Essays in Business Cycle Economics

by

Dana Galizia

B.A., The University of Western Ontario, 2005
M.A., McGill University, 2007

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

in

THE FACULTY OF GRADUATE AND POSTDOCTORAL STUDIES
(Economics)

The University of British Columbia
(Vancouver)

June 2015

© Dana Galizia, 2015
Abstract

This thesis contains three distinct chapters that contribute to our understanding of the causes and consequences of business cycles. Modern business-cycle models generally feature several different random shock processes that drive business cycles. Being able to reliably evaluate the individual importance of any one these shocks depends importantly on having accurate estimates of the variances of the shocks. In the first chapter, it is shown that when a model is a poor approximation to the data, typical variance estimates are biased upward. A simple procedure to identify and partially correct for these effects is proposed. Applying this procedure to a recent paper from the literature reduces the estimated variances by as much as a third of their respective naive estimates.

The second chapter explores a view of recessions (typically associated with Friedrich Hayek) whereby, after a period of rapid accumulation of houses, consumer durables and business capital, the economy goes through a period of needed liquidation that results in a decline in economic activity. An alternative (typically associated with Keynes) that is often contrasted with this liquidation view is that recessions are times of deficient demand. These two views have opposite implications for fiscal policy: in the first, fiscal policy simply prolongs the needed adjustment, while in the second fiscal policy can prop up demand. This chapter argues that the two views may be more closely linked than previously recognized, in that liquidations can produce periods where the economy is characterized by deficient demand.

The final chapter presents a model in which business-cycle booms and busts are inherently related, whereby a boom causes a subsequent bust, which in turn leads to another boom, and so on. In particular, it is shown how a purely deterministic model can produce fluctuations that persist indefinitely. These cycles exactly repeat themselves, while in the data business cycles are somewhat irregular. It is shown that by adding a small amount of random variation to the model, it is capable of replicating business cycle features in the data well, including their irregularity.
Preface

All materials in this dissertation are original, unpublished work. Chapter 2 is exclusively my independent work. Chapter 3 is joint work with Paul Beaudry and Franck Portier. Initial concept formation was by Paul Beaudry and Franck Portier. I participated in all stages of its realization. Paul Beaudry and I did most of the analysis of the model together. The manuscript was composed primarily by Paul Beaudry, with edits contributed by me and Franck Portier. I developed the formal proofs of the theoretical propositions, in most cases after extensive conversations with Paul Beaudry. I created all figures except those contained in Appendix B, which were created by Franck Portier.

The materials in Chapter 4 are primarily though not exclusively my independent work. The basic structure of the model used throughout the chapter, as well as portions of the text of Section 4.5.1 are used with permission from Beaudry et al. (2014), of which I am a co-author, and which forms Chapter 3 in this thesis. The remainder of Chapter 4 is my independent work, including the composition of the manuscript, the theoretical results of Section 4.5.3, and all quantitative results presented throughout the paper.
# Table of Contents

Abstract ................................................................. ii

Preface ................................................................. iii

Table of Contents ...................................................... iv

List of Tables .......................................................... vii

List of Figures .......................................................... viii

Acknowledgments ....................................................... ix

Dedication ............................................................... x

1 Introduction ........................................................... 1
   1.1 Business cycles ......................................................... 1
   1.2 Chapter one: Misspecification and the causes of business cycles ........................................ 1
   1.3 Chapter two: Reconciling Hayek’s and Keynes’ views of recessions ........................................ 2
   1.4 Chapter three: Can a limit-cycle model explain business cycle fluctuations? ................................ 2

2 Misspecification and the Causes of Business Cycles ......................................................... 4
   2.1 Introduction ............................................................. 4
   2.2 Basic framework and methodology .................................................. 7
      2.2.1 Example 1: Missing shocks ................................................. 10
   2.3 Formal framework ...................................................... 12
      2.3.1 Set-up ................................................................. 13
      2.3.2 Integral representation .................................................... 14
      2.3.3 Orthogonal decompositions ................................................ 15
      2.3.4 Discussion ............................................................. 16
   2.4 Further analysis ........................................................ 18
      2.4.1 Econometric model ...................................................... 18
## List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 2.1</td>
<td>Variances of true and smoothed shocks</td>
<td>26</td>
</tr>
<tr>
<td>Table 2.2</td>
<td>Corrected variances of smoothed shocks</td>
<td>26</td>
</tr>
<tr>
<td>Table 2.3</td>
<td>Sample correlations for smoothed shocks (case B)</td>
<td>27</td>
</tr>
<tr>
<td>Table 2.4</td>
<td>Corrected variances (case C)</td>
<td>29</td>
</tr>
<tr>
<td>Table 2.5</td>
<td>Naive variance decomposition for hours</td>
<td>30</td>
</tr>
<tr>
<td>Table 2.6</td>
<td>Corrected estimates (fraction of naive estimate)</td>
<td>36</td>
</tr>
<tr>
<td>Table 2.7</td>
<td>Corrected BCF variance decomposition for hours</td>
<td>37</td>
</tr>
<tr>
<td>Table 4.1</td>
<td>Parameter values</td>
<td>100</td>
</tr>
</tbody>
</table>
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Autocovariance function ($Cov(y_t, x_{t-k})$)</td>
<td>28</td>
</tr>
<tr>
<td>2.2</td>
<td>Correlation between $%\Delta C_t$ and $%\Delta I_{t+k}$</td>
<td>31</td>
</tr>
<tr>
<td>2.3</td>
<td>Maximum scaled MSRE for smoothed shocks</td>
<td>32</td>
</tr>
<tr>
<td>2.4</td>
<td>Correlation coefficients between smoothed shocks</td>
<td>33</td>
</tr>
<tr>
<td>2.5</td>
<td>Proportion of correlations outside 95% CI</td>
<td>34</td>
</tr>
<tr>
<td>3.1</td>
<td>Labor wedge as function of $X$</td>
<td>50</td>
</tr>
<tr>
<td>3.2</td>
<td>Consumption as function of $X$</td>
<td>51</td>
</tr>
<tr>
<td>3.3</td>
<td>Equilibrium determination</td>
<td>52</td>
</tr>
<tr>
<td>3.4</td>
<td>Cost of funds</td>
<td>59</td>
</tr>
<tr>
<td>3.5</td>
<td>Equilibrium determination (multiple equilibria)</td>
<td>63</td>
</tr>
<tr>
<td>3.6</td>
<td>Consumption as function of $X$ (multiple equilibria)</td>
<td>64</td>
</tr>
<tr>
<td>3.7</td>
<td>$X_{t+1}$ as a function of $X_t$</td>
<td>68</td>
</tr>
<tr>
<td>3.8</td>
<td>Response of economy to a noise shock</td>
<td>72</td>
</tr>
<tr>
<td>4.1</td>
<td>Hours worked data (1960-2012)</td>
<td>84</td>
</tr>
<tr>
<td>4.2</td>
<td>Conditions for a limit cycle</td>
<td>86</td>
</tr>
<tr>
<td>4.3</td>
<td>Static equilibrium determination</td>
<td>92</td>
</tr>
<tr>
<td>4.4</td>
<td>Deterministic model</td>
<td>101</td>
</tr>
<tr>
<td>4.5</td>
<td>Stochastic model</td>
<td>102</td>
</tr>
<tr>
<td>4.6</td>
<td>Autocovariance: Hours worked ($L$) and output ($y$)</td>
<td>103</td>
</tr>
<tr>
<td>4.7</td>
<td>Spectrum: Hours worked (alternative filters)</td>
<td>104</td>
</tr>
<tr>
<td>4.8</td>
<td>Hours worked in Smets-Wouters (2007)</td>
<td>105</td>
</tr>
<tr>
<td>4.9</td>
<td>Spectrum: Output (data and stochastic model)</td>
<td>107</td>
</tr>
<tr>
<td>B.1</td>
<td>The Model with Nash bargaining, consumption as function of $X$</td>
<td>148</td>
</tr>
<tr>
<td>B.2</td>
<td>The Model with Nash bargaining, equilibrium determination</td>
<td>149</td>
</tr>
<tr>
<td>B.3</td>
<td>The Model with Nash bargaining, equilibrium determination (multiple equilibria)</td>
<td>149</td>
</tr>
</tbody>
</table>
Acknowledgments

This thesis could not have been completed without the support graciously given to me by a number of people. I would like to express my deepest gratitude to my advisor Paul Beaudry for his unending supply of patience, guidance and encouragement, and for his generous financial support. I would also like to thank my other thesis committee members, Henry Siu and Yaniv Yedid-Levi, for their support and helpful advice. Thank you also to Franck Portier, Vadim Marmer, Francesco Trebbi, Matilde Bombardini and Amartya Lahiri for helpful discussions.

Thank you to my parents, who have never wavered in their support for me, and whose contributions to my arriving at this point have been both innumerable and indispensable. Finally, words cannot express my unending gratitude to my wife, Jocelyne, who has been by my side every step of the way. I will be forever thankful for the love she has given me and the sacrifices she has made so that I could reach this point. If I have accomplished anything, it’s because she made it possible.
For my loving wife, Jocelyne.
Chapter 1

Introduction

1.1 Business cycles
Since the early part of the twentieth century, a substantial part of economic research has been devoted to the study of business cycles (i.e., persistent but non-permanent fluctuations in economic aggregates such as output or unemployment). Despite the considerable amount of attention this topic has received, there are many questions that remain unanswered, and many more for which the typical answers are offered only tentatively.

This thesis consists of three distinct chapters, each of which seeks to contribute to answering the most fundamental questions considered by business-cycle economists: What ultimately causes business cycles, and how do those causes end up manifesting themselves as the patterns in the data that are typically thought of as characterizing the business cycle? The final two chapters can be viewed as exploring answers to these questions directly, while the first chapter contributes to our understanding of how to answer these questions.

1.2 Chapter one: Misspecification and the causes of business cycles
Modern macroeconomics models generally feature several different random shock processes that ultimately drive business cycle fluctuations. A standard tool in the quantitative macroeconomics toolbox for evaluating the individual importance of these shocks is a variance decomposition, by which the variance of any given variable is decomposed into individual portions attributable to each of the shocks in the model. The reliability of this tool depends importantly on having accurate estimates of the variances of the shocks.

In the first chapter of this thesis, I develop a novel framework and use it to show that when a model is misspecified—roughly speaking, when it is inherently incapable of matching the data to which it is applied—the shock variances as they are typically estimated will be biased upward. Next, using the same framework, I propose a simple procedure to identify and partially correct for the effects of model
misspecification on these variance estimates. As an example of its usefulness, I apply this procedure to a recent paper and find that it reduces the estimated variances of the shocks in the model by as much as a third of their respective naive estimates.

1.3 Chapter two: Reconciling Hayek’s and Keynes’ views of recessions

Recessions often happen after periods of rapid accumulation of houses, consumer durables and business capital. This observation has led some economists, most notably Friedrich Hayek, to conclude that recessions mainly reflect periods of needed liquidation resulting from past over-investment. According to the main proponents of this view, government spending should not be used to mitigate such a liquidation process, as doing so would simply result in a needed adjustment being postponed. In contrast, ever since the work of Keynes, many economists have viewed recessions as periods of deficient demand that should be countered by activist fiscal policy.

In the second chapter of this thesis, we reexamine the liquidation perspective of recessions in a setup where prices are flexible but where not all trades are coordinated by centralized markets. We show why and how liquidations can produce periods where the economy functions particularly inefficiently, with many socially desirable trades between individuals remaining unexploited when the economy inherits too many capital goods. In this sense, our model illustrates how liquidations can cause recessions characterized by deficient aggregate demand and accordingly suggests that Keynes’ and Hayek’s views of recessions may be much more closely linked than previously recognized. In our framework, interventions aimed at stimulating aggregate demand face the trade-off emphasized by Hayek whereby current stimulus mainly postpones the adjustment process and therefore prolongs the recessions. However, when examining this trade-off, we find that some stimulative policies may nevertheless remain desirable even if they postpone a recovery.

1.4 Chapter three: Can a limit-cycle model explain business cycle fluctuations?

In conventional models of the business cycle, all fluctuations are ultimately caused by the arrival of random shocks. As a result, individual booms and busts are largely unrelated phenomena. In the final chapter of this thesis, I explore an alternative to this viewpoint, which is that booms and busts are inherently related phenomena. According to this view, fluctuations are at least in part driven by deterministic cyclical forces, rather than random shocks.

The paper shows (1) how a purely deterministic general-equilibrium model can give rise to cyclical fluctuations that continue indefinitely, and (2) that this model can replicate business cycle features of the data once it includes a small amount of random variation. The deterministic cycle arises through a simple micro-founded mechanism in a rational-expectations environment, and does not rely on the existence of multiple equilibria or dynamic indeterminacy. Since these cycles would indefinitely repeat themselves in the absence of shocks, a simple productivity shock is introduced in order to create
irregularities. The model is estimated to match US hours data, and is shown to be able to match it closely. The productivity shock in the model is of a reasonable persistence and relatively small size, accounting for only around a fifth of the standard deviation of hours in the model. This highlights that models capable of generating deterministic fluctuations do not require the addition of large, persistent shocks in order to match patterns in the data, which is a common criticism of conventional models.
Chapter 2

Misspecification and the Causes of Business Cycles

2.1 Introduction

One of the central goals of macroeconomic research is to understand the root causes of business cycles. In recent years, much of the research devoted to this goal has been conducted in three steps. First, build a dynamic stochastic general-equilibrium (DSGE) model within which business-cycle fluctuations are generated by the random arrival of exogenous shocks\(^1\) that are then propagated through endogenous mechanisms. Second, estimate the parameters of this DSGE model using real-world data and some estimation algorithm (such as maximum likelihood or, increasingly, Bayesian methods). Third, within the estimated model, evaluate the quantitative importance of each of the shocks in generating business-cycle fluctuations.\(^2\)

If the true data-generating process (DGP) is contained in the set of DGPs spanned by the model constructed at step one above, then under some basic regularity conditions we can expect the parameters estimated at step two to converge to the true parameters, and in turn we may interpret the results of the third step as reflecting the actual importance of each type of shock in generating real-world economic fluctuations. However, when the true DGP is not contained in the set of DGPs spanned by the model—i.e., when the model is misspecified—this reasoning no longer holds. Since even the most optimistic economist would agree that all models are misspecified, this raises some important questions. Namely, what effect does misspecification have on shock variance estimates in a model? Does this effect depend on the severity of the misspecification? If misspecification is detected, can

---

\(^1\) Throughout this paper, what I refer to as shocks are the innovations to an exogenous process, rather than the process itself. For example, if TFP evolves according to an AR(1) process \(z_t = \rho z_{t-1} + \epsilon_t\), with \(\epsilon_t\) a sequence of i.i.d. random variables, then I refer to \(\epsilon_t\) as the (exogenous) TFP shock.

\(^2\) In the class of models considered in this paper, conditional on all other parameters there will be a one-to-one mapping between the variance of each shock and its respective quantitative importance. As such, I will use the terms “quantitative importance” and “shock variance” interchangeably.
some correction be made to the variance estimates? To my knowledge, no framework currently exists to help answer these questions. This paper attempts to fill that gap in the literature.

The impact of misspecification on understanding the root causes of business cycles is highly relevant. Over the last decade or so, DSGE models have become ever richer, with a typical model now possessing a variety features that (it is argued) allow it to fit the data reasonably well. Deriving some degree of confidence from this good fit, it has become common to ask the model which types of economic shocks appear to be important for driving real-world business-cycle fluctuations. Specifically, given a DSGE model that has been linearized around a non-stochastic steady state, and that contains structural shocks that are independent of one another both cross-sectionally and over time, it is a straightforward exercise to obtain the proportion of the variance of any endogenous variable attributable to an individual shock, i.e., to perform a variance decomposition. On the basis of a variance decomposition, one may then conclude, for example, that a particular shock is an important driving force for business cycle fluctuations. The usefulness of this variance decomposition tool in understanding the causes of real-world business cycles clearly depends fundamentally on the accuracy of the shock variance estimates. As I clarify in this paper, in the presence of misspecification these variance estimates will be systematically biased upward. While this poses a challenge to the conclusions typically drawn from a variance decomposition, as we shall see the news is not all bad. First, I show that as the severity of the misspecification becomes smaller (in a particular sense), this bias shrinks to zero. This reinforces the view that, even though a model may be misspecified, it may still yield useful conclusions. Second, I propose a simple method of adjusting the variance estimates that will reduce the degree of bias, at least in part. In combination, these two properties also imply that larger adjustments will typically be associated with more severe misspecification, so that this methodology can be used as an additional tool for measuring the severity of misspecification in a model. Since these adjustments will generally be different for each shock in the model, unlike other measures this one is unique in providing information to the modeller about misspecification at the level of individual shocks. As illustrated in an example below, this information can be useful in diagnosing the source of misspecification and, in turn, suggesting potential modifications to the model that may help to alleviate it.

The framework I propose revolves around the sequence of “smoothed shocks”, i.e., the sequence of realized shock values that, when fed into the model, allow it to exactly reproduce the observed data. As shown below, if the DSGE model (for a given set of parameter values) were the true DGP, then under certain regularity assumptions the sequence of smoothed shocks will asymptotically recover the true values of the shocks. Since the true shocks are (by construction) independent of one another

---


4 In practice there will typically be more than one sequence capable of reproducing the data, in which case the smoothed shocks are taken to be the sequence that is most likely to have occurred (i.e., the one for which the joint probability density function is maximized).
both intertemporally and cross-sectionally, this implies that if the model is the true DGP then the smoothed shocks should also be asymptotically independent of one another. This observation suggests a relatively simple diagnostic tool for detecting the presence of misspecification: if the smoothed shocks are correlated (beyond what can be reasonably explained by sampling error), then the model must be misspecified.\footnote{Note that, under the precise definition of "misspecification" adopted in this paper (see Section 2.3.1), the converse will not necessarily be true: it will be possible for the smoothed shocks to be uncorrelated, but for the model to nonetheless be misspecified.}

The basic intuition for this result is quite simple. Any model—including the (unknown) true DGP—can be thought of as a collection of mechanisms, where each mechanism corresponds to the combination of a shock and its impulse response function. A model is misspecified if it is missing one or more mechanisms that are present in the true DGP. If this is the case, the smoothing algorithm—that is, the algorithm that finds a sequence of shocks that exactly reproduces the data—must systematically mix together two or more of the mechanisms that are in the model in order to replicate the mechanisms that are missing. It is precisely this “mixing” process that leads to correlations in the smoothed shocks.

I next show that, when the model is misspecified, conventional shock variance estimates will tend to overstate the importance of one or more of the shocks. The intuition for this bias is again quite simple: since a mechanism in the model is being used to explain not only the fluctuations in the data actually caused by that mechanism, but also fluctuations caused by certain mechanisms that are not contained in the model, the smoothing algorithm attributes too much variation to the model mechanisms, and thus tends to overestimate the variance of the model shocks.

After highlighting this potential source of bias, I propose a simple procedure to (at least partially) correct for it. First, I orthogonally decompose a smoothed shock into two components: its true value in producing variation in the data, and an additional component related to misspecification. Next, I show that the true value of the shock should be unpredictable using values of other smoothed shocks. The true shock is therefore a component of the OLS residual obtained after regressing the corresponding smoothed shock on other smoothed shocks. The variance of this residual therefore represents a less biased estimate of the true variance of the shock.

To illustrate a practical application, I apply the above methodology to a recent paper by Justiniano et al. (2010), and show that the source of bias discussed above may indeed have significant implications. Justiniano et al. (2010) construct a medium-scale New Keynesian model featuring a variety of different shocks. The variance decomposition obtained from their estimated model indicates that the majority (roughly 60 percent) of the variance of hours worked at business-cycle frequencies can be attributed to the investment shock (a shock to the relative productivity of new investment vis-à-vis the existing capital stock). This shock is also found to account for the majority of business-cycle variation in output and investment. As I show, however, six of the seven smoothed shocks in their model (including the investment shock) are significantly correlated, suggesting that the model is misspecified and that the naive variance estimates may be overstating the importance of these shocks. Applying
the proposed correction procedure, I estimate that the true variance of the investment shock is smaller than the naive estimate by at least one-third.

I contribute here to two different bodies of literature. First, there is a substantial macroeconometric body of literature that explores issues of model fit and estimation/inference in potentially misspecified DSGE models (e.g., Gourieroux et al. (1993), Watson (1993), Canova (1994), Diebold et al. (1998), Schorfheide (2000), Hall and Inoue (2003), Dridi et al. (2007), Hnatkovska et al. (2012)). In addition, I also contribute to the vast literature that attempts to identify the sources of business cycle fluctuations. More specifically, I contribute to the recent literature that has considered investment shocks as a potential source of fluctuations, including the aforementioned Smets and Wouters (2003, 2007) and Justiniano et al. (2010), as well as Greenwood et al. (1988), Greenwood et al. (1997, 2000), Schmitt-Grohé and Uribe (2011, 2012), Justiniano et al. (2011), and Jermann and Quadrini (2012).

The remainder of the paper proceeds as follows. Section 2.2 introduces in a relatively informal way the basic framework and methodology of the paper, and presents a simple example to illustrate it. Section 2.3 then contains a formal presentation of the framework and methodology, while Section 2.4 provides some additional analysis. Section 2.5 then presents two additional examples in which the true DGP is known, before Section 2.6 applies the framework and methodology in a real-world example using the model of Justiniano et al. (2010). Finally, Section 2.7 concludes.

2.2 Basic framework and methodology

In this section I introduce the basic framework and methodology of the paper in a relatively informal way, with an eye towards conveying as clearly as possible the basic intuition. The ideas of this section are then discussed in formal detail in Section 2.3 below.

Suppose one has in hand an infinite sequence of past and future observations \( Y^\infty \equiv \{Y_t\}_{t=-\infty}^{\infty} \), where \( Y_t \) is some jointly normally distributed mean-zero \( m \)-variate data process satisfying basic stationarity and ergodicity assumptions. A model for \( \{Y_t\} \) is defined here as an expression for \( Y_t \) as a linear function of a (potentially infinite) history of \( r \)-dimensional i.i.d. normal random vectors, where any two different elements of that random vector are orthogonal to one another. That is, letting \( \tilde{Y}_t \) denote the model counterpart of \( Y_t \), a model is completely summarized by an MA(\( \infty \)) representation

\[
\tilde{Y}_t = \sum_{j=0}^{\infty} \tilde{\Psi}_j \varepsilon_{t-j}
\]

(2.1)

where \( \{\tilde{\Psi}_j\} \) is an absolutely-summable sequence of \( m \times r \) matrices and \( \varepsilon_t \) is an \( r \)-vector of i.i.d. normal random variables with diagonal covariance matrix, the \( l \)-th diagonal element of which is given by \( \tilde{\sigma}_l^2 \).\(^6\) Assuming that the true DGP can also be written in this way, there are two particular models

\(^6\) Note that the assumption of a diagonal covariance matrix here—i.e., of uncorrelated shocks—is without any loss of generality, as it is always possible to re-cast a model with correlated shocks in this form. For example, suppose we have a model in which shocks \( i \) and \( j \) are correlated with one another, with all other shocks uncorrelated. Then it is always possible
of interest: the econometric model (EM), and the true model (TM). Make the following important assumption about the EM.\footnote{Note that we do not make any such assumption about the TM, which need only in general satisfy (2.1).}

Assumption 2.1. (Invertibility) There exists some absolutely summable sequence \( \{ \psi_j \} \) of matrices such that, given an infinite sequence of past and future values of \( \tilde{Y}_t \) generated by (2.1), we may recover the sequence \( \{ \varepsilon_t \} \) from\footnote{Note that I define invertibility as the ability to recover the structural shocks from some combination of current, past and future values of \( \tilde{Y}_t \). This definition is more general than that used in the structural VAR literature (see, for example, Fernández-Villaverde et al. (2007), Sims (2012)), according to which the MA(∞) representation in (2.1) would be deemed invertible only if \( \varepsilon_t \) can be recovered from current and past values of \( \tilde{Y}_t \) alone.}

\[
\varepsilon_t = \sum_{j=-\infty}^{\infty} \psi_j \tilde{Y}_{t-j} \tag{2.2}
\]

Conditions under which Assumption 2.1 holds are fairly general and will be discussed in detail in Section 2.4.1. For now, we will simply take it as given. Under this assumption, the sequence of smoothed shocks, denoted \( \hat{\varepsilon}_t \), are obtained by applying the linear filter (2.2) to the data series \( Y_t \) (instead of its model counterpart \( \tilde{Y}_t \)). That is,

\[
\hat{\varepsilon}_t \equiv \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} \tag{2.3}
\]

It is straightforward to verify that if one substitutes the sequence \( \hat{\varepsilon}_t \) obtained from (2.3) into equation (2.1) for \( \varepsilon_t \) one recovers the data series \( Y_t \).\footnote{See Proposition 2.1 in Section 2.3 for a formal statement and proof of this fact.} Thus, the sequence \( \hat{\varepsilon}_t \) is indeed the sequence of smoothed shocks defined in Section 2.1. If the EM is correctly specified (i.e., if the EM and TM are one and the same), then one can interpret the sequence of smoothed shocks as capturing the true values of these shocks in the real world. In general, however, the TM will contain a number of shocks that are absent in the EM, in which case the sequence of smoothed model shocks, \( \hat{\varepsilon}_t \), would in general be different from their “true” values, which I denote by \( \varepsilon^*_t \).

At this point it is worth clarifying what I mean by the “true value” of a shock. This notion will be made precise in Section 2.3, but for now it will be sufficient to illustrate the concept with an example. In particular, consider the distinction between a real-world TFP shock, and a smoothed TFP shock inferred from some real-world data set\footnote{Assume for the sake of argument that this data set does not include the TFP process itself, so that the value of the TFP shocks must be inferred from the behavior of other endogenous variables in the system.} using a misspecified model. The former is the true value of the shock, and the one whose variance we would ultimately like to know, but it is not directly observable. On the other hand, the latter is observable, but it will in general differ from the true value, and thus its variance will be different as well.
Let $\sigma_l^{*2} \equiv Var(\varepsilon_{l,t}^*)$ denote the true variance of the $l$-th model shock. This is the quantity the econometrician would like to estimate. Further, let $\hat{\sigma}_l^2 \equiv Var(\hat{\varepsilon}_{l,t})$ be the variance of the corresponding smoothed shock. It can be verified that if the model is correctly specified then $\hat{\sigma}_l^2$ is the maximum-likelihood estimator of $\sigma_l^{*2}$. Since the model is implicitly assumed to be correctly specified when estimating the parameters via maximum-likelihood, $\hat{\sigma}_l^2$ is thus precisely the variance estimator that is typically used in the literature and that forms a key input into the variance decomposition.

When the model is misspecified, however, we will in general have $\hat{\sigma}_l^2 \neq \sigma_l^{*2}$. We are thus interested in the relationship between $\hat{\sigma}_l^2$ and $\sigma_l^{*2}$. Let $\nu_t \equiv \hat{\varepsilon}_t - \varepsilon_t^*$ denote the (unobserved) vector of “shock recovery errors”, and make the following assumption.

**Assumption 2.2. (Orthogonality)** $E[\varepsilon_t^*\nu_{t-k}'] = 0$ for all $k \in \mathbb{N}$.

The framework presented in detail in Section 2.3.1 will directly imply that Assumption 2.2 holds. Nonetheless, as with Assumption 2.1 we will for now simply take Assumption 2.2 as given. Under this assumption we can orthogonally decompose $\hat{\varepsilon}_t$ into two components: one capturing the true value of the shock ($\varepsilon_t^*$), and another entirely reflecting the fact that the EM is misspecified ($\nu_t$). The immediate consequence of this orthogonal decomposition is that

$$\hat{\sigma}_l^2 = \sigma_l^{*2} + Var(\nu_{l,t}) \geq \sigma_l^{*2} \quad (2.4)$$

Thus, when the model is misspecified the naive variance estimate $\hat{\sigma}_l^2$ will overstate the true variance of the shock; that is, misspecification causes the variance estimates to be biased upwards. This result is the first key contribution of this paper.

Next, note that the true values of the shocks are, by definition, independent of one another both cross-sectionally and intertemporally. That is, we have $Cov(\varepsilon_{l,t}^*, \varepsilon_{i,s}^*) = 0$ whenever $(l, t) \neq (i, s)$. It thus follows immediately that

$$Cov(\hat{\varepsilon}_{l,t}, \hat{\varepsilon}_{i,s}) = Cov(\nu_{l,t}, \nu_{i,s}) \quad (2.5)$$

Thus, if two smoothed shocks exhibit non-zero covariance, this can only be due to misspecification. This is the second key contribution of this paper. Finally, an implication of this result is that any component of $\hat{\varepsilon}_{l,t}$ that can be predicted using $\hat{\varepsilon}_{i,s}$ must be attributable to misspecification rather than to $\varepsilon_{l,t}^*$. This suggests a simple procedure to correct (at least in part) the variance estimates: Regress $\hat{\varepsilon}_{l,t}$ on date-$t$ values of shocks $i \neq l$ and on leads and lags of all shocks (including leads and lags of $\hat{\varepsilon}_{l,t}$). The error term from this regression is, by construction, the component of $\hat{\varepsilon}_{l,t}$ that cannot be predicted using the other shocks. The variance of this error term, which I denote $\bar{\sigma}_l^2$, therefore contains all of the variation in $\hat{\varepsilon}_{l,t}$ due to $\varepsilon_{l,t}^*$, plus an additional non-negative component. We will therefore have $\hat{\sigma}_l^2 \geq \bar{\sigma}_l^2 \geq \sigma_l^{*2}$, so that $\bar{\sigma}_l^2$ is a less-biased estimate of $\sigma_l^{*2}$ than the naive maximum-likelihood estimate. This result is the third key contribution of this paper.
2.2.1 Example 1: Missing shocks

Consider a simple economy populated by a representative household with preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t (C_t - \eta_t L_t)$$

Here, $C_t$ is consumption, $L_t$ is labor supplied, $\beta$ is the constant discount factor, and $\eta_t$, which captures the relative disutility of labor, follows the exogenous stochastic process

$$\log \eta_t = \varepsilon^{\eta, t}_{\eta}$$

where $\varepsilon^{\eta, t}_{\eta}$ is i.i.d. $N(0, \sigma^{\eta})$. The household produces output $X_t$ using technology $X_t = A_t L_t^{1-\alpha}$, where productivity $A_t$ follows exogenous stochastic process

$$\log A_t = \varepsilon^{A, t}_{A}$$

with $\varepsilon^{A, t}_{A}$ i.i.d. $N(0, \sigma^{A})$. In addition to consuming the output it produces, the household also receives a stochastic endowment $\varepsilon^{w, t}_{w}$ which is i.i.d. $N(0, \sigma^{w})$. Thus, the household budget constraint is given by

$$C_t = A_t L_t^{1-\alpha} + \varepsilon^{w, t}_{w}$$

Taking first-order conditions and log-linearizing the model around its non-stochastic steady state yields the following equations for consumption and labor

$$c_t = -\frac{1 - \alpha}{\alpha} \varepsilon^{\eta, t}_{\eta} + C^{-1} \varepsilon^{w, t}_{w} + \frac{1}{\alpha} \varepsilon^{A, t}_{A} \quad (2.6)$$

$$l_t = -\frac{1}{\alpha} \varepsilon^{\eta, t}_{\eta} + \frac{1}{\alpha} \varepsilon^{A, t}_{A} \quad (2.7)$$

where lower-case variables indicate log-deviations from steady state and $C$ is the steady-state value of $C_t$.

Suppose that the value of the parameter $\alpha$ was known to the econometrician, who wishes to estimate via maximum likelihood the variances of the stochastic processes using data on $c_t$ and $l_t$. Suppose, however, that the econometrician wrongly assumes that $\sigma^{A} = 0$, i.e., that productivity is constant. This econometrician would then estimate $\sigma \equiv (\sigma^{\eta}, \sigma^{w})'$ using the misspecified model

$$\tilde{c}_t = -\frac{1 - \alpha}{\alpha} \varepsilon^{\eta, t}_{\eta} + C^{-1} \varepsilon^{w, t}_{w} \quad (2.8)$$

$$\tilde{l}_t = -\frac{1}{\alpha} \varepsilon^{\eta, t}_{\eta} \quad (2.9)$$
The smoothed shocks $\hat{\varepsilon}_t \equiv (\hat{\varepsilon}_{\eta,t}, \hat{\varepsilon}_{w,t})'$ can be obtained as

$$\hat{\varepsilon}_t = \left( -\alpha \ell_t, C [c_t - (1 - \alpha) \ell_t] \right)$$

and the maximum-likelihood variance estimates are

$$\begin{pmatrix} \hat{\sigma}^2_{\eta} \\ \hat{\sigma}^2_{w} \end{pmatrix} = \begin{pmatrix} \alpha^2 \sigma^*_{\eta}^2 \\ C^2 \phi^* \end{pmatrix}$$

where

$$\phi^* \equiv Var \left[ c_t - (1 - \alpha) \ell_t \right] = \sigma^*_{\eta}^2 - 2 (1 - \alpha) \sigma^*_{cl} + (1 - \alpha)^2 \sigma^*_{l}^2$$

and $\sigma^*_{c}^2 \equiv Var (c_t), \sigma^*_{l}^2 \equiv Var (\ell_t)$ and $\sigma^*_{cl} \equiv Cov (c_t, \ell_t)$ are the relevant data moments. Plugging in the (known) true values for these data moments, we obtain

$$\begin{pmatrix} \hat{\sigma}^2_{\eta} \\ \hat{\sigma}^2_{w} \end{pmatrix} = \begin{pmatrix} \sigma^*_{\eta}^2 + \sigma^*_{A}^2 \\ \sigma^*_{w}^2 + C^2 \sigma^*_{A}^2 \end{pmatrix}$$

Thus, the maximum-likelihood estimates overstate the true variances by an amount that is increasing in the degree of misspecification, as captured by a non-zero value of $\sigma^*_{A}^2$.

Next, to obtain the corrected estimates, note that

$$\begin{pmatrix} \hat{\sigma}^2_{\eta} \\ \hat{\sigma}^2_{w} \end{pmatrix} = \begin{pmatrix} \sigma^*_{\eta}^2 \\ \sigma^*_{w}^2 \end{pmatrix}$$

Thus, we obtain the regression equations

$$\hat{\varepsilon}_{\eta,t} = -\frac{C \sigma^*_{A}^2}{\sigma^*_{w}^2 + C^2 \sigma^*_{A}^2} \hat{\varepsilon}_{w,t} + \xi_{\eta,t}$$

$$\hat{\varepsilon}_{w,t} = -\frac{C \sigma^*_{A}^2}{\sigma^*_{\eta}^2 + \sigma^*_{A}^2} \hat{\varepsilon}_{\eta,t} + \xi_{w,t}$$

where the $\xi$’s are standard OLS residuals that are orthogonal to the regressors. Some algebra yields

$$\sigma^*_{\eta}^2 \leq \hat{\sigma}^2_{\eta} \equiv Var (\xi_{\eta,t}) = \sigma^*_{\eta}^2 + \frac{\sigma^*_{w}^2}{\sigma^*_{w}^2 + C^2 \sigma^*_{A}^2} \sigma^*_{A}^2 \leq \hat{\sigma}^2_{\eta}$$

$$\sigma^*_{w}^2 \leq \hat{\sigma}^2_{w} \equiv Var (\xi_{w,t}) = \sigma^*_{w}^2 + \frac{\sigma^*_{\eta}^2}{\sigma^*_{\eta}^2 + \sigma^*_{A}^2} C^2 \sigma^*_{A}^2 \leq \hat{\sigma}^2_{w}$$

That is, the corrected shock variance estimates lie between the true variances and the maximum-likelihood variance estimates.

To understand these results, consider what happens when this economy is hit with a positive tech-
nology shock. There are two channels through which this shock affects the economy: the rise in productivity directly increases output and consumption, while the increase in the marginal product of labor leads the household to both work and consume more. As seen in equations (2.6)-(2.7), the combined effect of these two channels is that consumption and labor increase in the same proportion.

In the misspecified model (2.8)-(2.9), however, there does not exist a mechanism that can produce equiproportional changes in consumption and labor. A negative labor-disutility shock has a similar labor-leisure substitution effect as the technology shock, but it lacks the direct increase in output, and thus causes consumption to increase by relatively less than labor. Only by combining a negative labor-disutility shock with a positive endowment shock (which increases consumption but has no effect on labor) can the misspecified model get consumption and labor to increase in the same proportion. This “mixing” of shocks is what produces the negative covariance in (2.10). While maximum-likelihood estimation explicitly ignores the information conveyed by this mixing of shocks, the methodology proposed here uses it to correct, in part, the resulting overestimate of the shock variances.

2.3 Formal framework

Having introduced the framework and methodology in a relatively informal way in the previous section, this section presents a more rigorous foundation for several of the results of Section 2.2. In particular, I make precise the definition of the “true values” of the shocks in the EM, \( \epsilon^*_t \), which form a crucial part of the reasoning in Section 2.2, and show how the framework proposed here directly implies Assumption 2.2.

While formally stating the results of this section requires first setting up a fair amount of mathematical machinery, the basic intuition is fairly simple. Crucially, in order to make the concept of the “true value” of an EM shock well-defined, we must first answer the following questions: Under what circumstances and in what sense can we consider a shock in the EM and a shock in the TM to be fundamentally the same, even if the EM is misspecified? Under what circumstances should we view two such shocks as being fundamentally different? The framework presented in this section answers these as follows: Two such shocks are fundamentally the same if they are associated with impulse response functions (IRFs) that are identical up to a scaling and time-shift factor. Otherwise, they are different.

With these answers in hand, it then becomes straightforward to formally define the concept of the “true value” of a shock in the EM: If there is a shock in the TM that is fundamentally the same, then its value is the true value. If there is no such shock, then its true value is zero.\(^\text{11}\) Assumption 2.2 will then follow directly from the fact that shocks that are fundamentally different from one another are

\(^{11}\) There is a sense in which this definition is quite demanding, in that a shock in the EM must meet a high standard—its exact associated IRF must exist in the TM—in order for its “true” variance to be non-zero. Thus, there is a sort of discontinuity, in that if the IRF exists in the TM then the true variance is strictly positive, but if it differs by even an arbitrarily small amount from some IRF in the TM then the true variance is zero. However, as shall become clear in Section 2.5 below, in practice this discontinuity will not be important, in that the observed degree of misspecification—and therefore any adjustments made to the variance estimates—will approach zero as the IRFs in the EM approach those in the TM.
associated with independent white noise processes.

2.3.1 Set-up

This basic set-up of the framework proceeds as follows. First, the set of all stationary IRFs is partitioned into groups that are equivalent up to a particular scaling and time-shift factor, and to each cell of this partition we associate an independent univariate Gaussian white noise process. I then create a measurable space \((G, \Gamma)\) from the partition by choosing exactly one IRF from each cell of the partition to form a set \(G\), and then defining an appropriate \(\sigma\)-algebra \(\Gamma\) on that set.

**Step 1: Partition the IRFs and associate a white noise to each cell**

Let \(\ell^{1,m}\) denote the space of absolutely summable sequences in \(\mathbb{R}^m\) equipped with the product topology. We will interpret a typical element \(\Psi\) of \(\ell^{1,m}\) as an IRF, and write

\[
\Psi = (\Psi_j)_{j=0}^{\infty}
\]

Next, for \(\Psi \in \ell^{1,m}\), define \(J(\Psi) = \min \{ j : \Psi_j \neq 0 \}\) as the first non-zero element of \(\Psi\).\(^{12}\) Define an equivalence relation \(\sim\) on \(\ell^{1,m}\) by

\[
(\Psi \sim \bar{\Psi}) \iff (\exists a \neq 0 \text{ such that } \Psi_{J(\Psi)} + j = a \bar{\Psi}_{J(\bar{\Psi})} + j \text{ for all } j \geq 0)
\]

In words, if \(\Psi \sim \bar{\Psi}\), then \(\bar{\Psi}\) is a scaled, time-shifted version of \(\Psi\), in which case we will say that \(\Psi\) and \(\bar{\Psi}\) are indistinguishable. For convenience, when it exists we will denote the value of \(a\) in the above definition by \(a(\Psi, \bar{\Psi})\).\(^{13}\) For any \(\Psi \in \ell^{1,m}\), we define \([\Psi] \equiv \{ \bar{\Psi} \in \ell^{1,m} : \Psi \sim \bar{\Psi} \}\) as the equivalence class of \(\Psi\), and let \(Q\) denote the set of all equivalence classes in \(\ell^{1,m}\), which is a partition of that set. Thus, each element \(Q \in Q\) is a set of IRFs that are equivalent to one another according to \(\sim\). To each such \(Q \in Q\), we then associate an independent univariate white noise process \(\eta_t(Q)\), where \(\eta_t(Q) \sim N(0, 1)\) and \(E[\eta_t(Q) \eta_{t-k}(Q')] = 0\) for all \(k \in \mathbb{Z}\) when \(Q \neq Q'\).

**Step 2: Create a measurable space \((G, \Gamma)\) from the partition**

Let \(G \subset \ell^{1,m}\) be some collection of IRFs formed by choosing exactly one element from each \(Q \in Q\). Thus, any collection of IRFs in \(G\) will be distinguishable, and for any \(\Psi \in \ell^{1,m}\) there exists a unique element in \(G\) indistinguishable from it. Denote this element of \(G\) by \(G(\Psi)\). Next, the integral representation introduced in the subsequent section requires defining a measure on the subsets of \(G\). To do so requires first formally defining a \(\sigma\)-algebra on \(G\). For any \(\gamma \subset G\), let

\[
Q_\gamma \equiv \{ [\Psi] : \Psi \in \gamma \} \subset Q
\]

\(^{12}\) If no such element exists, i.e., if \(\Psi_j = 0\) for all \(j\), we set \(J(\Psi) = 0\).

\(^{13}\) Note that the order of the arguments matters here, and in particular, \(a(\Psi, \bar{\Psi}) = 1/a(\bar{\Psi}, \Psi)\).
Thus, \( \mathcal{Q}_\gamma \) is the set of equivalence classes spanned by the elements of \( \gamma \). We will say that \( \gamma \) is an open subset of \( G \) if and only if there exists some open set \( H \subset \ell^{1,m} \) formed by choosing exactly one element from each \( Q \in \mathcal{Q}_\gamma \). Letting \( \Gamma \) denote the Borel \( \sigma \)-algebra generated by these open sets, \((G, \Gamma)\) is a measurable space.

### 2.3.2 Integral representation

In this section, I show that, for any \( G \) as constructed above, any model of the form given in (2.1) can be equivalently recast as

\[
\tilde{Y}_t = \int_G \left( \sum_{j=0}^{\infty} \psi_j \eta_{t-j} (|\psi|) \right) d\sigma
\]

for some measure \( \sigma \) on \((G, \Gamma)\). I will refer to this representation for \( \tilde{Y}_t \) as its integral representation on \( G \).

To see how this representation can be constructed, fix \( G \) and a model (2.1). Write \( \tilde{\psi}_j = \left( \tilde{\psi}^{(1)}_j, \ldots, \tilde{\psi}^{(r)}_j \right) \), and define \( \tilde{\psi}^{(l)}(j) \in \ell^{1,m} \) for \( l = 1, \ldots, r \). Thus, \( \tilde{G} \equiv \left\{ \tilde{\psi}^{(1)}, \ldots, \tilde{\psi}^{(r)} \right\} \) is the set of IRFs contained in the model. For simplicity, I will restrict attention to the case where \( J(\psi) = 0 \) for all \( \psi \in G \cup \tilde{G} \). It is nonetheless straightforward to extend the argument to the general case.

Next, let \( \sigma \) be the (unique) measure on \((G, \Gamma)\) satisfying (1) \( \sigma (\{\psi\}) = a(\psi, G(\psi)) \tilde{\sigma}_l \) if \( \psi = G(\tilde{\psi}^{(l)}) \), and (2) for any \( \gamma \in \Gamma \) with \( \gamma \cap G(\tilde{G}) = \emptyset \),\(^{15}\) we have \( \sigma (\gamma) = 0 \). In words, \( \sigma \) is a measure that assigns weight \( a(\psi, G(\psi)) \tilde{\sigma}_l \) to the IRF in \( G \) that is indistinguishable from the \( \ell \)-th IRF in \( \tilde{G} \), and zero elsewhere. We then have

\[
\int_G \left( \sum_{j=0}^{\infty} \psi_j \eta_{t-j} (|\psi|) \right) d\sigma = \sum_{l=1}^{r} \sum_{j=0}^{\infty} a \left( \tilde{\psi}^{(l)}, G \left( \tilde{\psi}^{(l)} \right) \right) G \left( \tilde{\psi}^{(l)} \right) \tilde{\sigma}_l \eta_{t-j} \left( \left[ \tilde{\psi}^{(l)} \right] \right)
\]

Setting

\[
\varepsilon_{l,t} \equiv \tilde{\sigma}_l \eta_t \left( \left[ \tilde{\psi}^{(l)} \right] \right)
\]

this is clearly equivalent to the representation in (2.1).

\(^{14}\) We also assume throughout that \( \tilde{\psi}^{(l)} \sim \tilde{\psi}^{(l)} \) implies \( l = i \), and that \( \tilde{\psi}^{(l)} \neq 0 \) for any \( l \). If either of these assumptions are violated, the model itself contains a fundamental indeterminacy that can be fixed by, for example, eliminating \( \tilde{\psi}^{(l)} \) from the model in either case.

\(^{15}\) With slight abuse of notation, \( G(\tilde{G}) \) here denotes the unique set of elements of \( G \) that are indistinguishable from the elements of \( \tilde{G} \).
2.3.3 Orthogonal decompositions

In this section, using the integral form introduced above, I show that for any given EM we may obtain a decomposition of the TM into a component of $Y_t$ that is explained by IRFs in the EM, and an orthogonal component that is not. Further, I establish that, in combination with Assumption 2.1, this result implies Assumption 2.2.

In particular, fix a model, and choose $G$ such that $\tilde{\Psi}_l(l) \in G$ for all $l$ (and thus $G \left(\tilde{\Psi}(l)\right) = \tilde{\Psi}(l)$). Let $\sigma^*$ denote the measure associated with the integral representation of the TM on $G$, so that the TM can be written

$$Y_t = \int_G \left( \sum_{j=0}^{\infty} \Psi_j \eta_{t-j} (\Psi) \right) d\sigma^*$$

Letting $\tilde{G}^c$ be the complement of $\tilde{G}$ in $G$, we can write

$$Y_t = \int_{\tilde{G}} \left( \sum_{j=0}^{\infty} \Psi_j \eta_{t-j} (\Psi) \right) d\sigma^* + \int_{\tilde{G}^c} \left( \sum_{j=0}^{\infty} \Psi_j \eta_{t-j} (\Psi) \right) d\sigma^*$$

where $\tilde{\Psi}_j \equiv (\tilde{\Psi}^{(1)}_j, \ldots, \tilde{\Psi}^{(r)}_j)$, $\tilde{\epsilon}_{t,l} \equiv \sigma^* \eta_l (\tilde{\Psi}(l))$, $\sigma_l^* \equiv \sigma^* (\tilde{\Psi}(l))$, and $\tilde{\epsilon}^*_t \equiv (\tilde{\epsilon}^*_{1,t}, \ldots, \tilde{\epsilon}^*_{r,t})'$. More compactly, we can write this as

$$Y_t = \tilde{Y}_t^* + Z_t$$

(2.12)

$\tilde{Y}_t^*$ may be interpreted as the variation in $Y_t$ that is generated by the IRFs contained in the EM (or their equivalents according to $\sim$), and $Z_t$ as the variation in $Y_t$ generated by IRFs for which no equivalent exists in the EM. Note that the EM is correctly specified if and only if $Var(Z_t) = 0$; otherwise, it is misspecified. Note also that, irrespective of whether the model is the true DGP, $\mathbb{E} [\tilde{\epsilon}_t^* Z'_t] = 0$ for all $k$, and therefore in turn $\mathbb{E} [\tilde{Y}_t Z'_{t-k}] = 0$, so that (2.12) is an orthogonal decomposition of $Y_t$.

