Non-Abelian D5 Brane Dynamics

by

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Abstract

The goal of this thesis is to analyse the non-abelian dual model to the defect probe D7-brane embedding in $AdS_5 \times S^5$[1]. The D7-brane picture can be thought of as a large number (N_5) of D5-branes growing a transverse fuzzy two-sphere, called BIon. This non-abelian solution improves our knowledge of the system by incorporating deviations in $\frac{1}{N_5^2}$ in the number of flavors. Such corrections are important from the point of view of the AdS/CFT correspondance as the CFT dual to the probe system is a candidate model for graphene, which possesses an emergent $SU(4)$ symmetry. The main result of this work is the conductivity for the non-abelian D5 sytem. We find that quantum Hall states have a non-integer transverse conductivity that depends on the number of flavor branes in the model. This deviation scales in $\frac{1}{N_5^2}$ in the number of flavor branes and vanishes in the large N_5 limit.

Preface

Chapters 1 and 2 are a description of chosen material done by others and needed for this work.

Chapters 3 and 4 are based on research done during my Master's degree under the supervision of Professor Gordon Semenoff. This work was supported by funding from the Fonds Québécois de recherche en nature et technologies and the Natural Sciences and Engineering Research Council of Canada.

Table of Contents

Appendix

List of Figures

Chapter 1

Introduction

Dirichlet-branes have played a tremendous role in the construction of string theoretic models since it was suggested in [14] that they are an intrinsic part of the type IIB model of string theory. Not only they are the extended objects on which open strings can end, but they are also BPS objects acting as sources of quantized Ramond-Ramond flux. D-branes now have a life of their own as one can write down an action for them recreating the open string action found from scattering amplitudes.

Under the AdS/CFT correspondence D-branes have proven extremely useful in describing physical systems at strong coupling from their gravity dual using the Top-Down approach in String Theory. This method consists in building a specific configuration of D-branes in ten-dimensional spacetime and using the dictionary of the correspondence to understand its dual field theory. In distinction with the Bottom-Up approach, this method is blind to the four-dimensional dynamics on the gravity side. Even without knowing the four-dimensional gravity theory, the Top-Down approach remains a great tool from our good knowledge of brane dynamics. Surely the most famous example of such a D-brane based configuration is the Sakai-Sugimoto model[16] of AdS/QCD which from probe D8-branes in a D4-branes geometry reproduces important aspects of QCD such as chiral symmetry breaking.

In this chapter, we will review some material we will refer to in the subsequent sections. We start by introducing the Born-Infeld and Chern-Simons actions for general D-branes as well as some important features of branes such as BPS states and Dirac charge quantization. Then, we introduce the notion of non-abelian brane action[18] [12] and fuzzy spherical geometry.

1.1 BIons and BPS bound

We describe here the Born-Infeld action which accounts for the coupling of open strings to the Neveu-Schwarz background fields $G_{\mu\nu}, B_{\mu\nu}$ and Φ. For this discussion, we will work in the type II superstring framework.
Let us start with a space-filling D9-brane with gauge field strength F_{ab} in flat

space. The appropriate action that generalizes non-linear electrodynamics for this configuration is simply,

$$S \sim \int d^{10}\sigma \sqrt{\det\left(\eta_{ab} + 2\pi\alpha' F_{ab}\right)} \tag{1.1}$$

where we remember that α' is the inverse of the string tension $T = \frac{1}{2\pi\alpha'}$ and can be assimilated to the string length $\alpha' = l_s^2$.

Upon compactification of the 9th coordinate, T-duality tells us the Dirichlet boundary condition becomes a Neumann boundary condition and the D9-brane is transformed into a D8-brane. The now transverse coordinate is the compactified component of the gauge field $2\pi\alpha' A_9 = X_9$. By repeating this procedure, we obtain an action for a Dp-brane,

$$S \sim \int d^{p+1}\sigma \sqrt{\det\left(\eta_{ab} + \partial_a X^m \partial_b X_m + 2\pi\alpha' F_{ab}\right)} \tag{1.2}$$

We now need a way to implement the background fields in the action. We note that the Kalb-Ramond field $B_{\mu\nu}$ is not gauge invariant. Under a spacetime gauge transformation $\delta B = d\xi$, it picks up a term

$$\frac{1}{2\pi\alpha'}\int_M B + \delta B = \frac{1}{2\pi\alpha'}\int_M B + \frac{1}{2\pi\alpha'}\int_{\partial M} \xi \tag{1.3}$$

The change in B is only affected by the values of ξ on the worldsheet. We can see this as a signal that the correct gauge invariant way to implement B in the action is by mixing with the gauge transformations of the gauge field A. As A lives on the worldsheet, its gauge transformations are given by,

$$\int_{\partial M} A + \delta A \tag{1.4}$$

Therefore, the combination $B + 2\pi\alpha' dA$ preserves spacetime gauge invariance with the choice of gauge transformation $\delta A = -\frac{\xi}{2\pi\alpha'}$, where ξ is restricted to the worldsheet. This is good evidence that the Kalb-Ramond field should appear in the action with the gauge field strength in the form

$$B + 2\pi\alpha' F \tag{1.5}$$

The dilaton field must appear in the action as $e^{-\Phi}$ to respect the fact that we are dealing with open string tree level amplitudes and will be assimilated to the string coupling. Taking in account an arbitrary background metric,

we replace $\eta_{ab} + \partial_a X^m \partial_b X_m$ by G_{ab}. These changes yield the celebrated Born-Infeld action for a Dp-brane:

$$S_{BI} = T_p \int d^{p+1}\sigma e^{-\Phi} \sqrt{\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)} \tag{1.6}$$

With T_p being the tension of the brane:

$$T_p = \frac{1}{(2\pi)^p \sqrt{\alpha'}^{p+1}} \tag{1.7}$$

We will be interested in solutions to the Born-Infeld action. To demonstrate an important point, we will simplify the problem by considering a single Dp-brane in Minkowski space with the Dilaton and Kalb-Ramond fields set to zero. We will excite only one transverse direction ($X^9(r)$) and switch on an electric field on the brane with no magnetic components. The energy density in that configuration simplifies to:

$$\begin{aligned} \varepsilon^2 &= (1 + \vec{E} \cdot \vec{\nabla} X^9)^2 + (\vec{E} - \vec{\nabla} X^9)^2 \\ &= \Big(1 + \mid \vec{E} \cdot \vec{\nabla} X^9 \mid\Big)^2 + E^2 - 2 \mid \vec{E} \cdot \vec{\nabla} X^9 \mid + (\nabla X)^2 \end{aligned} \tag{1.8}$$

This leads to the BPS bound

$$\varepsilon \geq 1 + \mid \vec{E} \cdot \vec{\nabla} X^9 \mid \tag{1.9}$$

Which is saturated for $\vec{E} = \pm \vec{\nabla} X^9$. By solving the equations of motion for X^9, one obtains the Laplace equation $\nabla^2 X^9 = 0$, which is solved by the Coulomb solution,

$$X^9(r) = \frac{c_p}{r^{p-2}} \tag{1.10}$$

The energy of the Coulomb field is of course infinite and working where the bound is saturated, we impose a cutoff on short distances,

$$E(\delta) = T_p \Omega_{p-1} \int_\delta^\infty dr\ r^{p-1} (\vec{\nabla} X^9)^2 = T_p \Omega_{p-1} c_p (p-2) X^9(\delta) \tag{1.11}$$

This result is very interesting as when $\delta \to 0$ the BIon grows an infinite transverse spike with its base being a point charge. The spike has infinite energy but a constant energy per unit of length. This behaviour was first noticed in [2] and was demonstrated to correspond to an infinitely long fundamental string attached to the brane. This can be seen by setting the constant c_p to $c_p = \frac{1}{2\pi\alpha' T_p \Omega_{p-1}(p-2)}$ and the energy of the spike becomes

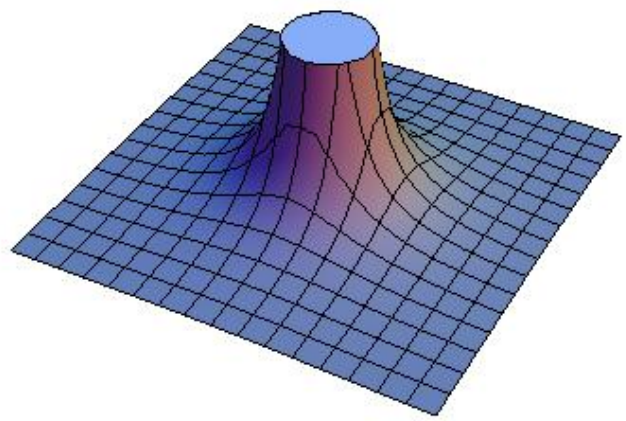

Figure 1.1: The brane grows a spike in the transverse direction corresponding to a fundamental string attached to the brane. The endpoint of the string is the Coulomb charge propagating the electric field on the brane

exactly the energy of a fundamental string of length $X(\delta)$. This value of c_p is not arbitrary but happens to exactly be the value required by charge quantization. This statement is easily verified in the case of a D1-brane and generalizable to higher dimensional branes by T-duality.

We wish to end this section by verifying that the energy bound we found above is truly BPS. This can be done by performing the following supersymmetry analysis. Under a supersymmetry transformation, there are three fermion fields (Gravitino ψ_M, Dilatino λ and Gaugino χ^a) whose variation we must consider. As discussed by Green, Schwarz and Witten [5], the variation of the dilatino is always zero when the Dilaton and Kalb-Ramond fields are turned off. The question for the gravitino is purely geometric and depends only the background geometry. We will solely consider here the variation of the gaugino field which is the only non-trivial part. Its variation under a supersymmetry transformation is,

$$\delta\chi = \Gamma^{\mu\nu} F_{\mu\nu} \epsilon \tag{1.12}$$

where the matrices $\Gamma^{\mu\nu}$ are defined from the 10-dimensional Dirac matrices Γ^μ by

$$\Gamma^{\mu\nu} = [\Gamma^\mu, \Gamma^\nu] \tag{1.13}$$

A BPS configuration is one for which $\delta\chi = 0$ for some ϵ. As only one transverse field X^9 is turned on and the electric field describes a point charge, the non-trivial values for $F^{\mu\nu}$ are

$$F_{0r} = F_{9r} = \frac{\partial}{\partial r}\frac{c_p}{r^{p-2}} \tag{1.14}$$

which leads to the supersymmetry equation,

$$\delta\chi = \left(\Gamma^0 + \Gamma^9\right)\Gamma^r\partial_r X\epsilon \tag{1.15}$$

The configuration is BPS if

$$\left(\Gamma^0 + \Gamma^9\right)\epsilon = 0 \tag{1.16}$$

By going to a Weyl basis for the gamma matrices, it is clear that this equation is satisfied for half of the possible choices of ϵ and the BIon configuration is truly a half BPS state.

1.2 Chern-Simons Action and Dirac quantization

To understand the Dirac quantization on the branes, we recall the usual development for a magnetic point charge. We first consider a magnetic monopole of charge μ_m at the origin and integrate the flux over a sphere surrounding the charge.

$$\int_{S_2} F = \mu_m \tag{1.17}$$

The gauge field respects the usual equation $F = dA$ except on a semi-infinite Dirac string ending on the charge. We first take the Dirac string to be on the positive region of the z-axis. An electric charge μ_e circling the sphere on a closed loop Γ will interact with the magnetic field and pick up a phase

$$e^{i\mu_e \int_\Gamma A} \tag{1.18}$$

As we contract Γ around the Dirac string, the phase becomes

$$e^{i\mu_e\mu_m} = e^{2i\pi n} \tag{1.19}$$

The last equality comes from the fact that the phase needs to be 1 so the Dirac string remains invisible even under different orientation of the string. We conclude that the product of the electric and magnetic charges satisfy $\mu_e\mu_m = 2\pi n$.

In the previous section, we mentioned that branes are BPS objects. From Noether's theorem, they must therefore carry conserved charges. The correct charges that couples to the branes are the Ramond-Ramond charges. The Dp-brane interacts with the RR potential $C^{(p+1)}$ via the so-called Chern-Simons action, which forms the second piece of the D-brane actions

$$S_{CS} = -\mu_p \int d^{p+1}\sigma P\,[\sum_n C^{(n)}]\; e^{2\pi\alpha' F} \tag{1.20}$$

Here, μ_p is the Dp-brane RR-charge and F is the gauge field strength (Not to be confused with the RR-field strength defined next line). The symbol $P[...]$ denotes the pull-back to the brane worldvolume.

