p,2}}{\partial\lambda}, & \frac{\partial\Phi_{p,2}}{\partial a}\\ \frac{\partial\Phi_{p,3}}{\partial\lambda}, & \frac{\partial\Phi_{p,3}}{\partial a}\end{bmatrix}.$$

By using (2.4.23),

$$N(\mathcal{M})=N(\mathcal{M}_1)+N(\mathcal{M}_2)=N(\mathcal{M}_1)+\mathrm{sgn}\Big(\frac{\partial\Phi_{p,2}}{\partial\lambda}\Big)+\mathrm{sgn}\Big(\frac{\partial\Phi_{p,3}}{\partial a}\Big),$$

Therefore,

$$\begin{aligned}\deg\Big(\Phi_p;S_1^*(p,w)\times S_{\rho_2}(p,w),0\Big)&=(-1)^{N(\mathcal{M})}=(-1)^{N(\mathcal{M}_1)}\times(-1)^{N(\mathcal{M}_2)}\\ &=(-1)^{N(\mathcal{M}_1)}\times\mathrm{sgn}\Big(\frac{\partial\Phi_{p,2}}{\partial\lambda}\Big)\times\mathrm{sgn}\Big(\frac{\partial\Phi_{p,3}}{\partial a}\Big).\end{aligned}$$

We first consider the last two terms on the right hand side of the above equality. For $\frac{\partial\Phi_{p,3}}{\partial a}$, it is easy to see that the sign of this value is positive. Therefore

$$\mathrm{sgn}(\frac{\partial\Phi_{p,3}}{\partial a})=1.$$

To compute $\frac{\partial \Phi_{p,2}}{\partial \lambda}$, we have

$$\frac{\partial \Phi_{p,2}}{\partial \lambda} = -8\pi \frac{\Delta H(p)}{\rho_1 h(p)} \lambda e^{-\lambda} + O(e^{-\lambda}).$$

Thus

$$\frac{\partial \Phi_{p,2}}{\partial \lambda} = -2(\rho_1 - 4\pi) + O(e^{-\lambda}).$$

It remains to compute $N(\mathcal{M}_1)$. According to the definition, we have

$$\left[\frac{\partial(\Phi_{p,1}, \Phi_{p,4})}{\partial(q,\psi)}\right] \begin{pmatrix} \nu \\ \Psi \end{pmatrix} = \begin{pmatrix} -\nabla^2 H \cdot \nu + \nabla \Psi(p) \\ -\mathcal{I}_0 \end{pmatrix}, \tag{2.4.26}$$

where

$$\begin{aligned} \mathcal{I}_0 = & -\Psi + (-\Delta)^{-1}\Big(2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} \Psi \\ & - 2\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\big(\int_M h_2 e^{w-4\pi G(x,p)}\big)^2} \int_M \big(h_2 e^{w-4\pi G(x,p)} \Psi\big) \\ & - 4\pi\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} \big(\nabla G(x,p)\cdot \nu\big) \\ & + 4\pi\rho_2 \frac{h_2 e^{w-4\pi G(x,p)}}{\big(\int_M h_2 e^{w-4\pi G(x,p)}\big)^2} \int_M \Big[h_2 e^{w-4\pi G(x,p)} \big(\nabla G(x,p)\cdot\nu\big)\Big]\Big). \end{aligned}$$

According to the definition of the eigenvalue for the linearized equation of (1.1.14), we can get $(-1)^{N(\mathcal{M}_1)}$ is exactly the number of the negative eigenvalue of the linearized equation of (1.1.14) when $\Delta H(p)$ has the same sign as $\rho_1 - 4\pi$. Therefore,

$$d_+^{(2)} - (-1)N_1 = d_-^{(2)} - (-1)N_2,$$

where N_1, N_2 represent the number of the negative eigenvalue for the linearized equation of (1.1.14) when $\rho_1 - 4\pi$, i.e., $\Delta H(p) > 0$ and $\rho_1 - 4\pi < 0$, i.e., $\Delta H(p) < 0$ respectively. It is easy to see that the summation of N_1 and N_2 is the total negative eigenvalues of the linearized equation of (1.1.14) for a given solution (p, w). Thus, by the definition of the topological degree for the solution to the shadow system (1.1.14), we have all the topological degree contributed by the bubbling solution of (1.1.10) equals to the negative of the topological degree of the shadow system (1.1.14). Hence, we proved Proposition 1.1.3.□

2.5 Proof Of Theorem 1.1.1

This section is devoted to prove Theorem 1.1.1. We first study the shadow system and give a proof the Lemma 1.1.1. As we mentioned in the first Chapter, we introduce a deformation to decouple the system (1.1.14).

$$(S_t)\begin{cases} \Delta w + 2\rho_2\left(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1\right) = 0, \\ \nabla\big(\log(h_1 e^{-\frac{1}{2}w\cdot(1-t)}) + 4\pi R(x,x)\big)\mid_{x=p} = 0. \end{cases} \tag{2.5.1}$$

It is easy to see that the system (2.5.1) is exactly (1.1.14) when $t = 0$, and will be a decoupled system when $t = 1$. During the deformation from (S_1) to (S_0), we have

Lemma 2.5.1. *Let $\rho_2 \notin 4\pi\mathbb{N}$. Then there is a uniform constant C_{ρ_2} such that for all solutions to (2.5.1), we have $|w|_{L^\infty(M)} < C_{\rho_2}$.*

Proof. Since $\rho_2 \notin 4\pi\mathbb{N}$, then we can see any solution for the following equation

$$\Delta w + 2\rho_2\Big(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1\Big) = 0 \tag{2.5.2}$$

is uniformly bounded above. By using the classical elliptic estimate, we have $|w|_{C^1(M)} < C$, where the constant C depends on ρ_2. □

Proof of Theorem 1.1.1. It is known that the topological degree is independent of h_1 and h_2 as long as they are positive C^1 functions. So we can always choose h_1 and h_2 such that the hypothesis of Theorem 2.2.1 holds.

Let d_S denote the Leray-Schauder degree for (1.1.14). By Lemma 2.5.1, computing the topological degree for (1.1.14) is reduced to computing the topological degree for system (2.5.1) when $t = 1$,

$$\begin{cases} \Delta w + 2\rho_2(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1) = 0, \\ \nabla[\log h_1 + 4\pi R(x,x)] \mid_{x=p} = 0. \end{cases} \tag{2.5.3}$$

Since this is a decoupled system, the topological degree of (2.5.3) equals the product of the degree of first equation and degree contributed by the second equation. By the Poincaré-Hopf Theorem, the degree of the second equation is $\chi(M)$. On the other hand, by Theorem A, the topological degree for the first equation is $b_m + b_{m-1}$, where b_k is given (1.1.7). Therefore,

$$d_S = \chi(M) \cdot (b_m + b_{m-1}). \tag{2.5.4}$$

Hence we get the topological degree of the shadow system (1.1.14), combined with Proposition 1.1.3, we can get Theorem 1.1.1. □

2.6 Proof Of Lemma 2.3.2 And (2.4.14)

This Section is devoted to prove Lemma 2.3.2. Let

$$\overline{v} := \frac{\int_{B_{r_0}(0)} \frac{e^\lambda}{(1+e^\lambda|y|^2)^2} v(y) \mathrm{d}y}{\int_{\mathbb{R}^2} \frac{e^\lambda}{(1+e^\lambda|y|^2)^2} \mathrm{d}y} = \frac{1}{\pi} \int_{B_{r_0}(0)} \frac{e^\lambda}{(1+e^\lambda|y|^2)^2} v(y) \mathrm{d}y.$$

Then we have the following Poincare-type inequality:

$$\int_{B_{r_0}(0)} \frac{e^\lambda}{(1+e^\lambda|y|^2)^2} \phi^2(y) \mathrm{d}y \le c(\|\phi\|^2_{H^1(B_{r_0}(0))} + \overline{\phi}^2) \tag{2.6.1}$$

for some constant c independent of λ. Using (2.6.1) we can prove the following result.

Lemma 2.6.1. *Let $U(x)$ be defined as in (2.2.3). Assume $\phi \in O^{(1)}_{q,\lambda}$. Then* *there is a constant $c > 0$ such that for large λ*

$$\int_{B_{r_0}(q)} e^U \phi \mathrm{d}y = O(\lambda^2 e^{-\lambda} \|\phi\|_{H^1}), \tag{2.6.2}$$

and

$$\int_M \left[|\nabla \phi|^2 - 2\rho_1 h(q) e^U \sigma(x)^2 \phi^2 \right] \geq c \int_M |\nabla \phi|^2. \tag{2.6.3}$$

For a proof, see [15].

Proof of Lemma 2.3.2. We start with part (1). Let $\phi \in O^{(1)}_{q,\lambda}$ and $\psi \in O^{(2)}_{q,\lambda}$. Recall $2v_1 = v_{q,\lambda,a} + 2\phi$, $\phi \in O^{(1)}_{q,\lambda}$ and $v_2 = \frac{1}{2}w + \psi$, $\psi \in O^{(2)}_{q,\lambda}$. We compute

$$\langle \nabla \big(2v_1 + T_1(v_1, v_2)\big), \nabla \phi_1 \rangle = -\langle \Delta \big(2v_1 + T_1(v_1, v_2)\big), \phi_1 \rangle.$$

Here we will use the decomposition of $\Delta\big(2v_1 + T_1(v_1, v_2)\big)$ in (2.3.23)-(2.3.25).

$$\begin{aligned} \langle \nabla \big(2v_1 + T_1(v_1, v_2)\big), \nabla \phi_1 \rangle = & \int 2\nabla \phi \cdot \nabla \phi_1 - \int_{B_{r_0}(q)} 4\rho_1 h(p) e^U \phi \phi_1 + \text{remainders} \\ := & 2\mathfrak{B}(\phi, \phi_1) + \text{remainders}. \end{aligned} \tag{2.6.4}$$

Clearly, $\mathfrak{B}$ is a symmetric bilinear form in $O^{(1)}_{q,\lambda}$ and by Lemma 2.6.1, $\mathfrak{B}(\phi, \phi) \geq c_0 \|\phi\|^2_{H^1(\Omega)}$ for some $c_0 > 0$. For the remainder terms, by $\phi_1 \in O^{(1)}_{q,\lambda}$ and (2.6.2), we have

$$\int_M \big(8\pi(a-1) + (8\pi - 2\rho_1)\big)\phi_1 = 0 \tag{2.6.5}$$

and

$$\Big| \int_{B_{r_0}(q)} \rho_1 h(p) e^U \phi_1 \Big| = O(\lambda^2 e^{-\lambda}) \|\phi_1\|_{H^1(M)}. \tag{2.6.6}$$

Since $|\nabla H(q)| \leq C\lambda e^{-\lambda}$ for $v_1 \in S_{\rho_1}(p, w)$,

$$\begin{aligned} \int_{B_{r_0}(q)} \nabla H \cdot (x - q) \rho_1 h(q) e^U \phi_1 = & O(\lambda e^{-\lambda}) \int_{B_{r_0}(q)} |x - q| e^U |\phi_1| \\ = & O(\lambda e^{-\lambda}) \|\phi_1\|_{H^1(M)}. \end{aligned} \tag{2.6.7}$$

Also, by Lemma 2.6.1, we have

$$\begin{aligned}
&\int_{B_{r_0}(q)} 2\rho_1 h(q)e^U(a-1)(U+s-1)\phi_1 \\
=&4\lambda\int_{B_{r_0}(q)} \rho_1 h(q)e^U(a-1)\phi_1 \\
&+2\int_{B_{r_0}(q)} \rho_1 h(q)e^U(a-1)(U-\lambda+O(1))\phi_1 \\
=&(a-1)O(\lambda^2e^{-\lambda})\|\phi_1\|_{H^1(M)} \\
&+(a-1)\Big(\int_{B_{r_0}(q)} e^U\big(U-\lambda+O(1)\big)^2\Big)^{\frac{1}{2}}\Big(\int_{B_{r_0}(q)} e^U\phi_1^2\Big)^{\frac{1}{2}} \\
=&O(|a-1|)\|\phi_1\|_{H^1(M)}=O(\lambda e^{-\lambda})\|\phi_1\|_{H^1(M)}. \qquad (2.6.8)
\end{aligned}$$

For E^+, we obtain

$$\begin{aligned}
\int_{B_{r_0}(q)} |E^+\phi_1| \leq &\Big(\int_{B_{r_0}(p)} |E^+|^2\Big)^{\frac{1}{2}}\Big(\int_{B_{r_0}(q)} \phi_1^2\Big)^{\frac{1}{2}} \\
&=O(\lambda e^{-\lambda})\|\phi_1\|_{H^1(M)}, \qquad (2.6.9)
\end{aligned}$$

where we used (2.3.14).

For E^-, we see that $E^-=2\rho_1 h(q)e^U(O(\varphi^2)+O(\beta^2))$, thus

$$\begin{aligned}
\int_{B_{r_0}(q)} |E^-\phi_1| \leq &\int_{B_{r_0}(q)} 2\rho_1 h(q)e^U(O(\varphi^2)+O(\beta^2))\phi_1 \\
=&O(\varepsilon_2)\Big(\int_{B_{r_0}(q)} e^U\varphi^2\Big)^{\frac{1}{2}}\Big(\int_{B_{r_0}(q)} e^U\phi_1^2\Big)^{\frac{1}{2}} \\
&+O(\frac{\lambda^3}{e^{2\lambda}})\Big(\int_{B_{r_0}(q)} e^U\phi_1^2\Big)^{\frac{1}{2}} \\
=&O(\varepsilon_2)(\|\phi\|_{H^1(M)}+\lambda e^{-\lambda})\|\phi_1\|_{H^1(M)}+O(\frac{\lambda^3}{e^{2\lambda}})\|\phi_1\|_{H^1(M)} \\
=&O(\lambda e^{-\lambda})\|\phi_1\|_{H^1(M)}, \qquad (2.6.10)
\end{aligned}$$

provided ε_2 is small.

For the term $\int_{B_{r_0}(q)} \rho_1 h(q) e^U \phi_1 \psi$, we have

$$\begin{aligned}
&\Big|\int_{B_{r_0}(q)} \rho_1 h(q) e^U \phi_1 \psi\Big| \\
=&\Big|\int_{B_{r_0}(q)} \rho_1 h(q) e^U \phi_1 (\psi - \psi(q))\Big| + \Big|\int_{B_{r_0}(q)} \rho_1 h(q) e^U \phi_1 \psi(q)\Big| \\
\le& C\Big(\int_{B_{r_0}(q)} \rho_1 h(q) e^U \phi_1^2\Big)^{\frac{1}{2}} \Big(\int_{B_{r_0}(q)} \rho_1 h(q) e^U (\psi - \psi(q))^2\Big)^{\frac{1}{2}} \\
&+ |\psi(q)| \Big|\int_{B_{r_0}(q)} \rho_1 h(q) e^U \phi_1\Big| \\
=& O(\lambda e^{-\lambda}) \|\phi_1\|_{H^1(M)},
\end{aligned} \tag{2.6.11}$$

where we used $\psi \in O^{(2)}_{q,\lambda}$ and (2.6.2). This finished the estimate in $B_{r_0}(q)$.

By Lemma [15, Lemma 2.2],

$$\int_{B_{2r_0}(q)\setminus B_{r_0}(q)} \Delta(v_q - 8\pi G(x,q))\phi_1 = O(\frac{\lambda}{e^\lambda}) \|\phi_1\|_{H^1(M)}. \tag{2.6.12}$$

For the nonlinear term in $\Delta T_1(v_1, v_2)$ in $M \setminus B_{r_0}(q)$, by (2.3.14), we have

$$\int_{M\setminus B_{r_0}(q)} |e^\varphi \phi_1| = O(\int_{|\varphi|\ge\varepsilon_2} |e^\varphi \phi_1| + \int_{|\varphi|\le\varepsilon_2} |e^{\varepsilon_2}\phi_1|) = O(1)\|\phi_1\|_{H^1(M)}.$$

Because $\int_M h_1 e^{2v_1 - v_2} \sim e^\lambda$,

$$\int_{M\setminus B_{r_0}(q)} \frac{2\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}} |\phi_1| = O(e^{-\lambda}) \int_{M\setminus B_{r_0}(q)} e^\varphi |\phi_1| = O(e^{-\lambda}) \|\phi_1\|_{H^1(M)}. \tag{2.6.13}$$

Combining (2.6.4)-(2.6.13), we reach at

$$\begin{aligned}
\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla\phi_1\rangle =& 2\langle\nabla\phi, \nabla\phi_1\rangle - \int_{B_{r_0}(q)} 4\rho_1 h(q) e^U \phi\phi_1 \\
&+ O(\lambda e^{-\lambda})\|\phi_1\|_{H^1(M)}.
\end{aligned}$$

Next, we prove part (3). In $B_{2r_0}(q)$, by the setting of v_q, we have

$$\begin{aligned}\partial_\lambda v_q =&\Big(2 - \frac{\frac{\rho_1 h(q)}{2} e^\lambda |x-q|^2}{1+\frac{\rho_1 h(q)}{4} e^\lambda |x-q|^2} + \partial_\lambda\Big[\eta + \frac{\Delta H(q)}{\rho_1 h(q)}\lambda^2 e^{-\lambda}\Big]\Big)\sigma \\ =&(1+\partial_\lambda U)\sigma + O(\lambda^2 e^{-\lambda}). \qquad (2.6.14)\end{aligned}$$

In $M \setminus B_{2r_0}(q)$, $\partial_\lambda v_q = 0$. We compute $\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla \partial_\lambda v_q\rangle = -\langle \Delta(2v_1 + T_1(v_1, v_2)), \partial_\lambda v_q\rangle$ by using (2.3.23)-(2.3.25).