Next, substituting (2.12) into (2.3) we may obtain

$$\hat{\epsilon}_t = \sum_{j=-\infty}^{\infty} \psi_j \tilde{Y}_{t-j}^* + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

(2.13)

By Assumption 2.1, the first term on the right-hand side of this expression is the true value of the EM shocks $\tilde{\epsilon}_t^*$, and thus the second term is the recovery error $\nu_t$ defined in Section 2.2. Further, these two terms are clearly orthogonal to one another at all leads and lags, so that $\mathbb{E} [\tilde{\epsilon}_t^* \nu'_{t-k}] = 0$ for all $k$. That is, in combination with Assumption 2.1, the framework developed above necessarily implies that

---

16 This choice of $G$ is for simplicity of exposition only. With the appropriate modifications, the desired result will hold for any $G$.
Assumption 2.2 holds. We thus verify the upward-bias result from Section 2.2,
\[ \hat{\sigma}_l^2 = \sigma^*_l^2 + \text{Var}(\nu_{l,t}) \geq \sigma^*_l^2 \]
Furthermore, it is straightforward to see that when the model is misspecified (i.e., when \( \text{Var}(Z_t) \neq 0 \)), we will have \( \text{Var}(\nu_{l,t}) > 0 \) for at least one \( l \), and therefore \( \hat{\sigma}_l^2 > \sigma^*_l^2 \); that is, in the presence of misspecification the naive variance estimate will necessarily overstate the true variance of at least one shock. It can be easily verified that the other results of Section 2.2 also hold.

2.3.4 Discussion

It is worth clarifying in what contexts the framework proposed in this paper is appropriate to use. One of the key properties of the framework is that shocks are identified uniquely up to a scaling and time-shift factor for their IRFs. Conversely, any two shocks that are the same up to a scaling and time-shift factor (i.e., that are indistinguishable according to \( \sim \)) are not identified uniquely. Implicitly, the smoothing algorithm first determines which equivalence class (among those spanned by the EM) an imputed shock has come from using only information about the “shape” of the IRF from its first non-zero impact onwards. Once the equivalence class has been identified, the algorithm then chooses the appropriate date and value of the shock as determined by the specific IRF from that equivalence class that appears in the EM. A corollary to this procedure is that if one were to replace an IRF in the EM with a different one from the same equivalence class, one would in general obtain a different variance estimate for that shock. As such, this framework is not appropriate for addressing the question of which of two shocks from the same equivalence class—but with potentially different economic interpretations—are more relevant, since the algorithm views these two shocks as fundamentally the same. However, as can be easily verified, the estimate of the variance induced in the observable variables by these two different shocks would nonetheless be identical. To illustrate with a simple example, suppose the TM for output is given simply by \( X_t = A_t \), where \( A_t = \rho A_{t-1} + \varepsilon_t \), \( \varepsilon_t \) is an i.i.d. exogenous process with variance \( \sigma^2 \), and \( \rho \) is known. Using this TM as the EM and given sufficient data on \( X_t \), one would be able to recover the true sequence of shocks \( \varepsilon_t \), would obtain a variance estimate for the exogenous process equal to \( \sigma^2 \), and would find that the variance induced in \( X_t \) by this shock is equal to \( \sigma^2_X \equiv \sigma^2 / (1 - \rho^2) \). Now suppose one instead uses as the EM the same model, except that now it is assumed that \( X_t = \phi A_t \), where \( \phi \neq 0 \) is some (known) constant. In this case, one would recover the sequence of shocks \( \varepsilon_t / \phi \) and would obtain a variance estimate for the TFP shock of \( \sigma^2 / \phi^2 \), but would nonetheless find that the variance induced in \( X_t \) by this shock remains equal to \( \sigma^2_X \). As this example makes clear, then, one should view the framework in this paper not as seeking to find the variance of a shock, but as seeking to find the total variance in the observable variables induced by that shock (or by its equivalents according to \( \sim \)).

To anticipate one potential objection to the framework proposed in this paper, it should be noted that, if we were to find a process for the smoothed shocks that exhibits correlation, it may be the case
that the *true* shocks are in fact correlated. In this case, there may be a desire to view independence of the shocks as a modelling approximation made for convenience, rather than a property to be taken seriously (as it is in this framework). As such, one might argue, we should not be too concerned if we find evidence of its violation. I find this argument unconvincing for several reasons. First, if the shocks truly are correlated, there must be some underlying economic reason for this. Yet correlation among smoothed shocks is rarely acknowledged in the literature, let alone justified using some economic argument. Second, though it would increase the number of estimated parameters, it would be a relatively straightforward exercise to allow for correlation in the EM, but again, such an exercise appears to be rarely done in the literature. Third, if we do allow for such correlation in the model, rational expectations dictates that agents in the model know this. When there are intertemporal correlations between the shocks, this has the capacity to significantly alter agents’ behavior, since current shock realizations would carry information about future values of the shocks. There is little reason to believe *a priori* that estimating such a model would yield even qualitatively similar results to those obtained in the case where independence is assumed. Finally, as demonstrated in Section 2.5 below, to the extent that independence is a reasonable—though not necessarily perfect—approximation to the true structure of the exogenous processes, the bias-correction procedure proposed here would have a minimal effect on variance estimates. Thus, if resulting corrections are in fact large, this should be taken at the very least as clear evidence that independence is *not* a reasonable approximation.

Next, it may be helpful to view the framework presented in Sections 2.3.1-2.3.3 as a DSGE analogue to a typical regression setting. In the latter, the regression error term is, conceptually, the residual component of the dependent variable after as much of its variation as possible is explained by the regressors. In general, no economic meaning is assigned to this residual component. Rather, it is taken as arising from the econometrician’s ignorance about the full process underlying the dependent variable. In a DSGE setting, the shocks are often taken to be a similar residual component. Unlike in a regression setting, however, in a DSGE model the shocks are given an economic interpretation and are thus *a part of the econometrician’s explanation*, not residuals standing in for her ignorance. If we believe that factors beyond those captured by the model are of some relevance, this would motivate the explicit consideration of some orthogonal error term analogous to the regression residual. One may view the framework above as motivating such a residual (see, e.g., $Z_t$ in equation (2.12)).

To close this section, it is worth briefly mentioning here that there are, conceptually, two distinct types of misspecification that arise in the above framework. In the first type, the IRFs in the EM are also found in the TM, but there are IRFs in the TM that are not present in the EM. This is the case

---

17 Ingram et al. (1994) are a notable exception. They extend the canonical neoclassical growth model of King et al. (1988), which features only a shock to productivity, by adding two additional shocks. For several different calibrations of the model, they recover the implied series for the shocks using U.S. data on output, consumption and labor, and find that the shocks exhibit substantial correlation, both cross-sectionally and intertemporally.

18 In the above framework, this would in practice consist of adding one or more IRFs to the EM that are linear combinations of existing IRFs. In other words, this would effectively increase the number of shocks in the model.

19 There are some infrequent exceptions to this. For example, in Smets and Wouters (2007), innovations to TFP are also allowed to impact the exogenous spending shock.
in Example 1. In the second type of misspecification, there may be shocks in the EM and TM with the same economic interpretation but for which the associated IRF in the EM differs (however mildly) from the corresponding IRF in the TM. The framework introduced here does not distinguish between these two types of misspecification, despite the fact that there may be some desire to be more forgiving toward the second type. Nonetheless, as will be illustrated in the examples of Section 2.5 below, if the EM is a reasonable approximation to the TM, we will continue to find an important role for the EM shocks even if their precise IRFs differ somewhat from those in the EM.

2.4 Further analysis

In this section, I provide basic conditions on the EM and TM that are sufficient for the key results of Section 2.2 to hold, then re-state those results formally.

2.4.1 Econometric model

Consider the class of linearized DSGE models in state-space form as

\[ X_t = AX_{t-1} + B \varepsilon_t \]  

(2.14)

Here, \( X_t \) is an \( n \)-vector of (possibly unobservable) model variables, \( \varepsilon_t \) is an \( m \)-vector of i.i.d. normal random variables with diagonal covariance matrix, the \( l \)-th diagonal element of which is given by \( \sigma_l^2 \), and \( A \) and \( B \) are \((n \times n)\)- and \((n \times m)\)-matrix-valued functions of \( P \), respectively, where \( P \in \mathcal{P} \) is the vector of “deep” model parameters (excluding \( \sigma \equiv (\sigma_1, \ldots, \sigma_m)' \)). The eigenvalues of \( A \) are all strictly less than one in modulus, so that \( \{X_t\} \) is a covariance-stationary process.

For what follows, I assume that the parameter vector \( P \) is given. There is a substantial literature that considers methods for obtaining parameterizations in this class of models.\(^{20}\) I wish to sidestep this issue altogether and take as a starting point a particular parameterization (or a set of parameterizations, as the case may be) that the econometrician has identified as in some sense best for his or her purposes. The theoretical and methodological considerations that follow should thus be viewed as tools to analyze the implications of a particular value of \( P \) in the model, rather than of the model itself. Henceforth, for the sake of brevity, I shall generally suppress dependence of the analysis on the parameterization \( P \), with the tacit understanding that all discussion and results are related to that specific parameterization only.

The \( m \)-vector of observable data \( Y_t \) is presumed by the econometrician to be related to the model variables by

\[ \tilde{Y}_t = FX_t \]

(2.15)

where \( F \) is an \( m \times n \) matrix and, as in Section 2.2, \( \tilde{Y}_t \) is the model counterpart of \( Y_t \). \( F \) and \( B \) are

\(^{20}\) See, for example, Gourieroux et al. (1993), An and Schorfheide (2007), Canova (2007), Fernández-Villaverde (2010).
assumed to be such that the matrix $FB$ is invertible.\footnote{Note that I have implicitly assumed that the model contains the same number of shocks as observable variables; i.e., $m = r$ in the notation of Section 2.2. If we had $r > m$, we would be unable to invert the EM’s MA($\infty$) representation, so that Assumption 2.1 would not hold. On the other hand, if we had $r < m$, some subset of the elements of $\tilde{Y}_t$ would be linearly dependent; i.e., the implied autocovariance function would be singular. Since the actual data is unlikely to possess this feature, the model would be inconsistent with the data in that there would in general be no sequence of shocks that would allow the EM to exactly reproduce the data.}

### 2.4.2 Data process

Suppose we have a series of observations drawn from an $m$-variate process $\{Y_t\}$. I make the following assumption about $\{Y_t\}$.

**Assumption 2.3.** $\{Y_t\}$ is a mean-zero jointly normally distributed covariance-stationary process with $Var(Y_{i,t}) > 0$ for all $i = 1, \ldots, m$.

Note that we have not assumed that the TM for $Y_t$ is given by the EM (2.14)-(2.15). Rather, as argued in Section 2.3.3, we may write

$$Y_t = \tilde{Y}^*_t + Z_t$$

(2.16)

where

$$\tilde{Y}^*_t = FX^*_t$$

(2.17)

$$X^*_t = AX^*_{t-1} + B\varepsilon^*_t$$

(2.18)

Here, $\varepsilon^*_t$ and $Z_t$ have the same interpretation as in Section 2.3.3, and in particular $E[\varepsilon_t^* Z'_{t-k}] = E[\tilde{Y}^*_t Z'_{t-k}] = 0$ for all $k \in \mathbb{Z}$.

### 2.4.3 Theoretical results

This section formally presents the theoretical results underpinning the methodology discussed in Section 2.2. Let $C \equiv \begin{bmatrix} I_n - B (FB)^{-1} F \end{bmatrix} A$. I make the following assumption regarding the eigenvalues of $C$.

**Assumption 2.4.** None of the eigenvalues of $C$ lie on the complex unit circle.
In contrast, while we similarly require that the latent state vector be recoverable (so that we may in turn recover the structural shocks), we have no need to construct a backward-looking structural VAR representation. As such, there is no need for the state vector to be recoverable on the basis of past values of $Y_t$ only; it will be sufficient that some combination of past, present and future values of $Y_t$ reveal $X_{t-1}$, a property that is guaranteed by Assumption 2.4.23 This intuition is established formally in the following proposition.

**Proposition 2.1.** Suppose Assumptions 2.3-2.4 hold. Then there exists a sequence of absolutely summable $m \times m$ matrices $\{\psi_j\}_{j=-\infty}^{\infty}$ such that

$$
\varepsilon^*_t = \sum_{j=-\infty}^{\infty} \psi_j L^j (Y_t - Z_t) \quad (2.19)
$$

where $L$ is the lag operator and $\psi_j$ is a function of $A$, $B$, $F$ and $j$ only. Further,

$$
\left( \sum_{j=-\infty}^{\infty} \psi_j L^j \right) \left( \sum_{i=0}^{\infty} FA^i BL^i \right) = I_m = \left( \sum_{i=0}^{\infty} FA^i BL^i \right) \left( \sum_{j=-\infty}^{\infty} \psi_j L^j \right) \quad (2.20)
$$

**Proof.** All proofs are given in Appendix A.2.

Proposition 2.1 establishes conditions under which, given infinite sequences of observations $Y^\infty \equiv \{Y_t\}_{t=-\infty}^{\infty}$ and $Z^\infty \equiv \{Z_t\}_{t=-\infty}^{\infty}$, the true value of the structural shock, $\varepsilon^*_t$, may be recovered. The next proposition verifies that Assumptions 2.3 and 2.4 together imply Assumption 2.2.

**Proposition 2.2.** Suppose Assumptions 2.3-2.4 hold. Then

$$
\hat{\varepsilon}_t = \varepsilon^*_t + \nu_t \quad (2.21)
$$

where $E \left[ \varepsilon^*_t \nu_{t-k} \right] = 0$ for all $k \in \mathbb{N}$.

Proposition 2.2 orthogonally decomposes a smoothed EM shock into its true value and a residual. This decomposition result has several useful implications, summed up in the following corollary.

**Corollary 2.1.** Suppose Assumptions 2.3-2.4 hold. Then the following are true:

(a) $\hat{\sigma}_l^2 \equiv E \left[ \varepsilon^2_{t,l} \right] \geq \sigma^*_l^2$ for all $l$.

---

22 See, for example, Fernández-Villaverde et al. (2007).

23 The relaxation of the structural VAR assumption ($|\text{eig}(C)| < 1$) to Assumption 2.4 ($|\text{eig}(C)| \neq 1$) is an economically relevant one. As pointed out by, for example, Leeper et al. (2011) and Sims (2012), the requirement that all eigenvalues of $C$ lie strictly inside the complex unit circle often excludes models where agents’ information sets are strictly greater than the econometrician’s, such as in models with “news” and “noise” shocks (see, for example, Beaudry and Portier (2004), Schmitt-Grohé and Uribe (2012), Jaimovich and Rebelo (2009), Blanchard et al. (2013), Christiano et al. (2010a)). In contrast, Assumption 2.4 will not generally exclude this important class of models.
(b) If $E[\hat{\varepsilon}_{l,t}\hat{\varepsilon}_{i,s}] \neq 0$ and $(l, t) \neq (i, s)$, then the model is misspecified.

(c) If the model is misspecified, then there exists an $l$ such that $\hat{\sigma}_l^2 > \sigma_l^2$.

(d) For $(l, t) \neq (i, s)$, $E[\hat{\varepsilon}_{t}^*\hat{\varepsilon}_{i,s}] = 0$ irrespective of whether the model is misspecified.

Part (a) of Corollary 2.1 points out that the variance of smoothed shock $l$ is an upper bound for the true variance, which is itself unobservable. Parts (b) and (c) highlight that if the smoothed shocks exhibit correlation with one another, then the model is certainly misspecified, and thus the variance of at least one of the shocks will be overstated.

Part (d) forms the basis for the correction procedure that is the principal methodological contribution of this paper. In particular, for any $q \in \mathbb{N}$, we may write

$$\hat{\varepsilon}_{l,t} = \sum_{i \neq l} \Theta_{i,0}\hat{\varepsilon}_{i,t} + \sum_{j=1}^{q} (\Theta_{j}\hat{\varepsilon}_{t-j} + \Theta_{-j}\hat{\varepsilon}_{t+j}) + \xi_{l,t} \tag{2.22}$$

where $E[\xi_{l,t}\hat{\varepsilon}_{i,t+j}] = 0$ for any $(l, t) \neq (i, t+j)$ with $|j| \leq q$. Here, the $\Theta$’s have the interpretation of population regression coefficients, and $\xi_{l,t}$ as the corresponding OLS residual. $\xi_{l,t}$ captures the component of $\hat{\varepsilon}_{l,t}$ that cannot be predicted using $q$ past and future values of the recovered shocks (including the current values of shocks $i \neq l$). The following proposition establishes how a more accurate upper bound for $\sigma_l^2$ may be obtained from the regression equation (2.22).

**Proposition 2.3.** Suppose Assumptions 2.3-2.4 hold, and let $\xi_{l,t}$ be as in equation (2.22). Then

$$\sigma_l^{*2} \leq \hat{\sigma}_l^2 \equiv \mathbb{E}[\xi_{l,t}^2] \leq \hat{\sigma}_l^2 \tag{2.23}$$

Proposition 2.3 establishes that $\hat{\sigma}_l^2$ is an upper bound for the unknown value of $\sigma_l^{*2}$, and further that this upper bound is a (weakly) better estimator of $\sigma_l^{*2}$ than $\hat{\sigma}_l^2$.

### 2.4.4 Practical considerations

Assumption 2.4 and the availability of infinite past and future observations on $Y_t$ play an important role in the theoretical results of Section 2.4.3. In particular, in the case where the EM (2.14)-(2.15) is correctly specified, these properties together guarantee that the shocks may be exactly recovered as a linear combination of the observations on $Y_t$. This is crucial for the orthogonal decomposition of Proposition 2.2, and, since it relies on this result, also for the variance correction suggested at the end of Section 2.4.3.

In practice, however, Assumption 2.4 may fail to hold in important cases of interest (the application of Section 2.6 explores such an instance), and data sets are usually limited to at most a few hundred

\[\text{Note that the converse does not necessarily hold, i.e., if } E[\hat{\varepsilon}_{l,t}\hat{\varepsilon}_{i,s}] = 0 \text{ for all } (l, t) \neq (i, s), \text{ this does not necessarily imply that the model is correctly specified, so that the naive estimates } \hat{\sigma}_l^2 \text{ may still overstate the importance of one or more shocks.}\]
periods of observations. In either of these cases, the shocks may be recovered in expectation only, with a generally non-zero and non-diagonal error covariance matrix even when the model is correctly specified. Nonetheless, given the linear-Gaussian structure of the EM and TM, the expected values of the shocks conditional on the model and observed data (i.e., the smoothed shocks) will be linear combinations of the sequence of observations on $Y_t$. Thus, as in (2.3), we may express a smoothed shock as a linear filter applied to the time series of data.\textsuperscript{25} As such, using a similar argument as in the proof of Proposition 2.2, we may generalize the decomposition result as

$$\hat{\xi}_t = \hat{\xi}_t^* + \nu_t$$

where $\mathbb{E}[\hat{\xi}_t^* \nu_{t-k}'] = 0$, $\hat{\xi}_t^*$ is defined to be the value of the smoothed shock that would obtain if the model were correctly specified, and $\nu_t$ is the linear filter applied only to the sequence of $Z_t$'s (analogous to the second term on the right-hand side of equation 2.13). In general, the errors $\delta_t^* \equiv \hat{\xi}_t^* - \xi_t^*$ that would be made in recovering the shocks if the model were correctly specified will be correlated across different shocks. As a result, the actual smoothed shocks, $\hat{\xi}_t$, will in general exhibit non-zero correlation with one another even if the model is correctly specified. Whether the robust variance estimates proposed in Section 2.3 will continue to be an improvement on the naive ones under these circumstances depends on the quality of the approximation $\hat{\xi}_t^* \approx \xi_t^*$. While no analytical result is available to check the accuracy of this approximation, given that the researcher has in hand the fully specified model, it is nonetheless straightforward to check it numerically via simulation, as illustrated in the application of Section 2.6. Similarly, critical values for any relevant statistics may also be obtained via simulation.

\textbf{2.5 Further examples}

In this section, I present two additional fully-specified examples to illustrate the above methodology. In the first example, which is simple enough to be solved analytically, I highlight a type of misspecification that is conceptually distinct from the type present in the example of Section 2.2.1. In that earlier example, the EM correctly represents the dynamic impacts of the shocks it contains (i.e., the IRFs in the EM also exist in the TM), but is missing an additional source of exogenous variation. In contrast, in the first example presented in this section the economic interpretations of a shock in the EM and another in the TM are the same, but their IRFs are not, and so by the framework developed in Section 2.3 they are nonetheless considered fundamentally different.

Next, in the second example of this section, the TM is taken to be a variant of a standard medium-scale New Keynesian model (that of Smets and Wouters (2007)) that features a variety of real and nominal frictions and seven exogenous shock processes. The EM, on the other hand, will in general be missing several frictions and only contain three of the seven shock processes.\textsuperscript{26} This example most

\textsuperscript{25} Note that, when the data set is finite, the filter for $\hat{\xi}_t$ will generally depend on $t$.

\textsuperscript{26} Note that both types of misspecification highlighted above will be present in this example.
closely captures quantitative macroeconomic modelling in practice, whereby economists use highly
simplified models to fit data generated by substantially more complicated ones.

2.5.1 Example 2: Univariate AR(1) model

Suppose we have a sequence of data on TFP growth, $\gamma_t$, for which the TM is an AR(1) process

$$\gamma_t = \rho \gamma_{t-1} + \epsilon_{1,t}, \quad \epsilon_{1,t} \sim \text{i.i.d. } N(0, \sigma_1^2)$$

where $0 < |\rho| < 1$. The EM, meanwhile, is given by

$$\tilde{\gamma}_t = \delta \tilde{\gamma}_{t-1} + \tilde{\epsilon}_{2,t}, \quad \tilde{\epsilon}_{2,t} \sim \text{i.i.d. } N(0, \tilde{\sigma}_2)$$

with $|\delta| < 1$, where $\delta$ is some number taken as given by the econometrician and that may potentially
be different from $\rho$, in which case the EM would be misspecified. This EM can be easily inverted to
obtain $\tilde{\epsilon}_{2,t} = \tilde{\gamma}_t - \delta \tilde{\gamma}_{t-1}$. Substituting the data process $\gamma_t$ into the inverted EM for $\tilde{\gamma}_t$, we may obtain
the process for the smoothed shocks,

$$\hat{\epsilon}_{2,t} = (\rho - \delta) \gamma_{t-1} + \epsilon_{1,t}$$

and thus the naive estimate of the variance of the TFP shock is given by

$$\hat{\sigma}_2^2 = \frac{1 - 2\rho \delta + \delta^2}{1 - \rho^2} \sigma_1^2$$

When $\delta \neq \rho$, the IRF in the EM does not exist in the TM, and thus by the framework of Section
2.3 we have $\sigma_2^*^2 = 0$, which is clearly less than the naive estimate $\hat{\sigma}_2^2$. The conclusion that $\sigma_2^*^2 = 0$ is
rather stark here, since it will hold for all $\delta \neq \rho$ even as $\delta \to \rho$, while if $\delta = \rho$ we have $\sigma_2^*^2 = \sigma_1^2$.
However, this discontinuous behavior is of little practical importance. In particular, misspecification
is identified in this framework as non-zero autocovariance in the sequence of smoothed shocks. For
$k \neq 0$ we may obtain

$$\text{Cov} (\hat{\epsilon}_{2,t}, \hat{\epsilon}_{2,t-k}) = \frac{(\rho - \delta) (1 - \rho \delta)}{1 - \rho^2} \rho^{|k|-1} \sigma_1^2$$

Clearly, $\text{Cov} (\hat{\epsilon}_{2,t}, \hat{\epsilon}_{2,t-k}) \to 0$ as $\delta \to \rho$. Thus, even though the EM continues to be misspecified
as $\delta$ becomes arbitrarily close to $\rho$ (i.e., as the IRF in the EM becomes ever closer to the IRF in the
TM), the degree of observed misspecification (and thus any adjustment made using the bias-correction
procedure) will nonetheless shrink to zero.

Next, to apply the bias-correction procedure, we wish to find the component of $\hat{\epsilon};$ that cannot be
predicted using past and future values. In general, one can continue to improve the quality of the
adjusted estimate by including a greater number of leads and lags in the regression (i.e., by making $q$
arbitrarily large in equation (2.22)). For simplicity, I focus here on the simple case where $q = 1$. In
this case, the estimated regression equation is given by

$$
\hat{\varepsilon}_{2,t} = \beta (\hat{\varepsilon}_{2,t-1} + \hat{\varepsilon}_{2,t+1}) + \xi_t
$$

where

$$
\beta \equiv \frac{(\rho - \delta) (1 - \rho \delta) (1 - \rho^2) [1 - \delta (\rho - \delta)]}{(1 - 2\rho \delta + \delta^2)^2 - \rho^2 (\rho - \delta)^2 (1 - \rho \delta)^2}
$$

and $\xi_t$ is the OLS residual. The bias-corrected estimate of the shock variance is then given by $\bar{\sigma}_2^2 \equiv Var(\xi_t)$. Some simple algebra shows that

$$
\bar{\sigma}_2^2 = \hat{\sigma}_2^2 - 2\beta Cov(\hat{\varepsilon}_{2,t}, \hat{\varepsilon}_{2,t-1})
$$

It can be verified that $\beta Cov(\hat{\varepsilon}_{2,t}, \hat{\varepsilon}_{2,t-1}) \geq 0$ and therefore $0 = \sigma_2^*^2 < \bar{\sigma}_2^2 \leq \hat{\sigma}_2^2$. Thus, the corrected estimate lies between the true value and the naive estimate $\hat{\sigma}_2^2$. Furthermore, as expected given the discussion above, we have $\beta Cov(\hat{\varepsilon}_{2,t}, \hat{\varepsilon}_{2,t-1}) \to 0$ as $\delta \to \rho$, so that the size of the adjustment approaches zero as the IRF in the EM approaches the IRF in the TM. That is, notwithstanding the fact that the true variance is zero for all $\delta \neq \rho$, as the degree of misspecification approaches zero the bias-corrected estimate of the variance will approach the desired value of $\sigma_2^*^2$.

2.5.2 Example 3: Medium-scale new Keynesian model

In this example, I consider three different combinations of TM and EM. In all cases, the models will be variants of the widely-cited medium-scale New Keynesian model of Smets and Wouters (2007), which in its baseline form features a number of real and nominal frictions, as well as seven different exogenous shock process: exogenous spending, neutral TFP, investment-specific technology (IST), risk premium, price mark-up, wage mark-up, and monetary policy shocks.\(^{27}\) In particular, I consider the following combinations:

- **Case A**: The EM and TM both feature sticky prices and wages and only the first three shocks above (exogenous spending, neutral TFP, and IST).

- **Case B**: The EM and TM both feature sticky prices and wages. The EM contains only the first three shocks while the TM contains the full complement of seven shocks.

- **Case C**: The EM and TM both contain only the first three shocks. The EM features flexible prices and wages, while the TM features sticky prices and wages.

Note that, while the EM in Case A is correctly specified, Cases B and C correspond to two distinct types of misspecification. As in Example 1, the EM in Case B is missing important sources of exogenous variation due to the absence of the risk premium, mark-up and monetary policy shocks.

\(^{27}\) A summary of the equations characterizing the DGP and model is contained in Appendix A.1. For further details, see Smets and Wouters (2007).
Meanwhile, as in Example 2, in Case C even though the shocks in the EM have the same economic interpretation as shocks in the TM, their IRFs will be different since the EM does not feature sticky prices or wages. As such, these shocks are fundamentally different, and thus the true variance of each EM shock is zero. As we shall see, however, the “less misspecified” is the EM—that is, the more flexible are prices and wages in the TM—the smaller the degree of observed misspecification will be, with no observed misspecification in the limit as prices and wages become perfectly flexible. In turn, as in Example 2, the size of any bias adjustment will also shrink to zero.

I assume the econometrician estimates the variances of the EM shocks using data on three real variables: consumption, investment, and hours worked. For reasons discussed in Section 2.4.1, I assume that, aside possibly from the true degree of price and wage flexibility, the econometrician knows all of the parameter values of the TM except for the variances of the shock process innovations, which are then estimated using a large data set on consumption, investment and hours worked, themselves generated from the TM.\(^{28,29}\) The first column of data in Table 2.1 reports, for each of the three shock processes in the EM, the variance of the shock with that same economic interpretation in the TM.\(^{30}\) The remaining three columns report naive variance estimates for the three different cases.\(^{31}\)

For Case A, the EM is correctly specified, and as a result the naive variance estimates reported in the table are the same as the true values reported in the first column.\(^{32}\) For Case B, however, we have the first type of misspecification discussed above: the IRFs in the EM are the same as their TM counterparts, but the TM also includes other sources of exogenous variation. As we might have expected from (2.4), the shock variances are significantly overestimated in this case, ranging from over 13 times the true value for the investment shock, to well over 800 times for the exogenous spending and TFP shocks.

Next, I apply the bias-correction methodology to obtain revised estimates of the variances for Cases A and B. Specifically, I regress each of the three smoothed shocks on contemporaneous values of the other two smoothed shocks and on four leads and lags of all three shocks. I then compute the variance of the resulting residual. The results are presented in Table 2.2. For comparison purposes, the first column of data again reports the variance of the shock with that economic interpretation in the TM. Looking at the second column, the corrected estimates for Case A show no change relative to

\(^{28}\) Except where otherwise noted, the model parameters used for generating the data are taken as the mode of the posterior distribution reported in Smets and Wouters (2007).

\(^{29}\) In practice, I simulate 101,000 periods of data, discarding the first 1,000 to minimize the impact of the choice of the initial state vector—set equal to zero—on the results. I use this large number of data periods to avoid the complications (discussed in Section 2.4.4) that arise when samples are of a more realistic size.

\(^{30}\) Note that these variances are the same for each of the three cases.

\(^{31}\) Naive estimates are obtained by first running the Kalman smoothing algorithm (with shock variance parameters set equal to values from TM), then computing the sample variances of the resulting smoothed shocks. While these estimates will in general differ from estimates obtained via maximum likelihood, the two are asymptotically equivalent. Given the very large data set, it can be verified that the quantitative difference between the two approaches is negligible.

\(^{32}\) Technically, there are small discrepancies between the two sets of estimates due to (1) sampling error; and (2) the fact that, in a finite sample, the shocks cannot be exactly recovered. However, because of the large sample size employed here, such discrepancies are quantitatively unimportant.
Table 2.1: Variances of true and smoothed shocks

<table>
<thead>
<tr>
<th>Case</th>
<th>Shock</th>
<th>True</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>TM</td>
<td>Spending</td>
<td>0.27</td>
<td>0.27</td>
<td>221.52</td>
<td>2.18</td>
</tr>
<tr>
<td></td>
<td>TFP</td>
<td>0.21</td>
<td>0.21</td>
<td>179.27</td>
<td>0.61</td>
</tr>
<tr>
<td></td>
<td>IST</td>
<td>0.20</td>
<td>0.20</td>
<td>2.73</td>
<td>0.21</td>
</tr>
</tbody>
</table>

Notes: “True” column shows variance of shock in TM. Remaining columns show naive variance estimates for relevant TM and EM, obtained by first running Kalman smoothing algorithm (with shock variances set equal to values from TM), then computing sample variances of resulting smoothed shocks.

The naive case from Table 2.1. Intuitively, since in this case the model is the TM, the true sequence of shocks can be recovered (nearly) exactly. Since the true shocks are, by construction, uncorrelated with one another, so too are the smoothed shocks. Thus, none of the variation in a smoothed shock can be explained by variation in the other shocks, and we obtain an $R^2$ of essentially zero in the regression.

Table 2.2: Corrected variances of smoothed shocks

<table>
<thead>
<tr>
<th>Case</th>
<th>Shock</th>
<th>True</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>TM</td>
<td>Spending</td>
<td>0.27</td>
<td>0.27</td>
<td>0.55</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>TFP</td>
<td>0.21</td>
<td>0.21</td>
<td>0.44</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>IST</td>
<td>0.20</td>
<td>0.20</td>
<td>0.23</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Notes: “True” column shows variance of shock in TM. Remaining columns show bias-corrected estimates. For a given smoothed shock, set of regressors is contemporaneous values of other smoothed shocks plus four leads and lags of all smoothed shocks.

For Case B, on the other hand, the corrected estimates in Table 2.2 show a substantial improvement over the naive estimates from Table 2.1. For example, whereas the naive estimate of the neutral TFP variance was 875 times the true value, the corrected estimate is only a little more than twice the true value. The relative discrepancies are even smaller for the other two shocks. A clue for why the corrected estimates show such a large improvement in Case B can be seen from Table 2.3, which presents contemporaneous cross-correlations between the smoothed shocks (first three rows), as well as the first-order autocorrelation for each shock (final row). Clearly, the smoothed shocks exhibit a very high degree of contemporaneous correlation with one another, especially the exogenous spend-
ing and neutral TFP shocks, which are nearly collinear. The spending and TFP shocks also exhibit significantly negative first-order autocorrelation. Because there is such a high degree of correlation between the smoothed shocks, they are highly predictable using values of the other smoothed shocks, which in turn results in the regression residual being small.

<table>
<thead>
<tr>
<th>Table 2.3: Sample correlations for smoothed shocks (case B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spending(_t)</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>Spending(_t)</td>
</tr>
<tr>
<td>TFP(_t)</td>
</tr>
<tr>
<td>IST(_t)</td>
</tr>
<tr>
<td>Lagged variable</td>
</tr>
</tbody>
</table>

Notes: Table presents correlations among smoothed shocks for Case B. Upper three rows of data show unconditional correlation matrix. Final row shows first-order autocorrelation for that smoothed shock.

This high degree of correlation in the smoothed shocks stems from a number of features of the TM, one example of which I focus on here for illustrative purposes. Consider the risk premium shock. In the TM this shock accounts for over 80 percent of the variance of the one-step-ahead forecast error in consumption, while in the EM it is entirely absent. Thus, the EM is clearly missing an important factor in generating high-frequency variation in consumption. This is confirmed in Figure 2.1, which shows autocovariance functions (ACFs) for the three observable variables using the TM. Each panel of the figure corresponds to Cov\( (y_t, x_{t-k}) \) for two variables \( y_t \) and \( x_t \). Within each panel there are four lines plotted, each corresponding to the ACF that arises in the TM when all but one shock is shut down. The four plots correspond to the four “real” shocks (the exogenous spending shock, the two technology shocks, and the risk premium shock). The first five lags are shown, and to make the comparison as clean as possible, the variances of the shocks are normalized so that the unconditional variance of consumption is always equal to one.

Looking at the top-left panel of the Figure, we see that the autocovariance generated in the consumption process by the risk premium shock (the solid line) shows a relatively steep decline. In contrast, the spending, TFP and IST shocks (the dashed, dotted, and dash-dot lines, respectively) all generate highly persistent consumption dynamics. In order for the EM to be able to reproduce the high-frequency consumption variation that is a feature of the TM, it would therefore need to be the case that certain combinations of shocks tend to occur together or in a particular sequence (or both). Since the smoothing algorithm is precisely the process of finding a sequence of shocks that allow the EM to exactly reproduce the data, it should then come as no surprise that we observe correlations between them.

Next, consider Case C. In this case we have the second type of misspecification discussed above: the shocks in the TM and the EM have the same economic interpretation, but because the EM does not feature sticky prices and wages, the IRFs in the model are different from those in the TM. As such,
the true variances of the EM shocks are actually zero. As seen in Table 2.1, however, using naive estimation procedures we obtain non-zero estimates.\footnote{Note that, while it turns out the estimates for Case C are greater than the values reported in the first column, this need not always be true.}

Turning to the corrected estimates in Table 2.2, we find that they are quite close to their true value of zero. As in Example 2, however, there is some sense that this result is undesirable, since there are shocks in the TM with the same economic interpretation and that do actually account for a positive amount of variation. But as also illustrated in Example 2, the extremeness of the adjustment in this case is due to the very high degree of misspecification in the EM. In particular, there is a significant amount of price and wage stickiness in the TM: each quarter, only fractions 0.34 and 0.26 of price- and wage-setters, respectively, are allowed to re-set their prices/wages. In the EM, however, these fractions are incorrectly assumed to be equal to one. To see how the corrected estimates change as the
degree of stickiness in the TM and EM converge, Table 2.4 shows corrected variance estimates as the fraction of re-optimizers in the TM approaches the EM case of one (so that the EM is closer to being correctly specified). The first column of data in the table again reports the true variance estimates, while the second reproduces the adjusted levels from Table 2.2. The remaining columns report the results for three different values of the fraction of re-optimizers—0.95, 0.99 and 0.999—which show that the estimates steadily improve with the quality of the model. Thus, despite the stark fact that the true variances of the EM shocks remain zero even as the EM and TM converge, the observed degree of misspecification nonetheless converges to zero, and the adjusted shock values converge to the values from the TM. Put another way, one need only be concerned about this sort of “over-adjustment” if the EM is a poor approximation to the TM, which is precisely the case in which there should be a large adjustment so as to warn the econometrician that misspecification is likely to be a problem.

<table>
<thead>
<tr>
<th>Shock</th>
<th>True</th>
<th>Baseline</th>
<th>Fraction of re-optimizers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>.95</td>
</tr>
<tr>
<td>Spending, t</td>
<td>0.27</td>
<td>0.01</td>
<td>0.19</td>
</tr>
<tr>
<td>TFP, t</td>
<td>0.21</td>
<td>0.00</td>
<td>0.03</td>
</tr>
<tr>
<td>IST, t</td>
<td>0.20</td>
<td>0.01</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Notes: “True” column shows variance of shock in TM. “Baseline” column reproduces corrected estimates under baseline parameterization of TM from Table 2.2. Remaining columns show corrected estimates for several different fractions of re-optimizers in TM.

2.6 Application: Investment shocks

In this section, I apply the methodology introduced in this paper to the recent model of Justiniano et al. (2010) (henceforth JPT), itself a variant of the Smets and Wouters (2007) model discussed in Example 3 in Section (2.5).34

Following the Bayesian estimation procedure used by JPT,35 I re-estimate their model using quarterly U.S. data on output, consumption, investment, hours, wages, inflation and the nominal interest rate over the period 1954QIV-2004QIV (T = 201).36 Table 2.5 reports resulting variance decompositions for log-hours using the median parameter values from the posterior parameter distribution.37

---

34 While there are a number of differences between JPT and Smets and Wouters (2007), the main source of the divergence in their results is in the data used to estimate the models. Specifically, Smets and Wouters (2007) include consumer durables in their measure of consumption and exclude the change in inventories from their measure of investment. JPT, on the other hand, include consumer durables and the change in inventories in their measure of investment, with consumption including only non-durables and services. This produces a more volatile investment series and a less volatile consumption series, which largely drives their results.

35 See An and Schorfheide (2007) for a review of Bayesian estimation in the context of DSGE models.

36 See Appendix A.3 for details about data sources. Bayesian estimation of the DSGE model was done using Dynare (see Adjemian et al. (2011)).

37 The posterior statistics I obtained for the 35 estimated parameters were very close to those reported by JPT and are
Column (1) shows a standard decomposition of the unconditional variance. As noted by JPT, this decomposition indicates that the wage mark-up shock (typically interpreted as a labor supply shock) accounts for the majority (52 percent) of the unconditional variance of hours in the model. However, the estimated wage mark-up process is highly persistent (autoregressive parameter of 0.98), suggesting that the bulk of the variation induced by this shock may be of a long-run nature, and thus potentially less important for business cycle variation.

Table 2.5: Naive variance decomposition for hours

<table>
<thead>
<tr>
<th>Shock</th>
<th>(1) Unconditional</th>
<th>(2) BCF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monetary policy</td>
<td>0.03</td>
<td>0.06</td>
</tr>
<tr>
<td>Neutral technology</td>
<td>0.04</td>
<td>0.11</td>
</tr>
<tr>
<td>Government spending</td>
<td>0.08</td>
<td>0.02</td>
</tr>
<tr>
<td>Investment</td>
<td>0.25</td>
<td>0.61</td>
</tr>
<tr>
<td>Price mark-up</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>Wage mark-up</td>
<td>0.52</td>
<td>0.05</td>
</tr>
<tr>
<td>Patience</td>
<td>0.03</td>
<td>0.08</td>
</tr>
</tbody>
</table>

Notes: Table entries show naive variance decompositions for hours, obtained from JPT’s model using the median of the posterior parameter distribution. Column (1) decomposes the unconditional variance. Column (2) decomposes the BCF variance, defined as the variance associated with periodic fluctuations between 6 and 32 quarters. Columns may not add up to 1 due to rounding.

To address this issue, Column (2) of Table 2.5 reports the decomposition of the business-cycle-frequency (BCF) variance of log-hours in the model. Echoing the results found by JPT, the results indicate that the investment shock, rather than the wage mark-up shock, accounts for the majority (61 percent) of BCF fluctuations in hours. While not shown in the table, the investment shock was also found to account for large proportions of the BCF variance of output growth (56 percent) and investment growth (88 percent). On the basis of this evidence, one may be tempted to conclude, as JPT do, that “investment shocks are the leading source of business cycles.” (p. 137)

As JPT also note, however, the investment shock is found to be a negligible determinant of BCF fluctuations in consumption growth, accounting for a mere 6 percent of its variance, while the otherwise-irrelevant household patience shock accounts for 61 percent. Figure 2.2 plots the correlation between consumption growth and four leads and lags of investment growth in the data (solid line) and in the model (dashed line). The data indicates a positive and significant correlation between consumption growth and investment growth within a one-quarter lead/lag. In contrast, in the EM there is a small negative correlation between the two. This EM correlation is small because consumption and

---

38 BCF variances are obtained by integrating the spectrum of the model over the relevant frequencies (defined here, as in JPT, to be frequencies associated with periods between 6 and 32 quarters). This process effectively removes the variance associated with high- and low-frequency fluctuations, leaving only the medium-frequency fluctuations normally associated with business cycles. See Appendix A.4 for further details.
investment are driven primarily by two distinct (and orthogonal) shocks: the patience and investment shocks, respectively. Meanwhile, the correlation is negative primarily because a positive investment shock increases the return on investment with no change to current productive capacity, causing the household to substitute away from consumption and toward investment.\(^{39}\)

**Figure 2.2:** Correlation between \(\%\Delta C_t\) and \(\%\Delta I_{t+k}\)

To gain further insight, I proceed with analysis of the smoothed shocks. It can be verified in the case of JPT’s EM that the matrix \(C\) (as defined in Section 2.4.3) contains several eigenvalues equal to one, so that Assumption 2.4 fails to hold.\(^{40}\) To check that the approximation \(\hat{\varepsilon}_t^* \approx \varepsilon_t^*\) holds (see Section 2.4.4) in spite of the finite sample size and the violation of Assumption 2.4, I simulated data from the EM,\(^{41}\) then obtained the smoothed shocks from this simulated data. From this sequence of smoothed shocks, I then computed the mean squared recovery error (MSRE) for each shock in the EM (i.e., \(Var(\delta_{t,t}^*)\) in the notation of Section 2.4.4) at each date \(t\), then scaled the result by the (known) variance of that shock, \(\hat{\sigma}_l^2\). Figure 2.3 plots the maximum of this statistic across the seven different

\(^{39}\) The failure of many RBC-type models to generate positive comovement between consumption, investment and hours in response to shocks that affect the expected return to investment without affecting current production technology has been known since at least Barro and King (1984); see also Beaudry and Portier (2007).

\(^{40}\) Because JPT’s model features a stochastic trend in productivity, the data contains first-differences of several non-stationary variables, while the model state vector contains the corresponding variables in (stationarized) levels. Since a variable cannot be exactly recovered from even an infinite history of changes in that variable, and since the condition that \(C\) contain no eigenvalues of modulus one is precisely that needed to guarantee exact recovery of the state vector, it should be unsurprising that \(C\) contains eigenvalues equal to one for JPT’s model.

\(^{41}\) Simulated values throughout this section are based on \(N\) simulated data sets, each of \(T\) periods in length, where \(T = 201\) is the length of the actual data set.
shocks for each the first ten periods. As the figure shows, the smoothed shocks very quickly attain a high degree of accuracy, with the maximum MSRE equal to less than 2 percent of the variance of the shock by \( t = 4 \), and less than 1 percent by \( t = 5 \). Though not shown in the figure, accuracy continues to increase, with the maximum MSRE falling below 0.5 percent by \( t = 17 \) and remaining below this threshold for the remainder of the sample. Thus, beyond the first few periods, the approximation \( \hat{\varepsilon}_t \approx \varepsilon_t \) appears to be sufficient for the principal results of Section 2.4.3 to hold. To ensure that none of the results are driven by the relatively inaccurate recovery of the early shock values, I drop the first four periods of smoothed shocks in all subsequent computation.

**Figure 2.3:** Maximum scaled MSRE for smoothed shocks

\[
\text{Notes: Figure displays } \max_t \frac{\text{Var}(\delta_{t,t})}{\sigma_t^*}, \text{ computed by simulating 10,000 data sets. Data point for } t = 1 \text{ is off the scale in Figure 2.3, with a maximum MSRE equal to 80 percent of the variance of the shock.}
\]

Next, returning to the sequence of smoothed shocks obtained from the actual data, Figure 2.4 plots sample correlations between the smoothed investment shock at date \( t \) on the one hand, and each of the smoothed shocks at a number of leads and lags on the other. Black dots represent sample correlations, while the dashed lines show pointwise simulated 95 percent confidence intervals (based on 100,000 draws) for these correlations under the null hypothesis that the EM is correctly specified. Circles around dots highlight correlations that fall outside of the confidence intervals.

Of the 71 points shown in Figure 2.4, 15 (21 percent) are outside of the 95 percent confidence intervals.

---

\(^{42}\) The data point for \( t = 1 \) is off the scale in Figure 2.3, with a maximum MSRE equal to 80 percent of the variance of the shock.
Figure 2.4: Correlation coefficients between smoothed shocks

Notes: Dots indicate the correlation coefficient between the smoothed investment shock and the specified lead or lag of another smoothed shock. Confidence intervals were computed from 100,000 simulated data sets under the null hypothesis that the EM is correctly specified.

bands. If the EM were correctly specified, this would be an extreme result. Figure 2.5 presents a simulated probability mass function for the number of correlations between the investment shock and other shocks within 5 leads or lags that fall outside of the 95 percent confidence bands for the case when the EM is correctly specified. The median and mode of this distribution are both 3 and the mean 3.5. The maximum number of correlations outside of the confidence bands in 100,000 simulations was 14, a figure attained by only 2 (0.002 percent) of those simulations. 97 percent of the time, the number of such correlations was less than or equal to 7. In this context, 15 such correlations is clearly well outside what could be considered reasonable if the EM were correctly specified. While I focus on the investment shock in Figure 2.4 because of the important role ascribed to it by JPT, the extreme
number and degree of correlations are not limited exclusively to this shock. For example, the neutral technology and patience shocks each had 16 of 71 points outside of the 95 percent confidence bands, while the monetary policy shock had 15.

**Figure 2.5:** Proportion of correlations outside 95% CI

![Histogram showing proportion of correlations outside 95% CI](image)

*Notes:* Simulated probability mass function (based on 100,000 simulations of same number of periods as actual data set) for number of correlations between investment shock and other shocks (within 5 leads or lags) that fall outside of the simulated 95 percent confidence interval under null hypothesis that EM is correctly specified.

Note also that not only was the frequency of significant correlations in Figure 2.4 high, but the *degree* to which the correlations exceeded the bounds was in some cases also extreme. Most notably, the monetary policy shock at a two-quarter lead, the neutral technology shock contemporaneously and at a one-quarter lead and lag, and the household patience shock at a one-quarter lead all exhibit correlations with the investment shock that exceeded 0.27 in absolute value, well outside the 95 percent confidence bounds, which were all less than 0.15 in absolute value.

The sample correlation evident in Figure 2.4 between the smoothed investment and patience shocks is instructive. In particular, it was noted above that, in the estimated EM, investment was found to be driven by the former shock while consumption was driven by the latter, despite the fact that the EM was estimated using data that suggests that investment and consumption are correlated. This apparent puzzle is easily resolved if the two smoothed shocks themselves exhibit sample correlation, a property that Figure 2.4 clearly establishes. Intuitively, the estimation algorithm is “mixing” the investment and patience shocks together in a systematic way in order to reproduce the autocovariance patterns in the data. On the other hand, because in the model the shocks are assumed to be
orthogonal, all moments derived from the EM—including those implicit in the computation of variance decompositions—are obtained assuming no such mixing is taking place, leading to the apparent conflict between model and data.

