Regularity of Minimal Surfaces

A Self-contained Proof

by

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B.Sc., The University of Manitoba, 2013

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE AND POSTDOCTORAL STUDIES
(Mathematics)

The University Of British Columbia
(Vancouver)

April 2015

Abstract

In this thesis, a self-contained proof is given of the regularity of minimal surfaces via viscosity solutions, following the ideas of L. Caffarelli,X.Cabré [2], O.Savin[11][12], E.Giusti[7] and J.Roquejoffre[8], where we expand upon the ideas and give full details on the approach. Basically the proof of the program consists of four parts: 1) Density and measure estimates, 2) Viscosity solution methods of elliptic equations , 3) a geometric Harnack inequality and 4) iteration of the De Giorgi flatness result.

Preface

The topic of this thesis and the methodology used for proving the results were suggested by my supervisors Dr. Ailana Fraser and Dr.Young-Heon Kim. This thesis surveys a collection of known results and while not original the author has tried to present and organize it in a way unique to the author. Moreover the author has provided more details where needed as well as a few modifications to make things more clear than they were presented in their original form.

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Chapter 1

Introduction

This thesis focuses on the study of regularity of minimal surfaces. There are many results concerning the regularity of minimal surfaces. There is the work of De Giorgi which he presented his first results in 1952 developing the ideas of Caccioppoli. The De Giorgi[7] approach to studying minimal surfaces is to view them as boundaries of sets or as the interfaces between two fluids. A full proof that more closely resembles De Giorgi's original proof can be found in the book by Giusti [7] and the style of techniques used from Federer book[5]. Here we opt for a somewhat more modern approach which makes use of a geometric Harnack inequality and viscosity solutions based on the works of Caffarelli and Córdoba[3],Savin[11] and Roquejoffre[8]. The main results are the Gauss-Green formula for functions in the class of bounded variation, a geometric Harnack inequality for solutions to a quasi linear elliptic equation and the regularity of flat solutions to the minimal surface equation. In this chapter we provide an introduction to the topic of this thesis, and establish the framework in which one can work with general sets, allowing us to then set up the space of admissible sets in which the minimal sets can be found.

1.1 Preliminaries

The purpose of this first chapter is to introduce the preliminary definitions, conventions and notation as well as some of the basic theory of minimal surfaces and functions of bounded variation. We follow [4], [8], [11], [12] as our main references.

Definition 1. *Let U be an open subset of $\mathbb{R}^n$. A function $f \in L^1(U)$ is said to have bounded variation in a region U if*

$$\sup \left| \int_U f div\phi dx \right| \leq \infty,$$

where the supremum is taken over all test function ϕ in $C_c^1(U;\mathbb{R}^n)$ with $\|\phi\|_\infty \leq 1$. We write $BV(U)$ to denote the collection of functions with bounded variation in U.

Since we are primarily interested in presenting a regularity theory of minimal surfaces in the context of minimal boundaries, we are more interested in the case when f is the indicator function of some measurable set E. Therefore we define the perimeter of E in an open set U to be the total variation of E in U.

Definition 2. *For a Lebesgue measurable subset $E \subset \mathbb{R}^n$, we say that E has finite perimeter in U if*

$$P(E;U) = \sup \left| \int_E div\phi dx \right| < \infty,$$

where the supremum is taken over all test functions $\phi \in C_c^1(U)$ with $\|\phi\|_\infty \leq 1$.

By the classical form of the Gauss-Green theorem, we have that for a set E with C^1 boundary, $P(E;U)$ coincides with the classical notion of perimeter of ∂E, i.e $\mathscr{H}^{n-1}(\partial E)$.

Here we are able to gain some insight into the structure of sets and functions of bounded variation, and are able to associate a measure and "normal vector" to a given set E or function f.

Theorem 1 (Representation theorem for BV_{loc} functions)**.** *Let U be a open subset of $\mathbb{R}^n$. Then for $f \in BV_{loc}(U)$, there exists a Radon measure μ on U and a μ-measureable function $\sigma : U \to \mathbb{R}^n$ such that*

(i) $|\sigma(x)| = 1$ *μ-a.e. and*

(ii) $\int_U f\, div\, \phi\, dx = -\int_U \phi \cdot \sigma\, d\mu$

for all $\phi \in C_c^1(U;\mathbb{R}^n)$.

For $f \in BV_{loc}(U)$ we write $\|Df\|$ or $\|\nabla f\|$ to denote the measure μ given in the above theorem, and Df or ∇f to denote the vector-valued measure given by the measure with density σ with respect to $\|Df\|$. Now, if $f = \mathbb{1}_E$ and E is a set of locally finite perimeter in U, we will write $\|\partial E\|$ for the measure μ and $\nu_E = -\sigma$, where we think of ν_E as the outward normal on E.

We endow the space $BV(U)$ with the norm $\|\cdot\|_{BV}$ given by

$$\|f\|_{BV} = \|f\|_{L^1(U)} + \|\nabla f\|(U),$$

which makes the set $BV(U)$ into a Banach space. The fact that $\|\cdot\|_{BV}$ is a norm follows from the fact that $\|\cdot\|_{L^1}$ is a norm and that $\int_U |\nabla\cdot|$ is a seminorm. All that is left to prove is that the space is complete with this metric.

Theorem 2 (BV is a Banach Space)**.** *Let U be an open subset of $\mathbb{R}^n$. Then the set of functions $BV(U)$ with the norm*

$$\|f\|_{BV} = \|f\|_{L^1(U)} + \|\nabla f\|(U),$$

is a Banach Space.

Proof. Let f_i be a Cauchy sequence in $BV(U)$. Then from the definition of $\|\cdot\|_{BV}$, we have that the sequence is also Cauchy in the L^1-norm and hence, by completeness of $L^1(U)$, there is a candidate limit function $f \in L^1(U)$ such that $\|f_i - f\|_{L^1(U)} \to 0$ as $i \to \infty$. Now, since the sequence f_i is Cauchy in the BV-norm, we have that $\|f_i\|_{BV}$ is bounded which gives an upper bound on the variation measures. Now by theorem 3 (see below), $f \in BV(U)$, so all that is left to show is that the f_i converge to f in the BV-norm. Since f_i converges to f in L^1, it is enough to show that

$$\|\nabla(f_i - f)\|(U) \to 0.$$

Let $\varepsilon > 0$ and let N be large enough so that for $n,m > N$ we have

$$\|f_m - f_n\|_{BV} < \varepsilon.$$

This implies that

$$\|\nabla(f_m - f_n)\|(U) < \varepsilon.$$

Since $f_n \to f$ in L^1 we have that $(f_m - f_n) \to (f_m - f)$ in L^1. Thus, by theorem 3, we have that

$$\|\nabla(f_m - f)\|(U) \leq \liminf \|\nabla(f_m - f_n)\|(U) \leq \varepsilon.$$

Hence f_i converges to f in the BV-norm, hence $BV(U)$ is a Banach space. □

Theorem 3 (Semicontinuity of measures)**.** *Suppose that U is an open subset of $\mathbb{R}^n$ and that we have a sequence of functions $f_k \in BV(U)$ and $f \in L^1(U)$ such that $f_k \to f$ in $L^1_{loc}(U)$. Then f is in $BV(U)$ and we have the following estimate on its variation measure:*

$$\|\nabla f\|(U) \leq \liminf_{k\to\infty} \|\nabla f_k\|(U).$$

Proof. Take any $\phi \in C_c^1(U;\mathbb{R}^n)$ such that $\|\phi\|_\infty \leq 1$. Then, from theorem 1, we have that

$$\begin{aligned}\int_U f div\phi dx &= \lim_{k\to\infty}\int_U f_k div\phi dx \\ &= -\lim_{k\to\infty}\int_U \phi \cdot d\|\nabla f_k\| \\ &\leq \liminf_{k\to\infty}\int_U d\|\nabla f_k\| \\ &= \liminf_{k\to\infty}\|\nabla f_k\|(U).\end{aligned}$$

Thus taking supremum over all ϕ in C_c^1 gives the desired result. □

We will make use of the following approximation theorem which we will take on faith (see [4] section 5.2.2).

Theorem 4 (Approximation by smooth functions)**.** *For $f \in BV(U)$, there exist functions $\{f_k\}_{k=1}^\infty \subset BV(U)\cap C^\infty(U)$ such that*

1. *$f_k \to f$ in $L^1(U)$ and*
2. *$\|\nabla f_k\|(U) \to \|\nabla f\|(U)$ as $k \to \infty$.*

Since there is a well-known coarea formula for Lipschitz functions, one might ask if this result can hold for functions of bounded variation. We can extend this result to the space of *BV* functions since, in some sense, the space of BV functions is the closure of C^1 functions under the BV-norm with the variation measure being its derivative. The coarea formula will be needed in the proof of the ABP estimate in chapter 3. For $f : U \to \mathbb{R}$ and $t \in \mathbb{R}$, we define the superlevel set of the function f, E_t, by

$$E_t = \{x \in U | f(x) > t\}.$$

The following theorem is from [4] with the proof modified to make use of the preexisting coarea formula for Lipschitz functions.

Theorem 5 (The coarea formula for BV Functions)**.** *Let U be an open subset of $\mathbb{R}^n$ and suppose that $f \in BV(U)$. Then*

$$\|\nabla f\|(U) = \int_{-\infty}^{\infty} \|\partial E_t\|(U) dt.$$

Proof. By making use of the coarea formula for Lipschitz functions we can quite easily extend the result to the space of BV functions by taking a smooth approximating sequence. For $f \in BV(U)$, take a smooth approximating sequence f_n as in theorem 4. For this sequence, we know that each f_n is Lipschitz, hence the coarea formula will hold. If we denote the superlevel set of each f_n by $E_t^n = \{x \in U \mid f_n(x) > t\}$, then we have that

$$\begin{aligned}
\|\nabla f\|(U) &= \lim_n \|\nabla f_n\|(U) \\
&= \lim_n \int_U |\nabla f_n| dx \\
&= \lim_n \int_{-\infty}^{\infty} \|\partial E_t^n\|(U) dt \\
&\geq \int_{-\infty}^{\infty} \|\partial E_t\|(U) dt \quad \text{by Fatou's Lemma.}
\end{aligned}$$

Now, to establish the other inequality we need to work with the definition of the variation measure. Take $\phi \in C_c^1(U;\mathbb{R}^n)$ such that $\|\phi\|_\infty \leq 1$. Then we have that $\int_U f div\phi dx = \int_{-\infty}^{\infty} \int_{E_t} div\phi dx dt$, from which we see that we get the other inequality.
First suppose that $f \geq 0$, so that

$$f(x) = \int_0^{\infty} \mathbb{1}_{E_t}(x) dt \quad (a.e \;\; x \in U).$$

Thus,

$$\begin{aligned}
\int_U f div\phi dx &= \int_U \left(\int_{-\infty}^{\infty} \mathbb{1}_{E_t}(x) dt \right) div\phi dx \\
&= \int_0^{\infty} \int_U \mathbb{1}_{E_t}(x) div\phi(x) dx dt \\
&= \int_0^{\infty} \int_{E_t} div\phi dx dt.
\end{aligned}$$

Likewise, if $f \leq 0$,

$$f(x) = \int_{-\infty}^{0} (\mathbb{1}_{E_t}(x) - 1) dt,$$

and hence

$$\begin{aligned}\int_U f div\phi dx &= \int_U \left(\int_{-\infty}^0 (\mathbb{1}_{E_t}(x) - 1)\, dt\right) div\phi(x) dx \\ &= \int_{-\infty}^0 \int_U (\mathbb{1}_{E_t} - 1) div\phi(x) dx dt \\ &= \int_{-\infty}^0 \int_{E_t} div\phi dx dt.\end{aligned}$$

The general case follows if we express f in its positive and negative pieces as $f^+ - f^-$. So we have that E_t has finite perimeter *a.e.*. Now, for ϕ as above, we have

$$\int_U f div\phi dx \leq \int_{-\infty}^{\infty} \|\partial E_t\|(U) dt.$$

Hence,

$$\|\nabla f\|(U) \leq \int_{-\infty}^{\infty} \|\partial E_t\|(U) dt,$$

which gives the other inequality and therefore giving equality, and the desired result is proved. □

Here we recall the classical notion of perimeter as the Hausdorff measure and compare it to our current notion of perimeter (see [9] or [1] for more details on this topic).

