EXPLORING STUDENTS' MATHEMATICAL SENSE-MAKING THROUGH NON-ROUTINE PROBLEMS: VISUALIZATION, GESTURE, AND AFFECT.

by

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Abstract

This study explores graduate students' mathematical sense making through non-routine problems. I consider visualization, gesture and affect as integral cognitive aspects in the solution processes of participants. To analyze them, I introduce a suitable model, a think-aloud protocol coupled with meta-cognitive prompts. The study gives details of the solving of given non-routine problems by participants. It allows focusing on the relationship between visualization and gesture in conjunction with affective states in the process of sense making when solving non-routine problems in the absence of pre-determined mathematical procedures or algorithms. Visual imagery, gesture and particularly affective issues played a role in the solving processes of graduate students. As such these resources are seen as major ingredients in mathematics teaching and learning.

Preface

This dissertation is original, unpublished, independent work by the author, M. Medina. Dr. Susan Gerofsky provided suggestions in terms of methodology and approaches to data analysis for this study. For the study conducted in this thesis ethics approval was obtained from the Behavioural Research Ethics board under certificate number H12-00619.

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Dedication

This thesis is dedicated to mom and dad whose love and warmth made this journey possible.

Chapter 1: Introduction

1.1. Introduction

Mathematics is essentially the science of patterns (Resnik, 1981). It therefore follows that solutions to mathematical problems often express patterns. The process of exploring, understanding and expressing patterns whether formal or informal can be difficult and challenging. It requires focus, patience, resilience, creativity, and intuition - in essence, cognitive effort by the participant. One component in the process of thinking mathematically is sense making. *Sense making* can be considered as the cognitive process by which learners develop understanding of a concept by connecting it with existing knowledge (NCTM, 2009, p. 4). But how exactly does one make sense of an abstract mathematical idea so as to successfully solve a given problem? The process of making sense of an abstract mathematical idea and understanding it deeply depends a lot on the mathematical ability of an individual. But what exactly is mathematical ability? What is the nature of mathematical thinking? These questions are not easy to conceptualize and quantify. We may ask more fundamental questions such: What is the nature of mathematical ideas? What are they and how do humans think about them? I began contemplating about these questions and I found myself asking more focused questions such as: What cognitive factors are central to the process of sense making? How do learners make sense when solving a problem in the absence of predetermined and systematic algorithms or procedures? What does this growth of mathematical understanding and development look like? If insight into these questions can be obtained then researchers and educators can further explore, build and appreciate the richness of learners' sense-making abilities; but where to begin? I choose to start with my own teaching experience. After all, it is from personal experience that the problematic of this research is rooted - my teaching experience and personal interest.

1.2. Positionality of the Researcher

As an undergraduate student of mathematics I enjoyed and welcomed the challenge of solving math problems. In fact our lecturers emphasized the idea that the more math problems you solved, the more mathematically proficient you'd become. For me, I simply enjoyed the thrill of discovering, experiencing and making sense of patterns. This is what made me good at math.

More importantly, I had a deep desire to discover the reason and make sense for myself why a particular algorithm or proof worked. But I never quite understood this cognitive process - the process of coming to make sense of a given mathematical idea or pattern. This notion of sense making is what delivers those 'Aha! Moments' (Radford, 2009; Liljedahl, 2005) when one finally discovers the intricate details governing a mathematical idea or pattern. Through this process of sense making learners are able to create meaning for themselves and establish conceptual understanding (Liljedahl, 2005). But how does this process work? As one solves mathematical problems one often seldom, if ever, questions this cognitive process.

It wasn't till I took up a teaching position at the secondary level that I began to ponder more about what it meant to experience mathematics and create meaning for oneself by reasoning and making sense of the mathematics. After all, I was entrusted with the responsibility of ensuring that my students learn mathematics with understanding. As a teacher, I tried to emphasize and instill the desire to explore and probe a problem as deeply as possible. But this was never an easy task. As a teacher from Belize, I was constrained in this capacity. I had a syllabus and a workbook to complete. Most of the school year was dedicated to preparing students for the Caribbean Regional Exams (this exam is a measure of a schools' mathematics competency) that was the end goal. Essentially, given the limited time, teachers were forced to teach to the test. Teachers became focused on training students to be good test takers by developing test-taking skills even if it meant that conceptual understanding would suffer.

Yet my quest remains, to have students experience the enjoyment of exploring mathematical patterns using their own sense making abilities. There were few classroom instances where my students' thinking did intrigue me. These few classroom moments stimulated by non-routine problems made me realize that mathematical thinking and the process of sense making is diverse. These types of problems allowed me to observe some unique problem solving solutions. Of particular interest to me was visual thinking (i.e. the ability to think in terms of pictures) and the various visual methods (drawings, sketches, visual depictions of patterns) used to gain insight and successfully attain a non-standard solution. This mode of thinking also expressed ingenious ways of expressing complex patterns in very creative, elegant and schematic forms. Of course this wasn't always the case. But when used appropriately, these visual methods had the potential of capturing the essence of a pattern. I was fascinated by this approach to solving - I simply wanted to know more about the role of visualization in pattern exploration and expression.

After two years of teaching, I got the opportunity to pursue further studies. Graduate studies provided me the opportunity to study this topic more closely. Initially I considered exploring visualization in Geometry, as this is one area in mathematics where visualization is readily seen. However the more I read the more I realized that, as Giardina (2000) puts it, "to discuss the role of visualization in mathematics means to discuss [its] role in [relation] to other [cognitive] aspects that are central in mathematical activity" (p. 30). In other words, if I was going to explore the role of visualization in sense making I had to consider this cognitive process within the context of mathematical thinking, which brings into play other cognitive resources. In fact, as we shall see later on, the process of visualization does not exist as an isolated form of mathematical thinking ability but relies on other cognitive resources (speech, gesture, affect) central to the process of reasoning and sense making. Over time my research goal shifted from exploring visualization as an isolated thought process in problem solving to exploring the cognitive resources associated with sense making of which visualization is one central aspect. My focus gradually shifted to exploring participants' sense making processes through non-routine problems.

1.3. Statement of Problem: Issue

Teaching and learning mathematics for understanding are not easy activities. One reason is that mathematical notions are often treated as abstract objects subjected to rules of logic (Sfard, 1991). "Being capable of somehow 'seeing' these invisible objects appears to be an essential component of mathematical ability; lack of this capacity may be one of the major reasons because of which mathematics appears practically impermeable to so many well-formed minds" (Sfard, 1991, p. 3). Yet, characterizing mathematical understanding "in a way which highlights its growth, and identifying pedagogical acts which sponsor it, however, represent continuing problems" (Pirie & Kieren, 1994, p.165). The difficulty lies in the fact that mathematical thinking is a complex phenomenon involving many cognitive resources which themselves are not easy to conceptualize and operationalize.

Take for example making sense of a mathematical concept by thinking visually. Today most mathematicians value ‘visual’ thinking in their area of work and frequently emphasize its importance (Woolner, 2004). Most argue that students should work to develop this aspect within their own thinking. However, this belief has not gone unchallenged in mathematics education research. This perception is undermined by findings that point to a paradox between successful visualizing mathematicians and unsuccessful visualizing pupils. ‘Visualizers’, or visual thinkers, are not necessarily the most “successful performers in mathematics” (Woolner, 2004, p.450). Eisenberg and Dreyfus (1991), for example, claim that students rarely exploit visual approaches and are reluctant to visualize in making sense of mathematics. In addition, Presmeg’s (1997a) study shows that most mathematical difficulties encountered by ‘visualizers’ relate in one way or another to problems with generalization. Presmeg’s (1997a, 1992) findings show that some visualizers have difficulty using visual imagery to construct/generate more general mathematical ideas. For example learners who thought in vivid concrete detail (i.e. a static picture in the mind) often had difficulty using their imagery to generalize in contrast to those who used pattern imagery (pure relationships stripped of concrete details that could be moved or transformed) (Presmeg, 1992, p. 602). Pattern imagery was a strong source of generalization for the learner who used it in Presmeg’s study.

Now why is this the case? I argue that studies on visual thinking haven’t factored into the equation other cognitive resources and affective states (e.g. present moment mindedness, letting go) that may enable or constrain the process of sense making. Visual thinking has, for the most part, been studied as an isolated cognitive ability where the focus has been on apprehending the processes of constructing and using imagery to solve problems. The role of visual thinking in sense making needs to be studied within a theoretical framework of mathematical thinking taking into account other cognitive resources (for example gesture and affect) essential to sense making. As Edwards (2009) argues “[...] mathematical thinking, is embodied at multiple levels: through imagery, bodily motion and gesture [...]” (p.128). Thus it is important to consider those aspects of thinking (visualization, gesture, affect) that may enable or constrain a learner’s solution processes. Any theoretical model describing these aspects in thinking would need to be supported by empirical data derived from analyses of participants’ thinking.

It is important to consider that sense making is very much a personal cognitive process often laden with affect (Goldin, 2000; Presmeg, 1997a). This is one aspect of sense making that is often overlooked in mathematics problem solving research. Studies need to account for the personal and individual affective states associated with making sense, which may enable or constrain the mathematical solution processes of an individual (Presmeg, 1997a; Wheatley, 1997). For example, consider a student who dislikes mathematics. A past negative event or a series of negative events has made an impact on him/her as a learner. He/she may feel afraid or anxious at the thought of having to endure a class in mathematics where the goal is to solve. This affective state coupled with the fact that many school practices continue to emphasize rote memory work of fixed algorithms and procedures to solve (Schoenfeld, 1992) may constrain or downplay the learner's sense making abilities and the cognitive resources (visualization, gesture etc.) associated with it. On the other hand we can find another student who enjoys solving math problems and isn't afraid to tackle challenging ones. This student enjoys math and is able to use his/her own sense making abilities with success. Such students are not difficult to locate in a classroom (Liljedahl, 2005). As a researcher, this scenario piqued my interest. I wanted to explore the process of sense making in the context of non-routine problems that cannot be solved by a known algorithm. What cognitive resources would participants evoke in the absence of pre-determined algorithms, procedures or mathematical technique(s) to make sense when solving a math problem? Would they visualize? Would they gesture? What affective states would they bring to bear? How would these resources impact the process of sense making? This led me to the following research questions that are central to this study.

1.4. My Research Questions

(a) What role do visualization, gesture and affect play in supporting mathematical reasoning and sense making?

(b) How do these resources enable or constrain the process of sense making in problem solving in the absence of pre-determined algorithms?

Chapter 2: Literature Review

Mathematical thinking, or mathematics cognition, is "[...] a complex phenomenon, which involves many different cognitive resources" (Giardino, 2000, p. 29).

In this chapter, I will discuss literature related to the process of sense making in mathematical thinking. The discussion will be devoted to developing an account of the nature of sense making and the cognitive resources (visualization, gesture and affect) learners may bring to bear during this process. This section will be selective and illustrative with discussions and examples centered on the process of sense making.

The content is classified into three main parts. First, I'll attempt to frame mathematical thinking and sense making theoretically. In this section I'll review the theoretical model of mathematical thinking by Pirie & Kieren (1994). I made a reasoned choice in selecting this model - it treats the act "of generalizing and formalizing to be recursively connected to less sophisticated understandings, and not simply the product of acts of abstraction" (Martin, 2008, p. 64). It took into account my conjecture of sense making as a process connected to less formal/algorithmic forms of knowing where visualization is a central aspect. This model suited what I needed to frame the role of visualization in sense making.

To discuss visualization in relation to sense making, I draw insight from Lakoff & Núñez (2000) and Johnson's (1987) notion of *image schemata*. Since the focus of this study is on exploring the process of sense making and the cognitive resources associated with it, I devote the second part to a discussion of speech and gesture and aspects of affect ('letting go' and 'present moment mindedness') that learners evoked while making sense. In the final part, I situate my study within the field of mathematical thinking and problem solving.

2.1. Mathematical Thinking and Sense Making

An important goal of mathematics education is to understand the thinking involved in doing and learning mathematics (Núñez, Edwards & Matos, 1999, p. 45).

Mathematical thinking is an elusive, broad and complex phenomenon making it difficult to define. There is the Platonic view of mathematics as a science of abstract entities. Davis and Hersh (1981) muse "the typical working mathematician is a Platonist on weekdays and a formalist on Sundays" (p. 321). This Platonic viewpoint holds that mathematics is objective, universal, and not a product of human sense experience situated in context. Platonists characterize mathematical thought strictly as a mental activity where mathematical objects, patterns and relations can only be discovered, new connections made and structured through logic and reason alone (Resnik, 1981). It's assumed that mathematical "thinking is a pure mental activity - something *immaterial*, independent of the body, occurring *in* the head" (Radford, 2009, p. 111). On the other hand, there is the constructivist perspective that argues that mathematical activity is not only mental and mediated by the manipulation of formal written symbols, rather it is embodied, situated and mediated, "in a genuine sense, by actions, gestures and other types of signs" (Radford, 2009, p. 112). This alternative perspective states that rather "than positing a passive observer taking in a pre-determined reality, [this paradigm holds] that [mathematics] is constructed by the observer, based on non-arbitrary culturally determined forms of sense-making which are ultimately grounded in bodily experiences*"* (Núñez, Edwards & Matos, 1999, pg. 49).

Mathematicians often view mathematics with Platonic and formalist lenses (Davis & Hersh, 1981). But there is no way of knowing whether mathematical 'truths' do exist independent of the human mind-body system. It is more of a philosophical question to ask: Would mathematical objects or truths exist even if humans didn't? Such a question would lie in the realm of the metaphysics. What we could ask for the purpose of mathematics education is: What are the key cognitive aspects of mathematical thought? From these viewpoints of mathematical thinking lie two important aspects – reasoning and sense making.

2.1.1. What are Reasoning and Sense Making?

Reasoning in mathematics is the process of using axioms, definitions or stated assumptions to logically deduce conclusions (Resnik, 1981). Reasoning plays a fundamental role in mathematics and is understood to encompass formal and informal deduction. *Sense making* is defined as developing understanding of a new mathematical concept by connecting it with existing knowledge or previous experience (NCTM, 2009). Note, "practice, reasoning and sense making are intertwined across the continuum from informal observations to formal deductions, […], despite the common perception that identifies sense making with the informal end of the continuum and reasoning, especially proof, with the more formal end" (NCTM, 2009, p. 04). I adopt this perspective that reasoning and sense making are not mutually exclusive processes but are closely interrelated. Sense making is an integral aspect of the overall process of mathematical reasoning. It is this aspect of thinking that allows learners to experience mathematics, establishing meaning and understanding for themselves by actively reasoning and making sense of what is happening within the mathematics. But what cognitive resources enable or constrain this process?

In this literature review I'll explore the cognitive mechanisms of visualization, gesture, and affect as cognitive resources in the process of making sense. I choose to focus on these three aspects in thinking as "[...] mathematical thinking, is embodied at multiple levels: through imagery, bodily motion and gesture [...]" (Edwards, 2009, p.128). I consider the act of sense making within the context of mathematical thinking. To do this, I'll use Pirie and Kieren's (1994) model of mathematical reasoning as a framework.

2.1.2. A Model of Mathematical Thinking

There has been a wide variety of approaches attempting to capture the essence of the mathematical thinking (Pirie & Kieren, 1994). Consider the Concrete and Formal operational stages in Piaget's (1977) theory of cognitive development. At the concrete stage, learners (between the ages of seven and eleven) think logically but are concrete in their thinking. Learners are able to perform operations on mathematical objects but not entirely without concrete references or informal understandings. As learners grow older they develop abstract thought and are able to think logically in their mind. In other words, by a certain age, learners develop the capacity to perform logical deduction with little to no reliance on concrete or informal references. Piaget's (1977) model characterizes mathematical reasoning as a gradual linear process composed of stages or levels. Others have similarly characterized mathematical reasoning based on various polarized categories, for example: concrete and symbolic (Dreyfus, 1991), relational and instrumental (Skemp, 1976), and intuitive and formal (Fischbein, 1994).

However I found these perspectives to be limiting in that they segregate mathematical thinking into two distinct domains and exclude the process of sense making that learners evoke while thinking, specifically that interplay between concrete and formal abstract thought (NCTM, 2009). I initially based this assertion on my own experience of practicing mathematics and observations of students' thinking. I then discovered that Pirie & Kieren (1994) had theorized and modeled this aspect of thinking which they termed 'folding back' (I'll elaborate in the next section). I adopt the perspective that mathematical thinking involves individual cognitive abilities to create meaning when making sense of the mathematics.

These perspectives provide limited account of the process of sense making in thinking. Mathematical thinking needs to be viewed as a complex interplay of cognitive resources and not characterized solely as a composition of mutually exclusive categories. I adopted a model that would capture mathematical thinking as a non-linear process taking into account the interplay between concrete and abstract thought as a recursive process in constructing understandings. This criterion brought my search to Pirie and Kieren's (1994) model which describes the growth of mathematical thinking as a "whole, dynamic, leveled but non-linear, transcendently recursive process" (p.166). This model took into account my conjecture of sense making as an integral aspect of mathematical thinking.

Pirie and Kieren's (1994) theory and model of mathematical thinking is an "established and recognized theoretical perspective on the nature of mathematical understanding" (Martin, 2008, p. 64). Researchers (see for example, Davis & Simmt, 2003; Martin, 2008; Calvert, Zack & Mura, 2001) have used this model in their work. After reviewing other theoretical perspectives of mathematical thinking (e.g. Piaget, 1977; Dreyfus, 1991; Skemp, 1976) I made a reasoned choice in selecting this model. Despite its shortcomings, specifically the undeveloped notion of 'folding back', and limited understanding of how and why it occurred, this model emphasizes more localized ways of mathematical thinking (intuitive-ideas, concrete-representations) with visualization as a central aspect. The model treats the act "of generalizing and formalizing to be recursively connected to less sophisticated understandings, and not simply the product of acts of abstraction" (Martin, 2008, p. 64). This model mapped key aspects of sense making – it is a process that is connected to less formal/algorithmic forms of understandings. It is a process that involves use of imagery to develop understanding of patterns, ideas or concepts even when thinking formally. This model suited what I needed to frame the role of visualization in sense making.

I will take a few moments to review some of the interesting features of this model. The figure below captures Pirie and Kieren's (1994) model visually. I will be making reference to it in this section.