The RR potential has a field strength $F^{(p+2)} = dC^{(p+1)}$ which respects the Bianchi identity

$$d \wedge F^{(p+2)} = d \wedge *F^{(p+2)} = 0 \tag{1.21}$$

The Bianchi identity marks the duality between the field strength and their Hodge dual

$$*F^{(p+2)} = F^{(8-p)} = dC^{(7-p)} \tag{1.22}$$

This allows us to extend the point charge development by identifying the electric RR charge to be generated by a Dp-brane while the magnetic charge is spanned by a D(6-p)-brane transverse to the initial Dp-brane.
By following the same argument as for the magnetic point charge, we get that the integral of the magnetic flux on a sphere surrounding the monopole is the RR charge

$$\int_{S_{8-p}} *F^{p+2} = \mu_{6-p} \tag{1.23}$$

The Dirac string ending on the monopole is now enhanced to a Dirac sheet of 7-p dimensions. A Dp-brane circling the Dirac sheet picks up a phase $e^{i\mu_p\mu_{6-p}}$ and the requirement that the Dirac sheet is invisible leads to the quantization condition

$$\mu_p\mu_{6-p} = 2\pi n \tag{1.24}$$

1.3 Non-abelian brane action

The action ($S = S_{BI} + S_{CS}$) which was presented above for Dp-branes is truly the action of a single brane with $U(1)$ gauge field. Once one considers a stack of N_f coincident flavor branes, non-abelian effects are introduced and the action (1.6)+ (1.20) is not sufficient anymore. These effects can be seen

from the point of view of strings stretching between the branes. A string with its endpoints on different branes possesses a mass proportional to its length and can be integrated out of the Lagrangian in the low energy limit. But as we bring the branes together, such a string is no longer stretched and becomes massless. One can not distinguish which brane the endpoint is on and the $U(1)^{N_f}$ gauge symmetry is enhanced to a non-abelian $U(N_f)$ symmetry. The gauge field strength is modified with the usual non-abelian structure

$$F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b] \tag{1.25}$$

Non-commutative effects start to appear when we think of a Dp-brane as obtained from a D9-brane by T-duality. T-duality acting on a worldvolume coordinate transforms a Dp-brane into a D(p-1)-brane interchanging worldvolume gauge fields into scalars portraying transverse excitations of the brane.

$$A_p \to \Phi^p \tag{1.26}$$

These transverse coordinates Φ^i are $N \times N$ matrix-valued scalars with non-trivial commutation relations

$$[\Phi^i, \Phi^j] = \Theta^{ij} \tag{1.27}$$

The geometry now qualifies as being non-commutative and one can not resolve distances at small wavelengths.

In the following section, we wish to review the non-abelian extension of the Born-Infeld and the Chern-Simons action which we will later use to find interresting solutions to various probe-branes configurations.

1.3.1 Non-abelian Born-Infeld action

The extension of the Born-Infeld action to a fully non-commutative theory was first introduced by Tseytlin [18] but became extensively used in the context of Dielectric Branes [12]. In this section we will review the non-abelian formalism to obtain the Born-Infeld action and its main difference with its abelian counterpart.

The way to obtain the action is fairly straightforward. One starts by considering a D9-brane, for which there is no non-abelian structure in the geometry as there are no transverse directions, and starts T-dualizing as before. This time, non-abelian effects of the gauge field are taken into account during the procedure and the commutators of the transverse scalars appear in the action. The result of this approach is the non-abelian BI action (NBI)

given here,

$$S_{NDBI} = -T_p \int d^{p+1}\sigma \mathbf{STr}\left(e^{-\Phi}\sqrt{-\det(P[E_{ab} + E_{ai}(Q^{-1}-\delta)^{ij}E_{jb}] + 2\pi\alpha'\mathbf{F}_{ab})\det(Q^i_j)}\right) \tag{1.28}$$

Where we have defined $E = G + B$ and the matrix $Q^i_j = \delta^i_j + \frac{i}{2\pi\alpha'}[x^i, x^k]E_{kj}$. The latter is a purely non-commutative addition whose existence comes from T-duality of the gauge fields turned scalars.

We now simply wish to point out the main differences between the non-abelian result (1.28) and the commuting action (1.6).

First of all, from the construction of the action, the gauge field possesses obviously a non-abelian symmetry group with field strength given by (1.25).

Secondly, the pullback in (1.28) becomes non-abelian. This means that the derivatives need to be made covariant appropriately by taking into account the possible commutators:

$$\partial_a x^i \to D_a x^i = \partial_a x^i + i[A_a, x^i] \tag{1.29a}$$

$$P[E]_{ab} = E_{\mu\nu}D_a x^\mu D_b x^\nu = E_{ab} + E_{ai}D_b x^i + E_{ib}D_a x^i + E_{ij}D_a x^i D_b x^j \tag{1.29b}$$

Thirdly, the bulk supergravity fields are functions of all the spacetime coordinates and in general are functionals of the non-abelian scalars. The correct way to understand this statement was clarified by Taylor and Van Raamsdonk [17] and consists of expanding the bulk fields in a non-abelian Taylor expansion around the position at which the transverse coordinates vanish. For example, a general field $T_{\mu_1\ldots\mu_k}$ can be written as

$$T_{\mu_1\ldots\mu_k}(x^a, x^i) = \sum_{n=0}^{\infty}\frac{1}{n!}x^{i_1}\ldots x^{i_n}\left(\partial_{x^{i_1}}\ldots\partial_{x^{i_1}}\right)T_{\mu_1\ldots\mu_k}(x^a, x^i)\mid_{x^i=0} \tag{1.30}$$

Finally, the action depends non-linearly on the non-abelian scalars and we need a prescription for their precise ordering in the action in order to obtain a gauge invariant quantity. The symmetrized gauge trace is the most natural prescription as it specifies that one should take the symmetrized average over all orderings of the matrix-valued quantities $[x^i, x^j]$, $D_a x^i$, F_{ab} and x^i in the fundamental representation of $U(N_f)$.

This extension of the BI action is known to correctly predict the interactions of low energy superstring theory up to fourth order (F^4,$[x^i, x^j]^4$) but needs corrections at sixth orders.

It should be noted that this non-abelian action reproduces the results of the abelian system described in 1.6 when all the commutators $[x^i, x^j]$

vanish. This assures us that an abelian solution to the equations of motion is always a valid solution of the system competing with the non-commutative solutions.

1.3.2 Non-abelian Chern-Simons action

We will now review the non-abelian extension to the Chern-Simons action. The formalism to approach this problem was discussed in [12]. We will not sketch a proof of its derivation here but only mention that as for the NBI action, we can verify the consistency of the NCS action by starting from a D9-brane and T-dualizing $8-p$ coordinates to get the Dp+1 action. The full NCS term can be expressed as

$$S_{NCS} = \mu_p \int d^{p+1}\sigma \mathbf{STr}\left(P\left[e^{\frac{i}{2\pi\alpha'}\imath_x\imath_x}\sum_n C^{(n)}e^B\right]\wedge e^{2\pi\alpha' F}\right) \tag{1.31}$$

The symmetrized gauge trace prescription and the dependence of the bulk fields and gauge field strength on the non-abelian scalars follow the treatment of the NBI action. The important addition in the non-abelian extension of the CS action arises from the non-abelian interior product $\imath_x$. The interior product $\imath_x$ is an operator of degree -1 with its square acting on n-forms as:

$$\imath_x\imath_x C^{(n)} = x^i x^j C_{jix_3\ldots x_n} = \frac{1}{2}C_{ijx_3\ldots x_n}[x^i, x^j] \tag{1.32}$$

We notice that for usual commuting vectors v^i, the internal product reduces to the null operator as it is the non-commutativity of the x's that gives its structure. It is thus the exponential of interior products that generate the relevant non-abelian terms in the Chern-Simons action.

1.4 Review of the fuzzy sphere

In this section, we review the non-commutative extension of the definition of the sphere which will be a necessary tool in the following discussion to describe the action of a stack of D5-branes in Anti-de-Sitter geometry.

First of all, we consider the algebra of complex-valued functions on the sphere $\mathbf{C}(S^2)$. We expand an arbitrary function of this algebra in a polynomial expansion

$$\mathcal{A} = f_0 + f_i v^i + f_{ij}v^i v^j + f_{ijk}v^i v^j v^k + \ldots \tag{1.33}$$

On the two-sphere, we have three Cartesian coordinates and the index i runs from $i = 1, 2, 3$. We wish to generate the geometry of S^2 from a series of truncations of this algebra. We will call the truncation to order n $\mathcal{A}_n$.

We can start by truncating the algebra to its first term: $\mathcal{A}_1 = \mathbb{C}$. In this case, the algebra collapses to the complex numbers and the geometry of S^2 is seen as a point.

If we truncate the algebra at the linear term, we obtain a four-dimensional vector space spanned by f_0 and f_i. Note that $\mathcal{A}_2$ is not closed under multiplication for arbitrary vectors v^i and does not form an algebra itself. In order for the truncation to become an algebra itself, we impose a product rule such that the vectors v^i are defined as $v^i = \kappa\sigma^i$, where σ^i are the Pauli matrices and $\kappa^2 = \frac{r^2}{3}$. Under this condition, $\mathcal{A}_2$ closes and becomes the algebra of complex 2×2 matrices. The geometry can only distinguish the north and south poles of the two-sphere.

In general, a truncation to an arbitrary level n corresponds to approximating the sphere by a n^2 dimensional vector space. The truncation $\mathcal{A}_n$ closes when the vectors are in the $n = 2j + 1$ irreducible representation of $SU(2)$ as $v^i = \kappa L^i$. The normalization κ is chosen such that the radius of the sphere satisfies the usual definition

$$r^2 = \sum_{i=1}^{3} \left(v^i\right)^2 \tag{1.34}$$

The right-hand side of equation (1.34) is proportional to the Casimir invariant of $SU(2)$:

$$\sum_{i=1}^{3} \left(v^i\right)^2 = \kappa^2 \sum_{i=1}^{3} \left(L^i\right)^2 \tag{1.35a}$$

$$= C\kappa^2 \tag{1.35b}$$

$$= j(j+1)\kappa^2 \tag{1.35c}$$

The normalization factor κ is thus taken to be $\kappa = \frac{r}{\sqrt{j(j+1)}}$ and the coordinates on the sphere are

$$v^i = \frac{rL^i}{\sqrt{j(j+1)}} \tag{1.36a}$$

$$[v^i, v^j] = \frac{ir^2}{j(j+1)} \epsilon^{ij}{}_k L^k \tag{1.36b}$$

The truncation $\mathcal{A}_n$ is described by the algebra of complex $n \times n$ matrices M_n. For a small value of n, the commutators are large and the sphere is fuzzy. We can not resolve distances smaller than $d \sim \frac{r}{n}$. On the other hand, in the large n limit, the commutators vanish and the sphere becomes classical.

In the space M_n of $n \times n$ matrices, we can define a norm such that for any element $f \in M_n$,

$$\|f\|_n^2 = \frac{1}{n} Tr(f^* f) \tag{1.37}$$

The norm can be seen to generalize the notion of integration on the sphere,

$$\frac{1}{4\pi r^2} \int_{\Omega_2} f \to \frac{1}{n} Tr(f) \tag{1.38}$$

Where on the left-hand side, f denotes a function in $\mathbf{C}(S^2)$ while on the right-hand side, f is the associated matrix in M_n.

Chapter 2

Abelian Probe Branes

In this chapter, we review the abelian probe brane action of D3, D5 and D7-branes in $AdS_5 \times S^5$. We will work specifically with probes living on a 2+1 dimensional submanifold (defect) of AdS_5. We interest ourselves in these models as, when non-abelian effects are included, these systems have the potential to blow up to a D7-brane. The D7-brane configuration is relevant from gauge/gravity duality as, in the dual CFT, the corresponding system is a 2+1 dimensional gas of electrons exhibiting many interesting features as quantum Hall effect and can closely be related to graphene [9] [8].

2.1 Setup

From Maldacena's gauge/gravity correspondence [10], Anti-de Sitter geometry is realized as the near-horizon limit of a stack of N_c coincident D3 colour branes and equivalent to $\mathcal{N} = 4$ Super Yang-Mills theory in 3+1 dimensions. The range of validity of Maldacena's conjecture will be respected in the limit when the string theory is weakly interacting and the number of D3-branes is taken to be large $N_c \gg 1$. In this background geometry we add N_f flavor probe branes. Working in the probe limit entails that we can neglect the backreaction of the probes on the geometry. This corresponds to the limit $N_c \gg N_f$

Throughout this work, we will use the orientation of Table (2.1) for the D-branes. We will also use the following convention: Greek indices

Table 2.1: Orientation of D-branes

	0	1	2	3	4	5	6	7	8	9
$D3_c$	X	X	X	X						
$D3_f$	X	X	X		X					
D5	X	X	X		X	X	X			
D7	X	X	X		X	X	X	X	X	

$\mu, \nu, ... = 0, 1, ..., 9$ will refer to spacetime coordinates. Latin indices $a, b, ...$

will describe probe branes worldvolume coordinates and transverse directions to the probes will have the Latin indices from the middle of the alphabet $i, j, ...$ In Table 2.1 we made the distinction between the background colour D3-branes(D3$_c$) and the probe flavour branes (D3$_f$).

We will work in static gauge, i.e. the target space coordinate x^0 will be designated as the time evolution parameter τ. For reasons that will become clear later, we will use an $S^2 \times S^2$ embedding for the S^5. This means we parametrize the S^5 as the fibration of two S^2 over an interval $\psi \in [0, \frac{\pi}{2}]$. Both spheres are perpendicular to the probe D3-branes while the first two-sphere (unbarred) is wrapped by the D5-brane and the D7-brane wraps both (barred and unbarred) spheres. The appropriate metric on the S^5 is,

$$ds^2_{S^5} = d\psi^2 + \sin^2\psi \; d\Omega_2^2 + \cos^2\psi \; d\bar{\Omega}_2^2 \tag{2.1}$$

We will study these configurations with non-trivial magnetic field, charge density and temperature which, when the non-abelian structure is ignored, is known to lead to an interesting phase diagram with a BKT phase transition[4]. We turn on a gauge field strength as,

$$2\pi\alpha' \mathbf{F}_{ab} = \sqrt{\lambda}\alpha' \left(-\frac{d}{dr} a_0(r) dt \wedge dr + b dx \wedge dy \right) \tag{2.2}$$

This accounts for a charge density $A_t(r) = \frac{\sqrt{\lambda} a_0(r)}{2\pi}$ and aconstant magnetic field $B = \frac{\sqrt{\lambda} b}{2\pi}$.