Since $\phi \in O^{(1)}_{q,\lambda}$, we have

$$\int_M \nabla\phi \cdot \nabla \partial_\lambda v_q = 0. \qquad (2.6.15)$$

Direct computation yields,

$$\int_{B_{r_0}(q)} \partial_\lambda v_q = \int_{B_{r_0}(q)} (1+\partial_\lambda U) + O(\lambda^2 e^{-\lambda}) = O(\lambda^2 e^{-\lambda}). \qquad (2.6.16)$$

Hence,

$$\big(8\pi(a-1) + 8\pi - 2\rho_1\big)\int_{B_{r_0}(q)} \partial_\lambda v_q = O(\lambda^3 e^{-2\lambda}). \qquad (2.6.17)$$

Again by (2.6.14), we have

$$\begin{aligned}\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \partial_\lambda v_q =& \int_{B_{r_0}(q)} 2\rho_1 h(q) e^U (1+\partial_\lambda U + O(\frac{\lambda^2}{e^\lambda})) \\ =& \int_{\mathbb{R}^2} \frac{8}{(1+r^2)^2}(1+\frac{1-r^2}{1+r^2}) + O(\lambda^2 e^{-\lambda}) \\ =& 8\pi + \lambda^2 e^{-\lambda} \qquad (2.6.18)\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \Big[-2\log(1+\frac{\rho_1 h(q)}{4} e^\lambda |x-q|^2) \Big] \partial_\lambda v_q \\
&\quad = \int_{\mathbb{R}^2} \frac{8}{(1+r^2)^2} [-2\log(1+r^2)] \big(1 + \frac{1-r^2}{1+r^2} + O(\frac{\lambda^2}{e^\lambda})\big) + O(\frac{\lambda}{e^\lambda}) \\
&\quad = -8\pi + O(\frac{\lambda^2}{e^\lambda}). \qquad (2.6.19)
\end{aligned}$$

Combining (2.6.18) and (2.6.19), we get

$$\begin{aligned}
&\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U (U+s-1) \partial_\lambda v_q \\
&\quad = 16\pi\lambda - 16\pi + 16\pi \log \frac{\rho_1 h(q)}{4} + 64\pi^2 R(q,q) + O(\lambda e^{-\lambda}). \qquad (2.6.20)
\end{aligned}$$

By scaling, we compute

$$\begin{aligned}
&\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U O(|x-q|) \partial_\lambda v_q \\
&\quad = (a-1) \Big[O(R^{-1}) \int_{|z|\le r_0 R} \frac{16r}{(1+r^2)^3} \mathrm{d}z + O(\frac{\lambda^2}{e^\lambda}) \Big] \\
&\quad = O(e^{-\frac{1}{2}\lambda}) |a-1| = O(\lambda e^{-\frac{3}{2}\lambda}). \qquad (2.6.21)
\end{aligned}$$

Since $\nabla H(q) \cdot (x-q)$ is symmetry with respect to q

$$\begin{aligned}
&\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \nabla H(q) \cdot (x-q) \partial_\lambda v_q \\
&\quad = \int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \nabla H(q) \cdot (x-q) \big(1 + \partial_\lambda U + O(\frac{\lambda^2}{e^\lambda})\big) \\
&\quad = O\Big(\frac{\lambda^2}{e^\lambda}\Big) \int_{B_{r_0}(q)} e^U |x-q| = O(\lambda^2 e^{-\frac{3}{2}\lambda}). \qquad (2.6.22)
\end{aligned}$$

Next, we estimate the term $\phi\partial_\lambda v_q$ and $\psi\partial_\lambda v_q$. Since

$$\begin{aligned}0=\int_M \nabla\phi\cdot\nabla\partial_\lambda v_q &= -\int_M \phi\Delta(\partial_\lambda v_q)\\ &= -\int_{B_{r_0}(q)}\phi\Delta(\partial_\lambda U)+O(\lambda e^{-\lambda}\|\phi\|_{H^1(M)})\\ &= 2\rho_1 h(q)\int_{B_{r_0}(q)} e^U\phi\partial_\lambda U+O(\lambda e^{-\lambda}\|\phi\|_{H^1(M)}).\end{aligned} \tag{2.6.23}$$

Hence, by Lemma 2.6.1 and (2.6.23), we have

$$\begin{aligned}\int_{B_{r_0}(q)} 2\rho_1 h(p)e^U\phi\partial_\lambda v_q &= \int_{B_{r_0}(q)} 2\rho_1 h(q)e^U\phi(1+\partial_\lambda U+O(\lambda^2 e^{-\lambda}))\\ &= O(\lambda^2 e^{-\lambda})\|\phi\|_{H^1(M)}=O(\lambda^3 e^{-2\lambda}),\end{aligned} \tag{2.6.24}$$

and

$$\begin{aligned}\int_{B_{r_0}(q)} 2\rho_1 h(q)e^U\psi\partial_\lambda v_q &= \int_{B_{r_0}(q)} 2\rho_1 h(q)e^U\psi(q)\partial_\lambda v_q\\ &\quad+\int_{B_{r_0}(q)} 2\rho_1 h(q)e^U(\psi-\psi(q))\partial_\lambda v_q\\ &= 8\pi\psi(q)+O(\lambda e^{-\frac{3}{2}\lambda}),\end{aligned} \tag{2.6.25}$$

where we have used $|\psi-\psi(q)|\sim O(\lambda e^{-\lambda})|x-q|$. By (2.6.14), (2.6.2) and the Moser-Trudinger inequality,

$$\int_{B_{r_0}(q)} |E^+\partial_\lambda v_q|\le O(e^{-2\lambda}) \tag{2.6.26}$$

and

$$\begin{aligned}\int_{B_{r_0}(q)} |E^-\partial_\lambda v_q| &= \int_{B_{r_0}(q)\cap\{|\varphi\le\varepsilon_2|\}} 2\rho_1 h(q)e^U(O(\varphi^2)+O(\beta^2))\\ &= O(\lambda^3 e^{-2\lambda}).\end{aligned} \tag{2.6.27}$$

In $M \setminus B_{r_0}(q)$,

$$e^{2v_1-v_2} = O(e^{\phi}),\ \frac{2\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}} = O(e^{-\lambda})e^{\phi},\ \partial_\lambda v_q = O(\frac{\lambda^2}{e^{\lambda}}).$$

Hence, by the Moser-Trudinger inequality,

$$\int_{M\setminus B_{r_0(q)}} \frac{2\rho_1 h_1 e^{2v_1-v_2}}{\int_M h_1 e^{2v_1-v_2}} \partial_\lambda v_q = O(\lambda^3 e^{-2\lambda}). \tag{2.6.28}$$

By [15, Lemma 2.2] and $\partial_\lambda v_q = O(\lambda^2 e^{-\lambda})$,

$$\int_{B_{2r_0}(q)\setminus B_{r_0}(q)} \Delta(v_q - \overline{v}_q - 8\pi G(x,q)) \cdot \partial_\lambda v_q = O(\lambda^3 e^{-2\lambda}). \tag{2.6.29}$$

Combining (2.6.14) to (2.6.29), we obtain

$$\begin{aligned}
&\langle \nabla(2v_1 + T_1(v_1,v_2)), \nabla\partial_\lambda v_q\rangle \\
&\quad = -(a-1)\Big(16\pi\lambda - 16\pi + 16\pi\log\frac{\rho_1 h(q)}{4} + 64\pi^2 R(q,q)\Big) \\
&\qquad - 8\pi\Big(\frac{e^t}{\int_M h_1 e^{2v_1-v_2}} - 1\Big) + 8\pi\psi(q) + O(\lambda e^{-\frac{3}{2}\lambda}).
\end{aligned} \tag{2.6.30}$$

This proves part (3).

For the proof of part (4), we write

$$\begin{aligned}
\langle \nabla(2v_1 + T_1(v_1,v_2)), \nabla v_q\rangle &= \langle \nabla(2v_1 + T_1(v_1,v_2)), \nabla(v_q - \overline{v}_q)\rangle \\
&= -\langle \Delta(2v_1 + T_1(v_1,v_2)), (v_q - \overline{v}_q)\rangle.
\end{aligned}$$

First we note that

$$\int_M \big[8\pi(a-1) + 8\pi - 2\rho_1\big](v_q - \overline{v}_q) = 0.$$

By $\phi \in O^{(1)}_{q,\lambda}$,

$$\int_M \nabla\phi\cdot\nabla(v_q - \overline{v}_q) = 0.$$

In $B_{r_0}(p)$,

$$\begin{aligned} v_q - \overline{v}_q =& 2\lambda - 2\log(1 + \frac{\rho_1 h(q)}{4} e^{\lambda} |x-q|^2) + 8\pi R(q,q) + O(|y|) \\ &+ 2\log\frac{\rho_1 h(q)}{4} + O(\frac{\lambda^2}{e^{\lambda}}). \end{aligned} \tag{2.6.31}$$

We use (2.6.31) to compute $\int_M 2\rho_1 h e^U (U+s-1)(v_q - \overline{v}_q)$, after scaling, we have

$$\begin{aligned} &\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \log(1 + \frac{\rho_1 h(q)}{4} e^{\lambda}|x-q|^2) \\ =&\int_{\mathbb{R}^2} \frac{8}{(1+r^2)^2}\log(1+r^2) + O(\frac{\lambda}{e^{\lambda}}) \\ =&8\pi + O(\frac{\lambda}{e^{\lambda}}), \end{aligned} \tag{2.6.32}$$

$$\begin{aligned} &\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \Big[\log(1 + \frac{\rho_1 h(q)}{4} e^{\lambda}|x-q|^2)\Big]^2 \\ =&\int_0^{\infty} \frac{8}{(1+r^2)^2}[\log(1+r^2)]^2 2\pi r dr + O(\frac{\lambda^2}{e^{\lambda}}) \\ =&8\pi + O(\frac{\lambda^2}{e^{\lambda}}), \end{aligned} \tag{2.6.33}$$

and

$$\int_{B_{r_0}(q)} 2\rho_1 h e^U \Big[\log(1 + \frac{\rho_1 h(q)}{4} e^{\lambda}|x-q|^2)\Big] O(|x-q|) = O(e^{-\frac{1}{2}\lambda}). \tag{2.6.34}$$

Therefore, by (2.6.32)-(2.6.34),

$$\begin{aligned}
&\int_{B_{r_0}(q)} 2\rho_1 h e^U (U+s-1)(v_q - \overline{v}_q) \\
=&\int_{B_{r_0}(q)} 2\rho_1 h e^U \Big[4\lambda^2 - 8\log(1+\frac{\rho_1 h(q)}{4}e^{\lambda}|x-q|^2)\lambda \\
&\quad + \lambda\Big(32\pi R(q,q) + 8\log\frac{\rho_1 h(q)}{4} - 2\Big)\Big] + O(1) \\
=&\Big[256\pi^2 R(q,q) + 64\pi\log\frac{\rho_1 h(q)}{4} - 16\pi\Big]\lambda \\
&+ 32\pi\lambda^2 - 64\pi\lambda + O(1).
\end{aligned} \tag{2.6.35}$$

Similarly, we have

$$\begin{aligned}
\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U (v_q - \overline{v}_q) =& 16\pi\lambda - 16\pi + 64\pi^2 R(q,q) \\
&+ 16\pi\log\frac{\rho_1 h(q)}{4} + O(e^{-\frac{1}{2}\lambda}),
\end{aligned} \tag{2.6.36}$$

and

$$\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U O(|x-q|)(v_q - \overline{v}_q) = O(\lambda e^{-\frac{1}{2}\lambda}). \tag{2.6.37}$$

Since $\nabla H(q) = O(\lambda e^{-\lambda})$ and $\nabla H(q)\cdot(x-q)$ is symmetry with respect to q in $B_{r_0}(q)$, by (2.6.31), we have

$$\begin{aligned}
&\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \nabla H(q)\cdot(x-q)(v_q - \overline{v}_q) \\
=&\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \nabla H(q)\cdot(x-q) O(|x-q|) + O(\lambda^2 e^{-2\lambda}) \\
=&O(\lambda^2 e^{-2\lambda}).
\end{aligned} \tag{2.6.38}$$

By (2.6.2), and Lemma 2.6.1,

$$
\begin{aligned}
&\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \varphi(v_q - \overline{v}_q) \\
=&\int_{B_{r_0}(q)} 4\rho_1 h(q) e^U \phi\Big[\lambda + s - 2\log(1+\frac{\rho_1 h(q)}{4}e^\lambda|x-q|^2) + O(|x-q|)\Big] \\
&-\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \psi\Big[\lambda + s - 2\log(1+\frac{\rho_1 h(q)}{4}e^\lambda|x-q|^2) + O(|x-q|)\Big] \\
=&O(e^{-\frac{1}{2}\lambda})\|\phi\|_{H^1(M)} + (\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \phi^2)^{\frac{1}{2}} \\
&\times (\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \Big[\log(1+\frac{\rho_1 h(q)}{4}e^\lambda|x-q|^2) + O(|x-q|)\Big]^2)^{\frac{1}{2}} \\
&-\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \psi(q)\Big[\lambda + s - 2\log(1+\frac{\rho_1 h(q)}{4}e^\lambda|x-q|^2) + O(|x-q|)\Big] \\
&-\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U (\psi-\psi(q))\Big[\lambda + s - 2\log(1+\frac{\rho_1 h(q)}{4}e^\lambda|x-q|^2) + O(|x-q|)\Big] \\
=&O(1)\|\phi\|_{H^1(M)} + o(1)\|\psi\|_* - 16\pi\lambda_j\psi(q). \qquad (2.6.39)
\end{aligned}
$$

By a similar argument as in the proof of part (3),

$$
\int_{B_{r_0}(q)} E(v_q - \overline{v}_q) = O(\lambda^4 e^{-2\lambda}). \qquad (2.6.40)
$$

Since $v_q = O(1)$ in $M \setminus B_{r_0}(q)$, by [15, Lemma 2.2],

$$
\int_{B_{2r_0}(q)\setminus B_{r_0}(q)} \Delta(v_q - 8\pi G(x,q))(v_q - \overline{v}_q) = O(\lambda e^{-\lambda}). \qquad (2.6.41)
$$

For the integral outside of $B_{r_0}(q)$, we have $(\int_M h_1 e^{2v_1 - v_2})^{-1} = O(e^{-\lambda})$ and

$$
\begin{aligned}
&\int_{B_{2r_0}(q)\setminus B_{r_0}(q)} \frac{2\rho_1 h}{\int_M h e^{2v_1 - v_2}} e^{2v_1 - \psi}(v_q - \overline{v}_q) \\
=&\int_{B_{2r_0}(q)\setminus B_{r_0}(q)} O(e^{-\lambda}) e^{2\phi - \psi} = O(e^{-\lambda}). \qquad (2.6.42)
\end{aligned}
$$

Similarly

$$\int_{M\setminus B_{2r_0}(q)} \frac{2\rho_1 h}{\int_M h_1 e^{2v_1-v_2}} e^{2v_1-\psi}(v_q-\overline{v}_q) = O(e^{-\lambda}). \tag{2.6.43}$$

By (2.6.35)-(2.6.43), we have

$$\begin{aligned}
&\langle \nabla(2v_1+T_1(v_1,v_2),\nabla(v_q-\overline{v}_q)\rangle\\
=&-\Big(32\pi\lambda^2-64\pi\lambda+256\pi^2R(q,q)\lambda+64\pi\log\frac{\rho_1 h(q)}{4}\lambda\Big)(a-1)\\
&+\Big(16\pi\lambda-16\pi+64\pi^2R(q,q)+16\pi\log\frac{\rho_1 h(q)}{4}\Big)\Big(\frac{e^t}{\int_M h_1e^{2v_1-v_2}}-1-\psi(q)\Big)\\
&+O(1)\|\phi\|_{H^1(M)}+o(1)\|\psi\|_*+16\pi(a-1)\lambda+O(\lambda e^{-\lambda})\\
=&(2\lambda-2+8\pi R(q,q)+2\log\frac{\rho_1 h(q)}{8})\langle\nabla(v_1+T_1(v_1,v_2)),\nabla\partial_\lambda v_q\rangle\\
&+16\pi(a-1)\lambda+O(1)\|\phi\|_{H^1(M)}+o(1)\|\psi\|_*+O(\lambda e^{-\lambda}).
\end{aligned} \tag{2.6.44}$$

Finally, we prove part (2). We note that

$$\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla v_q\rangle=\langle\nabla(2v_1+T_1(v_1,v_2)),\nabla(v_q-\overline{v}_q)\rangle.$$

From $\phi\in O^{(1)}_{q,\lambda}$, we have

$$\langle\nabla\phi,\nabla\partial_q(v_q-\overline{v}_q)\rangle=0.$$

Since $\int_M(v_q-\overline{v}_q)=0$, we have $\int_M\partial_q(v_q-\overline{v}_q)=0$ and

$$\int_M(8\pi(a-1)+8\pi-2\rho_1)\partial_q(v_q-\overline{v}_q)=0.$$

In $B_{r_0}(q)$, by [14, Lemma 2.1]

$$\begin{aligned}
\partial_q v_q=&-\nabla_x U+\frac{\partial_q h(q)}{h(q)}\partial_\lambda U+\partial_q\Big(2\log h(q)+\frac{\Delta H(q)}{\rho_1 h(q)}\frac{\lambda^2}{e^\lambda}\Big)\\
&+8\pi\partial_q R(x,q)\mid_{x=q}+O(|x-q|)+O(\lambda^2e^{-\lambda}).
\end{aligned} \tag{2.6.45}$$

Since $\nabla_x U$ is symmetry with respect to q in $B_{r_0}(q)$,

$$\begin{aligned}&\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U (U+s-1)\nabla_x U\\ =&\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U O(|x-q|+\lambda^2 e^{-\lambda})\nabla_x U = 0.\end{aligned} \tag{2.6.46}$$

Hence by the fact $\partial_\lambda U$ is bounded,

$$\int_{B_{r_0}(q)} 2\rho_1 h(q)(U+s-1)\partial_q(v_q-\overline{v}_q) = O(\lambda). \tag{2.6.47}$$

For the other terms in (2.3.23), we need to estimate the following quantities:

$$\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \partial_q(v_q-\overline{v}_q) = O(1), \tag{2.6.48}$$

$$\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U O(|x-q|)\partial_q(v_q-\overline{v}_q) = O(1), \tag{2.6.49}$$

$$\begin{aligned}&\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \nabla H(q)\cdot(x-q)\nabla_x U\\ =&\nabla H(q)\Big(\int_{\mathbb{R}^2}\frac{8}{(1+r^2)^2}\frac{2r^2}{1+r^2}+O(e^{-\lambda})\Big)\\ =&(8\pi+O(\lambda e^{-\lambda}))\nabla H(q),\end{aligned} \tag{2.6.50}$$

and

$$\begin{aligned}&\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \nabla H(q)\cdot(x-q)\partial_q(v_q-\overline{v}_q)\\ =&\ (8\pi+O(e^{-\lambda}))\nabla H(q)+|\nabla H(q)|\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U O(|x-q|)\\ =&\ 8\pi\nabla H(q)+O(\lambda e^{-\frac{3}{2}\lambda}),\end{aligned} \tag{2.6.51}$$

where we have used $\nabla H(q) = O(\lambda e^{-\lambda})$ for $v_1 \in S_{\rho_1}(p,w)$.

