Restriction Theorems and Salem Sets

by

Kyle David Hambrook

A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

The Faculty of Graduate and Postdoctoral Studies
(Mathematics)

The University of British Columbia
(Vancouver)

July 2015

Abstract

In the first part of this thesis, I prove the sharpness of the exponent range in the L^2 Fourier restriction theorem due to Mockenhaupt [37] and Mitsis [35] (with the endpoint due to Bak and Seeger [1]) for measures on $\mathbb{R}$. The proof is based on a random Cantor-type construction of Salem sets due to Łaba and Pramanik [31]. The key new idea is to embed in the Salem set a small deterministic Cantor set that disrupts the restriction estimate for the natural measure on the Salem set but does not disrupt the measure's Fourier decay.

In the second part of this thesis, I prove a lower bound on the Fourier dimension of

$$E(Q, \Psi, \theta) = \{x \in \mathbb{R} : \|qx - \theta\| \le \Psi(q) \text{ for infinitely many } q \in Q\},$$

where Q is an infinite subset of $\mathbb{Z}$, $\Psi : \mathbb{Z} \to [0, \infty)$, and $\theta \in \mathbb{R}$. This generalizes theorems of Kaufman [29] and Bluhm [5] and yields new explicit examples of Salem sets. I also prove a multi-dimensional analog of this result. I give applications of these results to metrical Diophantine approximation and determine the Hausdorff dimension of $E(Q, \Psi, \theta)$ in new cases.

Preface

The main result in Chapter 2 of this thesis is a stronger version of the main result in the paper [23], published by Izabella Łaba and myself. The proof of the main result in Chapter 2 is a minor reformulation of the proof of the main result in [23].

Chapter 3 of this thesis is adapted from my paper [22], which has been submitted for publication. Some of the text of [22] has been used verbatim in this thesis. I am the sole author of [22].

Table of Contents

Acknowledgements

I thank my supervisor, Izabella Łaba, for her invaluable guidance throughout my Ph.D. She taught me how to be a mathematician.

I thank my supervisory committee members, Malabika Pramanik and Akos Magyar, for their advice and their time.

I thank my fiancée, Yumi Kondo, for her emotional support throughout my Ph.D. Every mathematical breakthrough I have had during my Ph.D. was made sitting by her side.

Dedication

To Yumi

Chapter 1

Introduction

We will begin this chapter by stating the main results of this thesis so that experts can quickly discern its contribution. The notation, conventions, and basic definitions used in this thesis are described in Section 1.1. Section 1.2 discusses the concepts of Hausdorff dimension, Fourier dimension, and Salem sets and describes the contribution of this thesis to the study of these concepts. Section 1.3 provides detailed background and motivation for the restriction problem. It also explains the contribution of this thesis to the study of the restriction problem.

This thesis consists of two main parts.

The first part concerns the following L^2 Fourier restriction theorem.

Theorem 1.0.1. *Let μ be a compactly supported probability measure on $\mathbb{R}^d$. Suppose there are $\alpha, \beta \in (0, d)$ such that*

$$\mu(B(x,r)) \lesssim r^{\alpha} \quad \forall x \in \mathbb{R}^d, r > 0, \tag{A}$$

$$|\widehat{\mu}(\xi)| \lesssim (1+|\xi|)^{-\beta/2} \quad \forall \xi \in \mathbb{R}^d. \tag{B}$$

Then for all $p \geq 2(2d - 2\alpha + \beta)/\beta$ we have

$$\|\widehat{f\mu}\|_p \lesssim \|f\|_{L^2(\mu)} \quad \forall f \in L^2(\mu). \tag{1.0.1}$$

Mockenhaupt [37] and Mitsis [35] independently proved Theorem 1.0.1 for $p >$

$2(2d-2\alpha+\beta)/\beta$. The endpoint case $p = 2(2d - 2\alpha + \beta)/\beta$ was proved by Bak and Seeger [1]. Theorem 1.0.1 generalizes the Tomas-Stein theorem, which is the following.

Theorem 1.0.2 (Tomas-Stein theorem)**.** *Let μ be the uniform probability measure on the unit sphere S^{d-1} in $\mathbb{R}^d$, $d \geq 2$. Then for all $p \geq (2d+2)/(d-1)$ we have*

$$\|\widehat{f\mu}\|_p \lesssim \|f\|_{L^2(\mu)} \qquad \forall f \in L^2(\mu).$$

Theorem 1.0.1 implies the Tomas-Stein theorem because (A) and (B) hold with $\alpha = \beta = d - 1$ when μ is the uniform probability measure on S^{d-1}.

The range of p in the Tomas-Stein theorem is optimal in the sense that the theorem does not hold if $(2d + 2)/(d - 1)$ is replaced by any smaller number. The counterexample that shows this is called the Knapp example. In the Knapp example, f is taken to be the characteristic function of a small (hence almost flat) spherical cap (cf. [49, Chapter 7]).

Since Theorem 1.0.1 contains the Tomas-Stein theorem as a special case, the range of p in Theorem 1.0.1 is also optimal. However, one can still ask whether the range of p in Theorem 1.0.1 is optimal for specific classes of measures. The Tomas-Stein theorem and analogs for curved hypersurfaces besides the sphere were known for a long time (cf. [43]). The point of Theorem 1.0.1 is that it also applies to measures on fractal sets and to measures on the real line $\mathbb{R}$. Therefore, it is natural to ask whether the range of p in Theorem 1.0.1 is optimal for such measures. This question is difficult because there is no obvious analog of the Knapp example (with its almost flat spherical cap) for such measures.

The main result of the first part of this thesis is the following theorem, which shows that the range of p in Theorem 1.0.1 is optimal for measures μ on $\mathbb{R}$ that satisfy (A) and (B) with $\beta \leq \alpha$.

Theorem 1.0.3. *Given any $0 < \beta \leq \alpha < 1$ and given any $p_* < 2(2 - 2\alpha + \beta)/\beta$ there is a compactly supported probability measure on $\mathbb{R}$ that satisfies (A) and (B) but fails to satisfy (2.0.1) whenever $p \leq p_*$.*

Chapter 2 of this thesis is devoted to Theorem 1.0.3 (and a slightly stronger variant). Theorem 1.0.3 strengthens a result proved by Izabella Łaba and myself in 2013 [23]. The proof of Theorem 1.0.3 is a minor modification of the proof given in [23], but it gives a stronger result. More details on how the argument in this thesis differs from the argument in [23] are also given in Chapter 2. Background on restriction theorems and further motivation for Theorem 1.0.3 is given in Section 1.3 of the current chapter.

The second part of this thesis concerns the explicit construction of Salem sets.

A Salem set is a subset of $\mathbb{R}^d$ whose Hausdorff and Fourier dimension are equal. Every countable set in $\mathbb{R}^d$ is Salem set of dimension 0, and $\mathbb{R}^d$ itself is a Salem set of dimension d. The classic example of a non-trivial Salem set is a sphere in $\mathbb{R}^d$, which is a Salem set of dimension $d-1$. The existence of Salem sets in $\mathbb{R}$ of arbitrary dimension $\alpha \in (0,1)$ was proved by Salem [41] using a random Cantor-type construction. Kahane [27] showed that for every $\alpha \in (0,d)$ there is a Salem set in $\mathbb{R}^d$ of dimension α by considering the images of compact subsets of $\mathbb{R}$ under certain stochastic processes (see also Chapters 17 and 18 of [28]). Moreover, Kahane [28, Chapter 18] showed that the image of any compact subset of $\mathbb{R}^d$ under fractional Brownian motion is almost surely a Salem set, indicating that Salem sets are ubiquitous amongst random sets. Recently, other random constructions of Salem sets have been given by Bluhm [6] and Łaba and Pramanik [31]. These random constructions cannot produce any explicit examples of Salem sets. Kaufman [29] was the first to find an explicit Salem set of dimension $\alpha \notin \{0, d-1, d\}$. The set Kaufman proved to be Salem is

$$\left\{x \in \mathbb{R} : \|qx\| \leq |q|^{-\tau} \text{ for infinitely many } q \in \mathbb{Z}\right\},$$

where $\tau > 1$ and $\|z\|$ denotes the distance from $z \in \mathbb{R}$ to the nearest integer. That this set has Hausdorff dimension $2/(1+\tau)$ is the classical Jarník-Besicovitch theorem [26], [3]. Kaufman showed the Fourier dimension of this set is also $2/(1+\tau)$. In his thesis, Bluhm [5] generalized Kaufman's result by showing that the set

$$\{x \in \mathbb{R} : \|qx\| \leq \psi(|q|) \text{ for infinitely many } q \in \mathbb{Z}\}$$

is a Salem set for any decreasing function $\psi : \mathbb{N} \to (0, \infty)$. Using a theorem of Gatesoupe [19], explicit Salem sets in $\mathbb{R}^d$ of any dimension α between $d-1$ and d can be obtained by considering radial versions of the sets of Kaufman and Bluhm.

Besides those already mentioned, no other explicit constructions of Salem sets in $\mathbb{R}$ of dimension $\alpha \neq 0, 1$ or in $\mathbb{R}^d$ of dimension $\alpha \neq 0, d-1, d$ were known until now. Moreover, even now, no explicit sets in $\mathbb{R}^d$ (Salem or otherwise) with Fourier dimension strictly between 1 and $d-1$ are known.

The main results of the second part of this thesis generalize the theorems of Kaufman [29] and Bluhm [5] and exhibit many new explicit Salem sets in $\mathbb{R}$. To be more specific, let Q be an infinite subset of $\mathbb{Z}$, let $\Psi : \mathbb{Z} \to [0, \infty)$, and let $\theta \in \mathbb{R}$. We prove a lower bound on the Fourier dimension of the set

$$\{x \in \mathbb{R} : \|qx - \theta\| \leq \Psi(q) \text{ for infinitely many } q \in Q\}$$

and prove that it is Salem set for many sets Q and functions Ψ (regardless of the value of θ). We also prove a multi-dimensional analog of this result and exhibit new explicit sets in $\mathbb{R}^d$ with Fourier dimension larger than 1. We give applications of these results to metrical Diophantine approximation. Chapter 3 of this thesis is devoted to these results. A paper on these results was submitted for publication in 2014 (see [22]).

1.1 Notation, Conventions, and Basic Definitions

In this section, we explain the notation and conventions we use that may be less than universal. We also give some basic definitions.

The expression $X \lesssim Y$ stands for "there is a constant $C > 0$ such that $X \leq CY$." The expression $X \gtrsim Y$ is analogous.

If $A \subseteq \mathbb{R}^d$ is a finite set, $|A|$ is the cardinality of A. If $A \subseteq \mathbb{R}^d$ is an infinite set, $|A|$ is the Lebesgue outer measure of A.

For any $A, B \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ we define

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\}, \\ c + A &= \{c + a : a \in A\}, \\ cA &= \{ca : a \in A\}. \end{aligned}$$

The diameter of a set $A \subseteq \mathbb{R}^d$ is $\operatorname{diam}(A) = \sup\{|x - y| : x, y \in A\}$. For $x \in \mathbb{R}^d$ and $r > 0$, $B(x, r)$ denotes the open ball in the Euclidean norm with center x and radius r.

We let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive integers and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. For $n \in \mathbb{N}$, let $[n] = \{0, \ldots, n-1\}$.

For $x \in \mathbb{R}$, $\|x\| = \min_{k \in \mathbb{Z}} |x - k|$ is the distance from x to the nearest integer.

The support of a function $f : \mathbb{R}^d \to \mathbb{C}$ is the closure of the set of points $x \in \mathbb{R}^d$ where $f(x) \neq 0$; it is denoted $\operatorname{supp}(f)$.

For $k \in \mathbb{N}$, we let $C^k(\mathbb{R}^d)$ denote the set of functions $f : \mathbb{R}^d \to \mathbb{C}$ that are k-times continuously differentiable. We let $C^\infty(\mathbb{R}^d)$ denote the set of functions $f : \mathbb{R}^d \to \mathbb{C}$ that are infinitely differentiable. For $k \in \mathbb{N} \cup \{\infty\}$, we let $C_c^k(\mathbb{R}^d)$ denote the set of functions $f \in C^k(\mathbb{R}^d)$ with compact support. When no confusion is possible, we write C^k for $C^k(\mathbb{R}^d)$ and C_c^k for $C_c^k(\mathbb{R}^d)$.

Every measure on $\mathbb{R}^d$ we consider will be assumed to be defined on a σ-algebra on $\mathbb{R}^d$ that contains the Borel σ-algebra on $\mathbb{R}^d$.

The support of a measure on $\mathbb{R}^d$ is the smallest closed set whose complement has measure 0. Thus any measurable set disjoint from the support has measure 0. Any set that contains the support of a measure is said to support that measure. We write $\operatorname{supp}(\mu)$ for the support of a measure μ on $\mathbb{R}^d$.

A probability measure on $\mathbb{R}^d$ is a measure μ with $\mu(\mathbb{R}^d) = 1$.

Let μ be a measure on $\mathbb{R}^d$. For any μ-measurable function $f : \mathbb{R}^d \to \mathbb{C}$,

$$\|f\|_{L^p(\mu)} = \left(\int_{\mathbb{R}^d} |f(x)|^p d\mu(x) \right)^{1/p} \qquad \text{if } 1 \le p < \infty,$$
$$\|f\|_{L^\infty(\mu)} = \inf\{C \ge 0 : |f(x)| \le C \text{ for } \mu\text{-almost every } x \in \mathbb{R}^d\}.$$

For $p \in [1, \infty]$, the space of μ-measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ with $\|f\|_{L^p(\mu)} < \infty$ is denoted $L^p(\mu)$. It is a complete normed vector space with norm $\| \ \|_{L^p(\mu)}$ (provided we identify functions that are equal μ-almost everywhere).

The Lebesgue measure on $\mathbb{R}^d$ will be denoted by λ. We will often use the notation $L^p(\mathbb{R}^d) = L^p(\lambda)$ and $\|f\|_p = \|f\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\lambda)}$.

The Fourier transform of an $L^1(\lambda)$ function $f : \mathbb{R}^d \to \mathbb{C}$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx \qquad \text{for all } \xi \in \mathbb{R}^d.$$

If μ is a measure on $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{C}$ is an $L^1(\mu)$ function, we define

$$\widehat{f\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) d\mu(x) \qquad \text{for all } \xi \in \mathbb{R}^d. \tag{1.1.1}$$

The Fourier transform of a finite complex-valued measure μ on $\mathbb{R}^d$ is defined by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} d\mu(x) \qquad \text{for all } \xi \in \mathbb{R}^d.$$

Let V and W be vector spaces with norms $\| \ \|_V$ and $\| \ \|_W$, respectively. A linear operator $T : V \to W$ is called bounded if there is a constant $C > 0$ such that

$$\|T(v)\|_W \le C\|v\|_V \quad \forall v \in V.$$

A linear operator $T : V \to W$ is continuous if and only if it is bounded.

1.2 Hausdorff and Fourier Dimension; Salem Sets

For $\alpha \geq 0$, the α-dimensional Hausdorff content of a set $A \subseteq \mathbb{R}^d$ is

$$H^\alpha(A) = \inf_{\mathcal{B}} \sum_{B \in \mathcal{B}} (\mathrm{diam}(B))^\alpha,$$

where the infimum is over all countable collections $\mathcal{B}$ of balls such that $A \subseteq \bigcup_{B \in \mathcal{B}} B$, and $\mathrm{diam}(C)$ denotes the diameter of a set $C \subseteq \mathbb{R}^d$.

The Hausdorff dimension of a set $A \subseteq \mathbb{R}^d$ is the supremum of all $\alpha \in [0, d]$ such that $H^\alpha(A) > 0$. It is denoted by $\dim_H(A)$.

The following lemma is due to Frostman [18] (cf. [34, Chapter 12], [49, Chapter 8]).

Lemma 1.2.1 (Frostman's lemma). *If $A \subseteq \mathbb{R}^d$ is a Borel set, the Hausdorff dimension of A is the supremum of all $\alpha \in [0, d]$ such that*

$$\mu(B(x, r)) \lesssim r^\alpha \quad \forall x \in \mathbb{R}^d, r > 0$$

for some probability measure μ on $\mathbb{R}^d$ with supp$(\mu) \subseteq A$.

The Fourier dimension of a set $A \subseteq \mathbb{R}^d$ is the supremum of all $\beta \in [0, d]$ such that

$$|\widehat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\beta/2} \quad \forall \xi \in \mathbb{R}^d$$

for some probability measure μ on $\mathbb{R}^d$ with $\mathrm{supp}(\mu) \subseteq A$. It is denoted by $\dim_F(A)$.

It is a well-known consequence of Frostman's lemma that

$$\dim_F(A) \leq \dim_H(A) \tag{1.2.1}$$

for all Borel sets $A \subseteq \mathbb{R}^d$ (cf. [34, Chapter 12], [49, Chapter 8]).

A set $A \subseteq \mathbb{R}^d$ with $\dim_F(A) = \dim_H(A)$ is called a Salem set.

Here are some examples of Salem sets. Every countable set in $\mathbb{R}^d$ is a Salem set of dimension 0. More generally, every Borel set in $\mathbb{R}^d$ of Hausdorff dimension 0 is a Salem set of dimension 0. The set $\mathbb{R}^d$ itself is a Salem set of dimension d. More generally, every set in $\mathbb{R}^d$ that contains a ball is a Salem set of dimension d. Every sphere in $\mathbb{R}^d$ is a Salem set of dimension $d-1$.

Here are some examples of non-Salem sets. All k-dimensional planes in $\mathbb{R}^d$ with $k < d$ have Hausdorff dimension k and Fourier dimension 0. The middle-thirds Cantor set in $\mathbb{R}$ has Hausdorff dimension $(\ln 2)/(\ln 3)$ and Fourier dimension 0. Körner [30] showed that for any given $0 \leq \beta < \alpha \leq 1$ there is a compact subset of $\mathbb{R}$ with Fourier dimension β and Hausdorff dimension α.

Note that the concepts of Fourier dimension and Salem sets depend on the ambient space, unlike the concept of Hausdorff dimension. For example, the real line $\mathbb{R}$ is a Salem set of dimension 1, but a line in $\mathbb{R}^2$ (which we can regard as a copy of $\mathbb{R}$ embedded in $\mathbb{R}^2$) is a non-Salem set with Hausdorff dimension 1 and Fourier dimension 0.

The existence of Salem sets in $\mathbb{R}$ for every dimension $\alpha \in (0,1)$ was first proved by Salem [41] using a random Cantor-type construction. The existence of Salem sets in $\mathbb{R}^d$ for every dimension $\alpha \in (0,d)$ was first proved by Kahane [27] (cf. Chapters 17 and 18 of [28])) who considered the images of compact subsets of $\mathbb{R}$ under certain stochastic processes. Other random constructions of Salem sets have been given by Bluhm [6] and Łaba and Pramanik [31]. These random constructions do not produce any explicit examples of Salem sets. In fact, it is quite difficult to find explicit Salem sets in $\mathbb{R}^d$ of dimensions other than 0, $d-1$, and d. The first person to do so was Kaufman [29]. He proved that the set

$$\left\{x \in \mathbb{R} : \|qx\| \leq |q|^{-\tau} \text{ for infinitely many } q \in \mathbb{Z}\right\},$$

where $\tau > 1$, is a Salem set of dimension $2/(1+\tau)$. Bluhm [5] showed that replacing $|q|^{-\tau}$ by $\psi(|q|)$ for any decreasing function $\psi : \mathbb{N} \to (0,\infty)$ also yields a Salem set. No other explicit examples of Salem sets in $\mathbb{R}$ with dimension $0 < \alpha < 1$ were known until now. A theorem of Gatesoupe [19] shows that for any Salem set E contained in $[0,1]$ of dimension $0 < \alpha < 1$, the set of points $x \in \mathbb{R}^d$ such that $|x| \in E$ is a Salem set of dimension $d-1+\alpha$. As it is not difficult to modify Kaufman's proof to show that

$$\left\{x \in [0,1] : \|qx\| \leq |q|^{-\tau} \text{ for infinitely many } q \in \mathbb{Z}\right\}$$

is a Salem set of dimension $2/(1+\tau)$ when $\tau > 1$, Gatesoupe's theorem leads to explicit Salem sets in $\mathbb{R}^d$ of every dimension $\alpha \in (d-1, d)$. No explicit examples of Salem sets in $\mathbb{R}^d$ of dimension less than $d-1$ are known. Moreover, no explicit sets (Salem or otherwise) in $\mathbb{R}^d$ of Fourier dimension strictly between 1 and $d-1$ are known.

In this thesis we generalize the theorems of Kaufman [29] and Bluhm [5]. To be more specific, let Q be an infinite subset of $\mathbb{Z}$, let $\Psi : \mathbb{Z} \to [0, \infty)$, and let $\theta \in \mathbb{R}$. We prove a lower bound on the Fourier dimension of the set

$$\{x \in \mathbb{R} : \|qx - \theta\| \leq \Psi(q) \text{ for infinitely many } q \in Q\},$$

and prove that it is Salem for many sets Q and functions Ψ (regardless of the value of θ). We also prove a multi-dimensional analog of this result and exhibit new explicit sets in $\mathbb{R}^d$ with Fourier dimension larger than 1. We give applications of these results to metrical Diophantine approximation.

1.3 Restriction Theorems

The restriction problem is one of the most important problems in harmonic analysis. It is connected to many other important problems in mathematical analysis, including the Kakeya conjecture, the Bochner-Riesz conjecture, the estimation of solutions to the wave, Schroedinger, and Helmholtz equations, and the local smoothing conjecture for partial differential equations (cf. [45]). It is even connected to Montgomery's conjecture in analytic number theory on the size of Dirichlet sums (see [9] and [10]).

The restriction problem was originally proposed by Stein in the late 1960s. It originates from the study of the boundedness of the Fourier transform as a linear operator between L^p spaces. To motivate it, we begin with the simplest case. Recall that the Fourier transform of a function $f \in L^1(\lambda)$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx \quad \text{for all } \xi \in \mathbb{R}^d. \tag{1.3.1}$$

It is easy to see that the Fourier transform operator $f \mapsto \widehat{f}$ is linear. Moreover, the inequality $|\int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx| \leq \int_{\mathbb{R}^d} |f(x)|dx$ implies

$$\|\widehat{f}\|_{L^\infty(\mu)} \leq \|f\|_1$$

for any measure μ, meaning the Fourier transform is always a bounded linear operator from $L^1(\lambda)$ to $L^\infty(\mu)$.