The correct way to implement non-zero temperature is via Hawking's radiation. The metric is then modified following [6] to include an AdS black-hole.

$$\begin{aligned} ds^2 = \sqrt{\lambda}\alpha' [& r^2 \left(-h(r) dt^2 + dx^2 + dy^2 + dz^2 \right) \\ & + \frac{dr^2}{h(r) r^2} + d\psi^2 + \sin^2\psi d\Omega_2^2 + \cos^2\psi d\bar{\Omega}_2^2] \end{aligned} \tag{2.3}$$

The black hole has radius r_h with the blackening function defined as $h(r) = 1 - \frac{r_h^4}{r^4}$ in the case of D3-branes. We remind the reader that α' is the Regge slope parameter giving the tension of the string $T = \frac{1}{2\pi\alpha'}$ and $\lambda = 4\pi g_s N_3$ with the radius of AdS given by $L^2 = \sqrt{\lambda}\alpha'$.

In this geometry, the Ramond-Ramond field strength is given by two parts, one on AdS_5 and one on S^5.

$$F^{(5)} = 4\lambda {\alpha'}^2 \left(r^3 dt \wedge dx \wedge dy \wedge dz \wedge dr + d\Omega_5 \right) \tag{2.4}$$

The RR potential is defined by $F^{(5)} = dC^{(4)}$ which results once again in two distinct parts,

$$C^{(4)} = \lambda\alpha'^2 \left(r^4 dt \wedge dx \wedge dy \wedge dz + \frac{1}{2}\left(\psi - \frac{\sin 4\psi}{4} - \frac{\pi}{2}\right) d\Omega_2 \wedge d\bar{\Omega}_2 \right) \tag{2.5}$$

Where $\frac{\pi}{2}$ is a gauge choice corresponding to the value of $\psi(r = \infty)$. Throughout this work we will be interested in solutions where ψ is allowed to vary with the radial coordinate $\psi = \psi(r)$, but where z is fixed and does not depend on r.

2.2 Probe brane action

2.2.1 D3 probes

We now introduce N_3 flavour probe D3-branes to the geometry following the orientation of table 2.1 and wish to compute the action for this system ignoring any possible interaction terms between the probes.

The Born-Infeld action for this system can be written as

$$S_{BI3} = -\frac{T_3}{g_s} N_3 \int d^4\sigma \sqrt{\det\left(P[G+B]_{ab} + 2\pi\alpha' F_{ab}\right)} \tag{2.6}$$

Where the tension of the D3-brane is given by $T_3 = \frac{1}{(2\pi)^3\alpha'^2}$. In this work, the Kalb-Ramond field $B_{\mu\nu}$ will be set to zero. This will be the case for each type of embeddings that we will consider. It should also be noticed that the action is multiplied by an overall factor of N_3. Of course, as the probes do not interact, they all contribute the same value to the action, hence this factor.

The induced metric on the probe worldvolume is

$$ds_4^2 = \sqrt{\lambda}\alpha' \left(r^2\left(-hdt^2 + dx^2 + dy^2\right) + dr^2\left(\frac{1}{hr^2} + \dot{\psi}^2\right)\right) \tag{2.7}$$

In our convention, the dot defines a derivative with respect to r. The BI action for the probe D3-brane is

$$S_{BI3} = -\mathcal{N}_3 N_3 \int dr \sqrt{\left(1 + hr^2\dot{\psi}^2 - \dot{a}^2\right)\left(r^4 + b^2\right)} \tag{2.8}$$

Here we have defined the constant

$$\mathcal{N}_3 = \frac{T_3}{g_s}\lambda\alpha'^2\ V_{2+1} \tag{2.9}$$

The second part of the action is given by the Chern-Simons (CS) term,

$$S_{CS3} = -\mu_3 N_3 \int d^4\sigma P[C^4] \tag{2.10}$$

Where the D3 RR charge is $\mu_3 = \frac{T_3}{g_s}$. The pullback picks up a term from the derivative of the z-direction,

$$S_{CS3} = -\mathcal{N}_{3'} N_{3'} \int dr\ r^4 \dot{z} \tag{2.11}$$

but as we mentioned above, we only interest ourselves in solutions where z does not depend on r which allows us to neglect the CS action for this configuration. The only relevant term is the BI action from which we find the equations of motion for $\psi(r)$. To do so, we start by removing the cyclic variable a from the Lagrangian via a Legendre transformation that takes us to the Routhian,

$$\mathcal{R}_3 = S_3 - \int dr\ \dot{a} \frac{\partial S_3}{\partial \dot{a}} \tag{2.12}$$

The result is,

$$\mathcal{R}_3 = -\mathcal{N}_3 N_3 \int dr\ \sqrt{(r^4 + b^2) + q_3^2} \sqrt{1 + hr^2 \dot{\psi}^2} \tag{2.13}$$

Here, q_3 is the constant of the equation of motion for the cyclic variable $\dot{a}$.

$$q_3 = \frac{1}{\mathcal{N}_3 N_3} \frac{\partial S_3}{\partial \dot{a}} = \frac{\dot{a}\sqrt{r^4 + b^2}}{\sqrt{1 + hr^2\dot{\psi}^2 - \dot{a}^2}} \tag{2.14}$$

This constant can be related to the total charge density via

$$\rho = \frac{1}{V_{2+1}} \frac{\partial S_3}{\partial \frac{d}{dr} A_t} = \frac{1}{V_{2+1}} \frac{2\pi}{\sqrt{\lambda}} \frac{\partial S_3}{\partial \dot{a}} = \frac{q_3 N_c N_3}{\pi\sqrt{\lambda}} \tag{2.15}$$

Where in the last equality, we have used the fact that $g_s = \frac{\lambda}{4\pi N_c}$. We now have the freedom to rescale the value of r in such a way that the magnetic field is taken outside of the integral. This is done while remembering that the Landau level filling fraction is the charge density per units of magnetic field,

$$\nu = \frac{2\pi\rho}{N_3 B} \tag{2.16a}$$

$$\frac{q_3}{b} = \frac{\nu\lambda}{4\pi N_3} = \frac{\pi\nu}{f_3} \tag{2.16b}$$

where f_3 was defined as $f_3 = \frac{4\pi^2 N_3}{\lambda}$.

Rescaling r to get rid of b in the integral, we get the Routhian,

$$\mathcal{R}_3 = -\frac{\mathcal{N}_3 N_3}{f_3}\left(\frac{2\pi B}{\sqrt{\lambda}}\right)^{3/2}\int dr\,\sqrt{f_3^2\left(r^4+1\right)+(\pi\nu)^2}\sqrt{1+hr^2\dot{\psi}^2} \tag{2.17}$$

In this particular case, it appears that ψ is also a cyclic variable and the action is trivial. The corresponding constant of the equations of motion is

$$0 = \frac{d}{dr}\left(\frac{hr^2\dot{\psi}\sqrt{f_3^2\left(r^4+1\right)+(\pi\nu)^2}}{\sqrt{1+hr^2\dot{\psi}^2}}\right) \tag{2.18}$$

Even though this system has no dynamical degrees of freedom, it becomes non-trivial once the full non-abelian action is considered, which we will investigate later.

2.2.2 D5 probes

Once we have done the D3 problem, embedding a stack of commuting D5-branes in this geometry is not a lot of work. Here, the probes live on a 2+1 dimensional defect on AdS_5 and wrap a two-sphere on the S^5.

The Born-Infeld action is written as

$$S_{BI5} = -\frac{T_5}{g_s}N_5\int d^6\sigma\sqrt{\det\left(P[G+B]_{ab}+2\pi\alpha' F_{ab}\right)} \tag{2.19}$$

with the D5-brane tension $T_5 = \frac{1}{(2\pi)^5\alpha'^3}$. This time, the induced metric receives a contribution on the wrapped sphere,

$$ds_6^2 = \sqrt{\lambda}\alpha'\left(r^2\left(-hdt^2+dx^2+dy^2\right)+dr^2\left(\frac{1}{hr^2}+\dot{\psi}^2\right)+\sin^2\psi d\theta^2+\sin^2\psi\sin\theta d\phi^2\right) \tag{2.20}$$

The BI action for this configuration is thus written as,

$$S_{BI5} = -\mathcal{N}_5 N_5\int dr\sqrt{4\sin^4\psi\left(r^4+b^2\right)}\sqrt{1+hr^2\dot{\psi}^2-\dot{a}^2} \tag{2.21}$$

Where $\mathcal{N}_5$ is defined in a similar fashion as $\mathcal{N}_3$ as

$$\mathcal{N}_5 = 2\pi\frac{T_5}{g_s}\left(\sqrt{\lambda}\alpha'\right)^3 V_{2+1} \tag{2.22}$$

The CS action for the probe D5-branes receives its only contribution from the term

$$S_{CS5} = -\mu_5 N_5 \int d^6\sigma P[C^{(4)}] \wedge 2\pi\alpha' F \tag{2.23}$$

As we did not turn on a flux for the gauge field strength on the two-sphere, this term cannot have a contribution on the D5 worldvolume and the CS action vanishes once again.

The full action is given only by the BI term to which we can apply the same procedure as before to remove the dependence on $\dot{a}$ and rescale the magnetic field appropriately. The corresponding Routhian is,

$$\mathcal{R}_5 = -\frac{\mathcal{N}_5 N_5}{f_5}\left(\frac{2\pi B}{\sqrt{\lambda}}\right)^{3/2} \int dr \sqrt{4\sin^4\psi f_5^2\left(r^4+1\right)+(\pi\nu)^2}\sqrt{1+hr^2\dot{\psi}^2} \tag{2.24}$$

Here, f_5 was obtained by the same procedure as before, defining the constant of the equation of motion for $\dot{a}$ as $\frac{q_5}{b} = \frac{\pi\nu}{f_5}$. In this case, we get $f_5 = \frac{2\pi N_5}{\sqrt{\lambda}}$.

Applying the Euler-Lagrange equations on the Routhian give us the equations of motion for the radius of the two-sphere $\psi(r)$,

$$\begin{aligned} 0 =& \frac{h\left(r\frac{d}{dr}\right)^2\psi}{1+hr^2\dot{\psi}^2} + 2\frac{r_h^4}{r^3}\left(1+\frac{1}{hr^2\dot{\psi}^2}\right) + hr\dot{\psi}\left(1+\frac{8f^2\sin^4\psi r^4}{4\sin^2\psi f^2(1+r^4)+(\pi\nu)^2}\right) \\ &- \frac{8\sin^3\psi\cos\psi f^2(1+r^4)}{4\sin^2\psi f^2(1+r^4)+(\pi\nu)^2} \end{aligned} \tag{2.25}$$

This time, the dynamics for ψ are not trivial and the system possesses very interesting solutions for different values of the parameters f, ν, r_h which we will look at closely in the next chapters.

2.2.3 D7 probes

Finally, we look at the embedding of a single probe D7-brane in the $AdS_5 \times S^5$ geometry. For this particular configuration, we will consider a worldvolume gauge field with flux on both two-spheres. Later on, the fluxes will be seen by the D7-branes as units of Dirac monopole generated by lower-dimensional branes. Adding a flux, the worldvolume field strength becomes,

$$2\pi\alpha'\mathbf{F}_{ab} = \sqrt{\lambda}\alpha'\left(-\frac{d}{dr}a(r)dt\wedge dr + bdx\wedge dy + \frac{f_7}{2}d\Omega_2 + \frac{\bar{f}_7}{2}d\bar{\Omega}_2\right) \tag{2.26}$$

The induced metric on the D7-branes is now,

$$ds_6^2 = \sqrt{\lambda}\alpha' \left(r^2 \left(-hdt^2 + dx^2 + dy^2\right) + dr^2 \left(\frac{1}{hr^2} + \dot{\psi}^2\right) + \sin^2\psi d\Omega_2^2 + \cos^2\psi d\bar{\Omega}_2^2 \right) \tag{2.27}$$

The BI action for the D7-branes is,

$$S_{BI7} = -\mathcal{N}_7 \int dr \sqrt{\left(f_7^2 + 4\sin^4\psi\right)\left(\bar{f}_7^2 + 4\cos^4\psi\right)\left(r^4 + b^2\right)} \sqrt{1 + hr^2\dot{\psi}^2 - \dot{a}^2} \tag{2.28}$$

with the definition $\mathcal{N}_7 = \frac{T_7}{g_s}\left(\sqrt{\lambda}\alpha'\right)^4 (2\pi)^2 V_{2+1}$. It is important to point out that

$$\mathcal{N}_7 = \frac{\mathcal{N}_3 N_3}{f_3} = \frac{\mathcal{N}_5 N_5}{f_5} \tag{2.29}$$

which will allow us to consider a single D7-brane as a stack of D3-branes growing two transverse fuzzy two-spheres (or one fuzzy four-sphere) or as a stack of D5-branes growing a transverse fuzzy two-sphere.