Definition 3. *Let (X,d) be a metric space. For any subset $U \subset X$, let diam U denote the diameter of the set U, that is*

$$diam\ U = \sup\{d(x,y) \mid x,y \in U\}.$$

Then, for s and δ positive, define the (s,δ)- dimensional outermeasure, $\mathscr{H}^s_\delta$, by

$$\mathscr{H}^s_\delta(U) = \frac{\omega_s}{2^s} \inf\left\{\sum_{i=1}^{\infty} (diam\ U_i)^s \mid U \subset \bigcup_{i=1}^{\infty} U_i,\ diam U_i < \delta\right\},$$

Where $\omega_s = \dfrac{\pi^{s/2}}{\Gamma(s/2+1)}$. Finally, the s-dimensional Hausdorff measure $\mathscr{H}^s$ is defined by taking δ to zero, that is

$$\mathscr{H}^s(U) = \sup_{\delta>0} \mathscr{H}^s_\delta(U) = \lim_{\delta\to 0} \mathscr{H}^s_\delta(U).$$

At first glance it would seem that trying to define the perimeter of a set E via the Hausdorff measure would restrict the definition to a smaller class of sets, but in fact this is not the case. Later we will state a result which tells us that the perimeter measure associated with a set E is nothing more than the restriction of the $n-1$ dimensional Hausdorff measure to the reduced boundary of the set E. In fact, when working in general metric measure spaces where there is no smooth or vector space structure like in Euclidean space, the perimeter measure of a set E is defined in terms of the restriction of the Hausdorff measure to the measure theoretic boundary of E.

In analysis and PDE, the integration by parts formula are both useful and important. Here we give results that can help extend the Gauss-Green formula to the class of BV functions. With the integration by part formula, we are able to validate some calculations using BV functions. First, we start by stating a series of density and measures estimates which can be used to show that the perimeter measure associated with a set is just $n-1$ dimensional Hausdorff measure restricted to its reduced boundary. The dimension of the Hausdorff measure makes sense since the perimeter should be a quantity of codimension 1. The presentation of the results below is based on that given in the book of Evans and Gariepy[4], with appropriate modifications made when needed.

Definition 4. *Let U be an open set in $\mathbb{R}^n$ and let E have finite perimeter in U. Then we say that a point $x \in U \cap supp(\|\partial E\|)$ is in the reduced boundary of E if the following hold*

(i) $\nu_E(x) := \lim_{r\to 0} \dfrac{\nabla E(B(x;r))}{\|\partial E\|(B(x;r))}$ *exists and*

(ii) $|\nu_E(x)| = 1.$

We denote the reduced boundary of E by $\partial^* E$, and note that the vector $\nu_E(x)$ is the normal to $\partial^* E$ at x.

One can think of the reduced boundary as the set of points where it makes sense to have a well defined unit normal vector ν_E to the boundary of the set E.

Lemma 6. *There exist positive constants $A_1, \dots, A_5$, depending only on n, such that, for each $x \in \partial^* E$,*

(i) $\liminf_{r\to 0} \dfrac{\mathscr{L}^n(B(x;r)\cap E)}{r^n} > A_1 > 0,$

(ii) $\liminf_{r\to 0} \dfrac{\mathscr{L}^n(B(x;r)\setminus E)}{r^n} > A_2 > 0,$

(iii) $\liminf_{r\to 0} \dfrac{\|\partial E\|(B(x;r))}{r^{n-1}} > A_3 > 0,$

(iv) $\limsup_{r\to 0} \dfrac{\|\partial E\|(B(x;r))}{r^{n-1}} \le A_4,$

(v) $\limsup_{r\to 0} \frac{\|\partial(E\cap B(x;r))\|(\mathbb{R}^n)}{r^{n-1}} \leq A_5$.

Results (i) and (ii) say that, for a given point $x \in \partial^* E$, the infinitesimal densities are nonzero. Results (iii) and (iv) say that the perimeter measure associated with E is comparable to $n-1$ dimensional Hausdorff measure.

The following theorem says that $\partial^* E$ has a very nice structure and that it is almost a smooth surface.

Theorem 7. *Assume that E has locally finite perimeter in* $\mathbb{R}^n$.

(i) Then

$$\partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N,$$

where

$$\|\partial E\|(N) = 0$$

and K_k is a compact subset of a C^1-hypersurface S_k. Furthermore,

(ii) $\nu_E|_{S_k}$ *is normal to* S_k *and*

(iii) $\|\partial E\| = \mathscr{H}^{n-1}|_{\partial^* E}$.

Now, if $\partial^* E$ is most of ∂E, then this would strongly suggest that ∂E is smooth in some sense. For example, if we were to randomly sample from E and then do some curve fitting on the data, then almost surely we would have that the fitted surface would be smooth. In theorem 12 we will show that for a minimal set E, in fact $\partial^* E$ is most of ∂E.

Definition 5. *Let $x \in \mathbb{R}^n$. We say that $x \in \partial_* E$, the measure theoretic boundary of E, if*

$$\limsup_{r\to 0} \frac{\mathscr{L}^n(B(x;r)\cap E)}{r^n} > 0$$

and

$$\limsup_{r\to 0} \frac{\mathscr{L}^n(B(x;r)\setminus E)}{r^n} > 0.$$

Then next lemma says that, for a BV set, the two types of boundaries, the reduced boundary and the measure boundary, are essentially the same. Here we present an enhanced and complete proof based on the one presented in the book of Evans and Gariepy[4].

Lemma 8. *(i)* $\partial^* E \subset \partial_* E$.

(ii) $\mathscr{H}^{n-1}(\partial_* E \setminus \partial^* E) = 0$

Proof. Assertion (i) follows from Lemma 6. The proof of assertion (ii) is very similar to Theorem 12, found in the next section, the only real difference is to establish a uniform lower bound on a measure estimate of $\|\partial E\|$ over balls. Since the mapping

$$r \mapsto \frac{\mathscr{L}^n(B(x;r)\cap E)}{r^n}$$

is continuous, if $x \in \partial_* E$, then there exists $0 < \alpha < \beta < 1$ and $r_j \to 0$ such that

$$\alpha \le \frac{\mathscr{L}^n(B(x;r_j)\cap E)}{\omega_n r_j^n} < \beta.$$

Thus

$$\min\{\mathscr{L}^n(B(x;r_j)\cap E), \mathscr{L}^n(B(x;r_j)\setminus E)\} \ge \min\{\alpha, 1-\beta\}\omega_n r_j^n,$$

so the isoperimetric inequality (see Evans and Gariepy[4] section 5.6.2 theorem 2) gives

$$\limsup_{r\to 0} \frac{\|\partial E\|(B(x;r))}{r^{n-1}} > 0.$$

For $x \in \partial_* E$, let

$$c(x) := \limsup_{r\to 0} \frac{\|\partial E\|(B(x;r))}{r^{n-1}} > 0.$$

Let $S = \partial_* E \backslash \partial^* E$ and define

$$L_k = \{x \in S : c(x) > \frac{1}{2^k}\}.$$

We have $S = \bigcup_{k=1}^{\infty} L_k$, and $\|\partial E\|(L_k) = 0$. Let $\varepsilon > 0$. Since $\|\partial E\|$ is a Radon measure, it is outer regular, so there is an open set $A_k^{\varepsilon} \subset \mathbb{R}^n$ containing L_k such that

$$\|\partial E\|(A_k^{\varepsilon}) \le \|\partial E\|(L_k) + \varepsilon 2^{-k} = \varepsilon 2^{-k}.$$

Let $\delta > 0$ and define the sets

$$\mathscr{F}_k := \left\{ B(x;r) \,|\, x \in L_k, B(x;r) \subset A_k^{\varepsilon}, 0 < r < \frac{\delta}{10}, \|\partial E\|(B(x;r)) \ge \frac{r^{n-1}}{2^k} \right\}.$$

$\mathscr{F}_k$ is a covering of L_k, so, by the Vitali covering lemma, there exists a countable disjoint collection

$\{B(x_i;r_i)\}_{i=1}^{\infty} \subset \mathscr{F}_k$ such that

$$L_k \subset \bigcup_{i=1}^{\infty} B(x_i;5r_i).$$

As $\operatorname{diam} B(x_i;5r_i) < \delta$, we see that this covering is one of the coverings which the infimum is taken over in the definition of $\mathscr{H}_\delta^{n-1}(L_k)$, so

$$\begin{aligned}
\mathscr{H}_\delta^{n-1}(L_k) &\leq \omega_{n-1} \sum_{i=1}^{\infty} (5r_i)^{n-1} &(1.1)\\
&\leq \omega_{n-1} 5^{n-1} 2^k \sum_{i=1}^{\infty} \|\partial E\|(B(x_i;r_i)) &(1.2)\\
&= \omega_{n-1} 5^{n-1} 2^k \|\partial E\|(\bigcup_{i=1}^{\infty} B(x_i;r_i)) &(1.3)\\
&\leq \omega_{n-1} 5^{n-1} 2^k \|\partial E\|(A_k) &(1.4)\\
&\leq \omega_{n-1} 5^{n-1} \varepsilon. &(1.5)
\end{aligned}$$

Letting $\varepsilon \to 0$ we get $\mathscr{H}_\delta^{n-1}(L_k) = 0$ and therefore

$$\mathscr{H}_\delta^{n-1}(S) = \mathscr{H}_\delta^{n-1}(\bigcup_{k=1}^{\infty} L_k) = \lim_{k\to\infty} \mathscr{H}_\delta^{n-1}(L_k) = 0.$$

Hence taking limits in δ gives $\mathscr{H}^{n-1}(S) = 0$.

□

Here we present the Gauss-Green formula from [4].

Theorem 9 (Generalized Gauss-Green Theorem)**.** *Let $E \subset \mathbb{R}^n$ have locally finite perimeter.*

(i) Then $\mathscr{H}^{n-1}(\partial_* E \cap K) < \infty$ for each compact set $K \subset \mathbb{R}^n$.

