The Finite Field Restriction Problem

by

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Abstract

This work studies the extension problem for subsets of finite fields. This remains an important unsolved problem in harmonic analysis, in both the Euclidean and finite field setting. We survey the partial results obtained to date, common techniques, and open conjectures. In the case of a homogeneous variety H over a *d*-dimensional finite field, the $L^2 \rightarrow L^4$ boundedness is proved whenever Hcontains no hyperplanes. This is accomplished by proving an incidence theorem for cones of this type, and applying a sufficient condition for $L^2 \rightarrow L^{2m}$ obtained by Mockenhaupt and Tao in their 2004 introductory paper. We moreover present counterexamples for particular cones when r < 4, establishing the 2 to 4 bound as optimal in r for general homogeneous varieties.

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Soli Deo gloria.

Chapter 1

Introduction

In the 1960s, Elias Stein posed a question concerning Fourier transforms of functions which have been restricted to a subset of Euclidean space, which is still unsolved. The classical version of this problem can be roughly thought of as an analysis of the boundedness of the Fourier transform operator \hat{f} on \mathbb{R}^d , for different classes of functions f. If f is an L^1 function, we know from the Riemann-Lebesgue lemma that its Fourier transform \hat{f} is continuous on \mathbb{R}^d , is bounded, and vanishes at infinity [3]. If we restrict f to some subset $E \subset \mathbb{R}^d$ (and denote this restriction by $f|_E$), in this case we know $\hat{f}|_E$ to be a continuous bounded function on E.

If we consider the class of L^2 functions, on the other hand, \hat{f} need not be continuous. From Plancherel's identity, we know that \hat{f} will be an L^2 function, but it is not necessarily defined everywhere. This means we cannot generally restrict such functions to a measure zero set.

We turn our attention to the intermediary cases, where $f \in L^p$ for some 1 . It is unclear how much can be said about the behaviour of the Fourier transform in this case - the Fourier transform here need not be continuous or bounded, but we may still have some form of good decay.

The Hausdorff-Young inequality (1.6) tells us that when $f \in L^p$ with $1 , <math>\hat{f}$ lies in the dual space $L^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ (so $2). We can certainly restrict <math>\hat{f}$ to any set E of positive measure. On the other hand, if E is a set of zero measure, it is not clear what we can expect.

This leads to the central motivating question, initially formulated by Elias Stein: For which values of $p \in [1,2]$ and which sets $E \subset \mathbb{R}^d$ may the Fourier Transform of a generic $L^p(\mathbb{R}^d)$ function be restricted to E?

We can reformulate this question by seeking a restriction estimate of the following form. Let S be a compact subset of some $E \subset \mathbb{R}^d$. We seek values of $1 and <math>1 \le r \le \infty$ such that for every Schwartz function f we have

$$\left\| \hat{f} \right\|_{S} \right\|_{L^{r}(S, d\sigma)} \lesssim \| f \|_{L^{p}(\mathbb{R}^{d})}$$

Here and throughout, $A \leq B$ denotes that there exists some constant C such that $A \leq CB$. The constant here may depend only on p, r and S. We denote the surface measure on S by $d\sigma$. In the Euclidean setting this is generally the usual Lebesgue measure on the set.

If such an estimate holds, we may use the density of the class of Schwartz functions in L^2 to obtain a restriction operator from $L^p(\mathbb{R}^d, dx)$ to $L^r(S, d\sigma)$ for general functions in L^p .

By duality and Parseval's identity (see Remark 1), we may equivalently seek values of p and r such that for all smooth functions f on S we have

$$\|(fd\sigma)\|_{L^{p'}(\mathbb{R}^d,dx)} \lesssim \|f\|_{L^{r'}(S,d\sigma)}.$$

This formulation of the problem is often referred to as the extension problem, as we take a function defined on S and attempt to extend the inverse Fourier transform to \mathbb{R}^d . This has become the more common expression of the problem in recent years, owing to numerous applications to partial differential equations.

Stein showed that such estimates do exist for non-trivial values of p and r, when the subset $E \in \mathbb{R}^d$ is a surface with sufficient curvature. The classical case of the restriction problem considers the sphere in \mathbb{R}^d . Partial results are known for a variety of sets, most notably spheres, cones and paraboloids. It is conjectured that an estimate exists for spheres and paraboloids if and only if

$$p < \frac{2d}{d+1}$$
, and $\frac{(d+1)}{p'} < \frac{d-1}{r}$

This is known when d = 2, but remains open for higher dimensions. The best known result to date is due to Tao [16]. The bilinear version of this result has been obtained by Wolff [20]. The necessary and sufficient conditions for cones are conjectured to be

$$p < \frac{2d-1}{d}, \ \frac{d}{p'} < \frac{d-2}{r}.$$

In 2003, Gerd Mockenhaupt and Terence Tao initiated a second phase of work on this question, which relocates the base field from *d*-dimensional Euclidean space to a vector space over a finite field in *d*-dimensions. The finite field setting provides us with a model of the Euclidean case that is more accessible in some respects. The finite field version of the problem is additionally connected to existing problems in algebraic geometry, combinatorics and additive number theory, so is of interest in its own right. The restriction problem is suspected to be more tractable in the finite setting, as we have access to the discrete methods offered by these fields. On the other hand, several techniques native to the Euclidean case are lost, for example Taylor expansions and dyadic scalings. In 2008, a similar adaptation of the Kakeya problem to finite fields was successfully solved by Zeev Dvir, using a polynomial method [2].

The Euclidean Kakeya conjecture is known to imply the Euclidean restriction conjecture. This does not hold it the finite case, and the finite field restriction problem has only been solved for particular low-dimension cases, as with its Euclidean counterpart. It is thought that further developments in the finite field setting will assist in our understanding of the Euclidean case.

In what follows we will express the finite field version of the problem formally, survey known methods and results, and present some new results for the cone in finite fields. We begin by introducing some basic notation and definitions for our problem.

1.1 Notation and Definitions in the Finite Field

Let \mathbb{F} be a finite field with q elements, and let d be the dimension of space. We write \mathbb{F}^* for the multiplicative group $\mathbb{F} \setminus \{\mathbf{0}\}$, and \mathbb{F}_* for the dual space of \mathbb{F} . We denote the cardinality of a set E in \mathbb{F}^d by |E|. For any abelian finite group G, we call a function $\chi : G \to \{z \in \mathbb{C} : |z| = 1\}$ a character if it is homomorphic; i.e. $\chi(g_1 \cdot g_2) = \chi(g_1)\chi(g_2)$, for all $g_i \in G$.

The set of all characters on G is itself an abelian group of size |G|, with respect to multiplication. We denote the group of characters over G by G^{\wedge} . Since we are working over a field, we can consider characters over the additive group \mathbb{F}^+ and over the multiplicative group \mathbb{F}^* . We generally denote additive and multiplicative characters by χ and ψ , respectively. Relevant results on characters are detailed in the following section.

Let f(x) be a complex-valued function on \mathbb{F}^d . Over finite fields integrals are replaced by finite sums, so that when dx is a counting measure we have

$$\int_E f(x)dx = \sum_{x \in E} f(x).$$

We consider complex valued functions f(x) defined on \mathbb{F}^d . In this context, we define the Fourier transform of f as

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)\chi(-x\cdot\xi)dx = \sum_{x\in\mathbb{R}^d} f(x)\chi(-x\cdot\xi).$$

Here dx is the counting measure on \mathbb{F}^d , $x \cdot \xi$ denotes the inner product $x \cdot \xi = \sum_{i=1}^d x_i \xi_i$, and χ is some non-trivial additive character (we may think of this as the equivalent of the function $e^{2\pi i x}$ in the Euclidean case). Note that \hat{f} is defined on the dual space of \mathbb{F}^d . We shall denote the dual space by \mathbb{F}^d_* , and equip it with the normalized counting measure $d\xi$. Integration over the dual space becomes

$$\int_{\mathbb{F}^d_*} g(\xi) d\xi := \frac{1}{q^d} \sum_{\xi \in \mathbb{F}^d_*} g(\xi).$$

As in the continuous case, we have an inversion formula for the Fourier transform:

$$f^{\check{}}(x) := \int_{\mathbb{F}^d_*} f(\xi)\chi(x\cdot\xi)d\xi = \frac{1}{q^d}\sum_{\xi\in\mathbb{F}^d_*} f(\xi)\chi(x\cdot\xi).$$

For $1 \leq p < \infty$ the L^p norm of f over \mathbb{F}^d is given by

$$||f||_{L^p(\mathbb{F}^d, dx)} = \left(\int_{\mathbb{F}^d} |f(x)|^p dx\right)^{\frac{1}{p}} = \left(\sum_{x \in \mathbb{F}^d} |f(x)|^p\right)^{\frac{1}{p}}.$$

When $p = \infty$, we have $||f||_{L^{\infty}(\mathbb{F}^d, dx)} = \sup_{x \in \mathbb{F}^d} |f|$. Note that L^p spaces are nested over fields of finite measure, as we will prove in the following section (Theorem 1.3.2).

Let S be a surface in \mathbb{F}^d . We endow S with a normalized surface measure $d\sigma$,

given by

$$d\sigma(x) = \frac{q^d}{|S|} \mathbf{1}_S dx,$$

where $\mathbf{1}_S$ is the characteristic function of S. This normalizes our integration such that the surface has total mass equal to one.

For a function f restricted to S, this yields

$$||f||_{L^p(S,d\sigma)} = \left(\int_S |f(x)|^p d\sigma(x)\right)^{\frac{1}{p}} = \left(\frac{1}{|S|} \sum_{x \in S} |f(x)|^p\right)^{\frac{1}{p}}.$$

and the corresponding inverse transform

$$(fd\sigma)\,\check{}\,(\xi) = \int_S \chi(\xi \cdot x) f(x) d\sigma(x) = \frac{1}{|S|} \sum_{x \in S} \chi(\xi \cdot x) f(x).$$

We express the restriction problem in this context as follows: given a subset E of \mathbb{F}^d , for which values of $1 \leq p, r \leq \infty$ do we have an estimate of the form

$$\|(fd\sigma)\|_{L^r(\mathbb{F}^d,d\xi)} \lesssim \|f\|_{L^p(S,d\sigma)},\tag{1.1}$$

independent of p and r, for every complex-valued function f on \mathbb{F}^d and every subset S of E. By duality (Remark 1), we can equivalently express this as the estimate

$$\left\| \hat{f} \right\|_{L^{p'}(S,d\sigma)} \lesssim \|f\|_{L^{r'}(\mathbb{F}^d,d\xi)}$$
(1.2)

where p' and r' are again used to denote dual (conjugate) exponents.

Let $R(p \to r)$ denote the smallest constant for which (1.1) holds. Note by duality that this is also the best constant under which (1.2) holds. The finite field restriction problem then seeks the values of p and r for which we have $R(p \to r) \leq 1$.

1.2 Characters and Gauss Sums

Over finite fields, the exponential function in the Fourier transform is replaced by a non-trivial character - we will often rely on results on character sums in order to obtain bounds on the norm of \hat{f} . We collect some basic definitions and relevant results on characters here. For proofs of these results, and further discussion of exponential sums, see [12]. For any character over a finite abelian group (G, \cdot) , we have $\chi(x)\chi(y) = \chi(x \cdot y)$, for all $x, y \in G$. Let 1_G denote the identity in (G, \cdot) . When $\chi(x) := 1_G$ for all $x \in G$, we call χ the trivial character (often denoted χ_0). For every character we have $\chi(1_G) = 1$, since $\chi(1_G)\chi(x) = \chi(1_G \cdot x) = \chi(x)$.

If G is a cyclic abelian group, with g as a generator, we have characters of the form $\chi_j(g^k) := e^{\frac{2\pi i jk}{q}}$ for every integer $0 \le j < q$. In fact, in this case G^{\wedge} consists exactly of the characters in this form.

We call the smallest positive integer k such that $(\chi(x))^k = 1$ for all x the order of χ . Since G is finite, note that for all $\chi \in G^{\wedge}$ and $x \in G$ we have $(\chi(x))^{|G|} = \chi(x^{|G|}) = \chi(1_G) = 1$, so every character is of finite order. When $k = 2, \chi$ is said to be a quadratic character.

Let \wp be the characteristic of the finite field \mathbb{F} . Then the prime field \mathbb{F}_{\wp} of size \wp (which is unique up to isomorphism) is contained in \mathbb{F} . Let Tr be the trace function from \mathbb{F} to \mathbb{F}_{\wp} . Since the trace function is an additive homomorphism, the function

$$\chi_1(x) := e^{\frac{2\pi i \operatorname{Tr}(x)}{\wp}}$$

is a character of \mathbb{F}^+ . We call this the canonical additive character.