The Chern-Simons action receives a contribution from the second Chern character,

$$S_{CS7} = -\mu_7 \int d^8\sigma \left(2\pi\alpha'\right)^2 P[C^{(4)}] \wedge F \wedge F \tag{2.30}$$

which this time is non-zero as the RR potential $C^{(4)}_{\theta\phi\bar{\theta}\bar{\phi}}$ lies on the D7-brane worldvolume. The result for the CS action is

$$S_{CS7} = -2b\,\mathcal{N}_7 b \int dr\ \dot{a}\ c(\psi) \tag{2.31}$$

The full action is still cyclic in $\dot{a}$ and after the now usual computations we get the following Routhian,

$$\mathcal{R}_7 = -\mathcal{N}_7 \left(\frac{2\pi B}{\sqrt{\lambda}}\right)^{3/2} \int dr \sqrt{\left(f_7^2 + 4\sin^4\psi\right)\left(\bar{f}_7^2 + 4\cos^4\psi\right)\left(r^4 + 1\right) + \left[\pi\nu + 2c(\psi)\right]^2} \sqrt{1 + hr^2\dot{\psi}^2} \tag{2.32}$$

From the Routhian, we find the equations of motion for ψ

$$
\begin{aligned}
0 =& \frac{h\left(r\frac{d}{dr}\right)^2\psi}{1+hr^2\dot{\psi}^2} + 2\frac{r_h^4}{r^3}\left(1+\frac{1}{hr^2\dot{\psi}^2}\right) \\
&+ hr\dot{\psi}\left(1+\frac{2r^4\left(f_7^2+4\sin^4\psi\right)\left(\bar{f}_7^2+4\cos^4\psi\right)}{(1+r^4)\left(f_7^2+4\sin^2\psi\right)\left(\bar{f}_7^2+4\cos^4\psi\right)+[\pi\nu+2c(\psi)]^2}\right) \\
&- \frac{8\sin\psi\cos\psi(1+r^4)\left(\sin^2\psi\bar{f}_7^2-\cos^2\psi f_7^2\right)+4\sin^3 2\psi\cos 2\psi(1+r^4)+2\sin^2 2\psi[\pi\nu+2c(\psi)]}{(1+r^4)\left(f_7^2+4\sin^2\psi\right)\left(\bar{f}_7^2+4\cos^4\psi\right)+[\pi\nu+2c(\psi)]^2}
\end{aligned}
\tag{2.33}
$$

2.3 Near-horizon limit

2.3.1 D5 probes

In the near-horizon geometry generated by the D3-branes 2.3, we remark that the worldvolume of the probe D5-branes is an $AdS_4 \times S^2$ submanifold of $AdS_5 \times S^5$[7]. This can be seen as the D5-branes sit at $z = 0,\ \psi = \frac{\pi}{2}$. The world-volume metric as seen by the probe D5-branes is

$$
ds_6^2 = \sqrt{\lambda}\alpha'\left(r^2\left(-hdt^2+dx^2+dy^2\right)+\frac{dr^2}{hr^2}+\sin^2\psi d\Omega_2^2\right) \tag{2.34}
$$

There are two types of excitations present in this limit: the closed type IIB strings propagating in $AdS_5 \times S^5$ and the stretched 3-5 open strings living in $AdS_4 \times S^2$. The most interesting feature of this construction is that the gauge/gravity duality acts twice. In the dual picture, we get an $\mathcal{N} = 4\ SYM$ theory in 3+1 dimensions at the boundary of $AdS_5 \times S^5$ while the probes generate a 2+1 ($z = 0$) defect CFT living at the boundary of $AdS_4 \times S^2$. 3-5 strings provide an interaction between the 3+1 $\mathcal{N} = 4\ SYM$ and the defect CFT.

The probe D5-branes break the usual $SO(2,4)$ symmetry of AdS_5 to $SO(2,3)$, the symmetry of AdS_4. It also breaks the $SO(6)$ symmetry of the S^5 to $SO(3) \times SO(\bar{3})$ as the D5-branes wrap an S^2 leaving the transverse $\bar{S}^2$ intact.

From the field theory perspective, the $SO(2,3)$ symmetry generated by the probes are associated to conformal symmetry in 2+1 dimensions. The defect is situated at $z = 0$ from the 3+1 dimensional perspective. The $SO(3) \times SO(\bar{3}) \cong SU(2)_{Vector} \times SU(2)_{Hyper}$ is the unbroken R-symmetry

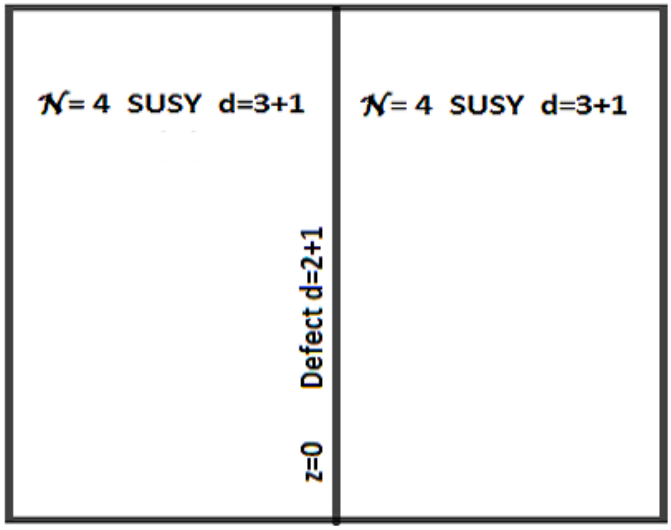

Figure 2.1: The defect occupies a 2+1 dimensional volume positioned at $z = 0$, corresponding to the position of the D-branes, in the 3+1 dimensional $\mathcal{N} = 4$ SUSY.

with 8 conserved supercharges. The original sixteen supercharge vector multiplet of 3-3 and 5-5 open strings transform respectively as a vector multiplet and a hypermultiplet under the eight preserved supercharges. The 3-5 strings transform as a bifundamental, the tensor product of the fundamental representation of each gauge group.

At weak coupling, the defect action in the field theory contains only fermions ψ and their scalar superpartners ϕ.

$$S = \int d^3x \sum_{\alpha=1}^{N_3} \sum_{\lambda=1}^{N_5} [\bar{\psi}_\alpha^\lambda i\gamma^\mu D_\mu \psi_\alpha^\lambda + D_\mu \bar{\phi}_\alpha^\lambda D^\mu \phi_\alpha^\lambda] \tag{2.35}$$

Under R-symmetry, the fermions, assimilated to electrons, transform under the spinor representation of $\bar{SO}(3)$ while the scalars are in the spinor representation of $SO(3)$. Both fermions and scalars transform under the fundamental representation of the gauge groups $U(N_5)$ and $SU(N_c)$.
Once we study the system under non-zero magnetic fields and density 2.2, the probe branes worldvolume is not exactly $AdS_4 \times S^2$ anymore as the parallel two-sphere is not maximal. Also, in general there can be a momentum in the z-direction. The line element on the D5 world-volume becomes the

following,

$$ds_6^2 = \sqrt{\lambda}\alpha' \left(r^2 \left(-hdt^2 + dx^2 + dy^2\right) + dr^2 \left(\frac{1}{hr^2} + \left(\frac{d\psi}{dr}\right)^2 + hr^4 \left(\frac{dz}{dr}\right)^2 \right) + \sin^2\psi d\Omega_2^2 \right) \tag{2.36}$$

2.3.2 D7 probes

From the point of view of the D7-branes, the story is almost the same. The worldvolume symmetry of the probes is $AdS_4 \times S^2 \times \bar{S}^2$ instead of $AdS_4 \times S^2$ as the second sphere is also on the worldvolume of the D7 but the discussion about the near-horizon limit remains similar.

A way to achieve a different model is by considering probe D7-branes that wrap an S^4 in S^5 instead of two two-spheres. The probe worldvolume geometry becomes $AdS_4 \times S^4$ and the main difference arising from this setup is that the original $SO(6)$ R-symmetry gets broken to $SO(5)$. The fermions are in the spinor representation of $SO(5)$ while the scalars remain unchanged under R-Symmetry. This configuration happens to be unstable as it was shown [15] to contain a tachyon which violates the Breitenlohner-Freedman bound. This is a good motivation for us to focus on $S^2 \times S^2$ embeddings for the rest of this work.

2.4 Correlators of the defect CFT

As it was pointed out in the previous section, the introduction of probe branes corresponds as the insertion of a 2+1 dimensional CFT located at $z = 0$. It is thus crucial to understand how the behavior of correlation functions in the CFT will be affected by the insertion of the defect. There will be two types of primary operators: those living in the 3+1 $\mathcal{N} = 4$ SYM denoted $\mathcal{O}_4(\vec{x}, z)$ and the defect primary operators $\mathcal{O}_3(\vec{x})$ situated at $z = 0$. We use the notation Δ_4, Δ_3 for the conformal dimension of primary operators $\mathcal{O}_4$ and $\mathcal{O}_3$ respectively. An interesting feature of the dCFT is that the defect introduced at $z = 0$ creates a "length scale" generating non-vanishing one-point function for the operator $\mathcal{O}_4$,

$$\langle \mathcal{O}_4(\vec{x}, z) \rangle = \frac{C_4}{z^{\Delta_4}} \tag{2.37}$$

while the operator $\mathcal{O}_3$, of course does not see that length scale and its one-point function vanishes.

We can also obtain non-trivial values for the product of two ambient operators of different conformal dimensions Δ_4, Δ_4',

$$\langle \mathcal{O}_4(\vec{x}_1, z_1)\mathcal{O}_4(\vec{x}_2, z_2)'\rangle = \frac{1}{z_1^{\Delta_4} z_2^{\Delta_4'}} f(\xi) \tag{2.38}$$

which is the usual behavior for a two point correlation function at the exception that the result is multiplied by an arbitrary function $f(\xi)$ with $\xi = \frac{(x_1-x_2)^2}{4z_1 z_2}$. It is easily verified that the parameter ξ is invariant under the 2 dimensional translations and $O(2,1)$ rotations that leave the boundary invariant allowing the introduction of the arbitrary function $f(\xi)$ in the two-point function.

We can also compute the two-point function between one ambient operator $\mathcal{O}_4$ and one defect operator $\mathcal{O}_3$. To do so we use the operator product expansion[11] of the ambient operator at the boundary,

$$\mathcal{O}_4(\vec{x}, z) = \sum_n \frac{C_{\Delta_4\Delta_n}}{z^{\Delta_4-\Delta_n}} \bar{\mathcal{O}}_{\Delta_n}(\vec{x}) \tag{2.39}$$

with the bar on the summed operators to make explicit the fact that these operators live at the boundary $z = 0$. The sum runs over all such possible operators and we write their conformal dimension as Δ_n. Using this boundary OPE, the two-point function is found from the contraction with an operator $\mathcal{O}_3(\vec{x})$

$$\langle \mathcal{O}_4(\vec{x}, z)\mathcal{O}_3(\vec{x}')\rangle = \frac{C_{\Delta_4\Delta_3}}{z^{\Delta_4-\Delta_3} d^{2\Delta_3}} \tag{2.40}$$

Here d^2 is the square of the distance between the position of the inserted operators $d^2 =\mid \vec{x} - \vec{x}' \mid^2 + z^2$

2.5 Minkowski vs black hole embedding

In the AdS-Schwarzschild background, there are two types of possible embeddings for the branes. The first one is the Minkowski embedding and corresponds to branes that end outside of the horizon. Such a scenario is possible if the D-brane worldvolume on the S^5 collapses to zero before reaching the black hole. When electric gauge fields are considered in a Minkowski embedding solution, at the point where the branes pinch-off there is nowhere for the field lines to go. However, we can include point charges in the form of strings stretching from the probe branes to the horizon which palliate

the problem. In the probe D5-brane system discussed above Minkowski embeddings arise when the filling fraction is set to zero. Then, the Routhian 2.24 vanishes when the angle $\psi(r)$ dynamically goes to zero. The probe D7-brane system also can have a Minkowski embedding but only when one of the sphere has no magnetic flux. When f_7 is zero, the same argument applies and the brane worldvolume vanishes at $\psi(r) = 0$ for filling fraction 1 in this case. This solution corresponds to an integer Quantum Hall state where the conductivity recreates the famous plateaus of graphene.

The second type of embedding is the Black Hole embedding which occurs when the worldvolume does not pinch-off and the branes reach the black hole. This is achieved in the D5 and in the D7 picture when the filling fraction is not an integer. Then, there is a residual charge preventing the energy to vanish and the brane has to reach the black hole horizon.

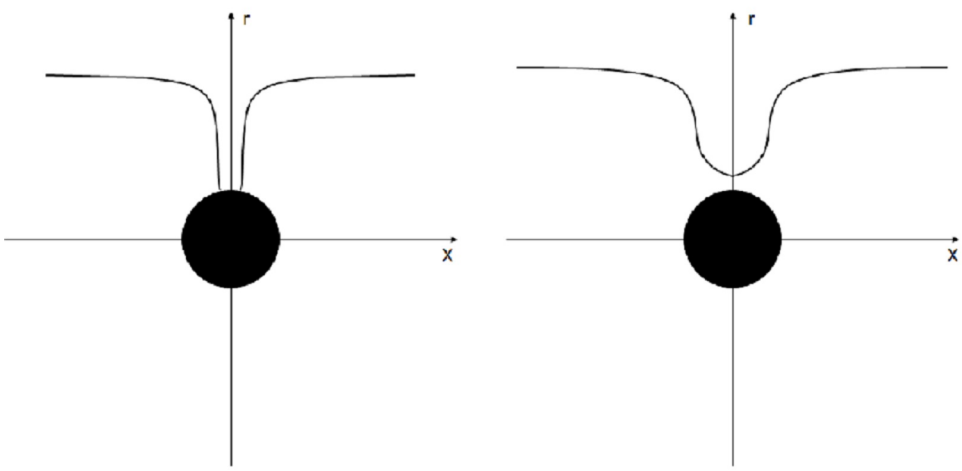

Figure 2.2: Two different types of embeddings. The circle in the middle represents a black hole with its horizon at $r = r_h$. The X-direction represents coordinates perpendicular to the D3-branes but parallel to the probe brane. (a) A black hole embedding. The probe brane extends in the radial direction until it reaches the horizon of the black hole (b) A Minkowski embedding. The probe brane pinches off at $r_0 > r_h$ and does not reach the horizon.