Next, as argued in Section 2.3, the correlations between the smoothed shocks are indicative of model misspecification and, as a result, the naive estimates of the variances of the shocks will be overstated. I turn now to obtaining bias-corrected estimates. In order to do so, a central issue that must be addressed is how to obtain estimates of the regression coefficients (i.e., the $\Theta$’s in equation (2.22)).

Given the relatively small sample size, using $q$ leads and lags of all the smoothed shocks (plus the other contemporaneous shocks) as regressors very quickly uses up degrees of freedom. However, as Figure 2.4 suggests for the case of the investment shock, only a small subset of these potential regressors will carry significant explanatory power, so that over-fitting is a serious concern. To address this issue, I use a simple selection rule, choosing only those regressors that meet the following criteria: (1) are within 5 leads or lags of the dependent variable, and (2) exhibit correlation coefficients with the dependent variable that are outside of a simulated 100 $(1-\alpha)$% confidence interval. This rule has the advantage of being both simple to apply and relatively flexible, in that the consequences of different values of $\alpha$ can be explored.

For a given value of $\alpha$ and this selection rule, I next compute the corrected estimate of the variance of shock $l$, $\hat{\sigma}^2_{l,\alpha}$, as the unbiased estimator of the variance of the regression residuals from equation (2.22). That is, $\hat{\sigma}^2_{l,\alpha} = (T_{l,\alpha} - k_{l,\alpha})^{-1} \sum \hat{\xi}_{l,t}^2$, where $\hat{\xi}_{l,t}$ is the fitted residual, $T_{l,\alpha}$ is the number of observation periods, and $k_{l,\alpha}$ is the number of regression parameters. Table 2.6 presents results for several different values of $\alpha$. For each value of $\alpha$, the table reports the corrected estimate, $\hat{\sigma}^2_{l,\alpha}$, as a fraction of the naive estimate, $\hat{\sigma}^2_l$. The smaller this fraction is, the more of the variance of the smoothed shock we may attribute to misspecification. For comparison purposes, the bottom row of the table reports simulated 90 percent lower bounds for the corresponding statistic obtained assuming the model is the true DGP.

Several things emerge from the results in Table 2.6. First, with the exception of the wage mark-up shock, the corrected variance estimates are well below the simulated lower bounds. Given the theoretical considerations of Section 2.3, this suggests that misspecification may be an important factor for this model. Second, for the most conservative case ($\alpha = 0.001$), the point estimates suggest that nearly one-third of the variances of the smoothed neutral technology and investment shocks are attributable to misspecification, while over 90 percent of the time this statistic would be zero if the model were the true DGP. Similarly, over one-fifth of the variance of the smoothed patience shock is attributable to misspecification for this level of $\alpha$. Third, for $\alpha = 0.005$, except again for the wage mark-up shock, the point estimates indicate that over one-fifth of the variance of every smoothed shock is attributable to misspecification, and over one-quarter when $\alpha = 0.02$. Again, these figures are

---

43 When all potential regressors within $q$ leads and lags are used, the number of estimated parameters will be $6 + 14q$, while the number of available observations is $197 - 2q$. The number of degrees of freedom is thus $191 - 16q$, which declines very rapidly with $q$. 
Table 2.6: Corrected estimates (fraction of naive estimate)

<table>
<thead>
<tr>
<th>Shock</th>
<th>$\alpha = 0.001$</th>
<th>$\alpha = 0.005$</th>
<th>$\alpha = 0.02$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monetary policy</td>
<td>0.86</td>
<td>0.70</td>
<td>0.69</td>
</tr>
<tr>
<td>Neutral technology</td>
<td>0.67</td>
<td>0.67</td>
<td>0.65</td>
</tr>
<tr>
<td>Government spending</td>
<td>0.86</td>
<td>0.76</td>
<td>0.75</td>
</tr>
<tr>
<td>Investment</td>
<td>0.68</td>
<td>0.68</td>
<td>0.67</td>
</tr>
<tr>
<td>Price mark-up</td>
<td>0.87</td>
<td>0.75</td>
<td>0.69</td>
</tr>
<tr>
<td>Wage mark-up</td>
<td>1.00</td>
<td>0.96</td>
<td>0.92</td>
</tr>
<tr>
<td>Patience</td>
<td>0.79</td>
<td>0.77</td>
<td>0.67</td>
</tr>
</tbody>
</table>

90% simulated lower bound: 1.00 0.94 0.90

Notes: Table entries show $\bar{\sigma}_{l,\alpha}^2 / \hat{\sigma}_l^2$, where $\bar{\sigma}_{l,\alpha}^2$ is corrected estimated of variance of shock $l$ using selection rule for $\alpha$ (see text) and $\hat{\sigma}_l^2$ is naive estimate. $\bar{\sigma}_{l,\alpha}^2$ is unbiased estimator of variance of regression residuals from equation (2.22). That is, $\bar{\sigma}_{l,\alpha}^2 = (T_{l,\alpha} - k_{l,\alpha})^{-1} \sum \hat{\xi}_t^2$, where $\hat{\xi}_t$ is fitted residual, $T_{l,\alpha}$ is number of observation periods, and $k_{l,\alpha}$ is number of regression parameters. Bottom row reports simulated 90 percent lower bounds for corresponding statistic when EM is correctly specified. Simulations are based on 10,000 draws of the same size as data set.

suggestive of an important degree of misspecification. It should also be noted again that the corrected variance estimates are upper bounds for the true value. Thus, the actual degree of misspecification may indeed be significantly higher than these figures suggest.

To check the implications for the hours variance decomposition, Table 2.7 reports the proportion of the variance of hours attributable to each of the shocks assuming the shock variances are the corrected estimates computed above.\(^{44}\) Again, 90 percent simulated lower bounds for the case where the EM is correctly specified are also reported (in parentheses). Column (1) reproduces the baseline naive BCF variance decomposition from Table 2.5, while columns (2)-(4) report corrected decompositions of the BCF variance of hours for different values of $\alpha$. The Table shows substantial declines in the estimated importance of the investment shock for the BCF variance of hours, with estimates decreasing by 19-20 percentage points relative to the naive case.

### 2.7 Conclusion

This paper makes several contributions to the literature. First, a novel framework is developed that can be used to analyze the implications of misspecification on estimates of model shock variances. Using this framework, I show that if a DSGE model is correctly specified, then under basic stability conditions the time-series process for the smoothed shocks should be a vector white noise with diagonal covariance matrix. Thus, if a realized process for the smoothed shocks does not possess this property, then the model must be misspecified.

\(^{44}\) Note that only the numerator of this fraction is different from the baseline case; the denominator remains unchanged.
### Table 2.7: Corrected BCF variance decomposition for hours

<table>
<thead>
<tr>
<th>Shock</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Naive</td>
<td>Corrected</td>
<td>Corrected</td>
<td>Corrected</td>
</tr>
<tr>
<td>α:</td>
<td></td>
<td>0.001</td>
<td>0.005</td>
<td>0.02</td>
</tr>
<tr>
<td>Monetary policy</td>
<td>0.06</td>
<td>0.05</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td></td>
<td>(0.06)</td>
<td>(0.06)</td>
<td>(0.06)</td>
<td>(0.06)</td>
</tr>
<tr>
<td>Neutral technology</td>
<td>0.11</td>
<td>0.07</td>
<td>0.07</td>
<td>0.07</td>
</tr>
<tr>
<td></td>
<td>(0.11)</td>
<td>(0.10)</td>
<td>(0.09)</td>
<td>(0.09)</td>
</tr>
<tr>
<td>Government spending</td>
<td>0.02</td>
<td>0.02</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.02)</td>
<td>(0.02)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>Investment</td>
<td>0.61</td>
<td>0.42</td>
<td>0.42</td>
<td>0.41</td>
</tr>
<tr>
<td></td>
<td>(0.61)</td>
<td>(0.58)</td>
<td>(0.55)</td>
<td>(0.55)</td>
</tr>
<tr>
<td>Price mark-up</td>
<td>0.06</td>
<td>0.05</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td></td>
<td>(0.06)</td>
<td>(0.06)</td>
<td>(0.05)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>Wage mark-up</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>(0.05)</td>
<td>(0.05)</td>
<td>(0.05)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>Patience</td>
<td>0.08</td>
<td>0.07</td>
<td>0.07</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td>(0.09)</td>
<td>(0.08)</td>
<td>(0.08)</td>
<td>(0.08)</td>
</tr>
</tbody>
</table>

*Notes: Columns (2)-(4) show corrected BCF variance decompositions for hours. Naive decomposition is reproduced in column (1). Figures in parentheses are 90 percent simulated lower bounds for case when EM is correctly specified.*

Next, I showed how, using this framework, one may orthogonally decompose a smoothed shock into two components: its true value, and an additional component related entirely to misspecification. Since the true values of any two shocks are independent by construction, any non-zero covariance between two distinct smoothed shocks must then be entirely attributable to misspecification. Further, if the smoothed shocks do exhibit non-zero covariance, using the sample variance of a smoothed shock as an estimator of the variance of the true shock will lead to estimates that are biased upward. To correct for this potential source of bias, I propose a fairly simple methodology that involves extracting the component of a given smoothed shock that is unpredictable using other shocks.

I apply this framework and methodology to a recent paper by Justiniano et al. (2010), and estimate that at least one-third of the variance of the investment shock—the leading driver of business cycle fluctuations in their model—can be attributed to misspecification, and as a result, the estimated importance of the investment shock in generating business cycle variation in hours declines by around 20 percentage points.
Chapter 3

Reconciling Hayek’s and Keynes’ Views of Recessions

3.1 Introduction

There remains considerable debate regarding the causes and consequences of recessions. Two views that are often presented as opposing, and which created controversy in the recent recession and its aftermath, are those associated with the ideas of Hayek and Keynes. The Hayekian perspective is generally associated with viewing recessions as a necessary evil. According to this view, recessions mainly reflect periods of liquidation resulting from past over-accumulation of capital goods. A situation where the economy needs to liquidate such an excess can quite naturally give rise to a recession, but government spending aimed at stimulating activity, it is argued, is not warranted since it would mainly delay the needed adjustment process and thereby postpone the recovery. In contrast, the Keynesian view suggests that recessions reflect periods of deficient aggregate demand where the economy is not effectively exploiting the gains from trade between individuals. According to this view, policy interventions aimed at increasing investment and consumption are generally desirable, as they favor the resumption of mutually beneficial trade between individuals.

In this paper we reexamine the liquidationist perspective of recessions in an environment with decentralized markets, flexible prices and search frictions. In particular, we examine how the economy adjusts when it inherits from the past an excessive amount of capital goods, which could be in the form of houses, durable goods or productive capital. Our goal is not to focus on why the economy may have over-accumulated in the past, but to ask how it reacts to such an over-accumulation once it is realized.

1 In response to the large recession in the US and abroad in 2008-2009, a high-profile debate around these two views was organized by Reuters. See http://www.reuters.com/subjects/keynes-hayek. See also Wapshott (2012) for a popular account of the Hayek-Keynes controversy.

2 See Caballero and Hammour (2004) for an alternative view on the inefficiency of liquidations, based on the reduction of cumulative reallocation and inefficient restructuring in recessions.

3 There are several reasons why an economy may over-accumulate capital. For example, agents may have had overly
As suggested by Hayek, such a situation can readily lead to a recession as less economic activity is generally warranted when agents want to deplete past over-accumulation. However, because of the endogenous emergence of unemployment risk in our set-up, the size and duration of the recession implied by the need for liquidation is not socially optimal. In effect, the reduced gains from trade induced by the need for liquidation creates a multiplier process that leads to an excessive reduction in activity. Although prices are free to adjust, the liquidation creates a period of deficient aggregate demand where economic activity is too low because people spend too cautiously due to increased unemployment risk. In this sense, we argue that liquidation and deficient aggregate demand should not be viewed as alternative theories of recessions but instead should be seen as complements, where past over-accumulation may be a key driver of periods of deficient aggregate demand. This perspective also makes salient the trade-offs faced by policy. In particular, a policy-maker in our environment faces an unpleasant trade-off between the prescriptions emphasized by Keynes and Hayek. On the one hand, a policy-maker would want to stimulate economic activity during a liquidation-induced recession because precautionary savings is excessively high. On the other hand, the policy-maker also needs to recognize that intervention will likely postpone recovery, since it slows down the needed depletion of excess capital. The model offers a simple framework where both of these forces are present and can be compared.

On a more general note, one of the contributions of this paper is to show why an economy can function quite efficiently in growth periods when it is far from its steady state, while simultaneously functioning particularly inefficiently when it is going through a liquidation phase near its steady state. When the economy is far from its steady-state level of capital, demand for capital is very strong and unemployment risk is therefore minimal. In contrast, when there is excessive capital, we show that reduced labor demand shows up at least in part as increased unemployment even if workers and firms bargain pair-wise efficiently on wages and hours-worked. The increased unemployment risk then causes households to increase precautionary savings, which in turn amplifies the initial fall in output and employment. The result is an over-reaction to the initial impetus induced by a need to liquidate capital. As a presentation device, we show how this process can be represented on a diagram somewhat similar to a Keynesian cross, but where the micro-foundation and many comparative statics differ substantially from the sticky-price interpretation commonly used to discuss multipliers. Moreover, by clarifying why this process does not depend on sticky prices, our analysis suggest that monetary policy may be of limited help in addressing the difficulties associated with a period of liquidation.

One potential criticism of a pure liquidationist view of recessions is that, if markets functioned efficiently, such periods should not be socially painful. In particular, if economic agents interact in optimistic expectations about future expected economic growth that did not materialize, as in Beaudry and Portier (2004), or it could have been the case that credit supply was unduly subsidized either through explicit policy, as argued in Mian and Sufi (2010) and Mian et al. (2010), or as a by-product of monetary policy, as studied by Bordo and Landon-Lane (2013).

It is now common in the macroeconomic literature to summarize the functioning of a model by indicating where and how it creates distortions or wedges, as exemplified by Chari et al. (2007). Accordingly, one way to view the working of our model economy is as generating an endogenous labor market wedge driven by unemployment risk, where the size of the wedge reacts to the extent to which inherited capital is above or below the steady state.
perfect markets and realize they have over-accumulated in the past, this should lead them to enjoy a
type of holiday paid for by their past excessive work. Looking backwards in such a situation, agents
may resent the whole episode, but looking forward after a period of over-accumulation, they should
nonetheless feel content to enjoy the proceeds of the past excessive work, even if it is associated with a
recession. In contrast, in our environment we will show that liquidation periods are generally socially
painful because of the multiplier process induced by precautionary savings and unemployment risk.
In effect, we will show that everyone in our model economy can be worse off when they inherit too
many capital goods from the past. This type of effect, whereby abundance creates scarcity, may appear
quite counter-intuitive at first pass. To make as clear as possible the mechanism that can cause welfare
to be reduced by such abundance, much of our analysis will focus on the case where the inherited
capital takes the form of a good that directly contributes to utility, such as houses or durable goods.
In this situation we will show why inheriting more houses or durables can make everyone worse off.
However, as we shall show, this result is a local result that is most likely to be present around an
economy’s steady state. In contrast, if we were to destroy all capital goods in our model economy,
this would always reduce welfare, as the direct effects on utility would out-weight the inefficiencies
induced by unemployment risk. Accordingly, our model has the characteristic that behavior can be
quite different when it inherits a large or small amount of capital from the past.

The structure of our model builds on the literature related to search models of decentralized trad-
ing. In particular, we share with Lucas (1990) and Shi (1998) a model in which households are
composed of agents that act in different markets without full coordination. Moreover, as in Lagos and
Wright (2005) and Rocheteau and Wright (2005), we exploit alternating decentralized and centralized
markets to allow for a simple characterization of the equilibrium. However, unlike those papers, we do
not have money in our setup. The paper also shares key features with the long tradition of macro mod-
els emphasizing strategic complementarities, aggregate demand externalities and multipliers, such as
Diamond (1982) and Cooper and John (1988), but we do not emphasize multiple equilibrium. Instead
we focus on situations where the equilibrium remains unique, which allows standard comparative
statics exercises to be conducted without needing to worry about equilibrium-selection issues. The
multiplier process derived in the paper therefore shares similarities with that found in the recent litera-
ture with strategic complementarities such as Angeletos and La’O (2013), in the sense that it amplifies
demand shocks. However, the underlying mechanism in this paper is very different, operating through
unemployment risk rather than through direct demand complementarities as in Angeletos and La’O
(2013).

Unemployment risk and its effects on consumption decisions is at the core of our model. The em-
pirical relevance of precautionary saving related to unemployment risk has been documented by many,
starting with Carroll (1992). For example, Carroll and Dunn (1997) have shown that expectations of
unemployment are robustly and negatively correlated with every measure of consumer expenditure
(non-durable goods, durable goods and home sales). Carroll et al. (2012) confirm this finding and
show why business cycle fluctuations may be driven to a large extent by changes in unemployment
uncertainty. Alan et al. (2012) use U.K. micro data to show that increases in saving rates in recessions appear largely driven by uncertainty related to unemployment.\(^5\) There are also recent theoretical papers that emphasized how unemployment risk and precautionary savings can amplify shocks and cause business cycle fluctuations. These papers are the closest to our work. In particular, our model structure is closely related to that presented in Guerrieri and Lorenzoni (2009). However, their model emphasizes why the economy may exhibit excessive responses to productivity shocks, while our framework offers a mechanism that amplifies demand-type shocks. Our paper also shares many features with Heathcote and Perri (2012), who develop a model in which unemployment risk and wealth impact consumption decisions and precautionary savings. Wealth matters in their setup because of financial frictions that make credit more expensive for wealth-poor agents. They obtain a strong form of demand externality that gives rise to multiple equilibria and, accordingly, they emphasize self-fulfilling cycles as the important source of fluctuations.\(^6\) Finally, the work by Ravn and Sterk (2012) emphasizes as we do how unemployment risk and precautionary savings can amplify demand shocks, but their mechanism differs substantially from ours since it relies on sticky nominal prices.

While the main mechanism in our model has many precursors in the literature, we believe that our setup illustrates most clearly (i) how unemployment risk gives rise to a multiplier process for demand shocks even in the absence of price stickiness or increasing returns, (ii) how this multiplier process can be ignited by periods of liquidation, and (iii) how fiscal policy can and cannot be used to counter the process.

The remaining sections of the paper are structured as follows. In Section 3.2, we present a static model where agents inherit from the past different levels of capital goods, and we describe how and why high values of inherited capital can lead to poor economic outcomes. The static setup allows for a clear exposition of the nature of the demand externality that arises in our setting with decentralized trade. We focus on the case where the inherited capital is in the form of a good which directly increases utility so as to make clear how more goods can reduce welfare. In Section 3.3, we discuss a set of extensions, including a discussion of the case where the inherited capital takes the form of a productive good. In Section 3.4, we extend the model to an infinite-period dynamic setting. We take particular care in contrasting the behavior of the economy when it is close to and far from its steady state. Finally, in Section 3.5, we discuss the trade-offs faced by a policy-maker when inheriting an excessive amount of capital from the past, while Section 3.6 concludes.

\(^5\) Using these empirical insights, Challe and Ragot (2013) have recently proposed a tractable quantitative model in which uninsurable unemployment risk is the source of wealth heterogeneity.

\(^6\) The existence of aggregate demand externalities and self-fulfilling expectations is also present in the work of Farmer (2010) and in the work of Chamley (2014). In a model with search in both labor and goods markets, Kaplan and Menzio (2013) also obtain multiple equilibria, as employed workers have more income to spend and less time to shop for low prices. As already underlined, and contrarily to those studies, our analysis is restricted to configurations in which the equilibrium is unique.
3.2 Static model

In this section, we present a very stripped-down static model in order to illustrate why an economy may function particularly inefficiently when it inherits a large stock of capital from the past. In particular, we will want to make clear why agents in an economy can be worse off when inherited capital goods are too high. For the mechanism to be as transparent as possible, we focus mainly on the case where the inherited capital produces services which directly enter agents’ utility functions. Accordingly, this type of capital can be considered as representing houses or other durable consumer goods. In a later section, we will discuss how the analysis carries over to the case of productive capital.

In our model, trades are decentralized, and there are two imperfections which cause unemployment risk to emerge and generate precautionary savings behavior. First, there will be a matching friction in the spirit of Diamond-Mortensen-Pissarides, which will create the possibility that a household may not find employment when looking for a job. Second, there will be adverse selection in the insurance market that will limit the pooling of this risk. Since the adverse selection problem can be analyzed separately, we will begin the presentation by simply assuming that unemployment insurance is not available. Later we will introduce the adverse selection problem which rationalizes this missing market, and show that all main results are maintained. The key exogenous variable in the static model will be a stock of consumer durables that households inherit from the past. Our goal is to show why and when high values of this stock can cause the economy to function inefficiently and possibly even cause a decrease in welfare. We will also explore the role of government spending in affecting economic activity in our setup.

3.2.1 Setup

Consider an environment populated by a mass $L$ of households indexed by $j$. In this economy there are two sub-periods. In the first sub-period, households buy good 1, which we will call clothes, and try to find employment in the clothing sector. We refer to this good as clothes since in the dynamic version of the model it will represent a partially durable good. The good produced in the second sub-period, good 2, will be referred to as household services since it will have no durability. As there is no money in this economy, when the household buys clothes its bank account is debited, and when (and if) it receives employment income its bank account is credited. Then, in the second sub-period, households balance their books by repaying any outstanding debts or receiving a payment for any surplus. These payments are made in terms of good 2, which is also the numeraire in this economy.\footnote{We remain agnostic about the precise details of how good 2 is produced for the time being. One possible interpretation is discussed in the following sub-section.}

Preferences for the first sub-period are represented by

$$U(c_j) - \nu(\ell_j)$$

where $c$ represents consumption of clothes and $\ell$ is the labor supplied by households in the production
of clothes. The function \( U(\cdot) \) is assumed to be increasing in \( c \) and strictly concave with \( \lim_{c \to \infty} U' \leq 0 \) and \( U''' > 0 \). The dis-utility of work function \( \nu(\cdot) \) is assumed to be increasing and convex in \( \ell \), with \( \nu(0) = 0 \). The agents are initially endowed with \( X_j \) units of clothes, which they can either consume or trade. We assume symmetric endowments, so that \( X_j = X \forall j \).

Trade in clothing will be subject to a coordination problem because of frictions in the labor market. At the beginning of the first sub-period, the household splits up responsibilities between two members. The first member, called the buyer, goes to the clothes market to make purchases. The second member searches for employment opportunities in the labor market. The market for clothes functions in a Walrasian fashion, with both buyers and firms that sell clothes taking prices as given. The market for labor in this first sub-period is subject to a matching friction, with sellers of labor searching for employers and employers searching for labor. The important information assumption is that buyers do not know, when choosing their consumption of clothes, whether the worker member of the household has secured a match. This assumption implies that buyers will worry about unemployment risk when making purchases of clothes.

There is a large set of potential clothes firms in the economy who can decide to search for workers in view of supplying clothes to the market. Each firm can hire one worker and has access to a decreasing-returns-to-scale production function \( \theta F(\ell) \), where \( \ell \) is the number of hours worked for the firm and \( \theta > 0 \) is a technology shift factor. Production also requires a fixed cost \( \Phi \), in terms of the output good, so that the net production of a firm hiring \( \ell \) hours of labor is \( \theta [F(\ell) - \Phi] \). For now, we will normalize \( \theta \) to 1, and will reintroduce \( \theta \) in its general form when we want to talk about the effects of technological change and balanced growth. We will also assume throughout that \( \Omega(\ell) \equiv F'(\ell)\ell \) is increasing in \( \ell \).

Moreover, we will assume that \( \Phi \) is sufficiently small such that there exists an \( \ell^* > 0 \) satisfying \( F(\ell^*) - F'(\ell^*)\ell^* = \Phi \). These restrictions on the production technology are always satisfied if, for example, \( F(\ell) = \ell^\alpha \), with \( 0 < \alpha < 1 \).

Firms search for workers and, upon finding a worker, they jointly decide on the number of hours worked and on the wage to be paid. The fixed cost \( \Phi \) is paid before firms can look for workers. Upon a match, the determination of the wage and hours-worked within a firm is done efficiently though a competitive bargaining process. In effect, upon a match, one can view a Walrasian auctioneer as calling out a wage \( w \) that equilibrates the demand for and supply of labor among the two parties in the match. Assuming such a process for wage and employment determination has the feature of limiting within-pair distortions that could muddle the understanding of the main mechanisms of the model. In Appendix B.2 we show that the main results of the paper are robust to alternative bargaining protocols.

Note that we have deliberately chosen a random-matching – rather than a directed-search – framework

---

\*In what follows, we will drop the \( j \) index except where doing so may cause confusion.

\* Because we assume free-entry for clothes firms, the quantity \( \theta \Omega(\ell) \) will equal net output of clothes (after subtracting firms’ fixed costs) by a single employed worker. The assumption that this quantity is increasing in \( \ell \) is satisfied, for example, if \( F \) is a CES combination of labor and some other input in fixed supply, with an elasticity of substitution between these inputs of at least 1, which nests the case where \( F \) is Cobb-Douglas.
as we want to illustrate how inefficiencies in the labor market can interact with the liquidation process to create periods of deficient aggregate demand. Given the wage, the demand for labor from the firm is described by the marginal productivity condition

\[ pF'(\ell) = w \]

where \( p \) is the relative price of clothes in terms of the non-durable good produced in the second sub-period.\(^{10}\) The supply of labor is chosen optimally by the worker in a manner to be derived shortly.

Letting \( N \) represent the number of firms who decide to search for workers, the number of matches is then given by the constant-returns-to-scale matching function \( M(N, L) \), with \( M(N, L) \leq \min\{N, L\} \). The equilibrium condition for the clothes market is given by

\[ L \cdot (c - X) = M(N, L)F(\ell) - N\Phi \]

where the left-hand side is total purchases of new clothes and the right-hand side is the total available supply after subtracting search costs.

Firms will enter the market up to the point where expected profits are zero. The zero-profit condition can be written as \(^{11}\)

\[ \frac{M}{N} [pF(\ell) - w\ell] = \frac{M}{N} [pF(\ell) - pF'(\ell)\ell] = p\Phi \]

At the end of the first sub-period, household \( j \)'s net asset position \( a_j \), expressed in units of good 2, is given by \( w\ell_j - p(c_j - X) \).\(^ {12}\) We model the second sub-period so that it is costly to arrive in that sub-period with debt. For now, we can simply denote the value of entering the second sub-period with assets \( a_j \) by \( V(a_j) \), where we assume that \( V(\cdot) \) is increasing, with \( V'(a_1) > V'(a_2) \) whenever \( a_1 < 0 < a_2 \); that is, we are assuming that the marginal value of a unit of assets is greater if one is in debt than if one is in a creditor position. In the following sub-section we specify preferences and a market structure for the second sub-period that rationalizes this \( V(\cdot) \) function.

Taking the function \( V(a) \) as given, we can specify the household’s consumption decision as well as his labor-supply decision conditional on a match. The buyer’s problem in household \( j \) is given by

\[ \max_{c_j} U(c_j) + \mu V (w\ell_j - p (c_j - X)) + (1 - \mu) V (-p (c_j - X)) \]

where \( \mu \) is the probability that a worker finds a job and is given by \( \mu \equiv M(N, L)/L \). From this expression, we can see that the consumption decision is made in the presence of unemployment risk.

\(^{10}\) As will become clear, \( p \) can be given an interpretation as an interest rate.

\(^{11}\) We assume that searching firms pool their ex-post profits and losses so that they make exactly zero profits in equilibrium, regardless of whether they match.

\(^{12}\) Note that it is in general possible to have \( c_j - X < 0 \), i.e., a household may choose to sell off a portion of its endowment. However, because all households are symmetric in this environment and since aggregate production is subject to a non-negativity constraint, in equilibrium we will always have \( c_j - X \geq 0 \).
The worker’s problem in household $j$ when matched, taking $w$ as given, can be expressed as choosing a level of hours to supply in the first sub-period so as to solve

$$\max_{\ell_j} -\nu(\ell_j) + V(w\ell_j - p(c_j - X))$$

3.2.2 Deriving the value function $V(a)$

$V(a)$ represents the value function associated with entering the second sub-period with a net asset position $a$. In this subsection, we derive such a value function by specifying primitives in terms of preferences, technology and market organization. We choose to model this sub-period in such a way that if there were no friction in the first sub-period, there would be no trade between agents in the second sub-period. For this reason let us call “services” the good produced in the second period household, with preferences given by

$$\tilde{U}(\tilde{c}) - \tilde{\nu}(\tilde{\ell})$$

where $\tilde{c}$ is consumption of these services, $\tilde{U}(\cdot)$ is increasing and strictly concave in $\tilde{c}$, $\tilde{\ell}$ is the labor used to produced household services, and $\tilde{\nu}(\cdot)$ is increasing and convex in $\tilde{\ell}$.

To ensure that a unit of net assets is more valuable when in debt than when in surplus, let us assume that households in the second sub-period can produce services for their own consumption, using one unit of labor to produce $\tilde{\theta}$ unit of services. However, if a household in the second sub-period has to produce market services – that is, services that can be sold to others in order to satisfy debt – then to produce $\tilde{\theta}$ units of market services requires them to supply $1 + \tau$ units of labor, $\tau > 0$. To simplify notation, we can set $\tilde{\theta} = 1$ for now and return to the more general formulation when talking about effects of technological change. The continuation value function $V(a)$ can accordingly be defined as

$$V(a) = \max_{\tilde{c}, \tilde{\ell}} \tilde{U}(\tilde{c}) - \tilde{\nu}(\tilde{\ell})$$

subject to

$$\tilde{c} = \tilde{\ell} + a \text{ if } a \geq 0$$

and

$$\tilde{c} = \tilde{\ell} + a(1 + \tau) \text{ if } a < 0$$

It is easy to verify that $V(a)$ is increasing in assets and concave. If $\tilde{\nu}(\tilde{\ell})$ is strictly convex, then $V(a)$ will be strictly concave, regardless of the value of $\tau$, with the key property that $V'(a_1) > V'(a_2)$ if $a_1 < 0 < a_2$; that is, the marginal value of an increase in assets is greater if one is in debt than if one is in surplus.\(^\text{13}\) In the case where $\tilde{\nu}(\tilde{\ell})$ is linear, then $V(a)$ will be piecewise linear and will

\(^{13}\) To avoid backward-bending supply curves, we will also assume that $\tilde{\nu}(\cdot)$ and $\tilde{U}(\cdot)$ are such that $V''(a) \geq 0$. This assumption is sufficient but not necessary for later results. Note that a sufficient condition for $V''(a) \geq 0$ is that both $\tilde{U}'''(\cdot) \geq 0$ and $\tilde{\nu}''(\cdot) < 0$. 

45
not be differentiable at zero. Nonetheless, it will maintain the key property that $V'(a_1) > V'(a_2)$ if $a_1 < 0 < a_2$. We will mainly work with this case, and in particular, will assume that $\tilde{\nu}(\tilde{\ell}) = v \cdot \tilde{\ell}$, which implies that $V(a)$ is piecewise linear with a kink at zero.

3.2.3 Equilibrium in the first sub-period

Given the function $V(a)$, a symmetric equilibrium for the first sub-period is represented by five objects: two relative prices (the price of clothing $p$ and the wage rate $w$), two quantities (consumption of clothes by each household $c$ and the amount worked in each match $\ell$), and a number $N$ of active firms, such that

1. $c$ solves the buyer’s problem taking $\mu$, $p$, $w$ and $\ell$ as given.
2. The labor supply $\ell$ solves the worker’s problem conditional on a match, taking $p$, $w$ and $c$ as given.
3. The demand for labor $\ell$ maximizes the firm’s profits given a match, taking $p$ and $w$ as given.
4. The goods market clears; that is, $L \cdot (c - X) = M(N, L)F(\ell) - N \Phi$.
5. Firms’ entry decisions ensure zero profits.

The equilibrium in the first sub-period can therefore be represented by the following system of five equations:

\begin{align*}
U'(c) &= p \left\{ \frac{M(N, L)}{L} V'(w\ell - p(c - X)) \\
&\quad + \left[ 1 - \frac{M(N, L)}{L} \right] V'(-p(c - X)) \right\} \tag{3.1} \\
\nu'(\ell) &= V'(w\ell - p(c - X)) w \tag{3.2} \\
pF'(\ell) &= w \tag{3.3} \\
M(N, L)F(\ell) &= L(c - X) + N \Phi \tag{3.4} \\
M(N, L)[pF(\ell) - w\ell] &= N p \Phi \tag{3.5}
\end{align*}

In the above system,\textsuperscript{14} equations (3.1) and (3.2) represent the first-order conditions for the household’s choice of consumption and supply of labor. Equations (3.3) and (3.5) represent a firm’s labor demand condition and its entry decision. Finally, (3.4) is the goods market clearing condition.

\textsuperscript{14}To ensure that an employed worker’s optimal choice of labor is strictly positive, we assume that $\lim_{\ell \to 0} U'(c) > \lim_{\ell \to 0} \frac{\nu'(\ell)}{\nu'(\ell)}$. 

46
At this level of generality it is difficult to derive many results. Nonetheless, we can combine (3.1), (3.2) and (3.3) to obtain the following important expression regarding a characteristic of the equilibrium,

$$\frac{\nu'(\ell)}{U'(c)} \left\{ 1 + \left( 1 - \mu \right) \left[ \frac{V'(-p(c - X))}{V'(w\ell - p(c - X))} - 1 \right] \right\} = F'(\ell) \tag{3.6}$$

From equation (3.6), we see that as long as $\mu < 1$, the marginal rate of substitution between leisure and consumption will not be equal to the marginal productivity of work; that is, the labor market will exhibit a wedge given by

$$\left( 1 - \mu \right) \left[ \frac{V'(-p(c - X))}{V'(w\ell - p(c - X))} - 1 \right]$$

In fact, in this environment, the possibility of being unemployed leads to precautionary savings, which in turn causes the marginal rate of substitution between leisure and consumption to be low relative to the marginal productivity of labor. As we will see, changes in $X$ will cause this wedge to vary, which will cause a feedback effect on economic activity. Obviously, in this environment there would be a desire for agents to share the risk of being unemployed, which could reduce or even eliminate the wedge. As noted earlier, the reason that this type of insurance may be limited is the presence of adverse selection, an issue to which we will return.

Our main goal now is to explore the effects of changes in $X$ on equilibrium outcomes. In particular, we are interested in clarifying why and when an increase in $X$ can actually lead to a reduction in consumption and/or welfare. The reason we are interested in this comparative static is that we are interested in knowing why periods of liquidations – that is, periods where agents inherit excessive levels of durable goods from the past – may be socially painful.

To clarify the analysis, we will make two simplifying assumptions. First, we will assume that the matching function takes the form $M(N, L) = \min\{N, L\}$; that is, matches are determined by the short side of the market. This assumption creates a clear and useful dichotomy, with the economy characterized as being either in an unemployment regime if $L > N$ or in a full-employment regime if $N > L$. We will also assume that $V(a)$ is piece-wise linear, with $V'(a) = v \cdot a$ if $a \geq 0$ and $V'(a) = v \cdot a \cdot (1 + \tau)$ if $a < 0$, with $\tau > 0$ and $v > 0$. This form of the $V(\cdot)$ function corresponds to the case discussed in subsection 3.2.2 where the dis-utility of work in the second sub-period is linear. The important element here is $\tau$. In effect, $1 + \tau$ represents the ratio of the marginal value of an extra unit of assets when one is in debt relative to its value when one is in surplus. A value of $\tau > 0$ can be justified in many ways, one of which is presented in subsection 3.2.2. Alternatively, $\tau > 0$ could reflect a financial friction related to the cost of borrowing versus savings.
Under these two functional-form assumptions, the equilibrium conditions can be reduced to the following:

\[ U'(c) = \frac{\nu'(\ell)}{F'\ell} \left( 1 + \tau - \frac{\min\{N, L\}}{L} \right) \quad (3.7) \]

\[ \frac{\min\{N, L\}}{L} = \frac{c - X}{F'\ell} \quad (3.8) \]

\[ \frac{\min\{N, L\}}{N} [F(\ell) - F'(\ell)\ell] = \Phi \quad (3.9) \]

\[ w = \frac{\nu'(\ell)}{v} \quad (3.10) \]

\[ p = \frac{\nu'(\ell)}{vF'(\ell)} \quad (3.11) \]

This system of equations now has the feature of being block-recursive. Equations (3.7), (3.8) and (3.9) can be solved for \( c, \ell \) and \( N \), with equations (3.10) and (3.11) then providing the wage and the price. From equations (3.7) and (3.8), one can immediately notice the complementarity that can arise between consumption and employment in the case where \( N < L \) (the unemployment regime). From (3.7) we see that, if \( N < L \), agents will tend to increase their consumption if they believe there are many firms looking for workers (\( N \) expected to be large). Then from equation (3.8) we see that more firms will be looking to hire workers if they believe that consumption will be high. So greater consumption favors greater employment, which in turn reinforces consumption. This feedback effect arises as the result of consumption and employment playing the role of strategic complements. Workers demand higher consumption when they believe that many firms are searching to hire, as they view a high \( N \) as reducing their probability of entering the second sub-period in debt. It is important to notice that this multiplier argument is implicitly taking \( \ell \), the number of hours worked by agents, as given. But, in the case where the economy is characterized by unemployment, this is precisely the right equilibrium conjecture. In particular, from (3.9) we can see that if the economy is in a state of unemployment, then \( \ell \) is simply given by \( \ell^* \), the solution to the equation \( F(\ell^*) - F'(\ell^*)\ell^* = \Phi \), and is therefore locally independent of \( X \) or \( c \). Hence, in the presence of unemployment, consumption and firm hiring will act as strategic complements. As is common in the case of strategic complements, multiple equilibria can arise. This possibility is stated in Proposition 3.1.

**Proposition 3.1.** There exists a \( \bar{\tau} > 0 \) such that (a) if \( \tau < \bar{\tau} \), then there exists a unique equilibrium for any value of \( X \); and (b) if \( \tau > \bar{\tau} \), then there exists a range of \( X \) for which there are multiple equilibria.

The proofs of all propositions are presented in Appendix B.1.

\[ \bar{\tau} = -U'' \left( U'^{-1} \left( \frac{\nu'(\ell^*)}{F'(\ell^*)} \right) \right) \frac{F'(\ell^*)[F(\ell^*) - \Phi]}{\nu'(\ell^*)}. \]
While situations with multiple equilibria may be interesting, in this paper we will mainly focus on the case where the equilibrium is unique, as we believe this is more likely to be the empirically relevant case. Accordingly, Proposition 3.1 tells us that our setup will have a unique equilibrium if the marginal cost of debt is not too large. For the remainder of this section, we will assume that $\tau < \bar{\tau}$. Proposition 3.2 focuses on this case and provides a first step in the characterization of the equilibrium.

**Proposition 3.2.** When $\tau < \bar{\tau}$, there exists an $X^\star$ such that if $X \leq X^\star$ then the equilibrium is characterized by full employment, while if $X > X^\star$ it is characterized by unemployment. Furthermore, there exists an $X^{**} > X^\star$ such that if $X > X^{**}$, then employment is zero and agents simply consume their endowment (i.e., $c = X$).\(^{16}\)

The content of Proposition 3.2 is very intuitive as it simply states that if agents have a low endowment of the consumption good, then there are substantial gains from trade, and that will favor full employment. In contrast, if the endowment is very high, this will reduce the demand for the good sufficiently as to create unemployment. Finally, if $X$ is extremely high, all trade among agents will stop as people are content to simply consume their endowment.

Proposition 3.2 can also be used to provide insight regarding the relationship between the labor wedge in this economy and the inherited endowment of $X$, where the labor wedge is defined as $\left[ U'(c) - \frac{\nu'(\ell)}{F'(\ell)} \right] / \frac{\nu'(\ell)}{F'(\ell)}$. In Figure 3.1, we plot the labor wedge as a function of $X$. As can be seen, for $X < X^\star$,\(^{17}\) the labor wedge is zero, while for $X \in [X^\star, X^{**}]$, the labor wedge rises monotonically, reaching a peak at the point $X^{**}$ where trade collapses. Then, for $X > X^{**}$, we enter the no-employment zone and the wedge declines gradually until it reaches zero anew at a point where the no-employment outcome is socially optimal. This figure nicely illustrates that the degree of distortion in this economy varies with $X$, with low values of $X$ being associated with a more efficient economy, while higher values of $X$ generate a positive and growing wedge as long as trade remains present. From this observation, we can see how a higher inherited capital stock can increase inefficiency. Proposition 3.3 complements Proposition 3.2 by indicating how consumption is determined in each regime.

**Proposition 3.3.** When the economy exhibits unemployment ($X^{**} > X > X^\star$), the level of consumption is given as the unique solution to

$$c = U^{-1} \left( \frac{\nu'(\ell^*)}{F'(\ell^*)} \left[ 1 + \tau - \frac{c - X}{F'(\ell^*)\ell^*} \right] \right)$$

When the economy exhibits full employment ($X \leq X^\star$), consumption is the unique solution to

$$c = U^{-1} \left( \frac{\nu'(\Omega^{-1}(c - X))}{F'(\Omega^{-1}(c - X))} \right)$$

\(^{16}\) $X^\star = U^{-1} \left( \frac{\nu'(\ell^*)}{F'(\ell^*)} - F'(\ell^*)\ell^* \right)$ and $X^{**} = U^{-1} \left( \frac{\nu'(\ell^*)}{F'(\ell^*)(1 + \tau)} \right)$.

\(^{17}\) We assume here and throughout the remainder of this paper that $U' \left( F(\ell^*) - \Phi \right) > \frac{\nu'(\ell^*)}{F'(\ell^*)}$, so that $X^\star > 0$. 

49
Figure 3.1: Labor wedge as function of $X$.

Note: Labor wedge is defined as \[
\frac{U'(c) - \nu'(\ell) F'(\ell)}{\nu'(\ell) F'(\ell)}.
\]
Example is constructed assuming the functional forms $U(c) = \log(c)$, \(\nu(\ell) = \frac{\nu_\ell^{1+\omega}}{1+\omega}\) and $F(\ell) = A\ell^\alpha$, with parameters $\omega = 1$, $\nu = 0.5$, $\alpha = 0.67$, $A = 1$, $\Phi = 0.35$ and $\tau = 0.3$.

Finally, when $X \geq X^{**}$, consumption is given by $c = X$.

Given the above propositions, we are now in a position to examine an issue of main interest, which is how an increase in $X$ affects consumption. In particular, we want to ask whether an increase in $X$, which acts as an increase in the supply of goods, can lead to a decrease in the actual consumption of goods. Proposition 3.4 addresses this issue.

**Proposition 3.4.** If $X^{**} > X > X^{*}$, then $c$ is decreasing in $X$. If $X \leq X^{*}$ or $X > X^{**}$, then $c$ is increasing in $X$.

The content of Proposition 3.4 is illustrated in Figure 3.2. Proposition 3.4 indicates that, starting at $X = 0$, consumption will continuously increase in $X$ as long as $X$ is compatible with full employment. Then, when $X$ is greater than $X^{*}$, the economy enters the unemployment regime and consumption starts to decrease as $X$ is increased. Finally, beyond $X^{**}$ trade collapses and consumption becomes equal to $X$ and hence it increases with $X$. The reason that consumption decreases with a higher supply of $X$ in the unemployment region is precisely because of the multiplier process described earlier. In this region, an increase in $X$ leads to a fall in expenditures on new consumption, where we define expenditures as $e \equiv c - X$. The decrease in expenditures reduces the demand for goods as perceived by firms. Less firms then search for workers, which increases the risk of unemployment. The increase in unemployment risk leads households to cut their expenditures further, which further amplifies the initial effect of an increase in $X$ on expenditures. It is because of this
**Figure 3.2:** Consumption as function of $X$.

![Graph](image)

*Note: Example is constructed assuming the functional forms $U(c) = \log(c)$, $\nu(\ell) = \frac{\nu^{1+\omega}}{1+\omega}$ and $F(\ell) = A\ell^\alpha$, with parameters $\omega = 1$, $\nu = 0.5$, $\alpha = 0.67$, $A = 1$, $\Phi = 0.35$ and $\tau = 0.3$."

type of multiplier process that an increase in the supply of the good can lead to a decrease in its total consumption ($X + e$). Note that such a negative effect does not happen when the economy is at full employment, as an increase in $X$ does not cause an increase in precautionary savings, which is the key mechanism at play causing consumption to fall.

The link noted above between household $j$’s expenditure, which we can denote by $e_j \equiv c_j - X_j$, and its expectation about the expenditures by other agents in the economy, which can denote by $e$, can be captured by rewriting the relations determining $e_j$ implied by the elements of Proposition 3.3 as

$$e_j = Z(e) - X \quad (3.12)$$

with

$$Z(e) \equiv U'^{-1}(Q(e)) \quad (3.13)$$

and

$$Q(e) \equiv \begin{cases} \frac{\nu'(\ell^*)}{F'(\ell^*)} \left(1 + \tau - \frac{\tau e}{\sigma} \right) & \text{if } 0 < e < e^* \\ \frac{\nu'(\Omega^{-1}(e))}{F'(\Omega^{-1}(e))} & \text{if } e \geq e^* \end{cases} \quad (3.14)$$

Here, $e^* \equiv \Omega(\ell^*)$ is the level of output (net of firms’ search costs) that would be produced if all workers were employed, with hours per employed worker equal to $\ell^*$. In equilibrium we have the additional requirement that $e_j = e$ for all $j$.

The equilibrium determination of $e$ is illustrated in Figure 3.3, which somewhat resembles a Keynesian cross. In the figure, we plot the function $e_j = Z(e) - X$ for two values of $X$: a first value
Figure 3.3: Equilibrium determination

Note: Example is constructed assuming the functional forms $U(c) = \log(c)$, $\nu(\ell) = \frac{\nu_0^{1+\omega}}{1+\omega}$ and $F(\ell) = A\ell^\alpha$, with parameters $\omega = 1$, $\nu = 0.5$, $\alpha = 0.67$, $A = 1$, $\Phi = 0.35$ and $\tau = 0.3$. Values of $X$ used were $X = 0$ for the full-employment equilibrium and $X = 0.7$ for the unemployment equilibrium.

of $X$ which places the economy in an unemployment regime, and a second value of $X$ which places the economy in a full-employment regime. An equilibrium in this figure corresponds to the point where the function $e_j = Z(e) - X$ crosses the $45^\circ$ line. Note that changes in $X$ simply move the $e_j = Z(e) - X$ curve vertically.

There are several features to note about Figure 3.3. First, in the case where $X \in (X^*, X^{**})$, so that the equilibrium of the economy is in an unemployment regime with positive trade (i.e., $0 < e < e^*$), the diagram is similar to a Keynesian cross. We can see graphically how an increase in $X$ by one unit shifts down the $Z(e) - X$ curve and, since the slope of $Z(e) - X$ is positive and less than one, a multiplier process kicks in which causes $e$ to fall by more than one. Because of this multiplier process, total consumption of clothes, which is equal to $e + X$, decreases, which is the essence of the first part of Proposition 3.4. Second, when $X < X^*$, so that the economy is in a full-employment regime (i.e., the equilibrium is such that $e > e^*$), the diagram is different from the Keynesian cross. The most notable difference is the negative slope of the function $Z(e) - X$ for values of $e > e^*$. This reflects the fact that unemployment risk is not present in this regime. In fact, when $X$ is sufficiently small so that the economy is in the full-employment regime, an increase in $X$ by one unit leads to a decrease in $e$ that is less than one, compared to a decrease of greater than one as exhibited in the unemployment regime. Here, expenditure by others actually plays the role of a strategic substitute with one’s own expenditure – as opposed to playing the role of a strategic complement as is the case in the unemployment regime – through its effects on real wages and prices. Accordingly, in this region, an increase in $X$ leads to
an increase in total consumption of clothes. Another more subtle difference with the Keynesian cross is in how the intercept of $Z(e) - X$ is determined. The intercept is given by $U' - 1 + (1 + \tau)$. The $X$ term in the intercept can be interpreted as capturing a pure aggregate-demand effect, whereby higher values of $X$ reduce aggregate demand. However, the remaining term, $U' - 1 + (1 + \tau)$, reflects technology and preferences. In particular, we can generalize this term by re-introducing the technology parameter $\theta$, in which case the intercept becomes $U' - 1 + (1 + \tau)$. In this case, we see that an improvement in technology shifts up the intercept, and will lead to an increase in expenditures. This feature of the $Z(e) - X$ curve illustrates its equilibrium nature, which incorporates both demand and supply effects, as opposed to a Keynesian cross that only reflects demand effects.