(ii) Furthermore, for $\mathscr{H}^{n-1}$ a.e. $x \in \partial_* E$, there is a unique measure theoretic unit outer normal $\nu_E(x)$ such that

$$\int_E div\, \phi\, dx = \int_{\partial_* E} \phi \cdot \nu_E \, d\mathscr{H}^{n-1}$$

for all $\phi \in C_C^1(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. By theorem 1 we have that

$$\int_E div\phi dx = \int_{\mathbb{R}^n} \phi \cdot \nu_E d\|\partial E\|.$$

But, since the support of the measure is concentrated on $\partial^* E$, we have that $\|\partial E\|(\mathbb{R}^n \setminus \partial^* E) = 0$, and so from lemma 7 we find that

$$\|\partial E\| = \mathscr{H}^{n-1}|_{\partial_* E}.$$

Thus the Gauss-Green formula holds for domains that are of bounded variation. □

1.2 Minimal Sets And The Monotonicity Formula

In geometry and PDE, the monotonicity formulas have been proven to lead to very useful results such as the Morawetz and virial identities from nonlinear dispersive equations. Here we develop a monotonicity formula for minimal sets and use it to show that most of the content of ∂E is contained in $\partial^* E$. But first, to justify some calculations, we must introduce the trace class for functions of bounded variation. One can extend the notion of the trace operator from Sobolev functions to *BV* functions. To do this, assume that U is an open bounded set with Lipschitz boundary. Since ∂U is Lipschitz, it has a unit outer normal vector $\mathscr{H}^{n-1}$-a.e by Rademacher's theorem. In this setting we can define the trace class for functions of bounded variation. We follow [7] as a guide for this section.

Theorem 10. *Let U be an open bounded set with Lipschitz boundary. Then there exists a bounded linear operator*

$$T : BV(U) \to L^1(\partial U; \mathscr{H}^{n-1})$$

such that for $f \in BV(U)$ and for almost every $x \in \partial U$ we have

$$\lim_{r\to 0} ⨍_{B(x;r)\cap U} |f - Tf(x)| = 0.$$

The function Tf, which we will also denote by $tr(f)$, is called the *trace* of f on ∂U, and is unique up to sets of $\mathscr{H}^{n-1}|_{\partial U}$ measure zero.

Let E be a set of *BV* in B_1 with $0 \in \partial E$. Define the function

$$\phi_E(r) = r^{1-n} \int_{B_r} |\nabla E|.$$

Now this quantity is invariant under the scaling

$$\phi_E(r) = \phi_{\frac{1}{r}E}(1)$$

since, for any $g \in C_0^1(B_1)$, if we define $h(rx) = g(x)$, then we get

$$\begin{aligned}\int_{B_1} \mathbb{1}_{\frac{1}{r}E} div(g) dx &= r \int_{B_1} \mathbb{1}_E(rx) div(h)(rx) dx \\ &= r^{1-n} \int_{B_r} \mathbb{1}_E div(h) dx.\end{aligned}$$

Hence, taking the supremum over all test functions gives the invariance. In fact, if E is scale invariant, i.e. E is a cone, then ϕ is constant.

Below is a precise definition based on the informal one presented in [3]

Definition 6. *We say that a set E is a minimal surface in Ω if, for every open set $A \subset \Omega$which is relatively compact in Ω, we have*

$$P(E;A) \leq P(F;A)$$

whenever E and F coincide outside a compact set included in A.

The Theorem below is from [7].

Theorem 11. *If E has minimal perimeter in B_1 and $0 \in \partial E$, then ϕ_E is a increasing function. Further ϕ_E is constant if and only if E is a cone.*

Proof. Let f be a smooth approximation to E in B_1 and let

$$a(r) = \left| \int_{B_r} |\nabla f| - \int_{B_r} |\nabla E| \right|, \quad b(r) = \int_{\partial B_r} |tr(f) - tr(E)|.$$

Let $J(r) = \int_{B_r} |\nabla f|$ and $h(x) = f(x/|x|)$ be the radial inward extension of f from ∂B_1.
Now, if we let $\phi(x) = x/|x|$ and let $\nabla_T f$ denote the tangential component of ∇f, then we have the following calculation between the gradient of h and f,

$$
\begin{aligned}
(n-1)\int_{B_1} \nabla h(x) &= (n-1)\int_0^1 \int_{\partial B_r} \nabla (f\circ\phi) \\
&= (n-1)\int_0^1 \int_{\partial B_r} (\nabla f\circ\phi) D\phi \\
&= (n-1)\int_0^1 \int_{\partial B_r} (\nabla f)(\frac{x}{|x|})\cdot\left(\frac{\delta_{ij}}{|x|}-\frac{x_ix_j}{|x|^3}\right)_{ij} \\
&= (n-1)\int_0^1 \int_{\partial B_r} \frac{1}{|x|}\nabla f(\frac{x}{|x|}) - \frac{1}{|x|}\langle \nabla f, \frac{x}{|x|}\rangle\cdot\frac{x}{|x|} \\
&= (n-1)\int_0^1 \int_{\partial B_r} \frac{1}{|x|}\nabla_T f(\frac{x}{|x|}) \\
&= (n-1)\int_0^1 \int_{\partial B_1} \frac{1}{r}\nabla_T f(x) r^{n-1} \\
&= \int_{\partial B_1} \nabla_T f(x).
\end{aligned}
$$

Hence we have that

$$
(n-1)\int_{B_1} |\nabla h| = \int_{\partial B_1} |\nabla_T f|.
$$

From this we can infer that

$$
\begin{aligned}
J'(1) &= (n-1)\int_{B_1} |\nabla h| + \left(\int_{\partial B_1} |\nabla f| - |\nabla_T f|\right) \\
&\geq (n-1)\left(\int_{B_1} |\nabla f| - (a(1)+b(1))\right) + \left(\int_{\partial B_1} |\nabla f| - |\nabla_T f|\right) \\
&= (n-1)J(1) + \int_{\partial B_1} (|\nabla f| - |\nabla_T f|) - (n-1)(a(1)+b(1)).
\end{aligned}
$$

Now, making use of the simple inequality $1-\sqrt{1-t}\geq t/2$ for $|t|\leq 1$ with $t=\dfrac{(\nabla f\cdot x)^2}{|\nabla f|^2}$, we get that

$$
J'(1) \geq (n-1)J(1) + \int_{\partial B_1} \frac{|\nabla f|}{2}\frac{(\nabla f\cdot x)^2}{|\nabla f|^2} - (n-1)(a(1)+b(1)).
$$

Now, the left hand side is nothing more then just $\phi'(1)$ where

$$\phi(r) = r^{1-n} \int_{B_r} |\nabla f|.$$

Doing a rescaling gives us

$$\phi'(r) \geq \frac{1}{2} \int_{\partial B_r} \frac{(\nabla f \cdot x)^2}{|\nabla f| r^{n+1}} - C(n,r)(a(r)+b(r)).$$

Integrating this last expression from r_1 to r_2 and applying the Cauchy-Schwartz inequality gives

$$\phi(r_2) - \phi(r_1) \geq \frac{1}{2} \frac{\left(\int_{B_{r_2} \setminus B_{r_1}} \frac{\nabla f \cdot x}{|x|^n} \right)^2}{\int_{B_{r_2} \setminus B_{r_1}} \frac{|\nabla f|}{|x|^{n-1}}} - C(n, r_1, r_2) \int_{r_1}^{r_2} (a(r)+b(r)) dr.$$

Now, if $f_n \xrightarrow{L^1} \mathbb{1}_E$ and $\int_{B_1} |\nabla f_n| \to \int_{B_1} |\nabla E|$, then we have that $\int_{B_r} |\nabla f_n| \to \int_{B_r} |\nabla E|$, so $a_n(r) \to 0$ for a.e. r. As well, we have that $b_n(r) \to 0$ by the L^1 convergence of f_n to E with the fact that $f_n = tr(f_n)$ and $\mathbb{1}_E = tr(\mathbb{1}_E)$ for a.e. r. We are able to conclude that

$$\phi_E(r_2) - \phi_E(r_1) \geq \frac{1}{2} \frac{\left(\int_{B_{r_2} \setminus B_{r_1}} \frac{\nabla E \cdot x}{|x|^n} \right)^2}{\int_{B_{r_2} \setminus B_{r_1}} \frac{|\nabla E|}{|x|^{n-1}}} \geq 0,$$

and so ϕ_E is monotone. □

Now, with this new tool we can show that, for a minimal set E,the boundary of E is almost the graph of a function. The theorem below is a combination of one presented in [7] which makes use of ideas from [4].

Theorem 12. *If E is a minimal set in B_1, then*

$$\mathscr{H}^{n-1}(\partial E \setminus \partial^* E) = 0.$$

Proof. Let K be any compact set contained in $\partial E \setminus \partial^* E$. Then, since $|\nabla E|$ is supported on $\partial^* E$, we have that

$$\int_K |\nabla E| = 0.$$

Since $|\nabla E|$ is a Radon measure, it is outer regular, so for $\varepsilon > 0$ there exists an open set A_ε contained in B_1 containing K such that

$$\int_{A_\varepsilon} |\nabla E| \leq \varepsilon.$$

For $\delta > 0$ and $x \in K$, there exists $r < \delta$ such that $B(x,5r) \subset A_\varepsilon$. Hence, by the Vitali covering lemma we can choose a countable subset of x_i such that

1. $B(x_i,r_i) \cap B(x_j,r_j) = \emptyset$ if $i \neq j$.

2. $K \subset \cup_{i=1}^{\infty} B(x_i,5r_i)$.

This gives

$$\sum_{i=1}^{\infty} \int_{B(x_i,r_i)} |\nabla E| \leq \int_{A_\varepsilon} |\nabla E| < \varepsilon$$

Now, using Theorem 11 and the estimate from Lemma 6 with A_3 being is the constant from that Lemma, we have that

$$\int_{B(x_i,r_i)} |\nabla E| \geq A_3 r_i^{n-1}.$$

This gives

$$A_3 \sum_{i=1}^{\infty} r_i^{n-1} \leq \varepsilon.$$

Hence we have bounds on the δ-Hausdorff measure of the set K

$$\mathscr{H}_\delta^{n-1}(K) \leq 5^{n-1} \frac{A_3}{\omega_{n-1}} \varepsilon,$$

and so the taking limits in δ and then in ε gives

$$\mathscr{H}^{n-1}(K) = 0.$$

Now, since $\mathscr{H}^{n-1}$ is an outer measure and $\mathbb{R}^n$ is strongly Lindelöf, we can find a countable cover of compact subsets of $\partial E \setminus \partial^* E$. Thus, by subadditivity, we get that

$$\mathscr{H}^{n-1}(\partial E \setminus \partial^* E) = 0.$$

$\square$

, we know that for a minimal set E, $\partial^* E$ makes up most of ∂E, and from Theorem 7, ∂E is made up of pieces of C^1 hypersurfaces. This gives strong evidence that ∂E is at least C^1. Later we will show that it is in fact much smoother.

Chapter 2

Viscosity Solutions

A viscosity solution of a given elliptic PDE is a form of weak solution to the equation. The role that they play can be thought of being somewhat analogous to how distributions solve a given equation. Here we give a brief overview of viscosity solutions based on the definitions outlined in the book of Caffarelli and Cabré[2].

In this section we consider elliptic equations of the form

$$F(D^2u, Du) = f, \tag{2.1}$$

where u and f are functions defined on a bounded domain Ω in $\mathbb{R}^n$ and $F(M, p)$ is a real valued function defined on $\mathscr{S} \times \Omega$, where $\mathscr{S}$ is the space of $n \times n$ symmetric matrices.

