Points of Small Height on Plane Curves

by

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Abstract

Let K be an algebraically closed field, and let $\mathcal{C}$ be an irreducible plane curve, defined over the algebraic closure of $K(t)$, which is not defined over K. We show that there exists a positive real number c_0 such that if P is any point on the curve $\mathcal{C}$ whose Weil height is bounded above by c_0, then the coordinates of P belong to K.

Preface

This thesis was written by myself. The topic of this thesis and the methodology for proving the results were suggested by my supervisor, Professor Dragos Ghioca.

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Chapter 1

Introduction

1.1 Generalities and preliminaries

Throughout our study in number theory, we would like to determine the degree of complexity of numbers. The theory of heights provides us with the tools to study numbers in this setting. In order to understand what a height function is, we could consider the elementary exponential height on the rationals. This type of height may be extended to finite extensions of the rationals and even further to function fields over number fields. In the work ahead, we consider heights for points on affine varieties. We denote by $\overline{k}$ the algebraic closure of a field k. We assume that the reader has an understanding of general valuation theory [4], Galois theory [1], and basic algebraic geometry [1]. We will review valuation theory over function fields $K(t)$ and its finite extensions, as well as the basic definitions and properties of the height function in this setting. Throughout our study, we will assume the base field K to be algebraically closed.

1.2 Motivation

The main focus of study ahead regards that of plane curves. We define $k[X] = k[x_1, \ldots, x_n]$ for some field k and consider the following definition.

Definition 1.2.1. An affine variety V over a field k is the set of common zeros of a collection of polynomials $f_1(X), \ldots, f_m(X)$.

If $n = 2$ and $m = 1$, we define the variety V as a *plane curve* $\mathcal{C} = \{(x, y) \in \overline{k}^2 | f(x, y) = 0\}$. As long as the polynomial $f(X, Y) \in \overline{k}[X, Y]$ is non-constant, $\mathcal{C}$ is non-empty. The choice for the field k could be any field, in particular the field of rational functions $K(t)$ for some constant field K. We would like to study whether or not there exists a positive lower bound for the height of a point on a plane curve defined over the algebraic closure of a function field $K(t)$. For $k = \mathbb{F}_p(t)$, Ghioca proved in [3] the following:

Theorem 1.2.2. *Let X be an affine subvariety of $\mathbb{A}^n$ defined over $\overline{\mathbb{F}_p(t)}$. Let Y be the Zariski closure of $X(\overline{\mathbb{F}_p})$. There exists a positive constant C depending only on X such that if $P \in X(\overline{\mathbb{F}_p(t)})$ and $h(P) < C$, then $P \in Y(\mathbb{F}_p(t))$.*

This theorem states that within the varieties over $\mathbb{F}_p(t)$ the only variaties that contain many point of small height are the varieties defined over the constant field $\mathbb{F}_p$. So, it is natural to ask whether the above result can be extended to function fields with constant fields, such as $\mathbb{C}(t)$ or $\overline{\mathbb{Q}}(t)$. In order to do this, we define a new endomorphism to replace the role of the Frobenius in the results of [3], and obtain the main theorem of this paper.

Theorem 1.2.3. *Let K be an algebraically closed field and let $\mathcal{C} \subset \mathbb{A}^2$ be an irreducible plane curve defined over $\overline{K(t)}$ and not over K. For $(x, y) \in \mathcal{C}(\overline{K(t)})$, there exists a real number $c_0 > 0$ such that if $h(x, y) < c_0$, then $(x, y) \in \mathcal{C}(K)$.*

1.3 Outline of study

We begin our study with valuation theory over function fields. We then define the basic notion of heights over function fields and extend the definition to finite extensions of the base field. We provide the reader with a brief overview of the definitions and basic properties of heights over function fields, and then move on to studying plane curves defined over $\overline{K(t)}$. As our constant fields are arbitrary, we examine what we refer to as the σ-*endomorphism* on $K[t]$ and its interplay with heights over finite extensions of K(t). With the use of a lemma of Derksen and Masser [2], we prove preliminary lemmas that then lead to a proof of Theorem 1.2.3 over $K(t)$.

Aiming to extend this result to curves $\mathcal{C}$ defined over $\overline{K(t)}$, we reduce our proof to a curve $\mathcal{C}'$ which is defined over $K(t)$; $\mathcal{C}'$ is the union of the Galois conjugates of $\mathcal{C}$ over $K(t)$. Using the fact that this curve $\mathcal{C}'$ contains $\mathcal{C}$, we conclude that Theorem 1.2.3 holds for curves defined over $\overline{K(t)}$.

Chapter 2

Valuations

We assume the reader is familiar with general valuation theory. For our case of interest, we take $\mathcal{F} = K(t)$, where K is algebraically closed and obtain the following *discrete valuations* where $v(K(t)^*) = \mathbb{Z}$.

2.1 Valuations over $K(t)$ and its extensions

Definition 2.1.1. Let $f, g, \in K[t]$, so that $F = \dfrac{f}{g} \in K(t)$. We define the *place at infinity* as:

$$v_\infty(F) = \deg g - \deg f.$$

For any $\alpha \in K$, define the valuation associated to $t - \alpha \in K[t]$ as:

$$v_{t-\alpha}(F) = v_{t-\alpha}(f) - v_{t-\alpha}(g)$$

with $v_{t-\alpha}(f) = v_{t-\alpha}((t-\alpha)^d f_1) = d$ and $(t-\alpha) \nmid f_1$.

Proposition 2.1.2. *If v is a discrete valuation on a field $\mathcal{F}$, then the subring $\mathcal{O}_v = \{x \in \mathcal{F} | v(x) \geq 0\}$ is a valuation ring with unique maximal ideal $\mathcal{M}_v = \{x \in \mathcal{F} | v(x) \geq 1\}$.*

Proof. By definition, we have that $\mathcal{O}_v$ is a ring. Now, we wish to show that for any $x \in \mathcal{F}$, either x or x^{-1} is in $\mathcal{O}_v$. If $x \notin \mathcal{O}_v$, then $v(x) < 0$. So, $v(x^{-1}) = v(1) - v(x) = 0 - v(x) > 0$. Hence, $x^{-1} \in \mathcal{O}_v$, making $\mathcal{O}_v$ a valuation ring. Now, if $x \in \mathcal{O}_v^{\times}$, then we have that $x, x^{-1} \in \mathcal{O}_v$, which happens if and only if $v(x) = v(x^{-1}) = 0$. Hence, $\mathcal{M}_v$ consists of all non-unit elements of $\mathcal{O}_v$, so that $\mathcal{O}_v/\mathcal{M}_v$ is a field, making $\mathcal{M}_v$ the unique maximal ideal of $\mathcal{O}_v$. □

From now on, we denote by $\Omega_{K(t)}$ the set of all valuations over $K(t)$.

Claim 2.1.3. *Modulo taking multiples, the valuations of Definition 2.1.1 are the only valuations on $K(t)$.*

Proof. Let v be a discrete valuation of $K(t)$ and let p be its place. We we consider the two cases $v(t) \geq 0$ and $v(t) < 0$.

Suppose $v(t) \geq 0$. Then, we have that $K[t] \subseteq \mathcal{O}_v$ and $v(K(t)^*) \neq \{0\}$. For $J = K[t] \cap \mathcal{M}_v$, we have that J is a non-zero ideal of $K[t]$. Since $1 \notin J$, we have that $J \neq K[t]$. As $\mathcal{M}_v$ is maximal in $\mathcal{O}_p$, we have that J is prime in $K[t]$. Hence, $J = (t - \alpha)$ for some $\alpha \in K$.

