Macro-size drop encapsulation

by

Amir Maleki

B.Sc., Sharif University of Technology, 2011

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF APPLIED SCIENCE

in

The Faculty of Graduate and Postdoctoral Studies

(Mechanical Engineering)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

August 2014

© Amir Maleki 2014

Abstract

Viscoplastic fluids do not flow unless they are sufficiently stressed. While in some flows this leads to unwanted features, this property can also be exploited in order to produce novel flow features. One example of such flows are visco-plastically lubricated (VPL) flows, in which a viscoplastic fluid is used to stabilize the interface in a multi-layer flow, far beyond what might be expected for a typical viscous-viscous interface. Here we extend this idea by considering the encapsulation of droplets within a viscoplastic fluid, for the purpose of transportation, e.g. in pipelines. The main advantage of this method, compared to others that involve capillary forces is that significantly larger droplets may be stably encapsulated, governed by the length scale of the flow and yield stress of the encapsulating fluid. We explore this setup both analytically and computationally. We show that sufficiently small droplets are held in the unyielded plug of the Poiseuille flow. As the length or radius of the droplets increase the carrier fluid eventually yields, potentially breaking the encapsulation. We study this process of breaking and give estimates for the limiting size of droplets that can be encapsulated.

Preface

This dissertation is an original intellectual product of the author, Amir Maleki with the guidance of his supervisors, Drs. Ian Frigaard and Sarah Hormozi. The CFD code used in chapters 3-5, originally developed by two PhD students Andreas Putz and Ali Rostaei, is modified by the author according to the requirement of this study.

The results of this study are presented at the “Canadian Applied and Industrial Mathematics Society 2014 Annual Meeting” and the “Sixth Pacific Rim Conference on Rheology”.

Table of Contents

Abstract	ii
Preface	iii
Table of Contents	iv
List of Figures	vi
Acknowledgements	vii
Dedication	viii
1 Introduction	1
1.1 Viscoplastic fluids	2
1.1.1 Poiseuille flow of a viscoplastic fluid	5
1.1.2 Mathematical complexity of viscoplastic fluids	5
1.2 VPL flows: Origin of Macro-size drop encapsulation	9
1.3 Particles, bubbles and drops in viscoplastic fluids	11
1.4 Outline of thesis	15
2 Problem Statement	16
2.1 Physical description	16
2.2 Governing equations	18
2.3 Dimensionless governing equations	18
2.4 The plane Poiseuille solution $U_P(y)$	21
2.5 Inertial effects and the droplet flow	21
2.6 Why large droplets yield the plug	22
3 Asymptotic Solution	25
3.1 Rescaling	25
3.2 Leading order solution in the yielded region	27
3.3 $\mathcal{O}(\delta)$ solution in the yielded layer	28

Table of Contents

3.4	Size of the yield surface perturbation from $y_{y,0}$ and effects of inertia	30
4	Numerical Simulation	33
4.1	Assumptions and simplifications.	33
4.2	Rheolef	35
4.3	Mesh adaptation	35
4.4	Convergence	36
4.5	Effects of droplet spacing	39
5	Results and Discussion	41
5.1	Encapsulation and failure	41
5.2	Maximum size of encapsulated droplets	43
5.3	Encapsulation with fluids of different densities	47
6	Axisymmetric Geometry	51
6.1	Poiseuille flow solution	52
6.2	Slender drop	52
6.3	Examples of encapsulation and failure	55
6.4	Maximum size of encapsulated droplets	55
7	Conclusions	59
7.1	Summary of the results	59
7.2	Limitations	62
7.3	Future directions	63
	Bibliography	64
A	Algebraic Details	71
A.1	Channel Poiseuille flow of a Bingham fluid	71
A.2	Asymptotic form of Bingham constitutive relations	73
A.2.1	channel geometry	73
A.2.2	pipe geometry	74
A.3	Perturbations are $\mathcal{O}(\delta^2)$	75
A.3.1	$y_T - y_{y,0} \sim \mathcal{O}(\delta^2)$	75
A.3.2	$U_0 + \delta U_1 = U_P + \mathcal{O}(\delta^2)$	76
B	Unsuccessful attempts	78
B.1	Method of islands of broken plug of [23]	78
B.2	Finding an admissible form of stress	80

List of Figures

1.1	Examples and mathematical models of Viscoplastic fluids . . .	4
1.2	Illustration of VPL flows	10
1.3	“Frozen-Into” patterns in VPL flows	12
2.1	Schematic description of the problem	17
3.1	Example of asymptotic results, pressure and velocity and yield surface	30
3.2	Asymptotic results: yield surface	32
4.1	Schematic description of the problem	34
4.2	Mesh adaptation cycles	37
4.3	Code validation	38
4.4	Effect of spacing	39
4.5	Variation in the pressure drop with spacing between droplets	40
5.1	Speed distribution(u) as the droplet gets larger	42
5.2	Stress distribution as the droplet gets larger (slender drops) .	44
5.3	Stress distribution as the droplet gets larger (fat drops) . . .	45
5.4	Maximum size of encapsulated droplets	46
5.5	Effect of density difference on encapsulation	49
5.6	Stress distribution as the droplet becomes heavier	50
6.1	Stress distribution in the axisymmetric geometry as the droplet gets longer	56
6.2	Stress distribution in the axisymmetric geometry as the droplet height is increased	57
6.3	Maximum size of encapsulated droplets axisymmetric geometry	58
7.1	Encapsulation of drops with exotic shapes.	60
B.1	Methods of island of broken plug	79

Acknowledgements

My special thanks goes to my supervisor, professor Ian Frigaard for his knowledge, his constant supports and patience, his caring and efforts. It was a privilege for me to work under his supervision. I cannot assume to have any better scientific mentor. Thanks Ian!

I would like to thank a fabulous couple, Drs. Sarah Hormozi and Hamed Ghasvari, who changed my life with their guidance and advice. Sarah assisted me in the analytical part of my project as a co-supervisor. She is always a model of a hard-working, energetic and passionate researcher. We had many many insightful discussions with Ian and Sarah through the course of this project.

I also acknowledge kind helps of Ali Roustaei. Ali just simply knows everything in the world of computation. Whenever I faced a problem, I knew Ali has some hints. In addition, all my other officemates in Complex Fluids Lab, especially Ida without whom I do not have any idea how days of graduate studies would look like, thank you all.

Last but not the least, I express my gratitude to Dr. Green, the head of mechanical engineering department at UBC.

Dedication

To my ever-lasting friend, with whom I share childhood memories and grown-up dreams, who has been the never-ending source of love and kindness, to my brother,

To Amin

Chapter 1

Introduction

Encapsulation is a process of entrapping one substance in another one. The main function of encapsulation is to isolate an aggressive component from the environment and/or deliver it to a particular receptor [47]. This could be beneficial in transportation, coating, site-specific drug delivery, medical imaging as well as food, cosmetic and pharmaceutical product manufacturing [29, 30, 12, 76].

Various techniques of encapsulation are proposed in the literature; e.g. [43, 26, 12, 47, 39, 74, 73], but it is not our intention to review these here. A common feature of many current techniques is that they are limited to droplet sizes governed by capillary forces. This means that only small drop size (order of millimetre or even smaller) can be encapsulated successfully. Instead, here we propose a method of encapsulation focused at Macro-scale droplets, independent of capillary forces. The main advantage of this method is that significantly larger droplets may be stably encapsulated, governed by the length scale of the flow and yield stress of the encapsulating fluid. Unlike many encapsulation processes, here we specially mean that the droplet is held within the “unyielded plug region of a viscoplastic fluid”. These terms are defined later.

This will open up use of encapsulation in many large scale industrial applications. In particular, oil and gas, food, medical and cosmetic industries can benefit from this. One interesting example is a process called *Acidizing* which will be explained here.

Acidizing oil and gas reservoir: an application of Macro-size drop encapsulation

Production of an oil or gas well depends on the reservoir’s porosity and permeability. Porosity quantifies how much void area exists in the rock formation and permeability measures the ability of the rocks to transmit oil or gas. The more porous and permeable the well, the more hydrocarbon can be extracted. One of most common ways of enhancing oil production is to stimulate the reservoir by pumping acidic fluids down the well. This process

called *acidizing* increases both porosity and permeability through opening new cracks in the rock formation and by dissolving carbonate materials between the sediment grains of the reservoir rocks. The process of acidizing has been long practiced. The history of acid treatment of oil wells is discussed by [40]. One of the earliest studies on acidizing using hydrofluoric acid is the classic study of [69]. With the development of computational techniques, attempts to model this process started around 1970's; see the references reviewed in [46]. More recent studies involve multi-scale multi-dimensional modelling of the process [11, 24, 25, 63, 49].

In spite of all these developments, there are still some crucial problems:

- Hydrochloric acid (HCL) and hydrofluoric acid (HF) are the two most common acids in the acidizing process. These materials are detrimentally corrosive for the casing (metallic wall of the well).
- Acidic fluids should start reacting at a certain location down inside the well, meaning that acid reactions should be retarded.

As an example, [2] reported an unsuccessful acid job which induced significant damage to the oil well. Various mechanisms of inhibition and retardation are being developed in order to address these issues; e.g. see [62, 58, 75].

Potentially if we can encapsulate acid drops inside a friendly, neutral, non-corrosive fluid, we can pump the acid down the well. Where drops of acid reach the desired location, we can break the encapsulation and let them react with the rocks. Breaking may occur naturally, e.g. through high shear as the drops enter the fracture, or could be controlled chemically. This automatically solves both issues mentioned earlier. We have eliminated contact of acidic fluids with the metallic wall of the casing; in addition, the operator can decide when and where the reactions should start. Note that with the fluid volumes pumped in oil industry applications, use of micro-size encapsulation is out of question.

Utilizing a specific category of non-Newtonian fluids, called viscoplastic fluids, we will suggest a new method of encapsulation and study the feasibility of this method in accommodating larger drop size. Here we introduce viscoplastic fluids:

1.1 Viscoplastic fluids

In Newtonian fluids shear stress $\hat{\tau}_{ij}$ and strain rate $\hat{\gamma}_{ij}$ are linearly proportional. The constant of proportionality is called viscosity $\hat{\mu}$ which is a

physical property of the fluid.

$$\hat{\tau}_{ij} = \hat{\mu} \hat{\gamma}_{ij} \quad (1.1)$$

This model defines fluid as a material which does not resist shear stress. Once the fluid experiences a shear stress, it starts to flow. Assuming the temperature is kept constant, the viscosity of a Newtonian fluid does not change with time, shear rate or shear stress, and remains always constant.

However there are many fluids in our daily life as well as in industrial and engineering applications for which viscosity does change. These fluids are generally termed non-Newtonian fluids. The one important category that we are interested in is called “viscoplastic fluids”, or interchangeably, “yield stress fluids”. In contrast to Newtonian fluid, these materials do not deform unless a given yield stress ($\hat{\tau}_Y$) is exceeded. In other words, there is a minimum stress below which the fluid acts like a rigid solid, resisting the stress. But once yield stress is exceeded, the fluid starts to deform and flow. Examples of these fluids are dairy products, liquid chocolate, drilling mud and lava; see Figure 1.1a. Many mathematical relations are proposed to model behaviour of viscoplastic fluids. Two common ones are the “Bingham” and “Herschel-Bulkley” models defined as follows:¹

Bingham Model

$$\hat{\tau}_{ij} = \left(\hat{\mu} + \frac{\hat{\tau}_Y}{\hat{\gamma}} \right) \hat{\gamma}_{ij} \iff \hat{\tau} > \hat{\tau}_Y \quad (1.2a)$$

$$\hat{\gamma} = 0 \iff \hat{\tau} \leq \hat{\tau}_Y \quad (1.2b)$$

Herschel-Bulkley Model

$$\hat{\tau}_{ij} = \left(\hat{\kappa} \hat{\gamma}^{n-1} + \frac{\hat{\tau}_Y}{\hat{\gamma}} \right) \hat{\gamma}_{ij} \iff \hat{\tau} > \hat{\tau}_Y \quad (1.3a)$$

$$\hat{\gamma} = 0 \iff \hat{\tau} \leq \hat{\tau}_Y \quad (1.3b)$$

¹All through the thesis, we adopt the convention of denoting all dimensional variables with a “hat”, i.e. $\hat{\cdot}$, and dimensionless variables without

Figure 1.1: a) Examples of visco-plastic fluids. Top row (left to right); paint, toothpaste, hagfish mucus, Middle row (left to right); dairy products, liquid chocolate, diaper cream, Bottom row (left to right); basaltic lavas, polymer gels, drilling Fluids b) Relationship between shear rate ($\dot{\gamma}$) and shear stress (τ) for Bingham fluids (solid line) and Herschel-Bulkley fluids (dashed line). Both pictures are taken from [32].

in which

$$\hat{\gamma}_{ij} = \frac{\partial \hat{u}_i}{\partial \hat{x}_j} + \frac{\partial \hat{u}_j}{\partial \hat{x}_i} \quad (1.4a)$$

$$\hat{\gamma} = \sqrt{\frac{1}{2} \sum_{i,j=1}^3 |\hat{\gamma}_{ij}^2|} \quad (1.4b)$$

$$\hat{\tau} = \sqrt{\frac{1}{2} \sum_{i,j=1}^3 |\hat{\tau}_{ij}^2|} \quad (1.4c)$$

Here $\hat{\mu}$ and $\hat{\tau}_Y$ are the plastic viscosity and yield stress. The power law index is n and κ is referred to as the consistency, which corresponds to the plastic viscosity when $n = 1$. The Bingham model simply assumes a constant (tangent) viscosity when the fluid is yielded. However, in Herschel-Bulkley model the viscosity of the fluid after yielding is a function of shear rate. This models a shear thinning ($n < 1$) or shear thickening ($n > 1$) material with a yield stress; the case of shear thinning is shown in Figure 1.1b.

1.1.1 Poiseuille flow of a viscoplastic fluid

The fully developed laminar flow of a fluid inside a long pipe or between two infinitely-wide plates is called Poiseuille flow. Poiseuille flow of a Newtonian incompressible fluid results in a parabolic profile for velocity which takes its maximum at the center line. Due to the symmetry, the shear stress is zero at the center line and is maximum on the wall. Therefore if we use a viscoplastic fluid instead, because of the existence of a yield stress, an inner layer arises at the center of channel or pipe within which stress is below yield stress. Therefore fluid inside the layer does not deform, and is commonly called the “plug region”. Velocity within the plug region does not vary and the fluid resembles a moving solid object. We will analytically solve Poiseuille flow of a Bingham fluid inside a channel in §A.1.

1.1.2 Mathematical complexity of viscoplastic fluids

From a mathematical point of view, modelling viscoplastic fluids is intrinsically difficult. Here we explain what sort of complexities have to be dealt with:

Analytical difficulty

Thin film analysis and lubrication approximation have been widely utilized in order to study problem involving motion of particles, drops or bubbles; e.g. see [13]. However, when dealing with viscoplastic fluids, it is known that a *naïve* application of similar approximation methods can lead to solutions that exhibit velocity gradients in the plug region. This is inconsistent with the Bingham model which defines plug regions as strain-rate-free domains. The inconsistency is called the “lubrication paradox” which has led to many confusing and incorrect statements. Confronted by this paradox, [42] concluded that no true plug region can exist in the flow of Bingham fluid in complex geometries. They added that, in spite of approximately shear-free plug regions, yielding must happen everywhere. This issue also questioned the physical robustness of the Bingham fluid as mathematical model.

Generally, the lubrication paradox appears as a result of a wrong scaling inside the plug region. In a long thin geometry, inside the yielded region scales of pressure and shear stress are determined by balancing, to the leading order, the shear force and pressure gradient. As we enter the plug, these scales are likely to remain valid. However, the extensional stresses within the plug become of the same order as the shear stresses. Therefore, in order to resolve the lubrication paradox, different scales must be introduced inside the plug. This led to development of more accurate methods which confirmed existence of true plug regions in complex geometries. As examples:

- The earliest attempt to resolve the lubrication paradox was the work of [72] in which they studied the axial flow of a Bingham fluid in an eccentric annulus. They adopted a very systematic perturbation technique and cleverly defined a plug-like region where stress exceeds yield stress but only by a very small amount and called it “pseudo-plug”². With a proper definition of scales in the pseudo-plug³ they managed to show that true plugs exist at the widest part of the annular gap (as well as narrowest part).
- [3] considered the case of gravity driven flow of a Bingham fluid over

²The idea of plug-like region where stress just exceeds the yield stress was first used by [53]

³Inside the pseudo-plug velocity with the form of $w_{pp} = w_{pp0}(\theta) + \delta w_{pp1}(\delta, r) + \mathcal{O}(\delta^2)$ is chosen where δ is the asymptotic parameter of the problem. This allows $\mathcal{O}(\delta)$ strain rate inside the pseudo-plug. The pseudo-plug is an enclosed region which requires a third region where the stress has to gradually relax from $-B$ to B . This is where $\tau < B$ and the true plug exists.

a slope. Incorporating a similar approach to [72], they argued that no true plug could exist in such a geometry. However, there exists a pseudo-plug region. In fact, the leading order velocity here is determined by the film height and therefore, as the height of film changes, slow velocity gradients arise in flow direction. Thus, absence of the true plug is expected. The author concluded that existence of true plug regions in varying geometries is more likely in wall-bounded geometries.

- [23] examined flow of a Bingham fluid between two wavy plates. Their goal was to study channel Poiseuille flow where small perturbations are introduced on the wall of the channel; in particular, how the plug region changes in response to the perturbations. First of all, they showed that the conventional lubrication scaling ends up with the lubrication paradox (variation of velocity in principal direction within the plug). To resolve this inconsistency, two different scales are proposed inside the pseudo-plug depending on the relative configuration of the pseudo-plug and the true plug and finally the existence of true plug region is confirmed provided that the amplitude of perturbations is sufficiently small. With the aid of the proper scaling inside the plug, they also estimated a sufficient condition for the plug to yield. More recently, [61] revisited this problem and verified the asymptotic solution provided by [23].

Computational difficulty

Viscoplastic models are all singular when strain rate vanishes, which implies an infinite effective viscosity in unyielded flow regions. In addition, the position of plug interface is not known *in priori*. To deal with the singularity problem two different classes of methods are proposed: Regularisation methods and augmented Lagrangian algorithms. In general, viscosity regularization methods suffer from an uncertainty, especially in determining the plug interface, while augmented Lagrangian methods are more complicated in implementation and more costly computationally [15, 4].

- Class One, Regularisation Methods: In regularisation methods, we eliminate the singularity of stress by replacing the plug region with a highly viscous fluid (as opposed to the original model which considers the plug region a solid). With this assumption we get rid of the singular point (with infinite derivative) near origin, however, it still remains non-differentiable. The efficiency of different regularisation methods

are compared in [22]. As an example, the “simple” regularisation model is written as:

$$\hat{\tau}_{ij} = \left(\hat{\mu} + \frac{\hat{\tau}_Y}{\hat{\gamma}_1 + \epsilon \hat{\gamma}_0} \right) \hat{\gamma}_{ij} \quad (1.5)$$

where $\hat{\gamma}_0$ is a representative strain rate of the flow and in which $\epsilon \ll 1$. Thus, the regularisation is negligible if the strain rate is of order $\hat{\gamma}_0$. As the strain rate approaches zero, the effective viscosity approaches $\hat{\tau}_Y/(\hat{\gamma}_0\epsilon)$, remaining large but finite. Compared to Augmented Lagrangian method (introduced below), regularisation methods are computationally cheaper and easier to implement [15]. However, as $\epsilon \rightarrow 0$, the method becomes costly. Unfortunately in spite of many non trivial problems computed with this method, there is no theory which guarantees regularisation methods converge to the correct stress field.