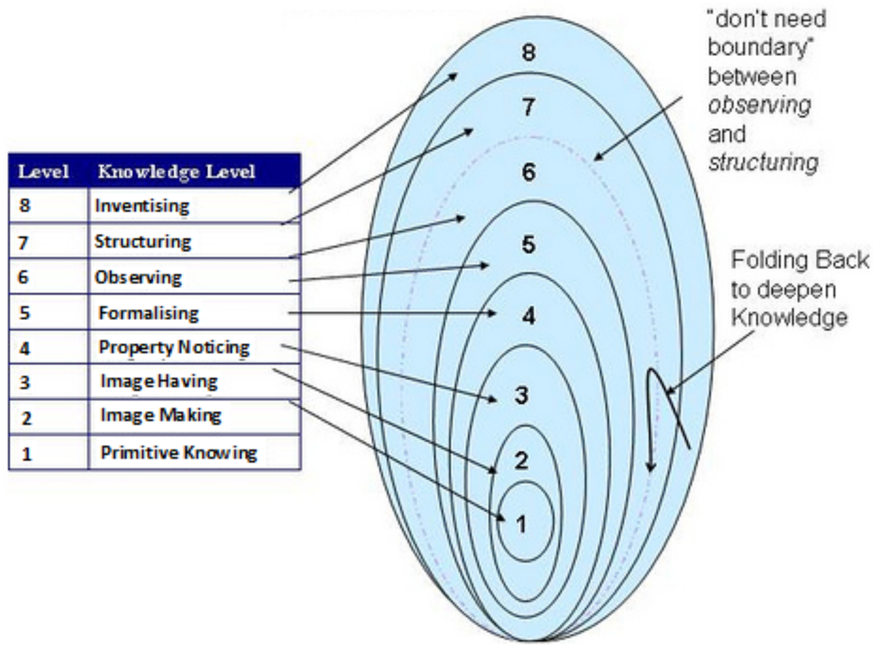

Figure 1: Pirie and Kieren's model of mathematical thinking (Pirie & Kieren, 1994)

The above figure illustrates one theory of mathematical thinking through eight embedded rings. Each inner ring is considered a different level or activity of knowing which give rise to successive outer levels of advance understandings. A unique feature, and a clear advantage of this model, is that each preceding level becomes embedded within the succeeding ones. As such, these forms of knowing are not treated as mutually exclusive categories but rather components of a dynamic process. According to Pirie & Kieren (1994), mathematical thinking starts off at the innermost level called *primitive knowing*. This level does not represent low-level mathematics, but rather an individual's initial understanding. It is what the teacher assumes the student can do and should be seen as the starting point of understanding (p.170). At this level "one cannot ever know what [the learner's] primitive knowledge is in full" (Pirie & Kieren, 1994, p.170). It is used simply to denote a learner's pre-knowledge, whatever the nature of this knowledge may be. It certainly would have been worthwhile for the authors to theorize about the nature of these primitive understandings. Do all learners possess equal constructs? Are these understandings unique and/or arbitrary? While this level describes the initial point of thinking, it limits insight into what learners come to rely on as a premise for reasoning and sense making. Later I will provide an account about the nature of this level of *Primitive Knowing*.

The second ring or level is what the authors refer to as *image-making* and represent an individual's ability to construct mental representations of mathematical objects, concepts or relations. The authors explain that at this level, learners use primitive knowing to construct suitable images for use in the third level *image-having*. These images form the base of understanding that learners fall back on and even modify as they move up the other levels with understanding. At the third level, *image-having*, learners now poses a suitable mental image or model that they can now use to abstract. This level, as Pirie and Kieren (1994) explain, marks the threshold of a learner's ability to abstract (p.170). The learner now possesses a suitable mental construct of a given concept that frees him/her from the need to rely on physical references, such as manipulatives. The third level leads into the fourth called *property noticing.* At this level, a learner is now capable of mentally manipulating images or aspects of his/her image to construct relevant mathematical properties by forming new connections and relations. In other words, the learner uses his/her mental image as a base from which to connect and make new relations, i.e. make sense. This is quite an interesting take as it highlights the growth of thinking by visual means. However, if at a certain point in their mathematical development children no longer require the use of manipulatives to handle, say, the mechanics of arithmetic or fractions, it is important to understand how exactly learners come to 'internalize' these properties and develop mental constructs for use in abstracting. How do learners come to develop these suitable mental models of, say, fraction mechanics? What is the nature of these mental constructs? What are they and how exactly are they constructed during the process of *image making*? The authors make no theoretical claim as to the nature of these mental models that learners come to possess. It is important to address the nature of these images and their impact in enabling or constraining the process of sense making.

I found myself wanting to know more about the process of *image making* and *image having* and how these lead up to the ability to notice and generate relevant mathematical properties and relations (i.e. make sense) without relying heavily on physical or concrete references. What cognitive resources do learners come to rely on here? In addition, what cognitive mechanism(s) allow learners to move back and forth between levels? The transition between these levels is not made explicit. How exactly do these levels evolve and iteratively change giving rise to the other outer levels of understandings? Filling in these gaps certainly would make this model more robust.

Now the strengths of this model are its built-in features that are worth mentioning: (1) *'Don't need boundaries' a*nd (2) *'Folding-back'*. The former represents thresholds at which a learner is able to abstract "without the need to mentally or physically reference specific images" (p.173). Take for example subsets. Once the learner possesses a deep intuitive feel for this concept, the extension of this idea becomes possible without having to heavily reference an image. The more crucial feature is that of *folding back* – a 'return-to' activity whereby a learner is capable of falling back to a preceding inner level in order to reconstruct and extend an initial, insufficient understanding in the process of making sense. This allows for the "recursive reconstruction" of knowledge upon return to the outer levels, because the knowledge, upon folding back, would have been influenced and shaped by the outer levels (p.173). Hence the authors' use of the term 'recursive process'. So if a learner is confused or uncertain of a given understanding he/she has the ability to fall back onto more concrete understandings to help reshape his/her thinking. This back and forth movement between the abstract and concrete is quite interesting as it shows that learners do make use of concrete knowledge even when abstracting. I like to think of these features as the recursive process of noticing and connecting new relations based on previous understanding which is the essence of sense making. But what cognitive mechanisms enable or constrain this process?

The way Pirie and Kieren theorize and model the growth of mathematical understanding based on case study observations of students engaged in mathematical activity shows that mathematical thinking is a non-linear recursive process. However, the model doesn't make explicitly clear the cognitive mechanism involved when learners transition from *primitive knowing* to *formalizing*. The transition between *primitive knowing*, *image making*, *image having* and *property noticing* are quite interesting in that it reveals the initial growth of understanding from a primitive intuitive sense of knowing to an abstract form when making sense via the use of images. Although this aspect of visualization is integrated into the Pirie-Kieren model, the process remains essentially undeveloped and unelaborated in their work. There was a lack of substantial examples and a limited understanding of when, how and why it occurred and its relationship to subsequent levels of knowing. Not much is said about the nature of these images, what they are, how learners construct and used them, or how they enable or constrain learners' understandings during sense making. Similarly when abstracting and formalizing mathematical ideas further along, *Property Noticing* and *Formalizing*, what cognitive mechanism or vehicle allows for that back and forth movement between these levels?

To help fill some of the gaps in Pirie and Kieren's (1994) model, I draw insight from Lakoff and Núñez's (2000) theory of embodied cognition to help explain what I like to refer to as *Primitive Knowing* of spatial relations. I theorize that learners use these relations as background knowledge in reasoning and sense making. In addition, I explore the notion of conceptual metaphors as another cognitive mechanism by which humans are able to conceptualize abstract concepts in concrete terms. This will be the focus of the next section.

2.1.3. Sense Making: A Perspective from the Theory of Embodied Cognition.

> [...] It appears that cognitive structure of advance mathematics makes use of the kind of conceptual apparatus that is the stuff of ordinary everyday thought such as image schemas, […], conceptual blends, and conceptual metaphor. (Núñez, 2000, p. 6)

According to Johnson (1987) and Lakoff & Núñez (2000) in their respective books: *Philosophy in the Flesh* and *Where Mathematics Comes From*, mathematical ideas are grounded in bodily experiences and shaped by the physical neural system of the human brain (Lakoff & Núñez, p.346). More specifically, "a great many cognitive mechanisms that are not specifically mathematical are used to characterize mathematical ideas" (Lakoff & Núñez, 2000, p.28). These everyday cognitive mechanisms are: *image schema* and *conceptual metaphors*. *Image schemata*, as used in this paper, are mental schematic structures that organize our physical experiences of spatial relations at a level that is more general and abstract (Johnson, 1987; Núñez *et. al.* 1999). For example the *container schema, balance schema, center-periphery schema, part-whole schema* are all "pre-conceptual structures, which arise, or are grounded in, human recurrent body movements through space, perceptual interactions and ways of manipulating objects" (p.01).

In conjunction, *conceptual metaphors* allow "human beings to map experiential structure from the 'imagistic' realms of sensory-motor experience (concrete) to non-imagistic (abstract) ones" (Johnson, 1987 in Grady, 2005, p.2). Lakoff & Núñez (2000) refer to the process of metaphorical thought as a "cross-domain mapping mechanism" (p. 6) that allows for the conceptual extension of bodily experiences into formal abstract concepts.

These two unconscious cognitive mechanisms are what characterize everyday ordinary ideas and are what learners come to rely upon when making sense. This theory provides an alternative and more fundamental view about the nature of mathematical ideas, how one thinks and make sense in contrast to the view that mathematical reasoning is strictly the mental manipulation of symbols based on rules to deduce universal truths or conclusions. It highlights "that individual concept (understanding and) construction involves the formation of image schemata and metaphors" (Rinvold, 2007, p. 178). This is one of the main reasons I selected this theoretical viewpoint; it provides deeper insight into the nature of sense making by grounding the construction of concepts in shared universal physical experiences as opposed to strict mental math.

Taking this embodied perspective, I theorize that the intuitive understandings of spatial relations form part of the innermost level, *Primitive Knowing*, in Pirie and Kieren's (1994) model. By drawing insight from Lakoff and Núñez's (2000) theory of *image schemata*, I'll explain how our shared biology and fundamental bodily experiences give rise to shared intuitive understandings of spatial relations – the premise from which other abstract relations are rooted.

2.1.4. Image-Schemata.

To illustrate what image schemata mean for sense making, consider the following mathematical notation $A \subset B$ which means set A is a subset of set B. But what exactly does this concept mean intuitively? A novice learner aiming to develop profound understanding may not be able to do so without some intuitive 'feel' of the notion when making sense of its meaning. This approach to learning is important because developing and having physical intuition of a concept is important for sense making and mathematical understanding. In fact, history shows that some of the great scientific breakthroughs were arrived at confidently by sheer intuitive 'feel'. For example, Einstein's intuitive insight of relativity came when he visualized himself riding on a beam of light or Archimedes in Syracuse discovering the laws of buoyancy while in his bathtub.

Now the origins of the notion of sets according to Johnson (1987), as with many mathematical ideas, is rooted in a universal felt bodily experience, in this case the physiological experience of being contained or enclosed by physical barriers. To further explain, imagine yourself sitting inside a room that is contained inside a larger one or picture an empty container contained inside another. Johnson refers to this internalized embodied physiological experience as the *container* *schema*. Johnson (1987) and Lakoff & Núñez (2000) explain that this physiological experience, structured as an image schema, serves as an intuitive repertoire in sense making and in the conceptualization of other mathematical ideas.

Since the notion of subsets depicts a spatial relation, the relation of 'in' and 'out'. The human mind is able to structure this spatial relation as an image schema as illustrated below.

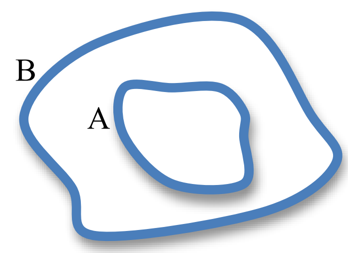

Figure 2: A ⊂ B

This schema is what Johnson (1987) calls the *container schema* and forms the basis from which new spatial relations such as: If $A \subset B$ and $B \subset C$ then $A \subset C$ is made possible. Below is a schematic illustration of this new relation, which makes sense:

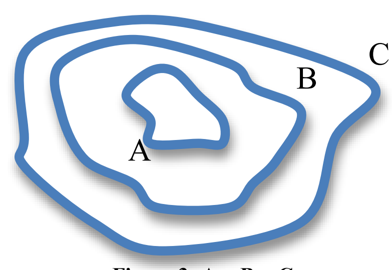

Figure 3: A ⊂ B ⊂ C

Taking another example, circles and points though considered abstract entities have been argued to be pictorial-mental schemata that humans construct by virtue of physical interaction within a spatial-physical world. This is made possible because the human visual system is intimately tied to the sensory motor system as one moves though space (Lakoff and Núñez, 2000) i.e. "image schemata are kinesthetic […]" (Lakoff and Núñez, 2000, p.34). However, the mechanism that leads to the conscious construction of image schemata by the individual is not made explicit by Lakoff & Núñez (2000) or Johnson (1987). If humans are able to internalize physical experiences and mentally structure them as schemata for use in mathematical reasoning and sense making, how are these schemas created, used and manifested in the process of sense making? This process may very well be unique and vary from one individual to another and any theoretical model of this process would need to account for individual cognitive ability.

I began to wonder about the cognitive ability at play here. Based on my observations of students' thinking, I suspect that proficient students, who are able to construct rich understandings of mathematical concepts, combine and utilize both visual and analytic ways of thinking. I conjecture that visualization is a key component of sense making. It is one cognitive mechanism by which image-schemata are effectively used in mathematical reasoning. Through my empirical study I tested this conjecture. One of my aims was to explore the visual-spatial component of intuitive sense making for mathematical understanding. I'll now attempt to examine the role of visualization and explain how this aspect of sense making helps shape understandings of spatial relations in *primitive knowing*, *image making* and *image having* in Pirie and Kieren's model.

2.1.5. Defining the Process of Visualization.

Review of the literature in this field indicates that there is no precise definition of the term *Visualization*. Neither researchers nor educators have an agreement about the terminology used in the field. For example, the terms: *visual processing*, *visual imagery*, *imagery processing, spatial reasoning* (including spatial skills and spatial abilities) and *visual intuition* are sometimes used interchangeably and other times take on different meanings. So for this review, it is necessary to clarify how the term *visualization* is used. First, consider Piaget and Inhelder's (1971) definition: "when a person creates a spatial arrangement there is a visual image in the person's mind, guiding this creation" (Presmeg 2006, p. 206). Pirie and Kieren refer to this stage as Image Having (Level 3 in their model). In the same vein, Presmeg (2006) establishes the following broader definition: "visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics" (p. 206). In other words, visualization is a mental process – the ability to reason or make sense of concepts in terms of images. I adopt Presmeg's (2006) definition as it suited my study on the basis of what I was able to observe from my participants. I also prefer to use the term visual intuition. More importantly, in this thesis, visualization will be treated as a cognitive process involving processes of constructing and transforming mental images that can be manifested through gestures, words (inscribed or spoken) or inscriptions (symbols, drawings/sketches) when making sense of mathematical ideas. Let us take a closer look at visualization and image schemata and its role in sense making.

2.1.6. Visualization and Image-Schemata.

Image schemata depict spatial relations in the form of mental imagery. As discussed before, Johnson (1987) explains that image schemata are schematic structures “that organize our mental representations (of spatial relations) at a level more general and abstract […]” (p.23, 24). Visualization is the more general cognitive process of constructing and transforming mental images that may be implicated in mathematical reasoning. Through the process of visualization an image schema can be combined with one or more to form new spatial relations (i.e. by forming new connections and relations from existing schemas). This ability is a powerful cognitive resource tool in the process of sense making as sense making involves creating new knowledge from existing ones. For example, Presmeg (1992) refers to *pattern imagery* (a cognitive process) as a type of imagery in which pure relationships are depicted in a spatial-scheme (Presmeg, 1992, p. 602). That is, images that get constructed as geometric patterns, patterns that are stripped of unnecessary concrete vivid details. Now the nature of these image schemata and how someone’s mind pictures or sees them may vary from one individual to another and would certainly worthwhile to research.

To summarize, image-schemata are abstract representations of shared bodily experiences of spatial perceptions. Through visual and analytic reasoning, we are able to connect and from new relations from these schemata. For example the logical statement: If $A \subset B$ and $B \subset C$ then $A \subset C$ based on basic premise of the container schema. How does the act of embodiment support the geometric-schematic patterning power of visualization in such a way that it lays down appropriate intuitions for analytical reasoning? Schematic imagery is an important ability to develop, as mathematics is the science of patterns (Rensik, 1981). I define embodiment as a means of cultivating visual intuition through spatial-kinesthetic experiences to aid *pattern imagery*. Visualization is thus one cognitive mechanism that may be activated in the process of sense making. Another mechanism through which the human mind is able to connect new ideas from existing ones is metaphorical thought.

2.1.7. Sense Making: Conceptual Metaphors.

> *"By allowing us to project beyond our basic-level experience, conceptual metaphor makes possible science, philosophy, and all other forms of abstract theoretical reasoning [including mathematical ideas]"* (Johnson 1987, p. 556).

In linguistics, a metaphor is the link we naturally make between two objects – usually something concrete (source domain) with something abstract (target domain) (Johnson, 1987; Sweetser, 1998; Turner & Fauconnier, 1995). In mathematics there are numerous metaphors. For example, we have the "numbers are points" metaphor that link numbers in arithmetic with points in geometry. This metaphor forms the basis of the number line and led to the birth of analytic geometry. These conceptual metaphors, as they are called, allow us to extend, better understand, think about and make sense of ideas by allowing us to map characteristics or properties of one domain onto another (Fauconnier & Turner, 2008). In the case of sense making, pointing out conceptual metaphors or blends of metaphors allows us to make better sense of abstract mathematical ideas from concrete ones.

Take for example the idea of the empty set. The familiar yet not so intuitive formal definition $\varnothing \subset A$, which is taken to mean: Given any set A, there is a unique set containing no members called the *empty set* $\varnothing$ which is a subset of A. Now, why is the 'empty' set, which contains no elements or members a subset of all sets? Why is this mathematical statement true? And why is it unique? Within the culture of those who practice mathematics professionally, the answer to these questions often follows from axioms, definitions and rules of logic. Yet, this approach to reasoning does not necessarily constitute genuine understanding for most. The formal definition of an empty set often makes little sense to most people. What makes the creation of this not so intuitive idea possible and why does it make sense? According to Lakoff & Núñez (2000), Sweetser (1998) and Turner & Fauconnier (1995), to understand why requires an analysis of the conceptual metaphors and blends that logically lead up to it. It is a cognitive process that involves linking ideas through metaphors (Sweetser, 1998). This cognitive mechanism often operates at an unconscious level making it difficult to study. However, linguistic science does offer specific techniques – *Mathematical Idea Analysis* (Lakoff & Núñez 2000; Núñez, 2000) to study this process. But that is beyond this paper. The point to be made here is that abstract mathematical ideas are, for the most part, constructed through a cognitive process that involves mapping, connecting, and relating concepts and ideas in order to generate new ones by means of conceptual metaphors (Fauconnier & Turner, 2008). Lakoff and Núñez (2000) in their theory of conceptual metaphors call this process *metaphorical mapping*. I conjecture that this mechanism is the missing gap in Pirie and Kieren's (1994) model of abstract reasoning.