3.2.4 Is there deficient demand in the unemployment regime?

In the case where $X$ is large enough for the economy to be in the unemployment regime ($X^* < X < X^{**}$), we have already noted that the marginal rate of substitution between consumption and leisure is greater than the marginal product of labor, with this distortion increasing the larger is $X$. In this sense, the economy is clearly working inefficiently in the unemployment regime. In this section, we want to examine whether this regime can also be appropriately characterized as suffering from deficient aggregate demand. In particular, suppose the structure of markets were not changed and $X^* < X < X^{**}$. Now suppose that all households deviated from their equilibrium strategies by increasing slightly their demand for consumption goods. If in this case the expected utility of the household would be increased, then it appears reasonable to characterize the situation as one of deficient demand. Using this definition, Proposition 3.5 indicates that the unemployment regime of our model is in fact characterized by deficient demand.

**Proposition 3.5.** When the economy is in the unemployment regime ($X^* < X < X^{**}$), a coordinated increase by households in the purchase of the first sub-period consumption good increases the expected utility of all households.

Proposition 3.5 can alternatively be interpreting as confirming that the consumption choices of individual households play the role of strategic complements in the unemployment regime.

3.2.5 Effects of changes in $X$ on welfare

We have shown that when $X$ is high enough, then the economy will be in the unemployment regime, where a local increase in $X$ causes consumption to fall. We now want to ask how expected welfare is affected in these cases, where expected welfare is defined as $U(c) + \mu [\nu(\ell) + V(w \ell - p(c - X))] + (1 - \mu) V(p(c - X))$. In particular, we want to ask whether welfare can decrease when the economy is endowed with more goods. Proposition 3.6 answers this question in the affirmative. Proposition 3.6

---

18 Recall that an increase in $\theta$ is associated with a proportional change in the search cost, so that $\ell^*$ remains unchanged.
Proposition 3.6. An increase in $X$ can lead to a fall in expected welfare. In particular, if either (i) $\tau$ is close enough to $\bar{\tau}$ or (ii) the average cost of work $\nu(\ell^*)$ is low enough relative to the marginal cost of work $\nu'(\ell^*)$, then there is always a range of $X \in [X^*, X^{**}]$ such that an increase in $X$ leads to a decrease in expected welfare.

Proposition 3.6 provides a step toward answering whether more goods can make everyone worse off. In effect, the proposition indicates that the economy can function in a very perverse fashion when households have inherited many goods. We saw from Proposition 3.4 that an increase in $X$ always leads to a decrease in consumption when we are in the unemployment regime. In comparison, Proposition 3.6 is weaker as it only indicates the possibility of a fall in welfare in the unemployment region when $X$ rises. In response to a rise in $X$ in the unemployment regime, there are three distinct channels through which expected welfare is affected. First, as discussed above, consumption falls, which tends to directly decrease welfare. Second, this fall in consumption is associated with a fall in the probability of being employed. It can be verified that the net benefit of being employed is strictly positive, so that this second effect also tends to decrease welfare. Finally, a rise in $X$ means that a given quantity of consumption can be obtained with a lower level of expenditure, which increases assets for the employed and decreases debt for the unemployed, and therefore tends to increase welfare. Whether this final effect is outweighed by the first two depends on the factors discussed in Proposition 3.6.

As noted in Proposition 3.6, the effects of an increase in $X$ on welfare depends, among other things, on the difference between the marginal utility cost of work and the average utility cost of work. This distinction is relevant because an important component of the net benefit of being employed is the utility value of wages earned, net of the value of foregone leisure. When the average cost of work is low, the net benefit of being employed is large, and therefore a rise in the unemployment rate caused by a rise in $X$ will have a larger negative effect on welfare (i.e., the second channel discussed above becomes more important). Hence, in our model, when employment is not perceived as very painful, and we are in the unemployment regime, then an increase in $X$ leads to decreased welfare.

3.2.6 Allowing for offers of unemployment insurance

In our analysis thus far, we have assumed that agents do not have access to unemployment insurance. It may be thought that allowing for the private provision of unemployment insurance would necessarily eliminate the mechanisms we have highlighted. For this reason, in this subsection we want to briefly indicate how our analysis can be extended rather trivially to include an adverse selection problem that...
will justify the absence of unemployment insurance, without changing the main results. In particular, suppose there is a fraction $\rho$ of households that behave as the households we have modeled to date, which we call participant households, and suppose the remaining $(1 - \rho)$ fraction of households, which we can call the non-participant households, are simply not interested in work within the period. These latter households are happy to consume their endowment without wanting to search for work. Now suppose that some private agent wanted to offer unemployment insurance before the matching process, but could not differentiate between the two types of households. In this case, an insurer will not be able to offer contracts that will only be attractive to the participant households, because any unemployment insurance contract with a positive net payment to unemployed individuals will be desirable to non-participants. Therefore, as indicated in Proposition 3.7, as long as $\rho$ is sufficiently low, this type of adverse selection problem implies that the only equilibrium outcome is one where no insurance is offered. Accordingly, in this setup, the mechanisms we have emphasized regarding how changes in $X$ affect outcomes will directly apply.

**Proposition 3.7.** In the presence of both participant households and non-participant households, if $\rho < \frac{1}{1+\tau}$, i.e., if the fraction of participant households is sufficiently low, then no unemployment-insurance contracts are traded in equilibrium.

### 3.2.7 Introducing government spending

We now turn to examining how changes in government spending can affect economic activity. To do this, we extend the model by simply adding a government to the first sub-period. The government undertakes two activities in this sub-period: it buys goods, and it taxes employed individuals. We assume that the government runs a balanced budget so that its expenditure on goods is equal to the lump-sum tax per employed worker times the number of employed workers. It turns out that the effects of government spending in this setup depend crucially on what the government does with the goods. Accordingly, we will consider two types of government purchases: wasteful, and non-wasteful. Wasteful government purchases, denoted $G_w$, are not valued by households, while non-wasteful purchases, denoted $G_n$, are assumed to directly affect agents’ utility by entering as a substitute to private consumption. Note that $G_w$ and $G_n$ are per-capita government expenditures. If we return to the set of equilibrium conditions given by equations (3.7) to (3.11), the only condition that changes with the introduction of a government is equation (3.8), the goods-market equilibrium condition. The other conditions remain the same once the variable $c$ is interpreted as total consumption including consumption of non-wasteful government purchases. The goods market equilibrium condition, equation (3.8),

---

20We can as well assume that they are valued by households but that utility is linearly separable in $G_w$. 

---
therefore has to be rewritten as \(21\)

\[
\min\{N, L\} = \frac{c - X + G_w}{F'(\ell)\ell} \tag{3.15}
\]

since \(c - X + G_w\) now represents the total purchases of clothes in the sub-period. If we again allow \(e\) to represent these total purchases \((e = c - X + G_w)\), then the determination of \(e\) takes a form almost identical to that described previously by equations (3.12)-(3.14). In fact, the determination of total expenditures \(e\) is now given by the solution to

\[
e = Z(e) - X + G_w \tag{3.15}
\]

where \(Z(e)\) was defined in equation (3.13).

There are two key things to notice about equation (3.15). First, non-wasteful government expenditure \(G_n\) does not enter into this condition, and therefore does not affect the equilibrium level of economic activity \(e\); that is, non-wasteful government expenditure crowds out private expenditure one-to-one. Second, in contrast, wasteful government expenditure will tend to stimulate activity in a manner parallel to a decrease in \(X\). To understand why non-wasteful government purchases do not affect activity, it is helpful consider how people would behave simply if they conjectured the outcome. In this case, since they would conjecture that unemployment risk is not changing, they would want to consume at the same overall level as before the increase in \(G_n\). But if they consume at the exact same overall level, it requires households to decrease their private purchases by exactly the same amount as the purchases made by the government. Hence, activity will not be increased and agents’ initial conjecture is rationalized. This is why non-wasteful government purchases do not affect activity in our setup, even when the economy exhibits unemployment. Note that this logic does not hold in the case of wasteful government purchases. If government purchases are wasteful, and people conjecture that unemployment risk is unaffected, their overall consumption will be unchanged, and, with no increased utility from government purchases, private purchases would also be unchanged. But total purchases – including those made by the government – would necessarily be increased. If the economy were in the unemployment regime, this additional demand would be met by a rise in the employment rate \(\mu\), and hence households’ conjecture that unemployment risk is unchanged would be false. Recognizing that unemployment risk in fact fell, households would reduce their precautionary savings and increase their private purchases, further increasing demand, and leading to a multiplier greater than one. If the economy had instead been in the full-employment regime, the additional demand would be met by a rise in hours per worker \(\ell\), which is associated with a rise in the price \(p\) and a corresponding fall in private purchases, mitigating to some extent the rise in demand caused by the government and leading to a multiplier less than one. These results are summarized in Proposition 3.8.

\(21\) We assume throughout this subsection that \(\tau < \bar{\tau}\), and that total government expenditures are sufficiently low so that the lump-sum tax on employed workers is not so large as to cause households to prefer to be unemployed.
**Proposition 3.8.** An increase in non-wasteful government purchases has no effect on economic activity. An increase in wasteful government purchases leads to an increase in economic activity. If the economy is in the unemployment regime, wasteful government purchases are associated with a multiplier that is greater than one, while if the economy is in the full-employment regime, wasteful government purchases are associated with a multiplier that is less than one.

From Proposition 3.8 we see that the multiplier associated with wasteful government purchases depends on the state of the economy and the type of purchases. In particular, the multiplier for wasteful government purchases is greater than one when the economy has a high level of $X$ and is therefore in the unemployment regime. In contrast, when the economy has a low level of $X$ and is therefore in the full-employment regime, the multiplier for wasteful government purchases is less than one. The interesting aspect of Proposition 3.8 is that it emphasizes why the effects of government purchases may vary drastically, from zero to more than one, depending on the circumstances.

While wasteful government purchases increase economic activity, this does not imply that they increase welfare. In fact, it can be easily verified that an increase in wasteful government purchases necessarily decreases welfare when the economy is in the full-employment regime, as it reduces private consumption and increases hours worked. On the other hand, when the economy is in the unemployment regime (due to a high value of $X$), the effect on welfare depends on a number of factors, in much the same way that the effect on welfare of a change in $X$ depends on a number of factors. For example, the change in welfare depends on the ratio of the average dis-utility of labor relative to the marginal dis-utility of labor. As discussed earlier, when this ratio is low, the net benefit to being employed is high, and since one of the effects of an increase in wasteful government purchases is to increase the employment rate, the resulting increase in welfare through this channel is also high. As such, welfare is overall more likely to increase when the average dis-utility of work is low.

It turns out that sufficient conditions under which an increase in wasteful government purchases increases welfare are given by those contained in Proposition 3.6 regarding the welfare effects of a change $X$. This is stated in Proposition 3.9.

**Proposition 3.9.** If the economy is in the unemployment regime and if $X$ is in the range such that a fall in $X$ would increase welfare, then an increase in wasteful government purchases will increase welfare.

### 3.3 Further discussions and relaxing of assumptions

#### 3.3.1 Relaxing functional-form assumptions

One of the important simplifying assumptions of our model is the use of a matching function of the “min” form. This specification has the nice feature of creating two distinct employment regimes: one
where there is unemployment and one where there is full employment. However, this stark dichotomy, while useful, is not central to the main results of the model. In fact, as we now discuss, the important feature for our purposes is that there be one regime in which expenditures by individual agents play the role of strategic substitutes, and another in which they play the role of strategic complements. To see this, it is helpful to re-examine the equilibrium condition for the determination of expenditure for a general matching function. This is given by

\[ U'(X + e_j) = vp(e) \left[ 1 + \tau - \frac{M(N(e), L)}{L}\tau \right] \tag{3.16} \]

where \( M(N, L) \) is a CRS matching function satisfying \( M(N, L) \leq \min\{N, L\} \). In (3.16), we have made explicit the dependence of \( N \) and \( p \) on \( e \), where this dependence comes from viewing the remaining four equilibrium conditions as determining \( N, p \) and \( \ell \) as functions of \( e \). Note that these other equilibrium conditions imply that \( p(e) \) and \( N(e) \) are always weakly increasing in \( e \). In (3.16) we have once again made clear that this condition relates the determination of expenditure for agent \( j, e_j \), to the average expenditure of all agents, \( e \). From this equation, we can see that average expenditure can play either the role of strategic substitute or strategic complement to the expenditure decision of agent \( j \). In particular, through its effect on the price \( p \), \( e \) plays the role of a strategic substitute, while through its effect on firm entry \( N \) and, in turn, unemployment, it plays the role of strategic complement. The sign of the net effect of \( e \) on \( e_j \) therefore depends on whether the price effect or the unemployment effect dominates. In the case where \( M(N, L) = \min\{N, L\} \), the equilibrium features the stark dichotomy whereby \( \partial p(e)/\partial e = 0 \) and \( \partial M(N(e), L)/\partial e > 0 \) for \( e < e^* \), while \( \partial p(e)/\partial e > 0 \) and \( \partial M(N(e), L)/\partial e = 0 \) for \( e > e^* \). In other words, for low values of \( e \) the expenditures of others play the role of strategic complement to \( j \)'s decision since the price effect is not operative, while for high values of \( e \) it plays the role of strategic substitute since the risk-of-unemployment channel is non-operative. This reversal in the role of \( e \) from acting as a complement to acting as a substitute is illustrated in Figure 3.4, where we first plot a cost-of-funds schedule for agents, defined by

\[ r = p(e) \left[ 1 + \tau - \frac{\min\{N(e), L\}}{L}\tau \right], \]

where \( r \) represents the total cost of funds to agent \( j \) when average expenditure is \( e \). Our notion of the total cost of funds reflects both the direct cost of borrowing, \( p(e) \), and the extra cost associated with the presence of unemployment risk. We superimpose on this figure the demand for \( e \) as a function of the total cost of funds, which is implicitly given by the function

\[ U'(X + e)/v = r. \]

This latter relationship, which can be interpreted as a type of aggregate demand curve, is always downward-sloping since \( U \) is concave. The important element to note in this figure is

\[ \nu' = vw \]

\[ pF'(\ell) = w \]

\[ M(N, L)\Phi(e - X) + N \Phi \]

\[ M(N, L)[pF(\ell) - w\ell] = Np\Phi \]

22 These remaining four equilibrium conditions can be written

\[ \nu'(\ell) = vw \]

\[ pF'(\ell) = w \]

\[ M(N, L)\Phi(e - X) + N \Phi \]

\[ M(N, L)[pF(\ell) - w\ell] = Np\Phi \]
that the cost-of-funds schedule \( r = p(e) \left[ 1 + \tau - \frac{\min\{N(e),L\}}{L} \tau \right] \) is first decreasing and then increasing in \( e \). Over the range \( e < e^* \), the cost of funds to an agent is declining in aggregate \( e \), since \( N \) is increasing while \( p \) is staying constant. Therefore, in the range \( e < e^* \), a rise in \( e \) reduces unemployment and makes borrowing less costly to agents. This is the complementarity zone. In contrast, over the range \( e \geq e^* \), the effect of \( e \) on the cost of funds is positive since the unemployment channel is no longer operative, while the price channel is. This is the strategic substitute zone. In the figure, a change in \( X \) moves the demand curve \( U'(X+e)/v = r \) without affecting the cost-of-funds curve. A change in \( X \) therefore has the equilibrium property \( \partial e / \partial X < -1 \) when \( e < e^* \) because the cost-of-funds curve is downward-sloping in this region, while \( \partial e / \partial X > -1 \) in the region \( e \geq e^* \) because the cost-of-funds curve is upward-sloping.

From the above discussion it should now be clear that our main results do not hinge on the “min” form of the matching function, but instead depend on the existence of two regions: one where the total cost of borrowing by agents at low levels of \( e \) is decreasing in \( e \) because the effect of \( e \) on unemployment risk dominates its effect on \( p \), with a second region where the price effect dominates the effect running through the unemployment-risk channel. It can be easily verified that a sufficient condition for this feature is that the elasticity of \( M(N, L) \) with respect to \( N \) tends towards one when \( N \) becomes sufficiently small, while simultaneously having this elasticity tending to zero when \( N \) is sufficiently large. This property is clearly captured by the “min” function, but is in fact also captured by a large class of matching functions, as the following proposition establishes.

**Proposition 3.10.** For any non-trivial matching function \( M(N, L) \)\(^{23}\) that is (i) non-decreasing and

\(^{23}\)By “non-trivial matching function” we mean a function satisfying, for any \( L > 0 \), \( M(N, L) > 0 \) for some \( N \).
weakly concave in \( N \) and (ii) satisfies \( 0 \leq M(N, L) \leq \min\{N, L\} \), the elasticity of \( M \) with respect to \( N \) approaches one as \( N \to 0 \) and approaches zero as \( N \to \infty \).

While this proposition guarantees under quite general conditions that the cost-of-funds locus will be negatively sloped at low levels of \( e \) and positively sloped at high values of \( e \), it is interesting to ask if this non-monotonicity property can be ensured by other means over a region where the matching function has a constant elasticity. In effect, this property can be ensured through assumptions on \( \nu(\ell) \) and \( F(\ell) \). In particular, if the elasticities of \( \nu'(\ell) \) and \( F'(\ell) \) with respect to \( \ell \) tend toward zero when \( \ell \) is sufficiently low – that is, if \( \nu(\ell) \) and \( F(\ell) \) become close to linear when \( \ell \) is low – this will guarantee a downward-sloping cost-of-funds schedule even if the matching function has a constant elasticity. Furthermore, if the elasticity of either \( \nu'(\ell) \) or \( F'(\ell) \) with respect to \( \ell \) tends toward infinity when \( \ell \) is large, this will guarantee that the cost-of-funds schedule will be upward-sloping at high values of \( e \). While it is an open empirical question whether any of these conditions are met in reality over an economically significant range, it appears at least plausible to us that for low values of activity (i) congestion effects in matching associated with increases in \( N \) are small, (ii) the returns to labor in production exhibit little decreasing returns, and (iii) the dis-utility of work is close to linear. All these conditions will favor a downward-sloping cost-of-funds curve at low levels of activity, which is what is needed for the main results of this paper to hold.

A second important functional-form assumption we have used to derive our results is that the dis-utility of work in the second sub-period be linear so as to obtain a piecewise linear \( V(a) \) function. This restriction is again not necessary to obtain our main results. However, if we depart substantially from the linearity assumption for second-sub-period dis-utility of labor, income effects can greatly complicate our simple characterizations.

### 3.3.2 A version with productive capital

We have shown how a rise in the supply of the capital good \( X \), by decreasing demand for employment and causing households to increase precautionary savings, can perversely lead to a decrease in consumption. While thus far we have considered the case where \( X \) enters directly into the utility function, in this section we show that Proposition 3.4 can be extended to the case where \( X \) is introduced as a productive capital good. To explore this in the simplest possible setting, suppose there are now two types of firms and that the capital stock \( X \) no longer enters directly into the agents’ utility function. The first type of firm remains identical to those in the first version of the model, except that instead of producing a consumption good they produce an intermediate good, the amount of which is given by \( \mathcal{M} \). There is also now a continuum of competitive firms who rent the productive capital good \( X \) from the households and combine it with goods purchased from the intermediate goods firms in order to produce the consumption good according to the production function \( g(X, \mathcal{M}) \). We assume that \( g \) is strictly increasing in both arguments and concave, and exhibits constant returns to scale. Given \( X \),
it can be verified that the equilibrium determination of $\mathcal{M}$ will then be given as the solution to

$$g_M(X, \mathcal{M})U'(g(X, \mathcal{M})) = Q(\mathcal{M})$$  \hspace{1cm} (3.17)

where $Q(\cdot)$ is defined in equation (3.14).

Note the similarity between condition (3.17) and the corresponding equilibrium condition for the durable-goods version of the model, which can be written $U'(X+e) = Q(e)$. In fact, if $g(X, \mathcal{M}) = X + \mathcal{M}$, so that the elasticity of substitution between capital and the intermediate good $\mathcal{M}$ is infinite, then the two conditions become identical, and therefore $X$ affects economic activity in the productive-capital version of the model in exactly the same way as it does in the durable-goods model. Thus, a rise in $X$ leads to a fall in consumption when the economy is in the unemployment regime. In fact, as stated in Proposition 3.11, this latter result will hold for a more general $g$ as long as $g$ does not feature too little substitutability between $X$ and $\mathcal{M}$.\(^{24}\)

**Proposition 3.11.** If the equilibrium is in the full-employment regime, then an increase in productive capital leads to an increase in consumption. If the equilibrium is in the unemployment regime, then an increase in productive capital leads to a decrease in consumption if and only if the elasticity of substitution between $X$ and $\mathcal{M}$ is not too small.

The reason for the requirement in Proposition 3.11 that the elasticity of substitution be sufficiently large relates to the degree to which an increase in $X$ causes an initial impetus that favors less employment. If the substitutability between $X$ and $\mathcal{M}$ is small, so that complementarity is large, then even though the same level of consumption could be achieved at a lower level of employment, a social planner would nonetheless want to increase employment. Since the multiplier process in our model simply amplifies – and can never reverse – this initial impetus, strong complementarity would lead to a rise in employment and therefore a rise in consumption, rather than a fall. In contrast, if this complementarity is not too large, then an increase in $X$ generates an initial impetus that favors less employment, which is in turn amplified by the multiplier process, so that a decrease in consumption becomes more likely.\(^{25}\)

Let us emphasize that the manner in which we have just introduced productive capital into our setup is incomplete – and possibly unsatisfying – since we are maintaining a static environment with no investment decision. In particular, it is reasonable to think that the more interesting aspect of introducing productive capital into our setup would be its effect on investment demand. To this end, we now consider extending the model to a simple two-period version that features investment. The

\(^{24}\) We assume throughout this section that an equilibrium exists and is unique. Conditions under which this is true are similar to the ones obtained for the durable-goods model, though the presence of non-linearities in $g$ makes explicitly characterizing them less straightforward in this case.

\(^{25}\) Note that a rise in $X$ also increases output for any given level of employment. To ensure that consumption falls in equilibrium, we require that the substitutability between $X$ and $\mathcal{M}$ be large enough so that the drop in employment more than offsets this effect.
main result from this endeavor is to emphasize that the conditions under which a rise in $X$ leads to a
fall in consumption are weaker than those required for the same result in the absence of investment.
In other words, our results from the previous section extend more easily to a situation where $X$ is
interpreted as physical capital if we simultaneously introduce an investment decision. The reason for
this is that, in the presence of an investment decision, a rise in $X$ is more likely to cause an initial
impetus in favor of less activity.

To keep this extension as simple as possible, let us consider a two-period version of our model
with productive capital (where there remains two sub-periods in each period). In this case, it can be
verified that the continuation value for household $j$ for the second period is of the form $R(X_2) \cdot X_{2,j}$,
where $X_{2,j}$ is capital brought by household $j$ into the second period and $X_2$ is capital brought into
that period by all other households. In order to rule out the possibility of multiple equilibria that could
arise in the presence of strategic complementarity in investment, we assume we are in the case where
$R'(X_2) < 0$. The description of the model is then completed by specifying the capital accumulation
equation,

$$X_2 = (1 - \delta)X_1 + i$$  \hspace{1cm} (3.18)

where $i$ denotes investment in the first period and $X_1$ is the initial capital stock, as well as the new
first-period resource constraint,

$$c + i = g(X_1, M)$$  \hspace{1cm} (3.19)

Given this setup, we need to replace the equilibrium condition from the static model (equation
(3.17)) with the constraints (3.18) and (3.19) plus the following two first-order conditions,

$$g_M(X_1, M)U'(c) = Q(M)$$  \hspace{1cm} (3.20)

$$U'(c) = R(X_2)$$  \hspace{1cm} (3.21)

Equation (3.20) is the household’s optimality condition for its choice of consumption, and is similar to
its static counterpart (3.17), while equation (3.21) is the intertemporal optimality condition equating
the marginal value of consumption with the marginal value of investment.

Of immediate interest is whether, in an unemployment-regime equilibrium, a rise in $X_1$ will pro-
duce an equilibrium fall in consumption and/or employment in the first period. As Proposition 3.12
indicates, the conditions under which our previous results extend are weaker than those required in
Proposition 3.11 for the static case, in the sense that lower substitution between $X$ and $M$ is possible.

**Proposition 3.12.** In the two-period model with productive capital,\(^\text{26}\) an increase in capital leads to
a decrease in both consumption and investment if and only if the elasticity of substitution between $X$
and $M$ is not too small. Furthermore, for a given level of equilibrium employment, this minimum
elasticity of substitution is lower than that required in Proposition 3.11 in the absence of investment

\(^\text{26}\) We are again assuming that the equilibrium exists, is unique, and is in the unemployment regime.
decisions.

The intuition for why consumption and investment fall when the elasticity of substitution is high is similar to in the static case. The addition of the investment decision has the effect of making it more likely that an increase in $X$ leads to a fall in consumption because the increase in $X$ decreases investment demand, which in turn increases unemployment and precautionary savings.

### 3.3.3 Multiple equilibria

Before discussing the welfare effects of changes in $X$, let us briefly discuss how multiple equilibria can arise in this model when $\tau > \bar{\tau}$. It can be verified that, when $\tau > \bar{\tau}$, the equilibrium determination of expenditures can still be expressed as the solution to the pair of equations $e_j = Z(e) - X$ and $e_j = e$. The problem that arises is that this system may no longer have a unique solution. Instead, depending on the value of $X$, it may have multiple solutions, an example of which is illustrated in Figure 3.5. In the figure, we see that, for this value of $X$, there are three such solutions.

**Figure 3.5:** Equilibrium determination (multiple equilibria)

![Equilibrium determination (multiple equilibria)](image)

*Note: Example is constructed assuming the functional forms $U(c) = \log(c)$, $\nu(t) = \frac{\nu t^{1+\omega}}{1+\omega}$ and $F(\ell) = At^{\alpha}$, with parameters $\omega = 1$, $\nu = 0.5$, $\alpha = 0.67$, $A = 1$, $\Phi = 0.35$, $\tau = 1.2$ and $X = 0.3$.*

Figure 3.6 shows how the set of possible equilibrium values of consumption depends on $X$ when $\tau > \bar{\tau}$. As can be seen, when $X$ is in the right range, there is more than one such equilibrium, with at least one in the unemployment regime and one in the full-employment regime. When this is the case, the selection of the equilibrium will depend on people’s sentiment. If people are pessimistic, they cut back on consumption, which leads firms to cut back on employment, which can rationalize the initial pessimism. In contrast, if households are optimistic, they tend to buy more, which justifies
many firms wanting to hire, which reduces unemployment and supports the optimistic beliefs. This type of environment featuring multiple equilibria driven by demand externalities is at the core of many papers. On this front, this paper has little to add. The only novel aspect of the current paper in terms of multiple equilibria is to emphasize how the possibility of multiple equilibria may depend on the economy’s holding of capital goods.

3.3.4 The role of beliefs

There is another aspect in which the current model differs from a Keynesian-cross setup, and that is with respect to the role of beliefs. The current setup should be thought of as part of the family of coordination games, and accordingly can potentially be analyzed with the tools and concepts used in the global games literature. Because of our assumption of homogeneity across households, we have not been very specific about agents’ beliefs up to now. Nonetheless, it is worth emphasizing that the type of multiplier process present in the unemployment regime is the equilibrium outcome of a simultaneous-move game, rather than the outcome of events occurring sequentially over time. As such, prior beliefs of the players in the game are potentially a key driving force in the multiplier process. To clarify the potential role of these beliefs in our setup, it is helpful to briefly consider the case where agents have different holdings of $X$. For example, suppose that each agent $j$ has an $X_j$ drawn from a distribution with mean $\chi$. The first-order condition for household $j$, assuming he thinks
he is in the unemployment regime, can then be stated as
\[ U'(e_j + X_j) = \frac{\nu'(\ell^*)}{F'(\ell^*)} (1 + \tau - E_j[\mu] \tau) \]

What is unknown to the household in this setup is the match probability \( \mu \), and therefore the expenditure decision, \( e_j \), depends on household \( j \)'s expectation of \( \mu \), which we write as \( E_j[\mu] \). But \( \mu \) in turn depends on firms’ entry decisions, which depends on firms’ expectation of aggregate consumption. This latter expectation can be expressed as \( E_f[\int e_i di] \), where the operator \( E_f[\cdot] \) represents expectations by firms, and \( \int e_i di \) is the aggregate level of expenditures. So the first-order condition for household \( j \) would be given by
\[ U'(e_j + X_j) = \frac{\nu'(\ell^*)}{F'(\ell^*)} \left( 1 + \tau - \frac{E_j[E_f[\int e_i di]/L]}{\Omega(\ell^*)} \tau \right) \]

We can now see that agent \( j \)'s consumption decision will depend on his expectation of firms’ expectation of the aggregate level of expenditure. This type of setup therefore involves forecasting the forecasts of others. If we assume that \( U(\cdot) \) is quadratic and all relevant random variables jointly normally distributed, then this problem can be solved analytically, and will lead agent \( j \) to have a decision rule for consumption which depends on both \( X_j \) and \( \chi \), the prior about the average level of \( X_i \) across all other agents. Hence, both actual \( X_j \)'s and beliefs regarding the average value of \( X_j \) in the economy will be main forces that drive expenditure and employment. For example, if agents believe that other agents have a high holding of \( X \), this will depress consumption for all agents regardless of the actual holdings of \( X \). Furthermore this effect can potentially be large because of the amplification mechanisms running through precautionary savings.

### 3.4 Dynamics

In this section we want to explore a dynamic extension of our static durable-goods model where current consumption contributes to the accumulation of \( X \). In particular, we want to consider the case where the accumulation of \( X \) obeys the accumulation equation
\[ X_{t+1} = (1 - \delta)X_t + \gamma e_t \quad 0 < \delta \leq 1 \quad , \quad 0 < \gamma \leq 1 - \delta \]  \hspace{1cm} (3.22)

where the parameter \( \gamma \) represents the fraction of current consumption expenditures, \( e_t = c_t - X_t \), which take the form of durable goods. Since we do not want to allow heterogeneity between individuals to expand over time, we will allow individuals to borrow and lend only within a period but not across periods; in other words, households are allowed to spend more than their income in the first
sub-period of a period, but must repay any resulting debt in the second sub-period. The problem facing a household in the first sub-period of a period is therefore to choose how much clothing to buy and, conditional on a match, how much labor to supply. We model the second sub-period as in sub-section 3.2.2, where households use labor to produce household services either for their own consumption or, at a level of productivity that is lower by a factor $1 + \tau$, for the consumption of others. In each second sub-period, then, the household chooses how much to consume of household services and how much to produce of household services to both satisfy his needs and to pay back any accumulated debt. In order to keep the model very tractable, we will continue to assume that dis-utility of work in the second sub-period is linear (i.e., equal to $v \cdot \tilde{\ell}$). Under this assumption, all households will choose the same level of consumption of household services in each second sub-period, while the production of household services will vary across households depending on whether they entered the sub-period in debt or in surplus. Since there are no interesting equilibrium interactions in second sub-periods, we can maintain most of our focus on equilibrium outcomes in the sequence of first sub-periods.

Relative to the static case, the only difference in equilibrium relationships (aside from the addition of the accumulation equation (3.22)) is that the first-order condition associated with the households’ choice of consumption of clothes is now given by the Euler equation

$$U'(X_t + e_t) - Q(e_t) = \beta \left[ (1 - \delta - \gamma)U'(X_{t+1} + e_{t+1}) - (1 - \delta)Q(e_{t+1}) \right]$$

(3.23)

where $Q$ is as defined in equation (3.14). In this dynamic setting, an equilibrium will be represented as a sequence of the previous equilibrium conditions (3.8) to (3.11) plus the accumulation equation (3.22) and the Euler equation (3.23).

There are many complications that arise in the dynamic version of this model, which makes characterizing equilibrium behavior difficult. In particular, there can be multiple equilibrium paths and multiple steady-state solutions. Luckily, the problem can be simplified if we focus on cases where $\delta$ is small; that is, on cases where the durability of goods is long. In addition to simplifying the analysis, focusing on the low-$\delta$ case appears reasonable to us, as many consumer durables are long-lived, especially if we include housing in that category. In the case where $\delta$ is sufficiently small, as stated in Proposition 3.13, the economy will have only one steady state and that steady state will have the property of exhibiting unemployment.

**Proposition 3.13.** If $\delta$ is sufficiently small, then the model has a unique steady state and this steady state is characterized by unemployment.

Proposition 3.13 is very useful, as it will allow us to analyze the equilibrium behavior around the steady state without worrying about equilibrium selection. Accordingly, for the remainder of this section, we will assume that $\delta$ is sufficiently small so that Proposition 3.13 applies. However, before

---

27 This lack of borrowing across periods can be rationalized if one assumes that the transaction cost of intermediating loans across periods is greater than $1 + \tau$. 66
examining local properties in some generality, we believe that it is helpful to first illustrate global equilibrium behavior for a simple case that builds directly on our static analysis. The reason that we want to illustrate global behavior for at least one example is to emphasize that local behavior in our setup is likely to differ substantially and meaningfully from global behavior. Moreover, the example will allow us to gain some intuition on how the latter local results should best be interpreted.

Before discussing the transitional dynamics of the model, we first briefly discuss the conditions under which the model would exhibit a balanced growth path. In particular, suppose production in the first sub-periods is given by \( \theta_t F(\ell_t) \) where \( \theta_t \) is a technology index that is assumed to grow at a rate \( g_{\theta} \). Then it is easy to verify that our economy will admit an equilibrium growth path where both \( e \) and \( X \) grow at rate \( g_{\theta} \) if the following three conditions are satisfied (i) the fixed cost of creating jobs grows at rate \( g_{\theta} \), (ii) the productivity of labor in the second sub-periods grows at rate \( g_{\theta} \), and (iii) the utility of consumption is represented by the log function. These conditions are not surprising, as they parallel those needed for a balanced growth path in many common macro models. The important aspect to note about this balanced-growth property is that the notion of high or low levels of capital should be interpreted as relative to the balanced growth path. In other words, the key endogenous state variable in the system should be viewed as the ratio of \( X_t \) to the growth component of \( \theta_t \).

### 3.4.1 Global dynamics for a simple case

The difficulty in analyzing the global dynamics for our model is related to the issue of multiple equilibria we discussed in the static setting. If the static setting exhibits multiple equilibria then the dynamic setting will likely exhibit multiple equilibrium paths. To see this, it is useful to recognize that our problem of describing equilibrium paths can be reduced to finding the household’s decision rule for consumption. Since the only state variable in the system is \( X_t \), the household’s decision rule for consumption will likely be representable by a relationship (which may be stochastic) of the form \( c(X_t) \). Given \( c(X_t) \), the equilibrium dynamics of the system are given by

\[
X_{t+1} = (1 - \delta - \gamma)X_t + \gamma c(X_t)
\]

If the relationship \( c(X_t) \) is a function, then equilibrium dynamics are deterministic. However, if we consider the case with \( \beta = 0 \) – so that households are not forward-looking and thus the dynamic equilibrium is simply a sequence of static equilibria – we already know that the household’s decision rule \( c(X_t) \) may not be a function. For example, if \( \tau > \bar{\tau} \), then the household’s decision rule may be a correspondence of the form given in Figure 3.6. Therefore, even for the rather simple case where \( \beta = 0 \) and \( \tau > \bar{\tau} \) we know that the equilibrium dynamics need not be unique, in which case some equilibrium-selection device will be needed to solve the model. In contrast, for the case where \( \beta = 0 \) and \( \tau < \bar{\tau} \), then we know from Proposition 3.3 that \( c(X_t) \) is a function. Hence, in the case where \( \beta = 0 \) and \( \tau < \bar{\tau} \), we can describe the global dynamics of the system rather easily, and this is what we will do in this section. In particular, when \( \beta = 0 \) and \( \tau < \bar{\tau} \), the stock of durables evolves according
to equation (3.24), with \( c(X_t) \) given by the value of \( c \) obtained using Proposition 3.3 with \( X_t \) in place of \( X \).

Figure 3.7 plots the equilibrium transition function for \( X \) for three cases; that is, it plots \((1 - \delta - \gamma)X_t + \gamma c(X_t)\) for different possible \( c(X_t) \) functions. The figure is drawn so that the steady state is in the unemployment region, which is consistent with a low value of \( \delta \) as implied by Proposition 3.13. As can be seen from the figure, when \( X_t \) is not too great (i.e., less than \( X^* \)) the economy is in the full-employment regime and \( X_{t+1} > X_t \). So if the economy starts with a low value of \( X_t \) it will generally go through a phase of full employment. During this phase, we know from Proposition 3.3 that consumption is also increasing. Eventually, \( X_t \) will exceed \( X^* \) and the economy enters the unemployment regime, at which point the dynamics depend on the derivative of the equilibrium decision rule, i.e., \( c'(X_t) \), where in this regime \( c(X_t) \) solves

\[
U'(c) = \frac{\nu'(\ell^*)}{F'(\ell^*)} \left( 1 + \tau - \tau \frac{c - X_t}{e^*} \right)
\]

If \(-c'(X_t) < \frac{1 - \delta - \gamma}{\gamma}\) when \( X^* < X_t < X^{**} \), then the transition function maintains a positive slope.
near the steady state and the economy will converge monotonically to its steady state. However, note that even if $X$ converges monotonically to its steady state in such a case, this will not be the case for consumption. Again, from Proposition 3.3 we know that consumption is decreasing in $X$ in the unemployment region. Hence, starting from $X = 0$, in this case consumption would initially increase, reaching a maximum just as the economy enters the unemployment regime, then decline towards its eventual steady-state level which is lower than the peak obtained during the transition. If instead $-c'(X_t) > \frac{1-\delta-\gamma}{\gamma}$, then the transition function for $X$ will exhibit a negative slope in the unemployment regime. In this case, $X$ will no longer converge monotonically to the steady state. In fact, if the slope of this function (which depends on the elasticity of $c$ with respect to $X$ at the steady state) is negative but greater than -1, the system will converge with oscillations. However, if this slope is smaller than -1, which can arise for very large negative values of $c'(X_t)$, then the system will not converge and instead can exhibit rich dynamics, including cycles and chaos. In general, however, even in the case where $c'(X_t)$ is very negative, the system will not necessarily be explosive, since once it moves sufficiently far away from the steady state, forces kick in that work to push it back. Such rich dynamics, with the possibility of limit cycles, are certainly intriguing, but we will not dwell on them since it appears unlikely to us that this type of configuration is relevant.

There are two main messages to take away from exploring the global dynamics in this special case with $\beta = 0$. First, the behavior of the state variable $X$ can be well-behaved, exhibiting monotonic convergence throughout. Second, the behavior of consumption (and therefore possibly welfare), can nonetheless exhibit interesting non-monotonic dynamics, with steady-state consumption actually being below the highest level it achieved during the transition. It is worth noting that if the steady state were to be in the full-employment regime (due, for example, to a higher $\delta$), then from $X = 0$ both $X_t$ and $c_t$ would always converge monotonically to the steady state when $\beta = 0$ and $\tau < \bar{\tau}$.

The most interesting aspect about the global dynamics in this case is that it allows us to illustrate the following possibility: If the economy is near its steady state, then a small reduction in $X_t$ will increase consumption and can potentially increase welfare, while a large decrease in $X_t$ will certainly decrease welfare. In this sense, the model exhibits behavior around the unemployment steady state that can differ substantially from behavior far away from the steady state, with the behavior far away from the steady state being more akin to that generally associated with classical economics, while behavior in the unemployment regime being more similar to that suggested by a Keynesian perspective.\textsuperscript{28}

### 3.4.2 Local dynamics in the general case

In this subsection, we explore the local dynamics of the general model when $\beta > 0$, still assuming that $\delta$ is sufficiently small so that the steady state is unique and in the unemployment regime. From our analysis of the case with $\beta = 0$, we know that local dynamics can exhibit convergence or divergence.

\textsuperscript{28} It is worth noting that this type of synthesis, which emphasizes differences between being near to the steady state versus far from the steady state, is substantially different from the new neo-classical synthesis, which emphasizes differences in the long run and the short run because of sticky prices.
depending on how responsive consumption is to $X$ around the steady state. The one question we could not address when $\beta = 0$ is whether dynamics could exhibit local indeterminacy. In other words, can forward-looking behavior give rise to an additional potential local source of multiple equilibria in our setup? Proposition 3.14 indicates that this is not possible; that is, the roots of the system around the unique steady state can not both be smaller than one.\footnote{In this section we only consider local dynamics around a unique unemployment-regime steady state. Nonetheless, it is straightforward to show that if the unique steady state is in the full-employment regime, then the local dynamics necessarily exhibit monotonic convergence.}

**Proposition 3.14.** The local dynamics around the steady state can either exhibit monotonic convergence in $c$ and $X$, convergence with oscillations, or divergence. Locally indeterminacy is not possible.

Proposition 3.14 is useful as it tells us that the decision rule for consumption around the steady state is a function.\footnote{This is a slight abuse of language since Proposition 3.14 does not rule out the existence of other equilibrium paths away from the steady state.} Accordingly, we can now examine the sign of the derivative of this function. The question we want to examine is whether the decision rule for consumption around the steady state has the property that a larger $X$ leads to a lower level of consumption, as was the case in our static model when in the unemployment regime. In other words, we want to know whether the results regarding the effect of $X$ on consumption we derived for the static model extend to the steady state of the dynamic setting with $\beta > 0$. Proposition 3.15 indicates that if $\tau$ is not too large, then local dynamics will exhibit this property. Note that the condition on $\tau$ is a sufficient condition only.

**Proposition 3.15.** If $\tau$ is sufficiently small, then in a neighborhood of the unique steady state, consumption is decreasing in $X$, with the dynamics for $X$ converging monotonically to the steady state.

From Proposition 3.15 we now know that, as long as $\tau$ is not too big, our model has the property that when the economy has over-accumulated relative to the steady state (i.e., if $X$ slightly exceeds its steady-state value), then consumption will be lower than in the steady state throughout the transition period toward the steady state, which we can refer to as a period of liquidation. In this sense, the economy is overreacting to its inherited excess of capital goods during this liquidation period, since it is reducing its expenditures to such an extent that people are consuming less even though there are more goods available to them in the economy. While such a response is not socially optimal, it remains unclear whether it is so excessive as to make people worse off in comparison to the steady state, since they are also working less during the liquidation phase. It turns out that, as in the static case, the welfare effect of such a liquidation period depends, among other things, on whether the average dis-utility of work is small enough relative to the marginal dis-utility. For example, if the average dis-utility of work is sufficiently low relative to its marginal value, then it can be verified that a liquidation period induced by inheriting an excess of $X$ relative to the steady state will make average
utility in all periods of the transition lower than the steady state level of utility. This result depends in addition on the unemployment rate not being too large in the steady state.

While we do not have a simple characterization of the global dynamics when $\beta > 0$, Propositions 3.14 and 3.15 suggest to us that the intuition we gained from the case where $\beta = 0$ likely extends to the more general problem as long as $\tau$ is not too large and $\delta$ is small. In particular, we take our analysis as suggesting that, starting from $X = 0$, the economy will generally go through a phase of full employment, with both $X$ and $c$ increasing over time. The economy then enters into the unemployment range once $X$ is large enough. Then, as long as $\tau$ is not too great, $X$ will continue to monotonically increase, converging toward its steady state. In contrast to $X$, upon entering the unemployment regime, consumption starts to decrease as unemployment risk leads to precautionary savings which depresses activity. Eventually, the economy will reach a steady state where consumption, employment, and possibly period welfare are below the peak levels reached during the transition.

In the above discussion of liquidation, we have taken the level of inherited capital as given and have only examined how the economy responds over time to a situation where $X$ is initially above its steady state. In particular, we have shown that such a liquidation phase can be associated with excessively low consumption, low welfare and high unemployment, all relative to their steady state values. While the focus of the paper is precisely to understand behavior during such a liquidation phase, it nonetheless remains interesting to ask how welfare would behave if we were to view the whole cycle, both the over-accumulation phase and the liquidation phase together. To briefly examine this issue, we build on the news-noise literature and consider a case where agents in an economy start at a steady state and then receive information about productivity. In particular, we have shown that such a liquidation phase can be associated with excessively low consumption, low welfare and high unemployment, all relative to their steady state values. While the focus of the paper is precisely to understand behavior during such a liquidation phase, it nonetheless remains interesting to ask how welfare would behave if we were to view the whole cycle, both the over-accumulation phase and the liquidation phase together. To briefly examine this issue, we build on the news-noise literature and consider a case where agents in an economy start at a steady state and then receive information about productivity. In particular, we have shown that such a liquidation phase can be associated with excessively low consumption, low welfare and high unemployment, all relative to their steady state values. While the focus of the paper is precisely to understand behavior during such a liquidation phase, it nonetheless remains interesting to ask how welfare would behave if we were to view the whole cycle, both the over-accumulation phase and the liquidation phase together. To briefly examine this issue, we build on the news-noise literature and consider a case where agents in an economy start at a steady state and then receive information about productivity.

In Figure 3.8 we report for illustration purposes two impulse responses associated with a simple calibration of such a noise-driven-boom-followed-by-liquidation model. We plot the dynamics for the stock of durables and the average period utility of households relative to the steady state. From the figure, we see that during the first period, when agents are acting on optimistic beliefs about productivity, their period welfare increases even if they are working hard to ramp up their stocks of durable goods. After one period, they realize their error since productivity has not actually improved, and consequently cut back on their expenditures to start a liquidation process. The welfare of households from the second period on is lower than in steady state because of the excessively cautious behavior of households, which stops the economy from taking advantage of the excessively high inherited capital stock.

It is interesting to contrast this path with that which would happen if unemployment risk were perfectly insured or if matching frictions were absent. In such a case, the news would still lead to a

---

31 See Beaudry and Portier (2013) for a survey of this literature.
Figure 3.8: Response of economy to a noise shock

Note: Impulse is associated with a 10% overly-optimistic belief by shoppers in the first sub-period of $t = 0$. $\hat{X}_t$ is the stock of durables and $\hat{u}_t$ is average period utility across all households, both expressed in deviations from steady state. Example is constructed assuming the functional forms $U(c) = \log(c)$, $\nu(l) = \nu_1 l^{1+\omega} / (1+\omega)$ and $F(l) = Al^\alpha$, with parameters $\beta = 0.9$, $\delta = 0.1$, $\gamma = 0.1$, $\omega = 1.2$, $\nu_1 = \nu_2 = 0.35$, $\alpha = 0.67$, $A = 1.2$, $\Phi = 0.5$ and $\tau = 0.3$.

boom, and the realization of the error would lead to a recession. However, the dynamics of period welfare would be very different. Instead of the boom being associated with high period welfare and the recession being associated with low period welfare, as in our model with unemployment risk, the opposite would happen. The boom would be associated with low period welfare, as agents would be working harder than normal, while in the recession welfare would be above the steady-state value since agents would take a vacation and benefit from past excess work. While evaluating welfare is certainly difficult, the path for period welfare in our model with unemployment risk appears to us as more in line with common perceptions about boom-bust cycles than that implied by a situation with no market frictions.