The restriction problem asks: For which measures μ on $\mathbb{R}^d$ and for which values of p and q is the Fourier transform a bounded linear operator from $L^p(\lambda)$ to $L^q(\mu)$? In other words, it asks: For which measures μ on $\mathbb{R}^d$ and for which values of p and q do we have an inequality of the form

$$\|\widehat{f}\|_{L^q(\mu)} \leq C_{p,q}\|f\|_p \tag{1.3.2}$$

for all functions f in $L^p(\lambda)$? Here $C_{p,q}$ is a constant. The inequality (1.3.2) is called a restriction inequality.

Before saying anything more, there is a technical issue in (1.3.2) we need to address. We haven't defined $\widehat{f}$ for functions f outside of $L^1(\lambda)$. In fact, it is not true that the integral in (1.3.1) exists for all f in $L^p(\lambda)$ when $p > 1$. However, the integral in (1.3.1) is certainly defined when $f \in C_c^\infty(\mathbb{R}^d)$. A standard application of Khinchin's inequality (cf. [49, Chapter 4]) shows that (1.3.2) cannot hold for all $f \in C_c^\infty(\mathbb{R}^d)$ when $p > 2$, so we can restrict attention to $p \leq 2$. If for some p and q we have (1.3.2) for all $f \in C_c^\infty(\mathbb{R}^d)$, then (since $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\lambda)$) the Fourier transform can be extended uniquely to all $f \in L^p(\lambda)$ in such a way that (1.3.2) holds for all $f \in L^p(\lambda)$. Moreover, the extension agrees with (1.3.1) when $f \in L^1(\lambda)$. In light of this discussion, it is technically cleaner to view the restriction problem as asking: When does an inequality of the form (1.3.2) hold for all $f \in C_c^\infty(\mathbb{R}^d)$?

One can view the restriction problem as the problem of generalizing the classical Hausdorff-Young theorem, which says that the Fourier transform is a bounded operator from $L^p(\lambda)$ to $L^q(\lambda)$ if and only if $1 \leq p \leq 2$ and $q = p' = p/(p-1)$.

To understand the term *restriction* in this context, note $\|\widehat{f}\|_{L^q(\mu)} = \|\widehat{f}\,\mathbb{1}_S\|_{L^q(\mu)}$, where S is the support of the measure μ. So (1.3.2) says the Fourier restriction operator that maps f to $\widehat{f}\,\mathbb{1}_S$ (i.e., maps f to the restriction of the Fourier transform of f to the set S) is a bounded operator from $L^p(\lambda)$ to $L^q(\mu)$.

As noted above, the Fourier transform is a bounded operator from $L^1(\lambda)$ to $L^\infty(\mu)$ for any measure μ. If μ is absolutely continuous with respect to Lebesgue measure, the proof of the Hausdorff-Young theorem can be modified to give a restriction inequality (1.3.2) with p and q depending on the regularity and support of the density function of μ.

The restriction problem gets interesting if μ is singular with respect to Lebesgue measure, so that the support of μ has Lebesgue measure 0. If $p > 1$, the Fourier transform of an $L^p(\lambda)$ function may be infinite on a set of Lebesgue measure 0. If the intersection of that set with the support of μ has positive μ-measure, the left-hand side of (1.3.2) will be infinite. Hence $f \mapsto \widehat{f}$ may not be a bounded operator from $L^p(\lambda)$ to $L^q(\mu)$ for any q when μ is singular.

So which singular measures should we consider? Well, if the support S of μ is contained in a hyperplane in $\mathbb{R}^d$, we can construct a function that is in $L^p(\lambda)$ for every $p > 1$ and whose Fourier transform is infinite on all of S. For instance, if $\psi : \mathbb{R}^{d-1} \to \mathbb{C}$ is any continuous compactly supported function, then the function f defined by

$$f(x_1, \ldots, x_d) = \frac{\psi(x_2, \ldots, x_d)}{1 + |x_1|}$$

is in $L^p(\lambda)$ for every $p > 1$, but its Fourier transform is infinite everywhere on the hyperplane $\{\xi \in \mathbb{R}^d : \xi_1 = 0\}$. Hence we must consider measures μ supported on sets that are not flat.

Perhaps the most natural singular measure to consider is the uniform probability measure on S^{d-1}, the unit sphere in $\mathbb{R}^d$. Indeed, this is the measure Stein considered when he first proposed the restriction problem in the late 1960s.

To discuss specific results, it will be convenient to restate the restriction problem in its equivalent dual form. We do so via the following lemma.

Lemma 1.3.1. *Let μ be a finite measure on $\mathbb{R}^d$. Let $p \in [1,\infty)$, $p' \in (1,\infty]$, and $q, q' \in [1,\infty]$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. The following statements are equivalent.*

$$\|\widehat{g}\|_{L^q(\mu)} \le C\|g\|_p \quad \forall g \in C_c^\infty(\mathbb{R}^d). \tag{1.3.3}$$

$$\|\widehat{h\mu}\|_{p'} \le C\|h\|_{L^{q'}(\mu)} \quad \forall h \in L^{q'}(\mu). \tag{1.3.4}$$

Proof. We show only that (1.3.3) implies (1.3.4) as the reverse implication is analogous. By Fubini's theorem, Hölder's inequality, and (1.3.3), we have

$$\left|\int_{\mathbb{R}^d} g(y)\widehat{h\mu}(y)dy\right| = \left|\int_{\mathbb{R}^d} \widehat{g}(x)h(x)d\mu(x)\right| \le \|\widehat{g}\|_{L^q(\mu)}\|h\|_{L^{q'}(\mu)} \le C\|g\|_p\|h\|_{L^{q'}(\mu)}$$

for all $g \in C_c^\infty(\mathbb{R}^d)$ and $h \in L^{q'}(\mu)$. By applying the formula

$$\|\widehat{h\mu}\|_{p'} = \sup\left\{\left|\int_{\mathbb{R}^d} g(y)\widehat{h\mu}(y)dy\right| : g \in C_c^\infty(\mathbb{R}^d), \|g\|_p = 1\right\}$$

(cf. [17, Chapter 6]), we obtain (1.3.4). □

Since μ is assumed to be finite in Lemma 1.3.1, $L^{q'}(\mu) \subseteq L^1(\mu)$ and $\widehat{h\mu}$ is well-defined by (1.1.1) for every $h \in L^{q'}(\mu)$. An analog of Lemma 1.3.1 can be proved without the hypothesis that μ is finite. However, to avoid complications, we consider only finite measures from now on.

In light of Lemma 1.3.1, it is clear an equivalent formulation of the restriction problem is the following. For which measures μ on $\mathbb{R}^d$ and for which values of p and q do we have an inequality of the form

$$\|\widehat{f\mu}\|_p \le C_{p,q}\|f\|_{L^q(\mu)} \tag{1.3.5}$$

for all functions f in $L^q(\mu)$?

Now we discuss some specific results on the restriction problem.

We start with a conjecture of Stein.

Conjecture 1.3.2 (Stein's restriction conjecture)**.** *If μ is the uniform probability measure on the unit sphere S^{d-1} in $\mathbb{R}^d$, then (1.3.5) holds for all $p > 2d/(d-1)$ when $q = \infty$.*

In 1970, Fefferman and Stein (see [16], [43]) proved the conjecture for $d = 2$, but it remains open for $d \geq 3$ (and is nonsense for $d = 1$).

Another foundational result on the restriction problem is the Tomas-Stein theorem of 1975 (see [43], [46], [47]).

Theorem 1.3.3 (Tomas-Stein theorem)**.** *If μ is the uniform probability measure on the unit sphere S^{d-1} in $\mathbb{R}^d$, then (1.3.5) holds for all $p \geq (2d+2)/(d-1)$ when $q = 2$.*

Many researchers have studied the restriction problem for hypersurfaces besides the sphere, the restriction problem for surfaces of lower dimension, and analogs of the restriction problem in finite fields and in the primes (cf. [2], [12], [20], [36], [38], [39], [45]).

Recently, there has been active research into the restriction problem for fractals, where μ is a natural measure on a fractal set.

In 2000-2002, Mockenhaupt and Mitsis independently generalized the Tomas-Stein restriction theorem to the fractal setting (see [35], [37]) with the following theorem.

Theorem 1.3.4. *Let μ be a compactly supported probability measure on $\mathbb{R}^d$. Suppose there are $\alpha, \beta \in (0, d)$ such that*

$$\text{(A)} \qquad \mu(B(x,r)) \lesssim r^{\alpha} \quad \forall x \in \mathbb{R}^d, r > 0,$$

$$\text{(B)} \qquad |\widehat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\beta/2} \quad \forall \xi \in \mathbb{R}^d.$$

Then (1.3.5) holds whenever $p \geq 2(2d - 2\alpha + \beta)/\beta$ and $q = 2$.

Note the endpoint case $p = 2(2d - 2\alpha + \beta)/\beta$ was not proved by Mockenhaupt or Mitsis; it was proved by Bak and Seeger [1] in 2011. Theorem 1.3.4 also holds for any compactly supported finite measure on $\mathbb{R}^d$, as any finite measure is a constant multiple of a probability measure. If μ is the uniform probability measure on the sphere in $\mathbb{R}^d$, then (A) and (B) hold with $\alpha = \beta = d - 1$, and so (1.3.5) holds for all $p \geq (2d+2)/(d-1)$ and $q = 2$. Therefore Theorem 1.3.4 contains the Tomas-Stein theorem as a special case.

We now explain how Theorem 1.3.4 is connected to fractals.

There is no consensus in the literature on the definition of a fractal, but we will not need a formal definition anyway. Informally, one may think of a fractal as a set in $\mathbb{R}^d$ that exhibits self-similarity, structure at arbitrarily small scales, and local and global irregularity.

The complicated structure of fractal sets is often summarized by various numerical dimensions, such as Hausdorff dimension and Fourier dimension. For example, the middle-thirds Cantor set (one of the most famous fractals) has Hausdorff dimension $(\ln 2)/(\ln 3)$ and Fourier dimension 0.

Condition (A) is closely connected to Hausdorff dimension. Indeed, recall that Lemma 1.2.1 says the Hausdorff dimension of a Borel set $A \subseteq \mathbb{R}^d$ is the supremum of all $\alpha \in [0, d]$ for which there exists a probability measure μ supported on A such that condition (A) holds. Consequently, if μ is a probability measure satisfying (A) for some $\alpha \in [0, d]$, then $\alpha \leq \dim_H(\text{supp}(\mu))$.

Condition (B) concerns the Fourier decay rate of μ and is connected to Fourier dimension. The original proofs of the Tomas-Stein theorem for the sphere and analogs of it for other surfaces hinge on deducing an estimate of the Fourier decay rate of the relevant measure from the curvature of the surface on which the measure is supported. By the definition of Fourier dimension in Section 1.2, a probability measure that satisfies condition (B) must satisfy $\beta \leq \dim_F(\text{supp}(\mu))$.

Complimentary to the Tomas-Stein theorem is the important fact that the range of p in the theorem cannot be enlarged. This is shown by the classic example of Knapp, in which one takes f to be the characteristic function of a small (hence almost flat) spherical cap (cf. [49, Chapter 7]). As Theorem 1.3.4 contains the Tomas-Stein theorem as a special case, the range of p in Theorem 1.3.4 also cannot be enlarged in general. However, the point of Theorem 1.3.4 is that it also applies to measures supported on fractal sets and to measures on $\mathbb{R}$. Therefore, it is interesting to ask whether the range of p in Theorem 1.3.4 is best possible for such measures. In 2013, Izabella Łaba and I addressed this question (see [23]).

To understand the result of Łaba and myself in [23], first observe that for a probability measure μ on $\mathbb{R}$ that satisfies (A) for some $\alpha \in (0,1)$ and satisfies (B) for every $\beta < \alpha$, the inequality (1.3.5) holds whenever $p > 2(2-\alpha)/\alpha$ and $q = 2$. The result Łaba and I established in [23] implies the following.

Theorem 1.3.5. *Given any $0 < \alpha < 1$ and given any $p_* < 2(2-\alpha)/\alpha$ there is a compactly supported probability measure on $\mathbb{R}$ that satisfies (A) and satisfies (B) for every $\beta < \alpha$ but does not satisfy (1.3.5) when $p \leq p_*$ and $q = 2$.*

The construction in [23] of the measure μ is based on the randomized Cantor-type construction of Salem sets by Łaba and Pramanik [31]. Łaba and I modified the construction so that the Cantor set that supports μ contains a smaller deterministic Cantor set. Care must be taken in including the deterministic Cantor set to avoid destroying the Fourier decay estimates on μ. The inequality (1.3.5) is shown to be violated when $p \leq p_*$ and $q = 2$ for any prescribed constant $C_{p,q}$ by taking f to be the characteristic function of a finite iteration of the deterministic Cantor set. The argument ultimately hinges on counting solutions to certain linear equations in the endpoints of the finite iterations of the larger Cantor set. To ensure the number of solutions is sufficiently large, the embedded deterministic Cantor set is constructed with the endpoints of each finite iteration in a generalized arithmetic progression.

In this thesis, we improve the result of [23]. In particular, we prove the following.

Theorem 1.3.6. *Given any $0 < \beta \leq \alpha < 1$ and given any $p_* < 2(2-2\alpha+\beta)/\beta$ there is a compactly supported probability measure on $\mathbb{R}$ that satisfies (A) and (B) but does not satisfy (1.3.5) when $p \leq p_*$ and $q = 2$.*

This is a restatement of Theorem 1.0.3. In Chapter 2, we discuss the minor differences between the proof of Theorem 1.3.6 (which we give in this thesis) and the proof of the result of [23].

Theorem 1.3.6 establishes the sharpness of range of p in Theorem 1.3.4 for the class of probability measures μ on $\mathbb{R}$ that satisfy (A) and (B) for some $0 < \beta \leq \alpha < 1$.

In particular, Theorem 1.3.6 covers those measures that satisfy (A) and (B) with $\alpha = \beta$. Theorem 1.3.5 does not cover such measures; it only covers those measures that satisfy (A) for some $0 < \alpha < 1$ and satisfy (B) for β arbitrarily close to (but strictly less than) α. Theorem 1.3.6 establishes sharpness for those measures that satisfy $0 < \beta < \alpha < 1$. Theorem 1.3.5 does not cover such measures because it only permits taking p_* up to the number $2(2-\alpha)/\alpha$, not up to the larger number $2(2-2\alpha+\beta)/\beta$.

Chen [11] modified the argument used by Łaba and myself in [23] to eliminate the dependence of the measure on p_*. In particular, Chen [11] proved

Theorem 1.3.7. *Given any* $0 < \beta \leq \alpha < 1$ *there is a compactly supported probability measure on* $\mathbb{R}$ *that satisfies (A) and (B) but does not satisfy (1.3.5) when* $p < 2(2-2\alpha+\beta)/\beta$ *and* $q = 2$.

Evidently, Theorem 1.3.7 implies Theorem 1.3.6. Theorem 1.3.7 is perhaps also a more satisfying sharpness statement than Theorem 1.3.6. The modification of the argument of [23] that we present in this thesis to prove Theorem 1.3.6 does not follow the modification of [11]. However, we do borrow some of his notation.

Chapter 2

The Sharpness of the Exponents in the Mockenhaupt-Mitsis-Bak-Seeger Restriction Theorem

In this chapter, we study the sharpness of the range of exponents in the following general L^2 restriction theorem of Mockenhaupt [37] and Mitsis [35] (with the endpoint due to Bak and Seeger [1]).

Theorem 2.0.1. *Let μ be a compactly supported probability measure on $\mathbb{R}^d$. Suppose there are $\alpha, \beta \in (0, d)$ such that*

$$\mu(B(x,r)) \lesssim r^{\alpha} \quad \forall x \in \mathbb{R}^d, r > 0, \tag{A}$$

$$|\widehat{\mu}(\xi)| \lesssim (1+|\xi|)^{-\beta/2} \quad \forall \xi \in \mathbb{R}^d. \tag{B}$$

Then for all $p \geq 2(2d-2\alpha+\beta)/\beta$ we have

$$\|\widehat{f\mu}\|_p \lesssim \|f\|_{L^2(\mu)} \qquad \forall f \in L^2(\mu). \tag{2.0.1}$$

We will consider the following question. Is the range of p in Theorem 2.0.1 sharp? In other words, does the theorem still hold if $2(2d-2\alpha+\beta)/\beta$ is replaced by a smaller number p_*?

This question, as stated, is too general. The Knapp example, where f is the characteristic function of a small spherical cap and μ is the uniform probability measure on the unit sphere in $\mathbb{R}^d$, shows the range of p is sharp (cf. [49, Chapter 7]). However, we can consider the sharpness for specific sets of measures. The novelty of Theorem 2.0.1 compared to earlier restriction theorems is that it also applies to measures on fractal sets and to measures on $\mathbb{R}$. Therefore, it is natural to consider the sharpness of the range of p in Theorem 2.0.1 for such measures.

In this chapter, we will show the range of p in Theorem 2.0.1 is sharp for measures μ on $\mathbb{R}$ that satisfy (A) and (B) with $\beta \leq \alpha$. In particular, I will prove the following theorem.

Theorem 2.0.2. *Given any $0 < \beta \leq \alpha < 1$ and given any $p_* < 2(2 - 2\alpha + \beta)/\beta$ there is a compactly supported probability measure on $\mathbb{R}$ that satisfies (A) and (B) but fails to satisfy (2.0.1) whenever $p \leq p_*$.*

Theorem 2.0.2 will be deduced from the following stronger result.

Theorem 2.0.3. *Let $0 < \beta_0 \leq \alpha_0 < 1$ with $\alpha_0 = \ln a / \ln c$ and $\beta_0 = (2\ln a - \ln b)/\ln c$ for some $a, b, c \in \mathbb{N}$. Given any $p_* < 2(2-2\alpha_0+\beta_0)/\beta_0$, there is a compactly supported probability measure on $\mathbb{R}$ that satisfies the following.*

$$\mu(B(x,r)) \lesssim r^{\alpha_0} \quad \forall x \in \mathbb{R}, \forall r > 0. \tag{2.0.2}$$

$$|\widehat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\beta_0/2}(\ln(1 + |\xi|))^{1/2} \quad \forall \xi \in \mathbb{R}. \tag{2.0.3}$$

There is a sequence of functions $f_l \in L^2(\mu)$ such that

$$\lim_{l \to \infty} \frac{\|\widehat{f_l \mu}\|_p}{\|f_l\|_{L^2(\mu)}} = \infty \quad \forall p \leq p_*. \tag{2.0.4}$$

In section 2.1, we show that Theorem 2.0.3 implies Theorem 2.0.2. The rest of the chapter constitutes the proof of Theorem 2.0.3.

In 2013, Izabella Łaba and I [23] proved (implicitly) the following theorem, which is the special case of Theorem 2.0.3 where $\alpha_0 = \beta_0$.

Theorem 2.0.4. *Let $0 < \alpha_0 < 1$ with $\alpha_0 = \ln a / \ln c$ for some $a, c \in \mathbb{N}$. Given any $p_* < 2(2-\alpha_0)/\alpha_0$, there is a compactly supported probability measure on $\mathbb{R}$ that satisfies the following.*

$$\mu(B(x,r)) \lesssim r^{\alpha_0} \quad \forall x \in \mathbb{R}, \forall r > 0. \tag{2.0.5}$$

$$|\widehat{\mu}(\xi)| \lesssim (1+|\xi|)^{-\alpha_0/2}(\ln(1+|\xi|))^{1/2} \quad \forall \xi \in \mathbb{R}. \tag{2.0.6}$$

There is a sequence of functions $f_l \in L^2(\mu)$ such that

$$\lim_{l\to\infty} \frac{\|\widehat{f_l\mu}\|_p}{\|f_l\|_{L^2(\mu)}} = \infty \quad \forall p \leq p_*. \tag{2.0.7}$$

Theorem 2.0.4 can be deduced from Theorem 2.0.3 by taking $a = b$.

Just as Theorem 2.0.3 implies Theorem 2.0.2, Theorem 2.0.4 implies the following theorem, which says the the range of p in Theorem 2.0.1 is sharp for measures μ on $\mathbb{R}$ that satisfy (A) for some $\alpha \in (0,1)$ and satisfy (B) for every $\beta < \alpha$.

Theorem 2.0.5. *Given any $0 < \alpha < 1$ and given any $p_* < 2(2-\alpha)/\alpha$ there is a compactly supported probability measure on $\mathbb{R}$ that satisfies (A) and satisfies (B) for every $\beta < \alpha$ but fails to satisfy (2.0.1) whenever $p \leq p_*$.*

Let us compare what Theorem 2.0.2 and Theorem 2.0.5 tell us.

Theorem 2.0.2 establishes the sharpness of range of p in Theorem 2.0.1 for the class of probability measures μ on $\mathbb{R}$ that satisfy (A) and (B) for some $0 < \beta \leq \alpha < 1$. In particular, it covers those measures that satisfy (A) and (B) with $\alpha = \beta$. Theorem 2.0.5 does not cover such measures. Theorem 2.0.5 only covers those measures that satisfy (A) for some $0 < \alpha < 1$ and satisfy (B) for all $\beta < \alpha$.

Furthermore, Theorem 2.0.5 does not establish the sharpness of the range of p in Theorem 2.0.1 for the class of probability measures μ on $\mathbb{R}$ that satisfy (A) and (B) for some $0 < \beta < \alpha < 1$, while Theorem 2.0.2 does. It is true that Theorem 2.0.5 produces, for any prescribed $0 < \beta < \alpha < 1$ and $p_* < 2(2-\alpha)/\alpha$, a probability measure μ that satisfies (A) and (B) but does not satisfy (2.0.1) when $p \leq p_*$. However, establishing sharpness for the class of measures under consideration requires exhibiting, for any prescribed $0 < \beta < \alpha < 1$ and $p_* < 2(2-2\alpha+\beta)/\beta$, a probability measure μ that satisfies (A) and (B) but does not satisfy (2.0.1) when $p \leq p_*$. Theorem 2.0.5 does not allow us to take p_* all the way up to $2(2-2\alpha+\beta)/\beta$. It only permits taking p_* up to the number $2(2-\alpha)/\alpha$, which is strictly less than $2(2-2\alpha+\beta)/\beta$ when $\beta < \alpha$.