Lemma 1.2.1 (Basic Character Sums). Let G be a finite abelian group, and let G^{\wedge} be the group of all characters over G. Then we have

$$\sum_{x \in G} \chi(x) = \begin{cases} 0 & \text{if } \chi \text{ is non-trivial} \\ q & \text{if } \chi = \chi_0 \end{cases}$$

and

$$\sum_{\zeta \in G^{\wedge}} \chi(x) = \begin{cases} 0 & \text{if } x \neq 1_G \\ q & \text{if } x = 1_G \end{cases}$$

Proof. First, note that when $\chi = \chi_0$, we have $\sum_{x \in G} \chi_0(x) = \sum_{x \in G} 1$, and when $x = 1_G$, $\sum_{\chi \in G^{\wedge}} \chi(1_G) = \sum_{\chi \in G^{\wedge}} 1 = |G|$.

If $\chi \neq \chi_0$, then there exists some $y \in G$ such that $\chi(y) \neq 1$. But

$$\chi(y)\sum_{x\in G}\chi(x) = \sum_{x\in G}\chi(y)\chi(x) = \sum_{x\in G}\chi(yx) = \sum_{x\in G}\chi(yx) = \sum_{x\in G}\chi(x),$$

so $\chi(y) \neq 1$ implies $\sum_{x \in G} \chi(x) = 0$.

Now, for a fixed non-identity $x \in G$, define the function \hat{x} such that $\hat{x}(\chi) = \chi(x)$, for all $\chi \in G^{\wedge}$. This is a non-trivial character on G^{\wedge} , since $\hat{x}(\chi)\hat{x}(\chi') = \chi(x)\chi'(x) = \chi\chi'(x) = \hat{x}(\chi\chi')$, and there exists some χ for which $\chi(x) \neq 1$. Then applying previous result, we have $\sum_{\chi \in G^{\wedge}} \chi(x) = \sum_{\chi \in G^{\wedge}} \hat{x}(\chi) = 0$.

An application of these sums yields the following orthogonality relations:

Theorem 1.2.2 (Orthogonality Relations for Characters). Let χ_a, χ_b be multiplicative characters over finite abelian group G. Then for every $c, d \in G$, we have the following relations:

1.
$$\sum_{c \in G} \chi_a(c) \overline{\chi_b(c)} = \begin{cases} 0 & \text{if } a \neq b \\ |G| & \text{if } a = b \end{cases}$$

2. If $a \neq 0$ (i.e. χ_a is non-trivial), then $\sum_{c \in F^*} \chi_a(c) = 0$

3.
$$\sum_{\chi \in G^{\wedge}} \chi(c)\overline{\chi(d)} = \begin{cases} 0 & \text{if } c \neq d \\ |G| & \text{if } c = d \end{cases}$$

Since we are working over a field, it will often be useful to consider sums over the product of multiplicative and additive characters. For an additive character χ and multiplicative character ψ , a sum of the form $\sum_{x \in \mathbb{F}^*} \chi(x)\psi(x)$ is called a Gaussian (or Gauss) sum. The above relations allow us to evaluate this sum as follows.

Theorem 1.2.3. Let $G(\chi, \psi) = \sum_{x \in \mathbb{F}^*} \chi(x) \psi(x)$ be a Gauss sum. Then G evaluates to

$$G(\chi, \psi) = \begin{cases} |\mathbb{F}| - 1 & \text{if } \chi = \chi_0, \psi = \psi_0 \\ -1 & \text{if } \chi = \chi_0, \psi \neq \psi_0 \\ 0 & \text{if } \chi \neq 0, \psi = \psi_0 \end{cases}$$

We also note that $|G(\chi, \psi)| \leq |\mathbb{F}| - 1$ for every choice of characters χ and ψ . If $\chi \neq \chi_0, \psi \neq \psi_0$, we have $|G(\chi, \psi)| = |\mathbb{F}|^{\frac{1}{2}}$.

When η is a quadratic multiplicative character, for every $y\in\mathbb{F}$ we have the Gauss sum

$$G_y(\chi,\eta) = \sum_{x \in \mathbb{F}^d} \chi(xy) \eta(x)$$

with size

$$|G_y(\chi,\eta)| = \begin{cases} q^{\frac{1}{2}} & \text{if } y \neq 0\\ 0 & \text{if } y = 0 \end{cases}$$

When y = 1, we obtain the following character sum:

Theorem 1.2.4. Let η be a quadratic multiplicative character, and let χ be the canonical additive character on \mathbb{F} . Then for every $a \in \mathbb{F}^*$ we have character sum

$$\sum_{s \in \mathbb{F}} \chi(as^2) = \eta(a)G_1(\eta, \chi)$$

Theorem 1.2.5. Let η and χ be quadratic and canonical characters respectively, on F^d . Let $t \in F$, and $\alpha, \beta \in \mathbb{F}^d$. Then

$$\sum_{\alpha \in \mathbb{F}^k} \chi(t\alpha \cdot \alpha + \beta \cdot \alpha) = \chi(\frac{\|\beta\|_2}{-4t})\eta^k(t)(G(\eta, \chi))^k$$
(1.3)

Theorem 1.2.6. Let ψ be a non-trivial additive character over \mathbb{F} , then

$$\left|\sum_{\xi,\eta\in\mathbb{F}\backslash\mathbf{0}}\psi(x\xi^2)\psi(-x\eta^2)\right| = |\mathbb{F}|.$$
(1.4)

1.3 Basic Inequalities and Identities

The finite field setting retains many of the inequalities and identities we make use of in the continuous case. The proofs in this setting often become simple manipulations of sums and applications of the orthogonality relation for characters. We collect some of these results here.

Lemma 1.3.1. Let f, g be complex-valued functions on \mathbb{F}^d . Then we have

- (i) Parseval's identity: $\int_{\mathbb{F}^d} f(x)\overline{g(x)}dx = \int_{\mathbb{F}^d_*} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi$
- (ii) Plancherel's identity: $\|f\|_{L^2} = \left\|\hat{f}\right\|_{L^2}$

Proof. (i) Writing out the right hand side in full and rearranging the integrals

yields

$$\begin{split} \int_{\mathbb{F}^d_*} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi &= \int_{\mathbb{F}^d_*} \int_{\mathbb{F}^d} f(x)\chi(-x\cdot\xi)dx \int_{\mathbb{F}^d} \overline{g(y)}, \chi(y\cdot\xi)dyd\xi \\ &= \int_{\mathbb{F}^d} \int_{\mathbb{F}^d} f(x)\overline{g(y)} \int_{\mathbb{F}^d_*} \chi(-x\cdot\xi)\chi(y\cdot\xi)d\xi dxdy \\ &= \int_{\mathbb{F}^d} \int_{\mathbb{F}^d} f(x)\overline{g(y)}\delta_0(x-y)dydx \qquad (1.5) \\ &= \int_{\mathbb{F}^d} \int_{\mathbb{F}^d} f(x)\overline{g(x)}dx \end{split}$$

where (1.5) is from Lemma 1.2.1.

(ii) Set
$$g = f$$
 in (i).

We retain Hölder's inequality, which states for all $1\leq p,p'\leq\infty$ such that $\frac{1}{p}+\frac{1}{p'}=1$ we have

$$\|fg\|_{L^1} \le \|f\|_{L^p} \, \|g\|_{L^{p'}}$$

or

$$\left(\sum_{x \in \mathbb{F}^d} |f(x)g(x)|\right) \le \left(\sum_{x \in \mathbb{F}^d} |f(x)|^p\right)^{\frac{1}{p}} \left(\sum_{x \in \mathbb{F}^d} |g(x)|^{p'}\right)^{\frac{1}{p'}}$$

for all functions f and g on which this is well-defined. For a proof of this on generic measure spaces, see [14].

From this one can prove Minkowski's inequality, which establishes the triangle inequality for L^p norms. For any $1 \le p \le \infty$ we have

$$\|f+g\|_p \le \|f\|_p + \|f\|_p$$

We also have the Hausdorff-Young inequality

$$\left\| \hat{f} \right\|_{L^{p'}} \le \|f\|_{L^p}, \text{ where } f \in L^p.$$

$$(1.6)$$

We define the convolution of two complex-valued functions f, g on \mathbb{F}^d as follows:

$$(f*g)(x) := \int_{\mathbb{R}^d} f(y)g(x-y)dy = \sum_{y \in \mathbb{R}^d} f(y)g(x-y).$$

Note that convolution is symmetric with respect to x and y. Moreover, we

calculate

$$\begin{split} \widehat{f * g}(\xi) &= \int_{\mathbb{F}^d} \left(\int_{\mathbb{F}^d} f(y) \overline{g(x - y)} dy \right) \chi(-x \cdot \xi) dx \\ &= \int_{\mathbb{F}^d} f(y) \int_{\mathbb{F}^d} \overline{g(x - y)} \chi((-x + y - y) \cdot \xi) dx dy \\ &= \int_{\mathbb{F}^d} f(y) \chi(-y \cdot \xi) \left(\int_{\mathbb{F}^d} \overline{g(x - y)} \chi(-(x - y) \cdot \xi) dx \right) dy \\ &= \int_{\mathbb{F}^d} f(y) \chi(-y \cdot \xi) \hat{g}(\xi) dy \\ &= \hat{f}(\xi) \hat{g}(\xi) \end{split}$$

so we have

$$\widehat{(f*g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi) \tag{1.7}$$

and

$$\widehat{fg}(\xi) = (\widehat{f} * \widehat{g})(\xi)$$

It is also straightforward to calculate the inversion formula. We define the inverse Fourier transform of $g: \mathbb{F}^d_* \to \mathbb{C}$ so that $g^{\check{}}(x) = \int_{\mathbb{F}^d_*} g(\xi) \chi(x \cdot \xi) d\xi$, then we have $(\hat{f})^{\check{}}(x) = f(x)$, for every $x \in \mathbb{F}^d$.

Over finite fields we have a monotonic nesting of L^p norms. This is an important distinction between the Euclidean and finite settings which we will make use of to develop some simple restriction results in Section 2.1.

Theorem 1.3.2 (Nesting of L^p -spaces). Suppose $1 \le p_1 \le p_2 \le \infty$. Let μ_x and μ_{σ} be a simple counting measure and a normalized counting measure respectively, on the finite space \mathbb{F}^d . Then

- (1) $||f||_{L^{p_2}(\mathbb{F}^d,\mu_x)} \le ||f||_{L^{p_1}(\mathbb{F}^d,\mu_x)}$
- (2) $||f||_{L^{p_1}(\mathbb{F}^d,\mu_\sigma)} \le ||f||_{L^{p_2}(\mathbb{F}^d,\mu_\sigma)}$

so we have $L^{p_1}(\mathbb{F}^d, \mu_x) \subset L^{p_2}(\mathbb{F}^d, \mu_x)$ and $L^{p_2}(\mathbb{F}^d, \mu_\sigma) \subset L^{p_1}(\mathbb{F}^d, \mu_\sigma)$.

