Thermodynamic and Transport Properties of a Holographic Quantum Hall System

by

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BSc, University of Alberta, 2012

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE AND POSTDOCTORAL STUDIES
(Physics)

THE UNIVERSITY OF BRITISH COLUMBIA
(Vancouver)

November 2014

Abstract

We apply the AdS/CFT correspondence to study a quantum Hall system at strong coupling. Fermions at finite density in an external magnetic field are put in via gauge fields living on a stack of D5 branes in Anti-deSitter space. Under the appropriate conditions, the D5 branes blow up to form a D7 brane which is capable of forming a charge-gapped state. We add finite temperature by including a black hole which allows us to compute the low temperature entropy of the quantum Hall system. Upon including an external electric field (again as a gauge field on the probe brane), the conductivity tensor is extracted from Ohm's law.

Preface

This dissertation is an elaboration and summary, with permission, of reference [5], of which I am an author. Figures 8.1, 8.4, 8.2, 8.5, and 8.3 are taken directly from that paper. All other figures and tables are my original work. The calculations reported in Chapter 8 and Appendix A were originally performed by Charlotte Kristjansen. Gordon Semenoff supervised the entire project, and was responsible for proposing and advising the computations I performed within.

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Acknowledgements

I would like to thank Gordon Semenoff for guiding this project. I have much gratitude for Charlotte Kristjansen, who not only was the first to perform the entire conductivity calculation, but also gave valuable insights for other parts of the work. Lastly, I thank Niko Jokela for sharing his insight into the D7 embedding coordinates, and NSERC for funding this research.

Dedicated to Julia Seymour, for her unwavering support.

Chapter 1

Introduction

For a long time, string theory has been hailed for its potential as a grand unifying theory and a consistent quantum description of gravity; all in all, a reductionist's dream. In the last fifteen years, however, some of the most promising string theory research has been done in more emergent phenomena contained in strongly coupled field theories, with application to quantum chromodynamics, and more recently, many condensed matter systems. This complete turnaround is due to the AdS/CFT correspondence, conjectured by Juan Maldacena in 1997 [1]. The correspondence establishes an equivalence between a highly symmetric gauge field theory ($\mathcal{N} = 4$ Super Yang-Mills) and a super-string theory (Type IIB) in a higher dimensional Anti-de Sitter space (AdS). The backbone of the correspondence really is a duality in the description of extended p-dimensional objects in string theory called *Dp branes*. In a sense, we can think of the gauge field theory as living on the boundary of AdS. Really, though, these two theories, the boundary field theory and interior string theory are two descriptions of the exact same thing. They contain exactly the same information, and as a result, there is an entire dictionary relating ingredients on each side. Recently, this correspondence has found application to certain condensed matter systems. The string theory model in AdS gives us a so-called *holographic* model of the physics on the boundary. What makes this duality incredibly useful is that it is a strong coupling to weak coupling duality. That is, the holographic models probe the strong coupling regime of the boundary field theory using only weakly coupled string theory. At low energies, the problem reduces to solving classical supergravity.

In this thesis, we study a particular condensed matter system in the strong coupling regime, that of a two-dimensional fermion gas in the presence of an external magnetic field. It is well known that the magnetic field causes the fermions to organize themselves into Landau levels, which are filled at integer values of the filling fraction, which is the ratio of charge density to magnetic field. At these values, the system is in a charge-gapped state and the transverse (Hall) conductivity is quantized. This, combined with impurity effects leads to the so-called *Quantum Hall effect*, wherein the conductivity forms plateaus at it's quantized values as a function of the filling fraction.

To find the appropriate holographic model to describe this system, we follow reference [2], in which a stack of D5 branes, which are treated as probes, are embedded in AdS to give fermionic flavour fields in the corresponding field theory. Finite density, and external electric and magnetic fields can be included by adding gauge fields to the probe brane, while finite temperature can be included by making a black hole in the interior of AdS. With the right background supergravity fields, a dielectric effect occurs in the D5 branes, causing them to blow up into a single D7 brane in the interior of AdS [3]. At integer filling fraction, this D7 is able to effectively absorb the field lines of the D5s and cap off before entering the black hole horizon of AdS. This is a charge gapped quantum Hall state.

The purpose of this thesis is to compute two essential properties of this holographic quantum Hall system. The first is the entropy of the system at low temperatures, and the second is the conductivity tensor. We will find that both of these exhibit discontinuities as a function of filling fraction, but the hallmark plateaus of the quantum Hall effect will be absent due to the simplifying assumption of translational symmetry in our model. As with all effective models this will not capture all of the physics of its intended system of study. However, we hope that this perspective will offer a new kind of intuition for condensed matter physicists studying quantum Hall systems, wherein the difficulties of understanding strongly correlated electrons is replaced by a rather elegant gravitational paradigm. Indeed, we will find some indication of the possibility of Hall plateau formation without impurities to break translational symmetry. This is an exciting alternative for the traditional explanation of conductivity plateaus.

The outline of this thesis is as follows. There are several essential ingredients that go into the AdS/CFT correspondence (and its extension to the whole paradigm of gauge/gravity duality). The first two are in the name: anti de-Sitter space, and conformal field theories. The third is supersymmetry, since both the gauge field theory on the boundary and the string theory in the interior are supersymmetric. Lastly, of course, is string theory. All of these subjects are covered somewhat briefly in chapter 2. The real focus of this list will be on non-perturbative objects of string theory, D branes, since these are really at the core of the AdS/CFT correspondence. Thus, chapter 3 is devoted to understanding the existence and dynamics of these objects. Chapter 4 will outline the actual correspondence and how it is used in general. In chapter 5, we explain in detail the quantum Hall system that motivates our study, specifically looking at its weak coupling behaviour. Chapter 6 is a construction of the giant D5 model, in which the stack of D5 branes blows up to form a D7. This chapter is effectively a summary of references [2] and [4]. Chapters 7 and 8 outline the computation of the entropy and conductivity tensor respectively. The details are relegated to the appendix. Lastly, in chapter 9, we discuss some implications and directions for future work.

We should note that throughout this thesis, we will work in natural units unless otherwise indicated.

Chapter 2

Preparations

In this chapter we discuss four main ingredients that go into the AdS/CFT correspondence: Anti-de Sitter space-time, conformal field theories, supersymmetry, and string theory.

2.1 Anti-de Sitter space

2.1.1 The boundary of AdS

Consider a flat $(p+1)$-dimensional metric with two time-like directions i.e. a signature (2,p-1) flat metric.

$$ds^2 = -dX_0^2 + dX_1^2 + \ldots + dX_{p-1}^2 - dX_p^2. \tag{2.1}$$

Now embed in this space-time a hyperboloid with a radius of curvature L (see figure 2.1).

$$-X_0^2 + X_1^2 + \ldots + X_{p-1}^2 - X_p^2 = -L^2. \tag{2.2}$$

This hyperboloid defines a p-dimensional Anti-de Sitter space (AdS_p), which is a maximally symmetric solution of Einstein's equations. Note that the two time-like directions allow for closed time-like curves that wrap around the hyperboloid. It is conventional to 'unwrap' the hyperboloid to avoid this. This amounts to taking the universal cover, allowing $-\infty \leq t \leq \infty$. Lastly, we can see from (2.2) that the isometry group of AdS_p is $SO(p-1,2)$, in analogy with the usual 4D Poincaré isometry group of Minkowski space $SO(3,1)$

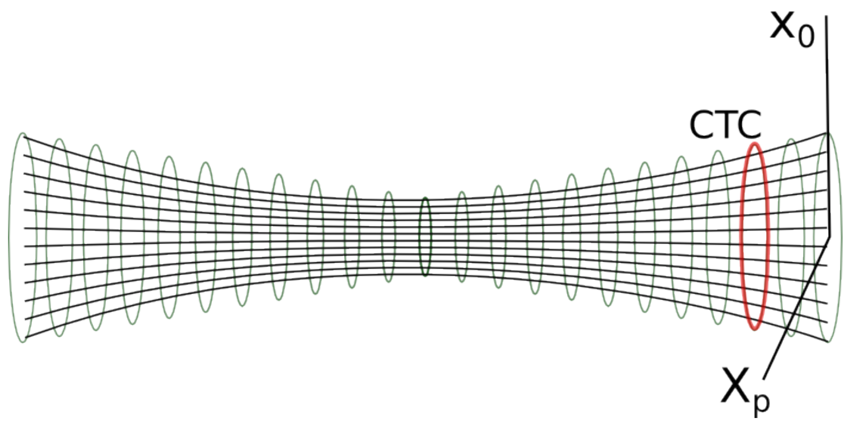

FIGURE 2.1: The hyperboloid of AdS_p embedded in Euclidean space. The red band illustrates a closed timelike curve.

Re-wrting the metric in 'hyper-polar' variables gives the *global* coordinates

$$ds^2 = L^2(-\cosh^2(\rho)d\tau^2 + d\rho^2 + \sinh^2(\rho)d\Omega_{p-2}^2), \tag{2.3}$$

where $d\Omega_{p-2}$ is the line element of a $(p-2)$-sphere. We call the location $\rho = 0$ the *Poincaré horizon*.

Often, it proves useful to view AdS as a Penrose diagram. To do so, we need to bring the boundary at $x = \infty$ to a finite coordinate. This can be done with the change of variables $\tan\theta = \sinh\rho$. In this case, the metric becomes

$$ds^2 = \frac{L^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{p-2}^2). \tag{2.4}$$

The Penrose diagram for AdS_2 is just a strip, and AdS_3 is cylinder (figure 2.2)[1]. This cylinder has boundary $\mathbb{R}_\tau \times S^{p-2}$. The boundary is very important to AdS/CFT, and we should note that there are many alternative descriptions of it.

We can immediately find one very interesting property of this space. If we consider a massless particle at a constant angle on the $(p-2)$-sphere, its geodesic satisfies $ds^2 = 0$. In global coordinates, this implies

$$\frac{d\rho}{d\tau} = \cosh\rho, \tag{2.5}$$

or in the Penrose coordinates,

$$\frac{d\theta}{d\tau} = \pm 1. \tag{2.6}$$

[1]The cylinder shows the important features of AdS_5 as well, we only need to tack on 2-spheres around the circumference.

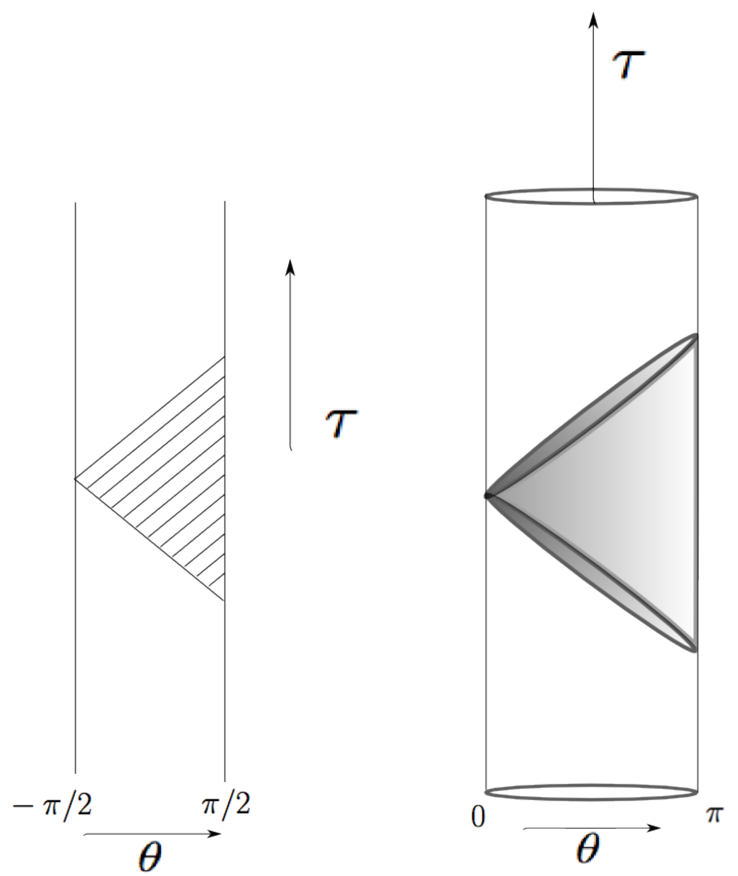

FIGURE 2.2: The left diagram shows AdS_2 in Penrose coordinates. The right diagram shows AdS_3. The shaded regions are the Poincarè patch.

In either coordinate, we can integrate this to get the proper time it takes to the particle to travel from the horizon to the boundary

$$\tau = \int_0^\infty \frac{d\rho}{\cosh(\rho)} = \frac{\pi}{2}. \tag{2.7}$$

By symmetry, the particle can travel to the boundary and back again in a proper time $\tau = \pi$. Now consider a stationary observer with $d\rho = 0$ located at the Poincaré horizon. They measure a time interval $d\tau' = L\cosh(0)d\tau$. So in the interval when the massless particle completes its round trip, the observer at the horizon measures a time

$$\tau' = L\pi. \tag{2.8}$$

Thus, even though the boundary is an infinite distance away, signals can be sent to and from the boundary in a finite time. This makes AdS space very unusual and very interesting. It means that boundary conditions at infinity can continually affect the dynamics deep in the interior of the space-time. It also means that Feynman diagrams are very different. Usually, in calculating Feynman diagrams, we assume asymptotic states that are plane waves at infinity. Now the physics on the boundary of AdS determines the asymptotic states.

We could also consider massive geodesics. Such particles have four-velocities U^μ that satisfy the time-like condition $U^\mu U_\mu = -1$, which implies that

$$\Rightarrow L^2\left[-\cosh^2(\rho)\left(\frac{d\tau}{d\lambda}\right)^2 + \left(\frac{d\rho}{d\lambda}\right)^2\right] = -1, \tag{2.9}$$

where λ is the proper time of the particle, and we have considered a constant position on the p-sphere. Note that the metric is independent of τ and therefore has time-translation symmetry and a corresponding Killing vector ∂_τ (or in components $k^\mu = \delta^\mu_\tau$). But for any Killing vector, we know that $k_\mu p^\mu = c$ for some conserved constant c. Thus,

$$k_\mu = -\cosh^2\rho\,\delta^\tau_\mu \tag{2.10}$$

$$\Rightarrow \frac{1}{2}\left(\frac{d\rho}{d\lambda}\right)^2 - \frac{c^2}{2\cosh^2(\rho)} = -\frac{1}{2L^2}. \tag{2.11}$$

(2.11) is an energy conservation equation. The first term on the left is the kinetic energy of the particle, the second term can be interpreted as a potential energy, and the total conserved energy is the quantity on the right side, which is always negative. Thus any massive particle will always be trapped by this potential, and will never be able to escape to infinity. This makes AdS a convenient box within which to do physics.

2.1.2 The Poincaré patch

While insightful, the global and Penrose coordinates defined above will not be used for calculations. Instead we transform to *Poincarè coordinates*. To do so, one simply follows the analogous transformations that take a Schwarzschild metric into Eddington-Finkelstein coordinates. We define

$$r = X_p + X_{p+1}, \quad v = X_{p-1} - X_p, \quad X^\mu = \frac{r}{L}x^\mu. \tag{2.12}$$

Note that the boundary is now at $r = \infty$. The equation of the hyperboloid (2.2) now reads

$$-L^2 = rv + \frac{r^2}{L^2}\eta_{\mu\nu}x^\mu x^\nu, \tag{2.13}$$

where $\eta_{\mu\nu}$ is a $(p-1)$ dimensional Minkowski metric. The induced metric on the hyperboloid is given by the pullback $g_{ij} = \eta_{ab}\frac{\partial X^a}{\partial x^i}\frac{\partial X^b}{\partial x^j}$, which simplifies to

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2. \tag{2.14}$$

These are the Poincaré coordinates for AdS_p. For this thesis, we will be focusing on the case of $p = 5$. These coordinates only a cover half of the hyperboloid, which is referred to as the *Poincaré patch* (shown in figure 2.2).

There is one last set of coordinates we will need. These are *Fefferman-Graham* coordinates given by the transformation

$$z = \frac{L^2}{r}, \tag{2.15}$$

which gives the transformed metric

$$ds^2 = \frac{L^2}{z^2}(\eta_{\mu\nu}dx^\mu dx_\nu + dz^2). \tag{2.16}$$

The boundary in these coordinates is at $z = 0$. In the opposite limit, $z \to \infty$, the timelike killing vector becomes null since at a stationary position, $d\tau = L/zdx^0$ Thus, the Poincaré horizon is located at $z \to \infty$.

We might also ask if there are any spherically symmetric solutions to Einstein's equations that are asymptotically AdS. Such a solution could describe a black hole embedded in the space-time. One can check that the following metric in Poincaré coordinates is exactly such a solution

$$ds^2 = \frac{r^2}{L^2}(-h(r)dt^2 + d\vec{x}^2) + \frac{L^2}{r^2}\frac{1}{h(r)}dr^2, \tag{2.17}$$

where $\vec{x} = (x^1, \ldots, x^{p-2})$, $h(r) = 1 - r_h^4/r^2$, and r_h is the radial coordinate of the black hole horizon.

2.2 Conformal field theory

Conformal symmetry enters the AdS/CFT correspondence in two important but very distinct ways. The conformal field theory, (CFT) in this correspondence is a quantum theory with conformal invariance which is used to describe some system in 3+1 dimensions. But conformal symmetry enters the AdS side as well, as an invariance on the two dimensional string world-sheet. The details of 2D conformal theories involve some subtleties, and so we will focus here on properties of theories with dimension greater than two.

2.2.1 Conformal transformations

Recall that conformal transformations are local scale transformations of the metric

$$g'_{\mu\nu} = \Omega(x) g_{\mu\nu}. \tag{2.18}$$

If we let $g_{\mu\nu}$ be the Minkowski metric, then the transformations with $\Omega(x) = 1$ gives the isometry group of Minkowski space, which is the Poincarè group. Thus, we see that the Poincarè transformations are a subset of conformal transformations. Conformal transformations have the special property that they preserve the angles between vectors.

Now if we make an infinitesimal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, then the metric transforms as

$$g_{\mu\nu} \to g_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu + O(\epsilon^2). \tag{2.19}$$

Demanding that this be a conformal transformation yields the equation

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = f(x) g_{\mu\nu} \tag{2.20}$$

$$\Rightarrow f(x) = \frac{2}{d} \partial^\mu \epsilon_\mu(x), \tag{2.21}$$

in d dimensions.

Let's work in Euclidean space with metric $\eta_{\mu\nu}$ for simplicity, then (2.20) implies that

$$\partial_\rho \partial_\mu \epsilon_\nu(x) + \partial_\rho \partial_\nu \epsilon_\mu(x) = \partial_\rho f(x) \eta_{\mu\nu} \tag{2.22}$$

$$\Rightarrow \partial_\nu f(x) \eta_{\nu\rho} + \partial_\mu f(x) \eta_{\nu\rho} - \partial_\rho f(x) = 2\partial_\mu \partial_\nu \epsilon_\rho \tag{2.23}$$

$$\Rightarrow 2\partial^2 \epsilon_\rho(x) = (2-d)\partial_\rho f(x). \tag{2.24}$$

$$\tag{2.25}$$

Taking a derivative of (2.24) and (2.20) gives

$$\Rightarrow 2\partial^\rho \partial^2 \epsilon_\rho(x) = (2-d)\partial^2 f(x), \quad \& \quad 2\partial^\mu \partial^2 \epsilon_\mu(x) = \partial^2 f(x) d, \tag{2.26}$$

which combine to give us the the condition for $f(x)$ to be conformal

$$(d-1)\partial^2 f(x) = 0. \tag{2.27}$$

We see that if $d = 1$, there is no constraint. Any smooth function $f(x)$ is conformal. However, if $d > 2$, we have

$$f(x) = A + B_\mu x^\mu \tag{2.28}$$

$$\Rightarrow \epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho. \tag{2.29}$$

a_μ is a just a regular translation, and $c_{\mu\nu\rho}$ constitutes so-called *special conformal transformations*. These are essentially an inversion, followed by a translation, followed by another inversion. To see the effect of $b_{\mu\nu}$, we plug it into (2.20)

$$\begin{aligned} [\partial_\mu(b_{\nu\rho}x^\rho) + \partial_\nu(b_{\mu\rho}x^\rho)] &= \frac{2}{d}\partial_\rho\eta^{\rho\sigma}\epsilon_\sigma\eta_{\mu\nu} && (2.30) \\ \Rightarrow b_{\nu\mu} + b_{\mu\nu} &= \frac{2}{d}b^\lambda_\lambda\eta_{\mu\nu}. && (2.31) \end{aligned}$$

If we write $b_{\mu\nu}$ as the sum of a symmetric and anti-symmetric part, $b_{\mu\nu} = S_{\mu\nu} + \omega_{\mu\nu}$, then this equation restricts the symmetric part to be

$$S_{\mu\nu} = \alpha\eta_{\mu\nu}, \tag{2.32}$$

and the anti-symmetric part is unrestricted. Thus we can write the linear transformation as

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + \omega_{\mu\nu}. \tag{2.33}$$

The first term is a *dilation* (sometimes called dilatation) or scale transformation, and the second is a rotation.

So we understand the affect of these transformations on coordinates, but what about fields? Recall that we can write an infinitesimal field transformation as $\phi \to \phi - iwG\phi$, where w is a parameter and G is a generator. For example, the dilation operation scales the coordinates and field via

$$\begin{aligned} x' &= \lambda x && (2.34) \\ \phi'(\lambda x) &= \lambda^{-\Delta}\phi(x). && (2.35) \end{aligned}$$

If the dilation is infinitesimal, we have

$$\begin{aligned} \phi'(x) &\approx \phi(x) + \frac{\partial\phi}{\partial\lambda}\lambda && (2.36) \\ &= \phi(x) + \partial_\mu\phi x^\mu\lambda, && (2.37) \end{aligned}$$

giving the generator $D \equiv -ix^\mu\partial_\mu$.

With this in hand, one can work out the conformal algebra, and with a suitable reorganizing of the generators, one can show that it satisfies the Lorentz algebra in $d+1$ dimensions. That is, the conformal group in $\mathbb{R}^{p,q}$ is isomorphic to $SO(p+1, q+1)$. In regular Minkowski space, this means the conformal group is $SO(4, 2)$.

A *primary* field is one which transforms as

$$\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x), \tag{2.38}$$

and Δ is referred to as the *scaling dimension* or *conformal dimension*. For a scalar field under a dilation, we can use (2.34), (2.35) to get

$$\left|\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right| = \lambda^d \tag{2.39}$$

$$\Rightarrow \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x), \tag{2.40}$$

so scalar fields are necessarily primary.

2.2.2 Stress-energy and conformal fields

Consider an infinitesimal conformal transformation $x^{\mu} \to x^{\mu} + \epsilon^{\mu}$. Under this transformation, we can look at the variation of the action using the definition of the Stress-Energy tensor

$$\delta \mathcal{S} = \frac{1}{2} \int d^d x T^{\mu\nu} \delta g_{\mu\nu}. \tag{2.41}$$

Using the conformal transformation of the metric, (2.19), gives

$$\delta \mathcal{S} = \frac{2}{d} \int d^d x T^{\mu}_{\mu} \partial_{\rho} \epsilon^{\rho}. \tag{2.42}$$

So if the stress-energy tensor has vanishing trace, then the theory is conformally invariant (at least classically). Of course, the converse is not necessarily true since we could have a transformation ϵ^{μ} with vanishing divergence. However, tracelessness of the stress-energy tensor turns out to be a useful smoking gun for conformal invariance.

We might ask what kind of field theories can be conformally invariant. Actually, we will rephrase this as what kind of field theories can be scale invariant, since it is very unlikely that a theory is scale invariant but not conformally invariant [6]. Consider a massless scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_{\nu}\phi)(\partial^{\nu}\phi), \tag{2.43}$$

and the transformation

$$x'^{\mu} = e^{\alpha} x^{\mu} \tag{2.44}$$

$$\phi'(x') = \phi(x) e^{-\Delta\alpha} \tag{2.45}$$

$$= \phi(x) - (\partial_{\mu}\phi(x) x^{\mu} + \Delta\phi(x))\alpha + O(\alpha^2). \tag{2.46}$$

Then the change in the Lagrangian is

$$\delta\mathcal{L} = -(\Delta(\partial^\mu\phi)^2 + (\partial^\mu\phi)^2 + (\partial^\mu\phi)x^\nu\partial_\mu\partial_\nu\phi). \tag{2.47}$$

This can only be written as a total divergence if $\Delta = 1$, in which case,

$$\delta\mathcal{L} = \partial_\mu\left(\frac{1}{2}(\partial\phi)^2 x^\mu\right). \tag{2.48}$$

Thus conformal invariance restricts Δ to be 1. The associated conserved current is

$$\begin{aligned} j^\mu(x) &= \phi\partial^\mu\phi + \partial^\mu\phi\partial_\nu\phi x^\nu + \mathcal{L}x^\mu & (2.49)\\ &\equiv \tilde{T}^\mu_\nu x^\nu, & (2.50)\end{aligned}$$

where $\tilde{T}^{\mu\nu}$ is a modified stress-energy tensor that preserves its properties.

Interestingly, if we worked out the change in the Lagranigan due a mass term $m^2\phi^2$, we would get

$$\delta\mathcal{L}_{m\phi^2} = \Delta m^2\phi^2 + \Delta m^2 x^\mu\phi\partial_\mu\phi, \tag{2.51}$$

which can only be written as a total divergence for $\Delta = 4$. Thus conformal field theories cannot include massive fields. This fits with our intuition that adding a mass introduces a preferred energy scale that breaks scale invariance. We could, however, make the scalar field interacting by including a ϕ^4 term without destroying scale invariance. Furthermore, we can include invariants built out of the space-time coordinates. Poincarè invariance requires that these be composed of separation vectors $|x_i - x_j|$. Scale invariance requires ratios of these $|x_i - x_j|/|x_k - x_l|$, and special conformal invariance upgrades these to cross-ratios involving at least four points. $|x_i - x_j||x_k - x_l|/(|x_i - x_k||x_j - x_l|)$.

Lastly, we should note that conformal symmetry can be (and often is) broken upon quantization of the field theory. Through the so-called *operator product expansion* [7], one can write products of operators at different points as (e.g. the stress-energy tensor):

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \tag{2.52}$$

The number c is the *conformal anomaly* and is set by the model and fields that are present. It tells you how the theory reacts to conformal symmetry breaking. If you introduce finite length scales via boundary conditions (e.g. placing the theory on a cylinder), then the free energy, vacuum expectation values, and correlation functions change according to c.

Conformal field theories are very different from other QFTs. For one thing, two-point functions are completely determined in terms of their scaling dimensions by conformal symmetry. For another, the S-matrix cannot be defined. Scale invariance prevents us from distinguishing between infinity and the interaction point, so asymptotic states cannot be well defined. We mentioned there is also an issue with using asymptotic states in AdS, and that is far from the extent of the marriage of these two subjects. Indeed, we can already see that the conformal group in $3+1$ dimensions is precisely the isometry group of AdS_5. This is the basis of what allows us to map degrees of freedom across the AdS/CFT correspondence.

2.3 Supersymmetry

The particular field theories used in AdS/CFT are supersymmetric. Supersymmetry (SUSY) exchanges bosons and fermions. For example, if we start with a simple kinetic Lagrangian for a scalar ϕ and a left-chiral Weyl spinor χ, we might guess that the supersymmetry transformation is $\phi \to \phi + \xi \cdot \chi$, where the parameter ξ must be Weyl spinor for Lorentz invariance. Then, dimensionality and invariance of the action determine the linear transformation of the fermionic field, $\chi \to \chi + (\partial_\mu \phi)\sigma^\mu \sigma^2 \xi^*$, where σ^μ are Pauli matrices.

Thus the SUSY transformations are parameterized by a two-component spinor ξ and its complex conjugate, giving a total of four supercharges denoted by the two left-chiral Weyl spinors Q, $Q^\dagger$. From the corresponding generator, one can compute the SUSY algebra [8]

$$\begin{aligned} \{Q_a, Q_b\} &= 0, \quad \{Q_a^\dagger, Q_b^\dagger\} = 0, \\ \{Q_a, Q_b^\dagger\} &= (\sigma^\mu)_{ab} P_\mu. \end{aligned} \tag{2.53}$$

This algebra works when the operators are applied to scalars, but for Weyl spinors, the algebra requires the Weyl equation in order to close. In other words, the SUSY algebra does not close off-shell. This is inconvenient, and can be worked around by adding an auxiliary field F that vanishes on-shell. The simple kinetic Lagrangian then becomes

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^\dagger + \chi^\dagger i \bar{\sigma}^\mu \xi + F F^\dagger. \tag{2.54}$$

The linear transformation of F is set by SUSY and Lorentz invariance to be $F \to F - i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi$, and the transformation of χ must be modified to $\chi \to \chi + (\partial_\mu \phi)\sigma^\mu \sigma^2 \xi^* + F\xi$.

How do these charges act on states? Recall that in the Poincarè group, there are operators that commute with all generators, called *Casimir Operators*. These are P_μ, and $W^\mu \equiv 1/2 \epsilon^{\mu\nu\rho\sigma} M_{\rho\sigma} P_\nu$. W^μ shares simultaneous momentum eigenstates with P^μ, and it's eigenvalue is the helicity quantum number h which is fixed for a given (massless) field (e.g. $h = \pm 1$ for a photon and $h = \pm 2$ for a graviton). If we include supercharges, then we find that

$$[Q_a, W^0] = -\frac{1}{2}(\sigma^3)_a^b Q_b P_3. \tag{2.55}$$

In which case,

$$\begin{aligned} W_0(Q_1|p,h\rangle) &= E(h+1/2)Q_1|p,h\rangle &(2.56)\\ \Rightarrow Q_1|p,h\rangle &= |p,h+1/2\rangle. &(2.57)\end{aligned}$$

So supercharges shift the helicity by 1/2, which we would expect since they are responsible for exchanging fermions with bosons.

The irreducible representations of the SUSY algebra are *supermultiplets*. From the algebra (2.53), we see that there must be a state of minimal helicity since $(Q_2)^2 = 0$. Then $Q_2^\dagger|p, h_{\min}\rangle = |p, h_{\min}+1/2\rangle$. But this must be the maximum helicity since $(Q_2^\dagger)^2 = 0$. Including CPT conjugates, we have the following possible states for massless fields

$h_{\min}$	Helicity States	Name
0	$\|0\rangle,\ \|0\rangle,\ \|1/2\rangle,\ \|-1/2\rangle$	Chiral multiplet
1/2	$\|1/2\rangle,\ \|1\rangle,\ \|-1/2\rangle,\ \|-1\rangle$	Vector/ gauge multiplet
1	$\|1\rangle, \|3/2\rangle,\ \|-1\rangle,\ \|-3/2\rangle$	Rarita-Schwinger fields (for $\pm 3/2$ states)
3/2	$\|3/2\rangle,\ \|2\rangle,\ \|-3/2\rangle,\ \|-2\rangle$	Graviton (for ± 2) and Gravitino (for $\pm 3/2$)

TABLE 2.1: $\mathcal{N} = 1$ supersymmetry states.

So far, we have only considered one set of supercharges Q_1, Q_2, $Q_1^\dagger$, $Q_2^\dagger$, but in principle there could be $\mathcal{N}$ such sets with the charges indexed as Q^I, $I = 1, 2, \ldots, \mathcal{N}$. The most general SUSY algebra is then

$$\begin{aligned} \{Q_a^I, Q_b^J\} &= (-i\sigma^2)_{ab} Z^{IJ} &(2.58)\\ \{Q_a^I, Q_b^{J\dagger}\} &= (\sigma^m u)_{ab}\delta^{IJ} P_\mu &(2.59)\\ \{Q_a^{I\dagger}, Q_b^{J\dagger}\} &= (-i\sigma^2)_{ab}(Z^{IJ})^*, &(2.60)\end{aligned}$$

where Z^{IJ} is called the *central charge*. It is anti-symmetric in I and J, and thus vanishes for $\mathcal{N} = 1$. For $\mathcal{N} > 1$, the supercharges may be transformed into one another via an $SU(\mathcal{N})$ rotation [9]. This symmetry is called *R-symmetry*.