If $f(t) \in K[t]$ is not divisible by $t - \alpha$, then $f(t) \notin \mathcal{M}_v$ and $v(f(t)) = 0$. For $g(t)$ non-zero in $K(t)$, we may write $g(t) = (t-\alpha)^{\beta}\dfrac{g_1(t)}{g_2(t)}$, where $\beta \in \mathbb{Z}$, $g_1, g_2 \in K[t]$, and $t - \alpha \nmid g_1, g_2$. So, $v(g(t)) = \beta v(t - \alpha) = m v_{t-\alpha}(g(t))$. Hence, $v \sim v_{t-\alpha}$.

Now, suppose that $v(t) > 0$. Then, $n = v(t^{-1}) > 0$ for $t^{-1} \in \mathcal{M}_v$. Take $f \in K[t]$ with $\deg(f) = d$. Then,

$$\begin{aligned} f(t) = \sum_{i=0}^{d} \alpha_i t^i &= t^d \sum_{i=0}^{d} \alpha_i x^{i-d} \\ &= t^d \sum_{i=0}^{d} \alpha_{d-i} t^{-i} \end{aligned}$$

$$= t^d(\alpha_d + \sum_{i=1}^{d} \alpha_{d-i} t^{-i})$$
$$= t^d(\alpha_d + g(t)),$$

where $g(t) \in \mathcal{M}_v$. Since we have that $\alpha_d \neq 0 \in K$, $v(\alpha_d) = 0$, so that $v(\sum_{i=0}^{d} \alpha_{d-i} t^{-i}) = 0$. Finally, we have that $v \sim v_\infty$ since

$$v(f(t)) = v(t^d) = -dv(t^{-1}) = kv_\infty(f(t)).$$

$\square$

With this claim, we may consider the following familiar lemma.

Lemma 2.1.4. *For $x \in K(t)^\times$, $\sum_{v \in \Omega_{K(t)}} v(x) = 0$.*

Proof. Let $x \in K(t)$ with $x = \frac{f}{g}$, $f, g \in K[t]$. By definiton of $v_{t-\alpha}$, if $P \nmid fg$, then $v_{t-\alpha}(x) = 0$. Thus, the only valuations we need to consider are those associated to $t - \alpha$ which divide f or g, as well as the infinite place. Letting $t-\alpha_1, \ldots, t-\alpha_n$ be all divisors of f and g, we have that $f = (t-\alpha_1)^{\beta_1} \ldots (t-\alpha_n)^{\beta_n}$ and $g = (t-\alpha_1)^{\gamma_1} \ldots (t-\alpha_n)^{\gamma_n}$. The valuation of a polynomial at $t - \alpha_i$ corresponds to the order of vanishing at $\alpha_i \in K$. So, we have that $\deg(f) = \sum_i \beta_i$ and $\deg(g) = \sum_i \gamma_i$. Hence,

$$\sum_{v \in \Omega_{K(t)}} v(x) = \sum_{t-\alpha} v_{t-\alpha}(x) + v_\infty(x)$$
$$= \sum_{t-\alpha} v_{t-\alpha}(x) + \deg(g) - \deg(f)$$

Now, we have that

$$\deg(g) - \deg(f) = \sum_i (\gamma_i - \beta_i)$$

and

$$\sum_{t-\alpha} v_{t-\alpha}(x) = \sum_i (v_{t-\alpha_i}(f) - v_{t-\alpha_i}(g)) = \sum_i (\beta_i - \gamma_i).$$

Combining these results,

$$\begin{aligned} \sum_{v \in \Omega_{K(t)}} v(x) &= \sum_{t-\alpha} v_{t-\alpha}(x) + \deg(g) - \deg(f) \\ &= \sum_i (\beta_i - \gamma_i) + \sum_i (\gamma_i - \beta_i) \\ &= 0 \end{aligned}$$

□

2.2 Residue fields for valuations of $K(t)$

$K(t)$ and its finite extensions L are fields satisfying Proposition 2.1.2 [4]. Using the definitions from that proposition, we have the following:

$$\mathcal{O}_{v_{t-\alpha}} = \{x = \tfrac{f}{g} \in K(t) | f, g \in K[t], t-\alpha \nmid g, \gcd(f,g) = 1\}$$
$$\mathcal{M}_{v_{t-\alpha}} = \{x = \tfrac{f}{g} \in K(t) | f, g \in K[t], t-\alpha | f, t-\alpha \nmid g, \gcd(f,g) = 1\}$$
and
$$\mathcal{O}_{\infty} = \{x = \tfrac{f}{g} | f, g \in K[t], g \neq 0, \deg(f) - \deg(g) \leq 0\}$$
$$\mathcal{M}_{\infty} = \{x = \tfrac{f}{g} | f, g \in K[t], g \neq 0, \deg(f) - \deg(g) < 0\}$$

We define the residue field $\kappa_v = \mathcal{O}_v/\mathcal{M}_v$ for each valuation $v \in \Omega_L$.

Proposition 2.2.1. *For $F = K(t)$ and place $v \in \Omega_F$, for $t-\alpha$ irreducible, the residue field $\kappa_{v_{t-\alpha}}$ is isomorphic to $K[t]/(t-\alpha)$ and $\kappa_\infty \cong K$.*

Proof. Let us first consider the place $v = v_{t-\alpha}$, for $\alpha \in K$. Consider the canonical projection $\mathcal{O}_{v_{t-\alpha}} \longrightarrow \kappa_{v_{t-\alpha}}$. We now define the map $\phi : K[t] \longrightarrow \kappa_{v_{t-\alpha}}$ as the restriction of the canonical map. The kernel of ϕ is $(t-\alpha)$, so we must show that ϕ is surjective.

Let $x = \dfrac{f(t)}{g(t)} \in \mathcal{O}_{v_{t-\alpha}}$, where $(t-\alpha) \nmid g(t)$ and $\gcd(f,g) = 1$. Since $t-\alpha$ is irreducible, we must have that $\gcd(t-\alpha, g(t)) = 1$. So, there exist $h(t), l(t) \in K[t]$ such that $(t-\alpha)h(t) + g(t)l(t) = 1$. Hence,

$$x = \frac{f(t)}{g(t)}(t-\alpha)h(t) + g(t)l(t) = \frac{(t-\alpha)h(t)f(t)}{g(t)} + f(t)l(t)$$

We then have that $\phi(x) = f(t)l(t)$. So, since both $f(t)$ and $l(t)$ are in $K[t]$ this proves that ϕ is surjective giving us the desired isomorphism $\kappa_{v_{t-\alpha}} \cong K[t]/(t-\alpha)$.

Now, let $v = v_\infty$. First, we note that for any $f(t) \in \mathcal{O}_\infty$, $f(t)$ may be written in the form

$$f(t) = \frac{b_n t^n + b_{n-1}t^{n-1} + \cdots + b_0}{t^n + c_{n-1}t^{n-1} + \cdots + c_0}$$

Where $n \geq 0$ and $b_i, c_i \in K$. Next, consider the map $\psi : \mathcal{O}_\infty \longrightarrow K$ where $\psi(f(t)) := b_n$. The map ψ is a ring homomorphism since for $f(t) = \dfrac{\sum_{i=1}^{n} b_i t^i}{\sum_{j=1}^{n-1} c_j t^j + t^n}$ and $g(t) = \dfrac{\sum_{i=1}^{m} d_i t^i}{\sum_{j=1}^{m-1} e_j t^j + t^m}$ we have:

$$\begin{aligned}
\psi(f(t)g(t)) &= \psi\left(\frac{\sum_{i=1}^{n} b_i t^i}{\sum_{j=1}^{n-1} c_j t^j + t^n} \cdot \frac{\sum_{i=1}^{m} d_i t^i}{\sum_{j=1}^{m-1} e_j t^j + t^m}\right) \\
&= \psi\left(\frac{\sum_{i+j=k}^{n+m} b_i d_j t^k}{\left(\sum_{j=1}^{n-1} c_j t^j + t^n\right)\left(\sum_{j=1}^{m-1} e_j t^j + t^m\right)}\right) \\
&= b_n d_m \\
&= \psi(f(t)) \cdot \psi(g(t))
\end{aligned}$$