Metaphorical mapping is a linguistic cognitive mechanism by which the human brain is able to extend primitive, intuitive mathematical ideas to advanced abstract concepts. Going back to our previous example of the container schema metaphor, the human mind is able to reference and use this spatial relation between objects (i.e. an object inside another object) as an intuitive basis when making sense about the notion of a set. Given this basic understanding (source domain), the idea of a set can then be further extended to construct new relations by logical reasoning, such as the empty set, using conceptual metaphors and/or blends. I find this idea to be a very interesting aspect in the process of sense making because it helps explain how metaphors "[...] ground our understandings of mathematics in terms of everyday (sense making) experience" (Lakoff & Núñez, 2000, p.10). For example when we use the container schema metaphor or treat number as points on the number line, we're mapping an embodied concept from one domain onto another. In this regard, "conceptual metaphors and blends permit the use of sensorimotor inference for abstract conceptualization and reasoning. This is the mechanism by which abstract reason is embodied" (Johnson, 1987, p. 556).

This cognitive process, I theorize, is the missing piece of the puzzle to help explain the mechanism by which one is able to extend intuitive notions into formal ideas in Pirie and Kieren's (1994) model when abstracting and finding new relation based on concrete understandings. Again, referencing the concept $A \subset B$ derived from the container schema metaphor, this metaphor is the basic premise used to establish the formal definition of an empty set. This metaphor along with blends is the link between concrete ideas and general formal ones. To summarize, metaphorical thought is the mechanism that allows mathematical ideas to be extended, formalize and generalized from concrete ones. It is a "cross-domain mapping mechanism" (p. 6) that allows for the conceptual extension of bodily experiences into formal abstract concepts (Lakoff & Núñez, 2000, p.6).

If so, then metaphors manifest through speech offer a window to understanding individual mathematical thinking and sense making. I wonder whether the ability to think metaphorically combined with the ability to think visually aids the process of sense making. How do metaphors expressed through speech and the use of image-schemata in visualization help support the process of sense making? What role do these play and how are they manifest in the process of sense making and reasoning? These are certainly worthwhile questions to consider for further research.

The argument has been made that mathematical reasoning, “is not only mediated by written symbols, but [...] also meditated, in a genuine sense, by actions, gesture (and speech) and other types of signs” (Radford, 2009, p.112). According to this viewpoint, mathematical thinking does not occur solely *in* the head but through a sophisticated coordination of body, mind, symbols and tools. In the case of metaphors, these are used in everyday language and expressed through speech and gesture and other means. Therefore to fully appreciate what it means to engage in sense making, one must look to those cognitive aspects that make such activity meaningful, fruitful and possible. Visualization is one of those cognitive aspects discussed already. I will now briefly look at speech, gesture and affect in the next two sections.

2.2. Speech and Gesture

The promotion of research into the role of gesture and mathematical reasoning, especially in building an establishing meaning, is in particular associated with names such as Alibali (2009), Goldin-Meadow and Singer (1999, 2003), Sabena (2008), McNeill (1992, 2000), Radford (2009), Nemirovsky and Ferrera (2009), Arzarello, Paola, Robutti and Sabena (2009), and Edwards (2009).

This section takes a look at the relation between gesture and talk in the constructing, manifesting and establishing mathematical understanding. For this thesis, I take and slightly add to Sabena’s (2008) working definition of gestures suitable to mathematical activity. Gesture includes all *those movements of hands and arms* [or other body parts] *that subjects [...] perform during their mathematical activities* [this includes sense making] *and which are not a significant part of any other action (i.e. writing, using a tool, ...)* (p. 21). Of course this act includes the use of talk. The act of gesturing according to McNeill (1992), e.g. hand/arm movements, first start off from a position of rest, move away from this position (hand/arms usually move away from the speaker) and then return to rest. During the movement, there is a central part called the *stroke* or *peak* where meaning is generally expressed. The space, usually a frontal plane, where this movement takes place is called the *gesture space*.

One important finding is that gestures are inherently linked to speech; they are complementary sides of the same coin (Sabena, 2008). Goldin-Meadow and Singer (2003) show that "when children explain their answers to a problem, they convey their thoughts not only in speech but also in the gestures that accompany that speech" (p. 509). The act of speech and gesture do not act independently during this process (McNeill, 1992). Speech and gesture are one system influencing each other in the process of sense making and meaning-making. Gesture not only reflects a child's understanding but also can play a role in shaping that understanding (Goldin-Meadow, 2003).

However, the synthesis of meaning and sense making through the use of talk and gesture should not be seen as linear but rather as a complex interplay. These processes are intimately related, "they are semantically and pragmatically co-expressive; they are essentially synchronous in time and meaning [...]" (Sabena, 2008, p. 23). In other words, they cannot be separated from one another and as such should be studied as one co-expressive system in sense making. McNeill suggests that through gestures, "people unwittingly display their inner thoughts and ways of understandings" (McNeill, 1992, p. 12). According to this approach, gesture and speech can be seen as a kind of window to access people's thinking – how they connect ideas and make sense (Alibali, 2005; Goldin-Meadow & Singer, 1999; 2003). Both acts serve as a medium through which to express ideas and thoughts as well as to make sense in the process of thinking. Therefore gesture and talk are essential cognitive aspects of the process of sense making in mathematical activity. The perspective that I'm advocating here is that talk and gesture are also linked to use of visual imagery and affective states of mind in the process of sense making.

2.3. Affect and Sense Making

In addition to the previously discussed cognitive resources for mathematical thinking, affect has "generally been seen as 'other' than mathematical thinking, not just a part of it" (Zan, Brown, Evans & Hannula, 2006, p. 113). In addition to understanding certain cognitive aspects (visualization, speech and gesture) in the process of sense making, affect is one important area that requires exploration. This is important as certain affective issues (e.g. fearing failure) may enable or constrain the sense making process of individuals. I'll briefly discuss the status of affect in mathematics education research here.

Emotion, attitudes, opinions, beliefs, interest, self-esteem, motivation, feelings, mood are only some of the words often used loosely and interchangeably to denote affect in mathematics education research (Leder & Forgasz, 2006; Liljedahl, 2005; Tobias, 1991; Mcleod, 1992). There is no set definition. What can be stated is that whatever above aspect is used and studied, as affect seems to encompass a wide range of concepts (Mcleod, 1992), none "can be observed directly; each needs to be inferred from behavior, speech or responses to specifically designed instruments (Leder & Forgasz, 2006, p. 404).

Most measuring instruments used to study these variables frequently take the form of surveys, questionnaires and scales. Responses are gathered and then assigned numerical scores to measure students' attitudes, beliefs and emotional responses on a continuum with positive responses being assigned higher scores (Leder & Forgasz, 2006, Mcleod, 1992). However this quantitative method limits reliability of results as assigning a numerical value to a subject's affect has a tendency to standardize and dehumanize subjects' 'felt' experience. Numbers are limited in their capacity to fully capture emotion as a phenomenon. Therefore qualitative methods (interviews and observations) are more appropriate in this aspect. Especially if the focus is on studying affectivity in sense making from the perspective of subjects, one aspect that is often overlooked in mathematics problem solving. Therefore for this study I aim to closely examine affective issue(s) or aspects that may enable or constrain the sense making process of individuals through analysis of participants' behaviors and responses/discussions as they solve none-standard problems.

In reading Leder and Forgasz (2006) and Zan *et. al.* (2006) comprehensive review entitled *Affect and Mathematics Education*, and *Affect in Mathematics Education* respectively, I found it interesting that the subject continues to elude researchers. The challenge it seems stems from the fact that the term affect isn't clearly defined or operationalized; perhaps this is because of its complexity and multidimensional nature. I argue that researchers need to be able to draw insight from fields such as social cognition, neuropsychology, performance science, psychology, etc. when studying specific aspects of human affect and its role in sense making. Not having a generalized definition or a working model of affect and its role in sense making could explain why the relationship between affect and the teaching and learning of mathematics continues to remain elusive even to this day. As Leder and Forgasz (2006) point out, it is difficult to find consistent findings across studies on affect and sense making (p. 416). Nonetheless, taking a qualitative approach to examining specific aspects of affect in problem solving can be informative with added insight from fields in cognitive science.

In the previous sections of this paper I attempted to offer a framework on the nature of sense making as a complex phenomenon that brings into play different cognitive resources – use of image-schemata, metaphors in speech, use of gesture in thinking and individual affective states. If the process of sense making through problem solving is to be explored, then studies need to, at the very minimum, take these points into consideration.

2.4. On the Status of Problem Solving and Sense Making

Today much of school mathematics, from elementary to graduate level, consists of giving definite, polished procedures to solve problems leaving little room for self-discovery, questioning or extension of ideas by reasoning and making sense of what is happening within the mathematics (Schoenfeld, 1992; Verschaffel, Greer, & de Corte, 2000). Teaching practice on problem solving has focused primarily on word problems of the type often emphasized in textbooks, tests, exams and other contemporary forms of knowledge assessment (Lyn D, Richard & Thomas, 2008; Borasi, 1986; Verschaffel, *et. al.*, 2000). These types of problems frequently characterize mathematical activity as the act of applying knowledge of predetermined algorithms or procedures where the end goal is supplying the 'right' answer (Verschaffel et. al., 2000). From my own experience of college level math, lecturers present a concept, show one or two examples of how to apply a particular mathematical technique, then give a list of problems to solve while hoping students develop profound understanding. Not much attention is given to the 'process' of sense making from the participant's mathematical point of view; rather the emphasis is on the end product i.e. supplying the right answer.

Along this same vein, Schoenfeld's (1992) points out:

> The impression given by this set of [mathematical] exercises, and the thousands like it that students work in school, is that there is one right way to solve the given set of problems - the method provided by the text or instructor. [...] students learn that

answers and methods to problems will be provided to them, the students are not expected to figure out the methods themselves (p. 343).

This is often the perspective students adopt, the perspective that there is a fixed method or rule to follow in solving a given problem. Consequently most students come to regard mathematics as the act (i.e. puzzle-like task) of applying rules and formulas to solve (Verschaffel *et. al.*, 2000). This of course does not constitute understanding (Skemp, 1976). Students need to experience mathematics and make meaning for themselves by reasoning and making sense of what is happening within the mathematics (NCTM, 2009). Perhaps it is time to revisit the notion of what it means to be engaged in mathematics and what it means to make sense by paying close attention to participants' sense making abilities.

Most studies relate sense making to the process of creating new knowledge from existing ones by forming relations and connections. Yet very few studies have taken into account the role of visualization, speech and gesture (a co-expressive system) and affective states in the process of sense making. Most focus on categorizing and describing heuristics rather than well defined processes. As such these heuristics or strategies possess descriptive rather than prescriptive power making attempts to teach students, for example Polya-style heuristics, generally unsuccessful (Schoenfeld, 1992). Studies on the role of visualization, gesture and affect as resources that may enable or constrain the process of sense making from the viewpoint of participants as they actively engage in solving non-routine problems in the absence of fixed algorithms are limited. Using Google Scholar, I was able to locate one study by Presmeg and Balderas - Cañas (2001) that looked specifically at the role of visualization, gesture and affect in mathematical thinking. However, this study focused primarily on analyzing participants' solution processes to standard problems rather than an exploratory approach to solving non-routine problems and examining the role of image schema, talk and gesture therein. How do these resources support the process of sense making in the absence of pre-determined mathematical algorithms or procedures? What role do these play and how are they manifested? It is this gap that this research aims to fill. I therefore ask:

(a) What role do visualization, gesture & speech, and affect play in supporting mathematical reasoning and sense making?

(b) How do these resources aid the process of sense making in problem solving in the absence of pre-determined algorithms?

A piece of research aimed at exploring participants' sense making as they attempt to solve non-routine problems in the absence of fixed formulas or algorithms could bring new insight about the role of visualization, speech and gesture, and affective states in sense making. In order to do this I draw on Pirie and Kieren's (1994) model of mathematical understanding as a framework in which to situate these cognitive resources for sense making. In addition I utilized a think-aloud protocol (Ericsson & Simon, 1993) coupled with meta-cognitive prompts (Anderson, Nashon & Thomas, 2009) in order to get participants to verbalize and discuss their thinking while solving. The next chapter will focus on the design of this study.

Chapter 3: Methodology

This chapter is divided into four main parts. First I explain why the choice of a qualitative approach. In second part, I describe the context of the research. Third, I discuss the research design followed by the procedure and methods used. In the final section I explain how the research data was analyzed.

3.1. A Qualitative Research Approach

The purpose of this research was to investigate the following overarching questions: (a) what role do visualization, gesture and affect play in supporting mathematical reasoning and sense making? (b) How do these resources enable or constrain the process of sense making in problem solving in the absence of pre-determined algorithms?

The goal of this research was to investigate the presence, role, and constraints of visualization, gesture and affective states in the sense making processes of participants as they solved non-routine problems. The study sought to examine closely how aspects of visualization, speech and gestures in conjunction with affective mental states may enable or constrain participants' sense making. One of my initial challenges was figuring how to investigate this aspect of mathematical thinking. What approach, procedure and methods would allow me to gain access to learners' thinking? Specifically, how do you investigate the process of sense making and the cognitive resources at play in problem solving? Clearly this is not an easy feat.

3.1.1. Choice of Depth over Breadth.

It's been stated already that mathematical thinking is "[...] a complex phenomenon, which involves many different cognitive resources" (Giardino, 2010, p.29). Mathematical activity is not superficial, it is elusive, complex and a challenge to make explicit. Moreover, mathematical thinking is a personal activity laden with meta-cognitive aspects, including affect, "that may enable or constrain the mathematical solution processes of an individual" (Presmeg & Balderas-Cañas, 2001, p. 290). Thus, to investigate learners' meta-cognitive and affective resources, it was necessary to develop deep inquiry into their process of mathematical thinking. I would need to investigate closely how learners think while solving. I needed an approach that would allow me to capture those moment of insights and cognitive aspects including what the learner felt while thinking. This, with the exploratory tone of the study, dictated the choice of depth over breadth.

3.1.2. Why Qualitative Methods.

The challenge still remained, what method would accompany this approach? Should a researcher interview participants and ask them about how the process (visualization, gesture, affective states)? Would one or more quantitative tests suffice? Or should the research encompass video recordings and analysis of external representations of participants' thinking? Would these measuring instruments and observations paint an accurate picture of the process of sense making or do we assume they do? Gray (1999) points out:

> The study of [thinking] in any context is fraught with difficulty. We make an assumption that report, description and external representation in the form of words, drawings and actions provide an indication of the nature of [mathematical thinking] (p.241).

Presmeg (2006) further adds: "There is no guarantee that the researcher's construction of the nature of [mathematical thinking] is accurate, nor that the thoughts of the individual were uninfluenced by the research process" (p.221).

The above statements led me further think about the measuring instruments and research methods used to study mathematical thinking. Woolner, (2004) and Kozhevnikov, Hegarty & Mayer (2002) used a quantitative method to investigate students' thinking. Their main research techniques include measuring participants' reasoning using one or more measuring instruments (e.g. MCT scores, Woolner, 2004). However, I found this method limited in its capacity to account for the personal affective aspects of thinking. You can't fully capture speech acts, gesture or affective states associated with thinking if participants sit in silence solving problems or performing tasks and then quantifying the results. These quantitative measurements only paint a partial picture making this method limited in its openness to let the unexpected emerge. In fact their research, as with many using this approach, served mainly to measure and categorize participants' mathematical ability. On the other hand, researchers such as Aspinwall, Shaw & Presmeg (1997), relied on qualitative methods (interviews, observations and document analysis) to comprehensively investigate how students think while solving problems. These latter studies helped informed my method of data collection.

Since speech and gesture can be video recorded and analyzed, I found it best to rely on capturing and studying the external manifestations of learners' thinking (gesture, speech, drawings etc.) to investigate the cognitive resources central to the process of sense making as participants explored non-standard problems. My aim was to develop depth of inquiry using the following research techniques: (1) present non-routine problems to be solved. The non-routine problems would allow for an exploratory approach and diverse ways of thinking rather than silent mental math. (2) Video record a think-aloud protocol coupled with meta-cognitive prompts. The video recordings would allow me to analyze in seconds the various gestures and speech acts including use of imagery in playback. The meta-cognitive prompts was also necessary as without probing, participants often fail to report the full extent of their use of imagery even when it represents an integral aspect of their thinking (Presmeg & Balderas-Cañas, 2001). Thus, the questioning was necessary. (3) Make field notes and observations and (4) follow up with post-interviews to clarify interesting points for the data analysis. I discovered that performing qualitative research on the nature of mathematical thinking could be insightful and informative.

3.2. Context of the Research

3.2.1. Participants.

The participants for this study included a total of 8 graduate level students who volunteered. Participants were recruited via a letter describing the research study in detail from the Faculty of Education and others from a graduate residential college at a large university in Canada**.** Two were initially recruited for the pilot study and the remaining six for the larger portion of the study. The pilot served as a means to test the think-aloud protocol and problem set before carrying out the larger part of the data collection.

There are several reasons why I made the deliberate choice to work with graduate students who are not math majors and had been away from mathematics for some time. There has been little research on the role of visualization in sense making by adult learners who are not research mathematicians (Presmeg & Balderas - Canas, 2001). There is also the claim in the literature that college level students are reluctant to visualize when they do mathematics (Eisenberg & Dreyfus, 1991). I wanted to collect further data to cast light not only on whether graduate students used visualization but also the role of gesture and affective states in sense making in exploring non-routine problems.

Since I wanted to explore the process of sense making other than formal math (i.e. pencil-and-paper), I worked with participants who had been away from math for some time and were open to explore problem solving through less formal algorithmic means and talk about the process. There were certain aspects about these graduate students' intellectual maturity that lend itself to this process: (1) They were introspective and thoughtful people. (2) They had confidence in their general abilities and were willing to explore problem solving informally and make sense without fear of having to remember algorithms or procedures. They didn't feel added pressure to produce a right answer as might have been in a regular classroom setting. (3) They were more open and less resistant to revisit the experience of problem solving which they had been away from for some time. There was a practical side to it as well. Being at the university these were the people who were readily available for my study. Researchers (for example, Presmeg & Balderas - Cañas, 2001) have similarly used the convenience of a university's population from which to recruit subjects.

The choice of limiting this study to a group of six participants followed the determination to look for depth rather than breadth while studying the process of mathematical thinking. I wanted to closely track how learners think. My aim was to collect data that could cast light on the process of sense making in problem solving, the cognitive resources and components of meta-affect at play. Therefore in regards to depth over breadth, this was a reasonable sample to observe and study.