Although supergravity theories will be important in this thesis, on the CFT side of the correspondence, we will restrict ourselves to field theories without gravity. That means, we cannot have states with helicity 2 (more discussion on this in 2.5.1). In fact, we will focus on multiplets with maximum helicity equal to one. It turns out that all multiplets of an $\mathcal{N} > 4$ theory contain helicities higher than one, so $\mathcal{N} = 4$ will be our maximal supersymmetry. We can then extend our table of massless multiplets as follows:

States	Multiplet Name	Number of Helicity States
$2\|0\rangle,\ \|1/2\rangle,\ \|-1/2\rangle$	$\mathcal{N}=1$ Chiral multiplet	4
$\|1/2\rangle,\ \|1\rangle,\ \|-1/2\rangle,\ \|-1\rangle$	$\mathcal{N}=1$ Gauge multiplet	4
$4\|0\rangle,\ 2\|1/2\rangle,\ 2\|-1/2\rangle$	$\mathcal{N}=2$ Hypermultiplet	8
$\|1\rangle,\ 2\|1/2\rangle,\ 2\|0\rangle,\ 2\|-1/2\rangle,\ \|-1\rangle$	$\mathcal{N}=2$ Gauge multiplet	8
$\|1\rangle,\ 4\|1/2\rangle,\ 6\|0\rangle,\ 4\|-1/2\rangle,\ \|-1\rangle$	$\mathcal{N}=3$ Gauge multiplet	16
$\|1\rangle,\ 4\|1/2\rangle,\ 6\|0\rangle,\ 4\|-1/2\rangle,\ \|-1\rangle$	$\mathcal{N}=4$ Gauge multiplet	16

Table 2.2: Extended supersymmetry states. The coefficient in front of each state is the number of those states in each multiplet.

Of particular importance to us will be the $\mathcal{N} = 2$ hypermultiplet and the $\mathcal{N} = 4$ gauge multiplet. For the $\mathcal{N} = 2$ hypermultiplet, we can combine the $|\pm 1/2\rangle$ states into two Weyl fermions, and the four $|0\rangle$ states into two complex scalars, with an overall $SU(2)$ R-symmetry. For the $\mathcal{N} = 4$ gauge multiplet, we have four Weyl fermions λ^i_a, $i = 1, \ldots, 4$, six real scalars X^j, $j = 1, \ldots, 6$ and a single gauge field A^μ, with an overall $SU(4)$ R-symmetry.

Things look a little different for massive representations. In that case, one can simplify the algebra by redefining Q^I in terms of linear combinations of supercharges to get

$$\{Q^{\bar{I}}_{a\pm}, Q^{\bar{J}\dagger}_{b\pm}\} = \delta^{\bar{I}}_{\bar{J}}\delta^b_a(M \pm Z^{\bar{I}\bar{J}}). \tag{2.61}$$

From this, one can see that unitarity enforces the so-called *BPS bound*

$$M \geq |Z^{\bar{I}\bar{J}}|. \tag{2.62}$$

Whenever this bound is saturated, the corresponding multiplet is shortened; the representation has its dimension decreased by a factor of 1/2 and is called a 1/2 *BPS state*.

From the above fields, we formulate Lagrangians that are invariant under supersymmetry. One particularly special Lagrangian is the $\mathcal{N} = 4$ Super Yang-Mills theory, which is a supersymmetric extension of a regular Yang-Mills theory to include the fields of the $\mathcal{N} = 4$ gauge multiplet. The Lagrangian is [10]

$$\mathcal{L} = \mathrm{Tr}\bigg(-\frac{1}{4g^2_{\mathrm{YM}}}F_{\mu\nu}\tilde{F}^{\mu\nu} - i\sum_i \bar{\lambda}^i\bar{\sigma}^\mu D_\mu\lambda_i - \sum_j D_\mu X^j D^\mu X^j + \text{non abelian terms}\bigg), \tag{2.63}$$

where D_μ is a gauge covariant derivative, and the non-abelian terms are composed of products of field commutators with $SU(4)$ R-symmetry (sometimes denoted $SU(4)_R$) matrices. The couplings g and θ_I have mass dimension zero, which makes this theory classically scale invariant. That makes this Lagrangian very special indeed, because it is has *superconformal* symmetry. In $3+1$ dimensions, this is $SO(4,2) \times SU(4)$. Incredibly, this additional conformal symmetry gets passed on to the quantum theory, where it is found that the beta function vanishes identically i.e. the coupling does not run with scale. It is an excellent example of a CFT.

2.4 Classical string theory

Take two tennis balls, tie them together with a string of length l, and spin them around. The tension acting on each ball is a constant, centripetal force:

$$T = \frac{pv}{l/2} \tag{2.64}$$

The balls have a combined angular momentum of

$$J = \frac{2p^2 v}{T}. \tag{2.65}$$

Now lets make these balls ultra-relativistic, so that $v \approx c$, and $p^2 = M^2$. We get

$$J \propto M^2. \tag{2.66}$$

With this little thought experiment in mind, it's no surprise that in the early 1960's, when exactly this relationship turned up for the spectrum of mesons, particle physicists contemplated the possibility that the quark anti-quark pair were bound together with some kind of string. Rapid progress was made in understanding the dynamics of this string, but the model eventually bowed out to the mathematically elegant formulation of quantum chromodynamics, in which the "string" that tied the quark pair together was seen to be nothing but a flux tube of gluons. For several years string theory lay dormant, and when it returned, it was under the new guise of a quantum theory of gravity. Ironically, string theory's connection to strongly coupled theories like QCD would turn out to be one of it's most powerful features. However, the real connection lies in the deep duality of the AdS/CFT correspondence.

2.4.1 The bosonic string

In any case, let us examine the dynamics such a one dimensional string. It's trajectory through space-time will fill out a two dimensional world sheet parametrized by coordinates σ, τ. For an object in Minkowski space we want to extremize an action proportional to the length of the object's world line. So for a string, it's natural that we use an action proportional to the area of the world sheet. Indeed, if we give this string a tension

$$T = \frac{1}{2\pi\alpha'},^{2} \tag{2.67}$$

and a length

$$l_s = \sqrt{\alpha'}, \tag{2.68}$$

then the dynamics should minimize the world-sheet area just as water with surface tension minimizes its surface area. This gives us the Nambu-Goto action

$$S = -T \int d^2\sigma \sqrt{-\det \gamma}, \tag{2.69}$$

where γ is the pullback of the metric $\eta_{\mu\nu}$ in D-dimensional space-time to the world sheet $\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}$ for embedding coordinates X^μ. This action is enticingly simple, containing only one free parameter (not bad for a potential theory of everything!), but we could write it in a computationally wiser way if we recognize that the D embedding coordinates are just scalars living on the world-sheet. So the action should just be the action for D scalar fields:

$$S = \frac{-1}{4\pi\alpha'} \int d^2\sigma \sqrt{-|g|} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{2.70}$$

This is the Polyakov action, and one can check that it gives the same equations of motion as Nambu-Goto. The difference is that now, the metric on the world-sheet $g_{\alpha\beta}$ has it's own dynamics and an equation of motion

$$T_{\alpha\beta} \equiv \frac{1}{\sqrt{-|g|}} \frac{\delta S}{\delta g^{\alpha\beta}} = 0 \tag{2.71}$$

$$\Rightarrow \begin{cases} \dot{X} \cdot X' = 0 \\ \dot{X}^2 + X'^2 = 0, \end{cases} \tag{2.72}$$

where dots are differentiation with respect to τ and primes are differentiation with respect to σ. These can be considered as constraints on the embedding equations of motion. The first one simply says that the motion of the string must be orthogonal to σ, that is, the string supports only transverse oscillations. Note that the Polyakov action has reparameterization invariance; we have a gauge freedom in how we write the world-sheet coordinates. In particular, the light-cone gauge, $\sigma^\pm = \tau \pm \sigma$, is convenient because it renders a simple equation of motion for X^μ,

$$\partial_+ \partial_- X^\mu = 0, \tag{2.73}$$

[2] α' is the Regge slope, the slope of the J vs M^2 plot you would find in the tennis ball experiment

and simple constraints (2.72)

$$(\partial_+ X)^2 = 0 = (\partial_- X)^2, \tag{2.74}$$

where $\partial_\pm \equiv \partial_{\sigma^\pm}$.

Moreover, the general solution is a sum of independent σ^+ and σ^- functions (left and right moving waves on the string).

If we consider a closed string satisfying $X^\mu(\sigma,\tau) = X^\mu(\sigma + 2\pi, \tau)$, then the solution is just $X_L^\mu + X_R^\mu$ with

$$\begin{aligned} X_L^\mu(\sigma^+) &= 1/2x^\mu + 1/2\alpha' p^\mu \sigma^+ + i\sqrt{\alpha'/2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^+} \\ X_R^\mu(\sigma^-) &= 1/2x^\mu + 1/2\alpha' p^\mu \sigma^- + i\sqrt{\alpha'/2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^-}. \end{aligned} \tag{2.75}$$

The first term is just the centre of mass position of the string. The second term is the centre of mass momentum which we could define to be the zero mode via $1/2\alpha' p^\mu = \sqrt{\alpha'/2}\alpha_0^\mu$. The remaining sum is just a Fourier sum of oscillation modes. The constraint equations then give a level matching condition $L_n = \tilde{L}_n = 0$ where

$$L_n = 1/2\sum_m \alpha_{n-m}\cdot\alpha_m, \tag{2.76}$$

$$\tilde{L}_n = 1/2\sum_m \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m. \tag{2.77}$$

The $n = 0$ constraint equations (2.74) provides the mass spectrum

$$\frac{\alpha'}{2}M^2 = \sum_{m>0}\alpha_{-m}\cdot\alpha_m. \tag{2.78}$$

If instead we consider an open string, then there are two natural choices of boundary conditions.

$$x'^\mu(\tau,0) = x'^\mu(\tau,\pi) \quad = \quad 0 \tag{2.79}$$

$$\dot{x}^\mu(\tau,0) = \dot{x}^\mu(\tau,\pi) \quad = \quad 0 \tag{2.80}$$

The first is a Neumann condition, and the second is a Dirichlet condition. The Dirichlet condition will be very important to us. Choosing it for any direction means that the string endpoints are at a fixed location in that direction. If we use Dirichlet conditions for the directions x^i, $i = 1, 2, \ldots, p$, then this means that the open string is fixed to a p-dimensional hyper-surface. We can similarly find a mode expansion for the open string spectrum analogous to (2.75).

2.5 Quantum string theory

2.5.1 A quantum theory of gravity

To quantize the theory, we can compute the conjugate momenta to the embedding coordinates X^μ and impose the equal-time Poisson bracket relations. Applying the canonical quantization to the Fourier expansion for these coordinates (2.75) gives the following commutation relations:

$$[x_0^\mu, p_0^\nu] = i\eta^{\mu\nu} \tag{2.81}$$

$$[x_0^\mu, x_0^\nu] = [p_0^\mu, p_0^\nu] = [\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0 \tag{2.82}$$

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}. \tag{2.83}$$

The α's define raising and lowering operators as to be expected since we are just dealing with free quantum fields. As usual, we can build up the Fock space by demanding that α_m^μ annihilates the vacuum for $m > 0$. The full Hilbert space should also be spanned by the momentum eigenstates so that

$$p_0^\mu|k;0\rangle = k^\mu|k;0\rangle \tag{2.84}$$

$$\alpha_m^\mu|k;0\rangle = 0. \tag{2.85}$$

So far we just have an infinite set of harmonic oscillators.

Focusing now on the open strings, the Hamiltonian can be calculated from the stress-energy tensor to give

$$H = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n = L_0. \tag{2.86}$$

When we promote this to a quantum operator, we have to worry about normal ordering which induces an extra term in the Hamiltonian,

$$\epsilon_0 \equiv \frac{d-2}{2}\sum_{n=1}^{\infty} n. \tag{2.87}$$

This is interpreted as the zero-point energy for open strings. Of course this vacuum energy is infinite, but can be renormalized using the analytic continuation of the Riemann-Zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{2.88}$$

$$\rightarrow \zeta(-1) = \frac{-1}{12}. \tag{2.89}$$

Then the renormalized zero-point energy becomes

$$\epsilon_0 = -\frac{d-2}{24}. \tag{2.90}$$

Now the level matching conditions from the previous section present us with a situation similar to that in QED where one imposes a gauge fixing condition to restrict the photon states to physical ones. This is the Gupta-Bleuler prescription, and it proceeds the same way for string theory. For a physical state $|\psi\rangle$, we must have $(L_0 + \epsilon_0)|\psi\rangle = 0$, which gives the quantum mass spectrum

$$m^2 = \frac{1}{\alpha'}(N + \epsilon_0), \tag{2.91}$$

where N is analogous to the harmonic oscillator number operator $N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n$.

Equation (2.91) is peculiar because the ground state, $N = 0$, has $m^2 = \frac{\epsilon_0}{\alpha'} < 0$. A negative mass-squared indicates a Tachyon, but it simple means that we are probing around a maximum of the potential energy, so this vacuum appears to be unstable. Fortunately, this mode is quite naturally removed by introducing supersymmetry. However, there is an even worse problem with the first excited states, $N = 1$. The mass-shell condition $m^2 = \frac{1}{\alpha'}(1 + \epsilon_0)$ yields massive states of negative norm (giving negative probabilities). However, the massless states are perfectly fine. Thus in order to circumvent these negative norms, we are forced to require these states be massless. Thus,

$$(1 + \epsilon_0) = 0 \tag{2.92}$$

$$\Rightarrow d = 26. \tag{2.93}$$

This is the critical number of dimensions for bosonic string theory. These first excited states describe polarizations of a massless spin 1 vector with U(1) gauge symmetry i.e. a photon field [11].

Turning now to the closed string, we find the mass spectrum

$$m^2 = \frac{4}{\alpha'}(N - 1), \tag{2.94}$$

which also includes a Tachyon mode for $N = 0$. For $N = 1$, $m^2 = 0$, so the first excited level is a massless mode. Like the photon, it has $d - 2 = 24$ polarizations but now must be a tensor product of left and right-moving states

$$\zeta_{\mu\nu}(\alpha^{\mu}_{-1}|k;0\rangle \otimes \tilde{\alpha}^{\nu}_{-1}|k;0\rangle), \tag{2.95}$$

where $\zeta_{\mu\nu}$ is the polarization tensor. Each of the states in the tensor product transform as a vector in SO(24). The polarization tensor can then be decomposed into the irreducible representations of SO(24): a symmetric rank 2 tensor, an anti-symmetric rank 2 tensor, and a trace

$$\zeta_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu} + \phi\eta_{\mu\nu}. \tag{2.96}$$

The anti-symmetric field $B_{\mu\nu}$ is called the *Kalb-Ramond* field. The scalar ϕ is the *dilaton* field that turns out to dynamically generate the string coupling g_s [12]. $g_{\mu\nu}$ is a massless symmetric spin-2 field. This must be a graviton field, which in a coherent state, forms the space-time metric.[3]

In hindsight, we could have expected that the closed string should contain a graviton since there is a mode of oscillation which is exactly the same as the oscillation of space-time points during the passing of a gravitational wave, as shown in figure 2.3.

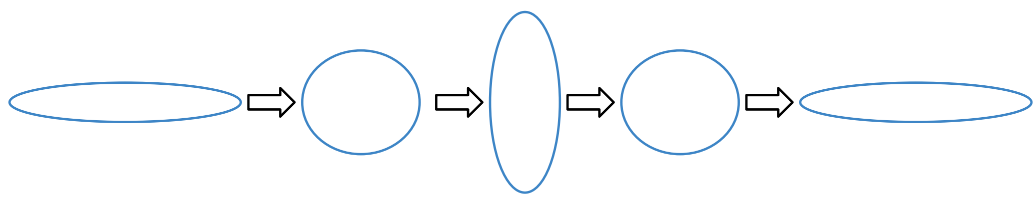

FIGURE 2.3: A graviton mode in the closed string spectrum.

Note that while the first excited states produced a gauge theory for the open string, they produced a gravitational theory for the closed string.

[3]How do we know this field corresponds to the graviton? Suppose we were to set out searching for the characteristics of a graviton candidate. We could determine its spin by process of elimination [13]. The graviton cannot be spin-0 since we cannot couple a scalar to a photon and so light would not bend around massive objects. It cannot be spin-1/2 or in fact any half-integer spin, since then the emission of a graviton would alter the internal angular momentum of the gravitating particle, and so could not produce a static force. Odd integers are eliminated also since they create potentials that are repulsive for like charges and attractive for unlike charges. Spin-2, however, produces a universally static potential to fit the bill. To make the force long range, the field must be massless. Furthermore, we have to eliminate the antisymmetric part which would appear in the Lagrangian as a copy of a Maxwell field. So our best candidate for the graviton is a massless symmetric spin-2 field. But in fact, the converse of this is also true [14]. Any massless interacting spin-2 field must have diffeomorphism invariance as a gauge symmetry. And so the $g_{\mu\nu}$ field we found above is in fact the graviton field!

2.5.2 Perturbation theory

We could carry on and work out Feynman diagrams and perturbation theory for the bosonic string, but we will be more interested in superstring theory and in particular, its non-perturbative elements. However, there are a few important things to mention about perturbative string theory. First, we use an operator formalism to replace excited states with *vertex operators*. This is done by mapping the closed string world-sheet cylinder to the complex plane, and the open string world sheet strip to the upper half-plane. Then incoming states are mapped to vertex operators at the origin, and creation operators α^{μ}_{-n} are given by residues of the vertex operator $\partial_z^n x^{\mu}(z)$.

The success of string perturbation theory, at least at the few loop level, lies in the fact that it does away with the UV divergences of quantum field theory that arise when the proper time of the internal virtual particle goes to zero, i.e. when an internal loops shrink to a point. For example, the one loop vacuum diagram is free to contract to a point in QFT, but the corresponding torus of the closed string diagram has a circumference bounded below by the size of the string. The path integral for each Feynman diagram reduces to an integral over the moduli space of that diagram's topology. The moduli space is a space of parameters defining the Riemann surface of the world-sheet modulo coordinate transformations and conformal transformations. In Euclidean signature, it is finite-dimensional [15].

2.5.3 Chan-Paton factors

In the next section we will introduce fermions on the world-sheet, but there is still something missing from our bosonic theory. Ultimately, we hope that string theory reproduces the standard model, including QCD. To have any chance of describing QCD, we need to include non-abelian fields. The $U(1)$ gauge field in the open string spectrum is not enough. We would like to upgrade this to a $U(N)$ gauge theory. Recall that open strings with Dirichlet boundary conditions have endpoints that are fixed on hyperplanes in the target space. Suppose there are N such hyperplanes. Since any string can end on any one of N possible hyperplanes, we need to index each string endpoint by assigning it a state i taking on N possible values. Then a general open string state $|k;a\rangle$ can be decomposed in a basis of $N \times N$ matrices λ called Chan-Paton factors. Our path integral will then include sums over all possible arrangements of the endpoints, which can easily be checked to be invariant under $U(N)$ rotations of λ

$$\lambda \to U\lambda U^{-1}. \tag{2.97}$$

Indeed, we will find that these factors describe part of the gauge symmetry of D branes.

2.5.4 The superstring spectrum

Clearly, we need to introduce fermionic fields if we want to reproduce the standard model particles. To do this, we a add Majorana spinors ψ^μ to the world-sheet[4]. Adding a kinetic Dirac term to the Polyakov action gives:

$$S = \frac{-1}{4\pi\alpha'}\int d^2\sigma(\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu), \tag{2.98}$$

where ρ^a is a 2×2 Dirac matrix. If we write the two component Majorana spinor as $\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$, then the constraints (2.72) become:

$$(\partial_\pm X)^2 + \frac{i}{2}\psi_\pm \cdot \partial_\pm \psi_\pm = 0. \tag{2.99}$$

In addition, there is a constraint on the conserved current J_μ associated with the supersymmetry of the action:

$$J_\pm \equiv \psi_\pm \cdot \partial_\pm X = 0. \tag{2.100}$$

The equations of motion for $\psi_\pm$ yield two consistent fermionic boundary conditions for the open string:

$$\psi_+^\mu(\tau,\pi) = \psi_-^\mu(\tau,\pi) \quad \rightarrow \text{Ramond Sector} \tag{2.101}$$

$$\psi_+^\mu(\tau,\pi) = -\psi_-^\mu(\tau,\pi) \quad \rightarrow \text{Neveu} - \text{Schwartz Sector}. \tag{2.102}$$

The (anti)periodic condition lets us write $\psi_\pm$ as the Fourier series

$$\psi_\pm(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_r \psi_r^\mu e^{-ir(\tau\pm\sigma)}, \tag{2.103}$$

where r is an integer for the Ramond (R) sector, and a half-integer for the Neveu-Schwartz (NS) sector.

For the closed string, we have to choose boundary conditions for the left and right moving modes individually, giving a total of four possible sectors of the theory: NS-NS,

[4]Recall that Majorana spinors are their own charge conjugate [16]

R-R, NS-R, R-NS. The Fourier expansion is

$$\psi^\mu_+(\tau,\sigma) = \sum_r \tilde{\psi^\mu_r} e^{-2ir(\tau+\sigma)} \tag{2.104}$$

$$\psi^\mu_-(\tau,\sigma) = \sum_r \psi^\mu_r e^{-2ir(\tau-\sigma)}. \tag{2.105}$$

Requiring non-negative norms now results in a critical dimension of $d = 10$, and a zero-point energy of:

$$\epsilon_0 = 0 \text{ (R)} \tag{2.106}$$

$$\epsilon_0 = -\frac{1}{2} \text{ (NS)}. \tag{2.107}$$

The mass-shell constraint is the same as for the bosonic case, only now the number operator counts fermionic modes as well [5].

Let us look at the open string spectrum first. The ground level is still tachyonic. The first excited level in the NS sector is obtained by acting on the ground state with one creation operator with the smallest value of r, $r = 1/2$, which gives $\psi^\mu_{-1/2}|k;0\rangle_{NS}$. As before, the first excited levels are massless, and it turns out this level is actually a space-time boson, one that transforms as a vector under $SO(8)$. We denote its representation by $\mathbf{8_v}$. We have simply rediscovered the transverse polarizations of the original gauge field A^μ, only now in ten space-time dimensions. In the R sector, the first excited level is $\psi^\mu_0|k;0\rangle_R$. The ψ^μ_0 vectors satisfy (up to an overall coefficient) the Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}, \tag{2.110}$$

and so give us spinors transforming in the representation $\mathbf{8_s}$ of $Spin(8)$, i.e. massless space-time fermions.

To produce the closed string spectrum, we take the tensor product of the above representations in each possible combination of left and right-moving modes, and then decompose

[5] The mode operator is now

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_{n-m}\cdot\alpha_m + \frac{1}{4}\sum_r (2r-n)\psi_{n-r}\cdot\psi_r. \tag{2.108}$$

and the number operator is

$$N = \sum_{n=1}^{\infty} \alpha_{-m}\cdot\alpha_n + \sum_{r>0} r\psi_{-r}\cdot\psi_r. \tag{2.109}$$

into the irreducible representations.

$$\begin{aligned}
\mathrm{NS-NS} \quad &: \quad \mathbf{8_v} \otimes \mathbf{8_v} \;\rightarrow\; g_{\mu\nu},\ B_{\mu\nu},\ \phi && (2.111)\\
\mathrm{NS-R} \quad &: \quad \mathbf{8_v} \otimes \mathbf{8_s} \;\rightarrow\; \psi_\mu,\ \lambda && (2.112)\\
\mathrm{R-NS} \quad &: \quad \mathbf{8_s} \otimes \mathbf{8_v} \;\rightarrow\; \psi'_\mu,\ \lambda' && (2.113)\\
\mathrm{R-R} \quad &: \quad \mathbf{8_s} \otimes \mathbf{8_s} \;\rightarrow\; \mathrm{R-R\ forms.} && (2.114)
\end{aligned}$$

The NS-NS sector reproduces a ten-dimensional version of the bosonic fields we saw before. The ψ_μ, and ψ'_μ are spin-3/2 fields called *gravitinos*, and the λ, λ' are spin-1/2 *dilatinos*. The R-R forms will be discussed shortly.

2.5.5 The GSO projection

We still have that annoying Tachyon hanging on in the ground state of the theory. To get rid of it, we define a projection operator

$$P_{\mathrm{GSO}} = \frac{1}{2}(1 - (-1)^F). \tag{2.115}$$

In the NS sector we define

$$(-1)^F \equiv (-1)^{\sum_{r>0} \psi_{-r}\cdot\psi_r}, \tag{2.116}$$

while in the R sector,

$$(-1)^F \equiv \pm\Gamma^0\Gamma^1 \ldots \Gamma^9 \cdot (-1)^{\sum_{r\geq 1} \psi_{-r}\cdot\psi_r}. \tag{2.117}$$

The higher dimensional gamma matrices are discussed in 3.1.2. The important point is that the operator $(-1)^F$, known as the *Klein operator*, has eigenvalue $+1$ for bosonic states and -1 for fermionic states (on the world-sheet). Each fermionic oscillator contributes one power of (-1), so even numbers of ψ^μ produce world-sheet bosons, and odd numbers produce world-sheet fermions [12]. Applying the projection operator P_{GSO} to all states removes states with an even number of ψ^μ excitations. Then in the NS sector, the Tachyon mode is removed, and as a bonus, the new ground state has an equal number of bosonic and fermionic degrees of freedom. Indeed, it turns out the full interacting theory is supersymmetric after this projection.

Chapter 3

D Branes

The purpose of this chapter is to illustrate the need for non-perturbative objects, D branes in the string theories we have just developed. Then, we will explain how to treat these things as dynamical objects in their own right.

3.1 The existence of D branes

3.1.1 T-duality

Superstring theory has left us with ten space-time dimensions to deal with. Reconciling these with the four dimensions we observe requires compactifying the remaining six. Compactification comes with its own uniquely stringy consequences. Suppose we compactify one dimension x^9 on a circle of radius R. Then the corresponding momenta must be quantized

$$p_0^9 = \frac{n}{R}, \quad n \in \mathbf{Z}. \tag{3.1}$$

A closed string will have a second quantum number w indicating the number of times the string winds around the compact dimension. This number, the *winding number*, has no QFT analog. The oscillator zero-modes then have a component in the x^9 direction given by

$$\alpha_0^9 = \sqrt{\frac{\alpha'}{2}}, \left(\frac{n}{R} + \frac{wR}{\alpha'}\right), \quad \tilde{\alpha}_0^9 = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right). \tag{3.2}$$

We expect the string mass to increase with both momenta and winding number. If the circle becomes very large, so that the curvature in x^9 approaches that of flat space, then the theory should become just like an unbounded QFT. The momentum states will approach a continuum of plane wave modes, and the winding states will cost too much energy to excite. In the opposite limit $R \to 0$, only very high frequency modes can satisfy periodicity along x^9, which cost too much energy. However, the winding modes are separated by less and less energy, forming a continuum. Explicitly, the spectrum for general R is now given by

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2), \tag{3.3}$$

and the level-matching condition is modified to

$$nw + N - \tilde{N} = 0. \tag{3.4}$$

The key insight found in equation (3.3) is that the tower of winding modes in the $R \to 0$ limit looks just like the tower of momentum modes in the $R \to \infty$ limit. This correspondence is an exact duality. The spectrum is invariant under the exchanges

$$R \leftrightarrow \frac{\alpha'}{R}, \quad n \leftrightarrow w, \tag{3.5}$$

which can be implemented by a space-time parity transformation on just the right-movers.

$$x^9_R(\tau - \sigma) \to -x^9_R(\tau - \sigma). \tag{3.6}$$

Because of this duality, closed strings do not "feel" circles of radius $R < \sqrt{\alpha'}$. Such circles are mapped onto large circles by the above duality known as *T-Duality*. Compactified geometries with regions smaller than $\sqrt{\alpha'}$ look very different to strings.

Mathematically, open curves have no winding number, so for open strings, no new tower of winding modes arises upon campactification of x^9. However, the interior of the open string is identical to the interior of a closed string, and so even in the limit $R \to 0$, the interior of the string is free to vibrate in the full ten dimensions. The endpoints, however, must be constrained to a nine-dimensional hyper-surface, a *D brane* [17]. Put another way, the limit $R \to 0$ causes the spectrum to appear as if another dimension is opening up [18].

If we impose Neumann boundary conditions on the string endpoints, then

$$\begin{aligned} \partial_\sigma x^9(\tau,\sigma)|_{\sigma=0,\pi} &= 0 & (3.7)\\ \Rightarrow [\partial_+ x^9_L - \partial_- x^9_R]_{\sigma=0,\pi} &= 0, & (3.8) \end{aligned}$$

since $\partial_+ x_R = 0 = \partial_- x_L$. After applying the T-dual transformation (3.6), this becomes

$$\begin{aligned} [\partial_+ x^9_L + \partial_- x^9_R]_{\sigma=0,\pi} &= 0 & (3.9)\\ \Rightarrow \partial_\tau x^9|_{\sigma=0,\pi} &= 0, & (3.10) \end{aligned}$$

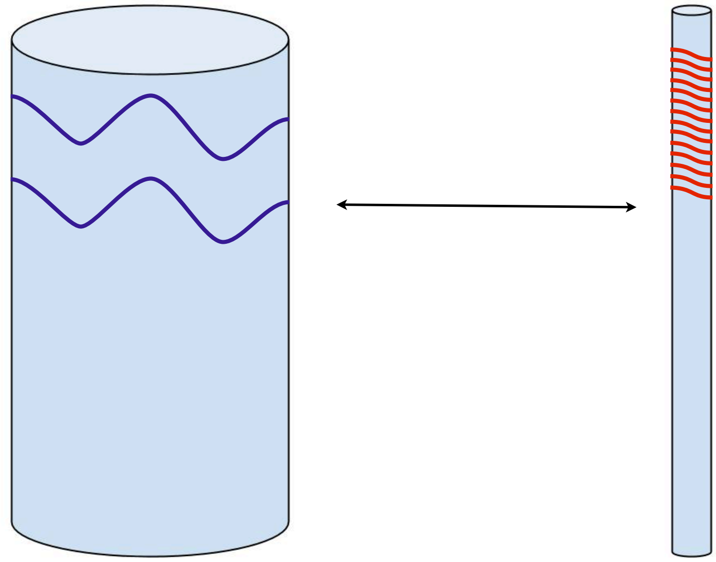

FIGURE 3.1: An example of T-duality. Large momentum (purple) strings with small winding number on a large cylinder are mapped to small momentum (red) strings with large winding number on a small cylinder.

which is a Dirichlet boundary condition that fixes the endpoints along one spatial direction. So T-duality interchanges Neumann with Dirichlet conditions on the string endpoints. Even if you start with only Neumann conditions, T-duality makes fixed open string endpoints truly unavoidable. Of course, we can "T-dualize" in more than one dimension, say $10-p$ dimensions, forcing Dirichlet conditions in p dimensions which form a p-dimensional hyper-surface, a Dp-brane.

Note that the transformation (3.6) must apply to fermions ψ as well since the worldsheet is supersymmetric, so that

$$\psi^9_-(\tau-\sigma) \to -\psi^9_-(\tau-\sigma) \tag{3.11}$$

under T-duality. Similarly, the Dirac matrix Γ^9 in the R-sector changes sign under T-duality, so that the whole Klein operator $(-1)^F$ changes sign.