and

$$\psi(f(t)+g(t)) = \psi\left(\frac{\sum_{i=1}^{n} b_i t^i \left(\sum_{j=1}^{m-1} e_j t^j + t^m\right) + \sum_{i=1}^{m} d_i t^i \left(\sum_{j=1}^{n-1} c_j t^j + t^n\right)}{\left(\sum_{j=1}^{m-1} e_j t^j + t^m\right)\left(\sum_{j=1}^{n-1} c_j t^j + t^n\right)}\right)$$

$$= \psi\left(\frac{(b_n + d_m)t^{n+m} + \sum_{i+j=k}^{m+n-1} \beta_k t^k}{\left(\sum_{j=1}^{m-1} e_j t^j + t^m\right)\left(\sum_{j=1}^{n-1} c_j t^j + t^n\right)}\right)$$

$$= b_n + d_m$$

$$= \psi(f(t)) + \psi(g(t)).$$

Certainly, ψ is surjective and $\ker(\psi)$ is the set of all f where b_n is zero, i.e when the degree of the numerator is less than the denominator. Hence, we have that $\ker(\psi) = \mathcal{M}_\infty$, proving that $\kappa_\infty \cong K$ as desired. $\square$

Proposition 2.2.2. *If the field K is algebraically closed, then for $v_{t-\alpha} \in \Omega_{K(t)}$, $\kappa_{v_{t-\alpha}} = K$. Moreover, if $L/K(t)$ is a finite extension with $w|v$, then $\kappa_w = K$.*

Proof. We must prove that the extension κ_w/κ_v is algebraic.

Given an algebraic extension $L/K(t)$, consider some element $x \in L \cap \mathcal{O}_w$. Since $L/K(t)$ is algebraic, x must be a root of some polynomial with coefficients in $K(t)$. That is, there exist $a_1, \dots, a_n \in K(t)$ where

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0.$$

Let v be a place of $K(t)$ lying below w. Then, v corresponds to either $t-\alpha$ or v_∞.

Suppose $v = v_{t-\alpha}$. Multiplying each a_i by $(t-\alpha)^{-\min_i v(a_i)}$, we have that each of the a_i now belongs to $\mathcal{O}_v$. With $a_i = \dfrac{f_i}{g_i}$, at least one of the a_i is a unit modulo $\mathcal{M}_v$, that is, $t-\alpha$ does not divide f_i or g_i. Now, reducing modulo $\mathcal{M}_w$, we have:

$$\bar{a}_i \bar{x}^i + \cdots + \bar{a}_0 = 0$$

where i is the largest index such that a_i is a unit modulo $\mathcal{M}_v$. As a remark, we note that $i \neq 0$ since otherwise $\bar{a}_0 = 0$ or $a_0 = 0$, modulo $\mathcal{M}_v$, a contradiction.

Now, suppose that $v = v_\infty$. We still have that x satisfies an equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ for $a_i \in K(t)$. We want the a_i in $\mathcal{O}_\infty$, so as each $a_i = \dfrac{f_i}{g_i}$, with $f_i, g_i \in K[t]$, we will multiply each a_i by $t^{\min_i \{\deg(g_i) - \deg(f_i)\}}$. Now we have that $\deg(f_i) \leq \deg(g_i)$ for every i, so that $v_\infty(a_i) \in \mathbb{Z}$ and at least one is a unit. This yields $a_i \in \mathcal{O}_\infty$. Reducing modulo $\mathcal{M}_w$, we have that

$$\bar{a}_i \bar{x}^i + \cdots + \bar{a}_0 = 0$$

Now in either case, we have that x is a root of an equation over κ_v since each $a_i \in \mathcal{O}_v/(\mathcal{M}_w \cap \mathcal{O}_v) = \mathcal{O}_v/\mathcal{M}_v = \kappa_v$. Hence, κ_w/κ_v is algebraic. In the first case $v = v_{P(t)}$, with $P(t) = t - \alpha$, and by Proposition 2.2.1, $\kappa_v = K[t]/(t-\alpha) = K$. But as κ_w is an extension of κ_v, we have that $\kappa_w = K$ as well.

Similarly, by Proposition 2.2.1, $\kappa_\infty = K$, yielding that $\kappa_w = K$ concluding the proof. $\square$

Chapter 3

Heights

Now that we have defined the valuations over $K(t)$, we may move to defining heights.

Definition 3.0.3. For $x \in K(t)$, the *local height* of x at $v \in \Omega_{K(t)}$ is

$$h_v(x) = -\min\{0, v(x)\} = \max\{0, -v(x)\}.$$

The *(global) height* of x is defined by $h(x) := h_{K(t)}(x) = \sum_{v \in \Omega_{K(t)}} h_v(x)$.

We may also define a multi-dimensional height, which will be useful in our study of plane curves. Let $(x, y) \in K(t)^2$. Then, define

$$h(x, y) = \sum_{v \in \Omega_{K(t)}} \max\{h_v(x), h_v(y)\}.$$

In the work ahead, we are concerned with the field $\overline{K(t)}$, so we may extend the height to a intermediate field $K(t) \subset L \subset \overline{K(t)}$, where $L/K(t)$ is finite. For any valuation v over $K(t)$, v extends to a valuation $w \in \Omega_L$ [4]. We say that $w \in \Omega_L$ *lies above* v if and only if $\mathcal{O}_v$ embeds into $\mathcal{O}_w$. If this is the case, then we have that $\kappa_v \hookrightarrow \kappa_w$ as well. So, we define $f(w|v) = [\kappa_w : \kappa_v]$. But, since our field K is algebraically closed, $f(w|v) = 1$, by Proposition 2.2.2.

Also, we define this valuation $w \in \Omega_L$ for $x \in K(t)$ as $w(x) = e(w|v)v(x)$, where $e(w|v) = w(u)$ for u a uniformizer of v in $K(t)$ (That is, $v(u) = 1$).

We also have that the sum formula from Lemma 2.1.4 holds over L. The general sum formula for extensions $L/K(t)$ is $\sum\limits_{w \in \Omega_L} n_w w(x) = 0$, where $x \in L*$ and $n_w = e(w|v)f(w|v)$ is the *local degree*. In our case though, $f(w|v) = 1$ as mentioned above, and $e(w|v)$ has already been absorbed into the valuation $w(x)$ since $w(x) = e(w|v)v(x)$. Thus, our local degrees $n_w = 1$ for every $w \in \Omega_L$ and $\sum\limits_{w \in \Omega_L} w(x) = 0$.

For $x \in L$ we define the *local height* of x at $w \in \Omega_L$ as $h_w(x) = \dfrac{1}{[L : K(t)]} \max\{-w(x), 0\}$ and the *global height* of x as $h(x) = \sum\limits_{w \in \Omega_L} h_w(x)$. Similarly, we may extend the defintion to points (x, y) in $\mathbb{A}^2$ with $h_w(x, y) = \dfrac{1}{[L : K(t)]} \max\{0, -w(x), -w(y)\}$ and $h(x, y) = \sum\limits_{w \in \Omega_L} h_w(x, y)$.

Remark: In the proof of the following, Proposition 3.0.4, we use the fact that the sum of all the ramification indices equals the degree of the extension, i.e. $\sum\limits_{w|v} e(w|v) = [M : L]$. The proof of this fact can be found in [5].