The proof of Theorem 2.0.3 is a straightforward modification of the proof of Theorem 2.0.4 given in [23]. It is based ultimately on the randomized Cantor-type construction of Salem sets by Łaba and Pramanik [31]. We modify the construction so that the Cantor set that supports μ contains a smaller deterministic Cantor set. The functions f_ℓ are the characteristic functions of the finite iterations of this deterministic Cantor set. We must be careful that the deterministic Cantor set is not so large that it destroys the Fourier decay estimates furnished by the randomness of the larger Cantor set. The divergence of $\|\widehat{f_\ell \mu}\|_p / \|f_\ell\|_{L^2(\mu)}$ relies on counting solutions to linear equations in the endpoints of the finite iterations of the larger Cantor set. The deterministic Cantor set is chosen with the endpoints of its finite iterations in a multi-scale arithmetic progression so that this count is sufficiently large to ensure divergence.

The difference between the proof of Theorem 2.0.3 and the proof of Theorem 2.0.4 given in [23] is the presence of $\beta_0 = (2\ln a - \ln b)/\ln c$ in addition to $\alpha_0 = \ln a/\ln c$. This is handled by simply using β_0 instead of α_0 in all estimates related to the Fourier decay of μ. The form of β_0 is chosen so that the length ($s = n^{\alpha_0 - \beta_0/2}$ in the terminology to be introduced in Section 2.2) of the single-scale arithmetic progression (which underlies the multi-scale arithmetic progression) is an integer.

2.1 Theorem 2.0.3 Implies Theorem 2.0.2

In this section, we show that Theorem 2.0.3 implies Theorem 2.0.2.

Let $0 < \beta \leq \alpha < 1$ and $p_* < 2(2-2\alpha+\beta)/\beta$ be given. Choose $\epsilon > 0$ small enough that $\alpha + \epsilon < 1$ and

$$p_* < \frac{2(2 - 2(\alpha + \epsilon) + (\beta + \epsilon))}{\beta + \epsilon}. \tag{2.1.1}$$

Choose $c \in \mathbb{N}$ large enough that

$$\frac{1}{\ln c} < \frac{\epsilon}{2}$$

Let $a \in \mathbb{N}$ be the smallest positive integer that

$$\alpha + \frac{\epsilon}{2} < \frac{\ln a}{\ln c}. \tag{2.1.2}$$

Then

$$\alpha + \frac{\epsilon}{2} \geq \frac{\ln(a-1)}{\ln c} > \frac{\ln a}{\ln c} - \frac{1}{\ln c} > \frac{\ln a}{\ln c} - \frac{\epsilon}{2}. \tag{2.1.3}$$

Therefore, by combining (2.1.2) and (2.1.3), we obtain

$$\alpha < \alpha + \frac{\epsilon}{2} < \alpha_0 < \alpha + \epsilon, \tag{2.1.4}$$

where

$$\alpha_0 = \frac{\ln a}{\ln c}.$$

Let $b \in \mathbb{N}$ be the largest positive integer such that

$$\frac{\ln b}{\ln c} < 2\alpha_0 - \beta. \tag{2.1.5}$$

Then

$$2\alpha_0 - \beta \leq \frac{\ln(b+1)}{\ln c} < \frac{\ln b}{\ln c} + \frac{1}{2\ln c} < \frac{\ln b}{\ln c} + \frac{\epsilon}{2}. \tag{2.1.6}$$

Therefore, by combining (2.1.5) and (2.1.6), we obtain

$$\beta < \beta_0 < \beta + \frac{\epsilon}{2} < \beta + \epsilon, \tag{2.1.7}$$

where

$$\beta_0 = 2\alpha_0 - \frac{\ln b}{\ln c} = \frac{2\ln a - \ln b}{\ln c}.$$

Since $\beta \leq \alpha$ and $\alpha + \epsilon/2 < \alpha_0$, it follows from (2.1.7) that

$$\beta_0 < \alpha_0.$$

By (2.1.1), (2.1.4), and (2.1.7), we have

$$p_* < \frac{2(2 - 2(\alpha + \epsilon) + (\beta + \epsilon))}{\beta + \epsilon} < \frac{2(2 - 2\alpha_0 + \beta_0)}{\beta_0}.$$

Theorem 2.0.3 now gives a compactly supported probability measure μ on $\mathbb{R}$ and a sequence of functions $f_l \in L^2(\mu)$ such that (2.0.2), (2.0.3), and (2.0.4) hold. It is clear that (2.0.4) implies the failure of (2.0.1) for every $p \leq p_*$. Since $\beta < \beta_0$, it is also clear that (2.0.3) implies (B). Since $\alpha < \alpha_0$, (2.0.2) implies

$$\mu(B(x, r)) \lesssim r^\alpha \quad \forall x \in \mathbb{R}, 0 < r < 1.$$

Since μ is a probability measure, we also have

$$\mu(B(x, r)) \leq \mu(\mathbb{R}) = 1 \leq r^\alpha \quad \forall x \in \mathbb{R}, r \geq 1.$$

This establishes (A) and completes the proof of Theorem 2.0.2 assuming Theorem 2.0.3.

2.2 The Numbers r, s, t, n, S_j, T_j, N_j, and L_j

We now begin the proof of Theorem 2.0.3. We start by defining some important numbers.

Let $0 < \beta_0 \leq \alpha_0 < 1$ with $\alpha_0 = \ln a / \ln c$ and $\beta_0 = (2\ln a - \ln b)/\ln c$ for some $a, b, c \in \mathbb{N}$. Let $p_* < 2(2 - 2\alpha_0 + \beta_0)/\beta_0$. Choose $r \in \mathbb{N}$ such that $r\beta_0 > 1$ and $2r \geq p_*$. Define $n = c^{2m}$, $t = a^{2m} = n^{\alpha_0}$, and $s = b^m = n^{\alpha_0 - \beta_0/2}$, where m is a fixed positive integer chosen so large that

$$p_* < \frac{2(2-2\alpha_0+\beta_0)}{\beta_0} - \frac{4\ln r}{\beta_0 \ln n}. \tag{2.2.1}$$

Define $S_j = s^j$, $T_j = t^j$, $N_j = n^j$, and $L_{j+1} = \ln(400(j+1)N_{j+1})$ for $j \in \mathbb{N}_0$.

2.3 The Sequences of Sets $(A_j)_{j=0}^\infty$ and $(P_j)_{j=0}^\infty$

In this section, we describe the sequences of sets $(A_j)_{j=0}^\infty$ and $(P_j)_{j=0}^\infty$.

Each A_j will be the set of left endpoints of the intervals in the j-th iteration of the Cantor set supporting μ. Each P_j will be the set of left endpoints of the intervals in the j-th iteration of the smaller deterministic Cantor set we embed within the Cantor set supporting μ.

Let $(P_j)_{j=0}^\infty$ be the sequence of sets defined by

$$P_0 = \{0\}\,, \quad P_{j+1} = P_j + \frac{1}{N_{j+1}}\{1,\ldots,s\} = \bigcup_{a\in P_j} a + \frac{1}{N_{j+1}}\{1,\ldots,s\}\,.$$

We will construct inductively a sequence of sets $(A_j)_{j=0}^\infty$ of the form

- $A_0 = \{0\}$
- $A_{j+1} = \bigcup_{a\in A_j} a + A_{j+1,a}$
- $A_{j+1,a} \subseteq \dfrac{1}{N_{j+1}}[n]$
- $|A_{j+1,a}| = T_{j+1}$
- $A_j \subseteq P_j$.

Note

$$|A_j| = T_j, \qquad A_j \subseteq \frac{1}{N_j}[N_j], \qquad A_{j,a} \subseteq \frac{1}{N_j}\{1,\ldots,s\}$$

for all $j \in \mathbb{N}$. The details of the construction of $(A_j)_{j=0}^\infty$ will be given in Section 2.8.

2.4 The Measures μ and μ_j; The Functions f_l

In this section, we define the probability measures μ and μ_j ($j \in \mathbb{N}_0$) on $\mathbb{R}$ and the functions $f_l : \mathbb{R} \to \mathbb{C}$ ($l \in \mathbb{N}_0$).

For $j \in \mathbb{N}_0$, define

$$E_j = A_j + [0, N_j^{-1}]$$

and define μ_j to be the uniform probability measure on E_j, i.e.,

$$d\mu_j = \frac{1}{|E_j|} \mathbb{1}_{E_j} dx = \frac{N_j}{T_j} \sum_{a \in A_j} \mathbb{1}_{[a, a+N_j^{-1}]} dx.$$

Define μ to be the weak limit of $(\mu_j)_{j=0}^{\infty}$, i.e.,

$$\mu = \lim_{j \to \infty} \mu_j.$$

We prove the weak limit exists in Section 2.5. A good reference on weak convergence is [4]. Note

$$\operatorname{supp}(\mu) = \bigcap_{j=1}^{\infty} E_j.$$

Let $j \in \mathbb{N}_0$ and $a \in A_j$. Then

$$\mu_j([a, a + N_j^{-1}]) = \frac{1}{T_j},$$

and in fact

$$\mu_k([a, a + N_j^{-1}]) = \frac{1}{T_j} \quad \forall k \in \mathbb{N}_0, k \geq j.$$

Consequently,

$$\mu([a, a + N_j^{-1}]) = \frac{1}{T_j}.$$

For $l \in \mathbb{N}_0$, define

$$F_l = P_l + [0, N_l^{-1}) \quad \text{and} \quad f_l = \mathbb{1}_{F_l}.$$

2.5 Existence of the Weak Limit

In this section, we prove the existence of the weak limit

$$\mu = \lim_{j\to\infty} \mu_j.$$

For all other sections in this chapter, F_j is the set defined in Section 2.4. However, in this section, F_j will be the cumulative distribution function of μ_j, i.e.,

$$F_j(t) = \mu_j((-\infty, t]), \quad \forall t \in \mathbb{R}.$$

Fix $j \in \mathbb{N}_0$. For $a \in A_j$, we have $F_{j+1}(a) = F_j(a)$ by the definitions of μ_{j+1} and μ_j. For $t \in \mathbb{R}$, let $a(t)$ be the smallest element of $N_j^{-1}[N_j]$ such that $a(t) \le t$. Then

$$\begin{aligned}
|F_{j+1}(t) - F_j(t)| &= |\mu_{j+1}([a(t), t]) - \mu_j([a(t), t])| \\
&\le \mu_{j+1}([a(t), t]) + \mu_j([a(t), t]) \\
&\le \mu_{j+1}([a(t), a(t) + N^{-j}]) + \mu_j([a(t), a(t) + N^{-j}]) \\
&\le \frac{2}{T_j}.
\end{aligned}$$

Therefore

$$\|F_{j+1} - F_j\|_\infty \le \frac{2}{T_j} \quad \forall j \in \mathbb{N}_0.$$

Since $T_j = n^{j\alpha_0}$ for $j \in \mathbb{N}_0$, we have $\sum_{j=0}^\infty T_j^{-1} < \infty$. It follows that $(F_j)_{j=1}^\infty$ is a uniformly convergent sequence of continuous cumulative distribution functions. Therefore the limit F is a continuous cumulative distribution function of a probability measure μ. Hence $(\mu_j)_{j=1}^\infty$ converges weakly to μ.

2.6 The Ball Condition

In this section, we will show that $\mu(|I|) \lesssim |I|^{\alpha_0}$ for all intervals $I \subseteq \mathbb{R}$. This is equivalent to (2.0.2) in Theorem 2.0.3.

Let I be an interval in $\mathbb{R}$. If $|I| > 1$, then

$$\mu(I) \leq \mu(\mathbb{R}) = 1 \leq |I|^{\alpha_0}$$

since μ is a probability measure. Now suppose $|I| \leq 1$. Choose $j \in \mathbb{N}_0$ such that $N_{j+1}^{-1} \leq |I| \leq N_j^{-1}$. Assume I intersects $\mathrm{supp}(\mu)$ (otherwise $\mu(I) = 0$). Note

$$\mathrm{supp}(\mu) = \bigcap_{j=1}^{\infty} E_j = \bigcap_{j=1}^{\infty} \bigcup_{a \in A_j} [a, a + N_j^{-1}].$$

Then I intersects an interval $[a, a + N_j^{-1}]$ for some $a \in A_j$. There are at most two such intervals, say J_1 and J_2, because $|I| \leq N_j^{-1}$ and $A_j \subseteq N_j^{-1}[N_j]$. Therefore

$$\begin{aligned}\mu(I) &= \mu(I \cap J_1) + \mu(I \cap J_2) \leq \mu(J_1) + \mu(J_2) \\ &= \frac{1}{T_j} + \frac{1}{T_j} = \frac{2n^{\alpha_0}}{N_{j+1}^{\alpha_0}} \leq 2n^{\alpha_0}|I|^{\alpha_0}.\end{aligned}$$

2.7 The Crucial Inequality

In this section, we state the crucial Fourier analytic inequality, explain the steps in its proof, and explain its use.

Here is the crucial Fourier analytic inequality:

$$\sum_{j=0}^{\infty} |\widehat{f_l \mu_{j+1}}(k) - \widehat{f_l \mu_j}(k)| \lesssim |k|^{-\beta_0/2}(\ln(e + |k|))^{1/2} \tag{2.7.1}$$

for all $k \in \mathbb{Z}$ with $k \neq 0$ and all $l \in \mathbb{N}_0$.

In the next three sections, we prove we can construct the sets $(A_j)_{j=0}^{\infty}$ so that (2.7.1) holds.

First we show we can construct the sets $(A_j)_{j=0}^{\infty}$ so that

$$\left|\frac{1}{T_j}\sum_{a\in A_j\cap F_l} Y_a(k)\right| \lesssim N_{j+1}^{-\beta_0/2}L_{j+1}^{1/2} \tag{2.7.2}$$

for all $k \in \mathbb{Z}$ and all $j, l \in \mathbb{N}_0, j \geq l$. Here

$$Y_a(k) = e^{-2\pi i k a}\left(\frac{1}{t}\sum_{b\in A_{j+1,a}} e^{-2\pi i k b} - \frac{1}{n}\sum_{b\in[n]/N_{j+1}} e^{-2\pi i k b}\right).$$

Next we show

$$|\widehat{f_l\mu_{j+1}}(k) - \widehat{f_l\mu_j}(k)| \leq 2\min\left\{1, \frac{N_{j+1}}{|k|}\right\}\left|\frac{1}{T_j}\sum_{a\in A_j\cap F_l} Y_a(k)\right| \tag{2.7.3}$$

for all $k \in \mathbb{Z}$, $k \neq 0$ and all $j, l \in \mathbb{N}$ with $j \geq l$.

Finally we show that

$$\sum_{j=0}^{\infty}\min\left\{1, \frac{N_{j+1}}{|k|}\right\} N_{j+1}^{-\beta_0/2}L_{j+1}^{1/2} \lesssim |k|^{-\beta_0/2}(\ln(e+|k|))^{1/2} \tag{2.7.4}$$

for all $k \in \mathbb{Z}, k \neq 0$.

Evidently (2.7.1) follows upon combining (2.7.2), (2.7.3), and (2.7.4).

Once (2.7.1) is established, we will use it to prove

$$|\widehat{\mu}(\xi)| \lesssim (1+|\xi|)^{-\beta_0/2}(\ln(e+|\xi|))^{1/2} \quad \forall \xi \in \mathbb{R}. \tag{2.7.5}$$

and

$$\lim_{j\to\infty}\|\widehat{f_l\mu_j}\|_{2r}^{2r} = \|\widehat{f_l\mu}\|_{2r}^{2r} \quad \forall l \in \mathbb{N}_0. \tag{2.7.6}$$

Note (2.7.5) is (2.0.3) in Theorem 2.0.3.

2.8 Proof of the Crucial Inequality. I.

In this section, we will define $(A_j)_{j=0}^{\infty}$ so that (2.7.2) holds. Recall that (2.7.2) is the inequality

$$\left|\frac{1}{T_j}\sum_{a\in A_j\cap F_l} Y_a(k)\right| \lesssim N_{j+1}^{-\beta_0/2} L_{j+1}^{1/2}$$

for all $k \in \mathbb{Z}$ and all $j, l \in \mathbb{N}_0, j \geq l$. Here (as elsewhere)

$$Y_a(k) = e^{-2\pi i k a}\left(\frac{1}{t}\sum_{b\in A_{j+1,a}} e^{-2\pi i k b} - \frac{1}{n}\sum_{b\in[n]/N_{j+1}} e^{-2\pi i k b}\right).$$

We will define $(A_j)_{j=0}^{\infty}$ inductively. Fix $j \in \mathbb{N}_0$. Assume A_j containing P_j is given. We will choose $A_{j+1} = \bigcup_{a\in A_j} a + A_{j+1,a}$ by choosing A_{j+1}, a for each $a \in A_j$.

We will need the following version of Hoeffding's inequality.

Lemma 2.8.1. *If $X_1, \ldots, X_n$ are independent complex-valued random variables with $\mathbb{E}X_i = 0$ and $|X_i| \leq 1$ for $i = 1, \ldots, n$, then*

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geq u\right) \leq 4\exp(-nu^2/4)$$

for all $u \geq 0$.

This version of Hoeffding's inequality for complex-valued random variables is deduced from the standard version for real-valued random variables in Appendix A.

Let $\mathcal{B}_{j+1}$ be the collection of all size t_{j+1} subsets of $N_{j+1}^{-1}[n]$. There are $\binom{n}{t}$ sets in $\mathcal{B}_{j+1}$. Moreover, each fixed $b \in N_{j+1}^{-1}[n]$ is contained in exactly $\binom{n-1}{t-1}$ sets in $\mathcal{B}_{j+1}$.

For each $a \in A_j$, choose a set $B_{j+1,a}$ independently and uniformly at random from $\mathcal{B}_{j+1}$. For each $a \in A_j$ and $k \in \mathbb{Z}$, define the random variable

$$X_a(k) = \frac{1}{2} e^{-2\pi i a k} \left(\frac{1}{t} \sum_{b \in B_{j+1,a}} e^{-2\pi i k b} - \frac{1}{n} \sum_{b \in N_{j+1}^{-1}[n]} e^{-2\pi i k b} \right).$$

For each $a \in A_j$, we have

$$\begin{aligned}
\mathbb{E}\left(\sum_{b \in B_{j+1,a}} e^{-2\pi i \xi b} \right) &= \frac{1}{\binom{n}{t}} \sum_{B \in \mathcal{B}_{j+1}} \sum_{b \in B} e^{-2\pi i \xi b} \\
&= \frac{1}{\binom{n}{t}} \sum_{b \in N_{j+1}^{-1}[n]} \binom{n-1}{t-1} e^{-2\pi i \xi b} \\
&= \frac{t}{n} \sum_{b \in N_{j+1}^{-1}[n]} e^{-2\pi i \xi b}.
\end{aligned}$$

It follows that $\mathbb{E} X_a(k) = 0$ for all $a \in A_j$ and $k \in \mathbb{Z}$. Clearly $|X_a(k)| \leq 1$ for all $a \in A_j$ and $k \in \mathbb{Z}$.

Fix $k \in [N_{j+1}]$. The random variables $(X_a(k))_{a \in A_j}$ are independent because of how we choose the sets $(B_{j+1,a})_{a \in A_j}$. Note

$$|A_j \cap F_l| = s^l t^{j-l} = (S_l/T_l) T_j \quad \forall l \in \mathbb{N}_0, l \leq j.$$

Define $u_{l,j} \geq 0$ by

$$u_{l,j}^2 = 4 (S_l/T_l)^{-1} T_j^{-1} \ln(400(j+1) N_{j+1}) \quad \forall l \in \mathbb{N}_0, l \leq j.$$

Lemma 2.8.1 implies

$$P\left(\left| \frac{1}{(S_l/T_l)T_j} \sum_{a \in A_j \cap F_l} X_a(k) \right| \geq u_{l,j} \right) \leq 4 \exp(-(S_l/T_l) T_j u_{l,j}^2/4) = \frac{1}{100(j+1)N_{j+1}}.$$

Then

$$P\left(\left|\frac{1}{T_j S_l^{-1}}\sum_{a\in A_j\cap F_l} X_a(k)\right| \geq u_{l,j} \text{ for some } k\in[N_{j+1}], l\in\{0,\dots,j\}\right) \leq \frac{1}{100}.$$

Therefore there is some realization of the randomly chosen sets $(B_{j+1,a})_{a\in A_j}$ such that

$$\left|\frac{1}{(S_l/T_l)T_j}\sum_{a\in A_j\cap F_l} X_a(k)\right| < u_{l,j} \tag{2.8.1}$$

for all $k\in[N_{j+1}]$ and all $l\in\mathbb{N}_0, l\leq j$. Fix such a realization. Since $X_a(k)$ is periodic with period N_{j+1} when $a\in A_j$, the inequality (2.8.1) holds for all $k\in\mathbb{Z}$.

We cannot simply define $A_{j+1,a}=B_{j+1,a}$ because then

$$A_{j+1}=\bigcup_{a\in A_j} a+A_{j+1,a}$$

may not contain

$$P_{j+1}=\bigcup_{a\in P_j} a+N_{j+1}^{-1}\{1,\dots,s\}.$$

We define $A_{j+1,a}$ for $a\in A_j$ as follows. For $a\notin P_j$, define $A_{j+1,a}=B_{j+1,a}$. For $a\in P_j$, define $A_{j+1,a}$ to be the set obtained by first removing s arbitrary elements of $B_{j+1,a}$ and then replacing them by the elements of $N_{j+1}^{-1}\{1,\dots,s\}$. Then $P_{j+1}\subseteq A_{j+1}$.

