Simultaneous optimization of vertical and horizontal road alignments

by

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Abstract

Optimization of three-dimensional road alignments is recognized as a nonlinear and nonconvex optimization problem. The development of models that fully optimize a three-dimensional road alignment problem is not yet successful, because there are many factors involved and complexities in the geometric specification of the alignment. At present, there are two optimization approaches, the models that simultaneously optimize horizontal and vertical alignments, and those employing two or more stages of optimization processes.

In this thesis, we develop a novel approach of solving a three-dimensional road alignment problem where the optimal horizontal and vertical alignments are determined simultaneously. We develop the surrogate cost model that approximate the earthwork cost and the pavement cost. The problem is modelled as a multiobjective optimization where the cost due to the length of the road, and the cost due to the volume of earthwork are found to be conflicting objectives. In order to study the proposed model, two case studies are tested and the numerical results are provided. The experimental results indicate that the problem is nonconvex, and that it is, indeed, a multiobjective optimization problem. Further developments and improvements in the area of cost penalty parameters are recommended for future work.
# Table of Contents

Abstract .............................................. ii  
Table of Contents ................................... iii  
List of Tables ....................................... v  
List of Figures ...................................... vi  
Acknowledgements ................................. vii  
Dedication ........................................ viii  

## Chapter 1: Introduction ......................... 1  
   1.1 Background and motivation .................. 1  
      1.1.1 The horizontal alignment .......... 2  
      1.1.2 The vertical alignment .......... 4  
      1.1.3 The three-dimensional alignment .. 5  
      1.1.4 Surrogate models ................. 6  
   1.2 Problem statement ......................... 7  
   1.3 Research approach ........................ 8  
   1.4 Multiobjective Optimization ............. 9  
   1.5 Thesis outline ............................ 10  

## Chapter 2: Road alignment optimization .... 11  
   2.1 Road design system ........................ 11  
   2.2 Horizontal alignment design ............ 11  
      2.2.1 Mathematical model of the horizontal alignment geometry ........ 13  
      2.2.2 Horizontal alignment geometric constraints .......... 16  
   2.3 Vertical alignment design ................. 17  
      2.3.1 Vertical alignment geometric constraint .......... 21
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3.2</td>
<td>Other constraints</td>
<td>21</td>
</tr>
<tr>
<td>Chapter 3: Surrogate model for the cost functions</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td>Surrogate cost model</td>
<td>23</td>
</tr>
<tr>
<td>3.2</td>
<td>Surrogate cost model for tangential road segment</td>
<td>24</td>
</tr>
<tr>
<td>3.2.1</td>
<td>Computing transition points</td>
<td>25</td>
</tr>
<tr>
<td>3.2.2</td>
<td>Length of the tangent road section</td>
<td>27</td>
</tr>
<tr>
<td>3.2.3</td>
<td>Volume of ground cut</td>
<td>28</td>
</tr>
<tr>
<td>3.2.4</td>
<td>Volume of ground fill</td>
<td>30</td>
</tr>
<tr>
<td>3.2.5</td>
<td>Procedure for classifying a grid cell as a region of cut, fill, both cut and fill, or neither</td>
<td>31</td>
</tr>
<tr>
<td>3.2.6</td>
<td>Cost calculation for the tangent road segment</td>
<td>31</td>
</tr>
<tr>
<td>3.3</td>
<td>Surrogate cost model for circular road segment</td>
<td>32</td>
</tr>
<tr>
<td>3.3.1</td>
<td>Calculation of x-boundary crosses</td>
<td>34</td>
</tr>
<tr>
<td>3.3.2</td>
<td>Length of circular road section</td>
<td>36</td>
</tr>
<tr>
<td>3.3.3</td>
<td>Volume of ground cut</td>
<td>36</td>
</tr>
<tr>
<td>3.3.4</td>
<td>Volume of ground fill</td>
<td>37</td>
</tr>
<tr>
<td>3.3.5</td>
<td>Cost calculation for a circular road segment</td>
<td>38</td>
</tr>
<tr>
<td>3.4</td>
<td>The overall surrogate cost function</td>
<td>38</td>
</tr>
<tr>
<td>Chapter 4: Multiobjective optimization model for 3D road alignment</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>4.1</td>
<td>Variable definition</td>
<td>40</td>
</tr>
<tr>
<td>4.2</td>
<td>Problem formulation</td>
<td>41</td>
</tr>
<tr>
<td>4.3</td>
<td>NOMAD: Nonlinear optimization with the MADS algorithm</td>
<td>41</td>
</tr>
<tr>
<td>4.4</td>
<td>Solution procedure</td>
<td>42</td>
</tr>
<tr>
<td>Chapter 5: Numerical results</td>
<td>43</td>
<td></td>
</tr>
<tr>
<td>5.1</td>
<td>Experimental setup</td>
<td>45</td>
</tr>
<tr>
<td>5.2</td>
<td>Case study 1</td>
<td>45</td>
</tr>
<tr>
<td>5.2.1</td>
<td>Experimental results 1</td>
<td>46</td>
</tr>
<tr>
<td>5.3</td>
<td>Case study 2</td>
<td>48</td>
</tr>
<tr>
<td>5.3.1</td>
<td>Experimental results 2</td>
<td>50</td>
</tr>
<tr>
<td>5.4</td>
<td>Implication of the results</td>
<td>52</td>
</tr>
<tr>
<td>Chapter 6: Conclusion and Future work</td>
<td>53</td>
<td></td>
</tr>
<tr>
<td>6.1</td>
<td>Future work</td>
<td>53</td>
</tr>
<tr>
<td>Bibliography</td>
<td>54</td>
<td></td>
</tr>
</tbody>
</table>
List of Tables

Table 5.1  Design constants and parameter values. . . . . . . . . 43
Table 5.2  Numerical results for case study 1. . . . . . . . . . 49
Table 5.3  Numerical results for case study 2 when $\theta = 0.5$. . . . 50
Table 5.4  Numerical results for case study 2 when $\theta = 0$. . . . . 52
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 1.1</td>
<td>An example a horizontal alignment.</td>
<td>3</td>
</tr>
<tr>
<td>Figure 1.2</td>
<td>An example vertical alignment, longitudinal profile.</td>
<td>4</td>
</tr>
<tr>
<td>Figure 1.3</td>
<td>An example 3D road alignment.</td>
<td>5</td>
</tr>
<tr>
<td>Figure 2.1</td>
<td>An example horizontal alignment for the model</td>
<td>12</td>
</tr>
<tr>
<td>Figure 2.2</td>
<td>Horizontal alignment geometry</td>
<td>14</td>
</tr>
<tr>
<td>Figure 2.3</td>
<td>Example of horizontal alignment obtained using formulas (2.1), (2.2), and (2.3)</td>
<td>16</td>
</tr>
<tr>
<td>Figure 2.4</td>
<td>An example section of vertical alignment.</td>
<td>18</td>
</tr>
<tr>
<td>Figure 2.5</td>
<td>Vertical alignment in $h_z$-plane</td>
<td>20</td>
</tr>
<tr>
<td>Figure 3.1</td>
<td>An example, the projection of a tangent road segment onto horizontal plane</td>
<td>26</td>
</tr>
<tr>
<td>Figure 3.2</td>
<td>An example cut cross-section.</td>
<td>28</td>
</tr>
<tr>
<td>Figure 3.3</td>
<td>An example, fill cross-section.</td>
<td>30</td>
</tr>
<tr>
<td>Figure 3.4</td>
<td>An example of a pave only cross-section.</td>
<td>31</td>
</tr>
<tr>
<td>Figure 3.5</td>
<td>Example of a horizontal curve section.</td>
<td>34</td>
</tr>
<tr>
<td>Figure 3.6</td>
<td>An example of road segments.</td>
<td>39</td>
</tr>
<tr>
<td>Figure 5.1</td>
<td>Three-dimensional view of the terrain.</td>
<td>44</td>
</tr>
<tr>
<td>Figure 5.2</td>
<td>The contour map.</td>
<td>44</td>
</tr>
<tr>
<td>Figure 5.3</td>
<td>Initial horizontal alignment in case study 1.</td>
<td>46</td>
</tr>
<tr>
<td>Figure 5.4</td>
<td>The optimal horizontal alignments for case study 1.</td>
<td>47</td>
</tr>
<tr>
<td>Figure 5.5</td>
<td>The optimal vertical alignments for $\theta = 1$.</td>
<td>48</td>
</tr>
<tr>
<td>Figure 5.6</td>
<td>The optimal vertical alignments for $\theta = 0$.</td>
<td>48</td>
</tr>
<tr>
<td>Figure 5.7</td>
<td>The horizontal alignments for case study 2.</td>
<td>49</td>
</tr>
<tr>
<td>Figure 5.8</td>
<td>Two optimal horizontal alignments computed from different starting alignments.</td>
<td>50</td>
</tr>
<tr>
<td>Figure 5.9</td>
<td>The optimal horizontal alignments.</td>
<td>51</td>
</tr>
<tr>
<td>Figure 5.10</td>
<td>The optimal horizontal alignments.</td>
<td>51</td>
</tr>
</tbody>
</table>
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Above all, I thank God for everything He provides me.
Dedication

This thesis is dedicated to my wife, Genet Alemayehu and my son, Solen Dessalegn.
Chapter 1

Introduction

1.1 Background and motivation

A good transport network is important in sustaining economic success, as it links people to jobs and delivers products to markets. It is also critical to domestic and cross-border trades and tourism activities. Road transport continues to be the dominant mode of transportation for moving goods between Canada and the U.S.A. [Rev]. In 2011, 56.5% of overall Canada-U.S.A. trade was transported by truck. These benefits come with cost related to road construction. In road construction projects, the main factors that determine the location of a road are the construction and maintenance costs, environmental requirements, and driver's safety. The cost items to be considered in the economic analysis of road projects are generally divided into two broad categories [JS04]: (1) supplier costs, which are directly incurred by road (highway) administrators, and (2) user costs incurred by road users. The supplier costs are further divided in three categories: (a) pavement, and other costs primarily depending upon the length of the alignment, (b) right-of-way costs including those costs associated with land and environmental impacts, and (c) earthwork costs. The user costs are divided into three categories: (a) travel-time cost, (b) vehicle-operating cost, and (c) accident cost [CGF89, JS04, SJK06].

Road design is a process for the determination of three-dimensional route location on the ground surface. It aims to connect two terminals (the start and end points) at minimum possible cost subject to the design, environmental, and social constraints [JM07].