In addition, regularisation methods may fail to give the correct position of the yield surfaces, and potential errors are largest in flows with long-thin geometries (as is our case); see [22]. This is probably the main drawback of regularisation methods [61]. For flows such as that here, where we investigate whether or not the droplet is fully encapsulated in unyielded fluid, regularization methods are unable to answer with certainty.

- Class Two, Augmented Lagrangian (AL) Algorithms: The alternative to regularization methods is algorithms such as the augmented Lagrangian method of [20, 27]. Through the augmented Lagrangian algorithms, the exact form of constitutive relation is implemented in the discretized equations which gives the true unyielded regions with exactly zero strain rate. As we want to study the perturbations of plug interface, and break of the plug, it is crucial to distinguish the true unyielded regions of the flow from those where the strain rate is merely small.

As an example, let us consider the augmented Lagrangian method which is developed for Bingham fluids. This method is based on the variational form of Stokes equation for Bingham plastic, derived e.g. by [19], which says that the velocity field of a Bingham fluid minimises the following variational form:

$$\hat{J}(\hat{v}) = \frac{\hat{\mu}}{2} \int_{\hat{\Omega}} \|\hat{\gamma}(\hat{v})\|^2 + \hat{\tau}_Y \int_{\hat{\Omega}} \|\hat{\gamma}(\hat{v})\| - \int_{\hat{\Omega}} \hat{g} \hat{v} \quad (1.6)$$

in which $\hat{\gamma}(\hat{v}) = \nabla \hat{v}$ and $\|\hat{\gamma}(\hat{v})\|$ is the norm of $\hat{\gamma}(\hat{v})$ and $\hat{g}$ represents a body force. $\hat{\Omega}$ is also the domain of the problem. The non-differentiable term of $\int_{\hat{\Omega}} \|\hat{\gamma}(\hat{v})\|$ hampers the use of the standard optimization techniques. Instead, two Lagrange multipliers ($\hat{\gamma}, \hat{\mathbf{T}}$) can be introduced to relax the problem into a saddle-point problem [20]:

$$\hat{J}_a(\hat{v}, \hat{\gamma}, \hat{\mathbf{T}}) = \hat{J}(\hat{v}) + \int_{\hat{\Omega}} (\hat{\gamma}(\hat{v}) - \hat{\gamma}) : \hat{\mathbf{T}} + \frac{\hat{a}}{2} \int_{\hat{\Omega}} \|\hat{\gamma} - \hat{\gamma}\| \quad (1.7)$$

where $\hat{a}$ is the augmented Lagrangian parameter. To find the saddle point, a Uzawa-type algorithm similar to the one in [67] is useful. It has been shown that $\hat{\mathbf{T}}$ and $\hat{\gamma}$ converge to “admissible” stress and strain tensors. A modern review of such methods is given by [28].

1.2 VPL flows: Origin of Macro-size drop encapsulation

The idea of Macro-size drop encapsulation stems from observations in a series of multi-layer flow studies called *VPL flows*. Viscoplastically Lubricated (VPL) flows are a specific arrangement of multi-layer flows where a yield stress fluid is used to stabilize the interface of the two layers. It is known that interfacial instabilities inherently occur in multi-layer flows as the physical properties of the fluids change across the interface. If we can position a visco-plastic fluid in the outer layers and tune the flow rates to keep the interfacial shear stress less than the yield stress (τ_Y), the fluid-fluid interface of the flow is converted to a fluid-solid interface, which does not allow instabilities to develop. In other words, a visco-plastic fluid adjacent to the wall lubricates a centrally positioned viscous fluid (see Figure 1.2).

This idea was originally introduced in [21] in which the author considered a two-layer flow of two Bingham fluids where the inner fluid has a smaller or zero yield stress. Through stability analysis, he demonstrated that this configuration of multi-layer flow does not suffer from the classical linear interfacial instabilities that are found for analogous multi-layer flows of Newtonian and many other non-Newtonian fluids. Instead, such flows first become linearly unstable to shear instabilities that are analogous to those found for single fluid flows. This is then continued by [51, 52, 36, 34] where they studied the nonlinear stability of a multi-layer shear flow, for Newtonian and non-Newtonian core fluid, and demonstrated that with careful choice of rheology, multi-layer flows can be achieved which are linearly and non-linearly stable even at significantly high Reynolds numbers and large

Figure 1.2: Illustration of VPL flows. a) Visco-plastically lubricated pipe flow; b) Cross-sectional view c) Schematics of the type of basic velocity profiles considered, with reduced rate of strain in the Newtonian fluid and an unyielded plug zone adjacent to the interface in the lubricating fluid; [32]

amplitudes of perturbation. The transient characteristics (entry, start up and development length after which the flow is parallel) of these flows are also studied in [37]. More interestingly, [36] computationally confirms that stable VPL flows can be achieved with more layers (e.g. seven layers). In addition, they have experimented numerically with different injection configurations in order to advect encapsulated “letters” and “write” in the flow. This is where a vague idea of encapsulation is introduced, but not treated very systematically.

[38] studied the feasibility of VPL flows experimentally. The schematics of their experimental setup is shown in Figure 1.3a. Xanthan⁴ as the inner layer and Carbopol as the outer layer⁵ flow in path 1 and path 2 respectively. Later [35] replaced the core fluid with Polyethylene oxide (PEO) which exhibits elastic behaviour to show that the VPL technique is also applicable for multi-layer flows with elasticity properties. The idea of VPL flows has been extended in different ways. For example, recently [18] and [33] have introduced periodic perturbations to the flow rates of initially stable VPL flows, resulting in stable patterned interfaces. The setup shown in Figure

⁴Xanthan has shear-thinning rheology.

⁵Carbopol is the most common viscoplastic fluid in experimental practices.

1.3a has changed such that a temporal profile of flow rate can be given to the inner or outer layer. Any sudden change in the flow rate produces some instabilities in the flow which could form frozen-into shapes within the interface of VPL flow. For example, Figure 1.3 demonstrates one typical experiment of [18] where a “diamond necklace” pattern is preserved.

We will extend this idea by considering the encapsulation of droplets within a visco-plastic fluid.

1.3 Particles, bubbles and drops in viscoplastic fluids

Although it is not directly relevant to our study, it is still useful to review the previous works of motion of particles, bubbles and drops in viscoplastic fluids. Motion of particles, bubbles and drops in viscoplastic fluids is becoming an ubiquitous process in many engineering and industrial applications such as oil and petroleum, pharmaceutical, cosmetic and medical industries. For instance:

1. Drilling mud used in the construction of oil and gas wells is considered a viscoplastic fluid. If the hydrostatic pressure of the mud is unexpectedly lower than the pore pressure of gas in the surrounding rock formation, gas enters and rises upward in the well. This phenomenon is called gas kick and can eventually cause a blowout at the surface with enormous environmental and economical costs.
2. In the processing and packaging stages of various products bubbles may either be unwanted or desirable, e.g., hair gel is sold by volume (bubbles included). In contrast, it is not always desirable to have air trapped in chocolate as aerated chocolate has a different taste than solid chocolate. Also bubbles may affect mechanical properties of structural materials like cement. [16, 15].
3. In materials such as cement or toothpaste which contain suspended particles, sedimentation breaks the homogeneity of the fluid and therefore alters the physical properties [54].

In the absence of fluid motions there are many studies of settling in stationary yield stress fluid whereby the yield stress holds a particle stationary

Figure 1.3: a) Schematic view of [18] experimental setup. b) Formation of a stable “diamond necklace” pattern in his experiments. Inner flow: PEO is initially flowing at a rate of $Q_{inner} = 15.4 \text{ ml/s}$, suddenly the flow rate is reduced to $Q'_{inner} = 2.6 \text{ ml/s}$, held at this flow rate for $T = 1 \text{ s}$ and then returned back to its initial value. These pattern repeats every 10 seconds. Outer flow: Carbopol flowing at the rate of $Q_{outer} = 8.6 \text{ ml/s}$.

until a certain critical buoyancy stress is exceeded. Thus, particles below a certain size are held in suspension. In the case of a spherical particle, [6] simulated the flow around the particle using a regularization method and interestingly, found out that the spherical particle falls only if

$$Y_g = \frac{\hat{\tau}_Y}{\hat{R}\hat{g}\Delta\hat{\rho}} \leq 0.095. \quad (1.8)$$

where $\hat{R}$ is the radius of sphere and $\Delta\hat{\rho}$ and $\hat{g}$ are density difference and gravitational acceleration respectively. Their results are later confirmed by many other studies showing only minor discrepancies; e.g. [8, 5, 44]. More recently, [60] analyzed Stokes flow around a particle in Bingham fluid with emphasis on the *resistance* and *motility* problem. Later, [59] performed careful experiment on the sedimentation of spherical particles. The main conclusion they draw was inaccuracy of common viscoplastic models in characterizing the rheological behaviour of viscoplastic material (in particular, Carbopol) which may lead to discrepancies between experimental and numerical data.

The motility bound given in (1.8) does not hold for a flowing yield stress fluid [54]. This arises another class of studies which consider particle settling induced by shear. [50] claimed that in pipe flows no settling occurs in a sheared yield stress fluid if $Y_g \geq 6$. However, based on their observations, [54] raised doubts as to the validity of this stability bound.

As for cylindrical particles, in a theoretical paper, [1] approximated the Bingham flow around a circular cylinder and indicated the main characteristics of velocity and stress fields. This problem has been later revisited by [14] and [71] who utilized regularisation methods to simulate the problem. The later carefully studied the effect of different numerical parameters on the convergence characteristics. In addition, they gave estimates for the drag coefficient of the particle as a function of Bingham number. With the aid of augmented Lagrangian method [64] computed this problem when the cylindrical particle is confined by two parallel plates. They investigated the flow characteristics, in particular shapes of yielded and unyielded regions, when the particle is asymptotically close to the wall. The results of this paper can be useful for the purpose of benchmarking and code validation.

Studies on the rise of bubbles in a viscoplastic medium are not unfortunately as clear. The mathematical analysis of this category of flow was initiated by [7] and then continued in [70]. They both employed perturbation analyses and provided solutions for a bubble moving in a Bingham fluid for small Bingham numbers. Utilizing variational analysis, [16] derived a sufficient condition (not necessary) for stopping of bubbles in Herschel

Bulkley fluids. Later, they [17] experimentally confirmed their analytical bounds and proposed less conservative empirical limits.

[68] simulated buoyancy driven flow induced inside a stagnant pool of Bingham fluid by a 2D bubble or drop. Using a regularization method, they computed the shape of plug region and argued the presence of rotating unyielded zones at the equator of the bubble. Accuracy of regularization methods in simulating bubble rise in a viscoplastic fluid has been studied by [15] who compared the regularization method introduced by [56] with the results of the augmented Lagrangian method in Herschel-Bulkey model. They showed that up to intermediate Bingham numbers the regularization model of [56] predicts the unyielded region accurately. However, for larger Bingham number extra caution is required in choosing the regularisation parameter.

In so far as viscous droplets are concerned, there are fewer studies in the literature. For instance, [57] simulated motion and deformation of a single drop in a Bingham fluid under gravity. They reported approaching a quasi-steady state after a relatively large transient period. The transient period and velocity magnitude depend strongly on Bingham number. They also observed stationary flows for sufficiently large Bingham numbers. [31] experimentally studied the fall of a Newtonian drop inside an otherwise stagnant Bingham fluid. Using a combined PTV and PIV technique, they determined the approximate position of plug interface as well as the wall effect on sedimentation velocity.

Interaction of multiple drops or bubbles in a viscoplastic media has also recently received attention [45, 57, 68, 41, 31]. In particular, [45] reported that the travelling particles in a viscoplastic media exhibit negligible interaction unless they are in close proximity (less than four sphere radii). [57] examined the interaction of two drops while they are falling under gravity in a Bingham fluid. For the case of two similar drops, they showed that within the proximity range defined in [45], the drops tend to approach each other and coalesce. More interestingly, [68] found in their simulations that upon certain arrangement, a group of bubbles can rise under conditions where a single drop is unable to move. In the context of our study, the proximity distance of [45] is relevant in that we also will determine a droplet spacing above which the droplets do not influence one another. However, note that our droplets do not yield the fluid encapsulating fluid, so the connection with the above studies is via locality of the stress field rather than the velocity field.

Finally, moving to large numbers of particles, bubbles or droplets we enter the realm of yield stress suspensions, foams and emulsions. Recent

work has addressed the question of estimating the bulk yield stress of a yield stress suspension, according to the fraction of the dispersed phase. [10] developed a theoretical model to estimate the overall yield stress of a suspension of noncolloidal and non-Brownian particles immersed in a yield stress fluid. They showed that such a suspension can be modelled as “a Herschel-Bulkley fluid with an exponent equal to that of the suspending fluid.” This agrees reasonably well with the experimental results of [55, 48]. Of course, in dealing with suspensions the dispersed phase length-scale is much smaller than that of the bulk flow, unlike here where the droplets are of the flow dimension and we focus on a continuous stream.

1.4 Outline of thesis

An outline of the thesis is as follows. In chapter 2 we describe the problem of study and derive the governing equations in dimensionless form. For simplicity we start with the channel geometry. Chapter 3 considers slender droplets, showing that the yield surface is barely perturbed by the presence of the droplets. In chapter 4 we describe the numerical approach that we took to compute the flow around an iso-dense droplet. Later, in chapter 5 we present the results of computational analysis. In particular, we determine the mechanisms of yielding as droplets grow large and find the maximal droplet size. We also explore the effect of density difference on encapsulation process. Next, in chapter 6 we generalize the foregoing results for plane channel encapsulation to the more practical pipe geometry. The thesis closes with a summary of contributions and future directions in chapter 7.

Chapter 2

Problem Statement

Here we describe the problem and specify the effective parameters. We simplify the problem by making some assumptions which help us solve it analytically and numerically.

2.1 Physical description

We consider an infinite number of drops of evenly spaced flowing within a yield stress fluid inside an infinitely vertically-oriented long channel. The width of channel is $2\hat{R}$ and length and width of drops are $2\hat{H}$ and $2\hat{L}$ respectively. Aspect ratio of drops is defined as $\delta = \frac{\hat{H}}{\hat{L}}$; see Figure 2.1 for a schematic view of the problem. The drops are to be considered Newtonian and the carrier fluid is modelled as a Bingham fluid (simplest model for a viscoplastic fluid). Both the bulk (carrier fluid) and disperse (drop) phase are incompressible and the areal flow is $2\hat{R}\hat{U}_0$, i.e. the mean velocity along the channel is $\hat{U}_0$. The droplets are spaced a distance $\hat{l}$ apart, each centred at $\hat{x} = k\hat{l} + \hat{U}_P\hat{t} : k \in \mathbb{Z}$, and have boundaries (interfaces) given by:

$$\hat{y} = \pm\hat{h}(\hat{x} - k\hat{l} - \hat{U}_P\hat{t}), \quad (2.1)$$

where $\hat{h}(0) = \hat{H}$.

As explained before in §1.1, in the absence of drops, velocity of the carrier fluid adopts a uniform symmetric Poiseuille profile, which in the case of Bingham fluid, contains an inner plug region in the middle of channel of width $2\hat{y}_{y0}$ and speed of $\hat{U}_P$ where shear rate is zero and the fluid is just like a moving solid. Insertion of drop inside the plug region will change the stress distribution within the plug region, and possibly may cause it to yield (total stress exceeds the yield stress $\hat{\tau}_Y$). We want to see when this plug region may encapsulate a fluid drop without the plug yielding. In this thesis, when we refer to encapsulation, we specifically mean that the droplet is held in an unyielded plug region.

Figure 2.1: Schematic description of the problem

2.2 Governing equations

Inside the drop continuity and momentum equations write:

$$0 = \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} \quad (2.2a)$$

$$\hat{\rho}_d \left(\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} \right) = -\frac{\partial \hat{p}}{\partial \hat{x}} + \hat{\mu}_d \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \hat{\mu}_d \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \hat{\rho}_d \hat{g} \quad (2.2b)$$

$$\hat{\rho}_d \left(\frac{\partial \hat{v}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} \right) = -\frac{\partial \hat{p}}{\partial \hat{y}} + \hat{\mu}_d \frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \hat{\mu}_d \frac{\partial^2 \hat{v}}{\partial \hat{y}^2} \quad (2.2c)$$

where $\hat{\rho}_d$ and $\hat{\mu}_d$ are the drop's density and viscosity. Similarly in the carrier fluid

$$0 = \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} \quad (2.3a)$$

$$\hat{\rho} \left(\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} \right) = -\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\partial \hat{\tau}_{xx}}{\partial \hat{x}} + \frac{\partial \hat{\tau}_{xy}}{\partial \hat{y}} + \hat{\rho} \hat{g} \quad (2.3b)$$

$$\hat{\rho} \left(\frac{\partial \hat{v}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} \right) = -\frac{\partial \hat{p}}{\partial \hat{y}} + \frac{\partial \hat{\tau}_{xy}}{\partial \hat{x}} + \frac{\partial \hat{\tau}_{yy}}{\partial \hat{y}} \quad (2.3c)$$

where $\hat{\rho}$ is the carrier fluid's density. In order to close (2.3) we need to incorporate the constitutive relations for a viscoplastic fluid. Taking carrier fluid as a Bingham fluid, the constitutive relation is (1.2) in which $\hat{\gamma} = \sqrt{\hat{\gamma}_{xy}^2 + \hat{\gamma}_{xx}^2}$, $\hat{\tau} = \sqrt{\hat{\tau}_{xy}^2 + \hat{\tau}_{xx}^2}$, and

$$\hat{\gamma}_{xy} = \hat{\gamma}_{yx} = \frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \quad (2.4a)$$

$$\hat{\gamma}_{xx} = -\hat{\gamma}_{yy} = 2 \frac{\partial \hat{u}}{\partial \hat{x}} = -2 \frac{\partial \hat{v}}{\partial \hat{y}} \quad (2.4b)$$

To simplify these sets of equations and find the effective dimensionless parameters, we introduce a scale for each parameter and nondimensionalize the governing equations. As expressions of boundary conditions in the dimensionless form look simpler, we explain them in the next section.

2.3 Dimensionless governing equations

We consider flows, periodic in $\hat{x}$ and symmetric with respect to the channel centerline $\hat{y} = 0$. We scale all lengths with $\hat{R}$, velocities with $\hat{U}_0$ and

2.3. Dimensionless governing equations

deviatoric stresses with $\hat{\mu}\hat{U}_0/\hat{R}$. The pressure is scaled as follows:

$$\hat{p} = \hat{p}_0 + \hat{\rho}\hat{g}\hat{x} + \frac{\hat{\mu}\hat{U}_0}{\hat{R}}p \quad (2.5)$$

i.e. p is the dimensionless modified pressure, subtracting off the static pressure $\hat{\rho}\hat{g}\hat{x}$ and a possibly time dependent pressure $\hat{p}_0(\hat{t})$.

It is assumed that the distance between droplets $\hat{l}$ is large enough for the carrier fluid to assume its undisturbed uniform Poiseuille profile between droplets, in which the plug moves with speed $\hat{U}_P$. We explore effects of varying $\hat{l}$ later in §4.5. To fully exploit the symmetry of the problem, we translate to a moving frame as follows:

$$(x, y) = \left(\frac{\hat{x} - \hat{U}_P\hat{t}}{\hat{R}}, \frac{\hat{y}}{\hat{R}} \right) \quad (2.6a)$$

$$\hat{t} = \frac{\hat{U}_P\hat{t}}{\hat{R}} \quad (2.6b)$$

$$(u, v) = \left(\frac{\hat{u} - \hat{U}_P}{\hat{U}_0}, \frac{\hat{v}}{\hat{U}_0} \right) \quad (2.6c)$$

In the moving frame, we consider the flow to be steady and due to periodicity consider only the domain: $(x, y) \in [-l/2, l/2] \times [-1, 1]$, where $l = \hat{l}/\hat{R}$. The droplet-encapsulating fluid interface is given by: $y = \pm h(x)$ with $h(0) = H = \hat{H}/\hat{R}$ and where $h(x) = 0$ for $|x| \geq L = H/\delta = \hat{L}/\hat{R}$.