Recall that for $a\in A_j$ and $k\in\mathbb{Z}$ we have

$$X_a(k)=e^{-2\pi ika}\left(\frac{1}{t}\sum_{b\in B_{j+1,a}} e^{-2\pi ikb}-\frac{1}{n}\sum_{b\in[n]/N_{j+1}} e^{-2\pi ikb}\right),$$

$$Y_a(k)=e^{-2\pi ika}\left(\frac{1}{t}\sum_{b\in A_{j+1,a}} e^{-2\pi ikb}-\frac{1}{n}\sum_{b\in[n]/N_{j+1}} e^{-2\pi ikb}\right).$$

For $a \notin P_j$, we have $Y_a(k) = X_a(k)$. For $a \in P_j$, $\sum_{b \in A_{j+1,a}} e^{-2\pi ikb}$ differs from $\sum_{b \in B_{j+1,a}} e^{-2\pi ikb}$ by at most s terms, hence

$$|Y_a(k) - X_a(k)| = \frac{1}{t} \left| \sum_{b \in A_{j+1,a}} e^{-2\pi ikb} - \sum_{b \in B_{j+1,a}} e^{-2\pi ikb} \right| \leq \frac{2s}{t}.$$

Therefore, for all $l \in \mathbb{N}_0, l \leq j$, we have

$$\begin{aligned}
\left| \frac{1}{T_j} \sum_{a \in A_j \cap F_l} Y_a(k) - \frac{1}{T_j} \sum_{a \in A_j \cap F_l} X_a(k) \right| &\leq \frac{1}{T_j} \sum_{a \in A_j \cap F_l} |Y_a(k) - X_a(k)| \\
&= \frac{1}{T_j} \sum_{a \in P_j} |Y_a(k) - X_a(k)| \\
&\leq \frac{|P_j|}{T_j} \frac{2s}{t} \\
&= \frac{2S_{j+1}}{T_{j+1}}.
\end{aligned}$$

Combining this with (2.8.1) yields

$$\left| \frac{1}{T_j} \sum_{a \in A_j \cap F_l} Y_a(k) \right| < (S_l/T_l)^{1/2} (4T_j^{-1} \ln(400(j+1)N_{j+1}))^{1/2} + \frac{2S_{j+1}}{T_{j+1}}.$$

Recall that $s = n^{\alpha_0 - \beta_0/2}$, $t = n^{\alpha_0}$, $S_j = s^j$, $T_j = t^j$, $N_j = n^j$, and $L_{j+1} = \ln(400(j+1)N_{j+1})$. It follows that

$$\left| \frac{1}{T_j} \sum_{a \in A_j \cap F_l} Y_a(k) \right| \lesssim N_{j+1}^{-\beta_0/2} L_{j+1}^{1/2}.$$

2.9 Proof of the Crucial Inequality. II.

In this section, we prove (2.7.3), i.e.,

$$|\widehat{f_l\mu_{j+1}}(k) - \widehat{f_l\mu_j}(k)| \le 2\min\left\{1, \frac{N_{j+1}}{|k|}\right\}\left|\frac{1}{T_j}\sum_{a\in A_j\cap F_l} Y_a(k)\right|$$

for all $k \in \mathbb{Z}$, $k \neq 0$ and all $j, l \in \mathbb{N}$ with $j \ge l$. Recall

$$Y_a(k) = e^{-2\pi i k a}\left(\frac{1}{t}\sum_{b\in A_{j+1,a}} e^{-2\pi i k b} - \frac{1}{n}\sum_{b\in[n]/N_{j+1}} e^{-2\pi i k b}\right).$$

Fix $j, l \in \mathbb{N}_0$ with $j \ge l$. For $a \in A_j$ and $x \in \mathbb{R}$, define

$$g_{a,j}(x) = N^j \mathbb{1}_{[a,a+N_j^{-1}]}(x) = N^j \mathbb{1}_{[0,1]}(N_j(x-a)).$$

Then

$$d\mu_j(x) = \frac{1}{T_j}\sum_{a\in A_j} g_{a,j}(x)dx,$$

and so

$$f_l d\mu_j(x) = \frac{1}{T_j}\sum_{a\in A_j} f_l(x) g_{a,j}(x)dx = \frac{1}{T_j}\sum_{a\in A_j\cap F_l} g_{a,j}(x)dx.$$

For $\xi \in \mathbb{R}$,

$$\widehat{g_{a,j}}(\xi) = e^{-2\pi i a\xi}\widehat{\mathbb{1}_{[0,1]}}(\xi/N_j).$$

Therefore

$$\widehat{f_l\mu_j}(\xi) = \frac{1}{T_j}\widehat{\mathbb{1}_{[0,1]}}(\xi/N_j)\sum_{a\in A_j\cap F_l} e^{-2\pi i a\xi}. \tag{2.9.1}$$

Likewise

$$\widehat{f_l\mu_{j+1}}(\xi) = \frac{1}{T_j}\widehat{\mathbb{1}_{[0,1]}}(\xi/N_{j+1}) \sum_{a\in A_{j+1}\cap F_l} e^{-2\pi i a\xi}. \tag{2.9.2}$$

We now rewrite (2.9.1) and (2.9.2). First, by the geometric sum formula,

$$\begin{aligned}
-2\pi i(\xi/N_j)\widehat{\mathbb{1}_{[0,1]}}(\xi/N_j) &= e^{-2\pi i\xi/N_j} - 1 = e^{-2\pi i(\xi/N_{j+1})n} - 1 \\
&= (e^{-2\pi i(\xi/N_{j+1})} - 1)\sum_{k\in[n]} e^{-2\pi i(\xi/N_{j+1})k} \\
&= -2\pi i(\xi/N_{j+1})\widehat{\mathbb{1}_{[0,1]}}(\xi/N_{j+1})\sum_{k\in[n]} e^{-2\pi i(\xi/N_{j+1})n} \\
&= -2\pi i(\xi/N_j)\widehat{\mathbb{1}_{[0,1]}}(\xi/N_{j+1})\frac{1}{n}\sum_{b\in N_{j+1}^{-1}[n]} e^{-2\pi i\xi b}.
\end{aligned}$$

Therefore

$$\widehat{\mathbb{1}_{[0,1]}}(\xi/N_j) = \widehat{\mathbb{1}_{[0,1]}}(\xi/N_{j+1})\frac{1}{n}\sum_{b\in N_{j+1}^{-1}[n]} e^{-2\pi i\xi b}.$$

Applying this to (2.9.1) gives

$$\widehat{f_l\mu_j}(\xi) = \frac{1}{T_j}\widehat{\mathbb{1}_{[0,1]}}(\xi/N_{j+1})\sum_{a\in A_j\cap F_l} e^{-2\pi i a\xi}\frac{1}{n_{j+1}}\sum_{b\in N_{j+1}^{-1}[n_{j+1}]} e^{-2\pi i\xi b}. \tag{2.9.3}$$

On the other hand, since

$$A_{j+1} = \bigcup_{a\in A_j} a + A_{j+1,a},$$

we have

$$\widehat{f_l\mu_{j+1}}(\xi) = \frac{1}{T_{j+1}}\widehat{\mathbb{1}_{[0,1]}}(\xi/N_{j+1}) \sum_{a\in A_j\cap F_l}\sum_{b\in A_{j+1,a}} e^{-2\pi i\xi(a+b)} \tag{2.9.4}$$
$$= \frac{1}{T_j}\widehat{\mathbb{1}_{[0,1]}}(\xi/N_{j+1}) \sum_{a\in A_j\cap F_l} e^{-2\pi i\xi a}\frac{1}{t}\sum_{b\in A_{j+1,a}} e^{-2\pi i\xi b}.$$

Fix $k\in\mathbb{Z}, k\neq 0$. Comparing (2.9.3) and (2.9.4) yields

$$\widehat{f_l\mu_{j+1}}(k) - \widehat{f_l\mu_j}(k) = 2\widehat{\mathbb{1}_{[0,1]}}(k/N_{j+1})\frac{1}{T_j}\sum_{a\in A_j\cap F_l} Y_a(k). \tag{2.9.5}$$

Since

$$|\widehat{\mathbb{1}_{[0,1]}}(k/N_{j+1})| \le \widehat{\mathbb{1}_{[0,1]}}(0) = 1$$

and

$$|\widehat{\mathbb{1}_{[0,1]}}(k/N_{j+1})| = \left|\frac{e^{-2\pi i k/N_{j+1}}-1}{-2\pi i k/N_{j+1}}\right| \le \frac{1}{\pi|k|/N_{j+1}},$$

it follows from (2.9.5) that

$$|\widehat{f_l\mu_{j+1}}(k) - \widehat{f_l\mu_j}(k)| \le 2\min\left\{1, \frac{N_{j+1}}{|k|}\right\}\left|\frac{1}{T_j}\sum_{a\in A_j\cap F_l} Y_a(k)\right|.$$

2.10 Proof of the Crucial Inequality. III.

In this section, we prove (2.7.4), i.e.,

$$\sum_{j=0}^{\infty}\min\left\{1, \frac{N_{j+1}}{|k|}\right\} N_{j+1}^{-\beta_0/2}L_{j+1}^{1/2} \lesssim (\ln(e+|k|))^{1/2}|k|^{-\beta_0/2}$$

for all $\forall k\in\mathbb{Z}, k\neq 0$.

Fix $k \in \mathbb{Z}, k \neq 0$. Let j_* be the smallest non-negative integer such that $N_{j_*+1} > |k|$. Then $N_{j_*} \leq |k|$. Our strategy is to the split the sum above into a sum over $0 \leq j < j_*$ and a sum over $j \geq j_*$ and estimate each separately. For any $j \in \mathbb{N}_0$,

$$L_{j+1} = \ln(400(j+1)N_{j+1}) \leq 10\ln(N_{j+1}).$$

If $0 \leq j < j_*$, then

$$L_{j+1} \leq 10\ln(N_{j_*}) \leq 10\ln|k|.$$

If $j \geq j_*$, then

$$\begin{aligned} L_{j+1} &\leq 10\ln(N_{j-j_*+1}N_{j_*}) \leq 10\ln(N_{j-j_*+1}|k|) \\ &\leq 20\ln(N_{j-j_*+1})\ln(e+|k|) = 20(j-j_*+1)\ln(n)\ln(e+|k|). \end{aligned}$$

For the sum over $0 \leq j < j_*$, we have

$$\begin{aligned} \frac{1}{|k|}\sum_{0\leq j<j_*} N_{j+1}^{1-\beta_0/2} L_{j+1}^{1/2} &= \frac{1}{|k|}\sum_{1\leq j<j_*}\left(\frac{N_{j_*}}{n^{j_*-j-1}}\right)^{1-\beta_0/2} L_{j+1}^{1/2} \\ &\leq (10\ln|k|)^{1/2}|k|^{-\beta_0/2}\sum_{1\leq j<j_*}\frac{1}{2^{(j_*-j-1)(1-\beta_0/2)}} \\ &= (10\ln|k|)^{1/2}|k|^{-\beta_0/2}\sum_{j=0}^{\infty}\frac{1}{2^{j(1-\beta_0/2)}}. \end{aligned}$$

For the sum over $j \geq j_*$, we have

$$\begin{aligned} \sum_{j\geq j_*} N_{j+1}^{-\beta_0/2} L_{j+1}^{1/2} &= \sum_{j\geq j_*}\left(N_{j_*+1}n^{j-j_*}\right)^{-\beta_0/2} L_{j+1}^{1/2} \\ &\leq (20\ln(n)\ln(e+|k|))^{1/2}|k|^{-\beta_0/2}\sum_{j\geq j_*}2^{-(j-j_*)\beta_0/2}(j-j_*+1) \\ &= (20\ln(n)\ln(e+|k|))^{1/2}|k|^{-\beta_0/2}\sum_{j=0}^{\infty}2^{-j\beta_0/2}(j+1). \end{aligned}$$

Combining the estimates yields

$$\sum_{j=0}^{\infty} \min\left\{1, \frac{N_{j+1}}{|k|}\right\} N_{j+1}^{-\beta_0/2} L_{j+1}^{1/2} \lesssim (\ln(e+|k|))^{1/2}|k|^{-\beta_0/2}.$$

2.11 The Fourier Decay of μ

In this section, we use the crucial inequality (2.7.1) to prove the inequality (2.7.5), which is

$$|\widehat{\mu}(\xi)| \lesssim (1+|\xi|)^{-\beta_0/2}(\ln(e+|\xi|))^{1/2} \quad \forall \xi \in \mathbb{R}.$$

This is (2.0.3) in Theorem 2.0.3.

Since μ_j converges to μ weakly, we have that $\widehat{\mu_j}$ converges to $\widehat{\mu}$ pointwise. Therefore

$$\widehat{\mu}(k) = \widehat{\mu_0}(k) + \sum_{j=0}^{\infty}(\widehat{\mu_{j+1}}(k) - \widehat{\mu_j}(k)) \quad \forall k \in \mathbb{Z}, k \neq 0. \tag{2.11.1}$$

Since $d\mu_0 = \mathbb{1}_{[0,1]}dx$, we have

$$|\widehat{\mu_0}(k)| = \left|\frac{e^{-2\pi ik}-1}{-2\pi ik}\right| \leq \frac{1}{|k|} \quad \forall k \in \mathbb{Z}, k \neq 0. \tag{2.11.2}$$

Since $f_0 = \mathbb{1}_{[0,1]}$, we have $f_0\mu_j = \mu_j$ for all $j \in \mathbb{N}_0$, so we can apply (2.7.1) and (2.11.2) to the series in (2.11.1) to obtain

$$|\widehat{\mu}(k)| \lesssim (\ln(e+|k|))^{1/2}|k|^{-\beta_0/2} \quad \forall k \in \mathbb{Z}, k \neq 0.$$

Since $(1+|k|)^{\beta_0/2} \leq 2|k|^{\beta_0/2}$ for $k \in \mathbb{Z}, k \neq 0$, and since $\widehat{\mu}(0) = 1$, it follows that

$$|\widehat{\mu}(k)| \lesssim (\ln(e+|k|))^{1/2}(1+|k|)^{-\beta_0/2} \quad \forall k \in \mathbb{Z}.$$

Then, by applying Lemma B.0.1 in Appendix B, we arrive at

$$|\widehat{\mu}(\xi)| \lesssim (1+|\xi|)^{-\beta_0/2}(\ln(e+|\xi|))^{1/2} \quad \forall \xi \in \mathbb{R}.$$

2.12 The Convergence of $\|\widehat{f_l\mu_j}\|_{2r}^{2r}$ to $\|\widehat{f_l\mu}\|_{2r}^{2r}$

In this section, we use the crucial inequality (2.7.1) to prove (2.7.6), i.e., to prove

$$\lim_{j\to\infty} \|\widehat{f_l\mu_j}\|_{2r}^{2r} = \|\widehat{f_l\mu}\|_{2r}^{2r} \quad \forall l \in \mathbb{N}_0.$$

Fix $l \in \mathbb{N}_0$. Clearly $f_l = \mathbb{1}_{F_l}$ is a bounded Borel measurable function. Since μ is continuous (i.e., the cumulative distribution function of μ is continuous), the finite set of points where f_l is discontinuous has μ-measure 0. Since μ_j converges to μ weakly, we have

$$\lim_{j\to\infty} \int_{\mathbb{R}} g d\mu_j \to \int_{\mathbb{R}} g d\mu$$

for all bounded Borel measurable functions $g : \mathbb{R} \to \mathbb{C}$ for which the set of points at which g is discontinuous has μ-measure 0. Applying this to $g(x) = e^{-2\pi i\xi x} f_l(x)$ gives that $\widehat{f_l\mu_j}$ converges to $\widehat{f_l\mu}$ pointwise as $j \to \infty$. Of course, $\widehat{f_l\mu_i}$ converges to $\widehat{f_l\mu}$ pointwise as $i \to \infty$ also. Therefore, for all $k \in \mathbb{Z}$ and $j \in \mathbb{N}_0$,

$$\widehat{f_l\mu}(k) - \widehat{f_l\mu_j}(k) = \sum_{i=j}^{\infty} (\widehat{f_l\mu_{i+1}}(k) - \widehat{f_l\mu_i}(k)).$$

Then (2.7.1) yields

$$|\widehat{f_l\mu}(k) - \widehat{f_l\mu_j}(k)| \lesssim |k|^{-\beta_0/2}(\ln(e+|k|))^{1/2}$$

for all $k \in \mathbb{Z}$, $k \neq 0$ and all $j \in \mathbb{N}_0$, $j \geq l$. Since $(1+|k|)^{\beta_0/2} \leq 2|k|^{\beta_0/2}$ for $k \in \mathbb{Z}, k \neq 0$, and since

$$\widehat{f_l\mu}(0) - \widehat{f_l d\mu_l}(0) = \mu(F_l) - \mu_l(F_l) = 0,$$

it follows that

$$|\widehat{f_l\mu}(k) - \widehat{f_l\mu_j}(k)| \lesssim (1+|k|)^{-\beta_0/2}(\ln(e+|k|))^{1/2}$$

for all $k \in \mathbb{Z}$ and $j \in \mathbb{N}_0$ with $j \geq l$. By applying Lemma B.0.1 from Appendix B to the measure $f_l(\mu - \mu_j)$, we obtain

$$|\widehat{f_l\mu}(\xi) - \widehat{f_l\mu_j}(\xi)| \lesssim (1+|\xi|)^{-\beta_0/2}(\ln(e+|\xi|))^{1/2}$$

for all $\xi \in \mathbb{R}$ and $j \in \mathbb{N}_0$ with $j \geq l$. Then since $r \in \mathbb{N}$ was chosen so that $r\beta_0 > 1$ and since

$$\lim_{j\to\infty}(\widehat{f_l\mu}(\xi) - \widehat{f_l\mu_j}(\xi)) \to 0$$

pointwise for every $\xi \in \mathbb{R}$, the dominated convergence theorem implies

$$\lim_{j\to\infty} \|\widehat{f_l\mu}(\xi) - \widehat{f_l\mu_j}(\xi)\|_{2r} = 0.$$

It follows by the triangle inequality that

$$\lim_{j\to\infty} \|\widehat{f_l\mu_j}\|_{2r}^{2r} = \|\widehat{f_l\mu}\|_{2r}^{2r}.$$

2.13 A Lower Bound on $\|\widehat{f_l\mu_j}\|_{2r}^{2r}$

In this section, we show that

$$\|\widehat{f_l\mu_j}\|_{2r}^{2r} \geq C_r \left(\frac{S_l}{T_l}\right)^{2r} \frac{N_l}{r^{l+1}S_l} \quad \forall j,l \in \mathbb{N}_0, j \geq l, \tag{2.13.1}$$

where $C_r > 0$ is a constant depending only on r.

Fix $j,l \in \mathbb{N}_0$ with $j \geq l$. Recall from (2.9.1) that

$$\widehat{f_l\mu_j}(\xi) = \frac{1}{T_j}\widehat{\mathbb{1}_{[0,1]}}(\xi/N_j)\sum_{a\in A_j\cap F_l} e^{-2\pi i\xi a}$$

for all $\xi \in \mathbb{R}$. Therefore

$$\widehat{f_l\mu_j}(\xi) = \frac{1}{T_j}\widehat{\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}}(\xi/N_j)e^{-\pi\xi/N_j}\sum_{a\in A_j\cap F_l} e^{-2\pi i\xi a}.$$

Then

$$\begin{aligned}
|\widehat{f_l\mu_j}(\xi)|^{2r} &= \frac{1}{T_j^{2r}}\left|\sum_{a\in A_j\cap F_l} e^{-2\pi i\xi a}\right|^{2r}\left|\widehat{\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}}(\xi/N_j)e^{-\pi i\xi/N_j}\right|^{2r} \\
&= \frac{1}{T_j^{2r}}\left|\sum_{a\in A_j\cap F_l} e^{-2\pi i\xi a}\right|^{2r}\left(\widehat{\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}}(\xi/N_j)\right)^{2r} \\
&= \frac{1}{T_j^{2r}}\sum_{a_1,\ldots,a_{2r}\in A_j\cap F_l} e^{-2\pi i\xi(\sum_{s=1}^{r} a_s - \sum_{s=r+1}^{2r} a_s)}\widehat{\mathbb{1}^{*2r}_{[-\frac{1}{2},\frac{1}{2}]}}(\xi/N_j).
\end{aligned}$$

Here we have used

$$\widehat{\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}}(\xi/N_j) = \frac{e^{-\pi i\xi/N_j} - e^{\pi i\xi/N_j}}{-2\pi i\xi/N_j} = \frac{\sin(\pi\xi/N_j)}{\pi\xi/N_j} \in \mathbb{R},$$

and the formulas $|z|^{2r} = z^r\,\overline{z}^{\,r}$ and $\left(\widehat{f}\right)^{2r} = \widehat{f^{*2r}}$, where f^{*2r} is the $2r$-fold convolution of f. Therefore

$$\begin{aligned}
\|f_l\mu_j\|_{2r}^{2r} &= \int_{\mathbb{R}} |\widehat{f_l\mu_j}(\xi)|^{2r}d\xi = N_j\int_{\mathbb{R}} |\widehat{f_l\mu_j}(N_j\xi)|^{2r}d\xi \\
&= \frac{N_j}{T_j^{2r}}\sum_{a_1,\ldots,a_{2r}\in A_j\cap F_l}\int_{\mathbb{R}} e^{-2\pi i\xi N_j(\sum_{s=1}^{r} a_s - \sum_{s=r+1}^{2r} a_s)}\widehat{\mathbb{1}^{*2r}_{[-\frac{1}{2},\frac{1}{2}]}}(\xi)d\xi.
\end{aligned}$$

By Fourier inversion,

$$\|f_l\mu_j\|_{2r}^{2r} = \frac{N_j}{T_j^{2r}}\sum_{a_1,\ldots,a_{2r}\in A_j\cap F_l}\mathbb{1}^{*2r}_{[-\frac{1}{2},\frac{1}{2}]}\left(N_j\left(\sum_{s=1}^{r} a_s - \sum_{s=r+1}^{2r} a_s\right)\right).$$

Since $\mathbb{1}^{*2r}_{[-\frac{1}{2},\frac{1}{2}]} \geq 0$, all the terms in the sum are non-negative. Therefore

$$\|f_l\mu_j\|_{2r}^{2r} \geq \frac{N_j}{T_j^{2r}} \mathbb{1}^{*2r}_{[-\frac{1}{2},\frac{1}{2}]}(0) M_{j,l,r},$$

where

$$M_{j,l,r} = \left| \left\{ (a_1, \ldots, a_{2r}) \in (A_j \cap F_l)^{2r} : \sum_{s=1}^{r} a_s = \sum_{s=r+1}^{2r} a_s \right\} \right|.$$

Take $C_r = \mathbb{1}^{*2r}_{[-\frac{1}{2},\frac{1}{2}]}(0)$.