Since the number of alternative routes connecting two end points is unlimited, a traditional route location analysis, which has relied heavily on human judgement and intuition, may overlook many good alternative [CGF89, LPZL13]. Thus, in order to reduce the workload for engineers, and to consider all possible route alternatives, an automated procedure that determines the road alignments and calculates the construction costs has been developed. The automation of road design problem reduces the tedious and error-prone manual tasks, most notably drafting [KJLS04]. In addition, this
1.1. Background and motivation

procedure allows the use of optimization techniques in search of a good alignment (an alignment with minimum cost) [OEC73]. Optimization techniques save much of designer’s time and provide the decision maker with a powerful tool that searches an alignment with minimum cost from a large number of alternative alignments. In fact, optimization of road alignment can yield considerable savings in construction costs when compared with unoptimized design procedures.

The problem of road design can be broken down into three interconnected stages: horizontal alignment, vertical alignment, and earthwork [HKL11]. Optimization of a road alignment is often defined as the minimization of the cost of the project satisfying certain requirements. Optimization of road design therefore implies the search for the optimum location of the vertical and horizontal alignments. Each stage is discussed further below.

1.1.1 The horizontal alignment

The Horizontal alignment is the projection of the three-dimensional road on the horizontal plane. A typical horizontal alignment is made up of a sequence of straights, circular curves, and transition curves. Transition curves have the property that the radius of curvature changes progressively along them. The design of a horizontal alignment mainly involves the design of horizontal curves and tangents. Curves are provided for smooth transition to avoid sudden changes in direction. The main considerations in horizontal alignment design are that, it should avoid lands which are expensive to purchase or restricted, obstacles which present engineering difficulties, and ground which may involve large earthwork or structure costs. Therefore, the cost of road construction for the horizontal alignment problem depends on the cost of acquiring land and on the output of the vertical alignment stage [HKL11].

The horizontal alignment optimization seeks a low cost route while adhering to the design standards and reducing environmental impacts [AH11]. In the literature, the following models have been developed for optimizing horizontal alignments: calculus of variation [Nic73], network optimization [Tri87], dynamic programming [OEC73], and genetic algorithms [JJS00]. Detailed discussion on the advantages and disadvantage of these methods can be found in [SJK06].

In optimization modelling using calculus of variation, the two end points are connected by a curve and integration of a cost function is minimized. This method can generate a smooth alignment and possibly yields a global optimum, but requires a continuously differentiable cost function, which
1.1. Background and motivation

may not exist over different land-use patterns and geographic features such as rivers and lakes [JJS00, SJK06]. In Network-optimization approaches, the alignment is represented by the arcs connecting the start and end points and the modified alignment problem have well-developed solution algorithms, but the resulting alignments are not smooth [JJS00]. The principal assumption in dynamic programming approach is that a problem can be divided into a number of sub-problems (or stages) and that the contribution to the objective function value from each sub-problem are independent and additive. For solving the horizontal alignment optimization problem, the stages are defined to be the evenly spaced lines perpendicular to the axis connecting the start and end points of the alignment [SJK06]. This approach may need less computer memory but also cannot yield a smooth alignment [JJS00]. A genetic algorithm integrated with the geographic information systems (GIS) can be used to solve the horizontal road alignment between two given end points. First, the model can generate smooth alignments based on highway design standards, and then the genetic algorithm approach can optimize very complex cost functions including user costs, which have been ignored in many existing models. Finally, the model directly exploits the information in a GIS database, which reduces data preprocessing time and allows the model to search through realistic and highly irregular spatial data [JJS00].

An example of a horizontal road alignment section composed of straights and circular curves is shown in Figure 1.3.
1.1. Background and motivation

1.1.2 The vertical alignment

The vertical alignment is the view of the centreline of the road when seen along the longitudinal cross section of the road. Generally, the vertical alignment is composed of straight sections known as vertical tangents or grades, and parabolic vertical curves, namely crest and sag curves, see Figure 1.2. Usually, the vertical alignment design is based on a pre-selected horizontal alignment. The main purpose of the vertical alignment is to provide point of elevations along the centreline of the horizontal alignment. The design requirements and other constraints, many of which are conflicting in nature, have made the design of a vertical alignment a rather complicated problem. Nonetheless, determination of the vertical alignment is a crucial step in the road design problem since it has important implications on road construction costs, traffic operations, vehicle fuel consumption, and safety [FCS02]. The need for optimization analysis in the selection of a desirable vertical alignment has long been recognized [FCS02].

![Figure 1.2: An example vertical alignment, longitudinal profile.](image)

In the vertical alignment optimization, one fits a road profile to the ground profile while respecting various grade constraints and other road specifications. The objective is to minimize the cost of construction and the negative impacts on the environment such as the natural landforms and soil [FCS02]. Many models are found for optimizing vertical alignments than for horizontal alignments [SJK06]. The vertical alignment optimization models include linear programming [MS81, Eas88], Numerical search [Hay70], state parametrization [GCF88], dynamic programming [GAA09, GCF88], genetic
1.1. Background and motivation

algorithm [FCS02, AH11], and mixed integer linear programming [HKL11]. See also [SJK06, GCF88, GAA09] for more references.

1.1.3 The three-dimensional alignment

A three-dimensional road alignment is the superimposition of two dimensional horizontal and vertical alignments. In essence, road alignment design is a three-dimensional problem represented in $X$, $Y$, and $Z$ coordinates. Figure 1.3 represents an example of a three-dimensional road.

The development of models that fully optimize a three-dimensional road alignment problem is not yet successful, because there are more factors involved and more complexities in the geometric specification of the alignment [SJK06, GAA09, ASAC05]. There are two basic approaches found in the literature: models that simultaneously optimize the horizontal and vertical alignments [ASAC05, JS03, CGF89], and models that employ two or more stages of optimization to get an optimized three-dimensional road alignment [Nic73, Par77]. Existing approaches for three-dimensional alignment optimization include genetic algorithms [JS03, KJSK07, KJS12], dynamic programming [LPZL13], neighbourhood search [CL06], and distance transform [DS06].

Figure 1.3: An example 3D road alignment.

Due to the rigorous geometric specifications of alignments and the com-
1.1. Background and motivation

Complexity of the problem, none of the existing methods have completely solved the problem of alignments optimization [LPZL13]. The model developed in [CGF89] yields the simultaneously optimized three-dimensional road alignment optimization problem. It uses a series of cubic spline functions to parametrize the road alignment, and the problem is formulated as a calculus of variations problem. The method of constraint transcription is used in optimal control theory is employed to transform constraints into a one-dimensional problem.

A dynamic programming model is used in [Nic73] to optimize the three-dimensional road alignment problem in two stages. In this approach, at the first stage, the model searches a relatively coarse grid of points for a preliminary alignment. Then a discrete variational calculus is adopted to refine the alignment so that the resulting alignment can deviate from the grid points. The resulting solution is rough due to the storage requirement for searching the initial grid.

1.1.4 Surrogate models

Engineering design heavily depends on computer simulations. The application of optimization to such systems, therefore, has to do with the objective functions that may come from large scale computer operations or computer simulations. Unfortunately, accurate and high-fidelity simulations are often computationally expensive, with evaluation times ranging from hours to days [KL13]. In many cases, optimization of such objectives by direct application of optimization routines is impractical [KY11]. One reason is that conventional optimization algorithms require tens, hundreds or even thousands of objective function calls per run, which makes the computational cost of the whole optimization process intractable [KY11, WPR05, QHS+05]. A second reason is that simulation-based objective functions are often analytically intractable, discontinuous, non-differentiable, noisy, or possibly fail to return a value [KY11]. Therefore, any technique that reduces the function evaluation count is crucially important. Feasible handling of these objective functions can be accomplished using surrogate models [KY11]. A surrogate \( s_f \) of the function \( f \) is a function that shares some similarities with \( f \), but is much cheaper to evaluate. A surrogate model is typically less accurate or has less quality than the true model, but it is cheaper to evaluate or consumes fewer computing resources.

For optimization problems, surrogate models can be regarded as approximation models for the objective function and the constraint set. The basic concept of surrogate-based optimization is that the direct optimization of
1.2 Problem statement

The computationally expensive model is replaced by an iterative process that involves the creation, optimization, and updating of a fast and analytically tractable surrogate model [KY11]. The surrogate should be a reasonably accurate representation of the true model, at least locally. The solution obtained by optimizing the surrogate model is verified by evaluating the solution of the true model which is used to update the surrogate model iteratively until some termination criterion is met [KY11].

The surrogate model generation is a key component of any surrogate-based optimization algorithm. There are two ways of generating the surrogate model [KY11]:

a) a model constructed from physically-based low-fidelity models, and

b) a model based on the function approximation of sampled data.

Since the physical surrogate is based on a particular knowledge of the physical system of interest, it can be formulated using analytical or semi-empirical formulas. Once the surrogate model is built, an optimization algorithms can be used to yield an approximation of the minimizer (or maximizer) of the true model.

1.2 Problem statement

Road alignment optimization based on cost minimization requires comprehensive formulation of costs and development of efficient solution algorithms. Among the major cost components that contributed in the construction of a road are the earthwork cost and pavement costs [CGF89].

In this thesis, we develop a surrogate model for the cost of ground cut, ground fill, material waste, and pavement based on the approximation of the physical terrain by planes.

These cost items account for more than 50% of the construction cost [CGF89]. Although the ground cut and fill costs depend on the varying nature of the soil, we regard it as a function of the volume of earth to be cut or filled. In the model, we have considered the volume of wasted material as the difference between the volume of ground cut and ground fill. The average-end-area method is widely used to estimate the volume of earthwork between two stations, however, it gives less accurate estimation when the terrain is irregular or mountainous [SJK06]. The pavement cost is computed using road length.
1.3 Research approach

This thesis seeks to solve a three-dimensional road alignment optimization problem. This problem is presented as a complex optimization problem in which both vertical and horizontal alignments are determined. As we have already discussed above, this problem can be solved in two approaches. The first approach solves both vertical and horizontal alignments simultaneously, and the second approach uses multi-stage (or multi-level) based optimization methods. Both approaches, however, suffer from the high computational effort and large memory requirements [SJK06]. Three-dimensional road alignment optimization is a problem having a constrained, nonlinear, and non-differentiable structure that cannot be efficiently solved by classic optimization techniques [ASAC05]. As a result, many heuristic based methods, such as genetic algorithm, simulated annealing, and tabu search, have been used to solve the problem.