3.2.2. Sampling Technique.

Research participants consisted of Masters and doctoral students recruited from the Faculty of Education and from a graduate residential college at a large university in Canada using a convenience-sampling model. The investigator enlisted the help of a third party (someone not connected to the study) to send out an initial email to graduate students with an attached letter of contact and consent form describing the research study in detail and participants' involvement (see Appendix C, D). This measure was taken to maintain an arm's-length relationship with potential participants during the recruitment process in order to minimize the possibility of students feeling pressured to join the study. Those interested were asked to contact the researcher directly via email or telephone for clarifications and further instructions. Those who volunteered were then asked to sign the consent forms, which the investigator collected and also made available on the first day of the problem solving sessions. I worked with the first 8 participants who volunteered regardless of their educational background.

3.2.3. Location.

The research took place on location at a university in Canada. Each problem solving session was held in the math lab in the Education building at the university at a time convenient to the participants. Several days later, post-interview were held in either the math lab in the Faculty of Education or at a location and time convenient to the participants. Names of participants were kept confidential. Pseudonyms were assigned to each participant in the research for identifying data sources and reporting of results.

3.3. Research Design

3.3.1. Researching and Selecting the Problem Set.

In researching the types of problems to present, I was guided by two main criteria. First, each problem had to be non-routine. Since the goal was to explore and probe learner's sense making through problem solving, I needed problems that were not too familiar to the participants. I also wanted to minimize the use of polished algorithms and route memory work. I wanted to invite participants to explore each problem by the approach that came most natural to them. The nature of non-routine problems would provide participants the opportunity to take on an exploratory approach to solving. Second, each problem had to be open-ended yet inviting and stimulating enough so as to allow for multiple approaches. In other words, each problem could be solved using different methods or strategies. In addition, the openness of these problems would allow for further extension or generalization of mathematical ideas and patterns. The mathematical problems that were considered for inclusion in the study are ones that required a general knowledge of high school mathematics. These criteria guided my search and selection of the problem set.

In searching for these non-routine problems, I appropriated several books that centered on the topic of problem solving. Sources for these non-routine problems included Mason, Burton and

Stacey (1985), Brown and Walter (2005), Brown and Walter (1993), Sawyer (1964), Polya (1957) as well as several research articles, for example see Campbell, Collis and Watson (1995), Presmeg and Balderas-Cañas (2001). These books and articles provided insight into the process of mathematical thinking and problem solving.

For each problem that was a potential candidate, I took time to work out the solutions myself. I took notes of my process. This exercise proved useful as it allowed me to take on the role of researcher, facilitator and participant. This activity allowed me to anticipate participants' thinking and to identify potential instances in their solution process where I could evoke their thinking further using specific meta-cognitive prompts. In solving each problem I made attempts to anticipate what strategies may be used, potential difficulties and what materials could aid the problem-solving process. Solving the problem myself also enabled me to categorize and gauge the level of difficulty of each problem. As I solved, I was inspired to think about how these problems would allow me to explore the process of mathematical thinking. To what extent would these problems influence my participants' thinking (the type, their structure, the order they're presented in, the general knowledge required to solve them, the participant's ability, etc.)?

It is important to mention that the extent to which these problems may have influenced the participant's thinking is not known (the nature of each problem, the type, and the order they were presented in). Aside from trying to avoid problems that had a fixed algorithm, there is no taxonomy of these types of problems. I had to start from scratch in selecting these. It was a trial and error process to find open-ended problems that would deliver, somewhat, the experience mathematicians undergo when solving a non-trivial problem for which there isn't a fixed algorithm, procedure or mathematical technique. There is potential for more work in this area. The more I thought about the process of meta-cognition and sense-making in problem solving, the more I realized that this activity wasn't something I could control entirely. The dynamics of this meta-cognitive activity would have to emerge from the participant's own engagement with each problem. The most I could do as a researcher was offer guidelines and look for entry points where I could probe their thinking.

For the problem set, I eventually settled upon two warm-up and three non-routine problems supplemented with additional questions just in case participants had already seen some of these (see Appendix A for the problem set). By solving each problem and taking notes, I was able to use this exercise as a stepping stone in designing the problem solving and post-interview protocols. What follows next is a discussion of these designs.

3.3.2. Designing the Interview Protocols.

One of the first challenges I faced early on in the design phase of my research was figuring out how to get participants to externalize and express their thoughts verbally, gesturally, diagrammatically in ways other than silent, pencil and paper mathematics. I wanted them to explore the problem freely and as deeply as possible and to discuss the process. I needed them to feel comfortable enough to express their thinking while they solved. How does one break this silent barrier that's been instilled over years of traditional school practice? I took into consideration the use of a think-aloud protocol (Ericsson & Simon, 1993).

3.3.3. The Think-Aloud Protocol.

Think-aloud protocols are a research method used to understand subject's cognitive processes via their verbal reports of their thoughts. For this study, the purpose of the protocol was to obtain information about participants' thinking by encouraging them to speak aloud while they solve. However, I came to realize that this alone would not be enough. According to Presmeg and Banderas-Cañas (2001), learners often fail to report the full extent of their thinking even when it forms an integral part of their solving process (p. 293). So as a researcher, I would need to be proactive when encouraging participants to express their thinking. To achieve this, I decided to supplement the think-aloud protocol with meta-cognitive prompts (Anderson, Nashon & Thomas 2009) (see Appendix B). In addition, during the problem solving sessions I would invite participants to freely use the materials provided (paper, colored pens, manipulatives, etc.) and to explore the space around them. I designed this protocol taking into consideration three aspects to the problem solving process - a before, during and after segment. What follows is a general outline of this protocol (see appendix B for a full description)

At the start of the problem solving session, participants were first briefed on the objective of the research study, their role and the protocol. To help participants understand the process of the protocol, two simple warm-up problems were presented. Participants were asked to solve these warm-up problems by the method that came most naturally, expressing their thinking verbally, gesturally or diagrammatically. Following this, participants then proceeded to solve the three non-routine problems (possibly using manipulatives and materials as well). The problems were presented typed on a sheet of paper. During this process, I looked for entry points in the participants' thinking and probed further using the meta-cognitive prompts. After the problem solving session was over, participants' questions, comments or thoughts were entertained.

3.3.4. Post-interview Protocol.

It became apparent that to improve the outcome of my data collection, I would need to follow-up on interesting points after conducting a pre-analysis of the video recordings. As such I designed a post-interview protocol to accompany my data collection (see Appendix B). After several days, I invited participants to a one-on-one interview to further clarify, discuss and explore points of interest revealed in their mathematical thinking. This added method of inquiry made it possible for me to closely investigate points that I had overlooked or could not otherwise follow up on during the problem solving sessions.

The design of this protocols followed a simple format: Brief the participants on the procedure, use snippets of the video recordings as a stimulus in the discussions, use some or all of meta-cognitive prompts to develop further inquiry. The questions that I asked came about after viewing each video recording repeatedly. These questions were designed to get participants to clarify, elaborate, and further discuss points in their mathematical thinking that I found interesting. These sessions were also video-recorded.

3.3.5. Designing the Meta-cognitive Prompts.

Finally, since I was going to investigate my participants' thinking through video-recorded think-aloud sessions, I also had the added task of designing the meta-cognitive prompts that would enable me as a researcher to gain entry into how, when and why participants used certain cognitive mechanisms to solve. This form of questioning was a necessary measure during these sessions as Presmeg and Balderas-Cañas (2001) points out:

> Unless the interviewer asks [subjects] about [their thinking], it may not be reported, even when it is present and constitutes an integral part of the problem-solving process. [...] The active involvement of the interviewer, [...], is required if [mathematical thinking] is the focus of the study. (p. 293)

I would therefore not only need to rely on meta-cognitive prompts to explore my participant's thinking, but more importantly, I would need to know what questions to ask and how to integrate them in the problem solving interviews. Generating effective meta-cognitive questions requires a certain level of domain knowledge and meta-cognitive skills as a researcher.

Through my introspection and notes in solving these problems I developed questions that could generate meaningful discussions. For example:

(a) Explain what your diagram, is trying to convey. What made you think of this particular diagram?
(b) Describe the strategy you are using. Walk me through it step-by-step.
(c) Explain what you mean by [...]? Can you elaborate a bit more?
(d) Tell me more about the association **X** you used here to make meaning/sense in your thinking. Can you talk about why you chose to use it?
(e) Were you asking yourself questions? Tell me about them.

I then categorized each question into a before, during or after segment that would form part of the problem solving session (see Appendix B). I kept in mind that these questions, based on emergent-grounded theory, would change and shift as I learnt more about my participants' mathematical thinking. As much as possible, the participants' responses guided my questioning.

In coming up with the meta-cognitive prompts to ask during the interviews, I took into consideration the following criteria: (1) each question needed to be open-ended. The open-ended nature of these prompts would give participants the opportunity to answer from their own frame of reference, i.e. responses needed to emerge from the participants. This is one reason why I choose not to administer structured, prearranged questions in the form of questionnaires or surveys, as these would limit flexibility and access to additional information, or details, achievable through probing. (2) They had to be semi-structure. The semi-structured questions allowed for flexibility in the questioning process. This format allowed me control over the interviewing situation and to stimulate discussion and obtain more information by not being too rigid.

One disadvantage with this questioning technique is that, as a researcher, I do not know to what extent asking my participants about their thinking influenced their thinking. For example, if I would ask “Can you draw a picture for me?” Would the participant have drawn one otherwise? I had to ask them if they would consider it. However, this form of probing was necessary if I was to elicit and encourage further information from the participants as they reported about their thinking. Therefore this form of questioning was necessary to probe the otherwise inaccessible thought processes associated with problem solving.

3.4. Materials and Data Collection

There were three main data sources: video recordings, participants' written work and the researcher’s observations and field notes. The data collection comprised of two main aspects: the video-recorded think-aloud problem solving sessions and the post-interview sessions. The materials that were used for this study mainly included the problem set, and all the additional resources in the math lab: desk, white board, smart board, manipulatives, paper, colored markers, etc.

3.4.1. Procedures.

For the problem solving session, participants worked in pairs on the non-routine problems through a think-aloud protocol coupled with meta-cognitive prompts in videotaped sessions. The problems were administered one at a time. To get participants comfortable with the protocol, I first presented them two warm-up problems. Before starting, each participant was asked if they were familiar with the problem, if so, an alternate problem was presented. During each problem, participants were encouraged to verbalize their mathematical thinking, make use of the manipulatives provided, draw, sketch and/or gesture as they solved. During and after each problem, participants’ thinking was probed using meta-cognitive prompts designed to gain insight into their thinking. Following the warm-up problems, participants then proceeded with the three non-routine problems under similar conditions.

Following the completion of the problem solving sessions and preliminary analysis of the recordings, the participants were then invited to a semi-structured post-interview session several days later. These sessions served to help clarify interesting points that arose during the preliminary analysis of recordings. Participants were interviewed using a series of questions in relation to instances captured during their problem solving session that I wanted to follow-up on. To help stimulate further discussion during these sessions, I showed participants snippets of their video recordings. At the end of each post-interview, participants were then encouraged to make any final comments, thoughts or questions. Some of these individual semi-structured interviews were video-recorded in the Faculty of Education building at the university while others at a location and time convenient to the participant. The sessions did not exceed one hour and every effort was made to ensure that participants felt comfortable to leave at anytime.

Before carrying out the larger portion of my data collection, I first conducted a pilot study in order to field test the problem-set, the interview protocol and to gain a feel for the dynamics of the interview sessions. This pilot proved useful as I took notes and made further adjustments to my problem set and interview protocol. From this experience, I added, subtracted and reorganized the problem set. I also modified slightly the problem session protocol based on feedback from my advisor. For this pilot study, I recruited two additional participants via the same recruitment protocol as stated earlier.

3.4.2. The Pilot.

The pilot did prove to be an exciting learning experience. The logistics component of this part of the research provided me with valuable insights into how to effectively coordinate with my participants. I kept entry logs in my field journal to help me keep organized and also to document my experience of the process of planning and carrying out this aspect of the data collection. Keeping a field note journal to record my thoughts and ideas as I moved forward with my study proved useful. I came to value the power of documenting my thoughts and taking notes consistently. This act alone helped me tremendously to stay focused, organized and on track with my work. By journaling my thoughts and ideas I was able to find answers to some of the obstacles and challenges I encountered.

One of the things I realized after the pilot session was that it is not easy to get individuals to express and externalize their thinking verbally, gesturally or diagrammatically. My participants did feel a bit reluctant. I recall walking out of the pilot session feeling a bit disappointed with the outcome. I felt I wasn't able to fully capture the external manifestations of my participants' thinking. I wrote down the following sentiment in my journal after the pilot session:

> Why were my participants so silent? How can I get them to verbalize, gesture, draw or sketch on paper what they are thinking? My study depends on studying how these

> cognitive resources are activated. How will I be able to provide an account of how these are manifested if I can't get my participants to project them?

Seeking advice, I decided to allow the participants to stand next to each about three to four feet apart facing the white-board or smartboard. This change in physical orientation gave participants the opportunity to discuss and explore the space around them more freely while collaborating. This shift in space was quite interesting given the usual custom of having to sit in silence solving with pencil and paper. In addition, I realized the importance of being confident while engaging my participants in further discussion. It became important for me to think of the interviews as a friendly human discussion rather than a rigid controlled scientific experiment.

In working with human subjects, as an aspiring educational researcher, I realized that this qualitative study is different from a controlled scientific experiment. It is not possible to control all the variables. Since I was working with human subjects, I viewed these sessions as a learning experience where I, the researcher, was learning and acquiring the skills needed to probe the process of sense making. As Lincoln (1985) expresses "It is […] the case that human instruments can be developed and continuously refined. [It's assumed that] humans [are one] major form of data collection device and that anyone committing him – or herself to this form of inquiry will have acquired, and will continue to hone, the skills needed in order to operate as an effective instrument" (p. 250).

The pilot did get me thinking about my role as a measuring instrument. Given that my goal was to explore the process of sense making through problem solving, I realized that my results and findings would, to a certain extent, be based primarily on my observations and the direction I would take after analyzing the data and writing up my findings. The types of questions I asked during the interviews are based on what I as a researcher deemed important or what I wanted to know more about (Lincoln, 1985).

3.4.4. Improving from the Pilot.

This pilot allowed me to refine my problem set and interview protocols based on my observations and field notes. The experience allowed me to better prepare for the second phase of my data collection. I redesigned the instructions of the think-aloud protocol in order to get participants to collaborate and explore the problem rather than sit in silence with pencil-and-paper. I modified some of the meta-cognitive prompts so as to yield interesting responses. I also gained a better sense of where to position the cameras. Below is a general outline of how I conducted the data collection.

I used two cameras for each session. One was placed on a tripod stand for a wide-shot and my assistant manned the second camera to get close-ups of the participants' activities, including their solutions. I encouraged my assistant to move freely around during the session and to not worry about appearing in the wide shot. I emphasized the need to videotape the finished work for each problem and to keep the camera rolling even after the session was over should any interesting thoughts or comments emerge.

For each of the three sessions, participants showed up in pairs to work on the problem set, except for one session where only one participant was able to attend. For each pair of participants I carried out the introductions and then we all sat together facing each other as I explained the goal of my research. I distributed to each participant a copy of the problem set and instructions and consent forms (see Appendix A). After briefing them about the think-aloud protocol and having addressed their questions and concerns, we got underway with the two warm-up problems. I repeated this preparation and introductory routine for all the other sessions.

As we began to record, I temporarily position myself near the wide-shot camera at the beginning of the session. I introduced each problem by asking one of the participants to read the problem out loud and then silently to themselves. As both carried forward working together on each problem I used my experience as a teacher to comfortably navigate the dynamics of each session. I moved around actively observing, eliciting and probing their thinking. I had in hand the list of meta-cognitive prompts as a guide (see Appendix B).

3.4.5. The Post-interviews.

I waited several days after the problem solving session before conducting the post-interviews. During the time leading up to these, I spent hours reviewing the recordings, analyzing and making copious notes. From here I selected specific points that I wanted to follow up on. These interviews took place either in the math lab in the Faculty of Education or at a location and time convenient to the participants. These sessions had a different dynamic feel. It was more of a relaxed, open discussion where participants freely elaborated and expand on the points I was interested in. I emphasized that they were free to ask questions at any point.

I used one camera during these discussions. Since these were one-on-one sessions, one camera with a medium-shot was sufficient. I sat next to the participant as we talked. Along with these questions I showed them video clips I hand selected of their mathematical activity on a laptop in order to stimulate further discussion. For each participant I interviewed, I tried to stay close to the topic being discussed and the questions I asked. Sometimes they would go off on a tangent and I humbly had to re-direct their attention to the specific questions I want to explore.

3.5. Analysis of Data

At the end of the fieldwork I had gathered a lot of data from different sources: problem solving video files (a total of three sessions, forty - sixty minutes long for each session), post-interview recordings (six sessions approximately thirty minutes each), researcher's observations and field notes, participants' written work (hard and soft copies), descriptive text and transcripts. Triangulation of the data sources was used to ascertain a comprehensive understanding of participants' cognition and sense making regarding their problem solving processes.

3.5.1. Coding and Analysis.

To ensure anonymity in reporting, each participant was assigned a pseudonym: Eve, Olga, Kara, Jon, Tom, and Paul. Each video recording, ranging from thirty to sixty minutes in length, was viewed using the computer software iMovie then analyzed. Video recordings were not transcribed in their entirety but were rather streamed in full and precise relevant snapshots of participants' mathematical thinking were observed, coded, and analyzed.

In analyzing each video recording, I looked for instances that revealed the use of visualization, gesture and speech, and affective states in the solution processes of each participant for each problem. I highlighted snapshots that were representative of particular ways of making sense and understanding. To keep track of these instances, I used the following letters: V, G, M, A to indicate the process of visualization (V), gesture (G), metaphorical thought (M) or when a particular segment indicated something in regards to the participant's affect (A). I also found it helpful to use symbols: note to self (NB), interesting points (!), theme (*), affective issue (*i*), among others, to flag interesting points in my notes as I viewed each file. Table 1 exemplifies the process of coding.