Chapter 3

Non-Abelian Probe Brane Action

The purpose of this chapter is to consider the full non-abelian action for D3 and D5 probe branes in $AdS_5 \times S^5$. In the previous chapter, we computed a simplified embedding of probe branes where the interactions between the branes were neglected, hence, the symmetries of the stack of probe branes was given by $U(1)^{N_f}$. But the full picture is somewhat different. It can be realized by bringing together a number of parallel non-coincident branes. An open string with both ends on the same brane describes a $U(1)$ massless gauge field which accounts for the $U(1)^{N_f}$ symmetry discussed previously. On the other hand, an open string with both ends on different branes is stretched and thus massive. As we bring the branes closer to one another, the stretched strings become massless and the gauge symmetry is enhanced to $U(N_f)$.

The first non-commutative BIon solution was discovered in [12] and computed the action of N_0 flavour D0-branes in a colour D2-branes background. It was shown that by letting three transverse directions to be excited the D0-branes could grow a two-dimensional transverse fuzzy sphere. This configuration, when the number of flavours is infinite, is dual to a single flavour D2-brane wrapping an S^2 with N_0 units of Dirac flux. The two configurations are dual in the sense that their energies are the same as well as their equilibrium radii. Once one examines the D0-branes configuration at finite values of N_0, $\frac{1}{N_0^2}$ corrections arise when compared to the D2-brane treatment. The non-commutative solution describes well the core of the BIon while the D2 configuration depicts the spike of the BIon precisely.

In the following sections, we wish to apply a similar treatment to different embeddings. This will allow us to improve our knowledge of some commutative setups of probe-branes by providing $\frac{1}{N_f^2}$ corrections for these embeddings. First, we look at non-commutative D5-branes growing a transverse fuzzy two-sphere. This should be dual to the D7-probe configuration mentioned in section 2.2.3 when only a flux on the sphere transverse to the

D5 is considered ($f_7 = 0$). Secondly, we look at non-commutative flavour D3-branes blowing up on a D5-brane or a D7-brane by exciting whether one or two transverse two-spheres. We dismiss the solution of D3-branes growing a fuzzy S^4 that would be dual to the S^4 embedding of a D7-brane because of the tachyon instability mentioned earlier.

3.1 D5-D7

Here, we enhance the solution of Section 2.2.2 to the full non-abelian solution where the stack of D5-branes will grow a transverse BIon spike in the form of a fuzzy sphere. An infinite number of terms will appear in the action and it will be necessary to make an expansion where the number of probe D5-branes is large. We will truncate the action to second order in $\frac{1}{N_5^2}$. This will allow us to find corrections to the D7-brane solution which we know should be the large N_5 limit of the non-abelian solution.

Non-abelian BI action

Let us start by remembering the non-abelian extension of the Born-Infeld action (NBI):

$$S_{NDBI} = -T_p \int d^{p+1}\sigma \mathbf{STr}\left(e^{-\Phi}\sqrt{-\det(P[E_{ab} + E_{ai}(Q^{-1}-\delta)^{ij}E_{jb}] + 2\pi\alpha'\mathbf{F}_{ab})\det(Q^i_j)}\right) \tag{3.1}$$

There are four transverse directions to the probe D5-branes and thus four coordinates become fuzzy and are included in the matrix $Q^i_j = \delta^i_j + \frac{i}{2\pi\alpha'}[X^i, X^k]E_{kj}$. The solution we are looking for is a stack of D5-branes growing a transverse two-sphere inside the S^5 geometry. To achieve this, we will make the ansatz where the coordinate z is proportional to the identity matrix and does not possess a non-abelian structure. To exhibit the fuzzy two-sphere geometry we go to Euclidean coordinates on the transverse $\bar{S}^2$

$$X_1 = \cos\psi \sin\tilde{\theta}\cos\tilde{\phi} \tag{3.2a}$$

$$X_2 = \cos\psi \sin\tilde{\theta}\sin\tilde{\phi} \tag{3.2b}$$

$$X_3 = \cos\psi \cos\tilde{\theta} \tag{3.2c}$$

This provides us with a better understanding of the fuzzy coordinates. The scalars are now seen as fuzzy distances interpreted as the transverse excitations of the brane. As mentioned above, the scalars are enhanced to matrices

in the $N_5 = 2j + 1$ irreducible representation[1] of $SU(2)$.

$$X_1 = \frac{2\cos\psi}{\sqrt{N_5^2 - 1}} L_x \tag{3.3a}$$

$$X_2 = \frac{2\cos\psi}{\sqrt{N_5^2 - 1}} L_y \tag{3.3b}$$

$$X_3 = \frac{2\cos\psi}{\sqrt{N_5^2 - 1}} L_z \tag{3.3c}$$

$$\cos^2\psi = X_1^2 + X_2^2 + X_3^2 \tag{3.3d}$$

Here, the transverse directions have been normalized following the development of 1.4 to span a fuzzy sphere of radius $\cos\psi$. To preserve the $SO(3)$ invariance on $\bar{S}^2$, we consider for ψ only functions of the radius coordinate, $\psi = \psi(r)$. In that coordinate system, following 1.30 the non-abelian Taylor expansion of the metric in the transverse directions becomes the unit matrix,

$$G_{ij} = \sqrt{\lambda}\alpha' \delta_{ij} \tag{3.4}$$

while the matrix Q_{ij} involving the commutators of Φ^i takes the form

$$Q_{ij} = \mathbb{I}_{ij} - \frac{4\sqrt{\lambda}\cos^2\psi}{2\pi(N_5^2 - 1)} \epsilon_{ijk} L^k \tag{3.5}$$

This is a $3N_5 \times 3N_5$ matrix and the determinant is taken over the indices on the sphere $i = 1, 2, 3$. Taking the determinant of this matrix results in an $N_5 \times N_5$ matrix which will later be traced on its $U(N_5)$ indices. To find this determinant we use the identity

$$\det(1 + A) = \exp\{\mathbf{Tr}_{3\times 3}(\log(1 + A)\} \tag{3.6}$$

This computation is detailed in the Appendix. The result is given as,

$$\det Q^i_j = \left(1 + \frac{16\lambda\cos^4\psi}{4\pi^2(N_5^2 - 1)^2} - \frac{i16\lambda^{3/2}\cos^6\psi}{3(2\pi)^3(N_5^2 - 1)^2}\right) \mathbb{I}_{N\times N} \tag{3.7}$$

where the last term on the right-hand side is imaginary. This might seem bothering at a first glance but this term comes from a product of three L^i matrices and all odd powers of L matrices vanish under the symmetrized

[1] A reducible representation could be decomposed as a sum of irreducible representations and would correspond to a solution with parallel D7-branes where the number of D7-branes would be one-to-one with the number of irreps in the decomposition

trace prescription. Upon tracing the $U(N_5)$ indices, the last term of 3.7 will drop off and only the first two terms will contribute to the action.

We also need the inverse matrix Q^{-1}

$$Q^{-1} - \delta = \frac{4\sqrt{\lambda}\cos^2\psi}{2\pi(N_5^2-1)}\epsilon_{ijk}L^k + \frac{16\lambda\cos^4\psi}{(2\pi)^2(N_5^2-1)^2}\epsilon_{ilk}\epsilon_{ljm}L^kL^m + ... \tag{3.8}$$

from which we can evaluate the second term in the pullback of equation 3.1. It appears that once again the odd powers of L matrices vanish under the symmetrized trace prescription. The computation is detailed in the appendix and up to second order in $\frac{1}{N_5}$, we get

$$P[G_{ai}(Q^{-1}-\delta)^{ij}G_{jb}] = -\frac{16\lambda\sin^2\psi\cos^4\psi\,\dot{\psi}^2}{(2\pi)^2(N_5^2-1)^2} \tag{3.9}$$

We can finally write the term in the first determinant of 3.1,

$$P[G_{ab} + G_{ai}(Q^{-1}-\delta)^{ij}G_{jb}] + 2\pi\alpha'\mathbf{F}_{ab} =$$
$$\sqrt{\lambda}\alpha'\begin{pmatrix} -hr^2 & 0 & 0 & -\dot{a} & 0 & 0 \\ 0 & r^2 & b & 0 & 0 & 0 \\ 0 & -b & r^2 & 0 & 0 & 0 \\ \dot{a} & 0 & 0 & \frac{1}{hr^2}+\dot{\psi}^2\left(1-\frac{16\lambda\sin^2\psi\cos^4\psi}{(2\pi)^2(N_5^2-1)^2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin^2\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin^2\psi\sin^2\theta \end{pmatrix} \tag{3.10}$$

Under the symmetrized trace prescription, the Lagrangian is proportional to the unit matrix and the full equation is multiplied by $N_5 = 2j+1$. We are now in possession of the complete non-abelian Born-Infeld action including corrections up to $\frac{1}{N_5^2}$.

$$S_{NBI5} = -2\mathcal{N}_5N_5\int dr\sin^2\psi\sqrt{\left(1+\frac{4\lambda\cos^4\psi}{4\pi^2(N_5^2-1)}\right)(r^2+b^2)}$$
$$\sqrt{\left(1+hr^2\dot{\psi}^2\left(1-\frac{16\lambda\sin^2\psi\cos^4\psi}{(2\pi)^2(N_5^2-1)^2}\right)-\dot{a}^2\right)} \tag{3.11}$$

We wish to remove the λ dependence and rewrite the action in terms of the

parameter $f_5 = \frac{2\pi N_5}{\sqrt{\lambda}}$.

$$S_{NBI5} = -2\frac{\mathcal{N}_5 N_5}{f_5}\int dr \sin^2\psi \sqrt{\left(f_5^2 + 4\cos^4\psi + \frac{4\cos^4\psi}{N_5^2-1}\right)(r^2+b^2)} \sqrt{\left(1 + hr^2\dot{\psi}^2\left(1 - \frac{16\sin^2\psi\cos^4}{(N_5^2-1)f_5^2}\right) - \dot{a}^2\right)} \tag{3.12}$$

By comparing this result to the D7-brane picture 2.28 we notice that both actions agree at order $\frac{1}{N_5}$ when we do not consider a flux on the first sphere S^2, corresponding to $f_7 = 0$. The non-abelian treatment provides a $\frac{1}{N_5^2}$ correction to the BI action as expected.

Non-abelian CS action

For the NCS term, the relevant RR potential is once again the $C^{(4)}$ (2.5) generated by the background D3-branes. In the abelian treatment, we found no Chern-Simons term as the only possible term $P[C^{(4)}] \wedge F$ could not be pulled-back to the worldvolume indices of the D5-branes. The non-abelian description is more general and allows new interactions through the internal product introduced in 1.3.2 by stripping off pairs of indices of the RR potential. For the D5-branes in a $C^{(4)}$ background, there are generally three terms that can arise from this procedure,

$$S_{NCS5} = (2\pi\alpha')\frac{T_p}{g_s}\int d^{p+1}\sigma \mathbf{STr}(P\left[-\frac{1}{2}\imath_x^4 C^{(4)}\right] \wedge F \wedge F \wedge F + iP\left[\imath_x^2 C^{(4)}\right] \wedge F \wedge F + P\left[C^{(4)}\right] \wedge F) \tag{3.13}$$

From our ansatz, $F^3 = 0$ as we did not put any gauge field on the wrapped sphere. Also, as the integral is taken on the D5-branes worldvolume, the last term of 3.13 vanishes as for the abelian CS action. The only resulting contribution comes from the second term of 3.13 by stripping off the two transverse components from the RR potential via the internal product.
To expose the $SO(3)$ symmetry, we once again go to Cartesian coordinates on $\bar{S}^2$ where the S^5 part of the Ramond-Ramond potential now takes the form

$$C^{(4)}_{ij\theta\phi} = \lambda\alpha'^2 \frac{c(\psi)}{2\cos^3\psi}\frac{\epsilon_{ijk}}{2}x^k dx^i \wedge dx^j \wedge d\Omega_2 \tag{3.14}$$

As the RR potential is a function of the transverse fields, we follow the argument discussed 1.30 and make a non-abelian Taylor expansion to compute its interior product. In this particular case, the expansion is trivial as the potential depends linearly on the fields:

$$\begin{aligned} \imath_x \imath_x C^{(4)}_{ij\theta\phi} &= x^j x^i x^k \partial_k C^{(4)}_{ij\theta\phi}(r) \\ &= \lambda \alpha'^2 \frac{ic(\psi)}{\sqrt{N_5^2 - 1}} d\Omega_2 \end{aligned} \tag{3.15}$$

The gauge trace simply gives a factor of N_5 as the interior product of the RR potential is proportional to the identity. The full Chern-Simons action for the configuration we are studying is:

$$S_{NCS5} = -\frac{\mathcal{N}_5 N_5}{f_5} \frac{2b}{\sqrt{1 - \frac{1}{N_5^2}}} \int dr\ \dot{a}\ c(\psi) \tag{3.16}$$

It should be noted that in a large j expansion of the Chern-Simons action the prefactor depending on j is $\frac{2}{\sqrt{1-\frac{1}{N_5^2}}} \sim 2 + \frac{1}{N_5^2}$. When compared to the D7-brane model, the NCS action receives no corrections at first order in a $\frac{1}{N_5}$ expansion.