Adopting these scales, the dimensionless form of (2.2) writes:

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad (2.7a)$$

$$\phi Re \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + m \frac{\partial^2 u}{\partial x^2} + m \frac{\partial^2 u}{\partial y^2} + \chi \quad (2.7b)$$

$$\phi Re \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + m \frac{\partial^2 v}{\partial x^2} + m \frac{\partial^2 v}{\partial y^2} \quad (2.7c)$$

here $\phi = \hat{\rho}_d/\hat{\rho}$ is the density ratio, $m = \hat{\mu}_d/\hat{\mu}$ is a viscosity ratio and χ is a dimensionless group:

$$\chi = \frac{(\hat{\rho}_d - \hat{\rho})\hat{g}\hat{R}^2}{\hat{\mu}\hat{U}_0}. \quad (2.8)$$

2.3. Dimensionless governing equations

representing the ratio of buoyant stresses to viscous stresses. The scaled equations of motion in the carrier fluid are:

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad (2.9a)$$

$$Re \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \quad (2.9b)$$

$$Re \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \quad (2.9c)$$

where the constitutive relations are:

$$\tau_{ij} = \left(1 + \frac{B}{\dot{\gamma}}\right) \dot{\gamma}_{ij} \iff \tau > B \quad (2.10a)$$

$$\dot{\gamma} = 0 \iff \tau \leq B \quad (2.10b)$$

in which $\dot{\gamma} = \sqrt{\dot{\gamma}_{xy}^2 + \dot{\gamma}_{xx}^2}$, $\tau = \sqrt{\tau_{xy}^2 + \tau_{xx}^2}$, and

$$\dot{\gamma}_{xy} = \dot{\gamma}_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (2.11a)$$

$$\dot{\gamma}_{xx} = -\dot{\gamma}_{yy} = 2\frac{\partial u}{\partial x} = -2\frac{\partial v}{\partial y} \quad (2.11b)$$

in which all shear rates are scaled by $\hat{U}_0/\hat{R}$. Two dimensionless groups introduced above are defined as: $Re = \rho \hat{U}_0 \hat{R}/\hat{\mu}$ is the Reynolds number and $B = \hat{\tau}_Y \hat{R}/\hat{\mu} \hat{U}_0$ is the Bingham number.

At the walls of the channel no-slip conditions are to be satisfied and at $x = \pm l/2$ the flow is fully developed and adopts the Poiseuille profile:

$$u(x, \pm 1) = -u_{p,0}, \quad v(x, \pm 1) = 0, \quad (2.12a)$$

$$u(\pm l/2, y) = U_P(y) - u_{p,0}, \quad v(\pm l/2, y) = 0, \quad (2.12b)$$

Here $U_P(y)$ denotes the fully developed plane Poiseuille profile, which has dimensionless plug speed $u_{p,0}$; see §2.4. At the interface between droplet and carrier fluid the velocity and the traction are continuous:

$$\llbracket \tau_{nt} \rrbracket = 0, \quad \text{at } y = \pm h(x), \quad (2.13a)$$

$$\llbracket -p + \tau_{nn} \rrbracket = 0, \quad \text{at } y = \pm h(x). \quad (2.13b)$$

Here the subscripts n and t on the deviatoric stress components refer to directions resolved normal and tangential to the interface. For simplicity we neglect capillary effects in the formulation as the aim is to study encapsulation via yield stress effects on a macro-scale, i.e. implicitly meaning that the surface tension is insignificant in comparison to the yield stress.

2.4 The plane Poiseuille solution $U_P(y)$

In §A.1 we straightforwardly show that channel Poiseuille flow solution of a Bingham fluid can be resolved as:

$$U_P(y) = \begin{cases} u_{p,0} & \text{if } |y| \leq y_{y,0}, \\ u_{p,0} \left[1 - \frac{(|y| - y_{y,0})^2}{(1 - y_{y,0})^2} \right] & \text{if } |y| > y_{y,0}, \end{cases} \quad (2.14)$$

in which $u_{p,0}$ is the plug velocity and is given by

$$u_{p,0} = \frac{B}{2y_{y,0}}(1 - y_{y,0})^2, \quad (2.15)$$

and $y_{y,0}$, the position of the yield surface, is found by this cubic equation known as *Buckingham Equation*:

$$y_{y,0}^3 - (3 + \frac{6}{B})y_{y,0} + 2 = 0, \quad (2.16)$$

provided that the unity flow rate is satisfied. It has been shown in §A.1 that 2.16 has a single root $y_{y,0}(B) \in (0, 1)$, and exhibits the following asymptotic behaviour:

$$y_{y,0} \sim \frac{B}{3} - \frac{B^2}{6} \quad \text{as } B \rightarrow 0; \quad y_{y,0} \sim 1 - \frac{\sqrt{2}}{B^{1/2}} + O(B^{-1}) \quad \text{as } B \rightarrow \infty \quad (2.17)$$

2.5 Inertial effects and the droplet flow

As discussed previously, the aim of our study is to understand situations under which there may exist a droplet fully encapsulated in the unyielded plug of the encapsulating fluid. What we consider below neglects inertial effects. We now examine this assumption.

Firstly with respect to the droplet flow, for any droplet that is fully encapsulated within the plug the velocity at the interface satisfies $(u, v) = 0$, as we have subtracted off the plug velocity. If we consider these conditions as the boundary conditions for the droplet domain the droplet flow problem effectively decouples from that of the carrier fluid. Using these conditions, the droplet flow has the (unique) Stokes flow solution $(u, v) = 0$, everywhere within the droplet, with

$$p = \chi x + \text{constant}. \quad (2.18)$$

2.6. Why large droplets yield the plug

The trivial solution also solves the steady inertial problem. Assuming that the plug remains unyielded we may even consider time dependent solutions for the droplet. As there is no driving force for the flow within the droplet, an energy stability analysis would show that any transients in the droplet decay to zero like $\sim \exp^{-(\phi Re/m)t}$, which is the usual viscous decay timescale of the droplet. There is no bound needed on the size of the initial condition from the perspective of the decay bound. However, transients within the droplet would induce stresses within the plug and hence *a priori* assumption of an unyielded plug could fail for sufficiently large initial conditions. Nevertheless, it is apparent that the trivial solution is likely to be stable for a reasonable range of $Re > 0$ and we therefore consider this to represent the droplet solution.

Since the fluid in the droplet is Newtonian, the trivial solution $(u, v) = 0$, implies vanishing of the shear stresses within the droplet. Equation (2.13) therefore simplifies to the following stress conditions to be satisfied by the Bingham fluid at the interface:

$$\tau_{nt} = 0, \quad \text{at} \quad y = \pm h(x), \quad (2.19a)$$

$$-p + \tau_{nn} = -\chi x + \text{constant}, \quad \text{at} \quad y = \pm h(x). \quad (2.19b)$$

Assuming symmetry and locating the droplet at the origin, we may consider the constant on the right-hand side of (2.19b) to be zero.

For the carrier fluid in the absence of the droplet, the Poiseuille flow solution is valid up to large Re , when the flow destabilizes due to turbulent transition. Departures from this solution in the droplet flow solution are likely to be associated with perturbation of the streamlines in the vicinity of the droplet. Conditions (2.19) are satisfied at the interface, which results in a perturbation of the stress field within the plug. This in turn may perturb the yield surface from its uniform value $y_{y,0}$ and the perturbed yield surface may then induce a velocity perturbation from the Poiseuille flow solution. If we suppose that the aspect ratio of the streamline perturbation is characterised by a parameter ε then the inertial terms in the x -momentum equation have size εRe and those in the y -momentum equation have size $\varepsilon^3 Re$. We may consider the inertial terms to be small if $\varepsilon Re \ll 1$, which we show later to be the case.

2.6 Why large droplets yield the plug

It might at first appear that the plug may sustain any droplet of size $h(x) < y_{y,0}$, since the rigid motion of the uniform plug around the droplet allows

2.6. Why large droplets yield the plug

the droplet to translate at uniform speed along the channel. However, this reasoning neglects the effects of the droplet on the stress field. We consider 2 simple examples that suggest how the stresses may act to break the plug region for sufficiently large droplets.

First let us consider an iso-dense droplet ($\chi = 0$; see later in §5.3 for $\chi \neq 0$). We have seen that the undisturbed flow (no droplets) is the Poiseuille flow of §2.4 above, in which the frictional pressure gradient satisfies:

$$\frac{\partial p}{\partial x} = -\frac{B}{y_{y,0}}, \quad (2.20)$$

Now let us introduce a droplet within the plug region, at $x = 0$ in the translating frame. If the droplet translates uniformly, the shear stresses in the droplet are zero and the pressure in the droplet is simply $p = \text{constant}$, (chosen to be zero). In contrast, within the yielded layer of the carrier fluid we have $p = -Bx/y_{y,0}$, varying by $\sim 2BL/y_{y,0}$ along the length of the droplet.

The stresses within the unyielded plug are indeterminate in a yield stress fluid. Stress continuity implies that the tangential stress must vanish at the interface, whereas the total normal stress must balance the constant droplet pressure. We see that the pressure imbalance between the droplet and the yielded layer of the encapsulating fluid is of size $\sim BL/y_{y,0}$, which must somehow be absorbed by the deviatoric stresses within the plug region. Consequently, a simple order of magnitude estimate for the length of droplet that can be sustained by a visco-plastic fluid is simply:

$$y_{y,0} \gtrsim L. \quad (2.21)$$

Putting this into dimensional terms we have:

$$\frac{\hat{\tau}_Y}{\hat{G}\hat{R}} \gtrsim \frac{\hat{L}}{\hat{R}}, \quad \Rightarrow \quad \frac{\hat{\tau}_Y}{\hat{G}} \gtrsim \hat{L}, \quad (2.22)$$

where $\hat{G}$ is the frictional pressure gradient of the uniform Poiseuille flow. Since $y_{y,0} < 1$ we see that (2.21) is fundamentally a restriction on droplet area (equivalently volume in three dimensions), i.e. we expect that $LH < y_{y,0}$.

A second example considers the tangential stress distribution. Consider a droplet with slowly varying height ($|\frac{\partial h}{\partial x}| \ll 1$), encapsulated inside the plug region. Inside the plug the stress distribution is undetermined, but has to satisfy the momentum equations in (2.9). Assuming the yield surface is

2.6. Why large droplets yield the plug

approximately $y_{y,0}$, we might linearly approximate:

$$\tau_{xy} \approx -B \frac{y - h}{y_{y,0} - h}, \quad (2.23)$$

i.e. satisfying the stress conditions at the droplet and yield surface. Ignoring the x -variation of h , from (2.9b) we find that $p - \tau_{xx} = -Bx/(y_{y,0} - h)$ within the plug. Outside the plug we have $p = -Bx/y_{y,0}$, and imposing continuity of normal stress across the yield surface:

$$-Bx/y_{y,0} = p - \tau_{yy} = p + \tau_{xx}, \quad \Rightarrow \quad \tau_{xx} = \frac{Bx}{2} \frac{h}{y_{y,0}(y_{y,0} - h)}. \quad (2.24)$$

Now observe that for fixed x , as $y_{y,0} - h \rightarrow 0$, $|\tau_{xx}| \rightarrow \infty$, i.e. suggesting that droplet widths may not approach arbitrarily close to the plug width.

Finally if we assume the distributions (2.23) and (2.24), on combining with the yield criterion ($\tau = B$) we predict that the plug yields along $y = \pm y_y(x)$:

$$\frac{y_y(x) - h}{y_{y,0} - h} \approx \sqrt{1 - \frac{x^2}{4} \left(\frac{1}{y_{y,0} - h} - \frac{1}{y_{y,0}} \right)^2} \quad (2.25)$$

Thus, also x is limited in such a stress distribution, i.e. $|L| < 2y_{y,0}(y_{y,0} - h)/h$, in order for the plug to remain unyielded around the droplet, or $LH < 2y_{y,0}(y_{y,0} - h)$. These two simple examples represent extremes of behaviour, indicating simply that we may expect limits on the size of encapsulated droplets. In practice the stress distributions within the unyielded plug can be any that satisfy the Stokes equations and boundary conditions. The idea of constructing admissible form of stress distributions inside the plug region is elaborated in more details in §B.

Chapter 3

Asymptotic Solution

A common class of flows for which it is possible to construct analytical solutions is that in which the streamlines are nearly one-dimensional. In the problem considered, non-uniformity only arises from the droplet shape and hence we look at droplets of small aspect ratio ($H/L = \delta \ll 1$). However, the analysis above leading to (2.21) suggests that (isodense) droplets can not be encapsulated if L is larger than $y_{y,0}$. Combined with the requirement of $\delta \ll 1$ implies that $H \sim \delta \ll 1$, i.e. we consider small-slender droplets.

Assuming $h(x) \sim O(\delta)$, we develop an asymptotic approximation to the flow in the case that $\delta \ll 1$. Evidently we expect that the axial velocity will have similar size to that of the droplet free flow and the appropriate scale for y remains the channel half-width. On the other hand, since the shear stress is zero at $y = h(x)$, this suggests an $O(\delta)$ perturbation of the stress field, which may induce velocities $v \sim \delta$. In order to balance the mass conservation equation, a rescaling of x is needed. Therefore, we consider the following rescaling:

3.1 Rescaling

$$x\delta = X, \quad y = Y, \quad u = U, \quad v = \delta V, \quad p\delta = P. \quad (3.1)$$

The deviatoric stresses are rescaled according to the velocity scales and implies:

$$\dot{\gamma}_{xy} = \dot{\gamma}_{XY}, \quad \dot{\gamma}_{xx} = \delta \dot{\gamma}_{XX}, \quad \tau_{XY} = \tau_{xy}, \quad \tau_{XX} = \delta \tau_{xx} \quad (3.2)$$

Note that this type of scaling leads to approximation based on the yielded flow region and driven by the flow geometry and as it has been discussed earlier in §1.1.2 can lead to the lubrication paradox. Ignoring the inertial

3.1. Rescaling

terms, the governing equations after rescaling (3.1) write:

$$0 = \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} \quad (3.3a)$$

$$0 = -\frac{\partial P}{\partial X} + \delta^2 \frac{\partial \tau_{XX}}{\partial X} + \frac{\partial \tau_{XY}}{\partial Y} \quad (3.3b)$$

$$0 = -\frac{\partial P}{\partial Y} + \delta^2 \frac{\partial \tau_{XY}}{\partial X} + \delta^2 \frac{\partial \tau_{YY}}{\partial Y} \quad (3.3c)$$

The constitutive relations are similar to (2.10) except with $\tau = \sqrt{\tau_{XY}^2 + \delta^2 \tau_{XX}^2}$ and $\dot{\gamma} = \sqrt{\dot{\gamma}_{XY}^2 + \delta^2 \dot{\gamma}_{XX}^2}$:

$$\dot{\gamma}_{XX} = 2 \frac{\partial U}{\partial X}, \quad \dot{\gamma}_{XY} = \frac{\partial U}{\partial Y} + \delta^2 \frac{\partial V}{\partial X}.$$

Adopting a regular perturbation expansions in δ of form:

$$(P, U, V) = (P_0, U_0, V_0) + \delta(P_1, U_1, V_1) + \delta^2(P_2, U_2, V_2) + \dots$$

it has been shown in Appendix A.2.1 that the asymptotic form of Bingham constitutive relations is resolved by:

$$\tau_{XY} = \left[\frac{\partial U_0}{\partial Y} + B \operatorname{sgn}\left(\frac{\partial U_0}{\partial Y}\right) \right] + \left[\delta \frac{\partial U_1}{\partial Y} \right] + \mathcal{O}(\delta^2) \quad (3.4a)$$

$$\tau_{XX} = 2 \left(1 + \frac{B}{\left| \frac{\partial U_0}{\partial Y} \right|} \right) \frac{\partial U_0}{\partial X} + \mathcal{O}(\delta) \quad (3.4b)$$

provided that the Bingham fluid is yielded. Assuming the above scaling, our periodic domain becomes $X \in [-l\delta, l\delta]$, $Y \in [-1, 1]$, although we may consider only $Y \geq 0$ through symmetry. The droplet occupies $X \in [-L\delta, L\delta] = [-H, H]$ and $Y \in [-h(X), h(X)]$. No slip boundary conditions are satisfied at the walls. In parts of the channel where there is a droplet, the two conditions (2.19a) & (2.19b) are satisfied at $Y = \pm h(X)$. In terms of the rescaled variables these become:

$$\tau_{XY} - \delta^2 \frac{2\tau_{XX}h_X}{1 + \delta^2 h_X^2} = 0, \quad (3.5a)$$

$$P + \delta^2 \frac{(1 - \delta^2 h_X^2)\tau_{XX} + 2\tau_{XY}h_X}{1 + \delta^2 h_X^2} = \chi X, \quad (3.5b)$$

where h_X is the derivative of $h(X)$ with respect to X . Ignoring the tip of the droplet $X \rightarrow \pm H$, we may assume that $|h_X| \sim O(1)$. Assuming that

3.2. Leading order solution in the yielded region

the droplet remains encapsulated in the plug region of the flow the stresses remain indeterminate within the plug, away from the interface. In order to apply these conditions to deriving a perturbation solution we need at least to understand the order of magnitude of the stress components. The scales used are derived from the yielded region, where they are determined by the velocity scales and a leading order shear flow balance for the pressure. As we enter the unyielded plug, the scaling for P and τ_{XY} is likely to remain valid. However as previously discussed in § 1.1.2, in long-thin geometries the plug yields the extensional stresses within the plug become of the same order as the shear stresses, i.e. this is the yielding mechanism. Therefore, although the leading order terms in (3.5a) & (3.5b) are clear:

$$\tau_{XY,0} = 0 \quad \text{at} \quad Y = \pm h(X), \quad (3.6a)$$

$$P_0 = \chi X \quad \text{at} \quad Y = \pm h(X), \quad (3.6b)$$

at first order there may be some adjustment close to breaking of the plug:

$$\tau_{XY,1} - 2\delta\tau_{XX,0} = 0 \quad \text{at} \quad Y = \pm h(X), \quad (3.7a)$$

$$P_1 + \delta\tau_{XX,0} = 0 \quad \text{at} \quad Y = \pm h(X). \quad (3.7b)$$

3.2 Leading order solution in the yielded region

To the leading order (3.3) and (3.4) lead to

$$\begin{aligned} 0 &= \frac{\partial U_0}{\partial X} + \frac{\partial V_0}{\partial Y} \\ 0 &= -\frac{\partial P_0}{\partial X} + \frac{\partial \tau_{XY,0}}{\partial Y} \\ 0 &= -\frac{\partial P_0}{\partial Y} \end{aligned}$$

We consider only $Y > 0$, as the flow is symmetric about the centreline. Integrating the X -momentum equation and satisfying (3.6a) gives

$$\frac{\partial U_0}{\partial Y} + B \operatorname{sgn}\left(\frac{\partial U_0}{\partial Y}\right) = \frac{\partial P_0}{\partial X}(Y - h) \quad (3.9)$$

The leading order position of the yield surface is denoted by $Y = y_y$. We know at the interface of plug region ($Y = y_y$) shear stress equates the yield stress:

$$\frac{\partial P_0}{\partial X}(Y_y - h) = B \operatorname{sgn}\left(\frac{\partial U_0}{\partial Y}\right) = -B. \quad (3.10)$$

3.3. $\mathcal{O}(\delta)$ solution in the yielded layer

Integrating one more gives the velocity:

$$U_0(Y) = \begin{cases} u_p - u_{p,0} & \text{if } |Y| \leq y_y, \\ u_p \left[1 - \frac{(|Y| - y_y)^2}{(1 - y_y)^2} \right] - u_{p,0} & \text{if } |Y| > y_y, \end{cases} \quad (3.11)$$

where the leading order plug velocity is given by:

$$u_p = \frac{B}{2(y_y - h(X))} (1 - y_y)^2. \quad (3.12)$$

In order to find y_y , we impose the flow rate constraint across the channel, i.e.

$$\int_{-1}^1 U_0 dY = 2 - 2u_{p,0}.$$

Note that the leading order velocity in the droplet is given by continuity at the interface as: $U_0 = u_p - u_{p,0}$. Integrating across the channel gives us:

$$y_y^3 - y_y \left(3 + \frac{6}{B} \right) + 2 + 6 \frac{h}{B} = 0. \quad (3.13)$$

This cubic equation is similar to (2.16) and gives y_y as a function of $h(X)$ and B . It follows that u_p is also a function of both B and x . The dependence of u_p on X via $h(X)$ contradicts the idea that the plug region is rigid, i.e. this is the so-called lubrication paradox of [42] discussed earlier in § 1.1.2. The inconsistency is resolved at the next order in δ .

