Multidimensional Lattice Walk Enumeration Through Coefficient Extraction Operators

by

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Abstract

In this thesis, we investigate the enumeration of lattice walk models, with or without interactions, in multiple dimensions, through the use of linear operators comprised of coefficient or term extractions. This is done with the goal of furthering our abilities to automate the derivation and solutions of the functional equations for the generating functions for the models. In particular, for a fairly large class of $d$-dimensional lattice walk models with interactions and arbitrary step sets, the generating function $Q$ satisfies the functional equation $(1 - t\Gamma S)Q = q$, where $\Gamma$ is an operator, and $S$ and $q$ are Laurent polynomials. We can automatically expand this equation to obtain an explicit functional equation satisfied by $Q$. For example, we derive an equation for $d$-dimensional lattice walks with interactions and small steps living in an orthant.

We also use this operator approach to unify and extend the algebraic and obstinate kernel methods through the use of a weighted orbit summation operator and substitutions. Other topics include: a partial classification of two-dimensional models with interactions and small steps on the quarter plane, explicit relations for $Q$ and partially-interacting versions of itself for some models, and an analysis of some of the more abstract properties of the operators involved.
Preface

My advisor, Andrew Rechnitzer, introduced me to the idea of studying lattice walk models with interaction, where he and his colleagues have studies some problem like this with varying success. He provided me with invaluable guidance and support along the way, and the occasional nudge to finish the thesis. I, Aleksandar Vlasev, did all the research, write-up and proofreading by myself, with some good discussions with Andrew about different issues. Along the way, I have used results and/or ideas from other mathematicians, but I have made a note of each instance, to the best of my knowledge. If a proof is missing, it is left as an exercise because it is either easy or trivial.
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Chapter 1

Introduction

The enumeration of lattice walks is a classical topic in combinatorics, but it has seen a resurgence in recent years, including the milestone of the full classification of the generating functions for models with small steps on the quarter plane. Work is now being done on the classification of three-dimensional models on an orthant, and other generalizations. We may consider models with larger step sizes set in different regions. A somewhat more difficult topic is that of self-avoiding walks, where we can study the properties of the generating functions but we do not have closed form expressions for all but the most basic models. It is natural to consider weaker models of walks that interact with themselves or with various points in the space.

While these topics seem related, the behaviour of different models and the tools used in their study can vary significantly. However, it seems that a lot of models can be related via the same framework. For example, there is some progress towards the automation of the analysis for a lot of models. In this thesis, we provide more evidence that such a unifying theory exists and we do so by recasting several methods of analysis into the language of operators. Furthermore, the change of perspective allows us to derive the functional equations for a broad class of models.

While lattice walk models have a rich mathematical theory, they also enjoy applications outside of math, where aspects of the physical world can be modelled via walk models. For example, proteins and polymers can be modelled as self-avoiding walks or walks with interactions, and their large-scale properties can be studied via parameters in the generating functions. And of course, the parts of the stock market can be modelled as a random walks, closely related to lattice walks.

In what follows, we briefly recall some of the more recent results in the history lattice walk enumeration.
1.1 Recent History

Some lattice walk models are easy to enumerate. For example, if the walks can go anywhere via any of their steps, the analysis does not even need generating functions — we just construct all possible sequences of steps from a set $S$ and then we find that the number of walks of length $n$ is just $|S|^n$. In more complicated models we make use of the generating function. To help us, we define a step polynomial $S$ keeping inventory of the step set $S$.

The next step up in difficulty is to consider walks on half-spaces, where walks are not allowed to venture past points with one coordinate equal to zero. These cases are not as simple, but various methods allow us to solve them. Check [3, 4, 5, 6, 4, 12] for work in one or more dimensions. It turns out that the generating functions we obtain are algebraic. Naturally, we are led to consider cases where the region is an intersection of half-spaces. Even for two dimensions, the problem has generated a rich theory, where some generating functions are not even D-finite, let alone algebraic.

The main object of study in these cases is a group $G$ of bi-rational transformations preserving the polynomial $K = 1 - ts$, called the kernel of the model. For a classification of two-dimensional walks with small steps in the quarter plane, see [8].

Through the classification done in [7, 8, 9, 10, 12], we can see that when $G$ is finite, we obtain a D-finite generating function. The main tool in the study of the finite group cases is the algebraic kernel method where we sum different functional equations produced by group actions by elements of $G$. This is done to cancel the majority of the terms in the functional equation and allow us to find out information about the model directly. We can also look beyond small steps. For example, see [11] for a treatment of models with finite groups and repeated steps.

There are 56 non-equivalent cases of quarter plane walks with an infinite group. For 51 of them, the generating function is not D-finite [13]. The remaining five cases required a fair bit more work [14, 15]. The main tool used to solve these five cases is the iterated kernel method where we use properties of the roots of the kernel $K$ to show the generating function is not D-finite.

With the classification completed, we are naturally led to investigate higher-dimensional models. Recently, the authors of [18] have classified lattice walk models with up to six steps by extending the algebraic kernel method to three dimensions. However, we do not need to restrict our analysis on the positive orthant. For example, see [16, 17] for analysis of models where the region of interest is a Weyl chamber.

For research on self-avoiding walks with various interactions, see [19, 20, 21, 22, 23, 24, 25]. The topic of self-avoiding walks does not lend itself as nicely to using generating functions like the ones considered so far, so we make simplifications. We drop the self-avoiding requirement but include interactions. For example, see [26, 27] for walks with interactions in an octant. See
[28, 29, 30] for walks with interactions on the plane, where the authors used the obstinate kernel method to find relationships between walks with interactions and walks without them. There are both examples of linear and nonlinear relations.

Also, there has been progress towards the automation of the classification of random walk models. For example, see [34] for some classification and [35] for analysis of some of the computational challenges involved in automation. There is some automation in the derivation of asymptotics for some models, for example [37, 36, 38, 39]. A lot of the other works mentioned so far have also discussed asymptotics. Since the enumeration of lattice walks is a combinatorial subject, there are connections with automation in combinatorics too. For examples, see [31, 32] for details on symmetric functions, and [33] for a holonomic approach to combinatorics.

We hope that there is a general theory unifying the various kernel methods for enumerating lattice walk models. By recasting some of the analysis in terms of operators, this thesis provides further evidence that such a theory may exist. Suppose that an algorithm exists where we can input an arbitrary lattice walk model and the algorithm can output data about the model: functional equation for the generating function, a solution of the equation or information about the function’s D-finiteness. Such an algorithm would not have difficulty with the particular model details and because of this, such an algorithm is no likely to exist.

There may exist lattice walk models which we cannot solve, but the idea of the existence of a general algorithm has been a strong motivator in our pursuit of generalizations. Our work succeeds in demonstrating that the first task is achievable — for a general class of lattice walk models with interactions we have the equation

\[(1 - t\Gamma S)Q = q,\]  

(1.1)

where \(\Gamma\) is an operator on the variables of \(Q\) and to find an explicit functional equation for \(Q\), we only need to determine how \(\Gamma\) acts on \(SQ\). For a lot of models, we may find a short explicit form of the operator and this allows us to find the equation quickly. Here \(S\) is the Laurent step polynomial keeping inventory of the steps in polynomial form, and \(q\) is a Laurent polynomial, serving as an initial condition. The generating function \(Q\) is fully determined by this equation, since we can invert it formally and obtain the series solution

\[Q = \sum_{k=0}^{\infty} (\Gamma S)^k(1) = 1 + t\Gamma(S) + t^2\Gamma(S\Gamma(S)) + \cdots.\]  

(1.2)
1.2 Roadmap

In this thesis we work towards a few different goals.

1. Perform a partial classification of walks with small steps on the quarter plane but with interactions with the axes. Almost half of all models are either trivial or easy to solve.

2. Extend the algebraic kernel method to interacting models with small steps in $d$ dimensions.

3. Extend the obstinate kernel method to interacting models with small steps in the quarter plane.

4. Automate the derivation of functional equations for higher-dimensional models with more general restrictions and interactions.

In Chapter 2, we study the enumeration of one-dimensional lattice walk models with small steps and interactions with the origin. We derive explicit generating functions for all models and the analysis includes some first examples of the methods that we use in later chapters. For example, we will see how operators make the derivations easier.

In Chapter 3, we use the powerful operator perspective to derive functional equations for the generating function enumerating higher-dimensional lattice walk models with small steps and interactions with the bounding hyperplanes. In the latter part of the chapter, we focus on the classification of two-dimensional models and some first solutions. This chapter contains most of the definitions relevant in the analysis.

In Chapter 4, we recast some of the algebraic kernel analysis in terms of operators and extend it to handle lattice walk models with interactions. We focus on two-dimensional models with finite groups and see that if the method extends, we have some tighter constraints. In the end, we are able to handle highly-symmetric models with group of order four, half of the models with group of order six, and all models with group of order eight.

In Chapter 5, we investigate the obstinate kernel method for the two-dimensional models with interactions and finite groups. Again, we recast the method in operator notation. For the interacting cases, the algebraic and the obstinate kernel methods are very similar. They can both be understood in the language of weighted orbital operators.

In Chapter 6, we investigate more general random walks models. That is, we find an operator functional equation for a large class of random walk models — those with general regional and marking constraints, general step sets and general starting conditions. The difficulty lies in finding a good representation for the operators involved.
1.3 Preliminaries

In this thesis, we work in the setting of formal power and Laurent series, where formal convergence suffices. Given a ring \( R \), we may construct several formal objects with a commutative indeterminate \( x \) with coefficients in \( R \).

- \( R[x] \): Ring of polynomials with monomials \( x^k \) for integers \( k \geq 0 \).
- \( R[x, 1/x] \): Ring of Laurent polynomials with monomials \( x^k \) for integers \( k \).
- \( R[[x]] \): Ring of formal power series with monomials \( x^k \) for integers \( k \geq 0 \).
- \( R((x)) \): Field of formal Laurent series with monomials \( x^k \) for integers \( k \).

We can also define their multivariate analogues \( R[x_1, \ldots, x_n] \) and \( R[x_1, x_1, \ldots, x_n, x_1] \), and their series counterparts \( R[[x_1, \ldots, x_n]] \) and \( R((x_1, \ldots, x_n)) \). For a thorough treatment of the subject, please see [1]. When we work with a combination of objects coming from different rings/fields over the indeterminate \( x \), we consider all objects as part of the most general ring/field at the time. If in doubt, we can always work in the field of formal Laurent series. Given a sequence \( (w_n) \), we may consider its generating function \( W(x) \), given by \( \sum_n w_n x^n \), which lives appropriately in one of the rings over \( x \) and/or \( \bar{x} \). The multivariate case for a sequence \( (w_{n_1, \ldots, n_m}) \) is defined analogously as

\[
W(x_1, \ldots, x_m) = \sum_{n_1 \ldots, n_m} w_{n_1, \ldots, n_m} x_1^{n_1} \cdots x_m^{n_m},
\] (1.3)

whenever this is a formal Laurent series.

A function \( W \) in one variable is a **polynomial** if it has only finitely many non-zero terms and it is **rational** if it can be expressed as a fraction of two polynomials. A more general class is that of **algebraic** functions. These are functions \( W \) that satisfy a polynomial equation

\[
a_0 + a_1 W + \cdots + a_n W^n = 0
\] (1.4)

with polynomial coefficients \( a_i \), where \( a_n \) is non-zero. Polynomials and rational functions are examples of algebraic functions. However, not all functions are algebraic and we need a more general class — that of **D-finite** functions. Those are functions that satisfy a finite order differential equation

\[
a_0 W + a_1 W' + \cdots + a_n W^{(n)} = 0
\] (1.5)
with polynomial coefficients \(a_i\), where \(a_n\) is non-zero. Clearly, algebraic functions are D-finite. The set of D-finite functions is closed under multiplication and taking linear combinations. However, not all functions are D-finite. The theory extends analogously to several variables.

We often need to extract particular coefficients from the formal polynomial/series when we solve enumeration problems via generating functions. We have the coefficient extraction operator \([x^n]\) that acts on functions \(W\) as

\[
[x^n]W(x) = w_n. \tag{1.6}
\]

The multivariate case is analogous with the caveat that, whenever the number of variables extracted is less than the number of variables present, we treat the remaining variables as parameters. For a more detailed discussion, please consult [2]. We construct more linear operators using coefficient extraction as building blocks. Here, a linear operator is a map between two vector spaces or modules that satisfies two properties — linear combinations of operators are operators as well and they can combine via composition to form new ones.

In the analysis of generating functions, we often work with functional equations. In this thesis, we discuss functional equations comprised of linear combinations of various coefficient extractions of a function \(Q\) with coefficients that may be rational or algebraic functions. These functional equations come from operator equations like \(LQ = 1\), where the operator is comprised of various coefficient extraction operators.

Finally, we use some basic group theory. We define groups \(G\) via generators \(g_1, \ldots, g_d\), and we construct orbits of functions under those group actions defined by the elements of \(G\). For example, we use the orbit stabilizer theorem in [Chapter 4].
Chapter 2

One-dimensional Lattice Walk Models

2.1 Derivation of the Functional Equation

Our analysis starts with the enumeration of one-dimensional lattice walk models to illustrate the methods and ideas we employ throughout the thesis. As part of the analysis, we use operators to simplify large portions of the work. We focus on models with interactions and small steps, that is, models where the step size is at most one and we count interactions with the origin. The step set $S \subseteq \{-1, 0, 1\}$ can be represented with the polynomial

$$S = S_{-1}x + S_0 + S_1x,$$

where $S_k$ is 1 or 0 if $k$ is in $S$ or not, respectively.

First, we look at a basic way of generating walks and then we find a way to encapsulate the process by an algebraic method. If we start with the empty walk at the origin, we can grow it by taking one of the steps from $S$ to obtain three walks. One of them ends on the negative axis, so we remove it. We can continue growing the remaining two walks, each time producing a few more walks and removing those that end on negative coordinates. At each step, the length of a walk increases by one and if the walk steps on the origin, we increment a counter tracking the interaction. We can generate all the lattice walks in this way.

We translate each of the steps in the process into a transformation on polynomials, and this enables us to use the powerful of generating functions to extract information about the properties of lattice walks. We track the length of walks via powers of $t$, their interactions via powers of $a$, and finally, their endpoints via powers of $x$. The empty walk has length zero and ends at the origin, without interacting. Therefore, its corresponding monomial is $a^0x^0t^0 = 1$. Each of the steps
changes the end coordinate and this corresponds to multiplying the monomial by one of \{\bar{x}, 1, x\}. The two surviving walks have monomials at and xt. It is a good exercise to check that the next iteration of this algorithm yields the monomials \(a^2t^2, axt^2, at^2, xt^2\) and \(x^2t^2\). If \(S\) is not the full set, we have fewer walks.

Instead of working with the monomials for individual walks, it is better to work with the whole set simultaneously. To make computations easier, we keep inventory of all the monomials inside a generating function \(Q(a, x; t)\), given by

\[
Q = Q(x, a; t) = \sum_{k=0}^{\infty} q_k(x, a) t^k,
\]

where \(q_k\) is the generating function for walks of length \(k\). Extending a walk corresponds to multiplying its monomial by a monomial for a step from \(S\). Let us see what happens when we multiply \(q_k\) by \(S\). When we expand this product, we obtain all combinations walks of length \(k\) and possible extensions via steps from \(S\). That is, we have extended all walks simultaneously.

As in the algorithm above, we need to eliminate the monomials for walks that step on negative coordinates and to multiply by \(a\) those for ones that step on the origin. Once we do that, we turn \(Sq_k\) into \(q_{k+1}\). The next important idea, and it is the kernel of this thesis, is to use some operator \(\Gamma\) for this task. Then we can simply write \(q_{k+1} = \Gamma(Sq_k)\) and sum over powers of \(t\). That is,

\[
\sum_{k=0}^{\infty} q_{k+1} t^{k+1} = t \sum_{k=0}^{\infty} \Gamma(Sq_k) t^k = t\Gamma \left( S \sum_{k=0}^{\infty} q_k t^k \right) .
\]

The expression on the left is \(Q - q_0 = Q - 1\) and the expression on the right is \(t\Gamma(SQ)\). This gives us a functional equation for \(Q\) in terms of \(S\) and the operator \(\Gamma\). That is, the equation is

\[
Q = 1 + t\Gamma SQ
\]

and we rewrite it as \((1 - t\Gamma S)Q = 1\) to highlight the operator form.

To find the operator \(\Gamma\), note that when we multiply \(q_k\) by \(S\), the farthest a walk can go on the negative axis is the coordinate \(x = -1\), so it is enough to remove terms from \(Sq_k\) with power of -1 of \(x\). For the interaction with the origin, we need to multiply the monomial without \(x\) by \(a\). All this can be achieved via the coefficient extraction operators \([\bar{x}]\) and \([x^0]\). We can write

\[
\Gamma = 1 - \bar{x}[\bar{x}] + (a - 1)[x^0],
\]
where $1$ is the identity operator. When we apply it on some Laurent polynomial $F$, the identity makes a copy of $F$, the $-\bar{\pi}[\pi]$ part removes the $\pi$ term from it, and the $(a-1)[x^0]$ part multiplies the $x^0$ coefficient by $a$. This completes our derivation of the functional equation. We make the extractions explicit later on, but for first, we analyze the operator $\Gamma$ in more detail.

Suppose that we decided to obtain the operator step-by-step by finding operators $E$ and $L$ that remove monomials with $x$ and mark monomials with $x^0$, respectively, and applying them sequentially. Then $\Gamma = LE$ or $\Gamma = EL$ by composition. The operators are given by

$$E = 1 - \bar{\pi}[\pi] \quad \text{and} \quad L = 1 + (a - 1)[x^0],$$

and if we let $b = a - 1$ and multiply them out, we find that

$$LE = (1 - \pi[\pi])(1 + b[x^0]) = 1 + b[x^0] - \pi[\pi] - \pi[x]b[x^0] = 1 + b[x^0] - \pi[\pi].$$

Therefore $\Gamma = LE$. Similarly, $\Gamma = EL$, so the operators commute. The last term in the calculation vanished because when we use $[x^0]$, the resulting term is constant with respect to $x$, and the extraction of the $\pi$ term yields nothing. Here is an example of how the operators work.

**Example 2.1.** Suppose that $S = x + \pi$. Starting with $Q_0 = 1$, we have

$$Q_1 = LES(1) = (1 + b[x^0] - \pi[\pi])(x + \pi) = x + \pi - x = x,$$

$$Q_2 = LES(Q_1) = (1 + b[x^0] - \pi[\pi])(x^2 + 1) = x^2 + 1 + b = a + x^2,$$

$$Q_3 = LES(Q_2) = (1 + b[x^0] - \pi[\pi])(a\pi + ax + x + x^3)$$
$$\quad = a\pi + ax + x + x^3 - a\pi$$
$$\quad = (1 + a)x + x^3.$$

Therefore,

$$Q = Q_0 + Q_1t + Q_2t^2 + Q_3t^3 + \cdots$$

$$= 1 + xt + (a + x^2)t^2 + (ax + x + x^3)t^3 + \cdots.$$  

Note that if we apply $\Gamma$ on $tSQ$, we obtain $Q$ with the first term missing.

The theory we have presented so far is consistent with present methods for finding the functional equation for $Q$ because once we compute the explicit extractions in $\Gamma SQ$, we obtain the classical explicit equations. The difference lies in the ease of use of operators versus the trickier inclusion-exclusion process that becomes unwieldy for models with interactions in higher dimensions.
Next, we perform the extractions to find the explicit equation. Let $Q_k = [x^k]Q$. Then,

\[
[x^0]SQ = [x^0]((S_{-1}[x] + S_0 + S_1[x])Q)
= (S_{-1}[x] + S_0[x^0] + S_1[\bar{x}])Q
= S_{-1}Q_1 + S_0Q_0,
\]

(2.12)

where last equality follows since $Q_{-1} = 0$. Here we used the property that as an operator $[x^i]x^j = [x^{i-j}]$. A similar computation shows that $[\bar{x}]SQ = S_{-1}Q_0$ and the explicit equation is given by

\[
1 = Q - t(SQ + b[S_0]SQ - \bar{x}[\bar{x}]SQ)
= Q - t(SQ + b(S_{-1}Q_1 + S_0Q_0) - \bar{x}S_{-1}Q_0).
\]

(2.13)

We can rewrite it in the following form:

\[
(1 - tS)Q - t(bS_0 - \bar{x}S_{-1})Q_0 - btS_{-1}Q_1 = 1.
\]

(2.14)

Here $Q = Q(x)$, $Q_0 = Q(0)$ and $Q_1 = Q'(0)$. It turns out that we can eliminate $Q_1$ by finding an equation for it via applying $[x^0]$ on both sides of the explicit equation. That is,

\[
1 = Q_1^0 - t(S_{-1}Q_1 + S_0Q_0) - tbS_0Q_0 - btS_{-1}Q_1
= (1 - atS_0)Q_0 - atS_{-1}Q_1,
\]

(2.15)

and substitute this solution for $tS_{-1}Q_1$ in the previous equation. After some simplification, we obtain the cleaner master functional equation

\[
(1 - tS)Q - (\bar{c} - t\bar{x}S_{-1})Q_0 = \bar{a}.
\]

(2.16)

The operator $L$ is actually invertible, so this allows us to take a shortcut and not have to perform the extraction and substitution. To undo the multiplication with $a$, we just need to multiply terms by $1/a$. Therefore, the inverse is $L^{-1} = 1 - c[x^0]$, where $c = 1 - \bar{a}$. This allows us to write the functional equation as

\[
(L^{-1} - tES)Q = \bar{a}
\]

(2.17)

and when we perform the extractions, we obtain Equation 2.16 directly, that is, the left-hand side is

\[
L^{-1}Q - tESQ = (Q - cQ_0) - t(SQ - \bar{x}S_{-1}Q_0) = (1 - tS)Q - (\bar{c} - t\bar{x}S_{-1})Q_0.
\]

(2.18)
It might seem like a similar amount of work, but in higher dimensions we need to solve $2^d$ equation by back-substitution to obtain shorter functional equation. For example, in \cite{30, 21} the authors had to perform the back-substitutions for a two-dimensional model. Using $L^{-1}$ provides a neat shortcut because we do not have to do any substitutions. We have the following result.

**Proposition 2.2.** Given a one-dimensional lattice walk model with small steps restricted to the non-negative axis, the generating function $Q$ satisfies the functional equations

\begin{align}
(L^{-1} - tES)Q &= \bar{\alpha}, \tag{2.19a} \\
(1 - tS)Q - (c - t\bar{x}S_{-1})Q_0 &= \bar{\alpha}. \tag{2.19b}
\end{align}

We call them the operator equation and the master functional equation for the model, respectively.

The operator notation/perspective is powerful. With it, we can derive functional equations for many lattice walk models in arbitrary dimensions with ease. Usually, derivations have employed inclusion-exclusion and careful book-keeping (Lemma 4 in \cite{8} p. 10, \cite{15} p. 58) to write master functional equations, or their precursors, in a direct manner. Unfortunately, that approach is difficult to generalize to arbitrary dimensions and more difficult restrictions/marking.

We can also do the coefficient extraction in a neater way. Instead of extracting the $x^k$ coefficient from the master functional equation, it is better to first go through the operator equation, since $[x^k]$ interacts with $L$ and $E$ in a nice way. We have

\begin{align}
[x^0]L^{-1} &= [x^0] - [x^0]\bar{c}[x^0] = [x^0] - c[x^0] = \bar{\alpha}[x^0], \tag{2.20a} \\
[x^0]E &= [x^0] - [x^0]\bar{\pi}[\bar{\pi}] = [x^0]. \tag{2.20b}
\end{align}

When we extract the $x^0$ term from the functional equation, we have

\begin{equation}
[x^0](L^{-1} - tES)Q = (\bar{\pi}[x^0] - t[x^0]S)Q = [x^0](\bar{\alpha} - tS)Q. \tag{2.21}
\end{equation}

Therefore, we have a functional equation for the first coefficient:

\begin{equation}
[x^0](1 - atS)Q = 1. \tag{2.22}
\end{equation}

We can perform the extraction immediately and it simplifies to

\begin{equation}
(1 - atS_0)Q_0 - atS_{-1}Q_1 = 1. \tag{2.23}
\end{equation}
Extracting higher coefficients is not difficult either. If \( k \geq 1 \), we have

\[
[x^k]L^{-1} = [x^k] - [x^k][x^0] = [x^k], \quad (2.24a)
\]
\[
[x^k]E = [x^k] - [x^k][\bar{\pi}] = [x^k]. \quad (2.24b)
\]

When we apply the extractions on both sides of the functional equation, we obtain

\[
[x^k](1 - tS)Q = 0, \quad (2.25)
\]

and after we expand and simplify the expression, we have

\[
(1 - S_0t)Q_k - t(S_{-1}Q_{k+1} + S_1Q_{k-1}) = 0. \quad (2.26)
\]

**Proposition 2.3.** Let \( S = S_{-1}\bar{x} + S_0 + S_1x \) be the step polynomial for a lattice walk model on the non-negative axis. Then the function \( Q_0 \) and \( Q_k \) for \( k \geq 1 \) satisfy the equations

\[
(1 - atS_0)Q_0 - atS_{-1}Q_1 = 1, \quad (2.27a)
\]
\[
(1 - S_0t)Q_k - t(S_{-1}Q_{k+1} + S_1Q_{k-1}) = 0. \quad (2.27b)
\]

### 2.2 Solving Functional Equations For Various Step Sets

If some steps in the step set of a model cannot be taken by walks, then we can remove them and obtain a simpler model with the same lattice walks and generating function. For one-dimensional models with small steps we can identify three major cases: \( S = \delta \) or \( S = \delta + x \) or \( S = \delta + \bar{x} + x \).

#### 2.2.1 The \( S = \delta \) or \( S = \delta + x \) Case

The condition allows us to find \( Q_0 \) directly from the \([x^0]\) equation. That is,

\[
Q_0 = \frac{1}{1 - a\delta t}, \quad (2.28)
\]

and we can substitute it into the master functional equation to derive the following result.

**Proposition 2.4.** Let \( S = \delta \) or \( S = \delta + x \). Then the generating function is

\[
Q(x, a; t) = \frac{\bar{a}}{1 - tS} \left( 1 + \frac{b}{1 - a\delta t} \right) = \frac{1 - \delta t}{(1 - a\delta t)(1 - tS)}. \quad (2.29)
\]
Something interesting starts to happen once we consider the relationship between $Q(x, a; t)$ and $\hat{Q} = Q(x, 1; t)$, its version without interactions. We compute that

$$\hat{Q} = \frac{1}{(1 - tS)}$$  \hspace{1cm} (2.30)

and we have the relation

$$(1 - a\delta t)Q - (1 - \delta t)\hat{Q} = 0,$$  \hspace{1cm} (2.31)

that we can rewrite as $B((1 - a\delta t)Q) = 0$ with the operator $B = 1 - [b^0]$, and similarly for $Q_0$. We find a similar relationships exist for higher-dimensional lattice walk models too.