In our research, we modelled the problem as a simultaneous optimization of vertical and horizontal alignment problems. In this approach the vertical alignment is composed of straight lines and the horizontal alignment is composed of straight lines and circular curves. The cost component considered are the earthwork cost. A surrogate model that calculates the cost of ground cut, ground fill, ground waste, and paving is developed. The design variables corresponding to the vertical and horizontal alignments are reflected in the surrogate cost model formulation. The surrogate model calculates the cost quickly, but provides a less accurate earthwork cost than other mixed integer linear programming (MILP) techniques.

Furthermore, in mountainous regions the cost function for ground cut and fill tends to favor a circuitous, perhaps the longest distance, alignments, while the paving cost function favors the shortest distance alignment. As a result, the two cost components conflict with each other. Therefore, the problem is presented as a multiobjective optimization problem. This problem is a nonlinear, nonconvex, non-differentiable constrained optimization problem. Consequently, the derivative-free optimization method, known as NOMAD (Nonlinear optimization with the MADS algorithm) is used to solve the problem. By minimizing the cost function, NOMAD generates the radius of curvature and coordinates of the intersection points for the horizontal alignment, and the coordinates of vertical intersecting points for the vertical alignments.
1.4 Multiobjective Optimization

In real life, decision makers or engineers are frequently faced with decision problems of several, perhaps conflicting, objectives. It is only natural to want all of the choices or decisions to be as good as possible, in other words, optimal. Single objective optimization technology is not sufficient to deal with problems with conflicting objectives. Thus, *multiobjective optimization* is an appealing alternative. In multiobjective optimization, several conflicting objective functions have to be optimized simultaneously over a feasible set determined by constraint functions. There is, usually, no unique solution that is simultaneously optimal for all objectives. As a result, one can only consider a trade-off among the objectives, and the primary goal of multiobjective optimization is to seek the best trade-off to support the decision maker in choosing a final preferred solution.

Unlike the single objective optimization, in the multiobjective context, the interest is often more in the objective space. For this reason, the notion of optimality depends on how decision alternatives are compared and ordered, that is, on the order relation in the objective space [Mie99]. The order relations in objective space can be defined by different dominance relations such as Pareto dominance, Geoffrions dominance, and Lexicographical dominance, of which the *Pareto dominance* is most commonly used [ASZ08]. Although there is no universally accepted solution concept for decision problems with multiple objectives, one would agree that a good solution must not be dominated by the other feasible alternatives [Yu74]. Generally, multiobjective optimization problems (MOP) solution techniques are classified depending on the moment when the engineer or decision maker is able to establish preferences relating the different objectives [CMVV11]. The preference informations may be classified as: no preference, a priori preference, progressive preference, and a posteriori preference.

Returning to horizontal alignment, in road design we have two conflicting objectives: (1) the cost that depends on the length of a route and (2) the cost that depends on the volume of earthwork. Different types of costs will favor different alignment configurations. For example, length-dependent costs and user costs tend to straighten the alignment, while location-dependent costs tend to favor more indirect and circuitous alignment [SJK06]. In this situation, there is a conflict between length of an alignment and the cost of an alignment. In this thesis, the surrogate cost model is formulated for two costs items, the cost that depends on the length of an alignment (paving cost) and the cost that depends on the volume of earthwork (cost due to ground cut, ground fill, and wasted material). Thus, a three-dimensional
alignment optimization problem is modelled as a multiobjective optimization problem.

1.5 Thesis outline

In Chapter 2, we present the mathematical models for the horizontal and vertical alignment geometries. The problem variables and constraints corresponding to the horizontal alignment and vertical alignment are also specified. Chapter 3 deals with the formulation of the surrogate model for the cost functions. The surrogate model corresponding to the tangent and circular sections of the horizontal alignment are presented. The model corresponding to the circular curve is more challenging. In Chapter 4, the multiobjective optimization model and NOMAD are discussed. In particular, a solution procedure for the three-dimensional optimization problem is presented. The numerical results are given in Chapter 5. Some concluding remarks are given in Chapter 6.
Chapter 2

Road alignment optimization

This chapter presents the mathematical formulation of the geometric design for horizontal and vertical road alignments. The geometric constraints related to each alignment is given. The mathematical formulation entails the specification of the design elements that also reflect the cost of the individual alignment.

2.1 Road design system

In the road alignment development process, the first stage (the planning stage) is the selection of a corridor along which the road is to pass. Following this stage is the road alignment design stage, which deals with the location, alignment, and shape of a road. The selection of a specific alignment involves the determination of horizontal and vertical alignments of a road profile, subject to a set of constraints and requirements [CGF89].

2.2 Horizontal alignment design

In designing a horizontal alignment geometry, a series of tangents and curved sections are joined. Restrictions on the horizontal road alignment geometry mainly come from code requirements and external limitations. These design codes require that the horizontal alignment of a road be composed of three types of design elements, namely line segments, circular arcs, and transition curves. The circular and transition curves are typically combined to form the curved section. The transition curves are often applied between tangents and circular curves for reducing a sudden change in the curvature.

In our model, the horizontal alignment is composed of the tangential segments and circular curves. In this approach the horizontal alignment geometry is required to satisfy two criteria: (1) the alignment should satisfy the orientation specified as $tangent - circle - tangent$, and (2) the start and end sections of the alignment should be tangent segment. The circular curves are placed between two adjacent tangents to mitigate the sudden changes in
2.2. Horizontal alignment design

the direction of the alignment. In this way the absence of transition curves
does not have a major effect on the applicability of the model. Thus, a
horizontal alignment for the model consists of two types of road segments:
straight lines (tangents) and circular curves, see Figure 2.1.

![Figure 2.1: An example horizontal alignment for the model](image)

The exact shape of the horizontal alignment is determined by the set of
intersection points and the radius of a circular curve. The circular curves
are inserted between tangents as defined by the intersection points. There
are three decision variables associated with each intersection point, namely
the $x$, $y$, and $r$, where $x$ and $y$ are the $x$–coordinate and $y$–coordinates
of intersection point, and $r$ is the radius of the circular curve. Thus, if the
number of intersection points is $N$, the horizontal alignment has to select $3N$
variables. The most important horizontal alignment constraints, which are
related to the geometric design and code requirements, are given as follows:

1. two adjacent circular segments should not overlap;
2. the radius of each circular curve should be greater than the minimum
turning radius (i.e. $r_{\text{min}} \leq r$);
3. the alignment should stay in the corridor (or design space).
2.2. Horizontal alignment design

2.2.1 Mathematical model of the horizontal alignment geometry

We design horizontal alignment as a curve that is described by a series of intersection points, $IP$, and radius of curvature $r$ that determine the design elements (the tangents and the circular curves).

Determination of horizontal alignment design elements

We assume we are given $N$ intersection points of the horizontal alignment $\{IP_1, IP_2, \cdots, IP_N\}$, and the corresponding set of radius of curvature $\{r_1, r_2, \cdots, r_N\}$, between the start and end points $S_h$ and $E_h$, where $S_h$ and $E_h$ are the horizontal component of the start and end points of the three-dimensional alignment. Then one can insert $N + 1$ tangential road segments and $N$ circular curves to get the exact shape of a horizontal alignment. The horizontal alignment geometry is determined by design elements, the tangents and circular curves. In this section, we calculate the points that determine the tangent section and circular curves, such as the center of the curve ($C_k$) and transition points $TC_k$ (from tangential section to circular section) and $CT_k$ (from circular section to tangential), see Figure 2.2.

We denote the vector from $IP_k$ to $IP_{k-1}$ and $IP_k$ to $IP_{k+1}$, respectively, by $IP_{k(k-1)}$ and $IP_{k(k+1)}$. Given intersection points $IP_{k-1}, IP_k$ and $IP_{k+1}$, and radius $r_k$, let $CIR_k$ be the circle of radius $r_k$ that is tangent to $IP_{k(k-1)}$ and tangent to $IP_{k(k+1)}$ at $TC_k$ and $CT_k$, respectively. We denote the centre of $CIR_k$ by $C_k = (x_{ck}, y_{ck})$, we denote the points $TC_k = (x_{tc_k}, y_{tc_k})$ and $CT_k = (x_{ct_k}, y_{ct_k})$. Let $\theta_k$ be the angle between $IP_{k(k-1)}$ and $IP_{k(k+1)}$, then we calculate the coordinates of $C_k$, $TC_k$, and $CT_k$. 


2.2. Horizontal alignment design

Calculation of the transition points $TC_k$ and $CT_k$

From the dot product of vectors $IP_{k(k-1)}$ and $IP_{k(k+1)}$, we have

$$\theta_k = \arccos\left(\frac{IP_{k(k-1)} \cdot IP_{k(k+1)}}{\|IP_{k(k-1)}\| \|IP_{k(k+1)}\|}\right),$$  \hspace{1cm} (2.1)

where

$$IP_{k(k-1)} = IP_{k-1} - IP_k = (x_{k-1} - x_k, y_{k-1} - y_k)$$
$$IP_{k(k+1)} = IP_{k+1} - IP_k = (x_{k+1} - x_k, y_{k+1} - y_k).$$

Also, we denote

$$DP = IP_{k(k-1)} \cdot IP_{k(k+1)} \text{ and } NP = \|IP_{k(k-1)}\| \|IP_{k(k+1)}\|,$$

where $DP$ and $NP$ means dot product and norm product, respectively. Let $L_k$ be the distances from $IP_k$ to $TC_k$. Then

$$\tan \frac{\beta_k}{2} = \frac{L_k}{r_k} \Rightarrow L_k = r_k \tan \frac{\beta_k}{2},$$

where $\beta_k$ is the central angle of the circular arc, see Figure 2.2.
2.2. Horizontal alignment design

Using the identity
\[
\tan \frac{\beta_k}{2} = \frac{1 - \cos \beta_k}{\sin \beta_k}
\]
and the fact that \( \beta_k = \pi - \theta_k \),
we have
\[
L_k = r_k \frac{1 + \cos \theta_k}{\sin \theta_k}.
\]
Notice that, because of the congruence of \( \triangle C_kTC_kIP_k \) and \( \triangle C_kCT_kIP_k \),
the length of line segments \( IP_kTC_k \) and \( IP_kCT_k \) are equal to \( L_k \).