3.3 $\mathcal{O}(\delta)$ solution in the yielded layer

We focus here on the yielded layer of encapsulating fluid, avoiding the tricky issue of the first order momentum balance within the unyielded plug. The idea is to correct the leading order solution, to account for the varying plug speed predicted. In this, we shall assume that $y_y(X)$ is an $\mathcal{O}(\delta)$ approximation to the position of the true yield surface at $Y = y_T$. The X -momentum balance at 1st order in the yielded layer is:

$$\frac{\partial P_1}{\partial X} = \frac{\partial^2 U_1}{\partial Y^2} \quad (3.14)$$

which requires two boundary conditions. One of these is no slip condition at the wall ($U_1(Y = 1) = 0$). To obtain the other condition notice that at $y = y_T$ we have

$$U(X, y_T) = u_{p,0} = U_0(X, y_T) + \delta U_1(X, y_T) + \mathcal{O}(\delta^2) \quad (3.15)$$

3.3. $\mathcal{O}(\delta)$ solution in the yielded layer

provided that the spacing between droplets is long enough and the plug remains unyielded (recall that we have subtracted the pure fluid plug velocity $u_{p,0}$ from the X -component of velocity in translating to the moving frame). (3.15) can be expanded around y_T and written as

$$u_{p,0} = U_0(X, y_y) + \delta \frac{\partial U_0}{\partial Y}(X, y_y) + \delta U_1(X, y_y) + \delta^2 \frac{\partial U_1}{\partial Y}(X, y_y) + \mathcal{O}(\delta^2) \quad (3.16)$$

We can argue that the second term in the right hand side of the equation is at most of $\mathcal{O}(\delta^2)$, because:

$$\delta \frac{\partial U_0}{\partial Y}(X, y_y) = \delta \frac{\partial U_0}{\partial Y}(X, y_T) + \delta^2 \frac{\partial^2 U_0}{\partial Y^2}(X, y_T) = \delta^2 \frac{\partial^2 U_0}{\partial Y^2}(X, y_T)$$

Note that we have imposed the condition at $y_y(X)$, which is known, rather than at $y_T(X)$. However, assuming that $y_y(X) - y_T(X) = \mathcal{O}(\delta)$, the discrepancy due to imposing at $y_y(X)$ is only at second order. Finally, the condition on U_1 is simply:

$$U_1(X, y_y) = \frac{1}{\delta}(u_{p,0} - u_p(X)) \equiv \eta. \quad (3.17)$$

This same value is adopted for all $Y < y_y$ in order to correct the plug velocity. We will show in §A.3 that $\eta = \mathcal{O}(1)$.

We integrate (3.14) impose the two boundary conditions and then impose the flow rate constraint ($\int_{-1}^1 U_1 dY = 0$) to find:

$$\frac{\partial P_1}{\partial X} = -6\eta \frac{y_y + 1}{(y_y - 1)^3} \quad (3.18)$$

and

$$U_1(X, Y) = -\eta \frac{(Y - 1)(3Y + 3Yy_y - 1 - 4y_y^2 - y_y)}{(y_y - 1)^3} \quad \text{if } Y > y_y \quad (3.19)$$

The last step is to find y_T . We find y_T by imposing zero shear rate at $Y = y_T$ and expanding about $Y = y_y$ with respect to δ :

$$\frac{\partial U_0}{\partial Y}(X, y_T) + \delta \frac{\partial U_1}{\partial Y}(X, y_T) = 0 \implies y_T(X) = y_y(X) - \delta \frac{\frac{\partial U_1}{\partial Y}(X, y_y(X))}{\frac{\partial^2 U_0}{\partial Y^2}(X, y_y(X))}. \quad (3.20)$$

An example of the perturbation solution is plotted in Figure 3.1a, showing $U_0(X, Y)$ and $y_T(X)$ for an elliptical droplet with $H = 0.2$ and $\delta = 0.1$

3.4. Size of the yield surface perturbation from $y_{y,0}$ and effects of inertia

Figure 3.1: Contours of the X -component of velocity obtained by the perturbation solution; $H = 0.2$, $\delta = 0.1$ and $B = 1$: a) leading order $U_0(X, Y)$; b) corrected velocity $U_0(X, Y) + \delta U_1(X, Y)$. Uncorrected yield surface position $y_y(X)$ in dashed line and corrected yield surface $y_T(X)$ in solid line. Pressure gradient for both solutions is also plotted.

at $B = 1$. Although H and δ are relatively large here (i.e. for what might be considered asymptotic), the idea is simply to amplify visually the features of the perturbation solution. Figure 3.1b shows the corrected solution $U_0(X, Y) + \delta U_1(X, Y)$ and $y_T(X)$. The corresponding variations in pressure gradients are also shown on the top of each graph.

3.4 Size of the yield surface perturbation from $y_{y,0}$ and effects of inertia

Having now derived an expression for $y_T(X)$ we wish to examine the size of the yield surface perturbation from the Poiseuille flow yield surface, $y_{y,0}$. As previously discussed in §2.5, it is this quantity that dictates the size of the inertial terms that we have neglected so far.

We note that our perturbation solution has been derived under the assumption that $h(X) \sim \mathcal{O}(\delta)$ and we now define $\bar{h}(X) = h(X)/\delta$. The calculation proceeds in 2 steps. First we examine the departure of y_y from $y_{y,0}$. We write $y_y = y_{y,0} + \delta \xi_1$ and show that ξ_1 is order 1. Substituting

3.4. Size of the yield surface perturbation from $y_{y,0}$ and effects of inertia

both y_y and $\bar{h}$ into (3.13) and expanding in powers of δ :

$$\mathcal{O}(1) : y_{y,0}^3 - y_{y,0}(3 + 6/B) + 2 = 0, \quad (3.21a)$$

$$\mathcal{O}(\delta) : 3y_{y,0}^2\xi_1 - \xi_1(3 + 6/B) + 6\frac{\bar{h}}{B} = 0, \quad \implies \quad \xi_1 = \frac{\bar{h}}{1 + 0.5B(1 - y_{y,0}^2)} \quad (3.21b)$$

The first expression is simply the Buckingham equation (2.16), as expected. The second expression defines ξ_1 . Now, since u_p is defined by y_y and h , straightforward expansion of (3.12) in powers of δ verifies that $\eta = \mathcal{O}(1)$; see Appendix A.3.

Secondly, combining (3.20) and (3.21b)

$$y_T(X) - y_{y,0} = \delta \left(\xi_1(X) - \frac{\frac{\partial U_1}{\partial Y}(X, y_y)}{\frac{\partial^2 U_0}{\partial Y^2}(X, y_y)} \right). \quad (3.22)$$

and show that (algebraic details can be found in Appendix A.3):

$$y_T - y_{y,0} = \delta \bar{h} \left(\frac{B(y_{y,0} + 1)(y_{y,0}^3 - y_{y,0}(3 + \frac{6}{B}) + 2)}{y_{y,0}(y_{y,0} - 1)(By_{y,0}^2 - B - 2)} \right) + \mathcal{O}(\delta^2). \quad (3.23)$$

We see that the leading order term on the right-hand side vanishes due to (3.21a), leading to: $y_T - y_{y,0} \sim \mathcal{O}(\delta^2)$. A similar exercise shows that $U_0 + \delta U_1 = U_p(y) + \mathcal{O}(\delta^2)$, i.e. the velocity field is given to $\mathcal{O}(\delta^2)$ by the Poiseuille solution of the droplet free flow (algebraic details can be found in Appendix A.3).

To summarise, insertion of an $\mathcal{O}(\delta)$ droplet within the plug region causes only an $\mathcal{O}(\delta^2)$ perturbation in both the yield surface position and the velocity field. However, the stress perturbation is of $\mathcal{O}(\delta)$, as is the pressure gradient perturbation. Additionally, the stress field within the unyielded plug region remains indeterminate. To explore this surprising feature, Figure 3.2 plots $y_T(X)$ and $y_y(X)$ for a range of different droplets at $B = 10$. As we see the corrected yield surface position is nearly constant and coincides with $y_{y,0}$ (equal to the end values of $y_y(X)$). Considering now the question of inertial effects, we see that for an $\mathcal{O}(\delta)$ encapsulated droplet as considered, the uniform Poiseuille flow gives an $\mathcal{O}(\delta^2)$ approximation to the velocity field in the region of the droplet (as well as away from the droplet). The yield surface perturbation is also of $\mathcal{O}(\delta^2)$. This suggest that the appropriate variation in the streamlines away from the uniform Poiseuille solution in the presence of an encapsulated droplet is in fact $\mathcal{O}(\delta^2)$ (instead of the $\mathcal{O}(\delta)$ used to derive and construct our perturbation approximation), and that consequently inertial effects may be legitimately neglected for $\delta^2 Re \ll 1$.

3.4. Size of the yield surface perturbation from $y_{y,0}$ and effects of inertia

Figure 3.2: Position of the yield surfaces $y_T(X)$ (solid line) and $y_y(X)$ (dashed line) for a range of different (elliptical) droplets at $B = 10$. Top rows $L = 0.5$, bottom row $L = 0.75$ and columns left to right $H = 0.05, 0.1$ and 0.2 . Note that $y_{y,0}$ is equal to the values of $y_y(X)$ at two ends. Due to the adopted scaling, all elliptic droplets are mapped to a circle with radius of H .

Chapter 4

Numerical Simulation

In this chapter we describe the detail of numerical analysis that we have conducted to solve the encapsulation problem without having any geometrical constrain on the drop's shape.

4.1 Assumptions and simplifications.

Motivated by the asymptotic results, we proceed under the assumption that an encapsulated droplet held within the plug will not significantly perturb the yield surface, and consequently that the inertial terms may be small even for significant Re . This we verify *a posteriori* from our numerical results. We also consider that the droplet and encapsulating fluid have the same density, i.e. $\chi = 0$.

As we consider a Stokes flow problem, we impose symmetry conditions at $x = 0$, $x = -l/2$ and $y = 0$, and solve only for: $(x, y) \in [-l/2, 0] \times [0, 1]$; (see Figure 4.1). In the encapsulating fluid the Stokes equations are

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad (4.1a)$$

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}, \quad (4.1b)$$

$$0 = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}. \quad (4.1c)$$

The constitutive laws are (2.10) & (2.12). The droplet domain is $y \leq h(x)$ and within the droplet the fluid is assumed to move uniformly at the speed of the unyielded plug. Thus, we solve only in the encapsulating fluid domain;

Figure 4.1: Schematic description of the problem

see Figure 4.1. The boundary conditions are:

$$-p(0, y) + \tau_{xx}(0, y) = 0, \quad v(0, y) = 0, \quad y \in [H, 1] \quad (4.2a)$$

$$u(x, 1) = 0, \quad v(x, 1) = 0, \quad x \in [-l/2, 0] \quad (4.2b)$$

$$-p(-l/2, y) + \tau_{xx}(-l/2, y) = Gl/2, \quad v(-l/2, y) = 0, \quad y \in [0, 1] \quad (4.2c)$$

$$\tau_{xy}(x, 0) = 0, \quad v(x, 0) = 0, \quad x \in [-l/2, -L] \quad (4.2d)$$

$$\tau_{nt}(x, h(x)) = 0, \quad -p(x, h(x)) + \tau_{nn}(x, h(x)) = 0, \quad x \in [-L, 0] \quad (4.2e)$$

Note in (4.2b) that for simplicity we compute in a fixed frame of reference instead of the moving frame, as translating by a constant $u_{p,0}$ can be imposed afterwards with no loss of generality. By imposing (4.2c) instead of the uniform Poiseuille flow we allow for the fact that the encapsulating fluid between droplets may not be fully developed. Indeed, below we study the required droplet spacing parameter l , needed for the droplets not to interact with each other. The parameter $Gl/2$ in (4.2c) is essentially the pressure drop along the computational domain. Each computation involves an iteration on G to ensure that the mean velocity is unity, i.e. we satisfy

$$\int_0^1 u(x, y) dy = 1, \quad (4.3)$$

which has a unique solution due to the monotonicity of the flow rate with respect to the pressure drop.

4.2 Rheolef

As mentioned earlier, regularisation methods do not predict the position of the yield surfaces with certainty. Since we investigate whether or not the droplet is fully encapsulated in unyielded fluid, it is essential to use augmented Lagrangian method to distinguish the true plug from regions with merely small strain rate. This necessity eliminates the options of using a commercial CFD software. The augmented Lagrangian algorithm is implemented within the code *Rheolef*, a C++ Finite Element Method library by P. Saramito and colleagues in Grenoble.⁶ In order to illustrate how powerful *Rheolef* is, here we show the first example of its tutorial. This emphasises that working with *Rheolef* eliminates all the hassle of code compiling. The following code solves $\Delta u = -1$ in Ω with $u = 0$ on $\partial\Omega$ when Ω is a given computational domain.

Program 4.1 Solving Poisson’s equation with Dirichlet boundary condition in *Rheolef*

```
int main (int argc, char** argv) {  
  environment rheolef (argc, argv);  
  geo omega (argv[1]);  
  space Xh (omega, argv[2]);  
  Xh.block ("boundary");  
  trial u (Xh); test v (Xh);  
  form a = integrate (dot(grad(u),grad(v)));  
  eld lh = integrate (v);  
  eld uh (Xh);  
  uh ["boundary"] = 0;  
  solver sa (a.uu());  
  uh.u = sa.solve (lh.u());  
  dout << uh;  
}
```

4.3 Mesh adaptation

A particular feature of our implementation in *Rheolef* is intensive mesh adaptivity, which focuses the mesh around the yield surfaces. It has been

⁶for more details visit <http://www-ljk.imag.fr/membres/Pierre.Saramito/rheolef/>

shown by [67] that the rate of dissipation defined by

$$\varphi = ||\dot{\gamma}||^2 + B||\dot{\gamma}|| \quad (4.4)$$

exhibits a sharp jump across the yield surface. This behaviour can be utilized in order to find the approximate location of plug interface and produce a finer mesh nearby. Details of implementation can be found in [67, 64, 61, 65]. Starting with an initial mesh, with a typical mesh size of say $d_0 \approx 0.02$, we compute the velocity and pressure using AL method. Then we calculate φ and regenerate the mesh such that the gradient of φ is equally distributed in all cells. This process is repeated typically four times until we get a desirable mesh configuration. Figure 4.2 shows cycles of the adaptation for one specific case. After the fourth cycle the typical mesh sizes are: in the plug: 0.02; near the yield surface: 0.001×0.004 ; near the wall: 0.002.

4.4 Convergence

Starting with a smaller d_0 results in a progressively smaller mesh at each step of the adaptivity and hence a better converged solution. The effects of adaptivity on convergence are also significant for the first few cycles. This is illustrated in Figure 4.3 where we have compared the numerical results of the code for plane Poiseuille flow with the exact solution from §2.4. This shows that $||e_u||$ and $||e_p||$, defined below, decay rapidly over the first few cycles of adaptation before leveling out, i.e. reducing the error after 4 cycles requires a smaller initial mesh d_0 .

$$||e_u|| = \frac{1}{N_{cells}} \sqrt{\sum_{N_{cells}} |\mathbf{u}_{comp} - \mathbf{u}_{exact}|^2} \quad (4.5a)$$

$$||e_p|| = \frac{1}{N_{cells}} \sqrt{\sum_{N_{cells}} (p_{comp} - p_{exact})^2} \quad (4.5b)$$

We are also able to compare our computed results against the asymptotic solutions over a range of small δ , e.g. fixed small H and varying L . This type of comparison verifies that the errors of leading order and corrected first order solutions are $\mathcal{O}(\delta)$ and $\mathcal{O}(\delta^2)$, respectively. The range of δ tested is however limited since for very small H the droplet becomes a *cut* in the encapsulating fluid, increasingly singular at the ends. On the other hand for larger L the encapsulating plug region breaks. As an aside, we

Figure 4.2: Mesh adaptation cycles for the case of $B = 20$, $H = 0.1$ and $L = 0.5$, where we start with an initial mesh scale, $d_0 \approx 0.02$: a) cycle 1, 3165 elements; b) cycle 2, 12689 elements; c) cycle 3, 21425 elements; d) cycle 4, 37290 elements. After the 4th cycle the typical mesh sizes are: in the plug: 0.02; near the yield surface: 0.001×0.004 ; near the wall: 0.002.

4.4. Convergence

Figure 4.3: Code Validation: Error of numerical results for channel Poiseuille flow compared to the exact solution; $\circ$: $B = 10$, $\diamond$: $B = 20$ and $\triangle$: $B = 50$. a) velocity and b) pressure

Figure 4.4: Example of the effects of droplet spacing for $B = 20$, $H = 0.5$ and $L = 0.6$: a) $l = 1.625$; b) $l = 1.8$; c) $l = 2.5$. The colour map shows the range of speeds in the flow and grey areas show unyielded fluid.

mention that the plane Poiseuille flow solution also gives an $\mathcal{O}(\delta^2)$ error when compared against the computed results and interestingly a smaller error than the corrected perturbation method.

4.5 Effects of droplet spacing

As we have already seen, the length of droplets has a significant effect on breaking of the encapsulating plug. With a restriction in L , if one is interested in the throughput of encapsulated droplets it is necessary to understand the effect of droplet spacing on the breaking process. Figure 4.4 shows an example sequence of computations for which the separation distance $l - 2L$ drops below a critical value and the plug regions start to yield. In this case, for $l = 1.625$ (Figure 4.4a) the plug has broken whereas for larger l an intact region exists. If the plug yields around the droplet the computed solution remains valid as solution to the Stokes flow, but in terms of an evolution problem the interface is free to deform and we would expect this deformation to induce a velocity field within the droplet.

Apart from the issue of plug breaking, a separate consideration is whether l is long enough for the flow to become fully developed between the droplets.

4.5. Effects of droplet spacing

Figure 4.5: Variation in the pressure drop upstream of the droplet plotted versus a scaled spacing between droplets. a) $B = 10$, ∇ : $H = 0.45$, $L = 0.6$; $\circ$: $H = 0.55$, $L = 0.15$; $\square$: $H = 0.05$, $L = 0.75$; $\star$: $H = 0.25$, $L = 0.25$ and $\diamond$: $H = 0.375$, $L = 0.5$ b) $B = 20$, ∇ : $H = 0.5$, $L = 0.6$; $\circ$: $H = 0.65$, $L = 0.2$; $\square$: $H = 0.1$, $L = 0.9$; $\star$: $H = 0.25$, $L = 0.25$ and $\diamond$: $H = 0.375$, $L = 0.5$. The broken lines show the pressure drop for Poiseuille flow.