To prove (2.13.1) now, it remains to establish the lower bound

$$M_{j,l,r} \geq \frac{T_j^{2r}}{N_j} \left(\frac{S_l}{T_l} \right)^{2r} \frac{N_l}{r^{l+1} S_l}. \tag{2.13.2}$$

Define

$$(A_j \cap F_l)^{\oplus r} = \left\{ \sum_{s=1}^{r} a_s : a_1, \ldots, a_r \in A_j \cap F_l \right\}.$$

We have

$$M_{j,l,r} = \sum_{b \in (A_j \cap F_l)^{\oplus r}} (g(b))^2,$$

where

$$g(b) = \left| \left\{ (a_1, \ldots, a_r) \in (A_j \cap F_l)^r : \sum_{s=1}^{r} a_s = b \right\} \right|$$

So by the Cauchy-Schwarz inequality

$$\left(\sum_{b \in (A_j \cap F_l)^{\oplus r}} g(b) \right)^2 \leq |(A_j \cap F_l)^{\oplus r}| M_{j,l,r}. \tag{2.13.3}$$

Clearly

$$\sum_{b\in(A_j\cap F_l)^{\oplus r}} g(b) = |(A_j\cap F_l)^r| = \left(\frac{S_l}{T_l}T_j\right)^r \tag{2.13.4}$$

To estimate $|(A_j\cap F_l)^{\oplus r}|$, note each $a\in A_j\cap F_l$ has a digit representation of the form

$$a=\sum_{k=1}^{l}\frac{a^{(k)}}{N_k}+\sum_{k=l+1}^{j}\frac{a^{(k)}}{N_k},$$

where $a^{(k)}\in\{1,\dots,s\}$ for $k=1,\dots,l$ and $a^{(k)}\in[n]$ for $k=l+1,\dots,j$. Therefore each $a\in A_j\cap F_l$ is of the form

$$a=\sum_{k=1}^{l}\frac{a^{(k)}}{N_k}+\frac{a^*}{N_l},$$

where $a^{(k)}\in\{1,\dots,s\}$ for $k=1,\dots,l$ and $a^*\in[N_j/N_l]$. Therefore each $b\in(A_j\cap F_l)^{\oplus r}$ is of the form

$$b=\sum_{k=1}^{l}\frac{b^{(k)}}{N_k}+\frac{b^*}{N_l}$$

where $b^{(k)}\in\{r,\dots,rs\}$ for $k=1,\dots,l$ and $b^*\in[rN_j/N_l]$. Hence

$$|(A_j\cap F_l)^{\oplus r}|\le (rs)^l\frac{rN_j}{N_l}=r^{l+1}S_l\frac{N_j}{N_l}. \tag{2.13.5}$$

Now (2.13.2) follows by combining (2.13.3), (2.13.4), and (2.13.5).

2.14 The Divergence of $\|\widehat{f_l\mu}\|_p/\|f_l\|_{L^2(\mu)}$

In this section, we show that

$$\lim_{l\to\infty}\frac{\|\widehat{f_l\mu}\|_p}{\|f_l\|_{L^2(\mu)}}=\infty \quad \forall p\le p_*. \tag{2.14.1}$$

This is (2.0.4) in Theorem 2.0.3.

Let $l\in\mathbb{N}_0$ and $1\le p\le p_*$ be given. Recall that $r\in\mathbb{N}$ was chosen so that $p_*\le 2r$. Since

$$|\widehat{f_l\mu}(\xi)|\le\mu(F_l)=\frac{S_l}{T_l}\quad\forall\xi\in\mathbb{R},$$

we have

$$\|\widehat{f_l\mu}\|_{2r}^{2r}=\int_{\mathbb{R}}|\widehat{f_l\mu}(\xi)|^{2r-p}|\widehat{f_l\mu}(\xi)|^p d\xi\le\left(\frac{S_l}{T_l}\right)^{2r-p}\|\widehat{f_l\mu}\|_p^p.$$

Therefore

$$\|\widehat{f_l\mu}\|_p^p\ge\left(\frac{S_l}{T_l}\right)^{p-2r}\|\widehat{f_l\mu}\|_{2r}^{2r}.$$

Since

$$\|f_l\|_{L^2(\mu)}^2=\mu(F_l)=\frac{S_l}{T_l},$$

it follows that

$$\frac{\|\widehat{f_l\mu}\|_p^p}{\|f_l\|_{L^2(\mu)}^p}\ge\left(\frac{S_l}{T_l}\right)^{(p/2)-2r}\|\widehat{f_l\mu}\|_{2r}^{2r}.$$

By combining (2.7.6) and (2.13.1), we have

$$\|\widehat{f_l\mu}\|_{2r}^{2r} \geq C_r \left(\frac{S_l}{T_l}\right)^{2r} \frac{N_l}{r^{l+1}S_l}.$$

Therefore

$$\frac{\|\widehat{f_l\mu}\|_p^p}{\|f_l\|_{L^2(\mu)}^p} \geq C_r \left(\frac{S_l}{T_l}\right)^{p/2} \frac{N_l}{r^{l+1}S_l}$$

Recall that $s = n^{\alpha_0-\beta_0/2}$, $t = n^{\alpha_0}$, $S_l = s^l$, $T_l = t^l$, and $N_l = n^l$. Note also that $r^l = n^{l\frac{\ln r}{\ln n}}$. Therefore we have

$$\frac{\|\widehat{f_l\mu}\|_p^p}{\|f_l\|_{L^2(\mu)}^p} \geq C_r n^{l(-\frac{1}{4}p\beta_0+1-\alpha_0+\frac{1}{2}\beta_0-\frac{\ln r}{\ln n})}.$$

Recall now that n was defined so that

$$p_* < \frac{2(2-2\alpha_0+\beta_0)}{\beta_0} - \frac{4\ln r}{\beta_0 \ln n}.$$

Using that $p \leq p_*$ and rearranging, we get

$$-\frac{1}{4}p\beta_0 + 1 - \alpha_0 + \frac{1}{2}\beta_0 - \frac{\ln r}{\ln n} > 0.$$

Therefore

$$\lim_{l\to\infty} \frac{\|\widehat{f_l\mu}\|_p^p}{\|f_l\|_{L^2(\mu)}^p} = \infty.$$

This proves (2.14.1) and completes the proof of Theorem 2.0.3.

Chapter 3

Explicit Salem Sets and Applications to Metrical Diophantine Approximation

In this chapter, we generalize theorems of Kaufman [29] and Bluhm [5] to construct new explicit examples of Salem sets in $\mathbb{R}$. We also prove a multi-dimensional analog of our result and obtain new explicit sets in $\mathbb{R}^d$ with large Fourier dimension. We give applications of our results to metrical Diophantine approximation.

In this chapter only, for $x \in \mathbb{R}^d$,

$$|x| = \max_{1 \le i \le d} |x_i|, \qquad |x|_2 = (\sum_{i=1}^{d} |x_i|^2)^{1/2}.$$

Let Q be an infinite subset of $\mathbb{Z}$, let $\Psi : \mathbb{Z} \to [0, \infty)$, and let $\theta \in \mathbb{R}$. Define $E(Q, \Psi, \theta)$ to be the set of all $x \in \mathbb{R}$ such that

$$\|qx - \theta\| \le \Psi(q) \text{ for infinitely many } q \in Q.$$

Recall that, for $x \in \mathbb{R}$, $\|x\| = \min_{k \in \mathbb{Z}} |x - k|$ is the distance from x to the nearest integer. We will always assume Ψ is bounded. Since $\|x\| \le 1/2$ for all $x \in \mathbb{R}$, assuming

Ψ is bounded results in no loss of generality. We will also always assume $\Psi(0) = 1$. This assumption is imposed only to avoid tedious notation. Since redefining Ψ at finitely many points does not change the set $E(Q, \Psi, \theta)$, assuming $\Psi(0) = 1$ results in no loss of generality.

For $M > 0$, define

$$\begin{aligned} Q(M) &= \{q \in Q : M/2 < |q| \leq M\}, \\ \epsilon(M) &= \min_{q \in Q(M)} \Psi(q). \end{aligned}$$

If $Q(M)$ is empty, $\epsilon(M)$ is undefined.

The main result of this chapter is the following theorem.

Theorem 3.0.1. *Suppose there is a number $a \geq 0$, an increasing function $h : (0,\infty) \to (0,\infty)$, and an unbounded set $\mathcal{M} \subseteq (0,\infty)$ such that*

$$|Q(M)|\epsilon(M)^a h(M) \geq M^a \quad \forall M \in \mathcal{M}. \tag{3.0.1}$$

Then there is a Borel probability measure μ supported on $E(Q, \Psi, \theta)$ such that

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-a} \exp\left(\frac{\ln|\xi|}{\ln\ln|\xi|}\right) h(4|\xi|) \quad \forall \xi \in \mathbb{R}, |\xi| > e. \tag{3.0.2}$$

Note the hypotheses $\mathcal{M} \subseteq (0,\infty)$ and (3.0.1) imply $Q(M)$ is non-empty and $\epsilon(M)$ is defined for all $M \in \mathcal{M}$.

We also have a higher-dimensional version of Theorem 3.0.1.

Let $m, n \in \mathbb{N}$, let Q be an infinite subset of $\mathbb{Z}^n$, let $\Psi : \mathbb{Z}^n \to [0,\infty)$, and let $\theta \in \mathbb{R}^m$. Define $E(m, n, Q, \Psi, \theta)$ to be the set of all points

$$(x_{11}, \ldots, x_{1n}, \ldots, x_{m1}, \ldots, x_{mn}) \in \mathbb{R}^{mn}$$

such that

$$\max_{1 \leq i \leq m} \|\sum_{j=1}^{n} q_j x_{ij} - \theta_i\| \leq \Psi(q) \text{ for infinitely many } q \in Q.$$

Clearly $E(1, 1, Q, \Psi, \theta) = E(Q, \Psi, \theta)$. As above, we will always assume Ψ is bounded and $\Psi(0) = 1$, and these assumptions result in no loss of generality.

For $M > 0$, define

$$\begin{aligned} Q(M) &= \{q \in Q : M/2 < |q_j| \leq M \quad \forall 1 \leq j \leq n\}, \\ \epsilon(M) &= \min_{q \in Q(M)} \Psi(q). \end{aligned}$$

If $Q(M)$ is empty, $\epsilon(M)$ is undefined.

Theorem 3.0.2. *Suppose there is a number $a \geq 0$, an increasing function $h : (0, \infty) \to (0, \infty)$, and an unbounded set $\mathcal{M} \subseteq (0, \infty)$ such that*

$$|Q(M)|\epsilon(M)^a h(M) \geq M^a \quad \forall M \in \mathcal{M}. \tag{3.0.3}$$

Then there is a Borel probability measure μ supported on $E(m, n, Q, \Psi, \theta)$ such that

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-a} \exp\left(\frac{\ln|\xi|}{\ln\ln|\xi|}\right) h(4|\xi|) \quad \forall \xi \in \mathbb{R}^{mn}, |\xi| > e. \tag{3.0.4}$$

Note the hypotheses $\mathcal{M} \subseteq (0, \infty)$ and (3.0.3) imply $Q(M)$ is non-empty and $\epsilon(M)$ is defined for all $M \in \mathcal{M}$.

We give some remarks on the proofs of Theorems 3.0.1 and 3.0.2 in Section 3.1. Sections 3.2 and 3.3 discuss motivations for Theorems 3.0.1 and 3.0.2. Section 3.4 contains applications of Theorems 3.0.1 and 3.0.2 to metrical Diophantine approximation. The combined proof of Theorems 3.0.1 and 3.0.2 constitutes Sections 3.5, 3.6, and 3.7. Section 3.8 contains the proof of Lemma 3.3.1.

3.1 Remarks on the Proof of Theorems 3.0.1 and 3.0.2

Since Theorem 3.0.2 is a generalization of Theorem 3.0.1, we give a single unified proof. The proof follows the methods of Kaufman [29] and Bluhm [5, 7]. We also found Wolff's variant of Kaufman's proof in [49] instructive. Kaufman [29] and Bluhm [7] considered $E(\mathbb{Z}, \Psi_\tau, 0)$, where $\Psi_\tau(q) = |q|^{-\tau}$ and $\tau > 1$. In [5], Bluhm considered $E(\mathbb{Z}, \Psi, 0)$ with $\Psi(q) = \psi(|q|)$ and $\psi : \mathbb{N} \to (0, \infty)$ decreasing. We needed relatively straightforward modifications in our proof to handle the mn-dimensional setting, more general functions Ψ, and the case $\theta \neq 0$. A more substantial modification was needed to handle general sets $Q \neq \mathbb{Z}$. We now explain this modification.

It is fairly easy to see that the proofs of Bluhm, Kaufman, and Wolff can be adapted to work for any set Q containing the set $\mathcal{P}$ of prime numbers. Their proofs use two properties of the primes. The first property is about their density:

$$|\mathcal{P} \cap (M/2, M]| \gtrsim \frac{M}{\ln M} \quad \forall M \in \mathbb{N}. \tag{3.1.1}$$

The second property is about their arithmetic structure:

$$|\{p \in \mathcal{P} : p \text{ divides } n,\ p \geq M\}| \leq \frac{\ln n}{\ln M} \quad \forall n, M \in \mathbb{N},\ 2 \leq M \leq n. \tag{3.1.2}$$

In our proof, we use the divisor bound of Wigert [48] (see Section 3.5) as a substitute for (3.1.2). This allows us to handle more general sets Q that may not have any special arithmetic structure. The cost is that we have the factor $\exp(\ln|\xi|/\ln\ln|\xi|)$ rather than $\ln|\xi|$ in Theorems 3.0.1 and 3.0.2. The analog in our proof of the density property (3.1.1) is essentially the assumption (3.0.1) (though (3.0.1) is a combined assumption about the density of Q and the size of Ψ).

3.2 Motivation: Explicit Salem Sets

The first motivation for our main result is the construction of explicit Salem sets and explicit sets with non-zero Fourier dimension. We start by recalling some definitions and basic facts mentioned in Chapter 1.

Let $A \subseteq \mathbb{R}^d$. For $\alpha \geq 0$, the α-dimensional Hausdorff content of A is

$$H^\alpha(A) = \inf_{\mathcal{B}} \sum_{B \in \mathcal{B}} (\mathrm{diam}(B))^\alpha,$$

where the infimum is over all countable collections $\mathcal{B}$ of balls such that $A \subseteq \bigcup_{B\in\mathcal{B}} B$. The Hausdorff dimension of A, denoted $\dim_H(A)$, is the supremum all of $\alpha \in [0, d]$ such that $H^\alpha(A) > 0$. The Fourier dimension of A, denoted $\dim_F(A)$, is the supremum of all $\beta \in [0, d]$ such that

$$|\widehat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\beta/2} \quad \forall \xi \in \mathbb{R}^d.$$

for some probability measure μ on $\mathbb{R}^d$ with $\mathrm{supp}(\mu) \subseteq A$. If A is a Borel set, Frostman's lemma implies

$$\dim_F(A) \leq \dim_H(A) \tag{3.2.1}$$

(cf. [34, Chapter 12], [49, Chapter 8]).

A set $A \subseteq \mathbb{R}^d$ with $\dim_F(A) = \dim_H(A)$ is called a Salem set. Every Borel set in $\mathbb{R}^d$ of Hausdorff dimension 0 is a Salem set, every set in $\mathbb{R}^d$ that contains a ball is a Salem set of dimension d, and every sphere in $\mathbb{R}^d$ is a Salem set of dimension $d - 1$.

Salem [41] proved the existence of Salem sets in $\mathbb{R}$ of arbitrary dimension $\alpha \in (0, 1)$ using a random Cantor-type construction. Kahane [27] showed that for every $\alpha \in (0, d)$ there is a Salem set in $\mathbb{R}^d$ of dimension α by considering the images of compact subsets of $\mathbb{R}$ under certain stochastic processes (cf. Chapters 17 and 18 of [28]). Moreover, Kahane [28, Chapter 18] showed that the image of any compact subset of $\mathbb{R}^d$ under fractional Brownian motion is almost surely a Salem set, indicating that Salem sets are ubiquitous amongst random sets. Recently, other random constructions of Salem sets have been given by Bluhm [6] and Łaba and Pramanik [31]. These random constructions do not produce explicit examples of Salem sets.

Kaufman [29] was the first to find an explicit Salem set of dimension $\alpha \notin \{0, d-1, d\}$. The set Kaufman proved to be Salem is $E(\mathbb{Z}, \Psi_\tau, 0)$, where $\Psi_\tau(q) = |q|^{-\tau}$ and $\tau > 1$. If $\tau \leq 1$, then $E(\mathbb{Z}, \Psi_\tau, 0) = \mathbb{R}$ by Dirichlet's approximation theorem, and so $E(\mathbb{Z}, \Psi_\tau, 0)$ is a Salem set of dimension 1. We are concerned with $\tau > 1$.

An easy and well-known argument (which we give in Section 3.8) gives

$$\dim_H E(\mathbb{Z}, \Psi_\tau, 0) \leq \frac{2}{1+\tau}.$$

Since $E(\mathbb{Z}, \Psi_\tau, 0)$ is a Borel set, (3.2.1) implies

$$\dim_H E(\mathbb{Z}, \Psi_\tau, 0) \geq \dim_F(\mathbb{Z}, \Psi_\tau, 0).$$

Kaufman showed that for every $\tau > 1$ there is a probability measure μ with support contained in $E(\mathbb{Z}, \Psi_\tau, 0) \cap [0,1]$ such that

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-1/(1+\tau)} \ln |\xi| \quad \forall \xi \in \mathbb{R}, |\xi| \geq 2.$$

which implies

$$\dim_F E(\mathbb{Z}, \Psi_\tau, 0) \geq \frac{2}{1+\tau},$$

and hence that $E(\mathbb{Z}, \Psi_\tau, 0)$ is a Salem set. See [7] for a detailed variation of Kaufman's argument. In his thesis, Bluhm [5] showed that $E(\mathbb{Z}, \Psi, 0)$ is Salem for any Ψ with $\Psi(q) = \psi(|q|)$ and $\psi : \mathbb{N} \to (0, \infty)$ decreasing. Technically, the results of Bluhm and Kaufman are for $E(\mathbb{N}, \Psi, 0)$, not $E(\mathbb{Z}, \Psi, 0)$, but it is easy to adapt their proofs to $E(\mathbb{Z}, \Psi, 0)$.

If $A \subseteq \mathbb{R}$ is a set of Fourier dimension $\alpha \in [0,1]$, then it is easy to see the product set $A^d \subseteq \mathbb{R}^d$ has Fourier dimension at least α by considering product measures. A theorem of Gatesoupe [19] implies that if $A \subseteq [0,1]$ has Fourier dimension $\alpha \in [0,1]$, then $\left\{x \in \mathbb{R}^d : |x|_2 \in A\right\}$ has Fourier dimension at least $d - 1 + \alpha$. Moreover, Gatesoupe's theorem implies that if $A \subseteq [0,1]$ is a Salem set of dimension $\alpha \in [0,1]$, then $\left\{x \in \mathbb{R}^d : |x|_2 \in A\right\}$ is a Salem set in $\mathbb{R}^d$ of dimension $d - 1 + \alpha$. Combining Gatesoupe's and Kaufman's results yields explicit examples of Salem sets in $\mathbb{R}^d$ of dimension α for every $\alpha \in (d-1, d)$. Of course, the unit sphere in $\mathbb{R}^d$ and $\mathbb{R}^d$ itself are explicit Salem sets with dimensions $d-1$ and d, respectively. Explicit examples of sets (Salem or otherwise) in $\mathbb{R}^d$ with Fourier dimension $\alpha \in (1, d-1)$ are unknown.

Theorems 3.0.1 and 3.0.2 generalize the theorems of Kaufman [29] and Bluhm

[5]. Theorem 3.0.1 gives many new explicit Salem sets in $\mathbb{R}$. Some particular Salem sets produced by Theorem 3.0.1 are discussed in Section 3.4. Theorem 3.0.2 gives new examples of explicit sets in $\mathbb{R}^d$ with Fourier dimension $\alpha > 1$.

3.3 Motivation: Metrical Diophantine Approximation

The second motivation for our main result comes from metrical Diophantine approximation, where there is considerable interest in the Hausdorff dimension of $E(Q, \Psi, \theta)$.

For $\tau \in \mathbb{R}$, define $\Psi_\tau : \mathbb{Z} \to [0, \infty)$ by $\Psi_\tau(q) = |q|^{-\tau}$. The classical Jarník-Besicovitch theorem [26], [3] is that

$$\dim_H E(\mathbb{Z}, \Psi_\tau, 0) = \min\left\{\frac{2}{1+\tau}, 1\right\}.$$

In the setting of restricted Diophantine approximation, where Q is not necessarily equal to $\mathbb{Z}$, Borosh and Fraenkel [8] showed that

$$\dim_H E(Q, \Psi_\tau, 0) = \min\left\{\frac{1+\nu(Q)}{1+\tau}, 1\right\},$$

where

$$\nu(Q) = \inf\left\{\nu \geq 0 : \sum_{\substack{q \in Q \\ q \neq 0}} |q|^{-\nu} < \infty\right\}.$$

Eggleston [15] previously obtained this result for certain sets Q with $\nu(Q) = 0$ or $\nu(Q) = 1$.

There are also several results for more general functions Ψ. For Ψ of the form $\Psi(q) = \psi(|q|)$ with $\psi : \mathbb{N} \to (0, \infty)$ decreasing, Dodson [14] showed that

$$\dim_H E(\mathbb{Z}, \Psi, 0) = \min\left\{\frac{2}{1+\lambda}, 1\right\},$$

where

$$\lambda = \liminf_{M\to\infty} \frac{-\ln \psi(M)}{\ln M}.$$

Hinokuma and Shiga [24] considered the non-monotone function $\Psi_\tau^{(HS)}(q) = |\sin q||q|^{-\tau}$ and proved that

$$\dim_H E(\mathbb{Z}, \Psi_\tau^{(HS)}, 0) = \min\left\{\frac{2}{1+\tau}, 1\right\}.$$

Dickinson [13] considered restricted Diophantine approximation with a function Ψ satisfying $\Psi(q) = \psi(|q|)$ with $\psi : \mathbb{N} \to (0,\infty)$ and

$$\lambda = \liminf_{M\to\infty} \frac{-\ln \psi(M)}{\ln M} = \limsup_{M\to\infty} \frac{-\ln \psi(M)}{\ln M}.$$

Dickinson deduced from the result of Borosh and Fraenkel above that

$$\dim_H E(Q, \Psi, 0) = \min\left\{\frac{1+\nu(Q)}{1+\lambda}, 1\right\}.$$

Rynne [40] proved a very general result that implies all of those above. Suppose only that $\Psi : \mathbb{Z} \to [0,\infty)$ is positive for all $q \in Q$. Let

$$\eta(Q,\Psi) = \inf\left\{\eta \geq 0 : \sum_{\substack{q\in Q\\ q\neq 0}} |q| \left(\frac{\Psi(q)}{|q|}\right)^\eta < \infty\right\}.$$

Rynne showed that

$$\dim_H E(Q, \Psi, 0) = \min\left\{\eta(Q,\Psi), 1\right\}.$$

The main result in the case of inhomogeneous Diophantine approximation (i.e, the case where θ is non-zero) is due to Levesley [32]. Levesley showed that if $\Psi(q) = \psi(|q|)$ with $\psi : \mathbb{N} \to (0,\infty)$ decreasing, and if

$$\lambda = \liminf_{M\to\infty} \frac{-\ln \psi(M)}{\ln M},$$

then

$$\dim_H E(\mathbb{Z}, \Psi, \theta) = \min\left\{\frac{2}{1+\lambda}, 1\right\}.$$

By an adaptation of Dickinson's argument from [13], the assumption that ψ is decreasing can be replaced by the assumption that

$$\lambda = \liminf_{M\to\infty} \frac{-\ln \psi(M)}{\ln M} = \limsup_{M\to\infty} \frac{-\ln \psi(M)}{\ln M}.$$

The main content of the formulas above is the lower bounds they give on the Hausdorff dimension of $E(Q, \Psi, \theta)$. The $\leq$-half of all the formulas for $\dim_H E(Q, \Psi, \theta)$ above are implied by the following lemma whose proof is well-known and straightforward. For completeness, we give the proof in Section 3.8.