Table 1
Coding Example

Session 01/Time	Olga	Jon
00:11 - 01:10	V (sketches a static image), G (uses hands to explain),	A (is reluctant to think informally), V (uses dynamic images)
02:00 - 03:00		
	NB (Transcribe this segment)	G (uses deictic gestures)
	A (Aha! Moment)	NB (describe this in your report)
45:00 - 47:00	! (Interesting approach)	NB (follow up on this theme)*
50:00 - 52:00	? (What happened here?)	NB (quote this statement)

Table 1: Coding Example

This coding system served as a blueprint to help me keep track of when the participant visualize, gestured or gave hits or clues to their thinking. In this way I could revisit and focus on what was said or done at a given interval in the video for each problem and across the whole problem battery. I used these records to readily reference specific points in the recordings as I analyzed the data over time. The gray colors indicate the time intervals when participants were working on a non-routine problem.

This method of studying the video recordings allowed for thematic analysis of participants' mathematical thinking for each problem. From these schematic snapshots I was then able to pinpoint, extract and index relevant video clips that allowed me to further: (1) analyze each participant's use of visualization (evidenced by drawings and verbal report such as "I saw in my head"), gestures, and affective states for each problem, (2) compare similarities and differences in participants' solution processes and thinking, (3) examine responses to particular meta-cognitive prompts, (4) extract and index digital stills that depicted any form of visual, gestural, or diagrammatic output, and (5) look for any emerging patterns or themes. For the purpose of depth of reporting, I focus on interesting cognitive and affective aspects, issues/themes that I observed regarding my participant's mathematical thinking. I then used these to follow up on in my post-interviews.

3.5.2. Analysis of the Post-interviews.

The post-interview recordings allowed me to follow up on affective states and other interesting points that participants exhibited in their thinking that was either insightful or needed further clarification. During the problem solving sessions, I wasn't able to pin point and follow up on these directly. The one-on-one post-interviews offered responses to specific designed questions in relation to the participant's sense making. I viewed these recordings using the software iMovie in playback. I then reviewed, extracted and transcribed relevant segments for use as complimentary data in my reporting of thematic findings.

3.5.3. Putting the Pieces Together.

Some clips provided rich quotes and a descriptive account of the mathematics done by the participants. I used each video clip and its associated description watching for confirmations or contradiction and to become familiar with what the participants did and said. I then put these snapshots into perspective along with my field notes and observations, participants' written work (which I collected and digitized or extracted from the recordings), including the data from the post-interviews. This resulted in the emergence of cognitive and affective aspects, issues and themes in the solution processes of participants that eventually became the focus of my results and findings.

The goal of this study was to present depth, rather than a broad, descriptive account of the cognitive resources used by the participants in order to gain insight and make sense while solving non-routine problems. In ascertaining an understanding of this process of problem solving, I adopted the epistemological paradigm that knowledge and understanding is a construct of human activity within a situated context (Glaser & Strauss, 1967 in Mills, Bonner & Francis, 2006). This perspective guided my method of inquiry by helping me understand that, in relation to the process of sense making, each participant's thinking emerged from the context they worked in. They constructed understandings and meaning by drawing on their own sense making and cognitive abilities (visualization, gesture and affect) when encouraged to take an exploratory approach to solving these non-standard problems. I carried out a thematic analysis of the data to ascertain an understanding of the role of visualization, gesture and affect in participants' problem solving processes. The themes and issues that emerged from the data became the focus of my findings and results.

3.5.4. Strengths and Limitations.

This research study was limited to a small group of participants: two were recruited for the pilot and six took part in the larger portion of the data collection. In addition, sampling was based on a convenience model thus limiting generalizability of results. However, given the exploratory nature of this study, it was acknowledged from the start that analysis of data examples was appropriate in yielding thick, rich description of the process of participants' mathematical thinking. In fact the goal was to zero in and look more closely at individuals' sense making processes and the role of visualization, gesture and affect therein.

Chapter 4: Results and Findings

In the previous chapters, a theoretical framework of mathematical reasoning and sense making, cognitive aspects central to the process of sense making, and the methodology of the study were presented and discussed. In this chapter, I provide a descriptive account of participants' sense making in relation to their solution processes to specific problems. For depth of reporting I focused my findings on observations of those problems that were more revealing and insightful. In selecting the problems to discuss in this chapter, I closely reviewed each video file over several days. I looked for instances where participants visualized, gestured and revealed affective issues in their thinking. I then selected the video clips that were more revealing and telling to help answer my research questions.

This chapter follows a thematic analysis of participants' solution processes in regards to visualization, gesture and affect. I explore the role of these resources by providing a descriptive account of participants' thinking as data examples. I put these examples of data into perspective using my field notes and observations, participants' written work and transcripts from both the problem solving sessions and post-interviews. For each of these sections, I provide further discussion, analysis and my interpretations of these findings and results.

4.1. Participants' Background

Here is a brief background of the participants whom I'll be reporting on in this section:

Olga is a second year student in the Faculty of Education finishing up her Masters degree in mathematics education. She had taught high school mathematics for a number of years in her native country of Turkey.

Jon is in his mid-twenties, is from Toronto, Canada and was about to graduate with a Masters degree in public health. He had expressed interest in the study via email and was glad to participate.

Kara is from Russia and a second year PhD student in Sociology at the university. From our introductory talk I gathered that she enjoys playing chess in her free time. She expressed that she hadn't been doing math for quite some time so she was a bit unsure if she'd be able to successfully solve each problem. I explained that the goal was not the answer but rather that I wanted to explore the way participants think and about the process of sense making through non-routine problems.

Paul is a PhD student from Austria studying international relations at the university's Faculty of Law. He similarly explained that he has been away from mathematics for quite some time. Despite moments of uncertainty at certain instances in his solution processes, he did display confidence and enthusiasm throughout the session.

Eve is in her early twenties and was studying music theory at the university for nearly two years. She was keen on starting her PhD in the United States in the fall semester. After the briefing she expressed eagerness to see what these non-routine problems looked like.

Tom is a third year PhD student in the Faculty of Education with research focus on the use of technology in education. He had taught at the secondary level for a number of years before seeking further studies.

4.2. Do Learners Visualize?

To facilitate a deeper understanding of the issues of when, how and with what effect participants' visualized when making sense of non-standard problems, my interpretation of participants' solving processes of given problems is described in this section.

Session One; Olga & Jon; Warm-up Problem A

Problem A: If you add together odd numbers from one upwards you will always get a perfect square. 1+3 = 4; 1+3+5 = 9; 1+3+5+7 = 16 ... Can you think of a way to prove that this must always be the case? (From Puzzlegrams, 1989, p.113)

Early in this session it became apparent that both participants felt a bit reluctant to talk through their solution processes. I encouraged them to explore the problem using the process that came most naturally to them. This is where my active questioning as the interviewer became necessary. Over the course of the session both gradually became comfortable exploring the problems through open discussions and collaboration. The following extract illustrates the methodological approach each used to solve *Problem A*.

After reading *Problem A*, both Olga (**O**) and Jon (**J**) go silent for about one minute.

R: Can you justify why this pattern works?

O: Maybe it's about uhmm … (pauses for a moment to think) interesting.

She then looks at Jon (**J**).

O: Do you want to go with geometry? Cause these are squares right? Two-by-Two.

Jon acknowledges that they are dealing with squares. Both then walk over to the white board. Olga grabs the marker and comments:

O: Aw! (Olga has an Aha! moment.) Yeah, I have something ... I'm not sure, but let's see.

She starts drawing a sketch of a single square and then subdivides it into four quadrants. Olga uses the square to explain:

O: This is a two by two, (she points to one of the squares using the marker and then spirals out clockwise) one plus three makes four!

Jon catches on right away and confirms.

J: Yeah! (Jon smiles) And if you add another square to it its five square being all around it right?

Olga proceeds to draw unit squares all around the original two by two square. Both then start counting together.

O & J: One, two, three, four, five.

They both appear to be very excited at this stage in the solution process.

O: Yeah, no, it's five... yeah it goes like that.

Olga then outlines another set of unit squares around this newly formed one. They then both decided to redraw the sketch to make it more clearly using different color markers. Jon assists with the sketch, see figure 4 below.

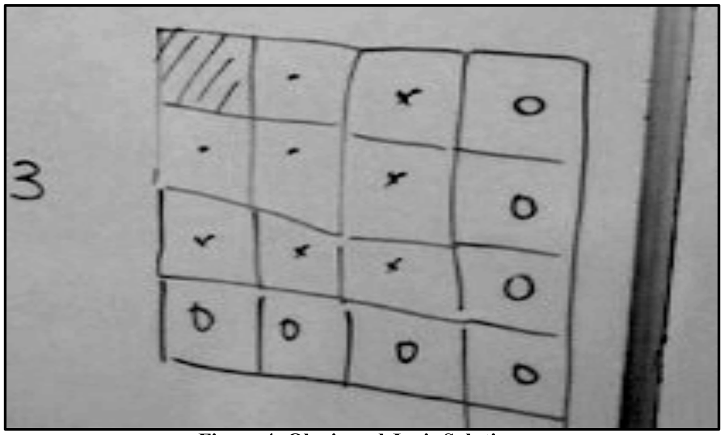

Figure 4: Olga's and Jon's Solution

I was curious to see if both were thinking along the same lines. It became obvious at the very start that Olga had provided the original insight. So I asked Jon if he initially saw this image.

R: How exactly did you (Jon) started out with this problem? Did you see this solution readily in your head, like an image just popped up? Or did you have to think through for a bit?

J: I looked at the equations that we were given, … just the difference between each equation.

R: So you kind of saw one, then you saw probably three, then five and kept adding those?

J: No, I saw the solutions. So I saw four, nine and sixteen. And then saw that the difference between them was five and seven.

J: Yeah, but I didn’t see anything visual at first. ... I kind of (chuckles) followed what she (Olga) was doing. I didn’t come up with the solution straight away.

I asked Jon if he wouldn’t mind elaborating on the approach he would have taken with this problem. He proceeded with writing down the numbers four, nine and sixteen in sequence on the whiteboard. He then took the difference between adjacent numbers and wrote down the results above and between the adjacent squares using arrows to depict these ‘gaps’ or ‘jumps’ (see Figure 5 below).

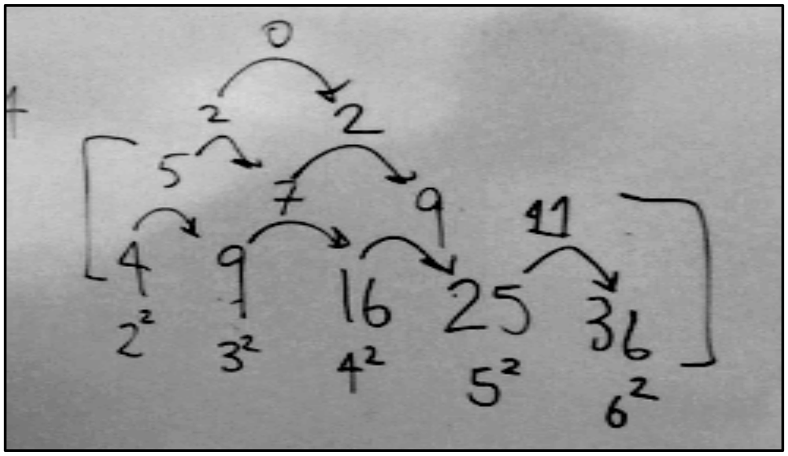

Figure 5: Jon's Imagery

His attempt to prove why this statement is true incorporated a visual and numeric component. I gathered that Jon's approach was to deconstruct and explore the pattern using a numerical sequence by first taking the difference between neighboring square numbers and then doing the same with the new results. He repeated this pattern to see where it would lead him. He commented, "I don't know, … I'm just working." He didn't seem entirely sure where all this subtraction would lead. In the end he was convinced that his approach did not seem to cast light or new insight to sufficiently prove the statement. He explains, "But in terms of proving why this is always the case, then maybe (points to line 4 in his work, see figure 5 from top to bottom) this is the solution." From my observations, he seemed to have taken on a numerical-analytic approach to explore this pattern given the premise of subtracting adjacent square numbers. I do believe he noticed that he was onto something. If he had extended his results and looked more closely at the repeating pattern depicted in lines two, three and four (see figure 5 above from top to bottom) he might have come up with perhaps a different image or approach to accompany the pattern he obtained through subtraction to make things more clear. It was not possible to investigate his reasoning beyond his imagery and written numerical sequence to justify why this pattern holds true.

These data examples shows two slightly different visual approach at exploring *Problem A*. Olga was convinced of her visual solution while Jon attempted to use a more formal numeric/visual analytic approach to try to obtain a proof but fell short in finding a general justification. These data examples highlight the importance of knowing what to look for visually and analytically. Olga's solution neatly captures the pattern visually through the use of squares in contrast to Jon's approach, though still visual. The data examples suggest to me that experience in knowing what to look for in a schematic diagram and how to use such visual image with non-visual analysis are defining factors in enabling or constraining the solution process of an individual.

It is well known that some people have difficulty constructing and using imagery in reasoning (Lowrie, 2000). There are several reasons for this. One reason is that visual thinking is made up of independent cognitive components and that individuals perform differently on these components (Presmeg, 2006). Take one main component of visual thinking that is problem representation. The ability to represent a problem visually depends on one's ability to process information, an important stage of the problem-solving process (Schoenfeld, 1992). Olga represented all the necessary information to *Problem A* in her imagery in contrast to Jon's. Another reason is that most students have "inadequate experience of [a given] concept to provide appropriate intuitions" (Tall, 1991, p.106). In the case of Olga, the appropriate intuition that led her to construct a square served useful. This was also the case for Paul as he solved this problem.

Sometimes an image paints a thousand words, or in this case, captures a given pattern elegantly. Other times an image form only a prerequisite to finding a solution. It is important to be able to recognize the strengths and limitations when using an image, what to look for and how to use it. By the slightly different approaches to *Problem A* taken by Jon, we see the importance in knowing how to represent information visually, what to look for in a diagram, how to use it, and its limitations.

The next problem again shows the distinction in mathematical thinking between Olga and Jon.

Problem B: A man has 2 sons. The sons are twins; they are the same height. If we add the man's height to the height of 1 son, we get 10 feet. The total height of the man and the 2 sons is 14 feet. What are the heights of the man and his sons? (Sawyer, 1964, p. 40)

Below is an account of participants' solution processes.

R: Can one of you kindly read the problem out loud?

As Olga begins to do so she goes for the blocks as manipulatives and begins to explain,

O: Let’s get two sons (places the block side by side). The sons are twins so let’s change this (swaps one block for an equally sized one).

After Olga finishes reading the entire problem she exclaims,

O: This is an easy one.

She continues to manipulate the blocks as she reasons and re-reads relevant segments of the problem. Jon at this point confirms without a doubt that the two sons are 4 feet and the father is 6 feet. I wanted to know whether they manipulated these blocks mentally to get to the answer or whether the arithmetic they used had a visual component to it. So I asked,

R: When you both started, did you have an image in your mind? If so, can you draw a picture?

Jon goes to the board and does so. Pretty soon Olga assists by using vertical lines to represent the height of each stick figure Jon drew. Jon continued with his work and assigned numerical symbols to denote these respective heights. He follows through by writing down the equation $a + b = 10$ and $2a + b = 14$. From here he explains that the difference between these two formulas is $a = 4$, the height of one son. So logically the height of the man has to be $b = 6$. I challenged them to not focus on the algebra and to consider an alternative approach this problem. Olga quickly proceeded to draw vertical line segments explaining her reasoning as she does so.

O: Since it’s height I’m going up to down (gestures the movement with her right hand) and not horizontally.

(One line she labels as *man + son* and then exclaimed)

O: But in the other case it is going to be longer (draws and labels a longer line). Here you have man and two sons.

She then reasons that if you compare these lines side by side, the difference gives you the height of one son (see Figure 6 below).

O: If this is 10 and this is fourteen then the difference should be 4.

(Immediately as Olga finishes Jon exclaimed)

J: That’s kind of what I did but I did it in my head.

I went ahead and asked Jon to elaborate on what he did mentally.

R: When you said you did it in your head, what exactly did you do? Did you use variables or did you use some sort of diagram as what’s on the board?”

(Jon paused for a bit and then stated)

J: I just saw the numbers.

R: Numbers? What did they look like? Could you talk more about it?

J: It’s hard (goes to the board and begins to explain as he writes).

He writes down a simplified version of his previous equation and explains that it was not a visual diagram that he saw but that he only logically subtracted the two equations to get the answer. It is not know for certain what type of image, if any, was guiding his solution process. Below is a comparison of their written work.

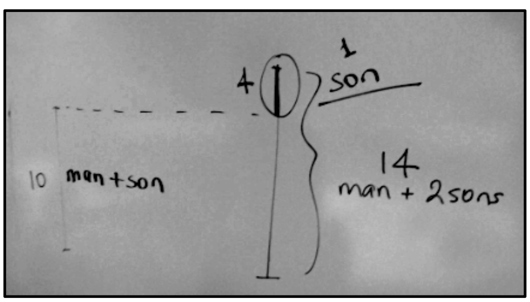

Figure 6: Olga’s Imagery

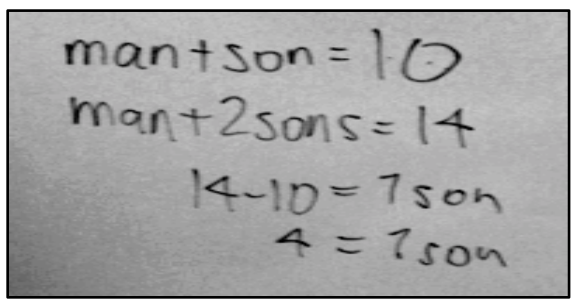

Figure 7: Jon’s Analytical Reasoning

Interestingly enough, these two warm-up problems were not the only ones that depicted the use of visual imagery, more specifically schematic imagery (Presmeg, 1987/1992), as an integral component in the solution process of individuals. Eve, Kara, and Paul used similar imagery as Olga (Figure 6). Apart from the two warm-up problems, non-routine *Problem 2* solved by the following participants also depicts the use of schematic images to gain insight and make sense.

These schematic images also guided the various hand gestures they used when explained their reasoning. I will now discuss Kara and Paul’s reasoning in the following section.

Problem 2 A 10 by 10 square depicts a field with just enough grass in it to feed 100 cows. Is it possible to have square field with just enough grass in it for 200 cows? Can you prove this?

(in Melrose study, p. 3)

Kara.

I came across this problem in a discussion about non-trivial solutions to math problems in one of my graduate courses. Captivated by one child's solution in our discussion, I was curious to see how others would solve this problem. So I decided to present it in my study. With *Problem 2*, Kara started off by drawing two 10 by 10 squares side by side (see Figure 8 below).

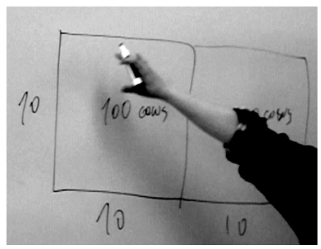

Figure 8: Kara's Explanation – Part 1

K: This newly formed rectangle fits 200 cows. (Kara outlines its dimensions with her hand.)