The method we have chosen to study this compares the mean pressure gradient behind the drop with the analytical solution for plane Poiseuille flow:

$$\Delta p = \frac{p_l - p_L}{l - L},$$

where p_l and p_L are the pressures at $x = -l/2$ and $x = -L$, respectively; see Figure 4.5. For a range of droplet shapes we observe that at sufficiently large l the encapsulation of the droplet is intact and the Δp approaches the analytic value for a yield stress fluid. In fact Δp is always very slightly below the analytical value, due to a reduction very close to $x = -L$, where presumably the fluid senses the presence of the droplet. As a rule of thumb, if l is at least 2.5 times $\max\{H, L\}$ we find that the pressure drop values remains near constant.

Chapter 5

Results and Discussion

5.1 Encapsulation and failure

We have computed approximately 400 encapsulation flows in channel geometry, as described. These cover the approximate ranges: $5 < B < 200$, $0.025 < H < 0.9$ and $0.025 < L < 1.25$. Figure 5.1 illustrates some of the general trends observed. For each row of computations the Bingham number is held constant. The left-hand column has L kept constant and H varied; the right column has variable L as H stays fixed. The colourmap plots the speed of the fluid, with the grey area denoting the unyielded plug. We observe that there essentially are two different regimes of plug breaking:

Slender droplets : If the droplet aspect ratio $\delta = H/L$ is small (in other words, the length of drop is much larger than its width) then the plug appears to yield initially close to the two tips of the droplet (left column of Figure 5.1).

Fat droplets : If the width of the droplet is similar or larger than the length, the plug appears to yield closer to $x = 0$, in the widest part of the droplet (right column of Figure 5.1).

Essentially this suggests that increasing either L or H will eventually yield the plug around the droplet, as might be expected.

The other remarkable observation is that the yield surface position is approximately constant for the simulations shown in Figure 5.1, up to and directly after the plug breaks (although of course with breaks in it after yielding). This implies that the stress distribution to the outside of the plug is not greatly affected by plug breaking. Thus, the deduction from the asymptotic analysis of the past section: of an $\mathcal{O}(\delta^2)$ perturbation of the yield surface due to the droplet, appears to be more widely valid even for larger aspect ratios $\delta = H/L$.

To examine the breaking mechanism, we have plotted the various stresses in Figure 5.2 for one sequence of slender drops ($B = 20$, $H = 0.2$ and increasing L). The first noticeable feature is that as L increases through the

5.1. Encapsulation and failure

Figure 5.1: Speed distribution (u) as the droplet gets larger. Rows represent constant Bingham number; $H = 0.2$ over left column (*slender drop*) and $L = 0.5$ over right column (*fat drop*). a) $B = 5$ and $L = 0.55, 0.6, 0.65, 0.675$ and 0.7 ; b) $B = 5$ and $H = 0.3, 0.34, 0.375, 0.4$ and 0.425 ; c) $B = 20$ and $L = 0.8, 0.85, 0.9, 0.925$ and 1 ; d) $B = 20$ and $H = 0.5, 0.55, 0.575, 0.61$ and 0.625 ; e) $B = 50$ and $L = 0.95, 1, 1.05, 1.15, 1.2$ f) $B = 50$ and $H = 0.6, 0.65, 0.69, 0.725$ and 0.75

breaking transition the distribution of stresses remains qualitatively similar. Thus, the breaking itself does not affect the overall pattern of stresses which are dominated primarily by changes in droplet shape. Upstream and downstream of the droplet the stresses develop relatively quickly (within a channel width) and adopt distributions close to fully developed. The vertically oriented yielded regions (cracks?) coincide with a region within which the shear stress is focused: τ_{xy} becomes large and negative near the tips of the droplet. This is combined with significant extensional stresses $\tau_{xx} = -\tau_{yy}$. The basic mechanism appears to be that outlined in §2.6. In the absence of extensional stresses, the pressure must transit from its shear-layer values to zero at the droplet surface. This driving force is evidently larger at the tips of the droplet. One way of reducing the burden on the change in p is by taking up a part of the pressure within the extensional stresses. For example, on the interface where it is nearly flat the normal stress condition is $p = \tau_{yy}$ and we can observe that $|\tau_{yy}|$ becomes significant, changing sign towards the yield surface. In the centre of the droplets it appears that the distribution of τ_{xy} is shifted upwards by $h(x)$, but near to the ends of the droplet the stresses adapt to the far-field in a relatively narrow region.

The stress distributions in the case of fat droplets are illustrated in Figure 5.3. for $B = 20$, $L = 1.4$ and increasing H . Although the far-fields are similar to the slender droplet, the mechanism of breaking appears to be different. Again we have a transition as the geometry is increased. However, this time as H increases, note that from the momentum balance (and observation that the yield surface position is unchanged), we expect

$$\frac{\partial \tau_{xy}}{\partial y} \sim \mathcal{O}\left(\frac{B}{y_y - H}\right),$$

in the centre of the droplet (within a layer that is getting progressively short). If we suppose that the pressure gradient is partly constrained by the far-field plane Poiseuille flow pressure gradient, then we might expect that the y -derivative of τ_{xy} is balance by x -derivatives in τ_{xx} which we observe in Figure 5.3. The yielding appears to occur at the ends of the central shear layer.

5.2 Maximum size of encapsulated droplets

By successively iterating with respect to H and L we are able to approximately determine the limiting size of droplet that can remain encapsulated within the plug region. The procedure that we took here was to simulate

5.2. Maximum size of encapsulated droplets

Figure 5.2: Stress distribution as the droplet gets larger. $B = 20$, $H = 0.2$ and; $L = 0.8, 0.85, 0.9, 0.925$ and 1 . a) speed; b) pressure; c) τ_{xy} ; d) τ_{xx} ; e) $\sigma_{xx} = -p + \tau_{xx}$ and f) $\sigma_{yy} = -p + \tau_{yy}$

5.2. Maximum size of encapsulated droplets

Figure 5.3: Stress distribution as the droplet gets larger. $B = 20$, $L = 0.7$ and $H = .5, 0.55, 0.575, 0.61$ and 0.625 . a) speed; b) pressure; c) τ_{xy} ; d) τ_{xx} ; e) σ_{xx} and f) σ_{yy}

Figure 5.4: Maximum size of encapsulated droplets, computed for five different B . $\circ$: $B = 5$; $\square$: $B = 10$; $\triangle$: $B = 20$, $\diamond$: $B = 50$ and $\star$: $B = 200$

flow around a drop and find the plug region. If the plug region is intact, we enlarge the drop until the plug starts to yield. The accuracy of increments is 0.01 in both directions, H and L . Although this process might seem straightforward, some issues have to be dealt with. To explain these issues, note that to determine the surface of plug regions, we could take two similar approaches: calculate either strain rate ($\dot{\gamma}$) or stress (τ). In either case a comparison with a constant value (0 or B , respectively) is necessary here to identify the surface of plug regions. Generally comparing equality is not a nice operation in computational analysis, due to small numerical errors. For example, assume we have a computational geometry with typical mesh size of d . If the round-off error is ϵ_1 , then the order of error we might have for the strain rate is $\frac{\epsilon_1}{d}$. Thus, we need to allow for such errors in finding the actual surface of plug region and our comparison algorithm is as follows:

$$\begin{cases} |\dot{\gamma}| < \epsilon_2 & \text{or} & |\tau| < B(1 + \epsilon_2) & \text{plug region} \\ \text{otherwise} & & & \text{yielded region} \end{cases}$$

where $\epsilon_2 = \frac{\epsilon_1}{d}$.

Figure 5.4 shows the critical values in the (L, H) -plane for five different

values of Bingham number, B . For a given Bingham number, any drop lying below the associated curve does not break the plug region. We see that for any Bingham number (no matter how large) we have a maximum value for the length and height of the droplet. The maximum height is mostly limited by the plug width. Although there is no similar simple physical limit for the maximum length, we still see that the growth of stresses for long droplets yields the plug. In fact the shape of the curves at small H indicates that for slender droplets the encapsulation becomes very sensitive to changes in the length, i.e. small increases in the length can break the plug (see also Figure 5.2a, c & e).

A last observation here is that the increase in size of droplet with B is much less than linear. Therefore, just increasing the yield stress of the encapsulating fluid does not allow correspondingly large droplets. The underlying reasons why seem to be captured in the simple analysis of §2.6. Long droplets break near the ends. Breaking arises from the need to bridge the discrepancy between non-zero (frictional) pressures in the yielded fluid layers and zero normal stress on the droplet surface. This discrepancy increases with L but also with B , i.e. the frictional pressure scales like $xB/y_{y,0}$.

5.3 Encapsulation with fluids of different densities

We now explore the effect of allowing a density difference between the fluids. Assuming the encapsulated fluid remains in steady rigid motion, the pressure field within the droplet is: $p = \chi x + \text{constant}$, where we recall that $\chi = (\hat{\rho}_d - \hat{\rho})\hat{g}\hat{R}^2/(\hat{\mu}\hat{U}_0)$ represents the competition between buoyant and viscous stresses. Considering the preceding discussion and the simple analysis in §2.6, an intuitive notion is that by selecting $\chi = -B/y_{y,0}$ the pressure gradient in the droplet would match that in the yielded fluid layers. This would eliminate the need for normal stress gradients across the plug, and hence presumably reduce τ .

We test the above notion computationally. The code described earlier is adapted by discretizing both the droplet and carrier fluid domains. The additional body force term χ can be added to the droplet x -momentum equation, (or instead $-\chi$ added to the carrier fluid equations). Different fluid properties are assigned in each domain (e.g. $B = 0$ is inside the droplet) and the Stokes flow solution is computed on both domains simultaneously using the augmented Lagrangian algorithm described earlier.

Firstly, we address the question of whether the density difference might

significantly affect breaking of the plug region. Figure 5.5a shows the maximum height H before the plug yields, for two droplets with different L at $B = 20$. We observe that the shorter droplet ($L = 0.4$) is relatively insensitive to χ , whereas the longer droplet ($L = 1.75$) shows remarkable sensitivity. Sensitivity to L *per se* is to be expected as the effect of χ naturally amplifies with x (or L), due to the channel orientation. We see that if the droplet is heavier than the carrier fluid ($\chi > 0$), encapsulation fails at a smaller sizes for the drop. On the other hand, for lighter droplet ($\chi < 0$) buoyancy tends to stabilize the encapsulation, allowing larger drops to be successfully encapsulated. For the case shown in Figure 5.5a, in which $B = 20$, we see that maximum size of the drop happens at $\chi \approx -27$ to -28 . This approximately coincides with the frictional pressure gradient of the plane Poiseuille flow, (for $B = 20$ we find $\frac{\partial p}{\partial x} = -\frac{B}{y_{y,0}} \approx -27.8$). Thus, the notion that selecting $\chi \approx -B/y_{y,0}$ may provide an optimal size of encapsulated droplet appears reasonable. Essentially buoyancy acts to balance the frictional pressure losses for $\chi > -B/y_{y,0}$, but on decreasing χ still further strong buoyancy forces act to exert stress on the plug.

Note that although the normal stress discrepancy may be reduced for $\chi \approx -B/y_{y,0}$, the tangential stresses in the plug are still perturbed by the presence of the droplet. Thus, the droplet may still break the plug. Figure 5.5b shows the critical values of H and L for $B = 20$, above which the plug yields for three different $\chi = -20, 0$ and 20 . Again we see favourable density difference (lighter droplets) allows the encapsulation of longer drops, whereas unfavourable density difference (heavier droplets) weakens the encapsulation capabilities.

Figure 5.6 explores variations in the stress fields as the density of the droplet is varied. The plug close to the central part of the droplet is largely unaffected by χ , but on increasing $|x|$ we observe generation of extensional stresses towards the ends of the droplet within the plug (Figure 5.6d). Increasing χ amplifies the size of extensional stresses and of the pressure in these regions, (Figure 5.6b & d). The stresses combine so that σ_{yy} shows little variation in y (Figure 5.6f), which implies that the pressure variation in y is negatively amplified in τ_{xx} . The net effect is that the front of the droplet is stretched and the rear is compressed (in the x -direction); see Figure 5.6e. These normal stress gradient results in significant shear stresses (Figure 5.6c). The underlying pattern is not changed by $\chi > 0$, but simply amplified.

5.3. Encapsulation with fluids of different densities

Figure 5.5: $B = 20$ (a) Variation of maximum height (maximum H which does not break the plug) for two different length of drop $\square$: $L = 0.4$ and $\circ$: $L = 1.75$ with density difference (χ). (b) Maximum size of drop for for three different χ . $\circ$: $\chi = 20$; $\square$: $\chi = 0$ and $\triangle$: $\chi = -20$.

5.3. Encapsulation with fluids of different densities

Figure 5.6: Stress distribution as the droplet becomes heavier: $B = 20$, $L = 0.9$ and $H = .2$. From top to bottom in each panel: $\chi = -10, -1, 0, 1$ and 10 . a) speed; b) pressure; c) τ_{xy} ; d) τ_{xx} ; e) σ_{xx} and f) σ_{yy} .

Chapter 6

Axisymmetric Geometry

For the purposes of practical application we now consider the analogous flow to that described in §2.1, but within a vertical pipe of radius $\hat{R}$. We describe the problem in a cylindrical coordinate system, assuming that the flow is axisymmetric with droplets traveling along the pipe axis. As in §2.1 a moving coordinate system is adopted. Lengths and velocities are scaled with $\hat{R}$ and $\hat{W}_0$ respectively where $\hat{W}_0$ is the mean velocity along the pipe, i.e. the flow rate is $\pi\hat{R}^2\hat{W}_0$. Denoting by $\hat{W}_p$ the fully developed plug velocity of the Hagen-Poiseuille flow (with no droplets), the variables are as follows:

$$(r, z, \theta) = \left(\frac{\hat{r}}{\hat{R}}, \frac{\hat{z} - \hat{W}_p \hat{t}}{\hat{R}}, \theta \right); \quad (u, w) = \left(\frac{\hat{u}_r}{\hat{W}_0}, \frac{\hat{u}_z - \hat{W}_p}{\hat{W}_0} \right), \quad (6.1)$$

and $t = \hat{W}_p \hat{t} / \hat{R}$. The deviatoric stresses are scaled with $\hat{\mu}\hat{W}_0/\hat{R}$ as is the modified pressure (on subtracting the static pressure of the carrier fluid).

$$\hat{p} = \hat{p}_0 + \hat{\rho}\hat{g}\hat{z} + \frac{\hat{\mu}\hat{W}_0}{\hat{R}}p \quad (6.2)$$

Exploiting axisymmetry of the geometry, the dimensionless form of continuity and momentum equations governing carrier fluid are

$$\frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0 \quad (6.3a)$$

$$Re \left(u \frac{\partial w}{\partial r} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z} \quad (6.3b)$$

$$Re \left(u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\partial \tau_{rz}}{\partial z} - \frac{\tau_{\theta\theta}}{r} \quad (6.3c)$$

The constitutive relations are similar to (2.10), but now $\dot{\gamma} = \sqrt{\dot{\gamma}_{rr}^2 + \dot{\gamma}_{zz}^2 + \dot{\gamma}_{\theta\theta}^2 + 2\dot{\gamma}_{rz}^2}$

is the rate of strain, and $\tau = \sqrt{\tau_{rr}^2 + \tau_{zz}^2 + \tau_{\theta\theta}^2 + 2\tau_{rz}^2}$ is the deviatoric stress.

The components are given by:

$$\dot{\gamma}_{rr} = 2\frac{\partial u}{\partial r}; \quad \dot{\gamma}_{\theta\theta} = 2\frac{w}{r}; \quad \dot{\gamma}_{zz} = 2\frac{\partial w}{\partial z}; \quad \dot{\gamma}_{rz} = \dot{\gamma}_{zr} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}. \quad (6.4)$$

6.1. Poiseuille flow solution

Reynolds number and Bingham number are defined as $Re = \hat{\rho}\hat{W}_0\hat{R}/\hat{\mu}$ and $B = \hat{\tau}_Y\hat{R}/\hat{\mu}\hat{W}_0$.

As before, we consider a single droplet from our periodic train of droplets. The scaled droplet length is L and the maximal radius is H . The droplet surface is steady within the moving frame of reference and is denoted $r = h(z)$. Within the droplet we must satisfy:

$$\frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0 \quad (6.5a)$$

$$\phi Re \left(u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + m \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) + m \frac{\partial^2 w}{\partial z^2} + \chi \quad (6.5b)$$

$$\phi Re \left(u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial r} + m \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + m \frac{\partial^2 u}{\partial z^2} - m \frac{u}{r^2} \quad (6.5c)$$

where $\chi = \frac{(\hat{\rho}_d - \hat{\rho})\hat{g}\hat{R}^2}{\hat{W}_0\hat{\mu}}$. Similar boundary and interfacial conditions are satisfied as for the plane channel problem earlier. As before, it is assumed that the droplet translates uniformly within the moving plug.

6.1 Poiseuille flow solution

Taking a similar approach to that of §A.1, we can readily find that in a fixed frame of reference the Hagen-Poiseuille solution in pipe geometry is resolved by:

$$W_p(r) = \begin{cases} w_{p,0} & \text{if } r \leq r_{y,0}, \\ w_{p,0} \left[1 - \left(\frac{r-r_{y,0}}{1-r_{y,0}} \right)^2 \right] & \text{if } r \geq r_{y,0}, \end{cases} \quad (6.6)$$

where $w_{p,0} = \frac{B}{2r_{y,0}}(1-r_{y,0})^2$ is the dimensionless plug velocity. On imposing the flow rate constraint we find that the yield surface position satisfies the following Buckingham equation:

$$r_{y,0}^4 - 4r_{y,0} \left(1 + \frac{3}{B} \right) + 3 = 0, \quad (6.7)$$

which has a unique zero in the interval $(0, 1)$.