Let \( V_k \) and \( U_k \) denote the vectors directed from \( IP_k \) to \( TC_k \) and \( IP_k \) to \( CT_k \), respectively. That is,
\[
V_k = TC_k - IP_k \quad \text{and} \quad U_k = CT_k - IP_k.
\]
Then
\[
V_k = TC_k - IP_k = L_k \frac{IP_{k-1}}{\|IP_{k-1}\|}, \quad \text{and} \quad U_k = CT_k - IP_k = L_k \frac{IP_{k+1}}{\|IP_{k+1}\|}.
\]
Thus, we get
\[
TC_k = IP_k + L_k \frac{IP_{k-1}}{\|IP_{k-1}\|}, \quad \text{and} \quad CT_k = IP_k + L_k \frac{IP_{k+1}}{\|IP_{k+1}\|}. \quad (2.2)
\]

Calculation of the center: \( C_k \)

Let \( M_k \) be the midpoint of the line segment that connects \( TC_k \) and \( CT_k \). Clearly \( M_k \) lies on the line that passes through \( IP_k \) and \( C_k \). Then
\[
M_k = \frac{1}{2}(TC_k + CT_k) = IP_k + \frac{L_k}{2} \left( \frac{IP_{k-1}}{\|IP_{k-1}\|} + \frac{IP_{k+1}}{\|IP_{k+1}\|} \right).
\]
Now,
\[
C_k - IP_k = (r_k + d_k) \frac{M_k - IP_k}{\|M_k - IP_k\|},
\]
where \( d_k \) is the distance from \( IP_k \) to the circle \( CIR_k \); calculated as
\[
\sec \frac{\beta_k}{2} = \frac{r_k + d_k}{r_k} \Rightarrow d_k = r_k \sec \frac{\beta_k}{2} - r_k = r_k \csc \frac{\theta_k}{2} - r_k.
\]
Therefore,
\[
C_k = IP_k + r_k \csc \frac{\theta_k}{2} \left( \frac{M_k - IP_k}{\|M_k - IP_k\|} \right). \quad (2.3)
\]
2.2. Horizontal alignment design

**Example** 1. Given the following set of intersection points, radius of curvature, and the start and end points.

\[
S_h = (5, 2), E_h = (17, 17), IP_1 = (10, 7), IP_2 = (14, 5), IP_3 = (16, 10), IP_4 = (13, 13), r_1 = 1.8, r_2 = 2, r_3 = 2, r_4 = 1.8.
\]

Then, the horizontal design elements (tangent section and circular arcs) that are generated using the above formulations are depicted in Figure 2.3.

![Figure 2.3: Example of horizontal alignment obtained using formulas (2.1), (2.2), and (2.3)](image)

2.2.2 Horizontal alignment geometric constraints

The design of horizontal alignment geometry is restricted or constrained, mainly, by code requirements and external limitations such as control areas or restricted areas. The set of constraints for horizontal alignment are discussed below.

**Box constraints on the IP location**

In our model, the location of each intersection point is contained by a box. Let \( IP_k = (x_k, y_k) \), and let \( x_{u_k}, x_{l_k}, y_{u_k}, y_{l_k} \) be real numbers. Then the box constraint corresponding to \( IP_k \) is given as follows.

\[
x_{l_k} \leq x_k \leq x_{u_k}, \quad y_{l_k} \leq y_k \leq y_{u_k}, \quad k = 1, 2, \ldots, N.
\]
2.3. Vertical alignment design

The circular curves should not overlap

This constraint depends on the length of the tangent section between two adjacent circular curves. Two adjacent curves can meet only if the length of the tangent between them is zero. We note that, because of the requirements discussed in Section 2.2 the length of the first and last tangential segment cannot be zero. Thus, this constraint can be written, mathematically, as follows.

\[ 0 \leq \| TC_k - IP_{k-1} \| - \| CT_{k-1} - IP_{k-1} \|, \quad k = 1, 2, \ldots, N. \]  
(2.4)

The minimum turning radius

Given the minimum radius \( r_{\text{min}} \) of the circular curve, the optimal radius at each intersection point has to satisfy the minimal radius requirement set. Thus

\[ r_{\text{min}} \leq r_k, \quad k = 1, 2, \ldots, N. \]  
(2.5)

2.3 Vertical alignment design

Vertical alignment is the view of three dimensional road alignment when seen along the longitudinal cross section of the road. It is composed of straight lines known as vertical tangents and parabolic vertical curves. There are two forms of vertical curves: the crest vertical curve and sag vertical curve. Curves are used to provide a gradual change in elevation between successive tangents for smooth traverse. Therefore, the standard approach for designing vertical alignment is by the selection of proper grades for the tangent sections and proper curve lengths. However, since the length of vertical curve relative to the length of tangent section is small, in our model, the design of vertical alignment is composed of only vertical tangents. A typical section of vertical alignment is shown in Figure 2.4.

In this section, we present the mathematical model of vertical alignment design element, the tangent section. The vertical alignment is modelled as a continuous piecewise linear function. We begin the model by stretching a surface orthogonal to the \( xy \)-plane along the horizontal alignment. The surface is so stretched to be flat, so we call it the \( hz \)-plane, where \( h \) is the distance measured along the horizontal alignment. The projection onto the \( hz \)-plane of the three-dimensional road alignment is the vertical alignment.
2.3. Vertical alignment design

Given the following sets of intersection points and the horizontal tangent points, calculated in Section 2.2.1,

\[
\text{IP} = \{IP_1, IP_2, \cdots, IP_N\}, \quad (2.6) \\
\text{TC} = \{TC_1, TC_2, \cdots, TC_N\}, \quad (2.7) \\
\text{CT} = \{CT_1, CT_2, \cdots, CT_N\}. \quad (2.8)
\]

We partition the horizontal alignment between two adjacent tangent points \((CT_{k-1} \text{ and } TC_k)\) into \(m_k\) equally spaced points. Then, corresponding to these points, we design a set of vertical points whose projection on to the horizontal plane is required to stay on the line segment between \((CT_{k-1} \text{ and } TC_k)\), see Figure 2.5. The sole criterion for determining these vertical points is based on the minimization of the cost of earth cut, earth fill, earth balance, and the cost of paving. We define the set of the vertical points corresponding to horizontal tangent between \(CT_{k-1} \text{ and } TC_k\) as...
2.3. Vertical alignment design

follows:

\[ \mathbf{VP}_k = \{ VP_1, VP_2, \ldots, VP_{m_k} \}, \quad (2.9) \]

where \( VP_j = (x_{k_j}, y_{k_j}, z_{k_j}) \). The mathematical details of the design road elevation \( z_{k_j} \), and the \( xy \)-coordinates \( x_{k_j} \) and \( y_{k_j} \), is given below.

Note that since the length of the curved section of the horizontal alignment is typically much smaller than the tangent section, we do not have vertical point corresponding to the circular curve.

Next, we calculate the coordinates of the vertical point \( VP_j \). Given the horizontal tangent points \( CT_{k-1} \) and \( TC_k \). Partition the line segment between \( CT_{k-1} \) and \( TC_k \) into \( m_k \) equally spaced points, and let \( p_{j-1} = (x_{k_{j-1}}, y_{k_{j-1}}) \) and \( p_j = (x_{k_j}, y_{k_j}) \) be consecutive points.

For \( j = 1, 2, \ldots, m_k \)

\[ x_{k_j} = x_{ct_{k-1}} + \frac{j}{m_k} (x_{tc_k} - x_{ct_{k-1}}), \quad (2.10a) \]
\[ y_{k_j} = y_{ct_{k-1}} + \frac{j}{m_k} (y_{tc_k} - y_{ct_{k-1}}), \quad (2.10b) \]
\[ x_{k_{j-1}} = x_{ct_{k-1}} + \frac{j-1}{m_k} (x_{tc_k} - x_{ct_{k-1}}), \quad (2.10c) \]
\[ y_{k_{j-1}} = y_{ct_{k-1}} + \frac{j-1}{m_k} (y_{tc_k} - y_{ct_{k-1}}). \quad (2.10d) \]

Let \( z_{k_{j-1}} \) and \( z_{k_j} \) be the design road elevations corresponding to \( p_{j-1} \) and \( p_j \), respectively. Now we parametrize the line segment between \( p_{j-1} \) and \( p_j \) using the parameter \( s \) as follows. For \( j = 1, 2, \ldots, m_k \)

\[ \frac{j-1}{m_k} \leq s \leq \frac{j}{m_k}. \]

Then, the coordinates of a point \( p = (x, y, z) \) between \( VP_{k_{j-1}} \) and \( VP_{k_j} \) are parametrized as follows,

\[ x(s) = x_{k_{j-1}} + (x_{k_j} - x_{k_{j-1}})(m_k s + 1 - j) = x_{ct_{k-1}} + s(x_{tc_k} - x_{ct_{k-1}}), \quad (2.11a) \]
\[ y(s) = y_{k_{j-1}} + (y_{k_j} - y_{k_{j-1}})(m_k s + 1 - j) = y_{ct_{k-1}} + s(y_{tc_k} - y_{ct_{k-1}}), \quad (2.11b) \]
\[ z(s) = z_{k_{j-1}} + (z_{k_j} - z_{k_{j-1}})(m_k s + 1 - j). \quad (2.11c) \]
2.3. Vertical alignment design

Figure 2.5: Vertical alignment in $h_z$-plane

The horizontal distance between $VP_{kj-1}$ and $VP_{kj}$, denoted as $d_{kj}$ is given by

$$d_{kj} = \sqrt{(x_{kj} - x_{kj-1})^2 + (y_{kj} - y_{kj-1})^2}.$$ 

The calculation of the parameter $s$ is discussed in the next chapter.

The number of vertical alignment variables, and the precision of earthwork volume estimation depends on the magnitudes of $m_k$. The number of vertical alignment variables on each horizontal tangent section, except for the first and last horizontal tangents, is $m_k + 1$. The first and last horizontal
2.3. Vertical alignment design

tangent section each have $m_0$ and $m_N$ variables, respectively, because the start and the end of a road are not variables.

Suppose the number of horizontal intersection points is $N$. Then, the number of variables of the vertical alignment is

$$M = m_0 + m_N + \sum_{k=1}^{N-1} (m_k + 1).$$

2.3.1 Vertical alignment geometric constraint

The vertical tangent section is constrained by the standard design limits. Thus, the design of vertical alignment is constrained by the maximum gradient requirement and the vertical elevation boundaries.