Proof. (1) Let f be such that $||f||_{L^{p_1}(\mathbb{F}^d,\mu_x)} = 1$. It suffices to show that

 $||f||_{L^{p_2}(\mathbb{F}^d,\mu_x)} \leq 1$ in this case. We have $||f||_{L^{\infty}} \leq 1$, and

$$\begin{split} \|f\|_{L^{p_2}} &= \left(\int_{\mathbb{F}^d} |f|^{p_2} d\mu_x\right)^{\frac{1}{p_2}} \\ &= \left(\sum_{x \in \mathbb{F}^d} |f(x)|^{p_2}\right)^{\frac{1}{p_2}} \\ &= \left(\sum_{\mathbb{F}^d} |f|^{p_1} |f|^{p_2 - p_1} d\mu_x\right)^{\frac{1}{p_2}} \\ &\leq \left(\|f^{p_2 - p_1}\|_{L^{\infty}(\mathbb{F}^d, \mu_x)} \sum_{x \in \mathbb{F}^d} |f|^{p_1}\right)^{\frac{1}{p_2}} \\ &= \|f\|_{L^{\infty}}^{\frac{p_2 - p_1}{p_2}} \|f\|_{L^{p_1}}^{\frac{p_1}{p_2}} \\ &\leq \|f\|_{L^{p_1}}^{\frac{p_1}{p_2}} = 1 \end{split}$$

(2) Since μ_{σ} is a normalized measure, we have $\|1\|_{L^p} = 1$, for all $1 \leq p \leq \infty$. Applying Hölder's inequality to the L^{p_1} norm, we see that

 $\|$

$$\begin{split} f\|_{L^{p_1}} &= \left(\int_{\mathbb{F}^d} |f(x)|^{p_1} d\mu_{\sigma} \right)^{\frac{1}{p_1}} \\ &= \left(\int_{\mathbb{F}^d} 1 \cdot |f|^{p_1} d\mu_{\sigma} \right)^{\frac{1}{p_1}} \\ &\leq \left(\left(\int_{\mathbb{F}^d} 1^{\frac{p_2}{p_2 - p_1}} d\mu_{\sigma} \right)^{\frac{p_2 - p_1}{p_2}} \left(\int_{\mathbb{F}^d} |f|^{p_1 \frac{p_2}{p_1}} \mu_{\sigma} \right)^{\frac{p_1}{p_2}} \right)^{\frac{1}{p_1}} \\ &= |1||_{L^{p_1}}^{\frac{p_2 - p_1}{p_2}} \left(\int_{\mathbb{F}^d} |f|^{p_2} d\mu_{\sigma} \right)^{\frac{1}{p_2}} \\ &= \|f\|_{L^{p_2}}, \text{ since we have a normalized measure.} \end{split}$$

Remark 1. If we define the operator T from functions on \mathbb{F}^d to those on E such that $Tf := \hat{f}|_E$, then the adjoint operator T^* is given by the extension map $T^* : g \mapsto (gd\sigma)^{\check{}}$. This follows easily from Parseval's identity — we note that $\langle f, T^*g \rangle = \int_{\mathbb{F}^d} f(\overline{gd\sigma})^{\check{}} = \int_{\mathbb{F}^d} \hat{f}(\overline{gd\sigma}) = \int_{\mathbb{F}^d} \hat{f}|_E \overline{gd\sigma} = \langle Tf, g \rangle$. This

establishes the equivalence of our dual statements (1.1) and (1.2), and will also allow us to develop the Tomas-Stein argument for finite fields in Section 2.3.

1.3.1 Riesz-Thorin interpolation

Riesz-Thorin interpolation provides one of the principal tools for developing ranges of exponents p and r. When two estimates are known for a given subset E, we may interpolate between the two sets of exponents in order to produce a range of p and r for which (1.2) will hold. Riesz-Thorin Interpolation is a result on generic linear operators, and we present it in its general form before turning to our particular case. A proof can be found in [8].

Theorem 1.3.3 (Riesz-Thorin Interpolation). Let $1 \le p_1, p_2, r_1, r_2 \le \infty$. For all $0 < \theta < 1$ we define p, r such that

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2} \quad and \quad \frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}$$

Let $T: (L^{p_1} + L^{p_2}) \rightarrow (L^{r_1} + L^{r_2})$ be a linear operator such that

$$||Tf||_{L^{r_1}} \le C_1 ||f||_{L^{p_1}}$$
 for every $f \in L^{p_1}$

and

$$||Tf||_{L^{r_2}} \leq C_2 ||f||_{L^{p_2}}$$
 for every $f \in L^{p_2}$

then

$$||Tf||_{L^r} \le C_1^{1-\theta} C_2^{\theta} ||f||_{L^p}$$
 for every $f \in L^p$.

1.3.2 Young's inequality

We can bound the convolution of f and g by norms of the original functions according to the following inequality. We will make use of this fact to develop Tomas-Stein type results in Section 2.3.

Theorem 1.3.4 (Young's inequality). Let $f \in L^p(\mathbb{F}^d)$, $g \in L^r(\mathbb{F}^d)$ be complexvalued functions. Let s be such that

$$\frac{1}{p} + \frac{1}{r} = \frac{1}{s} + 1.$$

then we have

$$\|f * g\|_{L^{s}(\mathbb{F}^{d}, dx)} \le \|f\|_{L^{p}(\mathbb{F}^{d}, dx)} \|g\|_{L^{r}(\mathbb{F}^{d}, dx)}$$

Proof. Let $f \in L^p(\mathbb{F}^d)$. The inequality is easily obtained via Reisz-Thorin interpolation of the operator given by convolution with f. We define T such that Tg := f * g, and interpolate the following two bounds:

(i) Let $g \in L^{p'}(\mathbb{F}^d)$. Then $f * g \in L^{\infty}$ and

$$||Tg||_{L^{\infty}(\mathbb{F}^d, dx)} \le C_1 ||g||_{L^{p'}(\mathbb{F}^d, dx)}$$

(ii) Let $g \in L^1(\mathbb{F}^d)$. Then $f * g \in L^p(\mathbb{F}^d)$ and

$$||Tg||_{L^{p}(\mathbb{F}^{d},dx)} \leq C_{2} ||g||_{L^{1}(\mathbb{F}^{d},dx)},$$

where $C_1 = C_2 = ||f||_{L^p(\mathbb{F}^d, dx)}$. Riesz-Thorin interpolation yields

$$\|Tg\|_{L^{s}(\mathbb{F}^{d},dx)} \leq C_{1}^{\theta}C_{2}^{1-\theta}\|g\|_{L^{r}(\mathbb{F}^{d},dx)} = \|f\|_{L^{p}(\mathbb{F}^{d},dx)}\|g\|_{L^{r}(\mathbb{F}^{d},dx)}$$

for any $\theta \in (0, 1)$, where r and s are given by

$$\frac{1}{r}=\frac{1-\theta}{1}+\frac{\theta}{p'}=1-\frac{\theta}{p}$$

and

$$\frac{1}{s} = \frac{1-\theta}{p} + \frac{\theta}{\infty} = \frac{1-\theta}{p}$$

Let $\theta = 1 - \frac{p}{s}$. Note that $\frac{p}{s} > 0$, so $1 - \frac{p}{s} < 1$. If $\frac{1}{s} = \frac{1}{p} + \frac{1}{r} - 1$, we have $0 < 1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{s}$, so p < s and $0 < 1 - \frac{p}{s}$. Thus we have

$$||f * g||_{L^{s}(\mathbb{F}^{d}, dx)} \le ||f||_{L^{p}(\mathbb{F}^{d}, dx)} ||g||_{L^{r}(\mathbb{F}^{d}, dx)}$$

with $\frac{1}{r} = 1 - \frac{1 - \frac{p}{s}}{p} = 1 - \frac{1}{p} + \frac{1}{s}$, so $\frac{1}{p} + \frac{1}{r} = \frac{1}{s} + 1$, as required.

It remains to show the bounds (i) and (ii). Proof of (i): Let $f \in L^p(\mathbb{F}^d)$, and $g \in L^{p'}(\mathbb{F}^d)$. Then

$$|f * g| = |\int f(x - y)g(y)dy|$$

$$\leq \int |f(x - y)||g(y)|dy$$

$$\leq ||f||_{L^{p}(\mathbb{F}^{d}, dx)} ||g||_{L^{p'}(\mathbb{F}^{d}, dx)}$$
(1.8)

where (1.8) is from Hölder's inequality. It follows that also

$$\|Tg\|_{L^{\infty}(\mathbb{F}^{d},dx)} \|f * g\|_{L^{\infty}(\mathbb{F}^{d},dx)} \le \|f\|_{L^{p}(\mathbb{F}^{d},dx)} \|g\|_{L^{p'}(\mathbb{F}^{d},dx)}$$

Proof of (ii): $\overline{\text{Let } f \in L^p(\mathbb{F}^d)} \text{ and } g \in L^1(\mathbb{F}^d).$ We have $||f * g||_{L^p(\mathbb{F}^d, dx)}^p = \int |\int f(y)g(x-y)dy|^p dx$, and

$$\begin{split} |\int f(y)g(x-y)dy| &\leq \int |f(y)|^{\frac{1}{p'}}|f(y)|^{\frac{1}{p}}||g(x-y)|dy\\ &\leq (\int |f(y)|dy)^{\frac{1}{p'}}(\int |f(y)||g(x-y)|^{p})^{\frac{1}{p}} \qquad \text{(H\"older)}\\ &= \|f\|^{\frac{1}{p'}}_{L^{1}(\mathbb{F}^{d},dx)}(\int |f(y)||g(x-y)|^{p}dy)^{\frac{1}{p}} \end{split}$$

so by Fubini (which amounts to changing the order of summation), we have

$$\begin{split} \|Tg\|_{L^{p}(\mathbb{F}^{d},dx)}^{p} &= \|f*g\|_{L^{p}(\mathbb{F}^{d},dx)}^{p} \leq \int_{\mathbb{F}^{d}} \|f\|_{L^{1}}^{\frac{p}{p'}} \int_{\mathbb{F}^{d}} |f(y)||g(x-y)|^{p} dy dx \\ &= \|f\|_{L^{1}(\mathbb{F}^{d},dx)}^{\frac{p}{p'}} \int \int |f(y)||g(x-y)|^{p} dy dx \\ &= \|f\|_{L^{1}(\mathbb{F}^{d},dx)}^{\frac{p}{p'}} \int |f(y)| \left(\int |g(x-y)|^{p} dx\right) dy \\ &\leq \|f\|_{L^{1}(\mathbb{F}^{d},dx)}^{\frac{p}{p'}} \|f\|_{L^{1}(\mathbb{F}^{d},dx)} \|g\|_{L^{p}(\mathbb{F}^{d},dx)}^{p} \end{split}$$

and

 $\begin{aligned} \|f * g\|_{L^{p}(\mathbb{F}^{d}, dx)} &\leq \|f\|_{L^{1}(\mathbb{F}^{d}, dx)}^{\frac{1}{p'}} \|f\|_{L^{1}(\mathbb{F}^{d}, dx)}^{\frac{1}{p}} \|g\|_{L^{p}(\mathbb{F}^{d}, dx)} = \|f\|_{L^{1}(\mathbb{F}^{d}, dx)} \|g\|_{L^{p}(\mathbb{F}^{d}, dx)}, \\ \text{as required. This completes the proof.} \end{aligned}$

Chapter 2

Progress Thus Far

2.1 Some Trivial Estimates

A simple application of the results in Section 1.3 yields certain bounds immediately, for any choice of the subset E. When $r = \infty$ it follows from $|\chi(x)| = 1$ that

$$\left\| (fd\sigma)^{\check{}} \right\|_{L^{\infty}(\mathbb{F}^{d}_{*},d\sigma)} = \max_{\xi} \left| \frac{1}{|S|} \sum_{x \in S} \chi(\xi \cdot x) f(x) \right| \le \|f\|_{L^{1}(\mathbb{F}^{d},d\xi)}$$

so we have $R(1 \to \infty)$ for any subset of \mathbb{F}^d . When r = 2, we use Plancherel to calculate

$$\begin{split} \|f\|_{L^{2}(\mathbb{F}^{d}, dx)} &= \|\hat{f}\|_{L^{2}(\mathbb{F}^{d}_{*}, d\xi)} \\ &= \left(\frac{1}{|\mathbb{F}|^{d}} \sum_{\xi \in S} |\hat{f}|^{2}\right)^{\frac{1}{2}} \\ &= \left(\frac{|S|}{|\mathbb{F}|^{d}}\right)^{\frac{1}{2}} \left(\frac{1}{|S|} \sum_{\xi \in S} |\hat{f}|^{2}\right)^{\frac{1}{2}} \\ &= \left(\frac{|S|}{|\mathbb{F}|^{d}}\right)^{\frac{1}{2}} \|\hat{f}\|_{L^{2}(S, d\sigma)}. \end{split}$$

From our nesting result in Theorem 1.3.2, this yields

$$\|\hat{f}\|_{L^{p'}(S,d\sigma)} \le \|\hat{f}\|_{L^{2}(S,d\sigma)} = \left(\frac{|\mathbb{F}|^{d}}{|S|}\right)^{\frac{1}{2}} \|f\|_{L^{2}(\mathbb{F}^{d},dx)}$$

for all $p' \leq 2$, which corresponds to $2 \leq p \leq \infty$. Hence we have the bound $R(p \to 2) \leq \left(\frac{|\mathbb{F}|^d}{|S|}\right)^{\frac{1}{2}}$ for every $2 \leq p \leq \infty$. Using $f(x) \equiv 1$ as a test function, we moreover see that $R(p \to 2) = \left(\frac{|\mathbb{F}|^d}{|S|}\right)^{\frac{1}{2}}$ for all such p. Next, suppose $1 \leq p_1 \leq p_2 \leq \infty$, and fix $1 \leq r \leq \infty$. Let C be such that

$$\|(gd\sigma)^{\check{}}\|_{L^r(\mathbb{F}^d,dx)} \le C \|g\|_{L^{p_1}(S,d\sigma)}.$$

Then from the nesting of L^p spaces with respect to a normalized measure, we have

$$\|(gd\sigma)\check{}\|_{L^r(\mathbb{F}^d,dx)} \le C \|g\|_{L^{p_2}(S,d\sigma)}$$

 \mathbf{SO}

$$R(p_2 \to r) \le R(p_1 \to r), \text{ for all } \le p_1 \le p_2 \le \infty.$$
(2.1)

Similarly, letting $1 \leq r_1 \leq r_2$, the monotonicity of $L^p(\mathbb{F}^d, dx)$ norms yields

$$R(p \to r_2) \le R(p \to r_1), \text{ for all } 1 \le p \le \infty.$$
(2.2)

Once a restriction bound is established, we can apply Riesz-Thorin interpolation (Theorem 1.3.3) with the trivial bound $R(1 \rightarrow \infty) \lesssim 1$ in order to obtain a set of acceptable exponents. This strategy combined with the above monotonicity result establishes a range of exponents for which the extension estimate holds.