She uses this image to further reason that in order to find the length of a square that would fit the 200 cows, one simply takes the square root of its area - 200. She explained,

K: The length of a perfect square that would fit 200 cows would need to be the root of 200 (she does so and obtains ten times root two).

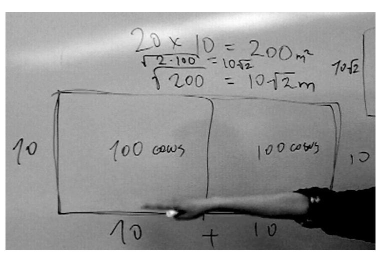

Figure 9: Kara's Explanation – Part 2

K: If the sides of the new square are of length ten root two then it can fit two hundred cows.

When asked to consider an alternative approach she thought for a while and then came up with following.

K: So we can also for example take a rope (chuckles) and just go around this field (refers to the rectangle in her diagram) and then try to adjust it in away that would become a square (see figure 10 below).

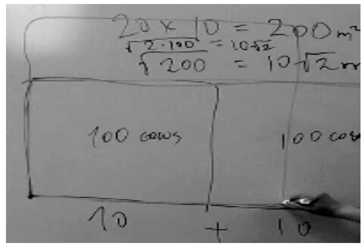

Figure 10: Kara's Explanation – Part 3

Kara's use of visualization occurred mainly in the preparation phase, to make sense of the relation between the two squares. She then used this image to reason through to a solution. She returned to the diagram in explaining her alternative approach using an imaginary rope, but her main confidence lay in her algebraic solution. For her, taking a rope would give the perimeter but not necessarily the area needed to accurately calculate the side of the square that would fit 200 cows. I infer that without the initial phase of constructing a schematic image to guide her sense making, getting to a known solution may not have been so trivial. What is quite interesting is that in explaining her alternative approach, her image and use of *deictic* gestures – i.e. the hand gesture of pointing to existing or virtual objects (McNeill, 1992) seem to work hand in hand in manifesting her spatial and perceptual understanding.

Paul.

In solving *Problem 2 – fields of cows*, Paul started similarly by drawing a 10 by 10 square. However his use of imagery was more dynamic (Presmeg, 1992). This data example explores how Paul was able to build meaning through the use of speech, gesture and diagrams.

P: I'm trying to visualize the initial 10 by 10 square. Maybe that will help me to figure out the next step.

He drew a 10 by 10 square and added two unit squares to represent two cows. He uses this diagram to visualize moving cows from one square field (has this image in his head) onto the other and arranging them to form a new square.

P: Can I basically add these cows here and here? (Refers to the upper and left side of his original square (see Figure 11 below).

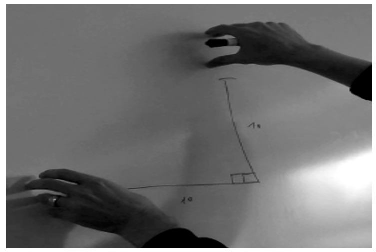

Figure 11: Paul's Explanation – Part 1

He explains his reasoning,

P: I want to take those 100 cows and add them to another field. So I will try to add some here and here (refers to upper and left side of initial square). So if that is going to be a square then … well I mean obviously the answer is like the root of 200 (that) would be the length of each (side).

R: Why the root of 200?

P: Because I am try to figure out how far I would need to go adding these cows here and here.

In re-explaining how he got his answer he said he visualized moving one square of 100 cows and neatly arranging these around the upper and left hand side of his initial square (see Figure 12, and 13 below). He gestured out this movement using his right hand (Figure 13). He then reasoned that the length of this new square field would be the root of 200. However, he realized that the square root of 200 would yield a length that is an irrational number (Figure 14). At a certain point in his solution process, Paul questioned whether the extra length needed to fit the 200 cows was practical. He was not convinced that in the real world, the extra length needed to fitting 200 cows in a square field could be measured.

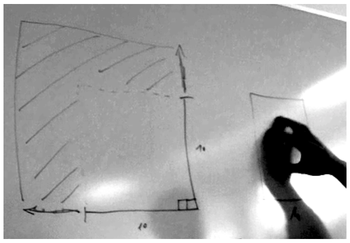
Figure 12: Paul's Explanation – Part 2

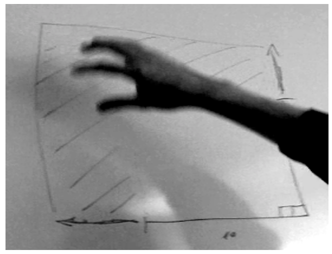
Figure 13: Paul's Explanation – Part 3

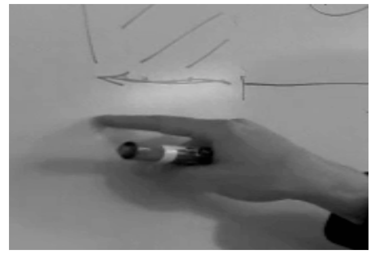
Figure 14: Paul's Explanation – Part 4

Paul's use of this type of dynamic imagery and accompanying gesture, i.e. the way he visualized transforming and superimposed one square onto the other was more dynamic in nature in comparison to Kara's static image. Paul visualized one square being malleable and fluid allowing him to manipulate its dimensions to combine both unit areas forming a new square of 200 cows with dimension ten-root-two. The use of dynamic imagery as in this case is very powerful in yielding insight as it structures information at a level that is more general and abstract (Presmeg, 1992, 1997). Thinking in this manner is very useful especially in areas of advanced mathematics such as analytic geometry, set theory and topology.

Discussion

The drawings and the gestures that accompanied them in these data examples highlight the role of imagery in sense making. From these schematic images, I observed that when visualizing, it is more helpful and more useful to process, represent and structure information in the most economic way possible. It is much useful to not think in terms of vivid details but rather in terms of geometric schematic patterns: lines, points, letters (see participants' sketches above) (Presmeg, 1992, 1997). The type of imagery most evoked in the thinking of Olga, Kara and Paul was *schematic imagery* (Presmeg, 1992). Some were static and one was dynamic (Paul's). These types of imagery are useful and sometimes desired as they are effective in providing insight during making sense (i.e. making connections and relations in an economic way through the use of imagery). We see this with Jon's imagery (Figure 11 to 13) to *Problem A* that could be further explored and generalized given the economic visual representation of the information.

However being able to visualize a schematic solution is not always the best method. For example, when faced with problems that require manipulating and keeping track of multiple arrangements of objects and keeping a mental record is not easy. I came to observe this aspect in the visual thinking of each participant during their initial approach to *Problem 3* - The milk-crate problem. This will be the focus next.

Problem 3: A certain square milk crate can hold 36 bottles of milk. Can you arrange 14 bottles in the crate so that each row and column has an even number of bottles? (Mason, Burton & Stacey, 1985, p. 181)

Each participant who worked on this problem started out sketching a two dimensional schematic image of an empty crate either on paper on the board. Some drew in marks to represent the bottles; others tried to arrange the bottles mentally. However, each participant after some time found it quite challenging to 'see' a possible arrangement. Everyone abandoned their initial approach of mentally manipulating the bottles to find a solution. Instead they switched to using the blocks as manipulatives along with their sketch of an empty crate. Through several trial and error attempts they eventually were able to obtain a solution. However their approach wasn't by brute force, each attempt had a visual/logical component to it. Below are two accounts.

After coming to realize that a pencil-paper approach wouldn't work, Kara decided to use the manipulatives. She arranged twelve blocks first on her six-by-six grid paper then sighed,

K: So I'm left with two (blocks), which I won't be able to place anywhere else.

K: I'm trying to arrange the blocks but I can't really see a solution.

After four attempts, Kara assumed she got a solution. From my observation, she started by placing four blocks in the first row and then another four in the second row, offset by one square to the right. She repeated this with the third row and then she placed the remaining two blocks at the center of the fourth row (Figure 15 below). From here her final step was to move the first-row-first-column block to the first-row-sixth-column space in order to satisfy the conditions of the problem (Figure 16 below).

X	X	X	X		
	X	X	X	X	
		X	X	X	X
		X	X		

Figure 15: Kara's Pattern

	X	X	X		X
	X	X	X	X	
		X	X	X	X
		X	X		

Figure 16: Kara's Solution

K: Aha! I got a solution but I'm not sure if it is all solutions.

Kara was able to attain what she thought was a solution after trial and errors multiple times. As seen in figure 15 above, there was a visual/logic component to her thinking. Given the pattern in figure 15, Kara reasoned that another solution would be to move the third-row-sixth-column block to the third-row-first-column space. She also visually reasoned that if you rotate the paper clockwise ninety degrees you would get another solution, their reflections would also be solutions. It was quite interesting to note how she was able to construct this pattern (Figure 15) from which to derive what she thought were other possible solutions.

Likewise, Tom & Eve after a several trial and error attempts fixated on a particular pattern (see figure 17 below) before obtain their solutions (figures 18, 19).

X X X

	X	X	X		
X	X				
X		X			
X			X		
X				X	

Figure 17: The Pattern Olga & Jon focused on

X	X	X	X		
X	X				
X		X			
X			X		
				X	X
				X	X

Figure 18: Olga & Jon's First Solution

X	X	X	X		
X	X				
X		X			
X			X	X	X
				X	X

Figure 19: Olga & Jon's Second Solution

It was also interesting to note the similarity in patterns participants made during their attempts. Below are three samples.

X X

X					X
	X			X	
		X	X		
		X	X		
	X			X	
X					X

Figure 20: Sample 1

X X

X	X	X			
X	X	X			
X	X	X			
X	X	X			

Figure 21: Sample 2

X X

X	X			X	X
		X	X		
		X	X		
X	X			X	X

Figure 22: Sample 3

It was interesting to note how everyone resorted to using the manipulative instead of a pencil-and-paper approach. Eve confirmed by saying,

E: The problem isn't easy; you need an external visual mechanism.

E: The blocks (manipulatives) make it easier to manipulate and keep track of a solution.

R: Do you think you would be able to find a solution without the manipulatives?

P: That would be harder. I'm kind of a visual learner and I think this makes it easier for me to see where I need to go.

These data examples show that pattern imagery can be a powerful cognitive resource but there are certain limitations to thinking visually. Being able to recognize this is important. The solver has to know how when to represent a problem visually and how to process information efficiently in order to be successful (Presmeg, 1992). In the case of Problem 3, using an external visual medium (i.e. the manipulatives) supported the process of visualization. Without this mediating medium, it would have been difficult to find a solution.

4.3. Do Participants Gesture?

The goal of this section is not to provide an explicit detailed account of mathematical sense making and the role of gesture therein. Instead the goal is will be to present an alternative sketch to the traditional pencil – and – paper view of math. By traditional math I'm referring to the application of memorized algorithms or procedure/techniques to solve problems often in silence. In this section I will illustrate how gesture and talk is not an isolated cognitive aspect of thinking (Arzarello, Paola, Robutti & Sabena, 2008). In isolation this aspect has limited scope, but against the backdrop of problem solving, gesture and words can support thinking processes of individual sense making and reasoning (Arzarello et. al., 2008; McNeill, 1992; Radford, 2009; Alibali, 2005; Edwards, 2009). I shall show how gesture and talk contribute to the process of sense making and further elaborate how imagery (drawing, sketches) provides an essential mediating role between the two.

Paul - Problem A:

One problem that shed light on the relation and role of gesture and visualization for mathematical thinking came in the last session and from warm-up *Problem A*. For *Problem A*, Paul started by reading it aloud. He then got up and went over to the whiteboard. He reread the problem and started to explain,

P: So first I would probably try to find a goal so I would need to get to some kind of square (draws a square) and now if I add odds ... So I'm just trying to see how this would work out.

He begins to label each side using the letter A, B respectively (see Figure 23 below).

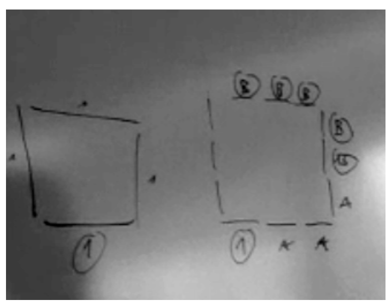

Figure 23: Paul's Initial Attempt

P: I'm just trying to visualize two examples so maybe that's not going to work.

Paul's first attempted to solve this problem was to keep an outline of a square diagram and then populate its outer dimensions using smaller line segments to represent the odd numbers 1, 3, 5. He soon realized this was not the best way and was encouraged to solve the task again. I pointed out that he was free to use any of the materials provided. He stood there for a while thinking in silence.

P: Yeah, so uhmm a square (speaks under his voice) the four is a square, the nine is a square and the sixteen is a square. "OH! OK (has an Aha! Moment).

He looked around (full of confidence) and decided to go for the manipulatives.

P: So now I'm thinking of adding squares. So it's like one (puts down one block) and then I have three … and then I add around it, three and so on (see figure 24 below).

He opts for similar colored blocks as manipulatives and places these around the unit block.

1	2	3
1	2	4
1	3	5

Figure 24: Paul's Insight

P: And now if I, Oh! OK, yeah, yeah, ok now it makes sense! So basically there's the one, then I add three around it (gestures with his figure in a clockwise motion) then the next around this side would be five, and then this should be seven. Yeah so it is always the length, yeah OK (refers to the outer blocks as the lengths).

During the session I asked,

R: What made you see the solution? At one point the solution seemed to suddenly appear to you.

P: I think all of a sudden I realized that … well I was thinking more of, really in terms of squares. So a square [...] and then perfect! Four. And then in my image, in my mind, all of a sudden the image appeared of four blocks (uses his fingers to gesture the appearance of his image, see figure 24 above).

R: Ah! Nice.

P: And, and the one plus three! I realized that it's actually like one and then three around it and then the next line.

He uses his left index figure to represent the unit block and then use the other in a single right and downward motion over this unit block (see Fig. 25, 26, 27, 28 below). He repeats this motion over the newly formed square and so on.

P: So it was visual somehow in my head.

Figure 25: Paul's Gesture 1

Figure 26: Paul's Gesture 2

Figure 27: Paul's Gesture 3

Figure 28: Paul's Gesture 4

It was interesting to note Paul's explanation of how the solution came to him in the form an image. First he saw a two-by-two square (Fig. 25) then all of a sudden something clicked in his mind and everything became clear. He realized that there were three other squares surrounding this bottom left square (see Fig. 24 above). He gestured using his right index finger in a clockwise motion to depict how he saw that the pattern repeating as the square becomes bigger. In comparison, Olga did the same in the form of a drawing except she went down and the across. I can only speculate that the orientation in space of these square figures and the gestures that accompanied them, i.e. Paul positioned the first block at the bottom left hand corner (see Fig. 24 above) while Olga positioned her first block in the upper left corner in her sketch (Fig. 4), can be attributed to how we perceive, internalize and structure spatial orientation and relations as images in our heads (Johnson, 1987). This evidence suggests that mathematical thinking does make use of embodied repertoire of images i.e. properties/characteristics of concrete objects that we internalize and store as image schemata. And the way we sketch or manifest these mental images shows that there are variations in the way we perceive these or how our mind's eyes see them. In this case, we see that Paul did make use of a mental two-by-two square (original insight) to make sense of the general pattern. He was able to further confirm this pattern using the blocks as manipulatives. This data example sheds light on how visualization is a resource tool in the process of making sense, analyzing and attaining further understanding of mathematical patterns that may not otherwise be so obvious.

The role of gesture in this example served mainly as a medium through which Paul was able to explain, justify and reinforce his reasoning. It was only when I asked Paul to explain how he got the solution that he began to gesture the image he saw. Initial he drew a schematic diagram on the board and then attempt to figure out why the pattern worked without gesturing. But it was until he had obtained his insight that he enthusiastically gestured while explaining the image he saw. What the analysis shows is the role of gesture in explaining and thinking. It suffices to say that explaining does involve thinking and so it can be argued that Paul gained further understanding of his reasoning by justifying why this pattern works through gesturing. The evidence indicates that Paul's image guided his finger gesture. The image and subsequent pattern he saw emanated though the movement of his fingers. In essence he was thinking through his actions and words given his image. There were instances during this gesturing process where the tone of his voice intensified. He became louder marking a change in variation signifying that 'aha! Moment,' followed by a more swift and subtle movement in his fingers. For example when he uttered "and the one plus three!" From Paul's activity we see the intricate relationship between visual memory, gesture and words working together to reinforce deeper understanding.

In the previous two sections I dealt with the role of visualization and gesture in establishing meaning. These example data show that visualization and gesture can be important resources in making sense of the mathematics. One can argue that these problems could have been explained or further understanding established without the need to gesture as evident by Jon and Kara's algebraic solution to *Problem B* and *Problem 2* respectively. I agree. However, the accounts discussed here shows that graduate students do visualize and gesture when reasoning. They do make use of the concrete. For these participants, use of visualization and gesture became resources in the process of sense making and reasoning. Interestingly enough, even research mathematicians with years of experience do visualize and gesture. Sinclair & Tobaghi (2010) have showed that mathematicians do use gesture when conveying dynamic diagrams (for example the notion of time, location and motion).

The last three sections of this chapter will now focus on affective states of mind that enabled or constrained the sense making process of individuals. I organize the discussion around themes that emerged from the problem solving sessions and post-interviews.

4.4. A case of Being Reluctant to Think Informally

Session 02; Kara:

This session was different from the others in that only one of my two participants was able to show up for that day. The other was unfortunately not able to make it. So for that day I went ahead and worked with this one participant. After briefing Kara on the think-aloud protocol we moved forward with the warm-up problems to get her comfortable expressing and externalizing her thinking. I asked Kara if she would kindly read the first *Problem A* aloud. She agreed and then she went silent for a couple of minutes reading the problem quietly while trying to understand what the problem was asking. Below is a description of her attempt to solve this problem.

K: (After about 2 minutes she sighed) "No, I don't really understand it. … So what, do you … do you need a proof of this? Uhmm.

Early in the interview it became apparent that she was puzzled about the question. So without trying to give away too much I explained.

R: It's just saying that there is a pattern here. Actually what do you think? … Have you seen this problem before?

K: No, no, it seems unusual.

R: Do you understand what the task is asking? (After a moment of silence she responded)

K: Yes, its asking, like say uhmm ... adding up the odd numbers ... uhmm gives a square, square. So …

I encouraged her to feel free to use the whiteboard if it would help. Without speaking further, she moved over to the board and began writing down the following: $2n - 1, n \in (1, 2, 3, ...)$. I noticed that Kara did have a solid math background from my observation of her formal- proof-style approach at starting this problem. This made things a bit more interesting in that I had to gently coax her to pause, step back, and explore the problem without relying heavily on formal mathematical procedures. I observed this tendency throughout the session. I asked if she would start off with a simple example instead. She agreed. After copying the pattern on the board she stared at it for quite some time without advancing.