3.1.2 R-R charges

What is the relevance of the sign of the Klein operator? Recall that there was a seemingly innocuous $\pm$ in the definition (2.117) for the Ramond sector. The sign of the Klein operator serves to define the space-time chirality of the fields. For the open string, this is just a matter of convention, but for the closed string, we are free to choose the sign for left and right-movers independently. The choice leads to two entirely different theories.

Type IIA: We choose opposite chiralities for left and right movers. These are exchanged under a parity operation, so the resulting theory is non-chiral.

Type IIB: We choose the same chirality for left and right-movers. This theory does not respect parity symmetry and so is chiral.

Both of these theories have $\mathcal{N} = 2$ space-time supersymmetry.

There are three additional superstring theories we can construct: *Type I*, which is a further projection of Type IIB, *Heterotic* $E_8 \times E_8$, and *Heterotic* $SO(32)$. The Heterotic theories mix bosonic string theory and superstring theory for the closed string [19]. However, we will restrict our attention to the Type II theories for this thesis.

Now we see that the sign change of the Klein operator under T-duality means that Type IIA and Type IIB string theories are swapped under T-duality.

We are now in a position to fill out 2.114, the R-R sector of the superstring spectrum. Recall that the mass-shell constraint reduced the degrees of freedom of our fields so that they transform in $Spin(8)$ which has irreducible representations $\mathbf{16}$ and $\mathbf{16'}$[1].

For Type IIA, we choose opposite chiralities for left and right-movers, giving us the representation $\mathbf{16} \otimes \mathbf{16'}$ which decomposes into irreducible representations that are even-rank differential forms (the Clifford algebra guarantees that the representations be antisymmetric). We can write the Weyl spinors in a basis of antisymmetric Dirac matrix products,

$$(\psi^L)_a (\psi^R)_{\bar{b}} = \sum_{n \text{ even}} F^{(n)}_{\mu_1 \ldots \mu_n} (\Gamma^{[\mu_1} \ldots \Gamma^{\mu_n]})_{a\bar{b}} \qquad \text{Type(IIA)} \tag{3.12}$$

$$(\psi^L)_a (\psi^R)_b = \sum_{n \text{ odd}} F^{(n)}_{\mu_1 \ldots \mu_n} (\Gamma^{[\mu_1} \ldots \Gamma^{\mu_n]})_{ab} \qquad \text{Type(IIB)}, \tag{3.13}$$

where the $F^{(n)}_{\mu_1 \ldots \mu_n}$ are n-forms. Now the $J_- = 0$ constraint (2.100) on the open string ground state reproduces the massless Dirac equation on the spinors

$$\alpha_0 \cdot \psi_0 |\psi^L\rangle \otimes |\psi^R\rangle = 0. \tag{3.14}$$

[1] We can build up the Dirac representations of the Clifford algebra 2.110 from tensor products of low dimensional representations (ie. regular four-dimensional Dirac matrices). In particular, each time we increase the space-time dimension by two, the size of the Dirac representation goes up by a factor of two. Thus in dimension $d = 4 + 2n$, the Dirac representation has dimension $4(2^n)$ (since in 4-d, Dirac spinors have four components). So for $d = 10$, $n = 3$, the Dirac representation is $\mathbf{32}_{\text{Dirac}}$, which can be decomposed into two independent irreducible representation Weyl spinors $\mathbf{16}$, $\mathbf{16'}$. When n is odd, each Weyl representation is self-conjugate and when n is even, they are conjugate to one another. $d = 10$ is special in that these Weyl spinors are also Majorana spinors [20]

This equation is more enlightening when written in terms of the “Field strength” $F^{(n)}_{\mu_1 \ldots \mu_n}$,

$$\partial_{[\mu} F^{(n)}_{\mu_1 \ldots \mu_n]} = 0. \tag{3.15}$$

This is just like the covariant form of the Maxwell equation of motion, i.e. the Bianchi identity, which defines the electromagnetic field strength in terms of a potential A^μ. In this case, there must also be a potential which is an $n-1$ form, $C^{(n-1)}_{\mu_1 \ldots \mu_{n-1}}$, satisfying

$$F^{(n)}_{\mu_1 \ldots \mu_n} = \partial_{[\mu_1} C^{(n-1)}_{\mu_2 \ldots \mu_n]}. \tag{3.16}$$

We call this $n-1$ form the *R-R potential*. This potential must be odd-rank for Type IIA and even-rank for Type IIB. Furthermore, just as Maxwell's equations imply that A^μ is invariant under a gauge transformation, equation (3.15) implies that these R-R potentials are gauge fields, with the following gauge transformation

$$C^{(p+1)}_{\mu_1 \ldots \mu_{p+1}} \to C^{(p+1)}_{\mu_1 \ldots \mu_{p+1}} + \partial_{[\mu_1} \Lambda^{(p)}_{\mu_2 \ldots \mu_{p+1}]}, \tag{3.17}$$

for any p-form $\Lambda^{(p)}$.

Now suppose we have a $p+1$-form potential $C^{(p+1)}$. Then it must minimally couple to a $p+1$ dimensional world-volume. Thus, we must have fundamental dynamic p-dimensional objects in the theory. We have only seen one possible candidate, the p-dimensional Dp branes that open strings end on. We have already seen that these surfaces are associated with non-abelian degrees of freedom in the Chan-Paton factors. Now we know they also carry R-R charge.

3.2 The D brane action

To summarize, we now have a description of Dp-branes as p-dimensional objects on which string endpoints are restricted to move upon, that are charged under a $p+1$ form R-R potential. For Type IIA, the irreducible representations of $\mathbf{16} \otimes \mathbf{16'}$ restrict these to be 0,2,4,6, or 8 dimensional, while for Type IIB, they can be -1,1,3,5,7 or 9 dimensional. The D(-1) brane is called a D-instanton and is localized in both space and time. The D3 is particularly important. It couples to a 4-form potential and thereby a 5-form field strength $F^{(5)}$, which is self-dual

$$*F^{(5)} = F^{(5)}. \tag{3.18}$$

3.2.1 D brane gauge fields

Recall the T-dual coordinate x^9 of the previous section. By computing its value at the string endpoints, we can determine the length of the string along the T-dual direction

$$x^9(\tau,\pi) - x^9(\tau,0) = \frac{2\pi\alpha' m}{R}, \quad m \in \mathbb{Z}. \tag{3.19}$$

But this is just an integer number of windings around the T-dual direction, i.e. m is the open string quantum number analogous to the closed string winding number. If we periodically identify positions separated by $\frac{2\pi\alpha'}{R}$ in the x^9 direction, then we see that open strings start and end on D branes at the same position in x^9. Now suppose we add Chan-Paton factors via the following constant gauge field only along the compact dimension

$$A_9 = \frac{1}{2\pi R}\begin{pmatrix} \phi_1 & & & 0 \\ & \phi_2 & & \\ & & \ddots & \\ 0 & & & \phi_N \end{pmatrix}, \tag{3.20}$$

i.e. part of an abelian subgroup of the usual $U(N)$ Chan-Paton factors. The addition of this field will introduce a term $-iq\int A_\mu \dot{x}^\mu$ in the action, that will shift the quantized canonical momentum along the compact dimension so that a string in state $|k; ij\rangle$ will have momentum

$$p^A_{ij} = \frac{n}{R} + \frac{\phi_j - \phi_i}{2\pi R}. \tag{3.21}$$

Putting this into the mode expansion, we find that the T-dual string length has changed,

$$\begin{aligned} x^9(\tau,\pi) - x^9(\tau,0) &= \frac{2\pi\alpha' m}{R} + (\phi_j - \phi_i)\frac{\alpha'}{R} && (3.22) \\ &= 2\pi\alpha'\left(\frac{m}{R} + (A_9)_{jj} - (A_9)_{ii}\right). && (3.23) \end{aligned}$$

In other words, the string endpoint in state i is now located at $2\pi\alpha'(A_9)_{ii}$ (up to a constant winding number in the T-dual direction). The string endpoints are still confined to D branes, so we see that the introduction of the gauge fields $(A_9)_{ii}$ and $(A_9)_{jj}$ shift the ith and jth D branes from their original positions. Reversing this logic, we find that setting $\phi_i = 0$ for all $1 \leq i \leq N$ restores the $U(N)$ Chan-Paton gauge symmetry and makes the N D branes coincident. This provides a remarkably simple string theory version of the Higgs mechanism. Suppose we start with massless open string states, that is, strings with zero winding number m and momentum number n, then we introduce a background abelian gauge field

$$A_9 = \frac{1}{2\pi R} \begin{pmatrix} 0 & & & 0 \\ & \phi_i & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}. \tag{3.24}$$

The mass-shell condition gains an extra term from the momentum p^A_{ii} (3.21), so that it reads

$$m^2_{ii} = \frac{1}{(2\pi\alpha')^2}(\phi_i)^2, \tag{3.25}$$

and depends solely on the separation of the D_i brane from the stack. So we started with N coincident D branes with $U(N)$ symmetry, and as one of the D branes, D_i, separates from the stack, this symmetry is broken $U(N) \to U(N-1) \times U(1)$. But during this process a massless open string connecting D_i to the stack acquires a mass as it is stretched. In section 7.1.2 we will examine this mass in more detail.

We could of course generalize A_9 to a gauge field A_μ with non-zero components ϕ_i in all dimensions. Additionally, we could make ϕ_i position-dependent $\phi_i \to \phi_i(x)$. As in the example above, the components of A_μ along the compact dimensions give the space-time position of the Dp-brane. These form a set of $9-p$ scalar fields which, for position-dependent ϕ_i, describe the shape or *embedding* of the Dp-brane. This is a small hint of gauge/gravity duality; a background gauge field describes the embedding of an energetic, and therefore gravitating, object in space-time[2]. The remaining components A_i, $i = 0, \ldots, p$ are just gauge fields living on the Dp-brane itself. Interestingly, N coincident D branes have embeddings described by matrices in the adjoint representation of $U(N)$, which are understood in terms of non-commutative geometry [21].

3.2.2 The Born-Infeld action

What is the appropriate effective action to describe these gauge fields A_μ? The above analysis shows us that we can interpret some of these fields in terms of the geometry of the D brane. If the D brane has tension, then the simplest action to consider would be one which serves to minimize the world-volume. Consider, for example, a D2-brane in Euclidean space with a constant background abelian gauge field

$$A_\mu = x^\nu F_{\nu\mu}. \tag{3.26}$$

[2] The D branes themselves have a tension

Now T-dualize the x^2 direction. If we T-dualize one direction along a Dp-brane that is already in place, then it is mapped to a D$(p-1)$-brane, since the corresponding Dirichlet condition in that direction is replaced with a Neumann condition. Thus in this example, we are left with a D1-brane with position

$$y^2 \equiv 2\pi\alpha' x^1 F_{12} \tag{3.27}$$

in the T-dual direction, according to the discussion following (3.23). Then for a D1-brane tension T_1, the action is just

$$\begin{aligned} S &= T_1 \int_{D1} ds & (3.28)\\ &= T_1 \int dy^1 \sqrt{(\partial_1 y^1)^2 + (\partial_1 y^2)^2} & (3.29)\\ &= T_1 \int dy^1 \sqrt{1 + (2\pi\alpha' F_{12})^2} & (3.30)\\ &= T_1 \int dy^1 \sqrt{\det(\delta_{ab} + 2\pi\alpha' F_{ab})}, & (3.31) \end{aligned}$$

where $a, b = 1, 2$. Of course, we can generalize this to a Minkowski metric and T-dualize in six directions to get

$$S = T_p \int d^4x \sqrt{-\det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}, \tag{3.32}$$

which is the *Born-Infeld* action [22]. The Born-Infeld action first appeared long before string theory, in 1934, as a way to tame the divergence of the self-energy of a point-like charge in classical electrodynamics [23]. The linearized equations of motion of (3.32) are precisely Maxwell's equations, and so it is a very natural generalization of the Maxwell action to non-linear electrodynamics. The action was developed by introducing a limit on the field strength in the same manner that one introduces a limit on velocity for a relativistic particle. This must have seemed to be an arbitrary maneuver in 1934, but in the context of D branes, it's perfectly natural. The abelian background gauge field gives us electrodynamics on the brane. Part of the gauge field strength describes fluctuations of the brane in space-time (from equation (3.27), $v_2 \equiv \partial_1 y^2 = 2\pi\alpha' F_{12}$). The speed of these fluctuations, v_2, is bounded by the speed of light, and thus the corresponding field strength is also bounded. It is this bound that eliminates the point-charge divergence.

Lastly, we expect the tension T_p to be generated by the strings ending on the D brane. Constructing a quantity with dimensions of mass per unit volume out the basic string tension parameter α' yields

$$T_p = \frac{1}{\sqrt{\alpha'}(2\pi\sqrt{\alpha'})^p}. \tag{3.33}$$

3.3 D brane dynamics

3.3.1 Dirac-Born-Infeld and super Yang-Mills

The Born-Infeld action we have written (3.32) assumes that the D brane is embedded in flat space. We should generalize this to a curved metric, and in fact allow the brane to couple to all the massless supergravity fields in the low energy limit we are interested in. For the NS-NS sector, (2.111), this amounts to including $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ in the effective action. At tree level, $e^{-\phi} = 1/g_s$, and gauge invariance under the transformations

$$A_\mu \rightarrow A_\mu + \frac{\Lambda_\mu}{2\pi\alpha'}, \tag{3.34}$$

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu, \tag{3.35}$$

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \partial_\mu\Lambda_\nu + \partial_\nu\Lambda_\mu, \tag{3.36}$$

restricts the appearance of these fields in the action to the following form:

$$S = \frac{-T_p}{g_s}\int d^{p+1}\xi\sqrt{-\det(g_{ab} + 2\pi\alpha'\mathcal{F})}. \tag{3.37}$$

Here ξ^a are the world-volume coordinates, g_{ab} is the pullback of the metric to the world-volume and $\mathcal{F} = F_{ab} - \frac{1}{2\pi\alpha'}B_{ab}$. This is the *Dirac-Born-Infeld* (DBI) action.

As with the Born-Infeld action, the linearized DBI action corresponds to the Maxwell action, i.e. a $U(1)$ Yang-Mills gauge theory[3]:

$$S_{\text{DBI}} = -\frac{T_P}{g_s}\frac{(2\pi\alpha')^2}{4}\int d^{p+1}\xi F_{ab}F^{ab} + O(F^4) \tag{3.38}$$

$$S_{YM} = -\frac{1}{4g_{\text{YM}}^2}\int d^{p+1}\xi F_{ab}F^{ab}. \tag{3.39}$$

To make this correspondence exact, we must set

$$g_{\text{YM}}^2 = \frac{g_s}{T_P(2\pi\alpha')^2}. \tag{3.40}$$

Here we see another reason why the D3 brane is special. From (3.33), for $p = 3$, the relationship between g_{YM} and g_s is independent of the energy scale α'.

The DBI action describes an isolated D brane, but what action should be used for N coincident Dp branes? The answer is provided by a full non-abelian action proposed by R. C. Meyers in [3]. It will not be instructive to include it here, but we will discuss

[3]More precisely, this is the dimensional reduction of 10d Yang-Mills to the world-volume of the Dp-brane.

a very interesting result of this action, the Myers effect, shortly. The low energy limit of the action naturally describes the low energy limit of the full $U(N)$ $\mathcal{N}=1$ super Yang-Mills theory [24]

$$S_{\rm YM} = \frac{1}{2g_{\rm YM}^2}\int d^{p+1}x[{\rm Tr}(F_{ab}F^{ab}) + 2i{\rm Tr}(\bar{\psi}\Gamma^{\mu}D_{\mu}\psi)]. \tag{3.41}$$

Recall that both Type II string theories have $\mathcal{N}=2$ space-time supersymmetry. Thus D branes preserve exactly half the supersymmetries of the bulk. Recall that such a state is a 1/2 BPS state. These saturate the BPS bound (2.62) and thus have a charge that is completely determined by their mass [25]. For D branes, this gives the exact (and very powerful) relation

$$T_p = q_p, \tag{3.42}$$

where q_p is the R-R charge of the Dp brane [24].

Lastly, we should note that if we were to take the dimensional reduction of (3.41) to a flat 6-dimensional torus, we would get 4-dimensional $\mathcal{N}=4$ super Yang-Mills [9].

3.3.2 Wess-Zumino terms

We know that the D brane necessarily couples to the R-R fields. To include this coupling in the effective D brane action, one can repeat the procedure in equations (3.28)-(3.31) for the term $\int_{Dp} C^{p+1}$. Using the T-dual transformations of the R-R potential [26], we arrive at the generic coupling of R-R potential to gauge fields for (possibly multiple) Dp branes. It is a Chern-Simons (topologically invariant) term

$$S_{\rm CS} = \frac{T_p}{2}\int_{Dp}\sum_q C^{(q)}\wedge {\rm Tr}(e^{2\pi\alpha'\mathcal{F}}), \tag{3.43}$$

where q runs over the dimension of all R-R potentials present. For Type IIB string theory, q is restricted to be even. In this case, expanding (3.43) to second order in $\mathcal{F}$ gives

$$S_{\rm CS} = \frac{N_pT_p}{2}\int_{Dp} C^{(p+1)} + \frac{T_p}{2}(2\pi\alpha')\int_{Dp} C^{(p-1)}\wedge{\rm Tr}(\mathcal{F}) + \frac{T_p}{4}(2\pi\alpha')^2\int_{Dp} C^{(p-3)}\wedge{\rm Tr}(\mathcal{F}\wedge\mathcal{F}), \tag{3.44}$$

where N_p is the number of Dp-branes. Note that the integral picks out contributions only from terms that are $p+1$-forms to match the dimension of the world-volume. This means the sum over q is limited to $q < p+1$. For an abelian configuration, each trace is replaced by a factor of N_p.

Later, we will see that the third term above is responsible for incompressible states in our model. The second term, however, deserves comment right now. It represents a coupling between a Dp brane and D$(p-2)$ branes. The interpretation of this coupling is provided by the *Myers Effect*.

3.3.3 The Myers effect

In reference [3], Robert Myers proposed a non-abelian extension to the DBI world-volume action (3.37) by including non-commuting scalars Φ_i, $i = p+1, \ldots 9$, that transform in the adjoint representation of the $U(1)$ gauge group of N coincident D branes (see section 3.2.1). In that paper, he considered N D0-branes in a background 4-form field strength

$$F^{(4)}_{tijk} = \begin{cases} -2f\epsilon_{ijk} & i,j,k \in \{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

He then worked out the corresponding potential for the adjoint scalars, and their equation of motion. The abelian configuration $[\Phi^i, \Phi^j] = 0$ for separated D branes satisfies the equation of motion with a potential $V(\Phi) = 0$. However, Myers showed that certain non-commutative $N \times N$ matrices Φ^i satisfy the equation of motion with a negative and therefore preferred potential. Thus N separated D branes are unstable to condensation in this background field strength. A remarkable feature of non-commutative geometry lets us interpret this solution as a D2-brane wrapping a 2-sphere with N D0s bound to it. To check this, one simply writes down the metric for a D2 world-volume wrapping a 2-sphere of radius R. Then include the same background R-R field (3.3.3), but instead add a $U(1)$ field strength $F_{\theta\phi} = \frac{f_0}{2}\sin\theta$. The single D2-brane is necessarily abelian, so one simply substitutes this solution into the regular DBI + Chern-Simons action. The corresponding potential has a unique stable extrema whose value matches that of the non-commutative solution for the D0 case, provided we identify f_0 with the number of D0 branes, N.

The field strength $F_{\theta\phi}$ may look familiar. Indeed its appearance far predates that of string theory. In 1931, in an attempt to complete electric-magnetic duality, Dirac considered an electron in the field of a hypothetical magnetic monopole [27]

$$\vec{B} = g\frac{\vec{r}}{r^3}. \tag{3.45}$$

But Maxwell's equations imply that no corresponding vector potential can be found globally for $\mathbb{R}^3$. If, however, we change to polar coordinates on S^2 and define a gauge field

$$A^{\pm} = \frac{f_0}{2}(\pm 1 - \cos\theta)d\phi \tag{3.46}$$

which changes by a gauge transformation at the equator, then the potential covers the whole sphere. But if this is the case, the corresponding wavefunction of the electron must undergo a gauge transformation $\psi_+ \to e^{ieg\phi/\hbar}\psi_-$ at the equator. The requirement that this wavefunction be single-valued yields Dirac's famous relation between quantization of electric and magnetic charge [28]. The magnetic monopole has a corresponding flux f_0 on the 2-sphere. If we calculate the field strength from $A^{\pm}$, we get

$$F_{\theta\phi} = \frac{f_0}{2}\sin\theta. \tag{3.47}$$

Thus, in the picture above, we can identify the number of D0 branes, N with the magnetic flux on the 2-sphere wrapped by the D2.

Note that the D0s we started with cannot carry net $C^{(3)}$ charge, and so neither can the D2. However, the D2 can carry a $C^{(3)}$ dipole and higher multipoles. So we see that the Myers effect is precisely the dielectric effect, but for R-R fields instead of electric fields. A neutral dielectric carries no charge, but when placed in an external electric field gains an induced polarization. Similarly, a stack of D0s is not charged under $C^{(3)}$, but when placed in an external $F^{(4)}$ field, a $C^{(3)}$ polarization is induced. Indeed in both the D0 and D2 picture, the leading term in the expansion of the R-R potential is $\propto \int dt F^{(4)}_{t123}$ which corresponds to the dipole of the R-R multipole expansion.

This remarkable relation between D0s and a D2 known as the Myers effect has generalizations to higher dimensional branes as well as holographic interpretations to go with them [29]. In particular, the focus of this thesis is on the holographic realization of D5 branes puffing out to form a D7 brane.

Chapter 4

The AdS/CFT Correspondence

It is clear from the previous chapter that string theory contains rich dynamics in the non-perturbative regime described by D branes. So far, we have approached things from the perspective of the string world-sheet (and the brane world-volume), but if we look for an effective theory over the whole space-time, we can elucidate the full non-linear dynamics. This should be expected, since at some level, the effective theory should give us classical general relativity with all of its non-linearity. In general relativity, solutions of the linearized equations of motion are but a tiny subset of the really interesting solutions. The others are *solitons*: localized, static, finite energy solutions like the Riesner-Nordström metric [30, 31]. These solutions cannot be found be perturbations about solutions of the linearized field equations. We will see that the identification of these gravitational solitons with the gauge theory description of D branes is what underpins the entire AdS/CFT correspondence.

4.1 The geometry of D branes

4.1.1 Strings in curved space-time

We can easily generalize the Polyakov action (2.70) to strings in a curved background by replacing $\eta_{\mu\nu}$ with $G_{\mu\nu}$ to get the so-called *Sigma model*

$$S_\sigma = \frac{-1}{4\pi\alpha'} \int d\sigma \sqrt{-|g|} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}. \tag{4.1}$$

One can check that the corresponding path integral is that of the Polyakov action with the insertion of an exponential of a graviton vertex operator i.e. a coherent state of gravitons. So we have not added any new ingredients by a including a curved background metric [18].

In addition to reparameterization invariance, the Polyakov action has another symmetry: *Weyl Invariance*, $g_{\alpha\beta} \to e^{\phi(\sigma,\tau)} g_{\alpha\beta}$. Weyl invariance is crucial to string theory and must be retained at the quantum level. This requires us to ensure that the beta functions for each of the coupling constants vanish, since otherwise, renormalized quantities would have a scale dependence that would violate Weyl symmetry. Alternatively, we can recall from section 2.2.2 that conformal symmetry is linked to tracelessness of the stress-energy tensor and impose

$$T^{\alpha}_{\alpha} = \frac{-1}{2\alpha'} \beta_{\mu\nu}(G) \partial X^{\mu} \partial X^{\nu} = 0, \tag{4.2}$$

where $T_{\alpha\beta} = \frac{4\pi}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\alpha\beta}}$ is calculated using a renormalized background metric $G_{\mu\nu}$, which is the renormalized coupling constant for the embedding scalars X^{μ}. Indeed, one finds that the coefficient $\beta_{\mu\nu}(G)$ above is identical to the beta function for this coupling constant. More exciting is the value of this beta function:

$$\beta_{\mu\nu} = \alpha' R_{\mu\nu}. \tag{4.3}$$

From this we see that Weyl invariance requires $R_{\mu\nu} = 0$, which are Einstein's equations in vacuum!

4.1.2 Supergravity actions

Vanishing of the beta functions for all massless fields in the Type II superstring theories can be shown to be equivalent to the equations of motion for the following (truncated) effective actions:

$$\begin{aligned} S_{IIA} &= S_0 - \frac{1}{2\kappa_0^2} \int d^{10}x \sqrt{-|G|} \Big[-\frac{1}{4}(dC^{(1)})^2 - \frac{1}{48}(dC^{(3)} + dB^{(2)} \wedge C^{(1)})^2 \Big] \\ &\quad - \frac{1}{4\kappa_0^2} \int B^{(2)} \wedge dC^{(3)} \wedge dC^{(3)} + \text{Fermion terms}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} S_{IIB} &= S_0 + \frac{1}{2\kappa_0^2} \int d^{10}x \sqrt{-|G|} \Big[-\frac{1}{12}(dC^{(1)} + C^{(0)} \wedge dB^{(2)})^2 - \frac{1}{2}(dC^{(0)})^2 \\ &\quad - \frac{1}{480}(dC^{(4)} + dB^{(2)} \wedge C^{(2)})^2 \Big] + \\ &\quad \frac{1}{4\kappa_0^2} \int (C^{(4)} + \frac{1}{2} B^{(2)} \wedge C^{(2)}) \wedge dC^{(2)} \wedge dB^{(2)} + \text{Fermion terms}. \end{aligned} \tag{4.5}$$

Here S_0 is the strictly NS-NS part;

$$S_0 = \frac{1}{2\kappa_0^2} \int d^{10}x \sqrt{-|G|} \{ e^{-2\phi} \Big[R + 4(\nabla\phi)^2 - \frac{1}{12}(dB^{(2)})^2 \Big], \tag{4.6}$$

with R being the Ricci scalar. κ_0 is related to Newton's gravitational constant G_N via

$$2\kappa_0^2 = 16\pi G_N/g_s^2. \tag{4.7}$$

However, these actions do not completely encode the dynamics of the R-R fields, and need to be supplemented with the self-duality of the five-form field strength (3.18). In these effective actions, we have restricted ourselves to a low energy $\alpha' \to 0$ truncation, and are thus left with two classical gravitational theories. The actions (4.4) and (4.5) are the Type II *supergravity* actions.

4.1.3 D3 geometry

As with Einstein's equations, we can set about looking for non-perturbative solutions to the equations of motion of the Type II supergravity actions by assuming a high degree of symmetry. Indeed, using the Riesner-Nordström metric as a guide, one could look for a similar charged black-hole solution, only with the electric charge replaced by an R-R charge. Then one would discover the following particularly nice family of solutions that are symmetric in p dimensions [32].

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H_p(r)}}\left\{-\left[1-\left(\frac{r_h}{r}\right)^{7-p}\right]dt^2 + \sum_{i=1}^{p} dx_i^2\right\} \\ &\quad + \sqrt{H_p(r)}\left\{\left[1-\left(\frac{r_h}{r}\right)^{7-p}\right]^{-1} dr^2 + r^2 d\Omega_{8-p}^2\right)\right\}, \qquad (4.8) \\ e^{2\phi} &= g_s^2\sqrt{H_p(r)^{(3-p)}}, \qquad (4.9) \\ C^{(p+1)} &= \frac{1}{g_s}(H_p(r)^{-1}-1)dt\wedge dx^1\wedge\ldots\wedge dx^p, \qquad (4.10) \end{aligned}$$

where we have defined

$$\begin{aligned} r_p^{7-p} &\equiv (2\sqrt{\pi})^{5-p}\Gamma\left(\frac{7-p}{2}\right)g_s N(\sqrt{\alpha'})^{7-p}, \qquad (4.11) \\ \alpha_p &\equiv \sqrt{1+\left(\frac{r_h^{7-p}}{2r_p^{7-p}}\right)^2} - \frac{r_h^{7-p}}{2r_p^{7-p}}. \qquad (4.12) \end{aligned}$$

The quantity N is the R-R charge [1].

$$N = \int_{S^{8-p}} *F^{(p+2)}. \tag{4.13}$$

[1]This is just an integrated form of Gauss' law: $d*F = *J \Rightarrow \int_{S^3} d*F = q_e \Rightarrow \int_{S^2} *F = q_e$

The time-like killing vector of this metric becomes null at $r = r_h$, so these solutions have a horizon there. Also, the singularity at $r = 0$ cannot be removed by a coordinate transformation; it is real. The sum over dx_i elements reveals that these solutions span the $p+1$ dimensions of $\mathbb{R}^{p+1}$ and are localized in the remaining $9-p$ dimensions. It is tempting to identify these metrics as those of a space-time distorted by a Dp-brane embedded in $9+1$ dimensions. This temptation is well placed, but for now we will just call the solutions *black p-branes*.

The case $p = 3$ is particularly interesting. For that we have

$$\begin{aligned} r_p^4 &= 4\pi g_s N \alpha'^2 \\ &= \lambda \alpha'^2, \end{aligned} \tag{4.14}$$

where we have defined the *'t Hooft* coupling $\lambda \equiv 4\pi g_s N$. Let us examine the near horizon $r \approx r_h$ regime and take the limit as $r \to 0$. We are going to take this limit in a very particular way. First, we fix the ratio $u \equiv \frac{r_p^2}{r} = \frac{\sqrt{\lambda}\alpha'}{r}$, and then let u get very large, i.e. $r_p >> r$. This corresponds to the so-called "throat" region of the space-time. In this limit,

$$\alpha_3 \to 1, \tag{4.15}$$

$$H_3(r) \to \left(\frac{r_p}{r}\right)^4 = \frac{\lambda\alpha'^2}{r^4}. \tag{4.16}$$

Before plugging this into the metric (4.8), it is convenient to make the transformation $r \to \frac{r}{\sqrt{\lambda}\alpha'}$, in which case,

$$H_3(r) \to \frac{1}{\lambda\alpha'^2 r^4}, \tag{4.17}$$

$$H_3^{1/2}(r) r^2 \to \sqrt{\lambda}\alpha', \tag{4.18}$$

yielding the metric

$$ds^2 = \sqrt{\lambda}\alpha' \left\{ r^2 \left[-\left(1 - \left(\frac{r_h}{r}\right)^4\right) dt^2 + \sum_{i=1}^{3} dx_i^2 \right] + \frac{1}{r^2}\left(1 - \left(\frac{r_h}{r}\right)^4\right)^{-1} dr^2 + d\Omega_5^2 \right\}. \tag{4.19}$$

The t, x_i and r coordinates appear precisely in the form of the AdS_5 black hole metric, (2.17), so the full geometry we have uncovered is that of $AdS_5 \times S^5$.