2.2**Necessary Conditions**

Let $S \subset (\mathbb{F}^d, dx)$ be an algebraic variety of size $|S| \sim q^k$ for some 0 < k < d(in finite fields, we think of this as the dimension of S in \mathbb{F}^d). When $r \geq 2$, Mockenhaupt and Tao show that $R(p \rightarrow r) \lesssim 1$ can hold only if

$$r \ge \frac{2d}{k}$$
 , and (2.3)

$$r \ge \frac{dp'}{k} = \frac{dp}{k(p-1)}.\tag{2.4}$$

Proof. We will use the trivial result (2.1). Let $r_1 \leq r_2$. Then $r'_2 \leq r'_1$, so the

nesting of normalized L^p spaces gives

$$||g||_{L^{r'_{2}}(\mathbb{F}^{d},d\xi)} \le ||g||_{L^{r'_{1}}(\mathbb{F}^{d},d\xi)},$$

that is,

$$\begin{split} \left(\frac{1}{|\mathbb{F}|^d} \int |g|^{r'_2}\right)^{\frac{1}{r'_2}} &\leq \left(\frac{1}{|\mathbb{F}|^d} \int |g|^{r'_1}\right)^{\frac{1}{r'_1}} \\ &\left(\int |g|^{r'_2}\right)^{\frac{1}{r'_2}} \leq |\mathbb{F}|^{d(\frac{1}{r'_2} - \frac{1}{r'_1})} \left(\int |g|^{r'_1}\right)^{\frac{1}{r'_1}} \\ &= |\mathbb{F}|^{d(\frac{1}{r_1} - \frac{1}{r_2})} \left(\int |g|^{r'_1}\right)^{\frac{1}{r'_1}}. \end{split}$$

which implies

$$R(p \to r_1) \le q^{d(\frac{1}{r_1} - \frac{1}{r_2})} R(p \to r_2), \text{ for all } 1 \le r_1 \le r_2 \le \infty.$$
(2.5)

Lastly we use our earlier calculation that

$$R(p \to 2) = \left(\frac{q^d}{|S|}\right)^{\frac{1}{2}}, \text{ when } 2 \le p \le \infty.$$
(2.6)

If $r \ge 2$, taking $r_1 = 2$ and $r_2 = r$ in (2.5) and moving the constant gives

$$R(p \to r) \ge q^{-d(\frac{1}{2} - \frac{1}{r})}R(p \to 2).$$

From (2.6), this yields

$$q^{-d(\frac{1}{2} - \frac{1}{r})} R(p \to 2) \ge q^{-d(\frac{1}{2} - \frac{1}{r})} R(\infty \to 2)$$
$$= q^{-d(\frac{1}{2} - \frac{1}{r})} \left(\frac{q^d}{|S|}\right)^{\frac{1}{2}}$$

Therefore $R(p \to r) \ge q^{-d(\frac{1}{2} - \frac{1}{r}) + \frac{d}{2}} |S|^{-\frac{1}{2}} = q^{\frac{d}{r}} |S|^{-\frac{1}{2}}$. By assumption $|S| \sim q^k$, so we have

$$R(p \to r) \ge q^{\frac{d}{r} - \frac{k}{2}}$$

but we require $R(p \to r) \leq C(p, r, d)$, so we must have $\frac{d}{r} - \frac{k}{2} \leq 0$, which rearranges to the necessary condition $r \geq \frac{2d}{k}$, as claimed.

We obtain the second necessary condition by producing a potential counterexample. Let f(x) be the indicator function $\mathbf{1}_{\eta}$, for some $\eta \in S$. Then (1.1) becomes

$$\left\| \left(\mathbf{1}_{\eta} d\sigma \right)^{\check{}} \right\|_{L^{r}(\mathbb{F}^{d}, dx)} \leq R(p \to r) \left\| \mathbf{1}_{\eta} \right\|_{L^{p}(S, d\sigma)}.$$

We calculate

$$\begin{split} \|\mathbf{1}_{\eta}\|_{L^{p}(S,d\sigma)} &= \left(\int_{S} \mathbf{1}_{\eta}^{p}\right)^{\frac{1}{p}} \\ &= \left(\int_{S} \mathbf{1}_{\eta}\right)^{\frac{1}{p}} \\ &= \left(\frac{1}{|S|} \sum_{\xi \in S} \mathbf{1}_{\eta}(\xi)\right)^{\frac{1}{p}} \\ &= |S|^{-\frac{1}{p}} \end{split}$$

 $\quad \text{and} \quad$

$$\begin{split} \|(\mathbf{1}_{\eta})^{\check{}}\|_{L^{r}(\mathbb{F}^{d},dx)} &= \left\| \frac{1}{|S|} \sum_{\xi \in S} \mathbf{1}_{\eta}(\xi) \chi(x \cdot \xi) \right\|_{L^{r}(\mathbb{F}^{d},dx)} \\ &= \left(\int_{\mathbb{F}^{d}} \left(\frac{1}{|S|} \sum_{\xi \in S} \mathbf{1}_{\eta}(\xi) \chi(x \cdot \xi) \right)^{r} dx \right)^{\frac{1}{r}} \\ &= \frac{1}{|S|} \left(\sum_{x \in \mathbb{F}^{d}} \left(\sum_{\xi \in S} \mathbf{1}_{\eta}(\xi) \chi(x \cdot \xi) \right)^{r} \right)^{\frac{1}{r}} \\ &= \frac{1}{|S|} \left(\sum_{x \in \mathbb{F}^{d}} \left(\chi(x \cdot \eta) \right)^{r} \right)^{\frac{1}{r}} \\ &= \frac{1}{|S|} \left(\sum_{x \in \mathbb{F}^{d}} \left(\chi(r(x \cdot \eta)) \right) \right)^{\frac{1}{r}} \\ &= \frac{1}{|S|} q^{\frac{d}{r}} \end{split}$$

In order for the bound (1.1) to hold for $\mathbf{1}_\eta$ we must have

$$|S|^{-1}q^{\frac{d}{r}} \le R(p \to r)|S|^{-\frac{1}{p}},$$

Using $|S| \sim q^k$, this becomes

$$|S|^{\frac{1}{p}-1}q^{\frac{d}{r}} \sim q^{\frac{k-kp}{p}+\frac{d}{r}} \le R(p \to r)$$

so we require $\frac{k-kp}{p} + \frac{d}{r} \le 0$, that is

$$r \ge \frac{dp}{k(p-1)} = \frac{dp'}{k}.$$

Mockenhaupt and Tao further refine the necessary conditions in the case where S contains some affine subspace. If $V \subset S$ is an affine subspace of dimension l (i.e. $|V| \sim q^l$), they show that the restriction estimate will fail unless

$$r \ge p' \frac{d-l}{k-l}.\tag{2.7}$$

Proof. To see this, set $f(x) = \mathbf{1}_V(x)$. Then (1.1) becomes

$$\|(\mathbf{1}_V d\sigma) \check{}\|_{L^r(\mathbb{F}^d, dx)} \le R(p \to r) \|\mathbf{1}_V\|_{L^p(S, d\sigma)},$$

where

$$\begin{split} \|\mathbf{1}_{V}\|_{L^{p}(S,d\sigma)} &= \left(\int_{S} (\mathbf{1}_{V}(x))^{p} d\sigma\right) \frac{1}{p} \\ &= \left(\frac{1}{|S|} \sum_{\xi \in S} (\mathbf{1}_{V}(\xi))^{p}\right)^{\frac{1}{p}} \\ &= |S|^{-\frac{1}{p}} \left(\sum_{\xi \in S} \mathbf{1}_{V}(\xi)\right)^{\frac{1}{p}} \\ &= |S|^{-\frac{1}{p}} \left(\sum_{\xi \in V} 1\right)^{\frac{1}{p}} \\ &= |S|^{-\frac{1}{p}} \left(\sum_{\xi \in V} 1\right)^{\frac{1}{p}} \end{split}$$

$$\sim |S|^{-\frac{1}{p}}q^{\frac{l}{p}}.$$

Since V is an *l*-dimensional subspace, we can form a basis $\{e_i\}_{i=1}^l \subset \mathbb{F}^d$, and

$$\begin{split} \| (\mathbf{1}_{V} d\sigma)^{\check{}} \|_{L^{r}(\mathbb{F}^{d}, dx)} &= \left\| \frac{1}{|S|} \sum_{\xi \in S} \mathbf{1}_{V}(\xi) \chi(x \cdot \xi) \right\|_{L^{r}(\mathbb{F}^{d}, dx)} \\ &= \left(\int_{\mathbb{F}^{d}} \left(\frac{1}{|S|} \sum_{\xi \in S} \mathbf{1}_{V}(\xi) \chi(x \cdot \xi) \right)^{r} dx \right)^{\frac{1}{r}} \\ &= |S|^{-1} \left(\sum_{x \in \mathbb{F}^{d}} \left(\sum_{\xi \in S} \mathbf{1}_{V}(\xi) \chi(x \cdot \xi) \right)^{r} \right)^{\frac{1}{r}} \\ &= |S|^{-1} \left(\sum_{x \in \mathbb{F}^{d}} \left(\sum_{\xi \in V} \chi(x \cdot \xi) \right)^{r} \right)^{\frac{1}{r}} \\ &= |S|^{-1} \left(\sum_{x \in \mathbb{F}^{d}} \left(\sum_{\xi \in V} \chi(x_{1}\xi_{1}) \chi(x_{2}\xi_{2}) \dots \chi(x_{l}\xi_{l}) \right)^{r} \right)^{\frac{1}{r}} \\ &= |S|^{-1} \left(\sum_{x \in \mathbb{F}^{d}} \left(\sum_{\xi_{1} \in \mathbb{F}} \chi(x_{1}\xi_{1}) \right)^{r} \dots \left(\sum_{\xi_{l} \in \mathbb{F}} \chi(x_{l}\xi_{l}) \right)^{r} \right)^{\frac{1}{r}} \end{split}$$

and each inner sum is equal to zero unless $x_i = 0$, so

$$= |S|^{-1} \left(\sum_{\substack{x \in \mathbb{F}^d \text{ s.t.} \\ x_1 = \ldots = x_l = 0}} \underline{q^r q^r \ldots q^r}_{l \text{ times}} \right)^{\frac{1}{r}}$$
$$= |S|^{-1} q^l q^{\frac{d-l}{r}}.$$

This yields the necessary estimate $|S|^{-1}q^l q^{\frac{d-l}{r}} \leq R(p \to r)|S|^{-1\frac{1}{p}}q^{\frac{l}{p}}$. Applying $|S| \sim q^k$, this rearranges to $q^{l+\frac{d-l}{r}-\frac{l}{p}-k+\frac{k}{p}} \lesssim R(p \to r)$, so we require

$$l + \frac{d-l}{r} - \frac{l-k}{p} + k \le 0,$$

i.e.
$$r \ge \frac{p(d-l)}{(1-p)(l-k)} = p'\frac{d-l}{k-l}.$$

2.3 Tomas-Stein Estimates

In the Euclidean case, the earliest non-trivial bound for the sphere is due to Tomas and Stein. It is known to hold for all smooth hypersurfaces of nonvanishing curvature on \mathbb{R}^d , when p = 2. Let d be the dimension of Euclidean space. It states that we have $R(p \to r) \leq 1$ for all p and r such that

$$r \ge \frac{2d+2}{d-1}$$
 and $r \ge \frac{p(d+1)}{(p-1)(d-1)}$.

