Logic in the *Tractatus*

by

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Abstract

What is logic, in the *Tractatus*? There is a pretty good understanding what is logic in *Grundgesetze*, or *Principia*. With respect to the *Tractatus* no comparably definite answer has been received. One might think that this is because, whether through incompetence or obscurantism, Wittgenstein simply does not propound a definite conception. To the contrary, I argue that the text of the *Tractatus* supports, up to a high degree of confidence, the attribution of a philosophically well-motivated and mathematically definite answer to the question what is logic.
Preface

It is something of an accident that I ended up writing a dissertation about the *Tractatus*. In 2009, I was casting around for a thesis topic and somehow found myself in Carl Posy’s seminar on Brouwer at The Hebrew University. He seemed like a pretty good person to ask. When we met, he took out his copy of the *Critique of Pure Reason* and read out a passage in which he discerned insights later conveyed in certain early 1970s lectures at Princeton on the philosophy of language. History—or, better, long conversation—brings a structure that he thought I needed. It also helped that Posy clearly enjoys talking about Brouwer. Then it did come down to early Wittgenstein.¹ In my first year as an undergraduate, my favorite course had been an introduction to logic taught by Sanford Shieh. I asked him what to read next, and he said, “well, the *Grundlagen* of course. Then then you might want to look at the *Tractatus*”. So I went out, found the *Tractatus*, and carried it around for a year or two. It ended up on my bookshelf and remained there for about ten years. Some feeling of personal failure lingers around that episode, which maybe a whole dissertation begins to redress.

When I first began to read the *Tractatus* as an adult, then, the first thing that struck me was the extreme implausibility of its characterization as a kind of logical atomism. It seemed just incomprehensible to me that Hume or Russell could say that each object has written into its nature all its possibilities of occurrence in states of affairs. But it also struck me as unpalatable to respond by arguing that the *Tractatus* is not atomist but holist, since that seems to amount to little more.

¹ Or Spinoza. So it could have been worse.
than consigning it to the opposing role in some Kantian antinomy. Anyway, this initial response was enough for a dissertation prospectus, but that was about it.

Then in 2012 I had the good luck to end up in Cambridge, MA, where, as it happened, Warren Goldfarb was giving a seminar on the *Tractatus*. A few weeks into the seminar, he pointed out that Wittgenstein tried to develop resources for analysis of the ancestral of a relation. Goldfarb said he was working on the question what is the complexity of the property of being a tautology in the *Tractatus* system.

I was intrigued by Goldfarb’s question, and spent much of that year trying to assign it a definite mathematical content. The basic idea behind the construction of propositions in the *Tractatus* is a process of inductive generation. Some elementary propositions are initially given. Furthermore, for any given bunch of propositions, some proposition is the joint denial of the propositions in that bunch. This construction has some fairly obvious mathematical and philosophical similarities with the cumulative hierarchy of sets. For the cumulative hierarchy of sets is described as the result of repeatedly applying, “as much as possible”, an operation of set-formation to multiplicities of items “already” obtained. Moreover, neither construction is an ordinary inductive definition which serves to single out objects from some antecedently given class of objects: rather, they are each inductions of the sort that Kleene (1952, 258) calls “fundamental”, purporting to generate the entities of the domain in the first place. However, this resemblance, while suggestive, should not be taken too flatfootedly. In particular, it is implausible and boring to suppose that for any multiplicity of propositions whatsoever, a proposition can be constructed which denies exactly them. What, by Wittgenstein’s lights, actually is the extent of the capacity to survey multiplicities?

At 5.501 Wittgenstein says that one can specify multiplicities of propositions by “stipulating the range of a propositional variable.” He gives “three ways” of fixing such ranges. The first two ways basically correspond to the syntax of first-

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2 He and a student couldn’t quite agree about whether it had been ten or eleven years since its previous offering.

3 References to entries of the *Tractatus* (Wittgenstein 1989) will take this standard form, sole exception being single-digit entries which will be prefaced by the letter T. (I will not refer to its preface.) Reference to entries of the *Prototractatus* will take the same form but prefixed by PT.
order logic. But the third way allows fixing the range of a variable by constructing a so-called “form-series”—roughly, an inductive specification of a multiplicity of formulas. Sadly, Wittgenstein says almost nothing about which operations can be used in such inductions.

Frustrated by Wittgenstein’s vagueness on this point, I decided to explore the hypothesis that one can form the joint denial of any effectively enumerable collection of formulas. Under this hypothesis, there then followed an analogy of the *Tractatus* system with the hyperarithmetical hierarchy, a technical construction which arose in mid 20th-century mathematical logic. Goldfarb wasn’t impressed. He urged, in particular, that according to Wittgenstein, we need to be able to “see” which things are subformulas of a given formula. So I tossed aside the arithmetization and tried just to say very carefully how Wittgenstein’s construction seemed to work. Rather than imposing external ideas about computability onto the collection of propositional signs, I tried instead to think about operations that one might think of as native to the realm of propositional signs itself. The idea that immediately suggested itself was that of substitution of a formula for a propositional variable in a formula. It quickly became clear that such a simple idea actually did the required technical work. That is, by forming a disjunction of the results of iterated substitution, it is fairly straightforward to express the ancestral of a given relation. In this way, I settled on what seems to me to be a plausible approximation of the system of logic developed in the *Tractatus*. It is an “approximation from below” in the sense that everything expressible in it is claimed to be expressible in the *Tractatus*. No converse claim is intended, but I doubt that significant strengthenings are interpretively defensible.

My original formulation of this analysis used a bizarre metalanguage: in particular Wittgenstein’s talk of variable-ranges was handled with plural variables. This made the formulation pretty hard to understand (according to people who tried to read it). So I recast the whole thing as a construction inside a particular standard model of the axiomatic set theory KPU. The result is a mathematically definite explication of the property of being a tautology which Goldfarb’s origi-
nal question addressed. This notion is definite despite avoiding any prejudgment whatsoever about the number and form of elementary propositions. By risking such prejudgment, one can instantiate the abstract notion to more concrete ones. Anyway, all this appears in Chapter 5, which embodies the main claim of the thesis: “here is what Wittgenstein thinks logic is”.
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Introduction

The aim of this thesis is to answer the following three questions about logic in the *Tractatus*.

(1) Is there such a thing?

(2) What is it?

(3) Why is it that?

The thesis addresses the questions in reverse order. Chapter 1 addresses question (3), chapters 2-4 address question (2), and Chapter 5 addresses question (1). In the present introductory chapter I’ll summarize these results, and in so doing advance what is the main claim of the thesis: that Wittgenstein in the Tractatus develops a conception of logic which is philosophically well-motivated and mathematically definite.

Pursuing question (3) we might ask what does the author of the *Tractatus* want out of a logico-philosophical engagement with logic itself? How did the subject, or discipline of logic capture his imagination; why, for that matter, does it warrant his respect? Here it is amusing to compare Wittgenstein’s attitude with that of Descartes. Like Wittgenstein, Descartes thought that logic was no great body of theoretical knowledge, but for Descartes, this was because he thought that logic was boring. Wittgenstein, to the contrary, felt that logic was rather too interesting, and that the expressive and clarificatory power of logic bewitches us into delusions about truth and thought. What explains the difference?

In 1879 Frege laid the foundation of the modern subject of logic in his book the *Begriffsschrift*. Frege’s preface cites a dream in Leibniz of a lingua characteristic, or characteristic language: a language adequate to the subject-matter, in such
a way that the marks of concepts correspond to the parts of the corresponding expressions of the language. If a problem arises in astronomy, or chemistry or jurisprudence—then we can take out our notebooks and calculate on the expressions of the concepts which are constitutive of that problem. Frege remarks that it is too much to expect all at once a universal characteristic language, a language adequate all at once to all subjects matter, but suggests that piecemeal progress had happened already in arithmetic, geometry, and chemistry. And he announces in BG the completion of another—the central piece, a notation adequate to the subject-matter of the science of logic. This piece of the puzzle is central because logic is what determines what counts as rational justification. The concepts of logic constitute the evidentiary structure internal to each special science.

But logic isn’t just the common portion of the special sciences but a science itself. Logic for Frege has a subject-matter of its own: the properly logica objects and concepts, objects and concepts or functions whose nature we acknowledge (if only partially) in any achievement of knowledge. So for example the True itself is a logical object, as is the False; negation is a logical function, is that function which takes the False to the True and everything else to the False. A basic law of logic is that the True is the negation of the negation of itself, for example. This law is essential to the use of nonconstructive reductio ad absurdum arguments in number theory, for example. Logic, for Frege, is a science, the science whose subject-matter is constitutive of rational justification.

So, logic for Frege is that science whose subject-matter constitutes rational justification and its truths articulate the general structure of rational justification. But, for Frege logic is also itself a rational science, consisting of a body of truths. Being rational, its truths stand in a kind of natural order of justification or grounding themselves, so that a given truth of logic will rest on other truths, and those on others, and at the bottom you have the basic laws of logic, the truths which are basic. The logician, now, makes evident, with respect to some proposition of logic, that it is in fact a truth, by making evident this relation of grounding which it bears to other already recognized truths. The logician does this by reducing the grounding relation down to a series of inferential steps, each itself evidently justificatory. This is the construction of a proof. What Frege himself does, is codify the concept of proof itself, by giving an axiomatic system. This is
a general recipe for proofs—one might say, the general form of the logically true, or at least, the general form of the theorem.

It’s this conception of logic as a science that Wittgenstein, in his book, sets out to destroy. For Wittgenstein, logic is not a science. There are no “logical truths”. So-called “logical truths” are really “tautologies”, i.e., redundancies or pleonasm, like saying the same thing somebody already said; they add no information. Nor are there any “logical objects”—logical vocabulary is mere punctuation. Logic has no subject-matter to describe, so it cannot have any truths of its own. Now, with respect to all this, it is not just that Wittgenstein wants to prove himself right in theory; he wants to effect meaningful social change, by getting people not to do what, under Frege’s program, they’d be doing.

So far this is mostly just what we might expect from the author of the Tractatus, some stylish opinionation. Can all this talk be substantiated? Wittgenstein’s work had better have some bones in it if it is to disrupt a program with the coherence and power of Frege’s. On my understanding, Wittgenstein takes the scheme of Frege’s program and transforms it.

Frege uses the word “proposition” to mean, roughly “theorem”, or “statement to be proven”, and therefore to mean something that is logically true. Wittgenstein, amusingly, says: there is no such thing as a proposition that is logically true. For Wittgenstein, a proposition is essentially not logically true. It is essentially logically possibly false, and logically possibly true. For Wittgenstein, propositions are mere possibilities.

Now, this of courses raises the question: what is a possibility? Is it possible that this dissertation not have been written? Or that when this sentence was written, it was 12:35pm in Vancouver? Or that it have been 12:35pm on the sun? Wittgenstein thinks that, in general, something makes something a possibility. Some possibilities Wittgenstein takes to be what I’ll called basic: their being possible has no further explanation (just as Frege’s being an axiom has no further explanation). Other possibilities are mere positions with respect to the obtaining and non-obtaining of the basic possibilities. Thus, the basic possibilities determine all possibilities, or, all possibilities supervene on the basic ones.

For Wittgenstein, this relationship of supervenience of possibilities on basic possibilities gets made evident through analysis. Analysis reduces an instance of
such supervenience of a possibility to the repeated truth-operations. Analysis thus represents a possibility as the result of repeatedly applying infinitary conjunction, disjunction, negation, etc., on the possibilities which are basic. What Wittgenstein himself does in the *Tractatus* is to codify the notion of analysis, by giving what he calls the general form of the truth-function. So, whereas Frege codifies the general form of the logically true and says, go find some proofs, Wittgenstein codifies the general form of the logically possible, and says go find some analyses. Whereas for Frege, logic gives knowledge of truths, for Wittgenstein logic gives only clarity.

(2) Let me now very briefly summarize my answer to the question what logic is supposed to be according the *Tractatus*. Wittgenstein intends his response to Frege to transform the character of logical activity, so that it is no longer directed toward recognition of truths of its own proper (incidentally, highest) species, but instead only toward sharper grasp of what it is for possibilities to obtain and not obtain. The heart of this response is the articulation of the general form of the proposition as the general form of the truth-function. On this articulation, there are some propositions, the basic or elementary ones, such that every proposition is just a truth-function of them, so that for a proposition to be true is no more than a matter of which elementary propositions are true. But how, precisely, do there get to be such things as truth-functions of elementary propositions?

Wittgenstein aims with the general propositional form (GPF) to give a general description of the propositions in any sign-language whatsoever. This goal of generality requires that the GPF contain only what is essential to any way of constructing truth-functions which a particular sign-language might exploit. Nonetheless, the GPF form is, like Frege’s axiomatizations, itself a kind of technical construction, which requires seemingly accidental hacks and manipulations. On my understanding, Wittgenstein’s construction can be made explicit by positing a scheme of notations, together with a method of associating truth-functions to notations, in such a way as to sustain the view that to be a truth-function is to be a truth-function associated to a notation in the scheme. The particular details of the scheme itself are to some extent inessential; the actual content of the description of propositions which the GPF embodies involves rather only the
t totality of associated truth-functions.

The scheme of notations can be characterized inductively. Thus, in particular, every basic or elementary proposition may be taken to be a notation itself. But second, suppose that we’ve constructed some bunch of propositional notations, which are, by hypothesis, expressions of possibility. Then, in general we can construct another expression of possibility by signalling agreement and disagreement with obtaining and nonobtaining of possibilities in the bunch. An expression constructed by such a signal is the result of a truth-operation on the expressions antecedently constructed. Wittgenstein initially decides that truth-operations themselves must be of finite number, and also in some sense transparently simple. But then he realizes that actually, it suffices to consider a single truth-operation, that of joint denial.

Thus, in general, a nonelementary propositional notation will consist of a sign of an operation $N$, together with an expression $A$ which indicates some other propositional signs $A_1, A_2, \ldots$. The propositional notation $NA$ is said (in my terminology) to deny “directly” the notations $A_1, A_2, \ldots$ which its associated pointer $A$ indicates. A notation $A_i$ may itself be an elementary proposition, or it may have the form $NB$ where $B$ in turn points to further notations $B_1, \ldots, B_k$ which are those directly denied by $B$. In this way, any nonelementary propositional notation stands at the origin of a tree generated from it via the direct denial relation. How does this feature of a notation determine that the notation corresponds to a truth-function of elementary propositions? It is naturally sufficient that the tree of direct denial associated to a notation must be well-founded, terminating only in notations which are elementary.

Thus, rather than thinking of propositions as “built up” from elementary propositions, instead we can think of the general propositional form as an insistence that all propositions can be arranged in a notational structure on which the direct denial relation is wellfounded. Analysis is then just a matter of finding such a notation for the sentences and thoughts whose truth-conditions we already grasp. Such a notation for the totality of sentences will induce a “predicted” consequence relation. As I argue in Chapter 1, an analysis is sound if the predicted consequence relation agrees with the actual one.

Chapter 2 addresses a puzzle in the desiderata for an understanding of logic
Wittgenstein insists that we can’t prejudge the question how many elementary propositions there are, or what is the way in which elementary propositions are built up out of names. Yet any sane explanation of first-order logic certainly prejudges this question. How, in particular, are we to render quantification, i.e., generalizations over objects, if we don’t know the structures of propositions? The solution to this problem lies in the 3.31s. Wittgenstein doesn’t there explain propositional functions in terms of “swapping” of names (or objects) in the way that Frege and Russell do. Rather, he explains them in terms of the curiously primitive notion of an “expression”, the result of somehow “turning a name into a variable”. Now, Wittgenstein’s point is this. An expression marks the sense of a proposition if it is the result of “removing” a name from that proposition. A propositional function presents an expression, and it ranges over those propositions whose sense that expression marks. But, now Wittgenstein’s funny “exclusive” understanding of the name-variables follows immediately. So the notion of expression yields the desired technical consequences. On the other hand, it also guarantees that the propositions in the range of a propositional function all contain the same number of names—hence, have the same “mathematical multiplicity”. In this way, Wittgenstein’s distinctive understanding of quantificational generality turns out to be deeply rooted in the idea of propositions as pictures. Moreover, this account requires no assumptions about propositions and names: the result of removing the name from that proposition is determined merely by the name alongside the proposition. So, Chapter 2 addresses the way of stipulating the range of a propositional variable which underlies objectual quantification.

Chapter 3 addresses Wittgenstein’s last listed method of fixing the range of a variable. This method is decidedly non-first-order, that of giving a form-series. Here, Wittgenstein’s ideas are much less developed. Moreover, since Wittgenstein seems to use something like this method to give the general propositional form, the status of the method is quickly implicated by any serious consideration of difficult interpretive questions about the point of the Tractatus as a whole. But there has been very little serious study of the details of the development of this idea in

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4One can also stipulate the range if the range is finite by simply listing its elements—alternatively, one simply write the elements of the range instead of using a variable at all.
Wittgenstein’s thinking. I devote Chapter 3 to this task, and several important interpretive results emerge. Wittgenstein sees himself in the *Tractatus*, in giving the general propositional form, as reconceptualizing the type-theoretical stratification of propositions found in Russell and Whitehead. The type of a proposition is merely aspectival, depending just on the complexity of a notational exhibition of its truth-functionality. Thus, Wittgenstein sees the form-series of 4.1252 as an example of movement from type to type in what becomes of the hierarchies of Russell and Whitehead, since the formulas in that form-series have increasing notational complexity. In this way, Wittgenstein replaces a system of type-theoretic constraints with a survey of notational freedoms.

(1) Finally, in Chapters 4 and 5 I turn to the question whether there is such a thing as that which Wittgenstein takes logic to be. The answer to this question is certainly not settled in the foregoing chapters, since they yield a philosophical characterization of logic which may for one reason or another resist any definite realization. I show that such a definite realization exists, by actually constructing one.

Chapter 4 is an informal introduction. Given that Wittgenstein never developed a systematic analysis of the device of formal series, one can only indulge in interpretively informed speculation how it might be used. I propose what must be the simplest possible approach which meets what is known of Wittgenstein’s expectations, and explore its use in conjunction with the understanding of objectual generality already developed in Chapter 2. In particular I show how to express the ancestral of a relation, and then also some related notions of interest in Wittgenstein’s analytical program.

Finally, in Chapter 5, I give the systematic reconstruction of Wittgenstein’s logical system. The reconstruction mimics the scheme of 4.51: “suppose I am given all elementary propositions: then I can simply ask what propositions I can construct out of them. And there I have all propositions, and that fixes their limits.” So, the reconstruction assumes as given a certain basis, $\mathfrak{B}$, which consists of the totalities of elementary propositions and of names, together with a relation which, in a sense to be made precise, determines how names characterize the sense of elementary propositions. The structure $\mathfrak{B}$ then determines a corre-
sponding class HF(\mathcal{B}) of hereditarily finite sets over the totalities of elementary propositions and names. I then explicitly construct, within HF(\mathcal{B}), the totality \mathcal{Z} of propositional notations. This shows that although Wittgenstein’s system has some semantic affinities with subsystems of infinitary logic, nonetheless its syntax is (almost) perfectly finitary. The semantic notions of truth and validity are another story. Truth becomes a relation between a formula and a subset of the class of elementary propositions. The truth-function associated to a notation is the class of those sets to which that notation bears the truth-relation. I conclude by suggesting that although Wittgenstein rejected full second-order quantification, nonetheless his conception of logic as elucidation of possibility commits him to realism about the notion of truth-possibility for elementary propositions.

A philosophical theme. The overarching aim of the dissertation is to try to answer the question what is logic in the *Tractatus*. However, the *Tractatus* being what it is, everything hangs together (or falls apart). Occasionally I found myself straying into logically inexcusable philosophical matters, sometimes even reaching for the same idea on multiple occasions. So in the end I decided to write some of these down, with apologies to the unexpecting reader.

Let me just mention here one salient example: throughout the unfolding of this project, I’ve been struck more and more by how deeply the conception of propositions as pictures pervades Wittgenstein’s thinking about logic itself. *A priori* it seemed to me almost impossible that the two topics had anything to do with each other—for one thing, at the outset I found only one of them interesting. Nonetheless, again and again in trying to understand Wittgenstein’s thinking about logic I found myself driven to appeal to some aspect or other of the idea of propositions as pictures. Moreover, in different theoretical contexts quite different aspects of the idea became salient. So it seems to me that there is no one single way in which propositions are like pictures; rather, it is a deep, multifaceted heuristic that guides almost all of Wittgenstein’s logical thinking.

In Chapter 1, the key influence of the picture theory is this: that in seeing a picture, we thereby see in the picture a possibility for the world. The proposition, in this way like a picture, “shows its sense”: the proposition shows how the world is if it is true. This aspect of pictoriality should not be understood psychologi-
cally. For example, it would be a mistake of the sort that Frege inveighed against in the *Grundlagen*, and which I take Wittgenstein to have appreciated deeply, to suppose that seeing something as a picture entails any sort of a mental episode. For example, it is that sort of mistake which leads to the old complaint that the negation of a picture can’t be a picture. (That is just like the complaint that it is impossible to visualize zero things.) Rather, seeing something as a picture means commanding its representational powers. Grasping a picture means being able to discern logical relations between it and other pictures, it means being able (at least in appropriate cases) to bring about the world that is as the picture depicts it to be, to hope or fear that the world is as depicted, to see the picture or its negation in the world itself, and so on. Like Frege, Wittgenstein never calls into question this power of grasping, never postulates some independent purchase on propositional structure out of which such questioning might gather strength. Rather, by reflection on the understanding of propositions, which their pictoriality makes possible, we might come to reach a clear view of their logical order. Talk of the pictorial likeness of propositions and the world thus recapitulates the idea that sentences of ordinary language are in logically perfect order as they are.

In Chapter 2, a quite different aspect of the pictoriality of propositions emerges to guide the discussion. Wittgenstein thinks of a picture as something like a model of reality, so that “names” are arranged in a picture as their bearers are thereby said to be arranged in the world. Thus, a picture, or a proposition, is in some sense a “hanging-together” of names, just as a possible fact is a hanging-together of objects. Now a fact, Wittgenstein thinks, has a definite “mathematical multiplicity”; roughly speaking, this is—perhaps among other things—a matter of how many objects it incorporates. A proposition expressing the fact must have the same multiplicity, i.e., must contain the same number of different names. Thus there cannot be different names of the same object. Nor does it really make sense to distinguish between different occurrences of a name in a proposition, any more than there is any natural way to talk about the number of times an object occurs in a fact. How many times does Obama occur in the fact that he is father of both Sasha and Malia? (Of course, people are not actually objects, but set that aside.) This question makes no sense. Nor, then, is there any more sense to the question how many times Obama’s name occurs in the proposition
that he is their father. As it turns out, all this metaphysical waxing has logical significance: for the idea of mathematical multiplicity is the key to Wittgenstein’s insight that the equality predicate is redundant. For, once we refuse to count different occurrences of a name in a proposition, then the so-called “exclusive” interpretation of the name-variables is practically forced on us.

Yet another aspect of the conception of propositions as pictures operates in Chapter 3. A big problem in that chapter is to try to figure out what justifies Wittgenstein’s appeal to this seemingly bizarre device of the form-series in his analysis of logical structure. What, in general, licenses the analysis of some proposition into others? Here, the guiding idea is that a proposition must have its logico-mathematical content in common with what it depicts. So, Wittgenstein thinks, even though “Socrates” isn’t really a name, still we must be able to reason with “Socrates” as though it were really a name: thus, he insists, the logic of *Principia Mathematica* is (pretty much) applicable to ordinary, unanalyzed sentences just as they are. But how can this be, if such application means treating as names items which are not really names at all? The proposition to be analyzed must bear a quite definite resemblance to the propositions into which analysis transforms it. Wittgenstein must then have felt that a form-series variable is like a propositional function, in bearing some such quite definite resemblance to its values.

So there is an interpretive claim: that Wittgenstein’s thinking about logical matters is shaped pervasively and multifariously by the conception of propositions as pictures. Soon I felt this feature of the form of Wittgenstein’s thinking to be reflected in the content of his thinking too, in what one might call a thesis of primacy of picturing. The point is that it is a mistake to think that picturing is something achieved with signs when signs are put to (appropriate?) use. Rather, as McCarty (1991, 72ff) puts it, “understanding is like standing”. Signs are an abstraction from pictures, and come into view only when something goes wrong. Like the one-sided face of illusions, signs are epiphenomenal. So it is backward to postulate a task of assignment of reference to names, because to be a name is to go proxy for an object in a picture. But it is also backward to think of facts and objects as “popouts” of antecedently determinate inferential practice: no inference runs free of the responsibility to truth and falsehood. On a very
schematic level, one might put the point as follows: that the formal unity be-
tween proposition and situation precedes both its decomposition into names and
its individuation as a nexus of the inferential web. In even more schematic form:
there is no horizontal falling apart without vertical unity.5

That, however, is about as far as I should go in an introduction. So let me
just conclude with a remark on the structure of exposition. The material of each
chapter was prepared for standalone presentation (containing advertisements for
other material). So, each chapter can be read independently of the others. In
particular, somebody who quickly wants the “technical core” may just read §1 of
Chapter 5, consulting Chapter 4 for motivation and examples. Chapters 1 and
2 are addressed to a historically informed philosophical audience. Chapter 3 is
primarily exegetical. Chapter 1 establishes the general interpretive outlook.

5No mortal interpreter could fail to find something of themselves, or of their teachers, in what
they read. Here I should acknowledge the influence of Roberta Ballarin and Ori Simchen, tor-
menting as usual the Carnap of Alan Richardson.
Chapter 1

The general propositional form

All the propositions of ordinary language are actually, as they are, in perfect logical order. (5.5563)

1.1 Analysis

At about the midpoint of the *Tractatus*, there appears an announcement:

> It now seems possible to give the most general propositional form: that is, to give a description of the propositions of *any* sign-language whatsoever in such a way that every possible sense can be expressed by a symbol satisfying the description, and every symbol satisfying the description can express a sense, provided that the meanings of the names are suitably chosen.

> The general form of the proposition is: This is how things stand. (T4.5)

After slogging through a tedious and slightly confusing exposition of truth-tables and of truth-functional validity, when the reader’s patience is running thin already, this elaborate buildup arises more or less out of the blue, ascending to its vaunted climax. The climax itself may then seem a little bit trite. Nonetheless,

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6 This translation is Pears-McGuinness. Ramsey-Ogden have here instead: “description of the propositions of some one sign-language”. The German has “irgend einer Zeichensprache”, with emphasis on the first two words.

7 As several commentators have observed.
I think that it nicely summarizes some of the early Wittgenstein’s fundamental ideas.

The mention of “things” is intended to bear no weight. Indeed, I think that whatever information there is here, it lies inside of the demonstrative “this”. One might as well read the entire sentence as an “ahem”, carefully enunciated as the speaker sweeps back a curtain to show you. What such a speaker would show you, behind the curtain, is how things are. Perhaps, in the best instance, the speaker, in revealing what is behind the curtain, shows you how things are, because, behind the curtain, there they are.

But, not every instance is of the best kind. For example, it might arise that, say, things are too far away, or, for that matter, that somebody will not look at them. For example, Hamlet resorts to hiring actors, who go proxy for the real villains and victims. Thereby the king is brought despite himself to see how things are: it suffices that he see the actors, arranged with respect to each other in this visceral narrative arc, and then, suddenly to realize whom they represent.

Unlike the king, of course, usually people do want to see how things are. They might be, so to speak, curious. A common practice, in such a case, is to ask somebody. But let’s consider this condition of wondering: what is it that’s wondered? Whatever it is, one can even ask of somebody if that’s how things are. So, it looks as though the general form of the proposition could also be put as a question: is this how it is? Once again there is just a demonstrative in the subject-position, with an accompanying “ahem” now modulated differently. But this alternative formulation brings something out: that, a priori, things might be as we’ve wondered, and, they might not be. As Wittgenstein put the point, “thought can be of what is not the case”. One can hardly pose a question by pointing the worldly circumstances that would form its answer, for it is ignorance of those worldly circumstances which animates the interrogation in the first place.

According to Wittgenstein, rather than simply pointing out how the world is, instead people resort to pictures: that is, to constructing a representation of some situation from a position outside of that situation. We can make ourselves

8In German, it is *Es verhält sich so und so.*
understood in this way, because in the picture, names are arranged as things are thereby said to be arranged. Thus, as Wittgenstein puts the point, a picture shows its sense; it shows how things stand if it is true. Since a picture is not what it portrays but a replica, there is room for two possibilities: that things be, and that they be otherwise than, the picture portrays them.

So, the *Tractatus* conception of pictures has two sides. The picture must be different from what is depicted, in order for the two to disagree. But the picture must be the same as what is depicted, in order for the picture to show how things stand if it is true. The difference arises because in the picture names go proxy for their bearers. The similarity arises because the picture and what it depicts have in common what Wittgenstein calls pictorial form.

Wittgenstein puts forward this conception of pictures as a completely general conception of thought. Thus, for Wittgenstein, every thought is, in some such way, a picture. The thought of things in a certain way, is very much like the sight of them that way. What is curious about this idea is that it seems, to put it mildly, like it might not be right.

To make matters worse, Wittgenstein goes on to say that a thought is a proposition with sense, and that the totality of propositions with sense is language (4, 4.001). It seems to me that somebody who wants to maintain that thinking is a kind of language will not help their case by maintaining that thoughts are like pictures, because, for example, sentences and pictures seem so unlike each other. As Frege puts the point, “It is striking that visible and audible things turn up here along with things which cannot be perceived with the senses. This suggests that shifts of meaning have taken place” (1984, 352). How could Wittgenstein’s conception of propositions as pictures even seem to be right?

As Frege points out, the problems with this conception are manifold:

Is a picture considered as a mere visible and tangible thing really true, and a stone or a leaf not true?

A picture is supposed to represent something. It might be supposed from this that truth should consist in correspondence of a picture to what it depicts. […]

Now a correspondence is a relative relation. But this goes against the
use of the word ‘true’, which is not a relative term. . . .

A correspondence, moreover, can only be perfect if the corresponding things coincide and so just are not different things. It is supposed to be possible to test the genuineness of a bank-note by comparing it stereoscopically with a genuine one. But it would be ridiculous to compare a gold piece stereoscopically with a twenty-mark note. (Frege 1984, 352).

In the following two sections, I’ll try to explain how the conception of propositions as pictures derives from Wittgenstein’s understanding of the nature of logic.

1.1.1 Logic and analysis

Frege’s 1879 *Begriffsschrift* opens with this remarkable passage:

In apprehending a scientific truth we pass, as a rule, through various degrees of certitude. Perhaps first conjectured on the basis of an insufficient number of particular cases, a general proposition comes to be more and more securely established by being connected with other truths through chains of inferences, whether consequences are derived from it that are confirmed in some other way or whether, conversely, it is seen to be a consequence of propositions already established. [. . .] The most reliable way of carrying out a proof, obviously, is to follow pure logic, a way that, disregarding the particular characteristics of objects, depends solely on those laws upon which all knowledge rests. (Frege 1967, 5)

Frege thus opens his book with a summary of the means of recognizing scientific truths. He distinguishes between the way that we happen to come to consider a proposition to be true, and the best possible foundation which the truth of that proposition could receive. How we happened to discover the truth may have practically nothing to do with the content of the truth itself. But, following the

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9Though this passage was published in 1918, its composition dates to Wittgenstein’s boyhood (Frege 1979, 127).
accident of its discovery, we begin to see the grounds in nature upon which it rests. The kind of grounds to which this deepening grasp will lead, depends on the particular nature of that truth.

According to Frege, it is obvious that the most reliable kind of proving of a scientific truth would disregard all particular characteristics of objects, and depend only on laws on which all knowledge rests. It would follow only “pure logic, a way that, disregarding the particular characteristics of objects, depends solely on those laws upon which all knowledge rests” (5). As is well known, Frege strove to uncover such a source for the truths of number. This would yield a rigorous proof of the laws of number on the sole basis of maximally general laws of logic. The value of such a proof would not consist in its raising our subjective confidence in the truth of the laws of arithmetic. Rather, it would reveal their place in a natural justificatory order. Basic logical laws would thus themselves stand at the root of a science of pure reason, which develops upward through the expression of further logical truths and blossoms into the science of number. Thus, in devising his ideography, Frege had in mind “right from the start”, the “expression of a content” (Frege 1979, 12): the contents of general logical laws, of arithmetic theorems, and finally of the application of all these rational truths throughout the special sciences of astronomy, chemistry, biology and so on.

Wittgenstein’s response is to declare that the propositions of logic are tautologies. Now, when he wrote the Tractatus, the word “tautology” was not a technical term, or not much of one. It meant something like “redundancy”, or “pleonasm”\textsuperscript{10} So as he immediately explains, the point is that “the propositions of logic say nothing”, and that “all theories that make a proposition of logic appear to have content are false”\textsuperscript{11}. Of course, by the time Wittgenstein composed this passage, Frege had published his Begriffsschrift and Grundgesetze, and Russell and Whitehead had published the Principia. There was a firmly established method of “proving” logical propositions, i.e., of deriving them by rules of inference from initially given propositions. Wittgenstein acknowledges the existence of this method as a datum:

The proof of logical propositions consists in the following process:

\textsuperscript{10}See Dreben and Floyd (1991) for a fascinating discussion.
\textsuperscript{11}6.11, 6.111.
we produce them out of other logical propositions by successively applying certain operations that always generate further tautologies out of the initial given ones. (6.126)

This passage encapsulates Wittgenstein’s ambivalent regard for the logical achievements of Frege and Russell. On the one hand, for example, at 3.325 Wittgenstein urges upon philosophers the use of a “sign-language that is governed by logical grammar—by logical syntax.” And, he says, “the conceptual notation of Frege and Russell is such a language, though, it is true, it fails to exclude all mistakes.” Wittgenstein finds that Frege and Russell correctly aim at constructing a sign-language which does not just ape the whims of words but respects the forms of logical syntax—even if, in pursuit of this aim, they sometimes fall short. For example, he complains that Frege and Russell “introduce generality in association with logical product or sum”, which “made it difficult to understand the propositions in which both ideas are embedded” (5.521). And, “it is self-evident that identity is not a relation between objects” (5.5301); “Russell’s definition of ‘=’ is inadequate” (5.5302). And again, according to 4.1273, the way in which Frege and Russell want to express the ancestral “contains a vicious circle”. Thus, Frege and Russell sometimes falter in the matters of detail. But, Wittgenstein can only pick nits like this because there is something about their projects which he does find illuminating, or seductive—the analysis of logical structure.

Thus, for example, Frege’s discovery of the quantifier reveals something of the essence of propositions. For, as Wittgenstein puts it, “in a proposition there must be exactly as many distinguishable parts as in the situation that it represents.” He considers various alternative notations for generality, for example, a notation which would express the universal generalization of a formula $F \times$ by simply prefixing the letter $G$ to it. Such a notation lacks the relevant “multiplicity” achieved by Frege’s quantifier notation, and therefore must be inadequate (4.0411). One cannot, in the expression of generality, “get away” from the kind of representational structure that the quantifier notation makes explicit. For this reason, the quantifier notation reveals something in the nature of representation itself. The quantifiers embody some kind of important insight.

Here, one might wonder: in the nature of all representation? Or just in representation of generality? But, Wittgenstein held that much ordinary representation is covertly general. Thus, my
So, Wittgenstein thinks that the Frege-Russell program of analysis of logical structure is somehow on the right track. But I said, he regards their logical achievements with a kind of ambivalence. The basic reason for this is that they subordinate the analysis of logical structure to a further aim, the development of an ultimate deductive system which would form the organon of a new science. Frege and Russell, so it seems, devoted years of their lives to the senseless transformation of empty formulas into empty formulas by emptiness-preserving rules—this in spite of Frege’s emphatic insistence that, “right from the start” he had as his aim with ideography the “expression of a content.” Now, Wittgenstein certainly concedes that an ideography like Frege’s can serve for the expression of a content, and indeed, can so serve as an improved, elucidatory form of expression. But, at least to the extent that the underlying logical analyses are free of defects, a sentence will have a proof in a system of Frege or Russell’s only if it fails to express a content. So there is certainly nothing to be found along the lines of a new and central rational deductive science that Frege, following Leibniz, envisioned. It now looks like Wittgenstein’s reading of Frege is that he single-handedly invented the piano, and even tuned it pretty well, but never envisioned its use to make music. Such an interpretation of Frege looks not implausible but incoherent: one cannot invent a piano but as a musical instrument. Wittgenstein must find it puzzling how it could have been the logicists who “brought about an advance in Logic comparable only to that which made Astronomy out of Astrology, and Chemistry out of Alchemy.”

There must be something—if not to how things must be if a tautology is true—at least, to the fact that a proposition is a tautology, which explains how such facts could have so entranced the inventors of serious logic.

The fact that the propositions of logic are tautologies shows the formal—logical—properties of language and the world.

claim would be at least: quantifier notation reveals something in the nature of ordinary representation. One might also argue that Wittgenstein, following Frege, finds the articulated structure of any representation, general or not, to be coeval with the relationship of that representation to the generalities it instantiates. However, Wittgenstein’s understanding of this relationship is somewhat different from Frege’s, so I won’t press the point here.

13 This irony is noted by Juliet Floyd (Floyd and Shieh 2001, 154).
14 Wittgenstein 1913, 351. Thanks to Goldfarb for goading me into this question.
The fact that a tautology is yielded by this particular way of connecting its constituents characterizes the logic of its constituents. If propositions are to yield a tautology when they are connected in a certain way, they must have certain structural properties. So their yielding a tautology when combined in this way shows that they possess these structural properties. (6.12)

The propositions of logic demonstrate the logical properties of propositions by combining them so as to form propositions that say nothing. This method could also be called a zero-method. In a logical proposition, propositions are brought into equilibrium with one another, and the state of the equilibrium then indicates what the logical constitution of these propositions must be. (6.121)

These passages present Wittgenstein’s error-theoretic interpretation of logicism. Since Wittgenstein holds that logical truths are not truths at all, there cannot be such a thing as a science whose proper truths are the propositions of logic. Nonetheless, this notion identified by Frege, being a proposition of logic, must have some kind of logical significance which explains how it could have so entranced him. In the passages just quoted, Wittgenstein sketches a kind of inquiry in which interest in the property of being a tautology takes the form it deserves. The deserved form of this interest is the following: that a tautology results from combining some propositions in a certain way tells us about the structure of those propositions. Wittgenstein gives examples of this sort of inference not just in the not 6.12s, but also in the 4.12s and the 5.13s. That conditionalizing \( A \) on \( B \) yields a tautology tells us that \( B \) entails \( A \); that a tautology results by negating

\[ p \rightarrow q \text{ and } p \land (p \rightarrow q) \rightarrow q \text{ is a tautology.} \]

The Prototractatus contains no seagulls; Wittgenstein seems to have dredged them up from some very old notes, perhaps in a fit of sentimentality (the editors of KS (144) cite the youthfully exuberant letter to Russell of 1903 (number 28 in McGuinness, ed., 2008), whose claim to a decision procedure for quantificational logic is explicitly retracted in 6.1203’s first sentence). Thus, the Tractatus numbering of 6.121 is appropriate, since 6.121 is a commentary not on 6.1203 but on 6.12.

\[ ^{15} \text{Note that in the Tractatus, 6.121 comes in the wake of the seagulls of 6.1203, which once gave me the impression that “this method” refers to the flight of the seagulls. But, that is not correct. In the Prototractatus, the antecedent of 6.121 is a commentary on the antecedent of 6.1221, which simply remarks that it says something about } p \rightarrow q \text{ and } p \text{ that } p \land (p \rightarrow q) \rightarrow q \text{ is a tautology. The Prototractatus contains no seagulls; Wittgenstein seems to have dredged them up from some very old notes, perhaps in a fit of sentimentality (the editors of } KS \text{ (144) cite the youthfully exuberant letter to Russell of 1903 (number 28 in McGuinness, ed., 2008), whose claim to a decision procedure for quantificational logic is explicitly retracted in 6.1203’s first sentence). Thus, the Tractatus numbering of 6.121 is appropriate, since 6.121 is a commentary not on 6.1203 but on 6.12.} \]
the conjunction of \( A \) with \( B \) tells us that \( A \) contradicts \( B \).\(^{16}\)

So, in the 6.12s, Wittgenstein sketches out a distinctive distribution of knowns and unknowns that characterizes a certain kind of inquiry. What we don’t know, and what we want to investigate, are the internal structures of some propositions, say \( A \) and \( B \). On the other hand, we know how to combine \( A \) and \( B \) into a single proposition (5.131) whose sense is a function of the senses of \( A \) and \( B \) (5.2341). And, moreover, since a tautology must itself show that it is a tautology (6.127), we can tell whether or not a tautology results by combining \( A \) and \( B \) into a single proposition. Consider a simple case, in which the analyst combines queried propositions \( A \) and \( B \) into the assembly \( \neg(A \land B) \), where, it turns out, this assembly is a tautology. The analyst knows that negation of \( A \land B \) is true if and only if \( A \land B \) is false, that \( A \land B \) is true if and only if \( A \) and \( B \) are both true, and furthermore, that a tautology is always, in a degenerate sense, true (4.46ff). In this way, that \( \neg(A \land B) \) is a tautology shows the analyst that \( A \) and \( B \) are never both true, or in other words, that \( A \) contradicts \( B \). So, for example, the analyst might record that “Ludwig is a bachelor” contradicts “Ludwig is unmarried”. Then, “Ludwig is a bachelor” might be rewritten “Ludwig is not married, and …”, or “Ludwig is married” might be rewritten “Ludwig is not a bachelor, and …” where in each case, the ellipses remain to be filled in by considering some concepts of gender. This is the sort of result that might be extracted from the information that a given proposition is a tautology, and, according to Wittgenstein, such results are what makes the property of being a tautology worth of attention. Such results, I take it, are the results of what Wittgenstein understands to be logical analysis.

Of course, the example I just gave is implausibly simplistic. Less trivial examples are readily found, like “if there are at least three cows and at least four hens, and if no cows are hens, and if all cows and hens are animals, then there are at least seven animals.” This and related examples might lead to a quantificational analysis of number-words of the sort that Frege considers in §55 of the Grundlagen;\(^{17}\) eventually, though, other examples involving arithmetical generalizations might justify Frege’s rejection of the quantificational analysis in favor of something else. In any case, more interesting analytical problems are ready to hand.

\(^{16}\)4.1211; 5.13-5.131.

\(^{17}\)Frege 1953, 67.
Still, there is something rather strained in this discussion. Why is it necessary at all to combine the queried propositions into a single one and ask whether it is a tautology? Can’t we simply discern between the queried propositions themselves whatever internal relations guarantee the tautologousness of their combination? That is, isn’t enough just to consider the relations between the constituent sentences “there are at least three cows”, “…at least four hens”, etc.? Wittgenstein puts the point as follows.¹⁸

If the truth of one proposition follows from the truth of others, this finds expression in relations in which the forms of the propositions stand to one another: nor is it necessary for us to set up these relations between them, by combining them with one another in a single proposition; on the contrary, the relations are internal, and their ex-

¹⁸Proops (2002, 288) describes the quoted remark as “darkly metaphorical”, and spends much of (2002) “unpacking” it. Proops’ view is that 5.132 attacks, in Frege anyway, a “proof-theoretic” account of logical consequence that might be extrapolated from the account of dependence-between-truths of in “Foundations of Geometry II” (Frege 1984, 333ff). I think, in contrast, that Wittgenstein’s point is much more direct. Given that tautologies don’t describe the world, and so lack sense, why did Frege and Russell keep on asserting them? Being logicians, they begin with a natural desire to understand logical consequence (and other internal relations between propositions). But since logical consequence finds no direct expression in their systems, they therefore sublimate the desire to understand it into a sterile enumeration of tautologous conditionals. The reflection of internal relations between propositions in the tautologousness of logical combinations of those propositions gets mistaken, by the logicist, for interesting facts which tautologies report. In this way, the predilection for tautologies is just one among many examples of the confusion of material relations with formal ones “which is very widespread among philosophers" (4.122). I see the drift of the 5.13s to be smoothly continuous with the 6.12s and diagnostic of obsession with all kinds of tautology, including disjunctions, biconditionals, negated sentences, etc., and correspondingly finding the actually interesting underlying facts to be not just entailments but also contradictions, equivalences, etc. So, on my reading, it is only because Frege asserts so many conditionals, i.e., only because his system contains a primitive sign for conditional rather than for disjunction, conjunction, etc., that Wittgenstein focuses on logical consequence in the 5.13s. In a way, then, my view is somewhat closer to the view of Ricketts (1985), according to which Wittgenstein attributes to Frege and Russell confusion about the concept of inference rule—though I don’t take Wittgenstein to attribute a Carroll-style confusion to Frege or Russell.

Proops finds that Wittgenstein attacks a Fregean theory of entailment that a scholar might extrapolate from the “Foundations II” passage and one highly compressed remark in Grundlagen (at §17). But since Frege doesn’t think the general concept of entailment is very important, and perhaps even doubts its cogency, it is not clear why he should be much bothered by criticism of his account of it—so much the worse for your concept, he might reply. In contrast, I take Wittgenstein’s target to be the elephant in the room, that Frege (and Russell) keep on making senseless assertions. On my reading, 5.132 fits naturally in the thematic nexus of the 4.12s-5.13s-6.12s, which form a systematic response to logicism.
istence is an immediate result of the existence of the propositions.

(5.131)

So, Wittgenstein thinks that it is a theoretically important datum that a proposition is a tautology. Nonetheless, this datum is only important because of its membership in a broader family of logical phenomena. It is just as interesting that a proposition be contradictory, or that some propositions entail or contradict another. The logicists’ enumeration of tautologies is misguided not only because of the underlying false pretense that the tautologies are the truths in the organon of some new science. It is moreover misguided because tautologousness is just one logically important phenomenon among many others. Some such phenomena are also just properties of single propositions, like, for example, logical contingency; but others are not properties but relations between propositions, like entailment, or joint contradictoriness or joint satisfiability.

As I understand Wittgenstein, then, the analyst begins with an appreciation of internal relations between propositions, such as entailment, exclusion, and so on. This appreciation embodies a grasp of how the world must be, if a proposition is true. Analysis then seeks to clarify thoughts which are obscure and confused, by setting propositions which express them into relations with each other, and noting the logical properties of the results. Thus, analysis is a matter of construction. But it is construction on ordinary propositions, proceeding upward from our ordinary, human appreciation of the force of what we say, of precisely how we, in picturing the world, merely thereby hold the world to yes or no. Rather than speaking of analysis as “top down”, then, I’d speak of it as “top up”.

In the distribution of knowns and unknowns which characterizes the predicament of analysis, the logical structure of propositions falls halfway in shadow and halfway in sunlight. On the one hand, Wittgenstein seems to take for granted that the significance of logical combinations of already understood propositions is completely clear. Indeed, he flamboyantly pushes this assumption to the brink, or, it’s often complained, past the brink of tenability, declaring, for example, that “proof in logic, is merely a mechanical expedient to facilitate the recognition

\[19\] Cf., among many, Potter 2009, 191.
of tautologies in complicated cases” (6.1262). Thus, returning to our simplistic example, in order that the tautologousness of \( \neg(A \land B) \) count as evidence that \( A \) contradicts \( B \), Wittgenstein must take for granted a grasp of the rules for evaluating negation and conjunction, a recognition of applicability of those rules to the expression \( \neg(A \land B) \), and the capacity to detect logical consequences of such applications of the evaluation rules. Taking all this for granted, the part of \( \neg(A \land B) \) which, as it were, exists over and above \( A \) and \( B \) themselves, is completely transparent. The presumption of transparency of logical construction is to be justified by appeal to the character of ordinary speaking, thinking and understanding. In particular, supposing two people each to have asserted propositions, then a third person may deny that they are both right, by uttering the words “you are not both right”. These words do not just dump the speaker into some undifferentiated role of naysaying, but construct an intellectual position explicitly, so that from the construction everybody recognizes, modulo what the first two speakers said, what it takes for the third speaker to be right or wrong.

On the other hand, propositions are given to the analyst as only superficially articulated wholes. There is nothing on the surface of “Ludwig is a bachelor” and “Ludwig is married” to indicate why one entails the other but not conversely, for on the surface these have the same structure.\(^{20}\) As Wittgenstein rather quaintly puts it, ordinary language is not designed to reveal the forms of thought underneath it, any more than clothing is designed to reveal the form of the body (4.002). So, one cannot infer from the surface structure of the sentence “Ludwig is a bachelor”, the form of the thought it expresses. Analysis therefore proceeds by almost Baconian poking and prodding of its subject matter.

We’ve supposed that, given as subject-matter the two propositions \( A \) and \( B \), a logical analyst constructs the assembly \( \neg(A \land B) \). In this complex assembly, the occurrence of \( A \) and \( B \) is manifest: indeed, Wittgenstein goes so far as to speak of \( A \) and \( B \) as occurring in \( \neg(A \land B) \) as “constituents” (Bestandteile), even though this severely violates his logico-grammatical sensibilities.\(^{21}\) Moreover, it is not just the

\[^{20}\text{Wittgenstein does acknowledge, for example, that “Ludwig is a bachelor” carries something on its surface, which explains, say, how the result of conjoining it with “Ludwig is a philosopher” entails “some philosopher is a bachelor”. However, this surface structure will presumably disappear on analysis, giving way to some other structure which legitimates the same inferences and more.}\]

\[^{21}\text{Cf. “a proposition cannot occur ‘in’ another”, } \textit{MN}, \text{p.116.}\]
occurrence of $A$ and $B$ but the logical meaning of this occurrence which $\neg(A \land B)$ makes plain: $\neg(A \land B)$ is, as it were by construction, that proposition which is true precisely when not both of $A$ and $B$ are true. Wittgenstein officially introduces this relationship between the construction and its raw materials as follows:

The structures of propositions stand in internal relations to each other. (5.2)

In order to give prominence to these internal relations, we can adopt the following mode of expression: we can represent a proposition as the result of an operation that produces it out of other propositions (which are the bases of the operation) (5.21).

In the particular case we’re considering, Wittgenstein represents a certain proposition, $C$, as the result of negating the conjunction of $A$ and $B$, by expressing it in the notation $\neg(A \land B)$. The fact that $C$ can be so represented shows that $C$ stands in a certain internal relation to $A$ and $B$. Now, because the proposition $C$ is a tautology, this internal relation of $A$ and $B$ to $C$ points out an internal relation between $A$ and $B$ themselves, namely that $A$ contradicts $B$. According to 5.21, it ought to be possible to give prominence to the contradiction of $A$ by $B$, by presenting $B$ as the result of an operation on $A$. The obvious move is to try representing $B$ simply as $\neg A$, so that “Ludwig is a bachelor” would be written as “Ludwig is not married”. But, $B$ can be so represented only if the formula $B \leftrightarrow \neg A$ is a tautology, i.e., only if the following is a tautology: that Ludwig is a bachelor iff Ludwig is not married. But Ludwig might be divorced, or not of marriageable age, etc. So one cannot simply represent the situation in which Ludwig is a bachelor as the situation which obtains when he is not married. Rather, his bachelorhood consists of his not being married together with further conditions, say that he’s of marriageable age, but has never been married; let’s write these as $A'$. Now, granting that $B \leftrightarrow \neg A \land A'$ is indeed a tautology, then $B$ itself can simply be written as $\neg A \land A'$. This rewriting of $B$ in terms of $A$ and $A'$ is another instance, perhaps even a paradigm, of the method described at 5.21.

Of course, following tradition, we might want to acknowledge that the predicate, being a bachelor, entails more than just being of marriageable age yet unmarried; someone does satisfy these conditions without being a bachelor if she
is a “she”. However, what is given for analysis is not the predicate, being a bachelor, but rather just the whole proposition that Ludwig is a bachelor. There is not much record of the early Wittgenstein having developed any particular views on the metaphysics of gender. But it seems likely that he would have held that “Ludwig is male” is itself a tautology, i.e., analytic, since he probably held, for example, that “Ludwig is in the room” logically entails “a male is in the room”. Then he would likewise have held that “Ludwig is unmarried” entails “Ludwig is male”. So, in the analysis of the single, whole proposition “Ludwig is a bachelor”, the extra conjunct suggested by the traditional analysis would be redundant. We could have anticipated that the suggestion of tradition may not be immediately applicable, because whereas the tradition aims at analyzing predicates (say, into species and differentia), Wittgenstein aims at analyzing whole propositions.

We can see how Wittgenstein takes analysis to illuminate logical structure of the propositions analyzed. For example, we began with the datum that a conjunction \(\neg(A \land B)\) was a tautology, which is not explained by the overt structure of \(A, B\). The datum indicates the obtaining of internal relations between \(A\) and \(B\), and in turn the possibility of rewriting \(B\) as the result of an operation on \(A\) (along with some auxiliary operands). Upon replacing the expression \(B\) with its analyzed form \(\neg A \land \neg A'\) in \(B\), we obtain a formula \(\neg(A \land \neg A \land A')\) whose validity is now completely independent of the forms of its as yet unanalyzed constituents. Thus, rewriting \(B\) as \(\neg A \land \neg A'\) explains the originally given datum that \(\neg(A \land B)\) is a tautology. Of course, we aren’t yet done, since, for example, \(A'\) was constructed from propositions to the effect that Ludwig has never been married, and that he is of marriageable age; the internal relations in which these stand to the proposition that he is married, remain to be articulated.

Analysis therefore begins by observing an internal relation between propositions which is not explained by their overt structure. By hook or by crook,

\[\text{22}^{\text{In NB2, he certainly held, for example, that if } x\text{ is a part of } y, \text{ then } “} y\text{ is in the room” logically entails “} x\text{ is in the room” (cf. e.g. 17.6.15f) Moreover, he seems to have held that propositions about } x\text{ entail propositions about the configurations of the parts of } x.\text{ The acceptance of such entailments is clearest in the Notebooks, but it surfaces in the Tractatus, particularly at 3.24. Such principles, together with some not unquestionable assumptions about the metaphysics of gender, do yield the conjectured entailment from “Ludwig is in the room” to “a male is in the room”.}\]

\[\text{23}^{\text{Here, he follows the path cleared by Frege. See in particular T3.3: “only in the context of a proposition has a name a meaning”.}\]
the analyst finds a way to rewrite the given propositions in terms which do then explain the observed internal relation. Now, a typical initial datum will be that two propositions cannot both be true, or that one cannot be true unless another is, and so on. Moreover, pursuit of the desired rewriting is constantly guided by knowledge of obtaining and nonobtaining of internal connections as well: for example, such knowledge is needed to reject the hypothesis that to be a bachelor is simply to be unmarried. So, analysis takes as its data not what is true but what is possible: as Wittgenstein puts it, “logic deals with every possibility and all possibilities are its facts” (2.0121). Thus, supposing $A$ to be analyzable as the result $O(A_1,\ldots,A_k)$ of applying operation $O$ to some other propositions, then there are no possibilities in which $A$ is true but $O(A_1,\ldots,A_k)$ false, or vice versa. Conversely, once $O$ and $A_1,\ldots,A_k$ are chosen so that no possibilities discriminate between $A$ and $O(A_1,\ldots,A_k)$, then nothing remains to distinguish between the logical positions of $A$ and $O(A_1,\ldots,A_k)$ at all, and so they are the same position. Absenting global theoretical considerations, nothing impeaches the analysis of $A$ as $O(A_1,\ldots,A_k)$. In the long run, however, the overarching aim of analysis is to make all covert logical structure explicit in the forms of propositional expression. This is a global program that addresses internal relations between all propositions; it is this responsibility to the global structure of language as a whole which determines what is to be analyzed in terms of what.