4.1.4 Ramond-Ramond 4-form solution

Before delving into the implications of (4.19), it is worth pausing to write out a more complete solution of the R-R 4-form for the choice $p = 3$. As mentioned before, the presence of an even-rank R-R form indicates that we are working with a Type IIB supergravity solution. From (4.10) and (4.17), we get

$$C^{(4)} = \frac{1}{g_s}(\lambda\alpha'^2 r^4 - 1)dt \wedge dx \wedge dy \wedge dz. \tag{4.20}$$

But recall that we can make any gauge transformation of R-R potentials according to (3.17). This allows us to replace the constant -1 term in (4.20) with a constant of our choosing. A convenient gauge choice is to make it equal to $\lambda\alpha'^2 r_h^4$ (for example by adding a gauge term $\Lambda_{\mu\nu\rho} = (1+r_h^4)t\epsilon_{\mu\nu\rho}$ with $\mu, \nu, \rho \in \{x, y, z\}$ in (3.17)). The resulting potential is then

$$C^{(4)} = \frac{\lambda\alpha'^2}{g_s}\left(1 - \frac{r_h^4}{r^4}\right)r^4 dt \wedge dx \wedge dy \wedge dz. \tag{4.21}$$

If this was all we could do with the 4-form, we would have no hope of seeing anything like a quantum hall effect. This is because the term of the D brane action that will ultimately be responsible for incompressible states is the last term in (3.44). For this term to be non-zero, there needs to be a piece of the R-R 4-form with volume elements independent of the AdS_5 piece of the metric. That only leaves the coordinates of the S^5. For the remainder of this thesis, we will fibrate the S^5 with two 2-spheres so that

$$d\Omega_5 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) + \cos^2\psi(d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2). \tag{4.22}$$

Then we will postulate a term in the 4-form potential $\frac{c(\psi)}{2} d\psi \wedge d\theta \wedge d\phi \wedge d\tilde{\theta} \wedge d\tilde{\phi}$. From this we can calculate the corresponding field strength

$$F^{\psi\theta\phi\tilde{\theta}\tilde{\phi}} = g^{\theta\mu}g^{\phi\nu}g^{\tilde{\theta}\sigma}g^{\tilde{\phi}\rho}g^{\psi\lambda}F_{\mu\nu\sigma\rho\lambda} \tag{4.23}$$

$$= \frac{(\sqrt{\lambda}\alpha')^5}{\sin^4\psi\sin^2\theta\cos^4\psi\sin^2\tilde{\theta}}\frac{\partial_\psi c(\psi)}{2}. \tag{4.24}$$

Then from the metric determinant

$$\sqrt{-|g|} = \sqrt{(\sqrt{\lambda}\alpha')^{10} r^6 \sin^4\psi\cos^4\psi\sin^2\theta\sin^2\tilde{\theta}}, \tag{4.25}$$

we can compute the Hodge dual of this field strength component

$$*F_{rtxyz} = \frac{1}{p!}\sqrt{|g|}\epsilon_{\nu_1\ldots\nu_p rtxyz}F^{\nu_1\ldots\nu_p} \tag{4.26}$$

$$= \frac{r^3}{\sin^2\psi\sin\theta\cos^2\psi\sin\tilde{\theta}}\frac{\partial_\psi c(\psi)}{2}. \tag{4.27}$$

Using the self-duality of $F^{(5)}$, (3.18), we can set this expression to be $F_{rtxyz} = 4(\sqrt{\lambda}\alpha')^2 r^3/g_s$, resulting in an equation for $c(\psi)$:

$$\partial_\psi c(\psi) = \frac{(\lambda\alpha'^2)}{g_s} 8\sin^2\psi\cos^2\psi\sin\theta\sin\tilde{\theta}. \tag{4.28}$$

Integrating this expression gives

$$c(\psi) = \frac{(\lambda\alpha'^2)}{g_s}\sin\theta\sin\tilde{\theta}\left(\psi - \frac{1}{4}\sin 4\psi - \frac{\pi}{2}\right), \tag{4.29}$$

where again the integration constant is a gauge choice. Thus we can write the full 4-form solution as

$$C^{(4)} = \frac{\lambda\alpha'^2}{g_s}\left[\left(1-\frac{r_h^4}{r^4}\right)r^4 dt\wedge dx\wedge dy\wedge dz + \frac{1}{2}\left(\psi - \frac{1}{4}\sin 4\psi - \frac{\pi}{2}\right)d\cos\theta\wedge d\phi\wedge d\cos\tilde{\theta}\wedge d\tilde{\phi}\right]. \tag{4.30}$$

4.2 The correspondence

4.2.1 $p =$ Dp

From (4.8), and our discussion of the AdS metric, we can identify the radius of curvature of the p-brane solution metric as

$$L^2 = \frac{1}{\sqrt{H_p(r)}r^2} = \sqrt{\lambda}\alpha'. \tag{4.31}$$

Then for the classical supergravity approximation to be valid, we need the curvature to be small compared to the scale of quantum corrections;

$$L >> l_s \tag{4.32}$$
$$\Rightarrow \lambda^{1/4}\sqrt{\alpha'} >> \sqrt{\alpha'} \tag{4.33}$$
$$\Rightarrow g_s N >> 1, \tag{4.34}$$

where we used (2.68) in the second line. Note that this is consistent with the $r_p >> r$ approximation we made earlier. Furthermore, to avoid string loop corrections, we must keep the coupling constant small, $g_s < 1$, so that we have $1 << g_s N << N$, i.e. N must be very large. Remember that we defined N as the total R-R charge. But of course, N has another interpretation. Recall from (3.42) and (3.44) that each D brane carries one unit of R-R charge and N coincident D branes carry N units of R-R charge. So N is both the charge of the supergravity solution and the number of coincident D branes. The classical supergravity limit then requires a large number of coincident D branes which carries a large N, $U(N)$ gauge theory. If on the other hand, we want to do open string perturbation theory, we would want a small coupling constant. For N coincident D branes, each string gets a Chan-Paton factor N, so that the coupling constant is g_sN, and perturbation theory requires $g_sN << 1$. Thus we have two different descriptions valid in two different regimes. For $g_sN << 1$ (weak coupling), we have open string perturbation theory with D brane boundary conditions. For $g_sN >> 1$ (strong coupling), we have classical Type II supergravity.

Recall that the $U(N)$ gauge theory in the open string description is broken when coincident D branes separate and strings connecting separated branes (which we will call W bosons) acquire a mass via the Higgs mechanism. In the supergravity description, there exists a multi-centered solution to the supergravity action, which is a generalization of the function $H_p(r)$ [33]:

$$H_p(r) = 1 + \alpha_p \sum_{i=1}^{k} \frac{r_{p_{(i)}}^{7-p}}{|\vec{r} - \vec{r_i}|^{7-p}}, \qquad r_{p_{(i)}} = 4\pi g_s N_i \alpha'^2, \tag{4.35}$$

where now each centre sources an R-R field so that we must integrate $*F_{p+2}$ over a sphere around each source to get N_i:

$$N_i = \int_{S_i^{8-p}} *F_{p+2}. \tag{4.36}$$

Now suppose we place an open string in this metric. Its endpoints must be attached to D branes, but there are none around. However, the strings could end at the singularities which are the centres $\vec{r_i}$. But then we may as well say that there are D branes located at these centres. Indeed, one can compare the W-boson mass (from a string stretched between two branes), to the mass of a string suspended between two centres in this metric and would find they are the same. We will see something similar to the latter of these two computations in section 7.1.2, and find that the mass is just $T_s|r_i - r_j|$ (in the static gauge $r = \sigma$, $t = \tau$). This helps to clarifies the connection between p-branes and Dp branes. The Dp brane is located at the centre $\vec{r} = \vec{r_i}$ of the supergravity solution, or equivalently, the metric (4.8) gives the geometry near a Dp brane.

4.2.2 Maldacena's conjecture

Here is the story so far. We start with Type IIb string theory in ten dimensions. Then we insert N parallel D3 branes. At low energies, the full theory contains excitations off the brane, which must be from the closed string spectrum (2.111) - (2.114), and excitations on the brane, which are open strings with endpoints attached to the brane. The low energy closed strings are effectively described by the Type IIB supergravity action (4.5), while the open strings are effectively described by the D brane world-volume action which reduces to $\mathcal{N} = 4$ super Yang-Mills theory for the D3s in the low energy limit (3.39)[2].

For a linearized metric $g_{\mu\nu} = \eta_{\mu\nu} + \kappa_0 g_s h_{\mu\nu}$, we can expand the Type IIB action about $\alpha' = 0$ $(\kappa_0 = 0)$, to get

$$\mathcal{S}_{IIB} = \frac{1}{2\kappa_0^2 g_s^2} \int d^{10}x \sqrt{|g|} \mathcal{R} \tag{4.37}$$

$$= \int d^{10}x [(\partial h)^2 + \kappa_0 g_s (\partial h)^2 h + O((\kappa_0 g_s)^2)], \tag{4.38}$$

which is a free field theory at $\alpha' = 0$.

We can similarly look at the bulk-brane interaction in this limit. The leading term in this interaction comes from the world-volume coupling to the space-time metric in (3.37)

$$\mathcal{S}_{\text{int}} \propto \frac{-T_p}{g_s} \int d^{p+1}\xi \sqrt{|g|} \tag{4.39}$$

$$\propto \frac{-1}{g_s \alpha'^2} \int d^4\xi \sqrt{|g|} \tag{4.40}$$

for the D3. But from (4.19), we know that $\sqrt{|g|} \propto \alpha'^5$, and so the leading term in $\mathcal{S}_{\text{int}}$ is zero for $\alpha' = 0$. Thus as $\alpha' \to 0$, the bulk-brane interactions disappear and *we are left with $\mathcal{N} = 4$ super Yang-Mills gauge theory for the open strings, and free gravity for the closed strings.*

Now what kind of physics does an observer at infinity see deep inside the bulk? The metric (4.19) is time translation invariant and so has a timelike killing vector $\xi^\mu = (1, 0, \dots, 0)$. A static time-like observer with velocity U^μ has a conserved energy $E = U^\mu \xi_\mu$, and satisfies $U^\mu U^\mu = -1$. These two equations fix the observer's velocity, which depends on their radial position

$$U^\mu(r) = \left(\frac{1}{\sqrt{-g_{tt}}}, 0, \dots, 0 \right). \tag{4.41}$$

The observer will measure the frequency of a signal with wave-vector k^μ to be

$$\omega(r) = -k_\mu U^\mu(r). \tag{4.42}$$

[2]Recall that upon dimensional reduction, $\mathcal{N} = 1$ supersymmetery can become $\mathcal{N} = 4$. We know the D3 corresponds to $\mathcal{N} = 4$ specifically by comparing fields (see section 2.3). The D3 breaks $10 - 4 = 6$ translational symmetries and gains 6 corresponding scalars describing its transverse fluctuations. On the brane lives one gauge field A^μ. This matches only with the $\mathcal{N} = 4$ states, which also include four spin-1/2 fermions. This will all look slightly different when we introduce a probe D5 brane later.

If we put an emitter at position r, and an observer at infinity, we then get the ratio of the frequencies they measure for a signal to be

$$\frac{\omega_{\rm obs}}{\omega_{\rm emiit}} = \frac{\sqrt{-g_{tt}}|_r}{\sqrt{-g_{tt}}|_\infty} = \frac{1}{H_p^{1/4}(r)}, \tag{4.43}$$

where we have set $r_h = 0$ for simplicity. This ratio goes to zero as $r \to 0$, so as the emitter moves deeper into the bulk, the signal detected at infinity shifts toward the infrared. So there are two separate sectors in which an observer at infinity detects low energy excitations. One is the set of long wavelength massless fields propagating outside the throat. The other is the set of red-shifted fields propagating inside the throat, whose geometry is described by $AdS_5 \times S^5$. These two sets do not interact. The fields outside the throat have wavelengths too large to probe inside the throat, and the fields inside cannot overcome the gravitational well to get out. Depending on whether we think of D branes in terms of the open strings that end on them, or in terms of supergravity solutions, we get two different pictures in the $\alpha' \to 0$ limit. The first gives $\mathcal{N} = 4$ super Yang-Mills for open strings and free supergravity for closed strings. The second gives Type IIB fields in $AdS_5 \times S^5$ for one sector, and free supergravity for the other. Since in each viewpoint, the free supergravity sector is decoupled from the other sector. It is natural to identify the other sectors in each viewpoint. This is precisely the conjecture proposed by Juan Maldacena [1][3]

$\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory in $3+1$ dimensions is dual to Type IIB superstring theory in $AdS_5 \times S^5$

As a check, we can compare the symmetries on both sides of the correspondence. We saw in section 2.3 that $3+1$ dimensional $\mathcal{N} = 4$ super Yang-Mills is scale invariant even at the quantum level. It has the full conformal symmetry group $SO(4,2)$. We also saw in 2.1 that this is precisely the isometry group of AdS_5. The isometry group of the S^5 is $SO(6) \sim SU(4)_R$, which is also the R-symmetry group of the $\mathcal{N} = 4$ supersymmetry algebra.

Note that if we work in units where the radius of curvature is $L = 1$, then (4.31) gives

$$\alpha' = \frac{1}{\sqrt{\lambda}} \sim \frac{1}{\sqrt{g_s N}}. \tag{4.44}$$

[3]Recall that we said the gauge theory on the branes was $U(N)$. This can generally be split into a $U(1) \times SU(N)$ theory. The $U(1)$ multiplet describes the center of mass positions of the branes, which we can choose to ignore [9]

Thus, we are expanding about a classical gravity solution $\alpha' = 0$, with quantum gravity corrections appearing at $O(\alpha')$. On the field theory side, we are doing a large N expansion with corrections appearing at $O(1/\sqrt{N})$.

4.2.3 Large N expansion

Consider a toy Yang-Mills theory of fields ϕ^a_λ with adjoint (colour) index $1 \le a \le N^2-1$, and flavour index λ. The $SU(N)$ Yang-Mills Lagrangian contains three and four-point vertices, and looks like

$$\mathcal{L}(\phi) \sim \mathrm{Tr}[(\partial\phi)^2 + \phi^2 + g_{YM}\phi^3 + g_{YM}^2\phi^4]. \tag{4.45}$$

Under a rescaling $\tilde{\phi} = g_{YM}\phi$, we can rewrite this as

$$\mathcal{L}(\phi) \sim \frac{1}{g_{YM}^2}\mathrm{Tr}[(\partial\tilde{\phi})^2 + \tilde{\phi}^2 + \tilde{\phi}^3 + \tilde{\phi}^4], \tag{4.46}$$

so that each vertex of a Feynman diagram will carry a factor of

$$\frac{1}{g_{YM}^2} = \frac{1}{2\pi g_s} \tag{4.47}$$

$$\sim \frac{N}{\lambda}. \tag{4.48}$$

Meanwhile, the propagator will scale as

$$\langle \tilde{\phi}^j_i \tilde{\phi}^l_k \rangle \sim g_{YM}^2 \langle \phi^j_i \phi^l_k \rangle \tag{4.49}$$

$$= g_{YM}^2\left(\delta^l_i \delta^j_k - \frac{1}{N}\delta^j_i \delta^l_k\right), \tag{4.50}$$

which simplifies in the *'t Hooft* limit, $N \to \infty$, to $\frac{\lambda}{N}\delta^l_i\delta^j_k$ [34].

The index structure suggests a double line notation for the propagator indicating the direction of colour flow shown in figure 4.1. If we contract over l, k and i, j, we get

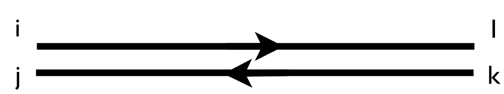

FIGURE 4.1

the loop shown in figure 4.2, and the trace gives a factor of N in the amplitude. Thus any Feynman diagram in this model with v vertices, p propagators, and l loops will

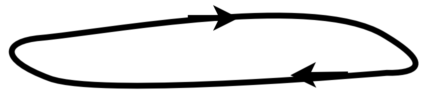

FIGURE 4.2: .

contribute an overall factor of

$$\left(\frac{\lambda}{N}\right)^p \left(\frac{N}{\lambda}\right)^v N^l \tag{4.51}$$

$$= N^{v-p+l}\lambda^{p-v} \tag{4.52}$$

to the amplitude.

Each Feynman diagram can then be associated with the two-dimensional surface that they can be plastered onto without any lines rising off the surface. Indeed, the exponent of N is just the Euler characteristic of the corresponding surface $\chi = 2 - 2g$, where g is the genus of the surface. So now perturbation theory in the large N limit gives a genus expansion

$$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda). \tag{4.53}$$

In this limit, the biggest contribution comes from genus 0 *planar* diagrams (called planar because they can also be plastered onto a flat surface). This expansion looks just like the perturbative expansion for closed strings provided we have $g_s \sim \frac{1}{N}$, which is exactly the case for fixed λ. Thus, it is important for the 't Hooft limit that we keep λ fixed as we take $N \to \infty$. The connection between string theory and gauge theory is ever-prevalent.

4.2.4 Different forms of the conjecture

In the 't Hooft limit above, we fixed λ. To make the correspondence more tractable, we will want to make λ large, since this means the radius of curvature $L \sim \lambda^{1/4}$ will be large. In all, we have three forms of the correspondence depending on how bold we wish to be. The statement 4.2.2, is the strongest form. Enforcing the 't Hooft limit gives us a slightly weaker but more reliable form as it uses a large N field expansion and a small g_s string perturbation expansion. The weakest form of the correspondence is the one we will use, the large λ limit. This yields a classical supergravity theory on the AdS side. On the field theory side, the coupling $g_{YM} \sim \sqrt{\frac{\lambda}{N}}$ becomes large so that we probe strong coupling physics in this limit. Thus Maldacena's conjecture, also known as the *AdS/CFT correspondence*, is an incredibly powerful conjecture, because in this version, it is a strong/weak coupling duality. It gives us a tool with which we can study strongly coupled problems using just classical supergravity. Of course, this property also makes it very hard to test.

Lastly, note that the decoupling of the two sectors discussed in 4.2.2 only requires that the space-time is asymptotically $AdS_5 \times S^5$, that is, we can allow for topology-changing objects deep in the throat. Thus, in its strongest form, the conjecture is a duality between the full Type IIB string theory on *any* asymptotically $AdS_5 \times S^5$ space and $\mathcal{N} = 4$ super Yang-Mills.

4.2.5 Holography

One of the biggest hints as to how the correspondence manifests itself comes from considering black hole information. The *Bekenstein bound* [35] states that the maximum entropy contained in a space with surface area A is precisely the entropy of a black hole whose event horizon has the same surface area

$$S_{\max} = S_{\mathrm{BH}} = \frac{A}{4G_N}. \tag{4.54}$$

The second equality above is established by the area theorem, which tells us that the area of the horizon does not decrease, so this is a natural quantity to associate with entropy [36]. The proof of the first equality is quite simple. Suppose otherwise, then the region has an entropy $S_1 > S_{\mathrm{BH}}$, so this region is not filled by a black hole. Now add matter to the region to form a black hole (or to enlarge any existing black hole). This process must not violate the second law of thermodynamics, so the new entropy is $S_2 \geq S_1$. But the new entropy is just that of the black hole formed with surface area B; $S_2 = B/4G_N$. Thus, $B \geq A$. But this is a contradiction; we have only increased the density of the region, so the black hole must have an area smaller than A.

This suggests a "Holographic Principle" wherein the degrees of freedom of a theory of quantum gravity in some volume should be contained on the boundary of that volume. Indeed if we consider $U(N)$ super Yang-Mills in $3+1$ dimensions with a UV cutoff δ, then the number of degrees of freedom should be $N^2 V_3/\delta^3$, since there are N^2 degrees of freedom in each pixel of side length δ. In, $AdS_5 \times S^5$, the holographic principle says that the number of degrees of freedom is proportional to the area. The area of a surface at radius $r = 1/\delta$ (δ should be small for a good UV cutoff, so we are looking near the boundary at $r = \infty$) is given by

$$A = \sqrt{|g|}|_{r=1/\delta} \tag{4.55}$$

$$\sim \frac{V_3 N^2}{\delta^3}, \tag{4.56}$$

In terms of AdS/CFT, this suggests that the CFT "lives" on the boundary. This is not to say that we supplement the AdS gravity theory with a theory on the boundary. Rather, from the discussion above, we should say the theory lives in the AdS bulk, but the information is completely encoded on the boundary (or vice versa).

4.2.6 Excerpts from the dictionary

It is time to decide exactly what is meant by *duality* in the AdS/CFT correspondence. We could just say there is an isomorphism between the Hilbert spaces of each theory, and be done with it, but that is both difficult to prove and not very useful. What we would like is a one-to-one mapping of fields on each side of the correspondence. However, CFTs like $\mathcal{N} = 4$ super Yang-Mills have no asymptotic states as we saw in section 2.2, so it is preferable to talk about operators on the field theory side. The precise mapping is provided by the celebrated GKPW rule [37]. It too is a conjecture, though perhaps more difficult to motivate than Maldacena's. We might try to equate partition functions on each side of the correspondence, except that there is reason to distinguish between the bulk of AdS and the boundary. This distinction should be manifest in the duality. The bulk requires data from the boundary in order for the Cauchy problem to be well-posed (recall that fields can propagate to $r = \infty$ and back in finite time). Furthermore, the holographic principle suggests that the CFT lives on the boundary. All of this persuades us to consider conformal operators on the boundary as sourcing the fields in the bulk (or equivalently, the boundary value of bulk fields source the CFT operators). In that case, the GKPW rule is rather intuitive

$$\langle e^{\int d^4x \phi_0(\vec{x})\mathcal{O}(\vec{x})} \rangle_{\text{SYM}} = \mathcal{Z}_{\text{string}}[\phi(\vec{x}, r)|_{r=\infty} = \phi_0(\vec{x})], \tag{4.57}$$

where $\phi(\vec{x}, r)$ is a Type IIB string field, $\phi_0(\vec{x})$ is its boundary value, $\mathcal{O}(\vec{x})$ is its dual CFT operator, and $\mathcal{Z}_{\text{string}}$ is the Type IIB partition function which is just $e^{iS_{\text{SUGRA}}}$ in the classical limit. With this, one can compute correlation functions of any operator $\mathcal{O}$ in super Yang-Mills. One just needs to identify the corresponding field ϕ, solve the equations of motion in the supergravity action, compute the on-shell action, and functionally differentiate with respect to its boundary value ϕ_0. Of course, the hard part is finding which operator corresponds to which field. We have already seen that AdS/CFT relates global symmetries of the field theory to local isometries of $AdS_5 \times S^5$. It turns out that the entire AdS/CFT dictionary relates global properties in the field theory to local properties on the gravity side. One place to start would be to decompose a field in $AdS_5 \times S^5$ into Kaluza-Klein towers, that is, write it in a basis of spherical harmonics on S^5

$$\phi(r,\vec{y}) = \sum_{\Delta=0}^{\infty} \phi_\Delta(r) Y_\Delta(\vec{y}), \tag{4.58}$$

where $\vec{y}$ are the coordinates on S^5. Then the Klein-Gordon equation fixes the mass of the field in terms of the conformal dimension Δ, which in turn fixes the conformal dimension of the CFT operator $\mathcal{O}$. For scalar fields in the bulk, these operators are traces over scalar fields in $\mathcal{N} = 4$ super Yang-Mills.

We will not work out the full dictionary, but simply tabulate some important entries below. Some of these will be justified later.

CFT Operator	Gravity Field
m	Δ
Scalar field ϕ	$\mathcal{O}_s$ Scalar operator
Dirac spinor ψ	$\mathcal{O}_f$ Fermionic operator
Global current J^μ	A^μ Maxwell field
Energy-momentum tensor $T^{\mu\nu}$	g_{ab} Metric
Spin	Spin
Temperature T	Hawking temperature
Chemical potential μ	$A^\mu(\infty)$
Energy scale	Radial coordinate

TABLE 4.1: AdS/CFT dictionary

The last entry in the table warrants a little more explanation. From the field theory point of view, an operator $\mathcal{O}$ sources a well-defined particle (well-defined in the UV). But it loses its definition as it interacts with the strongly coupled system until it can only be defined as part of a collective excitation in the IR. In the gravity description, $\mathcal{O}$ sources a supergravity field that propagates radially inward. We saw that as it moves deeper into the throat, its energy as observed at the boundary shifts towards the IR. In a sense this is just restating the equivalence of symmetries between the CFT and AdS. We can increase the scale of any of the boundary coordinates by a factor of l without affecting the CFT. We know this must be equivalent to an isometry of the AdS metric. But for the metric (4.8) to be unchanged, r must be scaled by a factor of l^{-1}. It is as if the extra radial dimension is imprinted on the boundary as the renormalization scale of the QFT. Indeed, we already saw that for the holographic principle to work, we had to identify $1/r$ with the UV cutoff.

4.2.7 Tests of AdS/CFT

There is an extensive body of work testing various aspects of the correspondence (e.g. [38–40]). Of course, direct tests are difficult, since phenomena on each side of the correspondence are worked out in opposite regimes of the coupling. Thus, tests are usually done on quantities that are independent of the coupling. These include certain correlation functions and the spectrum of chiral operators. Qualitative tests are more abundant where these so-called *holographic* string theory models are used as a framework for describing strongly coupled field theories. Some successes include: the discovery of a confining phase in super Yang-Mills at finite temperature, and the prediction that the entropy of this theory changes by a factor of 4/3 between strong and weak coupling. Incredible advances have been made in the application of AdS/CFT to super Yang-Mills hydrodynamics where it was discovered that the ratio of viscosity to entropy is $\eta/S = 1/4\pi$ [41]. Other accomplishments include: the computation of optical conductivity in $2+1$ dimensional super Yang-Mills at strong coupling [42], and multiple versions of holographic superconductivity [43, 44]. Finally, the confinement-deconfinement transition of super Yang-Mills has been shown to be equivalent to a Hawking-Page black hole transition [45].

This is but a small sampling of applications to come out of this field.

4.2.8 Finite temperature

Adding a finite temperature to the field theory introduces an energy scale which breaks conformal symmetry (and in fact supersymmetry)[4], however, as seen above, a slew of useful results have come from studying finite temperature AdS/CFT, so we will push on with a little willing suspension of disbelief.

In formulating black hole thermodynamics, it is natural to identify the surface gravity κ of a black hole with its temperature (in appropriate units). For one, thermal equilibrium occurs only if κ is constant over the horizon. For another, the change in energy of a stationary, non-spinning, neutral black hole is

$$dE = \frac{\kappa}{8\pi} dA, \tag{4.59}$$

[4]These are some of the issues that plague AdS/CMT (the application of the correspondence to condensed matter). Perhaps a bigger concern is that condensed matter usually deals with $U(1)$ and $SU(2)$ theories, which are not exactly "large N." Of course, if the strong form of the duality is true, this is no issue.

We have already identified the area as $A = 4S_{BH}$, so this would just be the TdS term in the first law of thermodynamics if we identify

$$\kappa = 2\pi T. \tag{4.60}$$

The surface gravity is defined from the time-like killing vector $\xi^a = (1, 0, 0, \ldots, 0)$ at the horizon:

$$\nabla^a(\xi^b \xi_b) = -2\kappa \xi^a. \tag{4.61}$$

Using, the killing equation, this can be written in a more convenient form: (see [46] for details)

$$\begin{aligned}
\kappa^2 &= -\frac{1}{2}(\nabla^a \xi^b)(\nabla_a \xi_b)|_{r \to r_h} && (4.62)\\
&= -\frac{1}{2} g^{a\mu} g_{b\nu} \Gamma^{\nu}_{\mu t} \Gamma^{b}_{at}|_{r \to r_h}. && (4.63)
\end{aligned}$$

Plugging in the D3 metric (4.19) and corresponding Christoffel symbols yields

$$\begin{aligned}
\kappa^2 &= \frac{(r^4 + r_h^4)^2}{r^6}|_{r \to r_h} && (4.64)\\
\Rightarrow T &= \frac{r_h}{\pi}. && (4.65)
\end{aligned}$$

This is the Hawking temperature for an AdS black hole. Since it is the only temperature around on the gravity side of the correspondence, it is natural to propose that this is the dual of the field theory temperature. As we turn on temperature in the CFT, a black hole emerges in the bulk. We can then follow the usual thermal field theory definition of a grand partition function $\mathcal{Z} = \exp(-\Omega/T)$, so that in the classical limit, we can identify Ω with the supergravity action. Note that we have used the symbol Ω in anticipation of including a finite density and chemical potential in the field theory. In that case, Ω is the grand potential for the grand canonical ensemble [47]. For our purposes, we will be interested in the canonical ensemble at fixed charge density, in which case we need to perform a Legendre transform of the supergravity action to get the Helmholtz free energy F. It is from the Helmholtz free energy that one can calculate the entropy of super Yang-Mills and get the 4/3 difference between strong and weak coupling mentioned earlier. It is an interesting unresolved question of how exactly the entropy interpolates between the strong and weak coupling values. Is there a discontinuous phase transition? See [48] for a discussion. We too will calculate the holographic entropy in chapter 7, but in a much more dressed up model.

4.2.9 Probes, flavours, and defects

Now we have a playground of possibilities to work with. If we start with a stack of N_c coincident D3 branes generating $AdS_5 \times S^5$ geometry, and then add N_q coincident Dq branes at radial infinity, we can effectively probe the supergravity solution provided $N_q << N_c$. To see the need for this limit, recall that when embedded in flat space-time (i.e. the boundary of AdS), the Dq branes will curve space-time according to the metric (4.8), with radius of curvature $L_q \sim H_q^{-1/4}$. Then for $N_q << N_c$, we have

$$H_q << H_c \tag{4.66}$$

$$\Rightarrow L_q >> L_c. \tag{4.67}$$

In this case, we can neglect the gravitational backreaction of the Dq branes and treat them as probes of the AdS geometry. Another nice thing about the probe limit, is that, provided we work to order $N_q N_c$, we can ignore quantum loop corrections to the fermion fields in what follows.

Strings suspended between the D3 stack and the Dq stack have a gravity description as strings stretched from the Dq stack to the Poincarè horizon. Though these strings have an infinite proper length, they have a finite energy. One can find a further test of the correspondence here if one lets $q = 3$, $N_q = 1$. Then the extra D3's world-volume action is the full DBI action (3.37) which contains higher derivative terms beyond Yang-Mills, only now the background metric that goes into the DBI action is AdS instead of flat space. But we have discussed this situation already. It is the Higgs symmetry breaking of $U(N_c + 1) \to U(N_c) \times U(1)$. The DBI action is precisely the effective action for the $U(1)$ fields. The strings connecting the lone brane to the N_c stack have an acquired a mass which is why we call them W-bosons. These have been integrated out in the DBI action, and now just source the $\mathcal{F}$ gauge fields living on the single brane. Indeed, if one calculates the one-loop corrections to the Higgs expectation value in this picture and in the field theory picture, one would find that they match [49].

If instead we keep all the D3s together, then the strings within the stack transform in the adjoint representation of $U(N)$. But we can decompose the adjoint into fundamental and anti-fundamental representations. For example, for $SU(3)$ this is just the tensor product decomposition $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$, where $\mathbf{3}, \bar{\mathbf{3}}, \mathbf{8}$ and $\mathbf{1}$ are the fundamental, anti-fundamental, adjoint and trivial representations respectively. We can understand this as each string endpoint being a charge in the fundamental/ anti-fundamental representation. Now again, we can separate some D3s so that a string has one endpoint on two separate branes. As viewed from the separated D3 world-volume, this string would form a fundamental representation field. It turns out that it transforms as a vector under the Lorentz group, so the title of W-boson is indeed very appropriate. But how do we generate a fundamental representation spinor field?

Suppose instead we introduce a stack of N_5 coincident D5-branes. We have some choice in how they are oriented. We can preserve the most supersymmetries by having four or eight dimensions spanned by only one type of D brane [32]. Thus, we are left with the following unique orientation (unique up to relabelling of coordinates):

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	X	X	X	X						
D5	X	X	X		X	X	X			

Table 4.2: D5 orientation

Here, Xs denote directions along which the branes extend, and blanks are directions in which the branes are point like.

The addition of D5s breaks half the supersymmetries. If they are separated from the D3 stack, the strings connecting the two groups form the $\mathcal{N} = 2$ hypermultiplet, which consists of two complex scalars as well as left and right chiral Weyl spinors. Note that the two brane-stacks overlap only in $2 + 1$ dimensions. This means the massless hypermultiplet string endpoints are confined to these dimensions. We have uncovered a *defect* field theory. In the dual description, we still have $\mathcal{N} = 4$ super Yang-Mills in $3 + 1$ dimensions, but now we also have an $\mathcal{N} = 2$ hyermultiplet confined to a $2 + 1$ dimensional defect. For the strings connecting the two stacks, one end transforms in the fundamental of $U(N_c)$, and the other transforms in the fundamental of $U(N_5)$ (there is no need to distinguish between the fundamental anti-fundamental here). If we consider the string as a whole, we say that it is a *bifundamental* field, since it transforms non-trivially under two different gauge groups [50]. We can consider this set-up either from a D3 world-volume perspective in which $U(N_5)$ is a global symmetry, or from the D5 perspective in which it is a gauge symmetry. We will take the latter approach.