The maximum allowable slope (or gradient) of the road

Once the vertical alignment variables are set, the design constraints have to be checked before conducting cost calculations. In compliance with the design standards, the following requirements have to be satisfied in the design of vertical alignment. Therefore, the vertical alignment design variable is required to respect the maximum grade constraint

$$\left| \frac{z_{kj} - z_{kj-1}}{d_{kj}} \right| \leq G_{\text{max}}.$$

The maximum and minimum elevation of the road

While not strictly necessary, it is reasonable to place a maximum and minimum road elevation constraints. Given the vertical off-set $\bar{z}$ from the current ground elevation $z_g$,

$$z_g - \bar{z} \leq z_{kj} \leq z_g + \bar{z}, \text{ for all } k_j.$$

2.3.2 Other constraints

In addition to the horizontal and vertical alignments geometric constraints, we require the circular section of the road to have the same elevation (i.e. flat). This requirement is a constraint for the optimization problem which is given mathematically as follow,

$$z_{km_k} = z_{(k+1)1}, \quad k = 1, 2, \ldots, N.$$
Chapter 3

Surrogate model for the cost functions

In this chapter we provide the mathematical formulation of the surrogate cost model for the three-dimensional road design problem. The surrogate model is developed based on the approximation of the ground surface by a series of planar surfaces. The designed road is represented by a continuous piecewise linear function which is expressed as parametric equations of a parameter $t$, where $t$ is normalized to lie between 0 and 1. The horizontal plane in the region of interest is divided into grid cells of equal size, which are small enough that the terrain above the grid is approximated by a single linear function. A matrix format is employed to store important ground information for the region of interest. Three different matrices are used to store the data for the determination of the elevation of the ground at a specified point in the $xy-$plane.

The surrogate cost function is formulated as the sum of the costs due to the paving and the earthwork. The cost associated with earthwork is the cost of ground cut, ground fill, and cost of waste material, which depend on the volume of the material. The calculation of the volume of ground cut and ground fill depends on the elevation difference between the approximating plane and the design road. Since the volume of earthwork is approximated, the cost calculation is only an approximation of the true cost. The paving cost is computed based on the length of the road. Hereafter, we refer to the approximating plane elevation as the ground profile. Ground cutting costs are incurred when the design road profile is lower than the ground profile, while filling costs are calculated when the road profile is above the ground profile.

The three-dimensional road alignment optimization problem is based on the minimization of a parametrized surrogate cost function subject to alignment constraints. Penalty parameters for each of the cost components are introduced. The optimal road alignment obtained by solving the surrogate-based optimization problem can be calibrated to different optimal alignments by choosing different values of the penalty parameters. The objective
function, as the total cost $C$, may be expressed in a general form as:

$$C = C_c V_c + C_f V_f + C_p W L + C_w \| V_f - V_c \|,$$  \hspace{1cm} (3.1)

where
- $L$ is length of the road
- $V_c$ is volume of ground cut
- $V_f$ is volume of ground fill
- $W$ is width of the road
- $C_c$ is the ground cut penalty parameter
- $C_f$ is the ground fill penalty parameter
- $C_p$ is the paving penalty parameter
- $C_w$ is the wasted material penalty parameter.

Earthwork volume estimation is one of the most important components in formulating the road construction costs. In our approach, the earthwork volume is estimated by continuous integration of the cross-sectional area between two end points along the road. The target road should minimize the surrogate cost model and conform to the constraints on horizontal and vertical alignments.

### 3.1 Surrogate cost model

Consider a three-dimensional road alignment between the starting point $START = (x_s, y_s, z_s)$ and the ending point $END = (x_e, y_e, z_e)$. We regard the projection onto the $xy$–plane (or horizontal plane) of a three dimensional alignment as being composed of tangents and circular curves as discussed in Chapter 2. We first divide the $xy$-plane into a square grid. We denote the $(u,v)^{th}$ grid cell by $G_{uv}$. The terrain in each grid is then approximated by a linear function

$$z = A_{uv}x + B_{uv}y + C_{uv},$$

where $x, y \in G_{uv}$, and $A_{uv}, B_{uv}$ and $C_{uv}$ are matrices containing the linear function data.

We aim to calculate the cost of a road section (tangential section or circular section) which is the sum of costs corresponding to each grid cell along the section. To approximate the cost of a road segment, one requires
3.2. Surrogate cost model for tangential road segment

i) an estimate of the volume of earth cut \((V_c)\),

ii) an estimate of the volume of earth filled \((V_f)\),

iii) the total length of the road \((L)\).

We then apply the costing parameters \(C_c, C_f, C_w, C_p\) to create a parametrized cost function (3.4). For the collection \(S\) of tangential road sections and circular road sections, we define \(V_{c\xi}\) is the total volume of ground cut on road segment \(\xi \in S\), and \(V_{f\xi}\) is the total volume of ground fill on road segment \(\xi \in S\). The overall volume of cut, denoted \(V_c\) and volume of fill, denoted \(V_f\), is given as

\[
- V_c = \sum_{\xi \in S} V_{c\xi}, \text{ and } \\
- V_f = \sum_{\xi \in S} V_{f\xi}.
\]

The cost function formulations for the tangential segment and circular curve are given in the following sections.

### 3.2 Surrogate cost model for tangential road segment

Suppose \(\xi_k \in S\) corresponds to a tangential segment. That is, horizontal projection is the tangent line that connects \(CT_{k-1}\) and \(TC_k\). We denote this horizontal tangent by \(HT_k\). Suppose there are \(m_k\) grade lines or vertical tangents along \(HT_k\). For \(j = 1, 2, 3, \ldots, m_k\), the vertical points, denoted by \(VP_{k,j}\) and \(VP_{k,j-1}\), are given as

\[
VP_{k,j-1} = (x_{k,j-1}, y_{k,j-1}, z_{k,j-1}), \text{ and } VP_{kj} = (x_{kj}, y_{kj}, z_{kj}),
\]

where \(z_{k,j-1}, z_{kj}, x_{k,j-1}, x_{kj}, y_{k,j-1}, \text{ and } y_{kj}\) are calculated using equations equations (2.10a) to (2.10d). Then the parametric representation of a vertical tangent between \(VP_{kj-1}\) and \(VP_{kj}\) is given as

\[
r_t(s) = (x(s), y(s), z(s)),
\]

where \(\frac{j-1}{m_k} \leq s \leq \frac{j}{m_k}\), and the parametric equations \(x(s), y(s), \text{ and } z(s)\) are given in equations (2.11a) to (2.11c). Thus, we define the parametric road profile and the parametric equation of a plane that approximates the terrain in grid \(G_{uv}\) as

\[
z_r(s) = z(s), \text{ and } z_g(s) = A_{uv}x(s) + B_{uv}y(s) + C_{uv}.
\]
3.2. Surrogate cost model for tangential road segment

For \( x(s), y(s) \in G_{uv} \) we have

\[
z_r(s) = z_{kj-1} + (z_{kj} - z_{kj-1})(m_ks + 1 - j), \tag{3.2}
\]

and

\[
z_g(s) = A_{uv}x(s) + B_{uv}y(s) + C_{uv}
\begin{align*}
&= A_{uv}(x_{ctk-1} + s(x_{ctk} - x_{ctk-1})) + \\
&\quad B_{uv}(y_{ctk-1} + s(y_{ctk} - y_{ctk-1})) + C_{uv} \\
&= (A_{uv}x_{ctk-1} + B_{uv}y_{ctk-1} + C_{uv}) + \\
&\quad s(A_{uv}(x_{ctk} - x_{ctk-1}) + B_{uv}(y_{ctk} - y_{ctk-1})). \tag{3.3}
\end{align*}
\]

3.2.1 Computing transition points

Since it may be required to cut and fill the ground within a particular grid cell, we need to calculate the \( x \) and \( y \) values at the point of transition from cut to fill or fill to cut. Next, we define the sets of parameters \( T_{jk}^x, T_{jk}^y, T_{jk}^t \) for the \( x \)-boundary cross, \( y \)-boundary cross, and for the transition point. These parameters are used to calculate the road elevation in (3.2) and the ground elevation in (3.3). Let \( x_u, y_{v-1} \) be the \( x \) and \( y \) boundary points at which the horizontal tangent \( HT_k \) crosses the grid cell \( G_{uv} \), see Figure 3.1.

Then, for \( \frac{j-1}{m_k} \leq s \leq \frac{j}{m_k} \) we have

\[
x(s) = x_{ctk-1} + s(x_{ctk} - x_{ctk-1}) = x_u \Rightarrow s = \frac{x_u - x_{ctk-1}}{x_{ctk} - x_{ctk-1}}, \]

\[
y(s) = y_{ctk-1} + s(y_{ctk} - y_{ctk-1}) = y_{v-1} \Rightarrow s = \frac{y_{v-1} - y_{ctk-1}}{y_{ctk} - y_{ctk-1}}.
\]

Therefore,

\[
T_{jk}^x = \left\{ s \mid s = \frac{x_u - x_{ctk-1}}{x_{ctk} - x_{ctk-1}}, \frac{j-1}{m_k} \leq s \leq \frac{j}{m_k} \right\},
\]

\[
T_{jk}^y = \left\{ s \mid s = \frac{y_{v-1} - y_{ctk-1}}{y_{ctk} - y_{ctk-1}}, \frac{j-1}{m_k} \leq s \leq \frac{j}{m_k} \right\}.
\]
3.2. Surrogate cost model for tangential road segment

Figure 3.1: An example, the projection of a tangent road segment onto horizontal plane

If there exist a transition from cut to fill (or fill to cut) within \( G_{uv} \), the parameter of transition \( s \) is calculated by solving the equation \( z_g(s) = z_r(s) \). Using equations (3.3) and (3.2), we solve for the transition point parameter \( s \).

\[
z_r(s) = z_g(s) \iff z_{k_{j-1}} + (z_{k_j} - z_{k_{j-1}})(m_k s + 1 - j) = (A_{uv} x_{ct_{k-1}} + B_{uv} y_{ct_{k-1}} + C_{uv}) + s(A_{uv}(x_{tc_k} - x_{ct_{k-1}}) + B_{uv}(y_{tc_k} - y_{ct_{k-1}}))
\]

\[
\iff (A_{uv} x_{ct_{k-1}} + B_{uv} y_{ct_{k-1}} + C_{uv} - z_{k_{j-1}} - (z_{k_j} - z_{k_{j-1}})(1 - j)) = s(m_k(z_{k_j} - z_{k_{j-1}}) - A_{uv}(x_{tc_k} - x_{ct_{k-1}}) - B_{uv}(y_{tc_k} - y_{ct_{k-1}}))
\]

\[
\iff s = \frac{\varphi}{\chi},
\]

where

\[
\varphi = (A_{uv} x_{ct_{k-1}} + B_{uv} y_{ct_{k-1}} - z_{k_j} + j(z_{k_j} - z_{k_{j-1}}) + C_{uv}),
\]

\[
\chi = (m_k(z_{k_j} - z_{k_{j-1}}) - A_{uv}(x_{tc_k} - x_{ct_{k-1}}) - B_{uv}(y_{tc_k} - y_{ct_{k-1}})).
\]
3.2. Surrogate cost model for tangential road segment