So, Wittgenstein conceives of analysis as introducing logical structure to explain antecedently grasped possibilities and necessities. Necessity, then, is the basis of logical structure. But now, the following is clear: Wittgenstein simply has no room for any kind of necessity but logical necessity. Necessities unexplained by logical structure can only indicate that analysis is incomplete. This result, that “there is only logical necessity” (6.37), is the reason for Wittgenstein’s account of propositions as pictures. But more on that in the next section.

1.1.2 What about reality?

As I’ve just argued, Wittgenstein’s conception of analysis leaves no room for any but logical necessity. But this doctrine is double-edged.\textsuperscript{24} It may look at first like

\textsuperscript{24}Thanks to Roberta Ballarin for this way of putting the point.
a purification of the concept of necessity, but at the same time it entangles the application of logic with metaphysics. For, as Wittgenstein acknowledged, the world is full of necessity.

A spatial object must be situated in infinite space. (A spatial point is an argument-place.)

A speck in the visual field, though it need not be red, must have some color: it is, so to speak, surrounded by color-space. Notes must have some pitch, objects of the sense of touch some degree of hardness, and so on. (2.0131)

A property is internal if it is unthinkable that its object should not possess it.

(This shade of blue and that one stand, eo ipso, in the internal relation of lighter to darker. It is unthinkable that these two objects should not stand in this relation.) (4.123)

[...] the simultaneous presence of two colors at the same place in the visual field is impossible, in fact logically impossible, since it is ruled out by the logical structure of color. (6.3751)

The next sentence of this entry is “(Here the shifting use of the word ‘object’ corresponds to the shifting use of the words ‘property’ and ‘relation’.).” From 4.122, it’s clear that by “shifting use” of “property” and “relation”, he means the “confusion between internal relations and relations proper”, as between internal and genuine properties. Since each of “internal relation” and “relation proper” expresses a formal concept, the slide is a confusion between two different formal concepts. Thus, the corresponding shift in use of the word “object” ought also to be a confusion between two different formal concepts. Now, Wittgenstein holds that the concept of simple object and the concept of complex object are distinct formal concepts. A sharp statement of this point occurs in 1913:

Russell’s ‘complexes’ were to have the useful property of being compounded, and were to combine with this the agreeable property that they could be treated like ‘simples’. But this alone made them unserviceable as logical types, since there would be significance in asserting, of a simple, that it was complex. But a property cannot be a logical type” (NL 100-101).

So, I conjecture that Wittgenstein has in mind at 4.123 a confusion between the use of “object” to refer to simples and its use to refer to complexes. (Such confusion will not be unfamiliar to students of NB2.) The objects mentioned at 4.123 and 2.0131 presumably support the necessities they do because they are not really “objects” in the sense of 2.02 (“objects are simple”); rather they are complexes, so that statements about them stand in internal relations to statements about their constituents (2.0201, 3.24). So, the last sentence of 4.123 suggests that the necessities cited in the text are purportedly genuine, rather than just metaphorical.
We are now in a position to summarize these results in an inference. Since the world is full of necessity, and since there is no necessity but logical, therefore logic fills the world.26

I have been promising for some time to explain why Wittgenstein thinks that propositions are pictures of reality. The picture theory is largely driven by two convictions about the nature of the logic. First, logic illuminates internal relations between propositions, internal relations which weave together all susceptibility to truth and falsehood. But second, because internal relations subsume the necessities of the world, therefore the world itself must somehow be full of logic. From these two convictions, Wittgenstein is driven to the conclusion that internal relations between propositions must reflect the necessities and possibilities of the world. It is by virtue of a likeness of propositions to the world that the necessities and possibilities of the world can be grasped through understanding of internal relations between propositions. This likeness of propositions to the world is the pictoriality of propositions: “sign and thing signified must be alike with respect to their total logical content” (4.9.14). Thus, Wittgenstein aims with the doctrine of pictoriality of propositions to secure an identity between the logical space of the truth and falsehood of propositions in language and the ontological space of the obtaining and nonobtaining of possible situations in the world.

One route to unifying the spaces of propositions and of situations was already trod by Russell:27 that is: just identify the propositions with the facts, so that a proposition simply is the fact that would obtain were it true. Suppose that a proposition says how things are, by specifying a way for things to be, and specifying which things are that way. The simplest way to do this is for the proposition simply to consist of those things, being that way. But then, a proposition cannot say how things are unless things are that way. So, a proposition cannot consist of the things which it’s about, standing in the relation in which the proposition represents them to stand.

26Wittgenstein does say at 5.61: “Die Logik erfüllt die Welt”. But he surely intends more in that passage than I will have to offer today.
27Having been blazed by Moore (1899).
Nonetheless, as the conception of analysis drives Wittgenstein to insist, there must be something in common between propositions and situations. Wittgenstein appears to settle for the next best thing to Russell’s identity theory. A proposition says of some things that they are in a certain way, by being some things, standing to each other in just that way—only in the proposition, we find not the constituents of the fact standing for themselves, but some other items going proxy for them. These proxies, or ultimate propositional constituents, are what Wittgenstein calls names. So, as Wittgenstein puts it:

In a proposition a situation is, as it were, constructed by way of experiment. [...] (4.031)

One name stands for one thing, another for another thing, and they are combined with one another. In this way the whole group—like a tableau vivant[lebendes Bild]—presents a state of affairs. [4.0311]

Wittgenstein thus conceives of propositions as pictures or models of reality. As I claimed, what motivates this conception is its potential to secure a unity between the spaces of propositions and of possibilities. For, on Wittgenstein’s view, it is part of what makes something a picture that the possibility of the picture be, somehow, at one with the possibility of what it depicts. Wittgenstein’s introductory remarks on the representational power of pictures make this motivation explicit:

That the elements of a picture are related to one another in a determine way represents that things are related to one another in the same way.
Let us call this connection of its elements the structure of the picture, and let us call the possibility of this structure the pictorial form of the picture. (2.15)

Pictorial form is the possibility that things are related to one another in the same way as the elements of the picture. (2.151)

These sentences are pretty fraught with verbal ambiguities. They do have the following ontological counterparts:
The determinate way in which objects are connected in a state of affairs is the structure of the state of affairs. (2.032)

Form is the possibility of structure. (2.033)

2.15b and 2.032a strike me as ambiguous between explaining “structure” as either the relation in which the elements of the complex stand to each other, or the circumstance that those particular elements stand in the relation in which they stand. The subsequent talk of the “possibility of structure” slightly favors the second reading, since it is not really what it means for a relation to be possible, unless this is understood as the possibility of its instantiation. So I take “structure” to be tantamount to “picturing fact”. This interpretation is also supported by Wittgenstein’s later usage of “structure” as for example in 4.1211, 5.13, etc.28

For example, the structures of $fa$ and $ga$ can show that $a$ occurs in both only if the structures involve $a$ itself. But, Wittgenstein also apparently intends that the sort of things shown by the structures of facts (or pictures or propositions) depend only on cross-reference to those things between the facts, rather than on the identities of the things—this is what makes them structural. So, the structure of a fact must involve elements of the fact in such a way as to make sense of the question whether the structures of a variety of facts have elements in common. But there is no need to suppose anything more to the involvement of elements in structures than what is required to decide whether and how various structures have elements in common.

Following the introduction of the notion of structure, 2.15b and 2.033 introduce a corresponding notion of form, as the possibility of structure. Accordingly, in the case of a state of affairs, form is the possibility that the elements of a state of affairs do stand in the relation in which they therein stand, and likewise in the case of a picture, form is the possibility that the elements of the picture stand in the relation in which they stand therein. An obvious interpretation of these remarks is that form is what makes structure possible. This interpretation is naturally elaborated by example. A structure might consist of a golf ball sitting in the hole of a donut. The golf ball and the donut are so shaped as to explain, jointly, that as it turns out, one fits the other, i.e., their two shapes then jointly

28 As pointed about by Ramsey (1923, 466).
explain the possibility of the structure. But, note that the possibility is explained only by the two shapes together. So, although the donut turns out to accommodate the golf ball, there is nothing to the donut in itself which would presage their graceful union. Wittgenstein sharply rejects such an understanding of objects: “if a thing can occur in a state of affairs, the possibility of the state of affairs must be written into the thing itself” (2.012). Thus, for Wittgenstein, it is not the case that objects have, on their own, forms which, considered alongside the forms of other objects, only then explain possibilities of structure.

Citing 2.012 and related passages, Michael Kremer remarks: “we seem to be driven inexorably towards a holistic reinterpretation of the picture theory, according to which it is not individual propositions but the whole system of language which ‘pictures’ the world” (1992, 421). Thus, although a proposition must, in order to picture reality, share its logical form with reality, it is not to the single proposition, but to the whole of language, that logical form should be ascribed. For Kremer, language as a whole is a “system of signs governed by rules of application”; these rules of application of signs are what constitute the form of language as a whole. Now, according to Kremer, we need an account not just of logical form of language as a whole, but, also, derivatively, of the logical forms of individual propositions. The logical form of one proposition will be given by its characteristic place in the “overall network of propositions”. For example, a disjunction might occupy a place as it were above each of its disjuncts, below each of which would be the conjunction of those two disjuncts. On the other hand, reality itself also has a logical form, in which situations are arranged with respect to relations of necessitation, exclusion, etc. A sameness of form between language and reality then induces a correspondence between propositions and situations; a proposition then depicts the situation to which it corresponds.

On this account, logical form, which Kremer tends to identify with logical multiplicity, is still what something must share with reality in order to represent reality. A system of notation, like those Wittgenstein scrutinizes at 4.0411, may “lack the necessary multiplicity to mirror the logical structure of the world, by lacking the wherewithal to reflect certain logical interrelationships between situations reflected in the standard notation” (1992, 422). Kremer’s account of multiplicity, like that of the Hintikkas (1986), appears to presuppose that there
is such a thing as the multiplicity of a system of notation, such that, with respect to this multiplicity, we can raise the question whether it is adequate to mirror reality. Kremer and Hintikka agree that being formally adequate to reality is a condition on language. Of course, Kremer differs from the Hintikkas over relative priority of the loci of form within language: the Hintikkas think that it flows from the parts to the whole, whereas Kremer thinks it flows in the other direction.²⁹

Apparently in elaborating the holistic nature of logical form, Kremer goes on repeatedly to compare language as a whole with a picture (421,422,423). It is not at all clear to me how to understand this comparison. Saying proposition is a picture entails distinguishing in the proposition the form it must share with reality, from the particular modification of that form which makes something that proposition. The former is that in virtue of which a proposition is susceptible to truth and falsehood, the latter is how reality must be if the proposition is true. But then what is it to call language as a whole a picture? What are the poles of truth and falsehood for language as a whole? What does reality have to do to say yes? I can’t see any other answer here than: that language and reality have the possibilities of agreement and disagreement with respect to their forms. Language as a whole depicts reality to be in a certain way—only, the kind of way that language-as-a-whole depicts for reality is a super-way, perhaps the structure of the system of coordinates at which the possibilities of obtaining and nonobtaining of states of affairs take their place.³⁰

In the context of this shift to holism, Kremer cites the following famous passage:

How can logic—all embracing logic, which mirrors the world—use such particular hooks and manipulations?³¹ Only because they are all connected with one another in an infinitely fine network, the great mirror. (5.511)

²⁹McCarty (1991, 52-53) warns against this hazard of “top-down” approaches to the Tractatus.
³⁰More in §1.3 on “pre-pictorial” structure.
³¹Pears-McGuinness translate Haken und Manipulationen as “crotchets and contrivances”, which I can’t do with a straight face. It may be that Wittgenstein alludes here to a means of weaving or crocheting together of the infinitely fine network mentioned in the next sentence.
³²This translation is not in Kremer.
There is a puzzle about this passage which Kremer doesn’t mention. A map, for example, can be evaluated for its adequacy by comparison with the region it purports to depict. This is because the map is a self-standing image, a structure which exists independently of that with which it is to be compared. On the other hand, a mirror is a piece of glass, one side of which is covered with a reflective material. The glass and the underlying reflective material do not themselves constitute a structure, such that the mirror is then to be evaluated for adequacy based on the resemblance of this structure to, say, Ludwig’s face. That is rather what we would say of something like a portrait. The mirror in itself is blank. The very image of “mirroring” undermines the conception of the relationship between language and reality which Kremer and the Hintikkas use it to illustrate.

Sullivan (2001) squares up to the problem that is bothering me. To restate: the problem is just, why do the logical properties of propositions have anything at all to do with the material necessities of the world—let alone, in their structure, reflect those material necessities completely? Or as Sullivan puts it: “what connects our picturing-propensities with the space of possible worlds?” (9). I take him to agree that part of the point of the metaphor of picturing is to gesture toward some connection of our saying-propensities and the space of possible worlds. But does the metaphor of picturing lead to any insight here? Or is it just palliative care?

Sullivan helpfully emphasizes Wittgenstein’s insistence on the connection between pictures and possibility. Thus he cites these passages:

The picture contains the possibility of the state of affairs it represents. (2.203)
The thought contains the possibility of the situation of which it is the thought. What is thinkable is also possible. (3.02)
The proposition determines a place in logical space. The existence of the logical place is guaranteed by the existence of the constituent parts alone, by the existence of the significant proposition. (3.4)

Following 3.4, it becomes clear in 3.41 that by “constituents” Wittgenstein means constituents of the proposition, and these are presumably names. So it is tempting to fall back on the idea that type-preserving referential links between names
and objects guarantee that every sentence does pick out a place in logical space (see 4.0312). But, for Sullivan, that referential links should by themselves secure coincidence of possibility seems like “dubiously grounded confidence” (12). Sullivan then suggests that in relying only on reference here, we omit any role for the notion of pictorial form. Thus, rather than accepting at face value 2.15’s apparent verbal stipulation that pictorial form just is the possibility of the picture, the suggestion is that pictorial form takes on some explanatory burden of its own.

This suggestion strikes me as exegetically plausible. Wittgenstein’s talk of form alludes to Kantian terminology of “forms of intuition”, apparently in particular to the rhetoric that ‘we’ cannot represent objects of intuition but as subject to the conditions which inhere in that form of intuition. This is pretty explicit at 2.0121: “Just as we are quite unable to think spatial objects outside of space, or temporal objects outside of time, so too there is no object that we can imagine excluded from the possibility of combining with others.” Such overtones are too strong to ignore, although their purpose is another matter.

Sullivan suggests that we should “think of the pictorial form of a picture as constituted by the form of its elements” (13). The elements of the picture, being significant expressions in a language, are therefore subject to distinguishing rules of use, which determine their potential for combining into propositions; this constitutive combinatory potential of an expression is its form. Any particular proposition is then an “actualization of the possibilities built into the forms of its constituents” (14). If, then, the combinatory potentials of names line up with the combinatory potentials of the objects for which they stand, then a proposition, actualizing as it does the the potentials of the constituents, would therefore guarantee the potential in objects for a corresponding reality. But why, Sullivan continues, should we assume any such alignment? Sullivan proposes that the alignment is simply a condition of picturing, and the hypothesized alignment cannot be a mystery—although he does then call it a “demand”, and starts looking

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33 I do not think that the Kantian rhetoric here should be taken to suggest that Wittgenstein is some kind of Kantian: Wittgenstein might well be turning Kantian rhetoric against Kant, as he clearly does at 6.233. I hope to investigate these matters elsewhere.

34 This formulation strikes me as both too much and too little. Too much, because the possibility of the proposition is already built into any one of its constituents. Too little, because some relations are neither symmetric nor asymmetric.
for motivation.

Sullivan uncovers a motivation for this demand which reveals some unease with his account of pictorial form as constituted by the forms of pictorial elements. On his official account of pictorial form, a spatial picture containing a red toy car and a blue toy car might have a different form from a spatial picture containing a red toy car and a red toy stop sign: stop sign and car have different combinatory potential. However, this official account seems to sit poorly with the text. The text seems to imply that all spatial pictures have one and the same form: the form of space. And Sullivan clearly registers this implication. A spatial picture, Sullivan acknowledges, may be made up of ordinary spatial objects, spatially arranged. Thus, as ordinary spatial objects, these elements—say, toy cars, have forms of objects, namely space, color, mass, etc. But, the forms of the picture and its elements, qua bald facts and objects, must be distinguished from the picture’s properly pictorial form. In a spatial picture, the toy cars go proxy for other items in reality, say cars. On the other hand, other parts of the picture do not go proxy for things distinct from themselves in reality, and yet contribute to the pictorial form: in the case of a spatial picture, the spatial arrangement of the toy cars simply is (modulo scaling factors) the arrangement of cars depicted. Finally, some aspects of the picture, say, that the toy cars are made of plastic rather than wood, have no pictorial significance at all. What is essential to the picture (to what Wittgenstein calls the “symbol”) is just the role of the pictorial elements to go proxy for objects, and those properties of the pictorial elements which the picture asserts of the corresponding objects.

Now, the crucial point for my purposes is this: merely in seeing something as a picture, we already discern in it some parts which go proxy for objects, and some properties of those parts which the picture asserts of those objects. Such power of discernment is prior to seeing something as a picture at all. Seeing something as a picture requires decomposition into constituent names and their assertoric assembly. Spatial pictures are unified by the existence of a single power

\[\text{Note that, of course, a spatial picture may also be, for example, a color-picture, if the colors of the elements of the picture are intended to say that the corresponding objects are so colored. But, a spatio-colored picture is at the same time a (merely) spatial one, and can therefore be grasped merely as such, by taking the colors to go proxy for themselves.}\]
of discernment, which is necessary and sufficient for seeing all spatial pictures as pictures.\(^{36}\) This power is the grasp of the spatial form of picturing. Wittgenstein holds that mere grasp of the spatial form of picturing suffices to see what is shown by any spatial picture, i.e., suffices to see in the picture how things must be if it is true. The basis of this conviction is that the spatial form of picturing just is spatial form, it is what is common to every spatial possibility, to every determination of space. Thus, an identity between all spatial pictures and all spatial possibilities underlies the capacity to see what a spatial picture shows, i.e., to see how things must be if a spatial picture is true. Conversely, it also underlies the converse capacity to see the picture in the possible fact.\(^{36}\) In sum, it underlies the capacity to picture spatial facts to ourselves.

We’ve been struggling with the question how what can be spatially pictured is just the same as what can happen in space. The tempting answer is that something about spatial pictures out to coincide with something about space. But, this answer is wrong. For it presupposes that there is something like a way that a purported spatial picture is, or a way that all purported spatial pictures are, which even makes room to wonder about a coincidence of possibilities. And there is supposed to be no room to wonder here. But why not? I think it may help to switch metaphors:

A gramophone record, the musical idea, the written notes, and the sound-waves, all stand to one another in the same internal relation of depicting that holds between language and the world. There is a general rule by means of which the musician can obtain the symphony from the score, and which makes it possible to derive the symphony from the groove on the gramophone record, and using the first rule, to derive the score again. That is what constitutes the inner similarity between these things which seem to be constructed in such entirely different ways. […] (4.014)

The possibility of all imagery [Gleichnisse], of all of our pictorial

\(^{36}\)This is an oversimplification. An accurate, though less evocative statement is this: the form of a picture consists in a single power of discernment, which is necessary for seeing all pictures of that form as pictures.

\(^{37}\)After all, picturing is a two-way street.
modes of expression, is contained in the logic of depiction. (4.015)

Recast in terms of the musical metaphor, our questions becomes something like this. What is the guarantee, not that a performance render the score faithfully, but just that the score can be rendered at all? Couldn’t there be an intrinsically unplayable musical manuscript? Conversely, couldn’t there be a musical performance which could not be written down? Could there be an inexpressible musical thought? These questions come into focus when we set aside the gramophone, and consider, say, 19th century European art music. Would it be possible to hear a Mahler symphony that could not be written down? Could Mahler write down a symphony that could not be played? Could you play one that couldn’t be heard? But now, as should be clear, the questions are getting silly. Their answer is something like "maybe, sort of". In one degenerate sense or another, there might be an unplayable score, but it would be a score only, as it were, by accident (hence the degeneracy). The connections between the realms of musical composition, and performance practice, and audience appreciation, all belong to the nature of (19th-century European art) music. The general unification of these media by itself precedes, and makes possible, any particular activities in one medium or the other. The different formats of music agree with each other, falling apart into discrete nodes with matching combinatory potentials, because they have a common form. This common form is (19th century European art) music. Without music, a score could be at best a piece of (20th century) graphic design. It would be logically possible, of course, to write a computer program which translates the patterns of the piece of graphic design into sound (say, as conceptual art). But the resulting structural resemblance between the ink patterns and the sounds would not be the internal relation between score and performance. In other words, to the question, how could the existence of a musical score guarantee the possibility of its performance, the answer is: the unity of all scores and performances precedes the isolated possibility of existence of any particular score or performance. There is no such thing as a score, unless there is such a thing as (19th century European art) music; nor is there such a thing as a performance. For otherwise: a score, or a performance, of what?
The previous paragraph is of course too picturesque to be really philosophical. But the basic point can be put a little more dryly: the unity between any given picture and the situation it depicts is prior to the articulation of anything as that picture. For, the possibility of the situation is already contained in the mere possibility of that picture. Or, better, these possibilities are one and the same. What makes both picture and situation possible is logical form, or the form of reality. The form of reality itself, prior to any of its accidental modifications, already underlies the unity between the attributes in which the modifications take place. So, the proposition is articulated, because it, like any possible thought, participates in logical form and thereby stands at one with the situation it depicts. A proposition is articulated, while a merely typographic entity is not. For a proposition to be articulated just is for it to participate in logical form. And to participate in logical form is to submit to the rule whereby a situation can be read off. It is likewise by participating in logical form that a world falls apart into facts to be read off in propositions. In slogan form: there is no horizontal falling apart without vertical unity.

1.1.3 Sign and symbol

I now want to clear up certain ontological or nomenclatural questions about linguistic expressions. Wittgenstein famously distinguishes between “sign” and “symbol”. The subtleties of this distinction are obviously implicated in difficult interpretive questions about the nature, or lack thereof, of nonsense, senselessness, and so on. I can’t do justice to these questions here, nor to the voluminous literature they’ve engendered. For present purposes, my interest in Wittgenstein’s account of linguistic expressions turns on the widespread attribution to Wittgenstein of a conception of logical truth as somehow basically “linguistic” in nature. I will aim to suggest, though certainly not prove, that Wittgenstein’s notion of a linguistic expression undermines that attribution.

It may be helpful to distinguish two kinds of questions. First, we might ask what is the Tractatus ontology of linguistic expressions. Second, we might ask
how various terms in the *Tractatus* are used in different contexts to pick out concepts in the ontology. After all, it is entirely possible, and I think, almost certainly true, that while Wittgenstein’s ontology is fairly clearly worked out, the usage of terminology is somewhat uneven.

As I argued in §1.1.2, seeing some bald circumstance as a picture presupposes some grasp of a general system or form of picturing. This claim applies in particular to occurrences of sentences of a natural language. That is, what embodies such an occurrence is, for all that’s sensibly perceptible, just a bald circumstance, and seeing it as a picture of reality requires a grasp of logical form. Similarly, a grasp of logical form is just as well required to see in some piece of reality which intentionally articulated actions depict it. In general, then, a bald circumstance “falls apart” into a picture. The picture itself is a nexus or hanging-together of names; it represents that the objects of the names stand to each other as the names stand to each other in a picture. This hanging together is what I’ll call the pictorially significant commonality between the picture and depicted situation, or, for short, the “common cement”.

Some kinds of picture share pictorially significant commonalities with what they depict which are material: for these material commonalities to be pictorially significant is for the picture to represent its objects as instantiating them. Thus for example, in a spatial picture of a cup on a table, the contiguity of the cup-image and table-image is a pictorially significant material commonality because the picture thereby represents the cup and image as instantiating contiguity. In contrast, logical pictures share no pictorially significant material commonalities with what they depict. Their purely logical “common cement” is a purely logical connection between their constituents (presumably something like instantiation of one by others).

Among the various bald circumstances that turn out to obtain, some of them constitute utterances or inscriptions. Such circumstances fall apart into articulated pictures of reality, and in particular, into pictures which are purely logical. The purely logical pictures into which utterance/inscription/etc.-constituting circumstances fall apart might be called sentences, propositions, or propositional signs. Now, as I’ve argued, it is essential to such a logical picture to enjoy a formal (or “vertical”) unity with a situation which is thereby the depicted situation. But,
just as nothing counts as performing one musical score unless that score is one
among many, likewise there can be no single isolated moment of picturing: as
soon as one picture is possible, many pictures are possible too. So, one circum-
stance can’t appear as a sentence unless lots of possible circumstances do so too:
for example, nothing can appear as a sentence until something else can appear as
that sentence, and something else can appear as its negation, and so on.

As a matter of psychological fact, however, it is not the case that endlessly
many merely possible circumstances could, in the fallible human eye, all at once
fall apart into pictures of distinct though logically related situations unless the
circumstances were constructed somehow systematically. How in practice we
manage to achieve this humanly necessary systematicity is a matter of empirical
psychology or anthropology. Certain unsurveyably complicated understandings
are involved already in securing that different circumstances are occurrences of
the same sentence. Of course, we need conventions not just to determine a same-
sentence relation, but also to determine when circumstances belong to logically
related sentences, i.e., sentences depicting logically related situations. Here is
where the need arise for the “hacks and manipulations” that Wittgenstein men-
tions at 5.511, logical signs like $\neg$, $\lor$, $\exists$, etc. These work together with repeated
exploitation of a same-sentence relation between circumstances to make psycho-
logically possible the simultaneous appearance of a whole infinite multiplicity of
logical positions. The resulting systematically structured realm of possibilities of
occurrence of sentences is what Wittgenstein calls a system of signs.

Much of the problems interpreting Wittgenstein’s nomenclature arise through
difficulties navigating the complicated structure of the system of possibilities of
occurrences of sentences (together, of course, with the unevenness of his usage).
But, relying on matters of empirical fact, it is possible to distinguish various
natural regularities in the structure. First, for example, it is to some extent ar-
bitrary which circumstances should be put forward as sentences, so in practice
there arises a multiplicity of distinct systems of sentencehood—English, French,
German, Chinese, etc. Consequently, some possibilities are given as sentences
according to English, others according to French, and so on, depending on the
intentions and history of the author. Wittgenstein sometimes distinguishes be-
tween sentences as sentences of one language as opposed to another and accord-
ingly requiring translations between them (4.025). But there are other distinctions which cut across languages. For example, languages exploit various systems of material regularities and internal relations between elements of these systems in order to secure that one and the same picture of reality can be, say, whispered to a lover, carried around in the pocket, stored in data warehouses in Colorado. Here Wittgenstein distinguishes between sentences that are written, versus spoken, etc (4.011). Yet another dimension, again somewhat orthogonal to the others, comes from the fact that the hacks and manipulations psychologically required for the expression of infinitely many distinct but logically related positions issue in redundancies so that structurally different patterns of hacking and manipulating converge on the same picture of reality. Here Wittgenstein is very fond of remarking that \[ p = \neg \neg p \] (4.0621, 5.41, 5.254). These three kinds of system, as I said, are at least somewhat orthogonal, so one could in principle construct from them several different notions of “type”. I don’t think that much philosophical profundity will be achieved by insisting that Wittgenstein must always mean one versus another of them. Taken flatfootedly, the passages cited in this paragraph together yield the amusing position that sentences are different in different languages, different if spoken or written down, but the same if one is the double negation of the other.

In fact, the situation is even more complicated still. As an anthropological fact, it is possible for human beings to pretend that some circumstance falls apart for them into a sentence, even though it does no such thing. For example, an actor in a play may sit in an armchair turning over some sheets of newsprint with sentence-like ink markings, and thereby act out the reading of a newspaper, although, its being (let’s suppose) not really a newspaper but only a prop, he can be doing no such thing.39 Now, such pretenses are intelligible, and so, it seems, we must have some concept not just of sentences but also of sentence-likenesses. For example, it is clear that (under our supposition) the ink markings on the actor’s prop are not really sentences, any more than the piece of plastic in his

39It has been speculated that such performances sometimes transpire in coffee shops, libraries, or philosophical seminars. It has even been speculated that it is possible to carry out such a performance without realizing this oneself. I do not advance any such speculations in the course of this thesis.
pocket is really a gun. Rather—and here we need to spell out the story a bit
more—they may be just blocks of randomly printed characters. This, of course,
can be discovered on closer inspection.

Thus, there are things which, in some sense appear to be sentences, yet which
turn out, after all, not to be sentences. But, this example is slightly far-fetched,
since actual newspapers make good props as newspapers. The example doesn’t
show that there is an interesting kind of thing, which we might call a “toy sen-
tence” (or “mock sentence”) in the way that there is indisputably the kind of
thing we call a toy gun. It shows only that sometimes, something seems to be
a sentence yet turns out not to be one. There is no general license to suppose
that, for a given kind of thing, there is another kind of thing, namely, the kind of
thing which seems like a thing of the first kind. Actually, two different posits are
conceivable here. One is the posit of a kind $K'$ such that things in $K'$ merely seem
like things in kind $K$ but actually aren’t; and the other is a posit of a kind $K''$ such
that things of kind $K''$ either are, or seem to be, things of kind $K$. Let’s call $K'$
the pseudo-$K$s, and $K''$ the quasi-$K$s. Are there such things as pseudosentences,
or a quasisentences?

Some commentators have supposed that the notions of pseudosentence or
quasisentence plays an important role in the Tractatus. For example, Michael
Kremer (1997, 98ff) argues that the very Tractatus usage of “sign” (Zeichen) gen-
erally refers to things belonging to a kind of quasisentence. More specifically,
Kremer argues that the usage of “sign” in the Tractatus should be explicated in
terms of a notion sign-design which he borrows from Wilfrid Sellars. Roughly,
given a printed inscription of a sentence, its associated Sellarsian sign-design is the
shape of the inked region (or, perhaps, the shape of the associated region of the
typeblock). Thus, a Sellarsian sign-design is a shape, and a phenomenon instanti-
ates a sign-design by in some sense “having” that shape. Now, it is clearly true that
under certain conditions, something can seem to be a sentence if it instantiates a
Sellarsian sign-design. It is perhaps not true under every condition—for example,
subterranean crystalization of boron, or libidinous meanderings of pond scum,
might in some sense “have” the shape that is the given sign-design, yet fail even
possibly to seem to be sentences. Of course, it may be that the shapes of sign-
designs are so sharply fitted to an actual occurrence of a sentence that the given

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sentence is practically guaranteed to be the thing that instantiates it. But then the
tingates that instantiates it is a sentence, not something that only seems like one.

The force of the analysis of signs as Sellarsian sign-designs is that there exists a
class of shapes such that, to seem like a sentence just is to have a shape in that class.
Now, in some sense, of course, such a class of shapes may platonistically exist.
Suppose it does. The resulting notion of quasi-sentence is that it has a shape in this
class. When, however, do two things belong to the same quasisentence? When
they have the same shape? But then, for example, it is impossible to transcribe
a quasisentence, or read it aloud, and so on. The concept of shape just doesn’t
bend in the right ways. The notion of Sellarsian sign-design doesn’t illuminate
the notion of sign in the *Tractatus*.

Wittgenstein does write, however:

A sign is what can be perceived of a symbol. (3.32)

So one and the same sign (written or spoken, etc.) can be common
to two different symbols—in which case they will signify in two dif-
ferent ways. (3.321)

A full interpretation of this passage runs beyond my purposes here. Nonetheless,
3.32 identifies a concept of sign which would organize occurrences of sentences
by perceptual similarity. It is presumably this identification which inspires the
Sellarsian analysis. However, that analysis rests on pretty implausible psychol-

ogy. Perception of language seems to be very special. For example, one can mem-
orize the contents of a document without ever noticing whether it is set in serif
or sans-serif. For that matter, it is possible to know what someone said, to the
word, without knowing whether this was uttered or written. A reasonable treat-
ment of “signs” in the sense of the 3.32s should, at the very least, follow a notion
of perceptual similarity which is characteristic of language processing rather than
appreciation of graphic design. Imagine, for example, a game of telephone where
instead of speaking people transmit the message through all different kinds of
formats—by typed document, flash drive, megaphone, semaphore, etc. Then, I
propose as a first pass, occurrences of sentences are occurrences of the same “sign”

\[^{40}\text{At least, such is the ideal of typographers.}\]

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if one is liable to lead to the other in such a game. Of course, this proposal has its defects, since the relation is not even symmetric—for example, some things are much more likely to be misheard than to be mishearings. But, the right account of “sign” in the sense of the 3.32s may result by some such extrapolation from 4.014.

The resulting notion of quasisentence, the “sign” of the 3.32s which so exercises Kremer, is linguistically epiphenomenal. As Wittgenstein insists in the 4.01s, what makes a something a picture of reality is the rule, or law of projection, by means of which it can be not just transmitted from the mouth of one person to another, or from a mouth to a pen, and so on, but also that it can be projected into the world itself, and back out from the world into language. It is laws of projection not just within language but between language and the world which distinguishes a sentence from a mere stone or a leaf. As a matter of anthropological happenstance, these laws of projection can be traced out in a richly complex tissue of practices. A proper part of the tissue, or perhaps one should say, an organ of the organism, may consist in, for example, a relay between different tokens of an electronic document. Such organs can be misaligned with respect to the patterns of sameness and difference they serve to maintain. The notion of “mere sign” serves only to draw attention to the possibility of such misalignments—it pertains, as it were, not to life but to sickness. Mere signs, like sicknesses, are privations, and they have no nature themselves. Just as for Aristotle, anyway, there is nothing that it is to be a sickness, similarly for Wittgenstein, there is nothing that it is to be a “mere sign”. Or to vary the metaphor a bit: to say “the sign is dead” is to say that it is a corpse. To say that a symbol is a sign together with its use is like saying that a human being is a corpse that is alive.

I’ve been arguing mainly about Wittgenstein’s ontology, and have acknowledged that the 3.32s point to frailties of the linguistic organism. So, then, I concede that in the 3.32s, the word “sign” means “mere sign”, provided that our understanding of “mere sign” is philosophically adequate. However: the usage of “sign” that predominates in the Tractatus raises no question whatsoever—indeed, forecloses the question—whether what it describes is really a sentence,

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41 Such an account would better suit Kremer’s interpretive aims, since it, unlike his own, could account for how philosophical illusions propagate from one victim to the next.
i.e., whether it really holds the world to yes or no. This is clear, for example, in the term “propositional sign”. If propositional signs are identified merely as signs, i.e., as quasisentences, what is the difference between a quasisentence and a quasiname? To which category should we assign, e.g., ALEX BURROWS? And as Frege demanded, what distinguishes it from a mere stone or leaf?

1.1.4 Propositions as truth-functions

The term “truth-function” has a standard usage in 20th century logic. For example, Shoenfield (1967) gives the following explanation:

A noteworthy feature of the formula \( A \land B \) is that in order to know whether \( A \land B \) is true or false, we only need to know whether \( A \) is true or false and whether \( B \) is true or false; we do not have to know what \( A \) and \( B \) mean. We can express this more simply by introducing some terminology. We select two objects, \( T \) and \( F \), which we call truth values. It does not matter what these objects are, so long as they are distinct from each other. We then assign a truth value to each formula as follows: we assign \( T \) to each true formula and \( F \) to each false formula. Then we see that the truth value of \( A \land B \) is determined by the truth values of \( A \) and \( B \).

A truth-function is a function from the set of truth-values to the set of truth-values. We can restate our remark as follows: there is a binary truth-function \( H_\land \) such that if \( a \) and \( b \) are the truth values of \( A \) and \( B \) respectively, then \( H_\land(a, b) \) is the truth-value of \( A \land B \).

(Shoenfield 1967, 11)

Shoenfield, like Frege, takes for granted that a sentence is true or false, and follows Frege’s procedure in identifying two objects with truth-values. Of course, Frege would not appreciate Shoenfield’s remark that “it does not matter what these objects are”. For Frege, a sentence stands to a truth-value as a name stands to an object, and so it surely matters which object is the True, just as it matters which object is the planet third closest to the sun, or which object is the number of planets closer to the sun than it. But, having corrected this insouciance, Frege

\[42\] And perhaps also the indifference to knowing what \( A \) and \( B \) “mean”.
would happily recognize Shoenfield’s subsequent explanation as clearly fixing the
meaning of the sign $\Lambda$, by determining which function the sign $\Lambda$ denotes.\(^\text{43}\)

Wittgenstein, of course, would be shocked by all this. First of all, regarding
the suggestion that the word “and” should be a name (of a function), he has this
to say:

My fundamental thought is that the so-called ‘logical constants’ are
not representatives. (4.0312)

To illustrate this thought, he devised a so-called “truth-table” notation, which
looks like this:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

And, he remarks:

It is clear that a complex of the signs ‘$F$’ and ‘$T$’ has no object (or
complex of objects) corresponding to it, just as there is none corre-
spanding to the horizontal and vertical lines or to the brackets.—
There are no ‘logical objects’. (4.441)

Now, in response to this, Frege-Shoenfield might reply: “well, your notation
is surely pedantic, but you pay too much attention to signs. The content is what
that matters, and a truth-table is just a name of a function. That is to say, that
notation is an open term, in the two free variables $p$ and $q$, which, under an
assignment of values to $p$ and $q$, denotes a truth-value.” Here, Wittgenstein might
say, Frege-Shoenfield is misled by the use of the letters $p$ and $q$. They are not “free
variables” at all, but schematic letters which contribute to a general illustration of
various possible uses of truth-tables. An actual use of a truth-table is a sentence,
or propositional sign (4.44); therefore it says something about the world and is
true or false depending on how the world is. For example, here is an expression

\(^\text{43}\)Given a stipulation for the case when not both of $a$ and $b$ are truth-values.
of “it’s raining and it’s sunny”:

<table>
<thead>
<tr>
<th>It’s raining</th>
<th>It’s sunny</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

This tabular expression says how the world is, by expressing agreement and disagreement with the various possibilities of truth and falsehood of the sentences in the upper row. If we drop the rightmost column, i.e., the column separated off to the right by a double line, then the resulting truncated table (as at 4.31) simply enumerates truth-possibilities for the entries in the uppermost row. Each of these rows “means” (4.3) a truth-possibility for the uppermost entries, by symbolizing this truth-possibility in a way that, Wittgenstein says, “can easily be understood” (4.31). By appending a T, after the double line, to such a row, we then signal agreement with that truth-possibility; by omitting such a T we signal disagreement. Thus, the truth-table for conjunction as a whole expresses agreement and disagreement with truth-possibilities for uppermost entries, which its various rows symbolize.

As usual with Wittgenstein’s criticisms of Frege, this one certainly tells us something about Wittgenstein. For Wittgenstein, the truth-functionality of

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44. Wittgenstein’s understanding of the tabular notation of 4.442 might be understood with reference to the seagull notation presented at 6.1203. In seagull notation, a truth-function of two elementary propositions can be written as follows. First, write T p F followed by T q F. Now, above these inscriptions, draw a seagull whose wingtips touch the T next to the p and the T of the q, and another whose wingtips touch the T of the P and the F of the q. Underneath, draw two more seagulls with wingtips covering the other two combinations. Finally, correlate a T with the bodies of some seagulls, and an F with the bodies of some others.

In this notation, the T and F around the p and q represent two poles of truth and falsehood which are intrinsic to them. These Ts and Fs correspond to the Ts and Fs in the rows representing truth-possibilities for p and q in a truth-table. On the other hand, the Ts and Fs which are correlated with them correspond to the Ts and Fs which represent agreement and disagreement with truth-possibilities. These outer Ts and Fs are expressions of agreement and disagreement with the Ts and Fs to which they are correlated. All the Ts and Fs are thus “poles” of some proposition or other. But, the innermost poles are not expressions of agreement and disagreement with the propositions to which they are correlated.

45. This motif is beautifully explored in Goldfarb (2001a).
a proposition is the functional dependence of its truth on the truth or falsehood of other propositions; these other propositions have their possibilities of truth and falsehood independently of each other. But, the possibilities of truth and falsehood of propositions simply are the possibilities of the obtaining and non-obtaining of situations. Thus, truth-functionality is a functional dependence of the obtaining or non-obtaining of one situation on some others. The sense of “function” in Wittgenstein’s usage of “truth-function” is therefore not the purified mathematical sense which appears in Frege, but rather the vulgate of the economist or engineer. Accordingly, for example, “this surface reflects white light” is a truth-function of “this surface reflects red light”, “…orange light”, “…yellow light”, etc., because whether or not a surface reflects white light is a function of whether or not it reflects wavelengths of light from throughout the visible spectrum. Likewise, statements of the form “the pressure in the tube is \( p \) p.s.i.” are truth-functions of statements of the form “the temperature in the tube is \( k \) degrees F” because pressure is a function of temperature.

1.2 Workings of the GPF

Obviously, the general propositional form is supposed to express what is common to all propositions. This has two halves: to be completely general, that is, to contain nothing not common to all propositions, and yet to characterize propositions, hence to characterize nothing other propositions. But, as we’ve seen, Wittgenstein already pointed out that to give such a form is a pretty trivial task: it suffices to do hardly more than clear your throat. Why, then, does Wittgenstein circle back and arduously build up again to

The general form of a truth-function is \([\overline{p}, \overline{\xi}, N(\overline{\xi})]\).

This is the general form of a proposition. (T6)

This time, he did away with the throat-clearing. But, where we once looked through the window of a demonstrative to see the image of proposition as picture, now the view is obstructed by this notational gadget. Why the obstruction of structure here, in what must be common to all propositions? The point is to show how it might turn out that all necessary, or, in the jargon, “internal” re-
lations between propositions are really just logical ones and hence inhere in the mere possibility of representation, which is the same as the possibility of what is represented.

The general strategy is to show that there’s some class of propositions amongst which no necessary connections obtain, such that all possibilities for the world are the results of truth-operations on propositions in this class, the class of so-called elementary propositions. A proposition expressed as a denial of some others depends, for its susceptibility to truth-and-falsehood, on the corresponding susceptibility of the propositions it denies. But, claims T6, similarly every proposition depends for its susceptibility to truth-and-falsehood on the susceptibility to truth-and-falsehood of the propositions that are elementary. Moreover, the detailed structure of this dependence will be the explanatory source of all necessary connections. So, propositions which are not necessarily connected to each other must be separable by some possibility for truth-and-falsehood amongst the propositions which are elementary. And conversely, any distribution of truth and falsehood over the elementary propositions represents a genuine possibility, so that all the separations apparently effected by truth-possibilities for elementary propositions are actually genuine.

At this point, Wittgenstein’s philosophical adventure leads into some rocky technical terrain. For he needs to show that propositions are the results of truth-operations on elementary propositions, or, in other words, that they are truth-functions of elementary propositions.

1.2.1 The one-many problem

To fix an image of the sort of technical situation we’ve now encountered, let me suggest an analogy from the very most elementary foundations of mathematics. The set-theoretic analyst of mathematics confronts as its data various mathematical objects and internal relations between them. For example, a vector space bears some internal relations to various particular vectors. These vectors stand in internal relations to each other—say, that one is a scalar multiple of another. Similarly, the vector space as a whole stands in internal relations to other vector spaces, and, for that matter, to other mathematical structures. The audacity
of set-theoretic foundations is to suppose that out of all the internal relations in mathematics, only one internal relation needs to be taken as fundamental, that of set-membership. All other internal relations are complex iterations of this single fundamental one. The question, then, is how can we understand a vector space as an object identifiable by the mere fact that exactly such-and-such objects bear to it the relation of set-membership, these other objects being so identifiable too? How, in such a way, can we reconstruct the immense variety of internal relations between mathematical structures with which we began? All such relations will have to be “reduced” to complex iterations of the now-fundamental relation of set-membership.

The technical predicament of the logical analyst in the *Tractatus* is somewhat similar. In particular, the analyst has given as data certain possible intellectual positions, say that Johnson flies, that nobody flies, that nobody flies while sleeping, etc. The proposition that nobody flies bears some internal relation to the proposition that Johnson flies, and another internal relation to the proposition that nobody flies while sleeping. The audacity of logical analysis as Wittgenstein envisages it is to suppose that out of all these internal relations only one needs to be taken as fundamental: this relation is the one which a proposition bears to some others, in virtue of which it can be uniquely identified as the denial of them.

Now, the data presented to the set-theoretic analyst of mathematics demand some device for ensuring that a set be somehow connected by the relation of set-membership to a large multiplicity of other sets. For example, a vector space may need to be eventually somehow connected by the set-membership relation to uncountably many individual vectors. To this end, the set-theorist makes certain existential assumptions, to the effect, say, that there exists a set to which infinitely many objects belong, and, say, that there exists a set containing as elements all the subsets of a given set, and so on. These assumptions are “external” or “transcendent” in the sense that they bear no immediate connection to the originally given data, and rather, in some sense, transcend the local mathematical subject-matter. But given these existential assumptions, the set-theoretic foundationalist builds up, using the postulated objects and constructional procedures, counterparts of the originally given mathematical data and their internal relations. A question
whether a mathematical structure and its set-theoretic counterpart are “the same” might arise, though it can be set aside.

The logical analyst of mental life as instructed by the *Tractatus* similarly encounters data which demand some device for connecting intellectual positions to large multiplicities of other intellectual positions. For example, the proposition that nobody flies is connected by the direct denial relation to the various propositions about Johnson, Moore, and so on. On my understanding, Wittgenstein’s procedure here departs somewhat that of the set-theoretic foundationalist. While the set theorist introduces as technical devices existential assumptions and constructional procedures which are external or transcendent, the devices of the Tractarian analyst are in some correspondingly vague sense, immanent. That is, the internal relation that Wittgenstein takes to be basic already appears among the logical data that we need to appreciate in order for there to be an analytical problem in the first place. Merely in virtue of the understanding of propositions we must already appreciate the possibility of representing a proposition as the direct denial of some others. So, rather than stepping outside, and formulating some “objective” existential postulations, the logical analyst simply makes manifest the logical relationships already present in the data.

Nonetheless, the essential point here is this. In purporting to describe the realm of mathematics solely by means of the relationship of set-membership, the set-theorist incurs a technical problem to explain how, in general, a set can be connected by this relation to extraordinarily many other objects. Wittgenstein similarly incurs a problem to explain how a proposition can be connected by the direct-denial relation to many other propositions. Indeed, eventually, all propositions must somehow or other be connected by iterations of direct denial. Moreover, in each case, the essential problem will be to find a method of representing multiplicities of items, in such a way that a given item can be represented as the item which bears the chosen fundamental relation to exactly those other items. Let’s call this problem the *one-many problem*. 
1.2.2 Some early-Russellian technology

So, Wittgenstein attempts to construct a general form of the proposition which would, in virtue of its structure, show that the arbitrary proposition gets its representational nature from its particular position in the network of internal relationships that inheres in the totality of propositions. For, the general propositional form shows that the representational nature of an arbitrary proposition is its functional dependence for truth on the truth and falsehood of propositions in a certain distinguished class, those which are elementary. And in particular, the arbitrary proposition is shown by the general propositional form to inherit its truth-functionality thanks to the fact that it can be grasped as the direct denial of some other propositions, whose truth-functionality would be already presupposed.

In this way, Wittgenstein’s presentation of the general propositional form demands substantial technical resources. For, it presumes the possibility of representing a proposition as connected in some cases, via this relation of direct denial, to some possibly infinite multiplicities of propositions. Thus, Wittgenstein incurred a burden here to demonstrate that this technical challenge of the one-many problem could be met, at least in principle. The understanding of this phase of the book therefore requires some grasp of the technical resources from which Wittgenstein could draw to meet this burden.

My historical hypothesis is that Wittgenstein silently drew the basic idea for his treatment of the one-many problem from Russell’s treatment of classes in his 1903 book *The Principles of Mathematics*. It is believed that Wittgenstein did read this work when in Manchester, as his interests began to shift from engineering to foundations of mathematics.

Nonetheless, this derivation of *Tractatus* technology from 1903 Russell is somewhat speculative. I do think that regardless of how Wittgenstein actually

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46 Russell (1951) writes confusingly that in Manchester “Wittgenstein became interested in the principles of mathematics”. McGuinness (2005) says only that Wittgenstein’s enthusiasm for *Principles* and *Grundgesetze* “leads one to suppose” he had read both books. Monk (1991, 30) asserts that Wittgenstein’s reading *PoM* in Manchester was a decisive event in Wittgenstein’s life, and that Wittgenstein spent two terms there mainly studying *PoM* and *Grundgesetze*. But Goldfarb justly remarks in (2001a) that he finds Monk’s claim “somewhat hyperbolic”. In any case, as Goldfarb pointed out to me, Wittgenstein clearly alludes to *PoM* at 5.5351.
got the ideas, the ideas are actually there in this early Russell.\footnote{Of course, it might be rash to rule out that actually all possible ideas (and, perhaps some impossible ones) are to be found in early Russell.} So even if the historical hypothesis is not quite correct, the marvelous lucidity of Russell’s exposition still helps to explain important parts of the *Tractatus*.

According to Russell, works on logic “distinguish between two standpoints, that of extension and that of intension” (66). But, according to him, “Symbolic logic has its lair” in some intermediate region between the two. In particular, Russell holds that classes must be understood extensionally, so that no two classes have the same elements. Actually, as Russell understands the concept of class, one simply cannot avoid this conclusion. In particular, he holds that one can define a class by simply listing its terms. For example, the numbers two and seven are a class. Similarly, the number two just by itself is a class, the class to which just the number two belongs.\footnote{So, Russell thinks, a class to which just one object belongs just is that object. To this idea there arises an objection (77) which appears also in *Grundgesetze* (Frege 1964, 48n). Consider, for example, the class \(((2,7))\), whose sole member is itself a class \((2,7)\) two members are the numbers two and seven. Since a class to which just one object belongs just is that object, therefore \((2,7)\) is just the same as \((2,7)\). But then, it follows that whatever belongs to \((2,7)\) must also belong to \(((2,7))\). Hence both 2 and 7 belong to \(((2,7))\). But, only one object belongs to \((2,7)\), namely the class \((2,7)\). Hence, 2 and 7 must both be identical to that object, and must therefore be identical to each other. To this objection, Russell rejects the assumption that just one object belongs to \((2,7)\). For, after all, the class \((2,7)\) is really two objects, the numbers 2 and 7. More generally, Russell allows, when the elements of a class are themselves classes, then a class can be “decomposed”, or represented as a bunch of items, in more than one way.} In other words, which, I’m sorry, are not grammatical, one might want to say: “a class to which some objects belong just is those objects.” Here, Russell observes, the grammatical difficulty is essential: the phrase “a class” is grammatically singular, but, a class is in general intrinsically many things, being defined by a phrase like “the numbers two and seven” which is grammatically plural. So, he concludes, the difficulty is “not removable by a better choice of technical vocabulary” (70).

Thus, while we, nowadays, are accustomed to thinking of there being one object, a set, to which many objects may bear the relation of set-membership, Russell’s corresponding conception of a class is primarily a conception just of those many objects. Russell does introduce a notion that corresponds more closely to our notion of a set, for which he uses the phrase “class as one”, in
contrast to this prior conception of “class as many”. But, I think, for Wittgenstein, the conception of “class as many” is prior. I’ll reserve “class” and “multiplicity” for this notion, and use “set” for Russell’s “class-as-one”. Where it is customary to denote a set by enclosing the names of its elements in curly braces as in \{a, b, \ldots\}, I’ll denote a class-as-many by enclosing the names of its elements in parentheses as in \( (a, b, \ldots) \). In particular, for example, there are three distinct sets \{a, \ldots, b, \ldots\}, \{a, \ldots, \{b, \ldots\}\}, and \{\{a, \ldots, b, \ldots\}\}. On the other hand, \((a, \ldots, b, \ldots), (a, \ldots, \{b, \ldots\}, ((a, \ldots, b, \ldots))\) are all the same class, which is just the objects \(a, \ldots, b, \ldots\).

So far, Russell’s account of classes has invoked only extensional resources: to define the class containing such-and-such items, one simply forms a list of the names of those items. But, as Russell observes, were we to try by enumeration to define an infinite class, “Death would would cut short our laudable endeavour before it had attained its goal” (69). Since in logic we do deal with infinite classes, logic must “find its lair” in the regions intermediate between intension and extension. For example, when considering a proposition about all numbers, we do not thereby consider each and every number, but rather only a concept, all numbers. Nonetheless, the proposition is in some sense about all numbers because, Russell says, the concept, all numbers, denotes exactly those things which are numbers (73). Thus, the “inmost secret of our power to deal with infinity” (73) is that an infinite collection can be denoted by a concept which is only finitely complex.

On Russell’s account, classes can contain items of any kind, anything that falls under his most general category of being: objects, propositions, propositional functions, variables, denoting concepts, and so on. But, then the notion of class has a particular special case, the notion of a “class of propositions of constant form” (90). Roughly speaking, such a class results from a proposition by allowing one or more of its objectual constituents to vary freely. Russell claims that this special case is more fundamental than the notion of class in general, for the general case can be defined in terms of the special one, though not conversely (89). In turn, a class of propositions-of-constant form is defined by the result of replacing an object in a proposition with a variable. For Russell, of course, a

\[\text{As far as I can make of it just now, through Tractatus-eyes.}\]

\[\text{He seems to have in mind a definition by appeal to the notion (?) of “such that” (93).}\]
proposition is not a linguistic or representational entity, but is rather, as it were, the fact that would obtain were the proposition true. So a variable, also, seems for Russell to be somehow nonlinguistic. A variable, in turn, presents a defining concept, so that the items falling under the concept are the variable’s values (91). In the fundamental case, of the “true” variable which has unrestricted range, the presented concept is, according to Russell, said to be formal (91). But in any case, it is in general by presenting a concept that a variable ranges over a multiplicity of items. Most importantly, a variable does not name this or that one of its values. Nor, on the other hand, does it plurally name the multiplicity of those values. Rather, through its associated concept, the variable indeterminately indicates or ranges over them. Now, when a constituent of a proposition is replaced by this variable, then the result is a propositional function. The propositional function determines a class of propositions of constant form, namely those propositions which result according as the variable in turn gives way to some one or other of its values.

1.2.3 Variables in the *Tractatus*

Wittgenstein’s approach to so-called one-many problem bears clear marks of Russell, and I want to emphasize some commonalities, but first let me mention some important differences. The first difference is in the general realization of the ideas. *PoM* is tentative and exploratory, seeming to run down every possible conceptual path, and Russell’s discussion is extremely complicated. In contrast, it seems to me, Wittgenstein thought these issues through to a fairly simple and systematic conception. Second, Wittgenstein distinguishes clearly between signs and what they express, or, as he puts it, between signs and the symbols to which they belong. Thus, Wittgenstein distinguishes between propositional signs, nominal signs, operators, etc., on the one hand, and propositions, names, operations, etc., on the other. In particular, a variable, for Wittgenstein, is just a sign. Just as the symbol to which a propositional sign belongs is a proposition, the symbol to which a variable belongs is a formal concept.

