Group Actions on Homotopy Spheres

by

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Abstract

In the first part of this thesis we discuss the rank conjecture of Benson and Carlson (5.2 in [7]). In particular, we prove that if G is a finite p-group of rank 3 and with p odd, or if G is a central extension of abelian p-groups, then there is a free finite G-CW-complex $X \simeq S^{n_1} \times ... \times S^{n_{rk(G)}}$; where $rk(G)$ is the rank of G.

We also treat an extension of the rank conjecture to groups of finite virtual cohomological dimension. In this context, for p a fixed odd prime, we show that there is an infinite group Γ satisfying the two following properties: every finite subgroup $G < \Gamma$ is a p-group with $rk(G) \leq 2$ and for every finite dimensional Γ-CW-complex $X \simeq S^n$ there is at least one isotropy subgroup Γ_σ with $rk(\Gamma_\sigma) = 2$.

In the second part of the thesis we discuss the study of homotopy G-spheres up to Borel equivalence. In particular, we provide a new approach to the construction of finite homotopy G-spheres up to Borel equivalence, and we apply it to give some new examples for groups of the form $C_p \rtimes C_{q^r}$.

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Chapter 1

Introduction

The origin of the study of group actions on spheres can be found in the spherical space form problem stated by H. Hopf in 1925. The problem concerns the classification of the finite groups that can act freely and smoothly on a sphere S^n.

The first result was due to P.A. Smith [35], who showed that if a finite group G acts freely on a sphere, then it must have periodic cohomology. Another necessary condition was found by Milnor [31]: in such a group, every element of order two must be central. Later, Madsen, Thomas and Wall [30] proved that these two necessary conditions are also sufficient: a finite group G acts freely and smoothly on a sphere if and only if G has periodic cohomology and every involution is central. From the homotopy point of view, Swan [38], proved that a finite group G has periodic cohomology if and only if there is a free finite G-CW-complex X homotopy equivalent to S^n.

These classical results have been extended in various directions. In this thesis we discuss and contribute to two such directions. In the first part we prove some results about the rank conjecture, while in the second part we discuss a new approach to the study of finite homotopy G-spheres [1] up to Borel equivalence. Despite dealing with similar objects, the two chapters are unrelated and can be read independently.

We begin by introducing the topic of the first part. It is a classical result that a finite group has periodic cohomology if and only if all of its abelian

[1] A homotopy G-sphere is a G-CW-complex X homotopy equivalent to S^n.

p-subgroups are cyclic. This last condition is commonly expressed using the notion of rank: The rank of a finite group G is the number $rk(G) = max\left\{k \in \mathbb{N}|\text{ there is a prime } p \text{ with } (\mathbb{Z}/p)^k < G\right\}$. Thus, a finite group G has periodic cohomology if and only if $rk(G) = 1$. From the aforementioned result of Swan, it follows that a group G has rank one if and only if there is a free finite G-CW-complex $X \simeq S^n$ [2]. Based on some of their own algebraic results, Benson and Carlson [7] suggested a first extension of Swan's theorem: the rank conjecture. The rank conjecture states that for any finite group G we have that $rk(G) = hrk(G)$, where $hrk(G)$ is the homotopy rank of G:

$$\min_{k \in \mathbb{N}} \{\text{there is a free finite dimensional G-CW-complex } X \simeq S^{n_1} \times ... \times S^{n_k}\}$$

With this notation, Swan's result says that $rk(G) = 1$ iff $hrk(G) = 1$.

It is worth noting that all the ingredients in the definition of $hrk(G)$ are essential for the conjecture to have a chance to be true. Every group G acts freely on some $EG \simeq *$ but EG has infinite dimension. The symmetric group Σ_3 cannot act freely on any sphere S^n ([31]), but it does act freely on some finite $X \simeq S^n$ ([38]), so that we require the G-CW-complex to only be homotopy equivalent to a product of spheres. Similarly the alternating group A_4 cannot act freely on any finite dimensional $X \simeq S^n \times S^n$ ([32]), but it does act freely on some finite $X \simeq S^n \times S^m$ ([2]), so that we don't require X to be homotopy equivalent to a product of equidimensional spheres.

Some results are known about the rank conjecture: Heller [25] showed that $(\mathbb{Z}/p)^3$ cannot act freely on a finite dimensional CW-complex homotopy equivalent to a product of two spheres. Adem and Browder [5] showed that if $(\mathbb{Z}/p)^m$ acts freely on $(S^n)^k$, then $m \leq k$. More recently, work of Adem and Smith [2] and Jackson [28], shows that if $rk(G) = 2$ then $hrk(G) = 2$ for a large family of groups. This includes the p-groups and the simple groups different from $PSL_3(\mathbb{F}_p)$, p odd. In this context, our main theorems are the following:

[2] Throughout the thesis the symbol "$\simeq$" will mean "homotopy equivalent".

Theorem 2.3.5. *For p an odd prime, every finite p-group of rank three acts freely on a finite CW-complex homotopy equivalent to the product of three spheres.*

Theorem 2.3.8. *Let G be a central extension of finite abelian p-groups. If $rk(G) = r$ then $hrk(G) \leq r$.*

Note that a converse to theorem 2.3.5 is given by Hanke in [23] in the sense that: if $(\mathbb{Z}/p)^r$ acts freely on $X = S^{n_1} \times ... \times S^{n_k}$ and if $p > 3dim(X)$, then $r \leq k$.

Another way of extending Swan's result is to consider infinite groups with periodic cohomology. In this case, results of Prassidis [33], Connolly and Prassidis [16] and Adem and Smith [2], show that a discrete group Γ acts freely and properly on $\mathbb{R}^n \times S^m$ if and only if Γ is a countable group with periodic cohomology.

Analogously, it is reasonable to ask for which other results (concerning the rank conjecture) can be extended to infinite groups.

The best candidates to study, are groups Γ with finite virtual cohomological dimension [3]. The reason is that for every such group Γ, there is a contractible finite dimensional Γ-CW-complex $\mathfrak{E}\Gamma$, with finite isotropy subgroups. The question for infinite groups seems more complicated than the one for finite groups for the following reason: every finite rank 2 p-group has a linear sphere with periodic isotropy subgroups. Our main result here states that, for infinite groups, the analogue property does not hold:

Theorem 2.4.1. *For p an odd prime, there is an infinite group Γ with finite virtual cohomological dimension, satisfying the two following properties: every finite subgroup $G < \Gamma$ is a p-group with $rk(G) \leq 2$ and for every finite dimensional Γ-CW-complex $X \simeq S^n$ there is at least one isotropy subgroup Γ_σ with $rk(\Gamma_\sigma) = 2$.*

[3] We say that Γ has finite virtual cohomological dimension if there is $\Gamma' < \Gamma$ with $|\Gamma/\Gamma'|$ finite and $H^n(\Gamma') = 0$ for all coefficients and for all n big enough.

We turn now our attention to the topic of the second part. Instead of requiring a free G-action and looking for a homotopy product of spheres that can sustain it; one can fix a homotopy sphere and ask for which different G-actions can occur for a fixed group G. This has been done, for example, by tom Dieck and Petrie [42] who initiated the study of homotopy representations. Another example is given by Dotzel and Hamrick in [18], where they show that, for a p-group G, each finite dimensional homotopy G-sphere is equivalent, in some sense, to a linear one. In this setting, we need a way of comparing G-spaces:

In general, for two G-spaces X and Y, to be equivariantly homotopy equivalent usually means the following: There are G-equivariant maps $f : X \to Y$ and $g : Y \to X$ and G-equivariant homotopies $F : X \times I \to X$ from $g \circ f$ to Id_X and $G : Y \times I \to Y$ from $f \circ g$ to Id_Y. Under this definition, the G-spaces EG and $*$ are not equivariantly homotopy equivalent because there is no equivariant map $* \to EG$. On the other hand, there is a G-equivariant map $EG \to *$ which is a homotopy equivalence.

With this is mind, we recall the following definition: two G-spaces X and Y are Borel equivalent if the Borel constructions $EG \times_G X$ and $EG \times_G Y$ are weak equivalent over BG. One can show that this happens if and only if there is a zig-zag of G-maps $X \to Z_1 \leftarrow ... \to Z_k \leftarrow Y$ each of which is a homotopy equivalence. Clearly, in this setting, EG is equivalent to $*$.

On the other hand, for any finite group G, Grodal and Smith [21] classified all the possible homotopy G-spheres up to Borel equivalence. To state the classification theorems in [21], we need to quickly introduce the following notation. (More details are in section 3.1). For a group G, we let Γ be the orbit category OrG. This is the category with $Ob(\Gamma) = \{G/H | H < G\}$ and $Mor(G/K, G/H) = \{f : G/K \to G/H | f \text{is a } G\text{-equivariant map}\}$. For a prime p, the category Γ_p, is the full subcategory of Γ defined by: $G/H \in Ob(\Gamma_p)$ if and only if H is a p-group.

We also need to recall that to each homotopy G-sphere X, we can associate a family of dimension functions $\left\{Dim^p_X(-)\right\}_{p||G|}$ in the following way: for all $p||G|$ and for all p-subgroups K, the homotopy fixed points $X^{hK} = Map_K(EK, X)$ have the mod p homology of a sphere. This yields dimension functions $Dim^p_X(-) : \Gamma_p \to \mathbb{N}$; $G/K \mapsto Dim(H^*(X^{hK}, \mathbb{F}_p))$ respecting fusion and satisfying the Borel-Smith condition [4]. With this notations, the first classification theorem of [21] is:

Theorem 1.0.1. *[21] Let X and Y be two homotopy G-spheres. Assume that for all $p||G|$ and for all p-subgroups $K < G$, we have that X^{hK} and Y^{hK} are connected. The space X is Borel equivalent to Y if and only if $Dim^p_X(-) = Dim^p_Y(-)$ for all $p||G|$. Moreover, every family of functions $\{D^p(-) : \Gamma_p \to \mathbb{N}\}_{p||G|}$ is realized as the dimension function family of a homotopy G-sphere, providing that:*

1. *$D^p(G/e) = D^q(G/e)$ for all $p, q||G|$;*
2. *$D^p(-)$ satifies the Borel Smith condition for all $p||G|$;*
3. *$D^p(-)$ respects fusion for all $p||G|$.*

In the second classification theorem, we denote by $C(\mathbb{F}_p)$ the category of chain complexes of left $\mathbb{F}_{(p)}$-modules and we write $C(\mathbb{F}_p\Gamma_p)$ for the category of contravariant functors $F : \Gamma_p^{op} \to C(\mathbb{F}_p)$.

Theorem 1.0.2. *[21] The family of functors $\{\Phi_p\}_{p||G|}$ defined by Φ_p : $\{$homotopy G-spheres$\} \to C(\mathbb{F}_p\Gamma_p)$, with $\Phi_p(X)(G/K) = C_*(Map_G(EG \times G/K, X), \mathbb{F}_p)$, satisfies the following properties:*

1. *For all X, the chain complex $\Phi_p(X)$ is quasi-isomorphic to a perfect $\mathbb{F}_p\Gamma_p$-chain complex.*
2. *Assume that for all $p||G|$ and for all p-subgroups $K < G$, we have that X^{hK} and Y^{hK} are connected. The chain complexes $\Phi_p(Y)$ and $\Phi_p(X)$ are quasi-isomorphic for all $p||G|$, if and only if X and Y are Borel equivalent.*

[4] A function $D^p(-) : \Gamma_p \to \mathbb{N}$ respects fusion, if $D^p(K) = D^p(K')$ whenever there is $g \in G$ with $gKg^{-1} = K'$.

We say that D^p satisfies the Borel-Smith condition if $D^p(-)|_P$ coincides with the dimension function of an orthogonal P-representation for all p-Sylow P.

Because of these theorems, it seems now possible to develop a finiteness obstruction theory for homotopy G-spheres up to Borel equivalence, in the realm of homological algebra over the orbit category. Such a problem has already been attacked by Jim Clarkson [14]. In particular, he was able to prove that all homotopy G-spheres are finite dimensional, up to Borel equivalence. Moreover, if $G = C_p \rtimes C_q$, he also showed that a homotopy G-sphere X is finite, up to Borel equivalence, if and only if $2q$ divides $Dim_X^p(G/e) - Dim_X^p(G/C_p)$. His method involves Dold's theory of algebraic Postnikov towers and relies on the assumption that $|G| = pq$.

Inspired by Clarkson's work [14], and using some of his results, we suggest a strategy where Postnikov towers are replaced by an arithmetic square. In particular, we provide a new approach to the construction of finite homotopy G-spheres, and we apply it to give new examples for groups of the form $C_p \rtimes C_{q^r}$.

Theorem 3.2.6. *Consider the group $G = C_p \rtimes C_{q^r}$ with faithful C_{q^r} action on C_p. For all $s \leq r$ and for all $j \geq 3$, there is a finite homotopy G-sphere X with:*

$$Dim_X^q(G/C_{q^t}) = \begin{cases} j + 2q^r & \text{if } t \leq s, \\ j & \text{otherwise.} \end{cases}$$

while $Dim_X^p(G/C_p) = j$.

Chapter 2

The Rank Conjecture

In the first section we introduce some background on equivariant obstruction theory. In the second section we use equivariant obstruction theory to prove the following auxiliary result: Let p be an odd prime, let G be a p-group and $S(V)$ a complex representation G-sphere. Then, for all integers $k \geq 0$, there exists a positive integer q such that the group $\pi_k(Aut_G(S(V^{\oplus q})))$ is finite.

We then incorporate this result in an outline of a known construction ([2], [16], [43]) that, in favourable conditions, gives a strategy to build group actions on products of spheres with controlled isotropy subgroups.

In the third section we use this construction twice: once to prove theorem 2.3.5 and once to generalize theorem 3.2 in [2] for p-groups G: if X is a finite dimensional G-CW-complex with abelian isotropy, we show that there is a free finite dimensional G-CW-complex $Y \simeq X \times S^1 \times ... \times S^{n_k}$. As a corollary we will be able to prove theorem 2.3.8.

Finally, in the fourth section, we discuss the extension of the rank conjecture to infinite groups and we prove theorem 2.4.1.

2.1 Equivariant obstruction theory

We now introduce some notions and results of equivariant obstruction theory that will be used in the sequel. In our outline we follow the classical references [40] and [41]. We include this section in order to make the proof of proposition 2.2.5 more readable. To this end, we are not going to state all the results in full generality.

Throughout this section, G will denote a finite group. A G-CW-complex is a CW-complex X together with a G-action such that:

1. For each $g \in G$ and each open cell E of X, the left translation gE is again an open cell of X,

2. If $gE = E$, then the induced map $E \to E$, $x \mapsto gx$ is the identity.

There are G-actions on CW-complexes which satisfy (a) but not (b). Usually, for a suitable subdivision, (b) is then satisfied. A pair of G-CW-complexes is a pair of CW-complexes (X, A) for which A and X are G-CW-complexes and the inclusion $A \to X$ is G-equivariant. The r-th skeleton of a pair (X, A) is the space $Sk_r(X, A) = Sk_r(X) \cup A$. To shorten the notation, we will write $X_r = Sk_r(X, A)$. Finally, we write $Dim(X, A)$ for the biggest dimension among the cells of X not in A.