K: (She then exclaim) There is a pattern. I don't actually know how to prove this.

I gently offered the use of the manipulative as an alternative to help her get started.

R: Here are some manipulatives if it helps.

K: Seriously? (She laughs a bit sarcastically) We are … we are actually going to start to form a formal proof using manipulative? I don't … I don't really understand exactly what I have to do. But...

R: Maybe if you could use these objects here it might help you to …

K: Prrooove? (She responds doubtfully).

She kept looking at the board for several minutes. Her activity did not move beyond a futile attempt to understand the conditions of the problem. She eventually resorted to using formal mathematical techniques but then stopped. She eventually exclaimed, "Damn (nodding her head), I'm stuck. I don't know." I gained the impression that Kara was very much attached to the formalistic approach to solving and was reluctant to explore the problem by considering the use of drawings, manipulatives etc. despite my encouragement. It seemed that meta-affect (feelings about her attachment) caused her to feel uncomfortable and constrained her ability to explore this problem in the absence of a formal language. It was not possible to investigate her exploratory approach to sense making in any phase beyond that of preparing and trying to figure out how to proceed with solving the problem.

Kara's reluctance to make sense of this problem using informal thinking may be associated with her understanding of what mathematics involves - her opinions and views of what mathematics is. It appeared that Kara considered the use of manipulatives or drawings as not a constituting a formal proof. Her reluctance to use them perhaps was a limiting factor in her ability to gain a more full understanding of the situation the problem presented.

This reluctance to use informal thinking does not mean that Kara does not know how to go about reasoning informally when attempting to make sense of these sorts of problems in the absence of a fixed algorithm. On the contrary, for *Problem B* and *Problem 2*, she did resort to a visual method but only when told to consider another approach other than algorithms. Although she realized that such approaches could be quite useful, she prefers or is unwilling to use informal thinking on her own accord. This was confirmed in the post-interview. Below are excerpts of her views and comments,

Kara: I did prove the first one using math induction. It was easy, but I needed the silence of my room and the formal math language that I understand. I tried to figure out the last one that day, but I'm not sure how it should be formalized for my brain to process. [...] I think for me the difficulty is that we usually are given instructions what to solve and what method to use in math so when I have to find the method I have to use without being told what it is - I get confused. I don't usually spend time thinking what does that relate to real world, what it is what we are trying to solve.

She also commented on her attempt to rigorously solve problem 3 - The milk-crate problem.

Kara: And I don't think I [...] remember anything from permutations computation so I do not think that I have enough knowledge to have a rigorous solution of the last one. However, I did some musing on paper, I will give it to you later if you want.

Below are her attempt at rigorously prove *Problem A* and *Problem 3*.

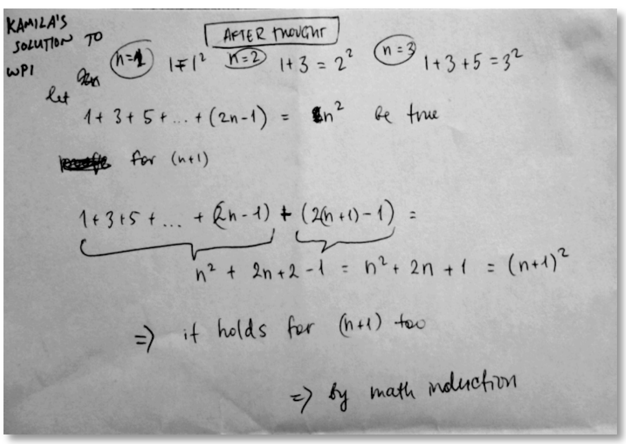

Figure 29: Kara's rigorous solution to *Problem A*

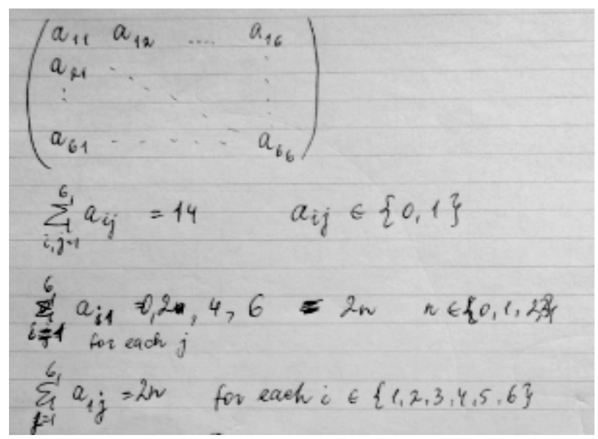

Figure 30: Kara's attempt at Solving *Problem 3*

In the first image (figure 29), Kara formalize and generalized the solution to *Problem A* using binomials and the technique of math induction. In the second image, she made attempts at using matrices and the sum of matrices (i.e. formal notations) to establish a rigorous proof to justify all possible solutions to Problem 3 – the milk-crate. She did not completely prove *Problem 3* but it was interesting to see her attempt at generalizing all possible solutions using the formal language of mathematics. Kara was one of the only participants to take this approach.

The reluctance to explore a problem without the use of a formal procedure has roots in the traditional view of what is mathematics. As discussed in the literature review, this cultural view is dogmatic. What I found interesting though was that this view has the potential of narrowing one's approach to solving. In the case of Kara, problem solving was one-sided, she held on to the view that she needed to think abstractly in order to solve rather than taking on an exploratory approach in order to make sense. For her, drawings and the use of manipulatives does not constitute mathematical thinking. Yet in contrast, as evident from others' exploratory approaches, informal thinking can provide insight - a pattern, which could then be extended and generalized in the process of making sense. Take for example Olga's diagram or Jon's written pattern to *Problem A*. Both of these writings can be coupled with numerical analysis to deduce a more formal proof.

This observation made me question the nature of mathematical thinking and the origins of mathematical ideas. Which comes first, formal thinking or informal/exploratory thinking? What are the consequences to teaching students the former? How do professionals practice mathematics? Is their thinking really all formal or is there an exploratory/informal component to their sense making? Sinclair & Tabaghi (2010) have found that "contrary to the formal, written mathematical discourse, mathematicians [do] use both language, [imagery] and gesture to convey a sense of motion and time in their thinking" (p.238). Certainly further empirical data on this topic could help inform teaching practices.

4.4.1. Views About the Nature of Formal Thinking and Sense Making.

One interesting and related theme that emerged from the research data were participants' perspective of formal vs. informal thinking in the process of sense making and how these may enable or constrain an individual's solution processes. I thank my camera assistant for capturing some of these moments by keeping the cameras rolling long after the participants had finished solving. A recurring pattern that I noticed was that after each session, some participants started commenting on how they enjoyed the idea of getting to explore a problem and not having to reference some memorized formula. Seeing this as an interesting mathematical point of view, I followed up on it through discussions in the post-interviews. Below is an excerpt of one participants' perspective.

Eve expressed the following sentiment:

E: I believe it is something almost pedagogical; teachers don't often emphasize teaching by taking a problem and having the student come up with the solution on their own. They just give you a bunch of formulas and tell you to apply them. Take for example the answer to the man and his two sons (*Problem A*), I never would have thought about solving this problem without using algebra, yet it is possible to do it! (Participant laughs) It wasn't until you told me not to use a formula that I began to visualize what was going on inside my head. The answer came to me quite easy and when I explain how I go it, the solution was so simple and it made complete sense to me!

R: Why do you think that is the case?

B: Well it might have to do with the fact that we aren't given enough time to think about a math problem long enough and being told not to use a formula.

Another participant in another session expressed a similar sentiment. Below are some of Tom's thoughts on the topic.

T: Hundreds of years ago, mathematics was done using simple tools, drawings in the sand. That was mathematics [...] today we think of algebra or trigonometry … as 'math' (uses fingers to quote the word 'math'). And playing in the sandbox is for kids or something (looks away a bit sad, folds arms). That's a different uhmm … culture.

R: It is a different cultural perspective. I agree.

T: And uhm... cause I use to get, be hard on kids who use to say 'I'm no good at math.' Well what is math to you? You seem to be pretty good on playing the piano, you seem to be pretty good at painting, perhaps if you see these statistics (referring to people's view and fear of math). It is a little misleading maybe (chuckles). ... How do you define mathematics?

R: You raise a good point and question actually. After reading, and even from my own experience as an educator, I now realize that kids have become accustomed to the idea of formulas, algorithms or procedures.

T: It's kind of being given an equation. …

R: An equation, that's math for them. And it made me think about why they would view math through those lens? … And now a large part of it is that our first contact with math is through schools.

T: Yeah and the curriculum. It's in a book that has a plus sign and a minus sign.

R: And so a lot of times we assume math is not 'playing in the sandbox' like what you said.

T: Yeah. But I think if you would've ... Archimedes. Archimedes for example who is able to estimate Pi pretty closely given what he had. Everything he did was in the sand, with pebbles.

E: I believe students who are good at math are those that figured out how to make use of their intuitive senses. And that is why they understand the algorithms better than most.

There is something about working in a 'sandbox' space that lets you play around and explore. It is a space where you can be playful and experimental. A critique on mathematics education is that there is so many structure and rigidness. Sometimes teaching to the test limits opportunities to be in that 'sandbox' space as Tom commented. This notion of a 'sandbox space' in computer science serves as a place to explore and try something without risk. It allows you to be in a space where you are working through your ideas. In a sense it is about teaching yourself. In some ways when you are in that space you are experimenting and exploring without being so attached to the end result (i.e. providing the right answer).

Most participants carried the sentiment that if they had been encouraged to explore math and to discover the methods on their own by reasoning and making sense, then they probably would have enjoy the process a lot more. This is so interesting, as a large percentage of the population don't like and even fear math (Tobias, 1991).

4.5. Affect: Present Moment Mindedness and Letting Go

One of the most fascinating themes that I came to observe from my research data on problem solving with regards to affective states was this notion of present-moment mindedness. I came to realize that my participants, in solving *Problem 3* (the milk-crate problem), were able to enjoy the process of solving without feeling anxious or fearing failure.

While solving this problem none wanted to give up finding a solution. They were persistent, focused, and when they finally found a solution – ecstatic. I wonder what conditions made this all possible? What made this affective state of present-moment mindedness possible? I certainly didn't know what to make of it. So I started to search for supplementary literature to support these observations. Over the next weeks, I certainly did locate some that stood outside the realm of mathematics education. The books I came across such as: *My Stroke of Insight: A Brain Scientist's Personal Journey* (Taylor, 2006), *The Power of Habit* (Duhigg, 2012), and *The Practising Mind* (Sterner, 2005) all shared common cognitive aspects: letting go, being in the present moment, being process-oriented and not goal-oriented. To get an idea of what these notions mean in the process of solving, consider any act of practice, for example the act of playing the videogame - Tetris. While playing, your entire focus and energy is devoted to the sole act of playing (Sterner, 2005), i.e. filling in the available spaces with the most appropriate block. In this moment, your mind is no longer wondering about any thoughts about the past or future, thoughts about what might happen or not happen you are in a space that allows you to try something without risk or failing to meet expectations. You even ignore the scores on the screen and your inner brain chatter (Taylor, 2006; Sterner, 2005). You're simply entrenched in the process. Most of your focus and energy is directed on the act itself, not the end result.

I soon found myself asking: What makes this possible? Is it just the nature of the problem itself? Can this mindset be consciously entered into when solving? In my quest to answer these questions I came across Sterner's (2005) perspective on the nature of this meta-cognitive affective state of mind. According to Sterner, a medium that allows you to let go of expectations helps to achieving this mindset. He associates four components to this medium. He calls these the four "S" words: *Simple*, *Small*, *Short*, and *Slow*. He explains these components as follows:

> *Simple*: For any complex task, keep it simple; break it down into very small systematic parts.
>
> *Small*: Make each sub-part small. The brain becomes more focused this way.
>
> *Short*: Ensure that the time taken to execute this small task or act is short.
>
> *Slow*: When engaged in the act itself, deliberately do it slow, don't rush. (p. 124 - 128)

I took these and compared it against the act of doing problem 3 and I realize that there is a certain degree of playfulness to this problem that helps achieve these conditions. There are certain elements that make this problem enjoyable. It has a visual, tactile and affect component to it.

Visually you are testing a possible solution by visually checking to make sure that the manipulatives are in the right spaces. It's tactile - you are physically moving and feeling (kinesthetically and visually) the manipulatives with each attempt at finding a solution. This process, I came to realize, is equivalent to moving pieces on a chessboard. Tom commented on this process saying, "I'm seeing and using horizontal and vertical lines to test to make sure that each row and column has an even number of objects." Chess players call this using 'lines of forces' (Presmeg, 1992). Finding a solution was not carried out by brute force, it was logical and the medium participants used as a space to move and test a possible solution carried with it a concrete-visual-tactile feel. I conjecture that this is one reason problems such as this are enjoyable.

Eve expressed the following sentiment, "I like this one because it is not necessarily numeric. It has a nice mixture of logic to it. This one has a visual and probably has something to do with factors. It makes you think logically and mathematically about finding out." The visual/tactile component of this problem provided participants a playful space where they could explore and try something logical without risk and not be so attached to having a fixed method to solve.

Letting go.

By letting go of the expectation of having the 'right' algorithm or procedures to attain 'the correct' answer, participants became more engaged in the process of solving *Problem 3*. In so doing they were able to gain a deeper appreciation and respect for their own mathematical thinking. Participants at certain instances experienced what Liljedahl (2005) refers to as an Aha! Moment. These moments are not only reserved for the upper echelons of practicing mathematicians (Liljedahl, 2005, p. 220). It is a very experiential and crucial part of mathematical thinking. The data example involving Olga, Jon and Kara's solution process to *Problem 3* reflects closely the experiential process mathematicians undergo when they discover patterns or ideas by their own accord. I conjecture that one prerequisite to this affective state of present moment mindedness is being in medium that allows you to freely explore and let go of the expectation of having 'the right' algorithm or being accountable for 'the right' answer. Problem 3 certainly presented that medium.

Of course one can argue that this act of manipulating objects does not involve solving a math problem. I argue that it can be once you start to analyze the various configurations of patterns that are possible and coming up with a mathematical explanation. Kara certainly tried this. But again what is important is the exploratory process and conditions that stimulate encouragement to then further explore and think abstract. What I observed from participants solving Problem 3 was calmness, excitement, enjoyment and ecstasy when the finally found a solution. It appeared as if though getting the 'right' answer no longer mattered, what mattered was the act of thinking itself. Perhaps any mental activity of this type does. This type of problem is equivalent to moving pieces on a chessboard or playing a game of Sudoku, it requires logic and visual reasoning.

Chapter 5: Discussion and Conclusion

In this final chapter I discuss the research questions in light of the results presented in the previous chapter, making further ties with the literature discussed. I also make some recommendations based on the evidence obtained for the teaching and learning of mathematics with understanding. I will end with recommendations for further research in the area of sense making and problem solving, highlighting possible avenues for further study.

5.1. Reflecting on My Research Questions

At the start of this research, I was guided by the desire to understand the process of sense making through the process of solving non-routine problems. I had a desire to answer the questions: a) What role do visualization, gesture, and affect play in supporting mathematical reasoning and sense-making? b) How do these resources aid the process of sense making in problem solving in the absence of pre-determined algorithms?

I therefore designed a study that would enable me as a researcher to explore learners' sense making when faced with non-routine problems that made little use of predetermined algorithms. The main objective was to identify the role of visualization and gesture and affective states that enable or constrain the process of sense making. To do this I explored the process of sense making through a think-aloud protocol (Ericsson & Simon, 1993) coupled with meta-cognitive prompts (Anderson, Nashon & Thomas 2009).

5.2. Answers to My Research Questions

Recent research in mathematical thinking and sense making has highlighted the significance of the body in the process of mathematics teaching. However, not much is offered about the cognitive components: visualization, gesture and affect in sense making. The goal of this study was to examine the role that these cognitive processes played. From the analysis of the data I gathered, I discovered the following:

Visualization: Pattern Imagery grounds how we perceive physical spatial relations in the real world as image schemata for use as a resource in the process of making sense of more abstract mathematical patterns. From the evidence provided in these data examples, we can see that participants do make use of certain images that have their origins in human bodily experiences

(Lakoff & Nunez, 2000; Johnson, 1987; Nunez, 2000). These images are referred to as image-schemata that organize our understandings of spatial relations at a level more general and abstract. For example, figures 4.1 and 5.0 as used by Olga and Paul respectively in solving *Problem A* depict the spatial relation of squares in the form of a pattern imagery which provided the initial insight to make better sense of the formal mathematical pattern. These images helped to ground their understanding of the pattern at a level that is more general and abstract.

I came to observe pattern and schematic imagery, images stripped of details, (Presmeg, 1986; 1992; Giardino, 2010; Presmeg & Balderas-Canas, 2001) throughout most of the sketches the participants made (see figures in chapter 4). In this regard, image-schemata played a large role in sense making by serving as a suitable visual mediator from which images could be drawn to gain insight about the general mathematical pattern of these non-routine problems. Without this pattern imagery insight into Problem A, for example, a solution may not have been attained. Therefore pattern imagery is a powerful cognitive resource in the process of sense making of abstract patterns by organizing spatial relations at a level that is more general and abstract.

Gesture: I was able to observe that speech and gesture do play a role in sense making. Gesture, as I observed it at certain instances in these examples from my data, aided the clarification and conceptual explanations of ideas. Kara and Paul's solution to *Problem 2* depicted the use of what McNeill (1992), and Nemirovsky and Ferrera (2009) refer to as *deictic* gestures (pointing to existing or virtual objects). Both made reference to a virtual object and location in their solution process to help clarify and make sense of their solution in hindsight. This type of gesturing allowed for an alternative medium to further explore, encode and organize spatial and perceptual information. By gesturing in this manner, both participants were able to further flesh out and confirm with certainty their solution to *Problem 2* given their images.

For example, Paul was able to further clarify his understanding of the pattern he saw to *Problem A* and his solution to *Problem 2* by gesturing and explaining. In this regard, visual imagery coupled with gesture and talk help to clarify and reinforce his understanding of the imagery that guided him to a solution. By gesturing and explaining he was able to convince himself why these solutions make sense. Use of gesture and speech acts in this regard offer a medium through which thought can be shaped, reinforced and meaning established (Nemirovsky & Ferrera, 2009;

Radford, 2009; Arzarello et. al., 2009; Alibali, 2005; Edwards, 2009). This is further evidence that gesturing does play an important role in sense making.