Wittgenstein proposes to represent a proposition as the joint denial of the values of a propositional variable (5.501). Thus, to explain how a proposition can
be logically related to infinitely many propositions, it would suffice to explain how a variable can assume infinitely many values. Echoing Russell, Wittgenstein says that if the range of the variable is supposed to be finite, then one can fix its range simply by enumerating the objects in its range; but, one cannot assume that the range of every variable is indeed finite (5.501). And so, one cannot assume the possibility in every case of fixing the range of a variable by bare enumeration. Instead, according to Wittgenstein, a variable secures its range by presenting a formal concept (4.1272); in turn the formal concept specifies some multiplicity of items as those which fall under the concept; this multiplicity is the range of the variable (5.501).

Just as in Russell, a variable in the *Tractatus* does not itself name any particular one of its values; nor does it plurally name the multiplicity of those values. Thus, for example, if ‘α’ is a variable which ranges over the totality of people, then sign-arrangement ‘α flies’ does not express any one definite proposition. Rather, this sign-arrangement serves only to mark off what is common to the various propositions to the effect that this or that element of the range of ‘α’ flies. However, according to Wittgenstein, one can convert the indefinitely indicating expression ‘α’ into a plural expression which does stand for things: by placing a bar over the variable ‘α’, one forms a plural term ‘α’ which stands at once for each and every value of ‘α’. Since, in our case, the range of the variable ‘α’ is the totality of people, the plural term ‘α’ stands for that totality: it stands for that class-as-many which, we are tempted to say, “just is” all of the people, i.e., the following logical formula would be correct:

\[ \alpha = (\text{Moore, Johnson, Russell,} \ldots) . \]

However, this example of the variable ranging over people is importantly misleading. For Wittgenstein holds that all variables “can be construed as propositional variables, even variable names”. So, the ground-level explanation of

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51 So, Russell reduces the general notion of class to the notion of class of propositions as constant form. But, this notion of class of propositions of constant form itself depends on the notion of variable, which takes as its range the extension of a formal concept. Such an extension (there may be only one such, the All) includes not just objects but propositions, etc. Thus, Russell does not hold in PoM that all variables can be construed as propositional variables.
logic will appeal only to variables which range over propositions, and all other forms of variation must be reduced to propositional variation. But, for example, how are we so to reconstrue such an apparently nominal variable as ‘α’ whose range is the people?

Well, how does the apparent existence of nominal variables arise in the first place? It looks like, for example, one can understand the variable name α to play some essential role in expressing a generalization over people, say in expressing the generalization that nobody flies. For example, suppose we begin with the proposition that Johnson flies, and then in this proposition regard Johnson as variable. Thus, rather than considering the proposition to say of Johnson that Johnson flies, we consider it to say of an arbitrary person that that person flies. This indication of an arbitrary person we might effect with the variably meaningful sign

\[ α \]

which, combined with the constantly meaningful sign

\[ \ldots \text{flies} \]

yields a corresponding variably meaningful sign

\[ α \text{ flies}. \]

In just the same way, a constantly meaningful sign

\[ \text{Johnson} \]

might be combined with the constantly meaningful expression

\[ \ldots \text{flies} \]

to yield the constantly meaningful

\[ \text{Johnson flies}. \]
But, Wittgenstein thinks, this explanation puts the cart before the horse. One cannot explain the meaning of an expression ‘Johnson flies’ by pointing out Johnson and pointing out (an instance of ?) flight, because “only in the context of a proposition has a name meaning.” For, as I’ve claimed, Wittgenstein proposes to understand the expressive power of a proposition by finding its place in the totality of propositions. So, similarly, one does not explain the function of a variably meaningful sign ‘α flies’ by stipulating that ‘α’ ranges over people and ‘…flies’ stands for flying. Rather, the complex expression ‘α flies’ is itself what is prior: it is a propositional variable which ranges over each and every proposition to the effect that this or that person flies. Thus, one could perhaps derivatively explain the range of the nominal variable ‘α’ by suggesting that ‘α’ stands to ‘Johnson’ as ‘α flies’ stands to ‘Johnson flies’. But, this explanation applies to ‘α’ only insofar as it occurs in the propositional variable ‘α flies’, and so it is not of much independent interest.

But, at this point a natural question arises. Russell’s notion of a class as many has two strong justifications. First, some notion of class appears to be essential to mathematics. Second, Russell’s particular fundamental realization of the concept of class appears to be perfectly familiarly implicit in the use of ordinary English noun phrases like ‘Johnson and Moore’ or ‘the people’. That is, we can’t but take for granted that sentences like ‘people need food’ somehow mean something; and the natural account is simply that ‘the people’ stands for all people. Moreover, as we’ve seen, the expression ‘the people’, can’t simply contribute to the proposition expressed all people; rather, it contributes a concept which in turn denotes the people whose organismic nature then decides the truth of the original proposition. Thus, to put the point in Tractatus notation, Russell’s account of ‘α’ and ‘\overline{α}’ is very naturally motivated. But it does not seem that Wittgenstein’s replacement concepts, of propositional variable and its extension, are similarly naturally motivated. In particular, while one might perhaps try to take an expression like ‘he or she flies’ as a model for the use of expressions like ‘α flies’, it seems hopeless to find any sort of English phrase which would somehow stand for the corresponding range of the propositional variable (i.e., to the extension of the formal concept the variable presents). In other words, it is not at all clear what ‘α flies’ is supposed to mean. For, Wittgenstein holds, propositions cannot
be named. Perhaps one could get some kind of approximation by simply listing,
as in ‘that Johnson flies, that Moore flies, . . . ’ But, if such a list works logically
as per Wittgenstein’s intention, then it isn’t to be distinguished from the simple
multiplicity of expressions ‘Johnson flies’, ‘Moore flies’, . . . So, it is probably fair
to say that the result of applying a bar operation does not appear to correspond
to any single naturally isolable English phrase.

But if the bar notation doesn’t correspond to any natural segments of English,
then what is Wittgenstein’s justification for introducing it in the basic account of
control over possibly infinite multiplicities? Consider, for example, the proposition
that nobody flies. Wittgenstein holds that this proposition bears an internal
relation to various other propositions: in particular, say, to the propositions that
Johnson flies, that Moore flies, and so on. In particular, the proposition that
nobody flies is constituted by its position as the joint denial of all such instances.
Thus, Wittgenstein thinks, this position itself embodies the relevant control over
that multiplicity. That is, for Wittgenstein, there is no one “object” or “set”
which collects together a multiplicity of propositions; rather, there is instead a
proposition which bears to the multiplicity a fixed logical relation—for example,
there is the joint denial of those propositions. And, this unification is generally
already implicit in representational practice. Thus, the analyst begins with the
datum that to affirm a proposition

\[ \text{nobody flies}, \]

is exactly to deny the propositions

\[ \text{Moore flies, Johnson flies, . . .} \]

Yet, since the original proposition does not manifestly contain these many propo-
sitions, etc., there must be a variable \( \xi \) such that

\[ \overline{\xi} = (\text{Moore flies, Johnson flies, . . .}) \]

so that in turn

\[ \text{nobody flies} = N(\overline{\xi}). \]
The question remains how to fix the values of the propositional variable ‘\( \xi \)’. This, Wittgenstein claims, is something “to be stipulated.” Such stipulations require technical resources for the construction of variables, which Wittgenstein outlines at 5.501. To describe those resources is the task of the next section.

This section began by laying out some Russellian resources for the control of possibly infinite multiplicities. We transposed these resources to the *Tractatus* framework, where it is insisted that all variables can be construed as propositional variables. But this meant that fundamental to the *Tractatus* framework is some notion of a multiplicity of propositions, which lacks any obvious form of unitary expression in ordinary English. The question then arose what could possibly be the intuitive source of this transposed technical notion. I suggest that just as, for Russell, the intuitive source of the notion of a class-as-many of objects is not any single representational unit but rather simply a corresponding multiplicity of names, similarly for the *Tractatus*, the intuitive source of the notion of a class-as-many of propositions is not any single representational unit but rather a corresponding multiplicity of propositions. Such multiplicity warrants our analytic attention precisely in case a proposition we want to understand can be represented as the result of applying a logical operation to the elements of that multiplicity.

### 1.2.4 Propositional variables: some details

Let me now turn to the details of Wittgenstein’s technical resources of stipulation of the value-ranges of variables. It is at 5.501 that Wittgenstein describes three Ways of fixing the values of a variable. Now, on Wittgenstein’s account, to fix the range of a variable is just to supply the formal concept under which falls exactly the items in the range, i.e., the formal concept whose extension is the range. Thus, at bottom the three Ways are really ways of fixing the use of a sign as a variable by specifying the formal concept to which it belongs. So in a way, the ways really embody commitments to certain formal-conceptual resources which are available to the analyst of language or thought. It is a difficult interpretive question what legitimates exactly these analytical resources. Naturally a constraint arises in the form of conditions of the possibility of representing a proposition
as the result of applying a logical operation to some bunch of other propositions: what makes it possible for a bunch to appear in this way? At 5.501 Wittgenstein does not give any principle but contents himself with three examples of legitimate stipulative procedures.

Way 1. Wittgenstein says that a variable presents a formal concept, and that the range of this variable is the totality of propositions which fall under the concept. But, as I’ve suggested, the theoretical appeal to the notion of concept and hence of variable derives from the analytical datum that a proposition might be logically related to some multiplicity of propositions which is not a priori finite. After all, as Wittgenstein supposes in the *Notebooks* (18.6.15f) and acknowledges in the *Tractatus* (4.2211), it might turn out that facts and states of affairs are infinitely complex.

Nonetheless, some propositions seem to be naturally representable without any immediate appeal to not a priori finite multiplicities. Thus for example we have quite naturally an analysis

\[
\text{Neither Moore nor Johnson flies} = N(\text{Moore flies}, \text{Johnson flies}).
\]

In this case, apparently, Wittgenstein holds that one can simply stipulate that the formal concept $\xi$ be that which is satisfied by just the two explicitly formulated propositions, so that $\overline{\xi} = (\text{Moore flies}, \text{Johnson flies})$.

Way 2. The second way of stipulating the values of a variable is the natural case we’ve already considered, and which basically amounts to Wittgenstein’s treatment of quantificational generality. This treatment is well-encapsulated by Wittgeinstein’s remark that unlike Frege and Russell, he dissociates generality from truth-function (5.52). As we’ve seen,

\[
\text{Nobody flies} = N(\text{Moore flies}, \text{Johnson flies}, \ldots)
\]

52 A natural suggestion is that one can “mechanically check” whether a proposition belongs to the bunch. But this presupposes a concept of computability over propositions.

53 To the second and third of which I devote entire chapters, and what follows is just an outline of those results.
and so we would like to construct \( \zeta \) such that

\[
\overline{\zeta} = (\text{Moore flies}, \text{Johnson flies}, \ldots).
\]

To this end, note that among these propositions in \( \overline{\zeta} \) there is a certain constant aspect and a variable aspect, and correspondingly such aspects in their expressions. Replacing the variable aspect with an indeterminately meaningful sign ‘\( \alpha \)’ gives a sign-arrangement

\[
\alpha \text{ flies}.
\]

This sign-arrangement seems to articulate a commonality amongst the verbal expressions of all the propositions in the desired range, because its indeterminately meaningful part corresponds to what is not constant to the expressions of the propositions in that range. We might therefore try to take the sign-arrangement itself to be a propositional variable. Since the sign-arrangement appears to signify the common content of the propositions in the desired range of the variable, the corresponding formal concept might be given as follows: to be marked by the content that “\( \alpha \) flies” presents. The generality of the original proposition then lies in the means by which a subordinate expression points out those propositions to which the operation of denial is then applied.54

Note, however, that the analysis just given is somewhat equivocal. For example, it is easy to imagine Frege or Russell asking: is

\[
\text{Tweetie flies}
\]

a value of “\( \alpha \) flies”? But if Tweetie is a bird (let us suppose), then, it seems, whether Tweetie flies cannot count against the truth of the proposition that nobody flies. Frege or Russell would take this to show that the proposition that nobody flies actually amounts to the denial of

\[
\text{Moore is a person and Moore flies, Johnson is a person and Johnson flies}, \ldots.
\]

---

54Wittgenstein’s account of quantificational generality is highly unusual in ways unmentioned here; but its eccentricities are deeply rooted. I defend this claim elsewhere.
Thus, the stipulation
\[ \xi = \alpha \text{ is a person and } \alpha \text{ flies} \]
would be taken to lead to the desired analysis
\[ \text{nobody flies} = N(\xi). \]

This “Fregean improvement” requires a revision in the original datum that it would be outright contradictory to maintain that nobody flies but that Moore flies. Clearly, Frege or Russell would indeed regard the original argument “nobody flies, so Moore doesn’t fly” as enthymematic. It is not really clear to me what to say about this from Wittgenstein’s point of view, since it is doubtful that he could regard “Moore is a person” as ruling out any possibilities.

Way 3. According to Wittgenstein, the third way of fixing the range of a variable was overlooked by Frege and Russell. Apparently, it has also been overlooked in most subsequent development of logic, and as yet its use has not been rigorously studied in the secondary literature. But, as is often the case in the Tractatus the basic idea is simple if peculiar: to incorporate into logical structure the concept “and so on”. That is, suppose given a proposition \( p \), and an operation \( O \) which generates a proposition from a proposition. Then, according to Wittgenstein, there exists a variable whose range consists of the totality of propositions that can be obtained by repeatedly applying the the operation \( O \) to the originally given proposition \( p \).

In developing this method, Wittgenstein might have had in mind some such

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55 The solution to this problem may lie in the near-certainty that, for Wittgenstein, “Moore”, “Johnson”, etc., are not really names since their bearers are mortal. But then, it may be that “Moore flies” and “Tweetie flies” have different logical forms. Then, the result of removing “Moore” from “Moore flies” would not be an expression which marks the sense of “Tweetie flies”. I am thinking of the passage in the Notebooks where Wittgenstein observes that one can say of the rod, but not of the ball, that it is leaning against the wall. Actually, an even closer example comes in the hopeless English translation of a similar passage to the effect that he cannot say of his watch that it is lying on the table. That is, in English, “the watch is lying on the table” does not, appearances (?) to the contrary, have the same form as “the client is lying on the table.” Thus, saying that nothing is the way the watch is said to be in the proposition that the watch is on the table, should entail nothing about the client.
example as:

None of Russell’s ancestors fly,

which, so it seems, is the denial of the propositions in the series

a parent of Russell flies

a parent of a parent of Russell flies

a parent of a parent of a parent of Russell flies

\vdots

We thus seek to construct a formal concept $\xi$ whose extension contains just these propositions. Such a construction can be written as

$$\xi = [\text{a parent of Russell flies, } \xi, \text{ a parent of } \xi].$$

According to T5.2522, the notation in the definiens works roughly like this. The whole thing is supposed to define a series of items. The first item in the series is indicated by the leftmost entry in the square brackets. The second and third entries together give an operation for generating an item from an item—here, a proposition from a proposition. Roughly speaking, they characterize the operation by means of “before and after shots”. So in this case, the operation, applied to a proposition, gives the result of plugging the proposition in for $\xi$ in the fragment: a parent of $\xi$. Now, the rest of the series consists of just what can be obtained by repeatedly applying the operation to the repeatedly given item. The terms of this series are thus, according to the definition, the values of $\xi$. So we now have once again

$$\text{None of Russell’s ancestors fly } = N(\overline{\xi}).$$

1.2.5 Insufficiency of local constraints

As we’ve seen, the analysis of any particular proposition has always the same pattern. Given the proposition $p$, we seek a formal concept $\xi$, such that $p$ is
the joint denial $N(\xi)$ of the propositions in the extension of $\xi$. Of course, this in itself is almost completely trivial. Since as we’ve seen, $N(N(p)) = p$, therefore to find $\xi$ such that $p = N(\xi)$ it suffices to take $\xi$ to characterize the single proposition $N(p)$. More generally, any one particular proposition has several possible analyses. To give another example, suppose we begin with a datum that the proposition $q$ is true iff both propositions $r$ and $s$ are true. Then, it will be tempting to conclude that the analysis

$$q = N(N(r), N(s))$$

must be the correct one, according to which $q$ is somehow ‘composed’ from $r$ and $s$ and therefore more complex than them—this will be particularly so if it appears that verbal expressions of $r$ and of $s$ are parts of a verbal expression of $q$. However, the datum by itself is equally consistent with the pair of analyses

$$r = N(N(q, t))$$

$$s = N(N(q, N(t)))$$

according to which actually $r$ and $s$ would be ‘composed’ of $q$ and some other randomly chosen proposition $t$.

The 5.2s observe that propositions stand in internal relations to each other, so that one proposition can be represented as the result of an operation on other propositions. This means, in practice, finding a propositional sign belonging to the first proposition by attaching the sign of an operation to a multiplicity of signs which belong to the other propositions. The analysis of propositions simply applies doggedly, toward a certain end, a certain restricted form of this practice. Thus, analysis always represents a proposition as the result of an operation of joint denial. Such representation is the syntactical result of prepending the letter ‘$N$’ to the enclosure in parentheses of a multiplicity of propositional signs. Since, however, in general it will be impracticable to inscribe that multiplicity outright, instead we construct a notation ‘$\xi$’ which plays the same logical role as that multiplicity.

So understood, analysis just applies a refined form of the ordinary practice of
expressing a proposition as the result of a truth-operation. But then, what we’ve so far said doesn’t suffice to pin down how analysis might be in any sense explanatory, elucidatory, or otherwise interesting. For, as we’ve observed, the practice can be applied to a given proposition in infinitely many divergent ways. One might observe that it must be, with respect to some given proposition, that it be true if and only if two others are true, and thus represent it as result of truth-operations on those others. But this only codifies notationally a relationship between three propositions, and various relationships between various propositions can be codified similarly. Nothing, then, makes a proposition in itself to be a conjunction as opposed to a disjunction, etc. We might try to insist, for example, that \( p \) is a conjunction if it has the form \( N(N(p), N(q)) \) for some proposition \( q \), but by picking \( q \) to be a tautology, then every proposition has this form. Thus, truth-operational articulatedness is an artifact only of this or that particular way of realizing a proposition in signs.

What underlies this essential relationality of logical structure is the idea of a proposition as a picture of reality. Consider, for example, a proposition with respect to some cup and table that the first is on the second. This proposition really depicts the cup and the table, so that as the cup and the table would be, palpably, combined in the world, so their proxies are combined in the proposition. Now, roughly speaking, the proposition that the cup is on the table is equivalent to the disjunction of two propositions, that the cup is on the left half of the table, and that it is on the right half. Hence, one might think, the original proposition is really the disjunctive result of gluing together two pictures of reality. Each of those two pictures contains a cup, and a table, and so the result of gluing them together contains two cups, and two tables, such that the one cup is on the left half of the one table, and the other cup is on the right half of the other table. But, this is clearly not what the original proposition is intended to depict; rather, the original proposition depicts just one cup, on just one table. Because there is no more than one cup and one table in the situation depicted, similarly there is no

\[56\]

The general idea here is well-expressed in the *Tractatus* by the remark that an operation is not a mark of the sense of propositions, but rather a mark of difference between senses of propositions. Thus, the operation of conjunction might be presented notationally by \( \langle \xi, \zeta, N(N(\xi), N(\zeta)) \rangle \) whose structure marks a “difference” or internal relation between three propositions.
In any case, since logical structure is essentially relational, a proposition is neutral with respect to logical structure when considered only intrinsically. But this means that we cannot understand analysis as unfolding "intrinsic" nature of propositions; one cannot simply "look inside" a proposition to see what it is made of, and then look inside what it is made of to see what that is made of, etc. Thus, local constraints on the adequacy of analysis do not tell us the order of analysis. The adequacy of analysis must depend on some further, "global" constraint. The point of T6 is to unfold this constraint, and that’s what we now turn to considering.

1.2.6 The general propositional form

In this section I just want to analyze the formulation of the T6-variable that Wittgenstein gives at T6:

\[ [p, \xi, N(\xi)]. \]

This formulation obviously resembles the notation explained at T5.2522. Many commentators have therefore supposed that the explanation at T5.2522 is intended to cover the use at T6. But, I think this is a mistake. The reason is that the explanation at T5.2522 simply does not explain this use. In particular, T5.2522 treats only a singular, constantly meaningful sign as its left entry, indicating the single initial term of the signified form-series. Similarly, it treats only a singular, variably meaningful sign as its middle entry, which together with the right-hand entry marks out an operation which, applied to an arbitrary single term of the series, yields another term. But, at T6, we have instead a left entry which is a plural constantly meaningful expression. Likewise, the middle entry at T6 is a plural, variably meaningful expression which, together with the right-hand entry, marks out an operation which yields a term of the series when applied to a multiplicity of such terms. Thus, the explanation at T5.2522 doesn’t specify the symbol at T6. Surprisingly, in Prototractatus Wittgenstein does explain use of the

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57I contend in Chapter 2 that this particular application of the thought of propositions as pictures is what drives Wittgenstein’s stipulation that in a generalized proposition, distinct variable names cannot assume the same value. Thus, the redundancy of the equality predicate is a mere corollary of the pictoriality of propositions.
sort that appears at T6:

We write the general term of a form-series like this:

\[ [\overline{x}_0, \overline{x}, O'(\overline{x})] \]

The \( \overline{x}_0 \) are the initial terms of the series, the \( \overline{x} \) are any of its terms, and \( O'(\overline{x}) \) is any term in the progression of the series which the operation \( O \) produces out of the \( \overline{x} \).

So, I take the summary import of the notation at T6 to be as follows. The square bracket notation as a whole is a variable, such that the range of this variable contains

- Every item of the multiplicity denoted by its barred left-hand entry;
- every result of applying the given operation to a value of the barred middle entry,
- and nothing else.

Just for an innocuous example, let’s use this method to construct a formal concept \( C \) whose extension \( C \) is the totality of natural numbers greater than 1. For this construction, we take as given the concept \( k \) of being a prime number. And, we take as given the concept \( \eta \) of: being a finite multiplicity. Finally, let \( \Pi(\eta) \) be the result of the operation to compute the product of the elements of the given instance of the concept \( \eta \). Then, the concept \( C \) whose range is the totality of numbers greater than 1 is

\[ [k, \eta, \Pi(\eta)]. \]

This is my probably pretty bad translation. The German is:

Schreiben wir das allgemeine Glied der Formenreihe so:

\[ [\overline{x}_0, \overline{x}, O'(\overline{x})] \]

Die \( \overline{x}_0 \) sind die Anfangsglieder der Reihe, die \( \overline{x} \) beliebige ihrer Glieder, und \( O'(\overline{x}) \) dasjenige Glied welches beim Fortschreiten in der Reihe durch die Operation \( O'(\overline{x}) \) aus den \( \overline{x} \) entsteht.

Note in particular that the German does contain the plurals.
Let’s now argue that 12 falls within $\overline{C}$. First note that 2 and 3 fall under the formal concept, $k$, of being a prime number, and therefore belong to the extension $\overline{k}$ of that concept. By the explanation of generalized form-series notation, it follows that 2 and 3 are initial terms of the presented form-series, and hence fall within $\overline{C}$. Now, being either 2 or 3 is a formal concept, say $e_1$, whose extension falls entirely within $\overline{C}$. So, again by explanation of form-series notation, it follows that the product $\Pi(e_1)$ of the elements of the extension of $e_1$ falls within $\overline{C}$ as well, that is, 6 falls within $\overline{C}$. Similarly, being either 2 or 6 is a formal concept, say $e_2$, whose extension we can now say falls entirely within $\overline{C}$. So the product $12 = \Pi(e_2)$ falls within $\overline{C}$.

Now, let’s turn to the particular use at T6. In that case,

- ‘$p$’ works as a constant denoting the multiplicity of initial terms of the series;
- ‘$\xi$’ works as a variable whose values are multiplicities of terms of the series;
- ‘$N(\xi)$’ works as a variable ranging over results of applying the operation marked by the letter ‘$N$’ to the elements of an arbitrarily chosen value of ‘$\xi$’.

And the import is that ‘$[p, \xi, N(\xi)]$’ works as a variable whose range consists of

- the multiplicity of items denoted by ‘$p$’;
- every value assumed by ‘$N(\xi)$’ when the value of ‘$\xi$’ is a multiplicity of items already in the range.

It remains, then, to identify the denotation of ‘$p$’ and the range of ‘$\xi$’ and the operation signified by ‘$\xi, N(\xi)$’. Generally, Wittgenstein uses the letter ‘$p$’ as a variable not just over elementary propositions, but over propositions in general.\(^59\) But, in the entry immediately after the introduction of T6-variable,\(^60\)

\(^59\) For brevity, I’ve carried out this explanation entirely within material mode. Carnap (1934) says that many sentences of Wittgenstein’s “which at first appear obscure become clear when translated into the formal mode of speech” (303).

\(^60\) Wittgenstein actually stipulates at 4.24 that $p, q, r, \ldots$ range over elementary propositions. But by my count, there are only two entries other than T6 where $p, q, r, \ldots$ clearly range over elementary propositions only: 4.31 and 5.101. I take these letters to be used as variables ranging over all propositions in the following passages: 5.12s, 5.13s, 5.1362, 5.141, 5.152, 5.151, 5.31, 6.1221.
Wittgenstein explains: “What this says is just that every proposition is a result of successive applications to elementary propositions of the operation $N'(\xi)$” (T6.001). This suggests that the initial terms of the intended form-series are precisely the elementary propositions. Since ‘$p$’ is intended to denote the multiplicity of initial terms, it is therefore natural to suppose that ‘$p$’ here works as a variable whose range is the totality of elementary propositions. Then, in turn, ‘$p$’ works as a constant which denotes the multiplicity of values of ‘$p$’, i.e., the totality of elementary propositions.

Anscombe remarks that “as in Frege, ‘$\xi$’ marks an informal exposition.” While I’d say that the entirety of the Tractatus is informal exposition (if not outright poetry), it does appear that Wittgenstein’s use of ‘$\xi$’ is pretty messy and diverse. That letter seems to serve in the Tractatus like the computer linguist’s “foo” as a catch-all for grammatical concepts too weird to deserve their own type of variable. But, the usage closest to that in T6 appears in the 5.5s, and in particular at 5.501, where he explains the bar notation itself. In that passage, it appears that ‘$\xi$’ is used, somewhat sloppily, as a variable which ranges over variables; but I think the intention is for it there to range over formal concepts. As we’ve seen, our aim is that ‘$\xi$’ be a variable which takes as its values certain multiplicities of terms of the form-series. Since the terms of the form-series in question are precisely the totality of propositions, therefore our aim is that ‘$\xi$’ range over multiplicities of propositions. However, it does not range over what we would call arbitrary sets of terms, because Wittgenstein does not take the notion of arbitrary set for granted. But, then: over which multiplicities? The natural suggestion is that 5.501 itself provides the answer: a multiplicity falls within the

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5152521, 55, 5501, 5502, 551, 552, 6, 6001, 601, 603, 61203. He tends for some reason to use it for bases of an operation; but some of these uses are singular and some plural. Possibly, Wittgenstein thinks of an expression like ‘$N(\xi)$’ as extremely incomplete, because it tries to signal an operation, but instead amounts only to a variable over all propositions (if such a variable exists), since every proposition is the negation of some propositions. That is, because an operation doesn’t characterize the sense of propositions it can’t be presented by a variable as in ‘$N(\xi)$’. Rather, the operation marks a difference between propositions, and is therefore presented by means of two variables, as in ‘$\xi, N(\xi)$’.

52That is: the cardinality of the range of this variable is, I contend, not greater than the cardinality of the form-series itself (unless the number of elementary propositions is finite). See the appendix for further discussion.
range of ‘\(\xi\)’ if and only if that multiplicity can be specified by some such formal concept as those of the three kinds introduced at 5.501.

We now have the following:

- ‘\(p\)’ ranges over the elementary propositions
- ‘\(\bar{p}\)’ denotes the totality of values of ‘\(p\)’
- ‘\(\xi\)’ ranges over formal propositional concepts
- ‘\(\bar{\xi}\)’ ranges over the extensions of formal propositional concepts.

Then, the range of the general propositional variable will include

- every elementary proposition;
- every joint denial of the extension of a formal propositional concept;
- and nothing else.

1.3 The GPF in action

1.3.1 A guide to analysis

So I take it that from a technical point of view, the workings of the notational form employed in T6 are, if a little delicate in detail, basically straightforward in broad brushstrokes. In other words, it is fairly easy to see how that kind of notation should in general secure the range of a variable, as, for example, in the case of the definition of the class of integers greater than 1. One begins with a bunch of items, and with an operation for constructing items from bunches of items. Repeatedly apply this operation until no new items result. The values of the variable so constructed are precisely those items which must eventually so appear.

However, while we can now understand certain uses of the pluralized square bracket notation, it is not clear that we can make sense of the particular use at T6. For, so far we have only established, with respect to pluralized square bracket notation, a result of the form “intelligible in, intelligible out”. That is, when
specifying the class of numbers greater than 2, we simply took for granted the notion of prime number, and the notion of finite product. Granted our grasp of these notions, the square bracket notation makes sense too. Now, I think that the operator $N$ and indeed, at least roughly, the means of specification of its bases, have been explained more or less adequately, and that these issues are no longer fundamentally in question. But, it is still not at all clear that we have any understanding of the use of ‘$p$’ in the T6 notation, where it is purported to vary over the totality of elementary propositions. Indeed, Wittgenstein goes out of his way to make sure that it would seem to be difficult to try to construct a variable with just the elementary propositions as its range:

We now have to answer \textit{a priori} the question about all the possible forms of elementary propositions. Elementary propositions consist of names. Since, however, we are unable to give the number of names with different meanings, we are also unable to give the composition of elementary propositions.

\begin{equation}
(5.55)
\end{equation}

If I cannot say \textit{a priori} what elementary propositions there are, then the attempt to do so must lead to obvious nonsense. \((5.5571)\)

As Brian McGuinness memorably said about the constituents of elementary propositions, these are just not the kind of thing we encounter on the street. We just don’t know what an elementary proposition is like: even if some thoughts are thoughts to the effect, for some elementary proposition or other, that it is the case, still, we have no way to tell, as it were by inspection, that this is actually so.

Thus, we cannot fix the values of the variable ‘$p$’ simply by listing: not only are there perhaps infinitely many of them, but Wittgenstein gives no suggestion how to recognize an elementary proposition if we saw it in the street. And, besides the outright listing, the only other way to specify a multiplicity of propositions is by the internal relationships of the elements of the multiplicity to some propositions already at hand. But such specification would require a grasp of the forms of elementary propositions, and this is precisely what we are not supposed to have, at least not \textit{a priori}. But then, it is just not clear how we can understand the function of ‘$p$’ in the notation for the general propositional form as a
propositional variable at all. For it is just not clear how its range is supposed to be fixed.

Wittgenstein does seem to acknowledge, in the midst of the pessimistic 5.55s, that in spite of its not a priori status, still there is something like a question what elementary propositions there are, and, of course, therefore also an answer to that question. But, he says, “the application of logic decides what elementary propositions there are” (5.557). So, to understand how the letter ‘p’ is supposed to take up its function as a variable ranging over the totality of elementary propositions, we need to understand Wittgenstein’s conception of the application of logic. In the remainder of this section, I will spell out what I take to be this conception, and thereby propose an answer to Sullivan’s question.

Of course, the problem is not just that we have no way to identify the elementary propositions. To identify a proposition as elementary requires identifying it as maximally early in the order of analysis. So if we just knew the order of analysis a priori, that would somehow be enough. But in fact, so far we have identified no principled way to determine which propositions are to be analyzed in terms of which. When any small bunch of logically interrelated propositions is considered in isolation, there will be various ways of analyzing some in terms of the others, and thus various possible conclusions as to which propositions in the bunch are more fundamental than the others in the order of analysis. Even something that looks like a conjunction might turn out to be prior, in the order of analysis, to each of its conjuncts.

At this point, I think it is essential to note that the general propositional form is supposed to secure the truth-functionality thesis. That is, the general form of the proposition is supposed to be the general form of truth-functions of elementary propositions. So, that every proposition is a truth-function of elementary propositions is supposed to follow from the fact that every proposition falls under the general form of such truth-functions. The question then arises, how is it that the general form of the proposition actually secures the truth-functionality of what falls under it?

The answer to this question rests, I think, on three key insights. First, truth-functionality explains the logical relationships of entailment and incompatibility between propositions. For according to the truth-functionality thesis, what-
ever elementary propositions turn out to be, every proposition is an expression of agreement and disagreement with truth-possibilities for them. Then, one proposition entails another provided that the first agrees only with those truth-possibilities with which the second proposition agrees. Similarly, for example, two propositions are incompatible provided that there is no truth-possibility with which they both agree.

The second insight characterizes the relationship between truth-operations and truth-functionality. Applied to elementary propositions, a truth-operation yields a truth-function of those original elementary propositions. But furthermore, the result of applying a truth-operation to truth-functions of elementary propositions is itself a truth-function of elementary propositions \[5.3\].

The third insight is due to Frege, that analysis discloses new forms of expression. That is, we might discover, through reflection on the nature of our thoughts, that their apparently immediate interconnections are actually mediated by heretofore unarticulated hypotheses.\(^{63}\) This may amount to the discovery of thoughts not even obviously expressible at all in the words we had to begin with (hence the exotic typography of *Begriffsschrift*). Thus, the domain of analysis cannot be just the thoughts which *a priori* we might try to bound by the language that is already familiar to us. Rather, it must include further means of expression whose use is to begin with merely possible. Of course, such signs with a merely possible use cannot be analyzed to begin with; so to begin with they can appear only as analysantia. But, on the other hand, one should not suppose that analysans must itself be unanalyzable, merely say, on account of its novelty or its aura of technical sophistication. For every analysans, like every analysandum, must itself be completely integrated into the system of signs, and this may in turn require that it too be analyzed.

Let’s now turn to the nature of application of logic, which begins with the totality of propositions. As we’ve just seen, this totality cannot be supposed to be bounded by the totality of actually formable sentences. So, considered concretely, the totality will not be covered by totality of *prima facie* understood actually formable sentences, but must include thoughts which can be formulated

\(^{63}\)Broadly speaking, this exposure of unarticulated hypotheses is the logicist strategy to establish the dispensability of intuition in mathematical argument.
only upon further reflection. Nonetheless, we do begin in the middle, amongst what we think we understand. And the aim is to construct an analysis.

I now turn to the question what is an analysis. For expository purposes, I want to separate the question what is an analysis from a further question when is an analysis good. In the most schematic form, an analysis is a function \( f \) such that, for some propositions \( A \), the function \( f \) associates to \( A \) a collection \( f(A) \) of other propositions. The import of this is that the analysis \( f \) says that \( A \) is the joint denial of the elements of \( f(A) \). That is, it is in the nature of an analysis \( f \) that for every \( A \) such that \( f \) analyzes \( A \), an insistence

\[ A = N(f(A)) \]

is a fundamental commitment of \( f \). By such an insistence, \( f \) consigns the assertibility conditions of \( A \) to the assertibility conditions of \( f(A) \). For, according to the insistence, a totally negative verdict on \( f(A) \) just is an affirmation of \( A \), and a total but not totally negative verdict on \( f(A) \) just is a denial of \( A \). Commitments so encapsulated can be thought of as the “local” commitments of \( f \). Whether or not \( f \) does well to undertake this or that such local commitment is a point to be investigated. Since some propositions are incompletely grasped at the outset, and indeed, since \textit{a priori} we have no idea what is to be analyzed in terms of what, we are certainly not at the outset in a position to evaluate the correctness of local commitments. We can only, at best, volunteer them tentatively, on a case-by-case basis. But as stipulated, the question whether something is an analysis at all is separate from the question whether, as an analysis it is good.

Now, among the commitments that an analysis undertakes, merely as an analysis, there are not just local commitments but also a global one. Note that it is, in general, only for some but not all propositions \( A \) that \( f \) returns a collection \( f(A) \) of propositions such that \( f \) alleges that \( A \) is their joint denial. These propositions we will say that \( f \) analyzes; the rest, that \( f \) leaves unanalyzed. Now, the global commitment of \( f \) as an analysis is this: to explain all necessary connections between propositions in terms of its analyses of propositions. Hence, \( f \) undertakes the commitment that no necessary connections obtain between the propositions it leaves unanalyzed. This means, however, that \( f \) must envisage
each total verdict of agreement and disagreement on unanalyzed propositions to constitute a genuine possibility. And conversely, since propositions left unanalyzed are supposed to be explanatorily ultimate with respect to the obtaining and nonobtaining of logical connections, \( f \) also undertakes a commitment that such verdicts on the propositions it leaves unanalyzed exhaust all possibilities. Let’s call such a total verdict, coherent or not, an \( f \)-envisaged possibility. Of course, \( f \) might be wrong about whether what it envisages is actually possible, and conversely whether it does envisage all the possibilities. But as already remarked, we will not have any way to tell this immediately since we probably do not have a good immediate understanding of the propositions that \( f \) leaves unanalyzed.

With this idea of an \( f \)-envisaged possibility in hand, we can now spell out what it means for \( f \) to secure the truth-functionality of every proposition: \( f \) must bear witness that every proposition is a truth-function of the propositions that \( f \) leaves unanalyzed. More precisely, \( f \) secures the truth-functionality of the proposition \( A \) provided that for every \( f \)-envisaged possibility \( M \), either \( f \) says that \( M \) affirms \( A \), or \( f \) says that \( M \) denies \( A \).

Given that \( f \) does secure truth-functionality, there is the further question whether \( f \) does so correctly. Now, \textit{a priori} we will probably have no understanding at all of the propositions that \( f \) leaves unanalyzed. Thus, nothing could immediately count for or against some hypothesis to the effect that such-and-such envisaged possibility does affirm (or does deny) such-and-such proposition. But, the fact that \( f \) bears witness to truth-functionality of all propositions commits \( f \) to further conclusions about logical relationships. This is because truth-functionality generates all logical relationships, and so \( f \), being an analysis, must acknowledge this. Thus, the testimony of \( f \) in regard to the relationship between possibilities and propositions commits \( f \) to verdicts on all questions as to entailment, consistency, and so on. For example, suppose that \( f \) envisages no possibility to affirm \( A \) and deny \( B \). Now, \( f \) pretends to envisage all genuine possibilities. Hence, \( f \) commits to the conclusion that there is no possibility under which \( A \) holds and \( B \) fails. But, is this right? Or does such a possibility exist? Conversely, suppose that \( f \) envisages some possibility to affirm \( A \) and deny \( B \). Since \( f \) pretends to envisage only those possibilities which are genuine, it follows that \( f \) commits to the existence of some such possibility, hence to the conclusion
that $f$ does not entail $B$. Is this right? Or actually does no such possibility exist?

In this way, we can see that the proposed analysis absorbs the structure of commitments that makes something an analysis in the first place. To be an analysis is to acknowledge that if propositions are such-and-such truth-functions then such-and-such logical relations hold. Thus, commitments about logical relationships between propositions some of which may be unknown require commitments about logical relationships between propositions all of which are known. It is because the commitments of analysis lift up into the light in this way that an analysis is correct or incorrect. The correctness condition on $f$ can now be stated more precisely as follows: that the answer to any logical question is what $f$ says it is. Thus, if $f$ is correct, then it must be if and only if some bunch of propositions does entail $A$ that $f$ says it does; and it must be if and only if such-and-such bunch of propositions is logically inconsistent that $f$ says it is, etc.

We’ve now spelled out what it is for an analysis $f$ to be good: $f$ must secure the truth-functionality of all propositions, and $f$ must do so correctly. This elaboration appeals crucially to questions about what $f$ says. In particular, we’ve observed that $f$ envisages such-and-such possibilities, and that $f$ might say that a given possibility affirms a proposition or that it denies it. We’ve also argued that if $f$ says stuff of that form, then $f$ commits also to saying things about entailment, consistency, and so on. However, we’ve dropped only hints about when $f$ must allow that a possibility $M$ affirms a proposition $A$. But the answers do follow from what has been said so far. Let $M$ be an arbitrary possibility envisaged by $f$. On the one hand, suppose that $f$ leaves $A$ unanalyzed. Recall that for $f$ to envisage $M$ is for $f$ to acknowledge $M$ as a total verdict of agreement and disagreement on propositions left unanalyzed by $f$. Since $f$ leaves $A$ unanalyzed, this means that either $f$ takes $M$ to affirm $A$ or $f$ takes $M$ to deny $A$. So, if $f$ leaves $A$ unanalyzed, we simply stipulate that what $f$ says $M$ does to $A$ is just what $f$ takes it to do. Now on the other hand, suppose that $f$ analyzes $A$. This means precisely that $f$ lets what $f$ says $M$ does to $A$ depend on what $f$ says $M$ does to the propositions into which $f$ analyzes $A$. This, after all, is what it is for $f$ to have given an analysis of $A$, i.e., for $f$ to have represented $A$ as the result of a truth-operation on other propositions. Specifically, $f$ represents $A$ as the joint denial of the propositions $f(A)$. Thus, $f$ says that $M$ affirms $A$ if $f$ says $M$ de-
nies B for all B in f(A); on the other hand, f says that M denies A if there’s at least one proposition in f(A) that f says M affirms and if for every proposition in f(A), f says either that M affirms it or that it denies it. This completes the stipulations on what f must say about M and A. We further insist that f be as quiet as possible, so that f does not say anything more about what M does to A than what is required by those stipulations.

Thus, we’ve determined that the goodness of f depends on what f says about affirmation and denial of propositions by the truth-possibilities f envisages. And we’ve now elaborated on just when f must say anything to that effect, at least inasmuch as f is an analysis at all. However, it does not follow from what we’ve said so far that f, being an analysis, actually does secure the truth-functionality of propositions. Hence, an analysis might fail to be good even before the question of correctness arises. Here are some examples—I’ll work the first in detail; the rest are similarly routine.

1. Suppose that there are three propositions A, B, and C, such that f(A) = (B), f(B) = (A), and f leaves C unanalyzed. Then, a possibility M which f envisages is simply a verdict on C, e.g., an affirmation of C. So, suppose f says that M affirms C but that f is otherwise silent; it remains to verify that f is an analysis. Clearly f says that M does to an unanalyzed proposition precisely what M does to it. And, if f analyzes a proposition, then f says that M affirms it whenever f says M denies all the propositions into which it is analyzed, i.e., never; and similarly f says M denies a proposition whenever f analyzes it into some propositions about all of which f says M issues a verdict but about at least one of which f says M affirms it. So, f says nothing to the effect that M affirms A or denies A. Thus, f is an analysis which does not secure truth-functionality of A.

2. Suppose there is a proposition A such that f(A) = A, and a C which f leaves unanalyzed.

3. Suppose that there is an infinite sequence of propositions A₀, A₁, A₂,… such that f(A₀) = (A₁), f(A₁) = (A₂), f(A₂) = (A₃), …, and that there is another proposition C left unanalyzed by f.
4. Consider each of examples 1-3, but without the C.

These examples all indicate that an analysis $f$ can fail to secure truth-functionality. But, note that all four examples have something in common, that in each case, applied to some proposition the analysis $f$ never halts. More precisely, let’s write $A \succ_f B$ provided that $B$ belongs to $f(A)$ for any $A$ and $B$. Then, in each case, it turns out that $f$ induces an “infinite descending chain”

$$A_0 \succ_f A_1 \succ_f A_2 \succ_f \cdots.$$  

For example, in the first case the chain is just $A \succ_f B \succ_f A \ldots$. To rule out these particular pathologies, it would suffice to insist that $f$ induces no infinite descending chain.

Let’s say that analysis $f$ is well-founded if it induces no infinite descending chain. The following question now arises. If $f$ is well-founded, then must $f$ secure the truth-functionality of all propositions? The answer is yes. To see this, suppose that $f$ is well-founded. Further suppose, toward contradiction, that $f$ does not secure truth-functionality of a proposition $A$. Then, $f$ envisages some verdict $M$ on propositions left unanalyzed by $f$ such that $f$ says neither that $M$ affirms $A$ nor that $M$ denies $A$. Now, since $f$ envisages $M$ as a total verdict on unanalyzed propositions, if $A$ were left unanalyzed, then $f$ would have to say either that $M$ affirms $A$ or that $M$ denies $A$. So, $f$ must analyze $A$. Generalizing, if $f$ is silent about what $M$ does to a proposition, then that proposition cannot be left unanalyzed. But now, since $f$ is an analysis, if $f$ is silent, with respect to $M$, about none of the propositions in terms of which $f$ analyzes a proposition, then $f$ cannot be silent about that proposition either. It follows that if $f$ is silent with respect to $M$ about a proposition, then $f$ must also be silent about one of the propositions in terms of which $f$ analyzes that proposition. Hence, there must exist an infinite descending chain of propositions, each of which $f$ analyzes into its successor.

We have thus given a sufficient condition for $f$ to secure the truth-functionality of all propositions: namely, that $f$ be well-founded. In a way, this condition is

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64This argument seems to require a weak form of the axiom of choice, but I am not sure if this is invariant under philosophically insignificant rearrangements of concepts.
quite natural. For example, suppose that we consider the analysis of a proposition in some sense to explain it. This explanation would itself introduce further ideas, each of which must either be understood, or be explained. An infinite descending chain of such explanations would leave the desire for understanding unsatisfied.

The concept of well-foundedness therefore appears to be of central interest to the understanding of the general propositional form. Historically, this concept was introduced by Mirimanoff (1917). And there is no reason to suppose Wittgenstein himself found it a clear articulation in the course of composing *Tractatus*. Nonetheless, it seems to me that the concept of wellfoundedness, and the structure of the induction used at T6 to fix the general propositional variable, are two sides of the same coin. If we put logic aside for a moment, and consider psychology instead, then, one may say that, so it happens, people who seek to understand the logical relationships between propositions do begin in the middle, with propositions that await analysis into propositions that are not immediately familiar. So, not as a matter of logic, but rather, as a matter of psychology, it might be said that the concept of well-foundedness better suits the predicament of the analyst.

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65 This remark is, of course, intended only as a motivational heuristic. I do think that Wittgenstein thought of analysis as, in some sense clarificatory. But, locally speaking it is not really “explanation” in any obvious sense.

66 So far as I know, the first mention of the concept in this connection is due to Goran Sundholm (1997), who suggests that the kind of order induced by the general propositional form is a well-founded partial order. One might introduce a genuine partial order relation on the totality of propositions, induced by the general propositional form, by taking the transitive closure of \( \prec \). But I don’t know of any place in the *Tractatus* where such an ordering is mentioned. There is an unfortunate tendency among some commentators to talk about the set of propositions as exhibiting this or that order type without spelling out which relation on the set so orders it.

67 But, as Goldfarb urged in conversation, it’s plausible that Wittgenstein would have learned the concept of wellordering from Russell (1903).

68 But, then again, maybe he does. As Goldfarb pointed out, 4.221 says: “it is obvious that analysis must arrive at elementary propositions, which consist of names in immediate combination”.

69 Indeed, there is a general result to bear this out. Now, let \( (\prec_f A) \) be the multiplicity of propositions \( B \) such that \( B \prec_f A \). Let \( I_f \) be the least subset of the totality of propositions such that \( I_f \) contains \( N(\prec A) \) for each proposition \( A \). It can then be shown, by an argument similar to what is given in the main text, that every proposition belongs to \( I_f \). The analogy with T6 is this. The propositions left unanalyzed by \( f \) are simply the propositions \( A \) such that \( (\prec A) \) is empty; these get put into \( I_f \) automatically and correspond to the elements of \( \bar{\Phi} \). If \( (\prec A) \) is not empty, then \( (\prec A) \) corresponds to the analysanta \( \bar{\xi} \) of an analyzed proposition, so that \( f \) can be understood to claim that \( A \) is \( N(\bar{\xi}) \).
Obviously, some mathematics remains to be explored here, but it is time to set that aside. For, we are now in a position to justify the following remark: as it is presented in T6, the general propositional form is simple, natural, and easy to understand. What T6 says is this. There is a correct analysis of the totality of propositions, which represents some propositions as joint denials of others; in turn, those others may also be so represented, or they may not be; but in any case, no such chain goes on forever. And that is all T6 says. Hence, the remark is justified. Now, given Wittgenstein’s account of analysis, it follows that an analysis of the sort described at T6 would secure that every proposition is a truth-function of the propositions left unanalyzed. Such an analysis, being correct, thereby gives the answer to every question as to which propositions follow from which others, as to which propositions are jointly inconsistent, and so on. So in particular, the analysis would be a description of all logically true propositions. But this result is unsurprising, because the answers to such questions are precisely what must be known in order that the analysis have been discovered in the first place.

1.3.2 Origins of the independence criterion

The style of my discussion in this chapter has been seemingly aprioristic. Nonetheless, I assert a definite interpretive claim: that no proposition is elementary in itself, but is elementary only in virtue of the analysis of the entirety of language. Analysis represents all propositions to be truth-operations of some fixed bunch propositions in such a way that the resulting iterative structure explains all necessary connections between states of affairs. Those propositions not represented as results of truth-operations are the elementary ones; it is a trivial corollary of the completeness of analysis, and the means by which analysis predicts the obtaining of necessary connections, that no necessary connections obtain between elementary propositions. But did I just make all this stuff up? In this section, I survey some discussions from Wittgenstein’s NB3, and claim to pinpoint the precise moment when Wittgenstein establishes this conception of elementariness.

Between April 1916 until October 1917 Wittgenstein produced his third wartime notebook. Unlike its predecessors, this volume contains signs of a concerted in-
vestigation of the general form of the proposition. It opens at 15.4.16 with an anticipation of T5.556: “We can only foresee what we ourselves construct”. This remark, I think, encapsulates a tension that pervades the Notebooks and maybe the Tractatus as well. What follows this remark is the very natural question: “But then where is the concept of a simple object still to be found?” For, Wittgenstein seems to think that to have a concept of the simple objects would entail foreseeing their forms or combinatory possibilities. Since, however, neither simple objects nor their forms are constructed by us, he’s therefore perplexed by the question how we could grasp the concept of simple object. But why does Wittgenstein think we know what a simple object is? In the previous notebook, and almost a year before (19.6.15), Wittgenstein had concluded that the concepts of thing, relation, property and so on cannot be derived from experience. Were they to depend on experience, then they could not be used in logic. But we do make use of them in logic, as with variables $x$ or function notations $\phi x$. At 15.4.16 the grounds for thinking we must grasp a concept of simple object are basically the same. He says: “We must be able to construct the simple functions because we must be able to give each sign a meaning. For the only sign which guarantees its meaning is function and argument.” Here, the point must be that one constructs a “simple function” by abstracting a name of a simple object from a simple proposition; hence, knowing what a simple object is would seem to be required to confer meaning on signs.

Wittgenstein’s reason for saying that the function-argument pattern guarantees its own meaning must be that the function-argument pattern is the pictorial form of a proposition which partakes of it. For, pictorial form would be what is common between the proposition and the situation depicted, so that once objects are assigned as meanings to the constituent names, then a combination of names under that form says something without further ado. Now, this assignment of meanings to names requires seeing (“simple”) propositions as functions of names in the first place, and thus requires the ability to “construct the simple functions”.

\footnote{Inasmuch as Wittgenstein’s question is what a proposition is, of course, everything he does is an investigation of the general propositional form. In particular, the persistent questions in Notebooks 1 and 2 about subject-predicate form in unanalyzed propositions seem to me to involve the fundamental issues of foreseeability which motivate conviction of the existence of g.p.f. But not really until Notebook 3 do we find concerted attempts to articulate the g.p.f.}
i.e., the functions which result from elementary propositions by turning a name into a variable.

The next day’s entry (16.4.16) begins, “Every simple proposition can be brought into the form $\phi x$”⁷¹. It seems clear that “simple proposition” is a terminological variant of “elementary proposition”. Since 15.4.16 anticipates 5.556, and the 5.55s address the question of foreseeing the number and forms of elementary propositions, it looks like this is what’s really at stake here early in NB3. So, the first sentence of 16.4.16 seems to be that the form $\phi x$ should allow us to specify the totality of elementary propositions. This, I think, is confirmed by next couple of paragraphs.

That is why we may compose all simple propositions from this form $[\phi x]$. Suppose that all simple propositions were given to me: then it can simply be asked what propositions I can construct from them. And these are all propositions and this is how they are bounded.

As the Notebooks’ editors observe, the last two sentences anticipate 4.51, itself a commentary on the first mention in the Tractatus of the general propositional form at 4.5. Thus, it is hard to resist the conclusion that at 16.4.16 Wittgenstein identifies two, jointly determinative sources of the general propositional form: namely first, the general form of what is not constructed, i.e., the general form $\phi x$ of simple propositions, and second, the general form of construction of propositions from propositions. Granted an analysis of possibilities of construction, a bound on the totality of propositions would derive from a bound on the elementary propositions. And so, Wittgenstein’s aim here seems to be to find a bound on the totality of propositions by identifying a common form $\phi x$ from which all elementary propositions “can be composed”. What expresses this common form would be a variable which takes the totality of elementary propositions as its range.

⁷¹On my understanding, Wittgenstein has said here only that it must somehow be possible to isolate a name in a proposition, but not in any way explained how this should takes place. (For example, presumably the constituents of ordinary sentences are not names.) While the mere possibility of such isolation is a purely logical result, the account of how the possibility is realized would be a result of the application of logic.
The details of this idea are not entirely clear to me. As a first pass one might speculate that Wittgenstein proposed that an elementary proposition

Quine respects Frege and Russell

is analyzable as the value of a function

$$\phi x$$

as applied to

( ) respects Frege and Russell

and

Quine.

Thus, the proposal somehow relies on a decomposition of elementary propositions into names and elementary propositional functions. This might be thought to be some kind of extension of the function-argument method of range-fixing described at T5.501. However, since at least in the _Tractatus_, Wittgenstein holds that it is not just in elementary propositions but in any proposition that a name can be turned into a variable. So it is not clear how function-argument structure could distinguish elementary propositions from arbitrary ones.

Despite the difficulties of understanding this discussion in detail, it seems clear that at 16.4.16 Wittgenstein tries to establish the existence of a general, distinguishing form of elementary propositions. This form would find expression in a variable, thereby subserving an articulation of the general form of all propositions. Seven months later, such a general strategy reappears in the following.

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72 Of course, I don’t claim the below is an elementary proposition, let alone one envisaged by Wittgenstein.

73 Speculation along these lines may draw some further support from the last of the April 1916 Notebook entries just now considered, namely 27.4.16. But there, it’s worth noting that the decomposition is said to issue in three arguments, $$\phi x, \phi(\_),$$ and $$\phi x$$ is presumably a descendant of the concept of logical form from Russell’s _Theory of Knowledge_ manuscript, and an ancestor of the _logische Urbild_ of 3.315.