**Proposition 2.5.** Let $B = 1 - [b^0]$ and let $Q$ be the generating function for a lattice walk model with $S = \delta$ or $S = \delta + x$. Then $Q$ and $Q_0$ both satisfy the equation $B((1 - a\delta t)F) = 0$.

**2.2.2 The $S = \delta + \vec{x} + x$ Case**

To derive the generating function in this case, we use the obstinate kernel method as detailed in Section 4.2 of [30, p. 12] and Section 4.1 of [21, p. 13] for two-dimensional models. In this method, we substitute an $x$ value in the functional equation that eliminates the term $(1 - tS)Q$. It is sufficient to find roots of the kernel $K = 1 - tS$. The equation $K = 0$ is quadratic in $x$ and has the roots

$$\sigma = \frac{1 - \delta t - \sqrt{(1 - \delta t)^2 - 4t^2}}{2t} \quad \text{and} \quad \sigma^* = \frac{1 - \delta t + \sqrt{(1 - \delta t)^2 - 4t^2}}{2t},$$  \hspace{1cm} (2.32)

which multiply to 1, so $\sigma = \sigma^*$. Assuming that the substitution eliminates $KQ$, we find that

$$-(c - t\sigma)Q_0 = \bar{a},$$  \hspace{1cm} (2.33)

and using this relation, we can solve for $Q$ explicitly. We compute

$$Q = \frac{\bar{a}}{1 - tS} \left( 1 - \frac{t(x - c)}{t\sigma - c} \right) = \frac{t}{1 - tS} \left( \frac{x - \sigma}{b - a\sigma t} \right).$$  \hspace{1cm} (2.34)

**Proposition 2.6.** Let $S = \delta + \vec{x} + x$. Then the generating functions for this model are

$$Q_0 = \frac{1}{a\sigma t - b} \quad \text{and} \quad Q = \frac{t(\sigma - \vec{x})}{(1 - tS)(a\sigma t - b)},$$  \hspace{1cm} (2.35)

where $\sigma = (1 - \delta t + \sqrt{(1 - \delta t)^2 - 4t^2})/(2t)$.

As before, we find a relation between functions for models with interactions and ones without.
Proposition 2.7. Let \( S = \delta + \bar{x} + x \). Then \( Q \) and \( Q_0 \) satisfy the equation \( B((a\bar{x}t - b)F) \).

In this case we can use the algebraic kernel method to obtain a linear relation for \( Q_0 \). This method is used extensively in the quarter plane model classification [8]. Here we find a polynomial \( f(a, t) \), for which \( B(fQ_0) = c \), instead of zero. Note that \( K \) is invariant under the map \(*: x \mapsto \bar{x}\) and we can write a second functional equation to complement the first. That is, we have the system

\[
(1 - tS)Q - (c - t\bar{x})Q_0 = \bar{a}, \\
(1 - tS)Q^a - (c - tx)Q_0 = \bar{a}.
\]  

(2.36) (2.37)

To eliminate the \( Q_0 \) term, we multiply the first equation by \( c - tx \), the second by \( c - t\bar{x} \), and subtract one equation from the other. We divide everything by \( 1 - tS \) and the equation that remains is

\[
(c - tx)Q(x) - (c - t\bar{x})Q(\bar{x}) = \frac{\bar{a}(\bar{x} - x)}{1 - tS} = \bar{a}F,
\]  

(2.38)

where \( F \) does not depend on \( a \). We apply \([x]\) on both sides to find a relation between \( Q_1 \) and \( Q_0 \):

\[
cQ_1 - tQ_0 = \bar{a}F_1.
\]  

(2.39)

If we set \( a = 1 \) here, we find that \( F_1 = -t\hat{Q}_0 \) and we can substitute that into the previous equation. After we multiply both sides by \( a \), we find that

\[
bQ_1 = t(aQ_0 - \hat{Q}_0).
\]  

(2.40)

Earlier we had the equation

\[
(1 - a\delta t)Q_0 = 1 + atQ_1,
\]  

(2.41)

which we can solve for \( Q_1 \) and insert the solution into the equation above. That is,

\[
b((1 - a\delta t)Q_0 - 1) = t^2a(aQ_0 - \hat{Q}_0).
\]  

(2.42)

Collecting coefficients and simplifying the result lead to the equation

\[
(c(1 - a\delta t) - at^2)Q_0 - (-t^2)\hat{Q}_0 = c,
\]  

(2.43)

and therefore, \( f(a, t) = c(1 - a\delta t) - at^2 \).

Proposition 2.8. Let \( S = \delta + \bar{x} + x \). Then \( B((c(1 - a\delta t) - at^2)Q_0) = c \).

We obtained a slightly different relationships upon applying the operator \( B \).
2.2.3 Conclusion

For one-dimensional models with small steps restricted to the non-negative axis, we were able to find explicit solutions for $Q_0$ and $Q$ and various relationships for $Q$ and $\hat{Q}$ with the operator $B$. In higher dimensions, the analysis is more difficult, but sometimes we see the same situation. In more difficult examples, we only find a relationship between the function for walks returning to the origin and its various partially-marked versions. Here is a table showing the results so far.

<table>
<thead>
<tr>
<th>$S$</th>
<th>$Q_0$</th>
<th>$Q$</th>
<th>$F(a, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>$\frac{1}{1 - a\delta t}$</td>
<td>$\frac{1}{1 - a\delta t}$</td>
<td>$1 - a\delta t$</td>
</tr>
<tr>
<td>$\delta + x$</td>
<td>$\frac{1}{1 - a\delta t}$</td>
<td>$\frac{1}{1 - \delta t}$</td>
<td>$1 - a\delta t$</td>
</tr>
<tr>
<td>$\delta + \sigma + x$</td>
<td>$\frac{1}{a\sigma t - b}$</td>
<td>$\frac{1}{t(\sigma - x)}$</td>
<td>$a\sigma t - b$</td>
</tr>
</tbody>
</table>

Table 2.1: Generating functions for walks with small steps on the non-negative half-line, where $B(FQ_0) = B(FQ) = 0$. 
Chapter 3

Higher-dimensional Lattice Walk Models

We begin the exploration of higher-dimensional models by making some useful definitions. Here are the main ingredients for a model set in \( \mathbb{Z}^d \).

- A fixed finite set \( S \) of steps for travel between points in the space. It can involve arbitrarily-many finitely-sized steps with multiplicities at least one. Its corresponding Laurent polynomial is denoted by \( S \).

- A region \( D \subseteq \mathbb{Z}^d \), where the walks are free to grow but not venture outside of the boundaries. This can be the whole space, an orthant, a semi-infinite strip, finite boxes, and so on.

- A subregion \( D_0 \) of \( D \), where interactions take place. To each point \( p \), there is a corresponding label \( a_p \) whose power we can increment upon visits to \( p \). Common subregions \( D_0 \) include the origin, the axes in two dimensions, various higher-dimensional hyperplanes, a cone, an infinite strip, the sides of a box and so on.

- A set \( Q \) of initial walks that we can evolve. It has a corresponding generating polynomial \( q \).

We grow lattice walks from an initial condition \( Q \), with steps \( S \), within \( D \) in \( \mathbb{Z}^d \), while counting interactions within \( D_0 \). The four sets determine the model completely.

Definition 3.1. A lattice walk model \( M \) is characterized by the following four parts: \( S, D, D_0, Q \).

Definition 3.2. The classical \( d \)-dimensional lattice walk model is one with small steps in the positive orthant in \( \mathbb{Z}^d \). The classical interacting model is one with interactions with the hyperplanes \( x_k = 0 \) bounding the non-negative orthant, with labels \( a_k \).
Example 3.3. The one-dimensional models from the previous chapter are examples of classical interacting models. Here $\mathcal{D}$ is the set of non-negative integers, $\mathcal{D}_0$ is the point at the origin with label $a$, and $Q$ is the empty walk. Finally, $\mathcal{S}$ is one of the subsets of $\{-1, 0, 1\}$.

Example 3.4. Consider classical models with interactions in two dimensions. Here $\mathcal{D} = \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1, x_2 \geq 0\}$ and $\mathcal{D}_0$ made up of the non-negative axes. Note that the origin is marked with both labels $a_1$ and $a_2$. As in the previous example, $Q$ is the empty walk and $\mathcal{S}$ is a set of small steps.

In order to enumerate lattice walks and their properties, we make use a generating function that tracks quantities like length and ending position. Since the sets $\mathcal{S}$ and $\mathcal{Q}$ are finite, the number of walks of any given length is finite as well. Therefore, the coefficients of $Q$ are Laurent polynomials.

Definition 3.5. The generating function $Q$ associated with a lattice walk model counts walks’ lengths with $t$, ending locations with $x_i$‘s, and interactions with $\mathcal{D}_0$ with some variables $a$. We write

$$Q = Q(x_1, \ldots, x_d, a; t) = \sum_{k=0}^{\infty} q_k(x_1, \ldots, x_d, a)t^k,$$  \hspace{1cm} (3.1)

where the coefficients $q_k$ are Laurent polynomials in the $x$ and $a$ variables.

Example 3.6. We examine the simplest kind of model — one where $\mathcal{D}$ is the whole space, there are no interactions, and $Q$ consists of the empty walk. Since the lattice walks can pick any step from the step set $\mathcal{S}$, the walks of length one are enumerated via $t\mathcal{S}$, the walks of length two, via $t^2\mathcal{S}^2$, and so on, where $\mathcal{S} = \mathcal{S}(x_1, \ldots, x_d)$. Then the full generating function is

$$Q(x_1, \ldots, x_d; t) = 1 + St + S^2t^2 + \cdots = \frac{1}{1 - St}. \hspace{1cm} (3.2)$$

Example 3.7. Let $\mathcal{S} = \{\text{NE}, \text{NW}, \text{SE}, \text{SW}\}$, that is, $\mathcal{S} = (x_1 + x_1)(x_2 + x_2)$. The only walk of length one has monomial $x_1x_2t$. When we grow it, the walk can take any of the four steps, so

$$Q = 1 + x_1x_2t + (a_1 + x_1)(a_2 + x_2)t^2 + \cdots. \hspace{1cm} (3.3)$$

We can extract coefficients too. For example, the walks ending on the $x_1$ axis are given by setting $x_2 = 0$ or extracting the $x_2^0$ coefficient. That is,

$$[x_2^0]Q = 1 + a_2(a_1 + x_1)t^2 + \cdots \hspace{1cm} (3.4)$$

and the walks returning to the origin are given by

$$[x_1^0x_2^0]Q = 1 + a_1a_2t^2 + \cdots. \hspace{1cm} (3.5)$$
It is difficult to do the analysis of classical higher-dimensional models without extra notation. In fact, the use of better notation paved the way to finding the operators and the functional equations. We usually see multi-index notation in the multivariate setting, but this is not sufficient for working with restricted regions, where some variables are missing. We have the following notation.

**Definition 3.8.** Let \( 0 := (0, \ldots, 0) \) and \( \pm 1 := (\pm 1, \ldots, \pm 1) \), where we have as many terms in these sequences as needed in the specific context. Also, let \([n] = \{1, \ldots, n\}\).

**Definition 3.9.** Let \( x_1, \ldots, x_d \) be indeterminates and let \( I = (i_1, \ldots, i_m) \) and \( J = (j_1, \ldots, j_m) \) be sequences of positive integers. Define the monomial \( x^I_J \) as

\[
x^I_J = x^{j_1}_{i_1} \cdots x^{j_m}_{i_m}, \tag{3.6}
\]

with special cases \( x^0_I = 1 \) and \( x^1_I = x_I \) and \( x^{-1} = 1/x_I = \bar{x}_I \). The **coefficient extraction** and **term extraction** operators are defined as

\[
[x^I_J]F = [x^{j_1}_{i_1} \cdots x^{j_m}_{i_m}]F = F^I_J \quad \text{and} \quad [[x^I_J]] = x^I_J[x^I_J], \tag{3.7}
\]

with special cases \( F^I_J = F \) if \( I \) is empty, and \( F^I_J = F_J \) if \( I \) is the full set of variables.

**Example 3.10.** In the previous example we had \( [x^0_2]Q = Q^0_2 \) and \( [x^0_1 x^0_2]Q = Q^0_1 Q^0_2 \). Furthermore, we have \( S^{\pm 1, \pm 1} = 1 \) and

\[
S^{-1} = (\bar{x}_1)(x_1 + \bar{x}_1)(x_2 + \bar{x}_2) = x_2 + \bar{x}_2. \tag{3.8}
\]

Because there are different variables involved, there may be some ambiguity in the result, depending on the order of extraction. We always perform coefficient extractions with respect to the \( x \) or \( a \) variables by first expanding any series in \( t \), and then applying the extraction on the coefficients.

**Definition 3.11.** Suppose that \( \Gamma \) is an operator involving \( x \) and \( a \) variables but not \( t \). We apply \( \Gamma \) on power series \( F = \sum_{k=0}^\infty f_k t^k \) in the following way:

\[
\Gamma F = \sum_{k=0}^\infty \Gamma(f_k) t^k. \tag{3.9}
\]

**Definition 3.12.** Given a label \( a \), we let \( b = a - 1 \) and \( c = b/a = 1 - \bar{a} \). Also, \( a_I \) extends like \( x_I \).

**Example 3.13.** Here we derive the two-dimensional functional equation for classical interacting models, without the use of inclusion-exclusion or using case-by-case analysis to find a shorter equa-
tion. As in the one-dimensional case, we can grow the lattice walks by multiplying $q_k$ by $S$. Then we use an operator $\Gamma$ that can fix the terms in $S q_k$, so we obtain the equation $(1 - t \Gamma S) Q = 1$.

To find $\Gamma$, we make use of the following observation. Since $E = 1 - [[x]]$ is idempotent, that is $E^2 = E$, this means that once we remove certain terms via $E$, we do not remove them again with repeated application. Therefore, we can remove terms with negative $x_1$ coordinate first, and ones with negative $x_2$ coordinate second, and the inclusion-exclusion is encoded inside the operators. We find that $E = E_1 E_2$, where $E_k = 1 - [[x_k]]$. Multiplying these out, we find that

$$E = 1 - [[x_1]] - [[x_2]] + [[x_1 x_2]].$$

(3.10)

Analogously, the marking operator is $L = L_1 L_2$, where $L_k = 1 + b_k [x_k^0]$, and finally, we have the equation $(L^{-1} - t E S) Q = \overline{a}_1 \overline{a}_2$ since $L$ is invertible. If we expand the expressions, we find that

$$Q = 1 + t SQ - t \tau_1 S_1^{-1} Q_1^0 - t \tau_2 S_2^{-1} Q_2^0 + t \tau_1 \tau_2 S_1^{-1} S_2^{-1} Q_{1,2}^{0,0}.$$

(3.11)

Contrast this notation to the one in [8, p. 10]. For $d$-dimensional models, one can write down the functional equation as a sum over subsets of $\{1, \ldots, d\}$, as in the the non-interacting case in Equation 5.1 in [15, p. 58], but this becomes difficult if we work with interactions. To add interactions, we have to modify the non-interacting functional equation. This has been done for two-dimensional models in Section 3 of [30, p. 7] and Section 3.2 of [21, p. 11], where the authors add terms like $(a_1 - 1) T$, where $T$ is an expression involving extractions of $Q$ and $S$. Instead of doing that, we use the operator notation to handle the problem in a more efficient manner.

### 3.1 Using Operators to Derive the Functional Equation

As we have seen in the previous example, it is not difficult to generalize the derivation of the functional equations if we use operators. The derivation of general case for $d$ dimensions is analogous to the one in the example, so we omit most of it. In $d$ dimensions, the interactions take place on the hyperplanes where $x_k = 0$, so we use the labels $a_k$ to mark those interactions. The operators here are $L = L_1 \cdots L_d$ and $E = E_1 \cdots E_d$, where order does not matter. We have $L^{-1}(1) = \tau_{[d]}$, since each operator $L_i^{-1}$ marks a constant term with $\overline{a}_i$. Or in operator notation,

$$L^{-1}(1) = (1 - c_1 [x_1^0]) \cdots (1 - c_d [x_d^0])(1) = \sum_{I \subseteq [d]} (-1)^{|I|} c_I = \overline{a}_{[d]}.$$  

(3.12)

The higher-dimensional functional equation for the classical interacting model is given as follows.
Theorem 3.14. The classical interacting lattice walk models in $d$ dimensions have generating functions $Q$ satisfying the functional equations

\[ (1 - tLES)Q = 1, \quad (3.13) \]
\[ (L^{-1} - tES)Q = \bar{a}[d], \quad (3.14) \]

where $L$, $E$ and $L^{-1}$ are of the form $\Gamma = \Gamma_1 \cdots \Gamma_d$.

The second functional equation has a lot fewer terms. To see this, we examine at the operator

\[ L_k E_k = 1 + b_k[x^0_k] - \bar{x}_k[\bar{x}_k]. \quad (3.15) \]

Once we apply it in the functional equation, the number of terms increases by a factor of three, so in the end we have $3^d$ different $Q$ terms. On the other hand, if we apply $L^{-1}$ on both sides of the equation, we only introduce $2^d$ new terms but they are the same as the ones generated by $E$. The resulting equation only has $2^d$ terms $Q$, one for each subset $I \subseteq [d]$. That is,

\[ L^{-1}Q = (1 - c_1[x^0_1]) \cdots (1 - c_d[x^0_d])Q = \sum_{I \subseteq [d]} (-1)^{|I|} c_I Q^0_I, \quad (3.16a) \]
\[ E\bar{S}Q = \sum_{I \subseteq [d]} (-1)^{|I|} \bar{x}_I[\bar{x}_I](\bar{S}Q) = \sum_{I \subseteq [d]} (-1)^{|I|} \bar{x}_I S^{-1}_I Q^0_I, \quad (3.16b) \]

where $[\bar{x}]S = S^{-1}_I Q^0_I$, since $Q^P_I = 0$ for any set $P$ with more than one negative term. Finally,

\[ \sum_{I \subseteq [d]} (-1)^{|I|} (c_I - t \bar{x}_I S^{-1}_I) Q^0_I = \bar{a}[d]. \quad (3.17) \]

Finally, we can write the master equation in $d$ dimensions.

Theorem 3.15. Let $K_I = c_I - t \bar{x}_I S^{-1}_I$ for subsets $I$ of $\{1, \ldots, d\}$. Then the generating function for a classical interacting $d$-dimensional lattice walk model satisfies the equation

\[ \sum_{I \subseteq [d]} (-1)^{|I|} K_I Q^0_I = \bar{a}[d]. \quad (3.18) \]

Note that $K_{\emptyset} = K$ here. Also, Equation 5.1 of [15, p. 58] is the non-interacting version of our equation, and in our notation it can be written as

\[ KQ - \sum_{\emptyset \neq I \subseteq [d]} (-1)^{|I|} \bar{x}_I S^{-1}_I Q^0_I = 1. \quad (3.19) \]
3.2 Coefficient Extractions

In this section, we show how the operators $L$ and $E$ interact nicely with basic coefficient extraction to produce more equations. It is easier to use the operator form of the functional equation in order to extract coefficients, than it is to use the explicit form. This makes for easier manual derivation, since we need to obtain functional equations for $Q_0^I$ for $I \subseteq [d]$ like in the one-dimensional case.

3.2.1 Using Operator Notation For Fast Coefficient Extraction

We can obtain further functional equations with less variables by extracting the $x_0^I$ coefficient from the master equation. The operator $[x_1^k]$ commutes with operators $M_j = L_j^{-1}$ and $E_j$ when $j \neq i$, so we only need to consider the action on $M_i$ and $E_i$. Similarly to Equation 2.20 and Equation 2.24, in the two-dimensional setting we have

$$\left[x_1^k\right]M_i = \overline{a}_I^{(k,0)} \left[x_1^k\right] \quad \text{and} \quad \left[x_1^k\right]E_i = \left[x_1^k\right]. \quad (3.20)$$

Therefore, the operator $[x_1^k]$ eliminates both $M_i$ and $E_i$ modulo $\overline{a}_I$ for $x_0^I$ coefficient from the functional equation for $I \subseteq [d]$, we have

$$\left[x_1^0\right]M_i = \overline{a}_I \left[x_1^0\right] \text{ and } \left[x_1^0\right]E_i = \left[x_1^0\right]. \quad (3.21)$$

If we use the more general operator $[x_1^0 x_P^J]$, where $P$ is a nonempty sequence of positive integers and $I \cup J = K$ with no overlaps, through the same process as before we find the equation

$$\left[x_1^0 x_P^J\right] \left(M_{[d]} - a t E_{[d]} S\right)Q = 0 \quad (3.23)$$

We perform the extractions explicitly in order to obtain equations like 3.16. First,

$$\left[x_1^0\right]SQ = S_I^0 Q_I^0 + R_I \quad (3.24)$$

where $R_I$ is a collection of $S_I^{-P} Q_I^P$ terms that involve sets $P$ with some non-zero entries. When we substitute this term into Equation 3.22, we find that
\[ \bar{a}_{[d]\setminus I} = M_{[d]\setminus I}[x^I_+]Q - a_I t E_{[d]\setminus I}[x^I_+](S Q) \]
\[ = M_{[d]\setminus I}Q^0_I - a_I t E_{[d]\setminus I}(S^0_I Q^0_I + R_I) \]
\[ = (M_{[d]\setminus I} - a_I t E_{[d]\setminus I}S^0_I)Q^0_I - a_I t E_{[d]\setminus I}R_I. \]  

(3.25)

We can rewrite this as

\[ (M_{[d]\setminus I} - a_I t E_{[d]\setminus I}S^0_I)Q^0_I = a_I(\bar{a}_{[d]} + tR'_I), \]

(3.26)

where \( R'_I = E_{[d]\setminus I}R_I \). We compute the terms

\[ M_{[d]\setminus I}Q^0_I = \sum_{J \subseteq [d]\setminus I} (-1)^{|J|} c_J Q^0_{I,J}, \]

(3.27a)

\[ a_I t E_{[d]\setminus I}S^0_I Q^0_I = \sum_{J \subseteq [d]\setminus I} (-1)^{|J|} a_I t \pi_J S^0_{I,J} Q^0_{I,J}, \]

(3.27b)

and rewrite the functional equation as

\[ \sum_{J \subseteq [d]\setminus I} (-1)^{|J|}(c_J - a_I t \pi_J S^0_{I,J} Q^0_{I,J}) Q^0_{I,J} = a_I(\bar{a}_{[d]} + tR'_I). \]

(3.28)

**Definition 3.16.** For positive integer sets \( I \) and \( J \) that do not share any elements, we have

\[ K_{J,I} = c_J - a_I t \pi_J S^0_{I,J}. \]

(3.29)

If \( I = \emptyset \), then \( K_{J,I} = K_J \). Equivalently, \( K_{J,I} = [x^0_J] K_{J,I}. \)

**Lemma 3.17.** Let \( I \) be a subset of \( \{1, \ldots, d\} \). Then the generating function \( Q^0_I \) for the classical interacting \( d \)-dimensional lattice walk model satisfies the functional equation

\[ \sum_{J \subseteq [d]\setminus I} (-1)^{|J|} K_{J,I} Q^0_{I,J} = \bar{a}_{[d]\setminus I} + a_I t R'_I, \]

(3.30)

where the \( R'_I \) term only involves higher-order coefficient extractions from \( Q \).

This is a clean way to write down all the equations for \( Q^0_I \) in a unifying way. To find \( R'_I \), we expand \( R'_I = E_{[d]\setminus I} \), where \( R_I \) is a sum of terms \( S^P_I Q^P_I \) for various sets \( P \) of size \( |I| \) with non-negative entries, where not all of them are zero. Also, note that \( Q^P_I = 0 \) whenever \( P \) has negative entries because \( Q^P_I \) enumerates valid walks only. Let \( sq = S^P_I Q^P_I \). Then
and thus,

\[ E_Ksq = \sum_{J \subseteq K} (-1)^{|J|} \bar{x}_{J} s_{J}^{-1} q_{J}^{0}. \]  

(3.32)

This allows us to derive the following explicit form of \( R'_{I}. \)

**Proposition 3.18.** The functions \( R'_{I} \) are given by

\[
R'_{I} = \sum_{J \subseteq [d] \setminus I} (-1)^{|J|} \sum_{P} S_{I,J}^{-P,-I} Q_{I,J}^{P,0},
\]

where the sum is over sets \( P \) with non-negative entries where some of them are positive.

Of course, when \( d = 1 \) or \( d = 2 \), we do not need the full generality of the formula, but for higher dimensions, it would be tedious to write the equations by hand without it.

### 3.2.2 Kernel Forms

We can write out the functional equations in a way that highlights the role of the kernel \( K: \)

\[ (1 - tLES)Q = LEKQ + (1 - LE)Q = LEKQ + (1 - L)Q, \]  

(3.34)

where \( EQ = Q \) since there are no walks outside of \( D \). Setting all the \( a_k \)'s to 1 in the last expression is equivalent to removing the marking operator, that is, we have the equation, \( E(KQ) = 1 \), for a non-interacting model, and its form suggests that we use the following substitution

\[ Q = \frac{W}{x_1 \cdots x_d K}. \]  

(3.35)

We multiply both sides of the equation by \( x_1 \cdots x_d \) and obtain

\[ x_1 \cdots x_d = x_1 \cdots x_d E(KQ) = x_1 \cdots x_d E(\bar{x}_1 \cdots \bar{x}_d W) = E'W, \]  

(3.36)

where \( E' = E'_{1} \cdots E'_{d} \) and

\[ E'_{k} = x_k E_{k} \bar{x}_k = x_k(1 - \bar{x}_k[\bar{x}_k]) \bar{x}_k = 1 - [x_k^{0}]. \]  

(3.37)
Proposition 3.19. Let $Q$ be the generating functions for a classical model in the positive orthant. Then the function $W = x_1 \cdots x_d K Q$ satisfies the equation

$$E'W = \sum_{I \subseteq [d]} (-1)^{|I|} W^0_I = x_1 \cdots x_d.$$  \hspace{1cm} (3.38)

and

$$Q = \frac{1}{x_1 \cdots x_d K} \left( x_1 \cdots x_d - \sum_{\emptyset \neq I \subseteq [d]} (-1)^{|I|} W^0_I \right).$$  \hspace{1cm} (3.39)

Example 3.20. In two dimensions, the equation the generating function is given by

$$Q(x_1, x_2) = \frac{x_1 x_2 + W^0_1(x_2) + W^0_2(x_1) - W^{0,0}_{1,2}}{x_1 x_2 (1 - t S)}.$$  \hspace{1cm} (3.40)

Example 3.21. Consider the one-dimensional model on the non-negative line with $S = \bar{x} + x$. Here the solution for the interacting version of the model is given by

$$Q = \frac{t(\bar{x} - x)}{(1 - t S)(a \bar{x} t - b)},$$  \hspace{1cm} (3.41)

so when we set $a = 1$ and simplify the expression, we find that the generating function for the non-interacting model is given by

$$\hat{Q} = \frac{x - \bar{x}}{x (1 - t S)},$$  \hspace{1cm} (3.42)

as predicted by the analysis of this section. Here $W_0 = \sigma$.