When p = 2, this gives the best possible estimate on the sphere. Tomas and Stein prove this by first rephrasing the problem in term of the Fourier transform on the surface measure $d\sigma$. For any finite measure μ , Tomas and Stein show that the estimate $\left\|\widehat{f\mu}\right\|_{L^p} \leq C \|f\|_{L^2(\mu)}$ for all $f \in L^2(\mu)$ is equivalent to the condition $\|\widehat{\mu} * h\|_{L^p} \leq C^2 \|h\|_{L^{p'}}$ for every Schwartz function h. They then prove the bound on the surface measure by applying a dyadic decomposition, and interpolating between bounds on the Bochner-Riesz kernel (which is essentially $(d\sigma)^{\check{}}$ bounded away from one). In the finite setting, we have no such dyadic decomposition, but analogous principles of restating the problem in terms of $(d\sigma)^{\check{}}$ and considering the Bochner-Riesz kernel can be put to use. Adapting the Euclidean approach in this way, Mockenhaupt and Tao prove that the Tomas-Stein estimate holds in the two-dimensional finite field case, when the subset is a paraboloid.

The finite field Tomas-Stein type approach uses the orthogonality of T and T^* (see Remark 1) to express the desired estimate in a form that deals with $(d\sigma)^{\check{}}$. If we remove the origin from $(d\sigma)^{\check{}}$, it is often possible to obtain good decay estimates on its size. We define the Bochner-Riesz kernel K(x), which is $(d\sigma)^{\check{}}$ with the origin removed.

2.3.1 Tomas-Stein type argument for even exponents

By analyzing decay of $(d\sigma)^{\check{}}$ away from the origin, Mockenhaupt and Tao obtain a set of sufficient conditions for restriction estimates from 2 to r, where r is even. The method employed relies on an analysis of the surface measure $d\sigma$, so gives some of the flavour of the Tomas-Stein type method for exponents, although no specific restriction estimates are obtained here.

Theorem 2.3.1 (Sufficient conditions for even exponents). Let r = 2m, m an integer, and let $S \subset \mathbb{F}^d$. Suppose that for every $\xi \in \mathbb{F}^d_*$, the size of

$$\{\{\xi_i\}_{i=1}^m \subset S : \xi_1 + \xi_2 + \ldots + \xi_m = \xi\}$$

is bounded by A. Then we have the bound

$$R(2 \to 2m) \le A^{\frac{1}{2m}} |F|^{\frac{d}{2m}} |S|^{-\frac{1}{2}}$$

Proof. We want to show

$$\|(fd\sigma)^{\check{}}\|_{L^{2m}(\mathbb{F}^d,dx)} \le (A^{\frac{1}{2m}}|F|^{\frac{d}{2m}}|S|^{-\frac{1}{2}})\|f\|_{L^2(S,d\sigma)}$$

Raising to the 2m, this becomes

$$\|(fd\sigma)^{\tilde{}}\|_{L^{2m}(\mathbb{F}^d,dx)}^{2m} \le (A|F|^d|S|^{-k})\|f\|_{L^2(S,d\sigma)}^{2m}$$

We use Plancherel and the intertwining of Fourier transforms and convolutions to calculate

$$\begin{aligned} \|(fd\sigma)^{\check{}}\|_{L^{2m}(\mathbb{F}^d,dx)}^{2m} &= \int \left((fd\sigma)^{\check{}}\right)^{2m} dx \\ &= \int \left(\left((fd\sigma)^{\check{}}\right)^m\right)^2 dx \\ &= \|((fd\sigma)^{\check{}})^m\|_{L^2(\mathbb{F}^d,dx)}^2 \\ &= \left\|(\widehat{(fd\sigma)^{\check{}}})^m\right\|_{L^2(\mathbb{F}^d,dx)}^2 \end{aligned}$$

and by repeated application of (1.7) to $((\widehat{fd\sigma})^{\check{}})^m$,

$$= \left\| \underbrace{\widehat{(fd\sigma)}^{*} * \dots * \widehat{(fd\sigma)}^{*}}_{m \text{ times}} \right\|_{L^{2}(\mathbb{F}^{d}_{*}, dx)}^{2}$$
$$= \left\| (fd\sigma) * \dots * (fd\sigma) \right\|_{L^{2}(\mathbb{F}^{d}_{*}, dx)}^{2}$$
$$= \int_{\mathbb{F}^{d}_{*}} ((fd\sigma * \dots * fd\sigma)(\xi))^{2} d\xi$$

By Cauchy-Schwartz we have

$$fd\sigma * fd\sigma = \left(\int f(\xi - \eta)d\sigma(\xi - \eta)f(\eta)d\sigma(\eta)\right)^2$$
$$= \left(\int f(\xi - \eta)(d\sigma(\xi - \eta))^{\frac{1}{2}}f(\eta)(d\sigma(\eta))^{\frac{1}{2}}(d\sigma(\xi - \eta)d\sigma(\eta))^{\frac{1}{2}}\right)^2$$
$$\leq \int f^2(\xi - \eta)d\sigma(\xi - \eta)f^2(\eta)d\sigma(\eta)\int d\sigma(\xi - \eta)d\sigma(\eta))$$
$$= (f^2d\sigma * f^2d\sigma)(d\sigma * d\sigma)$$

which generalizes to

 $(fd\sigma * \dots * fd\sigma)(\xi)(fd\sigma * \dots * fd\sigma)(\xi) \le (d\sigma * \dots * d\sigma)(\xi)(f^2d\sigma * \dots * f^2d\sigma)(\xi).$ Now $d\sigma(\xi) := \frac{q^d}{|S|} \mathbf{1}_S(\xi)$, so

$$d\sigma * d\sigma(\xi) = \int_{\mathbb{F}_*^d} d\sigma(\eta) d\sigma(\xi - \eta)$$

$$= \frac{1}{q^d} \sum_{\eta \in \mathbb{F}^d} \frac{q^d}{|S|} \mathbf{1}_S(\eta) \frac{q^d}{|S|} \mathbf{1}_S(\xi - \eta)$$

$$= q^d |S|^{-2} \sum_{\substack{\xi \in \mathbb{F}^d, \eta \in S\\ \xi\eta \in S}} 1$$

$$= q^d |S|^{-2} \sum_{\substack{\xi_1, \xi_2 \in S\\ \xi_1 + \xi_2 = \xi}} 1,$$

and we can show inductively that

$$\underbrace{d\sigma * \dots * d\sigma}_{k \text{ copies}} = q^d |S|^{-k} \sum_{\substack{\xi_1, \dots, \xi_k \in S \\ \xi_1 + \dots + \xi_k = \xi}} 1.$$

Now by assumption $\sum_{\substack{\xi_1,...,\xi_k \in S\\\xi_1+...+\xi_k=\xi}} 1 \le A$ for all $\xi \in \mathbb{F}^d$, so

$$(fd\sigma * \dots * fd\sigma)(\xi))^2 \le A|F|^d|S|^{-k}(f^2d\sigma * \dots * f^2d\sigma)(\xi)$$

and

$$\begin{aligned} \|(fd\sigma)^{\check{}}\|_{L^{2m}(\mathbb{F}^{d},dx)}^{2m} &\leq A|F|^{d}|S|^{-k} \int_{\mathbb{F}^{d}_{*}} (f^{2}d\sigma * \dots * f^{2}d\sigma)(\xi)d\xi \\ &= A|F|^{d}|S|^{-k}\|f\|_{L^{2}(S,d\sigma)}^{2m}, \text{ as required.} \end{aligned}$$

Note that this does not provide a restriction result as is, because the constant depends on the base field \mathbb{F} . However, it does give us a nice sufficient condition for proving estimates of the form $R(2 \to 2m)$. If we can bound the size of Stogether with the number of solutions to $\xi = \xi_1 + \ldots + \xi_k$ in S such that the size of the field is canceled out in the constant, we have our result. Such a method is used by Koh and Shen in [10] to prove the $R(2 \to 4)$ estimate for certain cones in 3-dimensions (2.3.2). We use this same approach to extend the $(2 \to 4)$ result to non-degenerate cones in \mathbb{F}^4 (Theorem 3.0.1).

2.3.2 A Tomas-Stein type estimate for the cone

In [10], Koh and Shen obtain the result $R(2 \rightarrow 4)$ for cones in 3-dimensional finite fields which do not contain any 2-dimensional affine spaces. They present two methods of proof — a geometric argument based on Section 2.3.1, and a Tomas-Stein type argument. The latter provides a good example for how such arguments work, so we detail it below:

Theorem 2.3.2. Let $P(x) \in \mathbb{F}[x_1, x_2, x_3]$ be a homogeneous polynomial such that $H := \{x \in \mathbb{F}^3 : P(x) = 0\}$ does not contain any planes passing through the origin, and $|H| \sim q^2$. Then we have $R(2 \to 4) \leq 1$.

Proof. We will make use of the bound

$$|(d\sigma)^{\check{}}(m)| \lesssim q^{-1}, \text{ for all } m \neq 0.$$
(2.8)

This was proved by Koh and Shen in [10] via Lemma 3.1.1, and an incidence

argument. Define the dual extension and restriction operators

$$T^*: L^p(H, d\sigma) \to L^r(\mathbb{F}^3, dx); T^*f := (fd\sigma)\check{}$$

and

$$T: L^{r'}(\mathbb{F}^3, dx) \to L^{p'}(H, d\sigma); Tg := \hat{g}\big|_H.$$

From Remark 1, we have $T^*T(g) = g * (d\sigma)$ for all g on (\mathbb{F}^3, dx) . We want to show

$$\|\hat{f}\|_{L^{2}(H,d\sigma)}^{2} \lesssim \|f\|_{L^{\frac{4}{3}}(\mathbb{F}^{3},dm)}^{2}$$

By orthogonality and Hölder's inequality we have

$$\begin{split} \|\hat{f}\|_{L^{2}(H,d\sigma)}^{2} &= \int_{H} \hat{f}\bar{f}d\sigma = \int_{\mathbb{F}^{3}} \hat{f}\big|_{H}\bar{f}\big|_{H}d\sigma \\ &= _{L^{2}(H,d\sigma)} = _{L^{2}(\mathbb{F}^{3},dm)} \\ &= _{L^{2}(\mathbb{F}^{3},dx)} \le \|f*(d\sigma)^{\check{}}\|_{L^{4}(\mathbb{F}^{3},dx)}\|f\|_{L^{\frac{4}{3}}(\mathbb{F}^{3},dx)} \end{split}$$

So it suffices to show

$$\|f * (d\sigma) \, \|_{L^4(\mathbb{F}^3, dx)} \lesssim \|f\|_{L^{\frac{4}{3}}(\mathbb{F}^3, dx)}.$$

This is where the analysis of $(d\sigma)$ comes in. We introduce the Bochner-Riesz kernel:

Let $K(x) := (d\sigma)^{\tilde{}}(x) - \delta_0(x)$, which is just $(d\sigma)^{\tilde{}}$ with the origin set to 0, since

$$(d\sigma)$$
 $(0) = \frac{1}{|H|} \sum_{x \in H} \chi(0, 0, 0) = 1.$

So

$$\begin{split} \|f*(d\sigma)\check{}\|_{L^{4}(\mathbb{F}^{3},dx)} &= \|f*(K+\delta_{0})\|_{L^{4}(\mathbb{F}^{3},dx)} \\ &\leq \|f*K\|_{L^{4}(\mathbb{F}^{3},dx)} + \|f*\delta_{0}\|_{L^{4}(\mathbb{F}^{3},dx)} \text{ for all } x \end{split}$$

For the second term, we can evaluate

$$(f*\delta_0)(x) = \int_{\mathbb{F}^3} f(y)\delta_0(x-y)dy = \sum_{y\in\mathbb{F}^3} f(y)\delta_0(x-y) = f(x)$$
, so by Theorem 1.3.2,

$$\|f * \delta_0\|_{L^4(\mathbb{F}^3, dx)} = \|f\|_{L^4(\mathbb{F}^3, dx)} \le \|f\|_{L^{\frac{4}{3}}(\mathbb{F}^3, dx)}$$

It remains to establish a bound on $||f * K||_{L^4(\mathbb{F}^3, dx)}$. We obtain the necessary bound $||f * K||_{L^4(\mathbb{F}^3, dx)} \lesssim ||f||_{L^{\frac{4}{3}}(\mathbb{F}^3, dx)}$ by interpolating the following two bounds:

- (I) $||f * K||_{L^4(\mathbb{F}^3, dx)} \lesssim q ||f||_{L^2(\mathbb{F}^3, dx)}$
- (II) $||f * K||_{L^{\infty}(\mathbb{F}^{3}, dx)} \lesssim q^{-1} ||f||_{L^{1}(\mathbb{F}^{3}, dx)}$

Setting $\theta = \frac{1}{2}$ in Riesz-Thorin interpolation then yields the bound for p and r such that

$$\frac{1}{p} = \frac{1 - \frac{1}{2}}{2} + \frac{\frac{1}{2}}{1} = \frac{3}{4}$$
$$\frac{1}{r} = \frac{1 - \frac{1}{2}}{2} + \frac{\frac{1}{2}}{\infty} = \frac{1}{4},$$

with constant $q^{\frac{1}{2}}q^{-\frac{1}{2}} = 1$, as required. To complete the proof, we show the bounds (I) and (II).