Proposition 3.0.4. *The definition of the height is well defined.*

Proof. Let $K(t) \subset L \subset M$ be a finite extension of fields. For $x \in L$, $v \in \Omega_L$, and $w \in \Omega_M$ we have:

$$\begin{aligned}
\sum_{w|v} \frac{1}{[M : K(t)]} w(x) &= \sum_{w|v} \frac{1}{[M : K(t)]} w(x) \\
&= \sum_{w|v} \frac{1}{[M : K(t)]} e(w|v)v(x)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{v(x)}{[M:K(t)]} \sum_{w|v} e(w|v) \\
&= \frac{v(x)}{[M:K(t)]} [M:L] \\
&= \frac{v(x)}{[L:K(t)]}.
\end{aligned}
$$

So, when considering the height and using the equality above, we have

$$
\begin{aligned}
h(x) = \sum_{v \in \Omega_L} h_v(x) &= \sum_{v \in \Omega_L} \frac{1}{[L:K(t)]} \max\{-v(x), 0\} \\
&= \sum_{v \in \Omega_L} \sum_{w|v} \frac{1}{[M:K(t)]} \max\{-w(x), 0\} \\
&= \sum_{w \in \Omega_M} \frac{1}{[M:K(t)]} \max\{-w(x), 0\} \\
&= \sum_{w \in \Omega_M} h_w(x).
\end{aligned}
$$

$\square$

Proposition 3.0.5. *For any $x \in L/K(t)$ and $n \in \mathbb{Z}$, $h(x^n) = nh(x)$.*

Proof.

$$
\begin{aligned}
h(x^n) &= \sum_{w \in \Omega_L} \frac{1}{[L:K(t)]} \max\{0, -w(x^n)\} \\
&= \sum_{w \in \Omega_L} \frac{1}{[L:K(t)]} \max\{0, -nw(x)\} \\
&= n \sum_{w \in \Omega_L} \frac{1}{[L:K(t)]} \max\{0, -w(x)\} \\
&= nh(x).
\end{aligned}
$$

$\square$

We conclude this section with a property of the two-dimensional height.

Proposition 3.0.6. *For $x, y \in L/K(t)$, $h(x,y) \leq h(x) + h(y)$.*

Proof. Recalling that $h(x,y) = \sum_{v \in \Omega_L} h_v(x,y) = \sum_{v \in \Omega_L} \max\{h_v(x), h_v(y)\}$, we note that $\max\{h_v(x), h_v(y)\} = h_v(x) + h_v(y)$ if and only if $h_v(x)$ or $h_v(y)$ equals 0. So, $h_v(x,y) \leq h_v(x) + h_v(y)$ for each place $v \in \Omega_L$, and taking the sum over all places, we obtain the result $h(x,y) \leq h(x) + h(y)$ as desired. $\square$

Chapter 4

Curves in Affine Space

When working in $\mathbb{A}^2$, the varieties that are of interest to us are plane curves. In practice, we consider polynomials $f(X,Y) \in \overline{K(t)}[X,Y]$ and are interested in the zero locus of this polynomial. For the equation $f(X,Y) = 0$, the solutions of it will represent a curve $\mathcal{C}$ in $\mathbb{A}^2$. If we consider a subfield $L \subset \overline{K(t)}$, we set $\mathcal{C}(L) = \{(x,y) \in L \times L : f(x,y) = 0\}$.

The main theorem for this paper is the following:

Theorem 4.0.7. *Let $\mathcal{C} \subset \mathbb{A}^2$ be an irreducible plane curve defined over $\overline{K(t)}$ and not over K. For $(x,y) \in \mathcal{C}(\overline{K(t)})$, there exists a real number $c_0 > 0$ such that if $h(x,y) < c_0$, then $(x,y) \in \mathcal{C}(K)$.*

4.1 The σ-endomorphism

In [3], the key endomorphism was the Frobenius, which one applies to the coefficients of the curve. Since we are now extending the result of Theorem 1.2.2 in $\mathbb{A}^2$ to arbitrary base fields, we need a different endomorphism.

Define $\sigma : K[t] \longrightarrow K[t]$, where $\sigma(K) = id_K$ and $\sigma(t) = t^m$, for some fixed $m \in \mathbb{N}$. The endomorphism σ then naturally extends to an endomorphism

$\overline{K(t)} \longrightarrow \overline{K(t)}$. In fact, we claim the following regarding the extension of σ.

Claim 4.1.1. *$\sigma : \overline{K(t)} \longrightarrow \overline{K(t)}$ is an automorphism.*

Proof. We know that $\sigma : \overline{K(t)} \longrightarrow \overline{K(t)}$ is an injective endomorphism of fields, so we wish to show that σ is surjective; that is, for every $f \in \overline{K(t)}$, there exists $g \in \overline{K(t)}$ such that $\sigma(g) = f$.

Let us start with an arbitrary algebraic element $f \in \overline{K(t)}$. So, f satisfies an equation $a_n f^n + a_{n-1} f^{n-1} + \ldots + a_0 = 0$ where $P(X) = a_n X^n + a_{n-1} X^{n-1} + \ldots + a_0$ is defined over $K(t)$ and irreducible.

We claim that $\sigma : K(t^{1/m}) \longrightarrow K(t)$ is a surjection, and in order to prove this all we need to show is that t has a pre-image in $K(t^{1/m})$. For $a_i = \sum\limits_{j=0}^{m} c_{ij} t^j$ and $\beta \in \overline{K(t)}$, define $t = \sigma(\beta)$. Then,

$$\sigma(\sum_{j=0}^{m} c_{ij}\beta^j) = \sum_{j=0}^{m} \sigma(c_{ij}\beta^j) = \sum_{j=0}^{m} c_{ij}\sigma(\beta)^j = a_i$$

We note that $\sigma(t^{1/m})^m = \sigma(t) = t^m$, so that $\sigma(t^{1/m}) = \zeta_m t$, for some primitive m^{th} root of unity $\zeta_m \in K$ (which exists since K is algebraically closed). Hence, we take $\beta = \zeta_m^{-1} t^{1/m}$, which is indeed contained in $K(t^{1/m})$.

Now that we know that $\sigma : K(t^{1/m}) \longrightarrow K(t)$ is a surjection. If we define $b_i = \sigma^{-1}(a_i)$ for $i = 0, \ldots, n$, we claim that $Q(X) = b_n X^n + \cdots + b_0 = 0$ has the same number of solutions as $P(X)$.

We may consider a complete splitting of $Q(X)$ over $\overline{K(t)}$ so that

$$b_n X^n + \cdots + b_0 = (X - x_1)^{e_1} \cdots (X - x_m)^{e_m}$$

Then, since $b_i = \sigma^{-1}(a_i)$, we have that

$$a_n X^n + \cdots + a_0 = (X - \sigma(x_1))^{e_1} \cdots (X - \sigma(x_m))^{e_m}$$

Hence, $P(X)$ and $Q(X)$ have the same number of roots over $\overline{K(t)}$. But σ maps a solution of $Q(X)$ to a solution of $P(X)$ and moreover, since

$\sigma : \overline{K(t)} \longrightarrow \overline{K(t)}$ is an endomorphism of fields, it must be injective. In particular, $f \in \overline{K(t)}$ has an image under σ, implying that $\sigma : \overline{K(t)} \longrightarrow \overline{K(t)}$ is surjective.

$\square$

Alternatively, we note that σ induces an automorphism of $\overline{K(t)}$ since σ maps K to K identically, and then it sends one transcendental element of $\overline{K(t)}$ (namely t) into a transcendental element (t^m) of $\overline{K(t)}$. So essentially σ maps an algebraic function $f(t) \in \overline{K(t)}$ into $f(t^m)$. This certainly means that σ is surjective (and thus an isomorphism) since $\sigma(g) = f$, where $g(t) = f(t^{1/m})$.

Definition 4.1.2. We say that a non-constant polynomial $f \in K[t][X, Y]$ is *reduced* if the coefficients a_i of f do not share a non-constant common factor in $K[t]$.

If $f_1, \ldots, f_k \in K[t]$, we define the greatest common divisor (gcd) of $f_1, \ldots, f_k$ as the unique monic polynomial of highest degree in $K[t]$ which divides all of the f_i. We define two different types of heights of non-zero elements of algebraic extensions $L/K(t)$.

Definition 4.1.3. For a place $w \in \Omega_L$ and $f \in L[X, Y]$ we have the *local height* $h_w(f) = \max_i \{h_w(a_i)\}$ and the *global height* $h(f) = \sum_{w \in \Omega_L} h_w(f)$.