6.2 Slender drop

For practical reasons it is advantageous to know if encapsulated droplets within a pipe flow also result in vanishing small perturbations of the flow

6.2. Slender drop

streamlines. Thus, similar to §3.1 we re-scale the problem assuming a small slender droplet:

$$r = R, \quad \theta = \Theta, \quad z\delta = Z, \quad u = \delta U, \quad w = W, \quad p\delta = P. \quad (6.8)$$

Upon rescaling and neglecting the inertial terms, the governing equations are:

$$\frac{1}{R} \frac{\partial}{\partial R} (RU) + \frac{\partial W}{\partial Z} = 0 \quad (6.9a)$$

$$0 = -\frac{\partial P}{\partial Z} + \frac{1}{R} \frac{\partial}{\partial R} (R\tau_{RZ}) + \delta^2 \frac{\partial \tau_{ZZ}}{\partial Z} \quad (6.9b)$$

$$0 = -\frac{\partial P}{\partial R} + \delta^2 \frac{1}{R} \frac{\partial}{\partial R} (R\tau_{RR}) + \delta^2 \frac{\partial \tau_{RZ}}{\partial Z} - \delta^2 \frac{\tau_{\Theta\Theta}}{R} \quad (6.9c)$$

where $\tau_{rz} = \tau_{RZ}$, $\tau_{rr} = \delta\tau_{RR}$, $\tau_{\theta\theta} = \delta\tau_{\Theta\Theta}$ and $\tau_{zz} = \delta\tau_{ZZ}$. The constitutive relations are similar to (2.10), except that now:

$$\tau = \sqrt{\tau_{RZ}^2 + \delta^2 \tau_{RR}^2 + \delta^2 \tau_{\Theta\Theta}^2 + \delta^2 \tau_{ZZ}^2} \quad (6.10a)$$

$$\dot{\gamma} = \sqrt{\dot{\gamma}_{RZ}^2 + \delta^2 \dot{\gamma}_{RR}^2 + \delta^2 \dot{\gamma}_{\Theta\Theta}^2 + \delta^2 \dot{\gamma}_{ZZ}^2} \quad (6.10b)$$

and

$$\dot{\gamma}_{RZ} = \frac{\partial W}{\partial R} + \delta^2 \frac{\partial U}{\partial Z} \quad (6.11a)$$

$$\dot{\gamma}_{RR} = 2 \frac{\partial U}{\partial R} \quad (6.11b)$$

$$\dot{\gamma}_{\Theta\Theta} = 2 \frac{W}{R} \quad (6.11c)$$

$$\dot{\gamma}_{ZZ} = 2 \frac{\partial W}{\partial Z} \quad (6.11d)$$

We adopt a regular perturbation expansion in δ : $(P, W, U) = (P_0, W_0, U_0) + \delta(P_1, W_1, U_1) + \dots$, substitute into the equations and derive the following leading order problem:

$$\frac{1}{R} \frac{\partial}{\partial R} (RU) + \frac{\partial W}{\partial Z} = 0 \quad (6.12a)$$

$$0 = -\frac{\partial P}{\partial Z} + \frac{1}{R} \frac{\partial}{\partial R} (R\tau_{RZ}) \quad (6.12b)$$

$$0 = -\frac{\partial P}{\partial R} \quad (6.12c)$$

6.2. Slender drop

Integrating the axial momentum equation from $R = h$ yields

$$\tau_{RZ} = \frac{\partial W_0}{\partial R} + B \operatorname{sgn} \left(\frac{\partial W_0}{\partial R} \right) = \frac{\partial W_0}{\partial R} - B = \frac{1}{2} \frac{\partial P}{\partial Z} \left(R - \frac{h^2}{R} \right) \quad (6.13)$$

and defining the yield surface $R = R_y$ as where $\tau_{RZ} = -B$ we find:

$$\frac{\partial W_0}{\partial R} = \frac{1}{2} \frac{\partial P}{\partial Z} \left(R - R_y - \frac{h^2}{R} + \frac{h^2}{R_y} \right) : \quad \frac{\partial P}{\partial Z} = -\frac{2BR_y}{R_y^2 - h^2} \quad (6.14)$$

We can reconstruct the velocity in the fixed frame, using the no-slip condition at $R = 1$:

$$W_0(r) + w_{p,0} = \begin{cases} w_p & \text{if } 0 \leq R \leq R_y \\ w_p \left[1 - \left(\frac{0.5(R-R_y)^2 + h^2(R-R_y)/R_y + h^2 \ln(R_y/R)}{0.5(1-R_y)^2 + h^2(1-R_y)/R_y + h^2 \ln R_y} \right) \right] & \text{if } R_y \leq R \leq 1 \end{cases} \quad (6.15)$$

where the plug velocity is given by

$$w_p = B \frac{R_y}{R_y^2 - h^2} \left(\frac{(1 - R_y)^2}{2} + \frac{h^2}{R_y} (1 - R_y) + h^2 \ln R_y \right) \quad (6.16)$$

Finally we use the unit flow rate condition to find the pressure gradient (equivalently R_y). This leads to the following Buckingham equation:

$$R_y^4 - 4(1 + 3/B)R_y + 3 + \frac{2h^2}{R_y} \left[(1 - R_y)^2(2 + R_y) + \frac{6}{B} \right] = 0, \quad (6.17)$$

which may be solved numerically for R_y .

Comparing the solution above with that in §6.1, we note that the leading order plug velocity, velocity profile and the Buckingham equation are all $\mathcal{O}(h^2)$ perturbations from the unperturbed Poiseuille flow solution, *c.f.* (6.15), (6.17), (6.6) & (6.7).

The implications of this are two-fold. Firstly, in terms of the perturbation procedure there is no need to seek a correction to the solution at first order in δ . In practical terms, this means that for any small slender drop of aspect ratio δ the streamlines in the yielded layer will be perturbed by $\mathcal{O}(\delta^2)$, suggesting that inertial effects on the flow are of $\mathcal{O}(\delta^2 Re)$ as for the channel flow. Secondly, this suggests that a perturbation approximation of the above form would only require a first order correction when $h \sim \delta^{1/2}$. As $\delta = H/L$ is the droplet aspect ratio, this suggests $L \sim \delta^{-1/2}$. This hints that longer and wider droplets may be accommodated in the pipe geometry.

6.3 Examples of encapsulation and failure

We now present computed solutions of droplets as either L or H are increased until breaking point. Figure 6.1 shows yielding of the “slender” droplet and Figure 6.2 shows the “fat” droplet. Superficially, the yielding mechanism are similar to the channel flow in where the plugs break (ends or near the centre). However, as suggested by the previous section, the plugs generally break at larger values of L .

The main reason is in the normal stresses. The pressure distributions found are qualitatively similar between pipe and channel geometries. At the yield surface in the channel $-\sigma_{yy}$ is continuous and approximately equal to the pressure. The normal stress must then vanish at the droplet surface which transfers some of the pressure into τ_{yy} . Since $\tau_{xx} + \tau_{yy} = 0$ we experience extensional stresses along the plug, i.e. $\tau_{xx} \neq 0$, and particularly near the ends of the droplet. These normal stress gradients promote shear stresses τ_{xy} , but additional shear stresses result directly in the widest part of the droplet where the stress gradients must increase.

The pipe differs firstly in that although normal stresses are generated at the yield surface, τ_{rr} can now be balanced by the hoop stress $\tau_{\theta\theta}$ as well as by τ_{zz} . Indeed, comparing Figs. 6.1e & f, we see that τ_{rr} and $\tau_{\theta\theta}$ largely cancel out over much of the plug, leaving only small bands where τ_{zz} is significant (Figure 6.1d). As a result, the shear stresses in much of the plug appear drastically reduced; see Figure 6.1c, occurring mostly close to $z = 0$.

6.4 Maximum size of encapsulated droplets

Following approximately 300 computations we have been able to quantify the maximal size of encapsulated droplets at four different Bingham numbers; see Figure 6.3. In order to have a better comparison between channel and pipe geometries, we include the channel data for $B = 10$. We see that in the range of slender droplets encapsulation is much easier in pipe geometry than in the channel although for short droplets the channel allows larger H . The computed results for the slender droplets are suggestive of $L \rightarrow \infty$ as $H \rightarrow 0$. Although consistent with the notion that $H \sim \delta^{1/2}$ and $L \sim \delta^{-1/2}$ (see §6.2), the slender geometries are not ideal for computations.

6.4. Maximum size of encapsulated droplets

Figure 6.1: Stress distribution in the axisymmetric geometry as the droplet gets longer: $B = 20$, $H = 0.4$ and $L = 1.2, 1.25, 1.3, 1.34$ and 1.45 . a) speed; b) pressure; c) τ_{rz} ; d) τ_{zz} ; e) τ_{rr} and f) $\tau_{\theta\theta}$.

6.4. Maximum size of encapsulated droplets

Figure 6.2: Stress distribution in the axisymmetric geometry as the droplet height is increased: $B = 20$, $L = 0.5$ and $H = 0.4, 0.45, 0.5, 0.525$ and 0.55 . a) speed; b) pressure; c) τ_{rz} ; d) τ_{zz} ; e) τ_{rr} and f) $\tau_{\theta\theta}$.

Figure 6.3: Maximum size of encapsulated droplets for four different Bingham numbers. $\circ$: $B = 5$; $\square$: $B = 10$; $\triangle$: $B = 20$, $\diamond$: $B = 50$. Included for comparison is a single curve showing the maximum size of droplet in the channel geometry for $B = 10$ (broken line).

Chapter 7

Conclusions

In visco-plastically lubricated flows, multi-layer flows are kept stable through the yield stress of the lubricating fluid, which prevents the interface from deforming; see e.g. [21, 51, 38, 36, 37]. In [33] this concept has been extended to multi-layer flows in which the interface is shaped, e.g. wavy, but again the yield stress prevents interfacial deformation; see [33].

In this thesis we have extended the concept in a different direction. Steady Poiseuille flows of yield stress fluids along uniform ducts are characterized by unyielded central plug regions, traveling at constant speed along the duct. Here we have investigated whether we may encapsulate droplets of a second fluid within the plug region of the yield stress fluid, with again the shape held constant by the yield stress of the fluid. The key advantage of the proposed methodology is that the yield stress holds the interface rigid, instead of relying on capillary forces as is common in many droplet encapsulation studies, allowing Macro-size drops to be encapsulated.

7.1 Summary of the results

In general, we have shown that the above methodology is feasible; i.e. Macro-size drops can be encapsulated inside the plug region of a poiseuille flow of a Bingham fluid in both Cartesian (channel) and cylindrical (pipe) coordinates. In particular,

- Introducing a droplet inside the plug region, travelling with the steady speed of plug does not deform the interface but does induce additional stresses within the carrier fluid. It is the latter that may lead to breaking of the plug.
- In both geometries (channel and pipe) the yield surface of the plug region containing the droplet is not significantly perturbed, apparently remaining near-planar as the droplet size is increased up to the point at which the plug breaks. We observed this in both analytical and computational results.

Figure 7.1: Encapsulation of drops with exotic shapes.

- From the asymptotic solution it appears that the deformation of the plug surface is only of second order ($\mathcal{O}(\delta^2)$) for slender droplets, where $\delta = H/L \ll 1$ represents the droplet aspect ratio.
- We can expect that inertial effects in the yielded part of the flow will scale like εRe , where ε is the aspect ratio of the streamlines, (i.e. $\varepsilon = \delta^2$ for slender droplets). Thus, we expect that the solutions and results we have computed here will retain their validity as approximations to the solution for significant Re , even though they are computed under Stokes flow assumptions.
- We know inertial terms are responsible for flow instability. Therefore, since our results suggest that the velocity perturbation due to the droplet is small, there is the possibility that these flows may be observable up to high Re .
- As for failure of encapsulation, two main modes are uncovered: droplets that become increasingly slender tend to break the plug at the ends of the droplet, whereas those that become increasingly wide tend to break near the centre of the droplet. Loosely speaking, the end mode of breakage appears to be driven by normal stress development whereas the central mode is driven by tangential stress gradients.

7.1. Summary of the results

- For elliptic (ellipsoidal) droplets we have computed the critical H & L at which breaking occurs for both channel and pipe geometries. Increasing the yield stress (via B) does increase the limiting droplet size, but not as much as might be expected. This is because the yield stress also increases the size of stresses found in the flow, i.e. considered at constant flow rate.
- The limiting values of H & L are different for channel and pipe, as might be expected. At the same Bingham number for short drops the channel may allow larger H for the same L . Here the limiting factor is anyway the yield surface position, which is similar as a function of B . The limitation in terms of length shows a more significant difference, with the pipe able to encapsulate much longer droplets than the channel.
- The pipe geometry uses the hoop stress to reduce normal stress effects.
- A different way in which droplet size may be increased is via introduction of a density difference between fluids. When pumping in the direction of gravity, lighter droplets increase encapsulation volume (heavier if pumping against gravity).
- The increase in encapsulation volume due to favourable density difference is not unbounded, but appears to reach a maximum approximately when the frictional pressure gradient of the base flow matches the buoyancy, i.e.

$$\frac{\partial p_0}{\partial x} = -\frac{B}{y_{y,0}} = \chi \quad \Rightarrow \quad \frac{\hat{\tau}_Y}{\hat{g}\hat{R}y_{y,0}} = \hat{\rho} - \hat{\rho}_d. \quad (7.1)$$

On decreasing the buoyancy beyond this limit, buoyancy stresses contribute to breaking of the plug. Simplistically, the main mechanism of buoyancy at the limit (7.1) is to compensate within the droplet for pressure variations in the yielded layer. This then reduces the need for normal stress gradients between the droplet interface and yield surface.

- The computational solutions reveal rather complex stress distributions inside the plug, not easily inferred from the asymptotic analysis.
- Although we have computed elliptical shaped droplets in this paper, in principle a yield stress allows any shape to be encapsulated. Figure 7.1 shows some exotic droplet shapes that do not yield the plug.

- The main results of this thesis have been submitted to Journal of Fluid Mechanics for publication: “Macro-size drop encapsulation, A. Maleki, S. Hormozi, A. Roustaie, I.A. Frigaard”.

7.2 Limitations

- The asymptotic solution predicts only the velocity and stress distributions outside the plug, but not the stress distribution inside the plug. Although we attempted to derive predictions of the (indeterminate) stress fields inside the plug, in order to predict breaking, we were unsuccessful. Details of unsuccessful attempts are listed in appendix §B.
- The asymptotic solution is unable to predict other flow features such as spacing between successive droplets.
- As we assume *a priori* that the droplet translates rigidly in the plug, there is no way to include the density difference in the asymptotic solution, i.e. we solve a 1 fluid problem with a perturbed droplet boundary.
- We have not considered surface tension effects in this thesis. Depending on the chosen droplet fluid there may anyway be no surface tension. We note that in using miscible fluids in many practical situations Péclet numbers (based on molecular diffusion) are very large and with no relative droplet motion dispersion is nullified. If the droplet is immiscible the validity of this neglect rests on the relevant stresses, and we expect our results to retain validity provided that

$$\hat{\tau}_Y \gg \frac{\hat{\sigma}_T}{\hat{\kappa}}, \quad (7.2)$$

where $\hat{\sigma}_T$ and $\hat{\kappa}$ are the surface tension coefficient and (a representative) radius of curvature of the droplet. Surface tension values between yield stress fluids and other fluids are not well known and reliable measurement techniques are still being developed, e.g. [9]. Nevertheless, for values typical of water and light oils, with yield stresses of ~ 10 Pa capillary effects are insignificant on lengthscales $\gtrsim 2$ mm, which represents many industrial scale pumping operations. This is not to say that interesting flow effects might not be produced by encapsulating immiscible fluid droplets. Indeed, these may have some advantages in the forming part of such a flow where it is likely that the plug is unyielded as a droplet is introduced to the main flow.

7.3 Future directions

A number of areas could be studied within this general direction.

- A challenging next step in order to validate/verify our results would be to set up macro-encapsulation experiments.
- One can consider more realistic models for the yield stress fluids, e.g. Herschel-Bulkley. However, we do not believe that this changes the characteristics of encapsulation dramatically from those found here.
- Considering other rheological effects such as visco-elasticity as well as including surface tension which may aid in the establishment of encapsulation.
- For practical purposes it is necessary to study the formation of droplets, either numerically or experimentally. The question here might be how to control droplet shape and frequency in the formation stage.

Bibliography

- [1] ADACHI, K. & YOSHIOKA, N. 1973 On creeping flow of a visco-plastic fluid past a circular cylinder. *Chem. Eng. Sci.* **28** (1), 215 – 226.
- [2] AL-MOHAMMAD, A.M., ALKHALDI, M.H., AL-MUTAIRI, S.H. & AL-ZAHRANI, A.A. 2012 Acidizing-induced damage in sandstone injector wells: laboratory testing and a case history. *SPE J.* **17** (3), 885–902.
- [3] BALMFORTH, N. & CRASTER, R. 1999 A consistent thin-layer theory for Bingham plastics. *J. Non-Newt. Fluid Mech.* **84**, 65–81.
- [4] BALMFORTH, N., FRIGAARD, I.A. & OVARLEZ, G. 2014 Yielding to stress: Recent developments in viscoplastic fluid mechanics. *Annu. Rev. Fluid Mech.* **46**, 121–146.
- [5] BEAULNE, M. & MITSOULIS, E. 1997 Creeping motion of a sphere in tubes filled with Herschel-Bulkley fluids. *J. Non-Newt. Fluid Mech.* **72**, 55–71.
- [6] BERIS, A.N., TSAMOPOULOS, J.A. & ARMSTRONG, R.C. 1985 Creeping motion of a sphere through a Bingham plastic. *J. Fluid Mech.* **158**, 219–244.
- [7] BHAVARAJU, S.M., MASHELKAR, R.A. & BLANCH, H.W. 1978 Bubble motion and mass-transfer in non-newtonian fluids .1. single bubble in power law and Bingham fluids. *AIChE J.* **24** (6), 1063–1070.
- [8] BLACKERY, J. & MITSOULIS, E. 1997 Creeping motion of a sphere in tubes filled with a Bingham plastic material. *J. Non-Newt. Fluid Mech.* **70**, 59–77.
- [9] BOUJLEL, J. & COUSSOT, P. 2013 Measuring the surface tension of yield stress fluids. *Soft Matt.* **9** (25), 5898–5908.
- [10] CHATEAU, X., OVARLEZ, G. & TRUNG, K.L. 2008 Homogenization approach to the behavior of suspensions of noncolloidal particles in yield stress fluids. *J. Rheol* **52** (2), 489–506.

- [11] COHEN, C.E., DING, D., QUINTARD, M. & BAZIN, B. 2008 From pore scale to wellbore scale: Impact of geometry on wormhole growth in carbonate acidization. *Chem. Eng. Sci.* **63** (12), 3088–3099.
- [12] COHEN, I., LI, H., HOUGLAND, J.L., MRKSICH, M. & NAGEL, S.R. 2001 Using selective withdrawal to coat microparticles. *Science* **292**, 265–267.
- [13] COX, B.G. 1964 An experimental investigation of the streamlines in viscous fluid expelled from a tube. *J. Fluid Mech.* **20** (2), 193–200.
- [14] DE BESSES, B.D., MAGNIN, A. & JAY, P. 2003 Viscoplastic flow around a cylinder in an infinite medium. *J. Non-Newt. Fluid Mech.* **115** (1), 27–49.
- [15] DIMAKOPOULOS, Y., PAVLIDIS, M. & TSAMOPOULOS, J. 2013 Steady bubble rise in Herschel-Bulkley fluids and comparison of predictions via the augmented Lagrangian method with those via the Papanastasiou model. *J. Non-Newt. Fluid Mech.* **200**, 34–51.
- [16] DUBASH, N. & FRIGAARD, I.A. 2004 Conditions for static bubbles in viscoplastic fluids. *Phys. Fluids* **16**, 4319–4330.
- [17] DUBASH, N. & FRIGAARD, I.A. 2005 Propagation and stopping of air bubbles in carbopol solutions. *J. Non-Newt. Fluid Mech.* **142**, 123–134.
- [18] DUNBRACK, G. 2013 Interfacial effects in visco-plastic lubrication flows. Master’s thesis, University of British Columbia.
- [19] DUVAUT, G. & LIONS, J.L. 1976 *Inequalities in Mechanics and Physics*. Springer-Verlag.
- [20] FORTIN, M. & GLOWINSKI, R. 1983 *Augmented Lagrangian methods: application to the numerical solution of boundary-value problems*. Elsevier Science Publishers B.V.
- [21] FRIGAARD, I.A. 2001 Super-stable parallel flows of multiple viscoplastic fluids. *J. Non-Newt. Fluid Mech.* **100** (1-3), 49–76.
- [22] FRIGAARD, I.A. & NOUAR, C. 2005 On the usage of viscosity regularisation methods for visco-plastic fluid flow computation. *J. Non-Newt. Fluid Mech.* **127** (1), 1–26.
- [23] FRIGAARD, I.A. & RYAN, D.P. 2004 Flow of a visco-plastic fluid in a channel of slowly varying width. *J. Non-Newt. Fluid Mech.* **123**, 67–83.