Hence, for \( \frac{j-1}{m_k} \leq s \leq \frac{j}{m_k} \), the set \( T^j_{tk} \) of the transition parameters is given by

\[
T^j_{tk} = \left\{ s \mid s = \frac{A_{uv}x_{ctk-1} + B_{uv}y_{ctk-1} - z_{kj} + j(z_{kj} - z_{kj-1}) + C_{uv}}{m_k(z_{kj} - z_{kj-1}) - A_{uv}(x_{tc_k} - x_{ctk-1}) - B_{uv}(y_{tc_k} - y_{ctk-1})} \right\}.
\]

(3.4)

We define the union of all sets of parameters as

\[
T^j_k = T^j_{zk} \cup T^j_{yk} \cup T^j_{tk}.
\]

Let \( K \) be the cardinality of \( T^j_k \). We sort \( T^j_k \) in an increasing order to create \( T^j_{ks} \) as

\[
T^j_{ks} = \{ s_1 < \cdots < s_{K-1} < s_K \}.
\]

Next, we compute the volume of ground cut \( V_{ck} \), volume of ground fill \( V_{fk} \), and length of the road \( L_{tk} \).

3.2.2 Length of the tangent road section

Given the points \( VP_{kj} = (x_{kj}, y_{kj}, z_{kj}) \) and \( VP_{kj-1} = (x_{kj-1}, y_{kj-1}, z_{kj-1}) \), where \( x_{kj}, x_{kj-1}, y_{kj}, \) and \( y_{kj-1} \) are calculated using equations (2.10a) to (2.10d).

The length of \( r_t(s) \), denoted \( L^j_{tk} \), is given by

\[
L^j_{tk} = \int_{\frac{j-1}{m_k}}^{\frac{j}{m_k}} dr_t(s)
\]

\[
= \frac{1}{m_k} \sqrt{(x_{kj} - x_{kj-1})^2 + (y_{kj} - y_{kj-1})^2 + (z_{kj} - z_{kj-1})^2}
\]

\[
= \frac{1}{m_k} \sqrt{\frac{1}{m_k^2}(x_{tc_k} - x_{ctk-1})^2 + \frac{1}{m_k^2}(y_{tc_k} - y_{ctk-1})^2 + (z_{kj} - z_{kj-1})^2}
\]

\[
= \frac{1}{m_k^2} \sqrt{(x_{tc_k} - x_{ctk-1})^2 + (y_{tc_k} - y_{ctk-1})^2 + m_k^2(z_{kj} - z_{kj-1})^2}.
\]

(3.5)

The length of tangential road segment is computed as

\[
L_{tk} = \sum_{j=1}^{m_k} L^j_{tk}.
\]

(3.6)
3.2. Surrogate cost model for tangential road segment

3.2.3 Volume of ground cut

Consider two consecutive parameters \( s_{i-1}, s_i \in T^j_{k_s} \). If the consecutive parameters bracket a cut, then the approximate volume of ground cut for the tangential road segment over \([s_{i-1}, s_i]\) is obtained by integrating the cross-section area over the effective length of the road in the specified grid cell. The elevation difference at \( s \in [s_{i-1}, s_i] \) is given by

\[
h_c(s) = z_g(s) - z_r(s) = \left( A_{uv}x_{ct_{k-1}} + B_{uv}y_{ct_{k-1}} + C_{uv} \right) + s \left( A_{uv}(x_{tc_k} - x_{tc_{k-1}}) + B_{uv}(y_{tc_k} - y_{tc_{k-1}}) \right) - \left( z_{k_{j-1}} + (z_{k_j} - z_{k_{j-1}})(m_k s + 1 - j) \right)
\]

Next, we calculate the cross-sectional area of the ground. We assume the cross-section of the ground to be cut a trapezoid. An example is shown in Figures 3.2.

\[
a_c = \frac{1}{2} h_c^2 \cot \theta_1 + W h_c + \frac{1}{2} h_c^2 \cot \theta_2 = h_c (W + \frac{1}{2} h_c \kappa)
\]

where \( W \) is the width of the road, \( \theta_1 \) and \( \theta_2 \) are side slope angles, and \( \kappa = \cot \theta_1 + \cot \theta_2 \).
The values of $\theta_1$ and $\theta_2$, and the value of $h_c$ are inter-dependent. Moreover, we assume that $0 < \theta_1 < \frac{\pi}{2}$ and $0 < \theta_2 < \frac{\pi}{2}$, which implies that $0 < \kappa < \infty$. Typically, for the surrogate model, we fix the values of $\theta_1$ and $\theta_2$. The approximate volume of ground cut over $[s_{i-1}, s_i]$ is given as

$$V^j_{c_k} = \int_{s_{i-1}}^{s_i} a_c(s)ds,$$

where $j = 1, 2, 3 \cdots, m_k$.

Assuming $s_{i-1}$ and $s_i$ bracket a cut, the cross-section area $a_c(s)$ is calculated as follows.

$$a_c(s) = h_c(s)(W + \frac{1}{2}\kappa h_c(s)) = Wh_c(s) + \frac{1}{2}\kappa h_c^2(s)$$

$$= W(z_g(s) - z_r(s)) + \frac{1}{2}\kappa(z_g(s) - z_r(s))^2$$

$$= W\left[\left(A_{uv}x_{ctk-1} + B_{uv}y_{ctk-1} - z_{k_j} + \frac{1}{2}s(z_{k_j} - z_{k_j-1})\right)^2\right] +$$

$$\frac{1}{2}\kappa\left[\left(A_{uv}x_{ctk-1} + B_{uv}y_{ctk-1} - z_{k_j} + \frac{1}{2}s(z_{k_j} - z_{k_j-1})\right)^2\right]$$

$$= W(\Omega + s\Theta) + \frac{1}{2}\kappa(\Omega + s\Theta)^2$$

$$= \left(W\Omega + \frac{1}{2}\kappa\Omega^2\right) + \left(W\Theta + \kappa\Omega\Theta\right)s + \frac{1}{2}\kappa\Theta^2s^2$$

(3.9)

where

$$\Omega = \left(A_{uv}x_{ctk-1} + B_{uv}y_{ctk-1} - z_{k_j} + \frac{1}{2}s(z_{k_j} - z_{k_j-1}) + C_{uv}\right),$$

$$\Theta = \left(A_{uv}(x_{ctk} - x_{ctk-1}) + B_{uv}(y_{ctk} - y_{ctk-1}) - m_k(z_{k_j} - z_{k_j-1})\right).$$

If $s_{i-1}$ and $s_i$ do not bracket a cut, then $a_c(s)$ is defined as 0. Integrating (3.8) using (3.9), we get

$$\text{Cutvolume} = (W\Omega + \frac{1}{2}\kappa\Omega^2)(s_i - s_{i-1}) + \frac{1}{2}(W\Theta + \kappa\Omega\Theta)(s_i^2 - s_{i-1}^2) + \frac{1}{6}\kappa\Theta^2(s_i^3 - s_{i-1}^3).$$
Therefore, we compute the volume as
\[ V^j_{ck} = \begin{cases} \text{Cut volume} & \text{if } s_{i-1}, s_i \text{ bracket a cut segment} \\ 0 & \text{otherwise} \end{cases} \quad (3.10) \]

We can now construct the volume of ground cut, \( V_{ck} \) for the tangential road segment as
\[ V_{ck} = \sum_{j=1}^{m_k} V^j_{ck}. \quad (3.11) \]

### 3.2.4 Volume of ground fill

The approximate volume of ground fill for the tangential road segment over \([s_{i-1}, s_i]\) is obtained by integrating the cross-section area over the effective length of the road in the specified grid cell, an example fill cross-section is shown in Figure 3.3. The approximate volume calculation is given as
\[ V^j_{fk} = \begin{cases} \int_{s_{i-1}}^{s_i} a_f(s) ds & \text{if } s_{i-1}, s_i \text{ bracket a fill segment} \\ 0 & \text{otherwise}, \end{cases} \quad (3.12) \]
where \( a_f(s) = -a_c(s) \) and \( j = 1, 2, 3 \cdots, m_k. \)

Therefore, the ground fill volume approximation is calculated using the same procedure as ground cut volume calculation.

![Figure 3.3: An example, fill cross-section.](image)

We can now construct the volume of ground fill, \( V_{fk} \) for the tangential road segment as
\[ V_{fk} = \sum_{j=1}^{m_k} V^j_{fk}. \quad (3.13) \]
3.2.5 Procedure for classifying a grid cell as a region of cut, fill, both cut and fill, or neither.

In order to determine whether or not the ground corresponding to a certain grid cell has to be cut or filled, one can check the following situations. For \( i = 1, 2, \ldots, K - 1 \),

1. if \( \left( z_g(s_i) - z_r(s_i) \right) \left( z_g(s_{i+1}) - z_r(s_{i+1}) \right) = 0 \), then there is no ground cut or ground fill required. So \( V^j_{ck} = V^j_{fk} = 0 \);

2. if \( \left( z_g(s_i) - z_r(s_i) \right) \left( z_g(s_{i+1}) - z_r(s_{i+1}) \right) \geq 0 \) but not both equal, then the grid cell is identified as cut region or fill region. Furthermore,
   - if \( z_g(s_i) - z_r(s_i) > 0 \), it is cut region,
   - else if \( z_g(s_i) - z_r(s_i) < 0 \), it is fill region,

3. if \( \left( z_g(s_i) - z_r(s_i) \right) \left( z_g(s_{i+1}) - z_r(s_{i+1}) \right) < 0 \), then the grid cell has both cut region and fill region. In this case, we calculate the point of transition from cut to fill or fill to cut using equation (3.4).