So we now have a supergravity description of fermions propagating along a $2+1$ dimensional defect. Note, however, that this theory necessarily includes scalar fields. What symmetries remain in such a model? We still have 3D translation and Lorentz invariance as well as $SU(4)$ R-symmetry. In addition, it is (at least classically) still conformally invariant. The D5 world-volume fills AdS_4 as well as two of the S^5 dimensions. We will always fibrate S^5 into two 2-spheres with an angle ψ between them. The D5 wraps one of these 2-spheres and is point-like on the other. Thus, the isometries of its world-volume must be $SO(3,2) \times SO(3) \times \tilde{SO}(3)$, where we distinguish between the isometry $SO(3)$ of the S^2 wrapped by the D5s, and $\tilde{SO}(3)$, the isometry of the unwrapped S^2.

The full theory is still 4-dimensional, and some work needs to be done to ensure that a set of fields transforms like the 3D hypermultiplet when confined to the defect [51]. It turns out that these symmetries are enough to completely specify the field theory action. The result has been worked out in [52]. It includes a variety of Yukawa couplings and potentials, but if we look at the weak coupling, massless regime (actually set the coupling to zero), it reduces to just a kinetic term

$$S_{\text{weak}} \int d^3x[(D^k q^m)^\dagger D_k q^m - i\bar{\psi}^i \rho^k D_k \psi^i], \tag{4.68}$$

where q is the complex scalar, ψ is a spinor representation of $\tilde{SO}(3)$, D_k is a gauge covariant derivative containing the 4-dimensional gauge field, and ρ^k are gamma matrices.

Now, we can ask what would happen if we made these fermions massive. We know this requires separating the D5 and D3 stacks from each other along a transverse direction. This would necessarily break the $\tilde{SO}(3)$ symmetry because it would single out one of the $\tilde{S}^2$ directions. The corresponding fermions arising from strings attached to each stack would gain a mass that breaks chiral symmetry. Thus, we can associate $\tilde{SO}(3)$ with the chiral symmetry of this model. It turns out that for our purposes, chiral symmetry breaking will be very important, but we will see how it can arise in a more organic way. For now, we set the fermion mass to zero, yielding a gapless, defect field theory.

Chapter 5

The Quantum Hall Effect

It is time to ask exactly what strongly coupled system we would like to probe with the gauge/gravity machinery we have developed. Our goal is to investigate the fate of the quantum Hall phenomenon as the coupling becomes strong. First let us look at the phenomenon in detail to see exactly which parts we can capture holographically.

5.1 Condensed matter approach

5.1.1 Semi-classical description

In the familiar setup of the Hall effect, one puts a current-carrying semi-conductor in a perpendicular magnetic field $B\hat{z}$. The current is deflected by the magnetic field until charge builds up on the edge of the semi-conductor and establishes a Hall electric field to precisely counteract the Lorentz force

$$\vec{E}_{\text{Hall}} + \vec{v}_d \times \vec{B} = 0, \tag{5.1}$$

where $\vec{v}_d$ is the drift velocity of the electrons.

For an initial current in the $\hat{y}$-direction, we have $J_y = \rho v_d$ (where ρ is the charge density), and the Hall equation becomes

$$\begin{aligned} E_x - v_d B &= 0 && (5.2)\\ \Rightarrow v_d &= \frac{E_x}{B} && (5.3)\\ \Rightarrow J_y &= \frac{\rho}{B} E_x. && (5.4) \end{aligned}$$

This is just Ohm's law, with an off-diagonal piece of the conductivity tensor

$$\sigma_{xy} = \frac{\rho}{B}. \tag{5.5}$$

For fixed magnetic field, the conductivity just increases linearly with charge density.

We could also turn off the current and consider electrons confined to a 2D surface. Then the Lorentz deflections will force the electrons into cyclotron motion with a well defined frequency $\omega = \frac{eB}{m}$. We might expect there to be a corresponding harmonic potential so that at the quantum level, the electron energy is quantized as the harmonic oscillator levels. We will see that this is indeed the case. Note that the edges of the sample will exert an impulse that prevents elections there from completing circles. These electrons will travel in semi-circles, inducing a current along the edge.

Suppose now we increase the charge density. Then the drift velocity will decrease and so will the cyclotron radius $r = \frac{mv_d}{eB}$, but the conductivity will increase. Eventually, the cycles will shrink to the point where impurity effects become important, whereupon the conductivity will plateau until the Fermi energy is raised sufficiently. To observe these effects requires small radii and thus strong magnetic fields, as well as low temperatures. This is why it was not observed until 1980 [53].

5.1.2 Quantized conductivity

To make more sense of the observed conductivity behaviour, we must move to a quantum picture. If we define a ratio

$$\nu = 2\pi\frac{\rho}{B}, \tag{5.6}$$

then it is observed that as one tunes a control parameter (ρ or B), the conductivity becomes stuck at $\frac{\nu}{2\pi}$ for integer values of ν. When electron interactions are important, there is an additional "fractional quantum Hall effect" for rational values of ν that we will not discuss here.

There are two important ingredients for the traditional integer Hall effect. One is that the electron has gapped states in its spectrum. The other is the presence of disorder to ensure that such states exist over a range of control parameter (forming a conductivity plateau). The Fermi energy is determined by the control parameters, and disorder gives a finite range of energies over which localized states exist. As we tune the parameter in the plateau region, the Fermi energy moves through this range. Later we will speculate on an alternative way in which plateaus may appear holographically that has nothing to do with disorder. In the next section, we will see that the presence of the charge gap and the quantized conductivity are due to the organization of electrons into Landau levels in the presence of the magnetic field.

5.1.3 2D electron gas model

The Hamiltonian for N electrons in a gauge potential $\vec{A}(\vec{r})$ is given by

$$\mathcal{H} = \frac{1}{2m^*}\sum_{i=1}^{N}(\vec{P}_i + e\vec{A}(\vec{r}_i))^2, \tag{5.7}$$

where m^* is the electron's effective mass.

With a magnetic field $\vec{B} = B\hat{y}$, we are free to choose the Landau gauge for the vector potential $\vec{A} = Bx\hat{y}$, in which case,

$$\mathcal{H} = \frac{1}{2m^*}\sum_{i=1}^{N}\Big(- \nabla_i^2 - 2iBx_i\frac{\partial}{\partial y_i} + e^2B^2x_i^2\Big). \tag{5.8}$$

Translational invariance in y gives the Schrödinger one-particle ansatz $\psi(t,\vec{x}) = \psi(x)e^{i(k_y y - Et)}$, in which case we can write the Schrödinger equation as

$$\Big(- \frac{1}{2m^*}\frac{d^2}{dx^2} + \frac{e^2B^2}{2m^*}(x + \frac{k_y}{eB})^2\Big)\psi(x) = E\psi(x). \tag{5.9}$$

But this is simply the Schrödinger equation for a one-dimensional harmonic oscillator centred at $x_0 = \frac{-k_y}{eB}$, with frequency $\omega = \frac{eB}{m^*}$, which is the cyclotron frequency we found earlier. The spectrum of energies are then just $E_n = \omega(n + 1/2)$. Each value of $n = 0, 1, 2, \ldots$ corresponds to a different Landau level. Let the sample have dimensions $L_x \times L_y$. Imposing periodic boundary conditions in y requires $k_y = 2\pi m/L_y$, $m \in \mathbb{Z}$. So neighbouring oscillators are separated by $\Delta x = \frac{2\pi}{L_y B}$. Then each Landau level contains $\frac{L_x}{\Delta x} = \frac{L_x L_y B}{2\pi}$ states. So the Landau levels are highly degenerate in strong magnetic fields. The number of states per unit area is $\frac{B}{2\pi}$, and we can define the filling fraction ν as the fraction of a Landau level that is occupied.

$$\nu = \frac{\text{electrons/area}}{\text{states/Landau Level/area}} = 2\pi\frac{\rho}{B}, \tag{5.10}$$

which is what we defined in (5.6).

Thus at integer ν, a Landau level is completely filled. There is then an energy gap of ω to the next Landau level. If the density is increased further, electrons begin to occupy the next Landau level and the chemical potential must jump across the gap. Thus, at integer filling fraction, the chemical potential has a discontinuity as a function of ρ. If we calculate the compressibility $\kappa = [\rho^2 \frac{d\mu}{d\rho}]^{-1}$, we see that it vanishes at these points and so the state is said to be *incompressible*. Away from these points it only takes an infinitesimal amount of energy to excite the electrons above the Fermi level, but at these points, it costs ω in energy.

The derivation of quantized conductivities is a triumph of a gauge argument due to Laughlin and Halperin [54, 55]. The quantum Hall effect is observed to be robust for all sample geometries, so we can imagine bending the sample to form an annulus. Now instead of inducing a current with an applied voltage, we can put a flux $\phi = -q\phi_0$, with $\phi_0 = \frac{h}{e}$, through the centre of the annulus. If a wave function extends around the annulus, it will pick up a phase $k_y L_y$ as y goes from 0 to L_y, as well as an Aharonov-Bohm phase $-q\phi_0$. The wave function is single-valued only if $q \in \mathbb{Z}$. If this is not the case, we do not get extended wave-functions.

Again we are free to change the geometry back to the strip of length L_y. We do so with an electric field along the width of the strip to get

$$\mathcal{H} = \frac{1}{2m^*}\left\{\left[\vec{P} + eB\left(c + \frac{q\phi_0}{BL_y}\right)\hat{y}\right]^2 + eE_x x\right\}. \tag{5.11}$$

If we slowly turn on one unit of flux, $q \to 1$, then each electron shifts by Δx, i.e. one oscillator site in the x-direction. Of course, only the *occupied* states move over, so this process is equivalent to moving one electron from each Landau level across the strip width. The net charge transferred is exactly $-ne$ where $n = \lfloor \nu \rfloor$ is the number of occupied Landau levels. Then using Faraday's law,

$$\int_c \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} \tag{5.12}$$

$$\Rightarrow \int_c dy \rho_{yx} j_x = -\frac{d\phi}{dt} \tag{5.13}$$

$$\Rightarrow \rho_{yx} \int dt J_x = -\phi_0, \tag{5.14}$$

where c is a contour enclosing the flux, and j_x is the current density. But $\int dt J_x$ is just the net charge transferred, $-ne$, so

$$\rho_{yx} = \frac{\phi_0}{ne} \tag{5.15}$$

$$\Rightarrow \sigma_{yx} = \frac{n}{h/e^2} = \frac{\lfloor \nu \rfloor}{2\pi}, \tag{5.16}$$

where in the last line, we switched to natural units, $h/e^2 = 2\pi$. From here on out, we will maintain natural units.

For wave-functions that are localized at scales well below the length L_y, there is no special flux required for single-valuedness, and in fact these states cannot carry current[1]. This is precisely what happens when disorder is introduced, producing random scattering off impurities. This scattering also broadens the Landau levels into bands (which we will assume are smaller than the energy gap). In fact, a theorem due to Philip W. Anderson [56], shows that arbitrarily weak impurities in a 2D system force states to be localized. Of course, there must be some extended state or else no current will flow. The presence of the magnetic field helps us out, and it turns out that at the centre of each Landau level, the localization scale diverges. Thus, as we move the Fermi energy (via a control parameter) through the localized part of the Landau level, the conductivity remains unchanged until the Fermi energy passes through the extended state energy, whereupon the conductivity jumps to the next plateau.

An argument of Lorentz covariance also tells us that disorder is necessary for the formation of plateaus at zero temperature. Suppose we did not have impurities. Then the system would have translation invariance. If we sit in a *partially* filled Landau level with charge density $\rho = \frac{\nu}{2\pi}B$, then by boosting into a frame with velocity v_i, we observe a current $J_i = \rho v_i$. The external magnetic field in this frame comes with a transverse electric field

$$E_i = -\epsilon_{ij} v_j B \tag{5.17}$$

$$= -\epsilon_{ij} \frac{2\pi}{\nu} J_j. \tag{5.18}$$

So the conductivity would just be the classical value. Since this is independent of the boost velocity, it should hold in all frames. Thus we cannot have plateaus in a translationally invariant system at zero temperature. Interestingly, for finite temperature, the system is coupled to a heat bath. This gives a preferred reference frame, destroying translation invariance and opening up the possibility of non-impurity driven plateaus.

5.2 D3-D5 system at weak coupling

Now we will see how this relates to the holographic D3-D5 system we began to discuss in the last chapter. We saw that the corresponding (one-particle) field theory for this system had a weak coupling action with a Dirac spinor ψ and complex scalar q

$$S_{\text{weak}} = \int d^3[|D_\mu q|^2 - i\bar{\psi}\gamma^\mu D_\mu \psi], \tag{5.19}$$

[1] One can show that the current operator is proportional to $\frac{\partial \mathcal{H}}{\partial q}$. Since the spectrum of a localized wave-function does not depend on q (there is no Aharonov-Bohm phase), these states cannot carry current.

with $D_\mu = \partial_\mu - iA_\mu$. If we work out the equations of motion for the spinors, we get

$$i\gamma^\mu(\partial_\mu + iA_\mu)\bar{\psi} = 0. \tag{5.20}$$

If we want to immerse the fermions in an external B-field, we use (in the Landau gauge) $A^\mu = (0, 0, Bx)$. Furthermore, we break up the Dirac spinor into two Weyl spinors $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$. The equations of motion are invariant under translations in y, so we can write the solution ansatz as $\psi_\pm = \psi_\pm(x)e^{i(k_y y - Et)}$. Now the equations of motion can be written in the Schrödingier form

$$\pm E\psi_\pm(x) = \pm i\sigma^1\partial_x\psi_\pm(x) \pm i\sigma^2(\partial_y + iBx)\psi_\pm(x), \tag{5.21}$$

where σ^i are Pauli matrices. Combining the $+$ and $-$ equations and defining $H_0 \equiv \sigma^1\partial_x + \sigma^2(\partial_x + iBx)$, we get,

$$\begin{aligned} -E\psi_-(x) &= -iH_0\psi_+ & (5.22)\\ &= \frac{1}{E}H_0^2\psi_-(x) & (5.23)\\ \Rightarrow E^2\psi_\pm(x) &= (\mathbb{I}\partial_x^2 + \sigma^1\sigma^2(iB) + \mathbb{I}(\partial_y + iBx)^2)\psi_\pm(x). & (5.24) \end{aligned}$$

Completing the square yields

$$(E^2 + B)\psi_\pm^1(x) = \left(\frac{d^2}{dx^2} - B^2\left(x - \frac{k_y}{B}\right)^2\right)\psi_\pm^1(x), \tag{5.25}$$

where we have focused on the first component of each Weyl spinor $\psi_\pm^1$ for simplicity.

This is again the same form as the 1D harmonic oscillator equation

$$\left(\frac{-1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)\psi = \tilde{E}\psi, \tag{5.26}$$

provided we make the following identifications.

$$m = -1/2; \;\; \omega^2 = 4B^2; \;\; \tilde{E} = E^2 + B. \tag{5.27}$$

The harmonic oscillator spectrum, $\tilde{E} = \omega(n + 1/2)$, $n = 0, 1, 2, \ldots$, then gives us the energy spectrum of these fermions

$$\begin{aligned} E^2 + B &= 2B(n + 1/2) & (5.28)\\ \Rightarrow E &= \sqrt{2Bn}. & (5.29) \end{aligned}$$

So the fermions in this theory also have Landau levels, but with an energy gap $\sqrt{2B}$. It is a promising candidate for quantum Hall states.

But we have neglected the scalars! We can repeat the same procedure for them. We get the equation of motion

$$\partial_\nu(\partial^\nu - iA^\nu q) = i(\partial_\mu q)A^\mu + A^2 q. \tag{5.30}$$

Using the same ansatz, $q = q(x)e^{i(k_y y - Et)}$, this can also be written in the form of a harmonic oscillator equation

$$E^2 q(x) = \left(-\frac{d^2}{dx^2} + B^2\left(x - \frac{k_y}{B}\right)^2 \right) q(x). \tag{5.31}$$

The parameters are the same as for the fermions, except now the energy correspondence is

$$E^2 = \tilde{E} \tag{5.32}$$

$$\Rightarrow E = \sqrt{B(2n+1)}. \tag{5.33}$$

We see that the bosons do not have a zero energy mode. If we only look at the first fermionic Landau level, ($n = 0$), then we will never excite the bosons since their threshold energy is $\sqrt{B}$.

Note that the oscillators are positioned exactly as in the 2D electron gas model, with the same spacing between centres. So we have the same number of oscillators per Landau level. However, in the D3-D5 theory, the spinors have a colour index from $SU(N_c)$, a flavour index from $SU(N_5)$ and an isospin index from $\tilde{SO}(3)$. Thus, the total Landau level degeneracy is $2N_5N\frac{B}{2\pi}$.

So we have a set of Landau levels for each of the N_5 flavour states and each of the two isospin states (ignore the colour index for now). Suppose we set $N_5 = 1$ and allow a small Coulomb interaction between the fermions. This will make an anti-symmetric spatial wave-function preferable. But the full fermionic wave-function must be anti-symmetric, so that means the fermions will prefer a symmetric isospin configuration. The result is a small gap between the first Landau level of each isospin state. This effect is known as *quantum Hall ferromagnetism*[2]. The resulting picture has broken $\tilde{SO}(3)$ chiral symmetry, so this effect may also be called magnetic catalysis of chiral symmetry breaking. Either way, it results in the formation of a chiral condensate and thus we expect the theory to be confining. So we should not consider the fermions as individual

[2]The sample is ferromagnetic in the sense that the isospins become spontaneously aligned in the presence of a magnetic field.

quarks, but rather as being bound together in groups of N to form baryons. This is why we ignored the colour index above. This also means we have to normalize the filling fraction. The expression in (5.6) gives the number of quarks in a Landau level divided by the total number of states in that level. Now we want the number of *baryons* in each level. Thus,

$$\nu = \frac{2\pi}{N}\frac{\rho}{B}. \tag{5.34}$$

This ferromagnetic effect means that the neutral ground state (where the first Landau level of one isospin state is full), $\nu = 0$ is gapped. The next gapped state occurs at $\nu = 1$, when both isospin states are full. Beyond that, we get the second Landau level ($n = 1$) with energy $\sqrt{2B}$. However, this is greater that the energy required to excite bosons. Once we let the bosons loose, there will be no more gapped states since they have no Pauli exclusion principle.

If we allow the full N_5 flavours, we can potentially get $2N_5 + 1$ Hall states with $\nu = 0, \pm 1, \ldots, \pm N_5$[3] provided the resolution of the degeneracy does not shift any of these Landau levels above the boson ground state energy. The corresponding quantized conductivities are then $\sigma_{xy} = \pm\frac{1}{2\pi}, \pm\frac{2}{2\pi}, \ldots, \pm\frac{N_5}{2\pi}$. In this thesis, we will find that these gapped states persist at strong coupling. Furthermore, the conductivity remains unchanged at integer filling fraction, but will deviate slightly for non-integer ν when the temperature is finite.

Finally, we should mention that quantum Hall ferromagnetism is observed in graphene, though it has an effective $SU(4)$ symmetry instead of $SO(3)$ [57]. At sufficiently strong magnetic fields, and pure samples, the four-fold degeneracy is completely lifted, introducing four new gapped staes and corresponding conductivity plateaus [58]. Of course, in arguing for this effect, we have assumed that the Coulomb interactions are weaky coupled so that we could ignore them in the Hamiltonian. But the fine structure constant of graphene is $\frac{e^2}{4\pi\hbar v_f} \approx \frac{300}{137}$, where v_f is the speed of light in graphene. This suggests that the Coulomb interaction is very strongly coupled. So there must be a framework which describes how this phenomenon persists at strong coupling. This is of course the gravitational description of the D3-D5 system that we develop in the next chapter.

[3] Particle-hole symmetry dictates that there should be gapped states with negative integer ν as well.

Chapter 6

A Giant D5 Model

Here we will outline the holographic giant D5 model developed in [2], that will be used to explore the fate of the D3-D5 field theory at strong coupling.

6.1 D3-D5 becomes D3-D7

6.1.1 D3-D5 at strong coupling

Exactly what ingredients do we need for a holographic quantum Hall model? We have seen that some of the important physics are captured by the D3-D5 model at weak coupling. Going to the gravity side, we will need to bring with us three important components. First, to look at thermodynamics, we need a finite temperature and therefore a finite horizon radius r_h in the AdS black hole geometry. Second, the magnetic field B is external to the system, and so will be maintained as is throughout the bulk. We will leave it in the Landau gauge, but change the normalization so that

$$A_y = \frac{1}{\sqrt{\lambda}\alpha'} Bx \equiv \frac{1}{2\pi\alpha'} bx. \tag{6.1}$$

Finally, we want to have a finite charge density ρ so that we can explore different values of ν. From the AdS/CFT dictionary, we know that the global $U(1)$ conserved current J^μ is dual to a local $U(1)$ gauge field A_μ. Thus to have a finite charge density $\rho = \langle J^t \rangle$, we need to include a temporal component A_t in this field. For a uniform charge density, this component cannot depend on the x and y coordinates, and there is no obvious need for it to vary over the 5-sphere, so we will assume it is independent of these coordinates.

It may, however, vary in the IR, so we had better keep a radial dependancy

$$A_t = \frac{\sqrt{\lambda}}{2\pi} a(r). \tag{6.2}$$

Now for the geometry. We start with the $AdS_5 \times S^5$ background metric generated by the D3 stack (4.19) and pullback to the D5 world-volume which spans $AdS_4 \times S^2$. That leaves embedding coordinates for the unwrapped sphere $\tilde{S}^2$, as well as z, and one more angular coordinate ψ. This is the defect orientation on page 70. We want to preserve chiral symmetry for now, so we will let the $\tilde{S}^2$ embedding be constant. Furthermore, we will keep the $SO(3)$ rotational invariance and translational symmetry, meaning that no embeddings will depend on x, y or the S^2 coordinates. This leaves two dynamical embedding functions $z(r)$ and $\psi(r)$. The $z(r)$ embedding will appear in the Lagrangian as a cyclic variable, and the solution to its equation of motion will come with a constant of integration that we are free to set to zero. Doing so will set $z(r)$ to be a constant and thus can be ignored it in the following. The resulting world-volume metric is (from here on primes will denote differentiation with respect to r):

$$\begin{aligned} ds^2 &= \sqrt{\lambda}\alpha' \Bigg[r^2(-h(r)dt^2 + dx^2 + dy^2) + \frac{dr^2}{h(r)r^2}\Big(1 + h(r)(r\psi'(r))^2 + h(r)^2(r^2 z'(r))^2\Big) \\ &\quad + \sin^2\psi(r)(d\theta^2 + \sin^2\theta d\phi^2)\Bigg]. \end{aligned} \tag{6.3}$$

In this setup with no Kalb-Ramond field, the world world-volume field strength is

$$2\pi\alpha'\mathcal{F} = \sqrt{\lambda}\alpha'(a'(r)dr \wedge dt + b dx \wedge dy). \tag{6.4}$$

But there is one more ingredient we need: an R-R 4-form $C^{(4)}$ coming from the D3 stack, or equivalently the Type IIB supergravity solution. We will use the most general form that we derived in (4.30). Then the DBI - Wess-Zumino action for N_5 of these D5 branes is

$$\mathcal{S}_5 = \frac{T_5}{g_s} N_5 \int d^6\sigma \left(-\sqrt{-\det(g + 2\pi\alpha'\mathcal{F})} + g_s 2\pi\alpha' C^{(4)} \wedge \mathcal{F}\right), \tag{6.5}$$

where $T_5 = \frac{1}{(2\pi)^5\alpha'^3}$, and $g_s = \frac{g_{YM}^2}{4\pi} = \frac{\lambda}{4\pi N_c}$.

For now the last term in (6.5) is totally innocuous, since the volume elements cannot match the world-volume element of the D5. Thus we can ignore it. However, for the D7 it will become very important.

Evaluating this action, we get

$$\mathcal{S}_5 = -2\pi \frac{T_5}{g_s}(\sqrt{\lambda}\alpha')^3 V_{2+1} N_5 \int_{r_h}^{\infty} dr 2 \sin^2\psi \sqrt{b^2+r^4}\sqrt{1+h(r)(r\psi'(r))^2 - a'(r)^2}, \quad (6.6)$$

where V_{2+1} is the volume of the defect. The temporal gauge field is cyclic, so its equation of motion is simply

$$a'(r) = \frac{q_5\sqrt{1+h(r)(r\psi'(r))^2}}{\sqrt{4\sin^4\psi(r)(b^2+r^4)+q_5^2}}, \quad (6.7)$$

where q_5 is a constant of integration. What is the interpretation of this constant? We know that A_t is dual to J^t, and that the precise relation is given by the GKPW rule to calculate correlation functions. We are told (in the classical supergravity regime) that for each J^t operator that appears in the correlation function, we differentiate the on-shell action with respect to the boundary value of the dual field once. $\rho = \langle J^t \rangle = \frac{1}{V_{2+1}} \frac{\delta S_5}{\delta A_t(\infty)}$. Variation of the action gives

$$\begin{aligned} \delta S &= \int_{r_h}^{\infty} dr \frac{\delta \mathcal{L}}{\delta(\partial_r A_t)} \delta \partial_r A_t & (6.8) \\ &= \delta A_t \frac{\delta \mathcal{L}}{\delta(\partial_r A_t)}\bigg|_{r_h}^{\infty} - \int_{r_h}^{\infty} dr \partial_r \frac{\delta \mathcal{L}}{\delta(\partial_r A_t)} \delta A_t. & (6.9) \end{aligned}$$

The last term vanishes by the equation of motion, and we get

$$\begin{aligned} \rho &= 2\pi \frac{T_5}{g_s}(\sqrt{\lambda}\alpha')^3 N_5 \frac{\delta \mathcal{L}}{\delta(\partial_r A_t)}\bigg|_{\infty} & (6.10) \\ &= q_5 \frac{2N_c N_5}{(2\pi)^2}. & (6.11) \end{aligned}$$

So q_5 defines the total charge density of the field theory.

As mentioned before, the on-shell action describes the grand potential. For the canonical ensemble, we want to find the Helmholtz free energy

$$F = \Omega + \mu N, \quad (6.12)$$

where $N = \rho V_{2+1}$ is the total number of baryons. μ is the chemical potential, and has the holographic dual $A_t(\infty)$[1] since this is the term that sources the charge density operator on the boundary. Thus we can write the free energy as

$$F_5 = \frac{2NN_5}{(2\pi)^2} V_{2+1}\left(\frac{\sqrt{\lambda}}{2\pi}\int_{r_h}^{\infty} dr \mathcal{L} + q_5\mu\right). \quad (6.13)$$

[1] This is not necessarily true for non-uniform charge density in the bulk [59]

Now the time-component of the gauge field must vanish at the horizon because the killing vector ∂_t vanishes there. Thus,

$$\mu = \frac{\sqrt{\lambda}}{2\pi}(a(\infty) - a(r_h)) \tag{6.14}$$
$$= \frac{\sqrt{\lambda}}{2\pi}\int_{r_h}^{\infty} a'(r)dr. \tag{6.15}$$

So then,

$$\begin{aligned} F_5 &= \frac{2NN_5}{(2\pi)^2}V_{2+1}\int_{r_h}^{\infty} dr\sqrt{1+h(r)(r\psi'(r))^2}\sqrt{4\sin^4\psi(b^2+r^4)+q_5^2} \\ &= \tilde{\mathcal{N}}_5\int_{r_h}^{\infty} dr\sqrt{1+h(r)(r\psi'(r))^2}\sqrt{4\sin^4\psi f^2(1+r^4)+(\pi\nu)^2}. \end{aligned} \tag{6.16}$$

In the last line, we scaled out b using

$$r \rightarrow r/\sqrt{b} \tag{6.17}$$
$$r_h \rightarrow r_h/\sqrt{b} = \pi\sqrt{\frac{\sqrt{\lambda}}{2\pi B}}T \tag{6.18}$$
$$\tilde{\mathcal{N}}_5 \equiv \frac{2\lambda^{1/4}N_c}{(2\pi)^4}V_{2+1}(2\pi B)^{3/2}. \tag{6.19}$$

We have also made use of the parameters $f \equiv \frac{2\pi}{\sqrt{\lambda}}N_5$, and the filling fraction $\nu = \frac{2\pi\rho}{N_c B}$.

The meat of the problem now is to determine the solution $\psi(r)$ from the Euler-Lagrange equation

$$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial\psi'(r)} = \frac{\partial\mathcal{L}}{\partial\psi}. \tag{6.20}$$

For this, it is convenient to define a function $V_5 = 4\sin^4\psi f^2(1+r^4)+(\pi\nu)^2$ so that we can write the Lagrangian as $\mathcal{L} = \sqrt{V_5}\sqrt{1+hr^2(\psi'(r))^2}$. Then the equation of motion reads

$$\frac{d}{dr}\left[\sqrt{V_5}\frac{h(r)r^2\psi'(r)}{\sqrt{1+h(r)r^2(\psi'(r))^2}}\right] = \frac{\partial_\psi V_5}{2\sqrt{V_5}}\sqrt{1+h(r)r^2(\psi'(r))^2}. \tag{6.21}$$

Consider the series expansion of $\psi(r)$ at infinity $\psi = \psi_\infty + \frac{c_1}{r} + \frac{c2}{r^2} + \ldots$. Plugging this into the equation above gives, to leading order

$$\left.\frac{\partial_\psi V_5}{\sqrt{V_5}}\right|_{r\rightarrow\infty} = 0, \tag{6.22}$$

This equation is satisfied by $\psi_\infty = \pi/2$, which is the condition we will use throughout the rest of the thesis. The series expansion above indeed satisfies the asymptotic equation of motion. By matching conformal dimensions for the constants c_1 and c_2, one finds that they correspond to the bare fermion mass (with conformal dimension equal to 1) and chiral condensate (with conformal dimension equal to 3) in the field theory. Again, we do not want to break chiral symmetry explicitly, so we use massless fields $c_1 = 0$, and let c_2 be determined dynamically. Upon solving the equation of motion numerically, it turns out that for the range of parameters we are interested in, one gets the non-constant solution $c_2 \neq 0$, and thus chiral-symmetry is spontaneously broken. This turns out to be true whenever $\nu < 1.68f$. The full phase transition was investigated in [60]. Thus, we see that quantum Hall ferromagnetism has a strong coupling holographic description as a D5 developing a non-trivial embedding.

6.1.2 D7 as a giant D5

So far we have a model of N_5 D5 branes with world-volume gauge fields coupled to an external R-R 4-form. But we know from our discussion of the Myers effect, that Dp branes in an external R-R potential can puff out to form a D$(p+2)$ brane. Thus we might expect that under certain conditions, the D5 stack will puff out from the boundary of AdS to form a D7 brane that wraps the extra 2-sphere[2]. We know that in the Myers effect, the lower dimensional brane is interpreted in the higher dimensional brane world-volume as a unit of magnetic flux. From (3.47), we know that this will appear in the gauge field strength as

$$F_{\theta\phi} = \frac{N_5}{2}\sin\theta, \tag{6.23}$$

in which case he full D7 field strength will appear as

$$2\pi\alpha'\mathcal{F} = \sqrt{\lambda}\alpha'\bigg(\frac{d}{dr}a(r)dr\wedge dt + bdx\wedge dy + \frac{f}{2}\sin\tilde{\theta}d\tilde{\theta}\wedge d\tilde{\phi}\bigg). \tag{6.24}$$

The DBI/ Wess-Zumino action is now

$$\mathcal{S}_7 = \frac{T_7}{g_s}\int d^8\sigma\bigg(-\sqrt{-\det(g+2\pi\alpha'\mathcal{F})} + g_s\frac{(2\pi\alpha')^2}{2}C^{(4)}\wedge\mathcal{F}\wedge\mathcal{F}\bigg). \tag{6.25}$$

Note that unlike the D5 case, the Wess-Zumino term for the D7 has a non-zero contribution

$$C^{(4)}\wedge\mathcal{F}\wedge\mathcal{F} = a'(r)bc(\psi)d\theta\wedge d\phi\wedge d\tilde{\theta}\wedge d\tilde{\phi}\wedge dr\wedge dt\wedge dx\wedge dy, \tag{6.26}$$

where $c(\psi)$ is defined in (4.29). The non-radial directions are integrated over to give a factor of $4V_{2+1}(2\pi)^4$.