(I) : We have

$$\begin{split} \|f * K\|_{L^{2}(\mathbb{F}^{3}, dx)} &= \|\widehat{(f * K)}\|_{L^{2}(\mathbb{F}^{3}, dx)} & \text{(Plancherel)} \\ &= \|\widehat{f}\widehat{K}\|_{L^{2}(\mathbb{F}^{3}, dx)} & \text{(by (1.7))} \\ &\leq \|\widehat{K}\|_{L^{\infty}(\mathbb{F}^{3}, dx)} \|\widehat{f}\|_{L^{2}(\mathbb{F}^{3}, dx)} \\ &= \|\widehat{K}\|_{L^{\infty}(\mathbb{F}^{3}, dx)} \|f\|_{L^{2}(\mathbb{F}^{3}, dx)} \end{split}$$

and $\hat{K}(x) = d\sigma(x) - \hat{\delta}_0(x) = \frac{q^3}{|H|} \mathbf{1}_H(x) - 1 \lesssim q$ for all x, since $|H| \sim q^2$, so $\|f * K\|_{L^2(\mathbb{F}^3, dx)} \lesssim q \|f\|_{L^2(\mathbb{F}^3, dx)}.$

(II) : Applying Young's inequality with $r = \infty$, we have

$$\|f * K\|_{L^{\infty}(\mathbb{F}^{3}, dx)} \leq \|f\|_{L^{p}(\mathbb{F}^{3}, dx)} \|K\|_{L^{p'}(\mathbb{F}^{3}, dx)}$$

in particular, $\leq \|f\|_{L^{1}} \|K\|_{L^{\infty}}$,

and for all $x \in \mathbb{F}^3$, we have $\|K\|_{L^{\infty}} \leq |(d\sigma)^{\check{}}(x)| \lesssim \frac{1}{q}$, so we have

$$||f * K||_{L^{\infty}} \lesssim q^{-1} ||f||_{L^1}$$
, as required.

2.4 Dyadic Pigeonholing and Incidence Arguments

Many of the known results for finite fields rely on arguments concerning the arithmetic structure of S, and bounds on the incidence of S with other subsets of \mathbb{F}^d . We outline the reasoning in these arguments here, which can also be found B.J. Green's lecture notes [4].

When r = 4, we can reduce the search for bound of the form (1.1) to a analysis of the structure of the subset S, and its incidence with other sets. We begin by obtaining an expression for the L^4 norm of $(fd\sigma)^{\sim}$ in terms of additive quadruples, when f is a characteristic function.

Lemma 2.4.1. Let E be any subset of the set $S \subset \mathbb{F}^d$ on which we are restricting \hat{f} . Then

$$\|(\mathbf{1}_E d\sigma)^{\tilde{}}\|_{L^4(\mathbb{F}^d, d\xi)}^4 = \frac{q^d}{|S|^4} \sum_{\substack{(\xi_1, \xi_2, \xi_3, \xi_4) \in E^4\\\xi_1 + \xi_2 = \xi_3 + \xi_4}} 1$$
(2.9)

Proof. We calculate

$$\begin{split} \|(\mathbf{1}_{E}d\sigma)^{`}\|_{L^{4}(\mathbb{F}^{d},d\xi)}^{4} &= \int |\int \chi(x\cdot\xi)\mathbf{1}_{E}(\xi)d\sigma(\xi)|^{4}dx \\ &= \int \left(\frac{1}{|S|}\sum_{\xi\in\mathbb{F}_{*}^{d}}\chi(x\cdot\xi)\mathbf{1}_{E}(\xi)\right)^{4}dx \\ &= \frac{1}{|S|^{4}}\sum_{x\in\mathbb{F}^{d}}\left(\sum_{\xi\in\mathbb{F}_{*}^{d}}\chi(x\cdot\xi)\mathbf{1}_{E}(\xi)\right)^{4} \\ &= \frac{1}{|S|^{4}}\sum_{x\in\mathbb{F}^{d}}\left(\sum_{\xi_{1},\xi_{2},\xi_{3},\xi_{4}\in E}\chi(x\cdot\xi_{1})\chi(x\cdot\xi_{2})\chi(x\cdot\xi_{3})\chi(x\cdot\xi_{4})\right) \end{split}$$

changing variables and applying additivity,

$$= \frac{1}{|S|^4} \sum_{x \in \mathbb{F}^d} \left(\sum_{\xi_1, \xi_2, \xi_3, \xi_4 \in E} \chi(x \cdot (\xi_1 + \xi_2 - \xi_3 - \xi_4)) \right)$$
$$= \frac{1}{|S|^4} \sum_{\xi_1, \xi_2, \xi_3, \xi_4 \in E} \left(\sum_{x \in \mathbb{F}^d} \chi(x \cdot (\xi_1 + \xi_2 - \xi_3 - \xi_4)) \right)$$

by Lemma 1.2.1,

$$= \frac{1}{|S|^4} \sum_{\substack{(\xi_1,\xi_2,\xi_3,\xi_4) \in E^4 \\ \xi_1 + \xi_2 = \xi_3 + \xi_4}} \sum_{x \in \mathbb{F}^d} 1$$
$$= \frac{q^d}{|S|^2} \sum_{\substack{(\xi_1,\xi_2,\xi_3,\xi_4) \in E^4 \\ \xi_1 + \xi_2 = \xi_3 + \xi_4}} 1$$

r	-	-	٦
L			1
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It is also straightforward to calculate

$$\|\mathbf{1}_E\|_{L^2(S,d\sigma)}^4 = \left(\frac{|E|}{|S|}\right)^2.$$

This implies that whenever we have the restriction result $R(2 \to 4)$ on $S \subset \mathbb{F}^d$, we must have

$$\sum_{\substack{(\xi_1,\xi_2,\xi_3,\xi_4)\in E^4\\\xi_1+\xi_2=\xi_3+\xi_4}} 1 \lesssim \frac{|S|^2|E|^2}{|\mathbb{F}|^d}.$$
(2.10)

For every $E \subset S$. This hints at the connection between restriction estimates over a subset S of a finite field and the arithmetic structure of the subset.

Indeed, if we can bound the number additive quadruples in (2.10) for any subset of S, this line of argument can be further extended to arrive at new restriction results when r = 4. We arrive at these by incorporating the bound in (2.10) into a dyadic pigeonholing argument which allows us to consider only characteristic functions of subsets E, instead of more general functions on S.

2.4.1 The dyadic pigeonholing argument

Let f be a bounded positive-valued function, supported on S. Without loss of generality, we may assume that $||f||_{L^{\infty}} = 1$. We divide the range of f into dyadic shells, by defining E_j to be the set of all x such that $2^{-j-1} < f(x) < 2^{-j}$, for every $j \in \mathbb{N}$. Let $f_j := f \cdot \mathbf{1}_{E_j}$, so that f is partitioned over these dyadic shells.

By an argument similar to (2.9), for each f_j we have

$$\begin{split} \|(f_j d\sigma)^{\check{}}\|_4^4 &= \frac{|\mathbb{F}|^d}{|S|^4} \sum_{\substack{(\xi_1, \xi_2, \xi_3, \xi_4) \in E^4\\\xi_1 + \xi_2 = \xi_3 + \xi_4}} f(\xi_1) f(\xi_2) f(\xi_3) f(\xi_4) \\ &\leq \frac{|\mathbb{F}|^d}{|S|^4} \sum_{\substack{(\xi_1, \xi_2, \xi_3, \xi_4) \in E^4\\\xi_1 + \xi_2 = \xi_3 + \xi_4}} 1 \end{split}$$

If we have good enough bound A_{E_j} on the sum of additive quadruples in terms of $|E_j|$, we can bound each $\|(f_j d\sigma)^{\check{}}\|_4$ by some L^p norm of f. Further, if we consider the tail of f, given by $g := f \cdot \mathbf{1}_{\{x: f(x) < 2^{-m}\}}$, we have the bound

$$\begin{split} \|(gd\sigma)^{\check{}}\|_{4}^{4} &= \frac{|\mathbb{F}|^{d}}{|S|^{4}} \sum_{\substack{(\xi_{1},\xi_{2},\xi_{3},\xi_{4}) \in S^{4}\\\xi_{1}+\xi_{2}=\xi_{3}+\xi_{4}}} g(\xi_{1})g(\xi_{2})g(\xi_{3})g(\xi_{4}) \\ &\leq A_{(\bigcup_{j=m}^{\infty}E_{j})} \frac{|\mathbb{F}|^{d}}{|S|^{4}} 2^{-4m}, \end{split}$$

for large enough m. Since $||f||_{L^{\infty}} = 1$, a good enough bound A_E will allow us to bound $||(gd\sigma)^{\check{}}||_4$ by the same L^p norm as above. Summing over this and the bound for the remaining f_j 's yields a restriction estimate for some $R(p \to 4)$, where p depends on the bound A_E .

An application of this method to the paraboloid in three dimensions is detailed by B.J.Green in [4], and is applied by Mockenhaupt and Tao in [17], and Iosevich and Koh in [6]. For an application to the sphere, see [7]. The key to proving exponents of this type becomes obtaining results on the arithmetic structure of S.

2.5 Summary of Existing Results

In Euclidean space, the restriction problem is traditionally considered with respect to hypersurfaces with non-vanishing Gaussian curvature — in particular the sphere, cones, and paraboloids. Each of these has been studied to some extent over finite fields. In their introductory paper, Mockenhaupt and Tao focus primarily on paraboloids and cones in low dimensions, and provide some counterexamples which serve as necessary conditions, as in (2.4) [17].

When S is given by the curve $\gamma(t) := (t, t^2, \dots, t^d)$, they use Theorem 2.3.1 to show $R(p \to r) \leq 1$ for every p, r such that $r \geq 2d, dp'$. The necessary conditions in (2.4) show that this estimate is sharp. This yields $R(2 \to 2d)$ as the best estimate possible for the curve.

2.5.1 Paraboloids

In the case of a paraboloid P, Mockenhaupt and Tao note that Theorem 2.3.1 proves $R(2 \to 4) \leq 1$ in all dimensions. We always have $|P| \sim q^{d-1}$, so (2.4) gives us the necessary conditions $r \geq \frac{2d}{d-1}, \frac{dp'}{d-1}$. When d = 2, the authors show that these are also sufficient, so the sharpest estimate is $R(2 \to 4) \leq 1$.

If d = 3, they restrict themselves to finite fields \mathbb{F} in which -1 is non-square (i.e. $\operatorname{char}(\mathbb{F}) \equiv 3 \mod 4$). Without this specification P may contain lines, in which case (2.7) gives us the more restrictive necessary condition $r \geq 3p'$.

If P does not contain any lines, they conjecture that the conditions from (2.4) are necessary and sufficient. In this case (2.3) and (2.4) with d = 3 yields $\left\{r \geq \frac{3p'}{2}, 3\right\}$, so the conjectured sharp estimate is $R(2 \to 3)$. This is not proved, but they improve upon the 2 to 4 estimate in both directions, by showing boundedness for $R(2 \to \frac{18}{5} + \epsilon) \forall \epsilon > 0$ and $R(\frac{8}{5} \to 4)$, up to a logarithmic factor of q.

In 2008, Iosevich and Koh showed that when $d \ge 3$ is odd and r = 4, the Tomas-Stein exponent is the best possible [6]. The authors obtain improved sufficient conditions for arbitrary finite fields with $d \ge 4$, d even, up to a logarithmic factor of q. They further improve on the $2 \rightarrow 4$ estimate in both directions when $d \ge 7$ is odd, with -1 non-square, again up a logarithmic factor. By applying a bilinear estimate in the stead of the dyadic argument, Lewko and Lewko obtain this exponent without the logarithmic factor in [11]. See Table 2.1 below for the specific results.

2.5.2 Quadratic Surfaces

We can generalize the paraboloid to other non-degenerate quadratic surfaces. For $x \in \mathbb{F}^d$ with $\operatorname{char}(F) > 2$, let Q(x) be a homogeneous polynomial of degree 2, expressible as $Q(x_1, \ldots, x_d) = \sum_{i,j=1}^d a_{ij} x_i x_j$, where $a_{ij} = a_{ji}$. If the matrix $\{a_{ij}\}$ is non-singular, we call Q(x) a non-degenerate quadratic form. For any nonzero j in \mathbb{F} , the non-degenerate quadratic surface Q_j is given by $Q_j = \{x \in \mathbb{F}^d : Q(x_1, \ldots, x_d) = j\}$. Using Gauss and Kloosterman sums (1.3 and 1.4) to derive incidence theorems for a Tomas-Stein type argument, Iosevich and Koh prove $R(2 \to 4) \leq 1$ for any non-degenerate quadratic surface in \mathbb{F}^d , $d \geq 2$, and $R(2 \to r) \leq 1$ for all $r \geq \frac{2d+2}{d-1}$. These are the only results on general quadratic forms to date [5].