The main object of study here is the homology and cohomology of such a pair (X, A). As usual, there is the cellular definition, suitable for defining the groups of the chain and cochain complexes, and there is the singular definition, suitable for the description of the differentials in the chain and cochain complexes. The two approaches agree, but we will not enter in the details of why they do so.

We begin with the more formal singular definition. Let (X, A) be a pair of G-CW-complexes with free G-action on $X \setminus A$. As usual, see for instance [24], intertwining the exact sequences of singular homology groups for the CW-pairs (X_n, X_{n-1}), we recover a chain complex of $\mathbb{Z}G$ modules $C_*(X, A)$:

$$\cdots \longrightarrow H_{n+1}(X_{n+1}, X_n) \longrightarrow H_n(X_n, X_{n-1}) \longrightarrow \cdots$$

If M is another $\mathbb{Z}G$-module, the cochain complex:

$$C^*_G(X, A; M) = Hom_{\mathbb{Z}G}(C_*(X, A), M)$$

yields cohomology groups $H^*_G(X, A; M)$.

We describe now the groups $C_n(X, A)$ in cellular terms. Since we assumed that the G-action on $X \setminus A$ is free, X_n is obtained from X_{n-1} as a pushout:

$$\begin{tikzcd} \coprod_{i\in J} G \times S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_{i\in J} G \times D^n \arrow[r] & X_n \end{tikzcd}$$

The corresponding characteristic map:

$$\phi : \coprod_{i\in J} G \times (D^n, S^{n-1}) \to (X_n, X_{n-1})$$

provides a canonical basis for the free $\mathbb{Z}G$-module $H_n(X_n, X_{n-1})$. It is given by the images of the canonical generators of $H_n(D^n, S^{n-1})$ under the j-th component of ϕ:

$$\{e\} \times (D^n, S^{n-1}) \longrightarrow G \times (D^n, S^{n-1}) \xrightarrow{\phi_j} (X_n, X_{n-1}) \, .$$

An element of $C^n_G(X, A; M)$ may thus be identified with a function on this basis with value in M.

Remark 2.1.1. *It is worth noting that the $\mathbb{Z}G$-module M can be thought of as a local coefficients system over $X/G\backslash A/G$. Consequently, $H^n_G(X, A; M) \cong H^n(X/G, A/G; M)$ (see section 8 of chapter 2 in [41]).*

We have so far talked about generic coefficients M. For our goals, the module M can be supposed to be of the following form: Let Y be a G-CW-complex. If Y is path-connected and n-simple, i.e. $\pi_1(Y, y)$ acts trivially on $\pi_n(Y, y)$, then the canonical map $\pi_n(Y, y) \to [S^n, Y]$ from pointed to free homotopy classes is bijective. The action of G on Y induces therefore a well defined action of G on $\pi_n(Y)$ and we can consider cochain complexes of the form $C^*_G(X, A, \pi_n Y)$. Obstruction classes live in such a complex. We will be interested in studying obstructions to extending homotopies. To this end, we first need to talk about obstructions to extending maps. The main result here is theorem 3.10 of [41]:

Theorem 2.1.2. *Let Y be a simply connected G-CW-complex. Let (X, A) be a path-connected pair of G-CW-complexes with G acting freely on $X \setminus A$. For each $n \geq 1$ there exists an exact obstruction sequence:*

$$[X_{n+1}, Y]_G \to Im([X_n, Y]_G \to [X_{n-1}, Y]_G) \xrightarrow{c^{n+1}} H_G^{n+1}(X, A; \pi_n Y)$$

which is natural in (X, A) and Y.

The exactness of this sequence means that each homotopy class $X_{n-1} \to Y$, which is extendable over X_n, has an associated obstruction element in the group $H_G^{n+1}(X, A; \pi_n Y)$. This obstruction element is zero if and only if the homotopy class $X_{n-1} \to Y$ is extendable over X_{n+1}.

Proof. Details can be found in [41]. Here we outline the construction of the map c^{n+1}, because it will be relevant to define the obstruction classes that we are interested in. Let $\phi : \coprod_{i \in J} G \times (D^{n+1}, S^1) \to (X_{n+1}, X_n)$ be the characteristic map and $e_j \in C_*(X, A)$ be the basis element corresponding to the j-th component ϕ_j. Fix an element $h \in [X_n, Y]_G$. The composition $h \circ \phi_j$ defines an element $c^{n+1}(h)(e_j) \in [S^n, Y] = \pi_n Y$. Extending by linearity, we recover an element $c^{n+1}(h) \in C_G^{n+1}(X, A; \pi_n Y)$.

This is not enough to define the announced map c^{n+1}, We also need to show that, if $[h_0]$ and $[h_1]$ are elements of $[X_n, Y]$ with the same image in $[X_{n-1}, Y]$, then $c^{n+1}(h_0)$ and $c^{n+1}(h_1)$ differ by a coboundary. For that purpose, choose a G-homotopy $k : I \times X_{n-1}$ with $k_i = h_i|_{X_{n-1}}$. Suppose that $\varphi : (D^n, S^{n-1}) \to (X, X_{n-1})$ is the characteristic map of an n-cell defining a basis element $e \in C_n(X, A)$. By composition, we recover a map:

$$\begin{array}{ccc} \{0\} \times D^n \cup I \times S^{n-1} \cup \{1\} \times D^n & \xrightarrow{\varphi \times Id} & \{0\} \times X_n \cup I \times X_{n-1} \cup \{1\} \times X_n \\ & & \big\downarrow{\scriptstyle (h_0, k, h_1)} \\ & & Y \end{array}$$

Composing with the standard homeomorphism $S^n \cong \partial I \times D^n \cup I \times S^{n-1}$, yields a homotopy class $x \in [S^n, Y] = \pi_n Y$. Setting $d(h_0, k, h_1) : C_n(X, A) \to \pi_n Y$; $e \mapsto x$ we recover an element $d(h_0, k, h_1) \in C^n_G(X, A; \pi_n Y)$. It turns out that $\delta d(h_0, k, h_1) = c^{n+1}(h_0) - c^{n+1}(h_1)$ as required. $\square$

As a by-product, the end of the proof also provides an obstruction cochain $d(h_0, k, h_1) \in C^n_G(X, A; \pi_n Y)$, defined for two maps $h_0, h_1 : X \to Y$ and a G-homotopy $k : I \times X_{n-1} \to X_{n-1}$, with $k_i = h_i|_{X_{n-1}}$. The properties of such an obstruction cochain are given in [41] and are summarized by:

Proposition 2.1.3. *With the notation above, the cochain $d(h_0, k, h_1)$ is a cocyle with homology class $\bar{d}(h_0, k, h_1) \in H^n_G(X, A; \pi_n Y)$. Moreover:*

1. *$\bar{d}(h_0, k, h_1) = -\bar{d}(h_1, k^-, h_0)$, where k^- is the inverse homotopy,*
2. *$\bar{d}(h_0, k, h_1) + \bar{d}(h_1, k', h_2) = \bar{d}(h_0, k + k', h_2)$,*
3. *$\bar{d}(h_0, k, h_1) = 0$ if and only if the G homotopy k extends to a G-homotopy $K : I \times X_n \to X_n$ with $K_i = h_i|_{X_n}$.*

We have being working under the assumption the G acts freely on $X \setminus A$, so far. Following [40] we are going to explain now why this is not a major restriction. As usual, G is a finite group, X a G-CW-complex and Y a simply connected G-CW-complex. Choose an indexing of the conjugacy classes of isotropy subgroups $\{(H_1), ..., (H_m)\}$ such that if $(H_j) < (H_i)$ then $i < j$. Consider the filtration $X_1 \subset ... \subset X_m$ given by $X_i = \{x \in X| \ (G_x) = (H_j)$ for some $j \leq i\}$. Such a filtration allows us to recover pairs $(X_i^{H_i}, X_{i-1}^{H_1})$ with free $WH_i = NH_i/H_i$-action on $X_i^{H_i} \setminus X_{i-1}^{H_i}$. Moreover:

Proposition 2.1.4. *[40] The $WH_i = NH_i/H_i$-action on $X_i^{H_i} \setminus X_{i-1}^{H_i}$ is free. Furthermore, given a G-map $k : X_{i-1} \to Y_{i-1}$, the extensions $K : X_i \to Y_i$ of f, are in bijective correspondence with the WH_i-extension $e : X_i^{H_i} \to Y_i^{H_i}$ of $k : X_{i-1}^{H_i} \to Y_{i-1}^{H_i}$.*

Proof. Set $X_{H_i} = \{x \in X | \ (G_x) = (H_i)\}$. Given K, we have $e = K^{H_i}$ and since $GX_{H_i} = X_i \setminus X_{i-1}$, the G map K is uniquely determined by K^{H_i}. Which shows injectivity. Conversely, suppose that we are given a WH_i-map $e : X_i^{H_i} \to Y_i^{H_i}$ extending k^H. We define a map $E : X_i \to Y_i$ by:

$$x \mapsto \begin{cases} e(x), & \text{if } x \in X_{i-1}; \\ ge(y), & \text{if } x = gy \text{ with } y \in X_i^H. \end{cases}$$

Following proposition 8.1.5 of [40], one can show that E is well defined and continuous. $\square$

We end the section by summarizing its application to the proof of 2.2.5: Consider two G-maps $f_1, f_2 : X \to Y$. Assume that there is i_0 such that $f_1|_{X_{i_0}} : X_{i_0} \to Y_{i_0}$ and $f_2|_{X_{i_0}} : X_{i_0} \to Y_{i_0}$ are homotopic. In order to know if f_1 and f_2 are homotopic, we want to know if the successive restrictions $f_1|_{X_i} : X_i \to Y_i$ and $f_2|_{X_i} : X_i \to Y_i$ are homotopic for all $i > i_0$.

Since homotopies are G-maps $F : I \times X \to Y$ with trivial G-action on I, we can consider $(I \times X)_i = I \times X_i$. A G-extension $K : I \times X_i \to Y_i$ of the given G-homotopy $k : I \times X_{i-1} \to Y_{i-1}$, exists if and only if there is a WH_i-extension of $k^{H_i} : I \times X_{i-1}^{H_i} \to Y_{i-1}^{H_i}$, by proposition 2.1.4.

By theorem 2.1.3, such a WH_i-extension exists, if and only if the difference cocycles $\bar{d}((f_1)_i, k^{H_i}, (f_2)_i) \in H_G^n(X_i^{H_i}, X_{i-1}^{H_i}; \pi_n Y_i)$ are all zero, for $n = 1, ..., Dim(X_i^{H_i}, X_{i-1}^{H_i})$. By virtue of remark 2.1.1, this homology groups satisfy: $H_G^n(X_i^{H_i}, X_{i-1}^{H_i}; \pi_n Y_i) \cong H^n(X_i^{H_i}/WH_i, X_{i-1}^{H_i}/WH_i; \pi_n Y_i)$, where $\pi_n Y_i$ is interpreted as a local coefficients system over $X_i^{H_i} \setminus X_{i-1}^{H_i}$.

In our case, the space Y will be a complex linear sphere, so that the local coefficients above are actually untwisted. For further reference, notice also that $X_{i-1}^{H_i} = \cup_{H > H_i} X^H$ while $X_i^{H_i} = X^{H_i}$.

2.2 A general construction

The main result of this section is the construction of proposition 2.2.6. A key ingredient of the construction is proposition 2.2.5, which says that under some conditions $\pi_k(Aut_G(S^n))$ is finite. We begin with some quick generalities. For a G-space X we write $Aut_G(X)$ for the monoid of equivariant self-homotopy equivalences of X. In other words, an element of $Aut_G(X)$ is an equivariant map $f : X \to X$ which is an homotopy equivalence and for which there exists an homotopy inverse $g : X \to X$ and homotopies $F, H : X \times I \to X$, between $f \circ g$ and Id_X, and between Id_X and $g \circ f$, which are all equivariant.

We are interested in the monoid $Aut_G(X)$ because, for a space Y, we have an injection of the G-equivariant X-fibrations over Y into $[Y, BAut_G(X)]$ (see [6]). In particular, in the construction of proposition 2.2.6, we will need to extend equivariantly a spherical fibration from S^{n-1} to D^n. Thus we want to study the groups $\left[S^{n-1}, BAut_G(S^m)\right] = \pi_{n-1}(BAut_G(S^m)) = \pi_{n-2}(Aut_G(S^m))$

The sequel of this section is structured with a series of lemmas and corollaries that we assemble into a proof of proposition 2.2.5. Lemmas 2.2.1, 2.2.2 and 2.2.4 are individual results needed in the proof of proposition 2.2.5. Lemma 2.2.3 serves the proof of lemma 2.2.4.

Lemma 2.2.1. *Let X be a G-CW-complex and let $Aut_G(X)$ be the monoid of G-equivariant self-homotopy equivalences of X. For each $f \in Aut_G(X)$ we write $Aut_G(X)_f$ for the path component of f. For $k > 0$, the map of unbased homotopy classes $\varphi : \left[S^k, Aut_G(X)_f\right] \to \left[S^k \times X, X\right]_G$ is injective and factors through:*

$$\begin{tikzcd} \left[S^k, Aut_G(X)_f\right] \arrow[r, "\varphi"] \arrow[d] & \left[S^k \times X, X\right]_G \\ \pi_k(Aut_G(X)_f) \arrow[ur] & \end{tikzcd}$$

In particular all G-equivariant homotopies $H : I \times S^k \times X \to X$ between maps representing the same element in $Im(\varphi)$ can be taken to satisfy $H(t, *, x) = H(t', *, x)$ for all $t, t' \in I$ and $x \in X$.