- [24] FURUI, K., BURTON, R.C., BURKHEAD, D.W., ABDELMALEK, N.A., HILL, A.D., ZHU, D. & NOZAKI, M. 2012a A comprehensive model of high-rate matrix-acid stimulation for long horizontal wells in carbonate reservoirs: part I-scaling up core-level acid wormholing to field treatments. *SPE J.* **17** (1), 271–279.
- [25] FURUI, K., BURTON, R.C., BURKHEAD, D.W., ABDELMALEK, N.A., HILL, A.D., ZHU, D. & NOZAKI, M. 2012b A comprehensive model of high-rate matrix-acid stimulation for long horizontal wells in carbonate reservoirs: part II-wellbore/reservoir coupled-flow modeling and field application. *SPE J.* **17** (1), 280–291.
- [26] GANAN-CALVO, A.M. 1998 Generation of steady liquid microthreads and micron-sized monodisperse sprays in gas streams. *Phys. Rev. Lett.* **80** (2), 285–288.
- [27] GLOWINSKI, R. 1984 *Numerical methods for nonlinear variational problems*. Springer-Verlag New York Inc.
- [28] GLOWINSKI, R. & WACHS, A. 2011 On the numerical simulation of viscoplastic fluid flow. *Handb. Numer. Anal.* **16**, 483–717.
- [29] GREF, R., MINAMITAKE, Y., PERACCHIA, M.T., TRUBETSKOY, V., TORCHILIN, V. & LANGER, R. 1994 Biodegradable long-circulating polymeric nanospheres. *Science* **263**, 1600–1603.
- [30] HILLERY, A.M., LLOYD, A.W. & SWARBRICK, J. 2001 *Drug delivery and targeting for pharmacists and pharmaceutical scientists*. Taylor and Francis Inc.
- [31] HOLENBERG, Y., LAVRENTEVA, O.M., LIBERZON, A., SHAVIT, U. & NIR, A. 2013 PTV and PIV study of the motion of viscous drops in yield stress material. *J. Non-Newt. Fluid Mech.* **193**, 129–143.
- [32] HORMOZI, S. 2011 Multi-layer flows with yield stress fluids. PhD thesis, University of British Columbia.
- [33] HORMOZI, S., DUNBRACK, G. & FRIGAARD, I.A. 2014 Transient behaviour and shaped interface formation in non-equilibrium multilayer flows. *Phys. Fluids* p. under publication.
- [34] HORMOZI, S. & FRIGAARD, I.A. 2012 Nonlinear stability of a viscoplastically lubricated viscoelastic fluid flow. *J. Non-Newt. Fluid Mech.* **169**, 61–73.

- [35] HORMOZI, S., MARTINEZ, D. M. & FRIGAARD, I.A. 2011c Stable core-annular flows of viscoelastic fluids using the visco-plastic lubrication technique. *J. Non-Newt. Fluid Mech.* **166** (23-24), 1356–1368.
- [36] HORMOZI, S., WIELAGE-BURCHARD, K. & FRIGAARD, I.A. 2011a Multi-layer channel flows with yield stress fluids. *J. Non-Newt. Fluid Mech.* **166** (5-6), 262–278.
- [37] HORMOZI, S., WIELAGE-BURCHARD, K. & FRIGAARD, I.A. 2011b Entry, start up and stability effects in visco-plastically lubricated pipe flows. *J. Fluid Mech.* **673**, 432–467.
- [38] HUEN, C.K., FRIGAARD, I.A. & MARTINEZ, D.M. 2007 Experimental studies of multi-layer flows using a visco-plastic lubricant. *J. Non-Newt. Fluid Mech.* **142** (1-3), 150–161.
- [39] JAWOREK, A. 2008 Electrostatic micro and nanoencapsulation and electroemulsification: A brief review. *J. Microencapsul.* **25** (7), 443–468.
- [40] KALFAYAN, L. 2008 *Production enhancement with acid stimulation*, 2nd edn. Penwell Corportation.
- [41] LAVRENTEVA, O.M., HOLENBERG, Y. & NIR, A. 2009 Motion of viscous drops in tubes filled with yield stress fluid. *Chem. Eng. Sci.* **64** (22), 4772–4786.
- [42] LIPSCOMB, G.G. & DENN, M. 1984 Flow of Bingham fluids in complex geometries. *J. Non-Newt. Fluid Mech.* **14**, 337–346.
- [43] LISTER, J.R. 1989 Selective withdrawal from a viscous 2-layer system. *J. Fluid Mech.* **198**, 231–254.
- [44] LIU, B.T., MULLER, S.J. & DENN, M.M. 2002 Convergence of a regularization method for creeping flow of a Bingham material about a rigid sphere. *J. Non-Newt. Fluid Mech.* **102**, 179–191.
- [45] LIU, B. T., MULLER, S. J. & DENN, M. M. 2003 Interactions of two rigid spheres translating collinearly in creeping flow in a Bingham material. *J. Non-Newt. Fluid Mech.* **113** (1), 49–67.
- [46] LIU, X.H., ORMOND, A., BARTKO, K., LI, Y. & ORTOLEVA, P. 1997 A geochemical reaction-transport simulator for matrix acidizing analysis and design. *J. Petrol. Sci. Eng.* **17** (1-2), 181–196.

- [47] LOSCERTALES, I.G., BARRERO, A., GUERRERO, I., CORTIJO, R., MARQUEZ, M. & GANAN-CALVO, A.M. 2002 Micro/nano encapsulation via electrified coaxial liquid jets. *Science* **295**, 1695–1698.
- [48] MAHAUT, F., CHATEAU, X., COUSSOT, P. & OVARLEZ, G. 2008 Yield stress and elastic modulus of suspensions of noncolloidal particles in yield stress fluids. *J. Rheol.* **52** (1), 287–313.
- [49] MAHESHWARI, P. & BALAKOTAIAH, V. 2013 Comparison of carbonate HCl acidizing experiments with 3D simulations. *SPE Prod. Oper.* **28** (4), 402–413.
- [50] MERKAK, O., JOSSIC, L. & MAGNIN, A. 2009 Migration and sedimentation of spherical particles in a yield stress fluid flowing in a horizontal cylindrical pipe. *AIChE J.* **55** (10), 2515–2525.
- [51] MOYERS-GONZALEZ, M.A., FRIGAARD, I.A. & NOUAR, C. 2004 Non-linear stability of a visco-plastically lubricated viscous shear flow. *J. Fluid Mech.* **506**, 117–146.
- [52] MOYERS-GONZALEZ, M.A., FRIGAARD, I.A. & NOUAR, C. 2010 Stable two-layer flows at all Re; visco-plastic lubrication of shear-thinning and viscoelastic fluids. *J. Non-Newt. Fluid Mech.* **165** (23-24), 1578–1587.
- [53] O'DONOVAN, E.J. & TANNER, R.I. 1984 Numerical study of the Bingham squeeze film problem. *J. Non-Newt. Fluid Mech.* **15** (1), 75–83.
- [54] OVARLEZ, G., BERTRAND, F., COUSSOT, P. & CHATEAU, X. 2012 Shear-induced sedimentation in yield stress fluids. *J. Non-Newt. Fluid Mech.* **177**, 19–28.
- [55] OVARLEZ, G., BERTRAND, F. & RODTS, S. 2006 Local determination of the constitutive law of a dense suspension of noncolloidal particles through magnetic resonance imaging. *J. Rheol.* **50** (3), 259–292.
- [56] PAPANASTASIOU, T.C. 1987 Flows of materials with yield. *J. Rheol.* **31** (5), 385–404.
- [57] POTAPOV, A., SPIVAK, R., LAVRENTEVA, O.M. & NIR, A. 2006 Motion and deformation of drops in Bingham fluid. *Ind. Eng. Chem. Res.* **45** (21), 6985–6995.

- [58] PRIYA, A. R. SATHIYA, MURALIDHARAN, V. S. & SUBRAMANIA, A. 2008 Development of novel acidizing inhibitors for carbon steel corrosion in 15% boiling hydrochloric acid. *Corros. Sci.* **64** (6), 541–552.
- [59] PUTZ, A., BURGHELEA, T.I., FRIGAARD, I.A. & MARTINEZ, D.M. 2008 Settling of an isolated spherical particle in a yield stress shear thinning fluid. *Phys. Fluids* **20** (3).
- [60] PUTZ, A. & FRIGAARD, I.A. 2010 Creeping flow around particles in a Bingham fluid. *J. Non-Newt. Fluid Mech.* **165**, 263–280.
- [61] PUTZ, A., FRIGAARD, I.A. & MARTINEZ, D.M. 2009 On the lubrication paradox and the use of regularisation methods for lubrication flows. *J. Non-Newt. Fluid Mech.* pp. 62–77.
- [62] QURAISHI, M.A. & JAMAL, D. 2000 Technical note: CAHMT - A new and eco-friendly acidizing corrosion inhibitor. *Corros. Sci.* **56** (10), 983–985.
- [63] RATNAKAR, R.R., KALIA N. & BALAKOTAIAH, V. 2013 Modeling, analysis and simulation of wormhole formation in carbonate rocks with in situ cross-linked acids. *Chem. Eng. Sci.* **90** (0), 179–199.
- [64] ROQUET, N. & SARAMITO, P. 2003 An adaptive finite element method for bingham fluid flows around a cylinder. *Comput. Methods in Appl. Mech. Eng.* **192**, 3317–3341.
- [65] ROUSTAEI, A. & FRIGAARD, I.A. 2013 The occurrence of fouling layers in the flow of a yield stress fluid along a wavy-walled channel. *J. Non-Newt. Fluid Mech.* **198**, 109–124.
- [66] RYAN, D.P. 2004 Slow flow of bingham fluid in a gap of slowly varying width. Master’s thesis, University of British Columbia.
- [67] SARAMITO, P. & ROQUET, N. 2001 An adaptive finite element method for viscoplastic fluid flows in pipes. *Comput. Methods in Appl. Mech. Eng.* **190** (40–41), 5391–5412.
- [68] SINGH, J.P. & DENN, M.M. 2008 Interacting two-dimensional bubbles and droplets in a yield-stress fluid. *Phys. Fluids* **20** (4).
- [69] SMITH, C.F. & HENDRICK, A.R. 1965 Hydrofluoric acid stimulation of sandstone reservoirs. *J. Petrol. Technol.* **17** (2), 215–222.

- [70] STEIN, S & BUGGISCH, H 2000 Rise of pulsating bubbles in fluids with a yield stress. *J. Appl. Math. Mech.* **80** (11-12), 827–834.
- [71] TOKPAVI, D.L., MAGNIN, A. & JAY, P. 2008 Very slow flow of Bingham viscoplastic fluid around a circular cylinder. *J. Non-Newt. Fluid Mech.* **154** (1), 65–76.
- [72] WALTON, I.C. & BITTLESTON, S.H. 1991 The axial-flow of a Bingham plastic in a narrow eccentric annulus. *J. Non-Newt. Fluid Mech.* **222**, 39–60.
- [73] WINDBERGS, M., ZHAO, Y., HEYMAN, J. & WEITZ, D.A. 2013 Biodegradable core-shell carriers for simultaneous encapsulation of synergistic actives. *J. Am. Chem. Soc.* **135** (21), 7933–7937.
- [74] ZHAO, Y., SHUM, H.C., ADAMS, L.A., SUN, B., HOLTZE, C., GU, Z. & WEITZ, D.A. 2011 Enhanced encapsulation of actives in self-sealing microcapsules by precipitation in capsule shells. *Langmuir* **27** (23), 13988–13991.
- [75] ZOU, C.J., LIAO, W.J., ZHANG, L. & CHEN, H.M. 2011 Study on acidizing effect of beta-cyclodextrin-PBTCA inclusion compound with sandstone. *J. Petrol. Sci. Eng.* **77** (2), 219–225.
- [76] ZUIDAM, N.J. & NEDOVI, V.A. 2010 *Encapsulation technologies for active food ingredients and food processing*. Springer.

Appendix A

Algebraic Details

A.1 Channel Poiseuille flow of a Bingham fluid

Here we want to study the plane Poiseuille solution. We assume the flow is fully developed and there is no x -gradient. We also neglect the gravity here assuming that the channel is horizontal. Due to the symmetry of the problem, without loss of generality, we can only consider $y > 0$. Finally, in order to find pressure gradient we take flow rate to be unity. With these assumptions 2.9b simplifies to

$$-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (\text{A.1})$$

Knowing that $p = p(x)$ and the stress has to be zero at the centerline due to symmetry, we write:

$$\tau_{xy} = -\frac{\partial p}{\partial x} y \quad (\text{A.2})$$

Yielding happens where $\tau = |\tau_{xy}| \geq B$ and therefore position of plug interface is characterised by $B = -y_{y,0} \frac{\partial p}{\partial x}$. In the yielded region $y \geq y_{y,0}$, $\frac{\partial U_P}{\partial y} < 0$ and therefore constitutive relations simplify to $\tau_{xy} = \frac{\partial U_P}{\partial y} - B$. Thus

$$\frac{\partial U_P}{\partial y} = \frac{\partial p}{\partial x} y - B = \frac{\partial p}{\partial x} (y - y_{y,0}) \Rightarrow U_P = \frac{\partial p}{\partial x} \left((y - y_{y,0})^2 - (1 - y_{y,0})^2 \right) \quad (\text{A.3})$$

where we satisfy no slip condition at the wall. This can be rearranged like:

$$U_P(y) = \begin{cases} u_{p,0} & \text{if } |y| \leq y_{y,0}, \\ u_{p,0} \left[1 - \frac{(|y| - y_{y,0})^2}{(1 - y_{y,0})^2} \right] & \text{if } |y| > y_{y,0}, \end{cases} \quad (\text{A.4})$$

in which $u_{p,0}$ is the plug velocity and is given by

$$u_{p,0} = \frac{B}{2y_{y,0}} (1 - y_{y,0})^2, \quad (\text{A.5})$$

A.1. Channel Poiseuille flow of a Bingham fluid

The position of the yield surface is found by demanding a flow rate of unity:

$$\int_0^1 U_P dy = \int_0^{y_{y,0}} u_{p,0} dy + \int_{y_{y,0}}^1 U_P(y) dy = 1 \quad (\text{A.6})$$

which leads to Buckingham equation:

$$f(y) := y^3 - (3 + \frac{6}{B})y + 2 = 0, \quad \text{and} \quad f(y_{y,0}) = 0 \quad (\text{A.7})$$

One can notice that for any $B > 0$, $f(0).f(1) < 0$ which means

$$\exists y_y \in (0, 1) : f(y_y) = 0 \quad (\text{A.8})$$

regardless of the value of B . Also $f'(y) < 0$ for $y \in (0, 1)$, so that there may be only a single root.

In limit when $B \rightarrow 0$, we can write $y_y = y_{y0} + By_{y1} + B^2 y_{y2}$. Plugging this expansion in the Buckingham Equation, collecting powers of B and equating them to zero yields:

$$\frac{6}{B} y_{y0} = 0 \quad \Rightarrow y_{y0} = 0 \quad (\text{A.9a})$$

$$y_{y0}^3 - 3y_{y0} - 6y_{y1} + 2 = 0 \quad \Rightarrow y_{y1} = \frac{1}{3} \quad (\text{A.9b})$$

$$B(3y_{y0}^2 y_{y1} - 3y_{y1} - 6y_{y2}) = 0 \quad \Rightarrow y_{y2} = -\frac{1}{6} \quad (\text{A.9c})$$

and finally we can write:

$$B \rightarrow 0 \quad y_y \sim \frac{B}{3} - \frac{B^2}{6} \quad (\text{A.10})$$

This is consistent with our expectations. In particular, for a Newtonian fluid ($B = 0$), the thickness of plug region is zero ($y_y = 0$).

On the other side of the spectrum we consider the case of $B \rightarrow \infty$. The zero order solution corresponds to the blockage of channel when the channel is entirely filled with plug region $y_y = 1$. Looking for the perturbations around this root, we can expand $y_y = 1 + \alpha B^p$; where p and α have to be found. Plugging into the Buckingham Equation:

$$3\alpha B^p + 3\alpha^2 B^{2p} - \frac{6}{B} - 3\alpha B^p - 6\alpha B^{p-1} + \mathcal{O}(3p) = 0 \quad (\text{A.11})$$

Given that $\alpha \neq 0$, the only possible value for p is $-1/2$. Then we can find $\alpha = -\sqrt{2}$ and therefore,

$$B \rightarrow \infty \quad y_y \sim 1 - \sqrt{\frac{2}{B}} \quad (\text{A.12})$$

A.2 Asymptotic form of Bingham constitutive relations

A.2.1 channel geometry

In this section we derive the asymptotic form of Bingham constitutive relations in Cartesian geometry. We adopt a regular asymptotic expansion in the form of

$$(u, v, p) = (u_0, v_0, p_0) + \delta(u_1, v_1, p_0) + \delta^2(u_2, v_2, p_2) + \dots \quad (\text{A.13})$$

where u and v represent dimensionless components of velocity and p is dimensionless pressure introduced in §3.1.⁷ We have

$$\dot{\gamma}_{xx} = 2 \frac{\partial u}{\partial x} = 2 \frac{\partial u_0}{\partial x} + \delta 2 \frac{\partial u_1}{\partial x} + \delta^2 2 \frac{\partial u_2}{\partial x} + \mathcal{O}(\delta^3) \quad (\text{A.14a})$$

$$\dot{\gamma}_{xy} = \frac{\partial u}{\partial y} + \delta^2 \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \delta \frac{\partial u_1}{\partial y} + \delta^2 \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_0}{\partial x} \right) + \mathcal{O}(\delta^3) \quad (\text{A.14b})$$

Substituting (A.14) in (1.4c) yields:

$$\begin{aligned} \dot{\gamma} &= \left(\frac{\partial u_0}{\partial y}^2 + \delta 2 \frac{\partial u_0}{\partial y} \frac{\partial u_1}{\partial y} + \delta^2 \left[\frac{\partial u_1}{\partial y}^2 + 2 \frac{\partial u_0}{\partial y} \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_0}{\partial x} \right) + 4 \frac{\partial u_0}{\partial x}^2 \right] \right)^{\frac{1}{2}} \\ &= \left| \frac{\partial u_0}{\partial y} \right| \left(1 + \delta 2 \frac{\frac{\partial u_1}{\partial y}}{\frac{\partial u_0}{\partial y}} + \delta^2 \frac{\frac{\partial u_1}{\partial y}^2 + 2 \frac{\partial u_0}{\partial y} \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_0}{\partial x} \right) + 4 \frac{\partial u_0}{\partial x}^2}{\frac{\partial u_0}{\partial y}^2} \right)^{\frac{1}{2}} \end{aligned} \quad (\text{A.15})$$

Knowing that $\delta \ll 1$, we can use Taylor expansion to find:

$$\dot{\gamma}^{-1} = \frac{1}{\left| \frac{\partial u_0}{\partial y} \right|} \left(1 - \delta \frac{\frac{\partial u_1}{\partial y}}{\frac{\partial u_0}{\partial y}} - \delta^2 \frac{4 \frac{\partial u_1}{\partial y}^2 + 8 \frac{\partial u_0}{\partial y} \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_0}{\partial x} \right) + 16 \frac{\partial u_0}{\partial x}^2 + 3 \frac{\partial u_1}{\partial y}^2}{8 \frac{\partial u_0}{\partial y}^2} \right) \quad (\text{A.16})$$

Now we have

$$\tau_{xy} = \left(1 + \frac{B}{\dot{\gamma}} \right) \dot{\gamma}_{xy} = \frac{\partial u_0}{\partial y} + B \operatorname{sgn}\left(\frac{\partial u_0}{\partial y}\right) + \delta \frac{\partial u_1}{\partial y} + \mathcal{O}(\delta^2) \quad (\text{A.17a})$$

$$\tau_{xx} = \left(1 + \frac{B}{\dot{\gamma}} \right) \dot{\gamma}_{xx} = 2 \left(1 + \frac{B}{\left| \frac{\partial u_0}{\partial y} \right|} \right) \frac{\partial u_0}{\partial x} + \mathcal{O}(\delta) \quad (\text{A.17b})$$

This analysis is taken from [66] with minor changes.