Routhian and equations of motion

Now that we are in possession of a non-abelian action for the D5-branes configuration we wish to obtain the equations of motion for the dynamical variable ψ. To do so, we once again find the Routhian for this action through a Legendre transformation to remove the cyclic variable a

$$\begin{aligned} \mathcal{R}_{N5} = -\frac{\mathcal{N}_5 N_5}{f_5} \left(\frac{2\pi B}{\sqrt{\lambda}}\right)^{3/2} \int dr \sqrt{1 + hr^2 \dot{\psi}^2 \left(1 - \frac{16 \sin^2\psi \cos^4\psi}{(N_5^2 - 1) f_5^2}\right)} \\ \sqrt{4 \sin^2\psi (1 + r^4)(f_5^2 + 4\cos^4\psi + \frac{4\cos^4\psi}{(N_5^2 - 1)}) + [\pi\nu + \frac{2}{\sqrt{1 - \frac{1}{N_5^2}}} c(\psi)]^2} \end{aligned} \tag{3.17}$$

Applying the Euler-Lagrange equations on the Routhian gives the equations of motion for ψ

$$
\begin{aligned}
0 =& \frac{hA(\psi)\left(r\frac{d}{dr}\right)^2\psi}{1+hr^2A(\psi)\dot{\psi}^2} + \frac{hr^2\frac{dA(\psi)}{d\psi}\dot{\psi}^2}{2(1+hr^2A(\psi)\dot{\psi}^2} \\
&+ hrA(\psi)\dot{\psi}\left(1+\frac{8\sin^4\psi r^4\left(f_5^2+4\cos^4\psi+\frac{4\cos^4\psi}{(N_5^2-1)}\right)}{4\sin^2\psi(1+r^4)(f_5^2+4\cos^4\psi+\frac{4\cos^4\psi}{(N_5^2-1)})+[\pi\nu+\frac{2}{\sqrt{1-\frac{1}{N_5^2}}}c(\psi)]^2}\right) \\
&- \frac{8\sin^3\psi\cos\psi f_5^2+4\sin^3 2\psi\cos 2\psi(1+r^4)+\frac{4}{\sqrt{1-\frac{1}{N_5^2}}}\sin^2 2\psi[\pi\nu+\frac{2}{\sqrt{1-\frac{1}{N_5^2}}}c(\psi)]}{4\sin^2\psi(1+r^4)(f_5^2+4\cos^4\psi+\frac{4\cos^4\psi}{(N_5^2-1)})+[\pi\nu+\frac{2}{\sqrt{1-\frac{1}{N_5^2}}}c(\psi)]^2}
\end{aligned}
\tag{3.18}
$$

Where we have defined,

$$
\begin{aligned}
A(\psi) &= 1-\frac{16\sin^2\psi\cos^4\psi}{(N_5^2-1))f_5^2} \\
\frac{dA(\psi)}{d\psi} &= \frac{32\sin\psi\cos\psi^3(3\sin^2-1)}{(N_5^2-1)f_5^2}
\end{aligned}
\tag{3.19}
$$

It is also useful to compute the Hamiltonian for this system as this will give us a tool to compare the energy of two different solutions. This can be obtained via a Legendre transformation:

$$
\mathcal{E}_{N5} = \frac{\mathcal{N}_5 N_5}{f_5}\left(\frac{2\pi B}{\sqrt{\lambda}}\right)^{3/2}\frac{\sqrt{4\sin^2\psi(1+r^4)(f_5^2+4\cos^4\psi+\frac{4\cos^4\psi}{(N_5^2-1)})+[\pi\nu+\frac{2N_5}{\sqrt{(N_5^2-1)}}c(\psi)]^2}}{\sqrt{1+hr^2\left(1-\frac{16\sin^2\psi\cos^4\psi}{(N_5^2-1)f_5^2}\right)\dot{\psi}^2}}
\tag{3.20}
$$

Let us note that the action and Routhian we found from the non-abelian D5 picture is in agreement with the D7 equations found in section 2.2.3 where only $\bar{f}_7$ is turned on and $f_7 = 0$ as we did not include a flux on the S^2 wrapped by the D5-branes. Turning on the flux on the sphere has very important physical differences with the present case. When a flux is turned on, the energy of the probes does not vanish anymore for the angle $\psi = 0$ which prevents the sphere from pinching off inside the bulk and prohibits Minkowski embeddings. This type of solutions can instead be portrayed by

D3-probe branes blowing up to a higher dimensional (D5 or D7) brane with flux. We will investigate the action for this phenomenon in the following section.

3.2 D3-D5

We now investigate the non-abelian action for a stack of coincident D3 branes blowing up to a D5-brane on a fuzzy two-sphere. In the dual picture, the D5-brane contains N_3 units of Dirac monopole flux on the two-sphere. The idea is similar to the D5-D7 system and we shall quickly go over the details to avoid redundancy.

3.2.1 D3

The NBI action is computed in a similar way. In this case, the transverse two-sphere which becomes fuzzy is taken to be the one with radius $\sin\psi$ as it is the one wrapped by the D5-brane. Following the same derivation, we find,

$$S_{NBI3} = -\mathcal{N}_3 N_3 \int dr \sqrt{(r^4+b^2)\left(1+\frac{4\lambda\sin^4\psi}{4\pi^2(N_3^2-1)}\right)}\sqrt{1+hr^2\dot{\psi}^2\left(1-\frac{16\lambda\sin^4\psi\cos^2\psi}{(2\pi)^2(N_3^2-1)^2}\right)-\dot{a}^2} \tag{3.21}$$

while the Chern-Simons action vanishes. Indeed, the RR potential $C^{(4)}_{\theta\phi\bar{\theta}\bar{\phi}}$ commutes on its two indices $\bar{\theta}$,$\bar{\phi}$ and one cannot strip off all of its indices with the internal product.

The full action is given by the NBI term only and following the usual procedure we find the Routhian and equations of motion for ψ.

$$\mathcal{R}_3 = -\mathcal{N}_3 N_3 \left(\frac{2\pi B}{\sqrt{\lambda}}\right)^{3/2} \int dr \sqrt{1+hr^2\dot{\psi}^2\left(1-\frac{16\sin^4\psi\cos^2\psi}{(N_3^2-1)f^2}\right)} \sqrt{(1+r^4)\left(1+\frac{4\lambda\sin^4\psi}{4\pi^2(N_3^2-1)}\right)+\left(\frac{q_3}{b}\right)^2} \tag{3.22}$$

where as usual, q_3 is the conserved quantity related to the cyclic variable $\dot{a}$,

$$q_3 = \frac{1}{\mathcal{N}_3 N_3}\frac{\partial S_3}{\partial \dot{a}} = \frac{\dot{a}\sqrt{(1+r^4)\left(1+\frac{4\lambda\sin^4\psi}{4\pi^2(N_3^2-1)}\right)}}{\sqrt{1+hr^2\dot{\psi}^2\left(1-\frac{16\lambda\sin^4\psi\cos^2\psi}{(2\pi)^2(N_3^2-1)^2}\right)-\dot{a}^2}} \tag{3.23}$$

3.2.2 D5 with flux

We now look at the commuting D5-brane system dual to the non-commutative D3-brane configuration above. Adding a Dirac monopole flux on the wrapped sphere, the worldvolume gauge field strength on the sphere is written as,

$$2\pi\alpha' F_{\theta\phi} = \frac{f}{2}\sin\theta d\theta \wedge d\phi \tag{3.24}$$

where the factor of 2 is for later convenience. The action is fully determined by the BI part,

$$\mathcal{R}_5 = -\mathcal{N}_5 \left(\frac{2\pi B}{\sqrt{\lambda}}\right)^{3/2} \int dr \sqrt{\left(f^2 + 4\sin^4\psi\right)(1+r^4) + \left(\frac{q_5}{b}\right)^2}\sqrt{1 + hr^2\dot{\psi}^2} \tag{3.25}$$

To make the contact between the two systems, we notice that

$$\frac{\mathcal{N}_3 N_3}{N_5} = \frac{2\pi N_3}{\sqrt{\lambda}} \tag{3.26}$$

and that the values of the charges can be rewritten in terms of the filling fraction,

$$\frac{q_3}{b} = \frac{\pi\nu}{f_3}, \quad f_3 = \frac{4\pi^2 N_3}{\lambda} \tag{3.27a}$$

$$\frac{q_5}{b} = \frac{\pi\nu}{f_5}, \quad f_5 = \frac{2\pi}{\sqrt{\lambda}} \tag{3.27b}$$

and both actions can be written is a similar way,

$$\mathcal{R}_3 = -\frac{\mathcal{N}_3 N_3}{f_3}\left(\frac{2\pi B}{\sqrt{\lambda}}\right)^{3/2} \int dr \sqrt{1 + hr^2\dot{\psi}^2\left(1 - \frac{16\sin^4\psi\cos^2\psi}{(N_3^2-1)f^2}\right)} \sqrt{\frac{4\pi^2}{\lambda}(1+r^4)\left(\frac{\lambda f_3^2}{4\pi^2} + 4\sin^4\psi + \frac{4\sin^4\psi}{(N_3^2-1)}\right) + (\pi\nu)^2} \tag{3.28}$$

$$\mathcal{R}_5 = -\frac{\mathcal{N}_3 N_3}{f_3}\left(\frac{2\pi B}{\sqrt{\lambda}}\right)^{3/2} \int dr \sqrt{\frac{4\pi^2}{\lambda}\left(f^2 + 4\sin^4\psi\right)(1+r^4) + (\pi\nu)^2}\sqrt{1 + hr^2\dot{\psi}^2} \tag{3.29}$$

Our results agree up to order $\frac{1}{N_3^2}$ as long as the flux on the sphere is normalized appropriately,

$$f = \frac{\sqrt{\lambda}}{2\pi} f_3 \tag{3.30}$$

3.3 D3-D7

This time we will look at a system where instead of growing a transverse fuzzy sphere, the D3-branes are excited in four transverse directions and the Myers effect happens on both fuzzy two-spheres at once, one of radius $\sin\psi$ and the other of radius $\cos\psi$. In the general case, the number of branes blowing up on each transverse two-sphere can be different. This will be accounted for by introducing a quantity N_3 and $\bar{N}_3$ which respectively correspond to the number of D3-branes blowing up on the fuzzy S^2 and $\bar{S}^2$. In the dual D7 picture, taking $N_3 \neq \bar{N}_3$ corresponds to having two different values of flux on each sphere which is translated as $f_7 \neq \bar{f}_7$

3.3.1 D3

The non-abelian Born-Infeld action is only modified from the previous systems through the Q^i_j matrix. Once both spheres are written in Cartesian coordinates,

$$\begin{aligned} X_1 &= \sin\psi\sin\theta\cos\phi \quad \bar{X}_4 = \cos\psi\sin\tilde{\theta}\cos\tilde{\phi} \\ X_2 &= \sin\psi\sin\theta\sin\phi \quad \bar{X}_5 = \cos\psi\sin\tilde{\theta}\sin\tilde{\phi} \\ X_3 &= \sin\psi\cos\theta \qquad\quad \bar{X}_6 = \cos\psi\cos\tilde{\theta} \end{aligned} \tag{3.31}$$

Q^i_j becomes a $(3N_3 + 3\bar{N}_3) \times (3N_3 + 3\bar{N}_3)$ block-diagonal matrix as the X_i commute with the $\bar{X}_i$ while each group has non-trivial commutation relation within each themselves.

This time, the symmetrized trace picks up two factors, N_3 and $\bar{N}_3$ accounting for all possible orderings of the matrices X and $\bar{X}$. The non-abelian BI action then take the form,

$$\begin{aligned} S_{NBI3} = -\mathcal{N}_3 N_3 \bar{N}_3 \int dr \sqrt{(r^4+b^2)\left(1+\frac{4\lambda\sin^4\psi}{4\pi^2(N_3^2-1)}\right)\left(1+\frac{4\lambda\cos^4\psi}{4\pi^2(\bar{N}_3^2-1)}\right)} \\ \sqrt{1+hr^2\dot{\psi}^2\left(1-\frac{16\lambda\sin^4\psi\cos^2\psi}{4\pi^2(N_3^2-1)^2}-\frac{16\lambda\sin^2\psi\cos^4\psi}{4\pi^2(\bar{N}_3^2-1)^2}\right)-\dot{a}^2} \end{aligned} \tag{3.32}$$

In this system, there is also a Chern-Simons contribution to the action which arises by taking off all indices of $C^{(4)}$ via the internal product.

$$S_{NCS3} = -\frac{T_3}{g_s}\int d^4\sigma\ \mathbf{Tr}\ (2\pi\alpha')^2 P[\imath^4 C^{(4)}] \wedge F \wedge F \tag{3.33}$$

Here, the internal product is easily taken as there are two distinct contractions,

$$\imath^4 C^{(4)} = [\Phi^i, \Phi^j][\bar{\Phi}^{\bar{k}}, \bar{\Phi}^{\bar{l}}] C^{(4)}_{ij\bar{k}\bar{l}} \tag{3.34}$$

finally, the CS action is written as follows

$$S_{NCS3} = -\frac{\mathcal{N}_3 N_3 \bar{N}_3}{f_3} \frac{2b}{2\sqrt{(1-\frac{1}{N_3^2})(1-\frac{1}{\bar{N}_3^2})}} \int dr\ \dot{a}\ c(\psi) \tag{3.35}$$

with f_3 picking up the new term $\bar{N}_3$,

$$f_3 = \frac{4\pi^2 N_3 \bar{N}_3}{\lambda} \tag{3.36}$$

Following the same procedure as usual, we find the Routhian,

$$\mathcal{R}_3 = -\frac{\mathcal{N}_3 N_3 \bar{N}_3}{f_{3'}} \left(\frac{2\pi B}{\sqrt{\lambda}}\right)^{3/2} \int dr \sqrt{1 + hr^2\dot{\psi}^2 \left(1 - \frac{16\lambda \sin^4\psi \cos^2\psi}{4\pi^2(N_3^2-1)^2} - \frac{16\lambda \sin^2\psi \cos^4\psi}{4\pi^2(\bar{N}_3^2-1)^2}\right)}$$
$$\sqrt{(1+r^4)\left(\frac{\lambda f_3^2}{4\pi^2 \bar{N}_3^2} + 4\left(1+\frac{1}{N_3^2-1}\right)\sin^4\psi\right)\left(\frac{\lambda f_3^2}{4\pi^2 N_3^2} + 4\left(1+\frac{1}{\bar{N}_3^2-1}\right)\cos^4\psi\right) + \xi_{3\bar{3}}^2} \tag{3.37}$$

with

$$\xi_{3\bar{3}} = \pi\nu + \frac{2}{\sqrt{(1-\frac{1}{N_3^2})(1-\frac{1}{\bar{N}_3^2})}} c(\psi) \tag{3.38}$$

The Routhian agrees up to $\frac{1}{N_3^2}$ or $\frac{1}{\bar{N}_3^2}$ corrections with the D7 probe brane action introduced in section 2.2.3 as long as the flux on the spheres in the D7 picture satisfies

$$f_7^2 = \frac{\lambda f_3^2}{4\pi^2 \bar{N}_3^2}, \quad \bar{f}_7^2 = \frac{\lambda f_3^2}{4\pi^2 N_3^2} \tag{3.39}$$

We notice that in the special case $f_7 = \bar{f}_7$, we get

$$f_3 = \frac{\lambda f_3^2}{4\pi^2 \bar{N}_3^2} = \frac{\lambda f_3^2}{4\pi^2 N_3^2} \tag{3.40a}$$

$$f_3 = f_7^2 = \bar{f}_7^2 \tag{3.40b}$$

Let us finish this section by noting that the non-abelian action for our configuration of D3-branes is not trivial as was its commuting counterpart. While the abelian action 2.2.1 had no dynamical degrees of freedom, the non-abelian action possesses interesting dynamics for the variable $\psi(r)$. The non-commutative solutions we analysed correspond to a single D5-brane or to a D7-brane depending if we allowed one or two two-spheres to be excited.