Interestingly enough, applying the derivatives $\partial_1$ and $\partial_2$ on both sides of the equation $E'W = x_1 x_2$ leads to the PDE $\partial_1 \partial_2 W = 1$ and a similar relationship exists in higher dimensions.

Proposition 3.22. The generating function for a classical $d$-dimensional model satisfies the PDE

$$\partial_1 \cdots \partial_d (x_1 \cdots x_d K Q) = 1.$$  \hspace{1cm} (3.43)

The results in this section are not difficult to generalize. We obtain a similar result if we use larger steps or include interactions. We just need to pre-multiply both sides of the equation by a higher order monomial $x^P_{[d]}$ and apply enough partial derivatives $\partial^N_I$. 

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3.3 First Solutions

3.3.1 Formal Neumann Series and Hadamard Products

Formally, we can expand the operator \((1 - t\Gamma S)\) in Neumann series and express the generating functions as

\[
Q = \sum_{k=0}^{\infty} t^k (\Gamma S)^k(1),
\]

(3.44)

where \((\Gamma S)^k(1)\) indicates repeated application of \(\Gamma S\) on 1. This gives us an explicit expression for \(Q\) that we can manipulate, but it is not a complete solution because we still need to compute the terms \((\Gamma S)^k\). However, this form allows us to derive a curious result that falls out of the operator notation — how we can build generating functions for some higher-dimensional models by Hadamard products of lower-dimensional ones.

In Section 5 of [18, p. 14], the authors find solutions for non-interacting models with step polynomials of the form

\[
S(x_1, x_2) = S_0(x_1) + S_1(x_1)S_2(x_2).
\]

(3.45)

Here we find a similar result for the simpler case \(S = S_1(x_1)S_2(x_2)\), but with interactions, and its generalization to higher dimensions.

**Example 3.23.** When \(S = S_1(x_1)S_2(x_2)\) the operator \(\Gamma S\) factors as

\[
\Gamma S = \Gamma_1\Gamma_2S_1S_2 = (\Gamma_1S_1)(\Gamma_2S_2),
\]

(3.46)

since \(\Gamma_2\) does not interact with \(S_1\) and the operators commute. Then

\[
(\Gamma S)^k = (\Gamma_1\Gamma_2S_1S_2)^k = (\Gamma_1S_1)^k(\Gamma_2S_2)^k
\]

(3.47)

and we use the Neumann series to find

\[
Q = \sum_{k=0}^{\infty} t^k (\Gamma S)^k = \sum_{k=0}^{\infty} t^k(\Gamma_1S_1)^k(\Gamma_2S_2)^k = \left(\sum_{k=0}^{\infty} t^k(\Gamma_1S_1)^k\right) \odot \left(\sum_{k=0}^{\infty} t^k(\Gamma_2S_2)^k\right),
\]

(3.48)
where \( \odot \) denotes the Hadamard product of two power series with respect to \( t \). By using

\[
Q_i(a_i, x_i; t) = \sum_{k=0}^{\infty} t^k (\Gamma_i S_i)^k
\]

(3.49)

and reversing the iteration, we reach the functional equations \((1 - t\Gamma_i S_i)Q_i = 1\) for each \( i \), where \( Q_i \) is the generating function for a one-dimensional model with step set \( S_i \). Now, \( Q_i \) is expressed as

\[
Q(a_1, a_2, x_1, x_2; t) = Q_1(a_1, x_1; t) \odot Q_2(a_2, x_2; t).
\]

(3.50)

**Example 3.24.** Once we know the form of the solution, we can prove that it works by a method that does not use the series solution. Suppose that \( Q_i \) is the solution of the functional equation

\[
(L_i^{-1} - tE_iS_i)Q_i = \bar{a}_i
\]

(3.51)

for \( i = 1 \) and \( i = 2 \). We show that \( Q = Q_1 \odot Q_2 \) is a solution of the functional equation

\[
(L^{-1} - tES)Q = \bar{a}_1 \bar{a}_2.
\]

(3.52)

Since the operators act by multiplication of terms, they can be moved over the Hadamard product if they do not affect the variables of one of the functions. That is,

\[
(L^{-1} - tES)Q = L^{-1}(Q_1 \odot Q_2) - tES(Q_1 \odot Q_2)
\]

\[
= (L_1^{-1}Q_1 \odot L_2^{-1}Q_2) - t((E_1S_1Q_1) \odot (E_2S_2Q_2)).
\]

(3.53)

Then we substitute \( L_i^{-1}Q_i = \bar{a}_i + tE_iS_iQ_i \) into the first term and expand it to find

\[
(L_1^{-1}Q_1 \odot L_2^{-1}Q_2) = (\bar{a}_1 + tE_1S_1Q_1) \odot (\bar{a}_2 + tE_2S_2Q_2)
\]

\[
= \bar{a}_1 \odot \bar{a}_2 + \bar{a}_1 \odot (tE_2S_2Q_2) + \bar{a}_2 \odot (tE_1S_1Q_1)
\]

\[
+ (tE_1S_1Q_1) \odot (tE_2S_2Q_2).
\]

(3.54)

The last term cancels and for each \( i \) we have

\[
\bar{a}_i \odot (tE_jS_jQ_j) = \bar{a}_i[t^0]tE_jS_jQ_j = 0,
\]

(3.55)

so we are done.
Lemma 3.25. Let $S = S_1S_2$, where $S_i = S_i(x_i)$ and consider the classical interacting models with those step sets and generating functions $Q$, $Q_1$ and $Q_2$. Then

$$Q(x_1, x_2, a_1, a_2; t) = Q_1(x_1, a_1; t) \odot Q_2(x_2, a_2; t).$$

(3.56)

This identity serves as the base case for the following general case.

Theorem 3.26. Let $S = S_1 \cdots S_r$, where $S_i$ and $S_j$ do not share variables when $i \neq j$, and consider the classical interacting models with those step sets and generating functions $Q$, $Q_1, \ldots, Q_r$. Then

$$Q = Q_1 \odot \cdots \odot Q_r.$$

Example 3.27. Consider the two-dimensional model with $S = (x_1 + \overline{\sigma})(x_2 + \overline{\sigma})$, which we discuss more in Chapter 5. The one-dimensional models are the same and have solutions

$$\tilde{Q}_0 = \frac{\sigma}{at - b\sigma} \quad \text{and} \quad \tilde{Q} = \frac{t(x - \sigma)}{x(1 - t(x + \overline{\sigma})(at - b\sigma)},$$

where

$$\sigma = \frac{1 - \sqrt{1 - 4t^2}}{2t}.$$  

(3.57)

(3.58)

We can expand and simplify $\tilde{Q}_0$ and $\tilde{Q}$ to find

$$\tilde{Q}_0 = \frac{2 - a - a\sqrt{1 - 4t^2}}{2(1 - a + a^2t^2)},$$

(3.59)

$$\tilde{Q} = \frac{(1 - a)x + (a - 2)t + 2ax^2t^2 + (at + (1 - a)x)\sqrt{1 - 4t^2}}{2(1 - a + a^2t^2)(x - t - tx^2)}.$$  

(3.60)

We have the expansion

$$\tilde{Q} = 1 + xt + (a + x^2)t^2 + x(1 + a + x^2)t^3 + \cdots$$

(3.61)

and finally,

$$Q(x_1, x_2, a_1, a_2; t) = \tilde{Q}(x_1, a_1; t) \odot \tilde{Q}(x_1, a_2; t)$$

$$= 1 + x_1x_2t + (a_1 + x_1^2)(a_2 + x_2^2)t^2$$

$$+ x_1x_2(1 + a_1 + x_1^2)(1 + a_2 + x_2^2)t^3 + \cdots.$$  

(3.62)

However, obtaining a closed form for the function via the Hadamard product seems to be difficult. The function is not algebraic since the unmarked variant is not algebraic either; see [8, p. 26]. However, the function $Q$ is D-finite since both functions in the product are.
3.3.2 Reducible Models and Groups For Walks With Small Steps

As observed in [8, p.5] and already mentioned in Chapter 2 if the model has some steps that cannot be used in the construction of lattice walks, then we can safely remove them from the step set, and we still have the same generating function enumerating the lattice walks. This makes it easy to find \( Q \) for many models. This idea works for any step set and any regional constraints, so we make it a definition.

**Definition 3.28.** For a model with step set \( S \), let \( S' \) consist of all steps that can be taken by walks in the model. We call \( S' \) a **minimal** step set. A model is **reducible** if \( S' \) is a proper subset of \( S \).

**Proposition 3.29.** The full generating function for a model is the same whether we use \( S \) or \( S' \).

It should be easier to find \( Q \) if we have more functional equations and cancel out some of the terms. One way to find more equations is to apply a transformation on the tuple \((x_1, \ldots, x_d)\) that leaves intact as many parts of the functional equation as possible but still adds some information in the form of a non-equivalent equation. This leads to the orbit summation method, a fruitful approach in the non-interacting case. See [8, p. 11] for details. We present it here as well.

We map each \( x_1, \ldots, x_d \) to some functions separately and then examine the group generated by such maps. We often eliminate the kernel in front of \( Q \) in order to find equations for the extracted forms of \( Q \), so it is useful to consider transformations that preserve the kernel. We call this group \( G(K) \), although it is not the full group of rational transformations preserving the kernel. For example, a lot of the time \((x_1, x_2) \mapsto (x_2, x_1)\) is not in the group, but it is a valid transformation preserving the kernel. If a transformation on \((x_1, \ldots, x_d)\) preserves the kernel \( K = 1 - tS \), then it must also preserve the step polynomial \( S \). We can write \( S \) as

\[
S = S_k^{-1} x_k + S_0^1 + S_k^1 x_k
\]

for some \( k \) from 1 to \( d \). We send \( x_k \mapsto y_k \) and solve \( S(x_k) = S(y_k) \) for \( y_k \) to determine the possible transformations we can perform. After some simplification, we obtain the equation

\[
(S_k^{-1} - S_k^1 x_k y_k)(y_k - x_k) = 0
\]

and we have three cases depending on which ones of \( S_k^{-1} \) and \( S_k^1 \) are zero.

1. If both are zero, the equation is trivially-true and all transformations preserve the kernel.
2. If only one is zero, then \( y_k = x_k \) and only the identity transformation preserves the kernel.
3. If neither one is zero, we have \( y_k = x_k \) or \( y_k \) is the involution \( y_k = \bar{S}_k S_k^{-1}/S_k^1 \).
Case one and two are mostly trivial and we find properties of the generating function right away.

1. If \( S_k^{-1} = 0 \) and \( S_k^1 = 0 \), then \( S = S_0^k \) and the model lives on the \( x_k = 0 \) hyperplane. Therefore, the model is equivalent to a lower-dimensional one with no \( x_k \) coordinate. If \( q(t) \) is the generating function in the latter case, then the earlier one is given by \( Q(x_k, a_k; t) = q(a_k t) \), where the rest of the variables have been suppressed. We can also find this solution from the functional equation directly. Let \( M = M' M_k \) and \( E = E' E_k \). Then by [Equation 3.22] and the observation that \([x_k^0]SQ = SQ\) and \([x_k^0]Q = Q\), we have

\[
(M' - a_k t E' S)Q = \overline{a}[d]_{\{k\}}.
\] (3.65)

This is just an equation for a \((d - 1)\)-dimensional model with an extra scaling of \( t \) by \( a_k \).

2. If \( S_k^{-1} \neq 0 \) and \( S_k^1 = 0 \), then the model is reducible to one with \( S = S_0^k \).

3. If \( S_k^{-1} = 0 \), \( S_0^k = 0 \) and \( S_k^1 \neq 0 \), then the walks in the model leave the \( x_k = 0 \) hyperplane and never return. Here we have an essentially lower-dimensional problem as well, since we have \( S = S_k^1 x_k \). If the generating function for the lower-dimensional problem is \( q(t) \), then the generating function for this model is \( Q(x_k, a_k; t) = q(x_k t) \), where the rest of the variables have been suppressed. We can also derive this result from the functional equation.

All the other cases are more complicated and require more careful analysis. We can define a group \( G_S \) of the nontrivial transformations.

**Definition 3.30.** Let \( G_S \) be the group generated by nontrivial \( g_k \) that preserve the polynomial \( S \).

It is easy to see that there are infinitely many such groups, just consider \( S = \prod_{k=1}^d (x_k + \overline{x}_k) \) with \( g_k(x_k) = \overline{x}_k \). The group here has order \( 2^d \). The orders of the finite groups set in 2 dimensions are 1, 2, 4, 6 and 8, from Theorem 3 in [8, p. 8]. If we were to consider walks with multiple copies of some steps, we could obtain even larger group orders as seen in Section 7 of [11, p. 11]. The \( d \)-dimensional groups defined above are Coxeter groups — a generalization of the dihedral groups encountered in two-dimensional models. Coxeter groups are defined as follows.

**Definition 3.31.** A Coxeter group is defined as

\[
G = \langle r_1, \ldots, r_d \mid (r_i r_j)^{m_{i,j}} = 1 \rangle,
\] (3.66)

where \( m_{i,i} = 1 \), and \( m_{i,j} \geq 2 \) for all \( i \neq j \). Here \( m_{i,j} \) can be infinite.
We call these \( r \)'s reflections and when \( m_{i,j} = 2 \), they commute. Also, \( m_{i,j} = m_{j,i} \). The rank of \( G \) is the number of generators in this specific representation of \( G \). Often \( G \) has more than one representation with different numbers of generators. We can draw a graph for each Coxeter group by going through the following steps.

- Label vertices with the generators of \( G \).
- Connect the vertices \( g_i \) and \( g_j \) if \( m_{i,j} \geq 3 \).
- Label the edges with \( m_{i,j} \) if \( m_{i,j} \geq 4 \).

If the graph has different components, the group factors as a direct product of Coxeter groups, represented by the individual components. We can see this in terms of the step polynomial \( S \).

**Proposition 3.32.** Let \( S = TR \) with corresponding groups \( G_S \), \( G_T \) and \( G_R \), where \( T \) and \( R \) do not share variables. Then \( G_S = G_T \times G_R \).

**Proof.** For some \( x_i \) in the polynomial \( T \), we have

\[
S_i^j = [x_i^j]S = R[x_i^j]T = RT_i^j
\]

and therefore,

\[
g_i(x_i) = \bar{x}_j \frac{T_i^{-1} R}{T_i R} = \bar{x}_i \frac{T_i^{-1}}{T_i}.
\]

Similarly, \( g_j(x_j) = \bar{x}_j R_i^{-1} / R_i \) where \( x_j \) is some variable in \( R \). Furthermore, \( g_i \) and \( g_j \) commute since they do not share variables. Since \( i \) and \( j \) are arbitrary, the generators from different sets commute and the Coxeter graph of the group has two components, one associated with each factor. That is, the group itself factors as stated. \( \square \)

### 3.4 Specializing to Two Dimensions

Until the end of the chapter we focus on two-dimensional models with interactions. We can set \( d = 2 \) in Equation 3.18 and Equation 3.30 to find the two-dimensional master functional equations.
Corollary 3.33. The generating functions for classical interacting two-dimensional models satisfy the functional equations

\[
KQ - K_1 Q_1^0 - K_2 Q_2^0 + K_{1,2} Q_{1,2}^0 = \bar{a}_1 \bar{a}_2 ,
\]

\[
(1 - ta_1 S_1^0) Q_1^0 - (c_2 - ta_1 \bar{x}_2 S_{1,2}^{0,-1}) Q_{1,2}^{0,0} - a_1 t (S_1^{-1} Q_1^1 - \bar{x}_2 S_{1,2}^{-1} Q_{1,2}^{1,0}) = \bar{a}_2 ,
\]

\[
(1 - ta_2 S_2^0) Q_2^0 - (c_1 - ta_2 \bar{x}_1 S_{1,2}^{-1,0}) Q_{1,2}^{0,0} - a_2 t (S_2^{-1} Q_2^1 - \bar{x}_1 S_{1,2}^{-1} Q_{1,2}^{0,1}) = \bar{a}_1 ,
\]

\[
(1 - a_1 a_2 t S_{1,2}^{0,0}) Q_{1,2}^{0,0} - ta_1 a_2 t (S_{1,2}^{-1,1} Q_{1,2}^{1,1} + S_{1,2}^{-1,0} Q_{1,2}^{1,0} + S_{1,2}^{0,1} Q_{1,2}^{0,1}) = 1 .
\]

3.4.1 First Examples in the Plane

While the classical two-dimensional models have been studied extensively, their interacting versions have not. We want to extend the non-interacting methods to the interacting setting and if they turn out to be insufficient, create new ones. Since the generating functions for the interacting models need to reduce to those for the non-interacting models, we do not expect to find closed form solutions for \( Q \) for all models. In fact, some of the classical models lead to non-D-finite generating functions, so their interacting counterparts are not D-finite either.

For the rest of the models, we hope to find one of the functions \( Q \) and \( Q_{1,2}^{0,0} \) explicitly, but this appears to be difficult in most nontrivial cases. Recent research has shown that sometimes relationships exist between \( Q \) or \( Q_{1,2}^{0,0} \) and their partially-marked versions \([30]\). As we saw in the previous chapter, the one-dimensional models admit a linear relationship with polynomial coefficients. There are also two other two-dimensional models that exhibit a rational relationship. However, there may be models where neither a linear nor a rational relationship exists. In this section we cover the first few examples of quarter plane models with interactions.

Our first task is to find all the reducible models and we can do this by looking at which of the steps \{N, NE, E\} they have in their step set.

1. There are 32 reducible models, where none of the three steps are present and walks cannot leave the origin.

Figure 3.1: Set of steps allowed in models that do not leave the origin.
2. There is a $2 \cdot 2^4 = 16$ reducible models with an $N$ step or an $E$ step, where the models are equivalent to one-dimensional ones.

![Sets of steps allowed in models with an N step or an E step.](image)

**Figure 3.2:** Sets of steps allowed in models with an N step or an E step.

3. There are two or more steps from {N, NE, E} or NE $\subseteq$ S. In this case, the models feature walks that leave both axes and once that happens, such walks can use any step from the step set. Therefore, these models are irreducible.

Cases 1 and 2 contain all of the quarter plane models reducible to one-dimensional models. We look at the case of a model equivalent to a one-dimensional model along the $x_2$ axis. From the analysis in the previous section, since both $S_1^{-1} = 0$ and $S_1^1 = 0$, we have the first situation. If the solution to the one-dimensional problem is $q(x, a; t) = q(x_2, a_2; t)$, then the solution to the models in cases 1 and 2 is $Q(x_1, x_2, a_1, a_2; t) = q(x_2, a_2; a_1 t)$. To see it in another way, consider the polynomial $S$ itself in the reduced models. Since $S$ factors, the function $Q$ is a Hadamard product between two one-dimensional models.

**Definition 3.34.** Let $B_k$ be the operator $1 - [b_k^0]$ and $B = B_1 \cdots B_d$.

Since the models are essentially one-dimensional, we also have a linear relationship using the operators $B_1$ and $B_2$. If $f(a, t)$ is the function that goes together with $q(x, a, t)$, then the function $F(a_1, a_2, t)$ that goes together with $Q(x_1, x_2, a_1, a_2, t)$ is given by

$$F(a_1, a_2, t) = f(a_2, a_1 t). \quad (3.70)$$

Then we have $B_2(FQ) = 0$ and consequently, $B_1 B_2(FQ) = 0$. We wrap this section up with the following proposition.

**Proposition 3.35.** All 64 reducible models are equivalent to ones confined to an axis. Their generating functions $Q$ and their functions $F$, such that $B(FQ) = 0$, are given by one-dimensional ones with $t$ scaled by one of $a_1$ or $a_2$. 

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3.4.2 Kernel Group Considerations

We can also partition the models with respect to their groups $G$, that is, we write “case $i.j$” when we have case $i$ for $g_1$ and case $j$ for $g_2$, and it is equivalent to case $j.i$. From previous sections, cases $1.j$ are comprised of reducible models, so we can skip them. We are left with the following cases.

Case 2.2: In this case both transformations are the identity, so for each $k$, $S_{1k}^{-1}$ or $S_{1k}$ is zero. If $S_{1k} = 0$ for some $k$, the model is reducible and therefore, already solved for. The only remaining case is when both $S_{1k}^{-1} = 0$ and $S_{1k} \neq 0$ for each $k$. The only valid models are ones with steps in \{N, E, NE\}, where we need to have positive $x_1$ and $x_2$ coordinates. Here are the models.

![Figure 3.3: Step sets for classical two-dimensional models coming from case 2.2.](image)

Case 2.3: In this case, the transformation for $x_1$ is trivial and the one for $x_2$ is $g_2$. Therefore, $S_{11}^{-1} = 0$ or $S_{11} = 0$, but not both. The first case is reducible, so we only consider the second. Furthermore, $S_{2k}^{-1} \neq 0$ and $S_{2k} \neq 0$, so there must be at least one step in the positive $x_2$ direction and one in the negative $x_2$ direction. If both NE and SE are in $S$, then we have 8 cases. If one of NE and SE is in $S$, but not both, then we have 8 more cases. Finally, if neither is in $S$, the other three steps must be present in order to satisfy the conditions. We have 17 cases in total and all the models are irreducible.

![Figure 3.4: Step sets for classical two-dimensional models coming from case 2.3.](image)

Case 3.3: This is the most interesting case, with two transformations for $x_1$ and $x_2$ each. Thus, there must be at least one step going in a positive and one going in a negative direction, for each of $x_1$ and $x_2$. If we look at which diagonal steps are present, similarly to case 2.2, we find that there are 145 such models. Previous results show that the group $G(K)$ has orders 4, 6, 8 and infinity, depending on the step set in question. All of these cases are irreducible.
### 3.5 Solving Basic Two-dimensional Models

#### 3.5.1 Case 2.2 – Group of Order One

The only irreducible models of this type are ones with \( S_1^{-1} = 0 \) and \( S_2^{-1} = 0 \). Here

\[
S = \delta + \alpha x_1 + \beta x_2 + \gamma x_1 x_2
\]

for some \( \alpha, \beta, \gamma \) and \( \delta \). The functional equation and its zeroth coefficient extractions are given by

\[
\begin{align*}
(1 - tS)Q - c_1 Q_1^0 - c_2 Q_2^0 + c_1 c_2 Q_{1,2}^{0,0} &= \pi_1 \pi_2, \\
(1 - ta_1(\delta + \beta x_2))Q_1^0 - c_2 Q_{1,2}^{0,0} &= \bar{\alpha}_2, \\
(1 - ta_2(\delta + \alpha x_1))Q_2^0 - c_1 Q_{1,2}^{0,0} &= \bar{\alpha}_1, \\
(1 - a_1 a_2 t\delta)Q_{1,2}^{0,0} &= 1.
\end{align*}
\]

This is a linear system for \( Q_{1,2}^{0,0}, Q_1^0, Q_2^0 \) and \( Q \), and by back-substitution,

\[
Q = \frac{1}{1 - tS} \left( \bar{\alpha}_2^2 + \frac{1}{1 - ta_1 a_2} \left( \frac{c_1}{1 - t a_1 S_1^0} + \frac{c_2}{1 - t a_2 S_2^0} - \frac{c_1 c_2}{1} \right) \right) \quad (3.73)
\]

\[
= \frac{1}{1 - tS} \left( \bar{\alpha}_2^2 + \frac{1}{1 - a_1 a_2 t\delta} (c_1 f_1 (a_1, t) + c_2 f_2 (a_2, t) - c_1 c_2) \right). \quad (3.74)
\]

Note how \( Q \) depends on functions \( f_i \) that are each missing one of the \( a \) variables. We shall see that the linear relation \( B(FQ) = 0 \) is satisfied by the function

\[
F(a_1, a_2, t) = 1 - a_1 a_2 t. \quad (3.75)
\]

First,

\[
\begin{align*}
B((1 - a_1 a_2 t) \bar{\alpha}_1 \bar{\alpha}_2) &= \bar{\alpha}_1 \bar{\alpha}_2 - \bar{\alpha}_2 - \bar{\alpha}_1 + 1 = c_1 c_2, \\
B(c_1 f_1 + c_2 f_2 - c_1 c_2) &= c_1 f_1 + c_2 f_2 - c_1 c_2 - c_2 f_2 - c_1 f_1 = -c_2 c_2. \quad (3.76)
\end{align*}
\]

Putting the two quantities together, along with dividing by \( 1 - tS \), shows that \( B(FQ) = 0 \). Of course, this works for \( Q_{1,2}^{0,0} \) too.
3.5.2 Case 2.3 – Group of Order Two

In this case only $S^{-1}_1$ is zero, so the functional equations are

\[
\begin{align*}
(1 - tS)Q - c_1Q^0_1 - (c_2 - t\bar{T}S^{-1}_2)Q^0_2 + c_1c_2Q^{0,0}_{1,2} &= \bar{a}_1\bar{a}_2, \\
(1 - ta_1S^0_1)Q^0_1 - (c_2 - ta_1\bar{T}S^{-1}_{1,2})Q^{0,0}_{1,2} &= \bar{a}_2, \\
(1 - ta_2S^0_2)Q^0_2 - c_1Q^{0,0}_{1,2} - a_2tS^{-1}_2Q^1_2 &= \bar{a}_1, \\
(1 - a_1a_2tS^{0,0}_{1,2})Q^{0,0}_{1,2} - ta_1a_2S^{-1}_{1,2}Q^{0,1}_{1,2} &= 1.
\end{align*}
\]

We can identify a subproblem that is essentially one-dimensional and therefore, already solved. To see this, take the second and fourth equations, and make the six substitutions $a_i = a$, $c_i = c$, $x_2 = x$, $a_1t = \tau$, $S^0_1 = T$ and $Q^0_1(x_2) = q(x)$. The two equations become

\[
\begin{align*}
(1 - \tau T)q - (c - \tau \bar{T}T^{-1})q_0 &= \bar{a}, \\
(1 - a\tau T_0)q_0 - a\tau T^{-1}q_1 &= 1.
\end{align*}
\]

We can see that this is a one-dimensional problem again. The solution here is $q(x, a; \tau)$, obtained by the methods in the previous chapter, so the original problem has solutions

\[
Q^0_1(x_2) = q(x_2, a_2; a_1t) \quad \text{and} \quad Q^{0,0}_{1,2} = q_0(a_2; a_1t).
\]

Since $Q^{0,0}_{1,2}$ and $Q^0_1$ are given by a one-dimensional model’s solution, they have a linear relationship with their partially-marked versions. Next, we rewrite the master equation as

\[
(1 - tS)Q - (c_2 - t\bar{T}S^{-1}_2)Q^0_2 = \bar{a}_1(\bar{a}_2 + b_1q - b_1c_2q_0) = \bar{a}_1\phi(x_2).
\]

Since $S^{-1}_1 \neq 0$ and $S^1_2 \neq 0$, we have a root $x_2 = \sigma_2$ that allows us to eliminate the $(1 - tS)Q$ term. That is, we are finding a root for $x_2$ again, since $x = x_2$ in the previous equations. We find that

\[
Q^0_2 = -\frac{\bar{a}_1\phi(\sigma_2)}{c_2 - t\bar{T}S^{-1}_2}
\]

and

\[
Q = \frac{1}{1 - tS} \left( \bar{a}_1\phi(x_2) - \frac{c_2 - t\bar{T}S^{-1}_2}{c_2 - t\bar{T}S^{-1}_2} \bar{a}_1\phi(\sigma_2) \right).
\]
We can simplify this form based on what $q$ and $q_0$ are from the one-dimensional setting. After a quick computation, we have $Q_2^0 = \bar{a}_1 a_2 \phi(\sigma_2)\hat{Q}_2^0$ and we can write

$$KQ = \bar{a}_1 \phi + K_2 \bar{a}_1 a_2 \phi(\sigma_2)\hat{Q}_2^0, \quad (3.85)$$

$$K\hat{Q} = \bar{a}_2 + K_2\hat{Q}_2^0. \quad (3.86)$$

Therefore, after expanding the definitions of the various quantities, we have

$$KQ - \bar{a}_1 a_2 \phi(\sigma_2)K\hat{Q} = \bar{a}_1(\phi - \phi(\sigma_2)) = c_1 (q - q(\sigma_2)) \quad (3.87)$$

Since $q = q(x_2, a_2; a_1 t)$, there is a function $f$ depending on $a$’s and $t$ only, for which $B_2(fq) = 0$. The same function works for $q(\sigma_2)$ as well. Finally, there is a linear relationship between $Q$ and its various partially-interacting versions. This time it seems that the relationship is not given by $B(FQ) = 0$ but rather a more complicated one, $B_2(fg(B_1(Q)/g)) = 0$, where $g = \bar{a}_1 a_2 \phi(\sigma_2)$. Finally, we can give the following theorem.