A continuous function u in a domain Ω is a *viscosity supersolution* of (4.1) if for every $x_0 \in \Omega$ and for any $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local minimum at x_0, we have

$$F(D^2\phi(x_0), D\phi(x_0)) \leq f(x_0).$$

We say that u is a *viscosity subsolution* if for every $x_0 \in \Omega$ and for any $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum at x_0, we have

$$F(D^2\phi(x_0), D\phi(x_0)) \geq f(x_0).$$

In the case of the minimal surface equation, we have that $f = 0$ and

$$F(M, p) = \frac{tr(M)}{\sqrt{1 + |p|^2}} - \frac{p^T M p}{(1 + |p|^2)^{3/2}}.$$

We will work in this more general framework of elliptic equations in $\mathbb{R}^n$ rather than just working with the minimal surface equation directly so as to make the proofs simpler. Let $\mathscr{S}$ again denote the set of symmetric $n \times n$ matrices. Then we will be looking at equations of the form

$$F : \mathscr{S} \times \mathbb{R}^n \to \mathbb{R}$$

with

$$F(D^2u, Du) = 0, \quad u : B_1 \to \mathbb{R}.$$

We shall assume that F satisfies the following properties:

1. F is elliptic i.e. if N is a symmetric positive semidefinite matrix;(written $N \geq 0$) then

$$F(M+N,p) \geq F(M,p) \tag{2.2}$$

2. F is uniformly elliptic in a neighborhood of $(0,0) \in \mathscr{S} \times \mathbb{R}^n$. This means that there exists $\delta > 0$ and λ, Λ such that if $|p| \leq \delta$ and $N \geq 0$ is a symmetric positive semidefinite matrix then

$$\lambda\|N\| \leq F(M+N,p) - F(M,p) \leq \Lambda\|N\| \quad \text{if } \|N\|, \|M\| \leq \delta. \tag{2.3}$$

3. Planes are solutions to the equation, that is

$$F(0,p) = 0. \tag{2.4}$$

4. F is linear in its first argument:

$$F(M+N,p) = F(M,p) + F(N,p). \tag{2.5}$$

Here $\|M\| = \max\{|\lambda_i| \mid \lambda_i \text{ is an eigenvalue of } M\}$. Further, from the above properties it will be shown that we have the following bound on F

$$\lambda\|M^+\| - \Lambda\|M^-\| \leq F(M,p) \leq \Lambda\|M^+\| - \lambda\|M^-\| \quad if \;\; \|M\|, |p| \leq \delta. \tag{2.6}$$

We will prove (2.6) in the special case of the minimal surface equation case and that the minimal surface equation satisfies the above four properties. The general case follows by replacing the minimal surface equation by a general F. For the minimal surface equation we have that F is given by

$$F(M,p) = \frac{tr(M)}{\sqrt{1+|p|^2}} - \frac{p^T M p}{(1+|p|^2)^{3/2}}.$$

Now, since the trace of a matrix is linear, and a bilinear form is linear in its matrix argument, we have that the minimal surface equation is linear in its matrix argument. Now to show that it is uniformly elliptic in a neighborhood, we clearly have that $F(N,p) \leq n\|N\|$. For the lower bound we make use of the fact that N is self-adjoint:

$$\begin{aligned}
F(N,p) &= \frac{tr(N)}{\sqrt{1+|p|^2}} - \frac{p^T N p}{(1+|p|^2)^{3/2}} \\
&\geq \frac{\|N\|}{\sqrt{1+|p|^2}} - |p|^2 \frac{\frac{p}{|p|}^T N \frac{p}{|p|}}{(1+|p|^2)^{3/2}} \\
&\geq \frac{\|N\|}{\sqrt{1+|p|^2}} - |p|^2 \frac{\|N\|}{(1+|p|^2)^{3/2}} \\
&= \|N\| \left(\frac{1}{(1+|p|^2)^{3/2}} \right) \\
&\geq \|N\| \left(\frac{1}{(1+|\delta|^2)^{3/2}} \right).
\end{aligned}$$

Since N is positive definite, we see that $F(N,p) \geq 0$. Hence this shows that the minimal surface equation is elliptic and also establishes that F is uniformly elliptic in a neighborhood of $p = 0$. For the third property it is clear that $F(0,p) = 0$.

Now, we want to write M as $M^+ - M^-$ where M^+ and M^- are both positive semidefinite matrices and $M^+M^- = 0$.We can do this by defining M^+ and M^- as follows: if we orthogonally diagonalize the matrix M as

$$M = O^T D O,$$

where O is an orthogonal matrix and D is a diagonal matrix and let D' be the diagonal matrix obtained by replacing each negative entry of D with a zero, then define

$$M^+ = O^T D' O.$$

Similarly, if we let D'' be the matrix obtained by replacing each positive entry of D with a zero then we define

$$M^- = O^T D'' O.$$

Also, we make note that, in general, the representation of M into two positive semidefinite matrices is not unique. However if we also add the condition that $M^+M^- = 0$, then the representation is unique and with this representation we have that $\|M^+\|, \|M^-\| \leq \|M\|$. Making use of the fact that the minimal surface equation is quasi-linear we have that

$$F(M,p) = F(M^+ - M^-, p) = F(M^+, p) - F(M^-, p),$$

and applying (2.3) gives us

$$\lambda\|M^+\| - \Lambda\|M^-\| \leq F(M,p) \leq \Lambda\|M^+\| - \lambda\|M^-\| \quad if \;\; \|M\|, |p| \leq \delta.$$

2.1 Boundary Defining Function

Before we can define what it means for an abstract set to satisfy the minimal surface equation, we must first try to define a function that encapsulates the main essence of ∂E. Here we make the assumption that E contains everything below a height $-\varepsilon$, that is $\{x \mid x_n \leq -\varepsilon\} \subset E$, or, at least locally, E should contain everything below some height in a small ball.

We now define the boundary defining function u of E. For this we will introduce some more notation. For a measurable subset A, we say that $A \subset_\lambda E$ if $\lambda^n(A \cap E) = \lambda^n(A)$, where λ^n is n-dimensional Lebesgue measure. Observe that if $B \subset A$ and $A \subset_\lambda E$, then we have that $B \subset_\lambda E$ since

$$\lambda^n(B) + \lambda^n(A \setminus B) = \lambda^n(A) = \lambda^n(E \cap A) = \lambda^n(E \cap B) + \lambda^n(E \cap A \setminus B) \leq \lambda^n(B \cap E) + \lambda^n(A \setminus B).$$

Hence the last inequality is an equality giving us that $B \subset_\lambda E$. Let $C_r(x', x_n)$ be the cylinder define by $C_r(x', x_n) := B'_r(x') \times [-\delta, x_n]$. Notice that if for some $r_0 > 0$ we have that $C_{r_0}(x', x_n) \subset_\lambda E$, then for every $r < r_0$ we have that $C_r(x', x_n) \subset_\lambda E$.

Define the function $u(x')$ by

$$u(x') = \sup\{x_n | \text{ there exists an } r > 0 \text{ such that } \; C_r(x', x_n) \subset_\lambda E\}.$$

Then u is lower semi-continuous and $(x', u(x))$ is in the measure theoretic boundary of E. To see that u is lower semi-continuous let $\{x'_k\}$ be a sequence of points that converge to x' and let $\varepsilon > 0$. Then there exists $r > 0$ such that $C_r(x', u(x') - \varepsilon) \subset_\lambda E$. Since x'_k converges to x', we have that there exists some k_0 such that, for $k > k_0$, $x_k \in B'_r(x')$. Let $r_k = r - |x' - x'_k|$ (i.e. $r_k = d(x'_k, \partial B'_r(x'))$). Then we have that $C_{r_k}(x'_k, u(x') - \varepsilon) \subset_\lambda E$, and so this gives $u(x'_k) \geq u(x') - \varepsilon$ by definition of $u(x')$. Thus taking limits in k gives that $\liminf_{y' \to x'} u(y') \geq u(x')$, i.e. u is lower semi-continuous. We also note that it is enough to show that u takes values in ∂E since, for a minimal set, we have that $\partial E = \partial^* E$.

To show that $(x', u(x'))$ is in the measure theoretic boundary, we just need to show that, for every $r > 0$, we have that $\lambda^n(E \cap B_r((x', u(x')))) > 0$ and $\lambda^n(E^C \cap B_r((x', u(x')))) > 0$. Let $r > 0$. Then there exists r' such that $C_{r'}(x', u(x') - r/3) \subset_\lambda E$. If $r'' = \min\{r, r'\}$, we have that $C_{r'}(x', u(x') - r/3) \cap B_r(x', u(x')) \supset B'_{r''}(x') \times [u(x') - r/2, u(x') - r/3]$, and so $\lambda^n(E \cap B_r((x', u(x')))) > 0$.
We get the other measure estimate by contradiction. Suppose that there is an $r > 0$ such that $\lambda^n(E^C \cap B_r((x', u(x')))) = 0$. Then $\lambda^n(E \cap B_r((x', u(x')))) = \lambda^n(B_r((x', u(x'))))$, so $B'_{r/100}(x') \times [u(x') - r/100, u(x') + r/100] \subset_\lambda E$. Then we can find an r' so that $C_{r'}(x', u(x') - r/99) \subset_\lambda E$. Finally, let $r'' = \min\{r, r'\}$. Then, for this choice of r'', we have that $C_{r''}(x', u(x') + r/100) \subset_\lambda E$, but this is a contradiction since then we would have that $u(x') + r/100 \leq u(x')$. This shows that $\lambda^n(E^C \cap B_r((x', u(x'))) > 0$ and so $(x', u(x'))$ is in the measure theoretic boundary of E.

Now we are in a position to make sense of what it means for an abstract set to satisfy the minimal surface equation.

Definition 7. *Let u be the boundary defining function of ∂E. Then we say that the boundary of a set ∂E satisfies the minimal surface equation*

$$F(D^2u, Du) := \frac{\Delta u}{\sqrt{1 + |\nabla u|^2}} - \frac{(\nabla u)^T D^2 u \nabla u}{(1 + |\nabla u|^2)^{3/2}} u = 0$$

in the viscosity sense if for any function $\phi \in C^2$ which touches u from below, we have that

$$F(D^2\phi, D\phi) \leq 0,$$

and for any function $\phi \in C^2$ which touches u from above we have

$$F(D^2\phi, D\phi) \geq 0.$$

If a set only satisfies the first inequality, we say that it is a supersolution, and if it satisfies the second one, then it is a subsolution.

It should be noted that if u is C^2, then we can write the minimal surface equation in divergent form as

$$F(D^2u, Du) = div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

This definition makes sense since we have shown that a minimal set is "almost" a graph of a function in a measure theoretical sense. Hence, the defining function u captures all relevant "information" of ∂E. Now, since we do not know whether u is smooth or not, we will take u and produce a new function which is close to u and has better regularity via a convolution.