Proof. The map $\varphi : \left[S^k, Aut_G(X)_f\right] \to \left[S^k \times X, X\right]_G$ is clearly well defined. To see that it is injective, consider a G-equivariant homotopy $H : I \times S^k \times X \to X$ from $\varphi(g_1)$ to $\varphi(g_2)$. Clearly $H|_{\{0\}\times\{x_0\}\times X} = \varphi(g_1)(x_0, -) = g_1(x_0) \in Aut_G(X)_f$. Which implies that $H|_{\{t\}\times\{x\}\times X} \in Aut_G(X)_f$for all $(t, x) \in I \times S^k$ because $H|_{\{t\}\times\{x\}\times X} \simeq H|_{\{0\}\times\{x_0\}\times X}$ via a path in $I \times S^k$ from $(0, x_0)$ to (t, x). As a result, H defines an homotopy from g_1 to g_2. To prove that φ factors through:

$$\begin{tikzcd}
\left[S^k, Aut_G(X)_f\right] \arrow[r, "\varphi"] \arrow[d] & \left[S^k \times X, X\right]_G \\
\pi_k(Aut_G(X)_f) \arrow[ur] &
\end{tikzcd}$$

we want to show that the map $\pi_k(Aut_G(X)_f) \to \left[S^k, Aut_G(X)_f\right]$ is a bijection. Observe that $Aut_G(X)$ is a monoid, thus an H-space so that $\pi_1(Aut_G(X)_{Id})$ acts trivially on $\pi_k(Aut_G(X)_{Id})$. The monoid $Aut_G(X)$ is very nice because all of its connected components are homotopy equivalent through maps of the form: $Aut_G(X)_{Id} \to Aut_G(X)_f$ with $g \mapsto f \circ g$. Consequently $\pi_1(Aut_G(X)_f)$ acts trivially on $\pi_k(Aut_G(X)_f)$ for all $f \in Aut_G(X)$. We conclude that $\pi_k(Aut_G(X)_f) \to \left[S^k, Aut_G(X)_f\right]$ is a bijection. The last claim directly follows from the diagram. □

Lemma 2.2.2. *Let G be a finite group acting on a space X. Let $H_1 < G$ be an isotropy subgroup maximal among isotropy subgroups. Set $X_1 = \{x \in X |\ G_x \in (H_1)\}$, where (H_1) denotes the conjugacy class of H_1. We then have that $Aut_G(X_1) \cong Aut_{WH_1}(X^{H_1})$ (here $WH_1 = NH_1/H_1$ is the Weil group).*

Proof. Let's begin by studying X_1. Clearly $X_1 \subset \cup_{H\in(H_1)} X^H$. Since $H \in (H_1)$ is supposed to be maximal, we must have that if $x \in X^H$, then $G_x = H$

so that $X_1 = \cup_{H \in (H_1)} X^H$. Similarly, if $x \in X^H \cap X^{H'}$, for H, H' in (H_1), then $H = G_x = H'$. As a result $X_1 = \cup_{H \in (H_1)} X^H$.

Observe next that a G-equivariant map $f : X_1 \to X_1$ restricts to a WH_1-equivariant map $f_1 : X^{H_1} \to X^{H_1}$ because $WH_1 = NH_1/H_1$ and H_1 acts trivially on X_1. The same holds for a G-equivariant homotopy $F : I \times X_1 \to X_1$, so that we have a well defined map $res : Aut_G(X_1) \to Aut_{WH_1}(X^{H_1})$.

One can then show that the map $res : Aut_G(X_1) \to Aut_{WH_1}(X^{H_1})$ has an inverse given by $res^{-1}(f)(x) = gf(g^{-1}x)$, where $g \in G$ is such that $g^{-1}x \in X^{H_1}$.

$\square$

Lemma 2.2.3. *Let G be a finite group and S^n a linear G-sphere. If $0 < k < n$ then $H^n(S^k \times S^n/G, \{*\} \times S^n/G, \mathbb{Z})$ is finite.*

Proof. Consider the long exact sequence of the pair $(S^k \times S^n/G, \{*\} \times S^n/G)$ with integer coefficients:

$$\begin{tikzcd}
H^{n-1}(S^k \times S^n/G) \arrow[r, two heads] & H^{n-1}(\{*\} \times S^n/G) \arrow[d] \\
H^n(S^k \times S^n/G) \arrow[d, "i^*"] & H^n(S^k \times S^n/G, \{*\} \times S^n/G) \arrow[l] \\
H^n(\{*\} \times S^n/G) &
\end{tikzcd}$$

Clearly $H^n(S^k \times S^n/G, \{*\} \times S^n/G) \subset Ker(i^*)$. But $H^n(S^k \times S^n/G) \cong H^n(\{*\} \times S^n/G) \oplus H^{n-k}(\{*\} \times S^n/G)$. Thus for $i^* : H^n(S^k \times S^n/G) \to H^n(\{*\} \times S^n/G)$ we have that $Ker(i^*) \cong H^{n-k}(\{*\} \times S^n/G)$. Finally, the groups $H^{n-k}(S^n/G)$ are finite for $0 < k < n$ because $H^{n-k}(S^n/G, \mathbb{Q}) = 0$ by the Vietoris-Begle theorem. $\square$

Lemma 2.2.4. *Let G be a finite group and $S(V)$ a linear G-sphere. For $H \leq G$ write $n_r(H)$ for the integer such that $S(V^{\oplus r})^H = S^{n_r(H)}$. For all*

$k > 0$ there is an integer $q > 0$ such that the groups:

$$H^{n_q(H_i)}(S^k \times S^{n_q(H_i)}/WH_i, \cup_{H>H_i} S^k \times S^{n_q(H)}/WH_i \cup \{\star\} \times S^{n_q(H_i)}/WH_i, \mathbb{Z})$$

are finite for all H_i with $n_1(H_i) > 0$.

Proof. Fix a subgroup $H_i < G$ such that $n_1(H_i) > 0$. If there is $H > H_i$ with $n_1(H) = n_1(H_i)$, then the required cohomology group is zero (it is of the form $H^{n(H_i)}(X, X, \mathbb{Z})$). Assume that for all $H > H_i$ we have $n_1(H) < n_1(H_i)$. In this case we want so show that we can take enough direct sums to be in the situation of lemma 2.2.3.

Let $n_{r,i} = max_{H>H_i} \{n_r(H)\}$ and $m_{r,i} = n_r(H_i) - n_{r,i} > 0$. Observe that $n_r(H) = rn_1(H) + (r-1)$ so that $n_{r,i} = rn_{1,i} + (r-1)$ and $m_{r,i} = n_r(H_i) - n_{r,i} = rn_1(H_i) + (r-1) - (rn_{1,i} + (r-1)) = rm_{1,i}$. Therefore there is a q_i big enough such that $m_{r,i} > k+2$. In other words $n_r(H_i) - k - 2 > n_{r,i}$. We have found an integer $q_i > 0$ such that all the cells τ of the CW-complex $S^{n_{q_i}(H_i)}$ of dimension $dim(\tau) \geq n_{q_i}(H_i) - k - 2$, are also cells of the relative CW-complex $(S^{n_{q_i}(H_i)}, \cup_{H>H_i} S^{n_{q_i}(H)})$.

We turn now our attention to the announced cohomology group. By our condition on the cells of $S^{n_{q_i}(H_i)}$, we have that the cells τ of the CW-complex $S^k \times S^{n_{q_i}(H_i)}/WH_i$ of dimension $dim(\tau) \geq n_{q_i}(H_i) - 2$, are also cells of the relative CW-complex $(S^k \times S^{n_{q_i}(H_i)}/WH_i, \cup_{H>H_i} S^k \times S^{n_{q_i}(H)}/WH_i)$. Henceforth: $H^{n_{q_i}(H_i)}(S^k \times S^{n_{q_i}(H_i)}/WH_i, \cup_{H>H_i} S^k \times S^{n_{q_i}(H)}/WH_i \cup \{\star\} \times S^{n_{q_i}(H_i)}, \mathbb{Z}) = H^{n(H_i)}(S^k \times S^{n_{q_1}(H_i)}/WH_i, \{\star\} \times S^{n_{q_i}(H_i)}/WH_i, \mathbb{Z})$. This last group is finite, by virtue of lemma 2.2.3. We conclude by observing that we can then set $q = max_{H_i<G} \{q_i\}$. □

Proposition 2.2.5. *Let G be a finite p-group. Let $S(V)$ be a complex representation G-sphere. For all integers $k \geq 0$ there exists an integer $q > 0$ such that $\pi_k(Aut_G(S(V^{\oplus q})))$ is finite.*

Proof. If $k = 0$, the result has been proven in [19]. Assume that $k > 0$. Before explaining how the proof proceeds, we recall some notation: Choose an ordering of the conjugacy classes of isotropy subgroups $\{(H_1), ..., (H_m)\}$ such that if $(H_j) < (H_i)$ then $i < j$. Consider the filtration $S(V)_1 \subset ... \subset S(V)_m = S(V)$ given by $S(V)_i = \{x \in S(V)|\ (G_x) = (H_j);\ j \leq i\}$. We have homomorphisms $R_i : \pi_k(Aut_G(S(V))) \to \pi_k(Aut_G(S(V)_i))$ and $S_i : \pi_k(Aut_G(S(V)_i)) \to \pi_k(Aut_G(S(V)_{i-1}))$. Here is how the proof runs. Look at the commutative diagram:

$$\begin{tikzcd}
\pi_k(Aut_G(S(V))) \arrow[r, "Id"] \arrow[ddr, "R_i"] \arrow[ddddr, "R_1"'] & \pi_k(Aut_G(S(V)_m)) \arrow[d] \\
& \cdots \arrow[d, "S_{i+1}"] \\
& \pi_k(Aut_G(S(V)_i)) \arrow[d, "S_i"] \\
& \cdots \arrow[d, "S_2"] \\
& \pi_k(Aut_G(S(V)_1))
\end{tikzcd}$$

Clearly to prove that $\pi_k(Aut_G(S(V)))$ is finite is the same as to prove that $Im(R_m)$ is finite. To prove that $Im(R_m)$ is finite, we will show by induction over i that $Im(R_i)$ is finite. Such an induction can be performed by showing that $Im(R_1)$ is finite and that $S_i^{-1}(R_{i-1}(f)) \cap Im(R_i)$ is finite for all i and for all $f \in \pi_k(Aut_G(S(V)))$. This outline can only be carried out up to replacing $S(V)$ with some power $S(V^{\oplus q})$.

We begin by showing that there is $q_1 > 0$ such that $\pi_k(Aut_G(S(V^{\oplus q_1})_1))$ is finite. In particular we will have that $Im(R_1) \subset \pi_k(Aut_G(S(V^{\oplus q_1})_1))$ is finite. For $H < G$ write $n_r(H)$ for the integer such that $S(V^{\oplus r})^H = S^{n_r(H)}$. Observe that $n_r(H) = rn(H) + (r-1)$. By lemma 2.2.2 we have that $\pi_k(Aut_G(S(V)_1)) = \pi_k(Aut_{WH_1}(S^{n_1(H_1)}))$. The WH_1-action on $S^{n_1(H_1)}$ is free because H_1 is maximal among isotropy subgroups. Therefore proposition 2.4 of [16] says that $\pi_k(Aut_{WH_1}(S^{n_1(H_1)}))$ is finite if $k < n_1(H_1) - 1$. If $k \geq n_1(H_1) - 1$, then there is a $q_1 > 0$ for which $k < q_1 n_1(H_1) + (q_1 - 1) - 1 = n_{q_1}(H_1) - 1$. As a result $\pi_k(Aut_G(S(V^{\oplus q_1})_1)) = \pi_k(Aut_{WH_1}(S^{n_{q_1}(H_1)}))$ is finite (always by proposition 2.4 of [16]).

Assume that we showed that $Im(R_{i-1})$ is finite. The inductive step is to prove that there is $q \geq q_{i-1}$ such that $S_i^{-1}(R_{i-1}(f)) \cap Im(R_i)$ is finite for all i and for all $f \in \pi_k(Aut_G(S(V^{\oplus q})))$. For that purpose we are going to use equivariant obstruction theory a la Tom Dieck (see [40] section 8 and [41] chapter 2). We begin with some preliminaries. As in lemma 2.2.4, let $q' > 0$ be such that the groups:

$$H^{n_{q'}(H')}(S^k \times S^{n_{q'}(H')}/WH', \cup_{H>H'} S^k \times S^{n_{q'}(H)}/WH' \cup \{\star\} \times S^{n_{q'}(H')}/WH')$$

are finite for all $H' < G$ with $n_1(H') > 0$. Let $q = max\{q_1, q'\}$. To simplify the notation we write $W = V^{\oplus q}$, $X = S^k \times S(W)$ and $\bar{X}^{H_i} = \cup_{H>H_i} X^H \cup \{\star\} \times S(W)^{H_i}$. With this notation we have that the group:

$$H^{n_q(H_i)}(X^{H_i}/WH_i, \bar{X}^{H_i}/WH_i, \pi_{n_q(H_i)}(S^{n_q(H_i)}))$$

is finite by lemma 2.2.4, while if $r \neq n_q(H_i)$ then the groups:

$$H^r(X^{H_i}/WH_i, \bar{X}^{H_i}/WH_i, \pi_r(S^{n_q(H_i)}))$$

are finite because they are finitely generated torsion abelian groups. (The fixed points of a complex representation spheres are odd-dimensional spheres whose homotopy groups are all but one finite).

A word of explanation is in order here: the space X^{H_i} is the one over which we want to extend a map already defined on $\bar{X}^{H_i}$ (see end of section 2.1). We have a map over the part of $\bar{X}^{H_i}$ given by $\cup_{H>H_i} X^H$, because of the inductive hypothesis. We have a map over the part of $\bar{X}^{H_i}$ given by $\{*\} \times S(W)^{H_i}$, because all the maps come from $\pi_k(Aut_G(S(W)))$ in the following way: By lemma 2.2.1 there is an injection $\pi_k(Aut_G(S(W)_i)) \rightarrow$

$\left[S^k \times S(W)_i, S(W)_i\right]_G$ yielding a diagram with injective columns:

$$\begin{tikzcd} \pi_k(Aut_G(S(W)_i)) \arrow[r, "S_i"] \arrow[d, "\varphi_i"'] & \pi_k(Aut_G(S(W)_{i-1})) \arrow[d, "\varphi_{i-1}"'] \\ {[X_i, S(W)_i]_G} \arrow[r, "s_i"'] & {[X_{i-1}, S(W)_{i-1}]_G} \end{tikzcd}$$

To prove that $S_i^{-1}(R_{i-1}(f)) \cap Im(R_i)$ is finite, it is enough to prove that $s_i^{-1}(\varphi_{i-1}(R_{i-1}(f))) \cap \varphi_i(Im(R_i))$ is finite. By abuse of notation we will keep on writing S_i and $R_{i-1}(f)$, but we will think of them as living in the bottom row of the diagram. Now, a homotopy h between $R_{i-1}(f)$ and $R_{i-1}(g)$ is constant over $\{*\} \times S(W)_{i-1}$ where it coincides with both $R_{i-1}(f)$ and $R_{i-1}(g)$. Consequently, h can be extended to a homotopy from $R_i(f)|_{X_{i-1} \cup \{*\} \times S(W)_i}$ to $R_i(g)|_{X_{i-1} \cup \{*\} \times S(W)_i}$.