3.5 Policy trade-offs

In this last section, we turn to one of our motivating questions and ask whether or not stimulative policies should be used when an economy is going through a liquidation phase characterized by high
unemployment. In particular, we consider the case where the economy has inherited from the past a level of \( X \) above its steady-state value and, in the absence of intervention, would experience a period of liquidation, with consumption below its steady-state level throughout the transition. Obviously, the first-best policies in this environment would be to remove the sources of frictions or to perfectly insure agents against unemployment risk. However, for a number of reasons, such first-best policies may not be possible. We therefore want to consider the value of a more limited type of policy: one that seeks only to temporarily boost expenditures. In particular, we are interested in asking whether welfare would be increased by stimulating expenditures for one period, knowing that this would imply a higher \( X \) tomorrow and therefore lower consumption in all subsequent periods until the liquidation is complete. This policy question is aimed at capturing the tension between the Keynesian and Hayekian prescriptions in recession. In answering this question, we will be examining the effects of such a policy without being very explicit about the precise policy tools used to engineer the stimulus, as we think it could come from several sources. However, it can be verified that the stimulus we consider can be engineered by a one period subsidy to consumption financed by a tax on the employed.

Examining how a temporary stimulus to expenditures affects welfare during a liquidation turns out to be quite involved. For this reason, we break down the question into two parts. First, we ask whether a temporary stimulus would increase welfare if the economy were initially in a steady state characterized by unemployment. Second, we ask whether the effect on welfare of such a stimulus would be greater if the economy were initially in a state of liquidation (i.e., with \( X_0 \) above its steady state) than in the case where it is initially at a steady state.

When looking at how a temporary boost in expenditures would affect welfare, one may expect it to depend on many factors, including the extent of risk-aversion and the dis-utility of work. However, since the level of expenditures represents a private optimum, the present discounted welfare effect of a temporary boost in expenditures turns out to depend on a quite limited set of factors. In particular, if the economy is initially at a steady state in the unemployment regime, then to a first-order approximation the direction of the cumulative welfare effect depends simply on whether the stimulus induces an increase or decrease in the presented discounted value of the output stream. This is stated in Proposition 3.16.

**Proposition 3.16.** Suppose the economy is in steady state in the unemployment regime. Then, to a first-order approximation, a (feasible) change in the path of expenditures from this steady state equilibrium will increase the present discounted value of expected welfare if and only if it increases the presented discounted sum of the resulting expenditure path, \( \sum_{i=0}^{\infty} \beta^i e_{t+i} \).

The logic behind Proposition 3.16 derives mainly from the envelope theorem. Since the consumption stream is optimally chosen from the individual’s perspective, most of the effects of a change in the consumption path are only of second order and can therefore be neglected when the change is small. Moreover, in the unemployment region, prices, wages and hours worked are invariant to changes in
expenditures. Hence the only effects needed to be taken into account for welfare purposes are the induced changes in the match probabilities times the marginal value of changing these match probabilities. When the economy is initially in a steady state, the marginal value of changing the match probabilities is the same at each point in time. Further, since the match probabilities are proportional to expenditures, this explains why welfare increases if and only if the perturbed path of expenditures has a positive presented discounted value. With this result in hand, it becomes rather simple to calculate whether, starting from steady state, a one-period increase in expenditures followed by a return to equilibrium decision rules results in an increase in welfare. In particular, recall that the law of motion for $X$ is given by

$$X_{t+1} = (1 - \delta)X_t + \gamma e(X_t) \quad 0 < \gamma < 1 - \delta$$

where the function $e(X_t)$ is the equilibrium policy function for $e_t$. Now, beginning from steady state, suppose at $t = 0$ we stimulate expenditures by $\epsilon$ for one period such that the stock at $t = 1$ is now given by

$$\tilde{X}_1 = (1 - \delta)X_0 + \gamma (e + \epsilon)$$

As a result of this one-period perturbation, the path for expenditures for all subsequent periods will be changed even if there is no further policy intervention. The new sequence for $X$, which we denote $\tilde{X}_t$, will be given by $\tilde{X}_{t+1} = (1 - \delta)\tilde{X}_t + \gamma e(\tilde{X}_t)$ for all $t \geq 1$. From Proposition 3.16, this perturbation increases present discounted welfare if and only if

$$\epsilon > -\sum_{t=1}^{\infty} \beta^t \left[ e(\tilde{X}_t) - e \right]$$

(3.25)

For $\epsilon$ small, we can use the linear approximation of the function $e(\cdot)$ around the steady state to make this calculation. Note that $e'(X) = -(1 - \delta - \lambda_1) / \gamma$, where $\lambda_1$ is the smallest eigenvalue of the dynamic system in modulus.$^{32}$ Thus, in this case, one may show that condition (3.25) becomes

$$\frac{1 - \beta(1 - \delta)}{1 - \beta \lambda_1} > 0$$

If the system is locally stable, then $\lambda_1 < 1$, and therefore this condition will always hold. Hence, if we are considering a situation where the economy is in an unemployment-regime steady state, and this steady state is locally stable, then a one-period policy of stimulating household expenditures will increase welfare. This arises even though most of the effect of the policy is to front-load utility by creating an initial boom followed by a liquidation bust.$^{33}$ While we knew that the initial steady state was sub-optimal, and that a policy that increases expenditures in all periods would likely be desirable, it is interesting to learn that a policy that favors expenditure today over expenditure tomorrow – when

$^{32}$ See the proof of Proposition 3.15.

$^{33}$ Note that this result does not depend on the welfare factors considered earlier in the static model, such as the magnitudes of $\tau$ and of the difference between the marginal and average disutility of work.
in the economy is in the unemployment regime – tends to increase welfare.

The question we now want to examine is whether the gains in welfare of a temporary stimulus are greater when the economy is initially in a liquidation phase than in steady state. We believe this is a relevant question since a case for stimulus during a liquidation can best be made if the gains are greater than when the economy is in steady state. Otherwise, there is no particular reason to favor stimuli more when unemployment is above normal than when it is at a normal level. Somewhat surprisingly to us, as long as $U'''$ is not too big, the answer to this question is negative, as stated in Proposition 3.17.

**Proposition 3.17.** Assuming the economy’s steady state is in the unemployment regime and $U'''$ is not too big, then, to a second-order approximation around the steady state, a temporary stimulus increases the presented discounted value of welfare less when implemented during a liquidation phase then when implemented at the steady state.

Although a period of liquidation is associated with a higher-than-normal level of unemployment, and the degree of distortion as captured by the labor wedge is higher in such periods when compared to the steady state, Proposition 3.17 indicates that the gains to a temporary stimulus are not greater during a liquidation period than in normal (steady-state) times. At first pass, one may be puzzled by this result, as one might have expected the gains to be highest when the marginal utility of consumption is highest. However, when the economy is in a liquidation phase, while the benefits from current stimulus are high, so are the costs associated with delaying the recovery. In fact, because consumption levels are at a private optimum, these two forces essentially cancel each other out. Moreover, when in the unemployment regime, the direct gain from employing one more individual – that is, the value of the additional production, net of the associated dis-utility of work – is the same regardless of whether unemployment is high or low. Hence, the only remaining difference between the value of stimulus in high- versus low-unemployment states relates to the net utility gain from employed workers entering the second sub-period in surplus rather than debt. In a lower-unemployment regime, households take less precaution, so that unemployed workers end up with more debt, which is costly. It is this force which makes postponing an adjustment particularly costly when in a liquidation phase.

With respect to the policy debate between the followers of Hayek and Keynes, we take our results are clarifying the scope of the arguments. On the one hand, we have found that a policy that stimulates current consumption at the cost of lower consumption in the future can often be welfare-improving when the economy features unemployment. However, at the same time, we have found that the rationale for such a policy does not increase simply because the level of unemployment is higher. Hence, if one believes that stimulus is not warranted in normal times (because of some currently un-modeled costs) and that normal times are characterized by excessive unemployment, then stimulus should not

---

34 Note that this condition on $U'''$ is sufficient but not necessary for this result.

35 There is an additional force at play here, which relates to the fact that the magnitude of the amplification mechanism will in general be different when the economy is away from the steady state. However, as long as $U'''$ is not too big, this effect can safely be ignored.
be recommended during liquidation periods. While this insight will likely not extinguish the debate on the issue, we believe it can help focus the dialogue.

### 3.6 Conclusion

There are three types of elements that motivated us to write this paper. First, there is the observation that most deep recessions arise after periods of fast accumulation of capital goods, either in the form of houses, consumer durables, or productive capital. This, in our view, gives plausibility to the hypothesis that recessions may often reflect periods of liquidation where the economy is trying to deplete excesses from past over-accumulation.\(^{36,37}\) Second, during these apparent liquidation-driven recessions, the process of adjustment seems to be socially painful and excessive, in the sense that the level of unemployment does not seem to be consistent with the idea that the economy is simply “taking a vacation” after excessive past work. Instead, the economy seems to be exhibiting some coordination failure that makes the exploitation of gains from trade between individuals more difficult than in normal times. These two observations capture the tension we believe is often associated with the Hayekian and Keynesian views of recessions. Finally, even when monetary authorities try to counter such recessions by easing policy, this does not seem to be fully effective. This leads us to believe that there are likely mechanisms at play beyond those related to nominal rigidities.\(^{38}\) Hence, our objective in writing this paper was to offer a framework that is consistent with these three observations, and accordingly to provide an environment where the policy trade-offs inherent to the Hayekian and Keynesian views could be discussed.

A central contribution of the paper is to provide a simple macro model that explains, using real as opposed to nominal frictions, why an economy may become particularly inefficient when it inherits an excessive amount of capital goods from the past. The narrative behind the mechanism is quite straightforward. When the economy inherits a high level of capital, this decreases the desire for trade between agents in the economy, leading to less demand. When there are fixed costs associated with employment, this will generally lead to an increase in unemployment. If the risk of unemployment cannot be entirely insured away, households will react to the increased unemployment by increasing saving and thereby further depressing demand. This multiplier process will cause an excess reaction to the inherited goods and can be large enough to make society worse off even if – in a sense – it is richer since it has inherited a large stock of goods. Within this framework, we have shown that policies aimed at stimulating activity will face an unpleasant trade-off, as the main effect of stimulus will simply be

\(^{36}\) Note that this is a fundamentalist view of recessions, in that the main cause of a recession is viewed as an objective fundamental (in this case, the level of capital relative to technology) rather than a sunspot-driven change in beliefs.

\(^{37}\) An alternative interpretation of this observation is that financial imbalances associated with the increase in capital goods are the main source of the subsequent recessions.

\(^{38}\) We chose to analyze in this paper in an environment without any nominal rigidities so as to clarify the potential role of real rigidities in understanding behavior in recessions. However, in doing so, we are not claiming that the economy does not also exhibit nominal rigidities or that monetary policy is ineffective. We are simply suggesting that explanations based mainly on nominal rigidities may be missing important forces at play that cannot be easily overcome by monetary policy.
to postpone the adjustment process. Nonetheless, we find that such stimulative policies may remain
desirable even if they postpone recovery, but these gains do not increase simply because the rate of
unemployment is higher. As noted, the mechanisms presented in the paper have many antecedents
in the literature, but we believe that our framework offers a particularly tractable and clear way of
capturing these ideas and of reconciling diverse views about the functioning of the macro-economy.
Chapter 4

Can a Limit-Cycle Model Explain Business Cycle Fluctuations?

4.1 Introduction

In conventional models of the business cycle, all fluctuations are ultimately caused by the arrival of random shocks. One implication of this is that a boom is usually caused by one collection of random events, and any subsequent recession is caused by another collection of random events. In this sense, in conventional models individual booms and busts are largely unrelated phenomena.

An alternative to this viewpoint is that booms and busts are inherently related, and that, for example, a protracted boom can sow the seeds for a subsequent recession, which can in turn lead to the next boom, and so on. According to this view, fluctuations are at least in part driven by deterministic cyclical forces that do not fit neatly into the usual shocks-and-propagation-mechanisms characterization of conventional models. While an earlier literature made an attempt to formalize the forces that can produce these deterministic fluctuations,¹ in recent decades this research area has gone dormant.

This paper revisits the idea of deterministic fluctuations, with the aim of showing (1) how a purely deterministic general-equilibrium model featuring strategic complementarity² near the steady state can give rise to a stable limit cycle,³ and (2) that this model can replicate business cycle features once it is augmented to include a small amount of variation from exogenous sources. The limit cycle in the model arises through a simple micro-founded mechanism in a rational-expectations environment. In

¹To fix terminology, I will say a dynamic system exhibits “deterministic fluctuations” if, in the absence of stochastic shocks, its state vector neither diverges to infinity nor converges to a single point. See Appendix C.3 for a more formal definition.

²Two agents’ decisions are strategic complements if a rise in any agent’s choice variable results in a rise in the marginal value of the other agent’s choice variable. Their decisions are strategic substitutes if the reverse is true, i.e., if a rise in any agent’s choice variable leads to a fall in the marginal value of the other agent’s choice variable.

³A “limit cycle” is a deterministic fluctuation that exactly repeats itself every $k$ periods. A “stable limit cycle” is a limit cycle that acts as an attractor for the system, so that nearby points converge to it over time. See Appendix C.3 for further detail.
contrast to most conventional models, the model I present does not require shocks in order to generate fluctuations, nor does it rely on the existence of multiple equilibria or dynamic indeterminacy. Cycles emerge endogenously and would indefinitely repeat themselves in the absence of shocks. Shocks are only introduced into the model to create irregularities in the cycles.

The key deterministic mechanism in the model, which is based closely on Beaudry et al. (2014), will center around the accumulation of a stock of capital, interpreted here as a stock of durable goods and/or housing. When this stock is high following a boom period, agents reduce their demand for new capital goods with the goal of running down the stock of capital, which leads to a bust. Because of a demand externality in the model, this bust is excessively large, and can lead new purchases of capital goods to become sufficiently depressed that the stock of capital overshoots its steady-state level. Once this happens, the capital stock is too low and the reverse mechanisms come into play, producing a boom and a subsequent capital stock that is too high, and so on.

The demand externality, meanwhile, arises because of two key imperfections in the model. First, there is a matching friction in the labor market in the spirit of Diamond-Mortensen-Pissarides, which creates the possibility that a household may not find employment when looking for a job. Second, households are unable to perfectly insure against this idiosyncratic unemployment risk. The combination of these two imperfections causes agents to react to a fall in the unemployment rate by increasing their demand for new goods, which in turn reduces the unemployment rate further. If sufficiently strong, this strategic complementarity—where an initial increase in one agent’s purchases leads all other agents to increase their own purchases—causes the economy to be very sensitive to small changes in the stock of capital goods. In turn, this causes the unique steady state of the model to be locally unstable, so that even in the absence of shocks the economy will in general fail to converge to the (unique) steady state. When the economy reaches full employment, however, the above demand externality is no longer operative. Instead, a rise in one household’s purchases causes the price of goods to increase, which tends to favor a decrease in expenditure by other households. As a result of this strategic substitutability, when far enough away from the steady state the economy exhibits locally the types of stabilizing forces that are present globally in conventional models. This prevents the system from exploding. The combination of this non-explosiveness property with an unstable steady state yields conditions under which a stable limit cycle may appear.

Because the model does not rely exclusively on shocks to drive fluctuations, it is capable of addressing one of the common criticisms of conventional business-cycle models, which is their frequent

---

4 Even though they may not require shocks to fundamentals to generate fluctuations, models featuring multiple equilibria or indeterminacy still require some time-varying equilibrium-selection device, and this device is usually taken to be some exogenous shock that affects agents’ beliefs in a coordinated way (see, e.g., Farmer and Guo (1994), Jaimovich (2007), Kaplan and Menzio (2014)). Thus, these models can still be seen as requiring shocks in order to generate fluctuations.

5 Because this mechanism operates through a demand channel, to make this channel as clear as possible the stock of capital enters directly into the household utility function in the model without affecting the production side of the economy. It is for this reason that I interpret the stock of capital here as including only durables and housing, and not productive capital. Nonetheless, subject to several technical restrictions the important elements of the model also extend to a productive-capital environment (see Beaudry et al. (2014)).
reliance on poorly motivated and/or implausibly large shocks. For example, the widely cited model of Smets and Wouters (2007) features two “mark-up” shocks that are highly persistent and together account for over half of the 40-quarter forecast-error variance (FEV) of output and hours worked. Outside of their apparent usefulness in helping the model fit moments of the data, however, little empirical justification for the size and persistence of these exogenous shocks is offered.

Mark-up shocks are just one example drawn from a large number of shocks proposed in the literature that may help a particular model fit the data, but that do not necessarily have any clear empirical justification. For example, investment shocks, liquidity shocks, news/noise shocks, risk shocks, financial shocks, government spending shocks, confidence shocks, intertemporal preference shocks, and ambiguity shocks have all been argued in recent years to be quantitatively important drivers of fluctuations, based primarily on their ability to help a particular model fit the data. Without a direct compelling argument for the empirical relevance of these shocks, though, the case can be made that, while the conventional class of models may have had some success in fitting the data, it has perhaps been less successful at explaining the data.

The second goal of this paper is therefore to show that the model can match fluctuations in US data with a minimal amount of exogenously-caused variation. With this in mind, I add a single shock—a simple TFP process—to the model, then estimate the model parameters to match as closely as possible the spectrum of hours worked in US data over the period 1960-2012. I focus on the hours spectrum here for two reasons. First, as discussed further in section 4.2, one of the main criticisms of earlier deterministic-fluctuations models was that they produced cycles that were far too regular. This regularity shows up clearly as a large spike in the spectrum. The ability for the model to match the much flatter spectrum found in the data will thus be an important test of its ability to generate realistic data. Second, as compared with other data series, hours is arguably less likely to be directly impacted by various exogenous shocks. For example, while GDP is directly affected by things like shocks to total factor productivity, over the business cycle we may expect hours to largely respond only indirectly to exogenous shocks. To the extent that this is true, variation in hours is more likely to be caused by the endogenous mechanisms that are the primary focus of this paper. Nonetheless, after estimating the model, I evaluate its ability to fit several other data series, including GDP.

The main quantitative results are as follows. First, I show that the purely deterministic version of the model (i.e., with the TFP shock shut down) is capable of generating cycles in hours of an empirically reasonable length. For example, at the baseline parameterization, the model generates cycles with a roughly 30-quarter period. This contrasts with the earlier literature on deterministic-fluctuations models, where cycles tended to be either far too short or far too long.

Second, these deterministic cycles are highly regular, as evidenced both by a clear repeating pat-
tern in the simulated path of hours as well as by a large spike in its spectrum. Including the TFP process, however, easily rectifies this problem, resulting in simulated data that appears realistically irregular and a spectrum that closely matches the one estimated from the data.

Third, the estimated TFP process is close to that estimated directly from TFP data, with an unconditional variance that is, if anything, slightly smaller. In many conventional business-cycle models, this TFP process on its own would generate fluctuations that are far smaller than those found in the data. In the model presented here, however, the bulk of the fluctuations come from the deterministic forces, which account for 79% of the standard deviation of hours. Instead, the TFP shock serves primarily to accelerate and decelerate the deterministic cycles at random, causing significant fluctuations in their length while only minimally affecting their amplitude. This result again highlights one of the more general insights of the paper: conventional models usually require large persistent shocks in order to produce the types of large persistent fluctuations found in the data. This is the primary apparent motivation for a large number of the shocks now found throughout the literature. Models capable of generating deterministic fluctuations, however, are able to produce large persistent fluctuations without any shocks whatsoever. Shocks are only required in order to help match the irregularity of the fluctuations found in the data, and this can be accomplished with shocks that are of a much more plausible type and size.

The remainder of the paper proceeds as follows. Section 4.2 briefly reviews an older deterministic-fluctuations literature and highlights some of its important failures. Section 4.3 discusses the key features of the data that the model will attempt to match. Section 4.4 discusses intuitively the basic properties an economic model must have in order to produce a limit cycle. Section 4.5 introduces the model and discusses its key properties, and includes several theoretical results establishing conditions under which a limit cycle may appear. Section 4.6 presents the main quantitative results of the paper, while section 4.7 concludes.

4.2 Literature: Deterministic fluctuations

Since at least as far back as Kaldor (1940), economists have considered models that are capable of producing deterministic fluctuations. In the 1970s and 1980s in particular, a large literature emerged that examined the conditions under which qualitatively and quantitatively reasonable economic fluctuations might occur in a purely deterministic setting (see, e.g., Benhabib and Nishimura (1979, 1985), Day (1982, 1983), Grandmont (1985), Boldrin and Montrucchio (1986), Day and Shafer (1987); for surveys of the literature, see Boldrin and Woodford (1990) and Scheinkman (1990)). By the early 1990s, however, this literature seemed to have largely gone dormant.

There appear to be several key reasons why interest in deterministic fluctuations may have waned, each of which are addressed in the present paper. First, the earlier literature on deterministic fluctuations can be broadly sub-divided into two categories: models with and without fully-microfounded,
forward-looking agents. The latter category, which were generally more capable of producing reasonable deterministic fluctuations than the former, likely fell out of favor as macro in general moved toward more microfounded models.

Second, in the category of models featuring forward-looking agents, the primary focus was on models with a neoclassical, competitive-equilibrium structure. Such models were often found to require relatively extreme parameter values in order to generate deterministic fluctuations. For example, the Turnpike Theorem of Scheinkman (1976) establishes that, under certain basic conditions met by these models, for a sufficiently high discount factor—i.e., for agents that are “forward-looking” enough—the steady state of the model is globally attractive, so that persistent deterministic fluctuations cannot appear. While in principle this does not rule out deterministic fluctuations completely, in practice the size of the discount factor needed to generate them was often implausibly low. For example, in a survey of deterministic-fluctuations models by Boldrin and Woodford (1990), discount factors for several of the models they discuss were on the order of 0.3 or less. As the present paper illustrates, however, if one departs from the assumptions of a neoclassical, competitive-equilibrium environment—for example, if there is a demand externality as in the model presented in section 4.5—then a discount factor arbitrarily close to one can relatively easily support deterministic fluctuations in equilibrium.

Third, as suggested above, models producing periodic cycles—that is, cycles which exactly repeat themselves every \( k \) periods—are clearly at odds with the data, where such consistently regular cycles cannot be found. This can be observed by looking at the spectrum of data generated by such a model, which will generally feature one or more large spikes at frequencies associated with \( k \)-period cycles. Spectra estimated on actual data generally lack such spikes, which suggests less regularity in real-world cycles. To address this issue, papers from the earlier literature largely sought to establish conditions under which such irregular cycles could emerge in a purely deterministic setting (i.e., via chaotic dynamics). While in a number of cases this was found to be possible, the conditions appear

---

8The first category includes, e.g., Benhabib and Nishimura (1979, 1985) and Boldrin and Montrucchio (1986), while the latter includes, e.g., Day (1982, 1983).
9While there are some exceptions, they are comparatively rare. Perhaps the clearest example is Hammour (1989), chapter 1, which is focused on deterministic fluctuations in an environment of increasing returns. Other exceptions include models in the search literature that are capable of generating deterministic fluctuations, such as Diamond and Fudenberg (1989), Boldrin et al. (1993), and Coles and Wright (1998). Note however that these search papers were mainly concerned with characterizing the set of possible equilibria for a particular model (which for some parameterizations included deterministic cycles), rather than being focused on deterministic cycles directly.
10See the discussion in section 4.5.3 for further details.
11It is possible in principle to rationalize such low discount factors by choosing a longer period length for the model. However, if households discount the future with a quarterly discount factor of 0.99 or greater—as is frequently the case in the business-cycle literature—a factor of 0.3 would be associated with a period length of 120+ quarters (30+ years). Since the minimum period length of a cycle is two periods, this would generate cycles on the order of 60+ years, well outside of what is normally thought of as the business cycle.
12See panel (b) of Figure 4.1 for an example.
13Informally, chaotic fluctuations are deterministic fluctuations (see footnote 1) that do not converge to periodic cycles and for which the paths emanating from two different initial points cannot be made arbitrarily close by choosing those initial points sufficiently close together. See, e.g., Glendinning (1994) for a formal definition.
to have been significantly more restrictive even than those required to generate simple periodic cycles. In contrast, rather than restricting attention to a purely deterministic setting, this paper embeds deterministic (but highly regular) cyclical mechanisms into a stochastic environment for which irregularity emerges naturally.

Finally, being inherently highly non-linear, economic models that are capable of generating deterministic fluctuations are often difficult to work with analytically beyond the very simplest of settings, and quantitative results often require computationally-expensive solution algorithms. Prior to relatively recent advances in computing technology, obtaining these quantitative results may have been infeasible and, as a result, a number of potentially fruitful areas of research—such as, for example, combining deterministic and stochastic cyclical forces—may have gone unexplored.

### 4.3 Data: Hours worked

As noted above, the focus of this paper is on attempting to explain patterns in hours worked. In this section, I discuss the key properties of the hours data series used as a focal point for both the qualitative and quantitative discussion that follows.

The hours series I use is the quarterly index of nonfarm business hours worked from the US Bureau of Labor Statistics, divided by civilian noninstitutional population obtained from the FRED database. The full sample period is from 1948Q1 to 2014Q1, though I focus on the 1960Q1-2012Q4 subsample. The series was transformed by taking logs and then running the result through a band-pass (BP) filter in order to remove long-run trend components, defined here as cyclical components with periods greater than 80 quarters (20 years). The filter was applied to the full 1948Q1-2014Q1 sample, after which observations outside of the 1960Q1-2012Q4 sub-sample were discarded.

Panel (a) of Figure 4.1 plots the resulting series, with NBER-dated recessions indicated by shaded areas. Two things should be noted from the figure. First, it confirms that the BP-filtered hours series exhibits fluctuations that correspond closely to conventional business cycle definitions, as evidenced by the large downward movements during NBER recessions. Second, with the exception of the 1971-1983 period where fluctuations were somewhat more frequent, over the past half-century a full cycle in hours appears to have taken anywhere from 8 to 11 years (32 to 44 quarters) to complete. This pattern, which suggests some degree of regularity, can be confirmed by looking at the spectrum, which is obtained by first orthogonally decomposing the BP-filtered data series into sinusoidal components of different period lengths, then computing the variance of each such component. Panel (b) of Figure 4.1

---

14 As alluded to below, prior to 1960 the business cycle appears to have been more irregular. The analysis considered here will be concerned with the more regular post-1960 period.


16 Understanding the mechanisms underlying these low-frequency fluctuations, such as cultural and demographic factors, are beyond the scope of this paper.

17 Note that none of the high-frequency fluctuations were removed, i.e., a BP(2,80) filter was used. Note also that the choice of this particular filter is not crucial. In section 4.6, I verify that the key results are robust to a number of alternative filtering choices.
Figure 4.1: Hours worked data (1960-2012)

Notes: Hours Worked series is the log of BLS nonfarm hours worked divided by population, detrended with a BP filter to remove fluctuations with periods greater than 80 quarters. In panel (a), shaded areas are NBER-dated recessions. In panel (b), raw spectrum in obtained as the squared modulus of the discrete Fourier transform of the data series (scaled so that the integral with respect to angular frequency over the interval $[-\pi, \pi]$ equals the variance of the series). Spectrum in figure is kernel-smoothed raw spectrum. Kernel is a Hamming window with bandwidth parameter 11.

plots the result, with the period of the component on the horizontal axis and the associated variance on the vertical one. From the figure, we see that the bulk of the variation in hours occurs at periodicities greater than 24 quarters, with a peak at around 40 quarters. It should be noted that cycles of this length are outside the typical range conventionally associated with business cycles. For example, in their Handbook article, Stock and Watson (1999) define business cycles as fluctuations between 6 and 32 quarters in length. While this range appropriately reflected the length of the business cycle over the broad historical time frame considered in that paper, Figure 4.1 suggests that over the more recent time frame considered here the business cycle has become longer and, apparently, somewhat more regular.

4.4 Conditions for a limit cycle

The basic conditions under which a limit cycle may appear are most easily understood by way of example. The example presented here is for a continuous-time bivariate system, but the basic intuition extends to discrete-time systems and to systems with an arbitrary number of state variables.¹⁹

¹⁸Stock and Watson (1999) report that of the 30 full cycles (peak to peak) identified by the NBER over the period 1858 to 1996, 90% were 32 quarters in length or shorter, with the shortest being six quarters. This observation formed the basis for their definition of the business cycle.

¹⁹There is a technical issue here that I will sidestep throughout this paper. Certain types of deterministic fluctuations share many of the basic qualitative features of a limit cycle, but never exactly repeat themselves. For example, in a bivariate
Suppose we have a bivariate system whose evolution is characterized by the differential equations
\[ \dot{z}(t) = g(z(t)) \] and the boundary condition \( z(0) = z_0 \), where (suppressing explicit dependence on \( t \) when no confusion will arise) \( z \equiv (x,y)' \), \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) is some function, and \( \dot{z} \) indicates the time-derivative of \( z \). It turns out to be more convenient for our purposes to re-cast this system into polar coordinates, expressing the state as \((\varphi,r)\)' , where \( \varphi \) is the angle between the vector \( z \) and the positive \( x \)-axis, and \( r \geq 0 \) is the magnitude of \( z \). Since it will not play an important role in the basic conditions for a limit cycle, assume for simplicity that \( \dot{\varphi} = \tilde{\theta} \) for some constant rate of rotation \( \tilde{\theta} \neq 0 \). Next, suppose we have
\[ \dot{r} = -r(ar + b) \]
for some parameters \( a \) and \( b \), with \( ab \neq 0 \). Note that, regardless of the values of \( a \) and \( b \), this system always has a unique steady state at \( r = 0 \), i.e., at the point \( x = y = 0 \). Thus, assuming \( r > 0 \), the vector \( z \) rotates around the origin (in \((x,y)\)-space) at a constant rate, but with a potentially fluctuating length. Depending on the signs of \( a \) and \( b \), however, the system will exhibit different qualitative properties.

Conventional economic models generally feature a globally stable steady state. This corresponds here to the case where \( \dot{r} < 0 \) for all \( r > 0 \), so that the length of \( z \) is always shrinking. This property holds when \( a, b \geq 0 \). Panel (a) of Figure 4.2 shows a phase plot of such a case. The individual arrows indicate the forces acting on the length \( r \) at that point in the state space. Also plotted are paths tracking the evolution of the system beginning from two different starting points: one close to the steady state and the other far away from it. We can see in this case that there are forces everywhere that tend to push the system inward towards the steady state. As a result, starting from any point the system converges to the steady state.

The case of a globally unstable steady state (i.e., \( \dot{r} > 0 \) for all \( r > 0 \)), meanwhile, corresponds to the case where \( a, b \leq 0 \), an example of which is plotted in panel (b) of Figure 4.2. Here we see that forces exist everywhere which push the system outwards, away from the steady state, so that if the system begins anywhere but at the steady state it will diverge to infinity.

The conditions under which a limit cycle may appear correspond to the remaining cases, i.e., where \( ab < 0 \). When this is true there now exists, in addition to the steady-state value \( r = 0 \), a second discrete-time system characterized by rotation around the unit circle by \( \theta \) radians per period, if \( \theta/\pi \) is irrational then the system will never return to the same point twice. I will ignore these uninteresting technicalities and apply the term “limit cycle” loosely, with the understanding that the term may not always strictly apply.

\[ \text{N.B.: The fact that } \dot{\varphi} \text{ does not depend on } r, \text{ and vice versa, will not generally hold for an arbitrary } g. \]
Figure 4.2: Conditions for a limit cycle

Notes: Panel (a): $a = 0.1, b = 0.2$. Panel (b): $a = -0.1, b = -0.2$. Panel (c): $a = 1, b = -0.5$. Panel (d): $a = -1, b = 0.5$. In all cases, $\overline{\theta} = 0.6\pi$.

(non-negative) solution to $\dot{r} = 0$, given by

$$r = \hat{r} \equiv -\frac{b}{a} > 0$$

When the system begins with $r = \hat{r}$, it will rotate in the plane at the constant rate $\overline{\theta}$, but move neither toward nor away from the steady state. The set of points $z$ such that $\|z\| = \hat{r}$ is the limit cycle of this system.

Of particular interest is the case where the limit cycle is stable, i.e., where neighboring points will
tend to converge to the limit cycle over time. This will occur if (1) when the system begins inside the limit cycle, forces tend to push it outwards; and (2) when the system begins outside the limit cycle, forces tend to push it inwards. This corresponds to the case where \( a > 0 \) and \( b < 0 \), so that when \( r \) is small \( \dot{r} \approx -br > 0 \), while for \( r \) large \( \dot{r} \approx -ar^2 < 0 \). An example of this is shown in panel (c) of Figure 4.2. Here we see that, regardless of where the system begins (as long as it is not exactly at the steady state) it will converge to the limit cycle.

For the sake of completeness, the final case where \( a < 0 \) and \( b > 0 \) is shown in panel (d) of Figure 4.2. This corresponds to an unstable limit cycle: if the system begins on the limit cycle (shown as the dashed circle in the figure), it will remain there forever, but if it begins off the limit cycle, it will either converge to the steady state (if it begins inside the limit cycle) or diverge to infinity (if it begins outside the limit cycle).

The preceding analysis highlights the two key properties needed to obtain a stable limit cycle in a general setting: (1) when the system is close to the steady state, there are forces which tend to push it away from the steady state (i.e., the steady state is unstable); and (2) when the system is far away from the steady state, there are forces which tend to push it towards the steady state (i.e., the system is non-explosive). A limit cycle then emerges as the set of points where these outward and inward forces precisely balance. A natural question to ask, then, is whether there are reasonable conditions under which a dynamic economy may exhibit these two key properties. It turns out that one possible set of such economic conditions is as follows. First, suppose that in a neighborhood of the steady state, individual agents’ actions are strategic complements. As is well known, strategic complementarity often leads to situations where small changes in state variables can lead to relatively large changes in equilibrium outcomes, which is precisely the state of affairs needed for a system to be (locally) unstable. Second, suppose that when far enough away from the steady state, individual agents’ actions act as strategic substitutes. When this is the case, small changes in state variables tend to produce small changes in equilibrium outcomes, and as a result the system tends to drift back towards the steady state. In the following section, an economic model is presented which possesses these two features and which, as a result, will have the potential to generate limit cycles.

### 4.5 The unemployment-risk model

In this section I present a simple economic model that is capable of generating limit cycles. In the model, which is based closely on Beaudry et al. (2014), households begin each period with a stock of durable goods and must decide how many additional goods to purchase in the goods market. Abstracting for the moment from forward-looking behavior, there are two key static factors that affect this decision. First, household demand is decreasing in the size of the current stock of durables: when their existing stock of durables is low, households want to purchase more, and vice versa. Second, because of a self-insurance motive, household demand is decreasing in the unemployment rate. There are two imperfections in the model that cause this self-insurance behavior to emerge. First, there is
a matching friction in the spirit of Diamond-Mortensen-Pissarides, which creates the possibility that a household may not find employment when looking for a job. Second, households are unable to perfectly insure against this idiosyncratic unemployment risk. The upshot is that an increase in the unemployment rate causes them to reduce their demand for new goods.

The combination of these two factors produces the following mechanism by which deterministic fluctuations emerge in the model: if households have an excess stock of durables, they reduce their demand for new goods. This fall in demand then increases the unemployment rate, which causes households to further reduce their demand, further increasing the unemployment rate, and so on, so that, in equilibrium, output falls by significantly more than the initial fall in demand. This multiplier mechanism—which occurs because of strategic complementarity in households’ purchasing decisions—drives the excess sensitivity in the dynamic system which is a pre-condition for local instability. Once the economy reaches full or zero employment, however, the self-insurance mechanism is not operative, and thus the excessive sensitivity that creates instability disappears. In its place, inward forces—arising because of strategic substitutability, which in turn operates through the price of new goods—emerge that prevent the economy from exploding. The combination of these locally-outward and globally-inward forces creates the conditions for a limit cycle to occur.

4.5.1 Static version

Before presenting the full dynamic model in detail, I begin by briefly presenting a simpler version of the model that is static in nature, highlighting the key properties that will be important in generating limit cycles in a dynamic setting. Further details and in-depth analysis of this static model can be found in Beaudry et al. (2014).

Consider an environment populated by a mass one of households. In this economy there are two sub-periods. In the first sub-period, households purchase consumption goods and try to find employment. As there is no money in this economy, when the household buys consumption goods its bank account is debited, and when (and if) it receives employment income its bank account is credited. As we shall see, households will in general end the first sub-period with a non-zero bank account balance. Thus, in the second sub-period, households resolve their net asset positions by repaying any outstanding debts or receiving a payment for any surplus. These payments are made in terms of a second good, referred to here for simplicity as household services. Household services are also the numeraire in this economy.

Preferences for the first sub-period are represented by

$$U(c) - \nu(\ell)$$

where $c$ represents consumption of clothes and $\ell \in [0, \bar{\ell}]$ is the labor supplied by households in the production of goods, with $\bar{\ell}$ the agent’s total time endowment. $U$ is assumed to be strictly increasing and strictly concave, while the dis-utility of work function $\nu$ is assumed to be strictly increasing and

88
strictly convex, with \( \nu(0) = 0 \). Households are initially endowed with \( X \) units of consumption goods, which they can either consume or trade. In the dynamic version of the model, \( X \) will represent a stock of durable goods and will be endogenous. Trade in consumption goods is subject to a coordination problem because of frictions in the labor market. At the beginning of the first sub-period, the household splits up responsibilities between two members. The first member, called the buyer, goes to the goods market to make purchases. The second member searches for employment opportunities in the labor market. The goods market functions in a Walrasian fashion, with both buyers and firms taking the price of these goods \( p \) (in units of household services) as given. The market for labor in this first sub-period is subject to a matching friction, with sellers of labor searching for employers and employers searching for labor. The important information assumption is that buyers do not know, when choosing how much to buy, whether the worker member of the household has secured a match. This assumption implies that buyers make purchase decisions in the presence of unemployment risk.

There is a large set of potential consumption goods firms in the economy who can decide to search for workers in view of supplying goods to the market. Each firm can hire one worker and has access to a decreasing-returns-to-scale production function \( F(\ell) \), where \( \ell \) is the number of hours worked for the firm.\(^{21}\) Production also requires a fixed cost \( k \) in terms of the output good, so that the net production of a firm hiring \( \ell \) hours of labor is \( F(\ell) - k \). Firms search for workers and, upon finding a worker, they jointly decide on the number of hours worked and on the wage to be paid. The fixed cost \( k \) is paid before firms can look for workers. Upon a match, the determination of the wage and hours worked within a firm is done efficiently though a competitive bargaining process,\(^{22}\) so that in equilibrium \( pF' (\ell) = w \), where \( w \) is the wage, expressed in terms of household services.\(^{23}\)

The labor market operates as follows. All workers are assumed to search for employment. Letting \( n \) represent the number of firms who decide to search for workers, the number of matches \( \phi \) is then given by the short side of the market, i.e., \( \phi = \min \{ n, 1 \} \). The equilibrium condition for the goods market is then given by

\[
c - X = \phi F(\ell) - nk
\]

where the left-hand side is total purchases of consumption goods and the right-hand side is the total available supply after subtracting firms’ fixed costs. Firms enter the market up to the point where

\(^{21}\)It is also assumed that \( F \) is such that both \( F' (\ell) \ell \) and \( [F (\ell) - F' (\ell) \ell] \) are strictly increasing functions of \( \ell \). This property is exhibited, for example, by the Cobb-Douglas function \( F(\ell) = A\ell^\alpha \).

\(^{22}\)By “competitive bargaining”, I mean any bargaining process such that the equilibrium outcome satisfies (1) that workers are paid their marginal product in a match, and (2) that, conditional on being matched, workers supply and firms hire the individually-optimal number of hours at the equilibrium wage. This can be microfounded by assuming, for example, that all “matched” firms and workers meet in a secondary labor market, and that this secondary market operates in a Walrasian fashion.

\(^{23}\)As discussed in Beaudry et al. (2014), the assumption of a competitive bargaining process is for simplicity. The main mechanisms are robust to alternative bargaining protocols.
expected profits are zero. This condition can be written as

$$\frac{\phi}{n} \left[ F(\ell) - \frac{w}{p} \ell \right] = k$$

At the end of the first sub-period, a household’s net asset position \( a \), expressed in units of household services, is given by \( a = w\ell - p(c - X) \) if the worker was employed, and \( a = -p(c - X) \) if the worker was unemployed. Rather than explicitly modelling the second sub-period, for simplicity assume that the continuation value function for the second sub-period, \( V \), is given by

$$V(a) = \begin{cases} 
va & \text{if } a \geq 0 \\
(1 + \tau)va & \text{if } a < 0 
\end{cases}$$

where \( v, \tau > 0 \) are parameters. This function is piecewise linear and concave, with a kink at \( a = 0 \).

Here, the marginal value of assets is given by \( v \) when assets are positive and \((1 + \tau) v \) when assets are negative. Since buyers in general face unemployment risk when making their purchase decisions, the wedge between the marginal value of assets when in deficit and that when in surplus generates self-insurance behavior, whereby a fall in the employment rate causes buyers to reduce their purchases out of increased concern that they will end up in the costly unemployment state. This mechanism is central to the strategic complementarity that emerges in the model, which in turn is what will allow the dynamic version of the model to generate limit-cycle behavior. The strength of this mechanism, meanwhile, is governed by the parameter \( \tau \). Given the above value function \( V \), the buyer’s problem is to choose \( e \) to maximize

$$U(X + e) + \phi [-\nu(\ell) + v(w\ell - pe)] - (1 - \phi)(1 + \tau) vpe$$

subject to \( e \geq 0 \), where \( e \equiv c - X \) is purchases of new goods. The worker’s problem, meanwhile, is to choose \( \ell \) to maximize \(-\nu(\ell) + v(w\ell - pe)\).

**Equilibrium**

Letting \( e_j \) denote purchases by household \( j \) and \( e \) the average level of purchases in the economy, one may show that household \( j \)’s optimal consumption-choice decision is characterized by

$$U'(X + e_j) = p(e) v [1 + \tau - \tau \phi(e)]$$

---

24As in Beaudry et al. (2014), assume that searching firms pool their ex-post profits and losses so that they each make exactly zero profits in equilibrium, regardless of whether they are matched with a worker.

25See Section 2.2 in Beaudry et al. (2014) for a discussion of how to microfound such a value function.

26As noted in Beaudry et al. (2014), what matters here is that the marginal value of assets be smaller in surplus than in deficit. The piecewise linearity property is assumed only for tractability.

27See Beaudry et al. (2014).
where $p(\cdot)$ and $\phi(\cdot)$ are the price of consumption goods and the employment rate, respectively, expressed as functions of aggregate purchases. The left-hand side of (4.1) is simply household $j$’s marginal utility of consumption. The right-hand side, meanwhile, captures buyer $j$’s expected marginal-utility cost of funds. When the economy is at full employment ($\phi(e) = 1$), this is simply equal to the price $p(e)$ of consumption goods in terms of household services, times the marginal value $v$ of those services when assets are non-negative. When there is unemployment, however, the buyer faces some positive probability of ending up in the negative-asset state, which is associated with a higher marginal value of assets (i.e., $(1 + \tau) v$). As a result, the expected marginal-utility cost of funds is higher and, all else equal, household $j$ would choose a lower level of purchases.

An equilibrium for this economy is given by a solution to (4.1) with the additional restriction that $e_j = e$. To understand how the equilibrium is affected by shifts in $X$, note the following properties of the equilibrium functions $p(\cdot)$ and $\phi(\cdot)$. First, one may show that $\phi(e) = \min \{e/e^*, 1\}$, where $e^*$ is the output (net of fixed costs) produced per firm when there is a positive level of unemployment.\(^{28}\) Second, one may show that $p(\cdot)$ is a continuous function of $e$, with $p'(e) = 0$ for $e < e^*$, and $p'(e) > 0$ for $e > e^*$.\(^{29}\) The consequences of these two properties for the marginal-utility cost of funds (i.e., the right-hand side of (4.1)) are illustrated by the curve labelled “cost of funds” in panel (a) of Figure 4.3. For $e$ sufficiently small, the curve is downward-sloping: as $e$ rises, output is increased along the extensive labor margin, lowering the unemployment rate and making purchases feel less expensive to households. Once $e$ reaches the full-employment level $e^*$, however, additional increases in output come via the intensive labor margin, which is associated with a rising price and thus an increased cost of funds.

The two regimes—unemployment and full employment—are associated with different equilibrium responses to a rise in the endowment $X$.\(^{30}\) Panel (a) of Figure 4.3 shows the case for the unemployment regime. The economy is initially in equilibrium at the level $e_1$ of purchases, which occurs at the intersection of the cost of funds curve and the solid marginal-utility function $U'(X + e)$. A rise in the endowment by $\Delta X$ then shifts this marginal-utility function to the left by $\Delta X$ units, as represented by the dashed curve in the figure. We see that the equilibrium level of purchases falls as a result of the rise in $X$, and furthermore that it falls by more than $\Delta X$ (so that total consumption $c = X + e$ falls). This amplified response is due to the strategic complementarity that exists in the unemployment regime: a rise in the endowment causes households to reduce their demand for new goods which, via

\(^{28}\)When there is unemployment, the “min” matching function and the firm’s zero-profit condition together imply $F(\ell) - F'(\ell) \ell = k$. Since $k$ is a constant, conditional on there being unemployment this implies that $\ell = \ell^*$, where $\ell^*$ solves this equation. Output net of fixed costs is then $e^* = F(\ell^*) - k$.

\(^{29}\)Combining the household’s labor supply condition and the firm’s labor demand condition, one may obtain $p = \nu'(\ell) / [vF'(\ell)]$. As pointed out in footnote 28, when $e < e^*$ we have $\ell = \ell^*$, so that $p = p^* \equiv \nu'(\ell^*) / [vF'(\ell^*)]$. Further, once the economy achieves full employment, a rise in output must come through the intensive margin of labor (i.e., through a rise in $\ell$), which causes $p(\cdot)$ to be increasing in $e$ on $e > e^*$.

\(^{30}\)As shown in Beaudry et al. (2014), if $\tau$ is sufficiently large there may be more than one equilibrium. While this is an interesting theoretical possibility, the evidence obtained from the quantitative exercise of section 4.6, though not conclusive, gives no indication that multiple equilibria are of concern. I therefore restrict attention throughout this paper to the case where the equilibrium is unique, i.e., where $\tau$ is not too large.
an extensive labor margin adjustment, lowers the employment rate $\phi$, which in turn raises the cost of funds, causing households to reduce purchases further, further lowering the employment rate, etc.

In contrast, panel (b) of Figure 4.3 shows the same experiment but beginning from the full-employment regime. In this case, we again see that a rise in $X$ is associated with a fall in equilibrium purchases, but in this case the fall is by less than $\Delta X$ (so that total consumption rises). This damped response occurs as a result of the strategic substitutability that exists when the economy is at full employment: a rise in the endowment causes households to reduce their demand for new goods which, via an intensive labor margin adjustment, lowers hours-per-worker, which lowers the price $p$, in turn lowering the cost of funds and causing households to increase their purchases.

The sensitivity of purchases to changes in $X$ in the unemployment regime because of strategic complementarity, and the corresponding insensitivity in the full-employment regime because of strategic substitutability, will play a crucial part in generating limit cycles in the dynamic version of the model. Note also that the sensitivity of $e$ to $X$ in the unemployment regime is increasing in the steepness of the slope of the cost of funds schedule in that regime. Since this steepness in turn depends positively on the parameter $\tau$, we see that $\tau$ captures the degree of strategic complementarity in the unemployment regime.

### 4.5.2 Baseline dynamic model

Consider now a dynamic version of the above economy. Time is discrete, and each period is divided into two sub-periods, with the economy operating in each such sub-period as in the static case. The principal difference from the static model is that the stock of durable goods brought into a period is
now endogenous, accumulating according to

\[ X_{t+1} = (1 - \delta) (X_t + \gamma e_t) \]  

(4.2)

where \( X_t \) is the stock of durables brought into period \( t \) and \( e_t \) is quantity of consumption-goods purchases in period \( t \). For simplicity, I assume that a constant fraction \( \gamma \in (0, 1] \) of these purchases are durable.\(^{31}\) \( \delta \in (0, 1] \) is the depreciation rate.

The household’s labor supply decision is entirely static and therefore the same as in the previous subsection (i.e., \( \ell_t \) is chosen each period to maximize \( -\nu(\ell_t) + v(w_t\ell_t - p_t e_t) \)). Buyers, meanwhile, face a dynamic optimization problem, choosing \( c_t \) and \( e_t \) to maximize the objective function

\[
\sum_{t=0}^{\infty} \beta^t \{ U(c_t) + \phi_t[-\nu(\ell_t) + v(w_t\ell_t - p_t e_t)] - (1 - \phi_t)(1 + \tau) v p_t e_t \} \]  

(4.3)

subject to \( c_t = X_t + e_t \) and the accumulation equation (4.2), and taking \( \ell_t \) as given.\(^{32}\)

I assume that a steady state for this economy exists and is unique. As is the case in the static model, it can be verified that this is true as long as \( \tau \) is not too large.\(^{33}\) I further assume that this steady state satisfies \( \ell < \bar{\ell} \), so that the household’s time constraint is not binding at the steady state.