74 I agree with, or will double down on—the suggestion in Ricketts (2012) that that this kind of approach does give an instance of genuinely quantificational generalization with respect to the predicate-position, and that it probably explains Wittgenstein’s use of such generalization at (5.5261).
The fact that it is possible to erect \textit{[muss sich aufstellen lassen]} the general form of proposition means nothing but: every possible form of proposition must be foreseeable. [21.11.16]

Thus, the foreseeability of logical forms guarantees not just that some form is common to all propositions, but moreover that the form explains how those forms could be known to us \textit{a priori}.\footnote{Note that the Prototractatus links foreseeability of logical form with determinacy of sense.}

So, in November 1916, Wittgenstein resumes his analysis of the general propositional form systematically. After declaring that the analysis ought to secure the foreseeability of all propositional forms, he writes: “We now need a clarification of the atomic function and the concept ‘and so on’”. This, evidently, recalls from 16.4.16 the dichotomy of determinative sources of the general propositional form. After concluding the 21.10.16 entry with discussion of the “and so on” half, he returns at 23.11.16 with the following suggestion:

What does the possibility of the operation depend on?
On the general concept of structural similarity.
As I conceive, e.g., the elementary propositions, there must be something common to them; otherwise I could not speak of them all collectively as the “elementary propositions” at all.
In that case, however, they must also be capable of being developed from one another as the results of operations. [23.11.16]

In this passage, Wittgenstein makes another attempt to construct a variable ranging just over the elementary propositions. The plan seems to be to specify them by means of a form-series variable. This, Wittgenstein seems to think, must be possible merely if elementary propositions have some commonly distinctive structural feature.\footnote{Such a linkage of formal features with formal series reappears in somewhat attenuated form in the Prototractatus, in particular in the PT5.00534s, though there formal commonality appears as a necessary but not sufficient condition for specifiability of some bunch of propositions by a form-series.} So, it seems we have here a clear use of an assumption I re-
jected in Ricketts, namely that the instances of a universal generalization could be given by means of a formal series.\textsuperscript{77,78}

In any case, it is now evident that actually in the Notebooks, Wittgenstein made two attempts to identify the common form of the elementary propositions. Throughout these attempts, Wittgenstein’s ultimate aim was to construct the form of all propositions. His strategy was to articulate the general propositional form in two stages. First, identify that form which is common to the elementary propositions. Second, identify the general form of construction of propositions from propositions. The ground for Wittgenstein’s aim to articulate the general propositional form was that the forms of all propositions must be foreseeable to us \textit{a priori}. The details of Wittgenstein’s strategy for articulating this form demonstrate that the commitment to the foreseeability of the forms of propositions involved a commitment to the foreseeability of the forms of elementary propositions.

Sullivan is correct in saying that nothing remains of these attempts in the \textit{Tractatus}. And as he observes, the 5.55s insist on their futility. Nonetheless, Wittgenstein’s presentation of the general propositional form in the \textit{Tractatus} presupposes some concept of the elementary proposition. But what, in the \textit{Tractatus}, is the basis of our grasp of this concept?

Wittgenstein winds down his second, 23.11.16 attempt at articulating the forms of elementary propositions with the following remark: “if there really is something common to two elementary propositions which is not common to an elementary proposition and a complex one, then this common thing must be capable of being given general expression in some way.” I think that the phrasing of the hypothesis here already evinces some doubt that some formal feature does characterize propositions as elementary. Indeed, I claim that a few days later, Wittgenstein rejected the hypothesis altogether. A few days later, in a single-line entry he writes:

Either a fact is contained in another one, or it is independent of it.

\textsuperscript{77}I suspect that Ricketts worked out his interpretation with an eye on these early passages. \textsuperscript{78}There is some reason to think that Wittgenstein may at the time have conceived of the second approach to be a refinement of the first approach. For, at 22.5.15 (which I’ll discuss in more detail shortly) Wittgenstein suggests that quantificational generality can be analyzed by means of form-series.
Rephrasing, this says that dependence of one fact on another is a sufficient condition for the containment of one by the other. I suggest that Wittgenstein hereby analyzes containment in terms of dependence. Granted that elementariness was antecedently characterized as containment of no further propositions, the remark would then reduce the concept of elementariness to the concept of dependence. As for the notion of dependence, it’s clear that in other passages in the Notebooks, as well as in the Prototractatus and Tractatus, Wittgenstein takes this to be a purely logical, and indeed purely truth-functional notion. Thus, I conjecture that at 28.11.16, Wittgenstein proposes a purely truth-functional characterization of the concept of elementariness itself. The details of such a characterization are not explicit there. But some such details are certainly implicit. For example, assuming Wittgenstein intended that elementary propositions do not contain each other, 28.11.16 implies outright that elementary propositions are truth-functionally independent.

A month or so later, some consequences of such a truth-functional characterization of elementariness emerge explicitly:

It is clear that the logical product of two elementary propositions can never be a tautology.

If the logical product of two propositions is a contradiction, and the propositions appear to be elementary propositions, we can see that in this case the appearance is deceptive. (E.g.: A is red and A is green.)

These remarks follow what I take to be the sharpest expression of the general propositional form in the Notebooks, in which for the first time Wittgenstein speaks of the totality of propositions as generated in terms of a single operation. I take the 8.1.17 entry to endorse an alternative account of elementariness, an account which remains in the Tractatus. On this new approach, nothing intrinsic to a proposition makes it elementary. Rather, to be elementary is simply to be independent of all other elementary propositions. (Cf. 4.211) Thus, the class of

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79 It’s clear that dependence has no conceivable nonlogical meaning in the Tractatus. As for dependence in NB, see for example 10.6.15a.
elementary propositions is characterized by the mutual independence of its elements. Of course, many classes are so characterized. However, not every such class is such that every proposition results by truth-operations from its elements, so that all necessary connections between propositions derive from the structure of the operational genesis. That, and only that, is what identifies a class of propositions as the elementary ones. It is from this conception of elementariness that there originates the idea of states of affairs, whose mutually independent possibilities of obtaining and non-obtaining explains the truth-susceptibility of elementary propositions and thereby, via the method of constructing propositions from propositions, also explains the structure of the totality of propositions. Only in arriving at this conception of elementariness at the end of NB3 does Wittgenstein at last make room for the remarks of the 5.55s, whose earliest known ancestors appear in the Prototractatus.

It follows from this conception of elementary propositions that, if it is in virtue of the unsurveyability of elementary propositions that the general propositional form is, somehow or other, transcendental, then the ultimate analysis of any one proposition is itself also transcendental. For, on this conception, what identifies a proposition as elementary is only its relationship to all other elementary propositions, together with the role that the entire class of elementary propositions plays within the global structure of the totality of propositions. So, distinguishing any proposition as elementary requires recognizing all elementary propositions as elementary. But some propositions are distinguished as elementary by a complete analysis of any one proposition. Hence, a complete analysis of any one proposition requires surveying the totality of elementary propositions. And surveying the totality of elementary propositions was what was supposed to prevent the general propositional form from appearing as a variable in the determination of sense of an ordinary proposition.

Thus, the 1s and 2s appear to present a dogmatic and aprioristic theory of the internal structure of elementary propositions, naïvely inspired by the surface structure of ordinary language. But these opening remarks are grounded in Wittgenstein’s conclusion at the end of NB3, that we can’t have knowledge of the sort to which they tempt us.
Chapter 2

Objectual generality

Don’t get involved in partial problems… (1.11.14n)

In the late 1970s Robert Fogelin sparked a little industry by defending a negative answer to the following apparent question: “is the system of the *Tractatus* adequate to First-Order-Logic-with-Equality?” In the early 1980s, Peter Geach and Scott Soames rebutted Fogelin’s argument. Actually, according to them, “the system is adequate.” It is now a rite of passage for would-be *Tractatus* scholars to promise to end the ensuing debate.

I would like to break with tradition, and argue that there is much more to be said than the disputants acknowledge—indeed, that there is more to be said than I could possibly have said here. It seems to me that the very phrasing “is the system adequate?” already begs the question against the *Tractatus* outlook on signs, on sign-systems, on what these things do for us, and what they do to us.

Juliet Floyd has suggested that one of Wittgenstein’s overarching aims in the *Tractatus* is to “break the metaphysical idolatry of notation”, and expressed doubt that Wittgenstein had any interest in constructing a smoothly functioning system of notation. I agree that the *Tractatus* aims to disrupt the sort of smooth functioning in logic or philosophy into which notational systems can seduce us. One source of value in the *Tractatus* is its power to reawaken us to questions that are silenced by the well-practiced movements of the muscles in our hands.

Now, let me be clear. I presume that it is a straightforward historical question whether Wittgenstein grasped the elementary syntactical functioning of quan-

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82 It seems to me that this is why Wittgenstein, as Goldfarb [[cite]] emphasizes, does not put forward any particular *begriffsschrift* of his own.
tificational logic, just as it is a straightforward question whether, say, Herbrand grasped it, and just as it is straightforward, if actually questionable, whether an undergraduate student in the exam last week grasped it. Fogelin has even raised doubts about this. It is notoriously difficult to resolve certain varieties of skepticism about matters of historical fact, and I will mainly leave this problem to professionals like Geach and Soames.

So although Fogelin’s provocation deserved its torrent of enthusiastic rebuttals, their very enthusiasm strikes me as somehow contrary to the spirit of the *Tractatus*. And perhaps it is discomfort with such choruses of affirmation that provides the main animus of this chapter. But let’s get down to some sharper points of doctrine and methodology.

First and foremost, these rebuttals of Fogelin all aim to present some smoothly functioning system of notation, a purported best possible candidate for being what Wittgenstein himself simply disdained to formulate, perhaps on account of his literary taste, or sense of station in life. However, it was part of the very idea of giving the general form of the proposition that it contain only what is essential to any system of signs whatsoever. Thus it was Wittgenstein’s aim precisely to prescind from any artifacts of notation, ever so apt as they eventually are to be mistaken for determinations of the subject matter. Perhaps Wittgenstein’s most characteristic criticism of Frege and Russell, after all, is that they pervasively mistake formal relations for material ones, widgets of notation for ways of the world. Fogelin’s respondents therefore fuel the very march of notational hegemony that Wittgenstein sought to disrupt. I will therefore seek to lay out Wittgenstein’s conception of quantificational generality in a way that gives no satisfaction to the lust for smooth functioning, and yet do so clearly. And I will also try to illustrate Wittgenstein’s strategy of prescinding from features of the sort that are liable to be confused with what is essential to all articulations of objectual generality whatsoever—trying to see, as McCarty (1991, 61) puts it, just what we can get away with, and what we cannot get away without.

A second, and related point, is that much of contemporary logical practice arose in a tradition that stems from Hilbert. Good logical training thus prompts

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83I have in mind in particular the remark that Ramsey was a “bourgeois thinker”.
the interpreter to foist onto the *Tractatus* a way of thinking about representation which is foreign to it. Signs become mute concatenations of surds, on which we confer referential powers by semantic interpretation. In the *Tractatus*, the vehicles of picturing become open formulas, wherein the natural proxies for objects are free variables which, lacking quantificational governance, require an imposition of value to contribute their share to the truth-bearing load. But what, in the context of the *Tractatus*, is the source of this awaited imposition? At this point, it is all too tempting to fall back on the suggestion of Peter Hacker: “a mechanism of a psychological nature is generated to project lines of projection onto the world” (quoted in Goldfarb 1981). In the context of *Tractatus*, it is precisely backwards to follow the order of explanation established by Hilbert and Tarski. Wittgenstein’s conception of logical structure begins with elementary picturing facts, arragements of objects which are *ipso facto* sayings that objects are so arranged in the world. Only then, once picturing has begun, does generality arise. “The truth or falsehood of a general proposition palpably depends on that of elementary propositions”. Generality is inflicted on picturing facts, through the selective denaturing of their nominal constituents. Thus, we have in the *Tractatus* not naming achieved by assignment of value to anonymous variable, but variation achieved through the anonymization of names.

As we go along, the points just mentioned will mainly advance themselves by rhetorical insinuation—the reader is hereby forewarned. But there is a third, more substantial claim, which is the eventual positive thesis of this chapter. To state the thesis, we need a little background. Wittgenstein famously insists that every elementary proposition has the possibilities both of truth and of falsehood. Moreover, the truth-value of every proposition is a function of the truth-value of elementary propositions. But, given these claims, it is not at all clear how to accommodate statements of identity. On the one hand, a given statement of identity does not seem to have the two poles of truth and falsehood, and so does not seem to belong among the elementary propositions. On the other hand, since Wittgenstein rejects the identity of indiscernibles (5.5302),

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84 The preceding few sentences are indebted to McCarty (1991).
85 Indeed, every genuine proposition has both possibilities, but I don’t need this subtler point here.
nor does identity seem to be explicable as a truth-function of elementary propositions. Yet, it seems to be in virtue of the relation of identity that, together with the quantifiers, we express such genuine facts as that there are two cups on the table. So, the story goes, Wittgenstein has a problem, which drives him into a deviant reinterpretation of the objectual generality, one which had surely never been palpably implemented in any rational discourse. This deviant reading, so it happens, enjoys the ideologically convenient feature that it allows all the “genuine facts” ordinarily expressible with the equality predicate to be expressed without the equality. Thus, Wittgenstein is backed up into a distortion of the concept of generality, a concept whose analysis was a fundamental project of his logicist predecessors—and indeed, a fundamental project of his own. This story radically underestimates Wittgenstein’s power as a philosopher. As argued in Chapter 1, Wittgenstein’s conception of analysis excludes all necessity but the logical, and the resulting need to explain necessities of the world leads him to a conception of propositions as picturing facts. I’ll now argue that the conception of propositions as picturing facts supports a distinctive account of propositional functions; in turn this account of propositional functions entails the redundancy of the equality predicate. Perhaps it is overstating the point to say that the redundancy of equality is merely a corollary to the idea that propositions are pictures. Rather, one may think of the role of relational structure and of identity in ordinary presentations of first-order logic as embodying one conception of the nature of mathematical multiplicity in representation. On this usual approach, although certain rules of inference or conventions of model-theoretic treatment single out the equality predicate as nominally logical, nonetheless identity manifests itself grammatically as a material relation. Wittgenstein reached another conception of mathematical multiplicity on which identity becomes genuinely formal, pertaining not to what a sentence says about things, but rather to what a sentence must have in common with things in order to say something about them. The conception of propositions as pictures embodies a deep and, I think, mostly unplumbed logical insight.\footnote{Another appreciation of the unity of Wittgenstein’s ideas about equality and the variable appears on Wehmeier (2012); there are also affinities with ideas of Kit Fine (2000).}

The chapter runs as follows. 2.1 and 2.2 build up some framework and termi-
nology for discussing first-order logic in the context of the *Tractatus*. 2.3 presents a precise, perhaps even idealized, formulation of Fogelinite skepticism; I suggest that this skepticism points to a philosophically important feature of Wittgenstein’s discussion. 2.4 summarizes the Geach-Soames response, concluding with some questions about the precise status of their interpretive claims. In 2.5, I begin by summarizing the Hintikka-Wehmeier reconstruction of Wittgenstein’s nonstandard interpretation of the variable. I then pose the problem of reconstructing Wittgenstein’s analysis using purely *Tractatus* resources, and contrast Wittgenstein’s account of propositional functions with the Fregean approach to which Wittgenstein’s has been recently likened by Thomas Ricketts (2012). In 2.6, I outline my own interpretation. The chapter concludes in 2.7 with an informal statement of the underlying mathematical framework I propose.

2.1 What is the issue?

I take it that nobody wants to contest whether there is, roughly speaking, such a thing as “the system” of Frege’s *Begriffsschrift*, or whether it is “adequate” to first-order logic. But it’s not quite obvious why this should be so. Frege jumps from truth-functional logic to second-order logic, never pausing to isolate first-order logic as a system in its own right. Moreover, he does not explicitly define a class of notations as the grammatical sentences of the system. Nor does he spell out completely the principles licensing passage from one judgment to another. And, of course, in 1879 Frege does not define any concept of semantic value of a formula, hence, also, no semantic concept of logical consequence. Still, it looks like it is safe to attribute to Frege in 1879 some sure grasp of the “essence” of first-order logic, because he demonstrates by example the structure of a sufficiently intricate system of proof.

In the *Tractatus*, however, no such system of proof appears. Indeed, this is no accident, for Wittgenstein holds that proof is merely a mechanical expedient for recognizing tautologies in complicated cases. In the development of astronomy the telescope becomes, at some point, practically necessary; and likewise, inasmuch as our judgments are, being judgments, suspended in logical interrelationships of implication and contradiction, proof may be practically necessary
to finding out just where we are. But these necessities are in each case no more than practical; they are the needs of people with limited powers of vision or understanding. When Frege advertises that the *Begriffsschrift* will illuminate the dependence of some truths on others lower down a tree at the base of which one finds truths which belong there because their being truths is sufficiently evident, Wittgenstein complains that Frege injects into logic a kernel of psychology.\(^87\)

So it is not really clear how the question of adequacy to first-order logic gets a foothold in the *Tractatus*. Where, in that book, is logic at all? Here, I think it might help to think not about proof, but instead about sense. First-order logic is, among other things, a recipe for language. Given some simple, primitive words—say, some names of individuals, and expressions of properties and relations between individuals, then the recipe guarantees a whole system of possibilities for distinguishing between ways in which the world might be. For example, given some names of Alice and Bob, and expressions for being a person and for loving: the recipe guarantees the possibility of saying that Alice loves Bob, or that she loves somebody who loves Bob, or that she loves only three people, and so on.

On the other hand, for example, the system does not secure the possibility of saying that Alice loves only finitely many people: whatever it secures which you can truly say of every situation in which Alice loves any finite number of people must be true of some other situations as well. Nor, for that matter, does it give a way to say only that she might’ve loved Bob.\(^88\)

At T4.5, Wittgenstein writes:

> It now seems possible to give the most general propositional form: that is, to give a description of the propositions of any sign-language whatsoever in such a way that every possible sense can be expressed by a symbol satisfying the description....

So our question begins to get a foothold. First-order logic is a recipe for a family of languages, which in turn guarantee the expressibility of many senses. On the other hand, in giving the general propositional form, Wittgenstein promises

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\(^87\) Now, I’m not sure if this is actually Wittgenstein’s complaint. Wittgenstein’s complaint “it is remarkable that a thinker as rigorous as Frege appealed to the degree of self-evidence as the criterion of a logical proposition” seems rather to object that the criterion makes logicality arbitrary.

\(^88\) Cf. 5.525.
to describe what is common to all the senseful expressions in all possible languages. This general propositional form had therefore better somehow accommodate first-order languages in particular. Now we do have some grounds for concern. For Wittgenstein takes the existence of the general propositional form to have the following consequence.

A proposition is a truth-function of elementary propositions (T5).

But, this looks like an implausible, if not just completely garbled, description of first-order formulas.

For Wittgenstein, something is an \(X\)-function of some other stuff provided the matter of its \(X\)hood depends only on the matter of the \(X\)hood of that other stuff. So in particular, T5 says that the truth-or-falsehood of a proposition depends only on the truth-or-falsehood of elementary propositions. This implies that some propositions are such that the truth-and-falsehood of any proposition is entirely determined by their truth-and-falsehood. As Wittgenstein remarks,\(^90\) every proposition is a truth-function of itself. So the teeth must be in the condition that the propositions in the decisive class are elementary.

By insisting that the truth-values of elementary propositions suffice to decide the truth-values of all propositions, Wittgenstein takes a crucial step away from Frege and Russell, and toward a clear notion of first-order logical validity. For under the assumption that every object has a name,\(^90\) a total verdict on the truth-and-falsehood of elementary propositions becomes the elementary diagram of a structure as understood in standard model theory. Wittgenstein’s truth-functionality thesis is therefore simply the thesis that what determines the truth-value of a proposition is the elementary diagram of a structure. Logical validity then becomes truth in all first-order structures over a given domain and signature. Although for Wittgenstein, it is possible to generalize with respect to every simple meaningful constituent of an elementary proposition, such generalizations yield no genuine higher-order resources.\(^91\) In particular, Wittgenstein’s approach

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89 At T6.

90 If this boggles—as it does Hugh Miller III (1995), then grant him what we grant his logic teacher, who held that objects appear in propositions as names of themselves. Alternatively, see Shoenfield (1967), or even Quine himself (1998).

91 For details on my account of Wittgenstein’s treatment of higher-order generality, see Chapter
invalidates the second-order comprehension axioms which allow Frege to define the ancestral (4.1252), and similarly invalidates Russell’s axiom of reducibility (6.1233). As Wittgenstein puts it elsewhere, the sense, or truth-functionality, of a proposition is its agreement and disagreement with truth-possibilities for elementary propositions. Far from obscuring the possibilities of first-order sense-making, the truth-functionality thesis opens a door to an account of them.

The truth-functionality thesis therefore supports the interpretive conjecture that in the *Tractatus*, Wittgenstein breaks from Frege and Russell by making room for an essentially first-order conception of logic. If the *Tractatus* has some problem with first-order logic, this must be a problem with the details of Wittgenstein’s account of the way in which propositions achieve their truth-functionality. For example, perhaps Wittgenstein holds that every proposition achieves truth-functionality because it is elementary. Since elementary propositions are logically independent, Wittgenstein would hold that no proposition contradicts another. But then he could not accept even an uncontested fragment of first-order logic.

The issue must therefore be as follows. According to Wittgenstein, some propositions are elementary. But, logical interdependence between propositions arises only with the introduction of propositions which are nonelementary. So Wittgenstein’s treatment of nonelementary propositions needs to account for the complexities of logical interdependence.

Allegedly, this account is defective. For example, maybe Wittgenstein implies, on purpose or not, that not all apparent possibilities of first-order sense-making are genuine. Or maybe what he says is just so obscure that nothing could determine whether it is incomplete or confused. In other words, the issue is not settled without a cogent reconstruction that also fits the text.

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2.2 Nonelementary propositions

Wittgenstein holds that the phenomenon of logical interdependence between propositions is already acknowledged, albeit falteringly, in the means by which we find them expression. As he puts it,

The structures of propositions stand in internal relations to one another. (T5.2)
In order to give prominence to these internal relations, we can adopt the following mode of expression: we can represent a proposition as the result of an operation that produces it out of other propositions (which are the bases of the operation). (T5.21)
An operation is what has to be done to the one proposition in order to make the other out of it. (T5.22)

Consider, for example, two people who in some respect disagree, so that things cannot at once be as the two think together. It may then arise that such disagreement can be traced to a single point. In particular, Dave may learn enough to find that he and Carol disagree when he hears Carol claim that Alice loves Bob. In Wittgenstein’s terminology, Dave finds, through this one claim by Carol, that his position lies outside of Carol’s. So, rather than expressing his contrary position *sui generis*, Dave now follows an alternative custom: he expresses his position as the result of denying what Carol said. Language already contains instruments for representing a proposition as the result of, so to speak, doing something to another proposition.

This talk of “doing something to a proposition” can mislead. What is actually carried out by Dave is a verbal construction, whereby he represents his own intellectual position as “lying outside” of the position expressed in Carol’s claim. It is correct that his position, or better, his occupying the position, is in some sense the result of denying Carol’s. But, there is nothing in the nature of the positions themselves such that one must be specified by means of the other and not vice versa. Maybe an analogy would be useful here. One might say that the Red

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93 Compare Frege’s talk of the Anwendung and Ergebnis of a Verfahren in Begriffsschrift’s Chapter III.
Sea is four hours’ drive from the Dead Sea. But, the Red Sea is not itself the result of a drive. Of course, one might happen to end up at the Red Sea, having driven from the Dead Sea. But one could also have driven, walked, or ridden a camel there from elsewhere, or perhaps even just be indigenous. Certainly, there is no interesting sense in which the generic circumstance of being at the Dead Sea is, as such, prior to the circumstance of being at the Red Sea. Similarly for the positions of Carol and Dave: you can get to each from the other. Note, furthermore, that this getting from each sea to the other by camel or by car does not make it the case that the seas have such-and-such geographical relationship. Rather, the geographical relationship between the seas makes it possible to travel between them. Similarly for the intellectual commitments achieved by Carol and Dave: it is that the propositional content of the one commitment be the contradictory of the propositional content of the other, which makes it possible for Dave to reach his commitment by expressing a denial of what Carol says.

Wittgenstein thus takes propositions to stand to each other in logical or internal relations. It is in virtue of such relations that, say, two propositions cannot both be true, or, more generally, that a proposition excludes the truth of each of several other propositions. To say that propositions bear such internal relations to each other is just to say that the propositions depend on each other for their truth and falsehood. These internal relations are the wellspring of Wittgenstein’s account of the truth-functionality of nonelementary propositions.

Toward a sketch of this account, we need first to develop one slightly tricky idea. Wittgenstein inherits from Frege and Russell a heuristic for starting with a relation between items, and reconceiving the relation as a procedure for locating an item by reference to the totality of items to which it bears that relation. In a little more detail, the idea is this. Say that a relation is extensional when no two items in its field agree exactly on the items to which they bear it. (Thus, being-mother-of is extensional, but being-parent-of is not.) Then an extensional relation has the feature that any item in its field can be seen as the unique item which bears the relation to exactly such-and-such items. It is for such a relation, that Wittgenstein plans to find an associated operation, such that the result of applying the operation to those items yields that item which bears the relation exactly to them.
Let’s return to Carol and Dave. Carol made a claim which, as it were, leaves no room for things to be how Dave takes them to be. Of course, there is no single specific way in which Dave takes the world to be, but, rather, his beliefs leave no room for things to be as Carol says they are. Thus, Carol’s claim stands in an internal or logical relation to Dave’s overall view. But, this relation, contrariness, is not extensional. For example, Dave’s position could be weakened or strengthened in various ways while preserving its relation of contrariness to Carol’s claim. However, a consequence of Dave’s position, that we have already mentioned, is another, weaker position, namely that things are not as Carol says. This position is not only contrary, but rather, in the jargon, contradictory, to what Carol says, because it is guaranteed to be correct whenever Carol’s claim is incorrect. Now, the relation of contradictoriness is extensional: there is only one position which allows the world to be in only those ways not allowed by any of some bunch of (other) positions. We now reach a means of characterizing any proposition in the field of the truth-exclusion relationship, namely, as that weakest position which excludes the truth of such-and-such propositions. But of course, Wittgenstein presumes that every proposition does have such a weakest contrary, or contradictory. This is obvious, because to find the denial, we just need to phone Dave. Thus, every proposition can be characterized as the denial of some other propositions.

Here, finally, is Wittgenstein’s audacious account of the truth-functionality of nonelementary propositions. Say that a conversation is maximal, provided that absolutely everything that could ever be said under any circumstances will at some point be said in that conversation. You are seated at a table with some other people, and you begin to talk. Actually, people may have already been sitting at this table, talking, perhaps for an unboundedly long time. You can say whatever you want (at last!). But, each contribution must always assume one of two forms. First, you may assert an elementary proposition. Second, you can deny a bunch of stuff that some other people said. (More precisely, a move of the second kind will find an expression “that is all bunk”, where the reference of the demonstrative has been rigorously specified). The claim of proposition 6 of the *Tractatus* is that if people actually do this, as much as possible, subject to those two constraints, then the resulting conversation is maximal. This claim of maximality
underpins Wittgenstein’s account of the truth-functionality of nonelementary propositions.

Let us now see how the idea just sketched is supposed to work. Consider a completely arbitrary proposition; we aim to show that it is a truth-function of elementary propositions. Since an elementary proposition is a truth-function of itself, we can assume this proposition to be nonelementary. By the assumed maximality of the table-talk, at some point this nonelementary proposition will have been asserted at the table. Thus, for the given proposition to be true is precisely for a truth to have been asserted at the table at that point. By the constraints on table talk, the nonelementary proposition must have been expressed in the form “that is all bunk”, where the reference of the demonstrative has been rigorously specified. Of course, this reference will be to some bunch of propositions that had been asserted antecedently. We have then the following: that the given proposition is true iff each of those other propositions is false. Each of those propositions must itself either be elementary, or be the joint denial of some other stuff. Since truth-functionality of each elementary proposition is presupposed, it suffices to continue to unfold the reference of the demonstrative in each of the denied nonelementary propositions. And so on.

Will this, actually, work? Of course, it is clear that we do not have here a proposed practical guide to achievement of understanding. Rather the question is: under the given conditions, must the truth-functionality of each proposition be fixed? The answer, for all that I’ve so far said, is no. For nothing I’ve so far said rules out that the proposition whose sense we interrogate be the origin of an infinite descending chain of denials.

Fine. We reject by fiat the existence of any infinite descending chain. That is, every process of unfolding the demonstrative of nonelementary propositions must, for each of its branches terminate in an elementary proposition. But now, the unfolding process is guaranteed to yield a determinate condition of agreement and disagreement with truth-possibilities for elementary propositions. That is, the truth-functionality thesis holds of everything expressed at the table. By the maximality assumption, we have therefore established the truth-

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94 By the principle of recursion on a well-founded relation.
functionality of everything that can be said.

In the account just sketched, we’ve slid over a big gap. Namely, when has the reference of a demonstrative been “rigorously specified”? Note that these are plural demonstratives we are dealing with here. Maybe the speaker will not have enough fingers to point to each of the referents. Suppose, more generally, only finitely many propositions can ever be specified as a basis of denial. Then, every proposition ever expressed in a maximal conversation will depend for its truth on only finitely many elementary propositions. In that case, Wittgenstein could not acknowledge the expressive possibility afforded by first-order logic of depending, for the correctness of what you say, on the truth or falsehood of *a priori* perhaps infinitely many elementary propositions. Then the *Tractatus* would indeed be, in the jargon, “inadequate”. At another extreme, maybe these conversants at the table can, as in a platonist dream world, for any collection of propositions whatsoever, fix on it and deny precisely its elements. It is routine to show that under such a hypothesis, for every class of structures over our background signature, i.e., for every class of total possible ways for the world to be, there would then be a proposition expressing exactly that the world is in some one or other of the ways in that class. The first-order definable classes of ways would be expressible too, piddlingly. But of course, we should remember who and what we are talking about here. This is Wittgenstein, student of Frege and Russell. Frege’s writings throughout his career are filled with amusing screeds about the purported concept of “aggregates.” Similarly, Russell’s response to the inconsistency of naïve comprehension was the opposite of the now-standard response—he felt that the underlying problem was with the notion of a class-as-one-in-extension, an alleged “unity constituted by a many”. Russell thus retreats to classes-in-intension, finding logical safety in constraints on propositional structure. Wittgenstein inherited this suspicion from his teachers, and, like Russell, took the prior and clearer notion to be that of a class-in-intension, the class as characterized by its defining property. Consequently, we just cannot take the notion of set for granted here.

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95 By Koenig’s lemma.
96 A basic problem with this suggestion is that it leaves completely obscure how a speaker communicates to their audience which collection has been chosen.
97 Thanks to Ori Simchen here.
The control of infinite multiplicities must be exerted in some other way.

It is at 5.501 that Wittgenstein attempts to fill the gap, by designating three Ways to specify the elements of the basis of an application of a truth-operation. The first Way is just to point out elements of the basis directly or list them yourself, but according to Wittgenstein this works only if the number of elements of the basis is finite. The second Way is Wittgenstein’s route to what ought to pass, in the wilds of the *Tractatus*, for what we should call quantificational generality. If the second Way leads into the mud, then that is where Wittgenstein’s analysis of quantificational expression leads.²⁸

2.3 Fogelin

Robert Fogelin argued, first in his (1976) book on Wittgenstein, that Wittgenstein’s analysis of quantificational expression does lead into the mud. Despite the rejoinders by Geach (1981) and Soames (1983), he stuck to his guns in the second (1986) edition. Now, let me first be clear: his conclusion is not credible. Nonetheless, I’m going to be somewhat patient, even charitable, toward Fogelin’s discussion. The reason is that Fogelin’s skepticism gives him a motivation to point out some peculiarities of the text which do not seem so interesting to readers with a more bullish attitude.

Fogelin opens his objection with a summary of the T6 presentation of the general propositional form. Then, however, he continues:

> It seems that every proposition that can be constructed using the recipe in proposition 6 will admit of a decision procedure, i.e., we will be able to determine in finitely many steps whether it is a tautology, contingency, or contradiction. . . .

According to Fogelin, on the one hand it is a basic epistemological commitment of the *Tractatus* that it always be effectively decidable whether or not this or that logical relationship holds between propositions.²⁹ Hence, according to Fogelin, it is Wittgenstein’s aim at T6 to construct an apparently quite general recipe for

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²⁸ There is a third Way too, which I’ll ignore here, but consider in chapters 3-5.
²⁹ Frascolla (2007, 141ff) agrees with this assessment.
propositional expression, such that the obtaining of any logical connections between propositions so expressible can be decided by an algorithm. Since, however, first-order logical validity is not decidable, Wittgenstein’s aim at T6 is incompatible with the acknowledgment of all first-order expressive possibilities. So, since Fogelin finds Wittgenstein’s account of propositional construction to be somewhat vague and underarticulated in the first place, he ventures as a philosophical interpreter to refine Wittgenstein’s account in such a way that the class of acknowledged propositions yields a decidable notion of validity.

Nonetheless, as Fogelin allows, Wittgenstein did not know that first-order logic was undecidable, and Wittgenstein fully intended at T6 to give an account of all first-order logical structure. However, according to Fogelin, since T6 ultimately acknowledges only a decidable fragment of first-order logic, this account must be defective. Fogelin then proceeds to try to diagnose the confusion concretely. The diagnosis begins with a fairly natural and I think textually well-grounded reconstruction. Accordingly, Wittgenstein describes a system of propositional signs, and then claims that the propositions are what is expressed those signs. The system of propositional signs is defined inductively in more or less the following way. An elementary proposition has its own primitively given sign; a propositional sign is also the result of prepending the fourteenth capital letter of the English alphabet to the enclosure in parentheses of (or, for short, the En of) a so-called propositional variable. In turn, the propositional variables are an auxiliary class of signs defined by induction simultaneously with the class of propositional signs as follows. A list of one or more propositional signs, or abbreviation thereof, is itself a propositional variable (a list); and what results from a propositional sign by replacing one of its constituent names with a variable name is a propositional variable (a prototype). We now define by induction the sense (or truth-condition) of each such sign. The sense of the sign for an elementary proposition is that proposition; and the sense of the En of a propositional variable is the joint denial of the values of that variable. The values of a list-variable are the propositions expressed by the signs listed; and the values of a prototype-variable are the propositions expressed by the results of replacing its constituent

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100 The Enning of a formula is supposed to correspond to an attachment of the sign of joint denial to that formula.
variable name with a constant name.

The Fogelinite now proceeds to identify a flaw in this framework. First, it is straightforward to show by the construction of signs that

(1) the En of the En of the result of replacing “Anna” with ex in “Anna loves Bob”

is a propositional sign. Moreover, by the fixing of sense, it follows that (1) expresses the denial of the joint denial of the senses of the results of substituting a constant name for ex in the result of substituting ex for “Anna” in “Anna loves Bob”.\(^{101}\) But on the other hand, consider

(2) The En of the result of replacing “Anna” with ex in the En of “Anna loves Bob”.

Clearly by the construction of signs this too is a propositional sign. Moreover, by the fixing of sense, it follows that (2) expresses the joint denial of each denial of the sense of a result of substituting a constant name for ex in the result of substituting ex for “Anna” in “Anna loves Bob”.\(^{102}\) So, more casually speaking, it follows that (1) says that somebody loves Bob, but that (2) says that everybody loves Bob. Thus, (1) and (2) say different things. But a straightforward syntactical argument shows that (1) and (2) are the same expression. So, in the account of logic just stated, to say that somebody loves Bob just is to say that everybody loves Bob. In other words, this account does not respect some reasonably familiar distinctions of first-order logic.

Now, we have just seen that the account as stated here is actually incoherent. More specifically, according to Fogelin, it breaks under the strain of trying to accommodate all first-order expressiveness while maintaining the possibility of some unspecified decision procedure for logical truth. Of course, at this point the interpreter might just stop here. But, Fogelin, acting in his rights as a philosophical interpreter, proposes to restore coherence by a minor adjustment. The root of the problem is that some signs have multiple formation histories, and the

\(^{101}\)\(\text{En(En([a/x]Rab))} = jd([\text{En([a/x]Rab})]) = jd(jd([a/x]Rab)) = jd(jd([Rcb] : c \in U))\).  
\(^{102}\)\(\text{En([a/x][En(Rab)])} = jd([a/x]En(Rab)) = jd([a/x]En(Rab)) = jd([En(Rcb) : c \in U] = jd(jd([Rcb]) : c \in U)).\)
interpretations generated by the different histories diverge. The fix is then just to cut down on the histories. Fogelin cites here Wittgenstein’s apparent explanation that when the denial sign is attached to an open formula, the result is to express the denial of each of the instances of that formula. So, he infers, any result of attaching a denial operator to an open formula should actually be regarded as a closed formula. Thus, the analysis (2) above violates Wittgenstein’s explanation, because it purports to find an subformula which is open, despite the fact that this subformula is the result of attaching the denial sign to a formula. More generally, then, Fogelin’s resolution is that an open formula is always considered to be closed by the innermost denial sign in whose scope it occurs. In this way, the Fogelinite settles on the conclusion that Wittgenstein’s account of logic is “expressively incomplete”. Although this looks unfortunate, we have have incidentally restored the metalogical coherence of Wittgenstein’s position, since the resulting expressible fragment of first-order logic is actually decidable.

As I said, there is some appearance of textual ground for this reconstruction in the text, especially with regard to what is most crucial for the account of first-order concepts, namely the role of propositional prototypes. In particular, note that in contrast to now-standard inductive presentations of first-order syntax, propositional prototypes are not introduced alongside elementary propositions by the base clause of the induction. Rather, a prototype must be constructed by a syntactical transformation on a proposition constructed antecedently. So, for example, on the now-standard approach, to find a way to say that somebody loves Bob, we begin with the result of substituting ex for “Anna” in “Anna loves Bob”, which may in turn be explained as the combination under a predicate of two terms, one constant and one variable. To this expression we then prepend further logical expressions, thereby forming a negated existential generalization with respect to the letter ex. In contrast, on the Fogelinite approach, one begins with the expression of some elementary proposition, say “Anna loves Bob”, constructs a propositional variable by replacing the word “Anna” with the

103 I’ll argue in 2.6 that this appearance trades on a use-mention confusion.
104 However, I’ll argue in 2.6 that Fogelin’s reconstruction trades on a use-mention confusion. Thus, Wittgenstein doesn’t describe attaching the En-operator to an open formula, but instead describes applying joint denial to the range of values of a propositional function. A similar allegation of use-mention confusion appears in Miller III (1995).
letter ex, and finally forms the En of this variable.

What the *Tractatus* itself says about the source of prototypes appears in an early passage:

Only propositions have a sense; only in the nexus of a proposition does a name have meaning. 3.3
I call any part of a proposition that characterizes its sense an expression. 3.31
An expression presupposes the forms of all the propositions in which it can occur. It is the common characteristic mark of a class of propositions. 3.311
It is therefore presented by means of the general form of the propositions that it characterizes.
In fact, in this form the expression will be *constant* and everything else *variable*. 3.312
Thus, an expression is presented by means of a variable whose values are the propositions that contain the expression.
(In the limiting case the variable becomes a constant, the expression becomes a proposition.)
I call such a variable a ‘propositional variable’. 3.313
An expression has meaning only in a proposition. All variables can be construed as propositional variables. (Even variable names.) 3.314
If we turn a constituent of a proposition into a variable, there is a class of propositions all of which are values of the resulting variable proposition. […] 3.315.

This passage is subtle and rich, even by *Tractatus* standards. But the thrust is clear. Proposition 3.3 rejects an account of the sense of propositional signs which would assume nonpropositional signs to have their meaning fixed independently. The 3.31s elaborate on this rejection, by developing a replacement for the concept of independently meaningful names. This replacement is the concept of an expression. Consider, first, the result of replacing “Anna” with ex in “Anna loves Bob”. Such a sign is a propositional variable, and it has no sense of its own. Rather, Wittgenstein holds, it serves to indicate several propositions by present-
ing their common mark, which is the so-called expression. Thus, the whole point of this stipulated concept of expression is to differ from the concept of a name precisely in manifestly explanatorily following, rather than apparently explanatorily preceding, the concept of proposition. Now, the general propositional form surveys the totality of propositions as the totality of what can be expressed by propositional signs. The Fogelinite therefore finds it only natural that the internal structure of the general propositional form should reflect the order of explanation which Wittgenstein advocates. Under the natural internal ordering exhibited through the general propositional form, a propositional variable does not arise in parallel with the propositional signs which express its values, but is constructed by means of them.

2.4 Geach-Soames

Fogelin seems to suggest that one can tell \textit{a priori} that something went wrong with Wittgenstein’s analysis of first-order logic, because Wittgenstein thought that logical truth is decidable. In particular, Fogelin concludes, Wittgenstein must have just confused himself about the elementary character of variable-binding. Now, in response to Fogelin, many people have pointed out that such confusion would be surprising in light of 4.0411, where Wittgenstein analyzes defective attempts to express quantificational generality, including an attempt with precisely the defect that Fogelin purports to identify. But, Fogelin might say, Wittgenstein took his eye off the ball, because he was trying to ensure decidability. Still, you might wonder: where does this kind of line stop? Should we also conclude that, say, Emil Post did not understand the basic formal patterns of first-order logic because he presumed that the \textit{Entscheidungsproblem} was solvable? Of course Post understood the formal patterns of quantificational logic, and it is well-established that Post spent many years seeking a decision procedure for truth in the system of \textit{Principia} (Davis 1965, 338). As for Wittgenstein, it seems fair to say that the sort of incompetence Fogelin alleges might have wearied Ramsey on his 1923 visit to Wittgenstein in Austria, where the two of them discussed the logic of the \textit{Tractatus} five hours a day for two weeks.\footnote{Monk, 212ff; this point is emphasized in Rogers and Wehmeier 2012).} Geach and Soames complain that the whole
issue of decidability is a red herring, corrupting not Wittgenstein’s treatment of quantification but Fogelin’s reconstruction of it.

According to Geach and Soames, the *Tractatus* actually does contain sufficient resources for analysis of first-order generality. These resources lie very close to the surface of the account of joint denial. For, joint denial has been explained by Wittgenstein in such a way as to apply to arbitrary classes of propositions. However, in the *Tractatus* framework, an ordinary quantified formula can be interpreted as expressing the result of some truth-operation on a class of propositions. This is because, roughly speaking, the *Tractatus* presupposes a “fixed-domain semantics”, according which the associated definition of logical consequence restricts consideration to structures with a common domain. For example, the closed universal generalization of some formula with respect to some variable term can be interpreted to express the joint affirmation of the propositions expressed by each closed instance of that formula. Just as universal and existential quantifiers are interdefinable via negation, so we might introduce a third quantifier, also interdefinable with those two, call it the En-quantifier, and attribute this to Wittgenstein. Having introduced the En-quantifier, and continuing to allow the secondary use of En to express finitarily-based joint denial, then it is obvious that the resulting notational system is pretty much indistinguishable from a standard first-order system.

So, according to Geach and Soames, it was Fogelin’s error to take Wittgenstein’s informal talk of “turning a constant name into a variable” to be a concrete notational proposal. The point of that talk was rather just to gesture toward one kind of multiplicity which can appear as basis of the denial operation. Nonetheless, the talk does license the construction of a notational system which would be expressively adequate. Such a system must, however, contain not just a sign for joint denial, but also a sign to indicate just where in a formation tree a free variable becomes bound. Geach treats the variable-binding device as an inflection of the denial sign. Soames arranges his syntax under an analogous constraint, so that a variable-binding device appears in the construction of a formula only when it is applied to a formula and this application is itself followed by another application of the denial sign.

Geach and Soames both hold that Wittgenstein’s informal talk can actually
be cashed out in a concrete notation, but that insodoing, the talk should not be taken too literally. Here are the details. An atomic formula consists of a predicate followed by a sequence (with appropriate length) of either constants or variables. Now, every atomic formula is a formula, and the result of prefixing the denial sign to a set-representative is a formula. The result of enclosing in parentheses some finite list of formulas is a set-representative, and the result of prefixing a variable to a formula is a set-representative, this prefix being understood to bind all unbound occurrences of the variable in that formula. A propositional sign is a closed formula.

Geach doesn’t explain systematically the determination of the sense of propositional signs so constructed, presumably relying instead on the obvious possibility of extending to this situation any ordinary approach to first-order semantics. Soames, however, does construct an explicit semantics. He defines, by simultaneous induction on complexity of closed formulas and of closed set representatives, the sense expressed by the closed formula and the class of propositions corresponding to the set representative. This all goes as one would expect. The sense of the result of attaching a denial sign to a set representative is the joint denial of the propositions belonging to the denoted set. The propositions corresponding to a parenthesized list of propositional signs are the propositions expressed by the signs listed; and the propositions corresponding to the result of prefixing a variable to a formula are the propositions expressed by the formula’s closed instances.

It seems to me that the aims of Geach and Soames are somewhat ambiguous. On the one hand, Wittgenstein himself tries to present a general propositional form. This means, in particular, answering the question which classes of sets of elementary propositions correspond to truth-conditions of propositions. The strategy, at T6, is to answer this question by something like an inductive definition, according to which nonelementary propositions are constructed from elementary propositions, by repeatedly applying joint denial to specifiable multiplicities of propositions already constructed. Wittgenstein’s attempt to give an inductive definition of the totality of sense is important in its own right from

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106 This version is due to Soames; I think it traces the text a little more tightly.
107 Soames presumes, as I do, that every object has a name.
the point of view of history of logic. No such attempt is even conceivable from within the frameworks of his teachers Frege or Russell, for example.⁹⁸ So, it is worth trying to understand how its details actually worked.

Now, it may be that Geach and Soames propose, with their notational reforms, to reconstruct Wittgenstein’s attempt to say which senses exist. On this view, Wittgenstein’s attempt to state the general propositional form requires a notational amendment because Wittgenstein’s strategy implicates a certain notation essentially. That is, Wittgenstein’s strategy would take two stages. First, he would define a class of notations, or indices, which would be inductively generated by constructive processes from the immediately given elementary propositions. Second, he would define, by induction on syntactical complexity of the class of indices, the class of sets of elementary propositions which corresponds to the truth-condition determined by an index. The totality of possible senses would then be the totality of truth-conditions which are determined in this way by indices.⁹⁹

On this interpretation of Wittgenstein’s strategy, a historically important question arises: what actually is the class of notations which Wittgenstein put forward as the indices of all possible senses? How did Wittgenstein himself try to define this class? Geach and Soames may be understood to propose that the class of notations identified by Wittgenstein is loosely isomorphic to the class of formulas of a first-order language. On this interpretation of the Geach-Soames projects, Wittgenstein’s notational proposals are essential to his articulation of the general propositional form. However, although Wittgenstein’s strategy, so understood, does require constructing a definite class of notations for the possible senses, his execution of the strategy is either incomplete or defective. In particular, Wittgenstein’s inductive definition of the class of notations provides for turning a name into a name-variable in a notation already constructed. But, this

⁹⁸That is, Frege and Russell develop no analysis of the notion of truth-condition in terms of the notion of truth-possibilities for elementary propositions, and so cannot form the general question which classes of truth-possibilities correspond to truth-conditions of propositions.
⁹⁹Such a procedure is analogous to the definition of hyperarithmetic sets, as, for example in Shoenfield (1967), 167ff.
¹⁰⁰Geach does acknowledge Way 3 variables, and sketches a way to deploy them along the lines of 4.1252.
provision does not yield the mathematical multiplicity required to express distinctions of scope. If Wittgenstein did pursue such a notation-first strategy, then the oversight in his construction would seem rather peculiar. It seems, after all, fairly obvious that, in constructing a statement that all values of a certain propositional function are false, one needs to say which function it is whose values are being considered. Such an oversight seems especially puzzling after Wittgenstein’s lectures about the importance of finding a notation that is logically correct (3.325, 4.0411).

On the other hand, another way of reading Geach or Soames is to take them to be reformulating interpreted first-order logic in such a way that it is then reasonably clear how its senses might be accommodated under the general propositional form. That is, they would be observing that every closed first-order formula can be interpreted as a joint denial of the senses of some closed first-order formulas of lower syntactical complexity. But then, it would be nice to know how the general propositional form itself is supposed to work. That is, consider an arbitrary propositional sign generated by the syntactical induction in Soames. We are told by Soames how the sense of such a sign is to be determined. But, how do we know that something actually answers to this predicted sense on Wittgenstein’s account? Must we just presuppose that Wittgenstein takes for granted some kind of second-order quantification over elementary propositions? But then aren’t we just back to the platonist dream that every set of propositions has a joint denial? In other words: how are we supposed to understand this apparent use of second-order quantification in T6?

2.5 Wehmeier and Ricketts

Although the papers just discussed are very nice, there is another interpretive problem that they pass over in silence. Wittgenstein does not just replace the ordinary universal and existential quantifiers with a joint-denial quantifier. Rather, actually he develops a dramatic revision of the logic of generality. To bring out the basic idea, notice that when Carol says that everybody loves Bob, then what she says entails that Bob loves Bob. That is, classically speaking, the variable name implicitly bound in Carol’s expression ranges over all people, over Bob in
particular. Wittgenstein, however, thinks that the proposition that everybody loves Bob does not entail that Bob loves Bob. It entails only that, as we would put it, everybody but Bob loves Bob, and as for Bob himself, well, he may or may not. More generally, the thought is that a variable ranges over all objects except for those which are mentioned in its scope. Such variables are sometimes referred to as “sharp”.\footnote{Thanks to Goldfarb for noting this.} Now, it turns out that when variables are read sharply, then the equality predicate becomes redundant: that is, whatever can be said with equality and normal variables you can also say without identity but with sharp variables.

Although this idea of Wittgenstein’s seems bizarre, it is not too hard to work out. Wittgenstein persuaded Ramsey that it worked in theory even if it is not especially practical. The idea also showed up occasionally in the history of the search for solvable cases of the decision problem for validity of first-order formulas.\footnote{See in particular Gödel (2002, 406 and 570; (thanks to Goldfarb for this reference)), and, speak of the devil, Geach and von Wright (1952).} Hintikka seems to have picked up the idea in the 1950s from Geach or perhaps von Wright, and produced the important (1956). More recently Kai Wehmeier has published on this topic extensively (2012 with Rogers, also 2012, 2008, 2004).

Somewhat more precisely, the claim of expressive equivalence between first-order logic with equality and sharp first-order logic without equality is more precisely this. Consider a purely relational first-order signature,\footnote{I.e., containing no constant or function symbols.} and now, for some arbitrary nonempty set, consider the class of first-order structures over this signature which have that set as domain. Call such a class a frame. Satisfaction concepts for each of the two logics being defined as above, the sense of a closed formula with respect to a frame is the class of structures in the frame with respect to which the formula is true. Then, relative to any frame at all, every sense of a closed formula of the language of first-order logic with equality over a purely relational signature is the sense of a closed formula in the language, over the same signature, of sharp first-order logic without equality.\footnote{See 5.2 for details. The first proof of this result appears in Hintikka (1956); a more elegant version appears in Wehmeier (2004).} This is so even though some open formulas in the first system determine functions from assignments

\footnote{Thanks to Goldfarb for noting this.}
\footnote{See in particular Gödel (2002, 406 and 570; (thanks to Goldfarb for this reference)), and, speak of the devil, Geach and von Wright (1952).}
\footnote{I.e., containing no constant or function symbols.}
\footnote{See 5.2 for details. The first proof of this result appears in Hintikka (1956); a more elegant version appears in Wehmeier (2004).}
to senses which no formulas determine in the second system—for example, an equation between distinct variables.

The reader may now be wondering why Wittgenstein thought it was a good idea to depart in this way from the basic idea of Frege and Russell. For example, was he just not paying attention when his logic teacher was explaining the basics? Many logic students often initially fail to appreciate that leading existential quantifiers can be witnessed by the same constant.

Actually, this question has a surprisingly good answer. The standard view of commentators is that Wittgenstein is driven to this reinterpretation of generality from an unexpected place. Recall that a fundamental doctrine of the *Tractatus* is that a proposition is a truth-function of elementary propositions. This means, at least, that the truth-values of elementary propositions determine the truth-value of every proposition.

But there is really big snarl around the equality predicate. Perhaps, with a sentence like “Alice is Alice” or “Alice is Bob” there is some feeling of language gone on holiday (to say nothing of Hesperus and Phosphorus), and the *Tractatus* might just dump such expressions as merely apparent. But equality appears in many sentences where language is hard at work, for example as in “Alice loves at least two people”; the sense of this sentence cannot be expressed in first-order logic without equality—that is, in the system that Geach and Soames had gotten pretty close to helping Wittgenstein to accommodate.

In response one might just wave a white flag and allow that the equality predicate may appear in propositions that are elementary. But this is out of the frying pan and into the fire. For every elementary proposition has possibilities both of truth and of falsehood. So the surrender leads to the even worse result that a proposition with respect to some object that it is the same as itself can be false.

There are various other stories in the literature about the way in which the equality predicate presents a problem for basic *Tractatus* doctrines, but nobody seems to disagree that it is a problem.

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115Michael Potter (2009, 290) says that a genuine relation of identity between objects would have to hold contingently, because objects are simple. Presumably this is on the grounds that if objects are simple and if identity is a genuine relation, then propositions to the effect that objects are identical must be elementary.
Now our surprisingly good to the question why Wittgenstein adopts the exclusive reading of the variable is this. It so happens that, under some proviso, everything that can be said by means of standard first-order logic plus equality can be said by means of sharp first-order logic without equality. Thus, at least up to the proviso, and provided you are willing to swallow the reinterpretation, the equality predicate is redundant. For example, if you want to say that Alice loves at least two people, you just follow the bad logic student: “there is an ex and a why such that Alice loves ex and Alice loves why.”

To summarize, on an amended standard account, Wittgenstein revises the semantics of first-order logic with equality by restricting the space of assignments to those which are injective. This revision, while dramatic, is also well-motivated. For, the expressiveness apparently afforded by the equality predicate presents a big problem for the truth-functionality thesis. Upon restricting the assignment space, the equality predicate becomes contextually eliminable, under the proviso that the language contains no constants. In this way, a seemingly unrelated ideological fiasco over the equality predicate finds its resolution in Wittgenstein’s tweak of the logic of generality.

One cause for concern is just that Wittgenstein, as a historico-philosophical personality, seems to have been highly averse to anything of an ad hoc or arbitrary flavor.\footnote{In particular, this sensibility seems to animate some of his nastier mentions of Russell.} Now, I would not wish to imply that this ingenious technical maneuver would be ad hoc by the standards of contemporary analytic metaphysics. Far from it. But, it might be ad hoc by the standards of Wittgenstein. And, here, we are doing Wittgenstein-interpretation, not contemporary analytic metaphysics. So there may also be some grounds to worry that, at least by Wittgenstein’s own standards, the sharp reading of the quantifiers is a flaw in the Tractatus. It appears to complicate the program to develop a philosophical understanding of the relationship between a generalization and its instances.

Ricketts (2012) acknowledges that Wittgenstein takes as a crucial philosophical problem to understand the link between a generalization and its instances. Hence Ricketts quite rightly finds Wittgenstein’s account of quantificational generality to be rooted in the 3.31s, which immediately follow the programmatic
endorsement of the context principle. However, Ricketts remarks that Wittgenstein’s account is “in its way Fregean”, referring specifically to Frege’s explanation of the function-argument segmentation in 2.9 of *Begriffsschrift*. Later on, Ricketts raises the problem of understanding from a *Tractatus* point of view the quantification into results of truth-operations on elementary propositions, and maintains that Wittgenstein approaches the issue from a Fregean direction.

Suppose we replace a name in an elementary sentence that occurs within the representation of the construction of a particular truth-function of that elementary sentence. The elementary sentence-function in the formula collects together its values so that the entire formula collects together expressions of that particular truth-function of values of the elementary sentence-function. (Ricketts 2012).