![Figure 3.4: An example of a pave only cross-section.](image)

3.2.6 Cost calculation for the tangent road segment

Given the cost penalty parameters \( C_p, C_c, C_f \) and \( C_w \), respectively, for the costs of paving, ground cut, ground fill, and waste material, then, each cost item is calculated as

\[
\begin{align*}
C^j_{pk} &= C_p W L^j_k \\
C^j_{ck} &= C_c V^j_{ck} \\
C^j_{fk} &= C_f V^j_{fk} \\
C^j_{wk} &= C_w |V^j_{ck} - V^j_{fk}|,
\end{align*}
\]

where \( C^j_{pk} \) is the paving cost, \( C^j_{ck} \) is ground cut cost, \( C^j_{fk} \) is the ground fill cost, and \( C^j_{wk} \) is cost of waste material.
3.3 Surrogate cost model for circular road segment

For \( j = 1, 2, \ldots, m_k \) the cost corresponding to the road segment that connects the points \( VP_{k,j-1} \) and \( VP_{kj} \) is the sum of all cost items related to earthwork and cost of paving. In particular, the earthwork cost is calculated as

\[
C_{Ek}^j = C_{ck}^j + C_{fk}^j + C_{wk}^j.
\]  

(3.14)

Thus, the cost calculation of the tangential road segment is divided into two parts, the cost of paving and cost of earthwork,

\[
T_{costE_k} = \sum_{j=1}^{m_k} C_{Ek}^j \quad (3.15a)
\]

\[
T_{costP_k} = \sum_{j=1}^{m_k} C_{Pk}^j. \quad (3.15b)
\]

3.3 Surrogate cost model for circular road segment

Suppose segment \( \xi \in S \) corresponds to circular arc, that is, a circle of radius \( r_k \) that connects \( TC_k \) and \( CT_k \), an example is shown in Figure 3.5. We denote the horizontal projection of this curve by \( HC_k \). The circular road segment is given in parametric form by

\[
r_c(s) = (x(s), y(s), z(s)),
\]

(3.16)

where,

\[
x(s) = x_{ck} + r_k \cos \theta_{ck} (s)
\]

\[
y(s) = y_{ck} + r_k \sin \theta_{ck} (s)
\]

\[
z(s) = z_{km_k} + (z_{(k+1)1} - z_{km_k})s,
\]

where \((x_{ck}, y_{ck})\) is the center of \( HC_k \) and \( \theta_{ck}(s) = \theta_{TC_k} + (\theta_{CT_k} - \theta_{TC_k})s \).

Let \( x_{u-1} \) and \( y_{v-1} \) be the \( x \) boundary and \( y \) boundaries at which the horizontal curve \( HC_k \) crosses the grid cell \( G_{uv} \), see Figure 3.5. Note that \( x_{u-1} = x_0 + (u-1)D_x \) and \( y_{v-1} = y_0 + (v-1)D_y \), where \( D_x \) and \( D_y \) are dimensions of each grid cell. Next we calculate the set of parameters \( T_{x_k}^k \) and \( T_{y_k}^k \) where the circular road section crosses the \( x \)-boundary and \( y \)-boundary,
3.3. Surrogate cost model for circular road segment

respectively.

\[ x_{u-1} = x_{c_k} + r_k \cos \theta_k(s) \Rightarrow \theta_k(s) = \arccos \left( \frac{x_{u-1} - x_{c_k}}{r_k} \right) \]

\[ y_{v-1} = y_{c_k} + r_k \sin \theta_k(s) \Rightarrow \theta_k(s) = \arcsin \left( \frac{y_{v-1} - y_{c_k}}{r_k} \right). \]

The parameters corresponding to the \( x \) boundary cross and the \( y \) boundary cross, denoted by \( s_x \) and \( s_y \), respectively, are calculated as

\[ s_x = \frac{1}{(\theta_{CT_k} - \theta_{TC_k})} \left( \arccos \left( \frac{x_{u-1} - x_{c_k}}{r_k} \right) - \theta_{TC_k} \right) \] \hspace{1cm} (3.17a)

\[ s_y = \frac{1}{(\theta_{CT_k} - \theta_{TC_k})} \left( \arcsin \left( \frac{y_{v-1} - y_{c_k}}{r_k} \right) - \theta_{TC_k} \right). \] \hspace{1cm} (3.17b)

Therefore, the sets \( T^k_x \) and \( T^k_y \) are defined as

\[ T^k_x = \{ s \mid s = s_x \} \] \hspace{1cm} (3.18)

\[ T^k_y = \{ s \mid s = s_y \}. \]

In order to calculate the earthwork cost for each grid cell, we need to determine the grid cells through which the circular curve passes. Given the intersection point \( IP_k = (x_k, y_k) \), and suppose that the coordinates of \( TC_k \) and \( CT_k \) are calculated as

\[ TC_k = (x_{tc_k}, y_{tc_k}), \text{ and } CT_k = (x_{ct_k}, y_{ct_k}). \]

First, we determine the points at which the circular curve intersects the grid cells, and then we sort them. The sorting technique is discussed next.

The Equation of a circle in rectangular coordinate system is defined as

\[ (x - x_{c_k})^2 + (y - y_{c_k})^2 = r_k^2. \] \hspace{1cm} (3.20)

We define \( x_{min}, x_{max}, y_{min} \) and \( y_{max} \) as

\[ x_{min} = \min \left\{ x_k, x_{tc_k}, x_{ct_k} \right\}, \quad x_{max} = \max \left\{ x_k, x_{tc_k}, x_{ct_k} \right\} \]

\[ y_{min} = \min \left\{ y_k, y_{tc_k}, y_{ct_k} \right\}, \quad y_{max} = \max \left\{ y_k, y_{tc_k}, y_{ct_k} \right\}. \]

We calculate the \( x \) boundary cross and \( y \) boundary cross in the ranges given below.

\[ x_{range} = [x_{min}, x_{max}] \text{ and } y_{range} = [y_{min}, y_{max}]. \]
3.3. Surrogate cost model for circular road segment

3.3.1 Calculation of $x$-boundary crosses

For each $x$ in the $x_{\text{range}}$ we start by defining

$$\bar{x} = D_x \left\lfloor \frac{x}{D_x} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ is the floor function which is defined below.

**Definition 1.** Let $x \in \mathbb{R}$. Then $\lfloor x \rfloor$ is defined as:

$$\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z} : \lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}.$$

That is, $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$. Consider the line segment $l$ that connects the points $q_1 = (\bar{x}, y_{\text{min}})$ and $q_2 = (\bar{x}, y_{\text{max}})$, which is given in parametric form as

$$l(t) = q_1 + (q_2 - q_1)t, \quad 0 \leq t \leq 1.$$

Essentially, $l$ is a line segment which is parallel to the $y$–axis, and which lies on the grid lines. We calculate the intersection point(s) of $l$ and the circle.
3.3. Surrogate cost model for circular road segment

in (3.20) by calculating the value(s) of the parameter \( t \). The value(s) \( t \) can be calculated by solving the equation

\[
(\bar{x} - x_{ck})^2 + (y_{\min} + (y_{\max} - y_{\min})t - y_{ck})^2 - r^2_k = 0, \quad 0 \leq t \leq 1, \quad (3.21)
\]

There are three possibilities about the solution of \( t \),

1) no real solution is found,
2) two distinct solutions are found,
3) exactly one solution is found.

If no real solution is found, then the line segment between \( q_1 \) and \( q_2 \) does not intersect the circular curve. However, if two distinct solutions are calculated, say \( t_1 \) and \( t_2 \) with \( 0 \leq t_1 \leq 1 \) and \( 0 \leq t_2 \leq 1 \), then we choose the right parameter by calculating the angle corresponding to each parameter. For example, if \( bp_1 = (\bar{x}, y_{\min} + (y_{\max} - y_{\min})t_1) \) is the point calculated using the parameter \( t_1 \). Then we determine the angle \( \theta_1 \) for \( bp_1 \), so that if

\[
\theta_{TC_k} \leq \theta_1 \leq \theta_{CT_k},
\]

\( \bar{x} \) is accepted as \( x \)-boundary cross and \( bp_1 \) is the boundary point, else it is rejected because it is not on the part of the curved road segment. Using the same procedure, we calculate the \( y \)-boundary cross and the corresponding boundary point.

Once the \( x \)-boundary cross \( \bar{x} \) and the \( y \)-boundary cross \( \bar{y} \) are determined, we calculate the corresponding parameters \( s_x \) and \( s_y \) using (3.17a) and (3.17b), respectively. Let \( BP_k \) be the set of distinct boundary points at which the circular curve crosses the grid cells, including the start and end points, and \( T_k \) be the set of parameters corresponding to \( BP_k \). Now we sort elements of \( BP_k \) and \( T_k \). However, since the circle, unlike the line segment, is not convex set, the sorting is not by increasing or decreasing order of elements of \( T_k \). The distance measurement from the elements of \( BP_k \) to \( TC_k \) is used for sorting.

Let \( M \) be the cardinality of \( T_k \), the sets after sorting are defined as

\[
T_{ks} = \{0 = s_1 < s_2 < s_3 < \cdots, s_{M-1} < s_M = 1\} \quad (3.22a)
\]

\[
BP_{ks} = \{bp_1^k, bp_2^k, \cdots, bp_M^k\}. \quad (3.22b)
\]
3.3. Surrogate cost model for circular road segment

3.3.2 Length of circular road section

The length of the circular road segment, denoted by $L_{c_k}$, is given by

$$L_{c_k} = r_k \theta_{c_k},$$

(3.23)

where $\theta_{c_k} = |\theta_{CT_k} - \theta_{TC_k}|$, measured in radians, is the magnitude of central angle of the curve.