Definition 8. *We define the inf-convolution of u by*

$$u^{\varepsilon}(x') = \inf_{y' \in B_1'} \{u(y') + \frac{1}{\varepsilon}|y'-x'|^2\}$$

It can be shown that u^{ε} is C^2 except on a set of measure zero, that it is semiconcave and that $u^{\varepsilon} \to u$ uniformly on compact sets of B_1' as $\varepsilon \to 0$. To see this, note that

$$\begin{aligned} u^{\varepsilon}(x') - \frac{|x'|^2}{\varepsilon} &= \inf_{y'}\left(u(y') + \frac{|y'-x'|^2}{\varepsilon} - \frac{|x'|^2}{\varepsilon}\right) \\ &= \inf_{y'}\left(u(y') + \frac{|y'|^2}{\varepsilon} - \frac{2x'\cdot y'}{\varepsilon}\right). \end{aligned}$$

Now this function is concave and upper semi-continuous as it is an infimum of continuous linear functions. By Alexandrov's theorem, this implies that there exists a set Z of measure zero so that for $x_0' \in B_1' \setminus Z$, there exists a matrix $D^2u^{\varepsilon}(x_0')$ and a vector $Du^{\varepsilon}(x_0')$ for which the following holds in a neighborhood of x_0'

$$u^{\varepsilon}(x') = u^{\varepsilon}(x_0') + Du^{\varepsilon}(x_0')\cdot(x'-x_0') + \frac{1}{2}(x'-x_0')^T D^2u^{\varepsilon}(x_0')(x'-x_0') + o(|x'-x_0'|^2).$$

In the next chapter we will show that if u is a viscosity supersolution to the minimal surface equation in the viscosity sense, then so is its inf-convolution.

Chapter 3

Minimal Surfaces Results

In this chapter, we introduce and prove all the necessary tools needed to prove the regularity of minimal sets, which will be done in the final chapter. The approach is based on [8], [10],[11], [12] and [3]. First, we need to recall the basics of curvature for C^2 objects in Euclidean space. These ideas will help us to use smooth approximation to extract important estimates for the minimal sets.

3.1 Curvature Results

We introduce some of the basic results on curvature taken from Gilbarg and Trudinger[6] which will be of use later. Let Ω be a open bounded simply connected subset of $\mathbb{R}^n$ having non-empty C^2 boundary $\partial\Omega$. The distance function d is defined by $d(x) := \text{dist}(x, \partial\Omega)$. Now, for $y \in \partial\Omega$, let $\hat{n}(y)$ denote the unit normal to $\partial\Omega$ at y. The curvatures of $\partial\Omega$ at a fixed point $y_0 \in \partial\Omega$ are determined as follows. Applying a rotation if necessary we may assume that the x_n direction lies in the same direction as $\hat{n}(y_0)$ in some neighborhood U of y_0. Then $\partial\Omega$ can be expressed as the graph of a function given by $x_n = \phi(x')$ where $x' = (x_1, \ldots, x_{n-1})$ with $\phi \in C^2$ and $D\phi(y_0') = 0$. The curvatures of $\partial\Omega$ at y_0 are then defined by the eigenvalues of the Hessian matrix $D^2\phi$ at y_0', denoted by $\kappa_1, \ldots \kappa_{n-1}$, and are called the principal curvatures of $\partial\Omega$ at y_0. The mean curvature of $\partial\Omega$ at y_0 is given by the Cesáro mean of the principal curvatures

$$H(y_0) = \frac{1}{n-1} \sum_{i=1}^{n-1} \kappa_i = \frac{1}{n-1} \Delta\phi(y_0').$$

The unit normal $\hat{n}(y)$ at a point $y = (y', \phi(y')) \in U \cap \partial\Omega$ is given by

$$\hat{n}_i(y) = \frac{-D_i\phi(y')}{\sqrt{1+|D\phi(y')|^2}}, \quad i = 1,\ldots,n-1, \quad \hat{n}_n(y) = \frac{1}{\sqrt{1+|D\phi(y')|^2}}.$$

Also, in a principal coordinate system we have that the derivatives of the normal at y_0 are given by

$$D_j\hat{n}_i(y'_0) = -\kappa_i\delta_{ij}, \quad i,j = 1,\ldots,n-1.$$

Lemma 13. *Let Ω be bounded and $\partial\Omega \in C^k$ for $k \geq 2$. Then there exists a positive constant μ depending on Ω such that $d \in C^k(\Gamma_\mu)$, where $\Gamma_\mu = \{x \in \mathbb{R}^n || d(x)| < \mu\}$.*

Lemma 14. *If Ω and μ are as above and $x_0 \in \Gamma_\mu$, then we have that*

$$\Delta d(x_0) = -\sum_{j=1}^{n-1} \frac{\kappa_i}{1-\kappa_i d(x_0)},$$

where κ_i is the i th principal curvature at the point $y_0 \in \partial\Omega$, where y_0 realizes x_0's distance to $\partial\Omega$. In particular, when we have $x_0 \in \partial\Omega$ and ∂E is the graph of a function a C^2 function ϕ, then $\Delta d(x_0) = -\sum\kappa_i = div\frac{D\phi}{\sqrt{1+|D\phi|^2}}(x_0)$.

3.2 Viscosity Solutions For The Minimal Surface Equation

In this section we make use of viscosity solution methods to develop some theory on minimal sets, the most important of which is theorem 18. This will set the stage in which one can prove a geometric Harnack inequality.

Theorem 15. *If E is minimal and u is the defining function of ∂E, then u is a viscosity supersolution of $div\frac{Du}{\sqrt{1+|Du|^2}} = 0$ in B_1.*

The following proof is based on the original given by Roquejoffre[8] but has been altered to make use of theorem 9 to make it simpler and shorter.

Proof. Suppose that u is not a viscosity supersolution of the MSE. Then there exists $\phi \in C^2(B'_1)$ such that $u-\phi$ has a maximum at $x'_0 \in B'_1$, but

$$div\frac{Du}{\sqrt{1+|Du|^2}}(0) > 0.$$

We may assume that $x'_0 = 0$ and that $u(0)-\phi(0) = 0$. Let Γ denote the graph of ϕ. Then, from lemma 14 we have that

$$\Delta d(\cdot,\Gamma)|_{x=0} = -div\frac{D\phi(0)}{\sqrt{1+|D\phi|^2(0)}}.$$

Now, since the map $x \mapsto d(x,\Gamma)$ is smooth in a neighborhood of 0, we can find a $\delta > 0$ such that

1. $(\Gamma + \delta e_n \geq \partial^* E \text{ over } B'_r)_{subgraph} \cup E$ is a perturbation of E in B_1 for some $r \in (0,1)$. Since $u - \phi$ is lower semi-continuous, for any $r \in (0,1)$ we could take any $\delta < \inf_{\overline{B_r}\setminus B_{r/2}}(u-\phi)(x')$. That is, we perturb E by pushing ϕ upwards through it.
2. $(\Gamma + \delta e_n)_{subgraph} \cap E^c$ has nonempty interior by lemma 6.
3. $div\frac{D\phi}{\sqrt{1+|D\phi|^2}} > 0$ in $\Omega = \{d(x,\Gamma) < \delta\} \cap E^c$ and, since we also have $div\frac{D\phi(0)}{\sqrt{1+|D\phi|(0)^2}} = -\Delta d(\cdot,\Gamma)|_{x=0}$, by further restricting δ we have that $-\Delta d > 0$ in Ω.

Now, by integrating over Ω and using Theorem 9 we get that

$$\begin{aligned}
0 > \int_\Omega \Delta d &= \int_\Omega div(Dd) \\
&= \int_{\partial\Gamma+\delta e_n} Dd \cdot n d\mathscr{H}^{n-1} + \int_{\partial E} Dd \cdot \nu_E d\mathscr{H}^{n-1} \\
&= \int_{\partial\Gamma+\delta e_n} 1\mathscr{H}^{n-1} + \int_{\partial E} Dd \cdot \nu_E d\mathscr{H}^{n-1}.
\end{aligned}$$

Hence we have that

$$\int_{\partial\Gamma} d\mathscr{H}^{n-1} < -\int_{\partial E} Dd \cdot \nu_e d\mathscr{H}^{n-1} \leq P(E \cap \Omega),$$

but this contradicts the fact that E was minimal. Hence u is a supersolution to the minimal surface equation. □

Corollary 16. *If E is minimal and u is the defining function of ∂E, then u is a viscosity subsolution of the minimal surface equation.*

Proof. All that was used in the above proof was that u took values in $\partial^* E$ so that we could apply lemma 6. So, if instead we assume that u is not a viscosity subsolution, then we can find a ϕ that touches u from above. Then, as above ϕ would give a perturbation of ∂E downwards. Applying the Gauss-Green formula to this perturbation gives the wrong inequality about the minimality of ∂E, which implies that u is a viscosity subsolution. □

Now that we know that u satisfies the minimal surface equation in the viscosity sense, we will show that its approximation u^ε, the inf-convolution of u, will be a supersolution to the minimal surface equation. This is quite important since the proof of the ABP estimate for u will depend on the regularity of u^ε. Recall that $u^\varepsilon(x) = \inf_{y\in\overline{B}}(u(x) + \frac{1}{\varepsilon}|x-y|^2)$.

Lemma 17. *For all $\varepsilon > 0$ small we have that u^ε is a super solution to the MSE.*

The following proof is based on the original given by Roquejoffre[8] but has been altered to have a more geometric flavor than the original algebraic proof.

Proof. Since u is lower semi-continuous, it realizes its minimum on compact sets. Let $y_{\varepsilon(x)}$ be the minimum for any x and ε in the definition of $u^\varepsilon(x)$. Then we have that $u^\varepsilon(x) = u(y_\varepsilon(x)) + \frac{1}{\varepsilon}|x - y_\varepsilon(x)|^2$.

Now, looking at u^ε, if we touch the graph from below by a C^2 function ϕ at some point x_0 and notice that u^ε lies below u, then by forming a translated copy of the graph of u by the mapping $(x,u(x)) \mapsto (x_0 - x^* + x, u(x) + \phi(x_0) - u(x^*))$ where x^* is the point $y_\varepsilon(x_0)$. Now this mapping is an isometry and hence the translated surface satisfies the minimal surface equation since the original graph did. Hence if a C^2 function ψ touches this translation from below at a point, then we have that $div\dfrac{D\psi}{\sqrt{1+|D\psi|^2}} < 0$. Now, since ϕ touched u^ε from below and the translation lies above u^ε, and this same ϕ also touches the the translate at this point, we have that $div\dfrac{D\phi(0)}{\sqrt{1+|D\phi|^2(0)}} < 0$, hence u^ε is a supersolution to the minimal surface equation in the viscosity sense. □

Now we need to establish an ABP measure estimate. Here we will do this in the general framework of elliptic equations in $\mathbb{R}^n$ that we gave in the previous chapter to make things simpler. The following proof of the ABP estimate is a based on a combination of the proofs given from Savin and Roquejoffre[8], with additional details on the estimates of the eigenvalues for the viscosity relation.

Proposition 18. *[ABP Estimate] Let u be a viscosity supersolution of the minimal surface equation in B'_1. Let $B \subset B'_1$ be a set of nonzero measure. For $a \le \delta/2$, let A be the set of contact points of u by sliding paraboloids of opening $-a$ and centers $y' \in B$ from below until they touch u for the first time. Then there exists a $q > 0$ that depends only on δ such that*

$$|A| \ge q|B|$$

provided that $A \subset\subset B'_1$. Here δ is chosen so that 2.6 is satisfied.