Lemma 3.3.1.

$$\dim_H E(Q, \Psi, \theta) \leq \min\{\eta(Q, \Psi), 1\}.$$

Because of (3.2.1), the Fourier analytic method of Theorem 3.0.1 stands as an alternative to the usual methods of proving lower bounds on the Hausdorff dimension of $E(Q, \Psi, \theta)$. In fact, Theorem 3.0.1 implies or implies special cases of all the results for $\dim_H E(Q, \Psi, \theta)$ above (details are given in Section 3.4). Moreover, Theorem 3.0.1 allows us to calculate the Hausdorff dimension of $E(Q, \Psi, \theta)$ in cases that (as far as we know) have not been treated previously in the literature, such as the case where $\theta \neq 0$ and $Q \neq \mathbb{N}, \mathbb{Z}$.

One particular advantage of the Fourier analytic method of Theorem 3.0.1 is the ease with which it handles the inhomogeneous case $\theta \neq 0$. In the proof of Theorem 3.0.1 it is trivial to accommodate $\theta \neq 0$, while Levelsey's proof of his result for $\theta \neq 0$ is a non-trivial extension of Dodson's proof for $\theta = 0$.

There are analogs of the formulas above for $\dim_H E(m, n, Q, \Psi, \theta)$. For example, Rynne [40] proved

$$\dim_H E(m, n, Q, \Psi, 0) = \min\{m(n-1) + \eta(Q, \Psi), mn\}$$

when $\Psi : \mathbb{Z}^n \to [0, \infty)$ is positive for all $q \in Q$ and

$$\eta(Q, \Psi) = \inf \left\{ \eta \geq 0 : \sum_{\substack{q \in Q \\ q \neq 0}} |q|^m \left(\frac{\Psi(q)}{|q|} \right)^\eta < \infty \right\}.$$

Theorem 3.0.2 (via (3.2.1)) provides a lower bound on $\dim_H E(m, n, Q, \Psi, \theta)$, but it does not reach the true value of $\dim_H E(m, n, Q, \Psi, \theta)$ for any known case with $mn > 1$.

3.4 Applications

In this section we will present several consequences of Theorem 3.0.1 that give new families of explicit Salem sets and imply formulas for $\dim_H E(Q, \Psi, \theta)$ discussed in Section 3.3. We will also present a typical consequence of Theorem 3.0.2 that yields explicit sets in $\mathbb{R}^{mn}$ ($mn > 1$) with Fourier dimension larger than 1.

Theorem 3.4.1. *Assume Ψ is of the form $\Psi(q) = \psi(|q|)$ with $\psi : \mathbb{N} \to (0, \infty)$ a decreasing function. Assume there is an increasing function $h : (0, \infty) \to (0, \infty)$ such that*

$$|Q(M)| \geq M/h(M) \quad \forall M \in \mathbb{N} \tag{3.4.1}$$

and

$$\lim_{x \to \infty} \frac{\ln h(x)}{\ln x} = 0$$

Then $E(Q, \Psi, \theta)$ is a Salem set of dimension $\min\{2/(1 + \lambda), 1\}$*, where*

$$\lambda = \liminf_{M \to \infty} \frac{-\ln(\psi(M))}{\ln M}. \tag{3.4.2}$$

Proof. Since Ψ is bounded, $\lambda \geq 0$.

Let $\lambda' < \lambda$ and $\delta > 0$. By (3.4.2), $\psi(M) < M^{-\lambda'}$ for all large $M \in \mathbb{N}$. It follows that

$$\sum_{\substack{q\in Q\\ q\neq 0}} |q| \left(\frac{\Psi(q)}{|q|}\right)^{(2+\delta)/(1+\lambda')} \lesssim \sum_{\substack{q\in Q\\ q\neq 0}} |q|^{-(1+\delta)} < \infty.$$

Therefore, by applying Lemma 3.3.1 and then letting $\lambda' \to \lambda$ and $\delta \to 0$, we have

$$\dim_H E(Q, \Psi, \theta) \leq \min\left\{\frac{2}{1+\lambda}, 1\right\}.$$

If $\lambda = \infty$, this argument shows $\dim_H E(Q, \Psi, \theta) = \dim_F E(Q, \Psi, \theta) = 0$.

Assume $0 \leq \lambda < \infty$. Let $\lambda' > \lambda$. By (3.4.2), there is an infinite set $\mathcal{M} \subseteq \mathbb{N}$ such that $\psi(M) > M^{-\lambda'}$ for all $M \in \mathcal{M}$. Since ψ is decreasing, it follows that $\epsilon(M) > M^{-\lambda'}$ for all $M \in \mathcal{M}$. Combining this with (3.4.1), we see that (3.0.1) holds with $a = 1/(1+\lambda')$. Since $\lambda' > \lambda$ is arbitrary, Theorem 3.0.1 gives

$$\dim_F E(Q, \Psi, \theta) \geq \min\left\{\frac{2}{1+\lambda}, 1\right\}.$$

$\square$

Theorem 3.4.1 implies the result of Dodson [14] for $\dim_H E(\mathbb{Z}, \Psi, 0)$ discussed in Section 3.3. Theorem 3.4.1 also implies the formula for $\dim_H E(\mathbb{Z}, \Psi, \theta)$ due to Levesley [32] mentioned in Section 3.3.

Theorem 3.4.2. *Assume* Ψ *is of the form* $\Psi(q) = \psi(|q|)$ *with* $\psi : \mathbb{N} \to (0, \infty)$. *Assume*

$$\lambda = \liminf_{M\to\infty} \frac{-\ln \psi(M)}{\ln M} = \limsup_{M\to\infty} \frac{-\ln \psi(M)}{\ln M} \tag{3.4.3}$$

and

$$\sum_{\substack{q\in Q\\ q\neq 0}} |q|^{-1} = \infty. \tag{3.4.4}$$

Then $E(Q, \Psi, \theta)$ *is a Salem set of dimension* $\min\{2/(1+\lambda), 1\}$.

Proof. Since Ψ is bounded, $\lambda \geq 0$.

The same argument as in the proof of Theorem 3.4.1 shows

$$\dim_H E(Q, \Psi, \theta) \leq \min\left\{\frac{2}{1+\lambda}, 1\right\}.$$

If $\lambda = \infty$, the argument shows $\dim_H E(Q, \Psi, \theta) = \dim_F E(Q, \Psi, \theta) = 0$.

Assume $0 \leq \lambda < \infty$. Seeking a contradiction, suppose

$$|Q(M)| < M/\ln^2(M)$$

for all large $M \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{\substack{q\in Q\\ q\neq 0}} |q|^{-1} = \sum_{k=0}^{\infty} \sum_{q\in Q(2^k)} |q|^{-1} &\lesssim \sum_{k=0}^{\infty} 2^{-k} \sum_{q\in Q(2^k)} 1 \\ &\lesssim \sum_{k=0}^{\infty} \frac{1}{\ln^2(2^k)} = \frac{1}{\ln^2(2)} \sum_{k=0}^{\infty} \frac{1}{k^2} < \infty, \end{aligned}$$

which contradicts (3.4.4). So there is an infinite set $\mathcal{M} \subseteq \mathbb{N}$ such that

$$|Q(M)| \geq M/\ln^2(M) \quad \forall M \in \mathcal{M}.$$

Let $\lambda' > \lambda$ be given. By (3.4.3), $\psi(M) > M^{-\lambda'}$ for all large $M \in \mathbb{N}$. Therefore $\epsilon(M) > M^{-\lambda'}$ for all large $M \in \mathbb{N}$. After removing finitely elements of $\mathcal{M}$, we have

$$\epsilon(M) > M^{-\lambda'} \quad \forall M \in \mathcal{M}.$$

Then (3.0.1) holds with $a = 1/(1+\lambda')$ and $h(x) = \ln^2(x)$. Since $\lambda' > \lambda$ is arbitrary, Theorem 3.0.1 gives

$$\dim_F E(Q, \Psi, \theta) \geq \min\left\{\frac{2}{1+\lambda}, 1\right\}.$$

$\square$

Theorem 3.4.2 implies the result of Dickinson [13] for $\dim_H E(Q, \Psi, 0)$ discussed in Section 3.3 in the case $\nu(Q) = 1$. Consequently, it also implies the results of Borosh and Fraenkel [8] and Eggleston [15] in the case $\nu(Q) = 1$. Theorem 3.4.2 implies the variation of the result of Levesley [32] for $\dim_H E(\mathbb{Z}, \Psi, \theta)$ mentioned in Section 3.3 that uses Dickinson's argument from [13].

As far as we know, Theorems 3.4.1 and 3.4.2 represent the first calculation of $\dim_H E(Q, \Psi, \theta)$ in the case where $\theta \neq 0$ and $Q \neq \mathbb{N}, \mathbb{Z}$.

Theorem 3.4.3. *Suppose* $\Psi_\tau^{(HS)} : \mathbb{Z} \to (0, \infty)$ *is defined by* $\Psi_\tau^{(HS)}(q) = |\sin q||q|^{-\tau}$. *Then* $E(\mathbb{Z}, \Psi_\tau^{(HS)}, \theta)$ *is a Salem set of dimension of* $\min\{2/(1+\tau), 1\}$.

Proof. For every $\epsilon > 0$,

$$\sum_{\substack{q \in \mathbb{Z} \\ q \neq 0}} |q| \left(\frac{\Psi(q)}{|q|} \right)^{2/(1+\tau)+\epsilon} = \sum_{\substack{q \in \mathbb{Z} \\ q \neq 0}} |q|^{-1-(1+\tau)\epsilon} |\sin q|^{2/(1+\tau)+\epsilon} < \infty.$$

So, by Lemma 3.3.1,

$$\dim_H E(\mathbb{Z}, \Psi_\tau^{(HS)}, \theta) \leq \min\left\{ \frac{2}{1+\tau}, 1 \right\}.$$

Let $Q = \{q \in \mathbb{N} : |\sin q| \geq 1/2\}$. Clearly

$$\min_{q \in Q(M)} \Psi_\tau^{(HS)}(q) \geq \frac{1}{2} M^{-\tau} \quad \forall M \in \mathbb{N}.$$

It is also easy to see that

$$|Q(M)| \geq \frac{M}{4\pi}$$

for all large M (for instance, by noting Q contains the nearest integer(s) to $(2k+1)\pi/2$ for every $k \in \mathbb{N}$). It follows that (3.0.1) holds with $a = 1/(1+\tau)$ and $h(x) = 4\pi$. Therefore Theorem 3.0.1 and the fact $E(Q, \Psi_\tau^{(HS)}, \theta) \subseteq E(\mathbb{Z}, \Psi_\tau^{(HS)}, \theta)$ implies

$$\dim_F E(\mathbb{Z}, \Psi_\tau^{(HS)}, \theta) \geq \min\left\{ \frac{2}{1+\tau}, 1 \right\}.$$

$\square$

Theorem 3.4.3 implies the formula for $\dim_H E(\mathbb{Z}, \Psi_\tau^{(HS)}, 0)$ due to Hinokuma and Shiga [24] mentioned in Section 3.3.

Finally, we give a typical consequence of Theorem 3.0.2 that yields explicit sets in $\mathbb{R}^{mn}$ with Fourier dimension larger than 1. The Hausdorff dimension of the sets is also determined for comparison.

Theorem 3.4.4. *Assume $\Psi : \mathbb{Z}^n \to [0, \infty)$ is of the form $\Psi(q) = \psi(|q|)$ with $\psi : \mathbb{N} \to (0, \infty)$ and*

$$\lambda = \liminf_{M\to\infty} \frac{-\ln \psi(M)}{\ln M} = \limsup_{M\to\infty} \frac{-\ln \psi(M)}{\ln M}. \tag{3.4.5}$$

Assume $h : (0, \infty) \to (0, \infty)$ is an increasing function such that

$$\lim_{x\to\infty} \frac{\ln h(x)}{\ln x} = 0. \tag{3.4.6}$$

Assume there is an unbounded set $\mathcal{M} \subseteq (0, \infty)$ such that

$$|Q(M)|h(M) \geq M^n \quad \forall M \in \mathcal{M}. \tag{3.4.7}$$

Then

$$\dim_H E(m, n, Q, \Psi, 0) = \min\left\{m(n-1) + \frac{m+n}{1+\lambda}, mn\right\}$$

and

$$\dim_F E(m, n, Q, \Psi, 0) \geq \min\left\{\frac{2n}{1+\lambda}, mn\right\}.$$

Proof. Since Ψ is bounded, $\lambda \geq 0$.

Let $\lambda' < \lambda$ and $\delta > 0$. By (3.4.5), $\psi(M) < M^{-\lambda'}$ for all large $M \in \mathbb{N}$. It follows that

$$\sum_{\substack{q\in Q\\ q\neq 0}} |q|^m \left(\frac{\Psi(q)}{|q|}\right)^{(m+n+\delta)/(1+\lambda')} \lesssim \sum_{\substack{q\in Q\\ q\neq 0}} |q|^{-(n+\delta)} < \infty.$$

Since $\lambda' < \lambda$ and $\delta > 0$ are arbitrary, we have

$$\sum_{\substack{q \in Q \\ q \neq 0}} |q|^m \left(\frac{\Psi(q)}{|q|} \right)^\eta < \infty \quad \forall \eta > \frac{m+n}{1+\lambda}.$$

Let $\lambda' > \lambda$ and $\delta > 0$. By (3.4.5), $\psi(M) > M^{-\lambda'}$ for all large $M \in \mathbb{N}$. Choose a sequence $(M_k)_{k \in \mathbb{N}}$ of numbers in $\mathcal{M}$ such that $M_k \geq 2M_{k-1}$. Then

$$\begin{aligned}
\sum_{\substack{q \in Q \\ q \neq 0}} |q|^m \left(\frac{\Psi(q)}{|q|} \right)^{(m+n-\delta)/(1+\lambda')} &\gtrsim \sum_{\substack{q \in Q \\ q \neq 0}} |q|^{-(n-\delta)} \gtrsim \sum_{k \in \mathbb{N}} \sum_{q \in Q(M_k)} |q|^{-(n-\delta)} \\
&\gtrsim \sum_{k \in \mathbb{N}} M_k^{-(n-\delta)} |Q(M_k)| \gtrsim \sum_{k \in \mathbb{N}} M_k^{\delta} (h(M_k))^{-1} \\
&\gtrsim \sum_{k \in \mathbb{N}} M_k^{\delta/2} = \infty
\end{aligned}$$

Since $\lambda' > \lambda$ and $\delta > 0$ are arbitrary, we have

$$\sum_{\substack{q \in Q \\ q \neq 0}} |q|^m \left(\frac{\Psi(q)}{|q|} \right)^\eta = \infty \quad \forall \eta < \frac{m+n}{1+\lambda}.$$

By the result of Rynne [40] for $\dim_H E(m,n,Q,\Psi,0)$ discussed in Section 3.3, we have

$$\dim_H E(m,n,Q,\Psi,0) = \min \left\{ m(n-1) + \frac{m+n}{1+\lambda}, mn \right\}.$$

If $\lambda = \infty$, this argument shows $\dim_H E(m,n,Q,\Psi,0) = m(n-1)$.

Assume $0 \leq \lambda < \infty$. Let $\lambda' > \lambda$. By (3.4.5), $\psi(M) > M^{-\lambda'}$ for all large $M \in \mathbb{N}$. Therefore $\epsilon(M) > M^{-\lambda'}$ for all large $M \in \mathbb{N}$. After removing finitely many elements of $\mathcal{M}$, we have $\epsilon(M) > M^{-\lambda'}$ for all $M \in \mathcal{M}$. Combining this with (3.4.7), we see that (3.0.3) holds with $a = n/(1+\lambda')$. As $\lambda' > \lambda$ is arbitrary, Theorem 3.0.2 implies

$$\dim_F E(m,n,Q,\Psi,0) \geq \min \left\{ \frac{2n}{1+\lambda}, mn \right\}.$$

$\square$

3.5 Proof of Theorems 3.0.1 and 3.0.2: The Key Function

We start with two basic definitions.

Suppose $f : \mathbb{R}^d \to \mathbb{C}$. If $f \in L^1(\mathbb{R}^d)$, the Fourier transform of f is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\cdot\xi}dx \quad \forall \xi \in \mathbb{R}^d.$$

If $f \in L^1([0,1]^d)$ (i.e., $\int_{[0,1]^d} |f(x)|dx < \infty$) and f is periodic for the lattice $\mathbb{Z}^d$ (i.e., $f(x) = f(x+k)$ for all $x \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$), the Fourier transform of f is defined by

$$\widehat{f}(\xi) = \int_{[0,1]^d} f(x)e^{-2\pi i x\cdot\xi}dx \quad \forall \xi \in \mathbb{R}^d.$$

There is no ambiguity with these definitions; if $f \in L^1(\mathbb{R}^d)$ and f is periodic for the lattice $\mathbb{Z}^d$, then $\widehat{f} = 0$ using either definition.

Now we begin the construction.

Let K be a positive integer with $K > mn + a$. Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be a non-negative C^K function with $\text{supp}(\phi) \subseteq [-1,1]^m$ and $\int_{\mathbb{R}^m} \phi = 1$. Then there is a $C_1 > 0$ such that

$$|\widehat{\phi}(\xi)| \leq C_1(1+|\xi|)^{-K} \quad \forall \xi \in \mathbb{R}^m. \tag{3.5.1}$$

For $\epsilon > 0$ and $x \in \mathbb{R}^m$, let $\phi^\epsilon(x) = \epsilon^{-m}\phi(\epsilon^{-1}x)$, and

$$\Phi^\epsilon(x) = \sum_{k\in\mathbb{Z}^m} \phi^\epsilon(x-k).$$

Note Φ^ϵ is C^K, is periodic for the lattice $\mathbb{Z}^m$, and satisfies

$$\widehat{\Phi^\epsilon}(k) = \widehat{\phi^\epsilon}(k) = \widehat{\phi}(\epsilon k) \quad \forall k \in \mathbb{Z}^m.$$

Therefore

$$\Phi^\epsilon(x) = \sum_{k\in\mathbb{Z}^m} \widehat{\phi}(\epsilon k)e^{2\pi i k x} \quad \forall x \in \mathbb{R}^m$$

with uniform convergence.

For $q \in \mathbb{Z}^n$, $\theta \in \mathbb{R}^m$, and $x = (x_{11}, \dots, x_{1n}, \dots, x_{m1}, \dots, x_{mn}) \in \mathbb{R}^{mn}$, define xq by identifying x with the $m \times n$ matrix whose ij-entry is x_{ij}, and define

$$\Phi^\epsilon_{q,\theta}(x) = \Phi^\epsilon(xq - \theta) = \sum_{k \in \mathbb{Z}^m} \widehat{\phi}(\epsilon k) e^{2\pi i k \cdot (xq - \theta)}.$$

Note $\Phi^\epsilon_{q,\theta}$ is C^K and is periodic for the lattice $\mathbb{Z}^{mn}$.

Assume from now on that $q_j \neq 0$ for all $1 \leq j \leq n$.

If $mn = 1$, then for $\ell \in \mathbb{Z}$ we have

$$\begin{aligned}
\widehat{\Phi^\epsilon_{q,\theta}}(\ell) &= \sum_{k \in \mathbb{Z}} \int_{[0,1]} \widehat{\phi}(\epsilon k) e^{2\pi i k (xq - \theta)} e^{-2\pi i \ell x} dx \\
&= \sum_{k \in \mathbb{Z}} e^{-2\pi i k \theta} \widehat{\phi}(\epsilon k) \int_{[0,1]} e^{2\pi i x (kq - \ell)} dx \\
&= \begin{cases} e^{-2\pi i q^{-1} \ell \theta} \widehat{\phi}(\epsilon q^{-1} \ell) & \text{if } q^{-1}\ell \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

In general, for $\ell = (\ell_{11}, \dots, \ell_{1n}, \dots, \ell_{m1}, \dots, \ell_{mn}) \in \mathbb{Z}^{mn}$ we have

$$\begin{aligned}
\widehat{\Phi^\epsilon_{q,\theta}}(\ell) &= \sum_{k \in \mathbb{Z}^m} \int_{[0,1]^{mn}} \widehat{\phi}(\epsilon k) e^{2\pi i k \cdot (xq - \theta)} e^{-2\pi i \ell \cdot x} dx \\
&= \sum_{k \in \mathbb{Z}^m} e^{-2\pi i k \cdot \theta} \widehat{\phi}(\epsilon k) \int_{[0,1]^{mn}} e^{2\pi i (k \cdot (xq - \theta) - \ell \cdot x)} dx \\
&= \sum_{k \in \mathbb{Z}^m} e^{-2\pi i k \cdot \theta} \widehat{\phi}(\epsilon k) \prod_{i=1}^{m} \prod_{j=1}^{n} \int_{[0,1]} e^{2\pi i x_{ij} (k_i q_j - \ell_{ij})} dx_{ij} \\
&= \begin{cases} e^{-2\pi i q_1^{-1} \ell_1 \cdot \theta} \widehat{\phi}(\epsilon q_1^{-1} \ell_1) & \text{if } q \in \widetilde{Q}(\ell), \\ 0 & \text{otherwise,} \end{cases}
\end{aligned} \tag{3.5.2}$$

where

$$\widetilde{Q}(\ell) = \left\{ q \in \mathbb{Z}^n : q_1^{-1} \ell_1 = \cdots = q_n^{-1} \ell_n \in \mathbb{Z}^m \right\}$$

and $\ell_j = (\ell_{1j}, \dots, \ell_{mj})$ for all $1 \leq j \leq n$.