We orient the D7-brane as shown in the table below, where x^3 is the z direction and x^9 is the ψ direction.

[2]Really this 2-sphere is an approximation to the non-abelian fuzzy sphere configuration of the D5 branes.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	X	X	X	X						
D7	X	X	X		X	X	X	X	X	

TABLE 6.1: D7 orientation

The $z(r)$ embedding is again satisfied by a constant, and the world-volume metric is then

$$\begin{aligned} ds^2 &= \sqrt{\lambda}\alpha'\Bigg[r^2(-h(r)dt^2+dx^2+dy^2)+\frac{dr^2}{h(r)r^2}\Big(1+h(r)(r\psi'(r))^2+h(r)^2(r^2z'(r))^2\Big) \\ &\quad +\sin^2\psi(r)(d\theta^2+\sin^2\theta d\phi^2)+\cos^2\psi(d\tilde{\theta}^2+\sin^2\tilde{\theta}d\tilde{\phi}^2))\Bigg]. \end{aligned} \tag{6.27}$$

Note that the radius of the second 2-sphere is $\cos^2\psi$. The full sphere coordinates must satisfy

$$x^1+x^2+x^3+x^4+x^5=1. \tag{6.28}$$

By choosing the radius of the first 2-sphere to be $\sin^2\psi$, we have $x^1+x^2+x^3=\sin^2\psi$. So in order to satisfy (6.28), we must have $x^4+x^5+x^6=\cos^2\psi$.

Now we can repeat the same analysis as above, for the D7. The action becomes

$$\begin{aligned} \mathcal{S}_7 &= -\frac{2\lambda N_c}{(2\pi)^4}V_{2+1}\int_{r_h}^{\infty}dr\Bigg(2\sin^2\psi\sqrt{(f^2+4\cos^4\psi)(b^2+r^4)}\sqrt{1+h(r\psi')^2-a'(r)^2} \\ &\quad +2a'(r)bc(\psi)\Bigg). \end{aligned} \tag{6.29}$$

The $a(r)$ equation of motion now reads:

$$\frac{2\sin^2\psi\sqrt{(f^2+4\cos^4\psi)(b^2+r^4)}a'(r)}{\sqrt{1+h(r\psi'(r))^2}}-2bc(\psi)=q_7. \tag{6.30}$$

Again, we Legendre transform the on-shell action to get the free energy and scale out b.

$$\begin{aligned} F_7 &= \tilde{\mathcal{N}_7}\int_{r_h}^{\infty}dr\sqrt{1+h(r\psi'(r))^2} \qquad (6.31)\\ &\quad \times\sqrt{4\sin^4\psi(f^2+4\cos^4\psi)(1+r^4)+(\pi(\nu-1)+2\psi-1/2\sin(4\psi))^2}, \end{aligned}$$

where $\tilde{\mathcal{N}_7}\equiv\frac{2\lambda N_c}{(2\pi)^4}V_{2+1}\left(\frac{2\pi B}{\sqrt{\lambda}}\right)^{3/2}=\tilde{\mathcal{N}_5}$.

The equation of motion for $\psi(r)$ is identical to the D5 case, (6.21), only with V_5 replaced by

$$V_7=4\sin^4\psi(f^2+4\cos^4\psi)(1+r^4)+\left(\pi(\nu-1)+2\psi-\frac{1}{2}\sin 4\psi\right)^2. \tag{6.32}$$

At large r, the equation of motion reduces to

$$\left.\frac{\partial_\psi V_7}{\sqrt{V_7}}\right|_{r\to\infty} = 0. \tag{6.33}$$

$\psi_\infty = \pi/2$ is also a solution to this equation, though it has another given by

$$f^2 + 4\cos^4\psi = 4\sin^2\psi\cos^2\psi \tag{6.34}$$

$$\Rightarrow \psi = \frac{1}{2}\arccos\left(\frac{1}{2}\sqrt{1-2f^2} - 1\right). \tag{6.35}$$

Since this solution only works for small values of f, ($f < 1/\sqrt{2}$), we will use the first solution which does not restrict our parameters. The latter solution has been investigated in [61]. More importantly, if we look at the the free energy, (6.31), the integrand becomes identical to that of the D5 stack for $\psi \to \pi/2$. So with this choice of boundary condition at infinity, the D7 brane reduces to N_5 D5s at the boundary of AdS. It really is like a giant D5 brane.

There is another remarkable thing to notice about (6.31), and that is $\nu = 1$ is a special value. It is the only value at which the D7 can cap off before entering the horizon at some radius r_0, in a so-called *Minkowski embedding.* To see this, note that for such an embedding, the profile $\frac{d\psi}{dr}$ will diverge at r_0. Then the only way to keep the free energy finite, is if $\nu = 1$ and $\psi(r_0) = 0$. The latter condition is indeed verified by numerically solving the equation of motion for $\psi(r)$. The Minkowski embedding enforces a minimum length for strings stretching from the probe brane to the horizon, and thus forms a charge gapped state. So we see that this system recovers the essential property of a quantum hall system that we were looking for. A gapped state occurs only at integer filling fraction. Note that this argument only works for $\nu = 1$. To get the higher integer Hall states one must consider multiple D7 branes with the D5 flux split between them. This is investigated in [2, 4].

6.1.3 D5-D7 phase diagram

To determine if the blow up to a D7 really occurs, we must know if the D7 system outlined above is energetically preferable to the D5 system outlined in section 6.1.1. This requires computing the on-shell free energy in each case. First one must compute the embedding solution $\psi(r)$ numerically. This can essentially be done with a shooting method. Some results are given in section 7.2. The integral over r in the free energies is formally divergent, but a comparison between F_5 and F_7 only requires their difference, which is finite. This difference was computed at zero temperature in [2] for a variety of parameters (ν, f). There it was found that as one tunes ν from 0 to 1, there is a critical value ν_c, dependent on f, where the blown up D7 configuration does have lower energy than the D5s. It continues to be favourable for $\nu > \nu_c$, and at $\nu = 1$, the favoured D7 gets a Minkowski embedding. In that paper, the authors confirmed that the D5 chiral symmetry breaking solution is stable below the bound $\nu/f < 1.68$. They also found that the D7 chiral symmetry breaking solution is not only stable beyond this regime, but is in fact, energetically preferable.

The D5-D7 phase diagram was extended to finite temperature in [4], where it was found that for large f, the D5-D7 transition occurs at $\nu = 0.5$.

To summarize, if we start at $\nu = 0$ and zero temperature, we have a D5 system with $SU(N_5)$ symmetry. As soon as the magnetic field is turned on, the $\tilde{SO}(3)$ symmetry is broken via magnetic catalysis of chiral symmetry breaking. Then if we increase ν to ν_c, the D7 takes over, breaking the $SU(N_5)$ symmetry. This continues to $\nu = 1$, where the D7 forms a gapped state. If, however, we turn up the temperature, we will eventually restore the $\tilde{SO}(3)$ symmetry. This chiral symmetry restoration phase was found to occur at

$$r_h \approx 0.4 \tag{6.36}$$

$$T \approx 0.32\frac{\sqrt{B}}{\lambda^{1/4}}. \tag{6.37}$$

Chapter 7

Entropy of the Giant D5 Model

In this chapter we investigate some thermodynamics of the model setup in the previous chapter. Specifically, we will find an analytic solution for the entropy of this system at very low temperatures.

7.1 Entropy calculation

7.1.1 Entropy of D5 branes

The entropy can be found from the first derivative of the free energy with respect to temperature. Starting with the setup outlined in section 6.1.1, we had the free energy given by (6.16).

Now, the entropy is just

$$\begin{aligned} S_5 &= -\frac{\partial F_5}{\partial r_h}\frac{\partial r_h}{\partial T} &(7.1)\\ &= -\frac{\pi\lambda^{1/4}}{\sqrt{2\pi B}}\frac{\partial F_5}{\partial r_h}. &(7.2)\end{aligned}$$

And,

$$\frac{\partial F_5}{\partial r_h} = -\tilde{\mathcal{N}}_5\Big(\mathcal{L}(r_h,r_h) - \int_{r_h}^{\infty} dr\partial_{r_h}\mathcal{L}(r_h,r)\Big), \tag{7.3}$$

where $\mathcal{L}(r_h, r)$ refers to the integrand of (6.16) (i.e. the transformed Lagrangian).

The first term in the bracket is this integrand evaluated at $r = r_h$,

$$\mathcal{L}(r_h, r_h) = \sqrt{4\sin^4(\psi(r_h))f^2(1+r_h^4) + (\pi\nu)^2}. \tag{7.4}$$

The second term is:

$$
\begin{aligned}
-\int_{r_h}^{\infty} dr \partial_{r_h}\mathcal{L}(r_h,r) &= -\int_{r_h}^{\infty} dr\left(\frac{\partial\mathcal{L}}{\partial h}\partial_{r_h}h+\frac{\partial\mathcal{L}}{\partial\psi'(r)}\partial_{r_h}\psi'(r)+\frac{\partial\mathcal{L}}{\partial\psi}\partial_{r_h}\psi\right)\\
&= -\int_{r_h}^{\infty} dr\left(\frac{\partial\mathcal{L}}{\partial h}\partial_{r_h}h+\frac{\partial\mathcal{L}}{\partial\psi'(r)}\partial_{r_h}\psi'(r)+\frac{d}{dr}\left(\frac{\partial\mathcal{L}}{\partial\psi'}\right)\partial_{r_h}\psi\right)\\
&= -\frac{\partial\mathcal{L}}{\partial\psi'(r)}\partial_{r_h}\psi\bigg|_{r_h}^{\infty}-\int_{r_h}^{\infty} dr\frac{\partial\mathcal{L}}{\partial h}\partial_{r_h}h. \qquad (7.5)
\end{aligned}
$$

In the second line above, we used the equation of motion for $\psi(r)$, and in the third, we integrated by parts.

Until now, we have ignored regularization, and the Lagrangian is divergent as $r\to\infty$. Often in AdS/CFT, this problem requires "holographic renormalization" through the addition of counter-terms to the boundary action [62], but here we can address the problem simply by subtracting a background Lagrangian that matches our Lagrangian asymptotically, so as to cancel the divergence. All of our energies will then be in reference to this background system. In this case, the obvious choice of background system to use is the same D5 model, but at zero charge density. $\mathcal{L}_{\rm BG}=\lim_{q_5\to 0}\mathcal{L}$. Looking at (6.16), we do indeed get $\mathcal{L}_{\rm BG}|_{r\to\infty}=\mathcal{L}$, so our free energy and entropy will be finite. With this background subtraction, the first term in (7.5) becomes

$$
-\left(\frac{\partial\mathcal{L}}{\partial\psi'(r)}-\frac{\partial\mathcal{L}_{\rm BG}}{\partial\psi'(r)}\right)\partial_{r_h}\psi\bigg|_{r_h}^{\infty}. \qquad (7.6)
$$

Near the boundary, we maintain the asymptotic form of the $\psi(r)$ solution with zero mass for the fundamental representation fields, $\psi(r)=\frac{\pi}{2}+\frac{c_2}{r^2}+\ldots$. We see that $\lim_{r\to\infty}\partial_{r_h}\psi(r)=0$, and so (7.6) is completely determined at the horizon where:

$$
\frac{\partial\mathcal{L}}{\partial\psi'(r)}\bigg|_{r=r_h}=\frac{hr^2\psi'(r)}{\sqrt{1+h(r\psi'(r))^2}}\sqrt{4\sin^4\psi(r)f^2(1+r^4)+(\pi\nu)^2}\bigg|_{r=r_h}=0=\frac{\partial\mathcal{L}_{\rm BG}}{\partial\psi'(r)}\bigg|_{r=r_h}, \qquad (7.7)
$$

since $\mathcal{L}$ and $\mathcal{L}_{\rm BG}$ have the same dependency on $\psi'(r)$.

All that remains to evaluate is the second term in (7.5). Here we appeal to a low temperature calculation. In particular, we assume that terms of order T^4 can be neglected in the free energy (or T^3 can be neglected in the entropy). In this case, we can expand the second term to second order in temperature (or equivalently r_h):

$$\begin{aligned}
& - \int_{r_h}^{\infty} dr \frac{\partial \mathcal{L}}{\partial h} \partial_{r_h} h \\
= & - \int_{r_h}^{\infty} dr \frac{\partial \mathcal{L}}{\partial h} \partial_{r_h} h \Big|_{r_h=0} \\
& + r_h \Bigg[\underbrace{\frac{\partial \mathcal{L}}{\partial h} \partial_{r_h} h \Big|_{r \to r_h}}_{\mathcal{L}_1} - \int_{r_h}^{\infty} \underbrace{\left(\frac{d}{dr_h} \left(\frac{\partial \mathcal{L}}{\partial h} \right) \partial_{r_h} h + \frac{\partial \mathcal{L}}{\partial h} \partial_{r_h}^2 h \right)}_{\mathcal{L}_2} dr \Bigg]_{r_h=0} \\
& + \frac{r_h^2}{2!} \left[\frac{d\mathcal{L}_1}{dr_h} + \mathcal{L}_2|_{r \to r_h} - \int_{r_h}^{\infty} \left(\frac{d^2}{dr_h^2} \left(\frac{\partial \mathcal{L}}{\partial h} \right) \partial_{r_h} h + 2 \frac{d}{dr_h} \left(\frac{\partial \mathcal{L}}{\partial h} \right) \partial_{r_h}^2 h + \frac{\partial \mathcal{L}}{\partial h} \partial_{r_h}^3 h \right) dr \right]_{r_h=0} \\
& + O(r_h^3).
\end{aligned} \tag{7.8}$$

Now,

$$\partial_{r_h} h = -4r_h^3/r^4$$

$$\frac{\partial \mathcal{L}}{\partial h} = \frac{1}{2}(r\psi'(r))^2 \sqrt{\frac{4\sin^4\psi f^2(1+r^4) + (\pi\nu)^2}{1 + h(r\psi'(r))^2}} \tag{7.9}$$

$$\mathcal{L}_1 = -2r_h \sqrt{(\pi\nu)^2 + 4f^2(1+r_h^4)\sin^4(\psi(r_h))}\psi'(r_h)^2 \tag{7.10}$$

$$\mathcal{L}_2 = \frac{-2r_h^2\sqrt{(\pi\nu)^2 + 4f^2(1+r^4)\sin^4(\psi(r))}\psi'(r)^2(3r^2 + (3r^4 - r_h^4)\psi'(r)^2)}{r(r^2 + (r^4 - r_h^4)\psi'(r)^2)^{3/2}} \tag{7.11}$$

After regularization, the integrals on the right hand side will be finite and the presence of the h derivatives will cause them to vanish in the limit $r_h \to 0$.

So the first two lines on the right hand side of (7.8) are zero and only the quadratic term contributes.

$$-\int_{r_h}^{\infty} dr \frac{\partial \mathcal{L}}{\partial h} \partial_{r_h} h = -4\psi'(0)^2 \sqrt{4\sin^4\psi(0) f^2 + (\pi\nu)^2} r_h^2 + O(r_h^3). \tag{7.12}$$

One can easily see that the embedding equation of motion, (6.21), requires $\psi(0) = 0$ for the D5, but the value $\psi'(0)$ will be determined later in section 7.2.

Thus we finally have all the terms in equation (7.3), and so the D5 entropy to second order in temperature (or equivalently horizon radius) is:

$$\begin{aligned}
S_5 & \approx \frac{\tilde{\mathcal{N}}_5 \pi^2 \nu \lambda^{1/4}}{\sqrt{2\pi B}} (1 - 4\psi'(0)^2 r_h^2) \\
& = \frac{\sqrt{\lambda} N_c N_5 V_{2+1} q_5}{(2\pi)^2} (1 - 4\psi'(0)^2 r_h^2),
\end{aligned} \tag{7.13}$$

where we have truncated the $\mathcal{L}(r_h, r_h)$ term, (7.4), to second order, assuming $\psi'(0)$ to be finite. There are two things to note about this expression. First, it is linear in ν even at finite low temperature. Second, it scales as $\sqrt{\lambda}$ at $T = 0$. This is strange because we are working in the regime of large λ. Such a large non-zero entropy at zero temperature is often a sign of degeneracy or an unstable solution. It may also be an artifact of the large N limit. Stability can be tested by looking at the equations of motion of fluctuations about the background fields, but we will not report on that here. This kind of $\sqrt{\lambda}$ behaviour of the entropy is often encountered in holographic models, and we will discuss its interpretation below. In any case, we can evaluate the bracket if we know the embedding solution (or rather it's derivative) at the horizon. However, we will postpone this until after discussing the D7 brane.

7.1.2 Comparison to single quark entropy

The above result has a nice interpretation from string theory following a similar analysis done by Herzog et al [63]. What is the free energy of a single string connecting a probe brane in AdS to the horizon? Consider the AdS_5 part of the black-brane supergravity solution

$$ds^2 = \left(-hr^2dt^2 + r^2dx^2 + r^2dy^2 + r^2dz^2 + \frac{dr^2}{hr^2} \right). \tag{7.14}$$

The Nambu-Goto action,

$$S = -T_s \int d\sigma d\tau \sqrt{-\text{det}g_{ab}}, \tag{7.15}$$

requires the pullback of the AdS_4 metric to the string world-sheet : $g_{ab} = \frac{\partial X^\mu}{\partial x^a}\frac{\partial X^\nu}{\partial x^b}G_{\mu\nu}$. Here X^μ are the space-time coordinates and x^a are the world-sheet coordinates σ, τ. Recall that the action is reparameterization invariant, so we are free to choose the static gauge $\sigma = r$, $\tau = t$. In this case,

$$\begin{aligned} g_{\sigma\sigma} &= \frac{1}{hr^2} + r^2x'^2 + r^2y'^2 + r^2z'^2 \\ g_{\tau\tau} &= -hr^2 + r^2\dot{x}^2 + r^2\dot{y}^2 + r^2\dot{z}^2 \\ g_{\sigma\tau} &= r^2\dot{x}x' + r^2\dot{y}y' + r^2\dot{z}z', \end{aligned} \tag{7.16}$$

where dots are derivatives with respect to τ and primes are derivatives with respect to σ. Then,

$$\begin{aligned} -\text{det}g_{ab} &= g_{\sigma\tau}^2 - g_{\sigma\sigma}g_{\tau\tau} \qquad (7.17) \\ &= r^4(\dot{x}x' + \dot{y}y' + \dot{z}z')^2 - \left(\frac{1}{hr^2} + r^2x'^2 + r^2y'^2 + r^2z'^2\right) r^2(-h + \dot{x}^2 + \dot{y}^2 + \dot{z}^2). \end{aligned}$$

We can then compute the quantity

$$\frac{\partial\sqrt{-\det g_{ab}}}{\partial \dot{X}^\alpha} = \frac{-T_s}{\sqrt{-\det g_{ab}}}[(\dot{X}\cdot X')X'^\nu - X'^2\dot{X}^\nu]G_{\alpha\nu}, \tag{7.18}$$

which allows us to find the canonical energy density:

$$\pi_t^\tau \equiv \frac{\partial\mathcal{L}}{\partial\dot{X}^t} = \frac{-T_s}{\sqrt{-\det g_{ab}}}hr^2\left(\frac{1}{hr^2} + r^2x'^2 + r^2y'^2 + r^2z'^2\right). \tag{7.19}$$

The free energy is then just the integral of this along the string

$$F = -\int d\sigma\pi_t^\tau. \tag{7.20}$$

We want one end of the string to act as a source for the gauge fields on the probe D brane. The other end cannot just end anywhere, and so must stretch to the black hole horizon (or the Poincaré horizon if there was no black hole). So the endpoints are at r_h and r_b, where r_b lies somewhere on the probe brane. One can easily check that the equations of motion for the string are satisfied by the constant solution

$$x(r,t) = x_0, \;\; y(r,t) = y_0, \;\; z(r,t) = z_0. \tag{7.21}$$

In which case,

$$\sqrt{-\det g_{ab}} = 1. \tag{7.22}$$

So in the static gauge,

$$F = T_s\int_{r_h}^{r_b} dr = T_s(r_b - r_h). \tag{7.23}$$

In the zero temperature limit, the free energy must recover the Lagrangian quark mass m, so $T_s r_b = m$. The string tension is related to the 't Hooft coupling via $T_s = \frac{\sqrt{\lambda}}{2\pi}$. Thus,

$$F = m - \frac{1}{2}\sqrt{\lambda}T, \tag{7.24}$$

and the entropy of a single quark is

$$S = \frac{\sqrt{\lambda}}{2}, \tag{7.25}$$

which is of course temperature independent. At zero temperature, the entropy we found for the D5 brane was (from (7.13)):

$$S_5 = \frac{\sqrt{\lambda}N_cN_5V_{2+1}q_5}{(2\pi)^2} = \frac{\sqrt{\lambda}V_{2+1}\rho}{2}. \tag{7.26}$$

So we see that, at least at zero temperature, the probe D5 stack entropy is precisely the entropy for a single quark times the total charge of the system.

7.1.3 Entropy of the D7 brane

We repeat the analysis of section 7.1.1 for the case of the blown up D7 brane outlined in 6.1.2.

To get the entropy, we again evaluate equation (7.3), only using the Lagrangian from F_7 (equation (6.31)). This time we have for the first term:

$$\mathcal{L}(r_h, r_h) = \sqrt{4\sin^4\psi(r_h)(f^2 + 4\cos^4\psi(r_h))(1 + r_h^4) + (\pi(\nu-1) + 2\psi(r_h) - 1/2\sin 4\psi(r_h))^2}. \tag{7.27}$$

The second term is evaluated using (7.5) where again the zero charge density background Lagrangian has the same $\psi'(r)$ dependancy, so that

$$\frac{\partial \mathcal{L}}{\partial \psi'(r)} \partial_{r_h}\psi \bigg|_{r_h}^{\infty} = \frac{\partial \mathcal{L}_{\mathrm{BG}}}{\partial \psi'(r)} \partial_{r_h}\psi \bigg|_{r_h}^{\infty} = 0. \tag{7.28}$$

The final term is evaluated using the r_h expansion, equation (7.8). For the D7,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial h} &= \frac{1}{2}\frac{(r\psi')^2\sqrt{4\sin^4\psi(f^2 + 4\cos^4\psi)(1+r^4) + (\pi(\nu-1) + 2\psi - 1/2\sin(4\psi))^2}}{\sqrt{1 + h(r\psi')^2}} \\
\mathcal{L}_1 &= -2r_h\bigg(4(1 + r_h^4)(f^2 + 4\cos^4(\psi(r_h)))\sin^4(\psi(r_h)) + (\pi(\nu-1) + 2\psi(r_h) \\
&\quad -1/2\sin(4\psi(r_h)))^2\bigg)^{1/2}\psi'(r_h)^2 \qquad (7.29)\\
\mathcal{L}_2 &= \frac{-2r_h^2\psi'(r)^2(3r^2 + (3r^4 - r_h^4)\psi'^2)}{r(r^2 + (r^4 - r_h^4)\psi'^2)^{3/2}}\bigg(4(1+r^4)(f^2 + 4\cos^4\psi)\sin^4\psi + (\pi(\nu-1) \\
&\quad +2\psi - 1/2\sin(4\psi))^2\bigg)^{1/2}. \qquad (7.30)
\end{aligned}$$

So the order T^0 and T^1 terms vanish using the same arguments as in the D5 case. The non-zero quadratic terms are:

$$\begin{aligned}
\frac{d\mathcal{L}_1}{dr_h}|_{r_h\to 0} &= -2\psi'(0)\bigg(4(f^2 + 4\cos^4(\psi(0)))\sin^4(\psi(0)) + (\pi(\nu-1) + 2\psi(0) \\
&\quad -1/2\sin(4\psi(0)))^2\bigg)^{1/2} \qquad (7.31)\\
\mathcal{L}_2|_{r\to r_h}|_{r_h\to 0} &= -6\psi'(0)\bigg(4(f^2 + 4\cos^4(\psi(0)))\sin^4(\psi(0)) + (\pi(\nu-1) + 2\psi(0) \\
&\quad -1/2\sin(4\psi(0)))^2\bigg)^{1/2}. \qquad (7.32)
\end{aligned}$$

We can now write out the entropy for the D7 brane.

$$\begin{aligned} S_7 &\approx -\frac{\tilde{\mathcal{N}}_7 \pi \lambda^{1/4}}{\sqrt{2\pi B}} \Big(\mathcal{L}(r_h, r_h) - 4r_h^2 \psi'(0)^2 \mathcal{L}(0,0) \Big) \\ &= -\frac{\sqrt{\lambda} N_c V_{2+1} B}{(2\pi)^2} \Big(\mathcal{L}(r_h, r_h) - 4r_h^2 \psi'(0)^2 \mathcal{L}(0,0) \Big). \end{aligned} \tag{7.33}$$

We need to expand $\mathcal{L}(r_h, r_h)$ to second order in r_h:

$$\begin{aligned} \mathcal{L}(r_h, r_h) &\approx \mathcal{L}(0,0) + \left.\frac{d\mathcal{L}(r_h, r_h)}{dr_h}\right|_{r_h=0} r_h + \left.\frac{d^2\mathcal{L}(r_h, r_h)}{dr_h^2}\right|_{r_h=0} \frac{r_h^2}{2} \\ &= \mathcal{L}(0,0) + \left(\frac{\partial \mathcal{L}(r_h, r_h)}{\partial \psi} \psi'(r_h) + \frac{\partial \mathcal{L}(r_h, r_h)}{\partial r_h} \right)_{r_h=0} r_h \\ &\quad + \left(\frac{\partial^2 \mathcal{L}(r_h, r_h)}{\partial \psi^2} \psi'(r_h)^2 + \frac{\partial \mathcal{L}(r_h, r_h)}{\partial \psi} \psi''(r_h) + \frac{\partial^2 \mathcal{L}(r_h, r_h)}{\partial r_h^2} \right)_{r_h=0} \frac{r_h^2}{2} \end{aligned} \tag{7.34}$$

But from (7.27) we can see that

$$\left.\frac{\partial \mathcal{L}(r_h, r_h)}{\partial r_h}\right|_{r_h=0} = 0 = \left.\frac{\partial^2 \mathcal{L}(r_h, r_h)}{\partial r_h^2}\right|_{r_h=0}. \tag{7.35}$$

We can use a trick to rewrite $\frac{\partial \mathcal{L}(r_h, r_h)}{\partial \psi}$. Using the equation of motion for ψ, we get

$$\begin{aligned} \frac{\partial \mathcal{L}(r_h, r_h)}{\partial \psi} &= \left(\frac{d}{dr} \frac{\partial \mathcal{L}(r, r_h)}{\partial \psi'} \right)_{r=r_h} \\ &= \left(\frac{d}{dr} \frac{\sqrt{4\sin^4\psi(f^2 + 4\cos^4\psi)(1+r^4) + (\pi(\nu-1) + 2\psi - 1/2\sin(4\psi))^2}}{\sqrt{1 + h(r\psi'(r))^2}} \right. \\ &\quad \left. \times h(r^2\psi'(r)) \right)_{r=r_h} \\ &= \sqrt{4\sin^4\psi(f^2 + 4\cos^4\psi)(1+r^4) + (\pi(\nu-1) + 2\psi - 1/2\sin(4\psi))^2} \\ &\quad \left. \times r^2\psi'(r)\partial_r h \right|_{r=r_h} \\ &= \mathcal{L}(r_h, r_h)(-4r_h\psi'(r_h)) \end{aligned} \tag{7.36}$$

And therefore,

$$\frac{\partial^2 \mathcal{L}(r_h, r_h)}{\partial \psi^2} = \mathcal{L}(r_h, r_h)(-4r_h\psi'(r_h))^2. \tag{7.37}$$

So the second order expansion of $\mathcal{L}(r_h, r_h)$ becomes

$$\mathcal{L}(r_h, r_h) \approx \mathcal{L}(0,0)\left[1 - 4r_h \Big(r_h\psi'(r_h)^2 \Big)_{r_h=0} + 2r_h^2 \Big(-r_h\psi'(r_h)\psi''(r_h) + 4r_h^2\psi'(r_h)^4 \Big)_{r_h=0} \right]. \tag{7.38}$$

Thus, the D7 entropy to second order in temperature is:

$$\begin{aligned} S_7 &\approx \frac{\sqrt{\lambda} N_c V_{2+1} B}{(2\pi)^2} \mathcal{L}(0,0) \Big\{ 1 - 4r_h \Big(r_h \psi'(r_h)^2 \Big)_{r_h=0} \\ &+ 2r_h^2 \Big(-2\psi'(r_h)^2 - r_h \psi'(r_h)\psi''(r_h) + 4r_h^2 \psi'(r_h)^4 \Big)_{r_h=0} \Big\}. \end{aligned} \tag{7.39}$$

To evaluate this requires knowing $\psi(0)$, $\psi'(0)$, and $\psi''(0)$, or at least whether they blow up at $r_h \to 0$, and this must be determined numerically. However, we can get an analytic expression if we consider the regime of large f. This is shown in the next section. Meanwhile, the numerical embedding solutions reveal that

$$\begin{aligned} \frac{d}{d\ln r}\psi(r)|_{r=r_h=0} &= 0 \\ \Rightarrow r_h \psi'(r_h)|_{r_h=0} &= 0. \end{aligned} \tag{7.40}$$

So only one of the quadratic terms in r_h above contribute, and the D7 entropy takes the same form as the D5 entropy

$$\boxed{S_7 \approx \frac{\sqrt{\lambda} N_c V_{2+1} B}{(2\pi)^2} \mathcal{L}(0,0) \Big\{ 1 - 4\psi'(0)^2 r_h^2 \Big\}}. \tag{7.41}$$

7.2 Embedding solutions

At the horizon, the equation of motion that ψ must satisfy is

$$\frac{\partial_\psi V}{V}\Big|_{r_h} = 0, \tag{7.42}$$

for the potentials

$$V_5 = 4\sin^4\psi f^2(1+r_h^4) + (\pi\nu)^2 \tag{7.43}$$

$$V_7 = 4\sin^4\psi(f^2 + 4\cos^4\psi)(1+r_h^4) + (\pi(\nu-1) + 2\psi - 1/2\sin 4\psi)^2. \tag{7.44}$$

The minimum of V_5 is simply $\psi = 0$ for which the entropy is given in equation (7.13). At finite temperature, the derivative $\psi'(r_h)$ is set by the equation of motion

$$\psi'(r_h) = \frac{-2\sin^3\psi(r_h)\cos\psi(r_h) f^2(1+r_h^4)}{r_h(4\sin^4\psi(r_h) f^2(1+r_h^4) + (\pi\nu)^2)}. \tag{7.45}$$

Since we know $\psi(0) = 0$, we can expand ψ in a Taylor series around $r_h = 0$, $\psi(r_h) = a_1 r_h + a_2 r_h^2 + \ldots$, and taking the limit of the above equation as r_h goes to zero gives $\psi'(0) = 0$. Thus, the second term in (7.13) vanishes, and the D5 entropy is constant up to third order in the temperature:

$$\boxed{S_5 = \frac{\sqrt{\lambda} N_c V_{2+1} B}{(2\pi)^2} \nu + O(r_h^3)} \tag{7.46}$$

Turning now to the D7 brane, the minimum of V_7 depends on the values of ν, f and must be solved numerically. We solve this in the range $0 < \nu < 1$ for different values of f ensuring that the equation of motion is also satisfied[1]. The embedding angle $\psi(0)$ drops continuously as ν is increased from 0 to 1, and as f is increased (see figure 7.1).