**Theorem 3.36.** Reducible classical two-dimensional models and ones with group order one or two have generating functions with linear relations with their partially-interacting variants.

**Miscellaneous Case**

There is one case with an infinite group that is both easy to solve and irreducible — the model with $S = x_1 x_2 + x_1 x_2$. The transformations are $g_1 = x_1 x_2^2$ and $g_2 = x_2 x_1^2$. If we apply them on $(x_1, x_2)$ in an alternating sequence, each time the absolute powers of $x_1$ and $x_2$ increase in one of the coordinates, and therefore, the group is infinite. However, this case is easy to solve because the model is essentially one-dimensional. It is equivalent to the model on the non-negative line with $S = x + x$ and marking visits to the origin with $a$, where $x = x_1 x_2$ and $a = a_1 a_2$. The solution is

$$Q(x_1, x_2, a_1, a_2; t) = q(x_1 x_2, a_1 a_2; t) \quad (3.88)$$

and we can add this model to the theorem as well.

**3.5.3 Counting Classes of Models**

Here we tally all the models on the quarter plane. We have 64 reducible models, 5 new cases with group of order one and 34 new cases with group of order two. Also, there are 29 models with group of order four, 6 with order six, and 4 with order eight, plus one easy case with infinite group. So far we have 143 cases of models we can solve easily or models with a finite group, some of which
we can solve easily. Next, by inspection of Table 4 in [8, p. 28], we see there are 120 models with infinite groups: 12 with three steps, 40 with four steps, 44 with five steps, and 24 with six steps. Adding all of those together yields 263 cases. There are 7 reducible models with an infinite group that we have counted twice. The total count is 256 and we are done.

Figure 3.5: Reducible classical two-dimensional models with an infinite group.

So far we have solved the 64 reducible models, the 39 other models with group of order one or two, and the single new case of easy-to-solve model with an infinite group, for a grand total of 104 trivial or easy models. Out of the remaining 152 models, 113 have an infinite group, and their analysis is not part of this thesis. Finally, we have 39 cases left, with a group of order four, six or eight. The rest of the thesis handles the analysis of those cases.

Figure 3.6: Step sets for classical two-dimensional models with a finite group.

Figure 3.7: Step sets for classical two-dimensional models with an infinite group. Unlisted step sets differ from these by one of the symmetries of the square.
Chapter 4

The Algebraic Kernel Method

In this chapter we explore the algebraic kernel method, as seen in Section 4 of [18, p. 9], and we extend it to handle models with interactions. First, we recast the non-interacting version of the method in a form that uses operators. Then we generalize the operators by giving them weights, and this allows us to investigate models with interactions.

4.1 Models Without Interactions

At the core of the algebraic kernel method we have the idea that we can apply transformations from a group $G$ onto the master functional equation and acquire different equations for each $g \in G$. After we sum these equations in a nice way, we manage to eliminate the functions $Q_0^0, Q_0^1, Q_0^2$ and their orbits, only leaving the orbit of $KQ$ and a term on the right-hand side. What makes this possible is pre-multiplying the whole equation by $x_1x_2$.

The analysis also works in higher dimensions, as we show later. Although there has been some generalization already, as in [8, p. 11] for the two-dimensional case and [18, p. 10] for the three-dimensional one, we present it in a compact and easy-to-generalize way. Instead of summing equations explicitly, like in [8], we use an operator $D_G$ that does the same thing. We focus on groups with a full set of $d$ generators.

**Definition 4.1.** For a classical $d$-dimensional model with group $G$ with $d$ generators, define the orbit summation operator $D_G$ as

$$D_G = \sum_{g \in G} (-1)^{|g|} g,$$

where $|g|$ is the number of generators in the representation of $g$. 

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In two dimensions, we can find the whole group by iterating over the two generators. That is, we apply \( g_1 \) and \( g_2 \) sequentially and the group is \( \{1, g_i, g_jg_i, g_ig_jg_i, \ldots\} \). However, we shall use the equivalent orbit \( \{1, g_i, g_ig_j, g_ig_ig_j, \ldots\} \), where the iteration is on the right. To declutter the notation, we use \( g_iji \) instead of \( g_ig_ig_jg_i \).

**Definition 4.2.** For a sequence \( I = (i_1, \ldots, i_n) \) of positive integers, define

\[
g_L = g_{i_1} \cdots g_{i_n} = g_{i_1} \cdots g_{i_n}.
\]

(4.2)

If the group is finite for a classical two-dimensional model, then \( g_1g_2^q n^p = 1 \) for some minimal \( n \), so we can rewrite \( D_G \) in a useful way [8]. The following lemma is a restatement of the orbit summation method, found in [8, p. 10], recast in operator form.

**Proposition 4.3.** For a classical two-dimensional model with group of order \( 2n \), generated by \( g_1 \) and \( g_2 \), we can write the operator \( D_G \) as

\[
D_G = (1 + g_{ij} + \cdots + (g_{ij})^{n-1})(1 - g_i).
\]

(4.3)

**Proof.** We can just check the identity by expanding it as

\[
D_G = (1 + g_{ij} + \cdots + (g_{ij})^{n-1})(1 - g_i) = 1 - g_i + g_{ij} - g_{ij} g_i + \cdots - (g_{ij})^{n-1} g_i,
\]

(4.4)

giving us the orbit summation resulting from the second form of the obit. \( \square \)

Here are some examples of small groups.

**Example 4.4.** When the order is four, the orbit is \( \{1, g_1, g_12 = g_{21}, g_2\} \) since \( g_1 \) and \( g_2 \) commute. The operator \( D_G \) is \( (1 + g_{12})(1 - g_1) \), but it also factors in a convenient way:

\[
D_G = 1 - g_1 - g_2 + g_{12} = (1 - g_1)(1 - g_2).
\]

(4.5)

When the order is six, the orbit is \( \{1, g_1, g_{21}, g_{121} = g_{212}, g_{12}, g_2\} \) and we can write the operator as

\[
D_G = (1 + g_{12} + g_{1212})(1 - g_1) = 1 - g_1 - g_2 + g_{12} + g_{21} - g_{121}.
\]

(4.6)

We can generalize the result from the lemma for groups of higher-dimensional models, but we need some results first.
Lemma 4.5. Consider a classical $d$-dimensional model with group $G$. The set $H$ of elements with even parity forms a subgroup.

Proof. The proof is analogous to the proof that the set of even permutations forms a subgroup of the set of permutations of $\{1, \ldots, n\}$. First, the identity $i$ of $G$ is in $H$, because $i^2 = i$ has even parity. Then $H$ is closed under the group operation because the product of two elements with even parity also has even parity. Finally, if $x$ has even parity, then its inverse $y$ has even parity, otherwise, $i = xy$ would not have even parity. Thus, $H$ is a subgroup by the two-step subgroup test.

The next corollary should be self-evident.

Proposition 4.6. Let $g_i$ and $g_j$ be generators of $G$. Then $g_{ij}(H)$ and $g_i(H) = g_j(H)$.

Lemma 4.7. For a classical $d$-dimensional model with a finite group, $D_G = D_H(1 - g_k)$, where $H$ is the subgroup of even elements.

Proof. First, note that $g_k$ is an involution that does not fix any element of $G$, that is $gg_k \neq g$ for any $g$ in $G$. Since applying $g_k$ on a group element changes its parity, this means that $g_k$ induces a partition of $G$ into two parts: a set of even elements and a set of odd elements in one-to-one correspondence. That is, we can partition $G$ as the union of $H$ and $Hg_k$. With the notation $e_h = (-1)^{|g|}$, we can immediately compute the form of $D_G$ from the statement of the lemma:

$$D_G = \sum_{g \in H} e_g g + \sum_{g \in H} (-1)^{|g|} e_g g_k = \left( \sum_{g \in H} e_g g \right) (1 - g_k) = D_H(1 - g_k). \quad (4.7)$$

Note that we could have also computed $D_G = (1 - g_k)D_H$ if we multiplied $g_k$ on the left.

With the previous generalization of the operator form, we obtain the $d$-dimensional generalization to Proposition 5 in [8, p. 11] and Lemma 52 in [15, p. 59].

Theorem 4.8. For a classical $d$-dimensional model with a finite group $G$,

$$D_G(x_1 \cdots x_d KQ) = D_G(x_1 \cdots x_d). \quad (4.8)$$

Proof. With the application of $D_Gx_1 \cdots x_d$, all terms except $KQ$ and 1 vanish since terms like $x_{[d]}x_I S^{-1} Q_I 0$ are annihilated by $D_G$ because they miss one or more variable $x_i$.

In [8, p. 11] and [18, p. 13], the authors have found that sometimes the algebraic kernel method fails. That is, $D_G(x_1 \cdots x_d) = 0$ and the equation does not give us new information. The operator form of the method allows us to find a more general instance of when this situation occurs.
Corollary 4.9. If an element \( h \) of \( G \) is an involution permuting the variables \( x_1, \ldots, x_d \), then the regular algebraic kernel method fails. That is, \( D_G(x_1 \cdots x_d KQ) = 0 \).

Proof. Since \( h \) is an involution, the same method applies as before and we can factor the operator \( D_G \) as \( D_H(1 - h) \) for some group \( H \) in \( G \). The action \( 1 - h \) eliminates the monomial \( x_1 \cdots x_d \).

Example 4.10. In two dimensions, the algebraic kernel method fails precisely for those models where we have a permutation as in the corollary. That is, \( h(x_1, x_2) = h(x_2, x_1) \) and \( h \) has order two. Proposition 6 in [8, p. 12] follows.

For models with six steps in three dimensions, the algebraic kernel method fails in 62 of them [18, p. 13]. Here \( G \) contains a permutation \( h(x_i, x_j) = (x_j, x_i) \). It is not clear if all cases of failure of the method can be attributed to such a situation.

Naturally, we need to find out if we can use \( D_H \) instead of \( D_G = D_H(1 - h) \)? The answer is yes. The operator \( D_H \) eliminates some of the \( Q \) variables and introduces others, but the result is a functional equation that is symmetric under permutation by \( h \). One may hope to perform a method analogous to the half-orbit summation method to prove algebraicity or D-finiteness.

Next, we generalize the orbit summation operator by including weights with each \( g \) in \( G \).

Definition 4.11. For a classical \( d \)-dimensional model with group \( G \) with \( d \) generators and given weights \( R_g \) for \( g \) in \( G \), define the weighted orbit summation operator \( D \) as

\[
D = \sum_{g \in G} e_g R_g g .
\]  

(4.9)

The normal orbit summation operator is one with weights \( R_g = 1 \). The action of such an operator can be seen in [21, p. 11], where the authors use an orbit summation with weights. We can capture various possible strategies in this notation.

Proposition 4.12. Pre-multiplying the functional equation by a function \( T \) is equivalent to applying a weighted orbital operator on the original equation. Application of more than one weighted operator is equivalent to applying a single weighted operator.

Proof. For the first statement, let \( T \) and \( f \) be some functions. Then we have

\[
D_G(T f) = \sum_{g \in G} e_g g(T f) = \sum_{g \in G} e_g g(T) g(f) = \sum_{g \in G} e_g R_g g(f) ,
\]  

(4.10)
where the weights are $R_g = g(T)$. For the second statement, we multiply two operators as

$$D_1 D_2 = \sum_{g \in G} \sum_{g \in G} e_g R_g g e_h T h h = \sum_{g \in G} \sum_{h \in G} R_g e_{gh} T_{gh} g h = \sum_{g \in G} \left( \sum_{h \in G} e_{gh^{-1}} R_{gh^{-1}} T_g \right) g,$$

so we obtain a new weighted operator. \hfill \Box

The second statement of the lemma leads to the following proposition.

**Proposition 4.13.** The set of weighted orbit summation operators associated with a group forms a monoid under composition.

**Proof.** Composition of operators is associative and in the last example we showed that the set is closed under composition. Thus we only need to show that the set has an identity. Let $R_g = 0$ for all $g$ in $G$ except for $R_1 = 1$. Then $D_G$ for these weights is the identity operator. \hfill \Box

Because of the preceding lemma, it does not matter if we try to use a cleverly-chosen function $T$ or operators $D_1$ and $D_2$, to circumvent the failure of the algebraic kernel method. We just need to focus on a weighted operator from the start. The procedure of eliminating terms from the functional equation requires that those terms are removed generically, that is, whenever $I \neq J$, the functions $Q^0_I$ and $Q^0_J$ are treated as independent variables and thus, each of them needs to be annihilated by $D$ on its own. Thus, we need $D'(t x_1 S^{-1}_I Q^0_I) = 0$ for each nonempty set $I$. This allows us to show that in general, when the method fails, all operators $D'$ that eliminate the unwanted terms in the functional equation must also eliminate the monomial $x_1 \cdots x_d$.

It is enough to work on the functional equation $x_1 \cdots x_d K Q = x_1 \cdots x_d$ with an operator $D$. Then we can find the original operator $D'$ from the relation $D' = Dx_1 \cdots x_d$, where $D'$ acts on the master functional equation by pre-multiplying it by $x_1 \cdots x_d$ and applying $D$. We show that all the weights of $D$ must be the same.

This observation turns the task into a linear algebra problem. We just need to apply $D$ on the functions $W_I = t x_I S^{-1}_I Q^0_I$, which are missing the variables indexed by $I$, collect the different results and set their coefficients equal to zero. That is, for every $f = W_I$ we find the orbit

$$\text{Orbit}_G(f) = \{g(f) | g \in G\} \quad (4.12)$$

and compute

$$0 = D(f) = \sum_{g \in G} R_g g(f) = \sum_{f' \in \text{Orbit}(f)} R^o_{g'} f', \quad (4.13)$$

where we set
Thus, we reach a linear system of $|\text{Orbit}_G(f)|$ equations in $|G|$ unknowns. We can do this for each of the $2^d - 1$ terms $W_i$ and solve the system of equation. Here is an example.

**Example 4.14.** Consider the classical two-dimensional model with $S = \bar{x}_1 + \bar{x}_2 + x_1x_2$, where the algebraic kernel method fails [8, p.12]. Here both of the generators map $x_i$ to $\bar{x}_1\bar{x}_2$ and the group is given by the orbit $\{1, g_1, g_2, g_{121}, g_{12}, g_2\}$ of $(x_1, x_2)$. The orbit of $1$ under $G$ is trivial and by symmetry of the group actions, the orbit of each of $W_1 = f_1(x_2)$ and $W_2 = f_2(x_1)$ is given by $\{f(x_1), f(x_2), f(\bar{x}_1\bar{x}_2)\}$. The relations we obtain are

\begin{align}
R - R_1 + R_{121} - R_{12} - R_2 &= 0, \\
(R - R_2)f(x_1) + (R_{21} - R_{121})f(x_2) + (R_{12} - R_1)f(\bar{x}_1\bar{x}_2) &= 0, \\
(R_{12} - R_{121})f(x_1) + (R - R_1)f(x_2) + (R_{21} - R_2)f(\bar{x}_1\bar{x}_2) &= 0,
\end{align}

and if we set the coefficients to zero, we find that all the weights are the same.

With a series of results, we now show that there are no clever weighted operators that let us circumvent the failure of the algebraic kernel method.

**Lemma 4.15.** The stabilizer of $x_k$ is given by the subgroup generated by actions $g_i$ for $i \neq k$.

**Proof.** We already know that $g_i$ fixes $x_k$ for every $i \neq k$. Consider an element $g = g_{a_1...a_n}$ in $G$ that fixes $x_k$. For a given $g$ we can sometimes obtain a shorter action $g'$ by the following procedure. If $a_1$ or $a_n$ is not $k$, then we can take $g_{a_1}gx_k = g_{a_2...a_{n-1}}x_k = g_{a_1}x_k$ and $g_{a_n}x_k = g_{a_n}x_k$. If both $a_1$ and $a_n$ are equal to $k$, we can take $g_kg_kx_k = g_kx_kg_k = x_k$. If we follow this procedure to its conclusion we either obtain $1$ or $g_k$. However, the latter one does not fix $x_k$ so we must arrive at $g = 1$ with this procedure, and we have just proved that every $g$ that fixes $x_k$ must have an even number of $g_k$’s in its generating sequence. Such actions are equivalent to ones with no $g_k$’s in their representation. That is, if $g$ fixes $x_k$, it has no $g_k$’s in its generating set and we are done. \qed

**Lemma 4.16.** The stabilizer of $W_k$ is given by $\{1, g_k\}$.

**Proof.** If $g$ stabilizes the generic function $W_k$, then it must also stabilize each $x_i$ for $i \neq k$. Then $g$ must be in the stabilizer of $x_i$ for every $i \neq k$. The intersection of these sets is precisely $\{1, g_k\}$. \qed
Lemma 4.17. The actions $h$ in $H$ lead to distinct functions $h(W_k)$.

Proof. First, because $h(W_k) = h g_k(W_k)$ for every $h$ in $H$, the orbit of $W_k^0$ is the same whether we use $H$ or $G$. By the orbit stabilizer theorem, the orbit of $W_k^0$ under $G$ is of size $|G|/2 = |H|$, so all the elements $h(W_k)$ must be distinct for different $h$’s.

Lemma 4.18. Consider a weighted orbital operator that annihilates functions $W_k$ for all $k$. Then its weights satisfy the equations $R_h = R_{h g_k}$ for all $h$ in $H$.

Proof. We just compute the action of the orbital operator as

$$D(W_k^0) = \sum_{g \in G} (-1)^{|g|} R_g g(W_k) = \sum_{h \in H} (R_h - R_{h g_k}) h(W_k)$$

and since the functions $h(W_k)$ are distinct, their coefficients must be identically zero as well.

Theorem 4.19. Consider a weighted orbital operator that annihilates functions $W_k$ for all $k$. Then all of its weights are the same.

Proof. From the lemma, we see that $R_h = R_{h g_k}$ for $k$ and all $h$ in $H$. That is, the weights are the same if their corresponding group elements differ by a generator $g_k$. Since we can reach any $g$ in $G$ by applying a sequence of generators, $R_g = R_1$ for all $g$, and the weights are all the same.

This has the implication that there is essentially one weighted orbital operator that annihilates all the functions $W_k^0$ and therefore, there is no clever weighted operator that allows us to circumvent the failure of the algebraic kernel method for classical models.

Corollary 4.20. We cannot circumvent the failure of the algebraic kernel method through the use of a weighted orbital operator.

4.2 Models With Interactions

The orbit summation method is not sufficient for extending the algebraic kernel method to models with interactions, but we are able to use weighted operators to make progress. The analysis for two-dimensional models is particularly nice because we end up with the same number of equations and unknowns, and we are able to find the required weights. The linear systems can be written as matrix equations, where the determinants tell us when we have a useful weighted operators.

In the end, the method does not extend fully because we do not eliminate the $Q_{1,2}^{0,0}$ term, although now the D-finiteness of $Q$ depends on the D-finiteness of the function $Q_{1,2}^{0,0}$ in a clearer way. Since
we use nontrivial generators only, the group has order 4, 6 or 8, and the functional equation in two dimensions is

\[ KQ - K_1Q_1^0 - K_2Q_2^0 + K_{1,2}Q_{1,2}^{0,0} = \alpha. \]  

(4.19)

4.2.1 Groups of Order Four

The orbit summation operator for the non-interacting case is

\[ D_G = 1 - g_1 - g_2 + g_1g_2 = (1 - g_1)(1 - g_2) = \Delta_1\Delta_2, \]  

(4.20)

but this operator does not eliminate the unknown functions in the interacting case. Instead, we use a weighted one and determine its weights. We find an appropriate linear system by applying group actions on the master equation. The unknowns are the functions \( Q_1^0 \) and \( Q_2^0 \), and their orbits under \( G \). If the associated matrix is invertible, the system has a unique solution and unfortunately, we cannot eliminate the variables. However, in some cases the determinant is zero and we can eliminate the unknowns.

Since the transformations commute, we have \( g_iQ_i^0 = Q_i^0 \) and \( g_ig_jQ_i^0 = g_jQ_i^0 \). The variables and their orbits are given by the following table.

<table>
<thead>
<tr>
<th>Group action</th>
<th>Effect on ( Q_1^0 )</th>
<th>Effect on ( Q_2^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Q_1^0 )</td>
<td>( Q_2^0 )</td>
</tr>
<tr>
<td>( g_1 )</td>
<td>( g_1Q_1^0 )</td>
<td>( Q_2^0 )</td>
</tr>
<tr>
<td>( g_2 )</td>
<td>( g_2Q_1^0 )</td>
<td>( Q_2^0 )</td>
</tr>
<tr>
<td>( g_{12} = g_2g_1 )</td>
<td>( g_2Q_1^0 )</td>
<td>( g_1Q_2^0 )</td>
</tr>
</tbody>
</table>

Table 4.1: Reductions of group actions on \( Q_1^0 \) and \( Q_2^0 \) for groups of order four.

Therefore, the only variables that we need to eliminate are in the set

\[ E = \{ Q_1^0, Q_2^0, g_2Q_1^0, g_1Q_2^0 \}. \]  

(4.21)

The situation looks promising since we also have four equations. We rewrite the functional equation in a way that highlights those variables:

\[ K_1Q_1^0 + K_2Q_2^0 = KQ + K_{1,2}Q_{1,2}^{0,0} - \alpha = \beta. \]  

(4.22)
Then we apply the transformations to produce the system of equations

\[ K_1 Q_1^0 + K_2 Q_2^0 = \beta , \]  
\[ g_1 K_1 Q_1^0 + g_1 K_2 Q_2^0 = g_1 \beta = \beta_1 , \]  
\[ g_2 K_1 g_2 Q_1^0 + g_2 K_2 Q_2^0 = g_2 \beta = \beta_2 , \]  
\[ g_{12} K_1 g_{12} Q_1^0 + g_{12} K_2 g_{12} Q_2^0 = g_{12} \beta = \beta_{12} . \]

Using the order of variables in \( E \), we can write the associated matrix \( M \) of the linear system as

\[
M_4 = \begin{bmatrix}
K_1 & K_2 & 0 & 0 \\
g_1 K_1 & 0 & 0 & g_1 K_2 \\
0 & g_2 K_2 & g_2 K_1 & 0 \\
0 & 0 & g_{12} K_1 & g_{12} K_2
\end{bmatrix}.
\]

Then

\[
\det M_4 = K_1 g_{12} K_1 g_1 K_2 g_2 K_2 - g_1 K_1 g_2 K_1 K_2 g_{12} K_2 \\
= K_1 g_{12} K_1 g_1 K_2 g_2 K_2 - g_1 (g_1 K_1 g_2 K_1 K_2 g_2 K_2) \\
= \Delta_1 (K_1 g_{12} K_1 g_1 K_2 g_2 K_2).
\]

The determinant must vanish if we are to eliminate variables, and that only happens when the expression in \( \Delta_1 \) is in the operator’s kernel. The models with groups of order four exhibit at least one symmetry across a line — either vertical or horizontal symmetry, or both. Without loss of generality, let the model have symmetry across the \( x_2 \) axis. That is, \( g_1 (x_1) = x_1 \), \( g_1 (S_1^i) = S_2^i \) and \( g_1 (S_1^i) = S_1^{-i} \), and we can compute some of the quantities. For example, \( g_1 K_2 = K_2 \) and \( g_1 g_2 K_2 = g_2 g_1 K_2 \). Therefore, \( \Delta_1 \) fixes both \( K_2 \) and \( g_2 K_2 \) and

\[
\det M_4 = \Delta_1 (K_1 g_{12} K_1 g_2 K_2) = K_2 g_2 K_2 \Delta_1 (K_1 g_{12} K_1).
\]

This quantity is zero if and only if \( \Delta_1 (K_1 g_{12} K_1) = 0 \), so we have

\[
\Delta_1 (K_1 g_{12} K_1) = K_1 g_{12} K_1 - g_1 K_1 g_2 K_2 \\
= c_1 t (\bar{x}_1 - x_1) (g_2 S_1^{-1} - S_1^{-1}) \\
= -c_1 t (\bar{x}_1 - x_1) \Delta_2 (S_1^{-1}).
\]

Therefore, this equation is true if and only if \( g_2 S_1^{-1} = S_1^{-1} \). We have
\[ g_2(S_{1,2}^{-1,1}x_2 + S_{1,2}^{-1,0} + S_{1,2}^{-1,1}x_2) = S_{1,2}^{-1,1}x_2\mu_2 + S_{1,2}^{-1,0} + S_{1,2}^{-1,1}x_2\mu_2 \]

\[ S_{1,2}^{-1,1}x_2\mu_2 + S_{1,2}^{-1,1}x_2\mu_2 = S_{1,2}^{-1,1}x_2 + S_{1,2}^{-1,1}x_2, \]

(4.31)

and we can write it succinctly as \( S_{1,2}^{-1,1} = \mu_2 S_{1,2}^{-1,1} \), where the \( S \)-terms are either both zero or they are both one. If both are one, since the model has vertical symmetry, then \( S_{1,2}^{-1} = S_{1,2}^0 \) and the model is highly-symmetric. If they are both zero, the vertical symmetry ensures that \( S_{1,2}^{1,1} \) and \( S_{1,2}^{1,-1} \) are also zero. That is the model has no diagonal steps and since the group has two nontrivial generators, we must have both \( \tau_2 \) and \( x_2 \) present in \( S \). Again, the model is highly-symmetric.