Henceforth, writing $X_{i-1} \cup S(W)_i$ for $X_{i-1} \cup \{*\} \times S(W)_i$, we can apply equivariant obstruction theory inductively over r to each of the diagrams:

$$\begin{tikzcd} {[Sk_{r+1}(X_i, X_{i-1} \cup S(W)_i), S(W)_i]_G} \arrow[r, "S_{i,r+1}"] \arrow[d, "Sk_r"'] & {[X_{i-1}, S(W)_{i-1}]_G} \\ {[Sk_r(X_i, X_{i-1} \cup S(W)_i), S(W)_i]_G} \arrow[ur, "S_{i,r}"'] & \end{tikzcd}$$

If $r = 0$, then $Sk_0(X_i, X_{i-1} \cup S(W)_i) = X_{i-1} \coprod \{x_0, ..., x_l\}$. Consequently, $S_{i,0}^{-1}(R_{i-1}(f)) \cap Sk_0(Im(R_i))$ depends on the connected components of the space $S(W)_{i-1}$. But $S(W)_{i-1}$ has finitely many connected components because it is a finite CW-complex, therefore $S_{i,0}^{-1}(R_{i-1}(f)) \cap Sk_0(Im(R_i))$ is finite. From now on, to simplify the notation, we are going to write $f_i = R_i(f)$ for all possible i and f. Assume that $S_{i,r}^{-1}(f_{i-1}) \cap Sk_r(Im(R_i)) = \left\{g_{i,r}^1, ..., g_{i,r}^t\right\}$ is finite of order t (i.e. $g_{i,r}^j \neq g_{i,r}^l$ if $j \neq l$). For each $g_{i,r+1} \in S_{i,r+1}^{-1}(f_{i-1}) \cap Sk_{r+1}(Im(R_i))$ there is a unique $g_{i,r}^j$ and a homotopy h from $g_{i,r} = g_{i,r+1}|_{Sk_r(X_i, X_{i-1} \cup S(W)_i)}$ to $g_{i,r}^j$ Notice that, by definition, we have $g^j, g \in \pi_k(Aut_G(S(W)))$ with $g_{i,r+1} = g_i|_{Sk_{r+1}(X_i, X_{i-1} \cup S(W)_i)}$ and $g_{i,r}^j = g_i^j|_{Sk_r(X_i, X_{i-1} \cup S(W)_i)}$.

We write $\bar{d}(g_{i,r}, h, g_{i,r}^j) \in H^{r+1}(X^{H_i}/WH_i, \bar{X}^{H_i}/WH_i, \pi_{r+1}((S^n)^{H_i}))$ for the homology class of the difference cocyle as in the preceding section. Now, if $\bar{d}(g'_{i,r+1}, h', Sk_{r+1}(g_i^j)) = \bar{d}(g_{i,r+1}, h, Sk_{r+1}(g_i^j))$, then we have:

$$\bar{d}(g'_{i,r+1}, h' + h^{-1}, g_{i,r+1}) = \bar{d}(g'_{i,r+1}, h', Sk_{r+1}(g_i^j)) + \bar{d}(Sk_{r+1}(g_i^j), h^{-1}, g_{i,r+1})$$

$$= \bar{d}(g'_{i,r+1}, h', Sk_{r+1}(g_i^j)) - \bar{d}(g_{i,r+1}, h, Sk_{r+1}(g_i^j)) = 0$$

so that $g_{i,r+1} \simeq g'_{i,r+1}$. We can therefore define an injection:

$$\begin{array}{c} S_{i,r+1}^{-1}(f_{i-1}) \cap Sk_{r+1}(Im(R_i)) \\ \big\downarrow \\ \coprod_{j=1}^{t} \left\{ (g_{i,r}^j) \right\} \times H^{r+1}(X^{H_i}/WH_i, \bar{X}^{H_i}/WH_i, \pi_{r+1}(S^{n_q(H_i)})) \end{array}$$

by setting $g_i \mapsto \left\{ g_i^j \right\} \times \bar{d}(g_{i,r+1}, h, Sk_{r+1}(g_i^j))$. Since we chose the integer q in order to have all the cohomology groups on the right hand side to be finite, we must have that the left hand side is finite as well.

Summarizing, by induction we have that $S_{i,r+1}^{-1}(f_{i-1}) \cap Sk_{r+1}(Im(R_i))$ is finite for all r. Since X is finite dimensional, this shows that $S_i^{-1}(f_{i-1}) \cap Im(R_i)$ is finite. We conclude as explained in the outline at the beginning of this proof. $\square$

We can now turn our attention to the main result of this section, the construction of proposition 2.2.6. We begin by giving a brief summary of the goal of the construction: We give conditions under which it is possible to "attach" a linear sphere to a finite dimensional G-complex X, in such a way that the final result is a finite dimensional G-CW-complex $Y \simeq X \times S^n$, whose isotropy groups are smaller than the one of the original space X. To keep track of the evolution of the isotropy subgroups in the process of attaching spheres, we introduce the following notation:

Let G be a finite group and X a G-CW-complex. We write:

$$rk_X(G) = max\,\{n \in \mathbb{N}| \text{ there exists } G_\sigma \text{ with } rk(G_\sigma) = n\}$$

Proposition 2.2.6. *Let G be a finite p-group and let X be a finite dimensional G-CW-complex. Assume that to each isotropy subgroup G_σ we can associate a representation $\rho_\sigma : G_\sigma \to U(n)$ such that $\rho_\sigma|_{G\tau} \cong \rho_\tau$ whenever $G_\tau < G_\sigma$. If ρ_σ is fixed point free for all G_σ with $rk(G_\sigma) = rk_X(G)$, then there exists a finite dimensional G-CW-complex $E \cong X \times S^m$ with $rk_E(G) = rk_X(G) - 1$. Moreover, if X is finite then E is finite as well.*

Proof. The proof follows [16]. We refer the reader to [43] for the details. Write S^{2n-1}_σ for the linear sphere associated to ρ_σ. We want to glue these spheres into a G-equivariant spherical fibration over X. We will proceed by induction over the skeleton of X. For every G-orbit of the 0-skeleton, choose a representative σ and define a map $G \times_{G_\sigma} S^{2n-1}_\sigma \to X^0$ by $(g, x) \mapsto g \cdot \sigma$. This defines a G-equivariant spherical fibration $S^{2n-1} \to E_0 \to Sk_0(X)$ whose total space is a finite dimensional G-CW-complex. Clearly if ρ_σ is fixed point free for all G_σ with $rk(G_\sigma) = rk_X(G)$, then $rk_{E_0}(G) = rk_X(G) - 1$.

The inductive step is next. Suppose given a G-equivariant spherical fibration over the $(k-1)$-skeleton $*^{q_{k-1}}S^{2n-1} \to E_{k-1} \to Sk_{k-1}(X)$ whose total space is a finite dimensional G-CW-complex. Assume also that if ρ_σ is fixed point free for all G_σ with $rk(G_\sigma) = rk_X(G)$, then $rk_{E_{k-1}}(G) = rk_X(G) - 1$. Now, for every G-orbit of a k-cell, choose a representative σ. The G_σ-equivariant fibration $*^{q_{k-1}}S^{2n-1} \to E_{k-1}|_{\partial\sigma} \to \partial\sigma$ is classified by an element $a_\sigma \in \pi_{k-2}(Aut_{G_\sigma}(*^{q_{k-1}}S^{2n-1}))$.

We want to have $a_\sigma = 0$: Observe that, in general, for two complex G-spheres $S(V)$ and $S(W)$, we have that $S(V \oplus W) \cong S(V) * S(W)$ as G-spheres. Therefore, by proposition 2.2.5, we can take enough Whitney sums of the fibration $*^{q_{k-1}}S^{2n-1} \to E_{k-1} \to Sk_{k-1}(X)$ to guarantee that $a_\sigma = 0$ (see lemma 2.3 and proposition 2.4 in [16]). We can then extend the G_σ-equivariant fibration $*^{q_k}S^{2n-1} \to E_{k-1}|_{\partial\sigma} \to \partial\sigma$ equivariantly across the cell σ. We define a G-equivariant spherical fibration over the orbit of σ by

$G \times_{G_\sigma} *^{q_k} S^{2n-1}_\sigma \to G\sigma$ with $(g, x) \mapsto g \cdot \sigma$.

Repeating the procedure for all the representatives of the G-orbits of k-cells, we recover a G-equivariant spherical fibration $*^{q_k} S^{2n-1} \to E_k \to Sk_k(X)$ with total space a finite dimensional G-CW-complex. Clearly if ρ_σ is fixed point free for all G_σ with $rk(G_\sigma) = rk_X(G)$, then $rk_{E_k}(G) = rk_X(G) - 1$. We conclude noticing that, by proposition 2.8 in [2], up to taking further fiber joins, we can assume that the total fibration $*^q S^{2n-1} \to E \to X$ is a product one.

For the last statement, one can observe that all the constructions take place in the category of finite CW-complexes, providing that the initial space X is a finite CW-complex. □

We end the section by connecting the construction to our problem. Observe first that $rk_X(G) = 0$ if and only if the action is free. Assume that $X \simeq S^{n_1} \times ... \times S^{n_t}$ is a finitely dimensional G-CW-complex with $rk_X(G) = rk(G) - t$. In this case we want to show that the conditions of proposition 2.2.6 are fulfilled in order to recover a suitable space $Y_1 \simeq S^{n_1} \times ... \times S^{n_t} \times S^{n_{t+1}}$ with $rk_{Y_1}(G) = rk(G) - (t + 1)$. Clearly, we then wish to apply proposition 2.2.6 again and again until we recover a suitable free space $Y_{rk(G)-t} \simeq S^{n_1} \times ... \times S^{n_t} \times S^{n_{rk(G)}}$.

In order to have the required G-space X to begin our process with, we consider the center $Z(G)$ of G. Since G is a p-group, $Z(G)$ is not trivial and acts freely on some $S^{n_1} \times ... \times S^{n_t}$ where each S^{n_i} is the linear sphere of a representation ρ_i of $Z(G)$. Consider the induced representations $\eta_i = Ind^G_{Z(G)} \rho_i$. We recover a G-space $X = S^{m_1} \times ... \times S^{m_t}$, where S^{m_i} is the linear G-sphere corresponding to η_i and $rk_X(G) = rk(G) - t$.

The question of deciding which p-groups satisfy the additional conditions given by repeatedly applying proposition 2.2.6, seems considerably harder. In the next chapter we exhibit two families of p-groups for which the conditions are satisfied.

2.3 Some p-groups

Before applying the strategy outlined at the end of the previous section to odd order rank three p-groups, we need some group theory. It comes from an unpublished work of Jackson [28] and we reproduce it here:

Lemma 2.3.1. *If G is a finite p-group with $rk(G) = 3$ and $rk(Z(G)) = 1$, then there exists a normal abelian subgroup $Q < G$ of type (p, p) with $Q \cap Z(G) \neq 0$.*

Proof. This follows from statements 4.3 and 4.5 in Suzuki [37]. $\square$

Proposition 2.3.2. *Let G be a finite p-group with $p > 2$, $rk(G) = 3$ and $rk(Z(G)) = 1$. Let Q be an abelian normal subgroup of type (p, p) as above. Suppose that $H < G$ with $H \cap Z(G) = 0$ and $|H| = p^n$. Then either H is cyclic, $H < C_G(Q)$, H is abelian of type (p, p^{n-1}) or $H \cong M(p^n) =< x, y | x^{p^{n-1}} = y^p = 1, y^{-1}xy = x^{1+p^{n-2}} >$.*

Proof. If $rk(H) = 1$ then H is cyclic since $p > 2$. Suppose that $rk(H) = 2$ and $H \cap Q \neq 0$. By assumption $Z(G) \cap Q = \mathbb{Z}/p$, $H \cap Z(G) = 0$ and $Q \cap H = \mathbb{Z}/p$. The map $c : H \to Aut(H \cap Q)$ given by $c_h(x) = hxh^{-1}$ is well defined because Q is normal. Since $|H| = p^n$ and $|Aut(H \cap Q)| = p - 1$, we have that the map c is trivial. As a result we have that $H < C_G(Q)$.

Assume now that $H \cap Q = 0$. In this case $H \cap C_G(Q) \neq H$ since otherwise we would have $rk(G) > 3$. Set $K = H \cap C_G(Q)$ and observe that K is cyclic (else we would have $rk(G) > 3$). Assume for a moment that $[G : C_G(Q)] = p$. In this case $[H : K] = p$, in other words H has a maximal cyclic subgroup. By [37] section 4, H needs then to be abelian of type (p, p^{n-1}) or $M(p^n)$.

We still have to prove that $[G : C_G(Q)] = p$. The group G acts on Q by conjugation and for each element q of Q not in center of G, we have that $G_q = C_G(Q)$. As a result $|C_G(Q)| = |G_q| = |G|/p$ since $Q \cong (Z/p)^2$ with the first coordinate in the center $Z(G)$. $\square$

Proposition 2.3.3. *Let G be a finite p-group with $p > 2$, $rk(G) = 3$ and $rk(Z(G)) = 1$. There exists a class function $\beta : G \to \mathcal{C}$ such that for any subgroup $H \subset G$, with $H \cap Z(G) = 0$, the restriction $\beta|_H$ is a complex character of H. If in addition H is a rank two elementary abelian subgroup, then the character $\beta|_H$ corresponds to an isomorphism class of fixed-point free representations.*

Proof. Define $\beta : G \to \mathcal{C}$ as follows:

$$x \mapsto \begin{cases} (p^2 - p)|G|, & \text{if } x = 0; \\ 0, & \text{if } x \in Z(G) \setminus 0; \\ -p|G|, & \text{if } x \in Q \setminus Z(G); \\ 0, & \text{if } x \in C_G(Q) \setminus Q; \\ -|G|, & \text{if } x \in G \setminus C_G(Q) \text{ of order p}; \\ 0 & \text{if } x \in G \setminus C_G(Q) \text{ of order greater than p.} \end{cases}$$

The map β is a class function because we have the following sequence of subgroups each normal in G:

$$0 < Z(G) < Q < C_G(Q) < G.$$

For the sequel of the proof, we set $\phi_k : \mathbb{Z}/p^n \to \mathcal{C};\ x \mapsto e^{2\pi ikx/n}$ while $\phi_k\phi_j : \mathbb{Z}/p^n \times \mathbb{Z}/p^n \to \mathcal{C};\ (x, y) \mapsto e^{2\pi i(kx+jy)/n}$. To lighten the notation, we drop the dependence in n on the ϕ_k's because it will be clear from the context. To understand correctly the characters that follow, it will be important to specify the generators of the elementary abelian subgroups treated.

Consider first an elementary abelian subgroup H of G of rank 2 and which intersects trivially the center $Z(G)$. If $H \cap Q \neq 0$, then $H \cap Q \cong \mathbb{Z}/p$ and $H \cong (H \cap Q) \times \mathbb{Z}/p$ so that:

$$\beta|_H = |G| \sum_{k=0}^{p-1} \sum_{j=1}^{p-1} \phi_k\phi_j$$

If $H \cap Q = 0$, then $\mathbb{Z}/p \cong H \cap C_G(Q)$ so that $H \cong (H \cap C_G(Q)) \times \mathbb{Z}/p$ and:

$$\beta|_H = (p-1)|G|/p \left(\sum_{k=0}^{p-1} \sum_{j=1}^{p-1} \phi_j \phi_k \right) + |G| \sum_{k=1}^{p-1} \phi_0 \phi_k.$$

Consider now a subgroup H of G with $H \cap Z(G) = 0$. We will proceed case by case using the classification above.

1. If $H \cap Q \neq 0$ then $H \subset C_G(Q)$ and $|K| = p$ with $K = Q \cap H$. Let ϕ be the character of K which is $p-1$ on the identity and -1 for each other element of K. Then:

$$\beta|_H = \frac{p|G|}{|H:K|} Ind_K^H \phi.$$

2. If $H \cap Q = 0$ and $H \subset C_G(Q)$ then H is cyclic and $\beta|_H = |G|/|H|(p^2 - p)\phi$ where ϕ is the character of H that is $|H|$ on the identity and 0 elsewhere.