For the term $\rho_1 h(q)e^U\phi\partial_q(v_q-\overline{v}_q)$,

$$\begin{aligned}&\int_{B_{r_0}(q)}2\rho_1 h(q)e^U\phi\partial_q(v_q-\overline{v}_q)\\ =&4\int_{B_{r_0}(q)}\rho_1 h(q)e^U\phi\partial_q(v_q-\overline{v}_q)\\ =&4\int_{B_{r_0}(q)}\rho_1 h(q)e^U\phi\partial_q U\\ &+\int_{B_{r_0}(q)}\rho_1 h(q)e^U\phi(O(\frac{\lambda^2}{e^\lambda})+O(|x-q|)).\end{aligned}\tag{2.6.52}$$

Using $\langle\nabla\phi,\nabla\partial_q(v_q-\overline{v}_q)\rangle=0$, it holds that

$$\begin{aligned}0=&\int_M\nabla\phi\nabla\partial_q v_q=-\int_M\phi\Delta(\partial_q v_q)\\ =&\int_{B_{r_0}(q)}2\rho_1 h(q)e^U\partial_q U\phi+\partial_q\log h(q)\int_{B_{r_0}(q)}2\rho_1 h(q)e^U\phi\\ &+O(\lambda^2e^{-\lambda})\|\phi\|_{H^1(M)}.\end{aligned}$$

By (2.6.2) and the above equality, we have

$$2\int_{B_{r_0}(q)}2\rho_1 h(q)e^U\partial_q U\phi=O(\lambda^2e^{-\lambda})\|\phi\|_{H^1(\Omega)}.$$

While for the term $\frac{e^t}{\int_M h_1e^{2v_1-v_2}}-1$ and ψ, we have

$$\begin{aligned}&\int_{B_{r_0}(q)}2\rho_1 h(q)e^U\Big(\frac{e^t}{\int_M h_1e^{2v_1-v_2}}-1-\psi\Big)\partial_q(v_q-\overline{v}_q)\\ =&\int_{B_{r_0}(q)}2\rho_1 h(q)e^U\Big(\frac{e^t}{\int_M h_1e^{2v_1-v_2}}-1-\psi(q)\Big)\partial_q(v_q-\overline{v}_q)\\ &-\int_{B_{r_0}(q)}2\rho_1 h(q)e^U(\psi-\psi(q))\partial_q(v_q-\overline{v}_q)\\ =&-8\pi\nabla\psi(q)+O\big(\frac{e^t}{\int_M h_1e^{2v_1-v_2}}-1-\psi(q)\big),\end{aligned}$$

where we used

$$\int_{B_{r_0}(q)} 2\rho_1 h(q) e^U \nabla\psi(q)(x-q)\nabla_y U = (8\pi + O(e^{-\lambda}))\nabla\psi(q),$$

and (2.6.48). Since $\partial_q(v_q - \overline{v}_q) = O(e^{\frac{1}{2}\lambda})$, as in the proof of part (3), we have

$$\int_{B_{r_0}(q)} E\partial_{\lambda_q}(v_q - \overline{v}_q) = O(\lambda^3 e^{-\frac{3}{2}\lambda}).$$

In $M \setminus B_{r_0}(q)$, $\partial_q(v_q - \overline{v}_q) = O(1)$. Hence by [15, Lemma 2.2],

$$\int_{B_{2r_0}(q)\setminus B_{r_0}(q)} \Delta(v_q - 8\pi G(x,q)) \cdot \partial_q(v_q - \overline{v}_q) = O(\lambda e^{-\lambda}).$$

Since $\int_M h_1 e^{2v_1 - v_2} = O(e^{-\lambda})$, the integral of the products of $\partial_q v_q$ and the nonlinear terms in (2.3.24) and (2.3.25) are of order

$$O(e^{-\lambda}) \int_M e^{\varphi} = O(e^{-\lambda}).$$

The estimates above imply

$$\begin{aligned}
&\langle \nabla(2v_1 + T_1(v_1, v_2)), \nabla\partial_q(v_q - \overline{v}_q)\rangle \\
=& -8\pi\nabla H(q) + 8\pi\nabla\psi(q) \\
&+ O\Big(\lambda|a-1| + \Big|\frac{e^t}{\int_M h_1 e^{2v_1 - v_2}} - 1 - \psi(q)\Big| + \frac{\lambda}{e^{\lambda}}\Big).
\end{aligned} \tag{2.6.53}$$

This proves the part (2) and hence the proof of Lemma is completed. □

Next, we give a proof of (2.4.14).

Proof of (2.4.14): For convenience, we denote

$$\mathfrak{E}_2 = \exp(w + 2\psi - \frac{1}{2}v_{q,\lambda,a}) - \exp\big(w + 2\psi - 4\pi a G(x,q)\big).$$

For $x \in M \setminus B_{r_0}(q)$. By [15, Lemma 2.2], we have

$$\Big|\frac{1}{2}v_{q,\lambda,a} - 4\pi aG(x,q)\Big| \leq \tilde{c}\lambda e^{-\lambda}$$

for some $\tilde{c}$ independent of c_1. Thus $|\mathfrak{E}_2| \leq c_5\lambda e^{-\lambda}$ in $M \setminus B_{r_0}(q)$.

For $x \in B_{r_0}(q)$, we note

$$\begin{aligned}
4\pi aG(x,q) - \frac{1}{2}v_{q,\lambda,a} =& 4\pi aG(x,q) - 4\pi aR(x,q) - a\log\big(\frac{\rho_1 h(q)}{4}\big) \\
&+ a\log\Big(1 + \frac{\rho_1 h(q)e^{\lambda}}{4}|x-q|^2\Big) - a\lambda \\
&- \frac{a}{2}\big(\eta + \frac{\Delta H(q)}{\rho_1 h(q_k)}\frac{\lambda^2}{e^{\lambda}}\big) + O(\lambda e^{-\lambda}) \\
=& a\log\Big(\frac{4}{\rho_1 h(q)e^{\lambda}|x-q|^2} + 1\Big) \\
&- \frac{a}{2}\big(\eta + \frac{\Delta H(q)}{\rho_1 h(q_k)}\frac{\lambda^2}{e^{\lambda}}\big) + O(\lambda e^{-\lambda}),
\end{aligned}$$

where we have used $\overline{v}_q = O(\lambda e^{-\lambda})$. Then, we have

$$\begin{aligned}
&\exp\big(w + 2\psi - 4\pi aG(x,q)\big) - \exp\big(w + 2\psi - \frac{1}{2}v_{q,\lambda,a}\big) \\
&= O(1)|x-q|^{2a}\Big(1 - \exp\big(4\pi aG(x,p) - \frac{1}{2}v_{q,\lambda,a}\big)\Big) \\
&= O(1)|x-q|^{2a}\Big(1 - \exp\Big[a\log\big(1 + \frac{4}{\rho_1 h(q)e^{\lambda}|x-q|^2}\big) \\
&\quad + O\Big(\eta + \frac{\Delta H(q)}{\rho_1 h(q)}\frac{\lambda^2}{e^{\lambda}}\Big) + O(\lambda e^{-\lambda})\Big] + O(\lambda e^{-\lambda})\Big).
\end{aligned} \tag{2.6.54}$$

When $|x-q| = O(e^{-\frac{1}{2}\lambda})$, we have

$$\exp\big(w + 2\psi - 4\pi aG(x,q)\big) = O(|x-q|^{2a}),$$

and

$$1 - \exp\Big(a\log\big(1 + \frac{4}{\rho_1 h(q)e^{\lambda}|x-q|^2}\big) + O(\lambda^2 e^{-\lambda})\Big) = O(e^{-a\lambda}|x-q|^{-2a}),$$

which implies

$$\mathfrak{E}_2 = O(\lambda e^{-\lambda}) \text{ for } |x-q| = O(e^{-\frac{1}{2}\lambda}).$$

When $|x-q| \gg e^{-\frac{1}{2}\lambda}$, then

$$\begin{aligned}
&1 - \exp\Big(a\log(1+\frac{4}{\rho_1 h(q) e^{\lambda}|x-q|^2}) + O(\eta + \frac{\Delta H(q)}{\rho_1 h(q)}\frac{\lambda^2}{e^{\lambda}})\Big) + O(\lambda e^{-\lambda}) \\
&= O\Big(\frac{1}{e^{a\lambda}|x-q|^{2a}} + \lambda e^{-\lambda}\Big).
\end{aligned}$$

As a result, we have the right hand side of (2.6.54) are of order $O(\lambda e^{-\lambda})$. Therefore

$$\mathfrak{E}_2 = O(\lambda e^{-\lambda}) \text{ for } x \in B_{r_0}(q).$$

Thus, we get (2.4.14). □

2.7 The Leray-Schauder degree

In this section, we provide a short introduction of the Leray-Schauder degree, namely the degree for the maps $T \in C(\mathcal{B}, \mathcal{B})$, where $\mathcal{B}$ is a Banach space and T is a compact perturbation of the identity $I = I_{\mathcal{B}}$.

Let D be an open bounded subset of the Banach space $\mathcal{B}$. We shall deal with compact perturbations of the identity, namely with operators $T \in C(\overline{D}, \mathcal{B})$ such that $T = I - K$, where K is a compact.

Let $p \notin T(\partial D)$. It is easy to check that $T(\partial D)$ is closed and hence

$$r := \operatorname{dist}(p, T(\partial D)) > 0.$$

It is known, see [10], that there exists a sequence $K_n \in C(\overline{D}, \mathcal{B})$ such that $K_n \to K$ uniformly in $\overline{D}$ and

$$K_n(\overline{D}) \subset E_n \subset \mathcal{B}, \text{ with } \dim(E_n) < \infty. \tag{2.7.1}$$

We shall define the degree of $I-K$ as the limit of the degrees of $I-K_n$, which we are going to introduce. Before that, we make the following preparations.

Let us consider a map $\phi \in C(\overline{\Omega}, \mathbb{R}^{m_1})$, where $\Omega \subset \mathbb{R}^{m_2}$ and $m_1 \leq m_2$. We can regard $\mathbb{R}^{m_1}$ as the subset of $\mathbb{R}^{m_2}$ whose points have the last $m_2 - m_1$ components equal to zero:

$$\mathbb{R}^{m_1} = \{x \in \mathbb{R}^{m_2} : x_{m_1+1} = \cdots = x_{m_2} = 0\}.$$

The above function ϕ can be considered as a map with values on $\mathbb{R}^{m_2}$ by understanding that the last $m_2 - m_1$ components are zero: $\phi_{m_1+1} = \cdots = \phi_{m_2} = 0$. Let $g(x) = x - \phi(x)$ and let $g_{m_1} \in C(\overline{\Omega} \cap \mathbb{R}^{m_1}, \mathbb{R}^{m_1})$ denote the restriction of g to $\overline{\Omega} \cap \mathbb{R}^{m_1}$. Let us show that if $p \in \mathbb{R}^{m_1} \setminus g(\partial\Omega)$ then

$$\deg(g, \Omega, p) = \deg(g_{m_1}, \Omega \cap \mathbb{R}^{m_1}, p). \tag{2.7.2}$$

Let $x \in \Omega$ be such that $g(x) = p$. This means that $x = \phi(x) + p$. Thus $x \in \Omega \cap \mathbb{R}^{m_1}$ and so $g_{m_1}(x) = g(x) = p$. This shows that $g^{-1}(p) \subset g_{m_1}^{-1}(p)$. Since the converse is trivially true, it follows that $g^{-1}(p) = g_{m_1}^{-1}(p)$. We can suppose that $\Omega \cap \mathbb{R}^{m_1} \neq \emptyset$, otherwise, $g_{m_1}^{-1}(p) = \emptyset$ and $g^{-1}(p) = \emptyset$. As usual, we can suppose that ϕ is of class C^1 and, moreover, that p is a regular value of g_{m_1}. Then according to the definition of degree, we get

$$\deg(g, \Omega, p) = \sum_{x \in g^{-1}(p)} \operatorname{sgn}[J_g(x)].$$

Now the Jacobian matrix $g'(x)$ is in triangular form

$$\begin{pmatrix} g'_{m_1}(x) & \cdot \\ 0 & I_{\mathbb{R}^{m_2 - m_1}} \end{pmatrix},$$

and hence $\operatorname{sgn}[J_g(x)] = \operatorname{sgn}[J_{g_{m_1}}(x)]$. As a consequence, we have

$$\deg(g, \Omega, p) = \sum_{x \in g^{-1}(p)} \operatorname{sgn}[J_g(x)] = \sum_{x \in g_{m_1}^{-1}(p)} \operatorname{sgn}[J_{g_{m_1}}(x)] = \deg(g_{m_1}, \Omega \cap \mathbb{R}^{m_1}, p),$$

which proves (2.7.2), provided p is a regular value. In the general case, we use the Sard Lemma to get the same conclusion.

The above discussion allows us to define the degree for a map g such that $g(x) = x - \phi(x)$, where $\phi(\overline{D})$ is contained in a *finite dimensional* subspace E of $\mathcal{B}$. Let $p \in \mathcal{B}, p \neq g(\overline{D})$. Let E_1 be a subspace of $\mathcal{B}$ containing E and p. We get $g_1 = g \mid_{\overline{D} \cap E_1}$ and define

$$\deg(g, D, p) = \deg(g_1, D \cap E_1, p). \tag{2.7.3}$$

Let us show that the definition is independent of E_1. Let E_2 be another subspace of $\mathcal{B}$ such that $E \subset E_2$ and $p \in E_2$. Then $E \cap E_1 \cap E_2$ and $p \in E_1 \cap E_2$. Applying (2.7.2) we obtain

$$\deg(g_i, D \cap E_i, p) = \deg(g \mid_{\overline{D} \cap E_1 \cap E_2}, D \cap E_1 \cap E_2, p), \ i = 1, 2.$$

This justifies the definition given in (2.7.3).

Now, let us come back to the map $T = I - K$, with K compact. Let $K_n \to K$ satisfy (2.7.1) and set $T_n = I - K_n$. Taking n such that

$$\sup_{x \in \overline{D}} \|K_n(x) - K(x)\| \leq \frac{\varepsilon}{2}, \tag{2.7.4}$$

we know that $p \notin T_n(\overline{D})$ and hence it makes sense to consider the degree $\deg(T_n, D, p)$ defined in (2.7.3).

Definition 2.7.1. *Let $p \notin T(\partial D)$, where $T = I - K$ with K compact. We set*

$$\deg(T, D, p) = \deg(I - K_n, D, p),$$

for any K_n satisfying (2.7.1) and (2.7.4).

Once more, we have to verify the definition, by showing that the degree does not depend on the approximation K_n. To prove this claim, let $T_i, i = 1, 2$, be such (2.7.1)-(2.7.4) hold. Let E_i be finite dimensional spaces such that $K_i(\overline{D}) \subset E_i$. If E is the space spanned by E_1 and E_2, we use the definition (2.7.3) to get

$$\deg(T_i, D, p) = \deg((T_i) \mid_{\overline{D} \cap E}, D \cap E, p), \; i = 1, 2. \tag{2.7.5}$$

Consider the homotopy

$$h(\lambda, \cdot) = \lambda(T_1) \mid_{\overline{D} \cap E} + (1 - \lambda)(T_2)_{\overline{D} \cap E}.$$

It is easy to check that h is admissible on $D \cap E$, i.e., $h(\lambda, \cdot) \neq p$ for all $(\lambda, x) \in [0, 1] \times \partial(\overline{D} \cap E)$. Thus,

$$\deg((T_1) \mid_{\overline{D} \cap E}, D \cap E, p) = \deg((T_2) \mid_{\overline{D} \cap E}, D \cap E, p).$$

This together with (2.7.5) proved that Definition 2.7.1 is well defined.

Chapter 3

The Lin-Ni Problem

3.1 Approximate Solutions

In this section, we construct suitable approximate solution, in the neighbourhood of which solutions in Theorem 1.2.1 will be found.