Chapter 4

Conductivities

Having found an action and equations of motion for various non-abelian probe branes systems, we are now able to compute interesting quantities for these configurations. In this section, we find the longitudinal and transverse conductivity for the non-abelian D3-D5 system which will be the corrected values of the D3-D7 system for a non-infinite number of flavors.

4.1 Minkowski embeddings

We advertised the D3-D5 configuration possesses Minkowski embeddings which are Quantum Hall states for particular values of the filling fraction ν. Such a claim needs to be supported by a computation of longitudinal and transverse conductivity as a QH state is one of charged matter in 2+1 dimensions with null longitudinal conductivity and quantized transverse conductivity. The correct procedure was elaborated for the D3-D7 scenario in [1] and we will use the same method here to find the conductivities.

Our goal is to find the associated conserved currents to the gauge fields

$$\langle J^i \rangle = \sigma_{ij} E^j \tag{4.1}$$

in the presence of a constant background electric field. We thus need add some extra gauge fields which will yield the needed conserved currents,

$$A_x(r,t) = \frac{\sqrt{\lambda}}{2\pi}\left(et + a_x(r)\right) \ , \ \ A_y(r,x) = \frac{\sqrt{\lambda}}{2\pi}\left(bx + a_y(r)\right) \tag{4.2}$$

These additional gauge fields introduce modifications to the action that are readily computed. For the D3-D5 system, these are,

$$S_{BI5} = -\frac{\mathcal{N}_5 N_5}{f_5}\int dr \sqrt{4\sin^4\psi\left(f_5^2 + 4\cos^4\psi + \frac{4\cos^4\psi}{N_5^2-1}\right)} \\ \sqrt{\left(r^4 + b^2 - \frac{e^2}{h}\right)\left(1 + hr^2\dot{\psi}^2\right) + \left(hr^4 - e^2\right)\dot{a}_y^2 + hr^4\dot{a}_x^2 - \left(r^4+b^2\right)\dot{a}_0^2 - 2be\dot{a}_0\dot{a}_y} \tag{4.3}$$

$$S_{CS5} = -\frac{\mathcal{N}_5 N_5}{f_5} \frac{2}{\sqrt{1-\frac{1}{N_5^2}}} \int dr \left(b\dot{a}_0 + e\dot{a}_y\right) c(\psi) \tag{4.4}$$

from which we find the equations of motion for the gauge fields,

$$d = \gamma \left(\left(r^4 + b^2\right)\dot{a}_0 + be\dot{a}_y\right) - \frac{2}{\sqrt{1-\frac{1}{N_5^2}}} bc(\psi) \tag{4.5a}$$

$$j_x = -\gamma h r^4 \dot{a}_x \tag{4.5b}$$

$$j_y = -\gamma \left(\left(hr^4 - e^2\right)\dot{a}_y - be\dot{a}_0\right) - \frac{2}{\sqrt{1-\frac{1}{N_5^2}}} bc(\psi) \tag{4.5c}$$

Where γ is defined for convenience as,

$$\gamma = \sqrt{\frac{4\sin^4\psi\left(f_5^2 + 4\cos^4\psi + \frac{4\cos^4\psi}{N_5^2-1}\right)}{\left(r^4+b^2-\frac{e^2}{h}\right)\left(1+hr^2\dot{\psi}^2\right) + \left(hr^4-e^2\right)\dot{a}_y^2 + hr^4\dot{a}_x^2 - \left(r^4+b^2\right)\dot{a}_0^2 - 2be\dot{a}_0\dot{a}_y}} \tag{4.6}$$

The equations for $\dot{a}_0$ and $\dot{a}_y$ are coupled but can be solved for each variable independently,

$$\dot{a}_0 = \frac{\tilde{d}\left(hr^4 - e^2\right) + be\tilde{j}_y}{\gamma r^4\left[h\left(r^4+b^2\right) - e^2\right]} \tag{4.7a}$$

$$\dot{a}_y = \frac{be\tilde{d} - \left(r^4+b^2\right)\tilde{j}_y}{\gamma r^4\left[h\left(r^4+b^2\right) - e^2\right]} \tag{4.7b}$$

where a tilde means that we added the term $\frac{2}{\sqrt{1-\frac{1}{N_5^2}}} bc(\psi)$. For example, $\tilde{d} = \gamma\left(\left(r^4+b^2\right)\dot{a}_0 + be\dot{a}_y\right)$. We can now put these back in our equation for γ and get,

$$\gamma = \frac{\sqrt{h\left(1+\frac{b^2}{r^4}-\frac{e^2}{hr^4}\right)\left(hr^4\Gamma + h\tilde{d}^2 - j_x^2 - \tilde{j}_y^2\right) - \left(hb\tilde{d} - e\tilde{j}_y\right)^2}}{\left(1+\frac{b^2}{r^4}-\frac{e^2}{hr^4}\right)\sqrt{1+hr^2\dot{\psi}^2}} \tag{4.8}$$

where the factor of Γ was defined to be

$$\Gamma = 4\sin^4\psi\left(f_5^2 + 4\cos^4\psi + \frac{4\cos^4\psi}{N_5^2-1}\right) \tag{4.9}$$

For the solution to be regular, when the branes pinch off the currents need to vanish: $\tilde{d}(r_0) = \tilde{j}_y(r_0) = j_x = 0$. From this, the longitudinal current vanishes directly,

$$\sigma_{xx} = \frac{J_x}{E_x} = 0 \tag{4.10}$$

while the transverse conductivity yields,

$$\sigma_{xy} = \frac{J_y}{E_x} = \frac{\nu}{2\pi} \tag{4.11}$$

Let us also remember that these Minkowski embeddings are found when the filling fraction ν is $\nu = \frac{1}{\sqrt{1-\frac{1}{N_5^2}}} \approx 1 + \frac{1}{2N_5^2}$. In this case, the conductivity is quantized in units of $\frac{1}{\sqrt{1-\frac{1}{N_5^2}}}$ instead of being integer. This is indeed a very surprising result as it postulates that for any finite number of probe D5-branes, the plateaus for the conductivity at strong coupling are slightly separated from their weak coupling counterpart which are found for each integer values of ν.

4.2 BH embeddings

When in the presence of a Black Hole Embedding, the probe branes fall inside the horizon and there is no gap between them and the background D3-branes. This represents a metallic state and we should expect that both longitudinal and transverse conductivity be non-zero.

Asking that the action should be real, the radicand in the numerator of (4.8) needs to be positive. At the specific value $r = r_*$,

$$h(r_*)\left(r_*^4 + b^2\right) = e^2 \tag{4.12}$$

the first term of the radicand vanishes. For the radicand to be positive, the last term needs to cancel out at that point. This corresponds to the relation,

$$\tilde{j}_y(r_*) = \frac{1}{e} h(r_*) b \tilde{d}(r_*) \tag{4.13}$$

To find a second relation is a bit trickier. An argument detailed in [13] tells us that the second piece of the first term of the radicand also vanishes at $r = r_*$. This can be seen from the fact that the second term is negative at the horizon and positive at the boundary. Therefore it possesses a zero. But if that zero does not coincide with the zero of the first term, then in the

region between the two zeros, the radicand is negative and the action does not satisfy the reality condition. This allows us to write our second relation,

$$j_x^2 + \tilde{j}_y(r_*)^2 = h(r_*)r_*^4\Gamma + h(r_*)\tilde{d}(r_*)^2 \tag{4.14}$$

These equations would be very hard to solve exactly but the conductivities are only given by the approximation of the linear response. This means that we can consider the electric field e as a small perturbation and expand the solution to first order. When $e = 0$, equation (4.12) tells us that $r_* = r_h$. Taking e to be non-zero, we are looking for a solution in the vicinity of the black hole radius of the type $r_* = r_h(1+x)$. Solving the quadratic equation for r_*^4 by neglecting any e^4 term, yields the solution,

$$r_*^4 = r_h^4\left(1 + \frac{e^2}{r_h^4 + b^2}\right) \tag{4.15}$$

Replacing this value of r_* in our equations for the conductivities (4.13), (4.14),

$$\sigma_{xy} = \frac{J_y}{E_x} = \frac{N_3}{2\pi^2}\left(\frac{b}{r_h^4 + b^2}\tilde{d}(r_h) + \frac{2}{\sqrt{1 - \frac{1}{N_5^2}}}bc(r_h)\right) \tag{4.16a}$$

$$\sigma_{xx} = \frac{J_x}{E_x} = \frac{N_3}{2\pi^2}\frac{r_h^2}{b^2 + r_h^4}\sqrt{\tilde{d}(r_h)^2 + (r_h^4 + b^2)\Gamma(r_h)} \tag{4.16b}$$

Both longitudinal and transverse conductivities are non-zero for the black-hole embedding as expected for a metallic state. Once again, the difference between the conductivities for the non-abelian D5-branes system only differs from the D7-brane system by a factor of $\frac{1}{\sqrt{1-\frac{1}{N_5^2}}}$ term.

Chapter 5

Conclusion

In this work, we evaluated the action for a stack of coincident probe flavor D5-branes embedded in $AdS_5 \times S5$ geometry. We chose an $S^2 \times S^2$ embedding on the S^5 and took into account the non-abelian $U(N_f)$ effects. We found a spherically symmetric solution to the system where the D5-branes grow a transverse fuzzy two-sphere spike. This solution, in the large number of flavors limit, is dual via Myers effect to a D7-brane formulation where a single D7-brane wraps the two-sphere transverse to the D5-branes with N_F units of RR flux on that sphere. The non-abelian D5-branes picture allows us to look for $\frac{1}{N_f}$ corrections to the D7 setup. We find that the initial probe D7-brane brane action does not receive any correction to order N_f^{-1}. The leading contribution to the difference between both setups is introduced at order N_f^{-2}.

We computed conductivities for the non-abelian solution and found that they deviate from the usual integer values by a factor of $\frac{N_f}{\sqrt{C}} = \frac{1}{\sqrt{1-\frac{1}{N_f^2}}}$ which depends strictly on the number of flavor branes probing the geometry. This result is the direct consequence of the change in the non-abelian Chern-Simons action. We wish to emphasize that this result is particularly surprising as CS actions are always quantized in integer units. We can point out that a similar result for the non-abelian CS action was obtained in [3] but from the point of view of D1-branes probing the geometry. One can argue that such a result is not physical. Indeed, in the weak coupling limit, the value of the transverse conductivity is integer and a no-renormalization theorem states that this results holds at any level in perturbation theory. Thus, the conductivity at strong coupling should also take integer values which disagrees with the non-abelian probe branes calculation.

It was suggested to us[2] that the change in the CS action might be cancelled by including fermions to the model. The Born-Infeld and Chern-Simons actions only take bosonic degrees of freedom into account and it would be possible that a fermionic one loop computation gives a factor that

[2] We are grateful to Rob Myers for pointing out this idea to us

would correct the CS action back to an integer value. We will leave the resolution of this question for a further work and remember that the results found here need to be taken with a grain of salt.