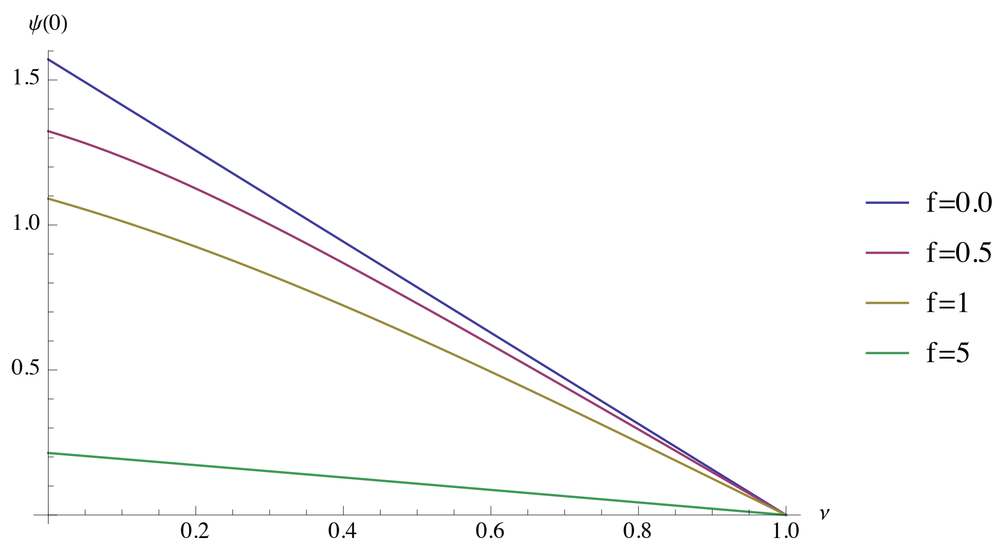

FIGURE 7.1: D7 brane embedding solutions at $r = r_h = 0$ for different values of f, ν.

The full solution $\psi(r)$ is found by iterating over a grid of initial values of $\psi(0)$. For a given initial value, the derivative $\psi'(0)$ is set by the equation of motion. We then integrate the solution up to a large value of r, and fit the result to the asymptotic form $\psi = \frac{\pi}{2} + \frac{m}{r} + \frac{c_2}{r^2}$. The solution with parameters closest to $\psi_\infty = \pi/2$ and $m = 0$ is then taken to be the embedding solution. In this way, c_2 is determined dynamically. The result is plotted for different values of r_h, in figure 7.2.

From these solutions, we confirm that $\lim_{r_h \to 0} r_h \psi'(r_h) = 0$, so that the D7 entropy to second order in temperature is

$$\begin{aligned} S_7 \quad \approx \quad & \frac{\sqrt{\lambda} N_c V_{2+1} B}{(2\pi)^2} \sqrt{4 \sin^4 \psi(0)(f^2 + 4\cos^4 \psi(0)) + (\pi(\nu - 1) + 2\psi(0) - 1/2 \sin 4\psi(0))^2} \\ & \times \left(1 - 4\psi'(0)^2 r_h^2\right). \end{aligned} \tag{7.47}$$

[1]This is only really a concern at $\nu = 1$ where the minimum of V_7 is also a root and the left hand side of the equation of motion above has a discontinuity.

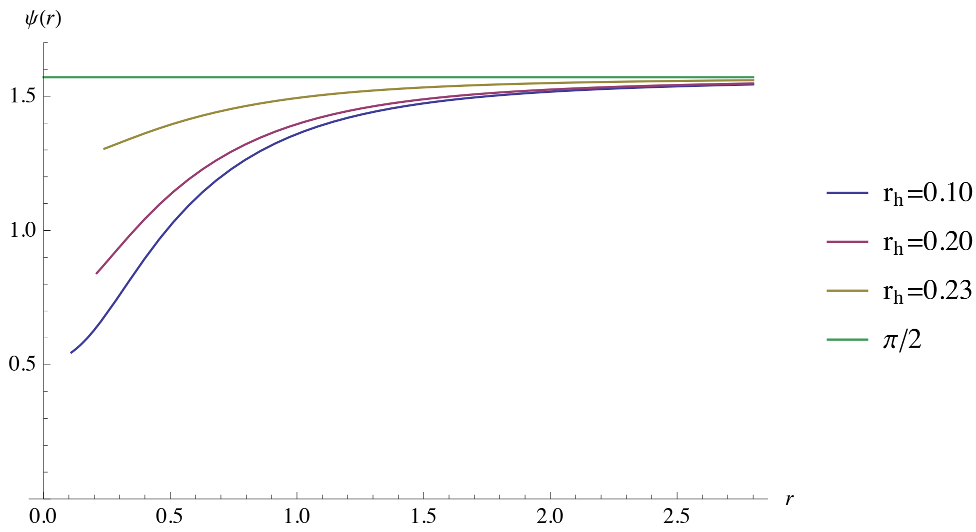

FIGURE 7.2: Embedding solutions at different temperatures for $\nu = 0.6$, $f = 1$ All solutions asymptote to $\pi/2$.

We can solve this analytically by considering the large f regime of the potential V_7. Then we have

$$V_7 \approx 4f^2 \sin^4 \psi + (\pi(\nu - 1) + 2\psi - 1/2 \sin(4\psi))^2, \tag{7.48}$$

which is always non-negative and also temperature independent. To minimize at large f, $\sin^4 \psi$ must be small and so we are forced to consider ψ near 0 or π. Then in the bracket, the linear term in ψ guarantees that ψ is near 0. For small ψ, the potential simplifies greatly,

$$V_7 \approx (\pi(\nu - 1))^2 + O(\psi^2). \tag{7.49}$$

But $V_7|_{r_h=0}$ is exactly what is under the square root of equation (7.47). So for large f, the zero temperature entropy becomes

$$S_7 \approx \frac{\sqrt{\lambda} N_c V_{2+1} B}{(2\pi)^2} \pi |\nu - 1|. \tag{7.50}$$

Comparing this to the D5 entropy, we see that the entropy is just a simple sawtooth function of ν with a transition between D5 and D7 occurring at $\nu = 0.5$. This is the last plot in figure 7.3. The other plots show the overall entropy for different values of f that have been found by plugging the numerical solutions for $\psi(0)$ into equation (7.47) at $r_h = 0$[2]. Note that the crossover between solutions takes place at lower filling fractions as f is decreased, and discontinuities appear.

[2]Using a small non-zero value of r_h would not visibly affect these plots since $\psi'(0) r_h$ is negligible

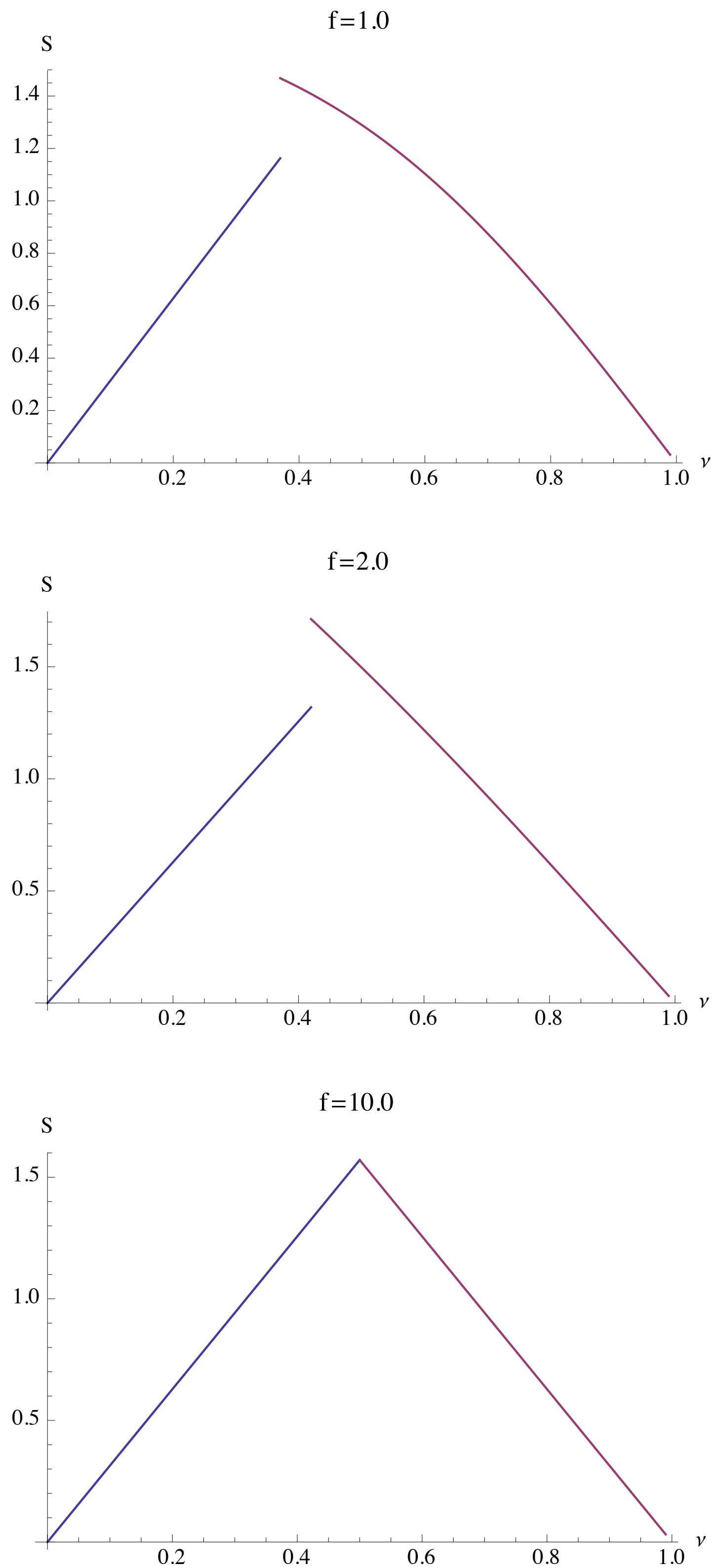

FIGURE 7.3: Low temperature entropy versus filling fraction for different values of f. The crossover is the D5-D7 transition ν_c where their Free energies are equal. The units of entropy are $\frac{\sqrt{\lambda} N_c V_{2+1} B}{(2\pi)^2}$.

We could also ask about the heat capacity, which is found from the entropy via

$$c_v = T\frac{\partial S}{\partial T}. \tag{7.51}$$

For the D5, we can guarantee that the leading order of the low temperature heat capacity is T^3 or higher. For the D7, the heat capacity goes as order T^2 or higher.

7.3 Weak coupling entropy

We may make a qualitative comparison to the zero-temperature entropy of the theory at weak coupling. The degeneracy of the Landau levels is $d \equiv 2N_5 N_c \frac{B}{2\pi}$, and the total fraction of levels filled is $\frac{\nu}{N_5}$. Thus, the number of micro-states available is

$$\Omega = \binom{d}{\frac{\nu}{N_5}d}, \tag{7.52}$$

and the entropy is

$$S = \ln\left(\frac{d!}{(\frac{\nu}{N_5}d)!(d-\frac{\nu}{N_5}d)!}\right). \tag{7.53}$$

Since the degeneracy is very large, we may use Stirling's approximation to write this as:

$$\begin{aligned} S &\approx d\ln\left(\frac{d}{d-\frac{\nu}{N_5}d}\right) + \frac{\nu}{N_5}d\ln\left(\frac{d-\frac{\nu}{N_5}d}{\frac{\nu}{N_5}d}\right) \\ &= d\left[\frac{-\nu}{N_5}\ln\left(\frac{\nu}{N_5}\right) + \left(\frac{\nu}{N_5}-1\right)\ln\left(1-\frac{\nu}{N_5}\right)\right], \end{aligned} \tag{7.54}$$

which is shown in figure 7.4.

Note the symmetry about $\nu = 0.5$ in both the weak coupling and $f = 10$ strong coupling case (last plot in figure 7.3). This is a reflection of particle-hole symmetry, since we can determine the entropy by calculating the number of occupied or unoccupied sites. Thus, the first two plots in 7.3 indicate that particle-hole symmetry is somehow broken as we lower f, and thereby lower the number of flavours in the strongly coupled field theory.

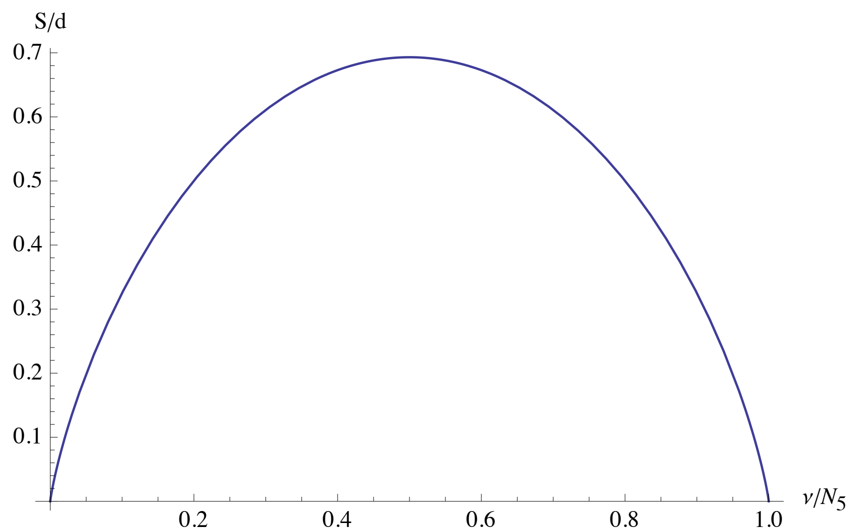

FIGURE 7.4: Zero-temperature entropy at weak coupling.

Chapter 8

Conductivity of the Giant D5 Model

In this chapter we compute the conductivity tensor of giant D5 model, and compare the Hall conductivity to the classical value.

8.1 D5 brane case

8.1.1 Setup

It turns out to be much more convenient to compute the conductivity in Fefferman-Graham coordinates. Recall from section 2.1 that the AdS metric in Fefferman-Graham coordinates was defined by

$$ds^2 = \frac{L^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2), \tag{8.1}$$

where we know that $L^2 = \sqrt{\lambda}\alpha'^2$. Comparing to our usual Poincaré AdS black hole coordinates, we see that the two coordinates should be related via

$$\frac{dz^2}{z^2} = \frac{dr^2}{r^2 h} \tag{8.2}$$

$$\Rightarrow dr = -\frac{r\sqrt{h}}{z}dz, \tag{8.3}$$

where we have chosen the sign so that z goes to zero at the AdS boundary. Integrating this result, with the appropriate choice of integration constant gives

$$z^2 = \frac{2}{r^2 + \sqrt{r^4 - r_h^4}}, \quad z_h = \frac{\sqrt{2}}{r_h} = \frac{\sqrt{2}\pi}{T}. \tag{8.4}$$

These can be inverted to get

$$r^2 = \frac{(z^4 + z_h^4)}{z^2 z_h^4}, \quad h(z) = 1 - \frac{4z^4 z_h^4}{(z^4 + z_h^4)^2}. \tag{8.5}$$

The world-volume metric of the D5 branes can now be written as

$$ds^2 = \sqrt{\lambda}\alpha'[-g_{tt}dt^2 + g_{xx}(dx^2 + dy^2) + g_{zz}dz^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{8.6}$$

With the above transformations, we can find the metric components:

$$g_{tt} = r^2 h = \frac{1}{z^2}\frac{(1 - z^4/z_h^4)^2}{1 + z^4/z_h^4} \tag{8.7}$$

$$g_{xx} = g_{yy} = r^2 = \frac{1}{z^2}(1 + z^4/z_h^4) \tag{8.8}$$

$$g_{zz} = \frac{1}{z^2}(1 + z^2\psi'(z)^2). \tag{8.9}$$

The 2-sphere metric remains unchanged, though now ψ is a function of z.

To find the conductivity we need to include an external electric field $\vec{E} = \frac{\sqrt{\lambda}}{2\pi}e\hat{x}$. Furthermore, we expect a non-zero current in the field theory $\langle J_x \rangle$, $\langle J_y \rangle$. These are dual to the gauge field components A_x, A_y in exactly the same way that $\langle J_t \rangle$ was dual to A_t (see section 6.1.1). This a gives a more elaborate gauge field on the probe branes,

$$2\pi\alpha'\mathcal{F} = \sqrt{\lambda}\alpha'[a'(z)dz \wedge dt + f_x'(z)dz \wedge dx + f_y'(z)dz \wedge dy + b dx \wedge dy - e dt \wedge dx]. \tag{8.10}$$

Again, Wess-Zumino terms will not contribute to the D5 action. Evaluating the remaining DBI action,

$$\mathcal{S}_5 = -\frac{T_5}{g_s}N_5 \int d^6\sigma \sqrt{-\det(g + 2\pi\alpha'\mathcal{F})}, \tag{8.11}$$

requires the determinant $\det(g + 2\pi\alpha'\mathcal{F}) = \det A \det B$, where

$$A = \begin{pmatrix} -g_{tt} & -e & 0 & -a'(z) \\ e & g_{xx} & b & -f_x'(z) \\ 0 & -b & g_{xx} & -f_y'(z) \\ a'(z) & f_x'(z) & f_y'(z) & g_{zz} \end{pmatrix}, \quad B = \begin{pmatrix} \sin^2\psi & 0 \\ 0 & \sin^2\psi\sin^2\theta \end{pmatrix}. \tag{8.12}$$

This yields,

$$\mathcal{S}_5 = \frac{2\sqrt{\lambda}N_cN_5}{(2\pi)^3}V_{2+1}\int_0^{z_h} dz\sqrt{\mathcal{S}}, \tag{8.13}$$

where

$$\begin{aligned}\mathcal{S} \;\; &\equiv \;\; 4\sin^4\psi\Big(2a'(z)bef_y'(z) - a'(z)^2(b^2+g_{xx}^2) - e^2(f_y^2+g_{xx}g_{zz}) \\ &\quad +g_{tt}[(f_x'(z)^2+f_y'(z)^2)g_{xx} + (b^2+g_{xx}^2)g_{zz}]\Big).\end{aligned} \tag{8.14}$$

Note that the limits $e,\ f_y'(z),\ f_x'(z) \to 0$, gives back the action we had earlier for the D5s.

Now just as $a(z)$ was cyclic in the case with no electric field, $f_x(z)$ and $f_y(z)$ are cyclic now, and their equations of motion each yield an integration constant which is a conserved charge q, q_x, q_y. These are related to the components of the current densities in the same way as before

$$q = \frac{(2\pi)^2}{2N_cN_5}\rho, \quad (q_x, q_y) = \frac{(2\pi)^2}{2N_cN_5}(J_x, J_y), \tag{8.15}$$

where

$$(J_x, J_y) = -\frac{1}{V_{2+1}}\left(\frac{\delta\mathcal{S}_5}{\delta f_x'(z)}, \frac{\delta\mathcal{S}_5}{\delta f_y'(z)}\right). \tag{8.16}$$

The gauge field equations of motion then read

$$q = \frac{-1}{\sqrt{\mathcal{S}}}4\sin^4\psi[bef_y'(z) - a'(z)(b^2+g_{xx}^2)], \tag{8.17}$$

$$q_x = \frac{1}{\sqrt{\mathcal{S}}}4\sin^4\psi[g_{tt}g_{xx}f_x'(z)], \tag{8.18}$$

$$q_y = \frac{1}{\sqrt{\mathcal{S}}}4\sin^4\psi[a'(z)be + f_y'(z)(g_{xx}g_{tt} - e^2)]. \tag{8.19}$$

8.1.2 Conductivity

In analogy with the Legendre transform we used earlier,

$$F_5 = \mathcal{S}_5 + A_t(\infty)\rho V_{2+1}, \tag{8.20}$$

we similarly do a Legendre transform for the other two gauge components to get the Routhian

$$\mathcal{R}_5 = \mathcal{S}_5 + A_t(\infty)\rho V_{2+1} + A_x(\infty)J_xV_{2+1} + A_y(\infty)J_yV_{2+1} \tag{8.21}$$

$$= \frac{-2N_cN_5}{(2\pi)^2}V_{2+1}\frac{\sqrt{\lambda}}{2\pi}\int_0^{z_h} dz[\mathcal{L}_5 + qa'(z) + q_xf_x'(z) + q_yf_y'(z)], \tag{8.22}$$

where again we assumed that the gauge fields vanish at the horizon (this will be consistent with their equations of motion). We get,

$$\mathcal{R}_5 = -\frac{2N_cN_5}{(2\pi)^2}V_{2+1}\frac{\sqrt{\lambda}}{2\pi}\int_0^{z_h} dz\frac{4\sin^4\psi}{\sqrt{\mathcal{S}}}[g_{zz}(g_{tt}b^2+g_{tt}g_{xx}^2-g_{xx}e^2)]. \tag{8.23}$$

The details of how we extract the conductivity from this Routhian are relegated to the appendix. However, the outline goes as follows. By solving for the gauge fields in terms of their conserved charges, we can write the Routhian in the schematic form

$$\mathcal{R}_5 = \frac{-2N_cN_5}{(2\pi)^2}V_{2+1}\frac{\sqrt{\lambda}}{2\pi}\int_0^{z_h} dz\sqrt{\frac{g_{zz}}{g_{tt}}}\frac{1}{g_{xx}}\sqrt{BC-A^2}, \tag{8.24}$$

where we have defined

$$\begin{aligned} A &\equiv qbg_{tt}-q_yeg_{xx} && (8.25)\\ B &\equiv g_{tt}g_{xx}^2+g_{tt}b^2-g_{xx}e^2 && (8.26)\\ C &\equiv 4\sin^4\psi g_{tt}g_{xx}^2+g_{tt}q^2-g_{xx}(q_x^2+q_y^2). && (8.27)\end{aligned}$$

Now we follow a beautiful little argument due to O'Bannon [64], by looking at the roots of A, B, and C. It turns out that reality of the Routhian requires that these all share a single common zero at some position z_*. That gives three equations $B(z_*)=0$, $A(z_*)=0$, and $C(z_*)=0$ with three unknowns z_*, q_x, and q_y. Solving for the latter two gives

$$q_y = \frac{qbe}{(g_{xx}(z_*)^2+b^2)}. \tag{8.28}$$

and

$$q_x = \frac{eg_{xx}(z_*)}{(g_{xx}(z_*)^2+b^2)}\sqrt{4\sin^4\psi(z_*)(g_{xx}(z_*)^2+b^2)+q^2}. \tag{8.29}$$

This two equations allow us to directly read off the conductivity tensor.

$$\begin{aligned} \langle J_i\rangle &= \sigma_{xi}E && (8.30)\\ \Rightarrow \sigma_{xi} &= \frac{2N_cN_5}{(2\pi)\sqrt{\lambda}}\frac{q_i}{e}|_{e=0}, && (8.31)\end{aligned}$$

where $i=x,y$. Written in terms of the r_h, ν and f variables from before we have the final expressions for the conductivity tensor

$$\boxed{\sigma_{xx} = \frac{2N_c}{(2\pi)^2}f\left(\frac{r_h^2}{1+r_h^4}\right)\sqrt{4\sin^4\psi(z_h)(1+r_h^4)+(\pi\nu/f)^2}}, \tag{8.32}$$

and

$$boxed\sigma_{xy} = \frac{N_c \nu}{2\pi}\left(1 - \frac{r_h^4}{1+r_h^4}\right). \tag{8.33}$$

Here we set $e = 0$ to look at the linear response, and we may simply insert the horizon embedding $\psi(r_h)$, that we found in the entropy calculation, into σ_{xx}.

8.2 D7 brane case

8.2.1 Setup

By looking at the actions for the D5 and D7 (6.6), (6.29), we see that we can get the D7 action from the D5 action by adding a Wess-Zumino term and using the replacement rule

$$4\sin^4\psi \to 4\sin^4\psi(f^2 + 4\cos^4\psi) \tag{8.34}$$

in the integrand $\mathcal{S}$.

The Wess-Zumino term with the new gauge fields is

$$T_7 \frac{(2\pi\alpha')^2}{2} C^{(4)} \wedge \mathcal{F} \wedge \mathcal{F} = \frac{\lambda N_c}{(2\pi)^3}\left(\frac{c(\psi)}{2}(ba'(z) - ef_y'(z)V_{2+1})\right). \tag{8.35}$$

At this point, for convenience, we have redefined $c(\psi) \to \int d\tilde{\theta} d\tilde{\phi} c(\psi)$.

The action is now,

$$\mathcal{S}_7 = \frac{-2\lambda N_c}{(2\pi)^4} V_{2+1} \int_0^\infty dz \left[\sqrt{\mathcal{S}(f^2 + 4\cos^4\psi)} + \frac{1}{2}c(\psi)(a'(z)b - ef_y'(z))\right]. \tag{8.36}$$

In fact, our replacement rule allows us to compute the conserved charges with ease. From,

$$q = \frac{-(2\pi)^4}{2N_c N_5 V_{2+1}} \frac{\delta \mathcal{S}_7}{\delta a'(z)}, \tag{8.37}$$

we note that the derivative with respect to $a'(z)$ gives a Wess-Zumino contribution of $-1/2bc(\psi)$, so that

$$q = \frac{4\sin^4\psi}{\sqrt{\mathcal{S}}}\sqrt{f^2 + 4\cos^4\psi}(bef_y'(z) - a'(z)(b^2 + g_{xx}^2)) - \frac{1}{2}bc(\psi). \tag{8.38}$$

Similarly, q_x receives no Wess-Zumino contribution since the functional derivative is with respect to $f'_x(z)$, and q_y receives a contribution of $1/2ec(\psi)$, yielding

$$q_x = \frac{4\sin^4\psi}{\sqrt{\mathcal{S}}}\sqrt{f^2+4\cos^4\psi}(g_{tt}g_{xx}f'_x(z)) \tag{8.39}$$

$$q_y = \frac{\sin^4\psi}{\sqrt{\mathcal{S}}}[a'(z)be+f'_y(z)(g_{xx}g_{tt}-e^2)]-\frac{1}{2}ec(\psi). \tag{8.40}$$

8.2.2 Conductivity

We can apply our replacement rule to find the Routhian as well. The Wess-Zumino terms (WZ) contribute to the Legendre transform via

$$-\int dza'(z)\frac{\delta(\mathrm{WZ})}{\delta a'(z)}-\int dzf'_x(z)\frac{\delta(\mathrm{WZ})}{\delta f'_x(z)}-\int dzf'_y(z)\frac{\delta(\mathrm{WZ})}{\delta f'_y(z)} = -\frac{1}{2}\int dzc(\psi)(a'(z)b-ef'_y(z)). \tag{8.41}$$

So the Legendre transform serves to precisely to remove the Wess-Zumino term from the action (of course it still affects the conserved charges), so we can write the Routhian as

$$\mathcal{R}_7 = -\frac{2\lambda N_c}{(2\pi)^4}V_{2+1}\int dz\frac{4\sin^4\psi}{\sqrt{\mathcal{S}}}\sqrt{f^2+4\cos^4\psi}[g_{zz}(g_{tt}b^2+g_{tt}g_{xx}^2-g_{xx}e^2)]. \tag{8.42}$$

This is exactly the same as the D5 Routhian integrand (modulo the $\sqrt{f^2+4\cos^4\psi}$ factor), so we can write it as

$$\mathcal{R}_7 = -\frac{2\lambda N_c}{(2\pi)^4}V_{2+1}\int dz\sqrt{\frac{g_{zz}}{g_{tt}}}\frac{1}{g_{xx}}\sqrt{BC-A^2}, \tag{8.43}$$

only we need to adjust the charges since

$$q^{D5} = q^{D7}+\frac{1}{2}bc(\psi),\ \ \&\ \ q_y^{D5} = q_y^{D7}+\frac{1}{2}ec(\psi), \tag{8.44}$$

so that the functions A and C change to

$$A \rightarrow (q+1/2bc(\psi))bg_{tt}-(q_y+2ec(\psi))eg_{xx} \tag{8.45}$$

$$C \rightarrow 4\sin^4(\psi)(f^2+4\cos^4\psi)g_{tt}g_{xx}^2+g_{tt}(q+2bc(\psi))-g_{xx}[q_x^2+(q_y+2ec(\psi))^2].$$

However, the function B is unchanged, and thus so is its zero z_*. The same argument applies as before; the equations $A(z_*)=0$ and $c(z_*)=0$, determine the charges to be

$$q_y = \left(\frac{bq-\frac{1}{2}c(\psi(z_*))g_{xx}(z_*)}{b^2+g_{xx}^2(z_*)}\right)e \tag{8.46}$$

$$q_x = \frac{g_{xx}}{b^2+g_{xx}^2(z_*)}\sqrt{4\sin^4\psi(z_*)(f^2+4\cos^4\psi(z_*))(b^2+g_{xx}^2(z_*))+(q+1/2bc(\psi(z_*)))^2}e.$$

Again, we can read off the conductivities at $e = 0$,

$$\boxed{\sigma_{xx} = \frac{N_c}{2\pi^2}\left(\frac{r_h^2}{1+r_h^4}\right)\sqrt{4\sin^4\psi(z_h)(f^2+4\cos^4\psi(z_h))(b^2+g_{xx}^2(z_h))+(q+1/2bc(\psi(z_h)))^2}} \tag{8.47}$$

$$\boxed{\sigma_{xy} = \frac{N}{2\pi}\left[\nu + \left((1-\nu)+\frac{\sin 4\psi(z_h)}{2\pi}-\frac{2}{\pi}\psi(z_h)\right)\left(\frac{r_h^4}{1+r_h^4}\right)\right]}, \tag{8.48}$$

where we have used $\lim_{e\to 0} z_* = z_h$.

8.3 Results

Comparing our results to the classical linear Hall conductivity $\sigma_{xy} = \frac{N_c\nu}{2\pi}$, we see that the D5 hall conductivity lies below this linear curve at finite temperature, and matches the classical case at zero temperature.

For the D7 Hall conductivity, note that figure 7.1 shows that $\psi(r_h) \to 0$ at $\nu = 1$ for all values of f. In this case, we get $\sigma_{xy}^{D7} = \frac{N_c\nu}{2\pi}$. So the D7 Hall conductivity always returns to the classical value at $\nu = 1$. Below $\nu = 1$, σ_{xy}^{D7} always lies above the classical curve. The amount of deviation of each conductivity tensor component from the linear curve is shown in figure 8.1 for $r_h = 0.2$ and in figure 8.2 for $r_h = 0.4$.

These deviations from the classical value hint at a plateau. Indeed, if we could take the large r_h limit, we would find that $\sigma_{xy}^{D5} \to 0$ throughout the range of ν over which the D5 dominates. Furthermore, in the limit of large f, $\psi(r_h) \to 0$ (see 7.1) and so

$$\sigma_{xy}^{D7} \to \frac{N\nu}{2\pi} + \frac{N(1-\nu)}{2\pi} \tag{8.49}$$

$$= \frac{N}{2\pi}. \tag{8.50}$$

In this case, the conductivity would exhibit perfect integer plateaus with a transition at the D5-D7 transition. Of course we cannot actually take this limit, because the temperature is numerically confined to $r_h \lesssim 0.4$. At higher temperatures, the chiral symmetry is restored. Pushing r_h to the edge of this limit, we see that the jump in conductivity is rather small. This is plotted with the classical Hall conductivity for comparison at large f in figure 8.3.