Let ε be as defined in (1.2.11). For any $Q \in \Omega_\varepsilon$ with $d(Q, \partial\Omega_\varepsilon)$ large, $U_{\Lambda,Q/\varepsilon} = \left(\frac{\Lambda}{\Lambda^2+|x-\frac{Q}{\varepsilon}|^2}\right)^{\frac{n-2}{2}}$ provides an approximate solution of (1.2.12). Because of the appearance of the additional linear term $\mu\varepsilon^2 u$, we need to add an extra term to get a better approximation. To this end, for $n = 4$, we consider the following equation

$$\Delta\bar{\Psi} + U_{1,0} = 0 \quad \text{in } \mathbb{R}^4, \quad \bar{\Psi}(0) = 1. \tag{3.1.1}$$

Then

$$\bar{\Psi}(|y|) = -\frac{1}{2}\ln|y| + I + O\Big(\frac{1}{|y|}\Big), \;\; \bar{\Psi}' = -\frac{1}{2|y|}\Big(1 + O\Big(\frac{\ln(1+|y|)}{|y|^2}\Big)\Big) \text{ as } |y| \to \infty, \tag{3.1.2}$$

where I is a constant. Let

$$\Psi_{\Lambda,Q} = \frac{\Lambda}{2}\ln\frac{1}{\Lambda\varepsilon} + \Lambda\bar{\Psi}\Big(\frac{y-Q}{\Lambda}\Big). \tag{3.1.3}$$

Then

$$\Delta\Psi_{\Lambda,Q} + U_{\Lambda,Q} = 0.$$

We note that

$$|\Psi_{\Lambda,Q}(y)|, \; |\partial_\Lambda\Psi_{\Lambda,Q}(y)| \le C\Big|\ln\frac{1}{\varepsilon(1+|y-Q|)}\Big|, \;\; |\partial_{Q_i}\Psi_{\Lambda,Q}(y)| \le \frac{C}{1+|y-Q|}. \tag{3.1.4}$$

For $n = 6$, let $\Psi(|y|)$ be the radial solution of

$$\Delta\Psi + U_{1,0} = 0 \text{ in } \mathbb{R}^n, \quad \Psi \to 0 \text{ as } |y| \to +\infty. \tag{3.1.5}$$

Then, it is easy to check that

$$\Psi(y) = \frac{1}{4|y|^2}(1 + O(\frac{1}{|y|^2})) \text{ as } |y| \to +\infty. \tag{3.1.6}$$

For $Q \in \Omega_\varepsilon$, we set

$$\Psi_{\Lambda,Q}(y) = \Psi(\frac{y-Q}{\Lambda}).$$

Then

$$\Delta\Psi_{\Lambda,Q}(y) + U_{\Lambda,Q} = 0 \text{ in } \mathbb{R}^6.$$

It is easy to see that

$$|\Psi_{\Lambda,Q}(y)|,\ |\partial_\Lambda\Psi_{\Lambda,Q}(y)| \le \frac{C}{(1+|y-Q|)^2},\ |\partial_{Q_i}\Psi_{\Lambda,Q}(y)| \le \frac{C}{(1+|y-Q|)^3}. \tag{3.1.7}$$

In order to obtain approximate solutions which satisfy the boundary condition, we define

$$\hat{U}_{\Lambda,Q/\varepsilon}(z) = -\Psi_{\Lambda,Q/\varepsilon}(z) - c_n\mu^{-1}\varepsilon^{n-4}\Lambda^{\frac{n-2}{2}}H(\varepsilon z, Q) + R_{\varepsilon,\Lambda,Q}(z)\chi(\varepsilon z), \tag{3.1.8}$$

where $R_{\varepsilon,\Lambda,Q}$ is defined by $\Delta R_{\varepsilon,\Lambda,Q} - \varepsilon^2 R_{\varepsilon,\Lambda,Q} = 0$ in Ω_ε and

$$\mu\varepsilon^2\frac{\partial R_{\varepsilon,\Lambda,Q}}{\partial\nu} = -\frac{\partial}{\partial\nu}\Big[U_{\Lambda,Q/\varepsilon} - \mu\varepsilon^2\Psi_{\Lambda,Q/\varepsilon} - c_n\varepsilon^{n-2}\Lambda^{\frac{n-2}{2}}H(\varepsilon z, Q)\Big] \text{ on } \partial\Omega_\varepsilon, \tag{3.1.9}$$

where $\chi(x)$ is a smooth cut-off function in Ω such that

$$\chi(x) = \begin{cases} 1 & \text{for } d(x,\partial\Omega) < \frac{\delta}{4}, \\ 0 & \text{for } d(x,\partial\Omega) > \frac{\delta}{2}. \end{cases}$$

We note (3.1.2) and (3.1.6), an expansion of $U_{\Lambda,Q/\varepsilon}$ and the definition of H imply that the normal derivative of $R_{\varepsilon,Q}$ is of order ε^{n-3} on the boundary of Ω_ε, from which we deduce that [2]

$$|R_{\varepsilon,\Lambda,Q}| + |\varepsilon^{-1}\nabla_z R_{\varepsilon,\Lambda,Q}| + |\varepsilon^{-2}\nabla_z^2 R_{\varepsilon,\Lambda,Q}| \le \begin{cases} C\Lambda, & n = 4, \\ C\varepsilon^2, & n = 6. \end{cases} \tag{3.1.10}$$

A similar estimate also holds for the derivatives of $R_{\varepsilon,\Lambda,Q}$ with respect to $\Lambda,\ Q$.

Now we are able to define the approximate bubble solutions. Since it is different in constructing the approximate solution for $n = 4$ and $n = 6$, we shall tact them respectively. For $n = 4$, let

$$\Lambda_{4,1} \le \Lambda \le \Lambda_{4,2},\ Q \in \mathcal{M}_{\delta_4} := \{x \in \Omega|\ d(x, \partial\Omega) > \delta_4\}, \tag{3.1.11}$$

where $\Lambda_{4,1} = \exp(-\frac{1}{2})\varepsilon^{\beta}$, $\Lambda_{4,2} = \exp(-\frac{1}{2})\varepsilon^{-\beta}$, β is a small constant with a generic constant δ_4, to be determined later. We write

$$\bar{Q} = \frac{1}{\varepsilon} Q,$$

and define our approximate solutions as

$$W_{\varepsilon,\Lambda,Q} = U_{\Lambda,Q/\varepsilon} + \mu\varepsilon^2 \hat{U}_{\Lambda,Q/\varepsilon} + \frac{c_4\Lambda}{|\Omega|}\mu^{-1}\varepsilon^2. \tag{3.1.12}$$

For $n = 6$, let

$$\begin{aligned} \sqrt{\frac{|\Omega|}{c_6}(\frac{1}{96} - \Lambda_6\varepsilon^{\frac{2}{3}})} \le & \Lambda \le \sqrt{\frac{|\Omega|}{c_6}(\frac{1}{96} + \Lambda_6\varepsilon^{\frac{2}{3}})}, \\ Q \in \mathcal{M}_{\delta_6} := & \{x \in \Omega|\ d(x, \partial\Omega) > \delta_6\}, \\ \frac{1}{48} - \eta_6\varepsilon^{\frac{1}{3}} \le & \eta \le \frac{1}{48} + \eta_6\varepsilon^{\frac{1}{3}}, \end{aligned} \tag{3.1.13}$$

where Λ_6 and η_6 are some constants that may depend on the domain, δ_6 is a small constant, which are determined later. Our approximate solution for

[2]For $n = 4$, we set the parameter Λ in a range that depends on ε, we have to take Λ into consideration, and we note that each component on the right hand side of (3.1.9) carry Λ as a factor.

$n=6$ is the following

$$W_{\varepsilon,\Lambda,Q,\eta}=U_{\Lambda,Q/\varepsilon}+\mu\varepsilon^2\hat{U}_{\Lambda,Q/\varepsilon}+\eta\mu^{-1}\varepsilon^4. \tag{3.1.14}$$

For convenience, in the following, we write $W,\ U,\ \hat{U},\ R,$ and Ψ instead of $W_{\varepsilon,\Lambda,Q},\ U_{\varepsilon,Q/\varepsilon},\ \hat{U}_{\Lambda,Q/\varepsilon},\ R_{\varepsilon,\Lambda,Q}$ and $\Psi_{\Lambda,Q/\varepsilon}$ respectively in the following. By construction, the normal derivative of W vanishes on the boundary of Ω_ε, and W satisfies

$$-\Delta W+\mu\varepsilon^2W=\begin{cases}8U^3+\mu^2\varepsilon^4\hat{U}-\mu\varepsilon^2\Delta(R_{\varepsilon,\Lambda,Q}\chi), & n=4,\\ 24U^2+\mu^2\varepsilon^4\hat{U}-\mu\varepsilon^2\Delta(R_{\varepsilon,\Lambda,Q}\chi)+\varepsilon^6(\eta-\frac{c_6\Lambda^2}{|\Omega|}), & n=6.\end{cases} \tag{3.1.15}$$

We note that W depends smoothly on $\Lambda,\ \bar{Q}$. Setting, for $z\in\Omega_\varepsilon$,

$$\langle z-\bar{Q}\rangle=(1+|z-\bar{Q}|^2)^{\frac{1}{2}}.$$

A simple computation shows

$$|W(z)|\le\begin{cases}C(\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}+\langle z-\bar{Q}\rangle^{-2}), & n=4,\\ C(\varepsilon^3+\langle z-\bar{Q}\rangle^{-4}), & n=6,\end{cases} \tag{3.1.16}$$

$$|D_\Lambda W(z)|\le\begin{cases}C(\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}+\langle z-\bar{Q}\rangle^{-2}), & n=4,\\ C\langle z-\bar{Q}\rangle^{-4}, & n=6,\end{cases} \tag{3.1.17}$$

$$|D_{\bar{Q}}W(z)|\le\begin{cases}C((\langle z-\bar{Q}\rangle^{-3}), & n=4,\\ C((\langle z-\bar{Q}\rangle^{-5}), & n=6,\end{cases} \tag{3.1.18}$$

and

$$|D_\eta W(z)|=O(\varepsilon^3),\ n=6. \tag{3.1.19}$$

According to the choice of W, we have the following error and energy estimates, we leave the proof in Section 6 of this Chapter.

Lemma 3.1.1. *We set*

$$S_\varepsilon[u] := -\Delta u + \mu\varepsilon^2 u - n(n-2)u_+^{\frac{n+2}{n-2}},\ u_+ = \max(u,0),$$

and introduce the following functional

$$J_\varepsilon[u] := \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 + \frac{1}{2}\mu\varepsilon^2 \int_{\Omega_\varepsilon} u^2 - \frac{(n-2)^2}{2}\int_{\Omega_\varepsilon} |u|^{\frac{2n}{n-2}},\ u \in H^1(\Omega_\varepsilon).$$

For $n=4$, we have

$$\begin{aligned} |S_\varepsilon[W](z)| \leq\ & C\Big(\langle z-\bar{Q}\rangle^{-4}\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}} + \langle z-\bar{Q}\rangle^{-2}\varepsilon^4(-\ln\varepsilon) \\ & + \frac{\varepsilon^4}{(-\ln\varepsilon)^{\frac{1}{2}}} + \frac{\varepsilon^4}{(-\ln\varepsilon)}\big|\ln\big(\frac{1}{\varepsilon(1+|z-\bar{Q}|)}\big)\big|\Big), \end{aligned} \tag{3.1.20}$$

$$\begin{aligned} |D_\Lambda S_\varepsilon[W](z)| \leq\ & C\Big(\langle z-\bar{Q}\rangle^{-4}\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}} + \langle z-\bar{Q}\rangle^{-2}\varepsilon^4(-\ln\varepsilon) \\ & + \frac{\varepsilon^4}{(-\ln\varepsilon)^{\frac{1}{2}}} + \frac{\varepsilon^4}{(-\ln\varepsilon)}\big|\ln\big(\frac{1}{\varepsilon(1+|z-\bar{Q}|)}\big)\big|\Big), \end{aligned} \tag{3.1.21}$$

$$\begin{aligned} |D_{\bar{Q}} S_\varepsilon[W](z)| \leq\ & C\Big(\langle z-\bar{Q}\rangle^{-5}\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}} + \langle z-\bar{Q}\rangle^{-3}\varepsilon^4(-\ln\varepsilon) \\ & + \langle z-\bar{Q}\rangle^{-1}\frac{\varepsilon^4}{(-\ln\varepsilon)}\Big), \end{aligned} \tag{3.1.22}$$

and

$$\begin{aligned} J_\varepsilon[W] = {}& 2\int_{\mathbb{R}^4} U_{1,0}^4 + \frac{c_4\Lambda^2}{4}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\ln\frac{1}{\Lambda\varepsilon} - \frac{c_4^2\Lambda^2}{2|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}} \\ & + \frac{1}{2}c_4^2\varepsilon^2\Lambda^2 H(Q,Q) + O\big(\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\Lambda^2\big) + O(\varepsilon^4(-\ln\varepsilon)^2). \end{aligned} \tag{3.1.23}$$

For $n = 6$, we have

$$S_\varepsilon[W](z) = -\varepsilon^6\big(24\eta^2 - \eta + \frac{c_6\Lambda^2}{|\Omega|}\big) + O(1)\varepsilon^3\langle z - \bar{Q}\rangle^{-4}, \tag{3.1.24}$$

$$|D_\Lambda S_\varepsilon[W](z)| = O(1)\big(\langle z - \bar{Q}\rangle^{-4}\varepsilon^3 + \varepsilon^6\big), \tag{3.1.25}$$

$$|D_\eta S_\varepsilon[W](z)| = O(1)\big(\langle z - \bar{Q}\rangle^{-4}\varepsilon^3 + \varepsilon^{6\frac{1}{3}}\big), \tag{3.1.26}$$

$$|D_{\bar{Q}} S_\varepsilon[W](z)| \leq \ C\langle z - \bar{Q}\rangle^{-5}\varepsilon^3, \tag{3.1.27}$$

and

$$\begin{aligned} J_\varepsilon[W] =& 4\int_{\mathbb{R}^6} U_{1,0}^3 + \Big(\frac{1}{2}\eta^2|\Omega| - c_6\eta\Lambda^2 + \frac{1}{48}c_6\Lambda^2 - 8\eta^3|\Omega|\Big)\varepsilon^3 + \frac{1}{2}c_6^2\Lambda^4\varepsilon^4 H(Q,Q) \\ &+ \frac{1}{2}\big(\eta - \frac{c_6\Lambda^2}{|\Omega|}\big)\varepsilon^4\int_\Omega \frac{\Lambda^2}{|x-Q|^4} + O(\varepsilon^5). \end{aligned} \tag{3.1.28}$$

3.2 Finite Dimensional Reduction

According to the general strategy used in Lyapunov-Schmidt reduction method, we first consider the linearized problem at W, and solve it in a finite-codimensional space, i.e., the orthogonal space to the finite-dimensional subspace generated by the derivatives of W with respect to the parameters Λ and $\bar{Q}_i$ in the case $n = 4$, and the orthogonal space to the finite-dimensional subspace generated by the derivatives of W with respect to the parameters Λ, $\bar{Q}_i$ and η in the case $n = 6$. Equipping $H^1(\Omega_\varepsilon)$ with the scalar product

$$(u,v)_\varepsilon = \int_{\Omega_\varepsilon} (\nabla u \cdot \nabla v + \mu\varepsilon^2 uv). \tag{3.2.1}$$

For the case $n = 4$. Orthogonality to the functions

$$Y_0 = \frac{\partial W}{\partial \Lambda},\ Y_i = \frac{\partial W}{\partial \bar{Q}_i}, 1 \le i \le 4, \tag{3.2.2}$$

in that space is equivalent to the orthogonality in $L^2(\Omega_\varepsilon)$, equipped with the usual scalar product $\langle \cdot, \cdot \rangle$, to the functions $Z_i, 0 \le i \le 4$, defined as

$$\begin{cases} Z_0 = -\Delta \frac{\partial W}{\partial \Lambda} + \mu\varepsilon^2 \frac{\partial W}{\partial \Lambda}, \\ Z_i = -\Delta \frac{\partial W}{\partial \bar{Q}_i} + \mu\varepsilon^2 \frac{\partial W}{\partial \bar{Q}_i},\ 1 \le i \le 4. \end{cases} \tag{3.2.3}$$

Straightforward computations provide us with the estimate:

$$|Z_i(z)| \le C(\varepsilon^4 + \langle z - \bar{Q} \rangle^{-6}). \tag{3.2.4}$$

Then, we consider the following problem: given h, finding a solution ϕ which satisfies

$$\begin{cases} -\Delta\phi + \mu\varepsilon^2\phi - 24W^2\phi = h + \Sigma_{i=0}^4 c_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi \rangle = 0, & 0 \le i \le 4, \end{cases} \tag{3.2.5}$$

for some numbers c_i.

While for the case $n = 6$. Orthogonality to the functions

$$Y_0 = \frac{\partial W}{\partial \Lambda},\ Y_i = \frac{\partial W}{\partial \bar{Q}_i}, 1 \le i \le 6,\ Y_7 = \frac{\partial W}{\partial \eta}, \tag{3.2.6}$$

in that space is equivalent to the orthogonality in $L^2(\Omega_\varepsilon)$, equipped with the usual scalar product $\langle \cdot, \cdot \rangle$, to the functions $Z_i, 0 \le i \le 7$, defined as

$$\begin{cases} Z_0 = -\Delta \frac{\partial W}{\partial \Lambda} + \mu\varepsilon^2 \frac{\partial W}{\partial \Lambda}, \\ Z_i = -\Delta \frac{\partial W}{\partial \bar{Q}_i} + \mu\varepsilon^2 \frac{\partial W}{\partial \bar{Q}_i},\ 1 \le i \le 6, \\ Z_7 = -\Delta \frac{\partial W}{\partial \eta} + \mu\varepsilon^2 \frac{\partial W}{\partial \eta}. \end{cases} \tag{3.2.7}$$

Direct computations provide us the following estimate:

$$|Z_i(z)| \leq C(\varepsilon^6 + \langle z - \bar{Q}\rangle^{-8}),\ 0 \leq i \leq 6,\ Z_7(z) = O(\varepsilon^6). \tag{3.2.8}$$

Then, we consider the following problem: given h, finding a solution ϕ which satisfies

$$\begin{cases} -\Delta\phi + \mu\varepsilon^2\phi - 48W\phi = h + \Sigma_{i=0}^7 d_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi\rangle = 0, & 0 \leq i \leq 7, \end{cases} \tag{3.2.9}$$

for some numbers d_i.

Existence and uniqueness of ϕ will follow from an inversion procedure in suitable weighted function space. To this end, we define

$$\begin{cases} \|\phi\|_* = \|\langle z - \bar{Q}\rangle\phi(z)\|_\infty,\ \|f\|_{**} = \varepsilon^{-3}(-\ln\varepsilon)^{\frac{1}{2}}|\overline{f}| + \|\langle z - \bar{Q}\rangle^3 f(z)\|_\infty,\ n = 4, \\ \|\phi\|_{***} = \|\langle z - \bar{Q}\rangle^2\phi(z)\|_\infty,\ \|f\|_{****} = \|\langle z - \bar{Q}\rangle^4 f(z)\|_\infty,\ n = 6, \end{cases} \tag{3.2.10}$$

where $\|f\|_\infty = \max_{z\in\Omega_\varepsilon}|f(z)|$ and $\overline{f} = |\Omega_\varepsilon|^{-1}\int_{\Omega_\varepsilon} f(z)\mathrm{d}z$ denotes the average of f in Ω_ε.