Proof. Since u^ε is semi-concave, by Alexandrov's theorem, we have that u^ε is C^2 almost everywhere. That is, there exists a set Z of measure zero such that, for all $x_0' \in B_1' \setminus Z$,

$$\begin{aligned}
u^\varepsilon(x') &= u^\varepsilon(x_0') + Du^\varepsilon(x_0')\cdot(x'-x_0') + \frac{1}{2}(x'-x_0')^T D^2u^\varepsilon(x_0')(x'-x_0') + o(|x'-x_0'|^2) \\
&= P(x',x_0') + o(|x'-x_0'|^2).
\end{aligned}$$

From the above we see that $P(x',x_0') - \frac{\eta}{2}|x'-x_0'|^2$ touches u^ε from below at x_0' for any $\eta > 0$. Aassume that $x_0' \in B\setminus Z$. Then, from the normal conditions, we have that

$$\begin{aligned}
Du^\varepsilon(x_0') &= D(-\frac{a}{2}|x'-y'|^2)|_{x'=x_0'} \\
&= -a(x_0'-y'),
\end{aligned}$$

which tells us that $|Du^\varepsilon(x_0')| \le 2a \le \delta$ and

$$D^2u^\varepsilon(x_0') \ge -aI.$$

Now, since u^ε is a viscosity super-solution, by applying the viscosity relation to $P(x',x_0') - \frac{\eta}{2}|x'-x_0'|^2$ at x_0' we get

$$F(D^2u^\varepsilon(x_0') - \eta I, p, u^\varepsilon(x_0'), x_0') \le 0.$$

Now we want to show that we can bound the eigenvalues of $D^2u^\varepsilon(x_0')$ from above by $\lambda(x_0') \le Ca$, where C is a universal constant. Suppose that there exists a unit eigenvector e with eigenvalue bigger then Ca. This combined with the previous matrix inequality gives that $D^2u^\varepsilon(x_0') \ge Cae\cdot e^T - aI$. Then, if Λ and λ denote the local elliptic constants in the δ-neighborhood of $(0,0) \in \mathscr{S}\times\mathbb{R}^{n-1}$, we have that

$$\begin{aligned}
0 &\ge F(D^2u^\varepsilon(x_0) - \eta I, Du^\varepsilon) \\
&= F\left(D^2u^\varepsilon(x_0) - Cae\cdot e^T + aI + Cae\cdot e^T, Du^\varepsilon\right) \\
&\ge F(Cae\cdot e^T - \eta I - aI, Du^\varepsilon) \quad \text{(by 2.2)} \\
&\ge \lambda(Ca - a - \eta) - (n-1)\Lambda a, \quad \text{(by 2.6)}
\end{aligned}$$

which is greater then zero if C is large enough, and this contradicts the face that u^ε is a supersolution. Hence we have that $D^2u^\varepsilon \leq CaI$.

Now, if we look at the contact condition of P and u^ε, i.e. the condition that the normal vectors coincide, we see that

$$-\frac{Du^\varepsilon(x_0')}{\sqrt{1+|Du^\varepsilon(x_0')|^2}} = -\frac{D(a/2|x_0'-y'|^2)}{\sqrt{1+|D(a/2|x_0'-y'|^2|^2}}$$
$$= a\frac{x_0'-y'}{\sqrt{1+a^2|x_0'-y'|^2}}.$$

So $Du^\varepsilon(x_0') = -a(x_0'-y')$ and so $y' = x_0' + \frac{1}{a}Du^\varepsilon(x_0')$. The map $\phi : A \mapsto B$ given by $\phi(x_0) = x_0' + 1/aDu^\varepsilon(x_0')$ is a surjection from A to B. From the bounds above we have that the differential map of ϕ is given by $D\phi = I + \frac{1}{a}D^2u^\varepsilon$, and so $\|D\phi\| \leq \frac{1}{a}\|D^2u^\varepsilon\| + 1 \leq C+1$. Also, we have that $0 \leq D\phi \leq (C+1)I$, so applying the coarea formula (theorem 5) we get that

$$|B| = \int_{\phi(A)} dx$$
$$\leq \int_A |detD\phi| dx$$
$$\leq (C+1)|A|.$$

This establishes the result for u^ε, and taking limits gives us the general result. If A_ε denotes the set of contact point for u^ε, then we have that

$$\limsup A_{1/k} = \cap_{m=1}^\infty \cup_{k=m}^\infty A_k \subset A.$$

Now each set $\cup_{k=m}^\infty A_k$ has measure at least $q|B|$, and since the sequence is decreasing by continuity of measure ,we have that $|A| \geq q|B|$.

□

For $a > 0$, let D_a denote the set of interior contact points with u, i.e

$$D_a = \{z' \in B_1' | \, \exists \, y' \in \overline{B}_1' \text{ such that } u(z') + \frac{a}{2}|z'-y'|^2 \leq u(x') + \frac{a}{2}|x'-y'|^2 \quad \forall x' \in B_1'\}.$$

Here we want to somehow increase the constant set obtained from the ABP estimate. The philosophy is that if we can touch the graph at one point with a paraboloid of curvature $1/a$, then, by changing the curvature a bit, we will touch the graph of u by larger amount. The proof below is based on the one given in [11] with more details on the eigenvalues computations and set containment argument.

Lemma 19. *(Bootstrap contact lemma) There exists positive universal constants C and c such that if $a \leq \delta/C$ and*

$$D_a \cap B'_r(x'_0) \neq \emptyset$$

for some ball $\overline{B}'_r(x'_0) \subset B'_1$, then

$$|D_{Ca} \cap B'_{r/8}(x_0)| \geq cr^{n-1}.$$

Proof. Let $z'_1 \in B'_r(x'_0) \cap D_a$ be a contact point and let by $y'_1 \in \overline{B'}_1$ be the center of the paraboloid $P(x', y'_1) = -\frac{a}{2}|x' - y'_1|^2 + u(z'_1) + \frac{a}{2}|z'_1 - y'_1|^2$ that touches u from below at z'_1. First we try to find a point $x'_1 \in \overline{B'}_{1/16}(x'_0)$ such that, for some universal constant C,

$$u(x'_1) - P(x'_1, y'_1) \leq Car^2.$$

Let $\phi : \overline{B'}_1 \to \mathbb{R}^+$ be the radially symmetric function

$$\phi(x') = \begin{cases} \frac{1}{\alpha}(|x'|^{-\alpha} - 1), & 1/16 \leq |x'| \leq 1 \\ \\ \frac{1}{\alpha}(16^{\alpha} - 1), & |x'| \leq 1/16. \end{cases}$$

where α is a large constant. Let

$$\psi(x') = P(x', y'_1) + ar^2 \phi(\frac{x' - x'_0}{r}).$$

We will show that ψ is a strict subsolution in the annular region $\frac{r}{16} < |x' - x'_0| < r$.

We have that $\|D\psi\| \leq \|DP\| + ar\|D\phi\| \leq \|DP\| + a\|D\phi\| \leq Ca \leq \delta$. For $r/16 < |x' - x'_0| < r$ we have that the hessian of $ar^2\phi(\frac{x' - x'_0}{r})$ in a basis $(e_r, e_{h_1}, \ldots, e_{h_{n-2}})$, where e_r is a radial vector and the e_{h_i} is an orthogonal frame to e_r, is given by

$$D^2 ar^2\phi(\frac{x'-x'_0}{r}) = \begin{pmatrix} a\phi''\left(\frac{|x'-x'_0|}{r}\right) & 0 \\ 0 & \frac{ar}{|x'-x'_0|}\phi'\left(\frac{|x'-x'_0|}{r}\right)I_{n-2} \end{pmatrix}$$
$$= \begin{pmatrix} a(\alpha+1)\left(\frac{|x'-x'_0|}{r}\right)^{-\alpha-2} & 0 \\ 0 & -ar^2\left(\frac{|x'-x'_0|}{r}\right)^{-\alpha-2} I_{n-2} \end{pmatrix}.$$

In this form we can directly read off the eigenvalues. So, by choosing α large enough, we have that $\|\psi^+\| = a((\alpha+1)\frac{|x'-x'_0|}{r}^{-\alpha-2} - 1)$, $\|\psi^-\| = a + ar^2\left(\frac{|x'-x'_0|}{r}\right)^{-\alpha-2}$, and

$$F(D^2\psi, D\psi) = F(-aI + aD^2\phi, D\psi)$$
$$\geq a\left(\lambda((\alpha+1)\frac{|x'-x'_0|}{r}^{-\alpha-2} - 1\right) - \left(a + ar^2\left(\frac{|x'-x'_0|}{r}\right)^{-\alpha-2}\right)\Lambda > 0.$$

Now we slide the graph of ψ from below until it touches u for the first time. That is, we are looking for the point x'_1 that obtains the minimum of

$$\inf_{\overline{B'}_r(x'_0)} (u - \psi).$$

We have that this minimum is negative since

$$u(z'_1) - \psi(z'_1) = P(z'_1, y'_1) - \psi(z'_1) = -ar^2\psi(\frac{z'_1 - x'_0}{r}) < 0.$$

Since for $x' \in \partial B'_r(x'_0)$ we have that

$$u(x') - \psi(x') \geq P(x', y'_1) - P(x', y'_1) \geq 0,$$

we see that $x'_1 \notin \partial B'_r(x'_0)$. Now the minimum can't be in $B'_r(x'_0) \setminus \overline{B'}_{r/16}(x'_0)$ because u is a supersolution and so we would have that $F(D^2\psi, D\psi) \leq 0$, but we just saw that $F(D^2\psi, D\psi) > 0$. This shows that the minimum is attained in $B'_{r/16}(x'_0)$. Since x'_1 is the minimum and its value is negative we get that

$$
\begin{aligned}
u(x_1') \leq \psi(x_1') &= P(x_1', y_1') + ar^2 \phi(\frac{x' - x_0'}{r}) \\
&\leq P(x_1', y_1') + Car^2.
\end{aligned}
$$

Now, by sliding the family of paraboloids

$$
P(x', y_1') - C' \frac{a}{2} |x' - y'|^2 + c_{y'}, \quad y' \in \overline{B'}_{r/64}(x_1'),
$$

from below until they touch the graph of u for the first time, where C' is a large constant which will be determined later, we will show that the set of contact points occur in $B'_{r/16}(x_0')$. Using the previous estimate can get an upper bound on $c_{y'}$ since

$$
u(x_1') - P(x_1', y_1') + C' \frac{a}{2} |x_1' - y'|^2 - c_{y'} \geq 0.
$$

Rearranging this gives us that

$$
c_{y'} \leq Car^2 + C' \frac{a}{2} (r/64)^2.
$$

Now if $|x' - x_1'| \geq r/16$ we have that

$$
\begin{aligned}
u(x') - P(x', y_1') + C' \frac{a}{2} |x' - y'|^2 - c_{y'} &\geq C' \frac{a}{2} |x' - y'|^2 - c_{y'} \\
&\geq C' \frac{a}{2} \left(|x' - x_1'| - |x_1' - y'| \right)^2 - Car^2 - C' \frac{a}{2} (r/64)^2 \\
&\geq C' \frac{a}{2} \left(\frac{3r}{64} \right)^2 - Car^2 - C' \frac{a}{2} (r/64)^2
\end{aligned}
$$

which is positive if C' is chosen to be large enough. So, we have that all the contact points occur inside $B'_{r/16}(x_1') \subset B'_{r/8}(x_0')$. Now all that is left to show is that the centers of the family of paraboloids given above are compactly contained in B'_1. We have that

$$\frac{a}{2}|x'-y'_1|^2+C'\frac{a}{2}|x'-y'|^2=(1+C')\frac{a}{2}|x'|^2-ax\cdot\left(y'_1+C'y'\right)+\frac{a}{2}(|y'_1|^2+C'|y'|^2)$$
$$=(1+C')\frac{a}{2}\left(|x'|^2-2x\cdot\left(\frac{y'_1}{1+C'}+\frac{C'}{1+C'}y'\right)\right)+\frac{a}{2}(|y'_1|^2+C'|y'|^2),$$

and so the centers are

$$\frac{y'_1}{1+C'}+\frac{C'}{1+C'}y'$$

with openings of $-(C'+1)a$. As $y'\in B'_{r/64}(x'_1)$ we have that, if r is small enough, the centers of the paraboloids range over a ball of radius cr where $c=\frac{C'}{64(1+C')}$. This ball will be compactly contained in B'_1 if r is small enough. We are now able to apply the previous ABP estimate which gives us

$$|D_{(C'+1)a}\cap B'_{r/8}(x_0)|\geq cr^{n-1}.$$

□

The following is a covering lemma that we will use to prove another measure estimate. The following lemma roughly says that if we continue to carve more and more out of a ball in a certain fashion, then we have to be carving measure out at an exponential rate. The proof provided below is modified from Savin[11].