⁷The uppercase letters of §3.1 are shown in lower case here.

A.2.2 pipe geometry

In this section we derive the asymptotic form of Bingham constitutive relations in cylindrical geometry. We adopt a regular asymptotic expansion to the form of

$$(u, w, p) = (u_0, w_0, p_0) + \delta(u_1, w_1, p_0) + \delta^2(u_2, w_2, p_2) + \dots \quad (\text{A.18})$$

where u and w represent dimensionless radial and axial velocities respectively and p is dimensionless pressure all introduced in §6.2. The uppercase letters of §6.2 are shown in lower case here. Using the identity:

$$\dot{\gamma}_{rr} + \dot{\gamma}_{\theta\theta} + \dot{\gamma}_{zz} = 0,$$

we rewrite (6.10b) as:

$$\dot{\gamma} = \sqrt{\dot{\gamma}_{rz}^2 - \delta^2(\dot{\gamma}_{rr}\dot{\gamma}_{\theta\theta} + \dot{\gamma}_{rr}\dot{\gamma}_{zz} + \dot{\gamma}_{rr}\dot{\gamma}_{zz})} \quad (\text{A.19})$$

In addition, we know

$$\dot{\gamma}_{rz} = \frac{\partial u_0}{\partial r} + \delta \frac{\partial u_1}{\partial r} + \delta^2 \left(\frac{\partial u_2}{\partial r} + \frac{\partial v_0}{\partial z} \right) + \mathcal{O}(\delta^3) \quad (\text{A.20a})$$

$$\dot{\gamma}_{rr} = 2 \frac{\partial v_0}{\partial r} + \mathcal{O}(\delta) \quad (\text{A.20b})$$

$$\dot{\gamma}_{\theta\theta} = 2 \frac{v_0}{r} + \mathcal{O}(\delta) \quad (\text{A.20c})$$

$$\dot{\gamma}_{zz} = 2 \frac{\partial u_0}{\partial z} + \mathcal{O}(\delta) \quad (\text{A.20d})$$

To go on, we need to calculate $\frac{1}{\dot{\gamma}}$, so we have

$$\dot{\gamma} = \left| \frac{\partial u_0}{\partial r} \right| \left[1 + \delta \left(2 \frac{\frac{\partial u_1}{\partial r}}{\frac{\partial u_0}{\partial r}} \right) + \delta^2 \left(\frac{2 \frac{\partial u_0}{\partial r} \left(\frac{\partial u_2}{\partial r} + \frac{\partial v_0}{\partial z} \right) + \frac{\partial u_1}{\partial r}^2 - 4 \left(\frac{v_0}{r} \frac{\partial v_0}{\partial r} + \frac{\partial v_0}{\partial r} \frac{\partial u_0}{\partial z} + \frac{v_0}{r} \frac{\partial u_0}{\partial z} \right)}{\frac{\partial u_0}{\partial r}^2} \right) \right]^{1/2} \quad (\text{A.21})$$

which results in

$$\dot{\gamma}^{-1} = \frac{1}{\left| \frac{\partial u_0}{\partial r} \right|} \left[1 - \delta \frac{\frac{\partial u_1}{\partial r}}{\frac{\partial u_0}{\partial r}} - \delta^2 \left(\frac{\frac{\partial u_0}{\partial r} \left(\frac{\partial u_2}{\partial z} + \frac{\partial v_0}{\partial z} \right) - 2 \left(\frac{v_0}{z} \frac{\partial v_0}{\partial r} + \frac{\partial u_2}{\partial z} \frac{\partial v_0}{\partial r} + \frac{v_0}{r} \frac{\partial u_0}{\partial z} \right) - \frac{5}{2} \frac{\partial u_1}{\partial r}^2}{\frac{\partial u_0}{\partial r}^2} \right) \right] \quad (\text{A.22})$$

Thus

$$\tau_{rz} = \frac{\partial u_0}{\partial r} + B \operatorname{sgn}\left(\frac{\partial u_0}{\partial r}\right) + \delta \frac{\partial u_1}{\partial r} + \mathcal{O}(\delta^2) \quad (\text{A.23a})$$

$$\tau_{rr} = 2 \frac{\partial v_0}{\partial r} \left(1 + \frac{B}{|\frac{\partial u_0}{\partial r}|}\right) + \mathcal{O}(\delta) \quad (\text{A.23b})$$

$$\tau_{\theta\theta} = 2 \frac{v_0}{r} \left(1 + \frac{B}{|\frac{\partial u_0}{\partial r}|}\right) + \mathcal{O}(\delta) \quad (\text{A.23c})$$

$$\tau_{zz} = 2 \frac{\partial u_0}{\partial z} \left(1 + \frac{B}{|\frac{\partial u_0}{\partial r}|}\right) + \mathcal{O}(\delta) \quad (\text{A.23d})$$

A.3 Perturbations are $\mathcal{O}(\delta^2)$

A.3.1 $y_T - y_{y,0} \sim \mathcal{O}(\delta^2)$

we define ξ_1 the departure of 0-order plug interface (y_y) from its Poiseuille counterpart : $y_y = y_{y,0} + \delta \xi_1$. Substituting this into (3.13) gives:

$$\mathcal{O}(1) : y_{y,0}^3 - y_{y,0}(3 + 6/B) + 2 = 0, \quad (\text{A.24a})$$

$$\mathcal{O}(\delta) : 3y_{y,0}^2 \xi_1 - \xi_1(3 + 6/B) + 6 \frac{\bar{h}}{B} = 0, \quad \implies \quad \xi_1 = \frac{\bar{h}}{1 + 0.5(1 - y_{y,0}^2)} \quad (\text{A.24b})$$

The first expression is simply the Buckingham equation (2.16), as expected. The second expression defines ξ_1 . Now, since u_p is defined by y_y and h , we straightforwardly expand 3.12 in powers of δ and write

$$\begin{aligned} \eta &= \frac{B}{\delta} (u_{p,0} - u_p(X)) = \frac{B}{2\delta} \left(\frac{(1 - y_{y,0})^2}{y_{y,0}} - \frac{(1 - (y_{y,0} + \delta \xi_1))^2}{(y_{y,0} + \delta \xi_1) - \delta \bar{h}} \right) \\ &= \frac{B}{2\delta} \left(\frac{(1 - y_{y,0})^2}{y_{y,0}} - \frac{(1 - y_{y,0})^2}{y_{y,0}} - \delta \frac{(1 - y_{y,0})(\xi_1 - \bar{h} - 3y_{y,0}\xi_1 + y_{y,0}\bar{h})}{y_{y,0}^2} \right) \\ &= \frac{B}{2y_{y,0}^2} ((1 - y_{y,0})(\xi_1 - \bar{h} + y_{y,0}\xi_1 + y_{y,0}\bar{h})) + \mathcal{O}(\delta) \end{aligned} \quad (\text{A.25})$$

A.3. Perturbations are $\mathcal{O}(\delta^2)$

This confirms that $\eta = \mathcal{O}(1)$ as has been claimed earlier. To evaluate y_T , we need to find $\frac{\partial U_1}{\partial y}(X, y_y)$ and $\left(\frac{\partial^2 U_0}{\partial Y^2}(X, y_y)\right)^{-1}$. From (3.19) we have:

$$\begin{aligned}\frac{\partial U_1}{\partial y}(X, y_y) &= -2\eta \frac{y_y + 2}{(y_y - 1)^2} = -2\eta \frac{(y_{y,0} + \delta\xi_1) + 2}{((y_{y,0} + \delta\xi_1) - 1)^2} \\ &= -2\eta \left(\frac{y_{y,0} + 2}{(y_{y,0} - 1)^2} - \delta\xi_1 \frac{y_{y,0} + 5}{(\xi_1 - 1)^3} \right) + \mathcal{O}(\delta^2)\end{aligned}\tag{A.26}$$

and from (3.11)

$$\left(\frac{\partial^2 U_0}{\partial Y^2}(X, y_y)\right)^{-1} = -\frac{y_y - \delta\bar{h}}{B} = -\frac{y_{y,0}}{B} + \mathcal{O}(\delta)\tag{A.27}$$

Substituting the expansion of y_y into (3.20) and incorporating (A.25), (A.26) and (A.27), we have:

$$y_T - y_{y,0} = \delta\left(\xi_1 - \frac{\frac{\partial U_1}{\partial y}(X, y_y)}{\frac{\partial^2 U_0}{\partial Y^2}(X, y_y)}\right) = \delta\left(\xi_1 - \frac{y_{y,0} + 2}{y_{y,0} - 1} \frac{y_{y,0}\xi_1 + y_{y,0}\bar{h} + y_{y_1} - \bar{h}}{y_{y,0}}\right) + \mathcal{O}(\delta^2)\tag{A.28}$$

and then with A.24b:

$$\begin{aligned}y_T - y_{y,0} &= \delta\bar{h} \frac{By_{y,0}^4 + By_{y,0}^3 - 3By_{y,0}^2 - 6y_{y,0}^2 - By_{y,0} - 6y_{y,0} + 2B}{y_{y,0}(y_{y,0} - 1)(By_{y,0}^2 - B - 2)} + \mathcal{O}(\delta^2) \\ &= \delta h \frac{B(y_{y,0} + 1)(y_{y,0}^3 - 3y_{y,0} - \frac{6}{B}y_{y,0} + 2)}{y_{y,0}(y_{y,0} - 1)(By_{y,0}^2 - B - 2)} + \mathcal{O}(\delta^2)\end{aligned}\tag{A.29}$$

Finally, taking Buckingham equation (2.16) into account, we see

$$y_T - y_{y,0} \sim \mathcal{O}(\delta^2)\tag{A.30}$$

A.3.2 $U_0 + \delta U_1 = U_P + \mathcal{O}(\delta^2)$

Similar to the previous section, we define ξ_2 the departure of 0-order velocity from poiseuille flow velocity: $U_0 = U_P + \delta\xi_2$. Our goal is to show that $\xi_2 + U_1 \sim \mathcal{O}(\delta)$.

Starting from 3.19 and expanding $y_y = y_{y,0} + \delta\xi_1$ we have:

$$U_1(X, Y) = -\eta \frac{(Y - 1)(3Y + 3Yy_{y,0} - 1 - 4y_{y,0}^2 - y_{y,0})}{(y_{y,0} - 1)^3} + \mathcal{O}(\delta)\tag{A.31}$$

A.3. Perturbations are $\mathcal{O}(\delta^2)$

and then substituting η from (A.25) :

$$U_1(X, Y) = B\bar{h} \frac{(Y-1)(3Y + 3Yy_{y,0} - 1 - 4y_{y,0}^2 - y_{y,0})(By_{y,0}^2 - 2y_{y,0}B - 2y_{y,0} + B)}{(y_{y,0} - 1)^2 y_{y,0}^2 (-2 - B + By_{y,0}^2)} \quad (\text{A.32})$$

From definition of ξ_2 , we can find

$$\xi_2 = \frac{1}{\delta}(U_0 - U_P) = \frac{B}{2y_{y,0}^2}(Y-1)(\xi_1 Y - Y\bar{h} + 2\bar{h}y_{y,0} + \xi_1 - \bar{h}) \quad (\text{A.33})$$

After substitution of ξ_1 from 3.21b and algebraic simplification we can show

$$\xi_2 + U_1 = \alpha \left(y_{y,0}^3 - y_{y,0} \left(3 + \frac{6}{B} \right) + 2 \right) + \mathcal{O}(\delta) \quad (\text{A.34})$$

in which α is a collection of numerous terms multiplied to each other. This then leads to the conclusion:

$$\xi_2 + U_1 \sim \mathcal{O}(\delta) \implies U_0 + \delta U_1 = U_P + \mathcal{O}(\delta^2) \quad (\text{A.35})$$

Appendix B

Unsuccessful attempts

We spent a lot of effort in order to find a mathematical relation which estimates the maximum size of encapsulated drop. Although the results were not successful, it is still worth mentioning the approach and the challenges that we faced, hoping that could be a guideline for further studies.

B.1 Method of islands of broken plug of [23]

In [23], the authors gave an estimate to the maximum of amplitude of a wavy channel which does not break the plug region. They performed this analysis by balancing continuity and momentum equations on the islands of the broken plug. We took the same approach here and find the average value of extensional stress along any vertical line inside the channel and supposedly, this has to be B/δ so that the plug yields.

With rearranging and combining the continuity and x-momentum equations together, and then using Green's theory one can straightforwardly show

$$\oint_{\partial A} J_1 dy + J_2 dx = 0 \quad (\text{B.1})$$

where

$$J_1 = -p + \delta^2 \tau_{xx} - \delta Re u^2 \quad (\text{B.2a})$$

$$J_2 = -\tau_{xy} + \delta Re uv \quad (\text{B.2b})$$

and A is any simple closed piecewise smooth curved in $x - y$ plane. We choose A as shown in Figure B.1 The integration in B.1 can be divided into four integration on four segments of the boundary (as shown in Figure). We know $y_T = y_{y,0} + \mathcal{O}(\delta^2)$, we can replace y_T by $y_{y,0}$ to make the analysis simpler. Therefore we have

$$\oint_{(1)} J_1 dy + \oint_{(2)} J_2 dx + \oint_{(3)} J_1 dy + \oint_{(4)} J_1 dy + J_2 dx = 0 \quad (\text{B.3})$$

Figure B.1: Methods of island of broken plug

$$\oint_{(1)} J_1 dy = \int_{h(x_b)}^{y_{y,0}} J_1(x_b, y) dy \quad (\text{B.4a})$$

$$\oint_{(2)} J_2 dx = \int_{x_b}^0 J_2(x, y_{y,0}) dx \quad (\text{B.4b})$$

$$\oint_{(3)} J_1 dy = \int_{h(0)}^{y_{y,0}} J_1(0, y) dy \quad (\text{B.4c})$$

$$\oint_{(4)} J_1 dy + J_2 dx = \int_0^{x_b} \left(h' J_1(x, h(x)) + J_2(x, h(x)) \right) dx \quad (\text{B.4d})$$

On $y = y_{y,0}$, we know that $\tau_{xy} = B$ and $v = 0$, so

$$\int_{x_b}^0 J_2(x, y_{y,0}) dx = \int_{x_b}^0 -B dx = Bx_b \quad (\text{B.5})$$

At $x = 0$, due to symmetry, $\tau_{xx}(0, y) = 0$ and $p = 0$ which gives

$$\int_{y_{y,0}}^{h(0)} J_1(0, y) dy = -\delta Re u_p^2 (h(0) - y_{y,0}) \quad (\text{B.6})$$

On the boundary of drop ($y = h(x)$), we know $\tau_{xy} = 0$ and $v = 0$, so

$$\int_0^{x_b} J_2(x, h(x)) dx = 0 \quad (\text{B.7})$$

B.2. Finding an admissible form of stress

and using integration by parts

$$\int_0^{x_b} h' J_1(x, h(x)) dx = h J_1(x, h(x)) \Big|_0^{x_b} - \int_0^{x_b} h(x) \frac{\partial J_1}{\partial x} dx = -\delta Re u_p^2 (h(x_b) - h(0)) \quad (\text{B.8})$$

Inside the plug region we know horizontal velocity in plug region is constant $u(x_b, y) = u_p$ and vertical velocity, up to $\mathcal{O}(\delta)$, is zero. Therefore,

$$\int_{h(x_b)}^{y_{y,0}} J_1(x_b, y) dy = -\delta Re u_p^2 (y_{y,0} - h(x_b)) - \int_{h(x_b)}^{y_{y,0}} p dy + \delta^2 \int_{h(x_b)}^{y_{y,0}} \tau_{xx}(x_b, y) dy \quad (\text{B.9})$$

substitution gives

$$\delta^2 \int_{h(x_b)}^{y_{y,0}} \tau_{xx}(x_b, y) dy = \int_{h(x_b)}^{y_{y,0}} p dy - B x_b \quad (\text{B.10})$$

We define $I := \delta \int_{h(x_b)}^{y_{y,0}} \tau_{xx}(x_b, y) dy$ and compare it with B as a measure to determine whether drop breaks the plug or not. To proceed we need to guess the variation of the normal stresses inside the channel. The pressure changes from $P_{pois} = -\frac{B}{y_{y,0}} x$ on the surface of plug region to 0 on the drop interface, assuming that there is no density difference. As an example, if we adopt a linear profile for pressure, and look for geometrical conditions which satisfy $I \leq B$, we find $\frac{3L}{2\delta} \leq 1$. This result is not intuitively consistent with our earlier results.

B.2 Finding an admissible form of stress

Inside the plug region the Bingham constitutive relations do not fully determine the stress behaviour. This means that any form of stress distribution, provided that it satisfies Stokes momentum equations, is admissible.

One idea that could lead to find an estimation of maximum size of drops is to find an admissible solution and play with the geometry of the drop until the total stress exceeds the yield stress (B).

We start with choosing a linear profile for τ_{xy} inside the plug region which satisfies the two boundary conditions that we have at the interface of drop and plug region ($\tau_{xy} = -B \frac{y-h(x)}{y_{y0}-h(x)}$). Using horizontal and vertical momentum equation, we can write

$$\frac{\partial \tau_{xy}}{\partial y} = -B \frac{1}{y_{y0} - h(x)} = -\frac{\partial \sigma_{xx}}{\partial x} = \frac{\partial p}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} \quad (\text{B.11})$$

B.2. Finding an admissible form of stress

and

$$\frac{\partial \tau_{xy}}{\partial x} = -B \frac{(y_{y0} - y)}{(y_{y0} - h(x))^2} h'(x) = \frac{\partial p}{\partial y} - \frac{\partial \tau_{yy}}{\partial y} = \frac{\partial p}{\partial y} + \frac{\partial \tau_{xx}}{\partial y} \quad (\text{B.12})$$

and then

$$2 \frac{\partial^2 \tau_{xx}}{\partial x \partial y} = B \frac{\partial}{\partial x} \left[\frac{y_{y0} - y}{(y_{y0} - h(x))^2} h'(x) \right] \quad (\text{B.13})$$

Knowing that at $x = 0$ $\tau_{xx} = 0$ and $h'(x) = 0$

$$\tau_{xx} = -\frac{B}{4} \left(\frac{y_{y0} - y}{y_{y0} - h(x)} \right)^2 h'(x) + f(x) \quad (\text{B.14})$$

From B.12 we can conclude that

$$p(x, y) = -\frac{B}{4} \left(\frac{y_{y0} - y}{y_{y0} - h(x)} \right)^2 h'(x) + g(x) \quad (\text{B.15})$$

which in turn results in

$$\sigma_{yy} = -p + \tau_{yy} = \frac{B}{2} \left(\frac{y_{y0} - y}{y_{y0} - h(x)} \right)^2 h'(x) - f(x) - g(x) \quad (\text{B.16})$$

we know that at $y = h(x)$, $\sigma_{yy} = 0$ and therefore $f(x) + g(x) = \frac{B}{2} h'(x)$. On the other hand, at $y = y_{y0}$, $\sigma_{yy} = -p_0(x) = -\frac{B}{y_{y0}} x$ and therefore $f(x) + g(x) = \frac{B}{y_{y0}} x$. It is not possible to satisfy both these two boundary conditions simultaneously.

An alternative method is to propose a different function for τ_{xy} in the form of $\tau_{xy} = -Bf(\eta)$ where $\eta = \frac{y-h(x)}{y_{y0}-h(x)}$ with the conditions of $f(0) = 0$ and $f(1) = 1$; e.g. $\tau_{xy} = -\frac{1}{2}B(\eta^2 + \eta)$. However, this method is also problematic as the integration required to find τ_{xx} makes the equations almost unsolvable analytically.