Before stating an existence result for ϕ in (3.2.5) and (3.2.9), we need the following lemma:

Lemma 3.2.1. *Let u and f satisfy*

$$-\Delta u = f,\ \frac{\partial u}{\partial\nu} = 0,\quad \bar{u} = \bar{f} = 0.$$

Then

$$|u(x)| \leq C\int_{\Omega_\varepsilon} \frac{|f(y)|}{|x-y|^{n-2}}\mathrm{d}y. \tag{3.2.11}$$

Proof. The proof is similar to [63, Lemma 3.1], we omit it here. □

As a consequence, we have

Corollary 3.2.1. *For $n = 4$, suppose u and f satisfy*

$$-\Delta u + \mu\varepsilon^2 u = f \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Then

$$\|u\|_* \leq C\|f\|_{**}. \tag{3.2.12}$$

For $n = 6$, suppose u and f satisfy

$$-\Delta u + c\mu\varepsilon^2 u = f \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon, \quad \overline{u} = \overline{f} = 0,$$

where c is an arbitrary constant. Then

$$\|u\|_{***} \leq C\|f\|_{****}. \tag{3.2.13}$$

Proof. For $n = 4$, integrating the equation yields $\bar{f} = \mu\varepsilon^2\bar{u}$, we may rewrite the original equation as

$$\Delta(u - \bar{u}) = \mu\varepsilon^2(u - \bar{u}) - (f - \bar{f}).$$

With the help of Lemma 3.2.1, we get

$$|u(y) - \bar{u}| \leq C\mu\varepsilon^2 \int_{\Omega_\varepsilon} \frac{|u(x) - \bar{u}|}{|x - y|^2} \mathrm{d}x + C \int_{\Omega_\varepsilon} \frac{|f(x) - \bar{f}|}{|x - y|^2} \mathrm{d}x.$$

Since

$$\langle y - \bar{Q}\rangle \int_{\mathbb{R}^4} \frac{1}{|x - y|^2} \langle x - \bar{Q}\rangle^{-3} \mathrm{d}x < \infty,$$

we obtain

$$\begin{aligned}
\|\langle y - \bar{Q}\rangle|u - \bar{u}|\|_\infty &\leq C\mu\varepsilon^2 \|\langle y - \bar{Q}\rangle^3|u - \bar{u}|\|_\infty + C\|\langle y - \bar{Q}\rangle^3|f - \bar{f}|\|_\infty \\
&\leq C\mu\|\langle y - \bar{Q}\rangle|u - \bar{u}|\|_\infty + C\|\langle y - \bar{Q}\rangle^3|f - \bar{f}|\|_\infty,
\end{aligned}$$

which gives

$$\|\langle y - \bar{Q}\rangle|u - \bar{u}|\|_\infty \leq C\|\langle y - \bar{Q}\rangle^3|f - \bar{f}|\|_\infty,$$

whence

$$\|\langle y-\bar{Q}\rangle u\|_\infty \le C\|\langle y-\bar{Q}\rangle\|_\infty|\bar{u}| + C\varepsilon^{-3}|\bar{f}| + \|\langle y-\bar{Q}\rangle^3 f\|_\infty \le C\|f\|_{**}.$$

Hence we finish the proof of the case $n=4$.

For $n=6$, by the help of Lemma 3.2.1,

$$|\langle y-\bar{Q}\rangle^2 u| \le C\int_{\Omega_\varepsilon}\frac{\langle y-\bar{Q}\rangle^2(|\mu\varepsilon^2 u|+|f|)}{|x-y|^4}\mathrm{d}x \le C(|\mu\ln\varepsilon|\|u\|_{***}+\|f\|_{****}),$$

where we used some similar estimates appeared in $n=4$. From the above inequality, we obtain $\|u\|_{***}\le\|f\|_{****}$. Hence we finish the proof. $\square$

We now state the main result of this section

Proposition 3.2.1. *There exist $\varepsilon_0>0$ and a constant $C>0$, independent of ε, Λ, $\bar{Q}$ satisfying (3.1.11) and independent of ε, η, Λ, $\bar{Q}$ satisfying (3.1.13), such that for all $0<\varepsilon<\varepsilon_0$ and all $h\in L^\infty(\Omega_\varepsilon)$, problem (3.2.5), (3.2.9) has a unique solution $\phi=L_\varepsilon(h)$ and the following estimates hold*

$$\begin{aligned}&\|L_\varepsilon(h)\|_*\le C\|h\|_{**},\ |c_i|\le C\|h\|_{**}\text{ for }0\le i\le 4,\\&\|L_\varepsilon(h)\|_{***}\le C\|h\|_{****},\ |d_i|\le C\|h\|_{****}\text{ for }0\le i\le 6.\end{aligned}\tag{3.2.14}$$

Moreover, the map $L_\varepsilon(h)$ is C^1 with respect to Λ, $\bar{Q}$ of the L^∞_*-norm in $n=4$ and with respect to Λ, $\bar{Q}$, η of the L^∞_{***}-norm in $n=6$, i.e.,

$$\|D_{(\Lambda,\bar{Q})}L_\varepsilon(h)\|_*\le C\|h\|_{**}\text{ in }n=4,\ \|D_{(\eta,\Lambda,\bar{Q})}L_\varepsilon(h)\|_{***}\le C\varepsilon^{-1}\|h\|_{****}\text{ in }n=6.\tag{3.2.15}$$

The argument goes the same as the Proposition 3.1 in [63], and we list the proof here. First, we need the following lemma

Lemma 3.2.2. *For $n=4$, assuming that ϕ_ε solves (3.2.5) for $h=h_\varepsilon$. If $\|h_\varepsilon\|_{**}$ goes to zero as ε goes to zero, so does $\|\phi_\varepsilon\|_*$. While for $n=6$, assuming that ϕ_ε solves (3.2.9) for $h=h_\varepsilon$. If $\|h_\varepsilon\|_{****}$ goes to zero as ε goes to zero, so does $\|\phi_\varepsilon\|_{***}$.*

Proof. We prove this lemma by contradiction and first consider $n = 4$. Assuming $\|\phi_\varepsilon\|_* = 1$. Multiplying the first equation in (3.2.5) by Y_j and integrating in Ω_ε we find

$$\sum_i c_i \langle Z_i, Y_j \rangle = \langle -\Delta Y_j + \mu\varepsilon^2 Y_j - 24W^2 Y_j, \phi_\varepsilon \rangle - \langle h_\varepsilon, Y_j \rangle.$$

We can easily get the following equalities from the definition of Z_i, Y_j

$$\begin{aligned} &\langle Z_0, Y_0 \rangle = \|Y_0\|_\varepsilon^2 = \gamma_0 + o(1), \\ &\langle Z_i, Y_i \rangle = \|Y_i\|_\varepsilon^2 = \gamma_1 + o(1), \ 1 \le i \le 4, \end{aligned} \tag{3.2.16}$$

where γ_0, γ_1 are strictly positive constants, and

$$\langle Z_i, Y_j \rangle = o(1), \ i \neq j. \tag{3.2.17}$$

On the other hand, in view of the definition of Y_j and W, straightforward computations yield

$$\langle -\Delta Y_j + \mu\varepsilon^2 Y_j - 24W^2 Y_j, \phi_\varepsilon \rangle = o(\|\phi_\varepsilon\|_*)$$

and

$$\langle h_\varepsilon, Y_j \rangle = O(\|h_\varepsilon\|_{**}).$$

Consequently, inverting the quasi diagonal linear system solved by the c_i's we find

$$c_i = O(\|h_\varepsilon\|_{**}) + o(\|\phi_\varepsilon\|_*). \tag{3.2.18}$$

In particular, $c_i = o(1)$ as ε goes to zero.

Since $\|\phi_\varepsilon\|_* = 1$, elliptic theory shows that along some subsequence, the functions $\phi_{\varepsilon,0} = \phi_\varepsilon(y - \bar{Q})$ converge uniformly in any compact subset of $\mathbb{R}^4$ to a nontrivial solution of

$$-\Delta\phi_0 = 24U_{\Lambda,0}^2\phi_0.$$

A bootstrap argument (see e.g. Proposition 2.2 of [68]) implies $|\phi_0(y)| \le$

$C(1+|y|)^{-2}$. As consequence, ϕ_0 can be written as

$$\phi_0 = \alpha_0 \frac{\partial U_{\Lambda,0}}{\partial \Lambda} + \sum_i \alpha_i \frac{\partial U_{\Lambda,0}}{\partial y_i},$$

(see [62]). On the other hand, equalities $\langle Z_i, \phi_\varepsilon \rangle = 0$ yield

$$\int_{\mathbb{R}^4} -\Delta \frac{\partial U_{\Lambda,0}}{\partial \Lambda} \phi_0 = \int_{\mathbb{R}^4} U_{\Lambda,0}^2 \frac{\partial U_{\Lambda,0}}{\partial \Lambda} \phi_0 = 0,$$
$$\int_{\mathbb{R}^4} -\Delta \frac{\partial U_{\Lambda,0}}{\partial y_i} \phi_0 = \int_{\mathbb{R}^4} U_{\Lambda,0}^2 \frac{\partial U_{\Lambda,0}}{\partial y_i} \phi_0 = 0,\ 1 \le i \le 4.$$

As we also have

$$\int_{\mathbb{R}^4} \left| \nabla \frac{\partial U_{\Lambda,0}}{\partial \Lambda} \right|^2 = \gamma_0 > 0,\ \int_{\mathbb{R}^4} \left| \nabla \frac{\partial U_{\Lambda,0}}{\partial y_i} \right|^2 = \gamma_1 > 0,\ 1 \le i \le 4,$$

and

$$\int_{\mathbb{R}^4} \nabla \frac{\partial U_{\Lambda,0}}{\partial \Lambda} \nabla \frac{\partial U_{\Lambda,0}}{\partial y_i} = \int_{\mathbb{R}^4} \nabla \frac{\partial U_{\Lambda,0}}{\partial y_i} \nabla \frac{\partial U_{\Lambda,0}}{\partial y_j} = 0,\ i \ne j,$$

the $\alpha_i' s$ solve a homogeneous quasi diagonal linear system, yielding $\alpha_i = 0, 0 \le i \le 4$, and $\phi_0 = 0$. So $\phi_\varepsilon(z - \bar{Q}) \to 0$ in $C^1_{loc}(\Omega_\varepsilon)$. Next, we will show $\|\phi_\varepsilon\|_* = o(1)$ by using the equation (3.2.5).

Using (3.2.5) and Corollary 3.2.1, we have

$$\|\phi_\varepsilon\|_* \le C(\|W^2 \phi_\varepsilon\|_{**} + \|h\|_{**} + \sum_i |c_i| \|Z_i\|_{**}). \tag{3.2.19}$$

Then we estimate the right hand side of (3.2.19) term by term. By the help of (3.1.16), we deduce that

$$|\langle z - \bar{Q} \rangle^3 W^2 \phi_\varepsilon| \le C(\varepsilon^4 (-\ln \varepsilon) \langle z - \bar{Q} \rangle^2 \|\phi_\varepsilon\|_* + \langle z - \bar{Q} \rangle^{-1} |\phi_\varepsilon|). \tag{3.2.20}$$

Since $\|\phi_\varepsilon\|_* = 1$, the first term on the right hand side of (3.2.20) is dominated by $\varepsilon^2(-\ln \varepsilon)$. The last term goes uniformly to zero in any ball $B_R(\bar{Q})$, and is dominated by $\langle z - \bar{Q} \rangle^{-2} \|\phi_\varepsilon\|_* = \langle z - \bar{Q} \rangle^{-2}$, which, through the choice of

R, can be made as small as possible in $\Omega_\varepsilon \backslash B_R(\bar{Q})$. Consequently,

$$|\langle z-\bar{Q}\rangle^3 W^2 \phi_\varepsilon| = o(1) \tag{3.2.21}$$

as ε goes to zero, uniformly in Ω_ε. On the other hand, we can also get

$$\begin{aligned}
\varepsilon^{-3}(-\ln\varepsilon)^{\frac{1}{2}}\overline{W^2\phi_\varepsilon} &\leq C\varepsilon(-\ln\varepsilon)^{\frac{1}{2}}\int_{\Omega_\varepsilon}\Big(\langle z-\bar{Q}\rangle^{-4}+\varepsilon^4(-\ln\varepsilon)\Big)|\phi_\varepsilon| \\
&\leq C\varepsilon(-\ln\varepsilon)^{\frac{1}{2}}\int_{\Omega_\varepsilon}\Big(\langle z-\bar{Q}\rangle^{-5}+\varepsilon^4(-\ln\varepsilon)\langle z-\bar{Q}\rangle^{-1}\Big)\|\phi_\varepsilon\|_* \\
&= o(1).
\end{aligned}$$

Finally, we obtain

$$\|W^2\phi_\varepsilon\|_{**} = o(1).$$

In view of the formula (3.2.4), we have

$$\langle z-\bar{Q}\rangle^3|Z_i| \leq C\Big(\langle z-\bar{Q}\rangle^3\varepsilon^4+\langle z-\bar{Q}\rangle^{-3}\Big) = O(1).$$

and

$$\varepsilon^{-3}(-\ln\varepsilon)^{\frac{1}{2}}\overline{Z_i} \leq C\varepsilon(-\ln\varepsilon)^{\frac{1}{2}}\int_{\Omega_\varepsilon}\Big|\langle z-\bar{Q}\rangle^{-6}+\varepsilon^4\Big|\mathrm{d}x = o(1).$$

Hence, $\|Z_i\|_{**} = O(1)$. Therefore, we have

$$\|\phi_\varepsilon\|_* \leq C\Big(\|W^2\phi_\varepsilon\|_{**}+\|h\|_{**}+\sum_i |c_i|\|Z_i\|_{**}\Big) = o(1), \tag{3.2.22}$$

which contradicts our assumption that $\|\phi_\varepsilon\|_* = 1$.

For $n=6$. We still assume that $\|\phi_\varepsilon\|_{***} = 1$. Using the similar arguments in previous case, we obtain the following

$$\begin{aligned}
&d_i = O(\|h\|_{****}) + o(\|\phi\|_{***}) \text{ for } 0 \leq i \leq 6, \\
&d_7 = O(\varepsilon^{-2}\|h\|_{****}) + O(\varepsilon^{-1}\|\phi_\varepsilon\|_{***})
\end{aligned} \tag{3.2.23}$$

and $\phi_\varepsilon(z-\bar{Q}) \to 0$ in $C^1_{loc}(\Omega_\varepsilon)$. Then, we will show $\|\phi_\varepsilon\|_{***} = o(1)$ by using the equation (3.2.9). At first, we write the equation (3.2.9) into the following

$$-\Delta\phi_\varepsilon + \mu\varepsilon^2(1-48\eta)\phi_\varepsilon = h + \sum_i d_i Z_i + 48U\phi_\varepsilon + 48\varepsilon^3\hat{U}\phi_\varepsilon. \tag{3.2.24}$$

Using Corollary 3.2.1 again, we have

$$\|\phi_\varepsilon\|_{***} \le C(\|(U+\varepsilon^3\hat{U})\phi_\varepsilon\|_{****} + \|h\|_{****} + \sum_i |d_i|\|Z_i\|_{****}). \tag{3.2.25}$$

From the formula of U and $\hat{U}$, it is not difficult to show

$$U + \varepsilon^3\hat{U} \le C\langle z - \bar{Q}\rangle^{-4}.$$

Similar to the case $n = 4$, we could show $\|\langle z-\bar{Q}\rangle^{-4}\phi_\varepsilon\|_{****} = o(1)$,

$$\|Z_i\|_{****} = O(1),\ 0 \le i \le 6 \text{ and } \|Z_7\|_{****} = O(\varepsilon^2).$$

Therefore, by the above facts and (3.2.23), we conclude

$$\|\phi_\varepsilon\|_{***} \le o(1) + C\|h\|_{****} + o(1)\|\phi_\varepsilon\|_{***} = o(1)$$

which contradicts the previous assumption that $\|\phi_\varepsilon\|_{***} = 1$. Hence, we finish the proof. □

Proof of Proposition 3.2.1. Since the proof of the case $n = 4$ and $n = 6$ are almost the same, we only give the proof for the former one. We set

$$H = \{\phi \in H^1(\Omega_\varepsilon) \mid \langle Z_i, \phi\rangle = 0,\ 0 \le i \le 4\},$$

equipped with the scalar product $(\cdot,\cdot)_\varepsilon$. Problem (3.2.5) is equivalent to finding $\phi \in H$ such that

$$(\phi,\theta)_\varepsilon = \langle 24W^2\phi + h, \theta\rangle, \quad \forall\theta \in H,$$

that is

$$\phi = T_\varepsilon(\phi) + \tilde{h}, \tag{3.2.26}$$

where $\tilde{h}$ depends on h linearly, and T_ε is a compact operator in H. Fredholm's alternative ensures the existence of a unique solution, provided that the kernel of $Id - T_\varepsilon$ is reduced to 0. We notice that any $\phi_\varepsilon \in \mathrm{Ker}(Id - T_\varepsilon)$ solves (3.2.5) with $h = 0$. Thus, we deduce from Lemma 3.2.2 that $\|\phi_\varepsilon\|_* = o(1)$ as ε goes to zero. As $\mathrm{Ker}(Id - T_\varepsilon)$ is a vector space and is $\{0\}$. The inequalities (3.2.14) follows from Lemma 3.2.2 and (3.2.18). This completes the proof of the first part of Proposition 3.2.1.

The smoothness of L_ε with respect to Λ and $\bar{Q}$ is a consequence of the smoothness of T_ε and $\tilde{h}$, which occur in the implicit definition (3.2.26) of $\phi \equiv L_\varepsilon(h)$, with respect to these variables. Inequality (3.2.15) is obtained by differentiating (3.2.5), writing the derivatives of ϕ with respect Λ and $\bar{Q}$ as linear combinations of the Z_i's and an orthogonal part, then we estimate each term by using the first part of the proposition. One can see [20],[34] for detailed computations. □

3.3 Finite Dimensional Reduction: A Nonlinear Problem

In this section, we turn our attention to the nonlinear problem, which we solve in the finite-dimensional subspace orthogonal to the Z_i. Let $S_\varepsilon[u]$ be as defined in Lemma 3.1.1. Then (1.2.13) is equivalent to

$$S_\varepsilon[u] = 0 \text{ in } \Omega_\varepsilon, \quad u_+ \neq 0, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon. \tag{3.3.1}$$

Indeed, if u satisfies (3.3.1), the Maximal Principle ensures that $u > 0$ in Ω_ε. Observing that

$$S_\varepsilon[W+\phi] = -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - n(n-2)(W+\phi)^{\frac{n+2}{n-2}}$$

may be written as

$$S_\varepsilon[W+\phi] = -\Delta\phi + \mu\varepsilon^2\phi - n(n+2)W^{\frac{4}{n-2}}\phi + R^\varepsilon - n(n-2)N_\varepsilon(\phi) \quad (3.3.2)$$

with

$$N_\varepsilon(\phi) = (W+\phi)^{\frac{n+2}{n-2}} - W^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}W^{\frac{4}{n-2}}\phi \quad (3.3.3)$$

and

$$R^\varepsilon = S_\varepsilon[W] = -\Delta W + \mu\varepsilon^2 W - n(n-2)W^{\frac{n+2}{n-2}}. \quad (3.3.4)$$

From Lemma 3.1.1 we get

$$\begin{cases} \|R^\varepsilon\|_{**} \le C\varepsilon\Lambda + \varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}},\ \|D_{(\Lambda,\bar{Q})}R^\varepsilon\|_{**} \le C\varepsilon, & n=4, \\ \|R^\varepsilon\|_{****} \le C\varepsilon^{2\frac{2}{3}},\ \|D_{(\Lambda,\bar{Q},\eta)}R^\varepsilon\|_{****} \le C\varepsilon^2, & n=6. \end{cases} \quad (3.3.5)$$

We now consider the following nonlinear problem: finding ϕ such that, for some numbers c_i,

$$\begin{cases} -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - 8(W+\phi)^3 = \sum_i c_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi\rangle = 0, & 0 \le i \le 4 \end{cases} \quad (3.3.6)$$

for $n=4$, and finding ϕ such that, for some numbers d_i,

$$\begin{cases} -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - 24(W+\phi)^2 = \sum_i d_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi\rangle = 0, & 0 \le i \le 7 \end{cases} \quad (3.3.7)$$

for $n=6$. The first equation in (3.3.6) and (3.3.7) reads

$$\begin{aligned} -\Delta\phi + \mu\varepsilon^2\phi - 24W^2\phi &= 8N_\varepsilon(\phi) - R^\varepsilon + \sum_i c_i Z_i, \\ -\Delta\phi + \mu\varepsilon^2\phi - 48W\phi &= 24N_\varepsilon(\phi) - R^\varepsilon + \sum_i d_i Z_i. \end{aligned} \quad (3.3.8)$$

In order to employ the contraction mapping theorem to prove that (3.3.6) and (3.3.7) are uniquely solvable in the set where $\|\phi\|_*$ and $\|\phi\|_{***}$ are small respectively, we need to estimate N_ε in the following lemma.