2.5.3 Spheres

Another important special case of quadratic surfaces occurs when we take $Q(x) = ||x||_2$. This is the finite field analog of the sphere. The above results for quadratic surfaces are valid here, and when d = 2 we already have the sharp estimate $R(2 \rightarrow 4)$. When $d \geq 3$ is odd, Iosevich and Koh find that the Tomas-Stein exponent is the best possible, in [7]. When $d \geq 4$ is even, they show $R(p \rightarrow 4)$ is bounded by a logarithmic factor of q whenever $p \geq \frac{12d-8}{9d-12}$, which is an improvement upon the Tomas-Stein exponent $p \geq \frac{4d-4}{3d-5}$.

2.5.4 Cones

When j is equal to 0, Q_j becomes the solution-set to a quadratic homogeneous variety, which is the finite field analog to the Euclidean cone. Koh and Shen study restriction estimates for these varieties in two and three dimensions ([9] and [10]). When d = 2 and the characteristic of \mathbb{F} is not equal to two, they show that the $(2 \to 4)$ holds on surfaces defined by homogeneous varieties of size O(q), provided the variety has no linear factor. Further, they show that interpolation with this bound and application of Theorem 1.3.2 yields the best range of exponents possible in this case. In [10], an application of Theorem 2.3.1 yields the same estimate for three-dimensional homogeneous varieties with size $O(q^2)$ which contain no planes passing through the origin. A counterexample from [17] shows that this is the best estimate possible for homogeneous varieties of this type.

	Known estimates	Necessary conditions					
Curves [17]							
General d $\gamma(t) := (t, t^2, \cdots, t^d)$	$2 \rightarrow 2d^*$	$\begin{array}{c} r \ge 2d, \\ r \ge dp' \end{array}$					
Non-Degenerate Quadr	atic Surfaces [5]	<u></u>					
General d	$2 \to 4$ $2 \to \frac{2d+2}{d-1}$	$r \ge 4,$ $r \ge \frac{2p}{p-1}$					
Paraboloids [6][11][17]		LL					
General d	$2 \rightarrow 4$	$r \ge \frac{2d}{d-1}, \frac{dp'}{d-1}$					
d = 2	$2 \rightarrow 4^*$	$r \ge 4, 2p'$					
d = 3, -1 nonsquare	$2 \xrightarrow{18}{5} + \epsilon$ $\frac{8}{5} 4$	$r \ge 3$ $r \ge \frac{3p'}{2}$					
d = 3, -1 square	$2 \rightarrow 4 $ (Best p given $r = 4$)	$r \ge 3, 2p'$					
$d \ge 3$, odd, -1 nonsquare	$\frac{8}{5} \rightarrow 4$						
$d \ge 4$, even	$\frac{\frac{4d}{3d-2} \rightarrow 4}{2 \rightarrow \frac{2d^2}{d^2-2d+2}}$	$r \ge rac{2d}{d-1}$ $r \ge rac{p(d+2)}{(p-1)d}$					
$d \ge 7, -1$ nonsquare,	$\frac{4d}{3d-2} \to 4$						
$q = \wp^l, 4 \nmid l(d-1)$	$2 \rightarrow \frac{2d^2}{d^2 - 2d + 2}$						
Spheres [7][17]							
d = 2	$2 \rightarrow 4$	$r \ge 4, 2p'$					
$d \ge 3, \text{ odd}$	$2 \rightarrow \frac{2d+2}{d-1}$	$r \ge \frac{2d}{d-1}, \frac{dp'}{d-1}$					
$d \ge 4$, even	$\frac{\frac{12d-8}{9d-12} \rightarrow 4}{2 \rightarrow \frac{2d+2}{d-1}}$	$r \ge \frac{2d}{d-1},$ $r \ge \frac{dp'}{d-1}$					
Cones [9][10]							
d = 2	$2 \rightarrow 4^*$	$r \ge 4, 2p'$					
$d = 3$ $ S \sim q^2, \text{ no planes}$	$2 \rightarrow 4^*$	$r \ge 4, \\ r \ge 2p'$					

Note: Results marked with an asterisk are optimal, in the sense that the value of r is the smallest possible, and p is the minimal possible value for this r.

Table 2.1: Summary of Existing Results for $p \to r$.

Chapter 3

Restriction Estimates for Cones over Finite Fields

In [10] Koh and Shen arrive at the $(2 \rightarrow 4)$ estimate for the three dimensional cone by proving incidence results on homogeneous varieties and applying them to Mockenhaupt and Tao's sufficient condition for even exponents (Theorem 1.3.2). In what follows we adapt their techniques to homogeneous in general dimensions.

Theorem 3.0.1. Let P(x) be a d-variate polynomial over \mathbb{F} , and $H = \{x \in \mathbb{F}^d : P(x) = 0\}$. Suppose that $|H| \sim q^{d-1}$, and that the homogeneous variety H is expressible as a disjoint (except at the origin) union of k-dimensional planes in \mathbb{F}^d for a fixed $k \leq d-2$, and does not contain planes of any dimension greater than k. Then we have the following restriction estimate on H:

$$R(2 \to 4) \lesssim 1.$$

Theorem 3.0.2. The restriction estimate $R(2 \rightarrow 4) \leq 1$ holds for any homogeneous variety $H \in \mathbb{F}^d$, provided $|H| \sim q^{d-1}$ and H does not contain a hyperplane passing through the origin.

3.1 Preliminary Lemmas

In the proof of Theorem 3.0.1, we will follow the methods of [9], generalized to the higher-dimensional setting. We will require the following lemmas.

Lemma 3.1.1 (Schwartz-Zippel). If P(x) is a d-variate polynomial over \mathbb{F} of degree $k \neq 0$, then

$$|\{x \in \mathbb{F}^d : P(x) = 0\}| \le kq^{d-1}.$$

Proof. This lemma was proven independently by Schwartz [15], Zippel [21] and Demillo-Lipton [1] from a probabilistic perspective, using inductive methods. We will follow a more recent proof by Dana Moshkovitz [13].

First note that when d = 1 this is just the factor theorem, and if $k \ge |\mathbb{F}|$, then the number of zeros is at most $|\mathbb{F}|^d \le k |\mathbb{F}|^{d-1}$, so the result is trivial. It suffices to consider $d \ge 2$ and $1 \le k < |\mathbb{F}|$. We will reduce these to the case d = 1, by restricting the function to one-dimensional objects.

Let $P(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_d]$ be a nonzero polynomial of degree $k \neq 0$. We can decompose $P(\mathbf{x})$ into $P(\mathbf{x}) = P_H(\mathbf{x}) + p(\mathbf{x})$, where $P_H(\mathbf{x})$ consists of all highestdegree monomials in $P(\mathbf{x})$. Note that $P_H(\mathbf{x})$ is nonzero and homogeneous, of degree k. For now we assume that there exists some nonzero element $\mathbf{y} \in \mathbb{F}^d$ such than $P_H(\mathbf{y}) \neq 0$ (this will be established in Lemma 3.1.2.) For such a \mathbf{y} , we can partition the elements of \mathbb{F}^d according to lines with direction \mathbf{y} , so that $\mathbb{F}^d = \bigsqcup_{x \in \mathbb{F}^{d-1}} \{(\underline{x}, x_d) + t\mathbf{y} | t \in \mathbb{F}\}$, where x_d is fixed.

Note that there are $|\mathbb{F}|^{d-1}$ such lines. Now consider the restrictions of P to such lines. $P(\mathbf{x} + t\mathbf{y})$ is a *k*th-degree univariate polynomial in t. Note that the coefficient of t^k is exactly the *k*-degree terms of P evaluated at \mathbf{y} , i.e. $P_H(\mathbf{y})$. Since $P_H(\mathbf{y}) \neq 0$, $P(\mathbf{x} + t\mathbf{y})$ is nonzero. By the d = 1 case, the number of zeros of P restricted to a given such line is at most k. The total number of zeros on \mathbb{F}^d is therefore at most $k |\mathbb{F}|^{d-1}$.

Lemma 3.1.2. Let $P(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_d]$ be a nonzero polynomial with degree $1 \leq k < |\mathbb{F}|$. Then there exists some $\mathbf{y} \in \mathbb{F}^d$ for which $P(\mathbf{y}) \neq 0$. Moreover, if P is homogeneous, \mathbf{y} is nonzero.

Proof. Note that when d = 1, P has at most $k < |\mathbb{F}|$ zeros, so the lemma holds. Suppose by way of contradiction that the lemma does not hold for general d. Let m be the smallest value for which the lemma fails. Then there exists $P \in \mathbb{F}[x_1, \ldots, x_m]$ such that $P(x_1, \ldots, x_m) = 0$ for every $(x_1, \ldots, x_m) \in \mathbb{F}^m$. For a fixed $a \in \mathbb{F}$, let $P_a(x_1, \ldots, x_{m-1}) := P(a, x_1, \ldots, x_{m-1})$. Since P_a is (m-1)variate, there exists $\mathbf{y} = (y_1, \ldots, y_{m-1})$ such that $P_a(\mathbf{y}) \neq 0$. But then for $\mathbf{y}_a := (a, y_1, \ldots, y_{m-1}) \in \mathbb{F}^m$ we have $P(\mathbf{y}_a) = P_a(\mathbf{y}) = 0$, a contradiction. Hence the lemma holds for all d. Finally, if P is homogeneous, $P(\mathbf{0}) = 0$. **Lemma 3.1.3.** Fix $k \leq d-1$. Suppose $H = \{x \in \mathbb{F}^d : P(x) = 0\}$ is a homogeneous variety, with $P(x) \neq 0$. Then for any k-dimensional plane $\Pi_M \not\subset H$, we have

$$|H \cap \Pi_M| \lesssim q^{k-1}.$$

Proof. Since Π_M is a k-dimensional plane in \mathbb{F}^d , without loss of generality we may assume there exists k - dimensional vectors $\{M_j\}_{j=k+1}^d$, and $M_0 \in \mathbb{F}^d$ such that $\Pi_M = \{M_0 + (x_1, \ldots, x_k, M_{k+1} \cdot (x_1, x_2, \ldots, x_k), \ldots, M_d \cdot (x_1, x_2, \ldots, x_k))\}$. Since every element of $H \cap \Pi_M$ is expressible in this form and also satisfies P(x) = 0, we have

$$|H \cap \Pi_M| = |y \in \mathbb{F}^k : P(M_0 + (y_1, \dots, y_k, M_{k+1} \cdot y, \dots, M_d \cdot y)) = 0|.$$

Now $P(M_0 + (y_1, \ldots, y_k, M_{k+1} \cdot y, \ldots, M_d \cdot y))$ is a k-variate polynomial. Since $\Pi_M \not\subset H$, it is not identically zero. The result therefore follows from Lemma 3.1.1.

The proof of Theorem 3.0.1 will rely on the following lemma, the details of which can also be found in [17] and [5].

Lemma 3.1.4. Let V be any algebraic variety in \mathbb{F}^d , $d \ge 2$, with $|V| \sim q^{d-1}$. Suppose that for every $\xi \in \mathbb{F}^d \setminus \{(0, \ldots, 0)\},$

$$\sum_{(x,y)\in V\times V: x+y=\xi} 1 \lesssim q^{d-2}$$

Then, we have

$$R(2 \to 4) \lesssim 1.$$

Proof. We apply the bound on $\{(x, y) \in V \times V : x + y = \xi\}$ to Theorem 2.3.1, with m = 2. This gives

$$R(2 \to 4) \lesssim (q^{d-2})^{\frac{1}{4}} q^{\frac{d}{4}} |V|^{-\frac{1}{2}}$$

and $|V| \sim q^{d-1}$, so

$$R(2 \to 4) \lesssim q^{\frac{d-2}{4} + \frac{d}{4} - \frac{d-1}{2}}$$
$$= q^{\frac{d-2+d-2d+2}{4}} = 1$$

so $R(2 \rightarrow 4) \lesssim 1$, independent of the base field.