For $M > 0$, define

$$F_M(x) = \frac{1}{|Q(M)|} \sum_{q \in Q(M)} \Phi_{q,\theta}^{\epsilon(M)}(x) \quad \forall x \in \mathbb{R}^{mn}.$$

This is the key function in the proof.

Note F_M is C^K and periodic for the lattice $\mathbb{Z}^{mn}$. Since $\operatorname{supp}(\phi) \subseteq [-1,1]^m$ and $\epsilon(M) = \min_{q \in Q(M)} \Psi(q)$, we have

$$\operatorname{supp}(F_M) \subseteq \{x \in \mathbb{R}^{mn} : \max_{1 \le i \le m} \| \sum_{j=1}^{n} x_{ij} q_j - \theta_i \| \le \Psi(q) \text{ for some } q \in Q(M)\}.$$

If $q \in Q(M)$ and $0 < |\ell| \le M/2$, then $q \notin \widetilde{Q}(\ell)$. So by (3.5.2) we have

$$\widehat{F_M}(\ell) = 0 \quad \text{for } 0 < |\ell| \le M/2. \tag{3.5.3}$$

As $\phi \ge 0$ and $\widehat{\phi}(0) = 1$, (3.5.2) also implies

$$|\widehat{F_M}(\ell)| \le \widehat{F_M}(0) = 1 \quad \forall \ell \in \mathbb{Z}^{mn}. \tag{3.5.4}$$

Choose $i_0 \in \{1, \ldots, m\}$ and $j_0 \in \{1, \ldots, n\}$ such that $|\ell| = |\ell_{j_0}| = |\ell_{i_0 j_0}|$. By (3.5.1) and (3.5.2), for all $\ell \in \mathbb{Z}^{mn}$ we have

$$\begin{aligned}
|\widehat{F_M}(\ell)| &\le C_1 \frac{1}{|Q(M)|} \sum_{q \in Q(M) \cap \widetilde{Q}(\ell)} |\widehat{\phi}(\epsilon(M) q_1^{-1} \ell_1)| \\
&\le C_1 \frac{1}{|Q(M)|} \sum_{q \in Q(M) \cap \widetilde{Q}(\ell)} (1 + \epsilon(M) |q_1|^{-1} |\ell_1|)^{-K} \\
&= C_1 \frac{1}{|Q(M)|} \sum_{q \in Q(M) \cap \widetilde{Q}(\ell)} (1 + \epsilon(M) |q_{j_0}|^{-1} |\ell_{j_0}|)^{-K} \\
&\le C_1 \frac{|Q(M) \cap \widetilde{Q}(\ell)|}{|Q(M)|} (1 + \epsilon(M) M^{-1} |\ell|)^{-K}.
\end{aligned}$$

To bound $|Q(M)\cap\widetilde{Q}(\ell)|$, first note that for $\ell\neq 0$ we have

$$\widetilde{Q}(\ell)\subseteq\left\{(q_1,\ldots,q_n)\in\mathbb{Z}^n:\frac{\ell_{i_0j_0}}{q_{j_0}}\in\mathbb{Z},\ q_j=\frac{\ell_{i_0j}q_{j_0}}{\ell_{i_0j_0}}\quad\forall 1\leq j\leq n\right\}.$$

It follows that

$$|Q(M)\cap\widetilde{Q}(\ell)|\leq|\widetilde{Q}(\ell)|\leq 2\tau(|\ell|),$$

where $\tau(v)$ denotes the number of positive integer divisors of the integer v. The divisor bound of Wigert [48] (cf. [21, p. 262]) says

$$\limsup_{v\to\infty}\frac{\ln\tau(v)}{\ln v/\ln\ln v}=\ln 2.$$

For $\zeta>0$, define

$$f_\zeta(x)=\exp\left(\frac{\zeta\ln x}{\ln\ln x}\right)\quad\forall x>e.$$

So for any $\zeta>\ln 2$ there is a $v_\zeta\in\mathbb{N}$ such that

$$\tau(v)\leq f_\zeta(v)\quad\forall v\in\mathbb{N},\ v\geq v_\zeta.$$

Therefore

$$|\widehat{F_M}(\ell)|\leq 2C_1\frac{f_\zeta(|\ell|)}{|Q(M)|}(1+\epsilon(M)M^{-1}|\ell|)^{-K}\quad\forall\ell\in\mathbb{Z}^{mn},|\ell|\geq v_\zeta. \tag{3.5.5}$$

3.6 Proof of Theorems 3.0.1 and 3.0.2: The Main Lemma

Define

$$g(\xi)=\begin{cases}1 & \text{if } |\xi|\leq e\\ |\xi|^{-a}f_1(|\xi|)h(4|\xi|) & \text{if } |\xi|>e\end{cases}$$

Lemma 3.6.1. *For every $\delta>0$, $M_0>0$, and $\chi\in C_c^K(\mathbb{R}^{mn})$, there is an $M_*=M_*(\delta,M_0,\chi)\in\mathcal{M}$ such that $M_*\geq M_0$ and*

$$|\widehat{\chi F_{M_*}}(\xi)-\widehat{\chi}(\xi)|\leq\delta g(\xi)\quad\forall\xi\in\mathbb{R}^{mn}.$$

The proof will show M_* can be taken to be any sufficiently large element of $\mathcal{M}$.

Proof. Since $\chi \in C_c^K(\mathbb{R}^{mn})$, there is a $C_2 > 0$ such that

$$|\widehat{\chi}(\xi)| \le C_2(1+|\xi|)^{-K} \quad \forall \xi \in \mathbb{R}^{mn}. \tag{3.6.1}$$

For every $p > mn$, we have

$$\sup_{\xi \in \mathbb{R}^{mn}} \sum_{\ell \in \mathbb{Z}^{mn}} (1+|\xi-\ell|)^{-p} < \infty. \tag{3.6.2}$$

Since F_M is C^K and periodic for the lattice $\mathbb{Z}^{mn}$, we have

$$F_M(x) = \sum_{\ell \in \mathbb{Z}^{mn}} \widehat{F_M}(\ell) e^{2\pi i \ell \cdot x} \quad \forall x \in \mathbb{R}^{mn}$$

with uniform convergence. Since $\chi \in L^1(\mathbb{R}^{mn})$, multiplying by χ and taking the Fourier transform yields

$$\widehat{\chi F_M}(\xi) = \sum_{\ell \in \mathbb{Z}^{mn}} \widehat{F_M}(\ell) \int_{\mathbb{R}^{mn}} \chi(x) e^{2\pi i(\ell-\xi)\cdot x} dx = \sum_{\ell \in \mathbb{Z}^{mn}} \widehat{F_M}(\ell) \widehat{\chi}(\xi - \ell)$$

for all $\xi \in \mathbb{R}^{mn}$. Then by (3.5.4) and (3.5.3) we have

$$\widehat{\chi F_M}(\xi) - \widehat{\chi}(\xi) = \sum_{|\ell| \ge M/2} \widehat{\chi}(\xi - \ell) \widehat{F_M}(\ell). \tag{3.6.3}$$

Case 1: $|\xi| \le M/4$. If $|\ell| \ge M/2$, then $|\xi - \ell| \ge M/4 \ge |\xi|$. Hence by (3.5.3), (3.6.1), (3.6.2) and (3.6.3) we have

$$\begin{aligned}
|\widehat{\chi F_M}(\xi) - \widehat{\chi}(\xi)| &\le C_2 \sum_{|\ell| \ge M/2} (1+|\xi-\ell|)^{-K} \\
&\le C_2(1+|\xi|)^{-a}(1+M/4)^{-(K-a-mn)/2} \\
&\qquad\qquad \sum_{|\ell| \ge M/2} (1+|\xi-\ell|)^{-mn-(K-a-mn)/2} \\
&\le \delta g(\xi)
\end{aligned}$$

for M sufficiently large.

Case 2: $|\xi| \geq M/4$. Using (3.6.3), write

$$\widehat{\chi F_M}(\xi) - \widehat{\chi}(\xi) = \sum_{\substack{|\ell| \geq M/2 \\ |\ell| \leq |\xi|/2}} \widehat{\chi}(\xi - \ell)\widehat{F_M}(\ell) + \sum_{\substack{|\ell| \geq M/2 \\ |\ell| > |\xi|/2}} \widehat{\chi}(\xi - \ell)\widehat{F_M}(\ell) = S_1 + S_2.$$

If $|\ell| \leq |\xi|/2$, then $|\xi - \ell| \geq |\xi|/2 \geq M/8$. Hence by (3.5.3), (3.6.1), and (3.6.2) we have

$$\begin{aligned}
|S_1| &\leq C_2 \sum_{\substack{|\ell| \geq M/2 \\ |\ell| \leq |\xi|/2}} (1 + |\xi - \ell|)^{-K} \\
&\leq C_2 (1 + |\xi|/2)^{-a} (1 + M/8)^{-(K-a-mn)/2} \sum_{\substack{|\ell| \geq M/2 \\ |\ell| \leq |\xi|/2}} (1 + |\xi - \ell|)^{-mn-(K-a-mn)/2} \\
&\leq \frac{\delta}{2} g(\xi)
\end{aligned}$$

for M sufficiently large.

Fix $\ln 2 < \zeta < 1$. By (3.0.1), (3.5.5), (3.6.1), (3.6.2) and because $K > a$ we have

$$\begin{aligned}
|S_2| &\leq 2C_1C_2 \sum_{\substack{|\ell| \geq M/2 \\ |\ell| > |\xi|/2}} \frac{f_\zeta(|\ell|)}{|Q(M)|} (1 + \epsilon(M)M^{-1}|\ell|)^{-K} (1 + |\xi - \ell|)^{-K} \\
&\leq 2C_1C_2 \sum_{\substack{|\ell| \geq M/2 \\ |\ell| > |\xi|/2}} |\ell|^{-a} f_\zeta(|\ell|) \frac{(\epsilon(M)M^{-1})^{-a}}{|Q(M)|} (1 + |\xi - \ell|)^{-K} \\
&\leq 2C_1C_2 (|\xi|/2)^{-a} f_\zeta(|\xi|/2) h(M) \sum_{\substack{|\ell| \geq M/2 \\ |\ell| > |\xi|/2}} (1 + |\xi - \ell|)^{-K} \\
&\leq \frac{\delta}{2} g(\xi)
\end{aligned}$$

for all sufficiently large $M \in \mathcal{M}$. $\square$

3.7 Proof of Theorems 3.0.1 and 3.0.2: The Measure

Let $\chi_0 \in C^K(\mathbb{R}^{mn})$ with $\int_{\mathbb{R}^{mn}} \chi_0(x)dx = 1$, $\operatorname{supp}(\chi_0) = [-1,1]^{mn}$, and $\chi_0(x) > 0$ for all $|x| < 1$. With the notation of Lemma 3.6.1, define

$$M_1 = M_*(2^{-1}, 1, \chi_0), \quad M_k = M_*(2^{-k-1}, 2M_{k-1}, \chi_0 F_{M_1} \cdots F_{M_{k-1}}) \quad \forall k \geq 2.$$

For $k \in \mathbb{N}_0$, define the measure μ_k by

$$d\mu_k = \chi_0 F_{M_1} \cdots F_{M_k} dx.$$

By Lemma 3.6.1,

$$|\widehat{\mu_k}(\xi) - \widehat{\mu_{k-1}}(\xi)| \leq 2^{-k-1} g(\xi) \quad \forall \xi \in \mathbb{R}^{mn}, k \in \mathbb{N}. \tag{3.7.1}$$

Since $g(\xi)$ is bounded , (3.7.1) implies $(\widehat{\mu_k})_{k\in\mathbb{N}_0}$ is a Cauchy sequence in the supremum norm. Therefore, since each $\widehat{\mu_k}$ is a continuous function, $\lim\limits_{k\to\infty} \widehat{\mu_k}$ is a continuous function. By (3.7.1) we have

$$|\lim_{k\to\infty} \widehat{\mu_k}(\xi) - \widehat{\mu_0}(\xi)| \leq \sum_{k=1}^{\infty} |\widehat{\mu_k}(\xi) - \widehat{\mu_{k-1}}(\xi)| \leq g(\xi) \sum_{k=1}^{\infty} 2^{-k-1} = \frac{1}{2} g(\xi) \tag{3.7.2}$$

for all $\xi \in \mathbb{R}^{mn}$. Since $\widehat{\mu_0}(0) = g(0) = 1$, it follows from (3.7.2) that

$$1/2 \leq |\lim_{k\to\infty} \widehat{\mu_k}(0)| \leq 3/2.$$

Therefore, by Lévy's continuity theorem, $(\mu_k)_{k\in\mathbb{N}_0}$ converges weakly to a non-trivial finite measure μ with $\widehat{\mu} = \lim\limits_{k\to\infty} \widehat{\mu_k}$ and

$$\operatorname{supp}(\mu) = \bigcap_{k=1}^{\infty} \operatorname{supp}(\mu_k) = \operatorname{supp}(\chi_0) \cap \bigcap_{k=1}^{\infty} \operatorname{supp}(F_{M_k}).$$

Because $M_k \geq 2M_{k-1}$ and because (as we noted in Section 3.5)

$$\operatorname{supp}(F_M) \subseteq \{x \in \mathbb{R}^{mn} : \max_{1\leq i\leq m} \| \sum_{j=1}^{n} x_{ij}q_j - \theta_i\| \leq \Psi(q) \text{ for some } q \in Q(M)\},$$

we have

$$\operatorname{supp}(\mu) \subseteq E(m, n, Q, \Psi, \theta).$$

Since $\chi_0 \in C_c^K$ and $K > a$, we have $|\widehat{\mu_0}(\xi)| \lesssim (1 + |\xi|)^{-a}$ for all $\xi \in \mathbb{R}^{mn}$. Combining this with (3.7.2) gives

$$|\widehat{\mu}(\xi)| \lesssim g(\xi) \quad \forall \xi \in \mathbb{R}^{mn}.$$

By multiplying μ by a constant, we can make μ a probability measure. This completes the proof of Theorems 3.0.1 and 3.0.2.

3.8 Proof of Lemma 3.3.1

Lemma 3.8.1 (Restatement of Lemma 3.3.1)**.**

$$\dim_H E(Q, \Psi, \theta) \leq \min \{\eta(Q, \Psi), 1\},$$

where

$$\eta(Q, \Psi) = \inf \left\{ \eta \geq 0 : \sum_{\substack{q \in Q \\ q \neq 0}} |q| \left(\frac{\Psi(q)}{|q|} \right)^{\eta} < \infty \right\}.$$

Proof. Since $\dim_H E(Q, \Psi, \theta) \leq \dim_H \mathbb{R} \leq 1$, we only need to prove

$$\dim_H E(Q, \Psi, \theta) \leq \eta(Q, \Psi).$$

Note $E(Q,\Psi,\theta)$ is invariant under translation by integers. Therefore

$$\begin{aligned}
\dim_H E(Q,\Psi,\theta) &= \dim_H \bigcup_{k\in\mathbb{Z}} E(Q,\Psi,\theta)\cap([0,1]+k)\\
&= \dim_H \bigcup_{k\in\mathbb{Z}} (E(Q,\Psi,\theta)-k)\cap[0,1]\\
&= \dim_H \bigcup_{k\in\mathbb{Z}} \dim_H E(Q,\Psi,\theta)\cap[0,1]\\
&= \dim_H E(m,n,Q,\Psi,\theta)\cap[0,1].
\end{aligned}$$

So it suffices to prove

$$\dim_H E(Q,\Psi,\theta)\cap[0,1] \le \eta(Q,\Psi).$$

Therefore, according to the definition of Hausdorff dimension, it will suffice to show that for all $\epsilon > 0$ and all $\eta > \eta(Q,\Psi)$ there is a countable collection $\mathcal{I}$ of intervals that covers $E(Q,\Psi,\theta)\cap[0,1]$ and satisfies

$$\sum_{I\in\mathcal{I}} (\mathrm{diam}(I))^\eta < \epsilon.$$

Let $\epsilon > 0$ and $\eta > \eta(Q,\Psi)$. Define $C = |\theta| + \sup_{q\in\mathbb{Z}} \Psi(q)$. Observe that

$$E(Q,\Psi,\theta)\cap[0,1] = \bigcap_{N\in\mathbb{N}} \bigcup_{\substack{q\in Q\\|q|\ge N}} \bigcup_{\substack{k\in\mathbb{Z}\\|k|\le C+|q|}} \{x\in[0,1] : |xq-\theta-k| \le \Psi(q)\}.$$

Let $N\in\mathbb{N}$. Then

$$\begin{aligned}
E(Q,\Psi,\theta)\cap[0,1] &\subseteq \bigcup_{\substack{q\in Q\\|q|\ge N}} \bigcup_{\substack{k\in\mathbb{Z}\\|k|\le C+|q|}} \{x\in[0,1] : |xq-\theta-k| \le \Psi(q)\}\\
&\subseteq \bigcup_{\substack{q\in Q\\|q|\ge N}} \bigcup_{\substack{k\in\mathbb{Z}\\|k|\le C+|q|}} I_{q,k},
\end{aligned}$$

where

$$I_{q,k} = [(\theta + k - \Psi(q))/|q|, (\theta + k + \Psi(q))/|q|].$$

We have

$$\begin{aligned}
\sum_{\substack{q\in Q\\|q|\geq N}} \sum_{\substack{k\in\mathbb{Z}\\|k|\leq C+|q|}} (\operatorname{diam}(I_{q,k}))^\eta &= \sum_{\substack{q\in Q\\|q|\geq N}} \sum_{\substack{k\in\mathbb{Z}\\|k|\leq C+|q|}} 2^\eta \left(\frac{\Psi(q)}{|q|}\right)^\eta \\
&\leq \sum_{\substack{q\in Q\\|q|\geq N}} (2(C+|q|)+1)2^\eta \left(\frac{\Psi(q)}{|q|}\right)^\eta \\
&\lesssim \sum_{\substack{q\in Q\\|q|\geq N}} |q| \left(\frac{\Psi(q)}{|q|}\right)^\eta .
\end{aligned}$$

The last sum converges because $\eta > \eta(Q, \Psi)$. So, by taking N sufficiently large, we can make the sum less than ϵ. $\square$

Chapter 4

Conclusion

In this chapter, we summarize the contributions of this thesis and discuss directions for future work.

4.1 Contributions

In this section, we summarize the contributions of this thesis.

The first major contribution of this thesis concerns the L^2 Fourier restriction theorem due to Mockenhaupt [37] and Mitsis [35] (with the endpoint due to Bak and Seeger [1]). We state the theorem again here for convenience.

Theorem 4.1.1. *Let μ be a compactly supported probability measure on $\mathbb{R}^d$. Suppose there are $\alpha, \beta \in (0, d)$ such that*

$$\text{(A)} \qquad \mu(B(x,r)) \lesssim r^{\alpha} \quad \forall x \in \mathbb{R}^d, r > 0,$$

$$\text{(B)} \qquad |\widehat{\mu}(\xi)| \lesssim (1+|\xi|)^{-\beta/2} \quad \forall \xi \in \mathbb{R}^d.$$

Then for all $p \geq 2(2d - 2\alpha + \beta)/\beta$ we have

$$\|\widehat{f\mu}\|_p \lesssim \|f\|_{L^2(\mu)} \quad \forall f \in L^2(\mu). \tag{4.1.1}$$

Restriction theorems are some of the main objects of study in harmonic analysis.

They represent an important way to understand the nature of the Fourier transform and the geometry of sets and measures. Restriction theorems are connected to many other important problems in mathematical analysis, including the Kakeya conjecture, the Bochner-Riesz conjecture, the estimation of solutions to the wave, Schroedinger, and Helmholtz equations, and the local smoothing conjecture for partial differential equations (cf. [45]).

Once a restriction theorem such as Theorem 4.1.1 has been established, it is natural to ask whether it can be improved. In this thesis, we considered whether we can enlarge the range of p in the theorem by replacing $2(2d-2\alpha+\beta)/\beta$ with a smaller number. We considered this question only for a particular class of measures. If we do not limit the class of measures considered, the well-known Knapp example (cf. [49, Chapter 7]) shows the range of p cannot be enlarged. Whereas the classic restriction theorems established before Theorem 4.1.1 applied only to smooth curved surfaces (like spheres) in $\mathbb{R}^d$, $d \geq 2$, the novel aspect of Theorem 4.1.1 is that it applies to measures supported on fractal sets and to measures on $\mathbb{R}$. In this thesis, we have considered the optimality of the range of p in Theorem 4.1.1 for measures on $\mathbb{R}$. In particular, we proved the following theorem.

Theorem 4.1.2. *Given any $0 < \beta \leq \alpha < 1$ and given any $p_* < 2(2-2\alpha+\beta)/\beta$ there is a compactly supported probability measure on $\mathbb{R}$ that satisfies (A) and (B) but does not satisfy (4.1.1) for every $p \leq p_*$.*

The theorem is important and interesting because there is no obvious analog of the Knapp example (with its small, almost flat, spherical cap) available to prove the optimality of the range of p in Theorem 4.1.1 when considering measures on the real line.

The second major contribution of this thesis is the construction of new explicit Salem sets in $\mathbb{R}$ and new explicit sets in $\mathbb{R}^d$ with large Fourier dimension.

There are many random constructions of Salem sets (cf. [41], [27], [28], [6], [31]). Moreover, Kahane [28, Chapter 18] showed that the image of any compact set in $\mathbb{R}^d$ under fractional Brownian motion is almost surely a Salem set, which illustrates that Salem sets are ubiquitous as random sets. These random constructions do not produce any explicit examples of Salem sets. Moreover, besides the classic examples of Salem sets in $\mathbb{R}^d$ of dimensions 0, $d-1$, and d, explicit examples of Salem sets are very difficult to find. The first (and essentially only) construction of explicit Salem sets with dimensions other than 0, $d-1$, or d is due to Kaufman [29], who showed that

$$\left\{x \in \mathbb{R} : \|qx\| \leq |q|^{-\tau} \text{ for infinitely many } q \in \mathbb{Z}\right\}$$

is a Salem set of dimension $2/(1+\tau)$ when $\tau > 1$. Bluhm [7] generalized this by showing that replacing $|q|^{-\tau}$ by $\psi(|q|)$ for any decreasing function $\psi : \mathbb{N} \to (0, \infty)$ still gives a Salem set. In this thesis, we further generalized the results of Kaufman and Bluhm. For $Q \subseteq \mathbb{Z}$ infinite, $\Psi : \mathbb{Z} \to [0, \infty)$, and $\theta \in \mathbb{R}$, we proved a lower bound on the Fourier dimension of

$$E(Q, \Psi, \theta) = \{x \in \mathbb{R} : \|qx - \theta\| \leq \Psi(q) \text{ for infinitely many } q \in Q\}.$$

Moreover, we showed that this set is Salem for many choices of Q and Ψ (regardless of the value of θ). Explicit examples of sets in $\mathbb{R}^d$ with Fourier dimension larger than 1 are also hard to find. We gave in this thesis some new explicit examples of such sets. Furthermore, we applied our results to compute the Hausdorff dimension of $E(Q, \Psi, \theta)$ in new cases, which is of interest in metrical Diophantine approximation.