Lemma 3.3.1. *There exists $\varepsilon_1 > 0$, independent of $\Lambda, \bar{Q}$, and C independent of $\varepsilon, \Lambda, \bar{Q}$ such that for $\varepsilon \le \varepsilon_1$ and*

$$\|\phi\|_* \le C\varepsilon\Lambda \text{ for } n = 4, \quad \|\phi\|_{***} \le C\varepsilon^{2\frac{2}{3}} \text{ for } n = 6.$$

Then,

$$\|N_\varepsilon(\phi)\|_{**} \le C\varepsilon\Lambda\|\phi\|_* \text{ for } n = 4, \quad \|N_\varepsilon(\phi)\|_{****} \le C\varepsilon\|\phi\|_{***} \text{ for } n = 6. \tag{3.3.9}$$

For

$$\|\phi_i\|_* \le C\varepsilon\Lambda \text{ for } n = 4, \quad \|\phi_i\|_{***} \le C\varepsilon^{2\frac{2}{3}} \text{ for } n = 6, \qquad i = 1, 2.$$

Then,

$$\begin{aligned} &\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{**} \le C\varepsilon\Lambda\|\phi_1 - \phi_2\|_* \text{ for } n = 4, \\ &\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{****} \le C\varepsilon\|\phi_1 - \phi_2\|_{***} \text{ for } n = 6. \end{aligned} \tag{3.3.10}$$

Proof. Since the proof of these two cases are similar, we only consider $n = 4$ here. From (3.3.3), we see

$$|N_\varepsilon(\phi)| \le C(W\phi^2 + |\phi|^3). \tag{3.3.11}$$

Using (3.1.16), we gain

$$\varepsilon^{-3}(-\ln\varepsilon)^{\frac{1}{2}}\overline{W\phi^2 + |\phi|^3} = \varepsilon(-\ln\varepsilon)^{\frac{1}{2}} \int_{\Omega_\varepsilon} (W\phi^2 + |\phi|^3),$$

where the integration term on the right hand side of the above equality can be estimated as

$$\begin{aligned}|W\phi^2+|\phi|^3| \leq & C(((\langle z-\bar{Q}\rangle^{-2}+\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}})|\phi|^2+|\phi|^3) \\ \leq & C(\langle z-\bar{Q}\rangle^{-4}+\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}\langle z-\bar{Q}\rangle^{-2})\|\phi\|_*^2+\langle z-\bar{Q}\rangle^{-3}\|\phi\|_*^3 \\ \leq & C\Big(\big(\varepsilon\langle z-\bar{Q}\rangle^{-4}+\varepsilon^3(-\ln\varepsilon)^{\frac{1}{2}}\langle z-\bar{Q}\rangle^{-2}\big)\Lambda\Big)\|\phi\|_*.\end{aligned}$$

As a consequence,

$$\varepsilon^{-3}(-\ln\varepsilon)^{\frac{1}{2}}\overline{W\phi^2+|\phi|^3}\leq C\varepsilon^2(-\ln\varepsilon)^{\frac{3}{2}}\Lambda\|\phi\|_*\leq C\varepsilon\Lambda\|\phi\|_*.$$

On the other hand,

$$\|\langle z-\bar{Q}\rangle^3(W\phi^2+|\phi|^3)\|_\infty\leq C\varepsilon\Lambda\|\phi\|_*.$$

and (3.3.9) follows. Concerning (3.3.10), we write

$$N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)=\partial_\vartheta N_\varepsilon(\vartheta)(\phi_1-\phi_2)$$

for some $\vartheta=x\phi_1+(1-x)\phi_2,\ x\in[0,1]$. From

$$\partial_\vartheta N_\varepsilon(\vartheta)=3[(W+\vartheta)^2-W^2],$$

we deduce that

$$\partial_\vartheta N_\varepsilon(\vartheta)\leq C(|W||\vartheta|+\vartheta^2) \tag{3.3.12}$$

and the proof of (3.3.10) is similar to the previous one. □

Proposition 3.3.1. *For the case $n=4$, there exists C, independent of ε and Λ, Q satisfying (3.1.11), such that for small ε problem (3.3.6) has a unique solution $\phi=\phi(\Lambda,\bar{Q},\varepsilon)$ with*

$$\|\phi\|_*\leq C\varepsilon\Lambda. \tag{3.3.13}$$

Moreover, $(\Lambda, \bar{Q}) \to \phi(\Lambda, \bar{Q}, \varepsilon)$ *is* C^1 *with respect to the* $*$*-norm, and*

$$\|D_{(\Lambda,\bar{Q})}\phi\|_{*} \leq C\varepsilon. \tag{3.3.14}$$

For the case $n = 6$, *there exists* C, *independent of* ε *and* Λ, η, Q *satisfying (3.1.13), such that for small* ε *problem (3.3.7) has a unique solution* $\phi = \phi(\Lambda, \eta, \bar{Q}, \varepsilon)$ *with*

$$\|\phi\|_{***} \leq C\varepsilon^{\frac{8}{3}}. \tag{3.3.15}$$

Moreover, $(\Lambda, \eta, \bar{Q}) \to \phi(\Lambda, \eta, \bar{Q}, \varepsilon)$ *is* C^1 *with respect to the* $***$*-norm, and*

$$\|D_{(\Lambda,\eta,\bar{Q})}\phi\|_{***} \leq C\varepsilon^{\frac{5}{3}}. \tag{3.3.16}$$

Proof. We only give the proof of $n = 4$, the other case can be argued similarly. In the same spirit of [20], we consider the map A_ε from $\mathcal{F}$=$\{\phi \in H^1(\Omega_\varepsilon) | \|\phi\|_* \leq C' \varepsilon\Lambda\}$ to $H^1(\Omega_\varepsilon)$ defined as

$$A_\varepsilon(\phi) = L_\varepsilon(8N_\varepsilon(\phi) + R^\varepsilon).$$

Here C' is a large number, to be determined later, and L_ε is given by Proposition 3.2.1. We note that finding a solution ϕ to problem (3.3.6) is equivalent to finding a fixed point of A_ε. On the one hand, we have for $\phi \in \mathcal{F}$. Then, using (3.3.5), Proposition 3.2.1 and Lemma 3.3.1,

$$\begin{aligned}\|A_\varepsilon(\phi)\|_* &\leq 8\|L_\varepsilon(N_\varepsilon(\phi))||_* + \|L_\varepsilon(R^\varepsilon)\|_* \leq C_1(\|N_\varepsilon(\phi)\|_{**} + \varepsilon)\\ &\leq C_2 C' \varepsilon^2\Lambda + C_1\varepsilon\Lambda \leq C'\varepsilon\Lambda\end{aligned}$$

for $C' = 2C_1$ and ε small enough, which implies that A_ε sends $\mathcal{F}$ into itself. On the other hand, A_ε is a contraction. Indeed, for ϕ_1 and ϕ_2 in $\mathcal{F}$, we write

$$\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* \leq C\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{**} \leq C\varepsilon\Lambda\|\phi_1 - \phi_2\|_* \leq \frac{1}{2}\|\phi_1 - \phi_2\|_*$$

for ε small enough. The contraction Mapping Theorem implies that A_ε has a unique fixed point in $\mathcal{F}$, that is, problem (3.3.6) has a unique solution ϕ

such that $\|\phi\|_* \leq C^{'}\varepsilon\Lambda$.

In order to prove that $(\Lambda, \bar{Q}) \to \phi(\Lambda, \bar{Q})$ is C^1, we remark that if we set for $\psi \in F$,

$$B(\Lambda, \bar{Q}, \psi) \equiv \psi - L_\varepsilon(8N_\varepsilon(\psi) + R^\varepsilon),$$

then ϕ is defined as

$$B(\Lambda, \bar{Q}, \phi) = 0. \tag{3.3.17}$$

We have

$$\partial_\psi B(\Lambda, \bar{Q}, \psi)[\theta] = \theta - 8L_\varepsilon(\theta(\partial_\psi N_\varepsilon)(\psi)).$$

Using Proposition 3.2.1 and (3.3.12) we write

$$\begin{aligned}\|L_\varepsilon(\theta(\partial_\psi N_\varepsilon)(\psi))\|_* \leq C\|\theta(\partial_\psi N_\varepsilon)(\psi)\|_{**} &\leq \|\langle z - \bar{Q}\rangle^{-1}(\partial_\psi N_\varepsilon)(\psi)\|_{**}\|\theta\|_* \\ &\leq C\|\langle z - \bar{Q}\rangle^{-1}(W_+|\psi| + |\psi|^2)\|_{**}\|\theta\|_*.\end{aligned}$$

Using (3.1.16), (3.2.10) and $\psi \in \mathcal{F}$, we obtain

$$\|L_\varepsilon(\theta(\partial_\psi N_\varepsilon)(\psi))\|_* \leq \varepsilon\|\theta\|_*.$$

Consequently, $\partial_\psi B(\Lambda, \bar{Q}, \phi)$ is invertible with uniformly bounded inverse. Then the fact that $(\Lambda, \bar{Q}) \mapsto \phi(\Lambda, \bar{Q})$ is C^1 follows from the fact that $(\Lambda, \bar{Q}, \psi) \mapsto L_\varepsilon(N_\varepsilon(\psi))$ is C^1 and the implicit function theorem.

Finally, we consider (3.3.14). Differentiating (3.3.17) with respect to Λ, we find

$$\partial_\Lambda\phi = (\partial_\psi B(\Lambda, \xi, \phi))^{-1}((\partial_\Lambda L_\varepsilon)(N_\varepsilon(\phi)) + L_\varepsilon((\partial_\Lambda N_\varepsilon)(\phi)) + L_\varepsilon(\partial_\Lambda R^\varepsilon)).$$

Then by Proposition 3.2.1,

$$\|\partial_\Lambda\phi\|_* \leq C(\|N_\varepsilon(\phi)\|_{**} + \|(\partial_\Lambda N_\varepsilon)(\phi)\|_{**} + \|\partial_\Lambda R^\varepsilon\|_{**}).$$

From Lemma 3.3.1 and (3.3.13), we know that $\|N_\varepsilon(\phi)\|_{**} \leq C\varepsilon^2$. Concerning the next term, we notice that according to the definition of N_ε,

$$|\partial_\Lambda N_\varepsilon(\phi)| = 3\phi^2|\partial_\Lambda W|.$$

Recalling that

$$|D_\Lambda W(z)| \leq C(\langle z - \bar{Q}\rangle^{-2} + \varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}),$$

which gives

$$\|\partial_\Lambda N_\varepsilon(\phi)\|_{**} \leq C\varepsilon.$$

Finally, using (3.3.5), we obtain

$$\|\partial_\Lambda \phi\|_* \leq C\varepsilon.$$

The derivative of ϕ with respect to $\bar{Q}$ can be estimated in the same way. This concludes the proof. □

3.4 Finite Dimensional Reduction: Reduced Energy

Let us define the reduced energy functional as

$$I_\varepsilon(\Lambda, Q) \equiv J_\varepsilon[W_{\Lambda,\bar{Q}} + \phi_{\varepsilon,\Lambda,\bar{Q}}] \tag{3.4.1}$$

for $n = 4$ and

$$I_\varepsilon(\Lambda, \eta, Q) \equiv J_\varepsilon[W_{\Lambda,\eta,\bar{Q}} + \phi_{\varepsilon,\Lambda,\eta,\bar{Q}}] \tag{3.4.2}$$

for $n = 6$. Then, We have

Proposition 3.4.1. *The function $u = W_{\Lambda,\bar{Q}} + \phi_{\varepsilon,\Lambda,\bar{Q}}$ is a solution to problem (1.2.13) for $n = 4$ if and only if $(\Lambda, \bar{Q})$ is a critical point of I_ε. The function $u = W_{\Lambda,\eta,\bar{Q}} + \phi_{\varepsilon,\Lambda,\eta,\bar{Q}}$ is a solution to problem (1.2.13) for $n = 6$ if and only if $(\Lambda, \eta, \bar{Q})$ is a critical point of I_ε.*

Proof. Here we only give the proof for the case $n = 6$. We notice that $u = W + \phi$ being a solution of (1.2.13) is equivalent to being a critical point of J_ε, which is also equivalent to the vanish of the d_i's in (3.3.7) or, in view of

$$\begin{aligned} &\langle Z_0, Y_0\rangle = \|Y_0\|_\varepsilon^2 = \gamma_0 + o(1), \\ &\langle Z_i, Y_i\rangle = \|Y_i\|_\varepsilon^2 = \gamma_1 + o(1),\ 1 \le i \le 6, \\ &\langle Z_7, Y_7\rangle = \|Y_7\|_\varepsilon^2 = \gamma_2 \varepsilon^3, \end{aligned} \tag{3.4.3}$$

where $\gamma_0, \gamma_1, \gamma_2$ are strictly positive constants, and

$$\langle Z_i, Y_j\rangle = o(1), i \neq j, 0 \le i, j \le 6, \quad \langle Z_i, Y_j\rangle = O(\varepsilon^3), i \neq j, i = 7 \text{ or } j = 7. \tag{3.4.4}$$

We have

$$J'_\varepsilon[W+\phi][Y_i] = 0, \quad 0 \le i \le 7. \tag{3.4.5}$$

On the other hand, we deduce from (3.4.2) that $I'_\varepsilon(\Lambda, \eta, Q) = 0$ is equivalent to the cancellation of $J'_\varepsilon(W+\phi)$ applied to the derivative of $W+\phi$ with respect to Λ, η and $\bar{Q}$. By the definition of Y_i's and Proposition 3.3.1, we have

$$\frac{\partial(W+\phi)}{\partial \Lambda} = Y_0 + y_0,\ \frac{\partial(W+\phi)}{\partial \bar{Q}_i} = Y_i + y_i,\ 1 \le i \le 6,\ \frac{\partial(W+\phi)}{\partial \eta} = Y_7 + y_7$$

with $\|y_i\|_{***} = O(\varepsilon^2),\ 0 \le i \le 7$. We write

$$-\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - 24(W+\phi)^2 = \sum_{j=0}^{7} d_j Z_j$$

and denote $a_{ij} = \langle y_i, Z_j\rangle$. It turns out that $I'_\varepsilon(\Lambda, \eta, Q) = 0$ is equivalent, since $J'_\varepsilon[W+\phi][\theta] = 0$ for $\langle \theta, Z_i\rangle = (\theta, Y_i)_\varepsilon = 0,\ 0 \le i \le 7$, to

$$([b_{ij}] + [a_{ij}])[d_j] = 0,$$

where $b_{ij} = \langle Y_i, Z_j\rangle$. Using the estimate $\|y_i\|_{***} = O(\varepsilon^2)$ and the expression of $Z_i, Y_i, 0 \leq i \leq 7$, we directly obtain

$$\begin{aligned}
&b_{00} = \gamma_0 + o(1), \quad b_{ii} = \gamma_1 + o(1) \text{ for } 1 \leq i \leq 6, \quad b_{77} = \gamma_2 \varepsilon^3, \\
&b_{ij} = o(1) \text{ for } 0 \leq i \neq j \leq 6, \quad b_{ij} = O(\varepsilon^3) \text{ for } i = 7 \text{ or } j = 7, i \neq j, \\
&a_{ij} = O(\varepsilon^2) \text{ for } 0 \leq i \leq 7, 0 \leq j \leq 6, \quad a_{i7} = O(\varepsilon^4) \text{ for } 0 \leq i \leq 7.
\end{aligned}$$

Then it is easy to see the matrix $[b_{ij}+a_{ij}]$ is invertible by the above estimates of each components, hence $d_i = 0$. We see that $I'_\varepsilon(\Lambda, \eta, Q) = 0$ means exactly that (3.4.5) is satisfied. $\square$

By Proposition 3.4.1, it remains to find critical points of I_ε. First, we establish an expansion of I_ε.

Proposition 3.4.2. *In the case $n = 4$, for ε sufficiently small, we have*

$$I_\varepsilon(\Lambda, \eta, Q) = J_\varepsilon[W] + \varepsilon^2 \sigma_{\varepsilon,4}(\Lambda, Q) \tag{3.4.6}$$

where $\sigma_{\varepsilon,4} = o(1)$ and $D_\Lambda(\sigma_{\varepsilon,4}) = o(1)$ as ε goes to 0, uniformly with respect to Λ, Q satisfying (3.1.11).

In the case $n = 6$, for ε sufficiently small, we have

$$I_\varepsilon(\Lambda, \eta, Q) = J_\varepsilon[W] + \varepsilon^4 \sigma_{\varepsilon,6}(\Lambda, \eta, Q) \tag{3.4.7}$$

where $\sigma_{\varepsilon,6} = o(1)$ and $D_{\Lambda,\eta}(\sigma_{\varepsilon,6}) = o(1)$ as ε goes to 0, uniformly with respect to Λ, η, Q satisfying (3.1.13).