3. If $H \cap Q = 0$ and H is cyclic with $H \cap C_G(Q) = 0$, then $|H| = p$ and $\beta|_H = |G|/p\phi$ where ϕ is $(p^3 - p^2)$ on the identity and $-p$ elsewhere.

4. If $H \cap Q = 0$ and H is cyclic with $H \cap C_G(Q) \neq 0$, then $\beta|_H = (p^2 - p)|G|/|H| \sum_{k=1}^{|H|} \phi_k$.

5. Assume that $H \cap Q = 0$ and that H is abelian of type (p, p^{n-1}). Write $H =< x, y \in H | x^p = y^{p^{n-1}} = 1, [x, y] = 1 >$. Notice that $< y >= H \cap C_G(Q)$. For each $1 \leq i \leq p$ set $H_i =< xy^{ip^{n-2}} >$. Clearly $|H_i| = p$, $H_i \cap C_G(Q) = 0$ and $H_i \cap H_j = 0$ if $i \neq j$.

   Let ϕ_i be the character of H_i which is $p-1$ on the identity and -1 elsewhere. Set:

$$\phi = \sum_{i=1}^{p} Ind_{H_i}^H \phi_i.$$

   Since $\phi(1) = |H|(p-1)$, $\phi(z) = -|H|/p$ for $z \in H \setminus < y >$ and $\phi(z) = 0$ for $z \in < y >$; we conclude that $\beta|_H = p|G|/|H|\phi$.

6. If $H \cap Q = 0$ and $H \cong M(p^n)$, we can write $H =< x, y | x^{p^{n-1}} = y^p = 1, y^{-1}xy = x^{1+p^{n+2}} >$. Let $N =< x^{p^{n-2}}, y >\cong (\mathbb{Z}/p)^2$ which is normal in H. Let ϕ be the character of N given by:

$$\beta|_H = (p-1)\left(\sum_{i=0}^{p-1}\sum_{j=1}^{p-1}\phi_j\phi_i\right) + p\sum_{i=1}^{p-1}\phi_0\phi_i.$$

Then we have $\phi(0) = p^2(p-1)$, $\phi((x^{kp^{n-2}}, 0)) = 0$, while $\phi((x^{kp^{n-2}}, y^l))$ $= -p$. Finally $\beta|_H = Ind_N^H|G|/p|H : N|\phi$.

□

We can now turn our attention to the topological problem:

Proposition 2.3.4. *For every odd order rank 3 p-group G, there is a finite dimensional G-CW-complex $X \simeq S^m \times S^n$ with cyclic isotropy subgroups.*

Proof. If $Z(G)$ is not cyclic, then it is enough to consider the linear spheres of representations of G induced from free representations of some $\mathbb{Z}/p \times \mathbb{Z}/p < Z(G)$.

Assume that $Z(G)$ is cyclic and let S^m be the linear G-sphere obtained by inducing from a free linear action of $Z(G)$. The isotropy subgroups for this action are the one described in proposition 2.3.2. The conditions of proposition 2.2.6 are fulfilled by proposition 2.3.3. The conclusion follows. □

As a direct consequence of theorem 3.2 in [2] we obtain:

Theorem 2.3.5. *For every odd order rank 3 p-group G, there is a free finite G-CW-complex $X \cong S^m \times S^n \times S^k$.*

A converse to theorem 2.3.5 is given by Hanke in [23] in the sense that: if $(\mathbb{Z}/p)^r$ acts freely on $X = S^{n_1} \times ... \times S^{n_k}$ and if $p > 3dim(X)$, then $r \leq k$.

Remark 2.3.6. *For $p = 2$ the situation is more complicated because of the classification of subgroups. A 2-group of rank 1 can be either cyclic or generalized quaternion. A 2-group with a maximal abelian subgroup can be cyclic, generalized quaternion, dihedral, $M(2^n)$ (see proposition 2.3.2) or $S_{4m} =< x, y|x^{2m} = y^2 = 1, y^{-1}xy = x^{m-1} >$ (see [37] section 4 chapter 4). For $p = 2$, the class function of proposition 2.3.3 does not restrict to characters over the subgroups, in general.*

Next, we present another family of p-groups for which we can recover a free action on the desired product of spheres. This is the family of central extensions of abelian p-groups. We first prove a stronger result, a generalization of theorem 3.2 in [2], and then we specialize it to central extensions of abelian p-groups.

Theorem 2.3.7. *Let G be a finite p-group and let X be a finite dimensional G-CW-complex with G_σ abelian for all cells $\sigma \subset X$. Then there is a free finite dimensional G-CW-complex $Y \simeq X \times S^{n_1} \times ... \times S^{rk_X(G)}$. Moreover, if X is finite, then Y is finite as well.*

Proof. We prove the theorem by induction over $rk_X(G)$. If $rk_X(G) = 1$, the theorem has been proven by A. Adem and J. Smith (3.2 in [2]).

The inductive step follows. By virtue of proposition 2.2.6, we only need to associate to each isotropy subgroup G_σ a representation $\rho_\sigma : G_\sigma \to U(m)$ such that $\rho_\sigma|_{G\tau} \cong \rho_\tau$ whenever $G_\tau < G_\sigma$ and such that ρ_σ is fixed point free for all G_σ with $rk(G_\sigma) = rk_X(G)$.

Consider the class function $\beta : G \to \mathcal{C}$ given by:

$$x \mapsto \begin{cases} |G|(p^{rk_X(G)} - 1), & \text{if } x = 0; \\ -|G|, & \text{if } o(x) = p; \\ 0, & \text{otherwise.} \end{cases}$$

To simplify the notation write $A = G_\sigma$ for an isotropy subgroup (which is abelian by hypothesis). We need to prove that $\beta|_A$ is a character which is fixed point free whenever $A \cong (\mathbb{Z}/p)^{rk_X(G)}$. Set $A_p = \{0\} \cup \{x \in A | o(x) = p\}$.

Since A is abelian we have $A_p \triangleleft A$. Fix an injection $f : A_p \to (\mathbb{Z}/p)^{rk_X(G)}$. Write $\rho_0 : (\mathbb{Z}/p)^{rk_X(G)} \to U(p^{rk_X(G)} - 1)$ for the reduced regular representation and let $\rho = \rho_0 \circ f$ be the representation $A_p \to (\mathbb{Z}/p)^{rk_X(G)} \to U(p^{rk_X(G)} - 1)$. Consider finally the representation of A given by $\eta = |G||A_p|/|A| Ind_{A_p}^{A} \rho$. Clearly $\eta(0) = |G|(p^{rk_X(G)} - 1)$, $\eta(x) = -|G|$ if $x \in A_p \backslash 0$ while $\eta(x) = 0$ if $x \notin A_p$. As a result $\beta|_A = \eta$. Let now $A \cong (\mathbb{Z}/p)^{rk_X(G)}$. Clearly $\beta|_A$ is a multiple of the reduced regular representation, thus fixed point free. □

Theorem 2.3.8. *Let G be a p-group. Assume that G is a central extension of abelians, then there is a free finite G-CW-complex $X \simeq S^{n_1} \times ... \times S^{n_{rk(G)}}$. The result in particular holds for extraspecial p-groups.*

Proof. Let $X = S^{n_1} \times ... \times S^{n_{rk(Z(G))}}$ be the product of the G-spheres arising from suitable representations of the center. Clearly $rk_X(G) = rk(G) - rk(Z(G))$ and G_σ is abelian. The conclusion follows. □

Remark 2.3.9. *For extraspecial p-groups, similar results have been obtained independently by Űnlű and Yalçin [44].*

2.4 Infinite groups

As pointed out in [16], there is a class of infinite groups which is worth considering, when studying the rank conjecture. This is the class of groups Γ with finite virtual cohomological dimension. Recall that, by definition, a group Γ has finite virtual cohomological dimension, if it has a finite index subgroup $\Gamma' < \Gamma$ with finite cohomological dimension (that is to say: $H^n(\Gamma') = 0$ for all coefficients and for all n big enough). Writing *vcd* for virtual cohomological dimension and *cd* for cohomological dimension, one can show that the number $vcd(\Gamma) = cd(\Gamma')$ is well defined. See for example [10] for background on groups with finite virtual cohomological dimension.

The crucial property that makes them interesting to us is the following: for any such group Γ there exists a finite dimensional Γ-CW-complex $\mathfrak{E}\Gamma$

with $|\Gamma_x| < \infty$ for all $x \in \mathfrak{E}\Gamma$.

It is already known that a group with finite virtual cohomological dimension, which is countable and with rank at most one finite subgroups, acts freely on a finite dimensional CW-complex $X \simeq S^m$ [16]. The next step would be to prove the analogue result for groups Γ with rank at most two finite subgroups.

The easiest examples to consider are amalgamated products $\Gamma = G_1 *_{G_0} G_2$, where G_i is a finite group for $i = 0, 1, 2$ and $G_0 < G_j$ for $j = 1, 2$. In this case, for every finite subgroup $H < \Gamma$, there is $\gamma \in \Gamma$ such that $\gamma H \gamma^{-1} < G_i$ for $i = 1$ or $i = 2$ (see [34]). In particular $rk(\Gamma) = max\,\{rk(G_1), rk(G_2)\}$.

The first attempt would be to find an effective Γ-sphere, i.e. a Γ-sphere with rank at most one isotropy subgroups. We exhibit now an amalgamation of two p-groups which doesn't have an effective Γ-sphere.

Theorem 2.4.1. *For p an odd prime, there is an infinite group Γ with finite virtual cohomological dimension, satisfying the two following properties: every finite subgroup $G < \Gamma$ is a p-group with $rk(G) \leq 2$ and for every finite dimensional Γ-CW-complex $X \simeq S^n$ there is at least one isotropy subgroup Γ_σ with $rk(\Gamma_\sigma) = 2$.*

Proof. Let E and E' be two copies of the extraspecial p-group of order p^3 and exponent p. (Such a group can be identified with the upper triangular 3×3 matrices over $\mathbb{F}_p$ with 1 on the diagonal). Consider the amalgamated product $\Gamma = E' *_{\mathbb{Z}/p} E$ given by $\mathbb{Z}/p = Z(E)$ and an injective map $f : \mathbb{Z}/p \to E$ with $f(\mathbb{Z}/p) \cap Z(E') = 1$. Clearly $rk(\Gamma) = 2$.

Let Γ act on a finite dimensional CW-complex $X \simeq S^n$. Consider the restriction of this action to E and E'. It is well known that the dimension function of a p-group action on a sphere is realized by a representation over the real numbers [18]. Therefore, an even multiple of the dimension functions for E and E' must be realized by characters χ_E and $\chi_{E'}$.

Clearly the dimension functions of χ_E and $\chi_{E'}$ must agree over $Z(E)$ and $f(\mathbb{Z}/p)$. Looking at the character table of E, we observe that every irreducible character α, giving rise to an effective sphere, vanishes outside $Z(E)$ while $\alpha(z) = m\zeta_p$ for all $z \in Z(E)\backslash\{0\}$ (here ζ_p is a p-root of the unity). Thus, χ_E and $\chi_{E'}$ cannot be both characters giving rise to effective spheres. We deduce that the original action must have some finite isotropy subgroups of rank 2. This provides an example of an infinite group, with rank 2 finite p-subgroups, not acting with effective Euler class on any sphere. $\square$

Remark 2.4.2. *This kind of behaviour cannot happen with finite groups: every rank 2 finite p-group, has a linear sphere with periodic isotropy subgroups.*

Let's try to approach the rank conjecture algebraically. For a finite group G, we have that $rk(G) = r$ if and only if there are r finite dimensional $\mathbb{Z}[G]$-complexes $\mathbf{K}_1, ..., \mathbf{K}_r$ such that $\mathbf{K} = \mathbf{K}_1 \otimes ... \otimes \mathbf{K}_r$ is a complex of projective $\mathbb{Z}[G]$-modules with $H^*(\mathbf{K}) \cong H^*(S^{n_1} \times ... \times S^{n_r})$ (see [7]).

In corollary 2.4.6 we prove a similar result: for every group Γ with $vcd(\Gamma) < \infty$, there are $rk(\Gamma)$ finite dimensional $\mathbb{Z}[\Gamma]$-complexes $\mathbf{C}_1, ..., \mathbf{C}_{rk(\Gamma)}$ such that $\mathbf{D} = C_*(\mathfrak{E}\Gamma) \otimes \mathbf{C}_1 \otimes ... \otimes \mathbf{C}_{rk(\Gamma)}$ is a complex of projective $\mathbb{Z}[\Gamma]$-modules with $H^*(\mathbf{D}) \cong H^*(S^{n_1} \times ... \times S^{n_{rk(\Gamma)}})$.

As a result, the group Γ introduced in theorem 2.4.1 satisfies the algebraic analogue of the rank conjecture but does not have an effective Γ-sphere. The geometric problem of knowing whether or not Γ acts freely on a product of two spheres is still open.