Conversely, if we started with a highly-symmetric model, we would have obtained \( \det M_4 = 0 \), and this leads to the following theorem.

**Proposition 4.21.** The determinant of \( M_4 \) is zero if and only if the model is highly-symmetric.

**Corollary 4.22.** The algebraic kernel method extends to the non-interacting case for groups of order four if and only if the model is highly-symmetric.

---

**Figure 4.1:** Highly-symmetric models on the plane with two nontrivial group generators

We are finally ready to use the orbit summation method. There are 8 highly-symmetric two-dimensional models, but only five of them have two nontrivial generators for the group \( G \). Furthermore, two of the models are reflections of each other, so we are really left with four distinct cases. From left to right in the figure above, they are the Cartesian model, the Diagonal Cartesian model, the model with two horizontal steps missing, and the full model. If we eliminate all the variables from the set \( E \) in the beginning of this section, we have a linear relation like

\[ \gamma\beta - \gamma_1\beta_1 - \gamma_2\beta_2 + \gamma_1\beta_1\beta_2 = 0. \]

(4.32)
Here the choice of signs is inspired by the orbit summation operators. Let $\bullet$ and $\ast$ stand for applications of $g_1$ and $g_2$, respectively. Note that we have $K_2^\ast = K_2$ and $K_1^\ast = K_1$. Next, we let
\[ x = Q_1^0, \quad y = Q_2^0, \quad u = Q_1^{0,\ast}, \quad v = Q_2^{0,\ast}, \]
(4.33)
\[ a = K_1, \quad b = K_1^\ast, \quad c = K_2, \quad d = K_2^\ast. \]
(4.34)

The original equations attain a simplified format given by
\[ \beta = ax + cy, \quad \beta_1 = bx + cv, \]
(4.35)
\[ \beta_2 = dy + au, \quad \beta_{12} = bu + dv. \]
(4.36)

Plugging these into the linear relation above yields
\[ 0 = (ax + cy)\gamma - (bx + cv)\gamma_1 - (dy + au)\gamma_2 + (bu + dv)\gamma_{12} \]
(4.37)
\[ = (a\gamma - b\gamma_1)x + (c\gamma - d\gamma_2)y + (b\gamma_{12} - a\gamma_2)u + (d\gamma_{12} - c\gamma_1)v, \]
and in order to eliminate the variables $x$, $y$, $u$ and $v$, we need all their coefficients to vanish. We obtain the equations
\[ a\gamma = b\gamma_1, \quad c\gamma = d\gamma_2, \quad b\gamma_{12} = a\gamma_2, \quad d\gamma_{12} = c\gamma_1. \]
(4.38)

If we set $\gamma = 1$, then $\gamma_1 = a/b$ and $\gamma_2 = c/d$ and $\gamma_{12} = (ac)/(bd)$. So we can multiply them out by $bd$ to find the relation
\[ bd\beta - ad\beta_1 - cb\beta_2 + ac\beta_{12} = 0, \]
(4.39)

where we have successfully eliminated $Q_1^0, Q_2^0, Q_1^{0,\ast}$ and $Q_2^{0,\ast}$. In terms of the $K$'s we have,
\[ K_1^\ast K_2^\ast \beta - K_1 K_2^\ast \beta^* - K_1^* K_2 \beta^* + K_1 K_2 \beta^{**} = 0. \]
(4.40)

We can rewrite this equation in the following nicer way.

**Theorem 4.23.** For a two-dimensional highly-symmetric model with a group with two nontrivial generators, we have the functional equation $\Delta(K_1^* K_2^* \beta) = 0$, where $\beta = KQ + K_{1,2} Q_{1,2}^0,0 - \alpha$.

This is a generalization of the result in Section 4.2 of [21, p. 16], where the authors use a weighted orbit sum to eliminate boundary terms. We can expand the functional equation to produce a different form. Let $K_0 = K_1^* K_2^*$. We find that
\[
\Delta(K_0 \beta) = \Delta(K_0(KQ + K_{1,2}Q_{1,2}^{0,0} - \alpha)) \\
= K \Delta(K_0 Q) + \Delta(K_0 K_{1,2})Q_{1,2}^{0,0} - \Delta(K_0) \alpha,
\]
where the second equality follows since \( \Delta \) commutes with \( K \) and functions with some variables \( x_i \) missing. All that remains is to expand the various quantities involved. Then we have

\[
\Delta g_i K_i = -\Delta K_i = -(c_i - t x_i S_i^{-1} - c_i + t \overline{x}_i S_i^{-1}) = -t(\overline{x}_i - x_i) S_i^{-1},
\]
\[
\Delta K_0 = \Delta_1(K_1^\ast) \Delta_2(K_2^\ast) = t^2 (\overline{x}_1 - x_1) (\overline{x}_2 - x_2) S_1^{-1} S_2^{-1}.
\]

Next,

\[
\Delta(K_0 K_{1,2}) = \Delta_1 K_1^\ast \Delta_2 K_2^\ast K_{1,2} = \Delta_1 K_1^\ast (K_2^\ast K_{1,2} - K_2 K_{1,2}^\ast),
\]
where, after expanding, the quantity in parentheses is

\[
K_2^\ast K_{1,2} - K_2 K_{1,2}^\ast = c_1 c_2 t S_2^{-1} (\overline{x}_2 - x_2) - c_2 t \overline{x}_2 S_2^{-1} (\overline{x}_2 - x_2) = c_2 t (\overline{x}_2 - x_2) (c_1 S_2^{-1} - \overline{x}_1 S_1^{-1}) .
\]

Finally,

\[
\Delta(K_0 K_{1,2}) = c_2 t (\overline{x}_2 - x_2) \Delta_1 K_1^\ast (c_1 S_2^{-1} - \overline{x}_1 S_1^{-1}) .
\]

To simplify this quantity, note that

\[
\Delta_1 K_1^\ast c_1 S_2^{-1} = c_1 S_2^{-1} \Delta_1 K_1^\ast = -c_1 t S_2^{-1} (\overline{x}_1 - x_1),
\]
\[
\Delta_1 K_1^\ast \overline{x}_1 S_1^{-1} = S_1^{-1} \Delta_1 K_1^\ast \overline{x}_1 = c_1 (\overline{x}_1 - x_1) S_1^{-1} .
\]

Once we substitute these back into the equation, we find that

\[
\Delta(K_0 K_{1,2}) = c_1 c_2 t (\overline{x}_1 - x_1) (\overline{x}_2 - x_2) (t S_1^{-1} S_2^{-1} - S_1^{-1} S_2^{-1}) .
\]

We can write down the following corollary after we combine some of the terms and simplify them.

**Corollary 4.24.** *For a two-dimensional highly-symmetric model with a group with two nontrivial generators and \( K_0 = K_1^\ast K_2^\ast \), we have the functional equation*

\[
K \Delta(K_0 Q) = \alpha t (\overline{x}_1 - x_1) (\overline{x}_2 - x_2) \left( t S_1^{-1} S_2^{-1} (1 - b_1 b_2 Q_{1,2}^{0,0}) + S_1^{-1} S_2^{-1} Q_{1,2}^{0,0} \right) .
\]
4.2.2 Groups of Order Six

The work here is similar to the previous case so we omit most of it. First, we tabulate all the unknowns $Q_0^i$ and their orbits under $G$. For example, $g_1Q_1^0 = Q_1^0$ and we write $\emptyset$. Also

$$g_{121}Q_1^0 = g_{212}Q_1^0 = g_{21}Q_1^0.$$  \hfill (4.51)

In the following table we write a sequence $L$ for group actions $g_L$ and reduced sequences that produce each element in the orbit.

<table>
<thead>
<tr>
<th>Sequence $L$</th>
<th>Alternative</th>
<th>$Q_1^0$</th>
<th>$Q_2^0$</th>
<th>$K_1$</th>
<th>$K_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>21212</td>
<td>$\emptyset$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>21</td>
<td>1212</td>
<td>2</td>
<td>21</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>121</td>
<td>212</td>
<td>12</td>
<td>21</td>
<td>121</td>
<td>212</td>
</tr>
<tr>
<td>2121</td>
<td>12</td>
<td>12</td>
<td>1</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>12121</td>
<td>2</td>
<td>$\emptyset$</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2: Reductions of group actions on $Q_i^0$ and $K_i$ for groups of order six.

After we apply the group actions, we find six different functions that need to be eliminated. Let

$$u_1 = Q_1^0, \quad u_2 = g_2Q_1^0, \quad u_3 = g_{12}Q_1^0, \quad v_1 = Q_2^0, \quad v_2 = g_1Q_2^0, \quad v_3 = g_{21}Q_2^0.$$

There are twelve different $K_1$’s and $K_2$’s that we label sequentially as $A_1, \ldots, A_6$ and $B_1, \ldots, B_6$. For example, $A_1 = K_1$ and $A_2 = g_1K_1 = K_1$ and $A_3 = g_{21}K_1$. The matrix associated with the linear system of equations is

$$M_6 = \begin{bmatrix}
A_1 & B_1 \\
A_2 & B_2 \\
A_3 & B_3 \\
A_4 & B_4 \\
A_5 & B_5 \\
A_6 & B_6 \\
\end{bmatrix}. \quad (4.54)$$
and it has determinant

$$\det M_6 = A_2A_4A_6B_1B_3B_5 - A_1A_3A_5B_2B_4B_6. \quad (4.55)$$

The even index A’s are found by applying $g_1$ on the odd index A’s, and vice versa for the B’s. Then

$$\det M_6 = \Delta_1(A_2A_4A_6B_1B_3B_5) = \Delta_1(g_1(K_1)g_{121}(K_1)g_{12}(K_1)K_2g_{21}(K_2)g_{12}(K_2)). \quad (4.56)$$

In the following table we show the relations between A’s and B’s for the six models with groups of order six. S11, S12 and S13 have one group and S21, S22 and S23 have the other. There are no easy relations for S13 and S23. It is easy to check that when a model has nontrivial relation, \( \det M_6 = 0 \). Also, using a a computer, we can check that the determinant is non-zero in the last two cases where there are no relations.

<table>
<thead>
<tr>
<th>Model</th>
<th>Name</th>
<th>A-relations</th>
<th>B-relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>S11</td>
<td>A_1 = A_6, A_2 = A_5, A_3 = A_4</td>
<td>B_1 = B_4, B_2 = B_3, B_5 = B_6</td>
<td></td>
</tr>
<tr>
<td>S12</td>
<td>A_1 = A_4, A_2 = A_3, A_5 = A_6</td>
<td>B_1 = B_2, B_3 = B_6, B_4 = B_5</td>
<td></td>
</tr>
<tr>
<td>S13</td>
<td>None</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>S21</td>
<td>A_1 = A_6, A_2 = A_5, A_3 = A_4</td>
<td>B_1 = B_2, B_3 = B_6, B_4 = B_5</td>
<td></td>
</tr>
<tr>
<td>S22</td>
<td>A_1 = A_4, A_2 = A_3, A_5 = A_6</td>
<td>B_1 = B_4, B_2 = B_3, B_5 = B_6</td>
<td></td>
</tr>
<tr>
<td>S23</td>
<td>None</td>
<td>None</td>
<td></td>
</tr>
</tbody>
</table>

**Table 4.3:** Relations for A = $K_1$ and B = $K_2$ for groups of order six.

For the models with zero determinant, we seek the same type of relation on $\beta$ and its orbit as in the case of groups of order four. We have the following linear relation:

\[
0 = \gamma \beta - \gamma_1 \beta_1 + \gamma_2 \beta_2 - \gamma_{121} \beta_{121} + \gamma_{12} \beta_{12} - \gamma_2 \beta_2 \\
= \gamma (A_1 u_1 + B_1 v_1) - \gamma_2 (A_6 u_2 + B_6 v_1) - \gamma_1 A_2 u_1 + B_2 v_2 \\
+ \gamma_{12} (A_5 u_3 + B_5 v_2) + \gamma_{21} (A_3 u_2 + B_3 v_3) - \gamma_{121} (A_4 u_3 + B_4 v_3). \quad (4.57)
\]
Expanding out and collecting terms produces the system of equations

\[ A_1 \gamma = A_2 \gamma_1, \quad A_3 \gamma_2 = A_6 \gamma_2, \quad A_5 \gamma_12 = A_4 \gamma_121, \quad (4.58) \]

\[ B_1 \gamma = B_6 \gamma_2, \quad B_5 \gamma_12 = B_2 \gamma_1, \quad B_3 \gamma_21 = B_4 \gamma_121. \quad (4.59) \]

It turns out that it does not have a nontrivial symbolic solution, so we need to use the relations here. There is a rational solution in each case, where \( \gamma \) is a free variable. We set \( \gamma \) to be the least common multiple of the denominators. We find that \( K_0 = \gamma \) in each of the four cases with relations, as illustrated in the following table.

<table>
<thead>
<tr>
<th>Model</th>
<th>S-polynomial</th>
<th>Weight ( \gamma = K_0 )</th>
<th>K-form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 \overline{x}_2 + \overline{x}_1 + x_2 )</td>
<td>( A_3A_5B_6 )</td>
<td>( g_21(K_1)g_12(K_1)g_2(K_2) )</td>
<td></td>
</tr>
<tr>
<td>( \overline{x}_1x_2 + x_1 + \overline{x}_2 )</td>
<td>( A_2A_5B_6 )</td>
<td>( g_1(K_1)g_12(K_1)g_2(K_2) )</td>
<td></td>
</tr>
<tr>
<td>( x_1x_2 + \overline{x}_1 + \overline{x}_2 )</td>
<td>( A_2A_3B_5B_6 )</td>
<td>( g_1(K_1)g_21(K_1)g_12(K_2)g_2(K_2) )</td>
<td></td>
</tr>
<tr>
<td>( \overline{x}_1\overline{x}_2 + x_1 + x_2 )</td>
<td>( A_2B_6 )</td>
<td>( g_1(K_1)g_2(K_2) )</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.4: Weight \( K_0 \) for the orbit summation operators.

Here we have picked the representations of \( A \)'s and \( B \)'s with the least number of transformations to reach the resulting expression. We also tried finding a possible \( K_0 \) by brute force for \( S13 \) and \( S23 \), but we did not find any among products of distinct \( A \)'s and \( B \)'s. It is likely that such a weight does not exist for these cases, and that the algebraic kernel method does not extend for them.

**Theorem 4.25.** The algebraic kernel method extends for a classical interacting model with group of order six if and only if \( \det M_6 = 0 \). If we let \( \beta = KQ + K_{1,2}Q_{1,2}^{0,0} - \alpha \), then we have the functional equation \( D_G(K_0\beta) = 0 \), where \( K_0 \) depends on the step set.
4.2.3 Groups of Order Eight

As before, we omit most of the analysis. Here are the results.

<table>
<thead>
<tr>
<th>Model</th>
<th>$S$-polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_1 + \overline{x}_1 + x_1\overline{x}_2 + \overline{x}_1x_2$</td>
</tr>
<tr>
<td></td>
<td>$x_1 + \overline{x}_1 + x_1x_2 + \overline{x}_1\overline{x}_2$</td>
</tr>
</tbody>
</table>

Table 4.5: Step polynomials for models with group of order eight.

<table>
<thead>
<tr>
<th>Sequence $L$</th>
<th>Alternative</th>
<th>$Q_0^0$</th>
<th>$Q_2^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2121212</td>
<td>$\emptyset$</td>
<td>1</td>
</tr>
<tr>
<td>21</td>
<td>121212</td>
<td>2</td>
<td>21</td>
</tr>
<tr>
<td>121</td>
<td>212121</td>
<td>12</td>
<td>121</td>
</tr>
<tr>
<td>2121</td>
<td>1212</td>
<td>212</td>
<td>121</td>
</tr>
<tr>
<td>12121</td>
<td>212</td>
<td>212</td>
<td>21</td>
</tr>
<tr>
<td>212121</td>
<td>121</td>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>1212121</td>
<td>2</td>
<td>2</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

Table 4.6: Reductions of group actions on $Q_i^0$ and $K_i$ for groups of order eight.

We name the $A = K_1$ and $B = K_2$ iterates as before. We have the same $K_0$ since the relations are the same in both models. That is,

$$A_1 = A_6, \quad A_2 = A_5, \quad A_3 = A_4, \quad A_7 = A_8,$$

$$B_1 = B_4, \quad B_2 = B_3, \quad B_4 = B_8, \quad B_6 = B_7.$$ (4.60)

Theorem 4.26. The algebraic kernel method extends for all classical interacting models with group of order eight. If $\beta = KQ + K_{1,2}Q_{1,2}^0 - \alpha$, then the functional equation is $D_G(K_0\beta) = 0$ and

$$K_0 = A_2A_3B_3B_6 = g_1(K_1)g_{21}(K_1)g_{212}(K_2)g_{1212}(K_2).$$ (4.62)
Chapter 5

The Obstinate Kernel Method

Next, we explore is the obstinate kernel method. For an application in a two-dimensional interacting model, see Section 4.2 of [30, p. 16]. The method is similar to the algebraic kernel method with a weighted operator, where underneath the operator notation, all we are doing is finding more valid equations. We apply kernel-preserving actions on the master equation in both methods. However, in the obstinate kernel method we also make a substitution with one of the roots \( x_k = \sigma_k(t) \) of the kernel. This is done in order to annihilate the \( KQ \) terms from the resulting equations. However, some transformations, taken together with the substitutions, do not lead to valid functional equations because some of the generating functions fail to remain power series in \( t \) after the substitution takes place. That is, one or more of the terms \( g(Q_i^0)|_{x_k=\sigma_k} \) is not a power series in \( t \) for some transformation \( g \) from \( G \) and some \( i \) and \( k \). This inspires use the following language.

**Definition 5.1.** A function \( f(x_1, \ldots, x_d; t) \) is **admissible** if it is a valid power series in the variable \( t \), that is, there are only non-negative orders of \( t \) with non-zero coefficients. A transformation \( g \) is **admissible** if it preserves the admissibility of \( f \) with respect to a root \( \sigma_k \) of the kernel \( K \), that is, if \( g(f)|_{x_k=\sigma_k} \) is admissible.

To recap, it is necessary that when we use a substitution \( x_k = \sigma_k \), we only use transformations \( g \) that preserve the admissibility of functions \( gQ_i^0 \) for all \( i \) in the master equation. We focus our work on models with groups of order four, six and eight, where we have two nontrivial generators \( g_1 \) and \( g_2 \). In this case, \( S_i^{\pm 1} \neq 0 \) for any \( i \) and choice of sign, and the roots of the kernel are

\[
\sigma_{k}^{\pm} = \frac{1 - tS_{k}^{0} \pm \sqrt{(1 - tS_{k}^{0})^2 - 4t^2 S_{k}^{1} S_{k}^{-1}}}{2t S_{1}^{1}},
\]

(5.1)

with the identities
\[ \sigma_k^+ \sigma_k^- = \frac{S_k^{-1}}{S_k^1} \quad \text{and} \quad \sigma_k^+ + \sigma_k^- = \frac{1 - t S_k^0}{t S_k^1}. \tag{5.2} \]

If we expand the roots in series of \( t \), we find that
\[ \sigma_k^- = S_k^{-1} t + O(t^2) \quad \text{and} \quad \sigma_k^+ = \frac{1}{S_k^1 t} - \frac{S_k^0}{S_k^1} - S_k^{-1} t + O(t^2), \tag{5.3} \]
where the higher order terms are the same.

Generally, only the \( \sigma_k^- \) substitution leads to admissible \( Q^0_i \) terms for all \( i \), but it also happens that \( \sigma_k^+ \) can do this too. For an example of this, see [30, p. 11]. We focus on the first case and leave the \( \sigma_k^+ \) root analysis for future work. If we do not have enough valid equations with the \( \sigma_k^- \) substitution, we would have to see if the \( \sigma_k^+ \) substitution can help. We only use \( \sigma = \sigma_k^- \).

It suffices to show that enough actions of \( G \) preserve the admissibility of the functions \( x_1 \) and \( x_2 \), and thus, the pair \( (x_1, x_2) \), with respect to a root, to show that the functions \( Q^0_k \) are also admissible under the actions and substitution. That is, we only need to check for which \( j, k \) and \( g \) is \( g(x_j)|_{x_k=\sigma_k} \) admissible. Luckily, some models pass this test and we obtain enough valid functional equations, so that we can use the obstinate kernel method. For those cases where we find less equations, it may be possible to show that the \( Q^0_k \) terms are power series in \( t \) via another method, like in [30].

**Proposition 5.2.** Consider a classical two-dimensional model with group of order four, six or eight. Models with the same group have the same admissibility properties for \( x_1 \) and \( x_2 \), and the pair \( (x_1, x_2) \). Models with group of order four all have the same admissibility properties.

**Proof.** All we need to do is compute the actions on the functions and check if substituting \( \sigma_k^- \) leads to admissible ones. This is tedious to do by hand but straightforward to do on a computer. The results are summarized in tables in the following sections. However, we can perform the analysis for groups of order four without difficulty.

The transformations \( 1 \) and \( g_2 \) fix \( x_1 \), but the transformations \( g_1 \) and \( g_1 g_2 \) change it to \( \overline{\sigma}_1 S_1^{-1}/S_1^1 \). Note that \( \overline{\sigma}_1 = \sigma_1^+ S_1^1/S_1^{-1} \), so \( g_1(x_1)|_{x_k=\sigma_1} = \sigma_1^+ \) and this function is not admissible. Therefore \( x_1 \)'s admissibility with respect to \( \sigma_1 \) is preserved only by the transformations \{1, \( g_2 \}\}.

Since models with groups of order four need to have one symmetry across an axis, without loss of generality, we can assume the symmetry is across the \( x_1 \) axis. That is, \( g_2(x_2) = \overline{x}_2 \), and the \( x_1 = \sigma_1 \) substitution trivially leads to an admissible function. Therefore, all of \( G \) preserves the admissibility of \( x_2 \) with respect to \( \sigma_1 \) and finally, only \( 1 \) and \( g_2 \) preserve the admissibility for both \( x_1 \) and \( x_2 \) with respect to \( \sigma_1 \). The analysis is analogous for the \( \sigma_2 \) substitution. \( \square \)
5.1 Initial Computations

This classification shows that we can treat all models with group of order four with the same method, but the analysis is more complicated for groups of order six or eight. Now, suppose that we have a set of valid equations for a substitution \( x_k = \sigma_k \), forming a linear system in the unknowns \( Q_{1,2}^{0,0} \), \( Q_1^0 \), \( Q_2^0 \), and their orbits under the valid group actions. We hope for redundancy in these equations, because it allows us to eliminate some of the unknowns. That is, we find a weighted operator that, paired with the substitution \( x_k = \sigma_k \), eliminates the unwanted terms. It turns out that it is enough to have \(|G|/2\) transformations preserving the admissibility for both \( x_1 \) and \( x_2 \).

5.1.1 Groups of Order Four

We start with the admissibility table for groups of order four.

<table>
<thead>
<tr>
<th>Root</th>
<th>( x_1(t) )</th>
<th>( x_2(t) )</th>
<th>( x_1(t) ) and ( x_2(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_1(t) )</td>
<td>1, ( g_2 )</td>
<td>All</td>
<td>1, ( g_2 )</td>
</tr>
<tr>
<td>( \sigma_2(t) )</td>
<td>All</td>
<td>1, ( g_1 )</td>
<td>1, ( g_1 )</td>
</tr>
</tbody>
</table>

Table 5.1: Admissibility-preserving transformations with respect to roots of the kernel with group of order four.

The different \( \sigma \) substitutions lead to essentially the same analysis, so we will only work with the first one, together with actions 1 and \( g_2 \). Let \( F^* \) denote \( g_2(F) \). The functional equations corresponding the the group actions are

\[
K Q - K_1 Q_1^0 - K_2 Q_2^0 + K_{12} Q_{1,2}^{0,0} = \alpha, \quad (5.4)
\]

\[
K Q^* - K_1^* Q_1^{0,*} - K_2^* Q_2^{0,*} + K_{12}^* Q_{1,2}^{0,0} = \alpha. \quad (5.5)
\]

Once we make the substitution, the function \( Q_2^0 \) is difficult to expand, so we want to eliminate it first. We multiply the equations by \( K_2^* \) and \( K_2 \), respectively, and subtract the second one from the first one. The resulting equation is

\[
K_2^* K Q - K_2 K Q^* - (K_2^* K_1 Q_1^0 - K_2 K_1^* Q_1^{0,*}) + (K_2^* K_{12} - K_2 K_{12}^*) Q_{1,2}^{0,0} - \alpha (K_2^* - K_2) = 0, \quad (5.6)
\]
where we can set \( x_1 = \sigma_1 \) to eliminate the \( K \) term. The substitution is implicit in the notation from now on. We have

\[
(K_2^* K_1 Q_1^0 - K_2 K_1^* Q_1^{0,*}) - (K_2^* K_{12} - K_2 K_{12}^*) Q_{1,2}^{0,0} + \alpha (K_2^* - K_2) = 0
\]  
(5.7)

and we can recast this elimination via the weighted orbital operator

\[
D_2 = g_2(K_2) \cdot 1 - K_2 g_2 = (1 - g_2) K_2^*.
\]  
(5.8)

If we apply it on the master equation, we obtain

\[
D_2(-K Q + K_1 Q_1^0 + K_2 Q_2^0 - K_{12} Q_{1,2}^{0,0} + \alpha) = 0,
\]

\[
-D_2(K Q) + D_2(K_1 Q_1^0) - D_2(K_{12} Q_{1,2}^{0,0}) + D_2(\alpha) = 0,
\]  
(5.9)

and once we make the \( \sigma_1 \) substitution, we find that

\[
D_2(K_1 Q_1^0) - D_2(K_{12}) Q_{1,2}^{0,0} + D_2(\alpha) = 0
\]  
(5.10)

and this is the same as Equation 5.6. The following notation is useful.