3.2 Proof of Theorems 3.0.1 and 3.0.2

3.2.1 Proof of Theorem 3.0.1

Let H be a homogeneous variety in \mathbb{F}^d , expressible as a union of k-dimensional planes intersecting only at origin, for some fixed $k \leq d-2$. Assume that |H| does not contain any planes of dimension greater than k, and $|H| \sim q^{d-1}$. Applying Lemma 3.1.4, it suffices to show that for every $\xi \in \mathbb{F}^d \setminus \{(0, \ldots, 0)\}$, we have

$$\sum_{(x,y)\in H\times H: x+y=\xi} 1 \lesssim q^{d-2}.$$

Equivalently, we must show $|\{x \in \mathbb{F}^d : P(x) = P(\xi - x) = 0\}| \leq q^{d-2}$. It thus suffices to prove $|H \cap (H + \xi)| \leq q^{d-2}$, for all $\xi \neq 0$. Fix $\xi \in \mathbb{F}^d \setminus \{(0, \dots, 0)\}$. By assumption, we have $H = \bigcup_{j=1}^n \prod_j$, where each \prod_j is a k-dimensional plane in \mathbb{F}^d . Note that each plane has size $\sim q^k$, so we have $n \sim q^{d-k-1}$. Since

$$|H \cap (H + \xi)| \le \sum_{j=1}^{n} |H \cap (\Pi_j + \xi)|,$$

it will suffice to consider the sets $H \cap (\Pi_j + \xi)$. If $\xi \in \Pi_j$, then $|H \cap (\Pi_j + \xi)| = |H \cap \Pi_j| \sim q^k$. Since $\xi \neq 0$ and the planes are disjoint except at the origin, this occurs at most once. We have

$$|H \cap (H+\xi)| \le q^k + q^{d-k-1} |H \cap (\Pi_l + \xi)|,$$

where Π_l is one of the k-dimensional planes comprising H and not containing ξ . If $\xi \notin \Pi_l$, then $\Pi_l + \xi$ does not intersect the origin, and hence is distinct from each Π_j comprising $H = \bigcup_{j=1}^n \Pi_j$. Since $0 \notin (\Pi_l + \xi)$, if $\Pi_l + \xi$ was contained in H, H would contain the (k + 1)-dimensional plane spanned by 0 and $\Pi_l + \xi$, a contradiction. Hence we have $(\Pi_l + \xi) \notin H$. From Lemma 3.1.3, it follows that

$$|H \cap (\Pi_l + \xi)| \lesssim q^{k-1},$$

so finally

$$\begin{split} |H \cap (H+\xi)| &\leq q^k + q^{d-k-1}q^{k-1} \\ &\lesssim q^{d-2}, \, \text{as required.} \end{split}$$

3.2.2 Proof of Theorem 3.0.2

The main result will follow as a consequence of Theorem 3.0.1. Let H be a homogeneous variety containing no hyperplanes, with $H \sim q^{d-1}$. We may decompose H into collections of planes of the same dimension, so that $H = \bigcup_{k \leq d-2} H_k$, where $H_k = \bigcup_j \prod_j$, such that $\{\prod_j\}$ are planes of dimension k, and H_k contains no planes of dimension greater than k. For all $k \neq l$, we have $H_k \cap H_l = \{(0, \ldots, 0)\}$. We have

$$\|f\|_{L^{2}(H,d\sigma)} \leq \sum_{k=1}^{d-2} \|f\|_{L^{2}(H_{k},d\sigma)}$$

$$\lesssim \sum_{k=1}^{d-2} \|\hat{f}\|_{L^{4}(H_{k},d\sigma)}$$

$$\lesssim \sum_{k=1}^{d-2} \|\hat{f}\|_{L^{4}(H,d\sigma)}$$

$$\lesssim \|\hat{f}\|_{L^{4}(H,d\sigma)}.$$
(3.1)
(3.2)

Where (3.2) follows from Theorem 3.0.1. Hence we have the estimate $R(2 \rightarrow 4)$ on H.

Remark 2. In general we would expect our estimate to worsen for more degenerate varieties. Hence if we impose stricter conditions on the highest-dimension planes present in H, we expect that a sharper restriction estimate is possible. In particular, when d = 4 it is believed that a sharper estimate will hold in the case where H contains no 2-dimensional planes.

3.3 Possible Improvements

If $S \subset \mathbb{F}^d$ has dimension k, Mockenhaupt and Tao proved the necessary conditions

$$r \ge \frac{2d}{k},\tag{3.3}$$

$$r \ge \frac{d}{k}p' \tag{3.4}$$

for $R(p \to r)$ bounded by O(1). In the case of a 3-dimensional cone in \mathbb{F}^4 , these yield the necessary conditions $r \ge 8/3$ and $p \ge \frac{3r}{3r-4}$. The first of these bounds will be improved to $r \ge 3$ in what follows, for general 3-dimensional cones in \mathbb{F}^4 . If we set r = 3, the second bound yields the necessary condition $p \ge 9/5$. Moreover, if S is known to contain a k-dimensional subspace, they prove the necessary condition

$$r \ge p' \frac{d-l}{k-l}.\tag{3.5}$$

Since every cone will contain 1-dimensional lines, in our case this implies $r \ge \frac{3}{2}p'$, so that $p \ge \frac{2r}{2r-3}$. When r = 4, we obtain a lower bound of $\frac{8}{5}$ on p. On the other hand, in the case that our cone contains a 2-dimensional plane, we obtain $r \ge 2p'$, which leads to $p \ge \frac{r}{r-2}$, and when r = 4, this gives the necessary condition $p \ge 4$. Hence the bound found in Theorem 3.0.2 gives the best possible range for p, in the case where H contains a 2-dimensional plane. If H only contains lines, (3.5) suggests it may be possible to reduced p to as low as $\frac{8}{5}$, when r = 4.

As we will see in Theorem 3.4.1, for general 3-dimensional cones in \mathbb{F}^4 we have the lower bound $r \geq 3$, an improvement upon (3.3). If H contains a 2-dimensional subspace, r = 3 implies $p \geq 3$. If H has no such planes, we may have r = 3, $p \geq 2$.

3.4 Upper Bounds on *r* for Specific Cones

In the case of the general three dimensional cone, Mockenhaupt and Tao produced a counterexample to the bound $(p \rightarrow r)$, for all r < 4, and all p [17]. We adapt this counterexample to the four dimensional case.

Theorem 3.4.1. Let $S := \{(x, y, z, w) \in \mathbb{F}^4 : x^2 + y^2 = zw\}$. (Note that by change of variables this is equivalent to the cone $x^2 + y^2 + z^2 - w^2 = 0$.) Then

for all $r \leq 3$, $1 \leq p \leq \infty$, $R(p \rightarrow r)$ is unbounded.

Proof. We consider $f = \mathbf{1}_X$, where X is a subset of S given by

$$X = \{(x', y', z', w') : z' \text{ is a square, } 4z'w' = x'^2 + y'^2\}.$$

Since S is a three dimensional cone, we have $|X| \sim |\mathbb{F}|^3$. We can make the change of variables $\{x' = tu, y' = su, z' = (t^2 + s^2)u, w' = u\}$ for all (x', y', z', w') on S, so we have

$$\hat{\mathbf{1}}_{X}(x',y',z',w') = \hat{\mathbf{1}}_{X}(tu,su,(t^{2}+s^{2})u,u)$$

$$= \sum_{z'\in Q} \sum_{x',y'\mathbb{F}} \chi(tux'+suy'+(t^{2}+s^{2})uz'+u(x'^{2}+y'^{2})/4z'),$$
(3.6)

where Q is the set of non-zero squares in \mathbb{F} , and χ is a non-trivial additive character of \mathbb{F} .

Rearranging and completing two squares, we have

$$\begin{split} \hat{\mathbf{1}}_{X}(x',y',z',w') &= \sum_{z'\in Q} \sum_{x',y'\in\mathbb{F}} \chi(\frac{u}{4z'}(4tx'z'+4sy'z'+4(t^{2}+s^{2})z'^{2}+x'^{2}+y'^{2})) \\ &= \sum_{z'\in Q} \sum_{x',y'\in\mathbb{F}} \chi(\frac{u}{4z'}((x'^{2}+4tx'z'+4t^{2}z'^{2})+(y'^{2}+4sy'z'+4s^{2}z'^{2}))) \\ &= \sum_{z'\in Q} \sum_{x',y'\in\mathbb{F}} \chi(\frac{u}{4z'}((x'+2tz')^{2})\chi(\frac{u}{4z'}((y'+2sz')^{2})) \\ &= \sum_{z'\in Q} [\sum_{x'\in\mathbb{F}} \chi(\frac{u}{4z'}((x'+2tz')^{2})\sum_{y'\in\mathbb{F}} \chi(\frac{u}{4z'}((y'+2sz')^{2})] \\ &= \sum_{z'\in Q} [\sum_{\xi\in\mathbb{F}} \chi(u\xi^{2})\sum_{\eta\in\mathbb{F}} \chi(u\eta^{2})], \end{split}$$

where $\xi = \frac{x'+2tz'}{2\sqrt{z'}}$, $\eta = \frac{y'+2sz'}{2\sqrt{z'}}$. Finally, applying the identity from Theorem 1.4 yields

$$\hat{\mathbf{1}}_X = |Q||\mathbb{F}|$$
$$= O(|\mathbb{F}|^2).$$

Since $|X| \sim |\mathbb{F}|$, we have $|\hat{\mathbf{1}}_X| \sim |X|^{\frac{2}{3}}$. Thus for all $1 \leq p \leq \infty$ we have

$$\|\hat{\mathbf{1}_X}\|_{L^{p'}}\gtrsim |\hat{\mathbf{1}_X}|\sim |X|^{\frac{2}{3}}>|X|_{L^{r'}},$$

for all r' > 3/2, which corresponds to r < 3.

We can follow much the same procedure for the four-dimensional cone of the form $S' := \{(x, y, z, w) \in \mathbb{F}^d : x^2 - y^2 = wz\}$. Making the change of variables $\{x = tu, y = tu, z = (t^2 - s^2)u, w = u\}$ on S' and considering the characteristic function of the set $X' := \{(x', y', z', w') : z' \text{ is a square, } 4z'w' = x'^2 - y'^2\}$ yields the above result.

Chapter 4

Conclusion

The preceding work adds the bound $2 \rightarrow 4$ for general dimensional cones without hypersurfaces to the body of known results, and demonstrates r = 4 cannot be improved in this case. As we have seen from this discussion as well as Table 2.1, lines of investigation remain in several directions.

Although Mockenhaupt and Tao completely solve the problem over the curve $\gamma(t)$, there are no results on curves of any other form thus far. In the case of the paraboloid, Mockenhaupt and Tao conjecture that their bounds in three dimensions can be improved to $3 \rightarrow 3$ when -1 is a square, and $2 \rightarrow 3$ otherwise. They also expect that counterexamples to these bounds exist in higher dimensions. Consulting Table 2.1, we note that counterexamples and refined results need to be sought for each of the quadratic surfaces under consideration, when d is large.

In the case of high-dimensional cones, we expect to obtain improvements upon the 2 to 4 bound when we restrict ourselves to cones of lower degeneracy. Over four-dimensional fields, if we assume that the homogeneous variety H does not contain any 2 or 3 dimensional planes passing through the origin, such an improvement should be possible. In in [6] and [7], Iosevich and Koh find incidence bounds via the bounds on Gauss sums found in Section 1.2. Applying these incidence results to a dyadic pigeonholing argument yields restriction results for certain cases of the paraboloid and sphere. Attempting these methods on the cone provides incidence bounds which are not sharp enough to lead to an improvement on the $2 \rightarrow 4$ result.

In the Euclidean case, it is sometimes possible to improve upon restriction results by passing to a bilinear estimate. These consider bounds on a product of two functions supported on non-overlapping subsets of S, so that (1.1) becomes

$$\|(f_1 d\sigma_1)(f_2 d\sigma_2)\|\|_{L^r(\mathbb{F}^d, d\xi)} \lesssim \|f_1\|_{L^p(S_1, d\sigma)} \|f_2\|_{L^p(S_2, d\sigma_2)}.$$

Recently, Lewko and Lewko adapted this approach to the finite field setting in order to obtain bounds for paraboloids and spheres that were previously only known up to a logarithmic factor of q [11]. However, it is unclear precisely what bilinear estimates can imply about linear estimates over finite fields in the case of the cone. An improved understanding of finite bilinear restriction theorems, of the form obtained by Tao and Vargas for the Euclidean setting (see [19], [18]), would be very useful. A result linking linear and bilinear results for the finite cone may lead to progress on this problem. Considering functions supported over sections of the cone that do not share any common directions may allow us to improve our incidence theorems, and arrive at an improved restriction estimate.

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