When considering the height of a point $x \in L/K(t)$, we would like a connection between the height of x and the height of $\sigma(x)$. The following observation, [2, Lemma 2.1], will be key in making this connection.

Lemma 4.1.4. *Let L be a finite extension of $K(t)$ and suppose that $x \in L$ satisfies $f(x) = 0$, for $f(X) = a_n X^n + \ldots a_0$ reduced and irreducible over $K(t)$. Then $h(x) = \dfrac{h(f)}{n}$.*

Remark: If the reader refers to [2, Lemma 2.1], she will notice a factor of $[L : K(t)]$. This arises due to the height in [2] being a normalized height. For our work, we do not need the normalized height and thus ignore the degree factor.

From this result, we may derive the following corollary:

Corollary 4.1.5. *For $l \in \mathbb{N}$, define $T = t^{1/l}$ and let L be a finite extension of $K(t)$. Suppose that $x \in L$ satisfies $f(x) = 0$, where $f(X) \in K(T)[X]$ is reduced and irreducible over $K(T)$. Then, $h(x) = \dfrac{h(f)}{l \cdot \deg(f)}$*

Proof. By Lemma 4.1.4, we have $h(x) = h_{K(T)} = \dfrac{h_{K(T)}(f)}{\deg(f)}$, where $h_{K(T)}$ is the height with respect to $K(T)$. But, we want the height h to be the usual height that is with respect to $K(t)$.

If $y \in K(T)$, then we note

$$h_{K(t)}(y) = \frac{1}{[K(T) : K(t)]} \sum_{w \in \Omega_{K(T)}} \max\{0, -w(y)\}$$

and

$$h_{K(T)}(y) = \sum_{w \in \Omega_{K(T)}} \max\{0, -w(y)\}.$$

Combining these statements, we find

$$h = h_{K(t)} = \frac{1}{[K(T) : K(t)]} h_{K(T)} = \frac{1}{l} h_{K(T)}.$$

By applying Lemma 4.1.4, we obtain our desired result of

$$h(x) = \frac{h_{K(T)}(f)}{l \cdot \deg(f)}$$

□

Claim 4.1.6. *For $x \in \overline{K(t)}$, $h(\sigma(x)) = mh(x)$.*

Proof. Let $\sigma : \overline{K(t)} \longrightarrow \overline{K(t)}$ be as we have defined. Then, for $x \in \overline{K(t)}$, let $f = a_n X^n + \ldots + a_0$ be its minimal polynomial over $L = \bigcup_{k=0}^{\infty} K(t^{1/m^k})$. Then, f is irreducible over L with $a_i \in K(t^{1/m^{k_i}})$, for some k_i. But then,

$a_i \in K(t^{1/m^k})$, for the maximum of the k_i. We may clear denominators so that we have $a_i \in K[t^{1/m^k}] = K[T]$. Thus,

$$\begin{aligned} 0 &= \sigma(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0) \\ &= \sigma(a_n)\sigma(x)^n + \sigma(a_{n-1})\sigma(x)^{n-1} + \ldots + \sigma(a_0) \\ &= \sigma(a_n)Y^n + \sigma(a_{n-1})Y^{n-1} + \ldots + \sigma(a_0), \text{ for } Y = \sigma(x). \end{aligned}$$

If we consider the restriction of σ as the map $K(t^{1/m^k}) \longrightarrow K(t^{1/m^{k-1}})$, this map is surjective. To see this, we note that $\sigma(t^{1/m^k})^{m^k} = \sigma(t) = t^m$. This yeilds $\sigma(t^{1/m^k}) = \zeta t^{1/m^{k-1}}$ for an m^{k-1}-th root of unity ζ. So, we take $\sigma(\zeta^{-1}t^{1/m^k}) = t^{1/m^{k-1}}$, proving surjectivity as desired. As $K(t^{1/m^k}) \longrightarrow K(t^{1/m^{k-1}})$ is an endomorphism of fields, it must be injective. By this, it is certainly true that $\sigma : L \longrightarrow L$ is an automorphism, which may be seen by the following chain:

$$K(t) \longrightarrow K(t^{1/m}) \longrightarrow K(t^{1/m^2}) \longrightarrow \cdots$$

We now want to show that $\sigma(a_n)Y^n + \ldots + \sigma(a_0) = 0$ is irreducible over L. Supoose on the contrary that this is not the case. Then,

$$\sigma(a_n)Y^n + \ldots + \sigma(a_0) = (b_m Y^m + \ldots + b_0)(c_k Y^k + \ldots + c_0) = 0,$$

where each $b_i = \sigma(b'_i), c_i = \sigma(c'_i)$, for some $b'_i, c'_i \in L$ since σ is an automorphism. Then, $\sigma(f) = \sigma(g)\sigma(h)$, yielding $\sigma(f - gh) = 0$. But once again, using that σ is injective, we have that $f = gh$ over L, a contradiction of the irreducibility of f over L.

Now that we have the irreducibility of $\sigma(a_n)Y^n + \ldots + \sigma(a_0)$, we may apply Corollary 4.1.5 and obtain:

$$h(\sigma(x)) = \frac{\max \deg_i \{\sigma(a_i)\}}{m^k n} = \frac{m \max \deg_i \{a_i\}}{m^k n} = \frac{mh(f)}{m^k n} = mh(x).$$

Hence, we have that $h(\sigma(x)) = mh(x)$. Additionally, in order to apply Corollary 4.1.5, we want that the $\gcd(\sigma(a_i)) = 1$. By assumption, we have that $a_i \in K[T]$ (which is a PID) for each i and $\gcd(a_i) = 1$. Hence, we know that there exist $\beta_i \in K[T]$ so that

$$1 = \sum_{i=0}^{n} \beta_i a_i = \sigma(\sum_{i=0}^{n} \beta_i a_i) = \sum_{i=0}^{n} \sigma(\beta_i)\sigma(a_i).$$

Since $\sigma(\beta_i) = c_i$ for some $c_i \in K[T]$, we have that $\gcd(\sigma(a_i)) = 1$, as desired. $\square$

Additionally, we have the following corollary which follows from Claim 4.1.6.

Corollary 4.1.7. *For* $(x, y) \in \overline{K(t)}^2, h(\sigma(x), \sigma(y)) \geq \frac{m}{2} h(x, y)$.

Proof. We have that:

$$\begin{aligned} h(\sigma(x), \sigma(y)) &\geq \max\{h(\sigma(x)), h(\sigma(y))\} \\ &= \max\{mh(x), mh(y)\}, \text{ by Claim 4.1.6} \\ &= m \max\{h(x), h(y)\} \\ &\geq \frac{m}{2}(h(x) + h(y)) \\ &\geq \frac{m}{2} h(x, y) \end{aligned}$$

The last inequality follows by Proposition 3.0.6 since $h(x, y) \leq h(x) + h(y) \leq 2 \max\{h(x), h(y)\}$. $\square$

4.2 Curves over $K(t)$

After defining and studying characteristics of the σ-endomorphism, we are now ready to consider the proof of Theorem 1.2.3. In this section we prove the following special case of Theorem 1.2.3.

Theorem 4.2.1. *Let* $\mathcal{C} \subset \mathbb{A}^2$ *be an irreducible plane curve defined over* $K(t)$ *and not over* K. *For* $(x, y) \in \mathcal{C}(\overline{K(t)})$, *there exists a real number* $c_0 > 0$ *such that if* $h(x, y) < c_0$, *then* $(x, y) \in \mathcal{C}(K)$.

In order to prove Theorem 4.2.1, we begin with the following lemma.