**Definition 5.3.** Let \( L \) be a concatenated list, \( \Delta_L = 1 - g_L \) and let \( X \) be some function. The operator \( D_L[X] \) is defined by its action on functions \( F \) as

\[
D_L[X](F) = \Delta_L(X F) = (1 - g_L)(X F) = X F - g_L(X F).
\]  
(5.11)

We drop the \( X \) from the notation if it is clear from context. In particular, \( D_L[1] = \Delta_L \).

For example, \( D_2 = D_2[K_2^*] = D_2[g_2(K_2)] \).

**Example 5.4.** If we use \( g_L(X) \) instead of \( X \) in the definition above, we obtain the Ore operator

\[
D_L[g_L(X)] = g_L(X) - X g_L.
\]  
(5.12)

We can expand some of the terms in the functional equation. First,

\[
D_2(\alpha) = \alpha (K_2^* - K_2)
\]

\[
= \alpha ((c_2 - t x_2 \bar{p}_2 S_2^{-1}) - (c_2 - t \bar{x}_2 S_2^{-1}))
\]

\[
= \alpha t (\bar{x}_2 - x_2 \bar{p}_2(\sigma_1)) S_2^{-1}(\sigma_1).
\]  
(5.13)
Then,
\[
D_2(K_{12}) = K_2^* K_{12} - K_2 K_{12}^*
\]
\[
= c_1 c_2 t(x_2 - x_2 \bar{\mu}_2) S_2^{-1} - c_2 t \bar{\sigma}_1 (x_2 - x_2 \bar{\mu}_2) S_{1,2}^{-1} \tag{5.14}
\]
\[
= c_2 t(x_2 - x_2 \bar{\mu}_2(\sigma_1))(c_1 S_2^{-1}(\sigma_1) - \bar{\sigma}_1 S_{1,2}^{-1}).
\]

Lemma 5.5. For models with groups of order four,
\[
D_2(K_1 Q^0) - D_2(K_{12}) Q_{1,2}^{0,0} + D_2(\alpha) = 0, \tag{5.15}
\]
where
\[
D_2(\alpha) = \alpha t(x_2 - x_2 \bar{\mu}_2(\sigma_1)) S_2^{-1}(\sigma_1), \tag{5.16}
\]
\[
D_2(K_{12}) = c_2 t(x_2 - x_2 \bar{\mu}_2(\sigma_1))(c_1 S_2^{-1}(\sigma_1) - \bar{\sigma}_1 S_{1,2}^{-1}). \tag{5.17}
\]

If the model is highly-symmetric, we see that \(g_2 S_1^{-1} = S_1^{-1}\) and therefore, \(g_2 K_1 = K_1\), and we can bring that factor outside of the operator, so \(D_2(K_1^* F) = K_1 D_2(F)\) for functions \(F\). It turns out that this happens for both substitutions if and only if the model is highly-symmetric, meaning that the analysis is easier only in those cases. This is easily verified using a computer since there are not that many models with group of order four. We investigate the highly-symmetric models in more detail in Section 5.2.

Proposition 5.6. Consider a classical model with a group of order four. If \(i \neq j\), then \(D_i[g_i K_j] = K_j \Delta_i\) in the \(\sigma_j\) substitutions if and only if the model is symmetric across the \(x_i\) axis. That is, the \(K\) functions factor out of the operators in both cases if and only if the model is highly-symmetric.

5.1.2 Groups of Order Six

We start with the admissibility tables for the two groups of order six.

<table>
<thead>
<tr>
<th>Root</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>((x_1, x_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_1)</td>
<td>1, (g_2, g_{21}, g_{121})</td>
<td>1, (g_1, g_2, g_{21})</td>
<td>1, (g_2, g_{21})</td>
</tr>
<tr>
<td>(\sigma_2)</td>
<td>1, (g_1, g_2, g_{12})</td>
<td>1, (g_1, g_{12}, g_{121})</td>
<td>1, (g_1, g_{12})</td>
</tr>
</tbody>
</table>

Table 5.2: Admissibility-preserving transformations with respect to roots of the kernel with group of order six for the models \(S11, S12, S13\).
Table 5.3: Admissibility-preserving transformations with respect to roots of the kernel with group of order six for the models $S_{21}, S_{22}, S_{33}$.

<table>
<thead>
<tr>
<th>Root</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$(x_1, x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1$</td>
<td>$1, g_{22}, g_{121}$</td>
<td>$1, g_1, g_{12}, g_{121}$</td>
<td>$1, g_{121}$</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>$1, g_{22}, g_{121}$</td>
<td>$1, g_1, g_{12}, g_{121}$</td>
<td>$1, g_{121}$</td>
</tr>
</tbody>
</table>

We focus on the first three models because they are amenable to the straight-forward method. The latter three need more work, since there are too few valid actions.

Figure 5.1: Step sets for models $S_{11}, S_{12}, S_{13}$.

For consistency, we work with $\sigma_1$ where the actions are 1, $g_2$, $g_{21}$. From the previous chapter, we know these actions reduce on the function $Q_0^0$.

Table 5.4: Reductions of group actions on $Q_1^0$ and $Q_2^0$, group of order four.

<table>
<thead>
<tr>
<th>Sequence $L$</th>
<th>$Q_1^0$</th>
<th>$Q_2^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varnothing$</td>
<td>$\varnothing$</td>
<td>$\varnothing$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$\varnothing$</td>
</tr>
<tr>
<td>21</td>
<td>21</td>
<td>21</td>
</tr>
</tbody>
</table>

As in the previous section, we wish to eliminate $X_2 = g_2Q_1^0$ and $Y = Q_2^0$ because they involve $\sigma$ once we make the substitution. Let

$$\beta = K_{12}Q_{1,2}^{0,0} - \alpha = \alpha(b_1b_2Q_{1,2}^{0,0} - 1) \tag{5.18},$$

so that we have

$$AX + BY = \beta, \tag{5.19}$$
$$A_2X_2 + B_2Y = \beta_2, \tag{5.20}$$
$$A_{21}X + B_{21}Y_{21} = \beta_{21}. \tag{5.21}$$
We want a linear combination of $\beta$, $\beta_2$ and $\beta_21$ that eliminates $X_2$ and $Y$. In this case, we can write

\begin{align*}
a\beta - b\beta_2 + c\beta_21 &= aAX + (aB - bB_2)Y + (cA_21 - bA_2)X_2 + cB_21Y_21 \\
&= aAX + cB_21Y_21 ,
\end{align*}

and for the elimination to work, we need $aB = bB_2$ and $cA_21 = bA_2$. One solution for $a$, $b$ and $c$ without denominators is

\begin{equation}
a = A_21B_2 , \quad b = A_21B , \quad c = A_2B .
\end{equation}

The elimination procedure yields

\begin{equation}
AA_21B_2Q_1^0 + A_2BB_21g_21Q_2^0 = A_21B_2\beta - A_21B\beta_2 + A_2B\beta_21 \\
= (A_21B_2 - A_21B + A_2B)\beta
\end{equation}

and we can recast the whole process equation into operator notation.

**Proposition 5.7.** Let

\begin{equation}
D(F) = A_21B_2F - A_21Bg_2(F) + A_2Bg_21(F) .
\end{equation}

Then

\begin{equation}
D(AX + BY) = AA_21B_2Q_1^0 + A_2BB_21g_21Q_2^0 = \beta D(1) .
\end{equation}

For each case, we can compute $D(1)$, but since the expressions are somewhat complicated and contain a square root, we do not list them here. Terms $AA_21B_2$ and $A_2BB_21$ contain roots too.

### 5.1.3 Groups of Order Eight

We start with the admissibility tables for the two groups of order eight.

<table>
<thead>
<tr>
<th>Root</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$(x_1, x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1$</td>
<td>$1, g_2, g_21, g_212$</td>
<td>All</td>
<td>$1, g_2, g_21, g_212$</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>$1, g_1, g_2, g_12, g_121, g_2121$</td>
<td>$1, g_1, g_12, g_121$</td>
<td>$1, g_1, g_12, g_121$</td>
</tr>
</tbody>
</table>

**Table 5.5:** Admissibility-preserving transformations with respect to roots of the kernel with group of order eight for model $S1$. 

60
Table 5.6: Admissibility-preserving transformations with respect to roots of the kernel with group of order eight for model $S2$.

We work on the model with step polynomial

$$ S = \overline{x}_1 + x_1 + \overline{x}_1 x_2 + x_1 \overline{x}_2, \quad (5.28) $$

since it has enough admissibility-preserving actions: $1, g_2, g_{21}, g_{212}$. Note that the model here is equivalent to the one seen in [30], because they have the same step set and master functional equation, so the analysis is the same. Using our notation and operators, we are able to redo some of the authors’ derivations in a cleaner way. First, we have the table of reductions for group action in this model.

<table>
<thead>
<tr>
<th>Sequence L</th>
<th>$Q^0_1$</th>
<th>$Q^0_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>21</td>
<td>2</td>
<td>21</td>
</tr>
<tr>
<td>212</td>
<td>212</td>
<td>21</td>
</tr>
</tbody>
</table>

Table 5.7: Reductions of group actions that preserve admissibility with respect to $\sigma_1$.

Similarly to previous sections, let $\beta = K_{12} Q^0_{1,2} - \alpha = \alpha (b_1 b_2 Q^0_{1,2} - 1)$ and $X = Q^0_1$ and $Y = Q^0_2$. Note that $\beta$ is left unchanged by group actions. We have the four equations

$$ AX + BY = \beta, \quad (5.29) $$
$$ A_2 X_2 + B_2 Y = \beta, \quad (5.30) $$
$$ A_{21} X_2 + B_{21} Y_{21} = \beta, \quad (5.31) $$
$$ A_{212} X_{212} + B_{212} Y_{21} = \beta, \quad (5.32) $$

where we wish to eliminate the functions $X_2$, $Y$ and $Y_{21}$ because they contain $\sigma_1$ as part of their arguments. To do this, we first use the first two and latter two equations to eliminate $Y$ and $Y_{21}$, respectively. That is, we compute $B_2$ times the first equation minus $B$ times the second equation,
and $B_{212}$ times the third equation minus $B_{21}$ from the last equation. Then we use the resulting two equations to eliminate $X_2$. The first step of the procedure is the same as applying the following two operators on the master functional equation and substituting $x_1 = \sigma_1$. They are
\begin{align}
D_2[B_2] &= B_2 - Bg_2, \quad (5.33) \\
-g_{212}D_2[B_2] &= g_{212}Bg_2 - g_{212}B_2 = B_{212}g_{21} - B_{21}g_{212}, \quad (5.34)
\end{align}
and we obtain the equations
\begin{align}
AB_2X - A_2BX_2 &= \beta(B_2 - B), \quad (5.35) \\
A_{21}B_{212}X_2 - A_{212}B_{21}X_{212} &= \beta(B_{212} - B_{21}). \quad (5.36)
\end{align}

To eliminate $X_2$, we multiply the equations by $A_{21}B_{212}$ and $A_2B$, add them together and obtain
\begin{align}
A_{21}B_{212}AB_2X - A_2BA_{212}B_{21}X_{212} &= \beta(A_{21}B_{212}(B - B_2) + A_2B(B_{212} - B_{21})). \quad (5.37)
\end{align}

This final equation is equivalent to applying the following operator on the master functional equation:
\begin{align}
D &= A_{21}B_{212}D_2[B_2] - A_2Bg_{212}D_2[B_2] \\
 &= (A_{21}B_{212} - A_2Bg_{212})D_2[B_2] \\
 &= (A_{21}B_{212} - g_{212}A_{21}B_{212})D_2[B_2] \\
 &= D_{212}[A_{21}B_{212}]D_2[B_2]. \quad (5.38)
\end{align}

From the master functional equation, we have
\begin{align}
D(\beta) &= \beta D_{212}[A_{21}B_{212}](B_2 - B), \quad (5.39) \\
D(AX + BY) &= D(AX) \\
 &= D_{212}[A_{21}B_{212}]D_2[B_2](AX) \\
 &= D_{212}[A_{21}B_{212}](AB_2X - A_2BX_2) \quad (5.40) \\
 &= D_{212}[A_{21}B_{212}](AB_2X) \\
 &= D_{212}[AA_{21}B_2B_{212}](X).
\end{align}

Note that $A_{212} = A$, so we can factor out $A$ from the operator. We have the following result.
 Proposition 5.8. For the first model with group of order eight,

\[ AD_{212}[A_{21}B_2B_{212}] (Q^0_1) = \beta D_{212}[A_{21}B_{212}](B_2 - B), \]  

(5.41)

where \( A_L = g_L(K_1), B_L = g_L(K_2) \) and \( \beta = \alpha(b_1b_2Q^0_{1,2} - 1) \), and the equality is valid only once we set \( x_1 = \sigma_1 \) in this expression.

To obtain an analogous statement for the \( \sigma_2 \) substitution, switch the \( A \)'s and the \( B \)'s, and the 1's and the 2's. It was found in [30] that the coefficients in front of the \( X \) and \( X_{212} \) terms become Laurent polynomials with the \( \sigma_k \) substitutions. The coefficient in front of \( \beta \) can be reduced to a function involving a square root. The calculations are complicated and would be tedious to do by hand. The square root part does not depend on \( a_2 \) in the first case, and \( a_1 \) in the second case. This observation allows the authors to find a relation for the different versions of \( Q^0_{1,2} \).

### 5.2 Highly-symmetric Two-dimensional Models

For highly-symmetric models, the first equation in Lemma 5.5 has the simplified form

\[ x_1 = \sigma_1 : \Delta_2(K^*Q^0_1) = \frac{1}{K_1} \left( \Delta_2(K_1^*K_2)Q^0_{1,2} - \alpha(K_1^* - K_2) \right), \]  

(5.42)

\[ x_2 = \sigma_2 : \Delta_1(K^1Q^0_2) = \frac{1}{K_2} \left( \Delta_1(K_1^*K_2)Q^0_{1,2} - \alpha(K_1^* - K_1) \right). \]  

(5.43)

It is enough to perform the analysis for one of the equations, since only the model with two horizontal or vertical steps removed requires working with both equations. In highly-symmetric models, \( \mu_2 = S^{-1}_2 / S^1_2 = 1 \) and

\[ \alpha(K_2^* - K_2) = \alpha t(x_2 - x_2)S_2^{-1}(\sigma_1), \]  

(5.44)

\[ \Delta_2(K_2^*K_2) = c_2 t(x_2 - x_2)(c_1S_2^{-1}(\sigma_1) - \sigma_1S_1^{-1,1}). \]  

(5.45)

To compute \( \Delta_2(K^*Q^0_1) \), we need to compute \( K_2(\sigma_1, x_2) \) and \( K_2^*(\sigma_1, x_2) \) given by

\[ K_2 = c_2 - t(x_2S_2^{-1}(\sigma_1)) \quad \text{and} \quad K_2^* = c_2 - tx_2S_2^{-1}(\sigma_1), \]  

(5.46)

where we can find that \( S_2^{-1}(\sigma) \) is a rational function given by

\[ S_2^{-1}(\sigma_1) = \frac{S_{1,2}^{1,1} + tS_{1,2}^{0,1}S_{1,2}^{0,0}}{t(S_{1,2}^{1,0} + S_{1,2}^{1,1}(x_2 + x_2))}. \]  

(5.47)
Here is a table summarizing the values of this function for the different models.

<table>
<thead>
<tr>
<th>Model</th>
<th>$S_{1,2}^{0,1}$</th>
<th>$S_{1,2}^{1,0}$</th>
<th>$S_{1,2}^{1,1}$</th>
<th>$S_2^{-1}(\sigma_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\times$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\bar{t}/(\bar{x}_2 + x_2)$</td>
</tr>
<tr>
<td>$\times$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\bar{t}/(\bar{x}_2 + x_2)$</td>
</tr>
<tr>
<td>$\times$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\bar{t}/(\bar{x}_2 + x_2)$</td>
</tr>
<tr>
<td>$\times$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$(1 + \bar{t})/(\bar{x}_2 + 1 + x_2)$</td>
</tr>
<tr>
<td>$\times$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$(1 + \bar{t})/(\bar{x}_2 + 1 + x_2)$</td>
</tr>
</tbody>
</table>

Table 5.8: Functions $S_2^{-1}(\sigma_1)$ for highly-symmetric models.

Lemma 5.9. For highly-symmetric models with $e = S_{1,2}^{1,1}$, $f = S_{1,2}^{1,0}$ and $g = S_{1,2}^{0,1}$, we have

\[ \frac{\alpha(K_2^* - K_2)}{K_1} = \frac{\alpha(\bar{x}_2 - x_2)(e + t f g)}{S_1^1 K_1}, \]
\[ \frac{\Delta_2(K_2^* K_{12})}{K_1} = \frac{c_2(\bar{x}_2 - x_2)(c_1(e + t f g) - e t \bar{x}_1 S_1^1)}{S_1^1(c_1 - t \bar{x}_1 S_1^1)}. \]

(5.48)
(5.49)

5.2.1 Cartesian Model – Steps N, S, E, W

Since $S_2^{-1}(\sigma_1) = 1$, this is the simplest case to analyze. We essentially repeat the calculations of Proposition 2.6, but for two dimensions. We obtain a rational relation for $Q_{1,2}^{0,0}$, instead of a linear one. Afterwards, we find an expression for $Q$ in terms of its partially-interacting versions. First,

\[ \alpha(K_2^* - K_2) = \alpha t(\bar{x}_2 - x_2) \quad \text{and} \quad \Delta_2(K_2^* K_{12}) = c_1 c_2 t(\bar{x}_2 - x_2). \]

(5.50)

Then, if we set $\beta = 1 - b_1 b_2 Q_{1,2}^{0,0}$ and use the fact that $K_1$ does not depend on $x_1$ or $c_2$,

\[ \frac{1}{K_1} (\alpha(K_2^* - K_2) - \Delta_2(K_2^* K_{12})) = \alpha \beta t(\bar{x}_2 - x_2) \frac{K_1}{K_1} = \alpha \beta f(x_2, a_1; t). \]

(5.51)

Then we can rewrite the functional equation as

\[ \Delta_2(K_2^* Q_1^0) + \alpha \beta f(x_2, a_1; t) = 0, \]

(5.52)
and once we expand the first using Equation 5.46, we have
\[(c_2 - tx_2)Q_1^0(x_2) - (c_2 - t\bar{x}_2)\bar{Q}_1^0(\bar{x}_2) + \alpha \beta f = 0.\] (5.53)

We extract the \(x_2\) and \(x_2^0\) terms to find the equations
\[c_2 Q_{1,2}^{0,1} - t Q_{1,2}^{0,0} + \alpha \beta f_2 = 0,\] (5.54)
\[-t\bar{Q}_{1,2}^{0,0} + \bar{a}_1 f_2 = 0.\] (5.55)

We can substitute the latter equation in the first one to obtain the identity
\[b_2 Q_{1,2}^{0,1} = t(a_2 Q_{1,2}^{0,0} - \beta \bar{Q}_{1,2}^{0,0}).\] (5.56)

The proof of the following result is similar to the one we see in Corollary 5.13, so we omit it.

**Lemma 5.10.** For non-negative integers \(m\) and \(n\),
\[b_2 Q_{1,2}^{m,n+1} = t(a_2 Q_{1,2}^{m,n} - \beta \bar{Q}_{1,2}^{m,n}),\] (5.57)
\[b_1 Q_{1,2}^{m+1,n} = t(a_1 Q_{1,2}^{m,n} - \beta \hat{Q}_{1,2}^{m,n}).\] (5.58)

From the first equation we have
\[b_2 Q_{1,2}^{1,1} = t(a_2 Q_{1,2}^{1,0} - \beta \bar{Q}_{1,2}^{1,0})\] (5.59)
and if we multiply it by \(b_1\) and use the relations in the lemma, we find that
\[b_1 b_2 Q_{1,2}^{1,1} = ta_2 t(a_1 Q_{1,2}^{0,0} - \beta \bar{Q}_{1,2}^{0,0}) - t\beta t(a_1 \bar{Q}_{1,2}^{0,0} - \beta \bar{Q}_{1,2}^{0,0})
\[= t^2(a_1 a_2 Q_{1,2}^{0,0} - a_2 \beta \bar{Q}_{1,2}^{0,0} - a_1 \bar{Q}_{1,2}^{0,0} + \beta^2 \bar{Q}_{1,2}^{0,0}).\] (5.60)

Let \(G = Q_{1,2}^{0,0}, \hat{G} = G_1\) and \(\tilde{G} = G_2\). From the \(x_1^0 x_2^0\) term of the master equation, we have
\[Q_{1,2}^{0,0} = 1 + ta_1 a_2(Q_{1,2}^{1,0} + Q_{1,2}^{0,1}).\] (5.61)

Multiplying it by \(b_1 b_2\) and using the relations above, leads to
\[b_1 b_2 G = b_1 b_2 + ta_1 a_2(b_2 t(a_1 G - \beta G_1) + b_1 t(a_2 G - \beta G_2)
\[= b_1 b_2 + t^2 a_1 a_2(b_2 a_1 G + b_1 a_2 G - b_2 \beta G_1 - b_1 \beta G_2).\] (5.62)
Using the relation for $\beta$ and simplifying the expression allows us to write $G$ as

$$G = \frac{b_1b - 2 - t^2a_1a_2(b_2G_1 + b_1G_2)}{b_1b_2 - t^2a_1a_2(a_2b_1 + a_1b_2 + b_1b_2(b_2G_1 + b_1G_2))}. \quad (5.63)$$

If we let $F = (b_2G_1 + b_1G_2)$ and perform the long division with respect to $F$, we find that

$$Q_{1,2}^{0,0} = \frac{1}{b_1b_2} \left( 1 - \frac{b_1b_2(1 - b_1b_2) - t^2a_1a_2(a_1b_2 + a_2b_1)}{b_1b_2 + t^2a_1a_2(a_1b_2 + a_2b_1 + b_1b_2F)} \right), \quad (5.64)$$

so $G$ is a rational function of $G_1$ and $G_2$. To find an equation for $Q$ we need to first take the second set of equations in Lemma 5.10, multiply them by $x_2^n$ and sum them up. The resulting equations are

$$b_1Q_1^{m+1} = t(a_1Q_1^m - \beta \hat{Q}_1^m) \quad (5.65)$$

and we can sum them with respect to $x_1$ after, multiply them by $x_1^{n+1}$, to find

$$b_1(Q - Q_1^0) = x_1t(a_1Q - \beta \hat{Q}). \quad (5.66)$$

We can solve this it for $Q_1^0$ as obtain

$$b_1Q_1^0 = (b_1 - a_1x_1t)Q + x_1t\beta \hat{Q}. \quad (5.67)$$

We substitute this expression, along with an analogous one for $Q_2^0$, into the original functional equation, and after some algebra, we find that

$$Q = \frac{(1 - b_1b_2G)(\alpha + tx_1\bar{b}_1K_1\hat{Q} + tx_2\bar{b}_2K_2\hat{Q})}{K - (1 - \bar{c}_1x_1t)K_1 - (1 - \bar{c}_2x_2t)K_2}. \quad (5.68)$$

We recognize some of the quantities as modified $K_i$ terms.