I’m sympathetic to Ricketts aim to see how Wittgenstein himself understood the notion of propositional function, and also to the thought that Wittgenstein’s understanding was somehow indebted to Frege (or Russell). However, it does seem that Wittgenstein’s explanation differs from Frege’s in a way that Ricketts does not record. Frege’s explanation is this:

If, in an expression [...], a simple or a compound sign has one or more occurrences and if we regard that sign as replaceable in all or some of these occurrences by something else (but everywhere by the same thing), then we call the part that remains invariant in the expression a function, and the replaceable part the argument of the function. (Frege 1967, 22)

On Frege’s account, then, a sign which occurs in a sentence may be regarded as replaceable by other signs. Frege’s informal examples make clear that by replacement he means paradigmatically, replacement of a name by a name. It is thus the replaceability of a name with another name in a sentence that underpins the segmentation of the expression into function and argument. On this conception, the values of the function are naturally those sentences which result by replacing that name with another name. Note, in particular, that Frege’s formulation explicitly allows for the possibility that, when an sentence contains many
occurrences of a single sign, then we may distinguish from among the totality of
occurrences some subcollection to be regarded as variable in contrast to the other
occurrences which remain considered fixed. Thus, Frege explicitly intends that
a name may be regarded as replaceable in just one fixed one of its occurrences
in a sentence ranges two occurrences of that name. The resulting function thus
indifferently encompasses values which contain two occurrences of that name,
alongside values which contain one occurrence of that name and one occurrence
of another name. In other words, the range of a function may in general con-
sist of sentences containing varying numbers of different names. As one would
expect, this conception is entirely consistent with normal mathematical practice.

Let me now summarize Wittgenstein’s own account in a way that should be
uncontroversial. As Ricketts says, Wittgenstein agrees with Frege that one can
begin with any whole proposition. Now, according to Wittgenstein, we may
take a name which occurs in the proposition, and turn this name into a variable.
By so turning a name into a variable, one obtains a propositional function. The
propositional function presents what Wittgenstein calls an Ausdrück. The Aus-
drück is the common mark of the sense of a multiplicity of propositions. The
values of the propositional function are the propositions whose sense is marked
by the Ausdrück which the propositional function presents.

There are two differences between the accounts of Wittgenstein and Frege.
Frege’s explanation begins by introducing the notion of replacement of a name
in a proposition with another name. In contrast, Wittgenstein does not talk
about replacing names with names. Rather, he talks about turning names into
variables. It is by turning names into variables that one obtains the presentation
of a common mark of the sense of propositions, an Ausdrück. Second, Frege’s
account explicitly allows for the possibility of distinguishing among many occur-
rences of a single name some particular occurrences to be regarded as replaceable.
Wittgenstein does not explicitly make such allowance. These two differences
have an important consequence. Frege allows that a function may take as its val-
ues sentences containing variable numbers of names. However, Wittgenstein’s
account precludes such variability. Any propositional function constructible on
Wittgenstein’s account takes as its range a multiplicity of propositions which all
contain the same number of names.
After claiming that Wittgenstein’s account of functions basically derives from Frege, Ricketts then introduces a scope-indicating device following Geach. Finally, he concludes, “there is a complication here that deserves mention”—the interpretation of propositional variables “in conformity with Wittgenstein’s views on identity” (20012, 12). On Ricketts’ Fregean explanation of the *Tractatus* account of propositional functions, a function collects together all results of replacing the considered-replaceable occurrences of a name with another name. Ricketts must then agree with the assessment of Rogers and Wehmeier that the *Tractatus* account is defective. For, as he acknowledges, Wittgenstein’s views on identity demand that the result of combining the sign of denial with a sign of a propositional function must be the denial of those values of the function which result from assigning the variables free in the function sign values which do not already appear as objects mentioned in the function sign. So, Ricketts must find that although Wittgenstein shares with Frege a project to explain the relationship between a generalization and its instances, Wittgenstein’s own attempted explanation seems marred by an *ad hoc* maneuver required to eliminate the equality predicate contextually. It might then be doubted whether Wittgenstein’s own attempt to explain the relationship between a generalization and its instances could be seen as successful.

So, it seems to me that the question why Wittgenstein reconstrues the workings of name-variables does have an *a priori* surprisingly good answer. But, this answer, at least insofar as it is developed in the literature, retains some interpretive and philosophical deficiencies. In particular, as I’ve argued, on this reading, Wittgenstein’s commitment to the independence of elementary propositions ultimately undermines his attempt to explain the relationship between a generalization and its instances. For, the independence of elementary propositions rules out an equality predicate. The ensuing adjustments to the concept of generality

117 Ricketts avoids attributing to Wittgenstein any distinction between occurrences of a name in an elementary proposition. To this extent, then, he does find a departure in Wittgenstein from Frege. But Ricketts does allow Wittgenstein to distinguish between occurrences of a name in a nonelementary proposition. Ricketts finds in the *Tractatus* a sharp distinction between “the picturing structure of elementary sentences and the iterative structure of portrayals of truth-functions.” In contrast, I hold that such a distinction cannot be reconciled with Wittgenstein’s treatment of generality. See 2.6.
do not square with normal mathematical practice. In fact, “normal” is an under-
statement. It is probably safe to say that no mathematician in recorded history
has seen the need for five different laws to express the associativity of multipli-
cation, depending purely on the pattern of identity and distinctness among the
arguments of the corresponding instances. The profundity of this aberration
makes it hard to take Wittgenstein’s approach seriously. Even if one can techni-
cally make sense of the variant system of logic, it can’t possibly be rooted in a
stable conception of representational structure. In particular, the concept of an
Ausdrück looks like a little greasepot of equivocation.

2.6 Pictoriality

Let me begin by setting aside for a moment the questions of 2.5, and return to a
question that lingers from 2.4. Did Wittgenstein really overlook the need distin-
guishing the possibilities of scope of a quantifier?

Recall that Fogelin’s puzzle addresses the relationship between

(1) the En of the En of the result of replacing “Anna” with ex in “Anna loves
   Bob”

(2) The En of the result of replacing “Anna” with ex in the En of “Anna loves
   Bob”.

In (1) a formula is presented so as to seem to want to say that somebody loves Bob.
In (2) a formula is presented as though it wants to say something else, namely
that everybody loves Bob. And yet (1) and (2) present the very same formula.
So something has gone wrong. But what? Fogelin infers that the phrasing of
(2) somehow tricks us into thinking that the formula says something other than
what it does say. Is that really the right diagnosis of the situation? That (1) and
(2) cannot say different things, because they are the same formula? That formula
(2) just tragically wants to say what it can’t say? Believe it or not, it is possible to
be even more pedantic. Consider

(1a) “Anna loves Bob”

Let alone, say, a physicist who requires several reformulations of Newton’s laws of motion.
(1b) The result of replacing “Anna” with ex in (1a).

(1c) The En of (1b).

(1d) The En of (1c).

(2a) “Anna loves Bob”

(2b) The En of (2a).

(2c) The result of replacing “Anna” with ex in (2b).

(2d) The En of (2c).

Clearly, so it seems, the last entries of the two sequences must say the same thing, because they too are the same formula. Is that really clear? I think that the answer is no. Our original naïve and allegedly incoherent Fogelinite semantics, proffered only as a diagnosis of confusion, was actually perfectly sound. To see this, recall the picture from 2.2 of an allegedly maximal conversation. The one sequence may be a single strand of this conversation, and the other sequence may be another strand. Of course, you have to actually follow the dynamics of the conversation in order to make out these implicit connections between the utterances, which are conveyed, in our metalinguistic report of the conversation, by nesting the label of one utterance in the description of the next utterance. In other words, the last entries of the two sequences are intrinsically indistinguishable, but what matters in logic is not intrinsic structure of any one thing that is said but the internal relations between all that is said.

We have now reached what so far as I know is a novel account of a system of propositional notations for the *Tractatus*. This system has exactly the formation rules predicted by Fogelin. However, the semantics would run not on individual propositional signs, but rather on trees of signs which record conversational origins. Although intrinsically indistinguishable signs may be assigned different meanings, these differences will always derive from differences in the underlying trees with respect to which the different interpretations were determined. Such an approach can readily be seen to recover the power of the Wehmeier-Soames
hybrid while avoiding adventitious postulation of a special scope-indicating device.

As many commentators have urged in response to Fogelin, Wittgenstein did not himself single out any particular notational system for a special role in the basic program. So, these commentators continued, there is no reason to suppose Wittgenstein had any particular commitment to the system of signs that Fogelin ascribes and indicts. I agree. We are essentially arguing about the form of words which participants in a maximal conversation must use in constructing a complex demonstration of a basis of for joint denial. It is not clear, a priori, why these people should have to talk one way or another. What is the purpose of the concept of the maximal conversation? i.e., of the general propositional form? That is a deep question I do not propose to settle here. However, one might notice that, in the relationalist sign-system just described, the only role for syntactical structure of a noninitial notation is to indicate the means by which it immediately arises from its predecessors on a tree. Every part of the structure of a noninitial notation which does not contribute to this role is logically inessential and really just a distraction. There is no good reason at all why the last entries of the two sequences described above have the same internal structure.

Nonetheless, as I argued in 2.3, it looks like there is a pretty good textual foundation for identifying a Fogelinite syntax in *Tractatus*. For, so it seems, the 3.31s talk about turning a name into a variable name to produce a propositional variable. And likewise, 5.501 explains that attaching an expression of joint denial to a variable name produces the joint denial of the values of that variable. I think that Fogelin has committed here a natural confusion of use and mention. The 3.31s do not describe turning a name into a variable in a propositional sign. Rather, they describe turning a name into a variable in a proposition, which yields a propositional function. Likewise, 5.501 talks about forming the joint denial of the elements of the range of a propositional function. These are movements in the realm of meaning—although, as I’ve emphasized, to speak of movement here is just to trace metaphorically the internal relations between senses which are themselves unchanging.

Thus, the relevant point of the 3.31s is not that in the allegedly maximal conversation, the result of replacing “Anna” with a variable in a sentence is a form of
words that a speaker uses. Rather, it is that the result of replacing Anna with a variable is a form of sense that a speaker means. What words might we suppose a speaker to utter in meaning such a thing? I give a formal syntax and semantics elsewhere, and my present concern is with interpretive justification. For present purposes, we may simply suppose that a speaker says, to express the joint denial of the values of the propositional function obtained by replacing Anna with a variable in a proposition already expressed by a speaker, something like this: “nothing is as you say Anna is.” In accordance with the program announced at T4.5, Wittgenstein’s analysis of this kind of expression involves resources not idiosyncratic to this or that particular system of signs, but rather only resources internal to the very possibility of a system of signs in general. The apparent eccentricities noted by Fogelin stem from the Wittgenstein’s insight that name-variables are not after all essential to the expression of quantificational generality.

Let’s now return to the questions that arose in 2.5-2.6. There we seemed to find that, setting aside the questions raised by Fogelin, Wittgenstein was driven into an account of quantificational generality that could not be grounded in any stable account of representational structure. For example, suppose what is said about Bob is that Anna loves him. Then, Wittgenstein must find that “nothing is as you say Anna is” is entirely consistent with Bob’s loving himself. That is, Wittgenstein needs the latter form of words to mean, as we would put it, that nobody other than Bob loves Bob. Now, our stipulation is that the words used by the speaker express the joint denial of the values of the propositional function obtained by replacing Anna with a variable name in the proposition that Anna loves Bob. But isn’t Bob himself a value of this variable name? But then isn’t the proposition that Bob loves Bob a value of the resulting propositional function? Should we just insist that he isn’t? That is capitulation.

Our question thus reduces to the question what are the values of a propositional function which results by replacing Anna with a variable in the proposition that Anna loves Bob. I suggest that Wittgenstein’s concept of a propositional function is rooted in the concept of what he calls somewhat stiltedly an “expression”, a logically meaningful commonality amongst various propositions. Informally, you may suppose an expression to be what a proposition says of something (or of some things). The key idea is that if the proposition that Bob loves Bob is a
value of the result of replacing Anna with a variable in the proposition that Anna loves Bob, then the proposition that Bob loves Bob says of Bob what the proposition that Anna loves Bob says of Anna. But consider the proposition which says of Anna what the proposition that Bob loves Bob says of Bob. This proposition says that Anna loves Anna. On the other hand, the proposition which says of Anna what the proposition that Anna loves Bob says of Anna is just the proposition that Anna loves Bob. And to say that Anna loves Bob is not to say that Anna loves Anna. Hence, the proposition that Bob loves Bob is not a value of the result of replacing Anna with a variable in the proposition that Anna loves Bob. Rather, the values of this expression are precisely those propositions which result from it by replacing the variable name with somebody other than Bob. In other words, I contend that it is Wittgenstein’s concept of an expression which underpins the sharp reading of the variable.\footnote{Strictly speaking, I’ll take expressions in the sense of the 3.31s to be notational items. Different signs may belong to the same proposition (i.e., may say the same thing); hence different signs may say the same thing of a given object. The expression associated to a propositional sign by a name is then a notation for what the sign says of the object. When time comes to get more careful, I’ll not speak of what a proposition says of an object, but of what a propositional sign says with respect to a name. However, saying-of-an-object is the fundamental notion. The idea is that a proposition says of something what the proposition that Bob loves Anna says of Bob if and only if it is expressed by some propositional sign which says with respect to a name of that thing what “Bob loves Anna” says with respect to “Bob”. Thanks to Roberta Ballarin for raising this issue.}

But this last characterization is still a little misleading. To bring this out, let me suggest a somewhat vulgar heuristic. The expected way of finding the range of a propositional function is just to plug random objects into the place where there was a variable. But Wittgenstein’s way of finding the range of a function is to pull an object out of a proposition and see if the result is that function.

Now, why can’t we just pull out the “first” occurrence of Bob in the proposition that Bob loves Bob? The problem lies, I think, in Wittgenstein’s departure from Russell’s conception of propositional structure. With respect to Russellian propositions, one can talk about the position of an object. Consider, for example, the conjunction that Bob loves Bob and Bob loves Bob. Russell thinks that the conjunction sign represents a relation between two propositions: thus, that the connective vocabulary expresses relations between propositions in the way that the amorous vocabulary expresses relations between people. Wittgenstein is
a philosophical rejectionist but he rejects this as much as anything. Connective vocabulary does not contribute representational structure. In particular, conjoining a proposition with itself does not take you beyond the original proposition. The duplication is idle, a samesaying (I would use the word “tautology” here, but that is already taken). But there seems to be no single inevitable identification of occurrences of Bob in the conjunction with occurrences of Bob in the proposition conjoined. Worse yet, there are more choices of occurrences of Bob in the conjunction than there are in its conjunct. I think that such an apparent proliferation of choices of segmentation is not really grounded, for Wittgenstein, in what the proposition says about Bob. Rather, one might say, it is not only an elementary proposition which is a hanging-in-one-another of objects. Every proposition is such a hanging-together. What we call logical composition does not, for Wittgenstein, multiply the “occurrences” of an object, any more than, if it turns out that you or me loves both of Bob and Anna, there will all of a sudden turn out to be two “occurrences” of you or me. Generality, for Wittgenstein, is rooted in the pictorial character of propositions, in which names hang together as the objects do. This grounding of generalizability in pictoriality is not, as both Fogelin and Ricketts were in their own ways led to suppose, intelligible only with respect to those propositions which are elementary. The apparent multiplicity of occurrences of an object, conjured by our means of expression, and the attendant appearance of possibilities of meaningful distinctions among those occurrences, is logically speaking an illusion. So, then, is any distinction between the internal structures of elementary and nonelementary propositions.

It is for this reason that by Wittgenstein’s lights, Russell’s way of determining the range of a propositional function is confused. For there are really two different concepts here. One concept is that of the class of propositions to which an expression is common. I.e., for any object, and any proposition, there is the class of propositions which say about some object what that proposition says about that object. For a proposition to belong to this class is for the expression to result from it by removal of the object. But then there is another concept, the class of results of “plugging” an object into the “gap”, “in” the propositional function. Here, I suppose, one takes out the binoculars and scans the savanna for candidates—any one will do, that is, if it fits. It is from such perspective that
one must never forget the taboo—perhaps, a taboo on incest, or onanism?—that an object which already occurs in the expression, even if it, so to speak, fits in the gap, does not belong there. In other words, it is only the image of the range of a propositional function as the results of plugging which demands an *ad hoc* restriction. As I contend, this image is specious. Rather, the range of a function is to be explained as the class of propositions to which an expression is common. An expression is like a necklace from which one link has been lost. When a link is torn out of a chain and left on the ground, it doesn’t just so happen to remain elsewhere on the chain.

Let me conclude this section by summarizing what I take to be the basic motivation for an alternative reading of Wittgenstein’s treatment of quantificational generality. It is standardly held that Wittgenstein is driven to the sharp treatment of name-variables by an ideological fiasco entangling the equality predicate. This gets the matter backward. As argued in Chapter 1, Wittgenstein’s basic convictions about the nature of logic stimulate the development of a conception of propositions as pictures of reality. And as I’ve just argued now, the conception of propositions as pictures demands the exclusive reading of name-variables. Thus, the redundancy of the equality predicate is a corollary to a deeper insight.

### 2.7 The picture

After all of this harangue, let me finally sketch in crudest outline my own understanding of Wittgenstein’s conception of quantification.

Given any proposition and any object we may ask what the proposition says about that object. What a proposition says about an object is, in Wittgenstein’s stipulative sense, an expression. Thus, the proposition that Bob loves Anna says of Bob that he loves Anna, and the proposition that Bob loves Bob says of him that he loves himself. This concept of expression extends in an obvious way to arbitrary finite sequences of names. In general, we expect that what a proposition says about some objects depends on the order in which the objects are presented.

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120 I said Russell, rather than Frege, because Frege’s account does not suffer the same asymmetry between abstraction and instantiation. Wittgenstein’s departure from Frege needs independent justification in terms of the pictorial conception of representational structure.
outside the proposition. For example, what the proposition that Bob loves Anna says of Bob and Anna isn’t what it says of Anna and Bob, though it may be what the proposition that Anna loves Bob says of Anna and Bob. Of course, we can also ask of the proposition that Bob loves Anna what it says of Carol, and of course the answer is that it says of Carol that Bob loves Anna.

Wittgenstein’s concept of an expression leads straightforwardly to an account of quantificational generality. Given an arbitrary proposition and name, there arises an expression, what the proposition says about the object named. An expression is thus a logically distinct part of the meaning of the given proposition, which the given proposition may have in common with other propositions. The expression marks the sense of another proposition as well, provided that there is some, presumably distinct, object, such that what the second proposition says of the second object is what the first proposition says of the first object. Consider now an expression determined by an arbitrary proposition and name. This in turn determines the totality of propositions whose sense the expression marks. A quantificational generalization is the result of a truth-operation on such a totality.

The reader may be excused for wondering what this talk of what a proposition says of an object really comes to. Can we make mathematical sense of it? Or must it wither in the never-never land of irreducibly metaphorical metaphysics? Let me conclude by sketching the mathematical underpinnings of my account. For expository simplicity, I will confine myself to that fragment of the Tractatus construction which has occupied the various commentaries discussed in this chapter.

Assume as primitively determined concepts the concepts of name and elementary proposition. It is natural to assume that no names are elementary propositions. Furthermore assume that there exists a relation of samesaying. This relation is assumed to answer every question whether what one elementary proposition says of some objects is the same as what another elementary proposition says of some other objects. We do stipulate that samesaying is an equivalence relation, but make no further assumptions about it. We also make no assumptions

\[121\] With the exception of one paragraph of Geach (1981, 170).
whatsoever about the structure of elementary propositions, other than that the extension of the samesaying relation be determinate.

In this way, we take as primitive every answer to a question when an expression marks the sense of a given elementary proposition. Given such primitives, we aim to construct all first-order definable truth-functions of elementary propositions. The key technical problem will be to show how to extend the concept of expression from elementary propositions to all propositions. The guiding principles are straightforward. For example, the denial of a proposition says of some list of objects what another proposition says of some other list of objects, provided that the second proposition denies a proposition which says of the second list of objects what the first denied proposition says of the first list of objects.

Now, recall the idea of a maximal conversation from 2.2. We introduce a new system of rules, permitting three kinds of move. An act of the first kind is the assertion of an elementary proposition. By an act of the second kind, somebody can single out any finite bunch of acts that people have performed already, and deny that any of propositions expressed by those other acts are true. In an act of the third kind, somebody can single out any one act that somebody already performed, emit a name, and then deny every proposition which says of something what the proposition expressed by the singled out act says of the thing named. Note that this second case allows for straw men, i.e., for the denial of propositions which nobody has yet asserted. I do not think that this phenomenon is incomprehensible or even actually unprecedented. However, for technical reasons it is useful to note that straw men can always be fleshed out in principle, for example by paying one’s friends to assert what one wishes to deny.\footnote{In this way, sufficiently wealthy speakers can overcome the obstacles raised by Fogelin (1987, 80). Just kidding. Fogelin does not manage to identify an obstacle in the first place, because this is a piece of pure mathematics. As Geach (1981) said, nobody actually has to list all the propositions denied. Mathematical constructions do not require construction workers to skolemize the quantifiers. I am just using the concept of conversation as an expository metaphor.}

Let us momentarily presume an answer to the question when a proposition says of something what another proposition says of something. We are then in a position to determine the sense of arbitrary propositions. Consider a conversation that really is maximal, so that every possible tree of moves descends from some act that has already been performed. We renew our insistence that a conver-
sation be wellfounded, so that although some speech act may point to an earlier one, and those to earlier ones, and so on, every single chain of indicated acts must eventually terminate in the assertion of an elementary proposition. Suppose also that the conversation contains no straw men, so that nobody ever denies something that was not yet asserted. We may assume the sense of propositions expressed by acts of the first kind to be fixed. Acts of the second and third kinds are denials. Since the conversation contains no straw men, the propositions they deny have been expressed already, and by the wellfoundedness assumption, we may assume their sense to have been fixed. But the sense of a denial of propositions with fixed senses is clearly fixed.123

Now let’s take up the task to extend the same-saying relation from elementary propositions to arbitrary propositions. Consider two speech-acts, Act One and Act Two, and two corresponding lists of names, say Team One and Team Two. We want to determine the answer to a question, whether or not what the proposition expressed in Act One says of Team One is the same as what the proposition expressed in Act Two says of Team Two. I will just use talk about what an act says of some objects as short for talk about what the proposition the act expresses says of them.

We first stipulate that Act One says of Team One what Act Two says of Team Two only if the two acts are the same kind of move. Now, note that if both Act One and Act Two are assertions of elementary propositions, then our question is already answered by the primitive part of the fixing of the behavior of the same-saying relation. So assume the Acts to be nonelementary. By the assumption of wellfoundedness, to determine the answer to the question, it suffices to reduce the question entirely to similar questions concerning acts singled out by Acts One and Two. The answer to our question is no unless the Acts are of the same kind, so suppose they are. Suppose the Acts are of the second kind, so that each singles out a finite bunch of prior acts. Then, Acts One and Two are the same expression if and only if the acts in the bunches can be paired off in such a way that, in each pair, the element drawn from the bunch indicated by Act One says of Team One what the element drawn from the bunch indicated by Act Two says of Team Two.

This paragraph is formalized in Definitions 5 and 6 of §5.1.
says of Team Two. Suppose on the other hand that Act One is of the third kind. Then each Act involves emitting a name and singling out one earlier act. Append to Teams One and Two the names emitted in Acts One and Two respectively. Now the answer to our question will be no unless the singled out acts say the same thing of the expanded teams, in which case the answer is yes.

There is another way to visualize the process of checking whether two acts say the same thing of some objects corresponding to each. Consider each act as an upside down tree. The tree is built up out of inch-long segments of copper wire connected by balls of clay. On the wire beneath a given ball of clay, there may or may not sit a colored bead. And at the end of each branch, there hangs a little mosaic consisting of colored beads glued to a piece of cardboard. To represent what an act says of some team of names, simply suppose a corresponding series of colored beads to sit on the topmost wire of the associated tree. Now, suppose that two trees can be laid on top of each other in such a way as to align every bead, ball of clay, and piece of wiring on the one tree with a bead, ball of clay and piece of wiring on the other tree. Consider two complete, linear branches on the two trees which correspond to each other under this alignment. These branches each bear a series of beads which, as we move downward, begins with Team One or Team Two, followed by further beads as they occur on the chain, followed finally by its terminating mosaic. With respect to any such chain, consider just its descending series of beads and the terminating mosaic. The mosaic is an elementary proposition, and the series of beads is an expanded team of names. We stipulate that two Acts say the same thing of their respective initial Teams of names if and only if each pair of corresponding decorated mosaics say the same thing of the expanded teams of names denoted by the series of beads above them. The question then remains when two mosaics say the same thing with respect to their corresponding series of beads.

The question what a mosaic says with respect to the beads above it is to be answered primitively. But for the reader’s amusement, let me fix an image. What a mosaic expresses with respect to one bead is simply a matter of the shape of the region of the mosaic which that bead occupies, together with the rest of the

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124 This paragraph is formalized in Definition 3 of §5.1.
contents of that mosaic. If we ask what the mosaic expresses with respect to a series of beads, then, roughly speaking, this is a matter of the series of shapes of the regions which are occupied by beads as we move up the series, together with the remnants of the contents of the mosaic. In conclusion, note that although a mosaic may appear to be made of many individual beads, what really stand for objects in a mosaic, i.e., the names of which the mosaic is really a nexus, are not the individual beads but rather their colors. As Wittgenstein put the point, “‘A’ is the same sign as ‘A’” (3.203). I have been urging that this point applies not just to names in elementary propositions but to names in all propositions. Every proposition is just a nexus of names. Propositions are pictures, and names are the colors of which they are formed. Of course, the intricacies of logical picturing can only really be grasped by examining the internal relations which makes possible the totality of logical pictures. For example, the beads on the wires do not contribute objects corresponding to the colors they bear, but denature the similarly colored beads in the mosaics beneath them, so that a mosaic contributes to the proposition none of the objects which correspond to the colors of the beads above it.\footnote{The metaphors of this concluding paragraph are cashed out in §5.2. I apologize for the delay, which arises because I'll cash out the metaphors in the context of a reconstructed system which includes variables constructed not just by Ways 1 and 2 but also Way 3.}
Chapter 3

Formal generality

In what is perhaps not the single obscurity of the *Tractatus*, Wittgenstein writes:

Only in this way is the progress from term to term of a form-series (from type to type in the hierarchies of Russell and Whitehead) possible. (Russell and Whitehead did not admit the possibility of such steps, but repeatedly availed themselves of it.) [5.252]

I aim to provide an interpretation of this remark.

The obvious question to arise is the following: what does Wittgenstein mean by “this way”? Now, it is an unfortunate fact of life in the *Tractatus* that one cannot always resolve the antecedent of a pronoun by straightforward appeal to preceding entries. For as is well known, Wittgenstein composed the *Tractatus* by assembling and rearranging individual sentences or paragraphs until he found an order which suited him. And sometimes this process produced grammatical possibilities of interpretation which he did not intend. Nonetheless, the relevant earlier entries are as follows:

The occurrence of an operation does not characterize the sense of a proposition.
The operation says nothing, only its result, and this depends on the bases of the operation.\(^{126}\)
(Operations and functions must not be confused with one another.) [5.25]

A function cannot be its own argument, whereas an operation can take one of its own results as its base. [5.251]

\(^{126}\)Here I've slightly modified Pears-McGuinness. The original runs: *Die Operation sagt ja nichts aus, nur ihr Resultat…*
According to 5.2521, an operation can take its own result as its basis. So, Wittgenstein’s point at 5.252 seems to be that this iterability of operations is what makes progress from term to term of a form-series possible. Specifically, the basis of the operation would be one term of the form-series, and the corresponding result would be the next term. Similarly, the result of the operation can itself be taken as a basis, which the operation takes to another result, the third term of the form-series. And so on. In this way, an operation takes us through an unending succession of terms; thereby a form-series is generated.

That much, I think, is relatively clear, and not in dispute. For, a page earlier, Wittgenstein has introduced the concept of operation. Our explanation so far fits well with some those remarks:

An operation is the expression of a relation between the structures of its result and of its bases. (5.22)

The operation is what has to be done to the one proposition in order to make the other one out of it. (5.23)

And that will, of course, depend on their formal properties, on the internal similarity of their forms. (5.231)

The internal relation by which a series is ordered is equivalent to the operation that produces one term from another (5.232)

Thus, I take the unparenthesized part of 5.252 to be at least verbally fairly clear. If we knew what an operation was supposed to be, then we would understand this much of 5.252 pretty well.

The very first example Wittgenstein gives of an operation is this:

Truth-functions of elementary propositions are results of operations with elementary propositions as bases. (These operations I call truth-operations.) (5.234)

[...] Negation, logical addition, logical multiplication, etc. etc. are operations. [...] (5.2341)

Part of the point of the idea of an operation, then, is to be marked off from the idea of a function: operations and functions must not be confused with each
other (5.21). Here, Wittgenstein has Russell in mind. For Russell as for Wittgenstein, a function is a distinguishable part of its values: if a proposition is a value of a function, then one can discern the function in the proposition by regarding certain objects in the proposition as variable. But Russell also thinks that negation is an example of such a function, and thus a distinguishable part of its values. Now Wittgenstein parts ways with Russell. For Wittgenstein, negation is not a function, but an operation. Russell thus confuses operations with functions. An operation, for Wittgenstein, is not a distinguishable part of its values. An operation can vanish: thus, for example, $\neg\neg p = p$ (5.254). Operations are ways of getting from one place to another. They are merely procedural: “operations cannot make their appearance before the point at which one proposition is generated out of another in a logically meaningful way” (5.233).

So, Wittgenstein’s idea of the operation is shaped by the Grundgedanke, by the fundamental thought that there are no logical objects, that “the ‘logical constants’ are not representatives” (4.0312). The ‘logical constants’ are merely means by which we represent a proposition as the result of an operation on other propositions (5.2). They present a logical place by giving directions to that place from another one. So the idea of the operation is shaped by the Grundgedanke, because operations are key to Wittgenstein’s understanding of truth-functionality. Results of truth-operations are truth-functions of their bases: the result of negating $p$ is a truth-function of $p$. Paradigm instances of operations are truth-operations, the counterparts in Wittgenstein’s thinking to the truth-functions of Principia.

Let’s now turn back to 5.252, to the idea that an operation, repeatedly applied to its own results, thereby enumerates a whole series of propositions. As it happens, we have skipped over the hard part. Parenthetically, Wittgenstein compares this repeated application to a passage from type to type in a Russell-Whitehead hierarchy. Thus, the parenthetical remark seems to imply that each term of a form-series is followed by a term of higher logical type.

It seems safe to say that we are now in the thick of a mystery. There is certainly no explicit endorsement in the Tractatus of the ramified theory of types. There is very little talk of higher-order quantification in general, and much of that is pejorative. Despite this near-silence, does Wittgenstein buy into some kind of theory of types after all?
Worse yet: wasn’t the whole point of the idea of operation that it doesn’t characterize the sense of its results? Doesn’t Wittgenstein illustrate this point at 5.254 by remarking that \( \neg\neg p \equiv p \)? But isn’t negation a paradigmatic operation, which should take us from term to term of a form-series \( p, \neg p, \neg\neg p, \ldots \)? How in the world can this form-series be seen as climbing through a system of Russell-Whitehead types? As 5.254 tells us, it goes around in a circle!

The questions raised by 5.252 and related passages are perhaps the hardest questions about the workings of logic in the *Tractatus*. But as we’ll see, the questions raised by 5.252 are immediately implicated by the question of the significance of the general propositional form. So one cannot really have a good grip on the book until these questions are properly answered. Among the few interpreters who have worked on these questions concertedly, some conclude that Wittgenstein must accept some sort of classical system of type theory.\(^{127}\) Yet others sharply deny this.\(^{128}\) Such a chasm between leading interpreters of the book indicates that there are serious problems here. One aim of this chapter is to do some of the textual spadework that’s essential to resolving the questions about higher-order logic in the *Tractatus*. Nonetheless, I do myself have a clear position, and in §1 I begin by staking it out: there is no interesting genuine higher-order quantification in the *Tractatus*. The *Tractatus* contemplates, and, in principle, can contemplate, only first-order quantification, perhaps of a multi-sorted variety. But, my position is, though clear, also only tentative. So I want to back off a little on the systematic considerations here, and try to look at the evidence honestly.

Let me conclude this introduction with an executive summary of the sections to follow.\(^{129}\)

It was essential to the programs of Frege and Russell that logic involve some kind of higher-order generality. For they considered logic to be a body of substantive truths which were applicable to all of science. This conception of logic already strongly supports quantification into predicate position. If quantifica-

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\(^{127}\)For example, Michael Potter (2008, 193ff) and Peter Sullivan (2004, 50).


\(^{129}\)The topic of this chapter involves a combination of conceptual and exegetical challenges which may tax the patience of all but the truly fanatical. Moderates might just read the executive summary.
tion reaches into object-positions alone, then the logical theorems which legit-

image inferences in the special sciences will mention properties special to those sci-

ences, and so not be really general. Another conceivable way to topic-neutrality
would be to abstract out this reference to properties, by regarding the predicates
as merely schematic. But Frege and Russell would resist this approach, since then
the theorems can no longer be regarded as substantive truths in their own right.
If, on the other hand, we allow theorems of logic to generalize over all objects
and properties whatsoever, then the theorems of logic can be seen as broadly ap-

plicable, substantive truths. Such truths would characterize the common portion
of all science: purely logical objects and functions like truth, falsehood, negation,
entailment, identity, and quantificational generality itself.  

So the philosophical understanding of logic as a body of broadly applica-
ble yet substantive truths is intertwined with the use of higher-order generality.
Of course, higher-order generality had another rationale too, because Frege and
Russell were logicists, aiming to show that arithmetic is just a highly developed
branch of logic. But, the logicist program is only remotely conceivable if logic
amounts to more than just the syllogism. It was conceivable because, to Frege
and Russell, logic looked very powerful.

Now, Wittgenstein learned logic from Frege and Russell. Thus, he inherited
their intuitions about logical expressiveness. In particular, the founding docu-
ment of the new logic, Frege’s Begriffsschrift, culminates in a logical analysis of
the ancestral of a relation. So Wittgenstein simply expected that the ancestral
was a logical notion. For what else could logic be? On the other hand, I
argue in §1, Wittgenstein has strong philosophical motivations of his own for
taking genuine quantificational generality to be generalization over objects, and
in particular over simple objects. First, his understanding of quantificational gen-
erality requires that the instances of a generalization receive their meaning di-
rectly by arbitrary convention, rather than meaning what they do as a matter
of what must be the case once the conventions are in place (6.124). So, the in-
stances of a generalization must be simple names. There can be quantification

\[ \text{This paragraph is heavily indebted to Goldfarb and Ricketts—see for example Goldfarb (2001) }
\text{and Ricketts (2012).} \]

\[ \text{For a discussion of the alternatives, see Wittgenstein (1913).} \]
into so-called “predicate-position”—or, more accurate, quantification into every simple, content-contributing articulated propositional segment; but such a generalization can only be instantiated by another such minimal segment. Second, as argued in Chapter 1, analysis as Wittgenstein conceives it aims to show that propositions are truth-functions of elementary propositions, by exhibiting them as resulting from the elementary propositions by successive applications of truth-operations. Quantificational generality is then to be understood as a particular means of specifying the basis of truth-operation; but then instances of a generalization must in some sense have lower complexity than the generalization itself; this at least rules out all higher-order quantification that is impredicative.

Nonetheless, the student of Frege and Russell did inherit their expectation that the ancestral was a logical notion. But, he rejected the resources Frege and Russell used to construct it. So, I argue in §2, Wittgenstein takes another approach: he defines the ancestral by means of a countably infinite disjunction. This raises a question of principle: just when can an infinite class of formulas be so taken as basis of a truth-operation? I argue that for Wittgenstein, the crucial constraint stems from the picture theory. On the one hand, an infinite multiplicity of propositions can’t itself be surveyed, so the elements of the multiplicity must be presented by a common mark. But, the picture theory requires a formal unity between the proposition and the situation it depicts. Hence, the common mark of propositions in the multiplicity must be a formal one, so that it is also common to the situations which the propositions in the multiplicity present. Now, Wittgenstein thinks that the propositions in a form-series do have such a common formal feature: this is what legitimates the analysis of the ancestral in terms of the concept of form-series.

In §§3 and 4, I argue that Wittgenstein envisaged two quite distinct kinds of application of the concept of the ancestral. In §3, I show that Wittgenstein invoked the notion of ancestral as early as 1914, toward the program of analysis of statements about complex spatial objects such as tables and chairs. In the development of this program, we find a striking illustration of a main thesis of Chapter I: a proposition borrows its internal structure from its formal unity with a situation. In §4, I summarize Wittgenstein’s early use of the ancestral in sketching the general propositional form, and observe that he imagined this use to have quite a
different logical status.

§5 gives my interpretation of 5.252. I discuss a notebook entry of 7.1.17 which is the first appearance of the general propositional form in something like the version we find on the first page of *Prototractatus*. This entry shows that Wittgenstein, as he completes the analysis of the general propositional form, sees the iterated application of truth-operations as the ultimate fate of the Russell-Whitehead stratification of the totality of propositions. In the view of the *Tractatus*, propositions do stratify into hierarchies, but this stratification is merely aspectival. No proposition has a single position in a hierarchy essentially, but only relative to this or that completed take on the totality of propositions. On this view, the \( N \)-operation does take us from one level of a hierarchy to another, and carries us up through the layers of the totality of propositions, just as Russell and Whitehead hoped to achieve off-the-books with their device of typical ambiguity. After Wittgenstein's rethinking, a hierarchy is no longer a system of logical constraints, but a survey of notational freedom.

In §6, I return to the tension that began to arise in §§3 and 4 between the two distinct applications of the form-series device. Wittgenstein does present the general propositional form itself by means of a form-series. Does this mean that he thinks that the totality of propositions can appear as the basis of truth-operations? I present two possible answers. According to one answer, Wittgenstein thinks that stipulable bases of truth-operations stand to the general propositional form as set-constituting multiplicities stand to the class of all sets. According to another answer, Wittgenstein meant what he said at 4.5, that the general propositional form is a variable, and thus available as stipulable basis for truth-operations. I tentatively favor the second conclusion, but find the textual evidence to be inconclusive, even, perhaps, disconcertingly muddied.

In §7, I present an alternative interpretation of 5.252, suggested by Thomas Ricketts in his valuable (2012). On Ricketts' view, 5.252 alludes to the possibility of using the form-series device to simulate higher-order predicative quantification. Ricketts also sketches a speculative reconstruction of this development of predicative quantification. Since as Ricketts agrees, any detailed account of the application of form-series must be somewhat speculative, differences here are to some extent a matter of taste. In any case, I hope that our readings that not sep-
parated by the chasm that almost surely separates the readings of two randomly chosen interpreters.

The critique of Ricketts is a kind of negative advertisement for a speculative reconstruction of my own, which I offer in the following chapter. I identify a very simple class of operations—essentially, substitution of a formula for a propositional variable in a formula—and show that this device suffices to express the ancestral and other $\Delta_1^1$ logical notions relevant to the program of logical analysis. The resulting system forms a recursive sublanguage of the finite-variable fragment of $\mathcal{L}_{\omega_1\omega}$.

3.1 Two problems with higher-order generality

As I’ve argued in Chapter 2, Wittgenstein’s account of the construction of propositional functions differs importantly from that of Frege. Frege’s account begins with the idea of replacing a name with a name in a sentence, thus leading us to consider the name as variable while the remainder of the sentence is fixed. By introducing the antecedently intelligible notion of replaceability, Frege purports to elucidate, though not to define, the new and questionable idea of something invariant under the replacements, namely the Fregean function. Wittgenstein’s account makes no primitive appeal to the idea of replacing a name with a name, for such an appeal leads to the standard fixing of the range of propositional functions, which Wittgenstein must reject. Does this mean that Wittgenstein just has to take the idea of propositional function as primitive? This would be unfortunate, since Wittgenstein’s concept of propositional function, unlike Frege’s, is not at all a natural generalization of the mathematical concept of function. So it stands even more in need of explanation. Rather than talking about replacing a name with a name in a proposition, and thereby coming to regard a propositional constituent as variable, Wittgenstein instead talks directly about turning a constituent into a variable. But what does this talk come to?

If we turn a constituent of a proposition into a variable, there is a class of propositions all of which are values of the resulting variable proposition. In general, this class too will be dependent on the meaning that our arbitrary conventions have given to parts of the original
proposition. But if all the signs in it that have arbitrarily determined meanings are turned into variables, we shall still get a class of this kind. This one, however, is not dependent on any convention, but solely on the nature of the proposition. […] (3.315)

Wittgenstein’s talk of arbitrary convention here is essential to his distinctive understanding of the variable, and thus of his construction of propositional functions. Suppose that we construct a “logical prototype” by beginning with a proposition turning all of its constituents into variables. Then, Wittgenstein says, the resulting prototype differs from the initially given proposition in that the prototype does not depend on any of the arbitrary conventions by which the constituents were assigned their meanings. Consider now another case, where we turn just a single constituent into a variable: the result then differs from the original proposition just in that it no longer depends on the convention which assigned a meaning to that one constituent.

The talk of arbitrary convention gestures toward the following idea. It is obviously possible to effect local changes to linguistic convention by drawing a chalk circle on the ground and stipulating that the name “Kant” when uttered in the circle means what “Spinoza” actually means. Such a stipulation really has two components: first, cancelling the actual meaning of the name “Kant” and second, assigning the then-meaningless word “Kant” a new meaning, that of “Spinoza”. But now suppose we enact just the first half of the stipulation, step into the resulting circle, and then utter “Kant was a great philosopher”. I take it that this is basically Wittgenstein’s understanding of how from “Kant was a great philosopher” one constructs the propositional function “x was a great philosopher”. This interpretation explains why Wittgenstein puts so much emphasis, in his account of generality, on the idea of “arbitrary convention”. Such rhetoric is, after all, rather peculiar: for example, so far as I know, it never appears in Frege or Russell’s expositions of corresponding ideas.

But this account also sets up a striking departure from Frege and Russell with respect to what parts of a proposition can be regarded as generalizable: namely, only those parts whose meaningfulness is expressly a matter of some arbitrary convention. Now, some aspects of an expression of a proposition do not con-
stitute parts in the relevant sense at all. Clearly signs which occur as names do constitute such parts. In contrast, for example, the “cat” in “cattle” could do so only under extraordinary circumstances. Wittgenstein also holds that since the sign of a truth-operation do not mark the sense of a proposition by means of which it’s expressed, also no part of a proposition corresponds to truth-operation signs either.\footnote{But, suppose we begin with a proposition, turn a name in it into a variable, and consider this resulting “leftover part”. Returning to the original proposition, can we turn the leftover part itself into a variable? To the extent that this can be done, it must be done by the selective local repeal of meaning-securing conventions. Thus for example we can surely turn into variables each of the names which occurs in the leftover part. But as far as I can see, that is pretty much all that we can do. And it is a poor substitute for genuine higher-order quantification. The essential problem is that the \textit{Tractatus} faithful must sharply distinguish between that which is fixed arbitrarily and what must be the case once things have been arbitrarily fixed (6.124). Inasmuch as the leftover part has a meaning at all, this is not thanks directly to something arbitrary but rather something that must be the case once we have begun to talk under the given conventions. One cannot simply single out for repeal some joint effect of a variety of interlocking conventions, but must repeal the conventions underlying the effect themselves. Wittgenstein’s occasional use of notation for quantifying into predicate position is shorthand for turning all but one of the names in the proposition into variables, thus a mere ersatz requiring nothing beyond ordinary polyadic first-order generalization.\footnote{I would be pleased to put it like this: since \( \sim \neg p \) is the same proposition as \( p \), therefore \( \neg \) doesn’t mark the sense of \( \sim \neg p \).} But, suppose we begin with a proposition, turn a name in it into a variable, and consider this resulting “leftover part”. Returning to the original proposition, can we turn the leftover part itself into a variable? To the extent that this can be done, it must be done by the selective local repeal of meaning-securing conventions. Thus for example we can surely turn into variables each of the names which occurs in the leftover part. But as far as I can see, that is pretty much all that we can do. And it is a poor substitute for genuine higher-order quantification. The essential problem is that the \textit{Tractatus} faithful must sharply distinguish between that which is fixed arbitrarily and what must be the case once things have been arbitrarily fixed (6.124). Inasmuch as the leftover part has a meaning at all, this is not thanks directly to something arbitrary but rather something that must be the case once we have begun to talk under the given conventions. One cannot simply single out for repeal some joint effect of a variety of interlocking conventions, but must repeal the conventions underlying the effect themselves. Wittgenstein’s occasional use of notation for quantifying into predicate position is shorthand for turning all but one of the names in the proposition into variables, thus a mere ersatz requiring nothing beyond ordinary polyadic first-order generalization.\footnote{This suggestion is due to Ricketts (2012); I’m absolutely committed to something like this by the argument of §3.1.}

The 3.31s may seem to suggest (as apparently to Potter (2009, 270)) that Wittgenstein countenances abstraction of expressions other than names. However, in my view, that is a misreading of the passage. Perhaps the crucial point is the first sentence of 3.315: “Verwandeln wir einen Bestandteil eines Satzes in eine Variable, so gibt es eine Klasse von Sätzen, welche Sämtlich Werte des so entstandenen variablen Satzes sind.”

In particular, we need to determine the intended meaning of the term “Bestandteil.” On my view, “Bestandteil” in this context evokes the Russellian terminology of “propositional constituent”; and propositional functions are not, according to Russell, constituents of their values.

The word “Bestandteil” occurs previously in the \textit{Tractatus} at 2011, 20201, and 324. 2011 says that it is essential to a thing to occur as “constituent” of a state of affairs. 20201 and 324 use the
Of course a there is a second and much better-recognized problem that Wittgenstein finds with the use of higher-order generality amongst the logicists. This problem specifically affects that higher-order generality which is impredicative. On the logicist treatment of arithmetic, cardinal numbers are identified with classes of equinumerous classes (or classes of equinumerous concepts); having defined the cardinal analog of the successor relation, one then marks out the finite numbers as those objects which bear the ancestral of the successor relation to the class with which zero has been identified. Now, it is second-order generality which lets us define the ancestral of a relation. On Frege’s version, \( a \) is said to bear the ancestral of \( R \) to \( b \) in case \( b \) has every \( R \)-hereditary property \( a \) has, i.e.,

\[
R'xy \iff \forall X(\text{Her}_R(X) \land Xa \to Xb),
\]

term to refer to the items into which analysis resolves a complex, thus presumably to simples. These antecedents suggest that Bestandteil denotes something simple, be it an object in a state of affairs, or a name in a sentence.

At 4.024 and 4.025 there are slightly more apposite occurrences. For example, 4.024 says that one understands a sentence upon understanding its Bestandteile. 4.025 says that translation of one language into another proceeds by translating only the Satzbestandteile. The use at 4.024 may be somewhat equivocal (depending on whether the point is that one needs to understand all its Bestandteile), but 4.025 is unequivocal: Satzbestandteile must refer to names rather than to arbitrary expressions.

Wittgenstein also uses “Bestandteil” in another sense, where what is under consideration is a proposition presented as the result of a truth-operation on other propositions. In this case, Wittgenstein occasionally refers to those other propositions, or rather to their signs in their occurrence in the presentation of the result of the truth-operation, as “Bestandteile” of the result. However, this usage only makes sense on the level of signs, since the bases of a truth-operation need not characterize the sense of its result. In contrast, the answer to the question whether a name characterizes the sense of a proposition must be invariant under truth-operational rephrasings and more generally independent of any particular means by which the proposition gets made sensibly perceptible.

Now, it’s true that the remainder of the paragraph 3.315 employs “Teil” anaphorically on “Bestandteil”. Moreover, 3.31 has “Jeden Teil des Satzes, der Seinen Sinn Charakterisiert, nenne ich einen Ausdruck (ein Symbol).” I think that the 3.315 occurrence of “Bestandteil” in contrast to “Teil” cancels any possible reference back to the 3.31 use of “Teil”. The hypothesis of such back-reference would be questionable anyway: according to McGuinness-and-Schulte, 3.315 derives from NL, whereas 3.31-3.313 derive from the PT3.20s (and a bit from NB2).

I think, then, that Wittgenstein intends no verbal link from the “Bestandteil” of 3.315 back to the “Ausdruck” of 3.31-3.313. “Bestandteil” means a propositional constituent, whereas “Ausdruck” means a part or aspect of a proposition. In the 3.31-3.313 use of “Ausdruck”, Wittgenstein refers not to what is removed, but to the leftover. There seems to me to be no clear implication in the 3.31s that the leftover can itself be removed.

This footnote is transcribed almost verbatim from an email to Ricketts. 134 I borrow this formulation from Goldfarb (naturally, his version is more elegant).
where a property is $R$-hereditary in case no thing with the property bears $R$ to something which lacks the property, i.e.,

$$\text{Her}_R(X) \leftrightarrow \forall x \forall y (X x \land Rxy \rightarrow Xy).$$

Supposing that only a few properties exist, say for example that there only exist properties with finite or cofinite extension, say, then $a$ and $b$ may satisfy Frege’s definition of the ancestral of $R$ even while $a$ does not bear the ancestral of $R$ to $b$. And, although of course Frege or Wittgenstein couldn’t say this, the same is true even if all first-order definable properties exist.

A natural response to this challenge is: “but all properties exist, including that of bearing the ancestral of $R$ to $a$”. Frege’s axiomatic implementation of his definition embodies this response, since it licenses instantiating the bound second-order variable in the definition to the property of bearing the ancestral of $R$ to $a$. Wittgenstein, perhaps evoking Poincaré,\textsuperscript{135} charges that such a response is viciously circular. As Ramsey and Gödel urged, this circularity seems really vicious only if one regards a property as being constructed by the definition, rather than as being merely singled out by it.\textsuperscript{136}

As with some other of Wittgenstein’s objections to Frege, however,\textsuperscript{137} the impredicativity complaint can also be seen to tell us something significant about Wittgenstein’s own thinking. As I’ve argued in Chapter 1, Wittgenstein sees logical structure as articulating the truth-functionality of propositions. In particular, a connective serves to present a proposition as the result of a truth-operation on some other propositions, with those propositions obtained by truth-operations on others and so on; this branching process terminates ultimately in the propositions that are elementary. After truth-operation has been separated from generality proper, first-order quantificational notation is then seen to articulate genuine propositional structure: since analysandum is represented as giving way to propo-

\textsuperscript{135}This suggestion is due to Goldfarb (2012).

\textsuperscript{136}Peter Sullivan pointed out in a discussion that Wittgenstein’s accusation of vicious circularity is directed not just against Frege but also against Russell, whose higher-order quantification is predicative. Wittgenstein may have been inured to this point by Russell’s adoption of the axiom of reducibility.

\textsuperscript{137}For example, 4.063.
sitions of a lower notational complexity, we can be sure of not going around in
a circle. But impredicative quantification cannot be so redeemed, because the
proposition to be analyzed will recur essentially in its analysis. So even if Frege
wouldn’t be much flustered by Wittgenstein’s borrowing from Poincaré, the pas-
sage does illustrate some central ideas of the *Tractatus*.

3.2 Wittgenstein’s alternative

Frege’s analysis of the ancestral thus conflicts with Wittgenstein’s truth-function-
ality thesis. At this point, Wittgenstein might simply have concluded that the
ancestral is not a logical notion. However, for someone who learned logic from
Frege and Russell, the definability of transitive closure of a relation might have
seemed as natively logical as the definability of, say, its symmetric closure. More-
over, it seems hasty to abandon the logicality of transitive closure because of
problems with one particular analysis. For example, a similar reaction might
inspire us to reject the logicality of symmetric closure because of the impredica-
tivity of the definition

\[
R^s xy \leftrightarrow \forall X (\forall u \forall v (Ruv \lor Rvu \rightarrow X uv) \rightarrow X xy).
\]

And this would be silly, since one has the unworrying alternative

\[
R^s xy \leftrightarrow Rxy \lor Ryx.
\]

The alternative analysis of symmetric closure is unworrying, of course, be-
cause as we’d now say, it is purely first-order. Unlike symmetric closure, tran-
sitive closure is not first-order definable, and so there is somewhat better evi-
dence that the concept of transitive closure intrinsically presupposes the notion
of arbitrary subset of the domain. But, this is still too hasty. From an extra-
Wittgensteinian point of view at least, the concept of transitive closure can be
coherently taken to be independent of such higher-order notions. For example,
the concept of transitive closure is closely tied with the concept of the order type
of the natural numbers, i.e., with the concept of finite ordinal. In particular,
once the finite ordinals are fixed, then concepts definable by induction on finite
ordinals would seem to be fixed as well. But the transitive closure of a relation is definable by induction on the finite ordinals. And, the concept of finite ordinal is acceptable from a point of view which would take for granted the totality of natural numbers but reject full second-order quantification. Now, it’s not clear that Wittgenstein himself could reason in this way, from the determinateness of the natural numbers to the determinateness of transitive closure. But, the existence of this line of reasoning indicates that one might coherently accept the determinateness, or perhaps even the logicality, of the concept of transitive closure, while rejecting as indeterminate or as nonlogical the use of impredicative quantification.

Since transitive closure is not first-order definable, we need some essentially not first-order logical device if transitive closure is to be a logical notion. From the Tractatus point of view, the first question to ask about putatively logical devices is whether they can be reconciled with the truth-functionality thesis. Intuitively, it should count as evidence that some device can be so reconciled if we can understand it simply to embody a new way of expressing agreement and disagreement with truth-possibilities for elementary propositions. But it is now worth recalling what every good student wants to say when asked to define “ancestor” in terms of “parent”, namely just that there is at least one truth among the following bunch of formulas:

$$Rab, \exists x(Rax \land Rx b), \exists x \exists y(Rax \land Rxy \land Ry b), \ldots .$$

This characterization certainly does seem to show that whether or not \(a\) stands to \(b\) in the ancestral of \(R\) is merely a truth-function of elementary propositions. For once the list is fixed, then each formula in the list is a truth-function of elementary propositions; but then it would seem that whether or not at least one of them is true is also just function of which elementary propositions are true. Truth-functionality, on Wittgenstein’s scheme, demands only that the result of introducing a new representational device preserves the possibility of tracing, from each constructed position, through chains of agreement and disagreement.

138 As in \(R'(0, x, y) \leftrightarrow Rxy\) and \(R'(s(k), x, y) \leftrightarrow \exists z(R'(k, x, z) \land Rzy)\), and then \(R''(x, y) \leftrightarrow \exists k \in \omega R'(k, x, y)\).
with other positions, back down all the way to a pattern of agreement and disagreement with truth-possibilities of elementary propositions. It seems entirely possible that some resources sufficient for rendering the naïve analysis could meet the truth-functionality constraint.

There is an analogy between the naïve approach to the ancestral and the Wittgenstein’s treatment of existential quantification. For example, Wittgenstein shows how “a bears R to something” is a truth-function of elementary propositions, by reducing its truth to the circumstance that there be at least one truth among the following bunch of formulas:

\[ Rab, Rac, Rad, \ldots \]

Thus, the naïve analysis of the ancestral and the *Tractatus* treatment of existential generalization both represent a proposition as a result of a truth-operation on some presumably infinite bunch of formulas. Moreover, this bunch is not a random set of formulas but the class of formulas instantiating some common feature. This analogy raises the question why existential quantification is a legitimate analytical resource. Perhaps the underlying principles license further resources that were overlooked by Frege and Russell, and which suffice to recover the ancestral.