Proof. We only consider the case $n = 4$ here, the other case can be argued similarly with minor changes. In view of (3.4.1), a Taylor expansion and the fact that $J'_\varepsilon[W + \phi][\phi] = 0$ yield

$$\begin{aligned}
I_\varepsilon(\Lambda, Q) - J_\varepsilon[W] =& J_\varepsilon[W + \phi] - J_\varepsilon[W] = -\int_0^1 J''_\varepsilon(W + t\phi)[\phi, \phi](t)\mathrm{d}t \\
&= -\int_0^1 \Big(\int_{\Omega_\varepsilon} (|\nabla \phi|^2 + \mu \varepsilon^2 \phi^2 - 24(W + t\phi)\phi^2) \Big) t\mathrm{d}t,
\end{aligned}$$

whence

$$I_\varepsilon(\Lambda, Q) - J_\varepsilon[W]$$
$$= -\int_0^1 \Big(8\int_{\Omega_\varepsilon}\Big(N_\varepsilon(\phi)\phi + 3[W^2 - (W+t\phi)^2]\phi^2\Big)\Big)t\mathrm{d}t - \int_{\Omega_\varepsilon} R^\varepsilon\phi. \quad (3.4.8)$$

The first term on the right hand side of (3.4.8) can be estimated as

$$\Big|\int_{\Omega_\varepsilon} N_\varepsilon(\phi)\phi\Big| \le C\int_{\Omega_\varepsilon} |\phi|^4 + |W\phi^3| = O(\varepsilon^4 \ln\varepsilon).$$

Similarly, for the second term on the right hand side of (3.4.8), we obtain

$$\Big|\int_{\Omega_\varepsilon} [W - (W+t\phi)]\phi^2\Big| \le C\int_{\Omega_\varepsilon} |\phi|^4 + |W\phi^3| = O(\varepsilon^4 \ln\varepsilon).$$

Concerning the last one, by using

$$\begin{aligned}|R^\varepsilon|_* = |S_\varepsilon[W]| =& O\Big(\varepsilon^4(-\ln\varepsilon)\langle z-\bar{Q}\rangle^{-2} + \varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}\langle z-\bar{Q}\rangle^{-4}\Big)\\ &+ O(\Lambda)\Big(\frac{\varepsilon^4}{(-\ln\varepsilon)}\big|\ln\frac{1}{\varepsilon(1+|z-\bar{Q}|)}\big| + \frac{\varepsilon^4}{(-\ln\varepsilon)}\Big),\end{aligned}$$

and $\|\phi\|_* = O(\varepsilon\Lambda)$, we have

$$\Big|\int_{\Omega_\varepsilon} R^\varepsilon\phi\Big| = O\Big(\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}\Lambda^2 + \varepsilon^3(-\ln\varepsilon)^{\frac{1}{2}}\Big).$$

This concludes the proof of the first part of Proposition (3.4.6).

An estimate for the derivatives with respect to Λ is established exactly in the same way, differentiating the right side in (3.4.8) and estimating each term separately, using (3.3.3), (3.3.5) and Lemma 3.1.1. □

3.5 Proof Of Theorem 1.2.1

In this section, we prove the existence of critical points for $I_\varepsilon(\Lambda, Q)$ and $I_\varepsilon(\Lambda, \eta, Q)$, thereby proving Theorem 1.2.1 by Proposition 3.4.1. According to the Proposition 3.4.2 and Lemma 3.1.1, we set

$$K_\varepsilon(\Lambda, Q) = \frac{I_\varepsilon(\Lambda, Q) - 2\int_{\mathbb{R}^4} U^4}{(-\frac{\ln\varepsilon}{c_1})^{\frac{1}{2}}\varepsilon^2} \tag{3.5.1}$$

and

$$K_\varepsilon(\Lambda, \eta, Q) = \frac{I_\varepsilon(\Lambda, \eta, Q) - 4\int_{\mathbb{R}^6} U^3}{\varepsilon^3} \tag{3.5.2}$$

When $n = 4$, we have

$$\begin{aligned} K_\varepsilon(\Lambda, Q) =& \frac{1}{4}c_4\Lambda^2 \ln\frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon}) - \frac{c_4^2\Lambda^2}{2|\Omega|} + \frac{1}{2}c_4^2\Lambda^2 H(Q,Q)(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}} \\ &+ O\big(\frac{\Lambda^2}{-\ln\varepsilon} + \varepsilon\big), \end{aligned} \tag{3.5.3}$$

and when $n = 6$, we have

$$\begin{aligned} K_\varepsilon(\Lambda, \eta, Q) =& \Big(\frac{1}{2}\eta^2|\Omega| - c_6\Lambda^2\eta + \frac{1}{48}c_6\Lambda^2 - 8\eta^3|\Omega|\Big) + \frac{1}{2}c_6^2\Lambda^4 H(Q,Q)\varepsilon \\ &+ \frac{1}{2}(\eta - \frac{c_6\Lambda^2}{|\Omega|})\varepsilon\int_\Omega \frac{\Lambda^2}{|x-Q|^4} + o(\varepsilon). \end{aligned} \tag{3.5.4}$$

Next, we consider $K_\varepsilon(\Lambda, Q)$, and find its critical point with respect to Λ, Q, and the critical point of $K_\varepsilon(\Lambda, \eta, Q)$ with respect to the parameters Λ, η, Q respectively.

First, we consider $K_\varepsilon(\Lambda, Q)$ for $n = 4$. For the setting of the parameters Λ, Q, we see that Λ, Q are located in a compact set. As a consequence, we can obtain a maximal value of $K_\varepsilon(\Lambda, Q)$. Then, we claim that:

Claim: The maximal point of $K_\varepsilon(\Lambda, Q)$ with respect to Λ, Q can not happen on the boundary of the parameters.

If we can prove this claim, then we could obtain an interior critical point of $K_\varepsilon(\Lambda,Q)$. Before proving the claim, we first consider

$$F_\varepsilon(\Lambda)=\frac{1}{4}c_4\Lambda^2\ln\frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon})-\frac{c_4^2\Lambda^2}{2|\Omega|}.$$

Note that

$$\frac{\partial}{\partial\Lambda}[F_\varepsilon(\Lambda)]=\frac{1}{2}c_4\Lambda\ln\frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon})-\frac{1}{4}c_4\Lambda(\frac{c_1}{-\ln\varepsilon})-\frac{c_4^2\Lambda}{|\Omega|},$$

Choosing $c_1=\frac{2c_4}{|\Omega|}$, we could obtain that there exists

$$\Lambda^*=\exp(-\frac{1}{2})\in(\exp(-\frac{1}{2})\varepsilon^{\beta},\exp(-\frac{1}{2})\varepsilon^{-\beta})$$

with some proper fixed constant $\beta\in(0,\frac{1}{3})$, such that

$$\frac{\partial}{\partial\Lambda}F_\varepsilon\mid_{\Lambda=\Lambda^*}=0.$$

It can be also found that such Λ^* provides the maximal value of $F_\varepsilon(\Lambda)$ in $[\Lambda_{4,1},\Lambda_{4,2}]$, where $\Lambda_{4,1}=\exp(-\frac{1}{2})\varepsilon^{\beta},\Lambda_{4,2}=\exp(-\frac{1}{2})\varepsilon^{-\beta}$. In order to prove the claim, we need to take Λ into consideration for the expansion of the energy, going through the first part of the Appendix, we have

$$\begin{aligned}K_\varepsilon(\Lambda,Q)=\ &\frac{1}{4}c_4\Lambda^2\ln\frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon})-\frac{c_4^2\Lambda^2}{2|\Omega|}+\frac{1}{2}c_4^2\Lambda^2H(Q,Q)(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}\\&+O(\frac{\Lambda^2}{-\ln\varepsilon}+\varepsilon).\end{aligned}$$

Now, we go back to prove of the claim, choosing $\Lambda=\Lambda^*$ and $Q=p$. (Here p refers to the point where $H(Q,Q)$ obtain its maximal value, it is possible to find such a point. Indeed, we notice a fact $H(Q,Q)\to-\infty$ as $d(Q,\partial\Omega)\to0$ see [63] and references therein for a proof of this fact. Therefore we could find such p.)

First, we prove that the maximal value can not happen on $\partial\mathcal{M}_{\delta_4}$. We choose δ_4 such that $d_2<\max_{\partial\mathcal{M}_{\delta_4}}H<d_1$ for some proper constant d_2,d_1 sufficiently negative, then we fixed $\mathcal{M}_{\delta_4}$. It is easy to see that $K_\varepsilon(\Lambda,Q)<$

$K_\varepsilon(\Lambda, p)$, where Q lies on the boundary of $\mathcal{M}_{\delta_4}$ and $\Lambda \in (\Lambda_{4,1}, \Lambda_{4,2})$. For $\Lambda = \Lambda_{4,1}$ or $\Lambda_{4,2}$, we go to the arguments below. Therefore, we prove that the maximal point can not lie on the boundary of $\mathcal{M}_{\delta_4} \times [\Lambda_{4,1}, \Lambda_{4,2}]$.

Next, we show $K_\varepsilon(\Lambda^*, p) > K_\varepsilon(\Lambda_{4,2}, Q)$. It is easy to see that

$$F_\varepsilon[\Lambda_{4,2}] \leq c\varepsilon^{-2\beta},$$

where $c < 0$. Then we can find $c_1 < 0$ such that $K_\varepsilon(\Lambda_{4,2}, Q) \leq c_1\varepsilon^{-2\beta}$ for any $Q \in \mathcal{M}_{\delta_4}$, since the other terms compared to $\varepsilon^{-2\beta}$ are higher order term. On the other hand, for the choice of Λ^*, p, we see that $K_\varepsilon(\Lambda^*, p) = O(1)$. Therefore, we prove that $K_\varepsilon(\Lambda^*, p) > K_\varepsilon(\Lambda_{4,2}, Q)$ for any $Q \in \mathcal{M}_{\delta_4}$.

It remains to prove that the maximal value can not happen at $\Lambda = \Lambda_{4,1}$. We choose $\Lambda = \varepsilon^{\beta/2}, Q = p$, direct computation yields.

$$K_\varepsilon(\varepsilon^{\beta/2}, p) = \frac{\beta c_4^2 \varepsilon^\beta}{4|\Omega|}(1 + o(1)), \quad K_\varepsilon(\Lambda_{4,1}, Q) = \frac{\beta c_4^2 \varepsilon^{2\beta}}{2|\Omega|}(1 + o(1)).$$

It is to see $K_\varepsilon(\varepsilon^{\beta/2}, p) > K_\varepsilon(\Lambda_{4,1}, Q)$ for any $Q \in \mathcal{M}_{\delta_4}$ when ε is sufficiently small. Hence, we finish the proof of the claim. In other words, we could obtain an interior maximal point in $[\Lambda_{4,1}, \Lambda_{4,2}] \times \mathcal{M}_{\delta_4}$. Therefore, we show the existence of the critical points of $K_\varepsilon(\Lambda, Q)$ with respect to Λ, Q.

For $n = 6$. We set $\eta = \frac{1}{48} + a\varepsilon^{\frac{1}{3}}$, $\frac{c_6\Lambda^2}{|\Omega|} = \frac{1}{96} + b\varepsilon^{\frac{2}{3}}$, then

$$K_\varepsilon(a, b, Q) := K_\varepsilon(\Lambda, \eta, Q) = \frac{1}{6912}|\Omega| + \big[F(Q) - (8a^3 + ab)|\Omega|\big]\varepsilon + o(\varepsilon), \tag{3.5.5}$$

where

$$F(x) = \frac{|\Omega|}{18432}\Big(|\Omega|H(x, x) + \frac{1}{c_6}\int_\Omega \frac{1}{|x - y|^4}\mathrm{d}y\Big),$$

$-\eta_6 \leq a \leq \eta_6$ and $-\Lambda_6 \leq b \leq \Lambda_6$.

We set $C_0 = F(p_0)$, p_0 refers to the point where $F(x)$ obtains its maximal value. Indeed, we have $H(Q, Q) \to -\infty$ as $d(Q, \partial\Omega) \to 0$ and $I(x) = \int_\Omega \frac{1}{|x-y|^4}\mathrm{d}y$ is uniformly bounded in Ω. Hence, we can always find such point

p_0. Let us introduce another five constants $C_i, i = 1, 2, 3, 4, 5$, with $C_2 < C_1 < C_0$, $0 < C_3 < C_4 < \eta_6$ and $0 < C_3 < C_5 < \Lambda_6$, the value of these five constants will be determined later.

We set

$$\Sigma_0 = \Big\{ -C_4 \le a \le C_4, \ \ -C_5 \le b \le C_5, \ Q \in \mathcal{N}_{C_2} \Big\}, \tag{3.5.6}$$

where $\mathcal{N}_{C_i} = \{q : F(q) > C_i\}, i = 1, 2$ and δ_6 is chosen such that $\mathcal{N}_{C_2} \subset \mathcal{M}_{\delta_6}$.

We also define

$$\begin{aligned} B &= \{(a, b, Q) \mid (a, b) \in B_{C_3}(0), \ Q \in \overline{\mathcal{N}_{C_1}}\}, \\ B_0 &= \{(a, b) \mid (a, b) \in B_{C_3}(0)\} \times \partial \mathcal{N}_{C_1}, \end{aligned} \tag{3.5.7}$$

where $B_r(0) := \{0 \le a^2 + b^2 \le r\}$.

It is trivial to see that $B_0 \subset B \subset \Sigma_0$, B is compact. Let Γ be the class of continuous functions $\varphi : B \to \Sigma_0$ with the property that $\varphi(y) = y, y = (a, b, Q)$ for all $y \in B_0$. Define the min-max value c as

$$c = \min_{\varphi \in \Gamma} \max_{y \in B} K_\varepsilon(\varphi(y)).$$

We now show that c defines a critical value. To this end, we just have to verify the following conditions

(T1) $\max_{y \in B_0} K_\varepsilon(\varphi(y)) < c, \ \forall \varphi \in \Gamma$,

(T2) For all $y \in \partial \Sigma_0$ such that $K_\varepsilon(y) = c$, there exists a vector τ_y tangent to $\partial \Sigma_0$ at y such that

$$\partial_{\tau_y} K_\varepsilon(y) \neq 0.$$

Suppose (T1) and (T2) hold. Then standard deformation argument ensures that the min-max value c is a (topologically nontrivial) critical value for $K_\varepsilon(a, b, Q)$ in Σ_0. (Similar notion has been introduced in [21] for degenerate critical points of mean curvature.)

To check (T1) and (T2), we define $\varphi(y) = \varphi(a, b, Q) = (\varphi_a, \varphi_b, \varphi_Q)$ where $(\varphi_a, \varphi_b) \in [-C_4, C_4] \times [-C_5, C_5]$ and $\varphi_Q \in \mathcal{N}_{C_2}$.

For any $\varphi \in \Gamma$ and $Q \in \mathcal{N}_{C_2}$, the map $Q \to \varphi_Q(a,b,Q)$ is a continuous function from $\mathcal{N}_{C_1}$ to $\mathcal{N}_{C_2}$ such that $\varphi_Q(a,b,Q) = Q$ for $Q \in \partial\mathcal{N}_{C_1}$. Let $\mathcal{D}$ be the smallest ball which contain $\mathcal{N}_{C_1}$, we extend φ_Q to a continuous function $\tilde{\varphi}_Q$ from $\mathcal{D}$ to $\mathcal{D}$ where $\tilde{\varphi}(Q)$ is defined as follows:

$$\tilde{\varphi}_Q(x) = \varphi(x),\ x \in \mathcal{N}_{C_1}, \quad \tilde{\varphi}_Q(x) = Id,\ x \in \mathcal{D} \setminus \mathcal{N}_{C_1}.$$

Then we claim there exists $Q' \in \mathcal{D}$ such that $\tilde{\varphi}_Q(Q') = p_0$. Otherwise $\frac{\tilde{\varphi}_Q - p_0}{|\tilde{\varphi}_Q - p_0|}$ provides a continuous map from $\mathcal{D}$ to $\mathbb{S}^5$, which is impossible in algebraic topology. Hence, there exists $Q' \in \mathcal{D}$ such that $\tilde{\varphi}_Q(Q') = p_0$. By the definition of $\tilde{\varphi}$, we can further conclude $Q' \in \mathcal{N}_{C_1}$. Whence

$$\begin{aligned}\max_{y \in B} K_\varepsilon(\varphi(y)) \geq & K_\varepsilon(\varphi_a(a,b,Q'), \varphi_b(a,b,Q'), p_0) \\ \geq & \frac{1}{6912}|\Omega| + (C_0 - C_6|\Omega|)\varepsilon + o(\varepsilon), \qquad (3.5.8)\end{aligned}$$

where $C_6 = 8C_4^3 + C_4C_5$ which stands for the maximal value of $8a^3 + ab$ in $[-C_4, C_4] \times [-C_5, C_5]$. As a consequence

$$c \geq \frac{1}{6912}|\Omega| + (C_0 - C_6|\Omega|)\varepsilon + o(\varepsilon). \qquad (3.5.9)$$

For $(a,b,Q) \in B_0$, we have $F(\varphi_Q(a,b,Q)) = C_1$. So,

$$K_\varepsilon(a,b,Q) \leq \frac{1}{6912}|\Omega| + (C_1 + C_7|\Omega|)\varepsilon + o(\varepsilon), \qquad (3.5.10)$$

where $C_7 = \max_{(a,b) \in B_{C_3}(0)} 8a^3 + ab < 8C_3^3 + C_3^2$.

If we choose $C_0 - C_1 > 8C_4^3 + C_4C_5 + 8C_3^3 + C_3^2 > C_6 + C_7$, we have $\max_{y \in B_0} K_\varepsilon(\varphi(y)) < c$ holds. So (T1) is verified.

To verify (T2), we observe that

$$\partial\Sigma_0 =: \{a,b,Q \mid a = -C_4 \text{ or } a = C_4 \text{ or } b = -C_5 \text{ or } b = C_5 \text{ or } Q \in \partial\mathcal{N}_{C_2}\}.$$

Since C_4, C_5 are arbitrary, we choose $0 < 24C_4^2 < C_5$. Then on $a = -C_4$ or $a = C_4$, we choose $\tau_y = \frac{\partial}{\partial b}$, on $b = -C_5$ or $b = C_5$, we choose $\tau_y = \frac{\partial}{\partial a}$.

By our setting on C_4, C_5, we could show $\partial_{\tau_y} K_\varepsilon(y) \neq 0$. It only remains to consider the case $Q \in \partial\mathcal{N}_{C_2}$. If $Q \in \partial\mathcal{N}_{C_2}$, then

$$K_\varepsilon(a, b, Q) \leq \frac{1}{6912}|\Omega| + (C_2 + C_7|\Omega|)\varepsilon + o(\varepsilon), \tag{3.5.11}$$

which is obviously less than c for $C_2 < C_1$. So (T2) is also verified.