**Theorem 5.11.** The Cartesian model has generating functions satisfying the equations

$$Q = \frac{(1 - b_1b_2Q_{1,2}^{0,0})(\alpha - \bar{b}_1K_1\hat{K}_1^*\hat{Q} - \bar{b}_2K_2\hat{K}_2^*\hat{Q})}{K - \bar{c}_1K_1K_1^* - \bar{c}_2K_2K_2^*}. \quad (5.69)$$

and

$$Q_{1,2}^{0,0} = G = \frac{1}{b_1b_2} \left( 1 - \frac{b_1b_2(1 - b_1b_2) - t^2a_1a_2(a_1b_2 + a_2b_1)}{b_1b_2 + t^2a_1a_2(a_1b_2 + a_2b_1 + b_1b_2(b_2\hat{G} + b_1\hat{G}))} \right). \quad (5.70)$$

This shows that $Q$ is built from $Q_{1,2}^{0,0}$ and from versions of itself, each with a single interaction.
5.2.2 Diagonal Cartesian Model – Steps NE, NW, SE, SW

This model is similar to a model of two interacting walks confined in a semi-infinite strip, covered in [21]. The models are different because there is an extra term in their functional equation. Through calculations like the ones in the previous subsection, we find that $Q_{1,2}^{0,0}$ has a linear relationship with its partially-marked versions. However we do not find an identity for $Q$. In this case,

$$\frac{\alpha(K_2^* - K_2)}{K_1} = \frac{\alpha(\bar{x}_2 - x_2)}{K_1(\bar{x}_2 + x_2)}, \quad (5.71)$$

and the functional equation becomes

$$K_2^* Q_1 - K_2 Q_1^* - \frac{c_2(\bar{x}_2 - x_2)}{\bar{x}_2 + x_2} Q_{1,2} + \alpha \frac{\bar{x}_2 - x_2}{(\bar{x}_2 + x_2)K_1} = 0. \quad (5.73)$$

Surprisingly enough, the only remaining term depending on $\sigma_1$ is $K_1$, so it is of no consequence to the rest of the analysis. To clear denominators, we multiply the equation by $\bar{x}_2 + x_2$. Here,

$$R = (\bar{x}_2 + x_2) \bar{K}_2 = c_2(\bar{x}_2 + x_2) - \bar{x}_2 = c_2 x_2 - \bar{x}_2 \bar{x}_2, \quad (5.74)$$

and this function satisfies the identities

$$R^* - R = v(K_2^* - K_2) = \bar{x}_2 - x_2, \quad (5.75)$$

and

$$\bar{x}_2 R^* - x_2 R = \bar{x}_2 c_2 v - 1 - x_2 c_2 v + 1 = c_2 v(\bar{x}_2 - x_2). \quad (5.76)$$

Finally, using the function $\delta_0 = \bar{a}_1(\bar{x}_2 - x_2)\bar{K}_1$, we can rewrite the equation as

$$R^* Q_1^0 - R Q_1^{0,*} - c_2(\bar{x}_2 - x_2) Q_{1,2}^{0,0} + \bar{a}_2 \delta_0 = 0. \quad (5.77)$$

Next, we work towards finding equations like those in Lemma 5.10 but for diagonal steps. To do that, we need to find higher order equations like (5.77). We start with the master functional equation

$$(1 - tS)Q - (c_1 - tv\bar{x}_1)Q_1^0 - (c_2 - t\bar{x}_2 S_2^{-1})Q_2^0 + (c_1 c_2 - t\bar{x}_1 \bar{x}_2)Q_{1,2}^{0,0} = \bar{a}_1 \bar{x}_2. \quad (5.78)$$

Upon extracting the $x_1^0$ coefficient, we find that

$$Q_1^0 = \bar{a}_2 + c_2 Q_{1,2}^{0,0} + a_1 tv Q_1^1. \quad (5.79)$$
We can extract higher order coefficients too. Let \( k \geq 1 \). Then the \( x_2^k \) coefficient is

\[
0 = Q_1^k - tS_1^{-1}Q_1^{k+1} - tS_1^0Q_1^k - tS_1^1Q_1^{k-1} - c_2Q_1^{k,0} \\
+ t\bar{x}_2(S_1^{-1,0}Q_1^{k+1,0} + S_1^{0,-1}Q_1^{k,0} + S_1^{1,-1}Q_1^{k-1,0}) \\
= Q_1^k - tvQ_1^{k+1} - tvQ_1^{k-1} - c_2Q_1^{k,0} + t\bar{x}_2(Q_1^{k+1,0} + Q_1^{k-1,0}) .
\]

(5.80)

We can rewrite the last one in a form more convenient for the next few steps:

\[
Q_1^k = tv(Q_1^{k+1} + Q_1^{k-1}) + c_2Q_1^{k,0} - t\bar{x}_2(Q_1^{k+1,0} + Q_1^{k-1,0}) .
\]

(5.81)

We have the equations

\[
Q_1^0 = \bar{a}_2 + c_2Q_1^{0,0} + a_1tvQ_1^1 ,
\]

(5.82)

\[
Q_1^{0,*} = \bar{a}_2 + c_2Q_1^{0,0} + a_1tvQ_1^{1,*} .
\]

(5.83)

Combining them with \( \text{Equation 5.77} \) gives

\[
0 = R^*Q_1^0 - RQ_1^{0,*} - c_2(\bar{x}_2 - x_2)Q_1^{0,0} + \bar{a}_2\delta_0 \\
= R^*(\bar{a}_2 + c_2Q_1^{0,0} + a_1tvQ_1^1) - R(\bar{a}_2 + c_2Q_1^{0,0} + a_1tvQ_1^{1,*}) \\
- c_2(\bar{x}_2 - x_2)Q_1^{0,0} + \bar{a}_2\delta_0 \\
= a_1tv(R^*Q_1^1 - RQ_1^{1,*}) + (R^* - R)(\bar{a}_2 + c_2Q_1^{0,0}) \\
- c_2(\bar{x}_2 - x_2)Q_1^{0,0} + \bar{a}_2\delta_0 \\
= a_1tv(R^*Q_1^1 - RQ_1^{1,*}) + \bar{a}_2((\bar{x}_2 - x_2) + \delta_0) ,
\]

(5.84)

and we have the equation

\[
R^*Q_1^1 - RQ_1^{1,*} + \bar{a}_2(\bar{x}_2 - x_2) + \delta_0 \\
+ a_1tv = R^*Q_1^1 - RQ_1^{1,*} + \bar{a}_2\delta_1 = 0 .
\]

(5.85)

This equation together with \( \text{Equation 5.77} \) form the base cases for the following result.

**Lemma 5.12.** Let \( m \) be an integer greater than zero. Then

\[
R^*Q_1^m - RQ_1^{m,*} - c_2(\bar{x}_2 - x_2)Q_1^{m,0} + \bar{a}_2\delta_m = 0 ,
\]

(5.86)

where \( \delta_m \) is not a function of \( a_2 \).
Proof. We prove this by induction. We have the base cases $m = 0$ and $m = 1$. The latter one fits the pattern because $Q^{1,0}_{1,2} = 0$, since we cannot have walks ending on $(1, 0)$. Assume that the formula is true for all cases up to and including $m$. Then we use Equation 5.81 to calculate

$$R^*Q^m_1 - RQ^{m,*}_1 = tv(R^*Q^{m+1}_1 - RQ^{m+1,*}_1) + tv(R^*Q^{m-1}_1 - RQ^{m-1,*}_1) + c_2(R^* - R)Q^{m,0}_{1,2} - t(x_2R^* - x_2R)(Q^{m+1,0}_{1,2} + Q^{m-1,0}_{1,2})$$

$$= tv(R^*Q^{m+1}_1 - RQ^{m+1,*}_1) + tv(c_2(x_2 - x_2)Q^{m+1,0}_{1,2} - \overline{a}_2\delta_{m-1})$$

$$= tv(R^*Q^{m+1}_1 - RQ^{m+1,*}_1) + c_2(x_2 - x_2)Q^{m,0}_{1,2} - c_2tv(x_2 - x_2)(Q^{m+1,0}_{1,2} + Q^{m-1,0}_{1,2})$$

Combining this expression with the equation from the lemma leads to

$$0 = R^*Q^{m+1}_1 - RQ^{m+1,*}_1 - c_2(x_2 - x_2)Q^{m+1,0}_{1,2} + \overline{a}_2\delta_{m+1},$$

where

$$\delta_{m+1} = \frac{\delta_m}{tv} - \delta_{m-1}$$

is independent from $a_2$. \( \square \)

Of course, there is an analogous statement for $Q^m_2$ if we follow the same analysis with the variable names switched. We can expand $R^*$ and $R$ to a form we can use:

$$0 = (c_2x_2 - \overline{a}_2x_2)Q^{m}_1(x_2) - (c_2x_2 - \overline{a}_2x_2)Q^{m}_1(x_2)$$

$$- c_2(x_2 - x_2)Q^{m,0}_{1,2} + \overline{a}_2\delta_m.$$ 

We extract the $x_2$ coefficient and obtain the equation

$$c_2Q^{m,2}_{1,2} - \overline{a}_2Q^{m,0}_{1,2} - c_2Q^{m,0}_{1,2} + c_2Q^{m,0}_{1,2} + [x_2^1]\overline{a}_2\delta_m = 0$$

$$c_2Q^{m,2}_{1,2} - \overline{a}_2Q^{m,0}_{1,2} + [x_2^1]\overline{a}_2\delta_m = 0.$$ 

We can also extract the $x_2^{n+1}$ coefficient for $n \geq 1$:

$$c_2Q^{m,n+2}_{1,2} - \overline{a}_2Q^{m,n}_{1,2} + [x_2^{n+1}]\overline{a}_2\delta_m = 0.$$ 

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This formula also contains the $n = 0$ case, so we can group them together. Set $a_2 = 1$ to find

$$-\tilde{Q}_1^{m,n} \left[ x_{2}^{n+1} \right] \delta_{m} = 0,$$

and substitute the resulting expression into the equation above to obtain

$$b_2Q_{1,2}^{m,n+2} = Q_{1,2}^{m,n} - \tilde{Q}_1^{m,n}.$$  

(5.94)

We finally have the following result.

**Corollary 5.13.** For integers $m$ and $n$ greater than zero, we have the following relations

$$b_1Q_{1,2}^{m+2,n} = Q_{1,2}^{m,n} - \tilde{Q}_1^{m,n},$$

(5.95)

$$b_2Q_{1,2}^{m,n+2} = Q_{1,2}^{m,n} - \tilde{Q}_1^{m,n}.$$  

(5.96)

We can sum up those equations to retrieve the following functional equations.

**Corollary 5.14.** For integers $m$ greater than zero, we have the following identities

$$Q_{1}^{m} - \hat{Q}_{1}^{m} = b_1Q_{1,2}^{m+2},$$

(5.97)

$$x_{1}^{2}(Q_{2}^{n} - \hat{Q}_{2}^{n}) = b_1(Q_{2}^{n} - Q_{1,2}^{1,n} - Q_{1,2}^{0,n}),$$

(5.98)

$$x_{1}^{2}(Q - \hat{Q}) = b_1(Q - x_{1}Q_{1}^{1} - Q_{1}^{0}),$$

(5.99)

where we can also set $a_2 = 1$ to obtain further identities.

In this case, we cannot solve for $Q$ simply in terms of partially-marked variants of $Q$ and $Q_{1,2}^{0,0}$ since the equations would have the terms $Q_{1}^{1}$ and $Q_{1,2}^{1}$. Now, we can move onto the main result about the diagonal walks — the relation between $Q_{1,2}^{0,0}$ and its non-interacting versions. We use the master equation to extract the $x_{1}^{0}x_{2}^{0}$ and $x_{1}^{1}x_{2}^{1}$ coefficients and obtain the following identities:

$$Q_{1,2}^{0,0} = 1 + ta_{1}a_{2}Q_{1,2}^{1,1},$$

(5.100)

$$Q_{1,2}^{1,1} = t(Q_{1,2}^{0,0} + Q_{1,2}^{0,2} + Q_{1,2}^{2,0} + Q_{1,2}^{2,2}).$$

(5.101)

We also have

$$b_1b_2Q_{1,2}^{2,2} = b_1(Q_{1,2}^{2,0} - \hat{Q}_{1,2}^{2,0}) = Q_{1,2}^{0,0} - \hat{Q}_{1,2}^{0,0} + \tilde{Q}_{1,2}^{0,0}.$$  

(5.102)
Theorem 5.15. The generating function $Q_{1,2}^{0,0}$ for the Diagonal Cartesian model satisfies the equation $B(fQ_{1,2}^{0,0}) = c_1c_2$, where $f(a_1, a_2, t) = c_1c_2 - a_1a_2t^2$.

5.2.3 The Full Model – All Steps

The situation here is more difficult because we no longer have a rational function in front of the $Q_{1,2}^{0,0}$ term and we are not sure how to proceed from there. We start with the following computation:

$$\frac{\alpha(K_2^* - K_2)}{K_1} = \frac{\alpha(x_2 - x_2)(1 + t)}{S_1^1K_1},$$

(5.108)

$$\frac{\Delta_2(K_2^* K_12)}{K_1} = \frac{c_2(x_2 - x_2)}{S_1^1}\left(1 - \frac{t}{c_1 - t\bar{c}_1S_1^1}\right).$$

(5.109)

To find a representation for that last fraction, we first compute the $\sigma_1$ roots as

$$\sigma_1 = \frac{u - \sqrt{u^2 - 4w^2}}{2w} \quad \text{and} \quad \bar{\sigma}_1 = \frac{u + \sqrt{u^2 - 4w^2}}{2w},$$

(5.110)
where \( u = 1 - tv \) and \( w = t(1 + v) \). Then we have

\[
\frac{t}{c_1 - \sigma_1 w} = \frac{2t}{2c_1 - u - \sqrt{u^2 - 4w^2}} = \frac{t(2c_1 - u + \sqrt{u^2 - 4w^2})}{2(c_1^2 - c_1u + w^2)}.
\] (5.111)

Let \( v = \bar{x}_2 + x_2 \). Then the denominator has the term

\[
c_1^2 - c_1u + w^2 = c_1^2 - c_1(1 - tS_1^0) + t^2(S_1^1)^2 = \bar{\alpha}_1^2b_1 - \bar{\alpha}_1b_1vt + (1 + v)^2t^2
\] (5.112)

and the expression under the square root factors as

\[
u^2 - 4w^2 = (1 - (3v + 2)t)(1 + (v + 2)t).
\] (5.113)

The first few terms of the square root are

\[
\sqrt{u^2 - 4w^2} = 1 - vt - 2(1 + v)^2t^2 - 2v(1 + v)^2t^3 + \cdots.
\] (5.114)

We can give a series representation of the square root via the series

\[
\sqrt{1 - z} = \sum_{n=0}^{\infty} \frac{(2n)!}{(1 - 2n)(n!)^24^n} z^n.
\] (5.115)

That is,

\[
\sqrt{(1 + (v + 2)t)(1 - (3v + 2)t)} = \left( \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!(v + 2)^n t^n}{(1 - 2n)(n!)^24^n} \right) \left( \sum_{n=0}^{\infty} \frac{(2n)!(3v + 2)^n t^n}{(1 - 2n)(n!)^24^n} \right)
\] (5.116)

\[
= \sum_{n=0}^{\infty} \frac{t^n}{4^n} \sum_{k=0}^{n} \frac{(-1)^k(2k)!(2n - 2k)!}{(1 - 2k)(1 - 2n + 2k)k!^2(n - k)!^2} (v + 2)^k(3v + 2)^{n-k},
\]

However, we are not sure if a closed form exists for these coefficients. Extracting \( x_2 \) coefficients from the overall expression is difficult because we would first need to expand it in series of \( t \). Now, \( K_2 \) and \( K_2^* \) have \( 1 + v \) in their denominators, so we can multiply the whole functional equation by \( 1 + v \) to clear out these denominators. We have

\[
(1 + v)K_2 = c_2(1 + v) - \bar{x}_2(1 + t) = (c_2 - 1 - t)\bar{x}_2 + c_2 + c_2x_2,
\] (5.117)

\[
(1 + v)K_2^* = c_2(1 + v) - x_2(1 + t) = c_2\bar{x}_2 + c_2 + (c_2 - 1 - t)x_2,
\] (5.118)
we can find a relation for $Q$ of $b$ of $Q$ where
the master functional equation for form of the equations, like those for $Q$
if we let $A$ and

$$f(a_1, x_2, t) = 1 - t \frac{c_1 - 1 + tv + \sqrt{(1 + (v + 2)t)(1 - (3v + 2)t)}}{c_1 - c_1 + c_1tv + (1 + v)^2t^2}.$$  

If we let $A = c_2 - (1 + t)$, we can write the functional equation as

$$\left( c_2\pi_2 + c_2 + Ax_2 \right)Q_1^1(x_2) - \left( A\pi_2 + c_2 + c_2x_2 \right)Q_1^0(\pi_2) = c_2(\pi_2 - x_2)f - \alpha g.$$  

As before, we try to find a relation for the walks returning to the origin via coefficients extractions.

We extract the $x_2$ coefficient from both sides to find that

$$c_2Q_{1,2}^{0,2} + c_2Q_{1,2}^{0,1} - (1 + t)Q_{1,2}^{0,0} = -\alpha g_2 + c_2(f_2^2 - f_2^0)Q_{1,2}^{0,0},$$  

and then we set $a_2 = 1$ on both sides to find that

$$-(1 + t)Q_{1,2}^{0,0} = -\pi_1 g_2^1.$$  

When we finally substitute this quantity back into the equation, it simplifies to

$$b_2(Q_{1,2}^{0,2} + Q_{1,2}^{0,1}) = (b_2(f_2^2 - f_2^0) - a_2(1 + t))Q_{1,2}^{0,0} - (1 + t)\pi_{1,2}^{0,0}.$$  

It appears that the quantity $f_2^2 - f_2^0$ has no closed form:

$$-(f_2^2 - f_2^0) = 1 + a_t + \frac{a_t^2}{b_1} (a^2_1 - a_1 - 4)t^3 + \frac{4a_t^2}{b_1^2} (a^3_1 - 3a_1^2 + 1)t^4 + \cdots.$$  

Unfortunately, we have not been able to go further. We can make some assumptions about the form of the equations, like those for $Q_{1,2}^{m,n}$ in the previous cases, but even then, it is not clear if we can find a relation for $Q_{1,2}^{0,0}$ like before. We would have six equations like $5.123$ plus six more where $b_i = 0$ for the $b_i$’s not present. Then we may use the four regular coefficient extractions from the master functional equation for $Q_{1,2}^{m,n}$ and $m, n \leq 1$ plus twelve more, where we set one or more of $b_1$ and $b_2$ to zero. We have a grand total of 28 equations in 36 unknowns, of which we need to eliminate 32. A quick attempt to do this via computer yielded no results.
Chapter 6

Generalizations

In this final chapter, we examine lattice walk models in more generality and we prove that a large class of models satisfy functional equations of the types we have seen already, namely $(1-t\Gamma S)Q = q$. There is already some work towards generalizations, but most of is on a smaller scale. For example, Definition 1.1 of [7, p. 2] has a functional equation for non-interacting lattice walks on the quarter plane with larger positive steps while the negative steps are limited to size at most one. For quarter plane models with small steps where multiple copies of steps are allowed, see [11]. For some more general regions, check [28, 21, 29] for walks on a semi-infinite strip. In [27] the author considers lattice walks in an octant with interactions along the diagonal. Our theory allow us to write down functional equations for all of these cases automatically.

We start by examining marking operators in more generality. Then we move onto finding functional equations for general models. Finally, we look at some examples and ideas generated from them. Unfortunately, time did not suffice to explore this area to its full potential.

6.1 Marking Operators And Functional Equations

We start with the observation that operators that remove monomials are just marking operators with labels set to zero, and therefore, we only consider marking operators for the rest of this work. A rather general operator is one that marks each point $p$ in $\mathbb{Z}^d$ with a unique label $a_p$. This serves as our most general definition. We work with marking functions first and then express their action on monomials via operators.

**Definition 6.1.** Given a set $A$ of labels and constants from some space over indeterminates, let $a : \mathbb{Z}^d \rightarrow A$ be a marking function.

It is sufficient that the labels and constants come from a complex space over the indeterminates.
Definition 6.2. Given a set of labels $A$, let $U_A$ be a set of marking functions on the space $\mathbb{Z}^d$. We define addition and multiplication term-wise on points of $\mathbb{Z}^d$, using the operations in the set $A$.

Proposition 6.3. The set of marking functions on $\mathbb{Z}^d$ together with addition and multiplication forms a commutative ring with identity. Furthermore, the ring is an integral domain. The set of invertible marking functions is comprised of functions with non-zero labels.

Proof. If $A$ is the whole space of labels and constants, the set $U$ is the same as a Cartesian product of infinitely many copies of $A$, and it is not difficult to see that the properties of a commutative ring and an integral domain are satisfied. Here the additive identity is the marking function whose labels are all 0, and the multiplicative identity is one whose labels are all 1.

Since we are working with marking operators on Laurent polynomials and series, we need a representation for marking operators that allow us to compute things easily.

Definition 6.4. Given a marking function $a$, let $\Gamma_a$ be a marking operator defined by its action on monomials: $\Gamma_a x^p = a(p)x^p$ for all $p$ in $\mathbb{Z}^d$. We extend this definition by linearity to formal Laurent series $u$, where $\Gamma_a u = \sum_p a(p)u_px^p$, and we write $\Gamma_a = \sum_p a(p)[[x^p]]$.

We need to check that this formal definition for marking operators behaves as expected and that it preserve the ring of functions $U_A$. Expanding the action $\Gamma_a u$ gives

$$\left(\sum_p a(p)[[x^p]]\right)\left(\sum_q u_q x^q\right) = \sum_{p,q} a(p)[[x^p]]u_q x^q = \sum_{p,q} a(p)u_q \delta_{p,q} x^q = \sum_p a(p)u_px^p. \quad (6.1)$$

Furthermore addition and multiplication work as expected:

$$\Gamma_1 + \Gamma_2 = \sum_p a_1(p)[[x^p]] + \sum_p a_2(p)[[x^p]] = \sum_p (a_1(p) + a_2(p))[x^p], \quad (6.2)$$

$$\Gamma_1 \Gamma_2 = \left(\sum_p a_1(p)[[x^p]]\right)\left(\sum_p a_2(p)[[x^p]]\right)$$

$$= \sum_{p,q} a_1(p)a_2(q)[[x^p]][[x^q]]$$

$$= \sum_{p,q} a_1(p)a_2(q)\delta(p,q)[[x^p]]$$

$$= \sum_p a_1(p)a_2(p)[[x^p]]. \quad (6.3)$$

We just need the following notation to allow us to express certain operators easily.
Definition 6.5. Let the term extraction operator be defined as

$$[[x^I]] = \sum_{K \in \mathbb{Z}^{d-n}} [[x^{P,K}_I]] ,$$

(6.4)

where \( I = \emptyset \) gives the identity operator \( 1 = \sum_p [[x^p]] \).

Proposition 6.6. The set \( U_A \) of marking operators with addition and multiplication, as defined in the preceding discussion, forms a commutative ring with identity. Furthermore, the ring is an integral domain.

Lemma 6.7. The marking operator \( \Gamma_a \), as defined by a sum, is well-defined and has the secondary form \( \Gamma = 1 + \sum_p (a(p) - 1)[[x^p]] \), where the sum runs over all points in \( \mathbb{Z}^d \) with \( a(p) \neq 1 \).

Proof. Assume that another operator \( \Gamma' \) behaves in the same way as in the definition. Then

$$\Gamma u - \Gamma'u = \left( \sum_p a(p)[[x^p]] \right) \left( \sum_q u_q x^q \right) - \sum_p a(p) u_p x^p$$

$$= \sum_p a(p) u_p x^p - \sum_p a(p) u_p x^p = 0,$$

(6.5)

and so the operator is well-defined. The second form follows by writing \( \Gamma = 1 + (\Gamma - 1) \) and expanding the latter quantity.

The following lemma is self-evident.

Lemma 6.8. For each point \( p \), the monomial \( x^p \) is an eigenfunction of \( \Gamma_a \) with eigenvalue \( a(p) \).

Proposition 6.9. The eigenspaces of formal Laurent series for each eigenvalue \( \lambda \) of \( \Gamma_a \) are generated by sums of monomials \( x^p \) with \( a(p) = \lambda \).

Proof. Let \( \Gamma u = \lambda u \) for some formal Laurent series. Then the only surviving terms in \( \Gamma u - \lambda u \) are those with \( a(p) \neq \lambda \). However, since the quantity is zero, they are also forced to add up to zero. Therefore, the series \( u \) is comprised of terms with monomials \( x^p \) for \( a(p) = \lambda \) in the first place.

Proposition 6.10. The space of Laurent series annihilated by \( \Gamma_a \) is generated by series of terms \( x^p \) where \( a_p = 0 \).

For all the problems so far, we had marking operators comprised of finitely many term extractions and the identity 1, where each term extraction has less variables than the dimension \( d \). As the following example shows, sometimes a marking operator comprised of infinitely many term extractions can sometimes be rewritten in terms of finitely many term extractions with less variables.

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Example 6.11. For example, consider the operator $\Gamma = \sum_{p} a[[x^p]]$ where all points receive the same label. Then $\Gamma = a$.

Sometimes, depending on the model, we may even have a smaller operator that does the same work as the full marking operator because of the geometry of the regions involved.

Example 6.12. Consider a classical one-dimensional model with no interactions. Because we do not want lattice walks to step on negative coordinates, the full marking operator is given by

$$\Gamma = 1 - \sum_{n=1}^{\infty} [[[x^n]]].$$

However, since the lattice walk can never reach negative coordinates beyond negative one, we have the equivalent form $\Gamma' = 1 - [[[x]]]$. Although the operators are different in form, they are essentially the same with regards to the problem at hand. We say the two operators are model-equivalent.

To make the equivalence work, we need to consider the whole model. With that in mind, we can think of the lattice walk models as living completely within the sumset $\mathcal{D} + \mathcal{S}$. This is how far walks from the model can possibly reach from within $\mathcal{D}$ by taking a single step. We call this sumset the range of the model. In the enumeration of lattice walks, we must eliminate all walks that take a step outside of $\mathcal{D}$, but they remain in $\mathcal{D} + \mathcal{S}$, so it is enough to remove walks in the complement of $\mathcal{D}$ inside the sumset. Note that the walks do not necessarily reach all points in $\mathcal{D} + \mathcal{S}$ as one can see in the case of reducible models.

Definition 6.13. Given sets $A$ and $B$, we let their sumset be

$$A + B = \{a + b | a \in A, b \in B\}.$$  \hspace{1cm} (6.7)

Definition 6.14. Let $\mathcal{D} + \mathcal{S}$ be the range of a lattice walk model with domain $\mathcal{D}$ and step set $\mathcal{S}$.

Definition 6.15. Let $\mathcal{M}$ be a lattice walk model in $\mathbb{Z}^d$ and let $\Gamma$ and $\Gamma'$ be two marking operators. Then $\Gamma \equiv_{\mathcal{M}} \Gamma'$ if and only if $\Gamma$ and $\Gamma'$ act identically on monomials from the range of $\mathcal{M}$. That is, the operators are model-equivalent if and only if $\Gamma x^p = \Gamma' x^p$ for all $p \in \mathcal{D} + \mathcal{S}$.

In the preceding example, the operators are equivalent because they act the same way on monomials from the range of the model, that is, monomials $x^k$ with $k \geq -1$. Similarly, in the two-dimensional problem we have an equivalence between the operator $E$ and the operator $E' = (1 - [[[x_1]])(1 - [[[x_2]]]).
Definition 6.16. A marking operator $\Gamma$ is **finite** with respect to a given model if it is equivalent to an operator $\Gamma'$ comprised of finitely many terms. That is,

$$\Gamma =_{\mathcal{M}} \Gamma' = \sum_{I \subseteq [d]} \sum_{J \subseteq \mathbb{Z}^d} a_{I,J} \left[ [x_I^J] \right], \quad (6.8)$$

where each set $J$ is finite and $a_{I,J}$ is a label. Also, we say $\Gamma$ is **infinite** if it is not finite.

Lemma 6.17. A finite product of finite marking operators is finite.

Definition 6.18. Given a $d$-dimensional model, let $L$ be an operator that marks locations in $\mathcal{D}_0$ and let $E$ be the operator that removes walks outside of $\mathcal{D}$. Then the marking operator corresponding to the model is $\Gamma_a = LE$ with marking function $a$ which sends points $p$ in $\mathcal{D}_0$ to their respective labels, and points outside of $\mathcal{D}$ to zero.

When we work within the framework set by a specific model, we do not distinguish between $\Gamma$ and $\Gamma'$. A natural question is to consider which models lead to finite operators. This depends on the region $\mathcal{D}$ and the nature of the marking of region $\mathcal{D}_0$. We have some cases where it is easy to say.

Lemma 6.19. Let $\mathcal{D}$ be a finite box and let $S$ be finite. Then the marking operator $\Gamma = LE$ is finite.