Wittgenstein’s conception of propositions as pictures entails that proposition and reality are alike, or one and the same, with respect to their total logical content. It is then in virtue of this likeness that a proposition makes evident how the world must be if it is true. Thus:

We can see this from the fact that we understand the sense of a propositional sign without having it explained to us. (4.02)

A proposition shows its sense.
The proposition shows, how things stand, if it is true. And it says, that they do so stand. (4.022)

So, a proposition is in this way like a picture, that there simply is no space between seeing a picture as a picture, and knowing how things are if they are as

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139 NB: “\(\exists x f x\) means ‘a true proposition \(f x\)’” (9.7.16).
140 The last comma does not appear in Pears-McGuinness.
depicted. This overtness of pictoriality is what decides just when, for Wittgenstein, an analysis may represent a proposition as the result of an operation on other propositions to be given not directly as constituents of the analyses but as values of a mere schematic representation. It is the very appearance manifest in the proposition which must itself be the way things are if the proposition is true, even though this appearance finds its position in logical space only thanks to certain contingent facts. So, as Wittgenstein puts it in the *Prototractatus*:

One must be able to see in the variable itself what it stands for. — There must be a wholly determinate resemblance between it and its values. (PT5.0054)

This I suggest, is the principle which underlies the possibility of representing a proposition as the result of a truth-operation on some possibly infinite multiplicity of propositions: that some common mark characterize exactly the propositions belonging to that multiplicity. The analysis of a proposition may then point not to the infinitude of other propositions directly, but instead to the characteristic mark itself. Such a mark is a formal or internal feature of propositions:

The expression for a formal property is a feature of certain symbols. 
So the sign for the characteristics of a formal concept is a distinctive feature of all symbols whose meanings fall under the concept. 
The expression of a formal concept is a propositional variable, in which this characteristic feature is constant. (4.126f-h)

Wittgenstein’s puzzling critique of Frege at 4.063 elaborates on his commitment to this position. There he complains that Frege finds that truth and falsehood are accidental properties of positions which are specifiable independently of appeal to the nature of those properties. Imagine, for example, a three-player game, in which two players are assigned to reconstruct a painting that is only visible to the third player. The first player calls out a coordinate on the painting; the third player replies with the color on the painting at that coordinate on the painting, and the second player paints in the corresponding region of the duplicating canvas. In principle, the first player might be deaf and blind, and have no idea what properties the second player records.

I think that this respect for appearances sharply separates Wittgenstein’s approach to analysis from that of Russell, and in particular leads him to retract his early pronouncement that “your theory of descriptions is quite certainly correct” (Letter 30 in McGuinness 2008). It is a misreading (cf. e.g., Hart (1973)) to suppose that the “indefiniteness” in 3.24 alludes to the scope ambiguity under negation which is predicted by Russell’s theory of definite descriptions.
It is only because the common mark of the values of a propositional variable is an internal or formal property that we can expect the analytical appeal to such marks to preserve the formal unity between proposition and situation. That is, because the mark is formal, its commonality amongst propositions just is its commonality amongst the situations those propositions depict. A mark that is not formal in this sense—for example, being expressed in English by a palindrome—justifies no expectation of any commonality between the corresponding multiplicity of presented situations, hence fails to explain how propositions constructed by appeal to the mark could be pictures of reality.143

Consider, for example, a propositional variable which results by denaturing a name in a given proposition. This variable presents a mark of those propositions from which that same mark arises by the denaturing of a name; the propositions so marked form its range. Propositions in the range thus palpably resemble the variable. It’s easy to see which are the propositions over which it ranges. They consist of the same number of names, arranged in the same way. The only difference between the propositions in its range arises through one-one recolorings of certain constituent names which are flagged in the variable’s notation.144

Now, Wittgenstein similarly holds that such a characteristic mark can be found for the multiplicity of formulas whose disjunction is the ancestral of a given relation. For, he suggests that there is a formal operation which takes propositions to propositions, such that the results of repeatedly applying the operation to some initially given proposition are precisely the elements of the multiplicity. It is stipulated that an operation is formal only if it produces a result from a given basis purely in virtue of an internal relation between the basis

143The first way of stipulating value-ranges at 5.501 is by enumerating a finite list of values. The propositions in the list can be chosen arbitrarily. Wittgenstein hesitates about whether the first method does yield a variable at all, for he writes: “instead of a variable we can simply write its values”. Presumably, the reason he hesitates is because he doesn’t think that a variable so stipulated presents a formal commonality which distinguishes its values, yet he explains the variable as presenting a commonality. He continues to refer to explicitly listed value-ranges as variables just because it is terminologically convenient to describe all presentations of multiplicities uniformly. Alternatively, since the list is finite, one could perhaps stipulate that the list itself is a kind of disjunctive formal concept.

144So, propositional functions as characterized in Chapter II meet Wittgenstein’s conception of the variable better than Russelian propositional functions do, since the values of the latter kind of function contain variable numbers of different names.
and result. A multiplicity of propositions so generated Wittgenstein calls a form-series. More formally, suppose \( O \) is the sign of an operation on propositions. Then, \([A, \xi, O(\xi)]\) is a propositional variable, whose values are the propositions \( A, O'(A), O'(O'(A)), \ldots \).

The form-series notation appears to allow construction of a propositional variable with the range needed to express the ancestral of a relation. That is, writing \( R^0ab \) for the formula \( Rab \), and \( R^{k+1}ab \) for the formula \( \exists x_k(R^k ax_k \land Rx_kb) \), then it suffices to find a formal procedure \( O' \) such that \( O : R^k ab \rightarrow R^{k+1} ab \). It seems obvious that under some reasonable interpretation of “formal procedure”, such an \( O' \) exists.

Now, Wittgenstein presupposes that the propositions in a form-series can be characterized by a distinguishing formal feature and so can also be presented by a variable. He seems to take the very notation \([A, \xi, O'(\xi)]\) of a form-series variable itself to show the common mark of the propositions in its range. I think it’s fair to say that however the concept of operation is explicated, the similarity of the propositions in the range of a form-series variable will not be nearly so palpable as the similarity between the values of a propositional function. Nonetheless, the concept of operation should somehow respect the constraint that it be “easy to see” whether a proposition falls in the range of a form-series variable which the operation serves to construct.

### 3.3 Form-series in analysis in the NB

So, Wittgenstein accepted the position of Frege and Russell that the notion of ancestral is part of logic. Of course, belonging to logic means something quite different to Frege and Russell than it does to Wittgenstein. In accepting that the ancestral is logical, Wittgenstein concluded that it was written into the very nature of picturing. The notion of ancestral, as analyzed by the device of the form-series, realizes certain unities of proposition and situation. I’ll now investigate the development through the *Notebooks* of this envisaged role of the ancestral, which, as we’ll see, appears already in 1914.

Consider a proposition to the effect that the watch is on the table: this is a mere appearance of a certain watch, on a certain table, and it must be the fact that
that watch is on that table which makes true the affirmation of that appearance. Yet there cannot be the possibility of such a fact unless some wheels and pins and sockets and housing and glass are assembled into that watch; and there cannot be the possibility of such an assembly unless incomprehensibly many intrinsically indiscernible material points cohere into the wheels, pins, sockets, etc. But how can all this look like, and, needless to say, be, a watch on the table? It is this fundamental explanatory constraint which guides the invocation of covert generality in analysis.

As Wittgenstein acknowledges, the innumerable indiscernibles constituting the constituents of the watch are not, in the bald fact of their coherence, the appearance of the watch. The appearance of the watch somehow conceals this fine structure. Moreover, there is another problem. Although these constituents so configured altogether are the watch, the equation does not reverse: the watch does not reduce to them uniquely. They are, one might say, sufficient but not necessary for the watch, or for its being on the table. As Anscombe puts it,

There are hundreds of different, more minutely statable, and incompatible states of affairs which would make that proposition true (Anscombe 1959, 34-35).

Despite these problems, Wittgenstein insists that a proposition like “the watch is on the table” must somehow genuinely depict a dyadic relation between two things. The ground of this conviction is not phenomenological but logical. That is, we can soundly reason with such propositions as though they were dyadic; but such reasoning could be sound only if the dyadicity were pretty much genuine.

But logic as it stands, e.g., in Principia Mathematica, can quite well be applied to our ordinary propositions, e.g., from “All men are mortal” and “Socrates is a man” there follows according to this logic “Socrates is a mortal” even though I equally obviously do not know what structure is possessed by the thing Socrates or the property of mortality. Hence they function just as simple objects. (22.6.15h)

In this way, the picture theory demands a logical unity between the proposition and the situation signified. Let’s call this need to account for unity of ordinary
propositions with the ultimate form of what they depict, the pictoriality con-
straint. The difficulty raised by this constraint is that logical structure must
explain all necessary connections, which requires that in the last analysis, the
apparent multiplicity of an unanalyzed proposition gives way to an incompre-
hesibly larger and more complicated logical structure. How can the apparently
simple representation be a logically adequate picture of the teeming monster un-
derneath? As Wittgenstein insists, “ordinary language is in perfect logical or-
der as it is.” Given this aim, an approach via Russell’s theory of descriptions
would be inadequate, because Russell’s theory purports to expose the apparent
subject-predicate structure as specious. A dramatic rendering of the pictoriality
constraint appears in the Notes to Moore:

When we say of a proposition of the form ‘aRb’ that what symbol-
izes is that ‘R’ is between ‘a’ and ‘b’, it must be remembered that
in fact the proposition is capable of further analysis because a, R,
and b, are not simples. But what seems certain is that when we have
analysed it we shall in the end come to propositions of the same form
in respect of the fact that they do consist of one thing between two
others. (NB111)

Anscombe and von Wright remark that this passage was “lightly deleted”; per-
his this was because of its aprioristic stringency. Nonetheless, Wittgenstein con-
tinues to pursue some account of how analysis could meet the pictoriality con-
straint. At 5.9.14, he proposes that a proposition ϕ[aRb] about a complex con-
sisting of a in the relation R to b might be analyzed into propositions which
retain the form ϕ, namely as: ϕ(a) ∧ ϕ(b) ∧ aRb. This proposal yields some
peculiar results: for example, from “my watch weights n grams” it hardly fol-
lows that the dial of my watch weighs n grams. Nonetheless, that Wittgenstein
retained this peculiar proposal can be seen from obvious kinship similarity to
the remark of 2.0201: “a statement about complexes resolves into a statement
about their constituents and into the propositions that describe the complexes
completely.”

145Wittgenstein’s use of “describe” (beschreiben) here is not Russell’s. Objects, in this sense, cannot be described, for a description, in the relevant sense, is a reflection in language of the complex-
Now, the 5.9.14 approach certainly could not be Wittgenstein’s conclusive treatment of complexes. For he maintains that the number of parts of a complex might be infinite. But, following Russell, he holds that analysis could not introduce infinitely complex propositions explicitly. So, we have a kind of trilemma. A situation might be infinitely complex. But analysis cannot present a proposition as infinitely complex. Yet the proposition is formally one with the situation it presents.

It is under these theoretical pressures that he introduces the notion of logical prototype. A prototype is an essentially general or schematic representation of a multiplicity of propositions, which nonetheless sufficiently resembles those propositions as to respect the pictoriality constraint on analysis. Propositional functions are one kind of example of a prototype, form-series variables are another kind, but Wittgenstein’s understanding is deliberately open-ended.

The appearance of prototypes in the underpinnings of truth-functionality of propositions is mainly tacit. But, Wittgenstein says:

When a propositional element signifies a complex, this can be seen from an indeterminateness in the propositions in which it occurs. In such cases we know that the proposition leaves something undetermined. (In fact the generality-sign contains a prototype.) (3.24)

What Wittgenstein means here by indeterminateness is precisely the appearance of necessary connections of the given propositions to other propositions which are unexplained by overt logical structure. It is precisely the contribution of the prototype to effect these connections. Let’s return to the case of the watch on the table. Although Wittgenstein himself never spells out what sort of prototype this case would involve, I speculate that the analysis might come to something like the following: “there are some objects, such that some of them are tablewise-arranged, the others watchwise-arranged, and each of the watchwise-arranged objects is above at least one of the tablewise-arranged objects, and some watchwise-

1461 take there to be two related senses of the term “prototype” (Ürbild) in the Tractatus. One is the sense of 3.315, which appears e.g. at 12.11.14 and descends from from Russell’s Theory of Knowledge manuscript. The other is the sense of 3.24, which pervades the struggles of NB2 (11.5.15, 19.6.15), and this is what’s in question here.
arranged-object is touching some tablewise-arranged object”. The terms “tablewise-arranged” and “watchwise-arranged” would presumably themselves be existential generalizations over various concepts of spatial configuration: for example, existential generalizations over possible positions of the hand on the watch’s dial. The analysis as a whole clearly requires prototypical specification of the bases of truth-operations. It is moreover plausible that such specification will require non-first-order resources, in particular the concept of ancestral. For example, saying that some objects to be arranged watchwise requires saying that they are connected.

Marie McGinn (2006, 124ff) helpfully suggests that prototypes may be understood as linguistic meanings of words, which determine a proposition only relative to the context of use. Thus, a prototype would contribute, alongside the meanings of other words in a sentence, to determining a function from contexts to propositions which would be the fixed linguistic meaning of the sentence. The fine structure of the prototypes themselves would be slurred over by the signs. This fits well with the following remarks:

The fact that there is no sign for a particular proto-picture does not show that that proto-picture is not present. Portrayal by means of sign language does not take place in such a way that the sign of a

147 As Ori Simchem pointed out, this analysis is compatible with a watch’s being embedded in a table. More generally, such a priori formalization seems pretty implausible. In the amusing entry 22.6.15e, Wittgenstein acknowledged that he could not give conditions for a watch’s being on the table. In the next couple of paragraphs, I’ll suggest that on Wittgenstein’s view, the logical structure of incompletely analyzed representation is parasitic on the structure of what is represented.

148 Throughout this passage I’m indebted to unpublished work of Thomas Ricketts.

149 Or “characters” in the sense of Kaplan (1977).

150 As McGinn puts it,

Rather, we should think of our mastery of the ordinary language sentence in which a sign for a complex occurs as grasp of the form of the proposition that would replace it on analysis. It is in this sense that the proposition “contains a prototype”—something that is not yet an expression with sense—which specifies the form of the proposition that the sentence can be used to express, but not its sense. The form or prototype can be described by a general proposition: \((Ex,Ey)Rx\). However, on a particular occasion of using the ordinary language sentence to express a thought, the variable signs of the prototype are replaced by constants and the speaker uses the resulting proposition to assert that a determinate possible state of affairs exists... (2006, 128).
proto-picture goes proxy for an object of that proto-picture.

The second sentence looks mysterious, but it fits fairly well with McGinn’s reading. In place of sign of proposition, Wittgenstein here considers the sign of a prototype. An object of a proposition would be an object denoted by one of its constituent names. But, a prototype doesn’t consist of names which go proxy for objects. Rather, it contains variables, which do not go proxy for objects but range over them.

On McGinn’s view, prototypes are aspects of linguistic meaning, so that their structures are fixed by the context-independent nature of the words in the language. The context serves only to fill in the identities of the constituents of the eventually expressed proposition. The form of the proposition expressed by the sentence is fixed by linguistic competence with the sentence-type itself, and context contributes only identities of propositional constituents. For example, the identities of the constituents of the watch would fill in the gaps in its prototypical representation, thereby determining a definite representation of the watch. It is not clear that this is quite Wittgenstein’s idea. In particular, the passage just quoted continues as follows:

The sign and the internal relation to what is signified determine the prototype of the latter; as the fundamental co-ordinates together with the ordinates determine the points of a figure. (8.5.15)

Applying this passage to the example we’re considering, what are signified are the watch, the table, and the possibility that the one is on the other. Thus, according to the passage, the signification determines the prototype of what is signified. It is the watch and the table that determine the structure of their prototypical appearances.

So, I agree with McGinn that the prototype articulates the internal structure of what it presents, so that one and the same prototype cannot present states of affairs with different internal structures. Moreover, McGinn is also right that according to Wittgenstein, the proposition a sentence expresses varies with context of use. But, McGinn furthermore holds that the sentence-type itself, independently of context, determines a prototype in such a way that, to obtain the

\[151\] Or “completely characterizes”, as at 16.6.15h of the previous footnote.
proposition expressed by some token in context, it suffices to resolve the values of the objectual variables in the prototype determined by the type. Thus, one and the same sentence-type, under its fixed linguistic meaning, could be used to assert the existence of different states of affairs in different contexts; but these states of affairs will differ only with respect to their objectual constituents and not with respect to their structures.

However, in the notebook passages under discussion, Wittgenstein seems to hold that not just the constituents but also the form of the proposition expressed by a sentence may vary from context to context. That is, context contributes not just content but also (“the real”) form.

When I say this watch is shiny, and what I mean by this watch alters its composition in the smallest particular, then this means not merely that the sense of the sentence alters in its content, but also what I am saying about this watch straightway alters its sense. The whole form of the proposition alters. [16.6.15g]

As Wittgenstein repeated obsessively, the structure of a watch, let alone of a mote in the air, is complicated and fluctuating. Moreover, a prototype articulates the internal structure of what it represents, so that one and the same prototype cannot present states of affairs with different internal structures. But then, prototypes are too complicated and variable to be determined by linguistic meaning of sentence-types alone.

Somehow, Wittgenstein’s idea must be that a prototype of the proposition expressed by “the watch is on the table” depends partly on the context of use of that sentence—in particular, on which watch is mentioned, and on what is the internal state of that watch. Oddly, Wittgenstein reformulates this dependence as follows:

That is to say, the syntactical employment of the names completely characterizes the form of the complex objects which they denote.

Now, if “x completely characterizes y” is taken to mean means (at least) “y supervenes on x”; at least this much of 16.6.15h does follow from 16.6.15g. The two

152 Another statement of this point appears at 20.6.15r.
153 Or “completely characterizes”, as at 16.6.15h of the previous footnote.
paragraphs seem to present different directions of explanation; perhaps the right
conclusion to draw is that since this relationship between representation and rep-
resented is logical, neither direction of explanation should be favored over the
other. In any case, the essential point for my purposes is that the structure of
the unanalyzed picture is somehow essentially tied to the particular situation it
depicts. The unity of proposition and situation is prior to the identity of the
proposition: it is that the watch has the structure it has which shows the proposi-
tion to have the structure it has, which shows the proposition to be what it is. It
is to the extent that a proposition depends for its nature on the articulable natures
of things that the proposition suffers indeterminateness. Analysis replaces such
covert dependence with overt articulation.

Let me just conclude this section by remarking that it was already in 1914
that Wittgenstein recognized both the need for non-first-order resources in the
analysis of propositions and moreover that already by this point he had devised
a way to supply them. The second edition of the *Notebooks* includes an appendix
with facsimiles of notebook extracts which the editors of that volume deemed
indecipherable. These extracts include the following:

\[ aRb \cdot bRc \cdot cRd \cdot dRe = \varphi(a,e) \]

\[ (\exists R^n ). aR^n e \]

together with Wittgenstein’s dating of 19.9.14. Another extract is assigned the
same date by the editors of the *Notebooks*, and transcribed in the body of their
text:

A proposition like “this chair is brown” seems to say something enor-
mously complicated, for if we wanted to express this proposition in
such a way that nobody could raise objections to it on grounds of
ambiguity, it would have to be infinitely long. (19.9.14)

These two extracts are all and only the material the editors date to 19.9.14. So,
it seems that they belong to the same entry. The reference to an infinitely long
sentence in the second passage must then refer to the formulation of the ancestral
in the first passage. Moreover, the first sentence of the next day’s entry is this:
That a sentence is a logical portrayal of its meaning is obvious to the uncaptive eye. (20.9.14)

So, I take the point of the discussion to be simple. In revealing the way in which an ordinary proposition secures the determinacy of its sense, analysis may resort to representing the proposition as the result of a truth-operation on the terms of a form-series. This conjecture receives some support from a later remark:

The mathematical notation for infinite series like

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

together with the dots is an example of that extended generality. A law is given and the terms that are written down serve as an illustration. In this way instead of $\forall x. f(x)$ one might write “$f(x). f(y) \ldots$.” [22.5.15]

There is no obvious grammatical antecedent of the phrase “that extended generality”. But, this passage appears in the context of a difficult discussion of the question how complex spatial objects could “seem to me to be essentially things—I as it were see them as things.—And the designation of them by names seems to be more than a mere trick of language” (13.5.15). Wittgenstein answers the question at T3.24, with the remark quoted above. But, already 15.5.15 anticipates this answer: “So much is clear, that a complex can only be given by means of its description; and this description will hold or not hold.” I conjecture that a week later, with the phrase “that extended generality” (22.5.15), Wittgenstein refers to the generality which a form-series variable may contribute to a proposition. Thus, the 22.5.15 entry points to a diversity of logical resources which would underpin the pictoriality of ordinary unanalyzed propositions. The concept of ancestral was held by Wittgenstein to be implicated in the very logic of picturing. Needless to say, this possibility would have been overlooked by Russell in his reliance on quantificational generality in the theory of descriptions.

Further support for the claim that “extended generality” at 22.5.15 means “formal generality” appears at 22.6.15, where Wittgenstein mentions Whitehead’s “Systematic statement of symbolic conventions” to explain how “the sense of the proposition ‘the watch is lying on the table’ is more complicated than the proposition itself.” [154]
3.4 Another role for form-series?

So by April 1916, when Wittgenstein takes up the third wartime notebook, the disjunctive analysis of the concept of ancestral was already firmly in hand. The ancestral is just one particularly natural special case of the extended conception of generality which analysis of the pictoriality of thought demands. It was already plausible that generality could be somehow so extended, since the ancestral would have appeared to the student of Frege and Russell as a paradigmatically logical notion.

The third notebook begins with an attempt to state the general form of the proposition, as the result from elementary propositions by applying a formal operation repeatedly. The ancestral then naturally reappears:

\[(p): p = aRx \cdot xRy \ldots zRb\]

Here, the relation \(R\) is that relation which the result of an operation would bear to its basis, a relation which Wittgenstein would later identify as “internal” (T5.232). This contrasts sharply with the earlier uses of the ancestral, whose underlying relation would be an ordinary external relation such as that of spatial contiguity. At 16.4.16, then, an ordinary analytical tool is pressed into an extraordinary application.

The form-series device is therefore not just another source of ordinary logical complexity, but also distinctively required to explain how “after the primitive signs have been given we can develop one sign after another ‘so on’” (21.11.16). Wittgenstein follows his sketch of this explanation at 16.4.16, by anticipating T5.252: “In this way, and in this way alone, is it possible to proceed from one type to another”. Thus, to judge from the context of this thought in the Notebooks, the reason that Wittgenstein says that only a form-series allows us to pass from type to type is that one must use a form-series to articulate the general propositional form. If there are hierarchies of types of propositions, then somehow or other the general propositional form must traverse them. It is therefore because of the involvement of a form-series in articulating the general propositional form that the form-series makes it possible to pass from type to type in a hierarchy of
propositions. The form-series which motivates the precursor to T5.252 is therefore not a form-series of the sort which would subserve the articulation of propositions involving the ancestral of ordinary external relations like contiguity or parenthood. Actually, the form-series in question isn’t supposed to belong to the articulation of a proposition at all:

The above definition can in its general form only be a rule for a written notation which has nothing to do with the sense of signs. But can there be such a rule?

The definition is only possible if it is itself not a proposition.

In that case a proposition cannot treat of all propositions, while a definition can. [17.4.16]

Here, Wittgenstein insists that the articulation of a general propositional form cannot be regarded as a proposition, presumably because otherwise a proposition might somehow talk about all propositions. He speculates that this use of a form-series might rather be understood to express a definition, provided that it makes reference only to signs, and not to their senses. This resembles some later remarks on propositional variables, especially in that segment of the 3.31s which derives from the PT5.00s. On my understanding, in the Tractatus, the point of assigning values to a propositional variable is to represent one proposition as the result of an operation as applied to the values of that variable; one thereby represents the proposition by means of the logical relations which it bears to others. Thus, the specification of values of a variable becomes a primary technique of analysis. Wittgenstein speaks of such specification of values as a matter of “definition”. In this vein, he says that a name, being an unanalyzable symbol, cannot be “pulled apart” by means of a definition (3.26; cf. also 3.24d). Its invocation of the concept of variable aside, I take this understanding of definition to be essentially secure by the end of the second notebook. At 17.4.16, then, Wittgenstein seems to envisage a realm of formulations of analysis of propositions, and to allow that formulations in this realm do not express propositions.

155Namely, 3.316-3.318.
156See in particular 21.6.15i; and compare with 3.24.
but must instead be understood as codifying rules of translation between systems of signs (3.343).\footnote{I don’t, at this point, propose to make sense of this notion of “codifying” which would somehow be contrasted with the “expressing” which ordinary signs do for propositions. I sometimes pretend to understand the “codification” of a rule as a kind of directive. But this pretense quickly gives way under pressure.}

So, the form-series device plays two roles for Wittgenstein. On the one hand, it underpins the logical structure of ordinary thinking about genealogy, geography, epidemiology, and so on. In this way, it underpins what one might call the structure of appearance, in particular the appearance of ordinary spatially complex things as, indeed, really things. For, such appearances are essentially appearances in signs, which partake of all the logical structure that there is. On the other hand, the very system of signs itself also invokes the concept of the “and so on”. It is in the context of laying out the unity of language in analysis that the form-series device finds another use. This second use forms the context in which Wittgenstein initially poses the remark that the form-series device allows us to pass from type to type in a hierarchy of propositions.

3.5 T5.252 and the evolving conception of hierarchy

Now, it seems to me that T5.252 is an example of a remark which means something different in the *Tractatus* than its precursor does in the *Notebooks*. In particular, I agree with Ricketts that in the *Tractatus*, the point of the remark is rather that the form-series device lets us pass from type to type in a hierarchy, not just in formulating the definitional assemblages like that of T6, but in this or that ordinarily senseful proposition of genealogy or geology or whatever. Yet in the *Tractatus* context the remark points to a deeper departure from Russell in Wittgenstein’s reception of the notion of hierarchy. This departure emerges in the gradual 1916 refinement of the account of how arbitrary propositions are generated from elementary ones by repeatedly applying an operation.

As Jinho Kang has argued (2005, 15), in the early part of NB3 Wittgenstein has not settled on the *Tractatus* idea that it is by means of just a single operation that all propositions are generated.\footnote{Indeed, at 11.5.16 he writes:}

\footnote{I think the internal evidence overwhelmingly supports Kang’s view, as against that of McGuin-}
There are also operations with two bases. And the “$|$”-operation is of this kind. $(x, y)_j$... is an arbitrary term of the series of results of an operation.

$(\exists x)\phi x$

Is $(\exists x)$ really an operation? But what would be its base?

Here, the $|$-operation would be the Sheffer stroke, which already appears in the Notes on Logic. This bears obvious resemblance to the $N$-operation of the Tractatus but also two differences. First, a result of $|$ is true when at least one of its
bases is false, whereas a result of \( N \) is true only when all of its bases are false. Second, and much more importantly, Wittgenstein only considers applying this operation to one or two bases. Indeed, the tone of the remark suggests that he had just hit on the idea that an operation might take more than one basis. At this stage, then, there cannot yet arise any question of replacing ordinary quantification with applications of \( \mid \). It is for this reason that Wittgenstein asks whether \((\exists x)\) is “really” an operation, as though this is a conclusion he’s forced to accommodate but can’t yet see how. Part of the difficulty seems to be that its basis is not \textit{a priori} finite. But Wittgenstein is also struck here by the fact that unlike \( p|q \), the notation \((\exists x).f x\) does not express a proposition as the result of operating on propositions which are themselves expressed by parts of that notation. This is why he asks what the bases of \((\exists x).f x\) are supposed to be—he cannot yet imagine that they are the instances of \( f x \), to all of which at once a single operation might be applied.

The \textit{Tractatus} account of quantification under the general propositional form requires Wittgenstein to distinguish truth-function from generality (5.521), and hence requires the full-fledged concept of propositional variable of 5.501 which, so far as I know, doesn’t appear until PT5.00s. At 9.7.16, Wittgenstein seems to approach the basic idea:

Don’t forget that \((\exists x)f x\) does not mean: There is an \( x \) such that \( f x \), but: There is a true proposition “\( f x \).” [9.7.16]

I suspect, though, that at this point what Wittgenstein intends is that the quantifier is an operation which takes as its base not a proposition but a prototype, or common form of a propositional multiplicity. For, he goes on a few days later to seem to despair of the possibility of finding a single general form of all operations:

If two operations are given which cannot be reduced to \textit{one}, it must at least be possible to set up a general form of their combination.
\[
\phi x, \psi x | \chi x, (\exists x)., (x). \quad [13.7.16]
\]

Here, the \( \phi x \) indicates the general form of the elementary proposition.\footnote{For it alludes to the 16.4.16 proposed specification of the elementary propositions by means of their function-argument structure.}
the existential and universal quantifiers in the presence of negation, although the reason for this is not clear to me.\footnote{In the same vein he speaks at 29.8.16 of the "usual small number of fundamental operations"; and similarly 26.11.16 has: "All operations are composed of the fundamental operations."} The last sentence of the quoted 13.7.16 entry anticipates the remark of 5.503, which alleges instead the easiness of spelling out how all propositions can be built up from a single operation.

So it must be somewhere between December 1916 and January 1917 that Wittgenstein concludes that after all only one operation need be taken as fundamental. For toward the end of the third notebook he writes:

In the sense in which there is a hierarchy of propositions there is, of course, also a hierarchy of truths and of negations \([\text{Verneinungen}]\), etc.

But in the sense in which there are, in the most general sense, such things as propositions, there is only one truth and one negation.

The latter sense is obtained from the former by conceiving the proposition in general as the result of the single operation which produces all propositions from the first level. Etc.

The lowest level and the operation stand for the whole hierarchy. \[7.1.17\]

This passage presents the closest approximation in the \textit{Notebooks} to the general propositional form as it appears in the \textit{Tractatus}. As we’ll see, it illuminates our eventual quarry, the meaning of the talk of levels in 5.252. Evidently, given its terseness and the sparseness of its context, its meaning must to some extent be a matter of speculation. But, it seems natural to suppose that when Wittgenstein refers to a hierarchy of truths and negations, he may refer to the distinction of levels of truth and falsehood which appears in \textit{Principia} (Whitehead 1910, 42).\footnote{Thanks to Sanford Shieh for thinking so too.}

Thus, propositions at different levels of a hierarchy may in turn have different levels of truth and falsehood proper to them. But for Russell, truth and falsehood, branching as he thinks they do into a variety of levels, thereby branch into a variety of levels of propositional function. This is because Russell takes truth and falsehood to be truth-functions, and takes truth-functions to be propositional

\footnote{Thanks to Sanford Shieh for thinking so too.}
functions. Now, I think that roughly speaking, Wittgenstein shares Russell’s assimilation of the apparent linguistic predications of truth and falsehood to what we would call the truth-functional connectives. So, transposed to *Tractatus* vocabulary, the passage might then be understood to be envisaging a stratification of types of affirmation and denial. Perhaps, by “the sense in which there are, in the most general sense, such things as propositions” he alludes to the existence of the general propositional form. Thus, the suggestion would be that one grasps theoretically the general propositional form by reconceiving Russell’s stratified truth-predicates (or, stratified connectives) as signalling truth-operations which do not mark the sense of their results. So understood, there would no longer be any need for many affirmations and denials, but a need only for one of each. The general form of the proposition still represents propositions as standing in a hierarchy, but the hierarchy is only the hierarchy of iterated applications of truth-operations, thus merely a hierarchy of notations and not of modifications of sense.

In the 7.1.17 statement of the general propositional form, then, Wittgenstein acknowledges a hierarchy of propositions. My contention is that this acknowledgement remains in the *Tractatus* and in particular reappears at T6 and at 5.2522. The grounds for this contention are that the entry at 7.1.17 simply is the attainment of the general propositional form as it appears in the *Tractatus*. Thus, the *Tractatus* acknowledges a hierarchy of propositions, namely, that hierarchy of ordered applications of the operation of denial by means of which all propositions are eventually generated. This hierarchy, however, is independent of reality (26.4.16; 5.5561). For, it assigns a rank to a proposition simply by measuring the number of times a truth-operation is applied to construct it. One and the same proposition will thus reappear at arbitrarily high levels, since it is expressed by the double negation of what expresses it. Indeed, I’ve already claimed that elementariness does not characterize the sense of propositions. For, all there is to the sense of a proposition is the way in which it depicts the world to be. And

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162 Since the very first page of the manuscript of the *Prototractatus* contains a variant of T6, the first page of the *Prototractatus* manuscript must have been produced sometime after 7.1.17.
163 At least relative to a completion of analysis, one can talk about the smallest number of applications of a truth-operation a proposition’s expression requires.
there are no truth-operators in the world, on whose application the geometry of notation forces well-foundedness. There is no hierarchy of propositions in the world, not just as a matter of what is, but as a matter of what could be: there are only internal relations between situations. But it is only in virtue of the structure of notation, of iterated application of truth-operators, that propositions take positions in a hierarchy.

We’re now in a position to see the point of the claim at 5.252. Let me repeat the passage:

Only in this way is progress from term to term of a form-series (from type to type in the hierarchies of Whitehead and Russell) possible. (5.252)

In the *Tractatus*, hierarchies characterize systems of propositional notation. A hierarchy is simply what we would call a measure of the truth-functional complexity of formulas, i.e., the supremum of the lengths of the branches of the formation trees. Now, at 5.501 Wittgenstein lists three kinds of propositional variable. The first kind of variable has its values fixed by a finite list; the second kind has its values fixed as the range of a propositional function. For a variable of either of these two kinds, some proposition in its range is such that the rank of the result of applying a truth-operation to all propositions in its range is the successor of the rank of that proposition. Thus, arbitrarily deep nestings of the first two kinds of variable only yield propositional notations of finite truth-operational rank. Contrast with this the use of the form-series variable in expressing the ancestral of a relation. The range of this variable is a series of propositions, such that for all \( n \), the \( n \)th proposition in the series is expressed by a notation containing \( > n \) nested occurrences of the \( N \)-operator. Since it is the number of nested occurrences of \( N \) in its notation which determines the rank of a proposition in a hierarchy, the form-series device therefore allows the construction of a series of terms each of which gives way to a successor of higher rank. Thus, through the use of the form-series device, the ranks of propositions enter the transfinite. Or, to say at last what I take Wittgenstein to mean at 5.252: through the use of a form-series device, one constructs a serial presentation of the bases for a truth-operation, a series which passes from level to level in the hierarchy of results of
truth-operational iteration.

Now, at this point it’s natural to ask: doesn’t Wittgenstein’s remark suggest that the form-series lets us move from one type to another in the hierarchies of Russell and Whitehead? Those hierarchies are not, or not all, themselves immediately intelligible as hierarchies of results of iterated truth-operations, ordered by number of iterations. Rather, most canonically, they are hierarchies of propositional functions, with a function assuming this or that position by means of the quantifier ranges it intensionally implicates. And, on my account, there’s little trace of this kind of structure left in the *Tractatus*, so it’s not clear how my explanation of 5.252 could be right.

Wittgenstein says that Russell and Whitehead repeatedly make use of formally generalized steps from one level of a hierarchy to the next, but that they do not admit the possibility of such steps. Perhaps another way to put this is: from time to time, Russell and Whitehead help themselves to formal generality, but their own principles preclude its official recognition. Now, if for “formal generality” we read “typical ambiguity”, then it becomes something Wittgenstein might have seen in *Principia*. Perhaps there is some significance to Wittgenstein’s mention of Whitehead: although it would already be natural for Wittgenstein to mention him as a co-author of *PM*, Wittgenstein shared the understanding that the theories of types are due to Russell. At 22.6.15 of Notebook 2, and in precisely the context which I’ve argued is a crucial occasion of Wittgenstein’s use of formal generality, he mentions Whitehead’s ‘Prefatory statement of symbolic conventions’, whose basic point is that “whatever can be proved for lower types… can also be proved for higher types. Hence … it is unnecessary to know the types of our variables, though they must always be combined within one definite type” (Whitehead and Russell 1910-1913 volume 2, IX).

It is then also clear to the *UNPREJUDICED mind* that the sense of the proposition “The watch is lying on the table” is more complicated than the proposition itself.

The conventions of our language are extraordinarily complicated. There is enormously much added in thought to each proposition and not said. (These conventions are exactly like Whitehead’s ‘Con-
The *Tractatus* concept of “formal generality” codifies this “generality of form” which Wittgenstein traces to Whitehead’s discussion. So, my understanding of Wittgenstein’s mention of Russell and Whitehead at 5.252 is this. The *Principia* presents a hierarchical arrangement of the totality of propositions and propositional functions. This hierarchy officially precludes generalizing over all propositions. Still, Russell and Whitehead take implicit appeal to particular such generalizations to be licensed by the device of typical ambiguity. Now, the *Tractatus* also presents a hierarchical arrangement of propositions, which threatens to interfere with some possibilities of logical construction. Formal generality lets us construct a proposition which is a truth-function of a whole series of propositions, each succeeding term resulting from its predecessor by a formal operation. In this way, formal generalizations are what lets a single proposition climb through Wittgenstein’s series of logical types.

Wittgenstein’s allusion to Russell and Whitehead elides what I take to be a philosophically significant technical departure for the *Tractatus*. For in Wittgenstein’s hands, the system of logical types is no longer a system of constraints erected for our (or God’s?) logical safety. So it would be wrong to think that formal generality is a kind of “system override” which can be safely exercised under certain conditions. Rather, for Wittgenstein, Russell and Whitehead simply do not acknowledge all possible ways of stipulating values of a propositional variable. A proposition can be a truth-function of elementary propositions in any way that it can possibly be—that is, in any way that can be articulated. We don’t suffer a system of type constraints, but enjoy the full expanse of notational possibilities.

### 3.6 The riddle of T6

By the end of *Notebook* 3, then, Wittgenstein has reached the following picture. Ordinary sentences stand in logical relations of entailment and contradiction because they express results of repeated truth-operations on elementary propositions and are therefore truth-functions of elementary propositions. In particular,
the totality of propositions can be exhibited notationally in a series of stages, each stage containing a joint denial of every specifiable multiplicity of propositions already constructed. Thus, the “theory of types” no longer describes a system of constraints on what can be said, but rather clarifies how things are said, by making notationally manifest the logical relationships which are constitutive of saying one thing rather than another.

Wittgenstein announces, in the Prototractatus: “the theory of types becomes clear.” This is one of the five or six percent of the Prototractatus entries which does not reappear in the Tractatus. The editors of the published version (Wittgenstein 1971) observe that its numbering as PT5.00 must be incomplete. However, they suggest that it probably belongs between or immediately after the following entries, which anticipate 5.251 and 5.252a:

A function cannot be its own argument, whereas an operation can take one of its own results as its base. [PT5.00161]
It is in this way and only in this way that the step from one type to another in the hierarchy is possible. [PT5.00162]

However, it’s also possible that the whole of the PT5.00s can be read as an elaboration of PT5.00. For, when he talks about the theory of types becoming clear, he refers to its transformation from a system of type constraints into an explanation of truth-functionality. And it is the PT5.00s which expound the results of this change.

To put my interpretive conjecture as a concrete counterfactual: had Wittgenstein preserved the PT5.00s in the Tractatus, then PT5.00 would have been numbered T5.001, and PT5.00x might have become T5.001x. It should be noted that PT5.00 is the only PT entry whose numbering is claimed to be incomplete by the editors. Moreover, Wittgenstein’s draft of PT contains many corrections of numbering. It therefore seems implausible that Wittgenstein simply overlooked the minor decision whether to append 1611 or 1621 to the string 5.00. A natural hypothesis is that he didn’t complete the numbering of 5.00 because of the

164One entry on manuscript page 101, which becomes T5.154, lacks a number altogether, but this only gives the somewhat trivial mention of an urn containing variously colored balls; moreover, the absence of a number there is flagged by Wittgenstein, whereas the numbering of PT5.00 is unflagged.
extent of induced renumberings of other entries. Now, no entries with indices of the form 5.0016... precede PT5.00 in the manuscript of PT, and PT contains only three such entries 5.00161, 5.00162, and 5.00163. So, numbering PT5.00 as PT5.0016... would not have induced much inconvenience of renumbering of other entries. On the other hand, the required renumbering predicted my hypothesis is quite extensive, since most of the PT5.00s (twenty-two entries) do precede PT5.00 in the manuscript.\[165\]

The Prototractatus is evidently a close ancestor of the Tractatus. Much of the difference between the texts consists in shifts of terminology or slight incrementations of numbering. So far as I can tell, there are only two really gross revisions. Michael Kremer (1997) has observed that Wittgenstein elevates his statement of the “context principle” from PT3.202 to T3.3, and argued that this indicates that the Tractatus is a “transitional work”, occupying a position of philosophical movement from Wittgenstein’s pre-Tractatus views to the views of the so-called “middle” or “later” Wittgenstein. But the grossest revision to the Prototractatus is not much discussed in the secondary literature.

The PT5.00s form a sustained exposition of Wittgenstein’s distinctive logical theory. This includes the explanation of the formula at T6, and the theory of propositional variables. Save for a familiar point on truth-operations which date to the Moore Notes, the known antecedents of the PT5.00s mainly appear in the attempts on the general propositional form from Notebook 3, which have been discussed already in this paper. The PT5.00s must culminate developments which followed the end of Notebook 3 in January 1917. But, what must been intended as a summary of these perfected developments disintegrates in the Tractatus, into an anonymous but distinctive diaspora which populates the 5.2s, the 5.50s, the 3.31s, and the 4.127s. My hypothesis is that the PT5.00s expound what Wittgenstein in the Prototractatus to be his “theory of types”. He broke up this

\[165\]In support of McGuinness-Schulte, it’s worth noting that the relationship of PT5.00 to PT5.00161 resembles the structure of various other continuous passages, including 3.332, PT3.00161, NB107, and NB96. This observation doesn’t refute my textual conjecture since it is surely PT3.00161 rather than PT5.00-PT5.00162 which is the antecedent of 3.332. But in any case, my interpretive claim doesn’t stand or fall with the textual conjecture. The point is just that the PT5.00s can be read as though they are a commentary on what is numbered as PT5.00. Reading them in this way illuminates 7.1.17, 5.252, and, I hope, ultimately T6
exposition for the same reason that he excised its lead remark, PT5.00. But what was this reason? Doesn’t the theory of types become clear in the *Tractatus*?

The theory of types does not become clear in the *Tractatus*, for the same reason that the formula at T6 is never properly explained. Many commentators (even quite good commentators such as McGinn 2006, 233) have taken Wittgenstein to explain the notation of T6 at T5.2522, and assumption has led them to concoct artificially “linear” orderings of the totality of propositions, under the conceit that the totality of propositions itself constitutes a form-series. However, this confuses two quite distinct roles into which Wittgenstein pressed the concept of iteration. For, 5.2522 explains only a form-series generated by an operation which takes just one proposition as base. In contrast, the purported form-series described at T6 would be generated by an operation taking a propositional multiplicity as its base. Thus, commentators confuse the articulation of the general form of the proposition, with a means of specifying a base for a truth-operation. Wittgenstein’s implementation of the truth-functionality thesis requires that these two functions of the iteration concept be distinguished. For, the determination of truth-value of the result of applying a truth-operation to the totality of propositions would depend on the result of that determination. Entry T5.2522 elaborates on the third way of stipulating the range of a variable, and therefore ought to follow the T5.51s and T5.52s, as elaborations of 5.501f.

The frustration of commentators’ attempts to concoct a linear ordering of the totality of propositions can be seen as a consequence of Wittgenstein’s attempt to enforce a separation between the two roles of the concept of iteration in his early philosophy of logic. In its use to specify bases of truth-operations, Wittgenstein only explains the iteration of unary operations. In contrast, Wittgenstein simply does not explicitly license the use of multigrade operations in specifying operational bases. However, a proposition can be seen as the result of a truth-operation on any multiplicity which is formally specifiable, since the range of a variable is purely a matter of stipulation. And there seems to be no sense of “formal specifiability” which would include form-series generated by unary but not by multigrade operations. Here it is striking to compare 5.2522 with its antecedent PT5.005351:
Let us write the general term of a series of forms like this:

\[ \bar{x}_0, \bar{x}, O'(\bar{x}) \]

The \( \bar{x}_0 \)'s are the initial terms of the series, the \( \bar{x} \)'s are terms arbitrarily selected from it, and \( O'(\bar{x}) \) is the term produced from the \( \bar{x} \)'s by means of the operation \( O'(\bar{x}) \) as the series proceeds.

In the Prototractatus, the explanation of multigrade-generated form-series appears in the place proper to explanation of a means of specifying the basis of a truth-operation, that is, as an elaboration of the antecedent of 5.501. But it appears that a multigrade-generated form-series is precisely the sort of multiplicity that the general propositional form would take as its range, were the general propositional form to be seen as a variable. We thus seem to have in the Prototractatus an unflinching embrace of impredicativity.

The change from PT5.005351 to T5.2522 is unquestionably deliberate, since it requires both a deletion of the bar superscripts and a change in number of the accompanying explanation. I think this change evinces an attempt on Wittgenstein’s part to distinguish between the presentation of the general propositional form and the form-series presentation of bases of truth-operations. Although I’m not exactly clear on how Wittgenstein wished to effect the distinction, let me suggest two hypotheses.

First, it is natural to suppose if \([\bar{p}, \bar{z}, N(\bar{z})]\) were really to be a variable, then it would involve an implicit generalization over all formal concepts. For, the middle term ranges over all specifiable multiplicities; together with the third term it signifies the operation which takes any multiplicity to its joint denial. However, since we’re supposing the bracket expression itself to be a variable, therefore it would itself have to be a value of the bound variable it contains. So as Sundholm (1992, 70) suggested, the T6 variable seems to involve some kind of circularity.

One tempting response, suggested by Peter Sullivan (2004), is to reject the statement of 4.53 that the general propositional form is a variable. Now, Sullivan doesn’t develop this proposal in detail. But here’s one way it might go. An ex-

\[166\text{I can’t resist quoting his remark: “given Wittgenstein’s harsh words against Frege and Russell concerning impredicativity (4.1273) some care on his part would not have been out of place here, if only to show that the author of the } \textit{Tractatus} \text{ was aware of the lurking impredicativity” (70).}\]

\[167\text{Well, Sullivan’s strategy is to argue that Wittgenstein tries to forestall the possibility of con-}\]
pression of generality is a variable which signifies a formal concept, and the values of the variable are the items which fall under it.\footnote{168} So, the general propositional form can be understood as a variable only if there is some common distinguishing mark of all formal concepts. Perhaps Wittgenstein implicitly rejects the required hypothesis that there is such a formal concept as that of formal concept itself.\footnote{169} This seems to be an insinuation of the phrasing of 5.501, with its emphasis on the entirely optional or even arbitrary character of catalogues of methods of stipulating the bases of truth-operations: “We \textit{can} distinguish three methods of description” (5.501, emphasis in original). Moreover, in the \textit{ProtoTractatus} antecedent of this sentence, the word “\textit{can}” is unemphasized, which (mildly!) suggests that in the revisions accompanying the dissolution of the 5.00s he decided to amplify the theme of openendedness of formality.

However, in the \textit{Tractatus}, Wittgenstein says that it should be possible to find an exact expression for all the ways in which propositions may arise by repeatedly applying the operation $N$ (5.503). The possibility of such an exact expression seems to me to sit ill with the thought that formality might not itself be formal. Frankly, the maneuver feels like a bit of a dodge. And if he really thought it was viciously circular say that the general propositional form is a variable, then why does he still say it at 4.53? I don’t know of any better way of developing Sullivan’s suggestion, but in the absence of something better, I don’t find the suggestion satisfactory.\footnote{170}

There is a simpler story. For Wittgenstein, analysis can represent a proposition as the result of a truth-operation on any multiplicity of propositions distinguished by some internal mark or feature. Any genuinely internal mark or feature is a legitimate means of specification, so long as it respects the logical unity between proposition and presented situation. The unity may be the minimal one: just, that \textit{this} is how things are. Of course, there is no reason to consider such

\footnote{168}This is 4.127, give or take a use-mention issue.

\footnote{169}On this view, available bases of truth-operations might stand to the totality of propositions as sets stand to the class of all sets.

\footnote{170}Sullivan, strangely, doesn’t mention the revisions of the PT5.00s.
a representation in itself to be in any respect revelatory. One needs somehow to fit every proposition into the system of all propositions, and this ultimately requires that analysis be wellfounded. Only if analysis constructs a wellfounded notational arrangement of propositions can it serve as an elucidation of thought. Thus, it is entirely possible to apply truth-operations to the totality of propositions, at least in the sense that, for example, it is possible to jointly deny a proposition and its negation. However, this possibility is irrelevant to the project of analysis, because it does not further the articulation of truth-functionality.

So, despite the revision from PT5.005.351 to 5.252.2, Wittgenstein maintains no in principle objection to specifying the bases of a truth-operation by means of a form-series variable which itself involves a variadic generating operation. However, he may have recognized no pressing practical need for such an extension of the form-series device in analysis. The change to 5.252.2 merely serves to clarify the distinction between the two appearances of form-series variable in Wittgenstein’s exposition. Wittgenstein may then have meant what he said, when he said “the general form of the proposition is a variable”.

### 3.7 An alternative reading

Thomas Ricketts (2012) presents a systematic reconstruction of the role of form-series in the *Tractatus* framework. Ricketts applies this reconstruction to some important interpretive questions, including the proper reading of 5.252. I’m sympathetic to much in Ricketts’ program. In this section, I want to point out some differences between our readings.

Ricketts’ paper opens by reviewing the importance of higher-order generality to the logicist projects of Frege and Russell, and then raises the question: “what becomes of higher-order generality in the *Tractatus*?” Ricketts maintains,

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171 There are some affinities to Wittgenstein’s response to the paradoxes in Gaifman (2000).

172 Naturally, I’m not completely confident of this conclusion. But the two hypotheses just sketched are the only ones I know of, and the former strikes me as tantamount to alleging a kind of intellectual dishonesty. My own limited experience with the *Tractatus*, unlike that with certain other books, has been that more and more of it becomes intelligible with time and patience. This makes me want to give its author the benefit of the doubt.

173 However, I’m thoroughly indebted to Ricketts’ work. The criticisms to follow should be taken as praise.
as do I, that Wittgenstein rejects the possibility of impredicative higher-order quantification altogether. Nonetheless, Ricketts also argues that the *Tractatus* contains resources to simulate Russellian predicative higher-order generalization. This “simulation” is achieved by means of formal generality, i.e., by means of the form-series device.

Let me now summarize Ricketts’ envisaged construction. Ricketts doesn’t concern himself with the details of specifying form-series themselves. Rather, he assumes,

wherever there is a procedure for recognizing members of a class of Russellian first-order sentences in terms of logically significant features of their construction, there is a formal law that generates a form-series whose members are those sentences. (Ricketts 2012, 18)

In particular, there will be a form-series $C_0(x), C_1(x), \ldots$ whose terms are exactly the first-order formulas in the free variable $x$. Now, $B$ be an atomic formula,

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174 Ricketts does allow that generality proper in the *Tractatus* might reach slightly beyond what can be achieved by purely first-order means (as I’ve identified them in Chapter II). Say that a (unary) elementary sentence function is the result of removing a name $a$ from an elementary proposition $A$. Ricketts says that such an elementary sentence function can itself be removed from $A$; then, according to him, the values of the resulting variable include all elementary propositions containing the name $a$. I hold that talk about “removal” or “replacement” must be explained in terms of abrogating some arbitrary formula conventions directly in virtue of which signs belong to the *bedeutungsvoll* symbols to which they do belong. But it is only genuinely simple signs, i.e., names, which receive their meaning directly from arbitrary conventions. So, I’m committed to the view that only genuine names can be “removed” or “turned into variables”. One can remove a succession of names from an elementary proposition, but the range of the resulting variable will be much more limited than what Ricketts envisages: it will include only those elementary propositions from which that variable can be constructed by removal of names. Thus, I individuate expressions (in the sense of 3.313) much more finely than does Ricketts. That is, Ricketts appears to hold that any name $a$ determines (or is?) a single expression $\chi(a)$ which marks the sense of all elementary propositions containing the name $a$, and then tries to explain how this expression could mark the sense of all propositions whatsoever that contain $a$. According to me, however, it is not correct to talk about a name as an expression or as determing an expression uniquely. Rather, an expression is only determined by a name together with a proposition. There will be (presumably) infinitely many different expressions $\chi(a, A_1), \chi(a, A_2), \ldots$ depending on the form of the proposition $A_i$. Each $\chi(a, A_i)$ will mark the sense of some propositions containing $a$, of course including $A_i$ itself. The classes of propositions correspondingly marked out by $\chi(a, A_i)$ then disjointly exhaust the propositions containing the name $a$. All propositions in the same $\chi(a, A_i)$ are instances of the same “logical prototype” in the sense of 3.315; but moreover, they will result from prototype by containing $a$ “in the same place”.

175 Ricketts’ construction therefore requires the existence of “free variables”. I think that the *Tractatus* conception of objectual generality excludes them, but do not press the point here.
presumably also in the free variable $x$, and let $A$ be an arbitrary containing at least one occurrence of $B$. Then, Ricketts assumes, for any selection from amongst the occurrences of $B$ in $A$, there is a form-series $D_0, D_1, \ldots$ such that each $D_i$ the result of uniformly replacing the selected occurrences of $B$ in $A$ with $C_i$. The disjunction of the $D_i$ will then express a generalization over all the first-order definable properties. For example, we could in this way generalize with respect to the first occurrence of $F$ in the formula $\forall x(Fx \rightarrow Fx)$, obtaining a formula $\forall X \forall x(Xx \rightarrow Fx)$.

Ricketts’ plan extends well beyond generalizations over first-order definable properties. A few refinements would open a path to generalization over first-order definable polyadic relations as well. But more importantly, we can now consider the totality of formulas in one free individual variable which involve simulated generalization over first-order definable relations. These formulas, Ricketts assumes, may themselves also be collected into a form-series, and so that generalization over them would be simulated as well. In this way, we might successively construct the hierarchy of Principian ‘orders’ of properties of individuals. Introducing variables of higher type, ranging over properties of properties, relations between properties and relations, etc. opens the prospect of simulated higher-type generalizations as well, with higher-type pseudoquantifiers similarly stratified into Principian ‘orders’. In this way, Ricketts speculates, the *Tractatus* may recover the expressiveness of the entire ramified hierarchy. Finally, Ricketts concludes, “I should like to think that this is what Wittgenstein has in mind, when he speaks in 5.252 of advancing from type to type in the hierarchies of Russell and Whitehead” (2012, 21).

Now, before presenting my response to all this, I should say that the foregoing summary has been tailored to highlight points of disagreement. On the whole, I find myself sympathetic with much of Ricketts’ philosophical-interpretive aims. But, let me turn to the disagreements. First, of course, is we read 5.252 quite dif-

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176 This formulation is a bit more general than that of Ricketts. But I suspect it should be liberalized somewhat further still. For example, we need to be able to accommodate generalizing with respect to both occurrences of $F$ in $Fa \rightarrow Fb$.