Lemma 20. *Assume the closed sets F_k satisfy*

$$F_0\subset F_1\subset F_2\subset\ldots F_k\subset\ldots\overline{B'}_{1/3}$$

with $F_0\neq\emptyset$. Suppose further for any r,x' with

$$B'_{r/8}(x')\subset B_{1/3},\quad \overline{B'}_r(x')\subset B'_1,$$

that

$$F_k\cap B'_r(x')\neq\emptyset$$

implies that

$$|F_{k+1}\cap B'_{r/8}(x')|\geq cr^{n-1}$$

Then we have that

$$|B'_{1/3} \setminus F_k| \leq (1-c_1)^k |B'_{1/3}|$$

for some constant c_1 depending on c.

Proof. For $x' \in B'_{1/3} \setminus F_k$ let $r_{x'} = dist(x', F_k)$. Since $\mathbb{R}^{n-1}$ is a strongly Lindelöf space, making use of Vitali covering lemma we can find a countable subset $I \subset B'_{1/3} \setminus F_k$ such that

1. $B'_{1/3} \setminus F_k = \bigcup_{x' \in I} B'_{r_{x'}/5}(x)$.

2. For $x' \neq y'$ we have that $B'_{r_{x'}/40}(x') \cap B_{r_{y'}/40}(y') = \emptyset$.

This collection satisfies the hypothesis, so

$$\begin{aligned}
|F_{k+1} \setminus F_k| &= |\cup_{x' \in I} \left(F_{k+1} \cap B'_{r_{x'}/5}(x') \right)| \\
&\geq |\cup_{x' \in I} \left(F_{k+1} \cap B'_{r_{x'}/40}(x') \right)| \\
&= \sum_{x' \in I} |F_{k+1} \cap B'_{r_{x'}/40}(x')| \\
&\geq c \sum_{x' \in I} |B_{r_{x'}/5}(x')| \\
&= c|B_{1/3} \setminus F_k|.
\end{aligned}$$

This estimate gives the desired estimate of the lemma since

$$\begin{aligned}
|B'_{1/3} \setminus F_k| &= |B'_{1/3} \cap F_k^C| \\
&= |B'_{1/3} \cap F_{k+1}^C| + |B'_{1/3} \cap F_k^C \setminus F_{k+1}^C| \\
&= |B'_{1/3} \setminus F_{k+1}| + |F_{k+1} \setminus F_k|.
\end{aligned}$$

Rearranging the above gives

$$|B'_{1/3} \setminus F_{k+1}| = |B'_{1/3} \setminus F_k| - |F_{k+1} \setminus F_k| \leq (1-c)|B'_{1/3} \setminus F_k|. \quad \square$$

$\square$

Now we only need one more measure estimate before we are able to prove the Harnack inequality. The proof is from the work of Savin[12], which we include for completeness.

Theorem 21. *Let $u : B'_1 \to \mathbb{R}$, $u \geq 0$ be a viscosity supersolution to $F(D^2u, Du) = 0$. Given $\mu > 0$, there exists $\varepsilon > 0$ and M depending on $n, \delta, \lambda, \Lambda$ such that if*

$$u(0) \leq \varepsilon$$

then

$$\mathscr{H}^{n-1}(\{u > Mu(0)\} \cap B'_{1/3}) < \mu.$$

Here δ is as in from 2.3 and λ, Λ are the corresponding elliptic constants.

Proof. Let $a = 18u(0)$ and define the set

$$F_k = D_{C^k a} \cap \overline{B'}_{1/3},$$

where C is the constant from Lemma 19. Since $u \geq 0$ in B'_1, the paraboloid of opening $-a$ and center 0 touches the graph of u for the first time in $B'_{1/3}$. To see this let z' be the contact point. Then we have

$$0 \leq u(x') - P(x', z') = u(x') + \frac{a}{2}\|x'\|^2 - \frac{a}{2}\|z'\|^2 + u(z').$$

Rearranging and setting $x = 0$ we find that

$$\|z'\|^2 \leq \frac{2}{a}u(0) = \frac{1}{9}.$$

Thus we have $F_0 \neq \emptyset$ and so from lemma 19 we see that the sets F_k satisfy the conditions of lemma 20 as long as $C^k a < \delta$. Hence

$$\mathscr{H}^{n-1}(B'_{1/3} \setminus D_{C^k a}) \leq (1 - c_1)^k \mathscr{H}^{n-1}(B'_{1/3}), \quad \text{if } C^k a < \delta.$$

Now, for $z' \in D_{C^k a}$ there exists a $y' \in \overline{B}'_1$ such that

$$u(z') + \frac{C^k a}{2}\|z' - y'\|^2 \leq u(x') + \frac{C^k a}{2}\|x' - y'\|^2 \quad \forall x' \in B'_1,$$

which gives $u(z') \leq u(0) + 2C^k a \leq 3C^k a := M$. Given μ, choosing k large enough so that $(1 - c_1)^k \leq \mu$ and ε so that $18C^k\varepsilon < \delta$, we get the desired result. $\square$

We make a note that these results only make use of the fact that $u \backslash \partial E$ is a super solution to the minimal surface equation we can also use the fact that it is a subsolution to get similar results which will be needed in the proof of the Harnack inequality in the next section.

Chapter 4

Regularity of Minimal Surfaces

In this section, we make use of all the tools developed in the last chapter to prove the desired regularity of minimal sets. We start by proving a geometric Harnack inequality, one of the major tools used to prove the De Giorgi flatness result. The De Giorgi flatness result says that if your set is contained in a flat strip on some scale, then, on a fixed smaller scale relative to the original scale in some new coordinate system, the set is contained in an even flatter strip. The proof of the Harnack inequality is based on the proof given by Savin in [12]. We have removed an implicit iterative covering argument that was in the original by taking r_0 to be small enough so that only two applications of Theorem 21 are needed.
We are now at the point where we can prove the Harnack inequality.

Theorem 22. *Assume that E is minimal in B_1 and that*

$$\partial E \cap B_1 \subset \{|x_n| \leq \varepsilon\}$$

with $\varepsilon \leq \varepsilon_0(n)$. Then

$$\partial E \cap B_{r_0} \subset \{|x_n| \leq \varepsilon(1-\eta)\},$$

where $r_0 = \frac{1}{2} * \frac{1}{1+3\sqrt{2}}$, and η and $\varepsilon_0(n)$ are small universal constants.

Proof. For this we assume that the result does not hold inside $B_{r_0} \cap \partial E$, that is, ∂E is not constrained by a slightly smaller set of planes. Then there exists points $x^1 = (x_1', y_1)), x^2 = (x_2', y_2) \in \partial E \cap B_{r_0}$ such that $y_1 = u(x_1') > \varepsilon(1-\eta)$ and $y_2 = u(x_2') < -\varepsilon(1-\eta)$. To apply theorem 21, we need to shift our coordinates system so that they are centered at x^1 and x^2 respectively. For the point x^2, we just shift up by ε and apply the theorem to the function $u(\frac{1}{1-r_0}(x'-x'_2))+\varepsilon$. For the other point x^1, we shift down by ε and then reflect to apply the theorem to the function $-u(\frac{1}{1-r_0}(x'-x'_1))+\varepsilon)$. Now theorem 21 gives us that $\mathscr{H}^{n-1}(\{u > M\eta\varepsilon-\varepsilon\}\cap B'(x'_2;(1-r_0)/3)) < C(r_0)\mu$ and $\mathscr{H}^{n-1}(\{u > \varepsilon - M\eta\varepsilon\}\cap B'(x'_1;(1-r_0)/3)) < C(r_0)\mu$. Now we want the two regions $\{u > \varepsilon - M\eta\varepsilon\}\cap B(x'_1;(1-r_0)/3)$ and $\{u > M\eta\varepsilon-\varepsilon\}\cap B(x'_2;(1-r_0)/3)$ to each have more than half the mass of B'_{r_0}. This will give us that the regions overlap which will lead to a contradiction. We compare the perimeter of ∂E inside the cylinder $C = B'_{r_0}\times[-1,1]\cap B_1$. Since the projection map is Lipschitz and has Lipschitz constant one, we have that

$$\mathscr{H}^{n-1}(\Pi_{e_n}(\partial E\cap C)) \le P(E;C).$$

If $M\eta < 1$, we have that the two sets are disjoint and by applying theorem 21 we find that each has $\mathscr{H}^{n-1}$-mass greater than

$$\frac{(1+C_2(r_0))}{2}\mathscr{H}^{n-1}(B'_{r_0}) - C(r_0)\mu$$

inside B'_{r_0} by the choice of r_0. So

$$P(E;C) > (1+C_2(r_0))\mathscr{H}^{n-1}(B_{r_0}) - 2C(r_0)\mu$$

On the other hand, if we preform a surgery on E and add the set $F = B'_{r_0}\times[-\varepsilon,\varepsilon]$ to it, we get a competitor to E. Since E contains everything below $\{x_n < -\varepsilon\}$, we can view this surgery as just adding a cap to the top side of ∂E over B'_{r_0}, and, by the minimality of E we find that

$$P(E;C) \le P(E\cup F;C) = \mathscr{H}^{n-1}(B'_{r_0}) + C\varepsilon \le \mathscr{H}^{n-1}(B'_{r_0}) + C_1(r_0)\varepsilon_0(n).$$

This gives a contradiction if we choose μ small and $\varepsilon_0(n)$ small enough so that

$$C_1(r_0)\varepsilon < C_2(r_0)\mathscr{H}^{n-1}(B'_{r_0}).$$

□

4.1 Flatness Theorem

Here we prove the final obstacle to showing the regularity of flat solutions to the minimal surface equation: the De Giorgi flatness theorem. Repeated applications of the De Giorgi flatness theorem to ∂E will imply that a solution must have a certain amount of smoothness. Our main reference for this section is [12].

Theorem 23 (Flatness theorem)**.** *Assume that E is minimal in $B_1, 0 \in \partial E$ and*

$$\partial E \cap B_1 \subset \{\|x_n\| \leq \varepsilon\}$$

with $\varepsilon \leq \varepsilon_0(n)$. Then there exists a unit vector ν_1 such that

$$\partial E \cap B_{r_0} \subset \{\|x \cdot \nu_1\| \leq \frac{\varepsilon}{2} r_0\},$$

where ε_0 and r_0 are the constants from theorem 22.