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Appendix A

A.1 Trace-log expansion

In this section we will compute the determinant of the matrix Q^i_j mentioned in 3.7.
Let us start with

$$Q^i_j = 1 - \frac{4\sqrt{\lambda}\cos^2\psi}{2\pi(N_5^2-1)}\epsilon_{ijk}L^k \tag{A.1}$$

Which we will write for later convenience as

$$Q = 1 + A \tag{A.2}$$

We use the identity:

$$\det(1+A) = \exp \mathbf{Tr}\log(1+A) \tag{A.3}$$

And replace the log by its Taylor expansion

$$\mathbf{Tr}\log(1+A) = \sum_{k=1}^{\infty}(-1)^k\frac{\mathbf{Tr}A^k}{k} \tag{A.4}$$

The problem is now simplified to finding the trace of powers of the matrix

$$A^i_j = -\frac{4\sqrt{\lambda}\cos^2\psi}{2\pi(N_5^2-1)}\epsilon_{ijk}L^k = \beta\epsilon_{ijk}L^k \tag{A.5}$$

Where we absorbed all the coefficients into the constant β. The matrix A is traceless so the first term will come from the square of A

$$\begin{aligned} A^2 &= \beta^2\epsilon_{ikl}\epsilon_{kjm}L^lL^m \\ &= \beta^2(-\delta_{ij}\delta_{lm} + \delta_{im}\delta_{lj})L^lL^m \\ &= \beta^2(-\delta_{ij}L^2 + L^jL^i) \end{aligned} \tag{A.6}$$

With trace

$$\mathbf{Tr}A^2 = -\frac{2\beta^2(N_5^2-1)}{4} = -\frac{4\lambda\cos^4\psi}{2\pi^2(N_5^2-1)} \tag{A.7}$$

The third power of A is computed in the same fashion

$$\begin{aligned}
A^3 &= \beta^3 \epsilon_{ikl}\epsilon_{kmn}\epsilon_{mjo} L^l L^n L^o \\
&= \beta^3(-\delta_{im}\delta_{ln} + \delta_{in}\delta_{lm})\epsilon_{mjo} L^l L^n L^o \\
&= \beta^3(-\epsilon_{ijo} L^2 L^o + \epsilon_{ljo} L^l L^i L^o) \\
&= \beta^3(-\epsilon_{ijo} L^2 L^o + \epsilon_{ljo} L^i L^l L^o + \epsilon_{ljo}[L^l, L^i] L^o) \\
&= \beta^3(-\epsilon_{ijo} L^2 L^o + \frac{\epsilon_{ljo}}{2} L^i [L^l, L^o] + i\epsilon_{ljo}\epsilon_{lik} L^k L^o) \\
&= \beta^3(-\epsilon_{ijo} L^2 L^o + \frac{i}{2}\epsilon_{ljo}\epsilon_{lok} L^i L^k + i(\delta_{ij}\delta_{ok} - \delta_{oi}\delta_{jk} L^k L^o)) \\
&= \beta^3(-\epsilon_{ijo} L^2 L^o - i L^i L^j + i\delta_{ij} L^2 - i L^j L^i) \\
&= \beta^3(\epsilon_{ijk}(1 - L^2) L^k + i\delta_{ij} L^2 - 2i L^j L^i)
\end{aligned} \tag{A.8}$$

The cube of A is now expressed in a simple form to take its trace. By doing so, the first term on the rhs vanishes as it is antisymmetric in i, j. We are left with

$$\mathbf{Tr} A^3 = \frac{i\beta^3(N_5^2 - 1)}{4} = -i\frac{16\lambda^{3/2}\cos^6\psi}{(2\pi)^3(N_5^2 - 1)^2} \tag{A.9}$$

The fact that the result is imaginary could seem disturbing at first but we remind the reader that the action is defined by the symmetrized trace of the determinants. Under the symmetrisation prescription, all odd powers of A cancel out in the expansion leaving us only with even powers of A which are all real valued. This can easily be seen by inverting, for example, L^1 and L^2 in A.8 providing an overall minus sign from the commutators.

There is no need in computing the expansion of the logarithm any further as the determinant is taken only over the three indices $i = 1, 2, 3$. Therefore, the result can be non-zero only up to third power in A. Let us put the terms together

$$\mathbf{Tr}\log(1 + A) = \frac{\beta^2(N_5^2 - 1)}{4} + \frac{i\beta^3(N_5^2 - 1)}{12} + ... \tag{A.10}$$

The determinant we are looking for is found by exponentiating the result of A.10

$$\det(1 + A) = 1 + \frac{\beta^2(N_5^2 - 1)}{4} + \frac{i\beta^3(N_5^2 - 1)}{12} \tag{A.11}$$

After taking the symmetrized trace, the only relevant terms are

$$\det(1 + A) = 1 + \frac{\beta^2(N_5^2 - 1)}{4} \tag{A.12}$$

And using the definition $f = \frac{2\pi N_5}{\sqrt{\lambda}} \sim 1$

$$\det(Q) = 1 + \frac{4\cos^4\psi}{f^2} + \frac{4\cos^4\psi}{(N_5^2 - 1)f^2} \tag{A.13}$$

A.2 Pullback computation

We present here the details of the computation of the quantity $P[G_{ai}\left(Q^{-1}-\delta\right)^{ij}G_{jb}]$ needed in the non-abelian Born-Infeld action. First of all, we recall that

$$\left(Q^{-1}-\delta\right)^{i}{}_{j} = \beta\ \epsilon_{ijk}L^k + \beta^2\ \epsilon_{ilk}\epsilon_{ljm}L^kL^m + ... \tag{A.14}$$

where $\beta = \frac{4\sqrt{\lambda}\cos^2\psi}{2\pi(N_5^2-1)}$.
As the metric is diagonal and the transverse directions depend only on the radial coordinate r, the lone term contributing in the pullback is

$$\begin{aligned} P[G_{ri}\left(Q^{-1}-\delta\right)^{ij}G_{jr}] &= \partial_r X^k G_{ki} G^{mj}\left(Q^{-1}-\delta\right)^{i}{}_{m}\partial_r X^l G_{jl} \\ &= \frac{4\sqrt{\lambda}\alpha'\sin^2\psi\ \dot{\psi}^2}{N_5^2-1}L^i\left(Q^{-1}-\delta\right)^{i}{}_{j}L^j \end{aligned} \tag{A.15}$$

where in the second line we have used $G_{ij} = \sqrt{\lambda}\alpha'\delta_{ij}$ and $X^i = \frac{\cos\psi}{\sqrt{j(j+1)}}L^i$.
The first term in the expansion is a product of three matrices,

$$\begin{aligned} \beta\epsilon_{ijk}L^iL^kL^j &= \beta\frac{\epsilon_{ijk}}{2}[L^i,L^k]L^j \\ &= i\beta\frac{\epsilon_{ijk}\epsilon_{ikl}}{2}L^lL^j \\ &= -i\beta L^2 \end{aligned} \tag{A.16}$$

Once again, the same argument allows us to dismiss this term as it vanishes under the symmetrized trace prescription.

The next term is even in powers of L and will therefore contribute to the action,

$$\begin{aligned} \beta^2\epsilon_{ilk}\epsilon_{ljm}L^iL^kL^mL^j &= \beta^2\left(-\delta_{ij}\delta_{km}+\delta_{im}\delta_{jk}\right)L^iL^kL^mL^j \\ &= \beta^2\left(L^iL^jL^iL^j - L^iL^jL^jL^i\right) \\ &= \beta^2L^iL^j[L^i,L^j] \\ &= i\beta^2\epsilon_{ijk}L^iL^jL^k \\ &= i\beta^2\frac{\epsilon_{ijk}}{2}L^i[L^j,L^k] \\ &= -\beta^2\frac{\epsilon_{ijk}\epsilon_{jkl}}{2}L^iL^l \\ &= -\beta^2L^2 = -\frac{\beta^2(N_5^2-1)}{4} \end{aligned} \tag{A.17}$$

This term is the only relevant term up to order $\frac{1}{N_5^2}$. To verify this assertion, let us compute the following term in the expansion. The next even term in powers of L is a product of six matrices,

$$\begin{aligned}
\beta^4\epsilon_{ikl}\epsilon_{kmn}\epsilon_{mop}\epsilon_{ojq}L^iL^lL^nL^pL^qL^j &= \beta^4\left(-\delta_{im}\delta_{ln}+\delta_{in}\delta_{lm}\right)\left(-\delta_{mj}\delta_{pq}+\delta_{mq}\delta_{pj}\right)L^iL^lL^nL^pL^qL^j \\
&= \beta^4\left(L^2L^2L^2-2L^2L^iL^jL^iL^j+L^iL^kL^iL^jL^kL^j\right) \\
&= \beta^4\left(-L^2L^i[L^j,L^i]L^j+L^i[L^k,L^i]L^jL^kL^j\right) \\
&= i\beta^4\left(-\epsilon_{jik}L^2L^iL^kL^j+\epsilon_{kil}L^iL^lL^jL^kL^j\right) \\
&= i\beta^4\left(-\frac{\epsilon_{jik}}{2}L^2L^i[L^k,L^j]+\frac{\epsilon_{kil}}{2}[L^i,L^l]L^jL^kL^j\right) \\
&= \beta^4\left(\frac{\epsilon_{jik}\epsilon_{kjl}}{2}L^2L^iL^l-\frac{\epsilon_{kil}\epsilon_{ilm}}{2}L^mL^jL^kL^j\right) \\
&= \beta^4\left(L^2L^2-L^kL^jL^kL^j\right) \\
&= -\beta^4\ L^k[L^j,L^k]L^j \\
&= -i\beta^4\ \epsilon_{jki}L^kL^iL^j \\
&= \beta^4\ \frac{\epsilon_{jki}\epsilon_{kil}}{2}L^lL^j \\
&= \beta^4\ L^2=\frac{\beta^4(N_5^2-1)}{4}
\end{aligned} \tag{A.18}$$

We notice that the factor β scales as $\frac{1}{N_5}$ and therefore the first term A.17 in the expansion is already a $\frac{1}{N_5^2}$ correction to the action while the second term A.18 scales as $\frac{1}{N_5^4}$ which we can neglect.

Putting everything together, we obtain,

$$\begin{aligned}
P[G_{ri}\left(Q^{-1}-\delta\right)^{ij}G_{jr}] &= -\sqrt{\lambda}\alpha'\sin^2\psi\ \dot{\psi}^2\frac{16\lambda^2\cos^4\psi}{(2\pi)^2(N_5^2-1)^2} \\
&\sim -\frac{16\sqrt{\lambda}\alpha'\sin^2\psi\cos^4\psi\ \dot{\psi}^2}{f^2(N_5^2-1)}
\end{aligned} \tag{A.19}$$

A.3 Conformal dimensions

To understand the asymptotic behavior of $\psi(r)$, we expand the action to linear order around the boundary value of ψ. We remind the reader that the potential for the non-abelian D3-D5 system is,

$$V(\psi) = \sqrt{4\sin^4\psi\,(r^4+b^2)\left(f_5^2+4\cos^4\psi+\frac{4\cos^4\psi}{N_5^2-1}\right)+\left(\pi\nu-\frac{2}{\sqrt{1-\frac{1}{N_5^2}}}bc(\psi)\right)^2} \tag{A.20}$$

Calling $\psi(r) = \psi_\infty + \phi(r)$ with $\phi(r)$ a small deviation from the value at infinity scaling as $\phi(r) \sim r^\Delta$, we expand the Routhian in the near-boundary region,

$$R_5 \sim -\frac{\mathcal{N}_5 N_5}{f_5}\left(\left(1+\frac{1}{2}hr^2\dot{\phi}^2(r)\right)V(r)\mid_{\psi_\infty} +\phi(r)\frac{\partial V(r)}{\partial\psi}\mid_{\psi_\infty} +\frac{\phi^2(r)}{2}\frac{\partial^2 V(r)}{\partial\psi^2}\mid_{\psi_\infty}\right) \tag{A.21}$$

Applying the Euler-Lagrange equation to the Routhian yields the equation of motion

$$\frac{d}{dr}\left(hr^2\dot{\phi}V(r)\right)-\frac{\partial V(r)}{\partial\psi}\mid_{\psi_\infty} -\phi(r)\frac{\partial^2 V(r)}{\partial\psi^2}\mid_{\psi_\infty}=0 \tag{A.22}$$

Considering only the leading term, this is

$$-\frac{\partial V(r)}{\partial\psi}\mid_{\psi_\infty}=0 \tag{A.23}$$

which gives the condition,

$$4\sin^3\psi_\infty\cos\psi_\infty\left(f_5^2+4\cos^4\psi_\infty+\frac{4\cos^4\psi}{N_5^2-1}-4\sin^2\psi_\infty\cos^2\psi_\infty\left(1+\frac{1}{N_5^2-1}\right)\right)=0 \tag{A.24}$$

This equation possesses trivial solutions ($\psi_\infty = 0, \frac{\pi}{2}$) and one complicated solution that depends on values of f_5 and N_5. We are interested in the solution $\psi_\infty = \frac{\pi}{2}$ for which the D5-branes wrap a maximal two-sphere at the boundary.

The first correction term to the equations of motion yields a quadratic equation for the power Δ,

$$\Delta(\Delta+3)V(r)\mid_{\psi_\infty}=\frac{\partial^2 V(r)}{\partial\psi^2}\mid_{\psi_\infty} \tag{A.25}$$

Evaluating the potential and its second derivative at $\psi = \frac{\pi}{2}$ gives us

$$V(r) \mid_{\psi=\frac{\pi}{2}} = 2r^2 f_5 \ , \ \ \frac{\partial^2 V(r)}{\partial \psi^2} \mid_{\psi=\frac{\pi}{2}} = -4r^2 f_5 \tag{A.26}$$

which gives the following relation for the conformal dimension,

$$\Delta^2 + 3\Delta + 2 = 0 \tag{A.27}$$

We find two solutions,

$$\Delta_+ = -1 \ , \ \ \Delta_- = -2 \tag{A.28}$$

as claimed above. The conformal dimensions indicate the behavior of the solution near the boundary. We can thus expand ψ for large values of r in the following way,

$$\psi(r_{max}) \sim \frac{\pi}{2} + \frac{m}{r} + \frac{c}{r^2} \tag{A.29}$$