**4.5.3 Limit cycles in the dynamic model**

The myopic case

Conditions under which limit cycles may appear in this model can be understood most easily in the myopic case where \( \beta = 0 \). In this case, we simply have a repeated sequence of the static economy discussed in section 4.5.1, with the only linkage between them being the inherited stock of durable goods. We may characterize the equilibrium evolution of the stock of durables over time as

\[ X_{t+1} = (1 - \delta) [X_t + \gamma e_t(X_t)] = g(X_t) \]

where \( e(X_t) \) expresses the equilibrium level of purchases at date \( t \) as a function of the only state variable, \( X_t \). This equilibrium is determined entirely as it was in Figure 4.3, with the unemployment regime characterized by strategic complementarity and the full-employment regime by strategic substitutability.

\(^{31}\)In the quantitative exercise below, I will interpret “durables” as including both conventional durable goods as well as residential investment, which is conceptually similar.

\(^{32}\)In order to avoid expanding heterogeneity between individuals over time, individuals are assumed to borrow and lend via their bank account balances only within a period but not across periods. In other words, households are allowed to spend more than their income in the first sub-period of a period, but must repay any resulting debt in the second sub-period. Similar assumptions were used in Lagos and Wright (2005) and Rocheteau and Wright (2005), and more recently in Kaplan and Menzio (2014), in order to avoid having to track the asset positions of all agents in the economy over time.

\(^{33}\)See Beaudry et al. (2014). See also the comments in footnote 30, which apply equally here.
Recall the two basic conditions discussed in section 4.4 which are required to generate a stable limit cycle: (1) a locally unstable steady state, and (2) global non-explosiveness. Letting $\bar{X}$ denote the steady state level of durables, these two conditions correspond mathematically to (1) $|g'(\bar{X})| > 1$, and (2) $|g'(X)| < 1$ for $|X - \bar{X}|$ sufficiently large, where

$$g'(X) = (1 - \delta) \left[ 1 + \gamma e'(X) \right]$$

It is straightforward to verify that the second condition necessarily holds here, as follows. Suppose $X$ is sufficiently small so that the economy is in the full-employment regime. As was shown earlier, a rise in $X$ in this regime is associated with a fall in $e$, but by less than the rise in $X$, i.e., $-1 < e'(X) < 0$. Thus, $(1 - \delta) (1 - \gamma) < g'(X) < 1 - \delta$, and therefore $|g'(X)| < 1$ clearly holds. Suppose instead that $X$ is very large. In this case it can be verified that the non-negativity constraint $e \geq 0$ binds, so that $e'(X) = 0$ and therefore $g'(X) = 1 - \delta$, and thus again $|g'(X)| < 1$ holds.

Next, suppose the steady state of the system is in the unemployment regime. Then from the analysis for the static model, we know that $e'(\bar{X}) < -1$, i.e., a rise in the stock of durables leads to a more than one-for-one fall in purchases. Whether or not $e$ falls sufficiently so that the first condition for a stable limit cycle holds will depend on the strength of the complementarity in this regime, i.e., on $\tau$. For smaller values of $\tau$, i.e., those for which

$$e'(\bar{X}) > \frac{2 - \delta}{\gamma (1 - \delta)} = \kappa$$

the complementarity is relatively weak, and thus $g'(\bar{X}) > -1$. In this case, the steady state is stable, so that a limit cycle will not appear. On the other hand, for larger values of $\tau$ (i.e., those for which $e'(\bar{X}) < \kappa$), we will have $g'(\bar{X}) < -1$, and thus the steady state is unstable. In combination with the fact that the system is non-explosive (as argued above), we see that in general a stable limit cycle will emerge in this case.

The general case

The previous subsection showed that, when $\beta = 0$, limit cycles can emerge in the unemployment risk model. While the myopic case was useful for building intuition, of more general interest is whether limit cycles may occur for an arbitrary $\beta$. It is not immediately obvious that this should hold, and indeed, as a “Turnpike Theorem” (due to Scheinkman (1976)) below highlights, in a class of models widely used in the literature, for $\beta$ sufficiently close to one limit cycles cannot occur.

In particular, consider a general deterministic dynamic economy with date-$t$ state vector $z_t \in \mathbb{R}^n$. Let $\mathcal{W}(z_t, z_{t+1})$ denote the period-$t$ return function when the current state is $z_t$ and the subsequent pe-
period’s state is $z_{t+1}$.$^{35}$ The following theorem characterizes the solution to the problem of maximizing lifetime utility $\sum \beta^t W(z_t, z_{t+1})$, where $\beta$ is the discount factor.

**Turnpike Theorem. (Scheinkman (1976))** If $W$ is concave, then there exists a $\bar{\beta} < 1$ such that if $\bar{\beta} \leq \beta \leq 1$ then the steady state is unique and globally stable.$^{36}$

The key property that ensures global stability in this theorem is the assumption that $W$ is concave. Since, all else equal, fluctuations are sub-optimal when $W$ is concave, when $\beta$ is sufficiently close to one it is in general optimal to take temporarily costly action in the present in order to avoid permanent fluctuations in the future. This in turn implies global convergence to the steady state, so that limit cycles cannot occur. Concavity of $W$ is a property that holds in a wide variety of economic models that have become standard in the literature, including nearly all quantitative models of the business cycle. As we shall see, however, in the unemployment-risk model discussed above, concavity of $W$ may be violated, in which case global stability may not obtain.

As a first step in establishing the potential for limit cycles in the unemployment-risk model, the following proposition verifies that the system satisfies the second condition needed for a stable limit cycle (i.e., non-explosiveness).

**Proposition 4.1.** Given any initial endowment of durables $X_0$, $\limsup_{t \to \infty} |X_t| < \infty$.

**Proof.** All proofs in Appendix C.1.

Proposition 4.1 ensures that in the limit the system either exhibits deterministic fluctuations (such as a limit cycle) or converges to a fixed point. The following proposition establishes that, in contrast to models for which the Turnpike Theorem applies, local instability is possible in this model for an arbitrarily high discount factor.

**Proposition 4.2.** There exists parameter values and functional forms such that, for some $\bar{\beta} < 1$, if $\bar{\beta} \leq \beta < 1$ then the (unique) steady state is locally unstable.

In combination with Proposition 4.1, Proposition 4.2 confirms that there are parameter values and functional forms for which the model will generate deterministic fluctuations even if $\beta$ is arbitrarily close to one. The reasons for the failure of the Turnpike Theorem to hold for this model can be clarified as follows. Suppose the steady state of the model is in the unemployment regime, and let $W(X_t, X_{t+1})$ be a period-$t$ return function such that the solution to the problem

$$\max_{\{X_{t+1}\}} \sum_{t=0}^{\infty} \beta^t W(X_t, X_{t+1})$$

(4.4)

$^{35}$Note that, in this formulation, $W$ implicitly incorporates any constraints and static-equilibrium outcomes, so that $W(z_t, z_{t+1})$ is the equilibrium period-$t$ return conditional on the current and next-period state being $z_t$ and $z_{t+1}$, respectively.

$^{36}$For a proof and more formal statement of the theorem, see Scheinkman (1976) Theorem 3.
implements the equilibrium of the model in a neighborhood of this steady state.\textsuperscript{37} If it turns out that $W$ is concave, then the Turnpike Theorem implies that the model cannot generate limit-cycle dynamics. The following proposition establishes that in fact $W$ may not be concave.

**Proposition 4.3.** There exists parameter values and functional forms such that, in the neighborhood of an unemployment-regime steady state, $W$ is not concave.

Intuitively, non-concavity of $W$ can arise as a result of a “bunching” mechanism in the model: because unemployment risk is low, when other agents are purchasing lots of goods it is a good time for an individual agent to purchase goods. Similarly, when other agents are purchasing few goods, it is a bad time for an individual agent to buy goods. If sufficiently strong, this bunching mechanism—which arises precisely because of the strategic complementarity in the unemployment regime—leads to a tendency to have periods of high durables accumulation alternating with periods of low durables accumulation, i.e., deterministic fluctuations.

The final proposition of this section clarifies the importance of the parameter $\tau$ in controlling the strength of this bunching mechanism, and therefore in influencing whether or not the economy will be able to generate limit-cycle dynamics.

**Proposition 4.4.** For $\tau$ sufficiently close to zero, the steady state is stable.

Proposition 4.4 thus confirms that, if $\tau$ is not sufficiently large, the degree of strategic complementarity is too small to produce an unstable steady state.

### 4.6 Quantitative exercise

This section presents the main quantitative results of the paper. I estimate a version of the dynamic model discussed above, with the primary goal of establishing that it is capable of matching the key quantitative features of the hours data discussed in section 4.3.

The baseline dynamic model presented in section 4.5 was constructed with an eye toward analytical tractability. As a result, that model lacks many of the features which are known to be helpful in quantitatively matching the data, and includes several others which, while not central to the key mechanisms, turn out to be restrictive in a quantitative setting. Since the main purpose of the exercise in this section is quantitative in nature, I make several adjustments to the model designed to help it in that regard.

First, as is well known, dynamic systems with a single state variable have considerable difficulty in producing deterministic fluctuations with the basic qualitative properties that we observe in macroeconomic aggregates. In particular, deterministic fluctuations in such models tend to be erratic, with the system often jumping back and forth from one side of the steady state to the other every few periods or less. Thus, if the unemployment-risk model is to have any chance of successfully replicating key

\textsuperscript{37}An example of such a $W$ is found in the proof of Proposition 4.3.
features of the data, it will require the addition of at least one other state variable. To this end, and following much of the quantitative business cycle literature, I now assume that the household exhibits internal habit-formation in consumption,\textsuperscript{38} so that its period utility for consumption is now given by

\[ U(c_t - hc_{t-1}) \]

Here, \( h \in [0, 1) \) is a parameter controlling the degree of habit persistence.

Second, the relatively simple structure of the baseline model produces a stark dichotomy, whereby in the unemployment regime all output adjustments occur along the extensive labor margin, while in the full-employment regime all adjustments occur along the intensive margin. In order to relax this stark dichotomy, in the quantitative version of the model I allow firms to be heterogeneous in terms of their fixed costs. That is, rather than assuming that all firms have fixed cost \( k \), I assume that the \( n \)-th firm has fixed cost \( k(n) \geq 0 \), where \( k(\cdot) \) is a non-decreasing function. This will allow for the possibility of there being regions where both extensive and intensive labor margin adjustments may occur.\textsuperscript{39,40}

Third, as discussed earlier and in contrast to what is observed in the data, purely deterministic models of economic fluctuations tend to yield cycles of a constant length. This can be observed either as a very regular pattern in a plot of time series data generated from the model, or as one or more large spikes in the spectrum estimated from that data.\textsuperscript{41} One of the key contributions of this paper is to show that by introducing a relatively small amount of randomness into a limit-cycle model it becomes possible to produce realistically irregular fluctuations. To this end, I also include in the model an exogenous random TFP process, \( \tilde{\theta}_t \).\textsuperscript{42}

\textsuperscript{38}The key desirable property for a second state variable here is that it introduces momentum into the dynamics of \( X \), so that movements from a high to a low level of \( X \) and back are gradual, rather than rapid as they are when \( X \) is the only state variable. Consumption habit exhibits this property by reducing period-to-period fluctuations in household demand, with the added advantages that it maintains tractability and keeps the model as close as possible to the baseline version discussed earlier. Nonetheless, there are likely a number of other choices (e.g., adjustment costs in investment or employment) that could have been made instead and that would have delivered similar qualitative dynamics.

\textsuperscript{39}To see this, note that the marginal firm entrant must earn zero expected profit, which in the unemployment regime is equivalent to the condition \( F(l) - F'(l) \ell = k(n) \), where \( n \) is the index of the marginal entrant. A rise in the employment rate is associated with a rise in \( n \), which (weakly) increases the right-hand side of this expression. Since the left-hand side of this expression is strictly increasing in \( l \), this then implies that a rise in the employment rate is in general also associated with a rise in hours-per-worker, i.e., both extensive and intensive labor margin adjustments occur.

\textsuperscript{40}The functional form chosen for this \( k(\cdot) \) (discussed below) will nest the baseline case of a constant fixed cost. Since the parameters of this function will be estimated, the data will ultimately choose the degree to which \( k(\cdot) \) is non-constant.

\textsuperscript{41}One may show that the spectrum associated with any limit cycle is infinitely high at a countable number of points (i.e., a countable sum of Dirac delta functions), and zero everywhere else.

\textsuperscript{42}For convenience, in order to retain certain analytical properties that are helpful in a computational setting, I assume that firms’ fixed costs and households’ second-sub-period value functions also fluctuate with the TFP process. Output, fixed costs, and the value function are thus given by \( \tilde{\theta}_t F(\cdot), \tilde{\theta}_t k(\cdot), \) and \( \tilde{\theta}_t^{-1}V(\cdot), \) respectively.
4.6.1 Functional forms, calibration and estimation

Production is assumed to be of the Cobb-Douglas form

\[ F(\ell) = A\ell^{\alpha} \]

Utility over consumption (net of habit) is assumed to be of the form

\[ U(C) = aC - \frac{b}{2}C^2 \]

while disutility of labor is taken to be of the form

\[ \nu(\ell) = \frac{\nu_1}{1 + \omega}\ell^{1+\omega} \]

The fixed cost of the \( n \)-th firm is assumed to be given by

\[
k(n) = \begin{cases} 
0 & n \leq n_0 \\
\frac{n-n_0}{\eta} \bar{k} & n_0 < n < n_0 + \eta \\
\bar{k} & n \geq n_0 + \eta 
\end{cases}
\]

where \( n_0, \eta \) and \( \bar{k} \) are parameters. This function is piecewise linear with three regimes: low-\( n \) firms have fixed cost zero, high-\( n \) firms have fixed cost \( \bar{k} \), and over the intermediate range \( k \) rises linearly from zero to \( \bar{k} \).\(^{43}\) Finally, I assume the TFP process is given by

\[ \theta_t \equiv \log(\hat{\theta}_t) = \rho \theta_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \left(\frac{\sigma}{100}\right)^2) \]

Several of the model parameters were directly calibrated. In particular, I set the labor share at a standard value of \( \alpha = 2/3 \). The inverse Frisch elasticity was calibrated at the widely used level \( \omega = 1 \). I set the depreciation rate and discount factor at standard values of \( \delta = 0.025 \) and \( \beta = 0.99 \), respectively, and normalize the maximum fixed cost and scale parameter in the second sub-period value function at \( \bar{k} = 1 \) and \( v = 1 \), respectively. Finally, the fraction of purchases entering the durables stock was calibrated at \( \gamma = 0.192 \), which is the average ratio of durables to total consumption in the National Income and Product Accounts data.\(^{44}\) The remaining parameters were estimated.

Solving the model for a particular parameterization was done using the parameterized expectations (PE) approach.\(^{45}\) Given this solution, a large data set (\( T = 100,000 \) periods in length) was

\(^{43}\)Quadratic utility and the piecewise-linear form for \( k(\cdot) \) were assumed for tractability and computational efficiency. None of the key properties of the model rely on these assumptions.

\(^{44}\)As noted above, I include the conceptually-similar residential investment under the heading of “durables”. The figure of 0.192 can thus be obtained from NIPA data as the average of (Durable goods + Residential investment)/(Consumption + Residential investment) over the sample period 1960Q1-2012Q4.

\(^{45}\)See, for example, den Haan and Marcet (1990) and Marcet and Marshall (1994). Details can be found in Appendix C.2.
simulated and, after taking logs of the resulting hours series and detrending it with the same BP filter as used for the data, the spectrum of log-hours was estimated. The non-calibrated parameters were then estimated so as to minimize the average squared difference between the model spectrum and the spectrum estimated from the data (see panel (b) of Figure 4.1). Further details of the solution and estimation procedure are presented in Appendix C.2.

Estimated parameter values are reported in Table 4.1. Several things should be noted. First, the TFP process is close to the process that would be estimated directly from productivity data. For example, using John Fernald’s (2014) measure of business-sector labor productivity growth over the sample period (1960Q1-2012Q4), after cumulating, linearly detrending, and fitting an AR(1) process, one obtains a persistence estimate of 0.974 and an innovation standard deviation of 0.713%, yielding an unconditional productivity standard deviation of 3.16%. The corresponding parameters estimated for the unemployment-risk model, meanwhile, are \( \rho = 0.969 \) and \( \sigma = 0.570 \), respectively, which yields an unconditional standard deviation of 2.30%. The fact that the model only features a single shock, and that the variance of that shock in the model is, if anything, smaller than its data counterpart highlights the more general observation that models featuring deterministic fluctuations may not require the presence of large amounts of exogenous variation in order to generate empirically reasonable business cycles.

The only other parameter with a clear comparator in the data or literature is habit persistence, which is estimated here to be \( h = 0.76 \), well within the range of standard estimates obtained elsewhere in the literature. For example, Smets and Wouters (2007) report a 90% confidence interval for habit of (0.64, 0.78), while Justiniano et al. (2010) report a 90% confidence interval of (0.72, 0.84).

The remaining parameters in Table 4.1 are composed mainly of uninteresting scale parameters, and parameters for which few if any precedents exist. The parameter \( \tau \), which captures the strength of the household’s desire to reduce spending in response to a rise in unemployment risk, falls into the latter category. Given its central role in the model, however, it deserves some comment. If interpreted narrowly as a one-period financial premium on debt vis-à-vis saving, the estimate of \( \tau = 0.27 \), or 27%, clearly exceeds typical borrowing-lending spreads as reported in the literature. However, there are several reasons to think this view of \( \tau \) may be overly restrictive. First, in order to avoid significantly complicating the model, conditional on the employment rate an individual worker’s probability of being employed is assumed to be independent from quarter to quarter. If the actual employment state of an individual exhibits persistence, then considering only one-period financial costs may understate households’ desire to reduce spending in response to an increase in unemployment. Second, borrowing-lending spreads that reflect average borrowing rates faced by all households may not accurately reflect rates faced by unemployed individuals, which are likely to be higher. Third, many unemployed individuals may in fact be unable to access financial markets at all, instead being forced

---


47Similar values are obtained when using Fernald’s TFP or utilization-adjusted TFP measures instead of labor productivity.
Table 4.1: Parameter values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated Parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>12.535</td>
<td>Marginal utility of consumption, intercept</td>
</tr>
<tr>
<td>$b$</td>
<td>2.247</td>
<td>Marginal utility of consumption, slope</td>
</tr>
<tr>
<td>$h$</td>
<td>0.761</td>
<td>Habit persistence</td>
</tr>
<tr>
<td>$\nu_1$</td>
<td>13.274</td>
<td>Labor disutility scaling factor</td>
</tr>
<tr>
<td>$\tau$</td>
<td>0.270</td>
<td>Premium on debt</td>
</tr>
<tr>
<td>$A$</td>
<td>3.199</td>
<td>Constant productivity factor</td>
</tr>
<tr>
<td>$n_0$</td>
<td>0.843</td>
<td>Measure of firms with zero fixed cost</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.091</td>
<td>Measure of firms over which fixed cost is rising</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.969</td>
<td>Persistence of TFP</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.570</td>
<td>$100 \times s.d. \ of \ innovation\ to \ TFP$</td>
</tr>
<tr>
<td>Calibrated Parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.667</td>
<td>Labor share</td>
</tr>
<tr>
<td>$\omega$</td>
<td>1</td>
<td>Inverse Frisch elasticity</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.025</td>
<td>Depreciation of durables</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.99</td>
<td>Discount factor</td>
</tr>
<tr>
<td>$\bar{k}$</td>
<td>1</td>
<td>Maximum firm fixed cost</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.192</td>
<td>Fraction of purchases entering durables stock</td>
</tr>
</tbody>
</table>

to rely on costly asset liquidations and/or reduced consumption levels in order to meet their obligations, the potential for either of which may cause households to strongly reduce their desired spending. To the extent that any or all of these factors should be subsumed into $\tau$, the value estimated here may not be unreasonable.

4.6.2 Main results

To illustrate the deterministic mechanisms, I first report results obtained when shutting down the TFP shock (i.e., setting $\sigma = 0$). In particular, I first obtained the PE coefficients from the full stochastic model. The simulation results for the deterministic model were then generated using these stochastic PE coefficients, but feeding in a constant value $\theta_t = 0$ for the TFP process. In other words, agents in the deterministic model implicitly behave as though they live in the stochastic world. As a result, any differences between the deterministic and the stochastic results in this section are due exclusively to differences in the realized sequence of TFP shocks, rather than differences in, say, agents’ beliefs about the underlying data-generating process.

This is equal to the length of the sample period of the data.
the abandonment of this literature. As this exercise demonstrates, however, unreasonable cycle lengths are by no means an unavoidable property of these models. Second, notwithstanding the reasonable cycle length, it is clear when comparing the simulated data in Figure 4.4 to the actual data in Figure 4.1 that the fluctuations in the deterministic unemployment-risk model are far too regular, a shortcoming shared by many earlier models of deterministic fluctuations.

**Figure 4.4: Deterministic model**

Notes: Panel (a) shows 212-quarter simulated sample (same size as data set) of BP-filtered log(hours worked) \((\phi_t \ell_t)\) generated from the deterministic model. Initial simulated series was 252 quarters long, with first and last 20 quarters discarded after BP-filtering. Details for computation of model spectrum in panel (b) can be found in Appendix C.2.

These properties of the deterministic model—i.e., a highly regular 30-quarter cycle—can also be seen clearly in the frequency domain. Panel (b) of Figure 4.4 plots the spectrum for the deterministic model (dashed line), along with the spectrum for the data (solid line) for comparison. Consistent with the pattern in the time domain, the spectrum exhibits a peak at around 30 quarters. Further, the regularity of the cycle is manifested as a large spike in the spectrum. In contrast, the spectrum estimated from the data is much flatter.

Re-introducing the TFP shock into the model, we see a markedly different picture in both the time and frequency domains. Panel (a) of Figure 4.5 plots a 212-quarter sample of log-hours generated from the stochastic model. While clear cyclical patterns are evident in the figure, it is immediately obvious that the inclusion of the TFP shock results in fluctuations that are significantly less regular than those generated in the deterministic model, appearing qualitatively quite similar to the fluctuations found in

---

50 Note that the cycles clearly do not exactly repeat themselves. As alluded to in footnote 19, this property is due to the discrete-time formulation of the model. In a continuous-time version of the model, the cycles would necessarily repeat themselves, a direct consequence of the Poincaré-Bendixson Theorem (see, e.g., Guckenheimer and Holmes (2002), p. 44).

51 Note that the model was not re-estimated after shutting down the TFP shock. As such, there may be alternative parameterizations of the deterministic model that are better able to match the spectrum in the data.
Figure 4.1 for actual data. This is confirmed by the spectrum, which is plotted in panel (b) of Figure 4.5 alongside the data spectrum. Also plotted is a pointwise 90% simulated confidence interval from the model for data sets of the same length as the data (i.e., 212 quarters). The stochastic model clearly matches the data quite well in this dimension, including possessing a peak near 40 quarters and, as compared to the deterministic model, lacking any large spike. The good fit of the model can also be seen by looking at the autocovariance function (ACF) of hours, i.e., $\text{Cov}(L_t, L_{t-k})$, where $k$ is the lag (in quarters). Panel (a) of Figure 4.6 plots the result for the first 40 lags for both the data and model, along with pointwise 90% confidence intervals. As the figure shows, the curves lie nearly on top of one another, indicating that the model matches the data very well in this dimension also.

**Figure 4.5: Stochastic model**

To verify that the good fit of the spectrum is not driven by the choice of filter, Figure 4.7 plots the data and model spectra for hours under four alternative filtering choices. Panels (a)-(c) present results for three alternative band-pass filters with different upper bounds (100, 60, and 40 quarters, respectively), while panel (d) plots spectra using a Hodrick-Prescott filter with parameter 1600. As the figure shows, the model fits the data very well in all cases.

---

52 That is, if the model were the true data-generating process, then at each periodicity the spectrum estimated from the data would lie inside the confidence interval 90% of the time.

53 Note that the ACF is simply the inverse Fourier transform of the spectrum. Since the spectrum of the model and data are similar, we would expect the ACF to be similar as well, a property clearly verified in Figure 4.6.

54 Note that the model spectra were obtained using the baseline model parameters as reported in Table 4.1.
Next, it should be emphasized that the exogenous shock process in this model primarily accelerates and decelerates the endogenous cyclical dynamics, causing significant random fluctuations in the length of the cycle while only modestly affecting its amplitude. For example, in the deterministic version of the model the standard deviation of log-hours is 0.026, while in the stochastic model it is 0.033, implying that 79% of the standard deviation of hours is due to deterministic mechanisms. In contrast, if this TFP process were the only shock process operating in the widely-cited model of Smets and Wouters (2007), for example, it would generate a standard deviation of log-hours of only 0.005. This again suggests the more general point that, if one is willing to consider the class of models capable of generating deterministic fluctuations, then a very parsimonious set of shocks that are small
Notes: Each panel plots corresponding data (solid) and model (dashed) spectrum using the reported filter instead of the baseline BP(2,80) filter. Dotted lines show pointwise 90% confidence intervals for the spectrum that would be estimated from a model-generated data set of the same length as the actual data set (i.e., 212 quarters).

in magnitude can potentially yield qualitatively and quantitatively reasonable fluctuations.

As a final exercise in this section, it is worth briefly further comparing the above results to those of Smets and Wouters (2007). Their model has received much attention in the literature for its ability to fit well a number of key macroeconomic data series. Panel (a) of Figure 4.8 shows the spectrum for hours worked as generated by the Smets and Wouters (2007) model at the reported median posterior parameter values. As suggested by the relatively close fit, their model also matches patterns in the hours data reasonably well, though not quite as well as the unemployment-risk model.\footnote{This should not be too surprising, as the unemployment-risk model was estimated to match only the hours series, while the Smets and Wouters (2007) was estimated to simultaneously match seven different data series (including hours).}

More insight into the drivers of fluctuations in the Smets and Wouters (2007) model can be obtained by looking at a spectral variance decomposition; that is, by decomposing the total variance at
each individual periodicity into the portions that are attributable to each of the shocks in that model. Panel (b) of Figure 4.8 presents such a decomposition. It is clear from the figure that, in the range of periodicities responsible for the bulk of the variance of hours, the two mark-up shocks (price and wage) in the Smets and Wouters (2007) model account for by far the largest portion. In fact, the proportion of the total hours variance that is explained by the mark-up shocks rises monotonically with periodicity, explaining around a third of the variance of hours by the 24-quarter periodicity and over half by the 36-quarter periodicity.\footnote{The importance of the mark-up shocks is not exclusive to hours within the Smets and Wouters (2007) model. For example, as reported in that paper, at a 40-quarter horizon the mark-up shocks together account for over half of the forecast-error variance (FEV) of output and over 80\% of the FEV of inflation.} In contrast, the unemployment-risk model presented here is equally capable of matching the spectrum in hours, but does so with only a reasonably-sized TFP shock and without relying on poorly motivated mark-up shocks.

### 4.6.3 Additional results

To this point, I have focused on the fit of the model with respect to the target series, hours worked. In this subsection, I evaluate how well the model performs in several other dimensions that were not directly targeted.

---

\textit{Notes:} Data spectrum is as in Figure 4.1. Spectrum for Smets-Wouters (SW) obtained by simulating 10,000 data sets of the same size as the actual data series. For each simulation, the data was de-trended and the spectrum estimated using the same procedures as for the actual data. A point-wise average was taken across all simulated spectra. Because the hours series used by SW for their estimation differs somewhat from the series used here, for purposes of comparability, in panel (a) the SW spectrum was scaled by a constant so that the total variance is the same as in the data. Panel (b) shows portion of variance at each periodicity attributable to each of the following shock groupings: “Mark-up” – price and wage mark-up shocks; “Bond Premium” – bond premium shock; “Technology” – TFP and investment-specific technology shocks; “Monetary policy” – monetary policy shock; “Gov’t spending” – government spending shock.
Panel (a) of Figure 4.9 compares the spectrum of output for the data and the stochastic model. As shown in the figure, the model spectrum matches the data reasonably well, though it is somewhat too large (indicating too much output variance in the model), and the average periodicity is somewhat too low. The second observation should not be too surprising, as the model does not include capital as a factor of production. Since productive capital tends to exhibit lower-frequency fluctuations than labor (the other factor of production), all else equal its omission from the model will cause the average periodicity of output to be too small. Panel (d) of Figure 4.6, meanwhile, plots the ACF for output, which confirms the first observation: the variance of output in the model (i.e., the autocovariance at lag $k = 0$) is slightly larger than in the data. Notwithstanding this, however, the spectrum and ACF for output in the data lies well within a 90% confidence interval for the model, suggesting a relatively good overall fit.

Next, panel (b) of Figure 4.9 plots the coherence between hours and output for the data and for the stochastic model. Coherence is analogous to a regression $R^2$, giving the proportion of the variance of hours that can be linearly predicted by output at a given periodicity. A coherence of one would thus indicate that hours and output are perfectly correlated at that periodicity, while a coherence of zero would indicate that hours and output are orthogonal. In the data (solid line in the figure), we see that at the lowest periodicities hours and output are modestly correlated, with coherence around 0.4-0.5.

As the periodicity rises, the coherence initially increases relatively rapidly, reaching a peak of 0.87 at around 13 quarters. Over this range, as indicated by the dashed line in the figure the model coherence matches the data very well. Beyond the 13-quarter periodicity, however, the data and model begin to diverge somewhat. The data coherence largely flattens out, with a gradual downward slope, reaching 0.82 at the 80-quarter periodicity. The model coherence, meanwhile, rises somewhat over this range. As with the spectrum of output, the discrepancy between the data and model coherences at higher periodicities can be explained by the lack of productive capital in the model. Notwithstanding this discrepancy, however, the basic qualitative properties of the relationship between hours and output in the data—namely, moderate correlation at higher frequencies but significant correlation at medium- to low frequencies (including the range of frequencies in which the bulk of variation occurs)—are well-captured by the model.

While coherence measures the strength of the relationship between two series at a given periodicity, it provides no information about the sign of this relationship or whether one series tends to lead the other. To address how well the model fits in these dimensions, panels (b) and (c) of Figure 4.6 plot the

---

57 Data series for output is the log of nominal GDP, deflated by population and the GDP deflator, then de-trended using a BP(2,80) filter using the same procedure as with hours worked (see section 4.3). Output in the model is the sum of wage earnings and firm profits, which is equal to total production net of fixed costs, i.e., $\hat{\theta}_t \left[ \phi_t F(\ell_t) - \int_0^{\ell_t} k(x) \, dx \right]$, where $n_t$ is the number of firm entrants at date $t$.  

58 The coherence at a periodicity $P$ is given by $|s_{L,y}(P)|^2 / \left[ s_L(P) \ s_y(P) \right]$, where $s_L$ is the spectrum of hours, $s_y$ is the spectrum of output, and $s_{L,y}$ is the cross-spectrum.  

59 Including capital would tend to reduce the coherence between output and hours by introducing another factor of production which is imperfectly correlated with hours. Since fluctuations in capital tend to be much more important at higher periodicities, the coherence would tend to fall by more at the upper end of the range of periodicities.
Figure 4.9: Spectrum: Output (data and stochastic model)

(a) Output Spectrum

(b) Hours–Output Coherence

(c) Empl., Hrs./Worker Spectrum

Notes: Data series for output is the log of nominal GDP, deflated by population and the GDP deflator. Data series for the employment rate is the log of the BLS’s index of nonfarm business employment divided by population. Data series for hours-per-worker is the log of nonfarm business hours divided by nonfarm business employment. All series were de-trended using a BP(2,80) filter using the same procedure as with hours worked. Output in the model is the sum of wage earnings and firm profits, which is equal to total production net of fixed costs, i.e., $\tilde{\theta}_t [\phi_t F (\ell_t) - \int_0^{\ell_t} k (x) dx]$, where $n_t$ is the number of firm entrants at date $t$. Spectrum for data and model computed as with hours. Raw coherence at a periodicity $p$ is given by $|s_{L,y} (p)|^2 / [s_L (p) s_y (p)]$, where $s_L$ is the spectrum of hours, $s_y$ is the spectrum of output, and $s_{L,y}$ is the cross-spectrum. Coherence was then kernel-smoothed using a Hamming window with bandwidth parameter 51. In panels (a) and (b), dotted lines show pointwise 90% confidence intervals for the spectrum and coherence, respectively, that would be estimated from a model-generated data set of the same length as the actual data set (i.e., 212 quarters).

cross-covariance function (CCF) for hours and output. Two things should be noted from these plots. First, hours and output are positively correlated in both the model and data. Second, in the model hours and output are in phase (i.e., the peak of the CCF occurs at a lag of $k = 0$), while in the data the peak occurs at the point where output leads hours by one quarter. Nonetheless, the CCF is close to flat.
in the data between its peak and $k = 0$, suggesting that any lead of output is weak at best. Further, as suggested by the reported 90% confidence intervals, over all the cross-covariance between output and hours is well-captured by the model.

Finally, while we have established that the model does a good job of matching patterns in total hours, consider the model’s implications for its two component parts, the employment rate, $\phi_t$, and hours-per-worker, $\ell_t$. Panel (c) of Figure 4.9 shows spectra for the data and stochastic model for these two series. From the figure, we see that the spectrum of the employment rate from the model matches fairly well the one from the data, and in particular the employment rate exhibits an overall level of volatility that is close to the volatility in the data. Thus, this model addresses one of the frequent criticisms of many models of unemployment in the literature, which is that they generate too little employment volatility.

On the other hand, the model does a relatively poor job of matching behavior in hours-per-worker. In particular, while the basic pattern of the model spectrum is close to that in the data, the model spectrum is in most places too small, especially beyond the lowest periodicities. This suggests that the model features too little in the way of movements along the intensive labor margin. To understand why, recall that when the economy moves into a region where the fixed-cost function $k(\cdot)$ is increasing, forces come into play which cause output fluctuations to occur on both intensive and extensive labor margins. Recall also that the former are associated with strategic substitutability (through changes in the price of goods), while the latter are associated with strategic complementarity (through changes in unemployment risk). If a given change in output occurs too much along the intensive margin (as is the case for this parameterization of the model), the associated strategic substitutability tends to push the economy back towards the steady state quickly, so that any change in hours-per-worker is relatively small and short-lived.

4.6.4 Multiple equilibria and indeterminacy

In the estimation exercise conducted above, I only considered parameter combinations for which (a) there exists a unique steady state, and (b) the probability of having multiple static equilibria (i.e., multiple equilibria in a period, conditional on the current state and on agents’ beliefs about the future) was negligible. As mentioned briefly above, these two constraints can be expressed as upper

---

60The peak of the data CCF is only 0.28% greater than it is at $k = 0$.
61Data series for the employment rate is the log of the BLS’s index of nonfarm business employment divided by population. Data series for hours-per-worker is the log of nonfarm business hours divided by nonfarm business employment. Both series were de-trended using a BP(2,80) filter using the same procedure as with hours worked.
63As Figure 4.9 shows, extensive-margin fluctuations are an order of magnitude larger than intensive-margin fluctuations in both the model and the data. As a result, even though the model does not capture well the intensive-margin fluctuations, this has little impact on the fit of total hours, which is driven primarily by extensive-margin fluctuations.
64One way to increase the variance of hours-per-worker is thus to have the upward-sloping part of the fixed-cost function be less steep. Since hours-per-worker was not a target of the estimation algorithm, however, there is no reason why it should have favored a flatter $k(\cdot)$. Improving the fit of the model in this dimension by including hours-per-worker information as part of the estimation objective function is a task for future work.
bounds on $\tau$. Intuitively, multiple steady states and multiple equilibria may arise in this model if the strategic complementarity between agents’ actions is too strong. Since $\tau$ governs the strength of this complementarity, ruling out multiple equilibria is equivalent to limiting the size of $\tau$.

In particular, define

$$\tau^* \equiv \frac{\alpha \bar{b}}{(1 - \alpha) \upsilon \pi^*}$$

$$\bar{\tau} \equiv \frac{\left[1 - \beta (1 - \gamma) (1 - \delta)\right] \left[(1 - \delta) \gamma + \delta\right] (1 - \beta h) (1 - h) \tau^*}{[1 - \beta (1 - \delta)] \delta}$$

The following proposition characterizes sufficient (though not necessary) conditions under which the steady state and static equilibria are unique.

**Proposition 4.5.** The steady state of the unemployment-risk model is unique if $\tau < \bar{\tau}$. The period-$t$ static equilibrium is unique if $\tau < \tilde{\theta}_t^2 \tau^*$.

Ex ante, it is not clear whether imposing the constraints on $\tau$ from Proposition 4.5 is restrictive in practice. The results from the estimation reported above, however, give no indication that these constraints are binding. In particular, at the parameter values reported in Table 4.1, we have $\bar{\tau} = 2.61$ and $\tau^* = 0.82$, both well above the value of $\tau = 0.27$. Clearly, the constraint ensuring a unique steady state is not binding at the optimal parameter values. The constraint ensuring a static equilibrium, meanwhile, depends on the level of productivity $\tilde{\theta}_t$, which can in principle be arbitrarily small, and thus the constraint may be violated with strictly positive probability. Nonetheless, given the size of the estimated TFP shock, this probability is negligible in practice. For example, in 100,000 simulated periods, the smallest value of $\tilde{\theta}_t^2 \tau^*$ that occurred was 0.69, still more than twice the value of $\tau$.

While there are relatively simple analytical conditions that can be obtained to ensure uniqueness of the steady state and of static equilibrium, verifying dynamic determinacy—that is, the presence of a unique path converging to the limit cycle for a given initial state—is more challenging, since no analytical results are available in general. Nonetheless, in numerical simulations I was unable to find any evidence of indeterminacy. In particular, given initial values for the state variables and arbitrary initial values for the jump variables (chosen in practice from some neighborhood of the PE solution), one may simulate a non-stochastic version of the model forward.\(^{65}\) If, for a given initial state, the system were to converge to the limit cycle for multiple combinations of initial jump variables, this would indicate the presence of indeterminacy. However, performing this experiment many times beginning from different initial conditions, in all cases the system eventually exploded, which suggests to me that indeterminacy is not likely to be an issue here.

### 4.7 Conclusion

Conventional models of the business cycle usually feature fluctuations driven exclusively by exogenous shocks around a unique stable steady state. In these models, booms and subsequent recessions are

\(^{65}\)See Appendix C.4 for details.
typically unrelated, each being driven by different independent realizations of the underlying shocks. A contrasting view is that booms and busts are inherently related, with a boom sowing the seeds of a subsequent bust, which then sets the stage for the next boom. This second view received some attention in an older literature, but formal attempts to model underlying mechanisms appear to have been largely abandoned due to the perception that implausible assumptions or parameter values were necessary to generate quantitatively reasonable fluctuations.

In this paper, I present a purely deterministic general-equilibrium model featuring strategic complementarity near the steady state and show that it can give rise to a stable limit cycle. The limit cycle arises through a simple micro-founded mechanism in a rational-expectations environment. Cycles emerge endogenously, and thus the model does not require shocks in order to generate fluctuations, nor does it rely on the existence of multiple equilibria or dynamic indeterminacy.

Since cycles would indefinitely repeat themselves in the absence of shocks, a TFP shock is introduced into the model in order to create irregularities. The model is then estimated to match the spectrum of US hours. In contrast to results suggested in the earlier literature, I find that the model is able to match this spectrum quite closely. The TFP shock in the model is also shown to be of a reasonable persistence and relatively small size, accounting for around a fifth of the standard deviation of hours in the model. This result highlights the important insight that models capable of generating deterministic fluctuations do not require the addition of large, persistent, poorly-motivated shocks in order to match the patterns in the data, which is a common criticism of conventional models.
Chapter 5

Conclusion

In this thesis, I have contributed to the literature devoted to understanding the causes and consequences of business cycles. In the first chapter, I sought to answer the questions, What effect does misspecification have on shock variance estimates in a DSGE model? Does this effect depend on the severity of the misspecification? If misspecification is detected, can some correction be made to the variance estimates? To answer these questions, I first developed a novel framework that allows a particular DSGE model to be compared in a meaningful and well-defined way to the “true” (but unknown) model. Using this framework, I then showed that if a DSGE model is correctly specified, then the smoothed shocks should follow a vector white noise process with diagonal covariance matrix, and that if an observed process for the smoothed shocks does not possess this property, then the model must be misspecified. I further showed that if the model is misspecified then the shock variance estimates are biased upward, before proposing a simple procedure to correct, in part, for this bias. Finally, I applied this framework and methodology to a recent paper by Justiniano et al. (2010), and found that at least one-third of the variance of the investment shock—the leading driver of business cycle fluctuations in their model—can be attributed to misspecification.

In the second chapter, my co-authors and I proceeded from three observations. First, most deep recessions arise after periods of fast accumulation of capital goods, suggesting that recessions may often reflect periods of liquidation, a viewpoint often associated with the economist Friedrich Hayek. Second, recessions appear to be socially painful phenomena, suggesting that there are mechanisms at play that are causing the economy to function especially inefficiently, a viewpoint typically associated with John Maynard Keynes. Finally, even when monetary authorities try to counter such recessions by easing policy, this does not seem to be fully effective, suggesting that nominal rigidities may not be the only important source of inefficiency.

Our paper has offered a framework that is consistent with these three observations, and that accordingly provides an environment where the policy trade-offs inherent to the Hayekian and Keynesian views can be discussed. The narrative underpinning the model is quite straightforward. When the economy inherits a high level of capital, this decreases the desire for trade between agents in the
economy, leading to less demand. When there are fixed costs associated with employment, this will generally lead to an increase in unemployment. If the risk of unemployment cannot be entirely insured away, households will react to the increased unemployment by increasing saving and thereby further depressing demand. This multiplier process will cause an excess reaction to the inherited goods and can be large enough to make society worse off even if—in a sense—it is richer since it has inherited a large stock of goods. Within this framework, we showed that policies aimed at stimulating activity will face an unpleasant trade-off, as the main effect of stimulus will simply be to postpone the adjustment process. Nonetheless, we find that such stimulative policies may remain desirable even if they postpone recovery, but these gains do not increase simply because the rate of unemployment is higher.

In the final chapter, I have investigated the potential role of deterministic mechanisms in generating business cycle fluctuations. Such mechanisms received some attention in an older literature, but formal attempts to incorporate them into the standard macroeconomic modelling paradigm appear to have been largely abandoned due to the perception that implausible assumptions or parameter values were necessary to generate quantitatively reasonable fluctuations. In this paper, I have shown that this need not be the case. In particular, I presented a purely deterministic general-equilibrium model featuring strategic complementarity near the steady state and showed that it can easily give rise to limit cycles under plausible parameter values. Furthermore, adding a simple and reasonably-sized source of exogenous variation (in the form of a TFP shock) to the model, I found that the model is able to match data on US business cycles very well. This result highlights the important insight that models capable of generating deterministic fluctuations do not require the addition of large, persistent, poorly-motivated shocks in order to match the patterns in the data, which is a common criticism of conventional models.
Bibliography


Challe, Edouard and Xavier Ragot (2013). “Precautionary Saving over the Business Cycle.” PSE Working Papers hal-00843150, HAL.


Appendix A

Appendix for “Misspecification and the Causes of Business Cycles”

A.1 Model details for Example 3 in Section 2.5 (Smets and Wouters (2007))

The equations for the Smets and Wouters (2007) model are reproduced here for convenience. See Smets and Wouters (2007) for specifics regarding the model set-up and the derivation of these equations. The resource constraint:

\[ y_t = c_y c_t + i_y i_t + z_y z_t + \varepsilon_t^g \]

The consumption Euler equation:

\[ c_t = c_1 c_{t-1} + (1 - c_1) E_t c_{t+1} + c_2 (l_t - E_t l_{t+1}) - c_3 \left( r_t - E_t \pi_{t+1} + \varepsilon_t^b \right) \]

The investment Euler equation:

\[ i_t = i_1 i_{t-1} + (1 - i_1) E_t i_{t+1} + i_2 q_t + \varepsilon_t^i \]

The value of the capital stock evolves according to:

\[ q_t = q_1 E_t q_{t+1} + (1 - q_1) E_t r_{t+1}^k - \left( r_t - E_t \pi_{t+1} + \varepsilon_t^b \right) \]

The aggregate production function equation:

\[ y_t = \phi_p \left[ \alpha k_t^s + (1 - \alpha) t + \varepsilon_t^q \right] \]

The capital utilization equation:

\[ k_t^s = k_{t-1} + z_t \]
The capital utilization first-order condition:

\[ z_t = z_1 r_t^k \]

The capital accumulation equation:

\[ k_t = k_1 k_{t-1} + (1 - k_1) i_t + k_2 e_t^i \]

The intermediate goods firm’s real mark-up:

\[ \mu_t^p = \alpha (k_t^s - l_t) + e_t^p - w_t \]

The New Keynesian Phillips curve:

\[ \pi_t = \pi_1 \pi_{t-1} + \pi_2 \pi_{t+1} - \pi_3 \mu_t^p + e_t^p \]

The firm’s capital-labor ratio:

\[ k_t^s - l_t = w_t - r_t^k \]

The labor market real mark-up:

\[ \mu_t^w = w_t - \sigma l_t - \frac{\gamma}{\gamma - \lambda} c_t + \frac{\lambda}{\gamma - \lambda} c_{t-1} \]

The wage Phillips curve:

\[ w_t = w_1 w_{t-1} + (1 - w_1) (E_t w_{t+1} + E_t \pi_{t+1}) - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w + e_t^w \]

Finally, the Taylor rule:

\[ r_t = \rho r_{t-1} + (1 - \rho) [r_\pi \pi_t + r_y (y_t - y_p^\rho)] + r_\Delta y \left [ y_t - y_t^p - y_{t-1} + y_{t-1}^p \right ] + e_t^r \]

Above, \( y_t \) is output, \( c_t \) is consumption, \( i_t \) is investment, \( z_t \) are capital utilization costs, \( l_t \) is hours worked, \( k_t \) is the end-of-period capital stock, \( k_t^s \) is employed capital, \( q_t \) is the shadow value of capital, \( \mu_t^p \) is the intermediate goods firms’ real mark-up, \( r_t^k \) is the rental rate on capital, \( w_t \) is the real wage rate, \( \mu_t^w \) is the real wage mark-up, \( r_t \) is the nominal interest rate, \( \pi_t \) is inflation from date \((t-1)\) to date \(t\), and \( y_p^\rho \) is potential output (as obtained from the RBC version of the model). All variables are log-linearized around the deterministic trend.
The RBC version of the model is obtained by setting
\[ \xi_w = 0 \]
\[ \xi_p = 0 \]
and removing the risk premium, two mark-up, and monetary policy shocks. All other features of the DGP remain unchanged, including all of the real frictions and the monopolistically competitive intermediate goods market. The New Keynesian Phillips curve, wage Phillips curve and Taylor rule are indeterminate and thus dropped from the system, while the intermediate goods and labor market mark-up equations are replaced, respectively, with
\[ 0 = \alpha (k_e^s - l_t) + \varepsilon_t^o - w_t \]
\[ 0 = w_t - \sigma_l l_t - \frac{\gamma}{\gamma - \lambda} c_t + \frac{\lambda}{\gamma - \lambda} c_{t-1} \]
The consumption and marginal value of capital Euler equations are also combined to obtain
\[ c_t = c_1 c_{t-1} + (1 - c_1) c_t^e + c_2 (l_t - l_t^e) - c_3 q_1 q_t^e - c_3 (1 - q_1) r_t^{ke} + c_3 q_t \]

We have thus reduced the system by four equations, and correspondingly drop four variables: \( r_t, \pi_t, \mu^p_t \) and \( \mu^w_t \).