177 The Principia meaning of “order” is different from what is now standard: properties at a fixed level, say properties of an individual, recursively stratify into orders, according to the level and order of the quantifiers required to express the properties.
ferrently. Second, while I think that Ricketts has the right applications in mind for the logical resources of the *Tractatus*, it seems to me that the underlying account of those resources leads to some technical problems, and also to some interpretive positions which I can’t quite follow all the way.

Ricketts says that 5.252 alludes to the possibility of constructing, or simulating, the ramified hierarchy in *Tractatus* framework. On Ricketts’ account, form-series serve to generate the bases of the infinitary disjunctions and conjunctions which simulate predicative higher-type quantification. Thus, for example, a form-series may contribute to simulating predicative generalization over Principian ‘first-order’ properties of an individual. Such form-series may appear nested inside one another, thus simulating nested higher-type quantifiers as well. However, Ricketts’ account never invokes a form-series such that each of its terms is followed by a term of higher type. And yet, this seems to be just what 5.252 says that the form-series are supposed to make possible. So Ricketts’ reconstruction doesn’t seem to fit the letter of the text.\(^{178}\)

On my reading, in contrast, by “operation” at 5.252, Wittgenstein’s meaning fits something as simple as the \(N\)-operation itself. Thus, an example of progress from “type to type” would be the series \(A, NA, NNA, \ldots\) which the variable \([A, \xi, N \xi]\) would generate. Of course, such a series adds no expressive power, but other series with the same property do, for example a series of formulas whose disjunction expresses the ancestral. The problem for me, as Ricketts would quickly point out, is that the step from \(A\) to \(NA\) (or to \(\neg A\)) is certainly not a step to a higher Russell-Whitehead type. However, as already mentioned, the 7.1.17 notebook entry explicitly mentions a hierarchy of “truths and negations”, and he says that a single operation generates all propositions in this hierarchy. So the only question is why Wittgenstein says “Russell and Whitehead”. I suggest that Wittgenstein sees the Russell-Whitehead stratification as dissolving into the notational stratification induced by iteration of the \(N\)-operator. In the *Tractatus* context, the form-series handles easily the transitions from level to level which aren’t possible in the context of ramified type theory.

\(^{178}\)I also don’t know of any place where Wittgenstein mentions simulating higher-type quantification, or envisages applying it. In contrast, there’s good evidence he envisaged direct application of the form-series device in analysis, as I argued in §3.
Let me now turn to the question of reconstructing Wittgenstein’s basic non-first-order resources. Here, Ricketts doesn’t bother with the details of specifying form-series. Rather, he assumes that formulas can be assigned a logically arbitrary lexicographic ordering, with respect to which there is a reasonable notion of effectiveness for infinite classes of formulas. Then, he stipulates that every “effective” class of signs may be generated as the terms of a form-series. It seems that Ricketts may envisage the possibility of arithmetizing the propositional signs of the *Tractatus*, because names, and eventually the signs of elementary propositions, are constructed in the progress of analysis. Thus, it is just up to us, in the progress of analysis, to stipulate a notation which makes sense of talk of effectiveness, perhaps simply by insisting on arithmetizability.\(^{179}\)

Ricketts’ construction also requires some auxiliary syntactical machinery. For example, it presupposes the existence of formulas containing variables of arbitrarily high type. Moreover, some principle must guarantee\(^ {180}\) that, for any form-series \(S\) of formulas in a free variable of some (ramified) type \(\alpha\), and any formula \(A\) containing a free variable of type \(\alpha/\beta\), there exists a form-series which generates every result of instantiating the free variable in \(A\) to a formula in \(S\). Moreover, the idea of instantiation here is not straightforward—even in the earliest stages, questions arise like how to spell out the instantiation of the \(X\) in \(Xa \rightarrow Xb\) to formulas like \(Fx\) or even \(Rxy\). The complications here seem to lead far from the characteristic forms and concerns of the *Tractatus*, and it is not clear what might be gained interpretively from spelling them out.

But let me return to the stipulation that every “effective” class of formulas is generated by a form-series. I am not sure that unqualified talk of effectiveness is naturally suited to the problem of stipulating bases of a truth-operation. For one thing, arbitrary effective classes of notations do not in general have any logically significant commonality. Being a palindrome, or having an even Gödel number, are natural examples of decidable properties of notations, yet they are not logically significant. On the other hand, the property of being a formula is not such a

\(^{179}\) Here I’m indebted to conversation with Ricketts.

\(^{180}\) Actually, Ricketts doesn’t need the full strength of the following statement. But as far as I can see, weakenings require further complication, and complication is the main worry here. Insofar as I have worries about excessive strength, these address the “base clause” of Ricketts’ characterization rather than this “closure clause”.

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natural or obvious property, but requires specially tailored construction. With a
general notion of effectiveness in hand (perhaps underpinned by a prior comand
of a theory of arithmetic?), one then needs to erect on top of this general notion a theoretical superstructure of “logical significance”. We might begin with
a theory of arithmetized syntax, and then extend this to some (hopefully still ef-
fective!) proto-Tarskian criterion of permutation-invariance. But, it is worry-
ing that it should take work to turn the basics of this analysis into something even
seems like logic, let alone like the *Tractatus*.

Another symptom of the decidedly nonlogical feeling of the blanket appeal
to effectiveness is that it seems to obviate the independent concept of objectual
generality. By generating the class of first-order formulas, we *a fortiori* generate
the class of first-order formulas which differ from each other with respect to the
name which appears at one particular position; and presumably this class can be
sifted out effectively. Any notion of effectiveness adequate to Ricketts’ purposes
will be satisfied by the classes of instances of objectual generalizations. It seems
to me that a notion of complexity according to which the class of formulas with
even Gödel number is simpler than the class of instances of a universal general-
ization is not a fundamentally logical notion.

In fairness to Ricketts, at 22.5.15 Wittgenstein does actually envisages the
possibility of generating the instances of a universal generalization by means of a
form-series. But, I find this and related passages in the *Notebooks* to be philo-
sophically problematic. There are two reasons for this.

First, such passages as 22.5.15 belong to a worrisome strand of Wittgenstein’s
thinking where he seems to assume that any class of propositions sharing a nat-
ural common feature can be generated by a formal rule. This is explicitly sug-

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181 Perhaps the clearest way for me to bring out this point is to refer the reader to Appendix
A, which develops a connection between a simplified version of the Ricketts proposal and the
hyperarithmetic hierarchy.

182 “The mathematical notation for infinite series like

\[ 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \]

together with the dots is an example of that extended generality. A law is given and the terms that
are written down serve as an illustration. In this way instead of \((x)f x\) one might write ‘\(f x.f y \ldots\)’.”

(22.5.15).

176
gested as late as 23.11.16, toward motivating a hypothesis that the class of elementary propositions might be generated by means of a form-series. The worrisome strand is most prominent in NB3. But it also seems to motivate the somewhat dispiriting 5.512, which says that a common rule governs the construction of all propositions which are equivalent to \( \neg p \).

Given the subtle persistence of the worrisome strand, it seems indisputable that Wittgenstein did not think his way through to any systematic account of the workings of the form-series device. But, there is a second problem with such passages as 22.5.15, which disappears from all mentions of form-series in the *Tractatus*. In the *Notebooks* we find repeated mention of form-series generated by operations which systematically introduce new names, or new elementary propositions, into their results. However, in the *Tractatus*, none of the three form-series-like examples I know of—appearing at 4.1252, 5.512, and 6—have this feature. That is, none of the form-series-generating-operations in the *Tractatus* introduce new objects or elementary propositions into the constructed series. Instead, they manipulate truth-operations and bound objectual variables. I take this shrinking application of the device to reflect a considered judgment on Wittgenstein’s part about the relationship between what we construct and foresee, and what we do not.

In particular, as Sullivan has pointed out, a proposition (or propositional sign, or formula) is supposed to be the result of an operation on another proposition in virtue of an internal relation between the two (2004, 52). But operations that successively introduce more and more new objects must distinguish between objects by introducing them in one order rather than another, and the forms of objects will not in general underwrite such distinctions. Now, perhaps in response to this observation, Ricketts stipulates that an operation can exploit “logically insignificant” features of the names of the objects—for example, perhaps the names are indexed by integers. Such features would then yield a way of successively sim-

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183 Ramsey (1923, 472) noted that this would “presuppose the whole of symbolic logic”. Thanks to Peter Sullivan for this observation.

184 The suggestion to simulate all predicative quantification, and a fortiori all objectual quantification, seems to me in particular to overstep this boundary. There ought to be not to be a presupposition that objects can be numbered and typed at the very outset of analysis. Yet the very first instance of simulated higher-type generalization will require this.
gling out each object of a given logical form. However, Wittgenstein holds that an object can be singled out from others only if it has properties the others don’t have (2.02331). Hence, the Ricketts account, together with 2.02331, requires that objects of the same form respect the identity of indiscernibles. But Wittgenstein rejects the identity of indiscernibles (5.5302).

Perhaps there’s a general interpretive moral I want to draw from this critique of Ricketts’ account of the form-series device. On Ricketts’ account, it appears that the names of objects are something that is stipulated. Insofar as the number and forms of objects are reflected in the numbers and forms of their names, it then follows that the numbers and forms of objects are themselves open to stipulation—or, more broadly, to some kind of anticipation. Although the Notebooks toy with anticipating the forms of elementary propositions, Wittgenstein turns this aside in the 5.55s of the Tractatus, a passage which almost entirely postdates the Notebooks.

The general drift of the 5.55s seems to be this. Analysis lays out internal relations between propositions. But, propositions are pictures of reality, and therefore participate together with it in a common form. But then analysis is responsible to reality, which we cannot foresee.

With this last remark I have in mind especially the presupposition in Ricketts of the countability of names. For Ricketts, analysis must in its very first steps survey the totality of names—not with the treatment of objectual quantification, but with the treatment of simulated predicative quantification. For me, the analysis of objectual quantification eventually requires the use of particular names, because I take objectual quantification to be represented by removal of a name from a proposition. However, in my framework, it may be possible to give an account of the idea of removing a complex from a proposition, in such a way that analysis can put off the question whether it is a name or a complex that has been removed. I speculate that it is this sort of problem which motivates the seemingly bizarre proposal at 5.9.14, although the treatment of quantification—in particular, the separation of truth-function and from generality—was not at that point fully in place.

In particular, I take it that in the Notebooks, Wittgenstein did contemplate the possibility that the forms of elementary propositions are constructed—in particular at the outset of the third notebook. The 5.55s ensue from the settling of his perspective on the role of elementary propositions in the statement of the general propositional form, which takes place between November 1916 and January 1917.

I don’t think that this talk of reality, or of “things out there” subverts “the notion that analysis proceeds, starting from whole judgments, being responsible to inferential patterns only” (Goldfarb 2001a, 192). Goldfarb’s remark is in reference to Frege rather than to Wittgenstein, so I don’t disagree with that remark. Wittgenstein’s setting is special, because of the formal unity between proposition and situation.
In the next chapter, I’ll sketch a lightweight, partial account of the concept of form-series, and present some applications of the resulting device. I disavow any claim that Wittgenstein himself considered this account, let alone that he carried out the constructions I exhibit. On the other hand, the account has the following virtues. First, it is simple and definite. Second, it requires only resources that are incontestably available in the framework of the *Tractatus*. Third, it respects, perhaps as strongly as possible, the requirement that it be “easy to recognize” whether something is a value of the resulting propositional variables. Fourth, it suffices for natural applications that Wittgenstein did envisage. For these reasons, the account might be considered an “existence proof” of the feasibility of Wittgenstein’s form-series idea.

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I do think, again perhaps not in disagreement with Goldfarb, that Wittgenstein does not, in the *Tractatus*, repudiate entirely the idea of the NB2, that “when a proposition is at least as complex as its reference, then it is completely analyzed” (9.5.15). His understanding of complexity seems, in 1915-1916, to rotate up, around this fixed point, to meet the idea that ordinary language is in logically perfect order as it is.
Chapter 4

Illustrations

In the *Tractatus* Wittgenstein says that all propositions can be expressed in a system of signs with a surprisingly simple underlying scheme of generation. According to this scheme, every logically complex sentence consists of a verdict $N$ and a pointer $\xi$. The pointer indicates some propositions, and the verdict is that all of them are false.

Clearly the power of the resulting system depends on just which multiplicities of propositions can be indicated by a pointer. If only finite multiplicities can be indicated, then the resulting system coincides expressively with finitary truth-functional logic. If every set of propositions can be indicated, then every class of truth-possibilities for elementary propositions corresponds to the truth-condition of some constructible sentence. Neither of these options is interesting, and neither is Wittgenstein’s intention.

Rather, Wittgenstein says that the aim of the pointer must be somehow “stipulated (*festgesetzt*)” (5.501). At 5.501 he gives three examples of such conditions. The first is to indicate a finite bunch of propositions by listing them all. The second is to give a propositional function whose values are the propositions indicated. The third is to write down a proposition and a procedure for generating propositions from propositions, thereby indicating the totality of propositions which can be constructed from the given one by repeatedly applying the procedure.

The first kind of pointer should be clear. On the other hand, the second and third kinds require further explanation. I’ve developed an analysis of the second kind of pointer in related work, and the aim of what follows is to present some results concerning pointers of the third kind. Since my account of Way 2 pointers is nonstandard, I will sketch it here quickly, and then proceed to Way 3. By the way, I’ll henceforth refer to pointers as “(propositional) variables”.
4.1 Ways One and Two, a sketch

On my view, Wittgenstein’s analysis of quantification is rooted in his conception of propositions as pictures, and in particular in the question how a picture portrays an object or some objects. I take Wittgenstein to think that for any object and any proposition, there is a way in which the object is thereby said to be. Wittgenstein’s analysis of quantification on my view takes as fundamental not the Tarskian concept of satisfaction sequences and so on but rather the identity and distinctness of the ways in which elementary propositions say that objects are. Elsewhere I show how to reduce the concept of way in which an arbitrary proposition says some objects are to the concept of how elementary propositions say some objects are.

The concept of the way in which a proposition says an object is should not be confused with a Russellian concept of propositional function. There is a family resemblance between the notions but there are fundamental differences as well, and these differences have mathematically significant manifestations in extension. For example, an elementary proposition $Rab$ with respect to $a$ that it bears $R$ to $b$. The totality of propositions which say this of something or other is the range of a propositional variable constructed from $Rab$ and $a$. For example, $Rcb$ says with respect to $c$ what $Rab$ says with respect to $a$ and so it belongs to the range of the variable. Same for $Rdb$, $Reb$, and so on. However, suppose $Rbb$ said of $b$ what $Rab$ said of $a$. Well, $Raa$ says with respect to $a$ what $Rbb$ says with respect to $b$. Hence $Raa$ says with respect to $a$ what $Rab$ says with respect to $a$. And that is absurd. Thus, in particular $Rbb$ does not fall in the range of the function determined by $Rab$.

This conception of the relationship between names, elementary propositions, and the resulting Ausdrücken determined by arbitrary propositions yields an account of quantification which is in a sense to be made precise expressively equivalent to first-order logic with equality. Making that precise is not my concern here. Instead here is a rough and ready guide to notation.

To construct variables of Way 1, simply enclose a list of propositional signs in parentheses. Thus $(A, B, C)$ indicates $A$, $B$ and $C$. As for Way 2, we will simply write $\hat{a}A$ to mean what $A$ says with respect to $a$. Thus, $\hat{a}A$ is a variable which
indicates all propositions which say that of something.

The construction of variables can be iterated. Thus, \((A, B, (C, D))\) ultimately indicates \(A, B, C, D\) because it immediately indicates \((A, B)\) and \((C, D)\), which respectively immediately indicate \(A, B\) and \(C, D\). Similarly, \((\hat{a}A, \hat{b}B)\) immediately indicates pointers \(\hat{a}A\) and \(\hat{b}B\) which in turn indicate propositions which say of something respectively what \(\hat{a}A\) says with respect to \(a\) and what \(\hat{b}B\) says with respect to \(b\).

Let’s now return to the interaction between pointers and verdicts. If \(A\) is a pointer which ultimately indicates \(B, C, D\), then the verdict \(NA\) says that all of \(B, C, D\) are false. We will take sentences to derivatively indicate themselves, so that if on the other hand \(A\) is a sentence, then \(NA\) expresses the falsehood of the proposition indicated by \(A\), to wit, the falsehood of \(A\)\(^{188}\). Note, crucially, that anything of the form \(NA\) must be a verdict, no matter whether \(A\) is a pointer or a sentence.

Pointers and verdicts combine to yield almost familiar idioms. In the special case where \(A, B, \ldots\) are sentences, then \(NN(A, B, \ldots)\) and \(N(NA, NB, \ldots)\) express the ordinary disjunction and conjunction of what \(A, B, \ldots\) say, so in such a case we will write instead \(A \lor B \lor \ldots\) and \(A \land B \land \ldots\). But now suppose \(A\) is a pointer. Then \(NN(A)\) expresses the denial of the proposition that every proposition ultimately indicated by \(A\) is false. Thus we abbreviate \(NN\) as \(\lor\). In particular, if \(B\) is a proposition, then \(\lor \hat{b}B\) asserts the disjunction of propositions which say of something what \(B\) says with respect to \(b\). On the other hand, since one can ask of an arbitrary proposition what it says with respect to \(b\), it follows that for a given proposition \(C\) one can ask what \(NC\) says with respect to \(b\). Having thereby constructed the variable \(\hat{b}NC\) we can form the denial \(N\hat{b}NC\) of every proposition which \(\hat{b}NC\) indicates, i.e., the simultaneous denial of the propositions which say of something that that thing is not how \(C\) says that \(b\) is. Thus, for every proposition \(C\) and every object \(b\), there is an expression of the simultaneous truth of all propositions which say of something what \(C\) says with respect to \(b\).

\(^{188}\)I think that probably Wittgenstein took expressions of the kind \((A, B, \ldots)\) not to be variables but rather to be themselves the multiplicity of bases of application of \(N\). That is, I take his attitude to have been that of Russell in PoM that only infinite multiplicities require the recourse to intensions.
to $b$. That is to say, there exists a uniform analogue to universal as well as existential quantification. We will not introduce an abbreviation here because we have not yet sufficiently surveyed the principles of pointer construction to prove that for each pointer $A$ to propositions $B, C, \ldots$, there is a proposition to the effect that their denials $NB, NC, \ldots$ are all false, though this result is evident for pointers constructed by iterations of Ways 1 and 2 in tandem with denial verdicts. I regard Fogelinite skepticism about universal quantification itself in this context to be completely out of the question.

Since the notation just presented is slightly peculiar, let me hint at some ways of projecting into it soundly some familiar principles of interpretation. In an expression $\hat{a}A$, the governing $\hat{a}$ should be regarded to bind all occurrences of $a$ in $A$ which are not already bound. Once namelike occurrences of $a$ become so bound, then their $a$-ness completely disappears. One might thus think of $\hat{a}$ an “anonymizer”. It turns the occurrences of the “constant name” $a$ into occurrences of a variable name which is indeterminately any of the names $a, b, c, \ldots$ which are appropriate to the context.

Upon falling under control of an abstract, a name $a$ loses its individual name-like aspect, and occurrences of name $a$ itself become merely apparent. Terms enjoying such an anonymous second life might well be supposed to have entered a new logical category. This change in category might emphasized syntactically. In particular, suppose we simply introduce a new syntactic kind of expressions $x, y, z, \ldots$, and replace the “merely apparent” occurrences of names $a, b, c, \ldots$ with occurrences of $x, y, z, \ldots$. Then, an indefinite propositional constituent, a variable name, gives way to a definite constituent, a name-variable.

The shift from variable names to name-variables underpins Wittgenstein’s more or less familiar usage of quantifier notation. Thus, for Wittgenstein

$$\lnot\exists F a \quad \text{becomes} \quad \exists x F x$$

$$\forall \hat{a} R a b \quad \text{becomes} \quad \exists x R x b, \text{ and}$$

$$\forall \hat{b} (R b a \land \lnot N c \langle R c b \rangle) \quad \text{becomes} \quad \exists x (R x a \land \lnot \exists y (R y x)).$$

However, the fact remains that for Wittgenstein, there is some $a$ such that $\lnot\exists F a$, 
viz., \( \neg \exists x F x \), is the denial of all propositions that say with respect to some name \( Fa \) says with respect to \( a \). This, of course, is just the denial of the propositions \( Fa, Fb, Fc, \ldots \). But then likewise, there is an \( a \) such that \( \bigvee \hat{a}Rab \) asserts the truth of at least one proposition among those which say with respect to some name what \( Rab \) says with respect to \( a \). And this is to assert the truth of at least one of \( Rab, Rcb, Rd b, \ldots \), passing over the question of \( Rbb \) entirely. Thus, not for Wittgenstein but for us,

\[
\neg \exists aA \text{ becomes } \neg \exists x F x
\]
\[
\bigvee \hat{a}Rab \text{ becomes } \exists x(x \neq b \land Rx b), \text{ and}
\]
\[
\bigvee \hat{b}(Rba \land \neg \exists c(Rc b)) \text{ becomes } \exists x(x \neq a \land (Rx a \land \neg \exists y(y \neq x \land Ry x))).
\]

In general, the idea is for us to gloss \( NN\hat{a}A \) as \( \exists x(B \land A''') \), where \( B \) is the conjunction \( x \neq a \land x \neq b \land \ldots \) with \( a, b, \ldots \) the constants occurring in \( A'' \), and \( A''' \) is the result of everywhere replacing \( a \) with a new variable \( x \) in the gloss on \( A \).

Thus, although Wittgenstein does introduce name-variables with apparently familiar syntax, their logical role is rooted in a distinctive conception of the relationship between names and propositions. The pictorial character of a proposition involves its exhibition of some definite mathematical multiplicity. Such multiplicity is, at least, a multiplicity of names, and the multiplicity of names in a proposition requires, at least, an answer to the question how many names it contains. A different answer is a different multiplicity. Wittgenstein’s name-variables receive a nonstandard reading because they signal results of denaturing of names in logical pictures. Familiar quantifier notation obscures the underlying patterns of logical construction as Wittgenstein conceives them. Since these patterns come under strain with the introduction of Way 3 variables, I’ll stick to a notation which keeps track of those patterns.

### 4.2 Way Three

Let’s now return to the original observation that Wittgenstein describes not two but three ways of fixing the aim of the pointer. We have just seen that the first two ways yield a variant of first-order logic. It’s therefore natural to suppose that
the point of Way 3 is to lead further to the expression of concepts which are not
first-order definable. At 4.1252 and 4.1273 Wittgenstein suggests that the Way 3
method allows the construction of a pointer which indicates the multiplicity of
sentences in the series

\[ Rab \]

\[ \exists x (R ax \land R x b) \]

\[ \exists x \exists y (R ax \land R xy \land R y b) . \]

Obviously, were \( A \) a pointer whose range were the sentences (1), then \( \lor A \) would
hold of \( a \) and \( b \) just in case \( a \) bears to \( b \) the ancestral of \( R \).

Wittgenstein’s explanation of the construction of Way 3 pointers is incom-
plete. He gives an official notation, roughly \([A, \xi, O(\xi)]\). In this notation, the
first entry is simply a formula; the second and third entries together signify an op-
eration \( O : \xi \rightarrow O(\xi) \) on formulas. The notation as a whole is a variable which
indicates the formulas \( A, O(A), O(O(A)), \ldots \) which result by repeatedly applying
\( O \) to \( A \). Relative to some determinate concept of operation, then, the working of
the variable should be transparent. Supposing \( O : \xi \rightarrow O(\xi) \) to be an operation
which takes the \( n \)th term of (4.1) to the \( n + 1 \)th, then \( \lor [R ab, \xi, O(\xi)] \) would
express the ancestral of \( R \) in the way that Wittgenstein intends.

However, Wittgenstein says little to determine the concept of operation itself.
He says that there is associated to an operation \( O \) some distinguished “formal”
relation \( O^* \) such that \( B \) is the result of applying \( O \) to \( A \) in virtue of the fact that
\( B \) bears \( O^* \) to \( A \). It’s not clear what is involved in characterizing a relation as
“formal”. One possibility is that whether or not the relation holds depends only
the forms (and perhaps also number) of objects, and does not distinguish between
one object and another of the same form. In other words, one might speculate—
although I wouldn’t insist—that formality implies some sort of invariance under
type-respecting permutations of names. On the other hand, he also says that
“it should be easy to see” whether \( B \) bears \( O^* \) to \( A \), and more generally whether
something is indicated by a variable. Thus it would appear that \( O \) should be in

\[ ^{189} \text{Cf. in particular 2.0233.} \]
some sense effective. However, the analysis of the concept of effectiveness in this
case raises difficult interpretive questions, indeed verging into realms where
Wittgenstein’s thought may not be entirely unconfused.

Rather than trying to determine the indeterminate concept of operation I
suggest considering a class of procedures that ought to count as operations even
under a very miserly construal. Suppose we enrich the generation of formulas so
that it proceeds not just from elementary propositions but also from a denumer-
ably infinite collection \( p, q, r, \ldots \) of schematic propositional letters. Let \( A \) be a
formula so generated which contains the single letter \( p \). And now consider the
operation which takes an arbitrary propositional sign \( B \) to the result of every-
where substituting \( B \) for \( p \) in \( A \). I propose we consider the procedure

\[
O: \xi \mapsto A[p/\xi]
\]

as an operation. Let’s call the formula \( A \) associated to \( O \) the “carried formula” of
\( O \). I will sometimes refer to an operation \( O \) as a “root substitution” because it
grafts the carried formula onto the root of the formation tree of the operand.\(^{195}\)
Slightly abusing notation, it will be convenient instead to write simply \([B, p, A]\)

Let’s now consider a simple example of the working of root-substitution. Let
\( A \) be the formula \( N p \lor C \) for some \( C \). Then \([B, p, A]\) is a variable whose values
are the formulas

\[
\begin{align*}
B \\
NB \lor C \\
N(NB \lor C) \lor C
\end{align*}
\]

and so on. (This method of typesetting the values of a Way 3 variable shows that
it is indeed “easy to see” whether a successor term of the determined form-series
bears the requisite internal relation to its predecessor.)

Way 3 variables interact with the \( N \)-operator just like the variables of Ways 1

\(^{195}\)One could also consider “leaf substitution” which grafts a carried formula onto the leaves of
the operand, but I will not do so here. (I agree that the terminology is hokey.)
and 2. For example, $\sqrt{[B, p, A]}$ amounts to the disjunction $\sqrt{(B, NB\lor C, N(NB\lor C) \lor C, ...)}$. It is obvious that results of combining Way 3 and Way 1 variables can say nothing that Way 1 variables alone cannot say. The interest of Way 3 variables lies rather in their interaction with Way 2 variables. We now turn to such applications.

4.3 The ancestral

To illustrate the power of Way 3 variables, let me first show how, with a little trickery, one can use this device to express the ancestral of an arbitrary expressible relation. Note, however, that since Wittgenstein understands relations differently, he understands the concept of ancestral differently as well. In particular, Wittgenstein must find the concept of ancestral of $R$ ambiguous between that dyadic relation by virtue of which something is $R$-connected to something else, and on the other hand, that monadic property which some object has in virtue of which it is $R$-connected to itself. Here, Wittgenstein must find our idea of the ancestral to be essentially a confusion, since nothing could be both a relation between two things and a property of one thing. Similarly, if the relation $R$ contains a parameter $c$, then it makes no sense to talk about something bearing $R$ to $c$, or about something being connected to something else via some $R$-connection through $c$. However, suppose for example that $R$ is the relation which holds between $a$ and $b$ in case $Sabc$. Then, one might allow sense to be made of our saying $Sacc$ or $Scbc$, since the sort of situation which would prompt such emissions might well be the standing of $a$ to $c$ in a dyadic relation $S'$, or the standing of $c$ to $b$ in the dyadic relation $S''$. Thus, Wittgenstein can accommodate this other difference by formulating the ancestral, with respect to $a$ and $B$, not of the relation $Rab = Sabc$ but rather with respect to the relation $R'ab = Sabc \lor S'ab \lor S''ab$. Although it’s fairly easy to see how such accommodations can be constructed on a case-by-case basis, their systematic development in full generality is rather gruesome, and I won’t bother with it.

Let’s first consider Wittgenstein’s own rather naïvely suggested series of for-
mulas the disjunction of which would express the ancestral of $Rab$:

\[
\begin{align*}
Rab & \quad \text{(4.2)} \\
\exists x (Rax \land Rx b) & \\
\exists x \exists y (Rax \land Rxy \land Ry b)
\end{align*}
\]

It is obvious that this series cannot be generated by root-substitution alone, since a term of the series is not in general even close to being a subformula of its successor. Note however that we can find a series to roughly the same effect, for example

\[
\begin{align*}
Rab & \\
\exists x (Rax \land Rx b) & \\
\exists y (\exists x (Rax \land Rxy) \land Ry b).
\end{align*}
\]

Here we are certainly somewhat closer to meeting the constraints entailed by restriction to root-substitution. But first let’s replace these familiar first-order notations with notations from the Way 2 canon. This gives

\[
\begin{align*}
Rab & \quad \text{(4.3)} \\
\bigvee \hat{c} (Rac \land R cb) & \\
\bigvee \hat{d} (\bigvee \hat{c} (Rac \land Rcd) \land Rd b)
\end{align*}
\]

Now a problem appears. Note that every propositional sign and every operator contains only finitely many names. But the result of applying an operation to a proposition can be expressed by a notation containing no more names than appear in the signs of the operation and the proposition. Hence we certainly cannot generate the series of notations (4.3) itself as the terms of a form-series. Consider, however, the fourth term of the series, namely

\[
\bigvee \hat{e} (\bigvee \hat{d} (\bigvee \hat{c} (Rac \land Rcd) \land Rde ) \land Reb). \quad \text{(4.4)}
\]

Here, while the outer abstract $e$ controls names which appear in the subformula
\[ \forall d (\forall \check{c}(Rac \land Rcd) \land Rde), \text{ it does not control any names which appear in the} \]
\[ \text{subsubformula } \forall \check{c}(Rac \land Rcd). \text{ Thus there is no reason to insist on the distinct-} \]
\[ \text{ness of } e \text{ from } c. \text{ Generally, the outermost abstract in the } n + 2 \text{nd term of the} \]
\[ \text{series does not control any names in the subformula which corresponds to the } \]
\[ n \text{th term of the series. Instead we can simply construct the series} \]
\[ \begin{align*}
Rab \\
\forall \check{c}(\bigvee \check{b}(Rab \land Rbc) \land Rcb) \\
\forall \check{c}(\bigvee \check{b}(\forall \check{c}(\bigvee \check{b}(Rab \land Rbc) \land Rbc) \land Rcb) \land Rcb)
\end{align*} \]
which has obviously got the subformula property required. Hence we may simply express it by the form-series variable \([B_1, B_2]\) where \(B_1 = Rab\) and
\[ B_2 = \forall \check{c}(\bigvee \check{b}(p \land Rbc) \land Rcb). \]

Of course, since in the above series, adjacently introduced variables \(b, c\) assume distinct values, (4.5) says that \(a, b\) are \(R\)-connected with an even number of intervening terms. Thus we need to combine the series (4.5) with another series
\[ \begin{align*}
\forall \check{c}(Rac \land Rcb) \\
\forall \check{c}(\bigvee \check{b}(\forall \check{c}(Rac \land Rcb) \land Rbc) \land Rcb) \\
\forall \check{c}(\bigvee \check{b}(\forall \check{c}(Rac \land Rcb) \land Rbc) \land Rcb) \land Rcb
\end{align*} \]
Setting \(B_2 = \forall \check{c}(Rac \land Rcb)\), then (4.6) is indicated by the variable \([B, p, A]\). A variable over the totality of terms of both (4.5) and (4.6) is now given by Way 1, namely \(([B_1, p, A], [B_2, p, A])\), and the ancestral of the relation between \(a\) and \(b\) in virtue of which there obtains the fact \(Rab\) can be written
\[ \forall ([B_1, p, A], [B_2, p, A]). \]
4.4 More fun with root-substitution

One might think that expressibility of the ancestral is something of a fluke, and that restricting ourselves to root-substitution won’t let us formulate other natural or amusing assertions. Some such limitations may appear to follow from the more general constraint that the number of variables be finite. T. Ricketts observed in correspondence that the existential quantifiers in the third entry of (4.6) will be satisfied in case there obtains a fact

\[ \text{Rab} \land \text{Rbc} \land \text{Rcd} \land \text{Rdc} \land \text{Rdb}. \]

So, granted that the variables in Wittgenstein’s form-series (4.2) are to be read in accordance with Wittgenstein’s logical principles, then my formulation of the ancestral is importantly different (4.2). To bring out the difference, consider a particular problem situation, where we are trying to describe a linear concatenation of discrete segments, and in particular want to insist that concatenation does not cycle around. If we have given a relation like “immediately to the left of” then this can be said simply by saying that no segment on the line instantiates the monadic-ancestral of the relation of being immediately-to-the-left-of. Suppose though that we have given only only the relation of adjacency. Then, on Wittgenstein’s formulation of the ancestral, a segment will be self-connected by the ancestral of adjacency only if the line contains a loop. This is because Wittgenstein’s formulation would express the existence of a chain connecting an object to itself using a distinct variable for each link of the chain. But on my formulation, every segment will be trivially self-connected regardless of whether the line contains a loop. One might therefore worry that difference is a essential shortcoming attendant on the availability of only finitely many variables in a given formula.

Let’s just adjoin \( a, b \) to say that \( a \) and \( b \) are line segments and that they are adjacent. The problem is then to use this primitive to say that \( a \) is connected to itself by a chain that contains no repetitions. Such a chain may be arbitrarily long, and hence would seem require a logical picture containing arbitrarily many distinct names. Our resources preclude this. However, definability is always
relative to the structure under consideration and in this case we are considering the relation of adjacency of segments on a line. A chain of such adjacent segments contains repetitions if and only if it contains a stagger-step $xyx$. Unlike arbitrary repetition, a stagger-step is a finitely bounded pattern we can break by simply using more names than appear in pattern’s prototype. That is, it should suffice to describe the chain using names $a, b, c, \ldots$ in such a way that objects at one remove from each other be denoted by logically distinct names. So, we need to build up our picture of the chain not from a series of pictures $ab, bc, cb, bc, \ldots$ which portray only distinct adjacent objects, but rather from a series of pictures $abc, bcd, cdb, dcb, \ldots$ which portray distinct objects at one remove from each other. Consider now the series

$$ab \land bc \land cd$$

$$\bigvee b(ab \land bc \land cd) \land db$$

$$\bigvee c\left(\bigvee b(ab \land bc \land cd) \land db\right) \land bc$$

$$\bigvee d\left(\bigvee c\left(\bigvee b(ab \land bc \land cd) \land db\right) \land bc\right) \land cd$$

$$\bigvee b\left(\bigvee d\left(\bigvee c\left(\bigvee b(ab \land bc \land cd) \land db\right) \land bc\right) \land cd\right) \land db$$

$$\bigvee c\left(\bigvee b\left(\bigvee d\left(\bigvee c\left(\bigvee b(ab \land bc \land cd) \land db\right) \land bc\right) \land cd\right) \land db\right) \land bc$$

$$\bigvee d\left(\bigvee c\left(\bigvee b\left(\bigvee d\left(\bigvee c\left(\bigvee b(ab \land bc \land cd) \land db\right) \land bc\right) \land cd\right) \land db\right) \land bc\right) \land cd$$

Each formula in this series asserts of the object $a$ that it bears a certain relation to three objects $b, c, d$. According to any of the formulas, $a$ initiates a chain of some fixed length, which is free of any stagger-step $xyx$, and which ends with $bcd$, or with $cdb$, or with $dbc$. Now, this series cannot itself be generated by a single operation, since the series is generated by cycling through three operations. For our present purposes, however, it may suffice to take the composition of those three operations, thus generating the subseries containing every third formula of
Thus, let’s fix a form-series variable \([A, p, B]\) where

\[
A = ab \land bc \land cd
\]
\[
B = (\bigvee \hat{d}(\bigvee \hat{e}(\bigvee \hat{b}(p) \land d b) \land bc) \land cd).
\]

Thus, the propositional sign \(\bigvee [A, p, B]\) asserts the obtaining of at least one fact to the effect that a nonstaggering chain begins with \(a\), followed by \(3k\) terms for some \(k \geq 0\), followed by \(bcd\). In turn, \(\bigvee [A, p, B] \land da\) says a nonstaggering chain begins with \(a\) followed by \(3k\) terms followed by \(bcda\). Then the existential generalization

\[
\bigvee \hat{b} \bigvee \hat{c} \bigvee \hat{d}(\bigvee [A, p, B] \land da)
\]

(4.9)
says that \(a\) is self-connected by a nonstaggering chain with \(3k + 3\) terms. Now, a line containing at least two segments contains such an \(a\) if and only if it suffers a loop. Hence, to express a line’s freedom from looplessness, it suffices to deny the existence of such an \(a\), as in

\[
N\hat{a} \bigvee \hat{b} \bigvee \hat{c} \bigvee \hat{d}(\bigvee [A, p, B] \land da).
\]

The key to expressing noncyclicity is to express the existence of a chain which contains no repetitions. Once we can describe such chains there is more stuff we can say. Consider the relation that holds between \(a, b, c\) on a line provided that \(a\) is between \(b\) and \(c\), again using just adjacency as primitive. (Again this problem is due to Ricketts.) With the promise of such delights in mind, we first turn to formulating the general relationship of connectedness by nonstaggering chain. This requires allowing chains of length not divisible by 3. Here we play the trick from the expression for the ancestral. That is, fix

\[
A_0 = ab \land bc \land cd
\]
\[
A_1 = \bigvee \hat{e}(ae \land eb \land bc \land cd)
\]
\[
A_2 = \bigvee \hat{f}(af \land fe \land eb \land bc \land cd)
\]
\[
B = (\bigvee \hat{d}(\bigvee \hat{e}(\bigvee \hat{b}(p) \land d b) \land bc) \land cd).
\]
Then
\[
\sqrt{\sqrt{\sqrt{a_0, p, B}, [A_1, p, B], [A_2, p, B], (ac \land cd), ad}}
\]  
(4.10)
says that \(a\) is connected to \(d\) by a nonstaggering chain. Write \(C[b, c]\) for an arbitrarily chosen sign which says with respect to \(b, c\) what (4.10) says with respect to \(a, d\). Clearly such a sign can be formed by reletterings. (Note though that if \(C[b, c]\) is prefixed by an abstract as in \(\overset{\wedge}{c}C[b, c]\) then the relettering will not vary with the values of \(c\).)

Let’s now try to use the construction of \(C\) to say that \(g\) is between \(a\) and \(d\). Note that of course we cannot simply take the conjunction \(C[a, g] \land C[g, d]\), since this amounts only to saying that \(a, d, g\) are segments of the same line. But take \(T_g\) be the formula \(\sqrt{\sqrt{\sqrt{ga \lor \neg ga}}\). And let \(C_g[a, d]\) be the result of replacing every expression \(\sqrt{\sqrt{x}}\) in the construction of \(C[a, d]\) with \(\sqrt{\sqrt{x}}(T_g \land (\cdots))\). Then \(C_g[a, d]\) says that \(a, d\) are connected by a nonstaggering chain which does not contain \(g\). Hence, \(C[a, d] \land \neg C_g[a, d]\) says that \(g\) is between \(a\) and \(d\).

4.5 Limitations

Let’s now consider another non-first-order definable concept, namely that a given dyadic relation \(R\) be a wellordering. As with the concept of ancestral, some minor adjustments are in order, but the basic idea is obviously invariant under them. In particular the condition of (ir-)reflexivity of a relation does not make sense in the present framework. Either \(R\) is not dyadic relation but a monadic property, or it is a relation which holds between a thing and something else. The conditions to the effect that \(R\) is a linear ordering are straightforward:

\[
\neg \overset{\wedge}{b} \overset{\wedge}{c} N(Rab \land Rbc \to Rac)
\]
\[
\neg \overset{\wedge}{b} \overset{\wedge}{c} N(Rba \land Rca \to Rbc \lor Rcb)
\]

But how about the non-first order condition, namely that \(R\) be well-founded? This is inexpressible in any reasonable extension of the present framework. For, Lopez-Escobar (1966) showed that the wellfoundedness of a relation is not expressible even in any infinitary language \(L_{\omega\omega}\). That is, even for any language.
containing arbitrary disjunctions of length $< \alpha$ for some ordinal $\alpha$, there is no formula whose models are precisely those in which $R$ is wellfounded. And it is clear that the form-series operator does not even take us out of $L_{\omega_1 \omega}$. 
Chapter 5

Effing it

In the following chapter, I will try to present my best understanding of Wittgenstein's account of logical structure. I don't venture that Wittgenstein's account is entirely coherent. Since only what is coherent can be understood, I therefore don't pretend to survey Wittgenstein's account in its entirety. But nonetheless, I do claim here to describe the mathematical core of Wittgenstein's account of logical structure. *A fortiori*, I claim that the core of Wittgenstein's account is actually coherent. Moreover, this coherent core extends far enough outward to distinguish Wittgenstein's conception of logic rather sharply.

The following reconstruction does involve some technicalities, which Wittgenstein himself preferred to pass over in vagueness. And of course, any rigorous mathematization of informally developed abstractions requires some arbitrary choices. Nonetheless, I take the following to establish that the core of Wittgenstein's account admits of mathematically definite realization. Moreover, it also seems to me that there is, after all, not much freedom of movement in the realization of this core. If these thoughts are on the right track, then in the end, there is a reasonably determinate answer to the question what is the formal structure of logic as Wittgenstein conceives it.

The determination of logical structure has philosophical consequence. As I'll argue, a top-down, constructivist approach to analysis in the *Tractatus* entails, upon pain of unsoundness of its deliveries, a variety of realism about the notion of truth-possibility. One cannot coherently be constructivist all the way through the *Tractatus*, not even all the way through the core.
5.1 Framework

Wittgenstein distinguishes, in the structure of the totality of propositions, between what we construct, and what depends only on the form of the world. The elementary propositions depend only the form of the world. Elementary propositions are concatenations of names. Thus the form of the world also determines the names themselves. Similarly, the form of the world determines the manner in which elementary propositions are characterized by the occurrence of names.

As Wittgenstein says, every proposition is a position of agreement and disagreement with truth-possibilities for elementary propositions. But, while the positions constituted by elementary propositions are given with the form of the world, the positions constituted by nonelementary propositions are constructed by us. Thus, the complexity of nonelementary propositions depends on our capacities for construction of positions of agreement and disagreement with respect to the truth-possibilities for elementary propositions. For example, given any one proposition, we can formulate disagreement with its truth, thereby constructing a position of agreement with exactly those truth-possibilities with which the given proposition disagrees. Similarly, given any bunch of propositions, we can formulate disagreement simultaneously with the truth of each of them, thereby constructing a position of agreement with just those truth-possibilities with which all the given propositions disagree. But by what means can multiplicities of propositions be given? Here Wittgenstein appeals to the nature of those constructional capacities which the form of the world secures.

I will try to trace mathematically this distinction between what is given and what is constructed. The general strategy is simply to assume as a given abstract structure $\mathcal{B}$, the totality $\mathcal{E}$ of elementary propositions and their characterization by names in the totality $\mathcal{N}$ of names. Given such a $\mathcal{B}$, we assume the possibility of

\[\text{Cf. 4.51. What Wittgenstein means by “construction” is not at all clear to me, in particular whether the notion of construction really does involve essential reference to our (or “my”) constructional capacities. In my opinion, 4.51 should be juxtaposed with 5.64 before reaching any conclusions on this question. The naïve treatment of the notion of construction in this chapter is intended only as a mathematical heuristic. I don’t assert that Wittgenstein only intends 4.51 heuristically, but he seems to put it there partly because it clarifies the development of 4.5 through 5.501 and 6. That’s the sole intention use here.}\]
constructing certain sets of elements of this structure, sets of items thus obtained, and so on.

I’ll begin with the crudest possible notion of construction: simply to assume the construction of every finite set of items already constructed. The result is the class HF(∈) of hereditarily finite sets over ∈. In other words, HF(∈) is the smallest collection of objects which contains every element of N and every element of E, and which contains every finite set of objects it contains. Somewhat more pedantically, we’ll actually take HF(∈) to be a first-order structure ⟨|HF(∈)|, N, E, ∼, ∈⟩. Thus, a class of elements of |∈| will be definable over ∈ provided that it is definable over this particular signature.

The main benefit of this rather flatfooted and simplistic notion of construction is the following. If ∈ is a familiar kind of syntactical structure, then the notion of formula that arises in HF(∈) can be coded into a notion of formula on the class HF of hereditarily finite sets. In turn the structure of the hereditarily finite sets can be coded into the structure of the natural numbers. This means that, given a convenient choice of ∈, the complexity of notions defined over HF(∈) can be recast in the more familiar terms of the arithmetical or analytical hierarchies. For example, a class of elements of ω is Δ₀ over HF iff the corresponding class of natural numbers is recursive, and Σ₁ iff the corresponding class of naturals is r.e.¹⁹² Furthermore, there’s a bijection f from HF to ω such that a class X of elements of HF is Δ₀ iff the class f(X) of images of elements of X is Δ₀.

5.1.1 Basis

We can now formulate abstractly what I take to be Wittgenstein’s a priori assumptions on what must be given for any logical constructions to be possible. Wittgenstein assumes the existence of two disjoint, nonempty classes E and N, which are respectively the classes of elementary propositions and of names. Moreover, he assumes there to be a determinate manner in which a given name characterizes a given elementary proposition. Or more generally, suppose that a and b are sequences of names. Then, Wittgenstein assumes that there is a definite answer to the question whether a characterize an elementary proposition A in pre-

¹⁹²See Barwise (1975), 48ff for details.
cisely the way that \( \vec{b} \) characterize the elementary proposition \( B \). Thus, Wittgenstein assumes to be given a relation \( \approx \) of same-saying (or perhaps better, same-predicating). To express the import of \( \approx \) in yet another way, \( \vec{a}A \approx \vec{b}B \) holds just in case what \( A \) says of \( \vec{a} \) is what \( B \) says of \( \vec{b} \).

Formally, we thus reach

**Definition 1.** A basis is a structure

\[ \mathcal{B} = (\mathcal{E}, \mathcal{N}, \approx) \]

such that \( \mathcal{E} \) and \( \mathcal{N} \) are nonempty and disjoint, and \( \approx \) is an equivalence relation on \( \{ \mathcal{N}^k \times \mathcal{E} : k \in \omega \} \).

Even more formally, it would be better to put \( \mathcal{B} = (\mathcal{E} \cup \mathcal{N}, \mathcal{E}, \mathcal{N}, \approx) \). Thus, considered as a first-order structure, the domain \( |\mathcal{B}| \) of \( \mathcal{B} \) is \( \mathcal{E} \cup \mathcal{N} \), and \( \mathcal{E}, \mathcal{N} \) and \( \approx \) are classes of (pairs of) elements of \( |\mathcal{B}| \).

We will make no further assumptions on \( \mathcal{B} \). Thus, for example, although \( \vec{a}A \approx \vec{a}A \) holds because \( \approx \) is an equivalence, we do not assume \( \vec{a} \) and \( A \) bear \( \approx \) to anything else. Likewise, for example, whether or not \( \vec{a}A \approx \vec{b}B \) holds may depend on the ordering within the lists \( \vec{a} \) and \( \vec{b} \), or again it may not.\(^{193}\) We’ll return to these issues in §5.2.

### 5.1.2 Formulas

With the help of names, we now extend the class \( \mathcal{E} \) of elementary propositions to a broader class \( \mathfrak{Z} \) of signs, or *formulas*. The class \( \mathfrak{Z} \) is to include a system of propositional signs in the sense of the *Tractatus*. But alongside the propositional signs it also includes various auxiliary notations. Among these, some are what Wittgenstein called *propositional variables*. But we also introduce some further auxiliaries that matter interpretively only to the extent that they contribute to the determination of the functioning of the propositional signs and propositional variables.

\(^{193}\)A more compelling assumption, which Wittgenstein may himself have accepted, is the following: that if \( \vec{a} \vec{b}A \approx \vec{a} \vec{b}B \), then \( \vec{b}A \approx \vec{b}B \). But this will not be required of \( \mathcal{B} \).
To construct the class $\mathfrak{Z}$ in $\text{HF}(\mathfrak{B})$, we assume a reasonable encoding in $\text{HF}(\mathfrak{B})$ of the notion of sequence of elements of $\text{HF}(\mathfrak{B})$ and of related notions of concatenation, projection, and so on. We treat concatenation of signs as concatenation of their codes. The analog of the class of (closed) atomic formulas will be the class $\mathfrak{E}$ of elementary propositions. We furthermore assume to be constructed the following syntactic inventory:

- some individual signs $(, ), [ , ]$, $N$;
- a 1-1 function which, given any name $a$, returns a name-abstract $^\hat{a}$, and
- a countably infinite set $p$ of propositional letters $p, q, r, \ldots$.

All these items should be assumed to exist as pure, hereditarily finite sets independently of $\mathfrak{B}$ (except for the course of values of the circumflexion function, which is nonetheless uniformly and effectively determined by $\mathfrak{N}$).

The definition of the notion of $\mathfrak{Z}$-formula now runs by induction as follows.

**Definition 2.** The class of formulas is the smallest class such that

- every element of $\mathfrak{E}$ belongs to $\mathfrak{Z}$, and
- every element of $p$ belongs to $\mathfrak{Z}$.

Second,

- $(A_1, \ldots, A_k)$ belongs to $\mathfrak{Z}$,
- $^\hat{a}A_1$ belongs to $\mathfrak{Z}$, and
- $[A_1, p, A_2]$ belongs to $\mathfrak{Z}$

for all formulas $A_1, \ldots, A_k$, names $a$, and letters $p$. And moreover, for any formula $A$,

- $NA$ is a formula.

A letter $p$ occurs free as the formula $p$ itself. In a context $[A_1, p, A_2]$, the middle occurrence $p$ binds all free occurrences of $p$ in $A_2$. Otherwise, in general,
if a complex formula $A$ is constructed immediately from $A_1, \ldots, A_k$, then the letters free in $A$ are those free in any of $A_1, \ldots, A_k$. We will say that a formula is \textit{proper} if no letter occurs free in it, and if, furthermore, in each of its subformulas, $[A, p, B]$, the formula $B$ itself contains no occurrence of $\cdots$. The second requirement of propriety has a purely technical motivation which is given after Proposition 2.

A \textit{propositional sign} is a proper formula $A$ which is either an elementary proposition, or of the form $NB$. If the proper formula $A$ is not a propositional sign, then $A$ has one of the forms $(\cdots), \tilde{a} \cdots$, and $[\cdots, p, \cdots]$; and in such a case, $A$ is a \textit{propositional variable}.

It is fairly straightforward that $\mathcal{Z}$ renders $\mathcal{Z}$ as a $\Delta$ class in $HF(\mathcal{B})$. Of course, $\mathcal{Z}$ does not itself exist as a set in $HF(\mathcal{B})$, but each of its elements does. Formalized, the definition takes two steps:

(i) for $x$ an urelement, $x \in \mathcal{Z}$ iff $x \in \mathcal{E}$, and

(ii) for $x$ a set, $x \in \mathcal{Z}$ iff $\Psi(x, z_x)$ where $z_x$ is the intersection of $\mathcal{Z}$ with the transitive closure of $x$.

The formula $\Psi(x, z_x)$ is a disjunction of three formulas $\Psi_0, \Psi_1, \Psi_2$, where

- $\Psi_0(x, z_x)$ says “$x$ is a finite sequence, each term of which belongs to $z_x$”,
- $\Psi_1(x, z_x)$ says “$x$ is a sequence of length 2, whose first term is a circumflected name, and whose second term belongs to $z_x$”, and
- $\Psi_2(x, z_x)$ says “$x$ is a sequence of length 3, whose first and third terms belong to $z_x$, and whose middle term belongs to $p$”.

Each of the $\Psi_i$ is clearly formalizable as a $\Delta$ formula.\textsuperscript{194} The basic Corollary I.6.6 of Barwise (1975) implies that the class $\mathcal{Z}$ can itself be defined by a $\Delta$ formula over $HF(\mathcal{B})$. Similar maneuvers show that “subformula” and “immediate subformula” are $\Delta$ relations, and thus that the notion of proper formula is $\Delta$ too.

\textsuperscript{194}In particular, “$x$ is a finite sequence” is formalized using “$x$ is a sequence and its length is a natural number”, where “$k$ is a natural number” is formalized “$k$ and all of its nonzero predecessors are successor ordinals”.

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We now record a principle of induction on complexity of formulas which simplifies the treatment of $\exists$. In the statement of the principle, we refer to the result of prefixing a list $\vec{a}$ of zero or more name-abstracts to a formula as an abstraction from that formula.\footnote{This principle is not a theorem of KPU, since the variable $\Phi$ is, by the lights of KPU, second-order. Rather, it's a result of about the particular models of KPU we're considering.}

**Proposition 1.** Suppose that

(i) every abstraction from an element of $\mathcal{E} \cup p$ has property $\Phi$, and

(ii) if every abstraction from each of $A_1, \ldots, A_k$ has property $\Phi$, then so do all abstractions from $(A_1, \ldots, A_k)$, $[A_1, p, A_2]$, and $N(A_1)$.

Then every $A \in \exists$ has property $\Phi$.

**Proof.** Let $X$ be the smallest set satisfying

(i) every abstraction from an element of $\mathcal{E} \cup p$ is an element of $X$, and

(ii) if all abstractions from $A_1, \ldots, A_k$ are elements of $X$, then so are all abstractions from $(A_1, \ldots, A_k)$, $[A_1, p, A_2]$ and $N(A_1)$.

Clearly, $X$ satisfies the claimed induction principle. But, as a routine induction shows, $\exists \subseteq X$, so $\exists$ satisfies the principle too. \hfill \square

### 5.1.3 Indication and denial

A propositional sign belongs to a proposition, and is *ipso facto* either true or false. Now, the sign of an elementary proposition enjoys this susceptibility intrinsically. On the other hand, a nonelementary proposition inherits its susceptibility to truth and falsehood by pronouncing on the truth and falsehood of other propositions. (Such, anyway, is Wittgenstein’s construction myth.) Wittgenstein presumes that every nonelementary proposition $A$ can be written as a denial $NB$ of what is expressed by the sign $B$. The sign $B$ may itself be a propositional sign, in which case $A$ is simply the denial of $B$. On the other hand, $B$ may rather be a propositional variable, serving to point out some propositions of which $A$ is the denial. But how does a variable point to propositions? A variable is a schematic
or structural presentation of other signs, a feature common to some signs which it indicates. The indicated signs may themselves express propositions or may indicate other signs in turn, but these chains of indication all terminate in propositional signs which together comprise the ultimate indication of the propositional variable. The values of the variable are then the propositions expressed by the signs which it ultimately indicates.