In conclusion, we proved that for ε sufficiently small, c is a critical value, i.e., a critical point $(a, b, Q) \in \Sigma_0$ of K_ε exists. Which means K_ε indeed has critical points respect to Λ, η, Q in (3.1.13).

Proof of Theorem 1.2.1 completed. For $n = 4$, we proved that for ε small enough, I_ε has a critical point $(\Lambda^\varepsilon, Q^\varepsilon)$. Let $u_\varepsilon = W_{\Lambda^\varepsilon, \bar{Q}^\varepsilon, \varepsilon}$. Then u_ε is a nontrivial solution to problem (1.2.13) for $n = 4$. The strong maximal principle shows $u_\varepsilon > 0$ in Ω_ε. Let $u_\mu = \varepsilon^{-1} u_\varepsilon(x/\varepsilon)$ and this is a nontrivial solution of (1.2.12) for $n = 4$. Thus, we get Theorem 1.2.1 for $n = 4$.

For $n = 6$, we proved that for ε small enough, I_ε has a critical point $(\Lambda^\varepsilon, \eta^\varepsilon, Q^\varepsilon)$. Let $u_\varepsilon = W_{\Lambda^\varepsilon, \eta^\varepsilon, \bar{Q}^\varepsilon, \varepsilon}$. Then u_ε is a nontrivial solution to problem (1.2.13) for $n = 6$. The strong maximal principle shows $u_\varepsilon > 0$ in Ω_ε. Let $u_\mu = \varepsilon^{-2} u_\varepsilon(x/\varepsilon)$ and this is a nontrivial solution of (1.2.12) for $n = 6$. Thus, we get Theorem 1.2.1 for $n = 6$. Hence, we finish the proof. $\square$

3.6 Proof Of Lemma 3.1.1

We divide the proof into two parts. First, we study the case $n = 4$. From the definition of W, (3.1.10) and (3.1.15), we know that

$$\begin{aligned} S_\varepsilon[W] =& -\Delta W + \mu\varepsilon^2 W - 8W^3 \\ =& \; 8U^3 + \varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)\hat{U} - \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\Delta(R_{\varepsilon,\Lambda,Q}\chi) - 8W^3 \\ =& \; O\Big(\varepsilon^4(-\ln\varepsilon)\langle z-\bar{Q}\rangle^{-2} + \varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}\langle z-\bar{Q}\rangle^{-4}\Big) \\ & + O(\Lambda)\Big(\frac{\varepsilon^4}{(-\ln\varepsilon)}\Big|\ln\frac{1}{\varepsilon(1+|z-\bar{Q}|)}\Big| + \frac{\varepsilon^4}{(-\ln\varepsilon)^{\frac{1}{2}}}\Big). \end{aligned}$$

The estimates for $D_\Lambda S_\varepsilon[W]$ and $D_{\bar{Q}} S_\varepsilon[W]$ can be computed in the same way.

We now turn to the proof of the energy estimate (3.1.23). From (3.1.15) and (3.1.16) we deduce that

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla W|^2 + \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\int_{\Omega_\varepsilon} W^2 =& \; 8\int_{\Omega_\varepsilon} U^3 W + \varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)\int_{\Omega_\varepsilon}\hat{U}W \\ & - \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\int_{\Omega_\varepsilon}\Delta(R\chi)W. \end{aligned} \tag{3.6.1}$$

Concerning the first term on the right hand side of (3.6.1), we have

$$\begin{aligned} \int_{\Omega_\varepsilon} U^3 W =& \int_{\Omega_\varepsilon} U^4 + \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\int_{\Omega_\varepsilon}\hat{U}U^3 \\ & + \frac{c_4\Lambda}{|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\int_{\Omega_\varepsilon} U^3. \end{aligned} \tag{3.6.2}$$

By noting that

$$\int_{\Omega_\varepsilon} U^4 = \int_{\mathbb{R}^4} U_{1,0}^4 + O(\varepsilon^4), \quad \int_{\Omega_\varepsilon} U^3 = \frac{c_4\Lambda}{8} + O(\varepsilon^2),$$

we get

$$\int_{\Omega_\varepsilon} U^3 W = \int_{\mathbb{R}^4} U_{1,0}^4 + \frac{c_4^2\Lambda^2}{8|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}} + \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\int_{\Omega_\varepsilon}\hat{U}U^3 + O\big(\varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\big).$$

For the third term on the right hand side of the above equality, we have

$$\begin{aligned}\int_{\Omega_\varepsilon}\hat{U}U^3 &= -\int_{\Omega_\varepsilon}\Psi U^3 - c_4\Lambda\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\int_{\Omega_\varepsilon}H(x,Q)U^3 + \int_{\Omega_\varepsilon}(R\chi)U^3 \\ &= -\frac{c_4\Lambda^2}{16}\ln\frac{1}{\Lambda\varepsilon} - \frac{c_4^2\Lambda^2}{8}\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}H(Q,Q) + O(\Lambda^2).\end{aligned}$$

Hence, we have

$$\begin{aligned}\int_{\Omega_\varepsilon} U^3 W = \int_{\mathbb{R}^4} U_{1,0}^4 &+ \frac{c_4^2\Lambda^2}{8|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}} - \frac{c_4\Lambda^2}{16}\ln\frac{1}{\Lambda\varepsilon}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}} \\ &- \frac{c_4^2\Lambda^2}{8}\varepsilon^2 H(Q,Q) + O\big(\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\Lambda^2 + \varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\big).\end{aligned} \tag{3.6.3}$$

For the second term on the right hand side of (3.6.1)

$$\int_{\Omega_\varepsilon}\hat{U}W = \int_{\Omega_\varepsilon}\hat{U}U + \varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\int_{\Omega_\varepsilon}\hat{U}^2 + \frac{c_4\Lambda}{|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\int_{\Omega_\varepsilon}\hat{U},$$

by using

$$\begin{aligned}&\int_{\Omega_\varepsilon}\hat{U}U = O\big(\varepsilon^{-2}\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\Lambda^2\big), \quad \int_{\Omega_\varepsilon}\hat{U}^2 = O\big(\varepsilon^{-4}(-\ln\varepsilon)\Lambda^2\big), \\ &\int_{\Omega_\varepsilon}\hat{U} = \varepsilon^{-4}\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}}\int_\Omega\frac{\Lambda}{|x-Q|^2} + O(\varepsilon^{-4}\Lambda),\end{aligned}$$

and $\int_\Omega G(x,Q) = 0$, we obtain

$$\varepsilon^4\big(\frac{c_1}{-\ln\varepsilon}\big)\int_{\Omega_\varepsilon}\hat{U}W = \frac{c_4\Lambda^2}{|\Omega|}\varepsilon^2\int_\Omega\frac{1}{|x-Q|^2} + O\big(\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\Lambda^2\big). \tag{3.6.4}$$

For the last term on the right hand side of (3.6.1),

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \Delta(R\chi)W \\
&= \varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{-\frac{1}{2}}\frac{c_4\Lambda}{|\Omega|}\int_{\Omega_\varepsilon}\Delta(R\chi) + O(\Lambda^2) \\
&= \Big(\frac{c_1}{-\ln\varepsilon}\Big)^{-1}\frac{c_4\Lambda}{|\Omega|}\int_{\Omega_\varepsilon}\Delta\Big(U - \varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{\frac{1}{2}}\Psi - c_4\Lambda\varepsilon^2 H\Big) + O(\Lambda^2) \\
&= \Big(\frac{c_1}{-\ln\varepsilon}\Big)^{-1}\frac{c_4\Lambda}{|\Omega|}\int_{\Omega_\varepsilon}\Big(-8U^3 + \varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{\frac{1}{2}}U + c_4\Lambda\varepsilon^4\frac{1}{|\Omega|}\Big) + O(\Lambda^2) \\
&= \Big(\frac{c_1}{-\ln\varepsilon}\Big)^{-\frac{1}{2}}\frac{c_4\Lambda^2}{|\Omega|}\int_{\Omega}\frac{1}{(\varepsilon^2\Lambda^2 + |x-Q|^2)} + O(\Lambda^2 + \varepsilon^2(-\ln\varepsilon)),
\end{aligned}
\tag{3.6.5}
$$

where we used (3.1.8). Using (3.6.3)-(3.6.5), we get

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_\varepsilon}\Big(|\nabla W|^2 + \varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{\frac{1}{2}}W^2\Big) \\
=&4\int_{\mathbb{R}^4} U_{1,0}^4 + \varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{-\frac{1}{2}}\frac{c_4^2\Lambda^2}{2|\Omega|} - \frac{c_4^2\Lambda^2}{2}H(Q,Q)\varepsilon^2 \\
&- \frac{c_4\Lambda^2}{4}\varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{\frac{1}{2}}\ln\frac{1}{\Lambda\varepsilon} + O\Big(\varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{\frac{1}{2}}\Lambda^2\Big) + O\Big(\varepsilon^4\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{-\frac{1}{2}}\Big).
\end{aligned}
\tag{3.6.6}
$$

Next, we compute the term $\int_{\Omega_\varepsilon} W^4$.

$$
\begin{aligned}
\int_{\Omega_\varepsilon} W^4 =& \int_{\Omega_\varepsilon} U^4 + 4\varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{\frac{1}{2}}\int_{\Omega_\varepsilon} U^3\hat{U} + 4\varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{-\frac{1}{2}}\frac{c_4\Lambda}{|\Omega|}\int_{\Omega_\varepsilon} U^3 \\
&+ O\Big(\varepsilon^4\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{-2}\Big) \\
=& \int_{\mathbb{R}^4} U_{1,0}^4 - \frac{c_4\Lambda^2}{4}\varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{\frac{1}{2}}\ln\frac{1}{\Lambda\varepsilon} - \frac{c_4^2\Lambda^2}{2}\varepsilon^2 H(Q,Q) \\
&+ \frac{c_4^2\Lambda^2}{2|\Omega|}\varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{-\frac{1}{2}} + O\Big(\varepsilon^2\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{\frac{1}{2}}\Lambda^2\Big) + O\Big(\varepsilon^4\Big(\frac{c_1}{-\ln\varepsilon}\Big)^{-2}\Big).
\end{aligned}
\tag{3.6.7}
$$

Combining (3.6.6) and (3.6.7), we obtain

$$
\begin{aligned}
J_\varepsilon[W] =& \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla W|^2 + \frac{\mu\varepsilon^2}{2}\int_{\Omega_\varepsilon} W^2 - 2\int_{\Omega_\varepsilon} W^4 \\
=& 2\int_{\mathbb{R}^4} U_{1,0}^4 + \frac{c_4\Lambda^2}{4}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\ln\frac{1}{\Lambda\varepsilon} - \frac{c_4^2\Lambda^2}{2|\Omega|}\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{-\frac{1}{2}} \\
&+ \frac{1}{2}c_4^2\Lambda^2\varepsilon^2 H(Q,Q) + O\Big(\varepsilon^2\big(\frac{c_1}{-\ln\varepsilon}\big)^{\frac{1}{2}}\Lambda^2\Big) + O(\varepsilon^4(-\ln\varepsilon)^2). \quad (3.6.8)
\end{aligned}
$$

In the following, we prove (3.1.24)-(3.1.28). By the definition of W, (3.1.10) and (3.1.15), we know that

$$
\begin{aligned}
S_\varepsilon[W] =& -\Delta W + \varepsilon^3 W - 24W^2 \\
=& \ 24U^2 + \varepsilon^6\hat{U} - \varepsilon^3\Delta(R\chi) + \varepsilon^6\big(\eta - \frac{c_6\Lambda^2}{|\Omega|}\big) - 24U^2 - 24\eta^2\varepsilon^6 \\
&+ O\big(\varepsilon^3\langle z-\bar{Q}\rangle^{-4}\big)
\end{aligned}
$$

We rearrange the right hand side of the above equality and obtain

$$
\begin{aligned}
S_\varepsilon[W] =& -\varepsilon^6\big(24\eta^2 - \eta + \frac{c_6\Lambda^2}{|\Omega|}\big) + O\big(\varepsilon^3\langle z-\bar{Q}\rangle^{-4}\big) \\
=& \ O\big(\langle z-\bar{Q}\rangle^{-3\frac{2}{3}}\varepsilon^3\big).
\end{aligned}
$$

The estimates for $D_\Lambda S_\varepsilon[W]$, $D_{\bar{Q}} S_\varepsilon[W]$ and $D_\eta S_\varepsilon[W]$ can be derived in the same way. Now we are in the position to compute the energy. From (3.1.15) and (3.1.16), we deduce that

$$
\begin{aligned}
\int_{\Omega_\varepsilon} |\nabla W|^2 + \varepsilon^3\int_{\Omega_\varepsilon} W^2 =& \int_{\Omega_\varepsilon} (-\Delta W + \varepsilon^3 W)W \\
=& \int_{\Omega_\varepsilon} \Big(24U^2 + \varepsilon^6\hat{U} - \varepsilon^3\Delta(R\chi) + \varepsilon^6\big(\eta - \frac{c_6\Lambda^2}{|\Omega|}\big)\Big)W.
\end{aligned}
\tag{3.6.9}
$$

Concerning the first term on the right hand side of (3.6.9), we have

$$\begin{aligned}\int_{\Omega_\varepsilon} U^2 W &= \int_{\Omega_\varepsilon} U^3 + \varepsilon^3 \int_{\Omega_\varepsilon} \hat{U} U^2 + \eta\varepsilon^3 \int_{\Omega_\varepsilon} U^2 \\ &= \int_{\mathbb{R}^6} U_{1,0}^3 + \frac{1}{24} c_6 \eta \Lambda^2 \varepsilon^3 - \frac{1}{24} c_6^2 \Lambda^4 \varepsilon^4 H(Q,Q) - \frac{1}{576} c_6 \Lambda^2 \varepsilon^3 + O(\varepsilon^5).\end{aligned} \tag{3.6.10}$$

For the second, third and fourth term on the right hand side of (3.6.9), following the similar steps as we did in case $n = 4$.

$$\varepsilon^6 \int_{\Omega_\varepsilon} \hat{U} W = \varepsilon^6 \int_{\Omega_\varepsilon} \hat{U}(U + \varepsilon^3 \hat{U} + \eta\varepsilon^3) = -\eta\Lambda^2\varepsilon^4 \int_\Omega \frac{1}{|x-Q|^4} + O(\varepsilon^5), \tag{3.6.11}$$

$$\begin{aligned}-\varepsilon^3 \int_{\Omega_\varepsilon} \Delta(R\chi) W =& \varepsilon^3 \eta \int_{\Omega_\varepsilon} \Delta(U - \varepsilon^3 \Psi - c_6 \varepsilon^4 \Lambda^2 H) + O(\varepsilon^5) = \varepsilon^6 \eta \int_{\Omega_\varepsilon} U + O(\varepsilon^5) \\ =& \eta\Lambda^2\varepsilon^4 \int_\Omega \frac{1}{|x-Q|^4} + O(\varepsilon^5),\end{aligned} \tag{3.6.12}$$

and

$$\varepsilon^6 \big(\eta - \frac{c_6\Lambda^2}{|\Omega|}\big) \int_{\Omega_\varepsilon} W = \big(\eta^2|\Omega| - c_6\eta\Lambda^2\big)\varepsilon^3 + \big(\eta - \frac{c_6\Lambda^2}{|\Omega|}\big)\varepsilon^4 \int_\Omega \frac{\Lambda^2}{|x-Q|^4} + O(\varepsilon^5). \tag{3.6.13}$$

Using (3.6.10)-(3.6.13), we have

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_\varepsilon} |\nabla W|^2 + \frac{\varepsilon^3}{2} \int_{\Omega_\varepsilon} W^2 \\ &= 12 \int_{\mathbb{R}^6} U_{1,0}^3 + \big(\frac{1}{2}\eta^2|\Omega| - \frac{1}{48} c_6 \Lambda^2\big)\varepsilon^3 - \frac{c_6^2\Lambda^4}{2} H(Q,Q)\varepsilon^4 \\ &\quad + \frac{1}{2}\big(\eta - \frac{c_6\Lambda^2}{|\Omega|}\big)\varepsilon^4 \int_\Omega \frac{\Lambda^2}{|x-Q|^4} + O(\varepsilon^5).\end{aligned} \tag{3.6.14}$$

Then,

$$\begin{aligned}
\int_{\Omega_\varepsilon} W^3 = & \int_{\mathbb{R}^6} U_{1,0}^3 + 3\varepsilon^3 \int_{\Omega_\varepsilon} U^2 \hat{U} + 3\varepsilon^3 \int_{\Omega_\varepsilon} U^2 \eta + 3\varepsilon^6 \int_{\Omega_\varepsilon} U \eta^2 + 3\varepsilon^9 \int_{\Omega_\varepsilon} \hat{U} \eta^2 \\
& + \varepsilon^9 \int_{\Omega_\varepsilon} \eta^3 + O(\varepsilon^5) \\
= & \int_{\mathbb{R}^6} U_{1,0}^3 + \frac{1}{8} c_6 \eta \Lambda^2 \varepsilon^3 - \frac{1}{192} c_6 \Lambda^2 \varepsilon^3 + \eta^3 |\Omega| \varepsilon^3 - \frac{1}{8} c_6^2 \Lambda^4 H(Q,Q) \varepsilon^4 \\
& + O(\varepsilon^5). \qquad (3.6.15)
\end{aligned}$$

Combining (3.6.14)-(3.6.15), we obtain the energy

$$\begin{aligned}
J_\varepsilon[W] = & \ 4 \int_{\mathbb{R}^6} U_{1,0}^3 + \Big(\frac{1}{2} \eta^2 |\Omega| - c_6 \eta \Lambda^2 + \frac{1}{48} c_6 \Lambda^2 - 8 \eta^3 |\Omega| \Big) \varepsilon^3 + \frac{1}{2} c_6^2 \Lambda^4 H(Q,Q) \varepsilon^4 \\
& + \frac{1}{2} (\eta - \frac{c_6 \Lambda^2}{|\Omega|}) \varepsilon^4 \int_\Omega \frac{\Lambda^2}{|x-Q|^4} + O(\varepsilon^5). \qquad (3.6.16)
\end{aligned}$$

Hence, we finish the whole proof of Lemma 3.1.1. □

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