A.2 Proofs of theoretical results

Proof of Proposition 2.1

From (2.16)-(2.18), we can write
\[ \varepsilon^*_t = (FB)^{-1} (Y_t - Z_t) - (FB)^{-1} FAX^*_t \]
(A.1)

Substituting this into equation (2.18) for \( \varepsilon^*_t \) yields
\[ X^*_t = CX^*_t - 1 + B (FB)^{-1} (Y_t - Z_t) \]
(A.2)

Next, we may write \( C = QJQ^{-1} \), where \( J \) is the Jordan normal form of \( C \) and \( Q \) is a matrix whose columns are the corresponding generalized eigenvectors of \( C \). In particular, \( J \) is an \( n \times n \) matrix with the eigenvalues of \( C \) on the main diagonal ordered by increasing modulus, and, if there is a non-zero non-diagonal entry, then it is equal to one, lies immediately above the main diagonal, and satisfies that the entry immediately to the left of it is equal to the entry immediately below it. Note that \( Q \) and \( J \) are functions of \( A, B \) and \( F \) only. Note also that, since \( J \) is an upper-triangular matrix, its
eigenvalues are its diagonal elements, and thus it has the same eigenvalues as \(C\). Under Assumption 2.4, \(J\) can therefore be partitioned as
\[
J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}
\]
where \(J_1\) is an \(n_1 \times n_1\) matrix having eigenvalues strictly less than one in modulus, and \(J_2\) is an \(n_2 \times n_2\) matrix having eigenvalues strictly greater than one in modulus, with \(n_1 + n_2 = n\). Note that \(J_2\) is non-singular by construction. Partition \(Q\) and \(Q^{-1}\) conformably as
\[
Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_1 & Q_2 \end{pmatrix}
\]
\[
Q^{-1} = \begin{pmatrix} Q_1^{-1} & Q_2^{-1} \\ Q_1^{-1} & Q_2^{-1} \end{pmatrix}
\]
Letting \(x_t \equiv Q^{-1}X_t^*\) and \(y_t \equiv Q^{-1}B (FB)^{-1} (Y_t - Z_t)\), premultiplying equation (A.2) by \(Q^{-1}\) yields
\[
x_t = Jx_{t-1} + y_t
\]
or, partitioning \(x_t\) and \(y_t\) conformably with \(J\),
\[
\begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} = \begin{pmatrix} J_1 x_{1,t-1} \\ J_2 x_{2,t-1} \end{pmatrix} + \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}
\]
We may thus write
\[
x_{1,t} = \sum_{j=0}^{\infty} J_1^j L^j y_{1,t}
\]
and
\[
x_{2,t} = \sum_{j=-\infty}^{-1} J_2^j L^j y_{2,t}
\]
where convergence of the sums follows from the fact that the eigenvalues of \(J_1\) and \(J_2^{-1}\) are all less than one in modulus. Thus
\[
X_t^* = Qx_t
\]
\[
= \sum_{j=0}^{\infty} Q_1 J_1^j Q_1 B (FB)^{-1} L^j (Y_t - Z_t) + \sum_{j=-\infty}^{-1} Q_2 J_2^j Q_2 B (FB)^{-1} L^j (Y_t - Z_t)
\]
Backing this expression up one period and substituting into equation (A.1) for \(X_{t-1}\) yields
\[
\varepsilon_t^* = \sum_{j=-\infty}^{\infty} \psi_j L^j (Y_t - Z_t)
\]
where

\[
\psi_j = \begin{cases} 
-(FB)^{-1} FAQ_2 J_2^{-1} Q^2 B (FB)^{-1} & \text{if } j < 0 \\
(FB)^{-1} \left[ I_m - FAQ_2 J_2^{-1} Q^2 B (FB)^{-1} \right] & \text{if } j = 0 \\
-(FB)^{-1} FAQ_1 J_1^{-1} Q^1 B (FB)^{-1} & \text{if } j > 0 
\end{cases}
\]

Again, since the eigenvalues of \( J_1 \) and \( J_2^{-1} \) are all strictly less than one in modulus, the sequence \( \{\psi_j\} \) is absolutely summable.

Finally, note that

\[
X_t^* = \sum_{i=0}^{\infty} A^i B L^i \varepsilon_t^*
\]

and thus

\[
\left( \sum_{i=0}^{\infty} FA^i B L^i \right) \varepsilon_t^* = Y_t - Z_t \quad (A.3)
\]

Pre-multiplying (A.3) by \( \sum_{j=-\infty}^{\infty} \psi_j L^j \) yields

\[
\left( \sum_{j=-\infty}^{\infty} \psi_j L^j \right) \left( \sum_{i=0}^{\infty} FA^i B L^i \right) \varepsilon_t^* = \sum_{j=-\infty}^{\infty} \psi_j L^j (Y_t - Z_t) = \varepsilon_t^*
\]

which confirms the first equality in (2.20). Substituting (A.3) into (2.19) for \( (Y_t - Z_t) \) yields

\[
\varepsilon_t^* = \left( \sum_{j=-\infty}^{\infty} \psi_j L^j \right) \left( \sum_{i=0}^{\infty} FA^i B L^i \right) \varepsilon_t^*
\]

which confirms the second equality in (2.20).

\[\square\]

**Proof of Proposition 2.2**

Since the EM can be inverted by Proposition 2.1, Proposition 2.2 follows directly from (2.13) and the subsequent discussion.

\[\square\]

**Proof of Corollary 2.1**

Part (a) follows obviously from independence of \( \varepsilon_t^* \) and \( \nu_t \). To see part (b), suppose the EM is correctly specified, so that \( \text{Var}(Z_t) = 0 \). Then \( \hat{\varepsilon}_t = \varepsilon_t^* \) and therefore \( \mathbb{E}[\hat{\varepsilon}_{l,t} \hat{\varepsilon}_{i,s}] = \mathbb{E}[\varepsilon_{l,t}^* \varepsilon_{i,s}^*] = 0 \) for all \( (l, t) \neq (i, s) \). To see part (c), suppose the EM is misspecified. Then \( \text{Var}(Z_t) \neq 0 \) and therefore there exists some \( l \) such that \( \text{Var}(\nu_{i,t}) \neq 0 \). From Proposition 2.2, this implies that \( \tilde{\sigma}_l^2 > \sigma_l^* \). Finally, part (d) follows from independence of \( \varepsilon_{l,t}^* \) on the one hand and both \( \varepsilon_{i,s}^* \) and \( \nu_{i,s} \) on the other.
Proof of Proposition 2.3

By independence of $\varepsilon_{l,t}^*$ and $\nu_{l,t}$ (Proposition 2.2) and of $\varepsilon_{l,t}^*$ and $\hat{\varepsilon}_{i,s}$ for $(l, t) \neq (i, s)$ (Corollary 2.1(d)), we have

$$
E \left[ \xi_{l,t}^2 \right] = \sigma_l^* + E \left[ \left( \nu_{l,t} - \sum_{i \neq l} \Theta_{i,0} \hat{\varepsilon}_{i,t} - \sum_{j=1}^q (\Theta_{j} \hat{\varepsilon}_{t-j} + \Theta_{-j} \hat{\varepsilon}_{t+j}) \right)^2 \left( \nu_{l,t} - \sum_{i \neq l} \Theta_{i,0} \hat{\varepsilon}_{i,t} - \sum_{j=1}^q (\Theta_{j} \hat{\varepsilon}_{t-j} + \Theta_{-j} \hat{\varepsilon}_{t+j}) \right) \right] \geq \sigma_l^* ^2
$$

The second part of the Proposition (i.e., that $E \left[ \xi_{l,t}^2 \right] \leq \hat{\sigma}_l^2$) follows directly from standard regression results.

A.3 Data for JPT model

Quarterly U.S. data spanning 1954QIV to 2009QI was constructed for seven observable variables: real per capita GDP growth, real per capita consumption growth, real per capita investment growth, (log of) per capita hours, real wage growth, the inflation rate, and the nominal interest rate.

Raw data were obtained from several different sources. NIPA data were obtained from the Bureau of Economic Analysis. Data on the federal funds rate and population (civilian noninstitutional) were obtained from the St. Louis Fed’s FRED database. An index of hours worked in the non-farm business sector was obtained from the Bureau of Labor Statistics, as was an index of compensation in the non-farm business sector.

Real per capital GDP was constructed as nominal GDP divided by the GDP deflator and population. Real per capita consumption was constructed as nominal purchases of non-durable goods and services, divided by the GDP deflator and population. Real per capita investment was constructed as nominal gross private domestic investment plus purchases of durable goods, divided by the GDP deflator and population. Per capita hours was constructed as the hours index divided by population, and normalized by a constant factor chosen so that the average of the log of the series over the period 1954QIV-2004QIV was zero. The real wage was constructed as the compensation index divided by the GDP deflator. The inflation rate was constructed as growth rate of the GDP deflator. Finally, the nominal interest rate was constructed as one-quarter of the federal funds rate in annual terms.
A.4 BCF variance decomposition

Let $\Phi_k \equiv \mathbb{E} [X_t X_{t-k}']$ denote the $k$-th autocovariance of $X_t$ in the EM of Section 2.4.1. The autocovariance-generating function (AGF) for $\{X_t\}$ is given by

$$G_X(z) = (I_n - Az)^{-1} B \Sigma B' (I_n - A' z^{-1})^{-1}$$

where $\Sigma \equiv \mathbb{E} [\varepsilon_t \varepsilon_t']$. The spectrum for $\{X_t\}$ at frequency $\omega$ is given by

$$s_X(\omega) = (2\pi)^{-1} G_X(e^{-i\omega}) = (2\pi)^{-1} (I_n - Ae^{-i\omega})^{-1} B \Sigma B' (I_n - A'e^{i\omega})^{-1}$$

(A.4)

where $i \equiv \sqrt{-1}$. It can be shown that, for all integers $k$

$$\Phi_k = \int_{-\pi}^{\pi} s_X(\omega) e^{i\omega k} d\omega$$

Since we are interested in doing a decomposition of the unconditional variances of the endogenous variables, of particular interest is the case where $k = 0$. i.e.

$$\Phi_0 = \int_{-\pi}^{\pi} s_X(\omega) d\omega$$

(A.5)

Let $s_{X}^{ij}(\omega)$ denote the $jj$-th element of the spectrum. Equation A.5 implies that the integral of $s_{X}^{ij}(\omega)$ over all frequencies $\omega \in [-\pi, \pi]$ yields the unconditional variance of $X_{j,t}$ (the $j$-th element of $X_t$).

For $\omega_0 \in [0, \pi]$, we can interpret $\int_{-\omega_0}^{\omega_0} s_{X}^{ij}(\omega) d\omega$ as the portion of the unconditional variance of $X_{j,t}$ attributable to fluctuations with frequencies between 0 and $\omega_0$. For $0 \leq \omega_0 \leq \omega_1 \leq \pi$, define

$$v_j(\omega_0, \omega_1) \equiv \int_{-\omega_1}^{\omega_1} s_{X}^{ij}(\omega) d\omega - \int_{-\omega_0}^{\omega_0} s_{X}^{ij}(\omega) d\omega$$

(A.6)

If we choose $\omega_0$ and $\omega_1$ so that the interval $[\omega_0, \omega_1]$ gives the set of BCFs, then $v_j(\omega_0, \omega_1)$ can be interpreted as the portion of the unconditional variance of $X_{j,t}$ attributable to BCFs, i.e. its BCF variance.

Next, letting $v_j^{(l)}(\omega_0, \omega_1)$ be the quantity in (A.6) obtained when the variances of all but the $l$-th shock are set to zero, it is straightforward to show that $\sum_{l=1}^{m} v_j^{(l)}(\omega_0, \omega_1) = v_j(\omega_0, \omega_1)$. Thus,

$$\frac{v_j^{(l)}(\omega_0, \omega_1)}{v_j(\omega_0, \omega_1)}$$

can be interpreted as the contribution of the shock $l$ to the BCF variance of $X_{j,t}$.

---

1 See, for example, Hamilton (1994).
In practice, the integral in (A.6) is approximated using 500 bins spanning the relevant frequencies. Note that the period of a cycle with frequency $\omega$ is given by $\frac{2\pi}{\omega}$. Following Stock and Watson (1999) and JPT, I define business cycle frequencies to be those with periods between 6 and 32 quarters. Thus, I set $\omega_0 = \frac{2\pi}{32}$ and $\omega_1 = \frac{2\pi}{6}$.
Appendix B

Appendix for “Reconciling Hayek’s and Keynes’ Views of Recessions”

B.1 Proofs of propositions

Proof of Proposition 3.1

We first establish that there always exists an equilibrium of this model. Substituting equation (3.8) into equation (3.7) and letting \( e \equiv c - X \) yields

\[
U'(X + e) = \frac{\nu'(\ell)}{F'(\ell)} \left( 1 + \tau - \frac{e}{\Omega(\ell)} \right)
\]

(B.1)

where \( \Omega(\ell) \equiv F'(\ell)\ell \) is output net of search costs per employed worker, which is assumed to be strictly increasing. When \( N < L \) (i.e., the full-employment constraint is not binding), equation (3.9) implies that \( \ell = \ell^* \), and equation (3.8) implies that \( e < e^* \), where \( e^* \equiv \Omega(\ell^*) \). On the other hand, when \( N > L \) (i.e., the full-employment constraint binds), equation (3.8) implies that \( \ell = \Omega^{-1}(e) \). Further, since \( \min\{N, L\} < N \) and \( F(\ell) - F'(\ell)\ell \) is assumed to be strictly increasing in \( \ell \), equation (3.9) implies that \( \ell > \ell^* \), and thus, by strict increasingness of \( \Omega \), we also have \( e > e^* \). Substituting these results into equation (B.1) yields that \( e > 0 \) is an equilibrium of this model if it satisfies

\[
U'(X + e) = Q(e)
\]

(B.2)

where the function \( Q(e) \), defined in equation (3.14), is the expected marginal utility cost of consumption when aggregate expenditures are \( e = c - X \). Note that \( Q \) is continuous, strictly decreasing on \([0, e^*] \), and strictly increasing on \([e^*, \infty) \).

Lemma B.1. If \( U'(X) \leq Q(0) \), then there is an equilibrium with \( e = 0 \).
Proof. To see this, suppose aggregate conditions are that \( e = 0 \). Then the marginal utility of consumption when the household simply consumes its endowment is no greater than its expected marginal cost, and thus households respond to aggregate conditions by making no purchases, which in turn validates \( e = 0 \).

Lemma B.2. If \( U'(X) > Q(0) \), then there is an equilibrium with \( e > 0 \).

Proof. We have that \( \min Q(e) = \nu'(\ell^*) / F'(\ell^*) > 0 \). Since we have assumed \( \lim_{c \to \infty} U''(e) \leq 0 \), it necessarily follows that for any \( X \), there exists an \( e \) sufficiently large that \( U'(X + e) < \min Q(e) \), and therefore, by the intermediate value theorem, there must exist a solution \( e > 0 \) to equation (B.2).

Lemmas B.1 and B.2 together imply that an equilibrium necessarily exists. We turn now to showing under what conditions this equilibrium is unique for all values of \( X \). As in equation (3.12), we may represent household \( j \)'s optimal expenditure when aggregate expenditure is \( e \) as

\[
e_j(e) = \frac{Q'(e)}{U''(U'^{-1}(Q(e)))}
\]

Note that \( e_j'(e) \) is independent of \( X \), strictly increasing on \([0, e^*]\) and strictly decreasing on \([e^*, \infty)\).

Lemma B.3. If

\[
\lim_{e \uparrow e^*} e_j'(e) < 1 \quad (B.3)
\]

then \( e_j'(e) < 1 \) for all \( e \).

Proof. Note first that \( e_j'(e) < 0 \) for \( e > e^* \), so that this condition is obviously satisfied in that case. For \( e < e^* \), note that

\[
e_j''(e) = \frac{Q''(e) - U'''(X + e_j(e)) \left[ e_j'(e) \right]^2}{U''(X + e_j(e))}
\]

Since \( Q''(e) = 0 \) on this range and \( U''' > 0 \), we have \( e_j''(e) > 0 \), and thus \( e_j'(e) < \lim_{e \uparrow e^*} e_j'(e) \), which completes the proof.

Lemma B.4. Inequality (B.3) holds if and only if

\[
\tau < \bar{\tau} \equiv -U'' \left( U'^{-1} \left( \frac{\nu'(\ell^*)}{f'(\ell^*)} \right) \right) \frac{f'(\ell^*) [f(\ell^*) - \Phi]}{\nu'(\ell^*)}
\]

Proof. We have that

\[
\lim_{e \uparrow e^*} e_j'(e) = \frac{\nu'(\ell^*) \tau}{-U'' \left( U'^{-1} \left( \frac{\nu'(\ell^*)}{f'(\ell^*)} \right) \right) f'(\ell^*) [f(\ell^*) - \Phi]}
\]
which is clearly less than one if and only if $\tau < \bar{\tau}$.

**Lemma B.5.** If $\tau < \bar{\tau}$, then there always exists a unique equilibrium regardless of the value of $X$. If $\tau > \bar{\tau}$, then there exists values of $X \in \mathbb{R}$ such that there are multiple equilibria.

**Proof.** We have already established that there always exists an equilibrium. Note that equilibrium occurs at the point where the $e_j = e_j(e)$ locus intersects with the locus characterizing the equilibrium condition, i.e., $e_j = e$. To see the first part of the lemma, suppose $\tau < \bar{\tau}$ so that inequality (B.3) holds. Then since the slope of the equilibrium locus is one, and the slope of the $e_j = e_j(e)$ locus is strictly less than one by Lemma B.3, there can be at most one intersection, and therefore the equilibrium is unique.

To see the second part of the lemma, suppose that $\tau > \bar{\tau}$ and thus (B.3) does not hold. Then by strict convexity of $e_j(e)$ on $(0, e^*)$, there exists a value $e < e^*$ such that $e_j'(e) > 1$ on $(e, e^*)$. Define $\tilde{X}(e) \equiv U' - 1(Q(e)) - e$, and note that $e$ is an equilibrium when $X = \tilde{X}(e)$. We show that there are at least two equilibria when $X = \tilde{X}(e)$ with $e \in (e, e^*)$. To see this, choose $e_0 \in (e, e^*)$, and note that, for $X = \tilde{X}(e_0)$, $e_j(e_0) = e_0$ and $e_j'(e) > 1$ on $(e_0, e^*)$. Thus, it must also be the case that $e_j(e^*) > e^*$. But since $e_j(e)$ is continuous everywhere and strictly decreasing on $e > e^*$, this implies that there exists some value $e > e^*$ such that $e_j(e) = e$, which would represent an equilibrium. Since $e_0 < e^*$ is also an equilibrium, there are at least two equilibria.

This completes the proof of Proposition 3.1.

**Proof of Proposition 3.2**

**Lemma B.6.** If $\tau < \bar{\tau}$ and $X$ is such that $e > 0$, then $de/dX < 0$.

**Proof.** Totally differentiating equilibrium condition (B.2) with respect to $X$ yields

\[
\frac{de}{dX} = \frac{U''(X + e)}{Q'(e) - U''(X + e)} \tag{B.4}
\]

From Lemma B.4, we see that $Q'(e) > U''(U'^{-1}(Q(e)))$. In equilibrium, $U'^{-1}(Q(e)) = X + e$, so that this inequality becomes $Q'(e) > U''(X + e)$, and thus the desired conclusion follows by inspection.

Given Lemma B.6 and the fact that the economy exhibits unemployment when $e < e^*$ and full employment when $e \geq e^*$, it is clear that the economy will exhibit unemployment if and only if $X$ is smaller than the level such that $e = e^*$ is the equilibrium; that is, if $X \leq X^*$, where

\[
X^* \equiv U'^{-1} \left( \frac{\nu'(\ell^*)}{F'('\ell^*)} \right) - F'(\ell^*)\ell^*
\]

This completes the proof of the first part of the proposition.
Next, from Lemma B.1, we see that there is a zero-employment equilibrium if and only if $U'(X) \leq \frac{\nu'(\ell^*)}{F'(\ell^*)}(1 + \tau)$, which holds when $X \geq X^{**}$, where

$$X^{**} \equiv U'^{-1} \left( \frac{\nu'(\ell^*)}{F'(\ell^*)}(1 + \tau) \right)$$

This completes the proof of Proposition 3.2.

**Proof of Proposition 3.3**

If $X < X^{**}$, we know from Proposition 3.2 that $e > 0$, and therefore $e$ solves equation (B.2). Substituting $e = c - X$ for $e$ yields the desired result in this case. From Proposition 3.2, we also know that if $X \geq X^{**}$ then $e = 0$, in which case $c = X$, which completes the proof.

**Proof of Proposition 3.4**

If $X > X^{**}$, so that the economy features zero employment and therefore $c = X$, then clearly $c$ is increasing in $X$. Thus, suppose $X < X^{**}$, so that $e > 0$. Totally differentiating the expression $c = X + e$ with respect to $X$ and using equation (B.4), we obtain

$$\frac{dc}{dX} = \frac{Q'(e)}{Q'(e) - U''(X + e)}$$

(B.5)

Since the denominator of this expression is positive (see the proof of Lemma B.6), the sign of $dc/dX$ is given by the sign of $Q'(e)$, which is negative if $e < e^{*}$ (i.e., if $X^* < X < X^{**}$) and positive if $e > e^{*}$ (i.e., if $X < X^{*}$). This completes the proof.

**Proof of Proposition 3.5**

Letting $U(e)$ denote welfare conditional on the coordinated level of $e$, we may obtain that

$$U(e) = U(X + e) + \mu(e) \left[ L^* - \frac{\nu'(\ell^*)}{F'(\ell^*)}e \right] - \left[ 1 - \mu(e) \right](1 + \tau)\frac{\nu'(\ell^*)}{F'(\ell^*)}e$$

where $\mu(e) = e/[F'(\ell^*)\ell^*]$ denotes employment conditional on $e$, and $L^* \equiv \nu'(\ell^*)\ell^* - \nu(\ell^*) \geq 0$. Using the envelope theorem, it is straightforward to see that the only welfare effects of a marginal change in $e$ from its decentralized equilibrium value are those that occur through the resulting change in employment. Thus,

$$U'(e) = \left[ L^* + \tau\frac{\nu'(\ell^*)}{F'(\ell^*)}e \right] \mu'(e) > 0$$

and therefore a coordinated rise in $e$ would increase expected utility of all households.
Proof of Proposition 3.6

Denote welfare as a function of \(X\) by

\[
U(X) \equiv U(X + e) + \mu \left[ -\nu(\ell) + V(w\ell - pe) \right] + (1 - \mu)V(-pe)
\]

If \(X < X^*\), so that the economy is in the full-employment regime, or if \(X > X^{**}\), so that the economy is in the zero-employment regime, we may show that \(U'(X) > 0\) always holds. Thus, we focus on the case where \(X \in (X^*, X^{**})\). When this is true, some algebra yields

\[
U(X) = U(X + e) + \left\{ \ell^* \left[ \nu'(\ell^*) - \frac{\nu'(\ell^*)}{\ell^*} \right] + \frac{\nu'(\ell^*)}{F'(\ell^*)} \tau e - \mu - \frac{\nu'(\ell^*)}{F'(\ell^*)} (1 + \tau)e \right\}
\]

Using the envelope theorem, we may differentiate this expression with respect to \(X\) to obtain

\[
U'(X) = U'(X + e) + \left[ \mathcal{L}^* + \frac{\nu'(\ell^*)}{F'(\ell^*)} \tau e \right] \frac{d\mu}{dX}
\]

where \(\mathcal{L}^* = \nu'(\ell^*)\ell^* - \nu(\ell^*) \geq 0\).

Lemma B.7. \(U''(X) > 0\) on \((X^*, X^{**})\).

Proof. Substituting the equilibrium condition (B.2) into (B.6) and using the fact that

\[
\frac{d\mu}{dX} = \frac{1}{F'(\ell^*)\ell^*} \frac{de}{dX}
\]

after some algebra, we obtain

\[
U'(X) = \frac{\nu'(\ell^*)}{F'(\ell^*)} \left[ 1 + \tau + \tau \mu \left( \frac{de}{dX} - 1 \right) \right] + \frac{\mathcal{L}^*}{F'(\ell^*)} \frac{de}{dX}
\]

(B.7)

From (B.4), we may also obtain that

\[
\frac{de}{dX} = \left( \frac{\nu'(\ell^*) \tau}{-U'''(X + e) [F'(\ell^*)]^2 \ell^*} - 1 \right)^{-1}
\]

\[
\frac{d^2e}{dX^2} = \frac{U'''(X + e)}{U'''(X + e)} \frac{de}{dX} \left[ \frac{dc}{dX} \right]^2 > 0
\]

and therefore

\[
U''(X) = \frac{\nu'(\ell^*)}{F'(\ell^*)} \tau \frac{d\mu}{dX} \left( \frac{de}{dX} - 1 \right) + \left[ \frac{\nu'(\ell^*)}{F'(\ell^*)} \tau \mu + \frac{\mathcal{L}^*}{F'(\ell^*)} \ell^* \right] \frac{d^2e}{dX^2}
\]

Since \(de/dX < 0\), \(d\mu/dx < 0\), and thus the first term is positive, as is the second term, and the proof is complete.
Lemma B.8. If
\[ \tau > \tau \equiv \frac{\nu(\ell^*)}{\nu'(\ell^*)} \left( \frac{\bar{\tau}}{1 + \bar{\tau}} \right) \]
then there exists a range of \( X \) such that \( U'(X) < 0 \).

Proof. Since \( U \) is convex by Lemma B.7, \( U'(X) < 0 \) for some values of \( X \) if and only if \( \lim_{X \downarrow X^*} U'(X) < 0 \). Taking limits of equation (B.7), and using the facts that
\[ \lim_{X \downarrow X^*} \frac{dX}{dX} = \frac{-\bar{\tau}}{\bar{\tau} - \tau} \]
and \( \lim_{X \downarrow X^*} \mu = 1 \), we obtain that
\[ \lim_{X \downarrow X^*} U'(X) = \frac{\nu'(\ell^*)}{F'(\ell^*)} \left( \frac{1 - \frac{\tau \bar{\tau}}{\bar{\tau} - \tau} - \frac{\mathcal{L}^*}{F'(\ell^*)} \left( \frac{\bar{\tau}}{\bar{\tau} - \tau} \right)} \right) \]
Substituting in from the definition of \( \mathcal{L}^* \), straightforward algebra yields that this expression is less than one if and only if \( \tau > \tau \).

Proof of Proposition 3.7
We suppose there is a competitive insurance industry offering a menu of unemployment insurance contracts. A typical contract is denoted \((h, q)\), where \( h \) is the premium, paid in all states, and \( q \) is the coverage, which the purchaser of the contract receives if and only if he is unemployed. Both \( h \) and \( q \) are expressed in units of good 1. Since insurance is only potentially useful when \( 0 < \mu < 1 \), we henceforth assume that this is true. Note also that zero profit of insurers requires that \( h = (1 - \mu \hat{\rho})q \), where \( \hat{\rho} \) is the fraction of purchasers of the contract that are participant households. This implies that non-participant households will not purchase any such zero-profit contract featuring \( q < 0 \).

Lemma B.9. In any separating equilibrium, no contracts are purchased by participant households.\(^1\)

Proof. Suppose there is a separating equilibrium, and let \((h_p, q_p)\) denote the contract purchased by participant households, and \((h_n, q_n)\) that purchased by non-participant households. From the insurer’s zero-profit condition, we must have \( h_p = (1 - \mu)q_p \) and \( h_n = q_n \). Since non-participant households will always deviate to any contract with \( h_p < q_p \), this implies that we must have \( q_p < 0 \) in such an equilibrium.

\(^1\) Technically, agents are always indifferent between not purchasing a contract and purchasing the contract \((0, 0)\). For ease of terminology, we will assume that the contract \((0, 0)\) does not exist.
Next, for any zero-profit separating contract, the assets of employed participant households are given by \( A_e = w\ell - p[(1 - \mu)q_p + e] \) and of unemployed participant households by \( A_u = p(\mu q_p - e) \). Note that, since \( q_p < 0 \) and from the resource constraint \( w\ell > pc \), we must have \( A_u < 0 < A_e \). Also, the derivative of the household’s objective function with respect to \( q_p \) along the locus of zero-profit contracts is given by

\[
\frac{\partial U}{\partial q_p} = p \mu (1 - \mu) \left[ V'(A_u) - V'(A_e) \right] > 0
\]

wherever such a derivative exists. Since \( A_u < 0 < A_e \), this derivative must exist at the candidate equilibrium, and therefore in a neighborhood of that equilibrium the objective function is strictly increasing on \( q_p < 0 \). Thus, given any candidate zero-profit equilibrium contract with \( q_p < 0 \), there exists an alternative contract \((h'_p, q'_p)\) with \( q'_p > q_p \) which satisfies that \( h'_p - (1 - \mu)q'_p \) is strictly greater than but sufficiently close to zero so that participant households would choose it over \((h_p, q_p)\), while non-participant households would not choose it, and therefore insurers could make a positive profit selling it. Thus, \((h_p, q_p)\) cannot be an equilibrium contract. Since this holds for all \( q_p < 0 \), it follows that no separating equilibrium exists in which contracts are purchased by participant households. \( \square \)

Next, consider a pooling equilibrium, so that \( \hat{\rho} = \rho \). As argued above, we must have \( q \geq 0 \) in any such equilibrium. Assets of an employed worker when choosing a zero-profit pooling contract \((h, q) = ((1 - \mu \rho)q, q)\) are given by \( A_e = w\ell - p[(1 - \mu \rho)q + e] \), while \( A_u = p(\mu \rho q - e) \) are those of an unemployed worker. Let \( U(q) \) denote the value of the household’s objective function when choosing such a zero-profit pooling contract.

**Lemma B.10.** If \( U(q) \) is strictly decreasing in \( q \) whenever \( A_e > A_u \), then a pooling equilibrium does not exist.

*Proof.* Note first that if \( A_e \leq A_u \), then being unemployed is always strictly preferred to being employed by participant households, so that this cannot represent an equilibrium. Furthermore, as argued above, we must have \( q \geq 0 \) in any pooling equilibrium. Thus, suppose \( A_e > A_u \) and \( q > 0 \). We show that such a \( q \) cannot represent an equilibrium. To see this, let \((h', q')\) denote an alternative contract with \( 0 < q' < q \) and \( h' = (1 - \mu \rho)q' \). Since \( U \) is strictly decreasing in \( q \), this contract is strictly preferred by participant households. Furthermore, since non-participant households would get net payment \( \mu \rho (q' - q) < 0 \) from deviating to this new contract, only participant households would deviate to it, and therefore the expected profit to an insurer offering it would be \((1 - \rho)\mu q' > 0 \). Thus, this deviation is mutually beneficial for participants and insurers, and so \( q \) cannot be an equilibrium. \( \square \)

**Lemma B.11.** If \( \rho < 1/(1 + \tau) \), then there is no equilibrium in which an insurance contract is purchased by participant households.

*Proof.* Note that \( U(q) \) is continuous, with

\[
U'(q) = p \mu \left[ (1 - \mu) \rho V'(A_u) - (1 - \mu \rho) V'(A_e) \right]
\]
wherever this derivative exists (i.e., whenever $A_e A_u \neq 0$). If $A_e A_u > 0$, then $V'(A_e) = V'(A_u)$, and therefore $U'(q) = -p\mu(1-\rho) V'(A_e) < 0$. Suppose on the other hand that $A_e A_u < 0$. If in addition $A_e > A_u$, we must have $A_u < 0 < A_e$, and therefore $U'(q) = -p\nu\mu\left[1-\rho(1+\tau(1-\mu))\right]$. Since $\rho < 1/(1+\tau)$, it follows that $U'(q) < 0$. Thus, $U(q)$ is strictly decreasing whenever $A_e > A_u$, and therefore by Lemma B.10, no pooling equilibrium exists. Since, by Lemma B.9, there does not exist a separating equilibrium either, no equilibrium exists.

**Proof of Proposition 3.8**

We may re-write the equilibrium condition (3.15) as

$$U'(X + e - G_w) = Q(e) \quad (B.8)$$

where $Q(e)$ is as defined in equation (3.14).

That non-wasteful government purchases have no effect on economic activity can be seen directly from the fact that $G_n$ does not appear in equation (B.8). Totally differentiating equation (B.8) with respect to $G_w$, we obtain

$$\frac{de}{dG_w} = \frac{-U''(X + e - G_w)}{Q'(e) - U''(X + e - G_w)}$$

Under the assumption that $\tau < \bar{\tau}$, the denominator of this expression is positive, and thus $de/dG_w > 0$. Further, if the economy is in the unemployment regime, then $Q'(e) < 0$ and therefore $de/dG_w > 1$, while if the economy is in the unemployment regime, then $Q'(e) > 0$ and therefore $de/dG_w < 1$, which completes the proof.

**Proof of Proposition 3.9**

First, note that a balanced budget requires that employed workers be taxed $(G_n + G_w)/\mu$. Letting $e_p = e - G_n - G_w$ denote private expenditures, we may therefore obtain welfare as a function of $X$, $G_n$ and $G_w$ as

$$U(X, G_n, G_w) = U(X + e_p + G_n)$$

$$+ \frac{\nu'(\ell^*)}{F'(\ell^*)} \left[ \frac{F'(\ell^*)}{\nu'(\ell^*)} \ell^* + \tau e_p \right] \mu - G_n - G_w - (1 + \tau)e_p \quad (B.9)$$

where as before $\ell^* \equiv \nu'(\ell^*) \ell^* - \nu(\ell^*)$. Taking derivatives with respect to $G_w$ and applying the envelope theorem, we may obtain that

$$U_3(X, G_n, G_w) = \frac{\nu'(\ell^*)}{F'(\ell^*)} \left[ \frac{F'(\ell^*)}{\nu'(\ell^*)} \ell^* + \tau e_p \right] \frac{d\mu}{dG_w} - 1 \quad (B.10)$$
Meanwhile, differentiating (B.9) with respect to $X$, applying the envelope theorem and using the equilibrium condition $U'(X + e_p + G_n) = Q(e_p + G_n + G_w)$, we may obtain that

$$\frac{F'_{\ell}(\ell^*)}{\nu'_{\ell}(\ell^*)}L^* + \tau e_p = \frac{F'_{\ell}(\ell^*)}{\nu'_{\ell}(\ell^*)} \left[ U_1(X, G_n, G_w) - \frac{\nu'_{\ell}(\ell^*)}{F'_{\ell}(\ell^*)} (1 + \tau - \tau \mu) \right] \left( \frac{d\mu}{dX} \right)^{-1} \tag{B.11}$$

We may also obtain from the equilibrium condition that $d\mu/dG_w = -d\mu/dX$. Substituting this and (B.11) into (B.10), we may obtain

$$U_3(X, G_n, G_w) = \frac{\nu'_{\ell}(\ell^*)}{F'_{\ell}(\ell^*)} (1 - \mu) \tau - U_1(X, G_n, G_w)$$

Since the first term on the left-hand side is positive, if $X$ is in the range such that $U_1(X, G_n, G_w) < 0$, then we necessarily have $U_3(X, G_n, G_w) > 0$, which completes the proof.

**Proof of Proposition 3.10**

Let $m(N) \equiv M(N, L)$ and note that the restrictions on $M$ imply, among other things, that (a) $m(0) = 0$, (b) $m'(0) \in (0, 1]$, and (c) $\lim_{N \to \infty} m'(N) = 0$. We have that

$$\lim_{N \to 0} \frac{m'(N) N}{m(N)} = \lim_{N \to 0} \frac{m'(N)}{[m(N) - m(0)] / N}$$

where we have used property (a). The limit of the numerator is clearly just $m'(0)$, while the limit of the denominator is, by definition, also equal to $m'(0)$ and thus, since by property (b) $m'(0)$ is non-zero and bounded, we have that

$$\lim_{N \to 0} \frac{m'(N) N}{m(N)} = \frac{m'(0)}{m'(0)} = 1$$

Next, suppose $\lim_{N \to \infty} m'(N) N/m(N) > 0$. Since $0 < \lim_{N \to \infty} m(N) < \infty$, this implies that $\lim_{N \to \infty} N/g(N) > 0$ where $g(N) \equiv 1/m'(N)$. This in turn implies that $g(N) = O(N)$ as $N \to \infty$, or, equivalently, that there exists an $N_0 > 0$ such that, for $N \geq N_0$,

$$\frac{g'(N)}{g(N)} \leq \frac{1}{N}$$

where the right-hand side of this inequality is simply the growth rate of $N$. We may therefore obtain,
for $N \geq N_0$,

$$g(N) = g(N_0) \exp \left\{ \int_{N_0}^{N} \frac{g'(s)}{g(s)} ds \right\}$$

$$\leq g(N_0) \exp \left\{ \int_{N_0}^{N} \frac{1}{s} ds \right\}$$

$$= \frac{g(N_0)N}{N_0}$$

and thus $m'(N) \geq m'(N_0)N_0/N$. But

$$m(N) = m(N_0) + \int_{N_0}^{N} m'(s) ds$$

$$\geq m(N_0) + m'(N_0)N_0 \int_{N_0}^{N} \frac{1}{s} ds$$

$$= m(N_0) + m'(N_0)N_0 \left[ \log(N) - \log(N_0) \right]$$

The expression on the last line above is clearly unbounded as $N \to \infty$, which would imply the same for $m(N)$, a clear contradiction of the requirement that $M(N,L) \leq L$. Thus, we cannot have $\lim_{N \to \infty} m'(N)N/m(N) > 0$, i.e., we must have $\lim_{N \to \infty} m'(N)N/m(N) = 0$.

**Proof of Proposition 3.11**

The following result will be useful.

**Lemma B.12.** Let $\mathcal{E}^g_{X,M}$ denote the elasticity of substitution between $X$ and $M$ embodied in $g$. Then

$$\mathcal{E}^g_{X,M} = \frac{g_X(X,M)g_M(X,M)}{g_{XM}(X,M)g(X,M)} \quad \text{(B.12)}$$

**Proof.** Letting $H^k$ denote homogeneity of degree $k$, note first that, since $g$ is $H^1$, for $a, b \in \{X, M\}$, $g_a$ is $H^0$ and $g_{ab}$ is $H^{-1}$.

Next, by definition, we have

$$\mathcal{E}^g_{X,M} = \left[ \frac{d \log \left( g_X(X,M)/g_M(X,M) \right)}{d \log (M/X)} \right]^{-1}$$
Letting \( \tilde{M} \equiv M/X \) and using \( H^0 \) of \( g_X \) and \( g_M \), we may obtain

\[
E^g_{XM} = \frac{g_X(1, \tilde{M})}{g_I(1, M)} \left[ \frac{d}{dM} \left( \frac{g_X(1, \tilde{M})}{g_M(1, \tilde{M})} \right) \right]^{-1} \tilde{M} \left[ g_{XM}(1, \tilde{M})g_I(1, \tilde{M}) - g_X(1, \tilde{M})g_{MM}(1, \tilde{M}) \right] = \frac{g_X(X, M)g_M(X, M)}{M \left[ g_{XM}(X, M)g_M(X, M) - g_X(X, M)g_{MM}(X, M) \right]}
\]

where the last line follows from \( H^0 \) of \( g_a \) and \( H^{-1} \) of \( g_{ab} \). Adding and subtracting \( g_{XM}(X, M)g_X(X, M)X \) in the denominator and grouping terms yields

\[
E^g_{XM} = \frac{g_X(X, M)g_M(X, M)}{g_{XM}(X, M)[g_X(X, M)X + g_M(X, M)M] - g_X(X, M)[g_{XM}(X, M)X + g_{MM}(X, M)M]}
\]

The first bracketed term in the denominator equals \( g(X, M) \) by \( H^1 \) of \( g \), while the second bracketed term equals 0 by \( H^0 \) of \( g_M \), and thus equation (B.12) follows.

Next, let \( W(X, M) \equiv U(g(X, M)) \). Then the equilibrium condition (3.17) can be written

\[
W_M(X, M) = Q(M) \tag{B.13}
\]

Note that

\[
W_{MM}(X, M) = [g_M(X, M)]^2 U''(g(X, M)) + g_{MM}U'(g(X, M)) < 0
\]

so that the left-hand side of equation (B.13) is strictly decreasing in \( M \). To ensure the existence of an equilibrium with \( M > 0 \), we assume that \( W_M(X, 0) > Q(0) \). We further assume that \( g_{MM}(X, M) \geq 0 \), which ensures that \( Q_{MM} > 0 \), and therefore, similar to in the durable-goods model, there are at most three equilibria: at most two in the unemployment regime, and at most one in the full-employment regime. Additional conditions under which we can ensure that there exists a unique equilibrium are similar in flavor to in the durable-goods case, though less easily characterized explicitly. We henceforth simply assume conditions are such that the equilibrium is unique, and note that this implies that

\[
W_{MM}(X, M) < Q'(M) \tag{B.14}
\]

at the equilibrium value of \( M \). Define also

\[
E^Q_M \equiv Q(M)M \frac{Q'(M)}{Q(M)}
\]

as the elasticity of \( Q \) with respect to \( M \).
Lemma B.13. \( dc/dX < 0 \) if and only if

\[-E^Q_{X,M} > 1\]  

(B.15)

Proof. Totally differentiating the equilibrium condition (B.13) with respect to \( X \) yields that

\[\frac{dM}{dX} = \frac{W_{X,M}(X,M)}{Q'(M) - W_{M,M}(X,M)}\]  \hspace{1cm} (B.16)

Doing the same with the equilibrium condition \( c = g(X,M) \) yields

\[\frac{dc}{dX} = g_X(X,M) + g_M(X,M) \frac{dM}{dX} = \frac{g_X(X,M) [Q'(M) - W_{M,M}(X,M)] + g_M(X,M)W_{X,M}(X,M)}{Q'(M) - W_{M,M}(X,M)}\]

where the second line has used (B.16). By (B.14), the denominator is positive, so that this expression is of the same sign as the numerator. Substituting in for \( W_{M,M} \) and \( W_{X,M} \) and using the equilibrium condition (B.13), we may obtain that \( dc/dX < 0 \) if and only if

\[-E^Q_{X,M} \frac{g_X(X,M) + g_M(X,M) \frac{dM}{dX}}{g_X(X,M)g_M(X,M)} M < -E^Q_{X,M}\]  \hspace{1cm} (B.17)

The term in square brackets, meanwhile, can be written as

\[\frac{g_{X,M}(X,M)}{g_X(X,M)} \frac{g_{M,M}(X,M)}{g_M(X,M)} - \frac{g_{M,M}(X,M)}{g_M(X,M)} g_X(X,M)g_M(X,M)M\]

By \( H^0 \) of \( g_M \), the second term in the numerator equals zero, and thus by \( H^1 \) of \( g \), we have that

\[\frac{g_{X,M}(X,M)}{g_X(X,M)} - \frac{g_{M,M}(X,M)}{g_M(X,M)} = \frac{g_{X,M}(X,M)g(X,M)}{g_X(X,M)g_M(X,M)M}\]

Substituting this into (B.17) and using (B.12) yields (B.15).

If the economy is in the full-employment regime, \( E^Q_{X,M} > 0 \) and therefore, since \( E^Q_{X,M} > 0 \), condition (B.15) cannot hold. Thus, from Lemma B.13, if the economy is in the full-employment regime, \( dc/dX > 0 \). If instead the economy is in the unemployment regime, then \( E^Q_{X,M} < 0 \), and therefore condition (B.15) can hold as long as \( E^Q_{X,M} \) is sufficiently large, which completes the proof of the proposition.

Proof of Proposition 3.12

Let \( y = g(X_1,M) \) denote output of the final good in the first period. Furthermore, let \( B(X_2) \equiv U''(R(X_2)) + X_2 \) denote the total resources (output plus undepreciated first-period capital) that
would be required for the choice \( X_2 \) to satisfy the constraints (3.18) and (3.19) as well as the intertemporal optimality condition (3.21), and note that

\[
B'(X_2) = \frac{R'(X_2)}{U''(c)} + 1 > 1 \tag{B.18}
\]

where the inequality follows from the assumption made that \( R'(X_2) < 0 \). Since total resources actually available are \((1 - \delta)X_1 + g(X_1, M)\), we have \( X_2 = B^{-1}((1 - \delta)X_1 + g(X_1, M)) \), and therefore from condition (3.20) equilibrium can be characterized by a solution to

\[
G(X_1, M) = Q(M) \tag{B.19}
\]

for \( M \), where \( G(X, M) \equiv g_M(X, M)R(B^{-1}((1 - \delta)X + g(X, M))) \). Note that

\[
G_M(X_1, M) = g_M(X_1, M)R(X_2) + \frac{R'(X_2)[g_M(X_1, M)]^2}{B'(X_2)} < 0
\]

Similar to in the static case, we assume that \( G(X, 0) > Q(0) \) so that there is an equilibrium with \( M > 0 \), and further, conditions are such that this equilibrium is unique, which implies that

\[
G_M(X_1, M) < Q'(M) \tag{B.20}
\]

at the equilibrium value of \( M \).

**Lemma B.14.** If \( dX_2/dX_1 < 0 \) then \( dc/dX_1 < 0 \) and \( di/dX_1 < 0 \).

**Proof.** Since in equilibrium \( c + X_2 = B(X_2) \), we have that

\[
\frac{dc}{dX_1} = [B'(X_2) - 1] \frac{dX_2}{dX_1}
\]

Since \( B'(X_2) > 1 \), if \( dX_2/dX_1 < 0 \) then \( dc/dX_1 < 0 \). Further, if \( X_2 \) falls when \( X_1 \) rises, from the capital accumulation equation (3.18) we see that \( i \) must also fall. \( \square \)

**Lemma B.15.** \( dX_2/dX_1 < 0 \) if and only if

\[
\left\{-\varepsilon_M^Q + \frac{(1 - \delta)g_{XM}(X, M)}{g_X(X, M)[g_X(X, M) + 1 - \delta]} \right\} \varepsilon_{XM} > 1 \tag{B.21}
\]

**Proof.** Totally differentiating the equilibrium condition (B.19) with respect to \( X_1 \) yields that

\[
\frac{dM}{dX_1} = \frac{G_X(X_1, M)}{Q'(M) - G_M(X_1, M)} \tag{B.22}
\]
Doing the same with \( y = g(X, M) \) yields

\[
\frac{dy}{dX_1} = g_X(X_1, M) + g_M(X_1, M) \frac{dM}{dX_1} \quad \text{(B.23)}
\]

while differentiating \( X_2 = B^{-1}((1 - \delta)X_1 + g(X_1, M)) \) yields

\[
\frac{dX_2}{dX_1} = \frac{1}{B'(X_2)} \left( 1 - \delta + \frac{dy}{dX_1} \right) = \frac{[1 - \delta + g_X(X_1, M)] [Q'(M) - G_M(X_1, M)] + g_M(X_1, M)G_X(X_1, M)}{B'(X_2) [Q'(M) - G_M(X_1, M)]}
\]

where the second line has used equations (B.22) and (B.23). Since the denominator of this expression is positive by (B.18) and (B.20), the sign of \( dX_2/dX_1 \) is given by the sign of the numerator. Substituting in for \( G_M \) and \( G_X \) and using (B.19), some algebra yields that this expression is negative if and only if condition (B.21) holds.

Lemmas B.14 and B.15 together indicate that \( dc/dX_1 < 0 \) and \( di/dX_1 < 0 \) both hold if and only if condition (B.21) holds. Further, for a given equilibrium level of \( M \), it is clear that the minimum level of \( E_{X,M}^G \) needed to satisfy (B.21) is (weakly) greater than that needed to satisfy (B.15) in the static case.