Lemma 4.2.2. *Let $f \in K[t][X,Y]$ be a reduced polynomial of total degree d. Let $m \in \mathbb{Z}$ be such that $m \geq 2h(f)$, if $(x,y) \in \overline{K(t)}^2$ satisfies $f(x,y) = 0$ then either $h(x,y) \geq \dfrac{1}{4d}$ or $f(\sigma(x), \sigma(y)) = 0$.*

Proof. Suppose $f(X,Y) = \sum_{i,j} a_{ij} X^i Y^j$ for $i, j \in \mathbb{N} \cup \{0\}$. We choose an integer m with respect to f so that $m \geq 2h(f)$ and construct the σ-endomorphism from this particular m. For each i, we will define $m_{ij} = x^i y^j$.

Assume $f(\sigma(x), \sigma(y)) \neq 0$. We wish to show that $h(x,y) \geq \dfrac{1}{4d}$. We define the extension $L = K(t, x, y, \sigma(x), \sigma(y))$. We note that for a given $f(x,y) \in K[t][X,Y]$, $\sigma(f(x,y)) = f(\sigma(x), \sigma(y)) \in L$. Define $[L : K(t)] = N$ for some $N \in \mathbb{N}$.

If $\xi = f(\sigma(x), \sigma(y)) \neq 0$, then, by the sum formula, $\sum_{w \in \Omega_L} \dfrac{1}{N} w(\xi) = 0$. Since $f(x,y) = 0$, we have:

$$\begin{aligned}
\xi &= \xi - \sigma((f(x,y))) \\
&= f(\sigma(x), \sigma(y)) - \sigma(\sum_{i,j} a_{ij} m_{ij}) \\
&= \sum_{i,j} a_{ij} \sigma(x)^i \sigma(y)^j - \sum_{i,j} \sigma(a_{ij} m_{ij}) \\
&= \sum_{i,j} a_{ij} \sigma(m_{ij}) - \sum_{i,j} \sigma(a_{ij}) \sigma(m_{ij}) \\
&= \sum_{i,j} (a_{ij} - \sigma(a_{ij})) \sigma(m_{ij})
\end{aligned}$$

Claim 4.2.3. *$t^m - t | \sigma(g) - g$, for any $g \in K[t]$.*

Proof of Claim. Let $g = \sum_{i=0}^{N} c_i t^i$, so that $\sigma(g) = \sum_{i=0}^{N} c_i t^{im}$. Then, since $t^m - t | t^{ml} - t^l$, for any $l \in \mathbb{N}$ and $\sigma(g) - g = \sum_{i=0}^{N} c_i t^{im} - \sum_{i=0}^{N} c_i t^i = \sum_{i=0}^{N} c_i (t^{im} - t^i)$, we have that $t^m - t | \sigma(g) - g$ as desired. ☐

So, from Claim 4.2.3, $\xi = (t^m - t)\sum_{i,j} b_{ij}\sigma(m_{ij})$, $b_{ij} = \dfrac{a_{ij} - \sigma(a_{ij})}{t^m - t} \in K[t]$. Define the set $\mathcal{S}$ as follows:

$$\mathcal{S} = \{w \in \Omega_L : w \text{ lies above an irreducible factor in } K[t] \text{ of } t^m - t\}.$$

As we are working over an algebraically closed field K, we may write the polynomial $t^m - t$ in terms of m irreducible linear factors. For an $m-1^{th}$ root of unity $\zeta \in K$, $t^m - t = t\prod_{\zeta^{m-1}=1}(t-\zeta)$. So, the places $w \in S$ lie above an $m-1^{th}$ root of unity in K or zero. We now consider a series of inequalities.

For each $w \in \mathcal{S}$:

$$\begin{aligned}
\frac{1}{N}w(\xi) &= \frac{1}{N}w((t^m - t)\sum_{i,j} b_{ij}\sigma(m_{ij}))\\
&= \frac{1}{N}[w(t^m - t) + w(\sum_{i,j} b_{ij}\sigma(m_{ij}))]\\
&\geq \frac{1}{N}w(t^m - t) + \frac{1}{N}\min_{i,j}\{w(b_{ij}\sigma(m_{ij}))\}\\
&= \frac{1}{N}w(t^m - t) + \frac{1}{N}\min_{i,j}\{w(b_{ij}) + w(\sigma(m_{ij}))\}\\
&\geq \frac{1}{N}w(t^m - t) + \frac{1}{N}w(b_{IJ}) + \frac{1}{N}w(\sigma(m_{IJ})), \text{ for some } I, J\\
&\geq \frac{1}{N}w(t^m - t) + \frac{1}{N}w(\sigma(m_{IJ})), \text{ since } w(b_{IJ}) \geq 0.
\end{aligned}$$

Now, for any valuation $w \in \Omega_L$,

$$\begin{aligned}
\frac{1}{N}w(\sigma(m_{IJ})) &= \frac{1}{N}w(\sigma(x^I)\sigma(y^J))\\
&= \frac{1}{N}(w(\sigma(x^I) + w(\sigma(y^J)))\\
&= \frac{1}{N}(I\sigma(x) + J\sigma(y))
\end{aligned}$$

$$\geq \frac{I+J}{N} \min\{w(\sigma(x)), w(\sigma(y)), 0\}$$

Since the degree of m_{IJ} is at most d, we have:

$$\frac{1}{N} w(\sigma(m_{IJ})) \geq \frac{d}{N} \min\{w(\sigma(x)), w(\sigma(y)), 0\}.$$

Now, recalling that $h_w(x,y) = -\frac{1}{N} \min\{w(x), w(y), 0\}$, we have that

$$\frac{1}{N} w(\sigma(m_{IJ})) \geq -d h_w(\sigma(x), \sigma(y)). \tag{4.2.1}$$

So,

$$\frac{1}{N} w(\xi) \geq \frac{1}{N} w(t^m - t) - d h_w(\sigma(x), \sigma(y)) \tag{4.2.2}$$

Next, taking $\xi = \sum_{i,j} a_{ij} \sigma(m_{ij})$ and considering valuations $w \in \Omega_L \setminus \mathcal{S}$,

$$\begin{aligned}
\frac{1}{N} w(\xi) &= \frac{1}{N} w(\sum_{i,j} a_{ij} \sigma(m_{ij})) \\
&\geq \frac{1}{N} \min_{i,j} \{w(a_{ij} \sigma(m_{ij}))\} \\
&= \frac{1}{N} w(a_{IJ} \sigma(m_{IJ})), \text{ for some } I, J \\
&= \frac{1}{N} w(a_{IJ}) + \frac{1}{N} w(\sigma(m_{IJ}))
\end{aligned}$$

Now, recall that $h_w(f) = -\frac{1}{N} \min\{w(a_{ij}), 0\}$ for any $w \in \Omega_L$. Since $w(a_{IJ}) \geq \min\{w(a_{IJ}), 0\}$, we have:

$$\begin{aligned}
\frac{1}{N} w(\xi) &\geq \frac{1}{N} w(a_{IJ}) + \frac{1}{N} w(\sigma(m_{IJ})) \\
&\geq \frac{1}{N} \min\{w(a_{IJ}), 0\} + \frac{1}{N} w(\sigma(m_{IJ})) \\
&\geq \frac{1}{N} \min\{w(a_{IJ}), 0\} - d h_w(\sigma(x), \sigma(y)), \text{ by 4.2.1}
\end{aligned}$$

$$= -h_w(f) - dh_w(\sigma(x), \sigma(y)).$$

Combining our results for all $w \in \Omega_L$:

$$\sum_{w \in \Omega_L} \frac{1}{N} w(\xi) \geq \sum_{w(t^m - t) > 0} \frac{1}{N} w(t^m - t) - d \sum_{w \in \Omega_L} h_w(\sigma(x), \sigma(y)) - \sum_{w \in \Omega_L \setminus \mathcal{S}} h_w(f).$$

Hence,

$$0 = \sum_{w \in \Omega_L} \frac{1}{N} w(\xi) \geq \sum_{w(t^m - t) > 0} \frac{1}{N} w(t^m - t) - dh(\sigma(x), \sigma(y)) - h(f) \quad (4.2.3)$$

Then, by our results from Proposition 3.0.4:

$$\sum_{w(t^m - t) > 0} \frac{1}{N} w(t^m - t) = \sum_{v \in \Omega_{K(t)}} v(t^m - t) = -v_\infty(t^m - t) = m$$

By Claim 4.1.6,

$$\begin{aligned}
0 &\geq m - dh(\sigma(x), \sigma(y)) - h(f) \\
&= m - d \sum_{v \in \Omega_L} h_v(\sigma(x), \sigma(y)) - h(f) \\
&\geq m - d(h(\sigma(x)) + h(\sigma(y))) - h(f) \\
&= m - d(mh(x) + mh(y)) - h(f) \\
&\geq m - d(mh(x, y) + mh(x, y)) - h(f) \\
&= m - 2dmh(x, y) - h(f)
\end{aligned}$$

This yields $h(x, y) \geq \dfrac{1}{2d} - \dfrac{h(f)}{2dm}$. But, as $m \geq 2h(f)$, we have that $h(x, y) \geq \dfrac{1}{4d}$, as desired.