Proof. The operator $L$ is finite since it marks only finitely many points inside the box. We find $E$ next. Let the box have $x_k$ coordinates in $[a_k, b_k]$. Suppose that the $x_k$ coordinates of the steps are bounded in size by $n_k$. If $n_k = 0$, the walks do not leave the $x_k = a_k$ hyperplane and there is nothing to remove. When $n_k > 0$, we need to remove all monomials $x_1^{P_1} \cdots x_d^{P_d}$ with $P_k \in [a_k - n_k, a_k - 1] \cup [b_k + 1, b_k + n_k]$. Then $E = E_1, \ldots, E_d$, where

$$E_k = 1 - \sum_{j=1}^{n_k} \left[ [x_k^{a_k-j}] \right] + \left[ [x_k^{b_k+j}] \right], \quad (6.9)$$

and $E_k = 1$ when $n = 0$. Each of these is finite and so is $E$. \qed

Note that the operators $E_k$ remove monomials outside of the range of the model too, but that allows us to write the operator down in a neater way.

Theorem 6.20. The operator $\Gamma = LE$ is finite for models with finite $\mathcal{D}$ and $S$.

Proof. The region $\mathcal{D}$ lies within a bounding box $B$, so we can write $E = E' E''$, where $E' = E_1 \cdots E_d$ from the lemma and $E''$ is the operator removing all monomials outside of $\mathcal{D}$ but inside the interior of $B$. $E''$ is finite since there are only finitely many points in that region. Since we are marking only finitely many points, $L$ is finite too, and finally, $\Gamma$ is finite. \qed

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Also, we can handle partially-infinite regions, corresponding to boxes extended to infinity on some sides, if we are marking only finitely many points. Here one or more of \(a_k\) and \(b_k\) is infinite and we actually lose parts of \(E_k\) since no terms need to be removed at infinity. We can push this some more and obtain the following result.

**Proposition 6.21.** If \(L\) is finite and \(D\) is a box, except for a finite region, then \(\Gamma = LE\) is finite too.

**Proof.** This follows from the observation that \(E_B\) is finite and \(E_B - E_D\) is finite as well. The second fact is true because the regions differ in a finite region. \(\square\)

**Definition 6.22.** A lattice walk model with finitely many steps is **finite** if its marking operator \(\Gamma = LE\) is finite, and its initial condition \(q\) has finitely many terms. Otherwise, the model is **infinite**.

**Example 6.23.** All the models so far have been finite, but not all models are finite. Consider a one-dimensional model on the whole line, where for each integer \(i\), we mark point \(x^i\) with parameter \(a_i\) using operator \(L\). Then all equivalent marking operators \(L'\) have to have a term like \((a_i - 1)\)\([x^i]\); otherwise, it does not act the same way on monomials in the model’s range \(\mathbb{Z}\). Therefore, \(L\) is not equivalent to a finite operator.

Finite models are nicer because they have functional equations with finitely many terms and there is a chance that they may be amenable to the sort of analysis we have performed for quarter plane walk models in the past.

**Theorem 6.24.** The generating function \(Q\) for a lattice walk model satisfies the equations

\[
(1 - tLES)Q = q, \tag{6.10}
\]

\[
(L^{-1} - tES)Q = L^{-1}(q) = \alpha. \tag{6.11}
\]

**Proof.** Because of how we have set up the notation so far, the proof is analogous to the one in Chapter 2 and Chapter 3. The second functional equation is obtained by applying \(L^{-1}\) to the first one, so it only remains to show that the first functional equation holds. When we apply \(LE\) to \(tSQ\) we remove all monomials for walks ending outside of \(D\) and we mark all monomials for walks ending in \(D_0\). We have almost obtained \(Q\) again, but the first term \(q_0 = q\) is missing, so we add it in. That is \(LE(tSQ) + q = Q - q + q = Q\) and we are done. \(\square\)

**Corollary 6.25.** When the model is finite, we can perform the coefficient extractions in the functional equations to obtain explicit equations relating \(Q\) to various \(Q_I^J\) terms.

This follows because there are only finitely many extractions in the marking operators. It is not clear if can have explicit functional equations for models with infinite operators, because there would be infinitely-many \(Q_I^J\) terms in the equation. We would need more theory in order to do that.
6.1.1 Several Examples of Generalized Models

In the first two examples, we see some more general analysis of models. After that we consider some more concrete situations.

Example 6.26. Typically, we have an operator \( \Gamma = LE \) where \( E \) removes unwanted terms. The problem is that \( E \) is not invertible. We can recast the analysis in a way that circumvents that but at some cost. Instead of using an operator \( E \) to remove walks, we can just just a marking operator that labels walks for later deletion, for example, with a variable \( z \). We have a new lattice walk model \( \tilde{\mathcal{M}} \) with \( \tilde{D} = \mathbb{Z}^d \) and \( \tilde{D}_0 = D_0 \cup D^c \), where we mark monomials in \( D^c \) with \( z \) and those in \( D_0 \) in the same way as in model \( \mathcal{M} \). Here \( \tilde{E} \) is the identity operator since we do not remove any walks. We have the functional equations

\[
(1 - \tilde{L}S)\tilde{Q} = q, \tag{6.12}
\]

\[
(\tilde{L}^{-1} - tS)\tilde{Q} = \tilde{\alpha}. \tag{6.13}
\]

The downside is that we have more extractions than if we split the operator as \( \Gamma = LE \) and only applied \( L^{-1} \). If we obtained a solution for \( \tilde{Q} \), then we could find the original \( Q \) by taking \( Q = [z^0]\tilde{Q} \).

Example 6.27. We can also keep track of the walks’ histories of where they have been. To do this, we let \( \Gamma \) give a generic label to each point in space, that is, \( \Gamma x^p = a_p \). In this setup, a growing lattice walk picks up a label at each point that it steps on. As in Example 6.23, the operator is not finite, so we would not have an explicit functional equation with finitely many terms. Nevertheless, this allows us to frame self-avoiding within our theory.

Suppose that \( Q \) is the generating function for walks with histories and let \( W \) be the generating function for self-avoiding walks on the same space. Then we can obtain \( W \) from \( Q \) by setting all higher powers of \( a_p \) to zero. This is equivalent to picking out the \( a_0^p \) and \( a_p \) coefficients from \( Q \) via some operator \( A \). Since there are only finitely many walks of a given length and they are bounded, for each step size, \( A \) reduces to a finite operator \( A_k \), that is, \( W = AQ = \sum_{k=0}^{\infty} A_k(q_k)t^k \).

Formally, we can write \( \Gamma = \prod_p \Gamma_p \) and \( A = \prod_p A_p \), where the products go over the whole space and \( \Gamma_p = 1 + (a_p - 1)[[x^p]] \) and \( A_p = [[a_0^p]] + [[a_p]] \). Now, suppose that \( W \) is already in the form that we desire, that is, there are only \( a_0^p \) and \( a_p \) terms for every \( p \). Now, we fix \( a = a_p \) and \( x = x^p \) for some \( p \) to declutter the next calculation. We have

\[
A_p = A_p \Gamma_p = ([[a_0^p]] + [[a]])(1 + (a - 1)[[x]]) \tag{6.14}
\]

\[
= [a^0] + a[a] + ([\pi] + a[a^0] - [a^0] - a[a])[x] \tag{6.15}
\]

\[
= (1 + (a - 1)[[x]])[a^0] + (1 - [[x]])a[a], \tag{6.16}
\]

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and we identify this operator as $\Lambda_p = \Gamma_p[[x_0^0]] + E_p[[a_p]]$ where $E_p$ removes the monomial $[[x^p]]$. We can interpret this result in terms of the actions of the operators themselves. Since $W$ contains self-avoiding walks only, when we take $tSW$, we only run into problems for points $p$ where each walk has been at $p$ at some earlier time of its growth. Thus, we may take all the walks ending on $p$ for the first time and mark them with $a$, and remove those that have been there already. This is exactly what the $\Lambda_p$ operator does. We can finally write a functional equation for the generating function for self-avoiding walks:

$$(1 - t\Lambda S)W = a_0 ,$$

(6.17)

where $\Lambda = \prod_p \Lambda_p$. This operator is no longer invertible, but we can write $W$ as the infinite series

$$W = \sum_{k=0}^{\infty} (\Lambda S)^{(k)}(a_0) ,$$

(6.18)

where $(\Lambda S)^{(k)}$ is finite because it only sees finitely many points.

For example, using $S = \overline{x}_1 + x_1 + \overline{x}_2 + x_2$, we can compute the first few terms of the series:

$$W = a_{0,0} + t(a_{-1,0}a_{0,0}\overline{x}_1 + a_{1,0}a_{0,0}x_1 + a_{0,-1}a_{0,0}\overline{x}_2 + a_{0,1}a_{0,0}x_2) \cdots .$$

(6.19)

The following examples look at particular models with small steps in the plane.

**Example 6.28.** Consider an infinite strip comprised of all the points with coordinates $(i, j)$ in $\mathbb{Z}^2$ with $0 \leq j \leq n$, where the model interacts with the lines $x_2 = 0$ and $x_2 = n$. The operators are

$$E = E_{-1}E_{n+1} = (1 - [[x_2]])(1 - [[x_2^{n+1}]]) = 1 - [[x_2]] - [[x_2^{n+1}]] ,$$

(6.20)

$$L = L_0L_n = (1 + b_0[[x_2^0]])(1 + b_n[[x_2^n]]) = 1 + b_0[[x_2^0]] + b_n[[x_2^n]],$$

(6.21)

and the functional equation for $Q$ is given by

$$KQ - (c_0 - t\overline{x}_2S_2^{-1})Q^0_2 - x_2^n(c_n - tx_2S_2^1)Q^n_2 = a_0 .$$

(6.22)

If we extract the $[x_2^0]$ term, we find that

$$(1 - a_0tS_2^0)Q^0_2 - tS_2^{-1}Q^1_2 = 1$$

(6.23)
and if we extract the \([x_2^n]n\) term, we find that

\[
(1 - a_n t S_2^0) Q_2^n = t S_2^1 Q_2^{n-1}.
\]

(6.24)

Similarly, we can extract the \(x_2^k\) coefficients for \(1 \leq k \leq n - 1\). We have \(n + 1\) equations where we need to solve for the \(n + 1\) unknown functions \(Q_2^0, \ldots, Q_2^n\). Of course, we only need \(Q_2^0\) and \(Q_2^n\) to obtain a solution for \(Q\). In the simplest case, \(n = 1\) and the term extraction equations become

\[
(1 - a_0 t S_2^0) Q_2^0 = 1 + t S_2^{-1} Q_2^1,
\]

(6.25)

\[
(1 - a_1 t S_2^0) Q_2^1 = t S_2^1 Q_2^0.
\]

(6.26)

The solutions are

\[
Q_2^0 = \frac{1 - a_1 t S_2^0}{(1 - a_0 t S_2^0)(1 - a_1 t S_2^0) - S_2^1 S_2^{-1} t^2},
\]

(6.27)

\[
Q_2^1 = \frac{S_2^1 t}{(1 - a_0 t S_2^0)(1 - a_1 t S_2^0) - S_2^1 S_2^{-1} t^2},
\]

(6.28)

and we can find \(Q\) explicitly from

\[
Q = \frac{1}{K} (\bar{a}_0 + (c_0 - t x_2 S_2^{-1}) Q_2^0 + x_2(c_1 - t x_2 S_2^1) Q_2^1).
\]

(6.29)

**Example 6.29.** Consider a finite rectangle comprised of all the points with coordinates \((i, j)\) in \(\mathbb{Z}^2\) with \(0 \leq i \leq m\) and \(0 \leq j \leq n\), for some positive integers \(m\) and \(n\). There are also interactions like in the previous example. We use the operators

\[
E_{i,j} = 1 - \lfloor x_i^j \rfloor \quad \text{and} \quad L_{i,j} = 1 + b_{i,j} \lfloor x_i^j \rfloor
\]

(6.30)

to find the functional equation. For the model itself, we have

\[
E = E_{1,-1} E_{1,m+1} E_{2,-1} E_{2,n},
\]

(6.31)

\[
L = L_{1,0} L_{1,m} L_{2,0} L_{2,n}.
\]

(6.32)

Once we obtain a functional equation, we find that it has terms of the form \(Q_{i,j}^i\) and \(Q_{1,2}^{i,j}\). If we perform the extractions \(x_i^j\) only, we have even more \(Q\) terms. However, if we perform the extractions \(x_i^j\) for all points \((i, j)\) in the rectangle, we obtain \((m + 1)(n + 1)\) equations in that many \(Q_{1,2}^{i,j}\) terms. Since there is a unique solution \(Q\), we can solve this system and write \(Q\) directly. The difficulty lies in solving this system symbolically.

This example paves the way towards the following result for finite models on finite regions.
**Theorem 6.30.** The generating function for a finite lattice walk model set on a finite region is a rational function in its variables.

**Proof.** Let \( n \) be the number of points in \( \mathcal{D} \). Then for each point \( P \), we can extract the \( x^P \) term from the functional equation \((1 - t\Gamma S)Q = q\) and we only have \( Q \) terms of the form \( Q^I \) for points \( I \) in \( \mathcal{D} \). We have a system of \( n \) equations in \( n \) unknowns, and since the function \( Q \) is unique, we can solve this system uniquely. Each of the coefficients is a rational function of the variables \( x_1, \ldots, x_d \), the labels \( a_P \), and \( t \), and therefore, the solution is rational too.

Of course, once the region grows to a moderate size, it becomes prohibitively complicated to solve this system. For a box of side \( m \), the number of equations is \( m^d \), making this a difficult way of solving the problem. If the size of the region is large and the number of steps in \( S \) is small, then each \( Q_{i,j}^{I} \) depends on a few neighbours only, so the resulting linear system is rather sparse. Still, the solution fills a whole page for a three-by-three square, so the result is mostly of theoretical value.

### 6.2 Further Properties of Marking Operators

Since the function \( Q \) is given by Neumann series, which is connected to integral equations, it may be possible to find an analogue to Fredholm theory. Also, it may be possible to find a sort of Sturm-Liouville theory for lattice walk enumeration problems. In the next few pages, we look at the first few steps of such a theory, although there was insufficient time to analyze this problem in more detail. For that we would need a suitable inner product that would allow us to see if certain operators are self-adjoint. Finally, we would need to find eigenfunction expansions for the functions involved.

The operators we have seen so far are marking operators, like \( L \) and \( E \), in combination of multiplication by another function, like \( \Gamma S \), or a combination of operators, like \( L^{-1} - tES \). We look at their eigenvalues, eigenvectors, kernels and invertibility.

#### 6.2.1 Marking Operators With Multiplication

**Definition 6.31.** We call an operator **inhomogeneous** if it consists of a marking operator \( \Gamma \) and a Laurent polynomial \( S \), and acts in one of the following ways

\[
\Gamma S(u) = \Gamma(Su) \quad \text{or} \quad S\Gamma u = S \cdot \Gamma u.
\]

(6.33)

The two definitions are equivalent when it comes to finding the eigenpairs.

**Proposition 6.32.** The pair \((\lambda, u)\) is an eigenpair of \( \Gamma S \) if and only if \((\lambda, Su)\) is an eigenpair of \( S\Gamma \). In particular, if \( \lambda = 0 \), then \( \text{ker}(\Gamma S) = \overline{S}\text{ker}(\Gamma) \).

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Proof. Note that $\Gamma Su = \lambda u$ is equivalent to $S\Gamma(Su) = \lambda(Su)$. Therefore, the operators have the same eigenvalues and their eigenfunctions are $u$ and $Su$, respectively. Next, we pick $\lambda = 0$ to find the kernels of the operators. We find that $u \in \ker(\Gamma S)$ if and only if $Su \in \ker(S\Gamma) = \ker(\Gamma)$. □

Since this characterizes the kernel of the operator, we assume that $\lambda \neq 0$ from here on. Next, we find some eigenpairs. Let $\Gamma = LE$ as before. Recall that $E^2 = E$, since once the terms are removed, they cannot be removed again. We can apply $E$ on both sides of $LESu = \lambda u$ to find that

$$E(\lambda u) = \lambda Eu = ELESu = LE^2Su = LESu = \lambda u,$$

and therefore, $Eu = u$.

Lemma 6.33. If $u$ is an eigenfunction of the operator $LES$, then $Eu = u$ and $u \in \ker E'$, where $E' = E - 1$ is the pure coefficient extraction part of $E$.

Suppose that $\Gamma = LE$. In the interest of performing fewer extractions, we rewrite the eigenproblem $\Gamma Su = \lambda u$ as $ESu = \lambda Mu$, where $M = L^{-1}$. Since each operator is the identity plus a sum of extractions, we rewrite the operators as $E = E - 1 + 1 = E' + M$ and $M = 1 + M'$, where $E'$ and $M'$ are purely extractions. Using these relations, we can rewrite the function $u$ as

$$u = \frac{(\lambda M' - E'S)u}{S - \lambda},$$

where if the operators are finite, the extraction in the numerator leads to a Laurent polynomial in the variables $x_1, \ldots, x_d$. Because we used $M'$ and $E'$ instead of the equivalent for $LE$, we have fewer coefficients in the numerator. The resulting eigenfunction $u$ depends on finitely many coefficients. We can expand $u$ in Laurent series around the origin, set the coefficients to zero, and then solve the resulting system of equations to find the terms in the numerator. Curiously, if we apply $D = \lambda M - ES$ on $u$ we should find that $Du = 0$, but here $u$ has a new form and we obtain some condition on the terms in the numerator.

Example 6.34. Consider one of the classical one-dimensional model with interactions, where $S = \pi + x$ and $M' = -c[x^0]$. First we use the full operator $E' = -\sum_{k=1}^{\infty}[[\pi^k]]$. Since $u$ has no terms $u_k$ for $k \leq -1$, we see that

$$-E'Su = \sum_{k=1}^{\infty} \pi^k([\pi^{k+1}] + [\pi^{k-1}])u = \sum_{k=1}^{\infty} \pi^k(u_{-k-1} + u_{-k+1}) = \pi u_0.$$  (6.36)
Then
\[ u = \frac{-\lambda cu_0 + xu_0}{x + x - \lambda} = \frac{(1 - c\lambda x)u_0}{1 - \lambda x + x^2} = u_0 + \lambda u_0 x + \cdots, \] (6.37)
where \( u_0 \) is a free variable, and there is an eigenfunction for every \( \lambda \). Next, we use the equivalent operator \( E = 1 - [[x]] \). We have no reason to believe that the operator has the same eigenfunctions, but it does. To show this, we first write \( u \) as
\[ u = \frac{(\lambda M' - E'S)u}{S\lambda} = \frac{-c\lambda u_0 + (u_{-2} + u_0)x}{x + x - \lambda} = \frac{u_{-2} + (1 - c\lambda x)u_0}{1 - \lambda x + x^2}, \] (6.38)
We apply the \( D \) on \( u \) to find that \( -b\lambda u_{-2} = 0 \), so \( u_{-2} = 0 \) and the eigenfunctions are the same.

It turns out that this is not a coincidence and we have the following striking result.

**Proposition 6.35.** The set of eigenpairs of the full marking operator associated with a lattice walk model is a subset of the set of eigenpairs for any equivalent but reduced operator. Here reduced means a change to the marking portion and a reduction of terms in the extraction portion.

**Proof.** Let \( \Gamma_1 = L_1E_1 \) be the full operator and \( \Gamma_2 = L_2E_2 \) be the reduced one. Here \( L_1 \) and \( L_2 \) agree on \( D \), and \( E_1 \) and \( E_2 \) agree on the range \( D + S \). Suppose that \( \Gamma_1Su = \lambda u \) for some eigenpair \((\lambda, u)\). By Lemma 6.33, \( u \) only contains terms from \( D \) and therefore, \( Su \) only contains terms from the range of the model. Therefore, \( \Gamma_1Su = L_1E_1Su = L_2E_2Su \) due to the agreement of operators, and finally, \( \Gamma_2Su = \lambda u \).

The proof here also shows us that \( \Gamma_2 \) may have other eigenfunctions \( v \), since here \( v \) may contain terms from outside of the range of the model. However, if \( \Gamma_2 \) is finite, we see that \( (\lambda M' - E'S)u \) has finitely many terms. Thus, each eigenvalue \( \lambda \) may lead to at most a finite-dimensional eigenspace.

In the end, we need to work with the operator \( \mathcal{L} = 1 - t\Gamma S \) since it is the one that is involved in the functional equation for \( Q \). If we are to find a theory where we can expand \( Q \) in terms of eigenfunction of operators, we need to obtain the same results whether we use the full or reduced form of the operators. That is, most likely, we do not need the extra eigenpairs that the reduced operator may have. We have the following definitions.

**Definition 6.36.** Let \( \mathcal{L} = 1 - t\Gamma S \) be a lattice walk operator.

**Definition 6.37.** The natural eigenpairs of a marking operator are the eigenpairs associated with the full operator. The natural eigenpairs of a lattice walk operator are those coming from the full form of the marking operator within it.
Let \((\lambda, u)\) be a natural eigenpair of \(L\). We find that

\[
(L - \lambda I)u = u - \lambda u - t\Gamma Su
\]
\[
t\Gamma Su = (1 - \lambda)u,
\]
and the eigenpairs of \(L\) are closely related to those of \(\Gamma S\). If \(\lambda = 1\), then \(u\) is in \(\ker(t\Gamma S) = \overline{S}\ker\Gamma\) and we already know how to find these functions. If \(\lambda \neq 1\), we can find the eigenpairs by the method in the previous section. Let \(\mu = 1 - \lambda\). Then

\[
u = \frac{((1 - \lambda)M' - tE'S)u}{\lambda - K},
\]

where \(K = 1 - tS\) and the numerator only features monomials for points within \(D\).

**Example 6.38.** Consider a classical interacting two-dimensional model. The functional equation is \((\mu - tLES)u = 0\) and we let \(\mu = t/\nu\). If we apply \(M\) on both sides, we find that the natural eigenpairs of \(L\) satisfy

\[
0 = (\nu M - ES)u = \sum_{I \in [d]} (-1)^{|I|}(\nu c_I - \nu S_{-1}^{-1}u_I) = \nu \sum_{I \in [d]} (-1)^{|I|}K_I(\nu)u_I,
\]

where \(K_I = K_I(x, \nu)\) are Laurent polynomials as in the master functional equation. So the eigenfunctions \(u\) satisfy an equation similar to the master functional equation for the associated lattice walk model. Take \(d = 1\), for example. Then \(u\) satisfies \(Ku - K_1u_0 = 0\) and it is given by

\[
u = \frac{K_1(x, \nu)}{K(x, \nu)} = \frac{c - \nu x S^{-1}u_0}{1 - \nu S}
\]

for every \(\nu \neq 0\). The constant \(u_0\) is arbitrary, so we set it to 1. If we let \(S^{\geq 0} = S - xS_{-1}\), we can expand \(u\) in terms of \(\nu\) as

\[
u(x, \nu) = c - \nu x(S - aS^{\geq 0})(1 + S\nu + S^2\nu^2 + \cdots).\]

For the model \(S = \overline{x} + x\) in Chapter 2, we can solve for \(Q\) in terms of \(u\). That is,

\[
u = \frac{1}{aK(x, t)} \left(1 + \frac{K_1(x, t)}{K_1(\sigma, t)}\right) = \frac{1}{aK(x, t)} + \frac{u(x, t)}{aK_1(\sigma, t)},
\]

so we have found a connection between \(Q\) and the eigenfunctions of the lattice walk operator. This is as far as we will go for now.
Chapter 7

Conclusion

In this thesis, we looked at the problem of generalizing methods for handling lattice walk enumeration in many dimensions and with more general constraints, as well as models with interactions. In the earlier chapters, we used refined notation and a new perspective of applying operators on functions and functional equations. We were able to derive the functional equations for classical $d$-dimensional models with ease. This allowed us to generalize the methods to derive a functional equation for a much more general class of models. This is one more step towards finding out if we can automate the solving and classification of more complex lattice walk models. However, one of the limitations of using operators is that it may be difficult to find finite equivalent operators for some rather tame-looking regions, like a wedge. It is not clear if this is due to a shortcoming of the operator notation or if this is due to some more fundamental differences between quarter plane models and those on wedges.

We were also able to extend the algebraic and obstinate kernel methods to lattice walk models with interactions, although there is still a fair amount of work that needs to be done. First, the algebraic kernel method does not work in a lot of cases of finite groups when the model has interactions, and in those cases that it does work, it does not succeed in eliminating the $Q_{0,0}^{0,0}$ term. The resulting formulae still need to be worked through. At best, we hope that in those cases, if the function $Q_{1,2}^{0,0}$ is D-finite, then the full generating function $Q$ is D-finite too.

In the extension of the obstinate kernel method, we were able to find relations between $Q_{1,2}^{0,0}$ and its various partially-marked versions for two nontrivial models — the Cartesian and Diagonal models. However, we were not able to find such a relation for the Full model. It also remains to be seen if the cases with fewer transformations that preserve the admissibility of the functions $Q_i^0$ can also be tackled. The problem of solving functional equations in a closed form for more generalized models is wide open.
7.1 Future Research

In closing, the research for this thesis produced many ideas for future work, and since there were too many to pursue, we list them here for the interested reader.

- **Algebraic kernel method.**
  - For finite group cases, can say that $Q$ is D-finite if $Q_{1,2}^{0,0}$ happens to be D-finite?
  - Determine if the method can be extended to higher-dimensional models with interactions and/or models with larger steps.
  - Consider annihilation of terms in the functional equations by other operators, like differential ones, and if this can be used to find out more about the generating functions.

- **Obstinate kernel method.**
  - Can identities like $B(FQ_{1,2}^{0,0}) = 0$ be explained combinatorially? The one-dimensional model relations would be a good place to start.
  - Find $Q$ for the Diagonal model in terms of partially-marked versions of $Q$ and $Q_{1,2}^{0,0}$.
  - Find out if there is a similar relationship for $Q_{1,2}^{0,0}$ in the Full model.
  - Solve some of the enumeration problems with other groups, like those of order six.

- **Kernel methods in general.**
  - Determine if the algebraic and obstinate kernel methods can be combined into one holistic approach. This can involve classifying methods on the number of substitutions made.
  - Can the methods be extended to more generalized models?

- **Connections with other theory.**
  - Using the Cauchy integral formula, we can rewrite extractions as complex integrals. This suggests that some of the equations can be turned into multivariable Fredholm integral equations. Perhaps methods from that area can be adapted to this one.
  - Can we adapt Sturm-Liouville theory to lattice walk enumeration? In particular, can we find a suitable inner product for Laurent series to test operators for self-adjointness and can we expand $Q$ in terms of eigenfunctions of the lattice walk operator?
  - Can we find asymptotics for lattice walks directly from the operator equations?
Bibliography


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