**Proof of Proposition 3.13**

It can be verified that the steady-state level of purchases \( e \) solves

\[
U' \left( \frac{\delta + \gamma}{\delta} e \right) = \zeta Q(e) \quad \text{(B.24)}
\]

where

\[
\zeta \equiv \frac{1 - \beta(1 - \delta)}{1 - \beta(1 - \delta) + \beta\gamma} \in (0, 1)
\]

**Lemma B.16.** For \( \delta \) sufficiently small, a steady state exists and is unique.

**Proof.** Similar to in the static case, we may express individual \( j \)'s optimal choice of steady-state expenditure \( e_j \) given aggregate steady-state expenditure \( e \) as

\[
e_j(e) = \frac{\delta}{\delta + \gamma} U'^{-1} (\zeta Q(e))
\]

As before, we can verify that \( e'_j(e) < 0 \) for \( e > e^* \), while \( e''_j(e) > 0 \) and \( e''_j(e) > 0 \) for \( e < e^* \). Thus, an equilibrium necessarily exists and is unique if \( e'_j(e) < 1 \) for \( e < e^* \), which is equivalent to the
condition that \(\lim_{e \uparrow e^*} e'_j(e) < 1\). This is in turn equivalent to the condition \(\tau < \tilde{\tau}\), where

\[
\tilde{\tau} = -\frac{\delta + \gamma}{\delta \zeta} U'' \left( U' \left( \frac{\nu'((\ell^*))}{F'((\ell^*))} \right) \right) \frac{F'((\ell^*)) [F(\ell^*) - \Phi]}{\nu'((\ell^*))} \tag{B.25}
\]

As \(\delta \to 0\), \(\tilde{\tau}\) approaches infinity, and thus it will hold for any \(\tau\), which completes the proof. \(\square\)

Note for future reference that if \(e'_j(e) < 1\) then

\[
(\delta + \gamma) \Uprime(X + e) < \delta \zeta \Qprime(e) \tag{B.26}
\]

**Lemma B.17.** For \(\delta\) sufficiently small, there exists a steady state in the unemployment regime.

**Proof.** Since \(U'(0) > \Q'(0)\) by assumption, we also have \(U'(0) > \zeta \Q'(0)\). Thus, if

\[
U' \left( \frac{\delta + \gamma}{\delta} e^* \right) < \zeta \Q(e^*)
\]

then by the intermediate value theorem, equation B.24 holds for at least one value of \(e < e^*\). Note that

\[
\lim_{\delta \to 0} \frac{\delta + \gamma}{\delta} e^* = \infty
\]

and \(\lim_{\delta \to 0} \zeta t = (1 - \beta)/(1 - \beta + \beta \gamma) > 0\). Thus, since \(\lim_{e \to \infty} U'(e) \leq 0\) by assumption, it follows that

\[
\lim_{\delta \to 0} U' \left( \frac{\delta + \gamma}{\delta} e^* \right) \leq 0 < \lim_{\delta \to 0} \zeta Z(e^*)
\]

and thus the desired property holds for \(\delta\) close enough to zero. \(\square\)

Lemmas B.16 and B.17 together prove the proposition.

**Proof of Proposition 3.14**

Linearizing the system in \(e_t\) and \(X_t\) around the steady state and letting variables with hats denote deviations from steady state and variables without subscripts denote steady-state quantities, we have

\[
\dot{X}_{t+1} = (1 - \delta) \dot{X}_t + \gamma \dot{e}_t
\]

\[
\dot{e}_{t+1} = -\frac{[1 - \beta (1 - \delta)(1 - \delta - \gamma)]U''X + e}{\beta [(1 - \delta)Q'(e) - (1 - \delta - \gamma)U''(X + e)]} \dot{X}_t
\]

\[
+ \frac{Q'(e) - [1 - \beta \gamma (1 - \delta - \gamma)]U''(X + e)}{\beta [(1 - \delta)Q'(e) - (1 - \delta - \gamma)U''(X + e)]} \dot{e}_t
\]
or
\[
\hat{x}_{t+1} = \begin{pmatrix} \hat{X}_{t+1} \\ \hat{e}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 - \delta & \gamma \\ a_{eX} & a_{ee} \end{pmatrix} \begin{pmatrix} \hat{X}_t \\ \hat{e}_t \end{pmatrix} = A\hat{x}_t
\]

where \(a_{eX}\) and \(a_{ee}\) are the coefficients on \(\hat{X}_t\) and \(\hat{e}_t\) in the expression for \(\hat{e}_{t+1}\). The eigenvalues of \(A\) are then given by
\[
\lambda_1 = \frac{1 - \delta + a_{ee} - \sqrt{(1 - \delta + a_{ee})^2 - 4\beta^{-1}}}{2}
\]
\[
\lambda_2 = \frac{1 - \delta + a_{ee} + \sqrt{(1 - \delta + a_{ee})^2 - 4\beta^{-1}}}{2}
\]

We may obtain that
\[
\lambda_1\lambda_2 = \beta^{-1} > 1\]

so that \(|\lambda_i| > 1\) for at least one \(i \in \{1, 2\}\). Thus, this system cannot exhibit local indeterminacy (see, e.g., Blanchard and Kahn (1980)), which completes the proof.

**Proof of Proposition 3.15**

Note for future reference that (B.27) implies that if the eigenvalues are real then they are of the same sign, with \(\lambda_2 > \lambda_1\).

**Lemma B.18.** The system is saddle-path stable if and only if
\[
|1 - \delta + a_{ee}| > \frac{1 + \beta}{\beta}\]

in which case the eigenvalues are real and of the same sign as \(1 - \delta + a_{ee}\).

**Proof.** To see the “if” part, suppose (B.28) holds, and note that this implies
\[
(1 - \delta + a_{ee})^2 > \left(\frac{1 + \beta}{\beta}\right)^2 > 4\beta^{-1}
\]

and therefore the eigenvalues are real. If \(1 - \delta + a_{ee} > (1 + \beta)/\beta\), then this implies that \(\lambda_2 > \lambda_1 > 0\), and therefore the system is stable as long as \(\lambda_1 < 1\), which is equivalent to the condition
\[
(1 - \delta + a_{ee}) - 2 < \sqrt{(1 - \delta + a_{ee})^2 - 4\beta^{-1}}\]

Since \(1 - \delta + a_{ee} > (1 + \beta)/\beta > 2\), both sides of this inequality are positive, and therefore, squaring both sides and rearranging, it is equivalent to
\[
1 - \delta + a_{ee} > \frac{1 + \beta}{\beta}\]

which holds by hypothesis. A similar argument can be used to establish the claim for the case that
\[-(1 - \delta + \alpha) > (1 + \beta)/\beta.\]

To see the “only if” part, suppose the system is stable. If the eigenvalues had non-zero complex part, then $|\lambda_1| = |\lambda_2| > 1$, in which case the system would be unstable. Thus, the eigenvalues must be real, i.e., $(1 - \delta + \alpha)^2 > 4\beta^{-1}$, which in turn implies that

$$|1 - \delta + \alpha| > 2\sqrt{\beta^{-1}}.$$ 

If $1 - \delta + \alpha > 2\sqrt{\beta^{-1}}$, then, reasoning as before, $\lambda_2 > \lambda_1 > 0$, and therefore if the system is stable then (B.29) must hold. Since $(1 - \delta + \alpha) > 2\sqrt{\beta^{-1}} > 2$, then again both sides of (B.29) are positive, and thus that inequality is equivalent to (B.30), which in turn implies (B.28). Similar arguments establish (B.28) for the case where $-(1 - \delta + \alpha) > 2\sqrt{\beta^{-1}}$. 

**Lemma B.19.** The system is saddle-path stable with positive eigenvalues if and only if

$$\frac{(1 - \delta - \gamma)U''(X + e)}{(1 - \delta)Q'(e)} < \frac{(1 - \delta)Q'(e)}{(1 - \delta - \gamma)U''(X + e)} \quad (B.31)$$

**Proof.** Note that the system is stable with positive eigenvalues if and only if (B.30) holds. We have that

$$1 - \delta + \alpha - \frac{1 + \beta}{\beta} = \frac{[1 - \beta(1 - \delta - \gamma)][\delta \zeta Q'(e) - (\delta + \gamma)U''(X + e)]}{\beta[(1 - \delta)Q'(e) - (1 - \delta - \gamma)U''(X + e)]}$$

Since the numerator is positive by (B.26), inequality (B.30) holds if and only if (B.31) holds. 

**Lemma B.20.** If

$$\tau < \tilde{\tau}^* \equiv -\frac{1 - \delta - \gamma}{1 - \delta} U'' \left( \frac{\zeta \nu'(\ell^*)}{\nu'(\ell^*)} \right) \frac{F'\ell^*)[F(\ell^*) - \Phi]}{F(\ell^*)}$$

then the system is saddle-path stable with positive eigenvalues.

**Proof.** Note that condition (B.31) always holds around a full-employment steady state. If the steady state is in the unemployment regime, then it can be verified that condition (B.31) holds if and only if

$$e_j'(e) < \frac{\delta}{\delta + \gamma} \frac{1 - \delta - \gamma}{1 - \delta} \in (0, 1)$$

where $e_j'(e)$ is as defined in Lemma B.16. As before, this condition holds for all $e$ if it holds for $\lim_{e \uparrow \ell^*} e_j'(e)$, which it can be verified is equivalent to the condition $\tau < \tilde{\tau}^*$. Note also that $\tilde{\tau}^* < \bar{\tau}$, where $\bar{\tau}$ was defined in equation (B.25), so that this condition is strictly stronger than the one required to ensure the existence of a unique steady state. 

Lemmas B.19 and B.20 together establish that, for $\tau$ sufficiently small (e.g., $\tau < \tilde{\tau}^*$), the system converges monotonically to the steady state. It remains to show that consumption is decreasing in
Assuming \( \tau \) is sufficiently small so that the system is saddle-path stable with positive eigenvalues, it is straightforward to obtain the solution

\[
\begin{align*}
\hat{X}_t &= \lambda_1^t \hat{X}_0 \\
\hat{e}_t &= \psi \hat{X}_t \\
\hat{c}_t &= (1 + \psi) \hat{X}_t
\end{align*}
\]

where \( \psi \equiv -\frac{(1 - \delta - \lambda_1)}{\gamma}. \) Thus, consumption is decreasing in the stock of durables if and only if \( \psi < -1. \)

**Lemma B.21.** If (B.31) holds and the steady state is in the unemployment regime, then \( \psi < -1. \)

**Proof.** We may write

\[
1 - \delta - \gamma - \lambda_1 = \frac{\sqrt{|a_{ee} + 2\gamma - (1 - \delta)|^2 + 4\beta^{-1}[\beta(1 - \delta - \gamma)(a_{ee} + \gamma) - 1] - |a_{ee} + 2\gamma - (1 - \delta)|}}{2}
\]

Now, \( a_{ee} + 2\gamma - (1 - \delta) > a_{ee} - (1 - \delta) > 0, \) so that \( 1 - \delta - \gamma - \lambda_1 \) is positive if and only if \( \beta(1 - \delta - \gamma)(a_{ee} + \gamma) > 1. \) We have

\[
\beta(1 - \delta - \gamma)(a_{ee} + \gamma) = \frac{[1 + \beta \gamma(1 - \delta)]Q'(e) - U''(X + e)}{\left(\frac{1 - \delta}{1 - \delta - \gamma}\right)Q'(e) - U''(X + e)}
\]

Note by earlier assumptions that this expression is strictly positive, and that

\[
1 - \delta - \gamma - [1 + \beta \gamma(1 - \delta)] = \gamma \frac{1 - \beta(1 - \delta)(1 - \delta - \gamma)}{1 - \delta - \gamma} > 0
\]

Thus, if \( Q'(e) < 0 \) (i.e., the steady state is in the unemployment regime) then \( \beta(1 - \delta - \gamma)(a_{ee} + \gamma) > 1, \) in which case \( 1 - \delta - \gamma - \lambda_1 > 0 \) and therefore \( \psi < -1. \) \( \square \)

**Proof of Proposition 3.16**

Without loss of generality, assume the alternative path begins at \( t = 0, \) and let \( \tilde{e}_t(\Delta) \equiv e + \Delta \cdot \epsilon_t \) denote the alternative feasible path of expenditures, where \( \epsilon_t \) is the change in the path of expenditures, and \( \Delta \) is a perturbation parameter, which is equal to zero in the steady-state equilibrium and equal to one for the alternative path. Let \( \tilde{X}_t(\Delta) \) denote the associated path for the stock of durables, and note that \( \tilde{X}_0(\Delta) = X, \) i.e., this alternative path does not affect the initial stock of durables. Welfare can
then be written as a function of $\Delta$ as

$$U(\Delta) = \sum_{t=0}^{\infty} \beta^t \left\{ U(\bar{X}_t(\Delta)) + \bar{e}_t(\Delta) \left[ -\nu(\ell^*) + V(w^* \ell^* - p^* \bar{e}_t(\Delta)) \right] 
+ \left( 1 - \frac{\bar{e}_t(\Delta)}{F'(\ell^*)} \right) V(-p^* \bar{e}_t(\Delta)) \right\}$$

From the envelope theorem, beginning from the steady state path (i.e., $\Delta = 0$), for a marginal change in $\Delta$ the net effect on welfare through the resulting changes in $U$ and $V$ in each period is zero. Thus, we need only consider effects that occur through changes in the employment rate term, $\bar{e}_t(\Delta)/[F'(\ell^*)\ell^*]$.

A first-order approximation to $U(1)$ around $U(0)$ is therefore given by

$$U(1) \approx U(0) + \frac{1}{F'(\ell^*)\ell^*} \left[ L^* + \tau \frac{\nu'(\ell^*)}{F'(\ell^*)} \right] \sum_{t=0}^{\infty} \beta^t \bar{e}_t'(0)$$

Substituting in $\bar{e}_t'(0) = \epsilon_t$, the desired result obtains.

**Proof of Proposition 3.17**

Let $\tilde{e}_t(\epsilon)$ and $\tilde{X}_t(\epsilon)$ denote alternative paths for expenditure and the stock of durables, with $\tilde{e}_t(\epsilon) \equiv e(\tilde{X}_t(\epsilon)) + \epsilon_t$ and $\tilde{X}_{t+1}(\epsilon) = (1 - \delta)\tilde{X}_t(\epsilon) + \gamma e(\tilde{X}_t(\epsilon))$. Here, $e(\cdot)$ is the equilibrium policy function for expenditures, while $\epsilon_0 = \epsilon$ and $\epsilon_t = 0$ for $t \geq 1$. Letting $U(X_0, \epsilon)$ denote the corresponding welfare as a function of $X_0$ and $\epsilon$, we may write a second-order approximation to this function around $(X_0, \epsilon) = (X, 0)$ as

$$U(X_0, \epsilon) \approx U(X, 0) + U_{X0} \Delta X_0 + U_{\epsilon\epsilon} \epsilon^2 + U_{X\epsilon} \Delta X_0 \epsilon$$

where variables with hats indicate deviations from steady state and partial derivatives of $U$ are evaluated at the point $(X_0, \epsilon) = (X, 0)$. Clearly, to a second-order approximation, the welfare effect of a temporary stimulus is smaller when the economy is in a liquidation phase if and only if $U_{X\epsilon} < 0$.

Next, using the envelope condition as in the proof of Proposition 3.16, it is straightforward to obtain that

$$U_\epsilon(X_0, 0) = \frac{1}{F'(\ell^*)\ell^*} \sum_{t=0}^{\infty} \beta^t \left[ L^* + \tau \frac{\nu'(\ell^*)}{F'(\ell^*)} \right] \tilde{e}_t'(0)$$

where $X_t = \tilde{X}_t(0)$ is the stock of durables that would occur in the absence of stimulus. One may also obtain that

$$\tilde{e}_t'(0) = \begin{cases} 1 & : t = 0 \\ \gamma e'(X_t) \left\{ \prod_{i=1}^{t-1} [1 - \delta + \gamma e'(X_{t-i})] \right\} & : t \geq 1 \end{cases}$$
so that
\[
\mathcal{U}_e(X_0, 0) = \frac{1}{F'(\ell^*)\ell^*} \left\{ \mathcal{L}^* + \tau \frac{\nu'(\ell^*)}{F'(\ell^*)} e(X_0) + \gamma \sum_{t=1}^{\infty} \beta^t \left[ \mathcal{L}^* + \tau \frac{\nu'(\ell^*)}{F'(\ell^*)} e(X_t(X_0)) \right] 
\right. 
\left. \right. 
\left. \right. 
\left[ e'(X_t(X_0)) \prod_{i=1}^{t-1} \left[ 1 - \delta + \delta e'(X_{t-i}(X_0)) \right] \right) \right\}
\]

where \(X_t(X_0)\) indicates the equilibrium value of \(X_t\) given \(X_0\). Taking the derivative of this expression with respect to \(X_0\) and evaluating at \(X_0 = X\) yields
\[
\mathcal{U}_e(X_0, 0) = \frac{1}{F'(\ell^*)\ell^*} \nu'(\ell^*) \cdot \frac{1 - \beta \lambda_1 (1 - \delta)}{1 - \beta \lambda_1^2} \psi + \Xi e''(X)
\]

where \(\psi \equiv e'(X) < 0\), which was computed above, and \(\Xi\) is some strictly positive number. Since \(\lambda_1 < 1\), the first term on the right-hand side of this expression is clearly negative. Thus, there is a strictly positive number \(\xi\) such that if \(e''(X) < \xi\) we will have \(\mathcal{U}_e(X_0, 0) < 0\), which is the desired result.

Letting \(\chi(X_t)\) denote the equilibrium value of \(X_{t+1}\) given \(X_t\), we may re-express the equilibrium equations governing the dynamics of the system (i.e., equations (3.22) and (3.23)) as
\[
\chi(X_t) = (1 - \delta)X_t + \gamma e(X_t)
\]

and
\[
U'(X_t + e(X_t)) - Q(e(X_t)) = \beta \left[ (1 - \delta - \gamma)U'\left(\chi(X_t) + e(\chi(X_t))\right) + (1 - \delta)Q(e(\chi(X_t))) \right]
\]

Taking derivatives of both sides of these equations twice with respect to \(X_t\), evaluating at \(X_t = X\) and solving for \(e''(X)\), we may obtain that \(e''(X) = bU'''(X + e)\), where \(b\) is some number that does not depend on \(U'''(X + e)\). Thus, if \(U'''\) is sufficiently close to zero, \(e''(X) < \xi\) and the desired result holds.

**B.2 Introducing Nash bargaining**

Here we consider the static model of section 3.2 and replace the “competitive” determination of \(w\) and \(\ell\) within a match by Nash bargaining.

The gain from a match for a firm is \(pF(\ell) - w\ell\) while outside option is zero. The gain for the household is \(-\nu(\ell) + V(w\ell - p(c - X))\) while the outside option is \(V(-p(c - X))\). Using the piecewise linear specification for \(V\), the Nash-Bargaining criterion \(\mathcal{W}\) is:
\[
\mathcal{W} = \left( pF(\ell) - w\ell \right) \psi \left( -\nu(\ell) + \nu w\ell + \nu \tau p(c - X) \right)^\psi
\]
Maximizing $W$ w.r.t. $\ell$ and $w$ gives the following F.O.C.:

$$\frac{\psi W}{pF(\ell) - w\ell} \left( pF'(\ell) - w \right) = \frac{(1 - \psi)W}{-\nu(\ell) + vw\ell + v\tau p(c - X)} \left( vw - \nu'(\ell) \right)$$

$$\frac{\psi W}{pF(\ell) - w\ell} = \frac{(1 - \psi)W}{-\nu(\ell) + vw\ell + v\tau p(c - X)}$$

Rearranging gives the two equations

$$vpF'(\ell) = \nu'(\ell)$$

$$vw\ell = (1 - \psi)pF(\ell) + \psi\nu(\ell) - \psi\nu'(\ell) + \tau(p(c - X))$$

Assuming that the matching function is “min”, the equilibrium is given by the five following equations:

$$u'(c) = \frac{\nu'(\ell)}{F'(\ell)} \left( 1 + \tau - \frac{\min\{N, L\}}{L} \tau \right)$$  \hspace{1cm} (B.32)

$$w\ell = (1 - \psi)pF(\ell) + \frac{\psi}{\nu}(\ell) - \psi\nu(p(c - X))$$  \hspace{1cm} (B.33)

$$vpF'(\ell) = \nu'(\ell)$$  \hspace{1cm} (B.34)

$$\min\{N, L\}F(\ell) = L(c - X) + N\Phi$$  \hspace{1cm} (B.35)

$$\min\{N, L\}(pF(\ell) - w\ell) = pN\Phi$$  \hspace{1cm} (B.36)

Equations (B.33) and (B.34) determine $p$ and $w$ once $N$, $c$ and $\ell$ are determined by the three other equations. After some manipulations, those three equations (B.32), (B.35) and (B.36) can be written:

$$u'(c) = \frac{\nu'(\ell)}{F'(\ell)} \left( 1 + \tau - \frac{\min\{N, L\}}{L} \tau \right)$$

$$\min\{N, L\} \frac{(c - X)}{L} = \frac{(c - X)}{(1 - \psi)pF(\ell) + \psi pF'(\ell) \frac{\nu'(\ell)}{\nu'(\ell)} - \psi\tau p(c - X)}$$

$$\psi \min\{N, L\} \frac{\nu'(\ell)}{N} = \Phi \left( F(\ell) - F'(\ell) \frac{\nu'(\ell)}{\nu'(\ell)} + \tau(c - X) \right)^{-1}$$

In the unemployment regime, those equations write

$$u'(c) = \frac{\nu'(\ell)}{F'(\ell)} \left( 1 + \tau - \frac{N}{L} \tau \right)$$

$$\frac{N}{L} \frac{(c - X)}{(1 - \psi)pF(\ell) + \psi pF'(\ell) \frac{\nu'(\ell)}{\nu'(\ell)} - \psi\tau p(c - X)}$$

$$\Phi \frac{\nu'(\ell)}{\psi} = \frac{\Phi}{\psi} \left( F(\ell) - F'(\ell) \frac{\nu'(\ell)}{\nu'(\ell)} + \tau(c - X) \right)$$

Main difference with the model of the main text is that (B.42) does not determine $\ell$ independently of
(B.40) and (B.41). But it is still the case that, assuming \( \ell \) is fixed, (B.40) implies that if \( N \) is high, \( c \) will be high and (B.41) implies that if \( c \) is high, \( N \) will be high. As far as (B.42) implies that \( \ell \) does not vary too much, subsequent results of section 3.2 hold. This can be illustrated with a numerical example that reproduces Figures 3.2, 3.3 and 3.5.

Consider the functional forms

\[
\nu(\ell) = \frac{\nu_1 \ell^{1+\omega}}{1+\omega}, \quad F(\ell) = \theta_1 A \ell^\alpha, \quad u(c) = \ln c, \quad V(a) = \frac{\nu_2}{\theta_2} \text{if } a \geq 0 \quad \text{and} \quad (1 + \tau) \frac{\nu_2}{\theta_2} \text{if } a < 0.
\]

Common parameters values are \( \psi = .5, \omega = 1.2, \nu_1 = .5, \alpha = .67, A = 1, \Phi = .35, L = 1 \). Solving for the equilibrium in such a case produce Figures B.1, B.2 and B.3, which are qualitatively similar to Figures 3.2, 3.3 and 3.5.

**Figure B.1:** The Model with Nash bargaining, consumption as function of \( X \).

![Graph of Figure B.1](image)

*Note: Example is constructed assuming the functional forms \( \nu(\ell) = \frac{\nu_1 \ell^{1+\omega}}{1+\omega}, \quad F(\ell) = \theta_1 A \ell^\alpha, \quad u(c) = \ln c, \quad V(a) = \frac{\nu_2}{\theta_2} \text{if } a \geq 0 \quad \text{and} \quad (1 + \tau) \frac{\nu_2}{\theta_2} \text{if } a < 0. \) Parameters values are \( \psi = .5, \omega = 1.2, \nu = v = .5, \alpha = .67, A = 1, \Phi = .35, L = 1 \) and \( \tau = .05. \)
**Figure B.2:** The Model with Nash bargaining, equilibrium determination

![Diagram](image)

Note: Example is constructed assuming the functional forms \( \nu(\ell) = \frac{\nu \ell^{1+\omega}}{1+\omega} \), \( F(\ell) = \alpha \ell^\alpha \), \( u(c) = \ln c \). \( V(a) \) is \( va \) if \( a \geq 0 \) and \( (1 + \tau)va \) if \( a < 0 \). Parameters values are \( \psi = .5, \omega = 1.2, \nu = v = .5, \alpha = .67, A = 1, \Phi = .35, L = 1 \) and \( \tau = .05 \). Values of \( X \) used were \( X = .3 \) for the full-employment equilibrium and \( X = 0.9 \) for the unemployment equilibrium.

**Figure B.3:** The Model with Nash bargaining, equilibrium determination (multiple equilibria)

![Diagram](image)

Note: Example is constructed assuming the functional forms \( \nu(\ell) = \frac{\nu \ell^{1+\omega}}{1+\omega} \), \( F(\ell) = \alpha \ell^\alpha \), \( u(c) = \ln c \). \( V(a) \) is \( va \) if \( a \geq 0 \) and \( (1 + \tau)va \) if \( a < 0 \). Parameters values are \( \psi = .5, \omega = 1.2, \nu = v = .5, \alpha = .67, A = 1, \Phi = .35, L = 1, \tau = .4, X = .6 \) or \( X = 1 \).
B.3 Noise shock extension

For the extension discussed at the end of Section 3.4.2, we re-introduce the first-sub-period $(\theta)$ and second-sub-period $(\tilde{\theta})$ productivity factors to the model, and assume that $\tilde{\theta}_t = \theta_t$. We assume that the economy is always in the unemployment regime, and that all agents come into the first sub-period of period $t$ with the same belief about the value of $\theta_t$, but that after the household splits to go to market, the true value is revealed to the workers and firms, while the shoppers retain their initial belief.

To abstract from issues relating to uncertainty about the true value of $\theta_t$, we assume that all agents are subjectively certain – though possibly incorrect – about the entire stream of productivity values $\theta_t$, only updating such a belief if they receive some information that contradicts it. One may verify that, in the unemployment regime, shoppers’ prior beliefs are never contradicted until re-uniting with the workers after making their purchases. We denote agents’ belief about $\theta_t$ at the beginning of date $s$ by $\tilde{\theta}_t|_s$.

In the example constructed, we assume that productivity is constant at $\theta_t = 1$ for all $t \in \mathbb{Z}$, but that at the beginning of $t = 0$, agents receive information such that $\tilde{\theta}_t|_0 = \theta > 1$ for all $t \geq 0$, i.e., that productivity has risen permanently. After the households split, workers and firms learn that in fact productivity has not changed, nor will it in the future. Shoppers do not receive this information until after making their purchases, so that for one shopping period they are overly optimistic. In all subsequent periods $s \geq 1$, however, we have $\tilde{\theta}_t|_s = \theta_t = 1$. 

150
Appendix C

Appendix for “Can a Limit-Cycle Model Explain Business Cycle Fluctuations?”

C.1 Proofs

C.1.1 Proof of Proposition 4.1

Recall that

\[ X_{t+1} = (1 - \delta) (X_t + e_t) \]

Since \(e_t \geq 0\), if \(\limsup_{t \to \infty} |X_t| = \infty\) then \(\limsup_{t \to \infty} X_t = \infty\). Suppose then that

\[ \limsup_{t \to \infty} X_t = \infty \]

Since \(\delta \in (0, 1]\), this necessarily implies that \(\limsup_{t \to \infty} e_t = \infty\). But \(e_t\) is bounded above by the level of output, the maximum feasible level of which occurs when \(\phi_t = 1\) and \(\ell_t = \bar{\ell}\), in which case total output is given by \(F(\bar{\ell}) < \infty\). Thus we clearly cannot have \(\limsup_{t \to \infty} e_t = \infty\), and thus we cannot have \(\limsup_{t \to \infty} |X_t| = \infty\).

\[\square\]

C.1.2 Proof of Proposition 4.2

The proof proceeds by example, showing that, for the case where \(\gamma = 1\) and \(U(c) = ac - \frac{b}{2}c^2\), there exists parameter values and functional forms such that for \(\beta\) close enough to one the steady state is unstable.

With \(\gamma = 1\) and \(U(c) = ac - \frac{b}{2}c^2\), we may characterize the evolution of this system by the
conditions

\[ a - b (X_t + e_t) = vp(e_t) [1 + \tau - \tau \phi(e_t)] - \beta (1 - \delta) vp(e_{t+1}) [1 + \tau - \tau \phi(e_{t+1})] \quad \text{(C.1)} \]

\[ X_{t+1} = (1 - \delta) (X_t + e_t) \quad \text{(C.2)} \]

where \( p(\cdot) \) and \( \phi(\cdot) \) are as in the static model. For a given state \( X_t \) and a given anticipated level of \( e_{t+1} \), a sufficient condition to ensure that (C.1) has a unique solution is given by

\[ b > vp^* \frac{\tau}{e^*} \equiv b_0 \quad \text{(C.3)} \]

where \( e^* \) is output per firm (net of fixed costs) when the economy is in the unemployment regime and \( p^* \) is the price in the unemployment regime, as described in section 4.5.1 for the static model (see footnote 29 regarding \( p^* \)). I henceforth assume that (C.3) holds.

Next, the steady-state level of \( e \) is given by the solution \( \bar{e} \) to

\[ a - \frac{b}{\delta} \bar{e} = [1 - \beta (1 - \delta)] \cdot vp^* (1 + \tau - \tau \phi(\bar{e})) \]

with the steady-state level of \( X \) then given by

\[ \bar{X} = \frac{1 - \delta}{\delta} \bar{e} \]

Note that a sufficient condition for the steady state to be unique is given by

\[ b > \delta [1 - \beta (1 - \delta)] b_0 \]

which is clearly implied by (C.3).

Next, note that, for any \( e \in (0, e^*) \) (i.e., in the unemployment regime), the level of \( a \) that implements \( \bar{e} = e \) is given by

\[ \frac{b}{\delta} e + [1 - \beta (1 - \delta)] \cdot vp^* \left( 1 + \tau - \frac{\tau e}{e^*} \right) \]

Note also that \( \bar{e} \) is continuous in \( \beta \). Thus, choose some \( \bar{e}_1 \in (0, e^*) \), and let \( a = a_1 \), where \( a_1 \) is the value of \( a \) that would implement \( \bar{e} = \bar{e}_1 \) when \( \beta = 1 \), i.e.,

\[ a_1 \equiv \frac{b}{\delta} \bar{e}_1 + \delta vp^* \left( 1 + \tau - \frac{\bar{e}_1}{e^*} \right) \]

Thus, if \( \beta = 1 \) the steady state is in the unemployment regime by construction, and by continuity of \( \bar{e} \) in \( \beta \) the steady state is also necessarily in the unemployment regime for \( \beta \) sufficiently close to one. This implies the existence of a \( \bar{\beta} < 1 \) such that the steady state is in the unemployment regime when \( \beta > \bar{\beta} \). Assume henceforth that \( \beta \in (\bar{\beta}, 1) \) and note that this implies that \( p'(\bar{e}) = 0 \) and \( \phi' (\bar{e}) = 1/e^* \).
Next, linearizing equations (C.2)-(C.3) around this steady state and solving, we may obtain in matrix form

\[
\begin{pmatrix}
\hat{X}_{t+1} \\
\hat{e}_{t+1}
\end{pmatrix}
= \begin{pmatrix}
1 - \delta & 1 - \delta \\
-\frac{b}{\beta(1-\delta)b_0} & -\frac{b-b_0}{\beta(1-\delta)b_0}
\end{pmatrix}
\begin{pmatrix}
\hat{X}_t \\
\hat{e}_t
\end{pmatrix}
\equiv A
\begin{pmatrix}
\hat{X}_t \\
\hat{e}_t
\end{pmatrix}
\]

Thus, the steady state is locally stable if and only if at least one of the two eigenvalues of \(A\) lies inside the complex unit circle. These eigenvalues are given by

\[
\lambda_i = \frac{1 - \delta - \frac{b-b_0}{\beta(1-\delta)b_0}}{2} \pm \sqrt{\left(1 - \delta - \frac{b-b_0}{\beta(1-\delta)b_0}\right)^2 - 4\beta^{-1}}
\]

Note that \(\lambda_1\lambda_2 = \beta^{-1} > 1\), so that if the eigenvalues are complex then both must lie outside the unit circle. Suppose

\[
b = \left[1 + q (1 - \delta)^2\right] b_0
\]

for some \(q > 0\), and note that as long as \(\delta < 1\), which I henceforth assume, such a value of \(b\) satisfies (C.3). One may then show that the eigenvalues are complex as long as

\[
(1 - \delta)^2 (\beta - q)^2 < 4\beta
\]

Clearly, for \(\beta\) close enough to \(q\) this condition necessarily holds, and thus, if \(q\) is close enough to one (e.g., if \(q = 1\)), then for \(\beta\) arbitrarily close to one the eigenvalues are complex and therefore outside the unit circle, in which case the steady state is unstable.

\[\square\]

C.1.3 Proof of Proposition 4.3

Let

\[
\mathcal{V}(e_t; X_t) \equiv U(X_t + e_t) - vp^* \left[(1 + \tau) e_t - \frac{1}{2} \tau e_t^2\right]
\]

where \(e^*\) is output per firm (net of fixed costs) when the economy is in the unemployment regime and \(p^*\) is the price in the unemployment regime, as described in section 4.5.1 for the static model (see footnote 29 regarding \(p^*\)). It can be verified that maximizing

\[
\sum_{t=0}^{\infty} \beta^t \mathcal{V}(e_t; X_t)
\]

subject to (4.2) implements the de-centralized equilibrium outcome in the neighborhood of an unemployment-regime steady state. Thus, using

\[
\mathcal{W}(X_t, X_{t+1}) \equiv \mathcal{V}\left(\frac{1}{\gamma (1-\delta)} X_{t+1} - \frac{1}{\gamma} X_t; X_t\right)
\]

153
in problem (4.4) satisfies the desired properties. Next, we may obtain
\[ W_{11}(\bar{X}, \bar{X}) = \frac{(1 - \gamma)^2}{\gamma^2} U''(\bar{X} + \bar{e}) + \frac{1}{\gamma^2} \frac{vp^* \tau}{e^*} \]
Thus, \( W_{11}(\bar{X}, \bar{X}) > 0 \) if \( \frac{vp^* \tau}{e^*} > - (1 - \gamma)^2 U''(\bar{X} + \bar{e}) \)
This condition can clearly hold for certain parameter values (e.g., for \( \gamma \) sufficiently close to one), in which case \( W \) is not concave.

### C.1.4 Proof of Proposition 4.4

We show that the steady state is locally stable when \( \tau = 0 \). By continuity of all relevant functions, it then follows that the steady state is locally stable for \( \tau > 0 \) sufficiently small.

When \( \tau = 0 \), equilibrium is characterized by the equations
\[ U'(X_t + e_t) - vp (e_t) = \beta (1 - \delta) (1 - \gamma) U'(X_{t+1} + e_{t+1}) - \beta (1 - \delta) vp (e_{t+1}) \]
\[ X_{t+1} = (1 - \delta) (X_t + \gamma e_t) \]
Assume the steady state is in the unemployment regime, so that \( p(e_t) = p^* \) in a neighborhood of the steady state.\(^1\) Linearizing around this steady state, assuming \( \beta (1 - \delta) (1 - \gamma) > 0 \) we may obtain in matrix form
\[
\begin{pmatrix}
\dot{X}_{t+1} \\
\dot{e}_{t+1}
\end{pmatrix} =
\begin{pmatrix}
1 - \delta & (1 - \delta) \gamma \\
\frac{1 - \beta (1 - \delta)^2 (1 - \gamma)}{\beta (1 - \delta)(1 - \gamma)} & \frac{1 - \beta (1 - \delta)^2 (1 - \gamma)}{\beta (1 - \delta)(1 - \gamma)}
\end{pmatrix}
\begin{pmatrix}
X_t \\
e_t
\end{pmatrix}
\equiv A
\begin{pmatrix}
X_t \\
e_t
\end{pmatrix}
\]
The steady state is locally stable if at least one of the eigenvalues of \( A \) lies inside the unit circle. It is straightforward to show that the smallest eigenvalue of \( A \) is given by \( \lambda_1 = (1 - \delta) (1 - \gamma) \), which is clearly less than one in modulus. Thus, the steady state is locally stable. If instead \( \beta (1 - \delta) (1 - \gamma) = 0 \), then \( \dot{e}_t = -\dot{X}_t \) and thus \( \dot{X}_{t+1} = \lambda_1 \dot{X}_t \), which is clearly a stable system as well.

\(^1\)It is straightforward to verify that a full-employment-regime steady state must be stable.
C.2 Solution and estimation

C.2.1 Solution

To solve the model for a given parameterization, letting \( \tilde{e}_t \equiv e_t / \tilde{\theta}_t \) equilibrium in the economy is characterized by the following equations:

\[
a - b \left( X_t + \tilde{\theta}_t \tilde{e}_t - hc_{t-1} \right) + (1 - \delta) \gamma \lambda_t = \tilde{\theta}_t^{-1} \frac{\mu_t}{\alpha A} [\ell (\tilde{e}_t)]^{\omega + 1 - \alpha} [1 + \tau - \tau \phi (\tilde{e}_t)] + \mu_t \tag{C.5}
\]

\[
\mu_t = \mathbb{E}_t \left\{ \beta h \left[ a - b \left( X_{t+1} + \tilde{\theta}_{t+1} \tilde{e}_{t+1} - hc_t \right) \right] \right\} \tag{C.6}
\]

\[
\lambda_t = \mathbb{E}_t \left\{ \beta \left[ a - b \left( X_{t+1} + \tilde{\theta}_{t+1} \tilde{e}_{t+1} - hc_t \right) \right] + (1 - \delta) \lambda_{t+1} - \mu_{t+1} \right\} \tag{C.7}
\]

\[
c_t = X_t + \tilde{\theta}_t \tilde{e}_t \tag{C.8}
\]

\[
X_{t+1} = (1 - \delta) \left( X_t + \gamma \tilde{\theta}_t \tilde{e}_t \right) \tag{C.9}
\]

Here, \( \phi (\tilde{e}) \) and \( \ell (\tilde{e}) \) are the equilibrium levels of the employment rate and hours-per-worker conditional on total purchases \( \bar{e} \), and are given by

\[
\phi (\tilde{e}) \equiv \begin{cases} 
\frac{1}{2} \left( n_0 + \sqrt{n_0^2 + 4\eta \bar{e}} \right) & \text{if } 0 < \tilde{e} \leq \bar{e} \\
\frac{\bar{e}}{\bar{e}_*} & \text{if } \bar{e} < \tilde{e} < e^* \\
1 & \text{if } \tilde{e} \geq e^*
\end{cases}
\]

\[
\ell (\tilde{e}) \equiv \begin{cases} 
\left[ \frac{2\bar{e}}{\alpha A \left( n_0 + \sqrt{n_0^2 + 4\eta \bar{e}} \right)} \right]^\frac{1}{\alpha} & \text{if } 0 < \tilde{e} \leq \bar{e} \\
\left( \frac{e^*}{\alpha A} \right)^\frac{1}{\alpha} & \text{if } \bar{e} < \tilde{e} < e^* \\
\left( \frac{\bar{e}}{\alpha A} \right)^\frac{1}{\alpha} & \text{if } \tilde{e} \geq e^*
\end{cases}
\]

where \( e^* \equiv \frac{\alpha}{1 - \alpha} \bar{e} \) and \( \bar{e} \equiv (n_0 + \eta) e^* \). Meanwhile, \( \mu_t \) and \( \lambda_t \) are the Lagrange multipliers on the definition of consumption and the durables accumulation equations ((C.8) and (C.9)), respectively.

Conditional on the state variables \( X_t, c_{t-1} \) and \( \theta_t \), and on values of the Lagrange multipliers \( \mu_t \) and \( \lambda_t \), equation (C.5) can be solved for \( \tilde{e}_t \). To obtain values of \( \mu_t \) and \( \lambda_t \), I employ the method of parameterized expectations as follows. Let \( Y_t \equiv (X_t - \bar{X}, c_{t-1} - \bar{c}, \theta_t)' \) denote the vector of state variables (expressed as deviations from steady state). The expectations in equations (C.6) and (C.7) are assumed to be functions only of \( Y_t \), i.e.,

\[
\mathbb{E}_t \left\{ \beta h \left[ a - b \left( X_{t+1} + \tilde{\theta}_{t+1} \tilde{e}_{t+1} - hc_t \right) \right] \right\} = g_{\mu} (Y_t)
\]

where
\[
\mathbb{E}_t \left\{ \beta \left[ a - b \left( X_{t+1} + \tilde{\theta}_{t+1}\tilde{e}_{t+1} - hc_t \right) + (1 - \delta) \lambda_{t+1} - \mu_{t+1} \right] \right\} = g_\lambda (Y_t)
\]

I parameterize the functions \( g_j (\cdot) \) by assuming that they are well-approximated by \( N \)-th-degree multivariate polynomials in the state variables. In particular, let \( Y_t^{(N)} \) denote the vector whose first element is 1 and whose remaining elements are obtained by collecting all multivariate polynomial terms in \( Y_t \) (e.g., \( X_t, c_{t-1}, \theta_t, X_t^2, X_t c_{t-1}, X_t \theta_t, c_t^2, c_t \theta_t, \text{etc.} \)) up to degree \( N \). I assume that

\[
g_j (Y_t) = \Theta_j' Y_t^{(N)}
\]

where \( \Theta_j \) is a vector of coefficients on the polynomial terms. Thus, given \( \Theta_\mu, \Theta_\lambda \) and the state \( Y_t, \mu_t \) and \( \lambda_t \) are obtained as

\[
\begin{align*}
\mu_t &= \Theta_\mu' Y_t^{(N)} \\
\lambda_t &= \Theta_\lambda' Y_t^{(N)}
\end{align*}
\]

These values and values for the state variables can be plugged into (C.5) to yield a solution for \( \tilde{e}_t \), which can then be replaced in (C.8) and (C.9) to obtain values for the subsequent period’s state. In practice, I use \( N = 2 \).\(^2\)

To obtain \( \Theta_\mu \) and \( \Theta_\lambda \), I proceed iteratively as follows. Begin with some initial guesses \( \Theta_{\mu,0} \) and \( \Theta_{\lambda,0} \),\(^3\) and generate a sample of length \( T = 100,000 \) of the exogenous process \( \theta_t \). Next, given \( \Theta_{\mu,i} \) and \( \Theta_{\lambda,i} \), assume that \( g_j (Y_t) = \Theta_{j,i}' Y_t^{(N)} \) and simulate the path of the economy for \( T \) periods. Given this simulated path, let \( Y^{(N)} \) denote the matrix whose \( t \)-th row is given by \( Y_t^{(N)}' \), and construct \( T \)-vectors \( \tilde{g}_\mu \) and \( \tilde{g}_\lambda \), the \( t \)-th elements of which are given respectively by

\[
\beta h \left[ a - b \left( X_{t+1} + \tilde{\theta}_{t+1}\tilde{e}_{t+1} - hc_t \right) \right]
\]

and

\[
\beta \left[ a - b \left( X_{t+1} + \tilde{\theta}_{t+1}\tilde{e}_{t+1} - hc_t \right) + (1 - \delta) \lambda_{t+1} - \mu_{t+1} \right]
\]

i.e., the terms inside the conditional-expectation operators in equations (C.6) and (C.7). Then update the guesses of \( \Theta_j \) via

\[
\Theta_{j,i+1} = \left( Y^{(N)}' Y^{(N)} \right)^{-1} Y^{(N)}' \tilde{g}_j
\]

and iterate until convergence.

---

\(^2\)I experimented with larger values of \( N \) and found that it resulted in a substantial increase in computational time without significantly affecting the results.

\(^3\)In practice, I set the first elements of \( \Theta_{\mu,0} \) and \( \Theta_{\lambda,0} \) to the steady-state values \( \bar{\mu} \) and \( \bar{\lambda} \), respectively, and the remaining elements to zero. This corresponds to an initial belief that the \( g_j \)'s are constant and equal to their steady-state levels.
C.2.2 Estimation

As discussed in section 4.6.1, estimation was done by searching for parameters to minimize $\overline{S^2}$, the average squared difference between the model spectrum and the spectrum estimated from the data.

To obtain $\overline{S^2}$ given a solution to the model for a parameterization, $T = 100,000$ periods of data were simulated. This simulated sample was then subdivided into $N_{\text{sim}} = 1,000$ overlapping subsamples. For each subsample, the log of hours was BP-filtered, after which 20 quarters from either end of the subsample were removed, leaving a series of the same length as the actual data sample. The spectrum was then estimated on each individual subsample in the same way as for the actual data, and the results then averaged across all subsamples to yield the spectrum for the model.

C.3 Definitions

A deterministic dynamic system characterizing the evolution of a state vector $z(t)$ over time can be expressed as a function $G : T \times Z \rightarrow Z$. Here, $T$ is the set of dates at which the system is defined (e.g., $T = \mathbb{R}$ in a continuous-time formulation, and $T = \mathbb{Z}$ in a discrete-time formulation), while $Z \subset \mathbb{R}^n$ is the $n$-dimensional state space. The function $G$ takes a date $t \in T$ and a date-0 state $z(0) = z_0$ as inputs and returns $z(t) = G(t, z_0)$.\(^4\) We focus here on time-invariant dynamic systems, i.e., systems for which $G(t + \Delta t, z_0) = G(\Delta t, G(t, z_0))$. We have the following definition.

**Definition C.1.** $G$ exhibits **deterministic fluctuations** if, for some $z_0$, the following hold.

(a) $\limsup_{t \to \infty} \|G(t, z_0)\| < \infty$.

(b) $\lim_{t \to \infty} G(t, z_0)$ does not exist.

In words, the system exhibits deterministic fluctuations if it neither diverges to infinity nor converges to a single point. Of particular interest for us will be one type of deterministic fluctuation, the limit cycle, defined as follows.

**Definition C.2.** A subset $L \subset Z$ is a **limit cycle** of $G$ with prime period $k > 0$ if the following hold.

(a) For any $z \in L$ and $\Delta t \in (0, k)$, we have $G(k, z) = z$ and $G(\Delta t, z) \neq z$.

(b) For any $z, z' \in L$ there is some $\Delta t \geq 0$ such that $G(\Delta t, z) = z'$.

(c) Let $\Gamma(t, z)$ denote the distance between $G(t, z)$ and the closest point in $L$.\(^5\) Then there exists some $z \notin L$ such that either $\lim_{t \to \infty} \Gamma(t, z) = 0$ or $\lim_{t \to -\infty} \Gamma(t, z) = 0$.

The first property here says that, if the system starts at a point in $L$, then it will return to that point $k > 0$ periods later (and no sooner). The second property says that as the system evolves beginning from any point in $L$ it will visit every other point in $L$ at some subsequent date. Finally, the third

\(^4\)Note that by definition $G(0, z_0) = z_0$.

\(^5\)Properties (a) and (b) ensure that $L$ is necessarily a closed set, so that such a closest point always exists.
property says that there is some point not in the limit cycle such that, beginning from that point, the system will eventually converge to the limit cycle as it moves either forward or backward through time.

C.4 Solving the model forward

In the non-stochastic case, we may re-arrange equation (C.5) to yield

$$a - bX_t + bh c_{t-1} + (1 - \delta) \gamma \lambda_t - \mu_t = \frac{\nu_1}{\alpha A} [l (e_t)]^{\omega + 1 - \alpha} [1 + \tau - \tau \phi (e_t)] + be_t \equiv H (e_t)$$

Thus, given the state variables $X_t$ and $c_{t-1}$ and current values of $\mu_t$ and $\lambda_t$, we may obtain

$$e_t = H^{-1} (a - bX_t + bh c_{t-1} + (1 - \delta) \gamma \lambda_t - \mu_t)$$

where the conditions in Proposition 4.5 ensure that $H$ is an invertible function. This value of $e_t$ then gives $c_t$ and $X_{t+1}$ via equations (C.8) and (C.9), respectively. From equations (C.6) and (C.7) we can then solve for $\mu_{t+1}$ and $\lambda_{t+1}$ as

$$\mu_{t+1} = \frac{a - bX_{t+1} + bh c_t - H \left( \frac{1}{b} \right) \left( a - bX_{t+1} + bh c_t - \frac{1}{b} \mu_t \right) - \gamma \left( \frac{1}{b} \mu_t - \frac{1}{b} \lambda_t \right)}{1 - \gamma}$$

$$\lambda_{t+1} = \frac{a - bX_{t+1} + bh c_t - H \left( \frac{1}{b} \right) \left( a - bX_{t+1} + bh c_t - \frac{1}{b} \mu_t \right) - \left( \frac{1}{b} \mu_t - \frac{1}{b} \lambda_t \right)}{(1 - \delta) (1 - \gamma)}$$