We begin by recalling some preliminaries concerning the cohomology of finite groups. We follow here [8] and [9]. Let G be a finite group. Consider $\zeta \in H^n(G, R) \cong Ext^n_{RG}(R, R) \cong Hom_{RG}(\hat{\Omega}^n R, R)$, where $\hat{\Omega}^n R$ is the nth kernel in a RG-projective resolution $\mathbf{P}$ of R. We choose a cocycle $\hat{\zeta} : \hat{\Omega}^n R \to R$ representing ζ. By making $\mathbf{P}$ large enough we can assume that $\hat{\zeta}$ is surjective. We denote L_ζ its kernel and form the pushout diagram:

$$\begin{tikzcd}
 & L_\zeta \arrow[r, "="] \arrow[d] & L_\zeta \arrow[d] & & & & & \\
0 \arrow[r] & \hat{\Omega}^n R \arrow[r] \arrow[d, "\hat{\zeta}"] & P_{n-1} \arrow[r] \arrow[d] & P_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & R \arrow[r] & P_{n-1}/L_\zeta \arrow[r] & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & R \arrow[r] & 0
\end{tikzcd}$$

We denote by $\mathbf{C}_\zeta$ the chain complex:

$$0 \to P_{n-1}/L_\zeta \to P_{n-2} \to \ldots \to P_0 \to R \to 0$$

formed by truncating the bottom row of this diagram. Thus we have that $H_0(\mathbf{C}_\zeta) = H_{n-1}(\mathbf{C}_\zeta) = R$ while $H_i(\mathbf{C}_\zeta) = 0$ if $i \neq 0, n-1$. A useful result is given in the proof of theorem 3.1 in [3]:

Proposition 2.4.3. *Let G be a finite group. For all positive integer r, there exist classes $\xi_1, \ldots, \xi_r \in H^*(G, \mathbb{Z})$ such that, for all $H < G$ with $rk(H) \leq r$, the complex $\mathbb{Z}[G/H] \otimes L_{\xi_1} \otimes \ldots \otimes L_{\xi_r}$ is $\mathbb{Z}[G]$-projective.*

Proof. See the proof of theorem 3.1 in [3] $\square$

Corollary 2.4.4. *Let G be a finite group. For all positive integer r, there exist finite dimensional $\mathbb{Z}[G]$-complexes $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_r$ such that $H^*(\boldsymbol{C}_1 \otimes \ldots \otimes \boldsymbol{C}_r) = H^*(S^{n_1} \times \ldots \times S^{n_r})$; with $\boldsymbol{C}_1 \otimes \ldots \otimes \boldsymbol{C}_r$ a complex of $\mathbb{Z}[H]$-projective modules for all $H < G$ with $rk(H) \leq r$.*

Proof. Let $\xi_1, \ldots, \xi_r \in H^*(G, \mathbb{Z})$ be the classes given in proposition 2.4.3. Consider the chain complex $\mathbf{C} = \mathbf{C}_{\xi_1} \otimes \ldots \otimes \mathbf{C}_{\xi_r}$. Clearly $H^*(\mathbf{C}) = H^*(S^{n_1} \times \ldots \times S^{n_r})$. For the second part of the claim, observe that all the modules in $\mathbf{C}_{\xi_i}$ are $\mathbb{Z}[G]$-projective except the module $P_{n_i - 1}/L_{\xi_i}$. Recall that the tensor product of any module with a projective module is projective, so that it remains to examine the module $P_{n_1-1}/L_{\xi_1} \otimes \ldots \otimes P_{n_r-1}/L_{\xi_r}$. Let

$H < G$ be such that $rk(H) \leq r$. Since $\mathbb{Z}[G/H] \otimes L_{\xi_1} \otimes ... \otimes L_{\xi_r}$ is $\mathbb{Z}[G]$-projective by proposition 2.4.3, we conclude that $\mathbb{Z}[G/H] \otimes P_{n_1-1}/L_{\xi_1} \otimes ... \otimes P_{n_r-1}/L_{\xi_r}$ is $\mathbb{Z}[G]$-projective as in 5.14.2 of [9]. It then easily follows that $P_{n_1-1}/L_{\xi_1} \otimes ... \otimes P_{n_r-1}/L_{\xi_r}$ is $\mathbb{Z}[H]$-projective. □

This is the end of the reminder about the cohomology of finite groups. We now apply it to the case of an infinite group Γ with $vcd(\Gamma) < \infty$ and rank r. Write Γ' for a torsion-free normal subgroup of Γ with $G = \Gamma/\Gamma'$ finite. We apply corollary 2.4.4 to Γ/Γ' with $r = rk(\Gamma)$. We recover a $\mathbb{Z}[\Gamma]$-complex $\mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r$ such that $H^*(\mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r) = H^*(S^{n_1} \times ... \times S^{n_r})$; with $\mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r$ a complex of $\mathbb{Z}[H]$-projective modules for all finite $H < \Gamma$. On the other hand, we have that the $\mathbb{Z}\Gamma$-complex $C_*(\mathfrak{E}\Gamma)$ is contractible. Therefore the complex $\mathbf{D} = C_*(\mathfrak{E}\Gamma) \otimes \mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r$ is such that $H^*(\mathbf{D}) = H^*(S^{n_1} \times ... \times S^{n_r})$.

Lemma 2.4.5. *With the notation above, the complex $\boldsymbol{D}$ is $\mathbb{Z}[\Gamma]$-projective.*

Proof. The complex $C_*(\mathfrak{E}\Gamma)$ decomposes as a graded direct sum of permutation modules: $C_*(\mathfrak{E}\Gamma) = \oplus_\sigma \mathbb{Z}[\Gamma/\Gamma_\sigma] = \oplus_\sigma \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} \mathbb{Z}$. Here σ spans the cells of $\mathfrak{E}\Gamma/\Gamma$ and the grading is given by the dimensions of the cells σ. Consequently $\mathbf{D} = \oplus_\sigma(\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} \mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r)$, so that we only need to prove that $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} \mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r$ is $\mathbb{Z}[\Gamma]$-projective. Let Q_σ be a graded $\mathbb{Z}[\Gamma_\sigma]$-module such that $(\mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r) \oplus Q_\sigma$ is $\mathbb{Z}[\Gamma_\sigma]$-free. We then have that $(\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} \mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r) \oplus (\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} Q_\sigma) = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]}((\mathbf{C}_1 \otimes ... \otimes \mathbf{C}_r) \oplus Q_\sigma)$ is $\mathbb{Z}[\Gamma]$-free. □

Consequently, the algebraic version of the rank conjecture holds for groups of finite virtual cohomological dimension:

Corollary 2.4.6. *For a group Γ with $vcd(\Gamma) < \infty$ and $rk(\Gamma) = r$, there exist a finite dimensional contractible complex $C_*(\mathfrak{E}\Gamma)$ and r finite dimensional $\mathbb{Z}[\Gamma]$-complexes $\boldsymbol{C}_1, ..., \boldsymbol{C}_r$ such that $\boldsymbol{D} = C_*(\mathfrak{E}\Gamma) \otimes \boldsymbol{C}_1 \otimes ... \otimes \boldsymbol{C}_r$ is a $\mathbb{Z}[\Gamma]$-projective complex with $H^*(\boldsymbol{D}) \cong H^*(S^{n_1} \times ... \times S^{n_r})$.*

Remark 2.4.7. *We thank professor D. Benson for pointing out to us, that a similar result has already been proved in [27].*

Chapter 3

Finite Homotopy G-Spheres up to Borel Equivalence

In this chapter we discuss the study of finite homotopy G-spheres up to Borel equivalence. That is to say, given a homotopy G-sphere X, we want to know if it is Borel equivalent to a finite one. As explained by the classification theorems of Grodal and Smith (here theorems 1.0.1 and 1.0.2), Borel equivalences are captured by dimension functions and quasi-isomorphisms of some chain complexes over the orbit category. In the first section we introduce the reader to the topic of homological algebra over orbit categories. In the second section, we use it to present a new approach to the construction of finite homotopy G-spheres. In particular, we give new examples for groups of the form $C_p \rtimes C_{q^r}$.

3.1 Homological algebra over the orbit category

The goal of this section is to introduce the reader to the theory of homological algebra over the orbit category. The main reference for this section is [29]. We need to begin by discussing what we mean by modules over the orbit category. In general, one may define the notion of modules over any category, and most of the results presented below hold for modules over any finite, ordered, E.I. category. Since we will only be concerned with the modules over the orbit category (which is, in particular, a finite ordered E.I. category) we will not present the following results in the most general setting.

Let G be a finite group and $\mathcal{F}$ a family of subgroups closed under the actions of conjugation and taking subgroups. The orbit category of G with respect to the family $\mathcal{F}$, denoted $Or_{\mathcal{F}}G$, has transitive G-sets as objects and G-maps as morphisms. More concretely, we may write the objects of the orbit category as the set:

$$\mathrm{Ob}(Or_{\mathcal{F}}G) = \{G/H|H \in \mathcal{F}\}$$

the set morphisms between two G orbits is given by:

$$Mor(G/H, G/K) = \{g \in G|H^g < K\}/K.$$

In what follows, we will write $\Gamma = Or_{\mathcal{F}}G$ for the full orbit category and Γ_p for the orbit category $Or_{\mathcal{F}}G$, where $\mathcal{F}$ is the family of subgroups of prime power order for some prime p dividing the order of G. Notice that Γ_p is a full subcategory of Γ. Let R be a commutative ring with unit. An $R\Gamma$-module M is a contravariant functor:

$$M : \Gamma^{op} \to R\mathrm{-Mod}$$

from the orbit category to the category of R-modules. A morphism of $R\Gamma$-modules is then a natural transformation of functors. The category of $R\Gamma$ modules is denoted by $R\Gamma\mathrm{-Mod}$. Since Γ is a small category and $R\mathrm{-Mod}$ is abelian, the category of $R\Gamma$-modules is abelian as well. We are able, therefore, to do homological algebra in the category $R\Gamma\mathrm{-Mod}$. For $R\Gamma$-modules, the terms exact, injective, surjective, etc. are determined object wise. For instance, the sequence of $R\Gamma$-modules:

$$L \to M \to N$$

is exact if and only if the sequence of R-modules:

$$L(G/H) \to M(G/H) \to N(G/H)$$

is exact for all $G/H \in Ob\Gamma$. As usual, a $R\Gamma$-module P is projective if and only if the functor:

$$Hom_{R\Gamma}(P, -) : R\Gamma\text{−Mod} \to R\text{−Mod}$$

is exact. For an orbit $G/H \in Ob\Gamma$, define the *free module generated at G/H* by:

$$F_{G/H}(G/K) = RMor(G/K, G/H)$$

for all $G/K \in Ob\Gamma$. Here $RMor(G/K, G/H)$ is the free R-module on the set of $R\Gamma$ morphisms from G/K to G/H. The $R\Gamma$-module $F_{G/H}$ is defined in such a way that maps of $R\Gamma$-modules $F_{G/H} \to M$ are determined by the image of $Id_{G/H} \in F_{G/H}(G/H)$ in $M(G/H)$. In particular:

$$Hom_{R\Gamma}(F_{G/H}, M) \cong M(G/H)$$

which says that $F_{G/H}$ is a projective $R\Gamma$-module for each $G/H \in Ob\Gamma$. We are now able to give a definition of a free $R\Gamma$-module:

Definition 3.1.1. *An $R\Gamma$-module M is called free if it is isomorphic to:*

$$\bigoplus_{G/H \in \lambda} F_{G/H}$$

for some collection of orbits $\lambda \subset Ob\Gamma$.

Just as in the case of modules over a ring, one can prove that a $R\Gamma$-module is projective if and only if it is a summand of a free module. Similarly, we say that a $R\Gamma$-module M is finitely generated if there is a finitely generated free $R\Gamma$-module covering it: $\oplus_{i=1}^{k} F_{G/H_i} \twoheadrightarrow M$. A chain complex is called perfect if it is a finite dimensional, finitely generated chain complex of projective modules. Next, we prove a result of homological algebra that will be needed in the following section.

Lemma 3.1.2. *Every perfect free $\mathbb{Q}\Gamma$-chain complex C is quasi-isomorphic to its homology $H(C)$.*

Proof. A $\mathbb{Q}\Gamma$-complex C is a family of $\mathbb{Q}$-complexes with maps between them. Clearly, each of these complexes is quasi-isomorphic to its homology as a $\mathbb{Q}$-complex. In order to motivate what follows, we recall why this is the case: Let D be a perfect $\mathbb{Q}$-chain complex. Using a scalar product in each degree, we can write D under the form:

$$0 \to ... \to D_n = Ker(d_n) \oplus Ker^{\perp}(d_n) \to ... \to 0$$

The projection map $D \to H(D)$ is then well defined and it is a quasi-isomorphism. In order for the same method to work in the case of a $\mathbb{Q}\Gamma$-complex, we want to be able to split the $\mathbb{Q}$-modules $C_n(G/H)$ in a functorial way. To this end, it is enough to prove that the maps $C_n(G/K) \to C_n(G/L)$ respect the scalar products. This is what we are going to show next.

Since C is $\mathbb{Q}\Gamma$-free, all the $\mathbb{Q}$-modules involved must necessarily be of the form $\mathbb{Q}Mor(G/H', G/H)$, for some $H', H < G$. Such a vector space comes with a canonical basis with respect to which we define a canonical scalar product.

To study $C_n(G/K) \to C_n(G/K)$, observe that all the self-maps of the module $\mathbb{Q}Mor(G/K, G/H)$ preserve such a canonical scalar product, because they are given by permutation matrices induced by maps of the form $g : Mor(G/K, G/H) \to Mor(G/K, G/H)$; $u \mapsto ug$ for $g \in WK = NK/K$, the Weil group. To study $C_n(G/K) \to C_n(G/L)$, notice that all maps $\mathbb{Q}Mor(G/K, G/H) \to \mathbb{Q}Mor(G/L, G/H)$ respect the canonical scalar products because they are given by set injections $g : Mor(G/K, G/H) \to Mor(G/L, G/H)$; $u \mapsto ug$ for elements $g \in Mor(G/L, G/K)$. (Each morphism $g \in Mor(G/L, G/K)$ being surjective makes the corresponding map $g : Mor(G/K, G/H) \to Mor(G/L, G/H)$ injective).

To summarize: we showed that for a perfect free $\mathbb{Q}\Gamma$-complex C, there are $\mathbb{Q}\Gamma$-modules Ker_n and $Ker_n^{\perp}$ such that $C_n = Ker_n \oplus Ker_n^{\perp}$. Moreover, $Ker_n(G/H) = Ker(C_n(G/H) \to C_{n-1}(G/H))$. As a consequence, we can define a map of $\mathbb{Q}\Gamma$-complexes $C \to H(C)$ which is a quasi-isomorphism.

□

We end by recalling the notation of obstruction theory for $R\Gamma$-complexes (see [29]). Let $K_0(R\Gamma)$ be the Groethendieck group of finitely generated projective $R\Gamma$-modules. Let F be subgroup of $K_0(R\Gamma)$ generated by free $R\Gamma$-modules. The reduced K-theory group, is the quotient $\widetilde{K}_0(R\Gamma) = K_0(R\Gamma)/F$.

For a perfect $R\Gamma$-complex P, the sum $o(P) = \sum_{i=1}^{\infty}(-1)^i P_i$ defines a class $\tilde{o}(P) \in \widetilde{K}_0(R\Gamma)$. If P is a perfect complex and if $f : P \to C$ is a quasi-isomorphism, then we set $\tilde{o}(C) := \tilde{o}(P)$. It is the obstruction for C to be free in the sense of proposition 11.2 in [29]:

Proposition 3.1.3. . *With the above notation, $\tilde{o}(C) = 0$ if and only if C is weakly equivalent to a perfect free $R\Gamma$-complex. If C is perfect, then $\tilde{o}(C) = 0$ if and only if C is homotopy equivalent to a perfect free $R\Gamma$-complex.*

We would like to be able to express the obstruction theory for $R\Gamma$-modules in terms of the obstruction theories of $R\,[WH]$-modules, for H some subgroups of G. The trouble here is that the functor $Res_H : [R\Gamma - Mod] \to [R\,[WH] - Mod]$; $M \mapsto M(G/H)$ does not take projective to projective!

We need a better functor: For a $R\Gamma$-modules M, following [29], let $M(G/H)_s$ be the R-submodule of $M(G/H)$ generated by all images of R-homomorphisms $M(f) : M(G/K) \to M(G/H)$ induced by all non isomorphisms $f : G/H \to G/K$. Observe that $M(G/H)_s$ is directly a $R\,[WH]$-submodule because for $f \in WH$, the composition $g \circ f$ is an isomorphism if and only if g is. Finally, we define the splitting functor $S_H : [R\Gamma - Mod] \to [R\,[WH] - Mod]$ by $M \mapsto M(G/H)/M(G/H)_s$. As in [29], one can then prove that:

Proposition 3.1.4. *The splitting functors S_H respect "direct sum", "finitely generated", "free" and "projective".*

To compute the obstruction class, we now have theorem 10.34 of [29]:

Theorem 3.1.5. *The reduced K-theory splits as:*

$$\widetilde{K}_0(R\Gamma) \cong \oplus_{(H)<G} \widetilde{K}_0(R\,[WH]).$$

Here the sum runs over the conjugacy classes of subgroups $H < G$. Furthermore, if P_* is a perfect chain complex, then the above isomorphism sends $\tilde{o}(P_*)$ to $\oplus_{(H)<G}\tilde{o}(S_H(P_*))$.