Beyond $\nu = 1$, we know that a composite system takes over and we will not discuss this here. Indications are that the conductivity tensor in the interval $1 \leq \nu \leq 2$ is qualitatively the same as the interval $0 \leq \nu \leq 1$.

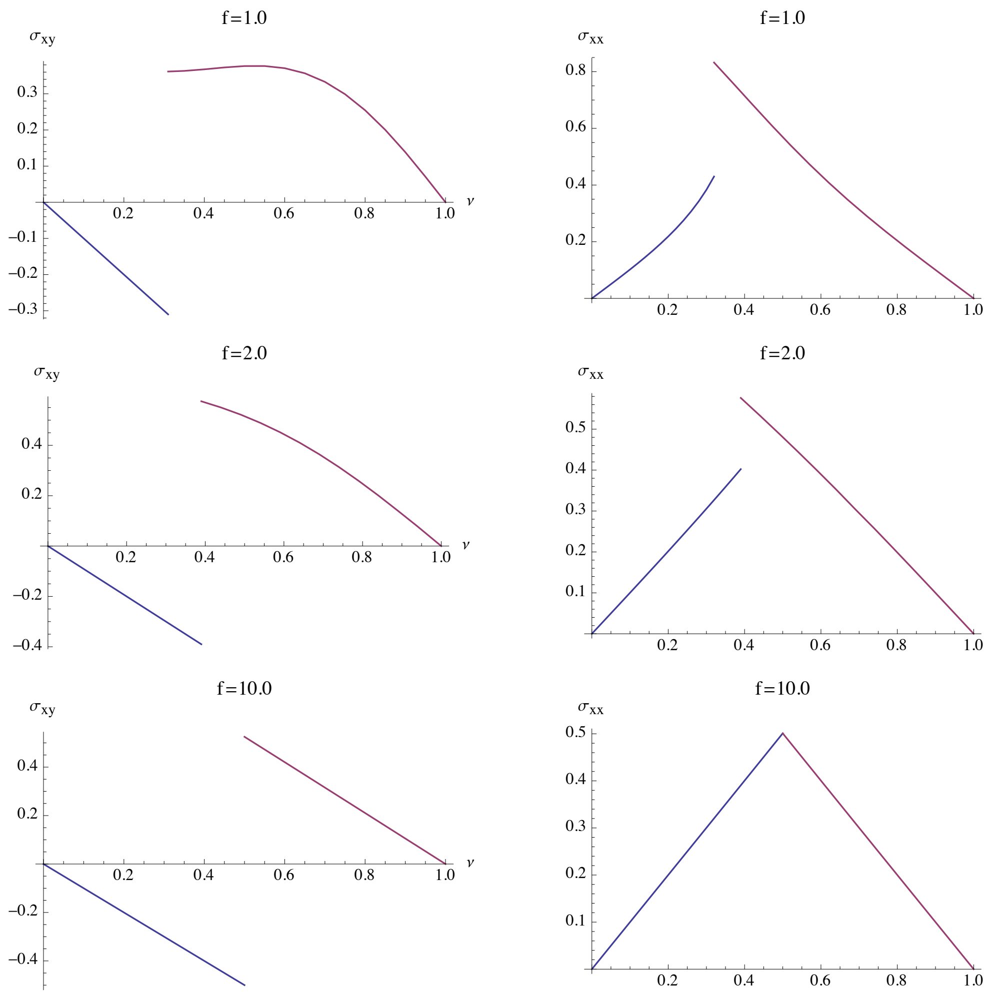

FIGURE 8.1: The first column shows the deviation of the holographic Hall conductivity from the classical one. The second column shows the holographic longitudinal conductivity. Each row corresponds to a different value of f. As we move along ν in the x-axis, a transition occurs between the D5 (blue) and the D7 (red). The units of the y-axes are $\frac{N_c}{2\pi} r_h^4/(1+r_h^4)$ for the first column and $\frac{N_c}{2\pi} r_h^2/(1+r_h^4)$ for the second column. The temperature for all of these plots corresponds to $r_h = 0.2$.

The ohmic conductivity σ_{xx} is just a temperature factor times the Routhian evaluated at the horizon, and thus is similar to the low-temperature entropy. Indeed, at low temperature and large f, the plot of σ_{xx} vs ν becomes identical to the entropy plot in this regime with the appropriate unit conversion (see the second column, third row of 8.1).

For convenience, we also compute the resistivity,

$$\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2}. \tag{8.51}$$

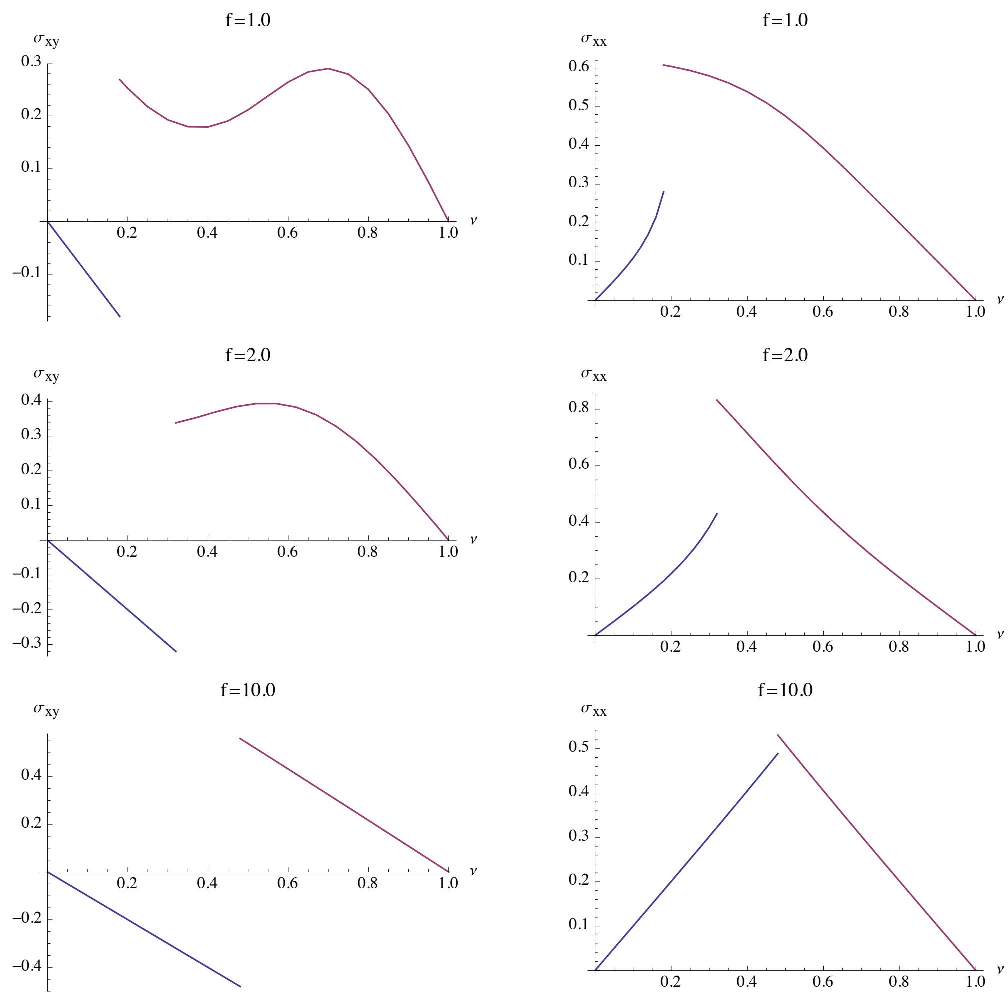

FIGURE 8.2: The first column shows the deviation of the holographic Hall conductivity from the classical one. The second column shows the holographic longitudinal conductivity. Each row corresponds to a different value of f. As we move along ν in the x-axis, a transition occurs between the D5 (blue) and the D7 (red). The units of the y-axes are $\frac{N_c}{2\pi} r_h^4/(1+r_h^4)$ for the first column and $\frac{N_c}{2\pi} r_h^2/(1+r_h^4)$ for the second column. The temperature for all of these plots corresponds to $r_h = 0.4$.

In the large f limit, we have

$$\sigma_{xx}^{D5} = \frac{r_h^2}{1+r_h^4}\frac{N_c \nu}{2\pi} \tag{8.52}$$

$$\sigma_{xy}^{D5} = \left(1 - \frac{r_h^4}{1+r_h^4}\right)\frac{N_c \nu}{2\pi} \tag{8.53}$$

$$\sigma_{xx}^{D7} = \frac{r_h^2}{1+r_h^4}\frac{N_c}{2\pi}(1-\nu) \tag{8.54}$$

$$\sigma_{xy}^{D7} = \frac{N_c}{2\pi}\left(\nu + (1-\nu)\frac{r_h^4}{1+r_h^4}\right), \tag{8.55}$$

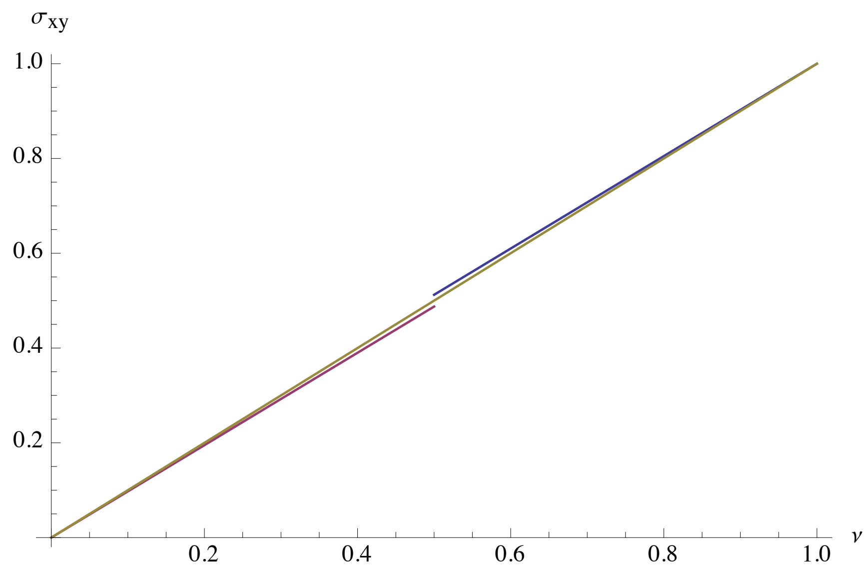

FIGURE 8.3: The Hall conductivity vs filling fraction for the D5 brane (red), the D7 brane (blue), and the classical case (green), for $f = 10$. The temperature corresponds to $r_h = 0.4$.

which gives,

$$\rho_{xx}^{D5} = \frac{2\pi}{N\nu} r_h^2 \tag{8.56}$$

$$\rho_{xx}^{D7} = -\frac{2\pi}{N_c} \frac{r_h^2(\nu - 1)}{r_h^4 + \nu^2}. \tag{8.57}$$

For smaller values of f, ρ_{xx} is shown in 8.4 and 8.5.

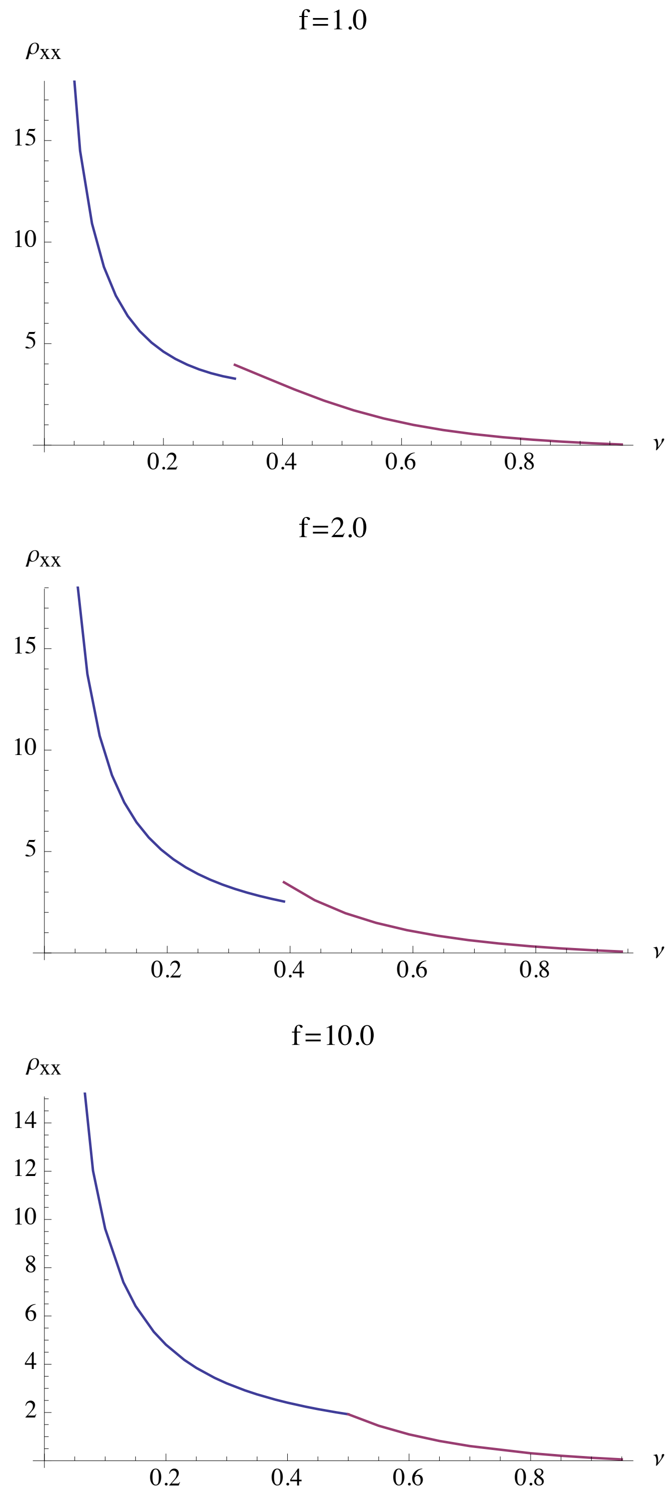

FIGURE 8.4: Longitudinal resistivity as a function of filling fraction. Each row corresponds to a different value of f. As we move along ν in the x-axis, a transition occurs between the D5 (blue) and the D7 (red). The units of the y-axes are $\frac{2\pi}{N_c} r_h^2$. The temperature for all of these plots corresponds to $r_h = 0.2$.

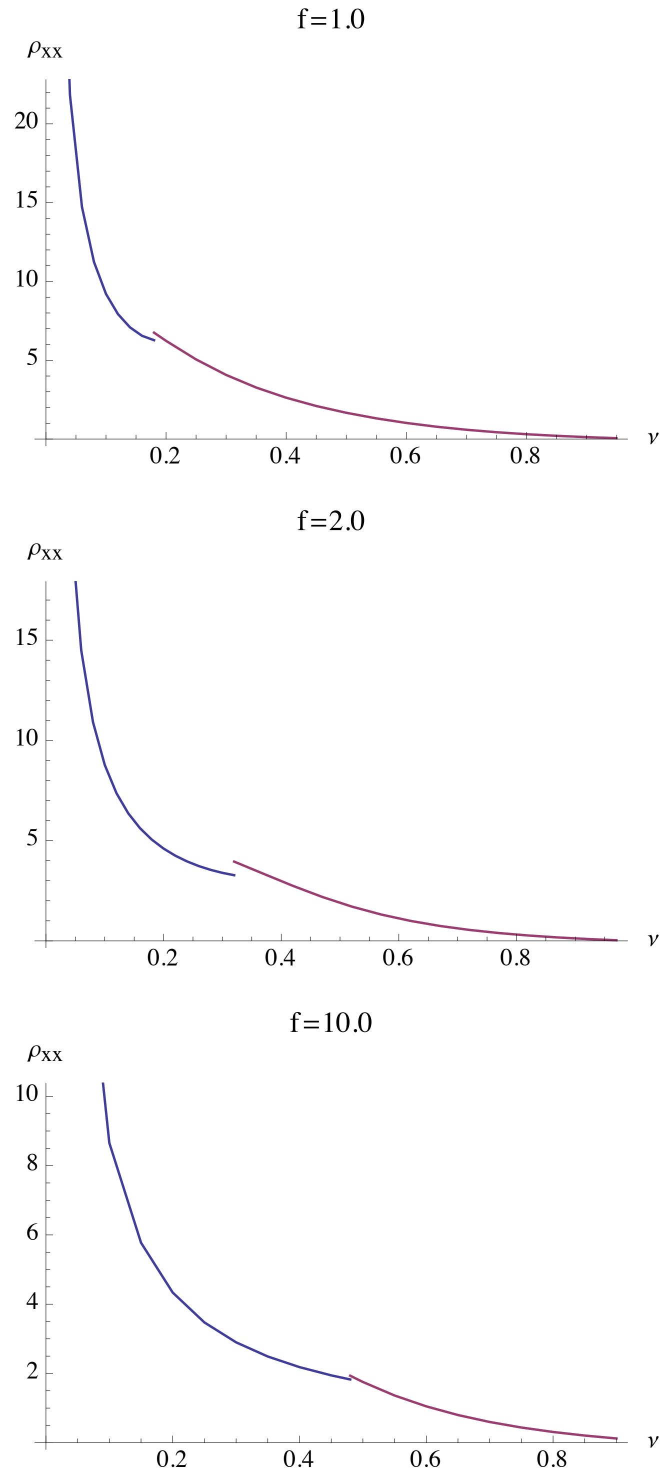

FIGURE 8.5: Longitudinal resistivity as a function of filling fraction. Each row corresponds to a different value of f. As we move along ν in the x-axis, a transition occurs between the D5 (blue) and the D7 (red). The units of the y-axes are $\frac{2\pi}{N_c} r_h^2$.The temperature for all of these plots corresponds to $r_h = 0.4$.

Chapter 9

Conclusion

We have seen a lot in this thesis, so let us review the salient points. We used gauge/gravity duality to investigate the charge transport, and thermodynamic properties a $2+1$ dimensional field theory of fermions in a magnetic field. At strong coupling, we were able to describe this field theory as a classical supergravity theory with additional ingredients, such as finite density, and external electric and magnetic fields, comprising gauge fields on a stack of probe D5 branes. At a certain filling fraction, the Ramond-Ramond supergravity solution facilitates a Myers effect in which the D5 branes blow up to form a D7 brane. At integer filling fraction, that D7 is then able to essentially dissolve the field lines from the D5 system, allowing it to cap off before entering the horizon. This yields a charge-gapped state in the dual theory, corresponding to an integer Hall state.

We studied the low temperature entropy of this system and found that below the D5-D7 transition, the entropy has a simple interpretation in terms of individual quarks i.e. strings stretched between the probe brane and the horizon.

We argued for the D5-D7 transition based on a comparison of their energies. This does not guarantee that the D7 is stable when it takes over. To check this, one would have to look at the dynamics of fluctuations in the probe brane geometry and gauge fields. The corresponding equations of motion are highly complicated and non-linear, and require a sophisticated numerical determinant method to solve [65, 66]. Furthermore, this cannot be done with the coordinates used in this thesis. When the D7 reaches a Minkowski embedding, it ends at some particular radial coordinate. For a complete fluctuation analysis, this radial coordinate must be allowed to fluctuate, and this cannot be done by simply allowing the embedding angle ψ to fluctuate. However, the equations of motion have been solved for a slightly different D7 system in transformed coordinates that account for this problem [44]. There it was found that for a careful choice of boundary conditions for the fluctuations, a superfluid mode results that has a field theory interpretation as an anyonic superfluid. There is a tantalizing possibility that such a mode exists for our model as well.

Lastly, we calculated the conductivity tensor for this system. As expected, translation invariance prevented any integer Hall plateaus from forming. However, hints of a plateau occurring at the D5-D7 transition appeared. This opens up the possibility of non-impurity driven plateaus forming in a strongly coupled quantum Hall system. Removing the homogeneity condition in the field theory dimensions of our model would be a good direction for future work. It is entirely possible that this system is unstable to inhomogenous condensates. Such an instability was found in another D7 system in [67], where it was observed that above a critical density, a striped phase formed. Perhaps a similar phenomenon in our system could yield integer Hall plateaus.

In any case, it is clear that holography has a exceedingly high potential as a tool for probing strongly-coupled condensed matter. For now, quantitative holographic predictions for the quantum Hall ferromagnet remain in the future, but the insight gained from viewing this system in terms of supergravity is tremendously illuminating.

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Appendix A

D5 Conductivity

We start with D5 Routhian in Fefferman-Graham coordinates (8.23)

$$\mathcal{R}_5 = -\frac{2N_c N_5}{(2\pi)^2} V_{2+1} \frac{\sqrt{\lambda}}{2\pi} \int_0^{z_h} dz \frac{4\sin^4\psi}{\sqrt{\mathcal{S}}} [g_{zz}(g_{tt}b^2 + g_{tt}g_{xx}^2 - g_{xx}e^2)]. \tag{A.1}$$

Now we can solve the q equations for the gauge fields. For $f_y'(z)$, this gives

$$1 = \frac{-q_y b e f_y'(z) + q_y a'(z)(b^2 + g_{xx}^2)}{q b e a'(z) + q f_y'(z)(g_{xx}g_{tt} - e^2)} \tag{A.2}$$

$$\Rightarrow f_y'(z) = -\frac{-b^2 q_y - q_y g_{xx}^2 + q b e}{q g_{xx} g_{tt} - q e^2 + q_y b e} a'(z). \tag{A.3}$$

Solving for $f_x'(z)$ gives,

$$f_x'(z) = [-g_{tt}g_{xx}(-q_x^2 + 4g_{tt}g_{xx}\sin^4\psi)]^{-1/2}\Bigg[-\Bigg(\Big(b^2 g_{tt} + g_{xx}(g_{tt}g_{xx} - e^2)\Big)q_x^2\Big(g_{zz}(-e^2 q + g_{tt}g_{xx}q + beq_y)^2 + a'(z)^2 g_{xx}(e^2q^2 - g_{tt}g_{xx}q^2 - 2beqq_y + (b^2 + g_{xx}^2)q_y^2)\Big)(-e^2 q + g_{tt}g_{xx}q + beq_y)^{-2}\Bigg]^{1/2} \tag{A.4}$$

And lastly, $a'(z)$ is,

$$a'(z) = -\sqrt{-g_{tt}}\sqrt{g_{zz}}\sqrt{b^2 g_{tt} + g_{xx}(g_{tt}g_{xx} - e^2)}(beq_y - e^2 q + g_{tt}g_{xx}q)\Bigg[-g_{xx}(b^2 g_{tt} + g_{xx}(g_{tt}g_{xx} - e^2))(4g_{tt}g_{xx}\sin^4\psi(b^2 g_{tt} + g_{xx}(g_{tt}g_{xx} - e^2)) + g_{tt}(g_{tt}g_{xx}q^2 - (b^2 + g_{xx}^2)(q_x^2 + q_y^2)) + 2beg_{tt}qq_y + e^2(g_{xx}q_x^2 - g_{tt}q^2))\Bigg]^{-1/2} \tag{A.5}$$

Inserting these identities into (8.23) gives

$$\mathcal{R}_5 = \frac{-2N_c N_5}{(2\pi)^2} V_{2+1} \frac{\sqrt{\lambda}}{2\pi} \int_0^{z_h} dz \sqrt{\frac{g_{zz}}{g_{tt}}} \frac{1}{g_{xx}} \sqrt{BC - A^2}, \tag{A.6}$$

where we have defined

$$\begin{aligned} A &\equiv qbg_{tt} - q_y e g_{xx} && \text{(A.7)} \\ B &\equiv g_{tt} g_{xx}^2 + g_{tt} b^2 - g_{xx} e^2 && \text{(A.8)} \\ C &\equiv 4\sin^4\psi g_{tt} g_{xx}^2 + g_{tt} q^2 - g_{xx}(q_x^2 + q_y^2). && \text{(A.9)} \end{aligned}$$

Now we follow the argument of [64]. First, we look to see of there are any roots of the function B at some value $z = z_*$. The equation $B(z_*) = 0$ has 16 solutions, four of which are real. Of those four, only one of those lies in the range $z \in (0, z_h)$. The solution is found as follows:

$$\begin{aligned} B(z_*) &= 0 \\ \Rightarrow e^2 z_*^4 (z_*^4 + z_h^4) &= \frac{(z_*^4 - z_h^4)^2 (z_*^8 + z_h^8 + z_h^4 z_*^4 (2 + b^2 z_h^4))}{z_h^8 (z^4 + z_h^4)} \\ \Rightarrow 4\left(\frac{z_*^4}{z_h}\right) \tilde{e}^2 \left(\frac{z_*^4}{z_h^4} + 1\right)^2 &= \left(\frac{z_*^4}{z_h^4} - 1\right)^2 \left(\left(\frac{z_*^4}{z_h^4} + 1\right)^2 + 4\tilde{b}^2 \frac{z_*^4}{z_h^4}\right). \end{aligned} \tag{A.10}$$

In the last line, we introduced the scaled fields $\tilde{e} \equiv \frac{z_h^2}{2} e$ and $\tilde{b} \equiv \frac{z_h^2}{2} b$. If we also let $Z \equiv z_*^4 / z_h^4$, and introduce the coefficients $a_2 \equiv 2(4\tilde{b}^2 + 4\tilde{e}^2 - 1)$ and $a_3 \equiv 4(\tilde{e}^2 - \tilde{b}^2)$, then this now amounts to finding the roots of a quartic

$$Z^4 + a_3 Z^3 + a_2 Z^2 + a_3 Z + 1 = 0. \tag{A.11}$$

Dividing by Z^2 and changing variables to $x = Z + 1/Z$, we get

$$\begin{aligned} x^2 + a_3 x + a_2 &= 0 \\ \Rightarrow \frac{Z^2 + 1}{Z} &= -2(\tilde{e}^2 - \tilde{b}^2) + \sqrt{4(\tilde{e}^2 - \tilde{b}^2)^2 - 2(4\tilde{b}^2 + 4\tilde{e}^2 - 1)} \\ \Rightarrow Z &= \tilde{e}^2 - \tilde{b}^2 + \sqrt{1 + (\tilde{e}^2 - \tilde{b}^2)^2 + 2(\tilde{e}^2 + \tilde{b}^2) + 1} \\ &\quad - \sqrt{[(\tilde{e}^2 - \tilde{b}^2) + \sqrt{(\tilde{e}^2 - \tilde{b}^2)^2 + 2(\tilde{e}^2 + \tilde{b}^2) + 1}]^2 - 1}, \end{aligned} \tag{A.12}$$

where we have selected the only root in $(0, z_h)$.

Now lets turn our attention to the function C. At the horizon, $g_{tt} \to 0$ and $g_{xx} \to 2/z_h^2$, so that

$$C(z_h) = -\frac{2}{z_h^2}(q_x^2 + q_y^2) < 0. \tag{A.13}$$

Meanwhile at the AdS boundary, we have $g_{tt}g_{xx}^2 \to 1/z^6$, $g_{tt} \to -1/z^2$, and $g_{xx} \to 1/z^2$, so that near $z = 0$,

$$C(z) \approx \frac{4}{z^6} - \frac{q^2}{z^2} - \frac{(q_x^2 + q_y^2)}{z^2}, \tag{A.14}$$

which is positive as $z \to 0$. Thus C must cross the axis at least once in the interval $(0, z_h)$. Suppose it had a zero at some $z_c \neq z_*$ (without loss of generality let $z_c > z_*$). Then for $z \in (z_*, z_c)$, the product BC would be negative. Looking at (A.6), we see this would cause the Routhian to become imaginary in that interval. Furthermore, A must also share the same zero at z_* since otherwise, the Routhian would be imaginary at that point. Thus demanding reality of the Routhian forces A, B, and C to share a zero at z_*, giving us three equations:

$$B(z_*) = 0 \quad \Rightarrow \quad g_{tt}(z_*) = \frac{g_{xx}(z_*)e^2}{(g_{xx}(z_*)^2 + b^2)} \tag{A.15}$$

$$A(z_*) = 0 \quad \Rightarrow \quad qbg_{tt}(z_*) = q_y e g_{xx}(z_*) \tag{A.16}$$

$$C(z_*) = 0 \quad \Rightarrow \quad \frac{4\sin^4\psi(z_*)g_{xx}^2(z_*)e^2 + q^2e^2\frac{g_{xx}(z_*)^2}{(g_{xx}(z_*)^2+b^2)}}{(g_x x(z_*)^2 + b^2)}. \tag{A.17}$$

Combining (A.15) and (A.16) gives

$$q_y = \frac{qbe}{(g_{xx}(z_*)^2 + b^2)}. \tag{A.18}$$

While combining (A.17) and (A.18) gives

$$q_x = \frac{e g_{xx}(z_*)}{(g_{xx}(z_*)^2 + b^2)}\sqrt{4\sin^4\psi(z_*)(g_{xx}(z_*)^2 + b^2) + q^2}. \tag{A.19}$$

Now note that

$$\begin{aligned} \left.\frac{z_*^4}{z_h^4}\right|_{\tilde{e}\to 0} &= -\tilde{b}^2 + \sqrt{\tilde{b}^4 + 2\tilde{b}^2 + 1} - \sqrt{(-\tilde{b}^2 + \sqrt{\tilde{b}^4 + 2\tilde{b}^2 + 1})^2 - 1} \\ &= 1 \qquad \text{(A.20)} \\ \Rightarrow g_{xx}(z_*)^2|_{\tilde{e}\to 0} &= \frac{4}{z_h^4}. \qquad \text{(A.21)} \end{aligned}$$

Equations (A.18), (A.19), and (8.15) directly give us the conductivity tensor.

$$\langle J_x \rangle = \sigma_{xx} E \tag{A.22}$$

$$\sigma_{xx} = \frac{2N_cN_5}{(2\pi)\sqrt{\lambda}}\frac{q_x}{e}|_{e=0}. \tag{A.23}$$

Here we take $e \to 0$ to eliminate non-linear terms in e, since we are interested in the linear response. Evaluating this, we get

$$\sigma_{xx} = \frac{2N_c N_5}{(2\pi)\sqrt{\lambda}} \left(\frac{2/z_h^2}{4/z_h^4 + b^2} \right) \sqrt{4 \sin^{\psi}(z_h)(4/z_h^4 + b^2) + q^2}. \tag{A.24}$$

To be consistent with the entropy results, let us convert back to $r_h = \sqrt{\frac{2}{b}} \frac{1}{z_h}$, $f = \frac{2\pi}{\sqrt{\lambda}} N_5$, and $q = \frac{\pi b \nu}{f}$. Then we get the longitudinal conductivity in it's final form,

$$\boxed{\sigma_{xx} = \frac{2N_c}{(2\pi)^2} f \left(\frac{r_h^2}{1 + r_h^4} \right) \sqrt{4 \sin^4 \psi(z_h)(1 + r_h^4) + (\pi\nu/f)^2}}. \tag{A.25}$$

Similarly, the Hall conductivity is found from

$$\langle J_y \rangle = \sigma_{xy} E_x \tag{A.26}$$

$$\Rightarrow \sigma_{xy} = \frac{2N_c N_5}{(2\pi)\sqrt{\lambda}} \frac{qb}{(4/z_h^4 + b^2)} \tag{A.27}$$

$$\Rightarrow \boxed{\sigma_{xy} = \frac{N_c \nu}{2\pi} \left(1 - \frac{r_h^4}{1 + r_h^4} \right)}. \tag{A.28}$$

Note that upon setting $e = 0$, q_x and q_y vanish and thus so do the gauge fields $f_x'(z)$, and $f_y'(z)$. Therefore, we may simply insert the horizon embedding $\psi(r_h)$, that we found in the entropy calculation, into σ_{xx}.