4.2 Directions for Future Work

In this section, we discuss directions for future work related to the content of this thesis.

We first note that Chen [11] has obtained the following improved version of Theorem 4.1.2 by modifying the argument of Łaba and myself from [23].

Theorem 4.2.1. *Given any $0 < \beta \leq \alpha < 1$, there is a compactly supported probability measure on $\mathbb{R}$ that satisfies (A) and (B) but does not satisfy (4.1.1) for every $p \leq 2(2d - 2\alpha + \beta)/\beta$.*

There are several interesting ways Theorems 4.1.2 and 4.2.1 could possibly be extended. We briefly discuss two of them.

First, one could try to obtain a version of Theorem 4.1.2 or 4.2.1 with $\beta > \alpha$. Since the Hausdorff dimension of a Borel set in $\mathbb{R}$ is the supremum of all $\alpha \in [0,1]$ for which the set supports some probability measure μ satisfying (A), it is tempting to think that the supremum of the numbers $\alpha \in [0,1]$ for which a *given* probability measure μ satisfies (A) is the Hausdorff dimension of the support of μ. This is wrong. The Hausdorff dimension of $\mathrm{supp}(\mu)$ can be strictly larger than the supremum of $\alpha \in [0,1]$ for which μ satisfies (A). Moreover, since the Fourier dimension of Borel set in $\mathbb{R}$ is the supremum of $\beta \in [0,1]$ for which the set supports some probability measure μ satisfying (B), it is tempting to think that the supremum of the numbers $\beta \in [0,1]$ for which a *given* probability measure μ satisfies (B) is the Fourier dimension of the support of μ. This is also wrong. The Fourier dimension of $\mathrm{supp}(\mu)$ can be strictly larger than the supremum of $\beta \in [0,d]$ for which μ satisfies (B). Finally, since the Fourier dimension of a Borel set is always less than or equal to the Hausdorff dimension of the set, it is tempting to think a probability measure μ on $\mathbb{R}$ that satisfies (B) for some β must satisfy (A) for some $\alpha \geq \beta$. This, again, is wrong. A probability measure μ may satisfy (B) for some β and fail to satisfy (A) for any $\alpha \geq \beta$. In other words, there are probability measures μ such that the largest α and β in $(0,d)$ for which (A) and (B) hold obey $\beta > \alpha$. Neither Theorem 4.1.2 nor Theorem 4.2.1 give the sharpness of the range of p in Theorem 4.1.1 for the class of such measures. It is likely a substantial new idea is needed to obtain such a sharpness result as it appears the current method cannot be modified to handle $\beta > \alpha$.

Second, it would be interesting to obtain a version of Theorem 4.1.2 or 4.2.1 for measures on $\mathbb{R}^d$ and $0 < \beta \leq \alpha < d$. The current method of proof does not seem to generalize easily to this situation. However, it seems likely that such a theorem can be obtained for α and β in a restricted range.

Concerning the results in this thesis on the construction of explicit Salem sets, we pose three questions that are interesting for future research.

What is the Fourier dimension of $E(Q, \Psi_\tau, 0)$ when $\nu(Q) < 1$? For example, consider Q as the set of squares (so that $\nu(Q) = 1/2$) or the set of powers of 2 (so that $\nu(Q) = 0$). Theorem 3.0.1 says nothing when $\nu(Q) < 1$. One natural conjecture is that $E(Q, \Psi_\tau, 0)$ is a Salem set, meaning its Fourier dimension is $\min\{(1+\nu(Q))/(1+\tau), 1\}$.

What is the Fourier dimension of $E(m, n, \mathbb{Z}, \Psi_\tau, 0)$? Theorem 3.0.2 implies the Fourier dimension is at least $\min\{2n/(1+\tau), mn\}$. It is natural to conjecture that the Fourier dimension is exactly $\min\{2n/(1+\tau), mn\}$. The verification of this conjecture would make $E(m, n, \mathbb{Z}, \Psi_\tau, 0)$ the first explicit example of a set in $\mathbb{R}^d$ ($d \geq 2$) with Fourier dimension strictly between 1 and $d-1$. Another natural conjecture is that $E(m, n, \mathbb{Z}, \Psi_\tau, 0)$ is a Salem set, meaning its Fourier dimension is $\min\{m(n-1) + (m+n)/(1+\tau), mn\}$.

What is the Fourier dimension of $E(m, n, Q, \Psi, \theta)$ when no additional restrictions are placed on the parameters? This is the most general question and therefore the most challenging.

Bibliography

[1] J.-G. Bak, A. Seeger, *Extensions of the Stein-Tomas theorem*, Math. Res. Lett. **18** (2011), no. 4, 767–781.

[2] A. Banner, *Restriction of the Fourier transform to quadratic submanifolds*, Princeton University Thesis, 2002.

[3] A. S. Besicovitch, *Sets of fractional dimension (IV); on rational approximation to real numbers*, J. London Math. Soc., **9** (1934), 126-131.

[4] P. Billingsley, *Convergence of Probability Measures*, 2ed, Wiley (1999).

[5] C. Bluhm, *Zur Konstruktion von Salem-Mengen*, Ph. D. Dissertation, Erlangen, 1996.

[6] C. Bluhm, *Random recursive construction of Salem sets*, Ark. Mat. **34** (1996), 51–63.

[7] C. Bluhm, *On a theorem of Kaufman: Cantor-type construction of linear fractal Salem sets*, Ark. Mat. **36** (1998), 307–316.

[8] I. Borosh, A. S. Fraenkel, *A generalization of Jarníks theorem on Diophantine approximations*, Indag. Math. **34** (1972), 193-201.

[9] J. Bourgain, *Remarks on Montgomery's conjectures on Dirichlet sums*, Geometric aspects of functional analysis (1989-90), 153-165, Lecture Notes in Math., 1469, Springer, Berlin, 1991.

[10] J. Bourgain, *On the distribution of Dirichlet sums*, J. Anal. Math. **60** (1993), 21-32.

[11] X. Chen, *Sets of Salem type and sharpness of the L^2-Fourier restriction theorem*, preprint, 2013. `http://arxiv.org/pdf/1305.5584v2.pdf`, visited 2015-04-26.

[12] M. Christ, *Restriction of the Fourier transform to submanifolds of low codimension*, Thesis, U. Chicago, 1982.

[13] H. Dickinson, *A note on the theorem of Jarník-Besicovitch*, Glasgow Math. J. **39** (1997), no. 2, 233-236.

[14] M. M. Dodson, *Hausdorff dimension, lower order and Khintchine's theorem in metric Diophantine approximation*, J. Reine Angew. Math. **432** (1992), 69–76.

[15] H. G. Eggleston, *Sets of fractional dimensions which occur in some problems of number theory*, Proc. London Math. Soc. **54** (1951), 42–93.

[16] C. Fefferman, Inequalities for strongly singular convolution operators, Acta Math. **124** (1970), 9–36.

[17] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., Wiley (1999).

[18] O. Frostman, *Potentiel déquilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Meddel. Lunds Univ. Math. Sem. **3** (1935) 1-118.

[19] M. Gatesoupe, *Sur un théorème de R. Salem*, Bull. Sci. Math. (2) **91** (1967), 125-127.

[20] B. Green, *Roth's theorem in the primes*, Annals of Mathematics, 161 (2005), 1609-1636.

[21] G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed., Oxford: Clarendon Press, 1975.

[22] K. Hambrook, *Explicit Salem Sets and applications to metrical Diophantine approximation*, preprint. `http://www.math.ubc.ca/~hambrook/research/Hambrook-Explicit-v2.pdf`, visited 2015-04-26.

[23] K. Hambrook, I. Łaba, *On the sharpness of Mockenhaupt's restriction theorem*, Geometric and Functional Analysis **23** (2013), no. 4, 1262-1277.

[24] T. Hinokuma, H. Shiga, *A remark on theorem of Jarník*, Ryukyu Math. J. **5** (1992), 1-6.

[25] W. Hoeffding, *Probability inequalities for sums of bounded random variables*, Journal of the American Statistical Association **58** (1963), no. 301, 13–30.

[26] V. Jarník, *Diophantischen Approximationen und Hausdorffsches Mass*, Mat. Sbornik, **36** (1929), 371–382.

[27] J.-P. Kahane, *Images d'ensembles parfaits par des séries de Fourier gaussiennes*, C. R. Acad. Sci. Paris Sr. A-B **263** (1966) A678–A681.

[28] J.-P. Kahane, *Some Random Series of Functions*, 2nd ed., Cambridge Univ. Press, 1985.

[29] R. Kaufman, *On the theorem of Jarník and Besicovitch*, Acta Arith. **39** (1981), 265–267.

[30] T. Körner, *Hausdorff and Fourier dimension*, Studia Math. **158** (2011), 37–50.

[31] I. Łaba, M. Pramanik, *Arithmetic progressions in sets of fractional dimension*, Geom. Funct. Anal. **19** (2009), no. 2, 429–456.

[32] J. Levesley, *A general inhomogeneous Jarnk-Besicovitch theorem*, J. Number Theory **71** (1998), 65-80.

[33] W. Littman, *L^p-L^q-estimates for singular integral operators arising from hyperbolic equations*, Partial Differential Equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971), pp. 479-481. Amer. Math. Soc., Providence, R.I., 1973.

[34] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, 1995.

[35] T. Mitsis, *A Stein-Tomas restriction theorem for general measures*, Publ. Math. Debrecen **60** (2002), 89–99.

[36] G. Mockenhaupt, *Bounds in Lebesgue spaces of Oscillatory integrals*, Habilitationsschrift, U. of Siegen, 1996.

[37] G. Mockenhaupt, *Salem sets and restriction properties of Fourier transforms*, Geom. Funct. Anal. **10** (2000), 1579–1587.

[38] G. Mockenhaupt and T. Tao, *Restriction and Kakeya phenomena for finite fields*, Duke Math. J. **121** (2004), no. 1, 35–74.

[39] E. Prestini, *Restriction theorems for the Fourier transform to some manifolds in* $\mathbb{R}^n$, Harmonic Analysis in Euclidean Spaces (Proc. Sympos. Pure Math. **35**, Amer. Math. Soc., 1979, vol I), 101-109.

[40] B. P. Rynne, *The Hausdorff dimension of sets arising from Diophantine approximation with a general error function*, J. Number Theory **71** (1998), no. 2, 166–171.

[41] R. Salem, *On singular monotonic functions whose spectrum has a given Hausdorff dimension*, Ark. Mat. **1** (1951), 353–365.

[42] E. M. Stein, *Maximal functions. I. Spherical means*, Proc. Nat. Acad. Sci. U.S.A. **73** (1976), no. 7, 2174–2175.

[43] E. M. Stein, *Harmonic Analysis*, Princeton Univ. Press, 1993.

[44] R. Strichartz, *Convolutions with kernels having singularities on a sphere*, Trans. Amer. Math. Soc. **148** (1970), 461–471.

[45] T. Tao, *Some Recent Progress on the Restriction Conjecture*, Fourier Analysis and Convexity, 217-243, Appl. Numer. Harmon. Anal., Birkhuser Boston, Boston, MA, 2004.

[46] P. A. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc. **81** (1975), 477–478.

[47] P. A. Tomas, *Restriction theorems for the Fourier transform*, in *Harmonic Analysis in Euclidean Spaces* (Proc. Sympos. Pure Math. **35**, Amer. Math. Soc., 1979, vol I), 111–114.

[48] S. Wigert, *Sur l'ordre de grandeur du nombre des diviseurs d'un entier*, Ark. Mat. **3** (1906/7), 1-9.

[49] T. Wolff, *Lectures on Harmonic Analysis*, Amer. Math. Soc., Providence, R.I. (2003).

Appendix A

Hoeffding's Inequality

The standard version of Hoeffding's inequality for real-valued random variables is the following.

Lemma A.0.1. *If* $X_1, \ldots, X_n$ *are independent real-valued random variables with* $\mathbb{E}X_i = 0$ *and* $|X_i| \leq 1$ *for* $i = 1, \ldots, n$*, then*

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geq u\right) \leq 2\exp(-nu^2/2)$$

for all $u \geq 0$.

For the proof of Lemma A.0.1, see [25].

The following version of Hoeffding's inequality for complex-valued random variables can be deduced from Lemma A.0.1.

Lemma A.0.2. *If* $X_1, \ldots, X_n$ *are independent complex-valued random variables with* $\mathbb{E}X_i = 0$ *and* $|X_i| \leq 1$ *for* $i = 1, \ldots, n$*, then*

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geq u\right) \leq 4\exp(-nu^2/4)$$

for all $u \geq 0$.

Proof. For $z \in \mathbb{C}$, if $|z| = (|\mathrm{Re}(z)|^2 + |\mathrm{Im}(z)|^2)^{1/2} \geq u$, then $|\mathrm{Re}(z)| \geq u/\sqrt{2}$ or $|\mathrm{Im}(z)| \geq u/\sqrt{2}$. Therefore

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geq u\right) \leq P\left(\left|\frac{1}{n}\sum_{i=1}^{n} \mathrm{Re}(X_i)\right| \geq \frac{u}{\sqrt{2}}\right) + P\left(\left|\frac{1}{n}\sum_{i=1}^{n} \mathrm{Im}(X_i)\right| \geq \frac{u}{\sqrt{2}}\right).$$

Applying Lemma A.0.1 on the right-hand side completes the proof. $\square$

Appendix B

A Fourier Decay Lemma

Lemma B.0.1. *Let μ be a complex-valued measure on $\mathbb{R}$ with compact support. Suppose there are positive constants b and c_1 such that*

$$|\widehat{\mu}(k)| \leq c_1(\ln(e + |k|))^{1/2}(1 + |k|)^{-b} \quad \forall k \in \mathbb{Z}. \tag{B.0.1}$$

Then there is a positive constant c_2 such that

$$|\widehat{\mu}(\xi)| \leq c_1 c_2(\ln(e + |\xi|))^{1/2}(1 + |\xi|)^{-b} \quad \forall \xi \in \mathbb{R}. \tag{B.0.2}$$

Proof. If the support of μ is contained in an interval $[-M, M]$, then the measure ν defined by

$$\int_{\mathbb{R}} f(x) d\nu(x) = \int_{\mathbb{R}} f(\tfrac{1}{4M}x + \tfrac{1}{2}) d\mu(x) \quad \forall f \in L^1(\mu)$$

has support contained in $[\frac{1}{4}, \frac{3}{4}]$ and satisfies $\widehat{\nu}(\xi) = \widehat{\mu}(\frac{1}{4M}\xi)e^{-\pi i \xi}$ for all $\xi \in \mathbb{R}$. Therefore it suffices to prove the lemma under the assumption that the support of μ is contained in $[\frac{1}{4}, \frac{3}{4}]$.

Let $\gamma : \mathbb{R} \to \mathbb{C}$ be a C^∞ function that is equal to 1 on $\mathrm{supp}(\mu)$ and has $\mathrm{supp}(\gamma) \subseteq [0, 1]$. For $a \in [-1, 1]$, define

$$\gamma_a(x) = e^{-2\pi i a x}\gamma(x) \quad \forall x \in \mathbb{R}.$$

We claim that for every $n \in \mathbb{N}$ there is a constant $C_{n,\gamma} > 0$ depending only on n and γ such that

$$|\widehat{\gamma_a}(k)| \leq C_{n,\gamma}(1+|k|)^{-n} \quad \forall k \in \mathbb{Z}, a \in [-1,1]. \tag{B.0.3}$$

We now prove this claim. Fix $n \in \mathbb{N}$. Note that for any compactly supported C^∞ function $f : \mathbb{R} \to \mathbb{C}$, integration by parts yields

$$\widehat{f^{(n)}}(k) = (2\pi i k)^n \widehat{f}(k) \quad \forall k \in \mathbb{Z}.$$

Therefore

$$\widehat{\gamma_a^{(n)}}(k) = (2\pi i k)^n \widehat{\gamma_a}(k) \quad \forall k \in \mathbb{Z}, a \in [-1,1]. \tag{B.0.4}$$

By direct computation, the n-th derivative of γ_a is

$$\gamma_a^{(n)}(x) = e^{-2\pi i a x} \sum_{j=0}^{n} \binom{n}{k} (-2\pi i a)^j \gamma^{(n-j)}(x) \quad \forall x \in \mathbb{R}.$$

Therefore there exists a constant $C'_{n,\gamma} > 0$ depending only on n and γ such that

$$|\gamma_a^{(n)}(x)| \leq C'_{n,\gamma} \quad \forall x \in \mathbb{R}, a \in [-1,1].$$

Since $\operatorname{supp}(\gamma_a^{(n)}) \subseteq [0,1]$ for every $a \in [-1,1]$, it follows that

$$|\widehat{\gamma_a^{(n)}}(k)| \leq \int_{[0,1]} |\gamma_a^{(n)}(x)| dx \leq C'_{n,\gamma} \quad \forall k \in \mathbb{Z}, a \in [-1,1].$$

Then (B.0.4) yields

$$|\widehat{\gamma_a}(k)| \leq \frac{C'_{n,\gamma}}{(2\pi)^n} |k|^{-n} \quad \forall k \in \mathbb{Z}, k \neq 0, a \in [-1,1].$$

This implies (B.0.3) because $(1+|k|)^n \leq 2|k|^n$ for $k \in \mathbb{Z}, k \neq 0$ and

$$|\widehat{\gamma_a}(0)| \leq \|\gamma_a\|_1 = \|\gamma\|_1 < \infty \quad \forall a \in [-1,1].$$

Now that the claim is proved, we can complete the proof of the lemma. Let $m \in \mathbb{Z}$ and $a \in [-1,1]$ be given. Since γ_a is C^∞ with $\mathrm{supp}(\gamma_a) \subseteq [0,1]$, we have

$$\gamma_a(x) = \sum_{k \in \mathbb{Z}} \widehat{\gamma_a}(k) e^{2\pi i k x} \quad \forall x \in \mathbb{R},$$

where the convergence is uniform for $x \in \mathbb{R}$. Then, as γ is equal to 1 on $\mathrm{supp}(\mu)$, we have

$$\begin{aligned}
\widehat{\mu}(m+a) &= \int_{\mathbb{R}} e^{-2\pi i a x} e^{-2\pi i m x} d\mu(x) = \int_{\mathbb{R}} e^{-2\pi i a x} \gamma(x) e^{-2\pi i m x} d\mu(x) \\
&= \int_{\mathbb{R}} \gamma_a(x) e^{-2\pi i m x} d\mu(x) = \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \widehat{\gamma_a}(k) e^{2\pi i k x} e^{-2\pi i m x} d\mu(x) \\
&= \sum_{k \in \mathbb{Z}} \widehat{\gamma_a}(k) \int_{\mathbb{R}} e^{-2\pi i (m-k) x} d\mu(x) = \sum_{k \in \mathbb{Z}} \widehat{\gamma_a}(k) \widehat{\mu}(m-k).
\end{aligned}$$

Write

$$\widehat{\mu}(m+a) = \sum_{|k| \leq \frac{1}{2}|m|} \widehat{\gamma_a}(k) \widehat{\mu}(m-k) + \sum_{|k| > \frac{1}{2}|m|} \widehat{\gamma_a}(k) \widehat{\mu}(m-k).$$

We bound each sum separately. Fix a positive integer $n > 1+b$ and define $C_b = \sum_{k \in \mathbb{Z}} (1+|k|)^{-n+b}$. If $|k| \leq \frac{1}{2}|m|$, then $\frac{1}{2}|m| \leq |m-k| \leq \frac{3}{2}|m|$, and so (B.0.1) and (B.0.3) give

$$\begin{aligned}
\sum_{|k| \leq \frac{1}{2}|m|} |\widehat{\gamma_a}(k)||\widehat{\mu}(m-k)| &\leq c_1 C_{n,\gamma} (\ln(e + \tfrac{3}{2}|m|))^{1/2} (1 + \tfrac{1}{2}|m|)^{-b} \sum_{|k| \leq \frac{1}{2}|m|} (1+|k|)^{-n} \\
&\leq c_1 C_b C_{n,\gamma} (\ln(e + \tfrac{3}{2}|m|))^{1/2} (1 + \tfrac{1}{2}|m|)^{-b}.
\end{aligned}$$

Define $C'_b = \sup_{x \geq 0} (\ln(e+x))^{1/2}(1+x)^{-b}$. Then by (B.0.1) and (B.0.3) we have

$$\begin{aligned}
\sum_{|k|>\frac{1}{2}|m|} |\widehat{\gamma}_a(k)||\widehat{\mu}(m-k)| &\leq c_1 C'_b C_{n,\gamma} \sum_{|k|>\frac{1}{2}|m|} (1+|k|)^{-n} \\
&\leq c_1 C'_b C_{n,\gamma} (1+\tfrac{1}{2}|m|)^{-b} \sum_{|k|>\frac{1}{2}|m|} (1+|k|)^{-n+b} \\
&\leq c_1 C_b C'_b C_{n,\gamma} (1+\tfrac{1}{2}|m|)^{-b} \\
&\leq c_1 C_b C'_b C_{n,\gamma} (\ln(e+\tfrac{3}{2}|m|))^{1/2} (1+\tfrac{1}{2}|m|)^{-b}.
\end{aligned}$$

Combining the bounds on the two sums yields

$$\begin{aligned}
|\widehat{\mu}(m+a)| &\leq 2c_1 C_b C'_b C_{n,\gamma} (\ln(e+\tfrac{3}{2}|m|))^{1/2} (1+\tfrac{1}{2}|m|)^{-b} \\
&\leq c_1 C_b C'_b C_{n,\gamma} 2^{b+\frac{3}{2}} (\ln(e+|m+a|))^{1/2} (1+|m+a|)^{-b}
\end{aligned}$$

for every $m \in \mathbb{Z}$ and $a \in [-1,1]$. This proves (B.0.2) because each $\xi \in \mathbb{R}$ can be written as $\xi = m+a$ for some $m \in \mathbb{Z}$ and $a \in [-1,1]$. $\square$

It is possible to prove a more general version of Lemma B.0.1 (cf. [28, p. 252]).