3.2 Finite homotopy G-spheres

Let X be a homotopy G-sphere. Recall that for all $p||G|$ and for all p-subgroups K, we have that the homotopy fixed points $X^{hK} = Map_K(EK, X)$ have the mod p homology of a sphere. In particular, for such a space, we have a family of dimension functions $\left\{Dim^p_X(-)\right\}_{p||G|}$. For us, each $Dim^p_X(-)$ is defined over Γ_p by $Dim^p_X(K) = Dim(H^*(X^{hK}, \mathbb{Z}_{(p)}))$. These dimension functions are preserved under Borel equivalence, since $Y^{hK} \simeq X^{hK}$, if X and Y are two Borel equivalent G-spheres.

Remark 3.2.1. *Elsewhere [14], [21], dimension functions have been defined by $Dim^p_X(K) = Dim(H^*(X^{hK}, \mathbb{Z}_{(p)})) + 1$. Such a definition behaves better with respect to join products and reduced homology. For our realization purposes, we find it more convenient to use a different convention. As usual $Dim^p_X(K) = -1$ means that $X^{hK} = \emptyset$.*

Given a homotopy G-sphere X, we want to know if we can find a perfect free $\mathbb{Z}\Gamma$-complex C, whose homology realizes the dimension functions $\left\{Dim^p_X(-)\right\}_{p||G|}$. If such a chain complex exists, we want to know if we can realize it as the chain complex of a space Y, which will be our finite model for the original sphere X. The next example shows that it is convenient to first look for perfect free $\mathbb{Z}_{(p)}\Gamma$-complexes C_p that realize the p-dimension function $Dim^p_X(-)$. The example also prepares the terrain for new and more general results, see theorem 3.2.6.

Example 3.2.2. *This example is a mix of example 4.4.2 and lemma 4.3.5 in [14]. Consider the symmetric group $G = \Sigma_3 = C_3 \rtimes C_2 =< \sigma, \tau | \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^2 >$. We order the category Γ as follows: $(G/G, G/C_3, G/C_2, G/e)$. We want to find chain complexes $D_3 \in \mathbb{Z}_{(3)}\Gamma$ and $D_2 \in \mathbb{Z}_{(2)}\Gamma$ realizing the dimension function $(-1, -1, 3, 3)$.*

We begin at the prime 3. We will need the fact that the number 2 is invertible in $\mathbb{Z}_{(3)}$, later in the construction. Notice that $H^*(\Sigma_3, \mathbb{Z}_{(3)})$ is two periodic, so that, by [38], there is an exact chain complex:

$$0 \longrightarrow \mathbb{Z}_{(3)} \longrightarrow M_3 \xrightarrow{f_3} M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \longrightarrow \mathbb{Z}_{(3)} \longrightarrow 0$$

with M_i a perfect free $\mathbb{Z}_{(3)}\Sigma_3$-module for $i = 0, ..., 3$. Since the free module generated at G/e is $F_{G/e} = (0, 0, 0, \mathbb{Z}_{(3)}\Sigma_3)$, we recover the following free $\mathbb{Z}_{(3)}\Gamma$-complex D_3':

$$\begin{array}{ccccc}
[3] & & & & M_3 \\
 & & & & \downarrow{\scriptstyle f_3} \\
[2] & & & & M_2 \\
 & & & & \downarrow{\scriptstyle f_2} \\
[1] & & & & M_1 \\
 & & & & \downarrow{\scriptstyle f_1} \\
[0] & & & & M_0 \\
 & & & & \\
 & \Sigma_3/\Sigma_3 & \Sigma_3/C_3 & \Sigma_3/C_2 & \Sigma_3/e
\end{array}$$

Its dimension function is $(-1, -1, -1, 3)$. Not quite what we want. Observe first that the map of G-sets $\pi : G \to G/C_2 \cong \overline{C}_3$ induces an exact sequence of $\mathbb{Z}_{(3)}\Sigma_3$-modules:

$$0 \longrightarrow Ker(\pi) \longrightarrow \mathbb{Z}_{(3)}\Sigma_3 \xrightarrow{\pi} \mathbb{Z}_{(3)}\overline{C}_3 \longrightarrow 0 \, .$$

The map $\mu : \mathbb{Z}_{(3)}\overline{C}_3 \to \mathbb{Z}_{(3)}\Sigma_3$, given by $\left[(\sigma^i, \tau^j)\right] \mapsto (\sigma^i, \tau^0)$ defines a $\mathbb{Z}_{(3)}C_3$-splitting of the exact sequence [5]*. As usual, since 2 is invertible in $\mathbb{Z}_{(3)}$, we can define a $\mathbb{Z}_{(3)}\Sigma_3$-splitting of the exact sequence, by setting $\mu_G : \mathbb{Z}_{(3)}\overline{C}_3 \to \mathbb{Z}_{(3)}\Sigma_3$, $\left[(\sigma^i, \tau^j)\right] \mapsto ((\sigma^i, \tau^0) + (\sigma^i, \tau^1))/2$. In particular, the free*

[5] C_3 is the subgroup of Σ_3 generated by σ, while $\overline{C}_3$ is the Σ_3-set given by Σ_3/C_2

module generated at G/e splits as $F_{G/e} = (0, 0, 0, \mathbb{Z}_{(3)}\overline{C}_3 \oplus Ker(\pi))$. Recall also that the free $\mathbb{Z}_{(3)}\Gamma$-module generated at G/C_2 is $(0, 0, \mathbb{Z}_{(3)}, \mathbb{Z}_{(3)}\overline{C}_3)$.

We can now modify D'_3 into a perfect free $\mathbb{Z}_{(3)}\Gamma$-complex D_3, which realizes the dimension function $(-1, -1, 3, 3)$. To describe the new complex, we use the symbol ρ_i to designate the projection of a direct sum into its i-th component. We leave out the columns Σ_3/Σ_3 and Σ_3/C_3 because they are trivial:

$$
\begin{array}{ccc}
[4] & & \mathbb{Z}_{(3)}\overline{C}_3 \oplus Ker(\pi) \\
 & & \Big\downarrow {\scriptstyle \rho_1\oplus 0\oplus\rho_2\oplus 0} \\
[3] & \mathbb{Z}_{(3)} & \mathbb{Z}_{(3)}\overline{C}_3 \oplus \mathbb{Z}_{(3)}\overline{C}_3 \oplus Ker(\pi) \oplus M_3 \\
 & \Big\downarrow {\scriptstyle 0} & \Big\downarrow {\scriptstyle 0\oplus\rho_2\oplus 0\oplus(f_3\circ\rho_4)} \\
[2] & \mathbb{Z}_{(3)} & \mathbb{Z}_{(3)}\overline{C}_3 \oplus \mathbb{Z}_{(3)}\overline{C}_3 \oplus Ker(\pi) \oplus M_2 \\
 & \Big\downarrow {\scriptstyle 1} & \Big\downarrow {\scriptstyle \rho_1\oplus 0\oplus\rho_3\oplus(f_2\circ\rho_4)} \\
[1] & \mathbb{Z}_{(3)} & \mathbb{Z}_{(3)}\overline{C}_3 \oplus \mathbb{Z}_{(3)}\overline{C}_3 \oplus Ker(\pi) \oplus M_1 \\
 & \Big\downarrow {\scriptstyle 0} & \Big\downarrow {\scriptstyle \rho_2\oplus(f_1\circ\rho_4)} \\
[0] & \mathbb{Z}_{(3)} & \mathbb{Z}_{(3)}\overline{C}_3 \oplus M_0 \\
 & & \\
 & \Sigma_3/C_2 & \Sigma_3/e
\end{array}
$$

Next, we construct a free $\mathbb{Z}_{(2)}\Gamma$-complex D_2, with the required homology. Again, we will need 3 to be invertible in $\mathbb{Z}_{(2)}$ for the construction to hold.

Consider first the perfect free $\mathbb{Z}_{(2)}\Gamma$-complex D_2':

$$
\begin{array}{lccccc}
[2] & & & & & \mathbb{Z}_{(2)}[\Sigma_3] \\
 & & & & & \downarrow{\scriptstyle f_2} \\
[1] & & & \mathbb{Z}_{(2)} & & \mathbb{Z}_{(2)}\overline{C}_3 \oplus \mathbb{Z}_{(2)}[\Sigma_3] \\
 & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f_1\circ\rho_2} \\
[0] & & & \mathbb{Z}_{(2)} & & \mathbb{Z}_{(2)}\overline{C}_3 \\
 & & & & & \\
 & \Sigma_3/\Sigma_3 & \Sigma_3/C_3 & \Sigma_3/C_2 & & \Sigma_3/e
\end{array}
$$

The maps f_1 and f_2 are given by:

$$
\begin{aligned}
f_1(e) &= \overline{\sigma} - \overline{\sigma}^2 \\
f_2(e) &= (\overline{\sigma}, e - \sigma\tau - \sigma^2\tau).
\end{aligned}
$$

Over the orbits G/Σ_3, G/C_3 and G/C_2, everything is clear. We need to study the $\mathbb{Z}_{(2)}\Sigma_3$-complex sitting over the orbit G/e. It is easy the check that $Im(f_1 \circ \rho_2) = Ker(\epsilon : \mathbb{Z}_{(2)}[\Sigma_3/C_2] \to \mathbb{Z}_{(2)})$, so that $H_0 = (0, 0, \mathbb{Z}_{(2)}, \mathbb{Z}_{(2)})$. Some elementary linear algebra, shows that we have $H_1 = (0, 0, \mathbb{Z}_{(2)}, \mathbb{Z}_{(2)})$ <u>because</u> the coefficients are in $\mathbb{Z}_{(2)}$. More in detail, after a suitable change of basis, the matrix of $f_2 : \mathbb{Z}_{(2)}\Sigma_3 \to Ker(f_1 \circ \rho_2)$ is given by:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 9 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Since 9 is invertible in $\mathbb{Z}_{(2)}$, we have that $Ker(0 + f_1)/Im(f_2) \cong \mathbb{Z}_{(2)}$ and $Ker(f_2) = 0$. Finally, by concatenating D_2' with itself, we obtain the desired *perfect free* $\mathbb{Z}_{(2)}\Gamma$*-complex* D_2 *realizing* $(-1,-1,3,3)$.

Similar observations and examples were already made by Clarkson in [14], where he used Postnikov towers to glue the p-local complexes together in the case of groups of the form $G = C_p \rtimes C_q$. His strategy relies on the fact that $|G| = pq$. In order to extend Clarkson's result (see theorem 3.2.6), we suggest a gluing method based on an arithmetic square, which does not need the condition $|G| = pq$. We will still need Postnikov towers in order to show that the output of the arithmetic square satisfies some properties. Following [22], we recall the main result concerning Postnikov towers of $R\Gamma$-complexes:

Given a projective $R\Gamma$ chain complex C, there is a sequence of projective chain complexes $C(i)$ together with maps $f_i : C \to C(i)$ inducing homology isomorphisms for dimension $\leq i$. Moreover, there is a tower of maps:

$$\begin{tikzcd}
 & C(i) \arrow[d] & \\
 & C(i-1) \arrow[r, "\alpha_i"] \arrow[d, dashed] & \Sigma^{i+1}P(H_i) \\
C \arrow[ruu] \arrow[ru] \arrow[r] \arrow[rd] & C(1) \arrow[r, "\alpha_2"] \arrow[d] & \Sigma^3 P(H_2) \\
 & C(0) \arrow[r, "\alpha_1"] & \Sigma^2 P(H_1)
\end{tikzcd}$$

such that $C(i) = \Sigma^{-1}C(\alpha_i)$, where $C(\alpha_i)$ denotes the algebraic mapping cone of α_i and $P(H_i)$ denotes a projective resolution of the homology module H_i. Recall also that the algebraic mapping cone of a chain map $f : C \to D$ is defined as the chain complex $C(f) = D \oplus \Sigma C$ with boundary map $\partial(x,y) = (\partial(x) + f(y), \partial y)$. Note that Σ^n is the shift operator for chain complexes given by $(\Sigma^n C)_i = C_{i-n}$. It then follows that:

Proposition 3.2.3. *Let* C *be a projective chain complex with finite homological dimension and such that* $H_i(C)$ *is finitely generated for all* i*. If there is an integer* n *such that* $Ext^i_{R\Gamma}(C, M) = 0$ *for all modules* M *and* $i > n$*,* *then C is homotopy equivalent to an n-dimensional, finitely generated projective chain complex.*

Proof. Let d be the homological dimension of C and consider the d-th stage of the Postnikov tower for C. By inspection of the above description of Postnikov towers, $C(d)$ is a finitely generated projective chain complex. The map $f_d : C \to C(d)$ is a homology isomorphism, so that C and $C(d)$ are homotopy equivalent. As a consequence, $Ext^i_{R\Gamma}(C(d), M) = 0$ for all modules M and $i > n$. By lemma 6.1 in [22], $C(d)$ is then homotopy equivalent to an n-dimensional, finitely generated projective chain complex. □

We can now state our gluing theorem:

Theorem 3.2.4. *Consider a finite family of primes $P = \{p_1, ..., p_n\}$. Assume that for every prime $p_i \in P$ we are given a perfect free $\mathbb{Z}_{(p_i)}\Gamma$-complex D_{p_i} such that $H(D_{p_i} \otimes \mathbb{Q})$ is isomorphic to $H(D_{p_j} \otimes \mathbb{Q})$ for all i and j. If there is a perfect free $\mathbb{Z}_{(1/P)}\Gamma$-complex $D_{1/P}$ such that $H(D_{1/P} \otimes \mathbb{Q})$ is isomorphic to $H(D_{p_j} \otimes \mathbb{Q})$, then there is a perfect free $\mathbb{Z}\Gamma$-complex D with $D \otimes \mathbb{Z}_{(p_i)}$ quasi-isomorphic to D_{p_i} for all $p_i \in P$ and $D \otimes \mathbb{Z}_{(1/P)}$ quasi-isomorphic to $D_{1/P}$.*

Proof. Consider the limit:

$$\begin{tikzcd}
 & & D = lim \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
D_{p_1} \arrow[drr] & \cdots \arrow[dr] & D_{1/P} \arrow[d] & \cdots \arrow[dl] & D_{p_n} \arrow[dll] \\
 & & H(D_{p_1} \otimes \mathbb{Q}) & &
\end{tikzcd}$$

Observe that the maps into $H(D_{p_1} \otimes \mathbb{Q})$ are given by lemma 3.1.2. As usual, we will use the flatness of $\mathbb{Q}$ to move the coefficients out of the homology. Assume first that $P = \{p\}$. The limit is then a pullback giving rise to a short exact sequence:

$$0 \to D \to D_p \oplus D_{1/p} \to H(D_p) \otimes \mathbb{Q} \to 0$$

which in turn yields a long exact sequence:

$$... \to H_n(D) \to H_n(D_p \oplus D_{1/p}) \to H_n(D_p) \otimes \mathbb{Q} \to ...$$

Applying the exact functor $- \otimes \mathbb{Z}_{(p)}$, gives an exact sequence:

$$... \to H_n(D)\otimes\mathbb{Z}_{(p)} \to (H_n(D_p)\oplus H_n(D_{1/p}))\otimes\mathbb{Z}_{(p)} \to H_n(D_p)\otimes\mathbb{Q}\otimes\mathbb{Z}_{(p)} \to ...$$

Since $\mathbb{Q}\otimes\mathbb{Z}_{(p)} = \mathbb{Q} = \mathbb{Z}_{(1/p)}\otimes\mathbb{Z}_{(p)}$, the map $H_n(D_{1/p})\otimes\mathbb{Z}_{(p)} \to H_n(D_p)\otimes\mathbb{Q}$ is surjective for all n, hence:

$$H_n(D) \otimes \mathbb{Z}_{(p)} \cong Ker\left[(H_n(D_p) \oplus H_n(D_{1/p})) \otimes \mathbb{Z}_{(p)} \to H_n(D_p) \otimes \mathbb{Q}\right]$$

Such a kernel is nothing but the pullback:

$$\begin{tikzcd}
H_n(D_p) \arrow[r] \arrow[d] & H_n(D_p) \arrow[d] \\
H_n(D_{1/p}) \otimes \mathbb{Q} \arrow[r, "\cong"'] & H_n(D_p) \otimes \mathbb{Q}
\end{tikzcd}$$

We readily conclude that $H_n(D)\otimes\mathbb{Z}_{(p)} \cong H_n(D_p)$. Similarly $H_n(D)\otimes\mathbb{Z}_{(q)} \cong H_n(D_{1/P}) \otimes \mathbb{Z}_{(q)}$ for all $q \notin P$, so that $H_n(D) \otimes \mathbb{Z}_{(1/P)} \cong H_n(D_{1/P})$, by the very essence of arithmetic squares.