□

Now that we have proven the lemma, we may move to the proof of our main theorem for curves defined over $K(t)$.

Proof of Theorem 4.2.1. Let $f(X,Y) = \sum_{i,j} a_{ij}X^iY^j = 0$ be the irreducible polynomial which defines our curve $\mathcal{C}$ over $K(t)$. As $f(X,Y)$ is irreducible, so is $\mathcal{C}$. In order to make use of Lemma 4.2.2, we will consider f in its reduced form and make sure that at least one of the coefficients a_{ij} is non-constant so that f is not defined over K. By Lemma 4.2.2, for $(x_0, y_0) \in \mathcal{C}$ and m such that $m \geq 2h(f)$, one of two cases must occur:

(1) $h(x_0, y_0) \geq \dfrac{1}{4d}$

(2) $f(\sigma(x_0), \sigma(y_0)) = 0 \Longleftrightarrow (\sigma(x_0), \sigma(y_0)) \in \mathcal{C}$

Let us take the first m we find such that $m \geq 2h(f)$ and assume that $(x_0, y_0) \notin \mathcal{C}(K)$. Our choice of m now determines our choice of σ corresponding to m. Also, denote by $\mathcal{C}^\sigma$ the zero locus of the curve $f^\sigma(X,Y)$, where $f^\sigma(X,Y) = \sum_{i,j} \sigma(a_{ij})X^iY^j$

Claim 4.2.4. *$\mathcal{C} \cap \mathcal{C}^\sigma$ is finite.*

Proof of Claim. Suppose on the contrary that $\mathcal{C} \cap \mathcal{C}^\sigma$ is infinite. Then, we must have that $\mathcal{C}$ is one of the irreducible components of $\mathcal{C}^\sigma$.

So, $f^\sigma \in Id(\mathcal{C}^\sigma) \subset (f) = Id(\mathcal{C})$, since $\mathcal{C}$ is an irreducible component of $\mathcal{C}^\sigma$. This implies that $f | f^\sigma$. Also, f and f^σ are of the same degree, so we must have that there exists some $\alpha \in K(t)$ such that $f^\sigma = \alpha f$. Now, $\alpha = \dfrac{\sigma(a_{ij})}{a_{ij}}$ for every i, j. But since f is reduced, by the remark after Definition 4.1.2, $\gcd(\{a_{ij}\}_{i,j}) = 1 \in K$. Hence, there exist $b_{ij} \in K[t]$ such that $\sum_{i,j} a_{ij}b_{ij} = 1$. Applying σ, we have that $\sigma(\sum_{i,j} a_{ij}b_{ij}) = 1 = \sum_{i,j} \sigma(a_{ij})\sigma(b_{ij})$. It follows that $\gcd(\{\sigma(a_{ij})\}_{i,j}) = 1$. By this, we must have that $\alpha \in K$; otherwise, we have a contradiction of the a_{ij} and $\sigma(a_{ij})$ being relatively prime. So, since

$\alpha = \dfrac{\sigma(a_{ij})}{a_{ij}} \in K$ for every i, j, we have that $\deg a_{ij} = \deg \sigma(a_{ij})$, so that $a_{ij} \in K$. Which then implies that f is defined over K. Contradiction.

$\square$

Similiarly, we define $\mathcal{C}^{\sigma^{-1}}$ as the zero locus of $f^{\sigma^{-1}}(X, Y) = \sum_{i,j} \sigma^{-1}(a_{ij})X^iY^j$. It is worth noting that $\mathcal{C}^{\sigma^{-1}}$ is not defined over K. If it were, $\mathcal{C} := \sigma(\mathcal{C}^{\sigma^{-1}})$ would be defineed over $\sigma(K) = K$, a contradiction. We claim that $\mathcal{C} \cap \mathcal{C}^{\sigma^{-1}}$ is finite. Suppose not, so that there exist infinitely many points $(x, y) \in \overline{K(t)}$ such that $f(x, y) = 0 = f^{\sigma^{-1}}(x, y)$. Then, by applying σ, we have

$$\sigma(\sum_{i,j} a_{ij}x^iy^j) = 0 = \sigma(\sum_{i,j} \sigma^{-1}(a_{ij})x^iy^j)$$
$$\sum_{i,j} \sigma(a_{ij})\sigma(x)^i\sigma(y)^j = 0 = \sum_{i,j} a_{ij}\sigma(x)^i\sigma(y)^j$$

So, there are infinitely many points in $\overline{K(t)}$ that satisfy $f(x, y) = 0 = f^{\sigma}(x, y)$. This contradicts Claim 4.2.4, proving that $\mathcal{C} \cap \mathcal{C}^{\sigma^{-1}}$ is finite.

Now, let $\{(x_1, y_1), \ldots, (x_N, y_N)\} \subset \mathcal{C} \cap \mathcal{C}^{\sigma^{-1}}$, be *all* the points which are *not* defined over K. For each j, we define $h_j = h(x_j, y_j)$. Note that $h_j > 0$ since the point is not defined over K. We let $c_0 = \min\{\frac{1}{4d}, \min_i h_i\}$, while if $N = 0$, i.e., all points in $\mathcal{C} \cap \mathcal{C}^{\sigma^{-1}}$ are defined over K, then simply let $c_0 = \dfrac{1}{4d}$.

For $(x_0, y_0) \in \mathcal{C}(\overline{K(t)})$ such that $h(x_0, y_0) < c_0$, we'll show $(x_0, y_0) \in \mathcal{C}(K)$. Indeed by Lemma 4.2.2, we have that $(\sigma(x_0), \sigma(y_0)) \in \mathcal{C}$. This means that $(x_0, y_0) \in \mathcal{C}^{\sigma^{-1}}$ and so, $(x_0, y_0) \in \mathcal{C} \cap \mathcal{C}^{\sigma^{-1}}$. But $h(x_0, y_0) < c_0 \leq h_i$ for each i, and therefore $(x_0, y_0) \in \mathcal{C}(K)$.

If $N = 0$, then automatically $(x_0, y_0) \in C \cap C^{\sigma^{-1}}$ yields that $(x_0, y_0) \in C(K)$ since all those points in the intersection are defined over K.

$\square$

Remark: Theorem 1.2.3 requires that the curve is not defined over K, so we will now see why this assumption is needed. Suppose our curve is defined by the polynomial $f(X, Y) = Y^2 - X^3 - 1$, which is defined over the base field K, and let $m \geq 3$.