Proof. We prove this by contradiction. Assume that there is a sequence of minimal surfaces ∂E_k such that

$$0 \in \partial E_k \cap B_1 \subset \{|x_n| < \varepsilon_k\},$$

with $\varepsilon_k \to 0$, for which the conclusion does not hold. Now, for each point $x_0 \in \partial E_k \cap B_r$, we apply the Harnack inequality at x_0 and obtain that in the x_n direction, the oscillation of $\partial E \cap B_r(x_0)$ is less then $2\varepsilon_k(1-\eta/2)$. We repeatedly apply the Harnack inequality as long as the hypothesis is satisfied, that is, as long as m satisfies $2^m \varepsilon_k (1-\eta/2)^{m-1} \leq \varepsilon_0(n)$, and obtain that in $\partial E \cap B_{r^m}(x_0)$, the oscillation is less than $2\varepsilon_k(1-\eta/2)^m$ in the x_n direction. We can let m go to infinity as ε_k goes to zero. Now, by dilating the situation in the x_n direction by a factor of $1/\varepsilon_k$, we have that the sets

$$A_k := \{(x', \frac{x_n}{\varepsilon_k}) | \ (x', x_n) \in \partial E \cap B_1\}$$

are included in $\{\|x_n\| \leq 1\}$. A compactness result with these functions gives us some flat limit function. For m as above we have that the oscillations of A_k inside $\|x' - x'_0\| < r^{-2m}$ is less than $2(1-\eta/2)^m$. Thus, each function f_k representing A_k is Hölder continuous with C^α norm bounded above by 2, and so, by Arzela-Ascoli, by passing to a subsequence if necessary, we get that the f_k converge to a Hölder continuous function $w(x')$ inside $\{\|x'\| \leq r\}$. Next we will show that w is harmonic by showing that it is harmonic in the viscosity sense. Assume that $P(x')$ is a quadratic polynomial that touches w from below at some point $(x'_t, w(x'_t))$. Then $\phi_\tau(x') = P(x') + c_m - \frac{\tau}{2}\|x'_t - x'\|^2$ will touch f_k from below at some point $(x'_{t,k}, f_k(x'_{t,k}))$, and by the convergence of f_k to w we have that $x'_{t,k} \to x'_t$. Since ϕ_τ touches A_k from below, $\varepsilon_k \phi_\tau$ touches ∂E_k from below, and since ∂E_k satisfies the minimal surfaces equation in the viscosity sense, by applying the viscosity relation to the touching paraboloids we have that

$$\frac{\varepsilon_k(\Delta P-\tau)}{\sqrt{1+\varepsilon_k|\nabla P-\tau(x'-x'_{t,k})|^2}}-\varepsilon_k^3\frac{(\nabla P-\tau(x'-x'_{t,k}))^T(D^2P-\tau I)(\nabla P-\tau(x'-x'_{t,k}))}{(1+\varepsilon_k|\nabla P-\tau(x-x_{t,k})|^2)^{3/2}}\leq 0.$$

Now by dividing by ε_k and letting $k\to\infty$, $\tau\to 0$ we get that $\Delta P\leq 0$ at x_t. We also get that $\Delta P\geq 0$ at x_t since if we touch w from above by P, then we would get that $\phi_\tau(x')=P(x')+c_m-\frac{\tau}{2}|x'_t+x'|^2$ would touch f_k from above, so $\varepsilon_k\phi_\tau$ would touch ∂E_k from above. Finally applying the viscosity relation gives

$$\frac{\varepsilon_k(\Delta P+\tau)}{\sqrt{1+\varepsilon_k|\nabla P+\tau(x'-x'_{t,k})|^2}}-\varepsilon_k^3\frac{(\nabla P+\tau(x'-x'_{t,k}))^T(D^2P+\tau I)(\nabla P+\tau(x'-x'_{t,k}))}{(1+\varepsilon_k|\nabla P+\tau(x-x_{t,k})|^2)^{3/2}}\geq 0.$$

Likewise taking limits gives $\Delta P\geq 0$, and so $\Delta w=0$ in the viscosity sense. However functions which are harmonic in the viscosity sense are harmonic in the classical sense, so we have that w is harmonic.

Now $w(0)=0$ since for every k we have that $0\in\partial E_k$. Doing a Taylor expansion we have

$$\|w(x')-x'\cdot\nabla w(0)\|\leq\frac{r_0}{4}\quad\text{if }\|x'\|\leq 2r_0$$

provided that r_0 is chosen small and universal.

Now since f_k converge to w, using the compactness of Arzelá-Ascoli we can make $\|f_k(x')-w(x')\|\leq\rho$ for any $\rho>0$ on $\|x'\|\leq 2r_0$ if k is large enough. So,

$$\|f_k(x')-x'\cdot\nabla w(0)\|\leq\|f_k(x')-w(x')\|+\|w(x')-x'\cdot\nabla w(0)\|\leq\rho+\frac{r_0}{4},$$

and choosing $\rho<r_0/4$ and recalling that $u_k=\varepsilon_k f_k$, multiplying the above inequality by ε_k gives us that

$$\|(x',u_k(x')\cdot(\varepsilon_k\nabla w(0),1)\|\leq\varepsilon_k\frac{r_0}{2}.$$

By normalizing the above vector, this gives us that ∂E_k satisfies the result of the theorem, i.e.

$$\partial E_k \cap B_{r_0} \subset \{\|x \cdot \nu\| \leq \frac{\varepsilon_k}{2} r_0, \}$$

which is a contradiction. □

4.2 Regularity Of Minimal Sets

Now we are at the point where we can show the regularity of ∂E. We present expanded details on the regularity process presented in [12]. By iterating Theorem 23, we obtain unit vectors v_k such that

$$\partial E \cap B_{r_0^k} \subset \{\|x \cdot v_k\| \leq \frac{\varepsilon}{2^k} r_0^k\}.$$

Now the v_k converge to a unique limit $v(0)$, for if there were two or more limit points, say ν_1 and ν_2, then we could take $k > l$ large enough so that $v_k \sim \nu_1$ and $v_l \sim \nu_2$. Then on B_{r^k} we would have that a portion of the constraining planes generated by v_k and v_l would be disjoint. Since ∂E must live in both of these regions, we get that it is not the graph of a function, a contradiction. Hence this gives uniqueness of the limiting normal vector. We want to see just how fast the unit vectors converge. To go from v_k to the next iteration we would have that ∂E is contained in both of the confined planes corresponding to v_k and v_{k+1}. To find the angle between the two vectors we can use the vertical length of both sets of constraining planes. The sum of these vertical lengths will be a very good approximation to the arc length of a circle formed by the intersection of the ball and the plane spanned by the two vectors. Since ∂E must be contained in both sets of planes, in the worst case we would have that the lower corner of one of the planes touches the corner of the other plane. Now the arc length of circle is given by $r_0^{k+1}\theta_k$, where θ_k is the angle between v_k and v_{k+1}. This gives us that

$$r_0^k \theta_k = \frac{r_0^k}{2^k}\varepsilon + \frac{r_0^{k+1}}{2^{k+1}}\varepsilon,$$

and so $\theta_k = \frac{\varepsilon}{2^k} + \frac{r_0}{2^{k+1}}\varepsilon$. This allows us to get an estimate on the tightness of v_k to v_{k+1}:

$$\begin{aligned} \|v_{k+1} - v_k\|^2 &= 2 - 2\cos\theta_k \\ &\leq \theta_k^2 \end{aligned}$$

so

$$\begin{aligned}\|v_{k+1}-v_k\| &\leq \theta_k \\ &\leq C(r_0)\frac{\varepsilon}{2^k.}\end{aligned}$$

Now, since $\sum_{k=1}^{\infty} v_{k+1}-v_k$ is absolutely summable by the above estimate and $\mathbb{R}^n$ is a Banach space, $\sum_{k=1}^{\infty} v_{k+1}-v_k$ exists and we have that $v(0)=v_1+\sum_{k=1}^{\infty} v_{k+1}-v_k$. This gives us that the rate of convergence of v_k to $v(0)$ is

$$\|v(0)-v_k\| \leq \sum_{n=k}^{\infty}\|v_{n+1}-v_n\| \leq 2C(r_0)\frac{\varepsilon}{2^k},$$

and so $\|x\cdot v(0)\| \leq \|x\cdot v_k\| + \|v(0)-v_k\| r_0^k \leq (C(r_0)+1)\varepsilon\frac{r_0^k}{2^k} = (C(r_0)+1)\varepsilon r_0^{k(1-\frac{\log 2}{\log r_0})}$, which gives us the $C^{1,\alpha}$ regularity of ∂E since

$$\partial E \cap B_{r_0^k} \subset \{\|x\cdot v(0)\| \leq (C(r_0)+1)\varepsilon r_0^{k(1-\frac{\log 2}{\log r_0})}.\}$$

Finally, we can apply the theory of DeGiorgi-Nash-Moser[13] to conclude that $u\in C^{2,\alpha}$. This together with classical Schauder theory, gives us that $u\in C^{\infty}$ by bootstrapping, since we can applying Schauder interior estimates to the solution of the minimal surface equation as found in Gilbarg and Trudinger PDE book [6].The necessary estimate and interior regularity result are as follows.

Theorem 24 (Interior Schauder Estimates)**.** *Let $\alpha\in(0,1)$. Suppose Ω is a domain and $u\in C^{2,\alpha}(\overline{\Omega})$ solves the elliptic equation*

$$Lu = a_{ij}D_{ij}u + b_iD_iu + cu = f \quad \text{in } \ \Omega, \tag{4.1}$$

where a_{ij}, b_i, c are in $C^{\alpha}(\overline{\Omega})$. Then for $\Omega' \subset\subset \Omega$,

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C\left(\|u\|_{C(\Omega)} + \|f\|_{C^{\alpha}(\Omega)}\right), \tag{4.2}$$

where $C = C(n,\alpha,L,\Omega',\Omega)\in(0,\infty)$.

Theorem 25 (Interior Regularity)**.** *Let $k\geq 0$ be an integer and $\alpha\in(0,1)$. Let $\Omega\subset\mathbb{R}^n$ be open and* *suppose that* $u \in C^2(\Omega)$ *satifies*

$$Lu = a_{ij}D_{ij}u + b_iD_iu + cu = f \quad \text{in } \Omega \tag{4.3}$$

for some elliptic operator L *with coefficients* $a_{ij}, b_i, c \in C^{k,\alpha}(\Omega)$ *and some* $f \in C^{k,\alpha}(\Omega)$*. Then* $u \in C^{k+2,\alpha}(\Omega)$

We see that, by applying theorem 25 to u for the elliptic equation induced by the minimal surface equation on u, we find that the a_{ij} and b_i are the derivatives of u. Since u is $C^{2,\alpha}$, we have that they are C^{α}, so the conditions of the theorem hold, hence u is in fact $C^{4,\alpha}$. Now we have that the coefficient's are $C^{2,\alpha}$ so we can apply the result again and get that u is $C^{6,\alpha}$. By proceeding inductively we do the so called bootstrapping procedure and find that u is $C^{k,\alpha}$ for every $k \in \mathbb{N}$, and hence u is C^{∞}.

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