For the general case it is enough to consider D as a sequence of pullbacks, and apply the above argument repeatedly.

Next we have to prove that D is quasi-isomorphic to a perfect $\mathbb{Z}\Gamma$-complex using 3.2.3. For that purpose, we need a well known algebraic fact [26]: a $\mathbb{Z}$-module M is finitely generated (resp. finite or trivial), if and only if all the $M \otimes \mathbb{Z}_{(p_i)}$ and $M \otimes \mathbb{Z}_{(1/P)}$ are finitely generated (resp. finite or trivial) $\mathbb{Z}_{(p_i)}$ and $\mathbb{Z}_{(1/P)}$ modules. In our case, it immediately follows that D has finite homological dimension and $H_i(D)$ is finitely generated for all i. Up to taking a projective resolution of D, we can assume that we have a projective chain complex. In particular, we can assume that all the quasi-isomorphisms above are homotopy equivalences.

To apply proposition 3.2.3, we also need to study the groups $Ext^k_{\mathbb{Z}\Gamma}(D, M)$ for all $\mathbb{Z}\Gamma$-modules M. As $\mathbb{Z}_{(p_i)}$ is flat over $\mathbb{Z}$, we have that $Ext^k_{\mathbb{Z}\Gamma}(D, M) \otimes \mathbb{Z}_{(p_i)} = Ext^k_{\mathbb{Z}_{(p_i)}\Gamma}(D \otimes \mathbb{Z}_{(p_i)}, M \otimes \mathbb{Z}_{(p_i)}) = Ext^k_{\mathbb{Z}_{(p_i)}\Gamma}(D_{p_i}, M \otimes \mathbb{Z}_{(p_i)})$. By assumption, D_{p_i} is perfect so that $Ext^k_{\mathbb{Z}_{(p_i)}\Gamma}(D_{p_i}, M \otimes \mathbb{Z}_{(p_i)}) = 0$ for all $k > dim(D_{p_i})$ and for all $p_i \in P$. Similarly, we have that $Ext^k_{\mathbb{Z}\Gamma}(D, M) \otimes \mathbb{Z}_{(1/P)} = Ext^k_{\mathbb{Z}_{(1/P)}\Gamma}(D_{1/P}, M \otimes \mathbb{Z}_{(1/P)}) = 0$ for all $k > dim(D_{1/P})$ As a result, $Ext^k_{\mathbb{Z}\Gamma}(D, M) = 0$ for all k big enough. By 3.2.3, D is equivalent to a perfect chain complex.

Now that we know that D is quasi-isomorphic to a perfect $\mathbb{Z}\Gamma$-complex, we can talk about its obstruction $\tilde{o}(D) \in \widetilde{K}_0(\mathbb{Z}\Gamma)$. Recall that $\tilde{o}(D) \otimes \mathbb{Z}_{(p_i)} = \tilde{o}(D \otimes \mathbb{Z}_{(p_i)}) = \tilde{o}(C_{p_i})$ for all $p_i \in P$ and of course $\tilde{o}(D) \otimes \mathbb{Z}_{(1/P)} = \tilde{o}(D \otimes \mathbb{Z}_{(1/P)}) = \tilde{o}(D_{1/P})$. This implies that D is quasi-isomorphic to a projective free $\mathbb{Z}\Gamma$-complex, provided that all the complexes D_{p_i} and $D_{1/P}$ are free. $\square$

Now that we can build perfect free $\mathbb{Z}\Gamma$-complexes with prescribed homology, we need a way of realizing them. This is done in theorem 8.10 of [22], for which we need some definitions: We call a function $\underline{n} : Ob(Or_{\mathcal{F}}G) \to \mathbb{N}$ monotone if $\underline{n}(G/K) \leq \underline{n}(G/H)$ whenever $(H) < (K)$. An $ROr_{\mathcal{F}}G$-complex C is called an $\underline{n}$-Moore complex, if it is connected and if $\widetilde{H}_i(C(G/H)) = 0$ for $i \neq \underline{n}(G/H)$.

Theorem 3.2.5. *[22] Let $\Gamma = Or_{\mathcal{F}}G$ and consider a perfect free $\mathbb{Z}\Gamma$-complex C. Suppose that C is an $\underline{n}$-Moore complex with $\underline{n}(G/H) \geq 3$ for all $G/H \in Ob(\Gamma)$. Suppose further that $C_i(G/H) = 0$ for all $i \geq \underline{n}(G/H) + 1$ and for all $G/H \in Ob(\Gamma)$. Then there is a finite G-CW-complex X such that $C(X)$ is homotopy equivalent to C as a $\mathbb{Z}\Gamma$-complex. Moreover $G_x \in \mathcal{F}$ for all $x \in X$.*

Now that we have explained the general strategy to build finite homotopy G-spheres, we are ready for some examples. Systematizing the methods of example 3.2.2 and using proposition 3.2.4, we obtain the following new result:

Theorem 3.2.6. *Consider the group $G = C_p \rtimes C_{q^r}$ with faithful C_{q^r} action on C_p. For all $s \leq r$ and for all $j \geq 3$, there is a finite homotopy G-sphere X with:*

$$Dim_X^q(G/C_{q^t}) = \begin{cases} j + 2q^r & \text{if } t \leq s, \\ j & \text{otherwise.} \end{cases}$$

while $Dim_X^p(G/C_p) = j$.

We prove first two technical lemmas.

Lemma 3.2.7. *Let G be the group $C_p \rtimes C_{q^r}$. For each $s \leq r$ there is a perfect free $\mathbb{Z}_{(p)}\Gamma$-complex T_p such that:*

$$H_i(T_p(G/K)) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } e \neq (K) < (C_{q^s}), \\ 0 & \text{otherwise.} \end{cases}$$

for $i = 0, 1$, while $H_i(T_p) = 0$ if $i > 1$.

Proof. The free $\mathbb{Z}_{(p)}\Gamma$-functor generated at G/C_{q^s} has value:

$$F_{G/C_{q^s}}(G/K) = \begin{cases} \mathbb{Z}_{(p)}C_{q^{r-s}} & \text{if } e \neq (K) < (C_{q^s}), \\ \mathbb{Z}_{(p)}\overline{C_p \rtimes C_{q^{r-s}}} & \text{if } e = K, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the map $F_{G/C_{q^s}} \to F_{G/C_{q^s}}$ induced by the group ring norm map $N : \mathbb{Z}_{(p)}C_{q^{r-s}} \to \mathbb{Z}_{(p)}C_{q^{r-s}}$, $t \mapsto t - t^2$; where t is a generator of $C_{q^{r-s}}$. This gives rise to a $\mathbb{Z}_{(p)}\Gamma$-complex T'_p with:

$$H_i(T'_p(G/K)) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } e \neq (K) < (C_{q^s}), \\ \mathbb{Z}_{(p)}\overline{C}_p & \text{if } e = K, \\ 0 & \text{otherwise.} \end{cases}$$

for $i = 0, 1$ and $H_i(T') = 0$ for $i > 1$. In the same fashion as example 3.2.2, one can prove that there exists a $\mathbb{Z}_{(p)}G$-module M such that $\mathbb{Z}_{(p)}G \cong \mathbb{Z}_{(p)}\overline{C}_p \oplus M$, as $\mathbb{Z}_{(p)}G$-modules. Following the construction of example 3.2.2 again, one can easily define a perfect free $\mathbb{Z}_{(p)}\Gamma$-complex T_p:

$$\mathbb{Z}_{(p)}\overline{C}_p \oplus M \longrightarrow \mathbb{Z}_{(p)}\overline{C}_p \oplus M \oplus F_{G/C_{q^s}} \longrightarrow F_{G/C_{q^s}}$$

with the required homology. □

Lemma 3.2.8. *The* $\mathbb{Z}_{(q)}G$*-module* $Ker(\epsilon : \mathbb{Z}_{(q)}\overline{C}_p \to \mathbb{Z}_{(q)})$ *is projective.*

Proof. The short exact sequence of coefficients:

$$0 \longrightarrow Ker(\epsilon) \longrightarrow \mathbb{Z}_{(q)}\overline{C}_p \xrightarrow{\epsilon} \mathbb{Z}_{(q)} \longrightarrow 0$$

induces a long exact sequence in homology:

$$\cdots \longrightarrow H_n(G, Ker(\epsilon)) \longrightarrow H_n(G, \mathbb{Z}_{(q)}\overline{C}_p) \xrightarrow{\cong} H_n(G, \mathbb{Z}_{(q)}) \longrightarrow \cdots$$

Since $H_n(G, \mathbb{Z}_{(q)}) = H_n(C_{q^r}, \mathbb{Z}_{(q)}) = H_n(G, \mathbb{Z}_{(q)}[G/C_{q^r}])$, we must have that $H_n(G, Ker(\epsilon)) = 0$, so that $Ker(\epsilon)$ is a projective $\mathbb{Z}_{(q)}G$-module. □

Here is the proof of theorem 3.2.6

Proof. In analogy to example 3.2.2, we begin by finding perfect free chain complexes $D_p \in \mathbb{Z}_{(p)}\Gamma$ and $D_q \in \mathbb{Z}_{(q)}\Gamma$ with the required homologies. Consider first the prime p. By [38], there is a chain complex D'_p:

$$0 \longrightarrow \mathbb{Z}_{(p)} \longrightarrow M_{2q^r-1} \xrightarrow{f_{2q^r-1}} \cdots \xrightarrow{f_1} M_0 \longrightarrow \mathbb{Z}_{(p)} \longrightarrow 0$$

with M_i a perfect free $\mathbb{Z}_{(p)}G$-module for all $i = 0, ..., 2q^r - 1$. We can look at this chain complex as a perfect free $\mathbb{Z}_{(p)}\Gamma$-complex, with zeroes away from the orbit G/e. Let $T_p^{*2q^r}$ be the concatenation of $2q^r$ copies of the complex T_p given in lemma 3.2.7. The required chain complex D_p is then found by concatenating $T_p^{*2q^r} \oplus D'_p$ with the trivial algebraic G-sphere of dimension j given by the chain complex:

$$C_i(S^j) = \begin{cases} F_{G/G} = \underline{\mathbb{Z}_{(p)}} & \text{if } i = 0, j; \\ 0 & \text{otherwise.} \end{cases}$$

We treat next the case of the prime q. Consider again the map $F_{G/C_{q^s}} \to F_{G/C_{q^s}}$ induced by the norm map $N : \mathbb{Z}_{(q)}C_{q^{r-s}} \to \mathbb{Z}_{(q)}C_{q^{r-s}}$. As in lemma 3.2.7, it defines a chain complex L, with homology:

$$H_i(L(G/K)) = \begin{cases} \mathbb{Z}_{(q)} & \text{if } e \neq (K) < (C_{q^s}), \\ \mathbb{Z}_{(q)}\overline{C}_p & \text{if } e = K, \\ 0 & \text{otherwise.} \end{cases}$$

for $i = 0, 1$ and $H_i = 0$ for $i > 1$. We add now free modules to modify this chain complex and reduce the homology at G/e from $\mathbb{Z}_{(q)}\overline{C}_p$ to $\mathbb{Z}_{(q)}$: Using lemma 3.2.8, the $\mathbb{Z}_{(q)}G$-module $Q = Ker(\epsilon : \mathbb{Z}_{(q)}\overline{C}_p \to \mathbb{Z}_{(q)})$ is summand of some free $\mathbb{Z}_{(q)}G$-module $F = Q \oplus B$. We can then easily produce a chain complex D'_q of the form $F \to F \oplus F_{G/C_{q^s}} \to F_{G/C_{q^s}}$ with homology:

$$H_i(D'_q(G/K)) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } (K) < (C_{q^s}), \\ 0 & \text{otherwise.} \end{cases}$$

for $i = 0, 1$ and $H_i(D'_q) = 0$ for $i > 1$ The required perfect free $\mathbb{Z}_{(q)}\Gamma$-complex D_q is found again by concatenating $(D')_q^{*2q^r}$ with the trivial algebraic G-sphere of dimension j.

We want next to apply proposition 3.2.4 to the complexes D_p and D_q. Observe that any of the constructions for either D_p or D_q, equally defines a perfect free $\mathbb{Z}_{(1/pq)}\Gamma$ complex $D_{1/pq}$ realizing the given dimension function. We conclude that 3.2.4 provides a perfect free $\mathbb{Z}\Gamma$-complex D realizing the same dimension function.

Navigating through the various constructions, we observe that the dimensional requirements of 3.2.5 are satisfied. Finally, we find a finite G-

CW-complex X with $C(X)$ quasi-isomorphic to D, by virtue of theorem 3.2.5. Since X is finite, the fixed points X^K are mod-p equivalent to the homotopy fixed points X^{hK} for all p-subgroups K and all $p||G|$. As a result, X is our finite G-sphere, Borel equivalent to all G-spheres Y with $\left\{Dim^p_X(-)\right\}_{p||G|} = \left\{Dim^p_Y(-)\right\}_{p||G|}$. $\square$

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