It is clear from the research data that in solving a problem in the absence of formulas, participants do resort to visual imagery specifically pattern imagery and use of gestures to break the gridlock when they encountered cognitive obstacles in their attempts to solve. In so doing the participants came to make more sense of the mathematics and in some instances, as in *Problem 3*, enjoyed and experienced the joy of accomplishment when they finally got a solution by their own means.

Affect: In terms of affective states that enable or constrain the process of sense making, participants' comments, from the discussion in chapter 4, highlights the role that formal abstract teaching may play in whether a student chooses to make meaning for themselves by making sense of what's happening within the mathematics. "I believe it is something almost pedagogical; teachers don't often emphasis teaching by taking a problem and having the student come up with the solution on their own. They just give you a bunch of formulas and tell you to apply them." Eve said. This is a common perspective that students adopt. "[...] students learn that answers and methods to problems will be provided to them, the students are not expected to figure out the methods themselves (Schoenfeld's 1992, p. 343). The overemphasis on use of pre-determined algorithms and procedures at all levels of schooling may be the reason why students are reluctant to explore problems for themselves and rely on their own sense making abilities – visualizing and gesturing. In the case of Kara, this debilitating meta-affect perspective was so strong that she could go no further in solving problem A.

In addition, I came to observe the affective state of present moment mindedness and the notion of 'letting go'. There is something liberating about having the freedom to explore by not being bounded to a fix formula, having the pressure of presenting a rigorous solution or the 'right' approach. By not focusing so much on a pre-determined formula or procedure, participants evoked their own sense making abilities which in turn help boost their confidence in solving even when doubt arose (e.g. Paul's solution of *Problem A*). Without the strong belief that an algebraic solution was required, participants were more open to explore a problem using their own sense making ability. This affective state of not feeling pressured in having to memorize and apply the 'right' formula to solve, helped participants build confidence when making sense of a problem by their own means.

In addition the visual/logical/tactile component to the milk-crate evoked that affective state of present moment mindedness that enable participants to keep at the problem until they had found a solution. The presence of this affective state in finding a solution to *Problem 3* was a central aspect of participants' sense making processes. This meta-affect was most noticeable in participants' persistence to not giving up and being enthusiastic about finding a solution. They all had an AHA! experience which had a transformative effect on participants affective domain, it instilled the positive beliefs about their abilities to do mathematics.

5.3. Recommendation and Implications for School Practice

One goal of this research was to inform my own practice of teaching and learning mathematics for understanding. The results obtained point to central cognitive resources important for success in problem solving: visualization, gesture and affect. Based on the observations I made, I would recommend the following,

1. Nurture students' ability to visualize

When it comes to mathematical reasoning and logic, it has been suggested that visualization, to some extent, plays a significant role (Presmeg, 1997). This is not surprising considering mathematics is a field containing graphs, schematic figures/diagrams, symbolic inscriptions, etc. Brown and Wheatley (1997) emphasize, the ability to visualize patterns and mathematical relationships are sometimes necessary for successful problem solving. Educators need to be aware that students do visualize and gesture when thinking mathematically. Visualization, gesture and affective states are cognitive components that enable or constrain the process of sense making.

As evident by some of the participants' solution processes, most made use of schematic imagery to yield insight. So when required to solve non-routine problems, students may be at a disadvantage if they are unable to construct and make use of appropriate visual intuition in the absence of predetermined mathematical procedures or algorithms. This study shows that there is value in being able to explore and think in terms of schematic and pattern imagery. It is important for students to be exposed to problems that help nurture and cultivate this aspect in their thinking and to develop understandings of how and when to reason with them. Teachers need to recognize the value and limitations of such images in order to assist students establish links between analytic and visual reasoning. In the case of Jon's approach to *Problem A*, knowledge of knowing what to look for in a schematic pattern and knowing how to couple this with analytic reasoning shows that experience with both imagery and analytical reasoning is a determining factor that may enable or constrain the solution process of an individual.

2. Use non-standard problems when possible.

Developing an exploratory approach to problems solving through a 'sandbox' space allows you to freely utilize your own sense making abilities without fear. Those participants who were able to find a solution, for example to problem 3 - the milk crate problem, didn't feel pressure of having to memorize a formula, algorithm or mathematical technique when searching for a solution. Fixed algorithms or procedures with little context have very little meaning to students. In fact, most of the times students have very little understanding of the inner workings of a given algorithm perhaps because they aren't the discoverer. Encouraging learners to discover mathematical patterns by providing students non-routine problems that carry with it a visual/logic/kinesthetic component helps evoke their own individual sense making abilities. In so doing they can get to appreciate the process of self-discovery and the AHA! experience associated with it. These non-routine problems should also be coupled with meta-cognitive prompts (Can you draw a picture? Explain what your thinking? Why does this approach work? Can you figure out another approach?).

It was interesting to note the excitement and joy my participants experienced while exploring these problems by not having to rely on a predetermined formula. Formal abstract thinking and writing can come after once learners are able to establish deep understanding by reasoning and making sense of what's happening within the mathematics. I'd like to conclude this section by stating that teacher, whenever possible, should take time to get to know how their students think, one problem, one moment at a time. After all, the more informed educators are about individual sense making the more they'll be able to help their students. Educators should make every effort to do so.

5.4. Thoughts for Future Research

This research offers a contribution to the field of mathematical thinking and problem solving by providing evidence through data examples that participants do visualize, gesture and experience certain affective states (present moment mindedness) while solving. It also highlights the role of these aspects in thinking in enabling or constrain the solution processes of individuals when making sense and synthesizing meaning. The findings show the close relationship between these cognitive resources in thinking. Especially the affective state of present moment mindedness associated with thinking, something that I consider to be relatively new to research on affect in mathematics education.

The findings in this study need to be corroborated and in some cases further developed and substantiated by more empirical data. For example, the notion of present-moment mindedness that was observed in *Problem 3* reflected the mindset of these participants. And though observations are limited to these problems, I am confident that analysis of different non-routine problems and a larger sample would yield similar themes. The data in general was collected under the conditions of a think-aloud protocol, a specific problem set and post-interview protocols thus limiting generalizability of findings. Nonetheless these allowed for in-depth analysis of participants' solution process in the form of data examples. I'm confident that should this model (think-aloud protocol, similar types of non-standard problems) be replicated with larger and more diverse sample, the findings would be close to the results obtained in this study – use of schematic imagery, use of gestures, the relationship between visualization and gesture, and the affective state of present moment mindedness. Developing taxonomy of non-standard problems (see Appendix A) that evokes an individual's visual/logical thinking and this affective state of mind is something to consider for further research.

Finally, in the case of some participants' reluctance to visualize and think informally, I pointed to possible reasons why this is sometimes the case. From this, future research can further explore more generally meta-cognition and meta-affect as to why this is the case. In so doing data could reveal more about the nature of meta-cognitive and meta-affect components central to the mathematical reasoning.

5.5. Final Thoughts

Within the educational perspective, I advocate the need to take into account the various individual cognitive resources learners come to rely on in the process of making sense. These cognitive components include visualization, gesturing and certain affective states. The interplay between these resources unfolded against the backdrop of an exploratory approach to solving rather than strict pencil-paper algorithm based math. The analysis presented in this study illustrates how individuals come to visualize, gesture and think while solving a given non-routine problem. Despite the small sample this research shows the prevalence of visualization, use of gesturing and the value of present moment mindedness and letting go.

More importantly, research in mathematics problems solving needs further empirical investigations on the role of visualization, gesture, speech and affect as cognitive dimensions of mathematical thinking through theory and analyses of participants' mathematical activities. If these resources are interrelated and flow back and forth into one another then understanding this process more generally can certainly help inform the nature of mathematical thinking.

I came to observe the notion of present-moment mindedness, a meta-affect component of thinking, responsible for minimizing fear of failure and spurring excitement and focus while solving. It would therefore be very exciting to study the nature of this mindset while solving similar types of problem as *Problem 3* and the conditions therein. Research in this area of meta-affect would certainly be a worthwhile endeavor especially considering the ample literature in cognitive psychology that supports this as a component of meta-cognition. This is one aspect of cognition that may enable or constrain success in mathematics. As such, research in mathematics problem solving should draw insight from fields like cognitive psychology, neuroscience, brain chemistry etc. when exploring this aspect of cognition. It was indeed quite interesting to observe how this aspect of cognition works to make problem solving a successful, enjoyable and rewarding experience. Further empirical data about this aspect of meta-cognition through analyses of students' problem solving process can inform our teaching practice.

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Appendix A

Problem Set

Warm-up Problems

A) “Perfect Squares”

If you add together odd numbers from one upwards you will always get a perfect square.

1+3 = 4; 1+3+5 = 9; 1+3+5+7 = 16 … Can you think of a way to prove that this must always be the case? (From Puzzlegrams, 1989, p.113)

B) “Father and sons”

A man has 2 sons. The sons are twins; they are the same height. If we add the man’s height to the height of 1 son, we get 10 feet. The total height of the man and the 2 sons is 14 feet. What are the heights of the man and his sons? (From Sawyer, 1964, p. 40)

Additional Warm-Up Question

1) “Hundred squares”

How few straight lines are required on a page in order to have drawn exactly 100 squares? (From Mason, Burton and Stacey, 1985, p. 175)

Non-routine Problems

1) “Summation”

What is the sum of the following series? 1 + 2 + 3 + … + 98 + 99 + 100. (Art of problem posing)

2) “Field of cows”

A 10 by 10 square depicts a field with just enough grass in it to feed 100 cows. Is it possible to have square field with just enough grass in it for 200 cows? Prove it. (in Melrose study, p. 3)

3) “Milk Crate”

A certain square milk crate can hold 36 bottles of milk. Can you arrange 14 bottles in the crate so that each row and column has an even number of bottles? (Mason, Burton & Stacey, 1985, p. 181)

Additional Question

1) “Desert crossing”

It takes nine days to cross a desert. A man must deliver a message to the other side, where no food is available, and then return. One man can carry enough food to last for 12 days. Food may be buried and collected on the way back. There are two men ready to set out. How quickly can the message be delivered without neither man going short of food? (Mason, Burton & Stacey, 1985, p. 165)

Appendix B

Interview Protocols

The Problem Solving Protocol

The goal of this protocol is to get participants to verbalize, gesture and diagrammatically express their thinking while solving a given problem. It is hoped that participants' thinking can be further probed using meta-cognitive questions.

It is anticipated that there will be three aspects to this interview process - a before, during and after segment. Before the think-aloud interview protocol, participants will be briefed as follows: The focus of this study is to document the process of sense making and your cognitive resources at play while you solve each non-routine problem. This does not mean that you need to be a "good" problem solver per se. The goal is to take an exploratory approach to solving rather than relying on fixed algebraic formulas. Exploring the problem may require the use of strategies and methods to generate insight and create meaning, and this process is what I hope to document and explore. If you find yourself being inclined to use a fixed formula, I want you to step back for a while and see if you can think outside of the "normal" algorithmic routine of solving. Please feel free to use the materials provided here. You can also sketch, gesture, move around, role-play etc. Any questions? To begin, let's have both of you stand about two arms length apart in front of the smartboard. We will start with two warm-up questions just to have you get comfortable with the expressing your thinking aloud. The goal is talk aloud through your thinking. Feel free to collaborate and remember, getting the right answer is not the end goal here but rather the process. So do not be discouraged.

Guiding Questions: As much as possible, participant's responses will guide the researcher's questioning.

Before:

1) Have you seen this type of problem before? If so, let's try another.

2) Do you already have a formula or a quick way to solve this? If yes, let try another problem.

3) Describe how you would begin approaching the problem. What are your initial thoughts?

During: During this segment, the researcher will encourage participants to use the method that comes most naturally to them. The researcher will serve as a facilitator during the problem solving process.

1) I invite you to describe aloud how you are solving the problem. Please describe the process.

2) Describe the strategy you are using. Walk me through it step-by-step.

3) Explain what you mean by [...]? Can you elaborate a bit more?

4) Tell me more about the association **X** you used here to make meaning/sense in your thinking. Why did you choose to use it?

After: Reflecting on the process

1) Have you answered the question? Is your answer reasonable?

2) Can you think of another way to solve it?

3) Can the result be generalized? Can extend the idea in any way? Explain how you would go about doing it.

4) Describe what you were thinking of at this step.

5) Describe how you managed to categorize and organize the information here? Is it something you saw readily? Describe what was going through your head. What were you 'thinking'?

6) Did you feel limited in your ability to gesture or draw what you were 'seeing' or thinking while solving this problem? Why?

7) What's the process like for you, going from 'seeing' the picture in your mind to sketching or gesturing it? Could you describe this process?

8) Do you feel that you could have solved the problem without the use diagrams, gesturing or the manipulatives?

9) If you were to explain how to solve this problem to someone else, explain how you would do it?

10) Is there anything else that I forgot to ask you or that you would like to add? If you need a minute to think about it, that's perfectly fine. It could be a comment or a question.

Post-Interview Protocol

After pre-analysis of the video recordings, the researcher will follow up on interesting points, emerging themes or issues requiring further investigation or clarification. Where needed, snippets of the video recordings will be shown to refresh participant's memory and to stimulate further discussion. All or some of the following questions may be asked.

Guiding Questions:

1) For problem **N**, you did something which I found quite interesting. Here, let's take a moment to see what you did. Here's what I would want to ask you [...].

a) Explain what your diagram, is trying to convey. What made you think of this particular diagram or why did you use that gesture?

b) Did this step come to you rather quickly like a bright idea, or did you have to think about it for a while?

c) Explain how the diagram you drew helped to clarify your thinking.

d) What's the process like for you, going from 'seeing' the picture in your mind to sketching or gesturing it? Could you describe this process?

2) I observed that at this point you [...]. Could you describe what you felt? What were you thinking of exactly? What was going through your head?

3) I recall that at one point you said something along the lines of [...]. Could you kindly elaborate a bit more on this?

4) In your thinking at this point, I noticed that [...].

5) How comfortable were you while solving this problem?

6) Was solving this problem challenging? What about it made it challenging?

7) I noticed that at this point were immersed in the process of solving that it seemed you had shut off all the tensions of the day and all the thoughts of what had to get done tomorrow? If so, can you describe what that moment felt like to you?

8) How did you feel about exploring the problem rather than having to recall a fixed formula or method? How did this process make you feel? Did this approach make you feel more comfortable?

9) Is there anything else that I forgot to ask you or that you would like to add? If you need a minute to think about it, that's perfectly fine. It could be a comment or a question.

Appendix C

The University of British Columbia

Department of Curriculum and Pedagogy

Faculty of Education

2125 Main Mall

Vancouver, B.C. Canada V6T 1Z4

September 17, 2012

Dear Participants:

I'm currently a graduate student in the Department of Curriculum and Pedagogy at the University planning to carry out a research project entitled: Exploring students' mathematical sense making through non-routine problems for my thesis paper under the supervision of Dr. Susan Gerofsky. This letter serves as an invitation to participate in this research project. Through this study, I hope to gain a better understanding of graduate students sense-making/meaning-making and the cognitive resources therein in coming to solve non-routine problems. I believe that you may benefit directly from reflection on your own problem solving processes.

If you are interested in participating, I ask that you kindly read the enclosed Consent Form and take time to carefully consider the details provided. After doing so, I ask that you return a complete consent form with your signature to me no later than one week after receipt. Only those who provide written consent will be included in the study.

Thank you in advance for your time and willingness to consider this invitation.

Sincerely,

Myron Medina, M.A. student

Department of Curriculum and Pedagogy.

Appendix D

Consent Form

Title: Exploring students' mathematical sense-making through non-routine problems.

Principal Investigator: Dr. Susan Gerofsky, Assistant Professor, Department of Curriculum & Pedagogy.

Co-Investigator: Myron Medina, M.A. Student, Department of Curriculum & Pedagogy.

Purpose: Since mathematical thinking is of a personal nature and this personal aspect is what enables or constraints the mathematical solution processes of an individual, through this study, I hope to gain a better understanding of how graduate students' make sense in coming to solve non-routine problems.

Study Procedures: Participants will engage, in pairs, in a think-aloud problem solving session in the Faculty of Education building at the University. Participants will attempt to solve a set of non-routine problems and will be interviewed while they do so. All written work will be collected for use as data for the study. The research also involves one post-interview session of about 40 to 60 minutes. The interview session will take place several days after preliminary analysis of data at a designated location and time convenient to the participant. With your permission, each session will be video recorded with a copy of the interview transcript provided to you upon request. I anticipate that you would need to commit 40 to 60 minutes per session for a total of two sessions.

Potential Risks and Benefits: There are no known risks associated with this study. I will be sensitive to your needs during the interview sessions and you are able to leave at any time. I believe that through your participation, you may benefit directly from the additional reflection on your meta-cognitive ability in solving non-routine problems.

Confidentiality: Names of participants will be kept confidential. Pseudonyms will be assigned to each participant in the research for identifying data sources and reporting of results. The results will only be used for academic publication(s). The data acquired will be stored in a password protected computer or a locked cabinet accessible only to the investigators for a period of five years after which all data will be destroyed.

Contact information: If you have any questions or desire further information about the study, feel free to contact Myron Medina (co-investigator) or Dr. Susan Gerofsky. If you have any concerns about your treatment or rights as a research participant, please telephone the Office of Research Services at the University.

Consent: Participation is completely voluntary. Participants can opt out at any point during the research, upon which time all data pertaining to the participant will be destroyed. Please indicate whether you agree to participate in this research project by signing the attached Consent Form and kindly returning it to the investigator. Kindly retain the attached copy for your records.

Consent Form

RESEARCH PROJECT: Investigating graduate students' sense making through non-routine problems.

I have read and retained a copy of the Consent Form and have had my concerns and questions answered to my satisfaction. Under the conditions outlined in the Consent Form, I agree to participate in this research project. I consent to the video recording during the think-aloud session and during the post-interview. I consent to have my written work and recordings related to the problem set to be used as research data. I understand that in any papers, publications or presentations from this study, my name will not be revealed.

____

Subject Signature   Date

____

Printed Name of Subject signing above

Phone: ______________________________

Address: ______________________________

Email: ______________________________

KEEP THIS COPY FOR YOUR RECORDS

Consent Form

RESEARCH PROJECT: Investigating graduate students' sense making through non-routine problems.

I have read and retained a copy of the Consent Form and have had my concerns and questions answered to my satisfaction. Under the conditions outlined in the Consent Form, I agree to participate in this research project. I consent to the video recording during the think-aloud session and during the post-interview. I consent to have my written work and recordings related to the problem set to be used as research data. I understand that in any papers, publications or presentations from this study, my name will not be revealed.

____

Subject Signature Date

____

Printed Name of Subject signing above

Phone: ______________________________

Address: ______________________________

Email: ______________________________

RETURN THIS COPY TO INVESTIGATOR