Wittgenstein gives three kinds of propositional variable. The first is simply a finite list of propositional signs. The second kind of variable is less trivial. Wittgenstein says that one can construct a variable ranging over a bunch of propositions, by forming the sign of a propositional function whose values are those propositions. Now, Wittgenstein’s notion of a propositional function is grounded in the notion of what a proposition says about an object. In particular, suppose we, so to speak, remove a name from a proposition, obtaining what Wittgenstein calls an Ausdruck. The Ausdruck is a common mark of the sense of many propositions, namely those from which that Ausdruck results upon removal of some name or other. Thus, Wittgenstein envisages that any name $a$ and propositional sign $A$ together determine an Ausdruck and thereby point out the range of propositions whose sense the Ausdruck marks. More precisely, these are the propositions $B$ such that, for some $b$, $B$ says about $b$ what $A$ says about $a$; or officially, the $B$ such that $\hat{b}B \approx \hat{a}A$ for some $b$. The idea is the same for the other connectives. Thus, to evaluate for $\sim$ in general, just drive the abstracts down the formation trees and check whether $\approx$ holds between corresponding leaves. Of course, we are interested in polyadic propositional functions as well, and so allow application of an abstract to arbitrary formulas, to obtain expressions like $\vec{a} \vec{b}A$.

The procedure for evaluation of $\sim$ is defined inductively by the method of Proposition III.

**Definition 3.** Pick $A,B \in \mathfrak{F}$. Then $\vec{a}A \sim \vec{b}B$ iff either

- $A = B = p$ for some $p \in p$, or
- $A,B \in \mathfrak{E}$ and $\vec{a}A \approx \vec{b}B$, or
- there are some $A_1,\ldots,A_k$ and $B_1,\ldots,B_k$ such that

$^{196}$Definition III cashes out the metaphor of §2.7.
- $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ or
- $A = [A_1, p, A_2]$ and $B = [B_1, p, B_2]$, or
- $A = NA_1$ and $B = NB_1$,

and $\bar{a}A_i \sim \bar{b}B_i$ for all $i \leq k$.

The method used to formalize Definition 4 here shows that $\sim$ is definable by a $\Delta$ predicate.

Besides the first and second kinds of variable there is a third, the form-series variable. One constructs such a variable by presenting a propositional sign and an operation on such signs; the range of the variable is the totality of propositions expressed by signs resulting from repeatedly applying the operation to the initially presented sign. I consider just a single, very simple kind of operation, and twist Wittgenstein’s notation slightly. Recall that the indicated occurrence of $p$ in $[A, p, B]$ is considered to bind all free occurrences of $p$ in $B$. This leads to a natural notion of substitution of a formula for a letter in a formula, analogous to the standard definition of “safe” substitution of terms in quantificational logic. We write $B[p/A]$ for the result of safely substituting $A$ for $p$ in $B$. An operation $O$ is determined by a propositional letter $p$ and a formula $B$ which contains the letter $p$; applied to a formula $A$ the operation $O$ returns the result $B[p/A]$ of substituting $A$ for all free occurrences of $p$ in $B$; applying $O$ to its result $B[p/A]$ yields $B[p/B[p/A]]$, and so on.

**Definition 4.** The $k$th term $[A, p, B]_k$ of the form-series indicated by $[A, p, B]$ is defined by induction on $k$ like this: $[A, p, B]_0 = A$, and $[A, p, B]_{k+1} = B[p/[A, p, B]_k]$.

A tedious formalization of the operation of substituting a $\exists$-formula for a propositional letter in a $\exists$-formula shows that $[A, p, B]_k$ is a $\Delta$ predicate of $A, p, B$ and $k$.

With these explanations of the ways of the variable in hand, the concept of immediate indication is now given by induction on complexity of variables.

**Definition 5.** $A$ immediately indicates $B$, i.e., $B \propto A$, iff either

- $A = (A_1, \ldots, A_k)$ and $B$ is one of the $A_i$, or
• $A = \hat{a}A_1$ and there’s a $b$ such that $\hat{b}B \sim \hat{a}A_1$, or

• $A = [A_1, p, A_2]$ and $B = [A_1, p, A_2]_k$ for some $k$.

Definition 5 does not obviously render $\alpha$ as $\Delta$. In the third clause, the existential generalization with respect to $k$ can be bounded by the rank of $B$, so it is no trouble. The second clause is more problematic. This is somewhat of a surprise since it corresponds to the relation of instance to generalization, which in usual developments of first-order logic would be obviously $\Delta$. The reason it would be $\Delta$ in a usual development is that atomic formulas themselves would then be coded by sets, so that the set which codes the “instantial term” would belong to the transitive closure of the set which codes the instantiating formula. But in the present case we are licensed no such assumption, since the counterparts of atomic formulas are not coded at all. For all we’ve said so far, infinitely many different names may characterize the sense of an elementary proposition, in which case it is far from obvious that the relation of generalization to instance should be in any sense “effective”. Formally, under that disaster scenario the generalization with respect to $b$ in the second clause above would then certainly not be bounded by a set which exists in $\text{HF}(\mathfrak{B})$.

If $\alpha$ is to be $\Delta$ at all, we must somehow constrain the character of $\mathfrak{B}$ to ensure that only finitely many names characterize the sense of an elementary proposition. There are various ways to do this. It is not clear to me that there is a way which makes sense no matter how $\mathfrak{B}$ is realized. But intuitively, $\hat{b}B$ represents “what $B$ says with respect to $b$”, which for Wittgenstein is simply “the result of turning $b$ into a variable in $B$”. Moreover, it is natural to assume, with respect to this informal idea, that $b$ characterizes $B$ if and only if the result of turning $b$ into a variable in $B$ is no longer $B$. Since $\hat{b}\bar{b}_1B \sim \bar{a}A$ formalizes “what $B$ says with respect to $b\bar{b}_1$ is what $A$ says with respect to $a\bar{a}$”, we are thus led to the definition $\text{Oc}(b, B) \leftrightarrow \hat{b}B \not\not B$. The assumption that only finitely

\begin{itemize}
\item Note: Wittgenstein clearly seems to wish to remain open to the idea that the number of names characterizing some elementary proposition might be infinite (4.2211). I neither assert nor presuppose the contrary. Rather, the point here is that if elementary propositions have always only finitely many constituents, then the relationship of a generalization to its instances is $\Delta$. I don’t commit to the converse, but do find it plausible.
\item Thus, the operation $B \mapsto \hat{b}B$ is quite different from $\lambda$-abstraction!
\end{itemize}
many names characterize the sense of an elementary proposition now amounts to the requirement that for \( B \) elementary, there are only finitely many names \( b \) such that \( \text{Oc}(b, B) \). Under this assumption, and using the definition of \( \text{Oc} \), it is straightforward to show by \( \in \)-induction that for each \( B \in \mathcal{Z} \), there is a (finite!) set \( x_B \) such that \( \forall y \in x_B \) \( \text{Oc}(b, B) \). And more generally, the formula \( \forall x \exists y \forall z (\text{Oc}(z, x) \iff z \in y) \) is true in \( \text{HF}(\mathcal{B}) \). This means that there is on \( \text{HF}(\mathcal{B}) \) a total function \( [x] \) of \( x \) such that \( z \in [x] \iff \text{Oc}(z, x) \) holds for all \( x \). Moreover, \( [x] \) is itself readily shown to be a \( \Sigma \) operation, which justifies its use in the construction of \( \Delta \) predicates. Finally, we may recast the second inset clause of Definition 5 so that if \( A = \hat{a}A_1 \), then \( B \varnothing A \) iff there’s a \( b \in [B] \) such that \( \hat{b}B \sim \hat{a}A_1 \), and \( \varnothing \) itself becomes \( \Delta \).

Of course, immediate indication is only a means to ultimate indication. At this point, we could simply take ultimate indication to be the ancestral of immediate indication. But, it will be useful to have the following

**Proposition 2.** *The relation \( \propto \) is well-founded on the proper formulas.*

*Proof.* It suffices to find a well-founded relation \( < \) on \( \mathcal{Z} \) such that if \( B \propto A \) then \( B < A \). Note that if \( A \) is either a Way 1 or a Way 2 variable, then \( B \propto A \) only if the syntactical complexity of \( B \) is less than that of \( A \). But, this need not hold in case \( A \) has the form \( \left[ A_1, p, A_2 \right] \). However, say that the form-series height of a formula is the length of the longest chain of successively nested occurrences of \( [\cdots] \) it contains. Since propriety demands that \( A_2 \) itself contains no occurrences of \( [\cdots] \), it follows that an indicated formula \( \left[ A_1, p, A_2 \right]_k \) has strictly lower form-series height than \( \left[ A_1, p, A_2 \right] \) for all \( k \in \omega \). On the other hand, if \( A \) is a Way 1 or Way 2 variable, than \( B \propto A \) only if \( B \) has no greater form-series height than \( A \). It therefore suffices to choose \( < \) so that \( B < A \) iff either \( B \) has lower form-series height than \( A \), or, \( B \) and \( A \) have the same form-series height but \( B \) has lower syntactical complexity than \( A \). \( \square \)

Note that this argument works only because we have stipulated that the second immediate subformula of a form-series variable contains no form-series variables itself. I have not been able to determine whether, without this restriction, the relation \( \propto \) is still well-founded.
Wittgenstein designates a variable’s (ultimate) indication by writing a bar over a variable: thus, if \( A \) is a propositional variable which (ultimately) indicates \( A_1, A_2, \ldots \), then \( \bar{A} \) is almost a kind of plural term (something like “the \( A_i \)”) which designates each of \( A_1, A_2, \ldots \). But, barred expressions aren’t really terms at all: rather, they stand to propositional signs as plural terms stand to singular terms. For example, one example of a barred expression might be rephrased “it’s raining and it’s snowing”, where the “and” does not express conjunction but rather works in something like the way it does in the noun phrase “Russell and Moore”. Perhaps it would better be phrased as “It’s raining. It’s snowing.”

In Wittgenstein’s account as I understand it, barred expressions need occur only in contexts of a very specific form: namely, in contexts of the form \( B \propto (\bar{A}) \). Thus, apparent reference to barred expressions serves only to formulate the circumstance that a proposition is indicated ultimately by the variable underlying the bar. And this circumstance is simply a relationship between a propositional sign and a variable. Thus, barred expressions may be handled syncategorematically.

**Definition 6.** \( A \) ultimately indicates \( B \), or \( B \propto (\bar{A}) \), iff either \( A \) is a propositional sign and \( B \) is the same sign as \( A \), or \( A \) is a variable and \( B \) is a propositional sign which stands to \( A \) in the ancestral of \( \propto \).

This definition simply enacts the pretense that a barred variable is a list of signs of the propositions that are values of the variable. Enclosing this pretended list in round brackets yields a purported variable of the Way 1 kind, its indication then to be ostensibly treated in the standard way. Thus, propositional multiplicities become logical fictions. The possibility of such an elimination is essential to Wittgenstein’s concept of the variable, motivated as it is by a broadly constructivist attitude toward the many.

Note that Definition 6 really has the form “\( B \propto \bar{A} \) iff \( B \propto^k A \)” for some \( k \), where \( \propto^k \) is defined by induction in the obvious way. However, it’s clear from Definition 6 that \( k \) may as well be bounded by the \( \varepsilon \)-rank of \( A \), and so ultimate indication is \( \Delta \) in \( \propto \) on \( \text{HF}(\mathcal{B}) \). Assuming \( \mathcal{B} \) chosen so that \( \propto \) itself is \( \Delta \) (as in remarks following Definition 6), then ultimate indication becomes \( \Delta \) on \( \text{HF}(\mathcal{B}) \) as well.

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Just as immediate indication is a means to indication, indication itself serves only to supply the bases of truth-operations. Wittgenstein considers only a single truth-operation, that of direct, joint denial: one performs this by pointing to some things people (might) have said and denying all of them. Formally,

**Definition 7.** Let $A$ and $B$ be propositional signs. Then $A$ directly denies $B$, or $B \prec A$, iff there is a formula $C$ such that $A = NC$ and $B \not\prec (\overline{C})$.

It is clear that the notion $\prec$ is $\Delta$ in the ancestral of $\prec$, for the existential quantifier $C$ in Definition 7 can be considered to be bounded by the transitive closure of $A$. So when $\mathfrak{B}$ is chosen so that $\prec$ is $\Delta$, then $\prec$ becomes $\Delta$ over $HF(\mathfrak{B})$.

Now as I’ve said, Wittgenstein’s construction myth is that only elementary propositions enjoy their truth-functionality intrinsically. One constructs a nonelementary proposition by denying, disjoining, conjoining, or otherwise truth-operating on propositions constructed already. But this makes sense only if we can find some truth-operational route from the constructed proposition back down to the elementary propositions. If, somewhere out in logical space we were to find two propositions, each the singular direct denial of the other, then we have not really found two propositions at all, because they do not occupy positions with respect to truth-and-falsehood of elementary propositions. Then again, there is the alleged matter of a proposition which denies itself. These considerations show that it is absolutely of the essence of this whole business that

**Proposition 3.** The relation $\prec$ is well-founded.

**Proof.** In the proof of Proposition 7, we obtained a transitive, well-founded relation $<$ such that $B \prec A$ only if $B < A$. We now argue that $B \prec A$ only if $B < A$. Suppose that $B \prec A$. Then we must have $A = NC$ for some $C$ such that $B \prec (\overline{C})$. If $C$ is a propositional sign, then $B = C$, and so $B < A$. On the other hand, suppose $C$ is a propositional variable. Then $B \prec (\overline{C})$ only if $B \prec B_1 \prec \cdots \prec C$ for some $B_1, \ldots$, and so $B < C$. But if $B < C$, then $B < NC$. So in any case, $B \prec NC$ only if $B < A$. Therefore since $<$ is well-founded, so is $\prec$.  

Now, the idea is that a propositional sign $A$ ultimately gets its susceptibility to truth and falsehood from the terminal nodes of the $\prec$-tree descending from $A$. 

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Those terminal nodes had better themselves be susceptible to truth and falsehood intrinsically.

**Proposition 4.** If A is a propositional sign, and if B is a ≺-minimal node of the ≺-tree descending from A, then B is an elementary proposition.

*Proof.* The argument is a tedious unspooling of definitions. Assume the hypothesis. By the Definition 7, it follows that B is a propositional sign. Now suppose B is not elementary. By the terminological explanations following Definition 2, it follows that B has the form NC for some proper formula C. If C is a propositional sign, then Definition 7 implies that C ⊺ NC = B, contra minimality of B. So suppose C is a variable. By Definition 8, every variable indicates something. So there must be a D such that D ⊺ C, and by Proposition 2, it follows that there is a ⊺-minimal E such that E bears the ancestral of ⊺ to C. Since a proper formula indicates only proper formulas, it follows that E is proper; and since E is ⊺-minimal, E must be a propositional sign. But then E ⊺ NC = B, a contradiction. □

5.1.4 Truth and possibility

To be susceptible to truth and falsehood is just to hold the world to yes or no: to submit the matter of truth and falsehood for an arbitrary and ultimate decision. Wittgenstein doesn’t say who is responsible for these decisions. But let’s suppose them all to be issued at once, in a single document, W, such that for each elementary proposition A, either W affirms A and does not deny A, or W denies A and does not affirm it. By reference to W, the truth and falsehood of each elementary proposition is thereby all at once decided. But then, the truth of every proposition is decided as well. For, by Propositions 3 and 4, it suffices to reduce the matters of affirmation and denial of A to the matters of affirmation and denial of the B ≺ A. And it is clear how to do this, for to deny everything A directly denies is just to affirm A, whereas to affirm something A directly denies is just to deny A.

**Definition 8.** Suppose A is nonelementary. Then

- W affirms A iff W denies all B ≺ A, and
• W denies A iff W affirms some B ≺ A.

In this analysis, the appeal to denial alongside affirmation seems to be essential. The reason is, roughly speaking, that the analysis claims that W affirms A when W does something which amounts to that. Simply withholding judgment on what is directly denied by A does not amount to the affirmation of A. Of course, the point is not, roughly speaking, that if W, like a cautious thinker, somehow didn’t pronounce on all the B ≺ A, then W would stay neutral on A. Rather, the point is that because the cautious thinker would have remained neutral on A by simply saying nothing about B, the absence of verdict on the B ≺ A does not by itself explain a verdict on A. One might simply stipulate, for example, that if W doesn’t affirm A, then W affirms NA. But that is what is to be explained.

This analysis gives yields some expected consequences. In the statement of each of the following, A is a propositional sign. Note that each argument presumes the wellfoundedness of ≺ recorded in Proposition 3.

**Proposition 5.** Either W affirms A or W denies A.

*Proof.* Suppose W doesn’t affirm A. Then by Definition 8, some B ≺ A is not denied by W. By induction hypothesis, W must affirm B, so W denies A. □

**Proposition 6.** W doesn’t both affirm and deny A.

*Proof.* Assume the contrary. Since W affirms A, W denies all B ≺ A. But since W denies A, W affirms some B ≺ A. This contradicts the induction hypothesis. □

**Proposition 7.** W denies A iff W affirms NA.

*Proof.* Suppose W denies A. By Definition 8, W affirms NA iff W denies all B ≺ NA. But, since A is a propositional sign, B ≺ NA iff B = A. Hence B affirms NA.

Conversely, if W affirms NA, then W denies some B ≺ A, and this B must itself be A. □

I’ve preferred the phrases “W affirms A” and “W denies A” because of their grammatical similarity to the model-theoretic predicate “satisfies”. Wittgenstein’s
preferred locutions are, rather “A agrees with W” and “A disagrees with W”. The entire explanation can be rephrased accordingly. For example, if A agrees with W, then so does NN(A, B). Similarly, NN(A, B) is satisfied if B agrees with W, regardless of whatever else W says. Thus, mere affirmation of A or of B suffices to win the agreement of NN(A, B). On the other hand, if neither A nor B are satisfied by W, then NN(A, B) must go unsatisfied too.

Here is just a bit more terminology and notation. Say that a frame $\mathfrak{F}$ is a class of verdicts on $\mathfrak{E}$. We write $|A|^\mathfrak{F}$ for the class of $W \in \mathfrak{F}$ such that $W$ affirms $A$, and refer to $|A|^\mathfrak{F}$ as the $\mathfrak{F}$-intension (or truth-condition) of $A$. Where $\mathfrak{F}$ is clear from the context, we drop the superscripted $\mathfrak{F}$.

Let’s defer momentarily the formalization of ideas of this subsection. But do note that we’ve taken a significant leap. For example, in general a verdict will not exist as a set inside of HF($\mathfrak{B}$); nor, or course, will arbitrary classes of verdicts exist in HF($\mathfrak{B}$) either.

5.2 Realizations of $\mathfrak{B}$

So far, the notions of elementary proposition and name have been treated abstractly, their requisite structure summarized in the general notion of a basis $\mathfrak{B}$. This approach has the interpretive advantage of respecting the position of the 5.55s that the forms of elementary propositions cannot be anticipated a priori. Nonetheless, this certainly doesn’t preclude examination of the effects of this or that particular choice of $\mathfrak{B}$. For example, it is natural to wonder about the relationship between truth-conditions expressible by notations in $\mathfrak{F}$ and truth-conditions expressed by first-order formulas over a given signature. But this curiosity presupposes a choice of $\mathfrak{B}$ with respect to which such a comparison makes sense at all. In this section, I’ll consider a particular choice of $\mathfrak{B}$ with respect to which the notion of “first-order logic in the Tractatus” makes sense.

But before going into this let me first give a silly construction of a basis $\mathfrak{B}$ to show that normalness is not essential. Names might be tokened by colored plastic balls and elementary propositions constructed by putting plastic balls in a bag, no more than one ball of each color per bag. The relation $\sim$ may say that bag $A$ says of the object designated by the color red what bag $B$ says of the object.
designated by green, provided that the same bag results by removing the red ball from $A$ and the green ball from $B$. In this case, of course, it would turn out that, for example, two objects could form exactly one state of affairs. So there would be no asymmetric relations, for example. Alternatively, $\mathfrak{B}$ might allow a bag to contain more than one ball of a given color, and with bags similar with respect to the designations of red and green provided that, in addition to the foregoing requirement, they moreover contain the same number of red and of green balls respectively. Such an alternative notation is easily seen to be capable of “encoding” another, more familiar notation, in which an elementary propositional sign is a sequence of names.

There are evidently some opportunities for amusement here, but let’s cut to the chase. Suppose, for example, that a state of affairs is something like a finite system of point-masses, each with its own inertia and gravitational influence on the others. Then any finite bunch of objects will be capable of forming infinitely many different states of affairs. However, any such state of affairs could be written in the form $R_{a_1 \ldots a_k}$, where a name’s position in the sequence following $R$ indicates the position of an object in the configuration that $R$ indicates.

The richness of the $R_{a_1 \ldots a_k}$ notation does bring with it a potential problem of surfeit, where states of affairs would exhibit symmetries which the notation apparently lacks. In such a case, the apparently diverse notations which turn out to represent the same state of affairs might simply be identified\footnote{This suggestion is due to Ricketts (2012).}. This is a comfortable philosophy for the pedestrian case of “$\{a, b\}$” versus “$\{b, a\}$”. But however the problem is handled, the key point is that apparent asymmetries of notation do not automatically imply asymmetries of subject-matter (any more than ♭ is more things than ♩). What matters is the extension of $\sim$, and that is up to $\mathfrak{B}$. This kind of thing is a familiar fact of life, like finding a distinctness of personal names not to entail distinctness of persons\footnote{Cf. 1.5.15e.}.

The choice of notation $R_{a_1 \ldots a_k}$ naturally determines the class $\mathcal{E}$ of elementary propositions and the class $\mathfrak{N}$ of names. What of the elementary sayings relation $\approx$? On what I take to be Wittgenstein’s account\footnote{Sketched in the 3.31s.}, a name $a$ together...
with a propositional sign $A$ determines, as a result of “denaturing” $a$ in $A$, the sign $aA$ of a so-called “Expression”$^{[202]}$. On the one hand, the sign $aA$ is a propositional variable, which indicates a multiplicity of propositions. On the other hand, an expression “marks the sense” of various propositions. Putting these two ideas together, a sign $aA$ of an expression is a propositional variable, which indicates those propositions whose sense is marked by the expression presented. To determine the range of $aA$, then, we need to say what is the expression which $aA$ presents, and when is the sense of a proposition marked by this expression. By Definition $^5$, we have already reduced these questions to the case of propositions which are elementary.

A natural approach to the idea of satisfaction is to reify syntactically the notion of variable name, which thus becomes a name-variable. More straightforwardly put, let’s introduce a series of name-variables $x, x_1, x_2, \ldots$. The result of “turning $a$ into a variable in $A$” can now be explicated as the result of everywhere replacing $a$ in $A$ with the first $x_i$ which does not yet occur in $A$. Thus, for example, the result of turning $a$ into a variable in $Rab$ is simply $Rx b$.

Now, which are the propositions whose sense is marked by the expression $Rx b$? The Russellian line is that they are the results of “determining the identity of $x$”. Thus presumably they are the propositions $Rab, Rbb, Rcb, \ldots$. However, as Wittgenstein says, $Rx b$ marks the sense of a proposition $A$ only if $Rx b$ is the result of turning a name into a variable in $A$. Clearly $Rx b$ marks the sense of $Rab$ and $Rcb$. However, $Rx b$ is not the result of turning a name into a variable in $Rbb$. Hence, $Rbb$ is not a value of $Rx b$.

### 5.2.1 Relational signatures

A relational signature $R$ consists of some relational predicates and some constant terms, together with a function mapping each relational predicate to a positive integer which is the arity of that predicate. Over a relational signature $R$, a closed atomic formula is a pair whose first entry is a relational predicate and whose second entry is a sequence of names whose length is the arity of that predicate.

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$^{[202]}$I’ll capitalize “expression” here to mark out what seems to be a stipulative usage also in Wittgenstein’s German.
We proceed to investigate the result of taking a basis \( \mathfrak{B} \) to be induced by \( \mathcal{R} \). Take \( \mathfrak{E} \) to be the set of closed atomic formulas of \( \mathcal{R} \) and take \( \mathfrak{M} \) be the set of constants of \( \mathcal{R} \). We now aim to construct a concept of *Ausdruck*, or of what is said by a proposition \( A \) of some objects \( a \).

To this end, introduce a countably infinite bunch \( x_1, x_2, \ldots, y_1, y_2, \ldots \) of variable names. The *Ausdruck* \( \chi(\vec{a}A) \) determined by the abstraction of names \( \vec{a} \) from an elementary proposition \( A \) is defined by induction on the length of \( \vec{a} \) as follows.

**Definition 9.**

- \( \chi(A) = A \) for \( A \) elementary, and
- \( \chi(\vec{a}A) = \chi(A)[a/x] \), where \( x \) is the first name-variable not in \( \chi(A) \).

The elementary samesaying relation \( \approx \) induced by \( \mathcal{R} \) is now defined as follows:

**Definition 10.** \( \vec{a}A \approx \vec{b}B \) iff \( \chi(\vec{a}A) = \chi(\vec{b}B) \).

Definition 9 now combines with Definition 3 to determine a samesaying relation \( \sim \) with respect to arbitrary signs in the class \( \mathfrak{Z} \) over the basis \( \mathfrak{B} \) induced by \( \mathcal{R} \).

In §7 of Chapter 2, I gave two informal methods of reducing the notion of samesaying with respect to arbitrary propositions to the notion of samesaying with respect to elementary propositions. One method described a language-game, the “maximal conversation”, and showed how the relation of samesaying on moves in the game is determined by the structure of the rules which license the moves. This method is essentially formalized in Definition 9 above. On the other hand, the second method described a physical realization of formulas constructed from copper wire, bits of clay, and glass beads. I’ll now formalize this second method, and then prove that resulting two analyses extensionally coincide.

In the following, we write \( I(A) \) for the set of immediate subformulas of \( A \), and \( K(A) \) for the main connective of \( A \). Note that the main connective of \( \vec{a}A \) is considered to be just \( \wedge \) so that \( K(\vec{a}A) = K(\vec{b}B) \).
**Definition 11.** The tree $\mathfrak{T}(A)$ of a formula $A$ is the set of sequences defined by induction on the complexity of $A$ so that

$$\mathfrak{T}(A) = \{\emptyset\} \cup \{A \rightsquigarrow s : \exists B \in I(A)(s \in \mathfrak{T}(B))\}.$$  

We write $s \leq t$ iff there’s a $u$ such that $s \rightsquigarrow u = t$. Indeed $\mathfrak{T}(A)$ is a tree in the sense that $t \leq t$ and $s \leq t$ implies $s \in \mathfrak{T}$.

**Definition 12.** A function $\theta : \mathfrak{T}(A) \simeq \mathfrak{T}(B)$ is an isomorphism of trees if $\theta$ is a bijection from $\mathfrak{T}(A)$ to $\mathfrak{T}(B)$ such that $\theta(s) \leq \theta(t)$ iff $s \leq t$ and $K(s_i) = K(\theta(s_i))$ for all $i \leq \text{length}(s)$.

**Definition 13.** For $s \in \mathfrak{T}(A)$, define $Q(s)$ by induction on the length of $s$ so that

$$Q(\emptyset) = \emptyset,$$

and

$$Q(A \rightsquigarrow s) = s$$

unless $A = \vec{a}A_1$ for some $A_1$, in which case

$$Q(A \rightsquigarrow s) = \vec{a} \rightsquigarrow Q(s).$$

Note that, by the construction of $\mathfrak{T}$, every $s \in \mathfrak{T}$ has finite length and hence a “last” entry, for which we write $e(s)$. A $p \in \mathfrak{T}(A)$ is said to be maximal if $e(p) \in \mathfrak{E}$.

Finally we can state

**Proposition 8.** Let $A, B \in \mathfrak{E}$. Then $A \sim B$ iff there’s a $\theta : \mathfrak{T}(A) \simeq \mathfrak{T}(B)$ such that

$$\chi(Q(p), e(p)) = \chi(Q(\theta(p)), e(\theta(p))).$$  \hspace{1cm} (5.1)

holds for all maximal $p \in \mathfrak{T}(A)$.

**Proof.** It suffices to prove the following result. Suppose $A$ and $B$ are not propositional abstracts. Then $\vec{a}A \sim \vec{b}B$ iff

$$\chi(\vec{a} \rightsquigarrow Q(p), e(p)) = \chi(\vec{b} \rightsquigarrow Q(\theta(p)), e(\theta(p))).$$  \hspace{1cm} (5.2)

holds for all maximal $p \in \mathfrak{T}(A)$ and all $\vec{a}, \vec{b} \in \mathfrak{N}$. We argue by induction on $A$.

First take $A \in \mathfrak{E}$, and suppose that $\vec{a}A \sim \vec{b}B$. We must then have $\chi(\vec{a}, A) = \chi(\vec{b}, B)$ with $B \in \mathfrak{E}$ and $\vec{a}, \vec{b}$ the same length. Now let $\theta : \mathfrak{T}(A) \rightarrow \mathfrak{T}(B)$ be such that $\theta(\emptyset) = \emptyset$ and $\theta(A) = B$. Clearly $\theta : \mathfrak{T}(A) \simeq \mathfrak{T}(B)$. Since $A, B \in \mathfrak{E}$, therefore...
5.2 \(Q(p) = \emptyset\) for all \(p \in \mathfrak{I}(A) \cup \mathfrak{I}(B)\). Moreover, if \(e(p) \in E\) for \(p \in \mathfrak{I}(A)\), then \(e(p) = p = A\) and so \(e(\theta(p)) = \theta(p) = B\). Thus

\[\chi(\vec{a}, A) = \chi(\vec{a} \sim Q(p), e(p))\] (5.3)

and

\[\chi(\vec{b}, B) = \chi(\vec{b} \sim Q(\theta(p)), e(\theta(p))).\] (5.4)

and so (5.2) holds for all maximal \(p \in \mathfrak{I}(A)\).

Conversely, suppose that there’s a \(\theta : \mathfrak{I}(A) \simeq \mathfrak{I}(B)\) such that (5.2) holds for all maximal \(p \in \mathfrak{I}(A)\). For such \(\theta\) and \(p\) we must have \(\vec{a} \sim Q(p) = \vec{a}, e(p) = A, \vec{b} \sim Q(\theta(p)) = \vec{b}\), and \(e(\theta(p)) = B\). This implies the equalities (5.3, 5.4), and therefore that \(\chi(\vec{a}, A) = \chi(\vec{b}, B)\) holds. Thus \(\vec{a}A \sim \vec{b}B\).

Now, suppose that \(A \not\in E\). By hypothesis, \(K_A\) must be one of (,), [], and \(E\), so that \(A = K_A(A_1, \ldots, A_k)\) for some \(A_1, \ldots, A_k\). In one direction, assume \(\vec{a}A \sim \vec{b}B\). Then there are \(\vec{b}\), of the same number as \(\vec{a}\), and there are \(B_1, \ldots, B_k\) such that \(B = \vec{b}(B_1, \ldots, B_k)\), and moreover \(A_i \sim B_i\) for all \(i \leq k\). By induction hypothesis we may assume there to be isomorphisms \(\theta_i : \mathfrak{I}(A_i) \simeq \mathfrak{I}(B_i)\) for all \(i \leq k\), each such that

\[\chi((\vec{a}) \sim Q(p'), e(p')) = \chi((\vec{b}) \sim Q(\theta_i(p')), e(\theta_i(p')))\] (5.5)

holds for all maximal \(p' \in \mathfrak{I}(A_i)\). Now, note that for each \(p \in \mathfrak{I}(A)\) there’s a unique \(A_i\) and \(p' \in \mathfrak{I}(A_i)\) such that \(p = A \sim p'\). So, define \(\theta : \mathfrak{I}(A) \to \mathfrak{I}(B)\) itself so that \(\theta(A \sim p') = B \sim \theta_i(p')\) and \(\theta(\emptyset) = \emptyset\). Clearly \(\theta : \mathfrak{I}(A) \simeq \mathfrak{I}(B)\). Moreover, for \(p = A \sim p'\), then

\[
Q(p) = Q(p') \land Q(\theta(p)) = Q(\theta_i(p')) \land \\
e (p) = e(p') \land e(\theta(p)) = e(\theta_i(p')).
\] (5.6)

But (5.5) and (5.6) entail (5.2). Since every maximal \(p \in \mathfrak{I}(A)\) is \(A \sim p'\) for some \(\mathfrak{I}(A_i)\)-maximal \(p'\), it therefore follows that every \(\mathfrak{I}(A)\)-maximal \(p\) satisfies (5.2).

Conversely, assume there’s a \(\theta : \mathfrak{I}(A) \simeq \mathfrak{I}(B)\) such that every \(\mathfrak{I}(A)\)-maximal \(p\) satisfies (5.2). Then \(B = K_A(B_1, \ldots, B_k)\) for some \(B_1, \ldots, B_k\). So for \(A_i \in I(A)\)
define \( \theta_i \) such that \( \theta_i(p') = \theta(A \rightarrow p') \). It follows that \( \theta_i : A_i \simeq B_i \) and moreover \( \theta_i \) satisfies (5.5) for all maximal \( p' \in \mathcal{Z}(A_i) \). By induction hypothesis, we now have \( \vec{a}A_i \sim \vec{b}B_i \) for all \( A_i \in I(A) \). By the definition of \( \sim \), this implies \( \vec{a}A = \vec{a}K_A(A_1, \ldots, A_k) \sim \vec{b}K_A(B_1, \ldots, B_k) = \vec{b}B \).

\[ \square \]

5.2.2 FOL

In some recent work, Kai Wehmeier (2004, 2008, 2012) has developed Hintikka’s (1956) analysis of Wittgenstein’s discussion of quantification in the 5.53s. Wehmeier and Hintikka argue that Wittgenstein’s discussion points toward a particular association of truth-conditions to formulas of first-order logic which rests on a nonstandard interpretation of objectual variables. On this so-called “weakly exclusive” interpretation, a variable ranges not over all objects, but exactly over those objects which are not mentioned in its scope. I think that the weakly exclusive interpretation is extensionally correct attribution to Wittgenstein, in the sense that it agrees with what Wittgenstein took to be the truth-conditions attached to a given first-order notation. Now, in Chapter 2 I argued that Wittgenstein describes a particular inductive procedure for associating truth-conditions to notations. This procedure differs from those of Russell or Tarski, appealing essentially as it does to the concept of Expression which is determined by a name and a propositional sign. Definitions 4 and 5 complete a formal reconstruction of what I claim to be Wittgenstein’s own actual procedure. In the following section, I prove that this alternative method of assigning truth-conditions to notations coincides extensionally with the weakly exclusive interpretation advanced by Wehmeier and Hintikka. These results illuminate the textual and interpretive foundation of the attribution to Wittgenstein of the weakly exclusive interpretation. They show that this interpretation, and thence the redundancy of the equality predicate, flow naturally from the conception of susceptibility to truth-and-falsehood which is encapsulated in the notion of logical picturing.

In a bit more detail, the following results evaluate the relationship between the truth-conditions expressible in \( \mathfrak{L}_{\Pi}(\mathfrak{R}) \) and those expressible in ordinary first-order logic. The main result is that in some sense, every intension of a first-order
formula is the intension of a propositional sign in $\mathcal{F}$. Appealing to analyses due to Hintikka and Wehmeier, we introduce a notion $\mathcal{L}_0$ of sharp first-order logic, which takes the syntax of an $\equiv$-free language but interprets objectual variable exclusively. We then introduce a partial mapping $\nu$ from expressions of $\mathcal{Z}_{B(R)}$ to formulas of $\mathcal{L}_1$. Proposition 9 says that images of $\mathcal{Z}_{B(R)}$-formulas under $\nu$ can be interpreted as a generalization of the concept of Expression from elementary propositions to arbitrary propositions. Proposition 10 then establishes that $\nu$ acts as a partial translation from $\mathcal{Z}_{B(R)}$ to $\mathcal{L}_1$. Conversely, in Proposition 11 we find an inverse mapping $\tau$ from $\mathcal{L}_1$ to $\mathcal{Z}_{B(R)}$ such that the composition $\tau \nu$ is a mere relettering of bound variables; this implies that every truth-condition expressible in $\mathcal{L}_1$ is the truth-condition of a formula in $\mathcal{Z}_{B(R)}$. Finally, in Proposition 13 we rehearse the result of Hintikka-Wehmeier, that every intension of an equality-free first-order formula over $\mathcal{R}$ under the sharp semantics is the intension of a formula of the corresponding system $\mathcal{F}$ of notations over the $\mathcal{B}$ induced by $\mathcal{R}$ (Proposition 12). All this together yields Proposition 14 that every first-order truth-condition is indeed expressible in $\mathcal{Z}_{B(R)}$.

I apologize in advance for the finickiness of the arguments; subsequent sections do not much depend on the technical concepts introduced in the rest of this subsection.

The concept of first-order relational signature induces a first-order syntax in an obvious way. Let the equality-free first-order language $\mathcal{L}$ over $\mathcal{R}$ be the smallest set containing every open atomic formula built up from a predicate together with constant and variable terms, along with every negation $\neg A$, disjunction $(A \lor B)$, and existential quantification $\exists x.A$ of formulas it contains. (Actually we take $\neg A$ to abbreviate $\neg \neg A$, $(A \lor B)$ as $\neg \neg (A \lor B)$ and $\exists x.A$ as $\neg \neg \exists x.A$.) The first-order language $\mathcal{L}^=\mathcal{E}$ over $\mathcal{R}$ is the smallest set satisfying the closure conditions for $\mathcal{L}$ together with the condition that it contain every formula $t = u$ for all $t, u$ among the constant and variable terms.

Let $\mathcal{F}$ be a frame over $\mathcal{E}$. We associate to each of $\mathcal{L}$ and $\mathcal{L}^=\mathcal{E}$ a canonical semantics as follows. Let $A$ be a formula of $\mathcal{L}^=\mathcal{E}$. We define by induction on complexity of closed formulas $A$ a first-order $\mathcal{F}$-intension $|A|_{\mathcal{F}}$. If $A$ is atomic, then $|A|_{\mathcal{F}} = \{ w \in \mathcal{F} : A \in w \}$. If $A$ is $a = b$, then $|A|_{\mathcal{F}} = \mathcal{F}$ if $a$ and $b$ are the same term, and $|A|_{\mathcal{F}} = \emptyset$ otherwise. If $A = \neg B$, then $|A|_{\mathcal{F}} = \mathcal{F} - |B|_{\mathcal{F}}$. If $A = (B \lor C)$,
then $|A|_0 = |B|_0 \cup |C|_0$. If $A \equiv \exists x_i B$, then $|A|_0 = \bigcup_{a \in \mathbb{N}} |B[x/a]|_1$. Now let $A$ be a formula of $\mathcal{L}$. The sharp first-order $\exists$-extension $|A|_1$ of a closed formula $A$ of the equality-free language $\mathcal{L}$ is similarly defined by induction on complexity of $A$, with the only changes being omission of the $=$-clause, and a different $\exists$-clause as follows: if $A \equiv \exists x_i B$, then $|A|_1 = \bigcup_{a \in \mathbb{N}} |B[x/a]|_1$, where $[A]$ is the set of constants and free variables occurring in $A$.

We begin by introducing a function $\nu$ which associates a canonical representative in $\mathcal{L}_1$ to each equivalence class induced by $\sim$ on $\mathfrak{M}_\mathcal{L}(\mathfrak{N})$.

- $\nu(A) = A$, if $A$ is elementary;
- $\nu(\bar{a}A_1) = \bar{x}\nu(A_1)[a/x]$ if $A = \bar{a}A_1$, and $x$ is the first name-variable not in $\nu(A_1)$.
- $\nu(A) = K_A(\nu(A_1), \ldots, \nu(A_k))$, if the main connective of $A$ is some other connective $K_A$.

**Proposition 9.** $\nu(\bar{a}A) = \nu([\bar{b}B])$ iff $\bar{a}A \sim [\bar{b}B]$.

**Proof.** Suppose $\nu(\bar{a}A) = \nu([\bar{b}B])$, so that

$$\bar{x}K_A(\nu(A_1), \ldots, \nu(A_k))[\bar{a}/\bar{x}] = \bar{y}K_B(\nu(B_1), \ldots, \nu(B_k))[\bar{b}/\bar{y}] \quad (5.7)$$

where $\bar{x}, \bar{y}$ are the earliest variables not in $\nu(A), \nu(B)$ respectively. Clearly we must have $\bar{x} = \bar{y}$, and moreover, $\bar{x}\nu(A_1)[\bar{a}/\bar{x}] = \bar{y}\nu(B_1)[\bar{b}/\bar{y}]$ for all $i \leq k$. Now, $\nu(A_i)$ and $\nu(B_i)$ must contain the same variables. For supposing $z$ to occur in $\nu(A_1)$, then $z$ must be distinct from the $\bar{x} = \bar{y}$, so when it occurs in $\nu(B_1)[\bar{b}/\bar{y}] = \nu(A_j)[\bar{a}/\bar{x}_j]$, this must be in virtue of its occurrence in $\nu(B_i)$. But then, taking $\bar{x}_j$ and $\bar{y}_j$ to be the earliest variables not in $\nu(A_j)$ and $\nu(B_i)$ respectively, we must have

$$\nu(\bar{a}A_j) = \bar{x}_j\nu(A)[\bar{a}/\bar{x}_j] = \bar{y}_j\nu(B)[\bar{b}/\bar{y}_j] = \nu(\bar{b}B_i). \quad (5.8)$$

By induction hypothesis, (5.8) implies $\bar{a}A_j \sim \bar{b}B_i$ for all $i \leq k$. Thus by Definition 8, it follows that $\bar{a}K_A(A_1, \ldots, A_k) \sim \bar{b}K_B(B_1, \ldots, B_k)$, i.e., $\bar{a}A \sim \bar{b}B$.

Conversely, suppose $\bar{a}A \sim \bar{b}B$. Then $\bar{a}A_i \sim \bar{b}B_i$ for all $i \leq k$. By induction hypothesis, it follows that $\bar{x}_i\nu(A)[\bar{a}/\bar{x}_i] = \bar{y}_i\nu(B)[\bar{b}/\bar{y}_i]$, so that $\nu(A_j)$ and
\(\nu(B,')\) must contain the same variables. But then \(\nu(A_1,\ldots,A_k)\) and \(\nu(B_1,\ldots,B_k)\) do too, so \(\vec{x} = \vec{y}\). So, each substitution \([\vec{x}/\vec{z}]\) which takes us from \(\nu(A)[\vec{a}/\vec{x}]\) to \(\nu(A)[\vec{a}/\vec{x}]\) is the same as the corresponding substitution \([\vec{y}/\vec{y}]\) which takes us from \(\nu(B)[\vec{b}/\vec{y}]\) to \(\nu(B)[\vec{b}/\vec{y}]\). This implies \((5.7)\), and hence that \(\nu(\vec{a}A) = \nu(\vec{b}B)\).

The following is the key result of this section. It says that in fact, the canonical \(L_1\)-representative \(\nu(A)\) of the \(J_{28(30)}\)-notation \(A\) receives the same intension as \(A\).

**Proposition 10.** \(|A| = \|A^v\|_1\).

**Proof.** Note that by Proposition 9 and the definition of \(\nu\), it follows that

\[
|\hat{a}A| = (|B| : \exists b(\hat{b} B \sim \hat{a} A)) \\
= (|B| : \exists b(\nu(\hat{b} B) \sim \nu(\hat{a} A))) \\
= (|B| : \exists b(\hat{y} B^v[b/y] = \hat{x} A^v[a/x]))
\]

where \(x\) and \(y\) are the earliest variables respectively new to \(A^v\) and \(B^v\). On the other hand,

\[
|\hat{a}A^v|_1 = |xA^v[a/x]|_1 \\
= (|A^v[a/b]| : b \notin [A^v[a/x]]).
\]

So by \((5.9)\) and \((5.10)\), it suffices to prove

\[
\hat{y} B^v[b/y] = \hat{x} A^v[a/x] \text{ iff } A^v[a/b] = B^v \wedge b \notin [A^v[a/x]] \tag{5.11}
\]

where \(x\) and \(y\) are the earliest variables respectively new to \(A^v\) and \(B^v\). First assume \(\hat{y} B^v[b/y] = \hat{x} A^v[a/x]\). Clearly, \(b \notin [A^v[a/x]]\). Moreover, we must have \(x = y\). But since \(x = y\), the assumption implies that \(B^v[b/y][y/a] = A^v[a/x][x/a]\). However, \(x, y\) occur neither in \(A^v[a/x]\) nor in \(B^v\), so it follows that \(B^v[b/y][y/a] = B^v[b/a]\), and \(A^v[a/x][x/a] = A^v\). Thus \(B^v[b/a] = A^v\) as desired.
Conversely, assume $A'[a/b] = B'[a/x]$. It follows by the right-hand conjunct that $A'[a/b][b/x] = A'[a/x]$, and from this together with the left-hand conjunct we obtain $A'[a/x] = B'[b/x]$. Since the left-hand conjunct entails that $A'$ and $B'$ contain the same variables, it follows that $A'[a/x] = B'[b/y]$.

We now construct a mapping $\tau$ from closed formulas of $\mathcal{L}$ to propositional signs of $\mathfrak{Z}$, which witnesses that $\nu$ is invertible up to logical equivalence. The definition of $\tau$ just mirrors that of $\nu$: thus $\tau(A) = A$ for $A$ atomic, $\tau(\neg A) = \neg(\tau(A))$, $\tau(A \lor B) = \neg \neg (\tau(A, \tau(B))$, and $\tau(\exists A) = \exists A^\tau$ where $a$ is a name which occurs neither as constituent nor as abstract in $A^\tau$.

**Proposition 11.** $|A^\tau|_1 = |A|_1$.

**Proof.** Let $B[x/a]$ be a closed formula. Suppose that $B[x/a]^\tau$ results from $B[x/a]$ by replacing each abstracted variable and the variables it binds with variables which do not occur its scope. Now,

$$(\exists x B)^\tau = \exists (B[x/a]^\tau)[a/y]$$

where $y$ does not occur in $B[x/a]^\tau$. Thus, $(\exists x B)^\tau$ results from $B$ by replacing each abstracted variable and the variables it binds with variables which do not occur its scope. It follows by induction on complexity of formulas that $A^\tau$ is the result of such a renumbering on $A$.

We can now complete the analysis of the relationship between $\mathfrak{Z}_{2[3]}$ and $\mathfrak{L}_1$ by asserting a converse of Proposition 10.

**Proposition 12.** $|A^\tau| = |A|_1$.

**Proof.** By Proposition 10, $|B| = |B'|_1$ for all $B$. So in particular, $|A^\tau| = |A'^\nu|_1$. But by Proposition 11, $|A'^\nu|_1 = |A|_1$. Thus $|A^\tau| = |A|_1$.

Finally we rehearse (one direction of) the result of Hintikka-Wehmeier which describes the relationship between “weakly exclusive” first-order logic without equality $\mathfrak{L}_1$ and “normal” first-order logic $\mathfrak{L}_0$. 220
Proposition 13 (Hintikka-Wehmeier). There’s a function $\theta$ such that $|A^\theta|_1 = |A|_0$ for all closed formulas $A$ of $\mathcal{L}^e$.

Proof. We first construct $\theta$ and then show that it has the desired properties. Note that the definition of $R$ implies that for each formula $A$ of $\mathcal{L}$, there’s a formula $\top \models A$ such that $\top \models A$ and $|\top \models A|_0 = \emptyset$. Now, define $\theta$ by induction on complexity of formulas as follows. If $A$ is atomic, then $A^\theta = A$; if $A$ is $t = u$ for some terms $t, u$ then $A^\theta$ is $T[A]$ if $t$ and $u$ are the same term, and is $\neg T[A]$ otherwise. Moreover, $A^\theta$ is $\neg B^\theta$ if $A$ is $\neg B$ and $A^\theta$ is $(B^\theta \lor C^\theta)$ if $A$ is $B \lor C$.

Finally, if $A$ is $\exists x B$ then $A^\theta$ is $\exists x B^\theta \lor \bigvee_{t \in B} B[x/t]^\theta$.

The verification that $|A^\theta|_1 = |A|_0$ goes by induction on complexity of closed formulas of $\mathcal{L}$. Every clause is obvious except for $\exists$. From the definition of $\theta$, it’s clear that $|\exists A| = |A|$, and that if $t \not\in |A|$, then $A^\theta[x/t] = A[x/t]^\theta$. Hence

$$|\exists A^\theta|_1 = |\exists A^\theta \lor \bigvee_{t \in |A|} A[t]^\theta|_1$$

$$= \bigcup_{t \not\in |A^\theta|} |A[t]^\theta|_1 \lor \bigcup_{t \in |A|} A[t]^\theta|_1$$

$$= \bigcup_{t \not\in |A|} |A[t]^\theta|_1 \lor \bigcup_{t \not\in |A|} A[t]^\theta|_1 \lor \bigcup_{t \not\in |A|} A[t]^\theta|_1$$

$$= \bigcup_{t \not\in |A|} |A[t]^\theta|_1 \lor \bigcup_{t \not\in |A|} A[t]^\theta|_1 \lor \bigcup_{t \not\in |A|} A[t]^\theta|_1$$

$$= \bigcup_{t \not\in |A|} |A[t]^\theta|_1$$

$$= \bigcup_{t \not\in |A|} |A[t]|_0$$

$$= |\exists A|_0.$$

And at last we obtain the result that every truth-condition expressible in first-order logic with equality is expressible in $\exists_{2\forall(\forall)}$.

Proposition 14. $|A^\theta^e| = |A|_0$.

Proof. Immediate from Propositions 13 and 12. 

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5.3 Reflection

Let’s now pick up the last of the main thread. In §5.1.4 I introduced the notions of arbitrary truth-possibility for elementary propositions, and the agreement or disagreement of propositions with truth-possibilities. The characteristic distribution of agreement and disagreement of a proposition with such truth-possibilities is its sense, intension, or truth-condition. These notions in turn determine notions of logical consequence and logical validity. A proposition is logically valid (or tautologous) if it disagrees with no truth-possibility for elementary propositions; a proposition follows from some others if it disagrees with no truth-possibility with which the others agree.

I’ve left open the cardinality of the sets of elementary propositions and of names, and in particular allowed for the possibility that these be infinite. In that case, the notion of a verdict may as well be, to all appearances, the notion of an arbitrary subset of an infinite set. Now, this does not on its face require a momentous leap in the strength of the metamathematical presuppositions. Some relations can be expressed between an element \( x \) and a subset \( X \) of a domain using a first-order formula in those two parameters. Someone who doesn’t accept the notion of arbitrary subset can interpret the parameter \( X \) to vary schematically through first-order formulas in the language of the structure. On the other hand, somebody else might read the same variable \( X \) to vary through arbitrary subsets of the domain. And of course those alternatives are not exhaustive. In other words, there is no difficulty in the mere fact that we consider a relation which holds formally between elements and subsets of a domain.

Definition 8 introduced the relation that holds between a verdict and a formula in case the verdict affirms the formula. This definition used induction on formulas, and so implicitly a second-order quantification over sets of formulas: roughly, “a formula \( A \) is true-in-\( W \) if \( A \) belongs to every set \( X \) such that (i) every element of \( W \) belongs to \( X \), and (ii) every nonelementary formula directly denying no formula in \( X \) also belongs to \( X \)”. It is not clear to me that there exists a first-order formula to the same effect. Roughly speaking, since \( W \) may be chosen isomorphic to \( \text{HF}(\mathcal{B}) \), there seems to arise a potential conflict with Tarski’s theorem.
The concepts of validity and consequence seem to be even more trouble. In Chapter 4, we found a method of expressing in \( \mathcal{Z} \) the ancestral of any expressible relation. Goldfarb has pointed out that this means the appropriate consequence relation is not arithmetically definable. For, given the expressibility of the ancestral, the structure of the natural numbers, together with operations of addition and multiplication, can be axiomatized up to isomorphism. The consequences of this axiomatization (in the relevant vocabulary) will be precisely the truths of arithmetic. Since the axiomatization is finite, it then follows that the concept of validity is not arithmetically definable either. So, it is also unlikely that the concept of validity will in general be first-order definable over HF(\( \mathcal{B} \)).

What kind of upper bound can we give on the complexity of the class of tautologies? This is a question for further research. The explicit form of the definitions of truth and of consequence renders these notions at worst \( \Pi_1^1 \) over HF(\( \mathcal{B} \)). I would guess that the upper bound is lower than that, and maybe just \( \Delta_1^1 \). But, this is a question for future research.

Let me conclude with a few remarks about what drives Wittgenstein into this situation. Definition 8 is supposed to give the condition under which any proposition is true, and therefore cannot be arbitrary. In particular, the susceptibility of propositions to truth and falsehood must be capable of being respected. Respect of the relevant kind will consist, for example, in not both affirming and denying a proposition; similarly it will entail committing upon denying the negation of a proposition to affirm that proposition. So, truth and falsehood have to be connected with affirmation and denial in such a way that respect for truth enjoins such exemplary dispositions with respect to affirmation and denial. This means that any definition of truth has to agree with Definition 8. For, Definition 8 is justified by appeal to those norms of affirmation and denial which are essentially characteristic of respect for truth. But then in this way, truth and falsehood become intelligible as a special case of affirmation and denial. Truths and falsehoods are singled out among all instances of affirmation and denial by a merely formal criterion, that the left relatum is a truth-possibility, i.e., a maximal-consistent set of elementary propositions and negations thereof.

At the same time, of course, it is essential to the existence of the general form of the truth-function that all patterns of affirmation and denial be reflected
in patterns of truth and falsehood, i.e., in patterns of affirmation and denial by truth-possibilities. Thus in particular, Wittgenstein maintains that a proposition $A$ affirms another proposition $B$ provided that no truth-possibility for elementary propositions affirms $A$ without affirming $B$. Similarly, he holds, if a proposition is affirmed by all truth-possibilities for elementary propositions, then it is affirmed by any bunch of propositions. Since affirmation and denial are antecedently understood, these are substantial mathematical conditions. It’s not clear to me under what philosophical hypotheses they can be met.

To bring this out, note that Wittgenstein says very little about which truth-possibilities for elementary propositions exist. I know of only one textually justified principle here: that if $W$ is a verdict and $A$ is an elementary proposition, then both $W \cup \{A\}$ and $W - \{A\}$ are verdicts. Such a weak assumption clearly does not sustain the relationship between affirmation and truth which Wittgenstein intends. For example, although the assumption guarantees for each elementary proposition that it might be false, and likewise guarantees the possible falsehood of arbitrary finite disjunctions of elementary propositions, it does not guarantee the possible falsehood of their infinite disjunctions. In particular it does not guarantee possible falsehood of any existential generalization of an elementary proposition. It’s hard to know whether to trust the tone of innocence at 4.45, a remark on combinatorics of truth-possibilities which entails no more than 1.21 itself.

Thus, on the one hand, we find in the *Tractatus* a broadly constructivist account of positions with respect to the possibilities of truth and falsehood of elementary propositions. In particular, the relationship of a proposition to the propositions it directly denies should be characterized constructively. But, the only thing which determines which are the propositions directly denied is the ultimately ensuing predicted consequence relation on the propositions we originally set out to answer. But generating this prediction seems to require some second-order quantification on an infinite domain, or in other words, commitment to the notion of arbitrary truth-possibility for elementary propositions. One might hope to find some constructive or predicative replacement for those second-order quantifiers, which would lift this commitment. But, my suspicion is that such a strategy wasn’t intended by Wittgenstein. In other words, the sus-
picion is that Wittgenstein envisaged the construction of positions of agreement and disagreement with respect to truth-possibilities for elementary propositions, but did not envisage the construction of truth-possibilities.
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