Now, suppose that we take $(x, y) = (t, \sqrt{t^3+1}) = p_0 \in \mathcal{C}(\overline{K(t)})$. For $p_1 = (\sigma^{-1}(x), \sigma^{-1}(y))$, we have that

$$\begin{aligned} f(\sigma^{-1}(x), \sigma^{-1}(y)) &= (\sigma^{-1}(\sqrt{t^3-1}))^2 - (\sigma^{-1}(t))^3 - 1 \\ &= \sigma^{-1}((\sqrt{t^3-1})^2 - t^3 - 1) \\ &= \sigma^{-1}(0) \\ &= 0. \end{aligned}$$

Thus, $p_1 \in \mathcal{C}(\overline{K(t)})$. Similarly for $p_n = (\sigma^{-n}(x), \sigma^{-n}(y))$, we have

$$\begin{aligned} f(\sigma^{-n}(x), \sigma^{-n}(y)) &= (\sigma^{-n}(\sqrt{t^3-1}))^2 - (\sigma^{-n}(t))^3 - 1 \\ &= \sigma^{-n}((\sqrt{t^3-1})^2 - t^3 - 1) \\ &= \sigma^{-n}(0) \\ &= 0 \end{aligned}$$

Hence, p_n belongs to the curve $\mathcal{C}$. Since $\sigma^n(p_n) = p_0$, we may apply Corollary 4.1.7 to see that $h(p_n) \leq \left(\frac{2}{m}\right)^n h(p_0)$. If we let n go to infinity, then $h(p_n)$ must go to zero since $m \geq 3$. Hence, for $(x, y) \in \mathcal{C}$ and again taking n to infinity, we have that $h(p_n)$ goes to 0. We cannot find a bound c_0 on the heights, and this shows that the conclusion of Theorem 1.2.3 can fail if $\mathcal{C}$ is defined over K.

4.3 Proof of the main theorem

Although we have been working with curves defined over $K(t)$, we can extend the result further to any curve defined over $\overline{K(t)}$. The fact that Theorem 1.2.3 is true over $K(t)$ will be a key tool in the extension to $\overline{K(t)}$.

Let L/K be a function field of transcendence degree 1. We will take an element $u \in L$ which is transcendental over K, so that L is finite over $K(u)$. Suppose we take a curve $\mathcal{C}$ defined over L, which is not necessarily irreducible. We define a new curve $\mathcal{C}' = \bigcup_{\tau \in \mathcal{G}} \mathcal{C}^{\tau}$, where $\mathcal{G}$ is defined as the set $\{\tau \in \operatorname{Aut}(\overline{L}) : \tau|_{K(u)} = id_{K(u)}\}$. From Galois theory, we know that for finite extensions $L/K(u)$, there exists some inseparable extension of degree $p^m, m \geq 0$ such that $K(u) \subseteq K(u^{1/p^m}) \subseteq L$, and $L/K(u^{1/p^m})$ is separable. We take $t = u^{1/p^m}$ and obtain the following proposition.

Proposition 4.3.1. *$\mathcal{C}'$ is defined and irreducible over $K(t)$.*

Proof. As all elements in $\mathcal{G}$ fix $K(u)$, they must fix $K(t)$ as well since any p^{nth}-root of unity in characteristic p is 1. Thus, the only elements of L fixed by all automorphisms in $\mathcal{G}$ are all the elements of $K(t)$. So, $\mathcal{C}'$ is defined over $K(t)$ and is fixed by any automorphism of $\operatorname{Gal}(K(t)^{sep}/K(t))$.

Now, we wish to show that $\mathcal{C}'$ is irreducible over $K(t)$. Suppose not: then there must exist distinct irreducible curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined over $K(t)$ and proper inside $\mathcal{C}'$ such that $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{C}'$. Since we have defined $\mathcal{C}' = \bigcup_{\tau \in \mathcal{G}} \mathcal{C}^{\tau}$, we must have

$$\mathcal{C}_1 = \bigcup_{\sigma \in \mathcal{S}_1} \mathcal{C}^{\sigma} \text{ and } \mathcal{C}_1 = \bigcup_{\gamma \in \mathcal{S}_2} \mathcal{C}^{\gamma}$$

for $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{G}$. Since $\mathcal{C}_1$ is defined over $K(t)$, it follows that $\mathcal{C}_1$ is fixed by any $\alpha \in \mathcal{G}$. Then, $\mathcal{C}_1 = {\mathcal{C}_1}^{\alpha} = \bigcup_{\tau \in \mathcal{G}} \mathcal{C}^{\alpha\sigma}$.

But this means that $\mathcal{C}^{\alpha\sigma} \subseteq \mathcal{C}_1$ for any $\sigma \in \mathcal{S}_1$. If we then vary α through every element in $\mathcal{G}$, it follows that $\mathcal{C}^{\tau} \subseteq \mathcal{C}_1$ for every $\tau \in \mathcal{G}$. Hence, $\bigcup_{\tau \in \mathcal{G}} \mathcal{C}^{\tau} =$

$\mathcal{C}' \subseteq \mathcal{C}_1$. Thus, $\mathcal{C}_1$ and $\mathcal{C}'$ are equal; the same follows for $\mathcal{C}_2$, proving that $\mathcal{C}'$ is irreducible as claimed. $\square$

Finally, we show that Theorem 1.2.3 holds for curves defined over $\overline{K(t)}$. But first, as we will make use of the curve $\mathcal{C}'$ above, we must verify that $\mathcal{C}'$ is not defined over K.

Suppose that $\mathcal{C}'$ were defined over K. Let ψ_α be the automorphism of $K(t)$ such that it is the identity on K and $\psi_\alpha(t) = t+\alpha$. Recalling that $\mathcal{C}' = \bigcup_{\tau\in\mathcal{G}} \mathcal{C}^\tau$, we must have that $\mathcal{C}^{\psi_\alpha}$ conincides one of the $\mathcal{C}^\tau$, for some $\tau : \overline{L} \longrightarrow \overline{L}$, fixing $K(u)$. Since there exist infinitely many $\alpha \in K$, by the pigeonhole principle there must be *distinct* $\alpha, \beta \in K$ such that

$$\mathcal{C}^{\psi_\alpha} = \mathcal{C}^\tau = \mathcal{C}^{\psi_\beta}$$

This means that $\mathcal{C}$ is fixed by ψ_γ, which is the automorphism of $K(t)$ such that ψ_γ is the identity on K and $\psi_{\beta-\alpha}(t) = t+\gamma$, $\gamma = \beta - \alpha \neq 0$. For the curve $\mathcal{C}$ being defined by $f(X,Y) = \sum_{i,j} a_{ij}X^iY^j$, the invariance of $\mathcal{C}$ under ψ_γ may be observed in the following.

$$\begin{aligned}
\sum_{i,j}\psi_\alpha(a_{ij})X^iY^j &= \sum_{i,j}\psi_\beta(a_{ij})X^iY^j \\
\psi_{-\alpha}(\sum_{i,j}\psi_\alpha(a_{ij})X^iY^j) &= \psi_{-\alpha}(\sum_{i,j}\psi_\beta(a_{ij})X^iY^j) \\
\sum_{i,j}a_{ij}X^iY^j &= \sum_{i,j}\psi_{\beta-\alpha}(a_{ij})X^iY^j
\end{aligned}$$

Because $\mathcal{C}$ is fixed by such a map, it must be defined over the fixed field for such an automorphism, which in the case of ψ_γ, is K; but $\mathcal{C}$ has been taken to *not* be defined over K. Contradiction.

Now, let $\mathcal{C}$ be a plane curve defined over $\overline{K(t)}$, and $\mathcal{C}'$ as in Proposition 4.3.1. By this proposition and Lemma 4.2.2 we know that there exists a real number $c_0 > 0$ such that if $(x,y) \in \mathcal{C}'(\overline{K(t)})$ and $h(x,y) \leq c_0$, then $(x,y) \in$

$\mathcal{C}'(K)$. So, let $(x, y) \in \mathcal{C}(\overline{K(t)})$ with $h(x, y) \leq c_0$. Since $\mathcal{C}(\overline{K(t)}) \subseteq \mathcal{C}'(\overline{K(t)})$, we may conclude that $(x, y) \in \mathcal{C}'(K)$. Thus, we must have that $(x, y) \in \mathcal{C}(K)$ as well.

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