3.3.3 Volume of ground cut

The parametric equation of the ground profile for a circular road section is given as

$$z_g(s) = A_{uv} x(s) + B_{uv} y(s) + C_{uv}$$

$$= A_{uv} (x_{c_k} + r_k \cos \theta_{c_k}(s)) + B_{uv} (y_{c_k} + r_k \sin \theta_{c_k}(s)) + C_{uv}$$

$$= (A_{uv} x_{c_k} + B_{uv} y_{c_k} + C_{uv}) + r_k (A_{uv} \cos \theta_{c_k}(s) + B_{uv} \sin \theta_{c_k}(s)).$$

Consider two consecutive parameters $s_{i-1}, s_i \in T_{k_x}$. If $s_{i-1}$ and $s_i$ bracket a cut, then the approximate volume of ground cut for the circular road segment over $[s_{i-1}, s_i]$ is obtained by integrating the cross-sectional area over the effective length of the road in the specified grid cell. The elevation difference at $s \in [s_{i-1}, s_i]$ is calculated by

$$h_c(s) = z_g(s) - z_r(s)$$

$$= (A_{uv} x_{c_k} + B_{uv} y_{c_k} + C_{uv}) + r_k (A_{uv} \cos \theta_{c_k}(s) + B_{uv} \sin \theta_{c_k}(s))$$

$$- (z_{km_k} + (z_{(k+1)_1} - z_{km_k}) s).$$

Recalling the constraint stated in section (2.3.2), we obtain

$$h_c(s) = (A_{uv} x_{c_k} + B_{uv} y_{c_k} - z_{km_k} + C_{uv}) + r_k (A_{uv} \cos \theta_{c_k}(s) + B_{uv} \sin \theta_{c_k}(s)).$$

The volume of ground cut for circular road segment is computed.

$$V_{c_k} = \sum_{i=1}^{M} \int_{s_{i-1}}^{s_i} a_c(s) ds,$$

(3.24)

where, $a_c(s) = Wh_c(s) + \frac{1}{2} \kappa h_c^2(s)$. 

36
3.3. Surrogate cost model for circular road segment

Assuming \( s_{i-1} \) and \( s_i \) bracket a cut, the cross-sectional area \( a_c(s) \) is calculated as follows. We denote \( \Lambda = (A_{uv}x_{ck} + B_{uv}y_{ck} - z_{km} + C_{uv}) \).

\[
a_c(s) = Wh_c(s) + \frac{1}{2} \kappa h_c^2(s)
\]

\[
= W\left[ \Lambda + r_k(A_{uv} \cos \theta_{ck}(s) + B_{uv} \sin \theta_{ck}(s)) \right] + \frac{1}{2} \kappa \left[ \Lambda + r_k^2(A_{uv} \cos \theta_{ck}(s) + B_{uv} \sin \theta_{ck}(s)) \right]^2
\]

\[
= [W\Lambda + \frac{1}{2} \kappa A^2] + [WA_{uv}r_k + \kappa A_{uv}r_k \Lambda] \cos \theta_{ck}(s) + [WB_{uv}r_k + \kappa B_{uv}r_k \Lambda] \sin \theta_{ck}(s) + \frac{1}{2} \kappa r_k^2 A_{uv}^2 \cos^2 \theta_{ck}(s) + \frac{1}{2} \kappa r_k^2 B_{uv}^2 \sin^2 \theta_{ck}(s) + \kappa A_{uv} B_{uv} r_k^2 \sin \theta_{ck}(s) \cos \theta_{ck}(s) \tag{3.25}
\]

If \( s_{i-1} \) and \( s_i \) do not bracket a cut, then \( a_c(s) \) is defined as 0.

We denote \( \Delta \theta_k = \theta_{CTk} - \theta_{TCk} \). Integrating (3.24) using (3.25), we get

\[
V_{ck}^i = [W\Lambda + \frac{1}{2} \kappa A^2](s_i - s_{i-1}) +
\]

\[
\frac{1}{\Delta \theta_k} [WA_{uv}r_k + \kappa A_{uv}r_k \Lambda] \left( \sin \theta_{ck}(s_i) - \sin \theta_{ck}(s_{i-1}) \right) -
\]

\[
\frac{1}{\Delta \theta_k} [WB_{uv}r_k + \kappa B_{uv}r_k \Lambda] \left( \cos \theta_{ck}(s_i) - \cos \theta_{ck}(s_{i-1}) \right) +
\]

\[
\frac{1}{8\Delta \theta_k} \kappa r_k^2 A_{uv}^2 \left[ 2(\theta_{ck}(s_i) - \theta_{ck}(s_{i-1})) + (\sin 2\theta_{ck}(s_i) - \sin 2\theta_{ck}(s_{i-1})) \right] +
\]

\[
\frac{1}{8\Delta \theta_k} \kappa r_k^2 B_{uv}^2 \left[ 2(\theta_{ck}(s_i) - \theta_{ck}(s_{i-1})) - (\sin 2\theta_{ck}(s_i) - \sin 2\theta_{ck}(s_{i-1})) \right] +
\]

\[
\frac{1}{\Delta \theta_k} \kappa r_k^2 A_{uv} B_{uv} \left[ \sin 2\theta_{ck}(s_i) - \sin 2\theta_{ck}(s_{i-1}) \right].
\]

The volume of ground cut for a circular section is then computed as

\[
V_{ck} = \begin{cases} 
\sum_{i=1}^{M} V_{ck}^i & \text{if } s_{i-1}, s_i \text{ bracket a cut segment} \\
0 & \text{otherwise} \end{cases} \tag{3.26}
\]

3.3.4 Volume of ground fill

Similar to the volume calculation for a ground cut, the ground fill volume is calculated using the integration of the cross-sectional area of a ground fill.

\[
V_{fk} = \begin{cases} 
\sum_{i=1}^{M} \int_{s_{i-1}}^{s_i} a_f(s)ds & \text{if } s_{i-1}, s_i \text{ bracket a cut segment} \\
0 & \text{otherwise} \end{cases} \tag{3.27}
\]
3.4. The overall surrogate cost function

where \( a_f(s) = -a_c(s) \).

3.3.5 Cost calculation for a circular road segment

Given the cost penalty parameters \( C_p, C_c, C_f \) and \( C_w \), respectively, for the costs of paving, ground cut, fill, and waste material. The cost items for the circular road section is given as

\[
C_{pk} = C_p W L_{ck}
\]
\[
C_{ck} = C_c V_{ck}
\]
\[
C_{fk} = C_f V_{fk}
\]
\[
C_{wk} = C_w \| V_{ck} - V_{fk} \|.
\]

Therefore, the cost corresponding to the circular road section is related to the earthwork cost and paving cost. In particular, the earthwork cost is calculated as

\[
C_{Ek} = C_{ck} + C_{fk} + C_{wk}.
\] (3.28)

Thus, the cost calculation for the circular road segment \( HC_k \) has two parts, the cost of paving and the cost of earthwork.

\[
C_{cost,Ek} = C_{Ek}
\] (3.29a)
\[
C_{cost,Pk} = C_{pk}.
\] (3.29b)

3.4 The overall surrogate cost function

In this section, we present the surrogate cost model used to calculate the cost of road construction from the start to the end of the road project using the cost formulas in (3.29a), (3.29b), (3.15a), and (3.15b). We note that, because of the assumption that is made in Chapter 2, the start and end section of the road are required to be tangential road sections. Now, we divide the road alignment, between the start and the end, into segments, see Figure 3.6.
3.4. The overall surrogate cost function

Each segment has a tangent part and a curve part, except for the last segment. The orientation of each segment is a tangent-curve. In this way, for \( k = 1, 2, 3, \ldots, N \), we have \( N + 1 \) segments including the last tangent segment. Therefore, for \( i = 1, 2, 3, \ldots, N + 1 \), the overall surrogate cost model is given as

\[
\text{Cost}_E = \sum_{k=1}^{N} (\text{Tcost}_{E_k} + \text{Ccost}_{E_k}) + \text{Tcost}_{E_{N+1}}, \tag{3.30}
\]

\[
\text{Cost}_P = \sum_{k=1}^{N} (\text{Tcost}_{P_k} + \text{Ccost}_{P_k}) + \text{Tcost}_{P_{N+1}}, \tag{3.31}
\]

where \( \text{Cost}_E \) and \( \text{Cost}_P \) are the overall earthwork cost and paving cost, respectively. It is easy to see that the surrogate cost calculation scales linearly with the number of intersection points and the number of grid cells.
Chapter 4

Multiobjective optimization model for 3D road alignment

In this chapter, we formulate the three-dimensional road alignment optimization as a multiobjective optimization problem. The surrogate cost model developed in Chapter 3 has two components, the cost due to the earthwork and the cost related to the length of the road. There is, usually, a conflict between the two cost components. Thus, the solution of road alignment optimization, based on cost minimization, should reflect the trade-off between the length of the road and the volume of earthwork. During the road alignment design process, an engineer can have different objectives that need to be satisfied. Some of the objectives may favour the shortest road possible, while others might favour an indirect path with a lower earthwork cost. In this situation, there is a trade-off between the choice of cost penalty parameters. The shortest path may incur more cost than an longer path.

The multiobjective optimization formulation include the formulation of the individual cost components, and the set of constraints. The paving costs and earthwork costs are formulated in (3.30) and (3.31). The set of constraints are also specified in subsections (2.2.2) and (2.3.1).

4.1 Variable definition

The costs are formulated as functions of decision variables, such as vertical intersection points, radius of curvature and the road design elevations at each vertical points. The parameters of the optimization model are the number of intersection points, the cost penalty parameters, and $\kappa$ which depends on the side slope of the road.

Let $X = (x_1, x_2, \ldots, x_N)$ and $Y = (y_1, y_2, \ldots, y_N)$ be the coordinates of the intersection points, $R = (r_1, r_2, \ldots, r_N)$ be the vector of radius of curvature, and $Z = (z_1, z_2, \ldots, z_M)$ be the vector of elevations of the design road. Define the objective functions $C_E$ and $C_P$,

$$C_E : \mathbb{R}^{3N+M} \rightarrow \mathbb{R} : (X, Y, R, Z) \mapsto \text{Cost}_E(X, Y, R, Z),$$
4.2 Problem formulation

The optimization model accepts the values of parameters and other constants as input and returns the values of decision variables. Given the start point \( \text{START} = (x_s, y_s, z_s) \), the end point \( \text{END} = (x_e, y_e, z_e) \), the maximum allowable gradient \( G_{\text{max}} \), and the minimum radius of curvature \( r_{\text{min}} \).

If \( \bar{z} \) is the vertical off-set from the current ground elevation \( z_g \), the simultaneous optimization of horizontal and vertical alignments is given as follows.

\[
\text{Minimize } \left\{ \text{Cost}_E(X,Y,R,Z), \text{Cost}_P(X,Y,R,Z) \right\}
\]

subject to:

**Horizontal alignment constraints**

For \( k = 1, 2, \ldots, N \),
\[
0 \leq ||TC_k - IP_{k-1}|| - ||CT_{k-1} - IP_{k-1}||,
\]
\[
r_{\text{min}} \leq r_k,
\]

**Vertical alignment constraints**

For \( k = 1, 2, \ldots, N \), and for \( j = 1, 2, \ldots, m_k \),
\[
|z_{kj} - z_{kj-1}| \leq G_{\text{max}} \sqrt{(x_{kj} - x_{kj-1})^2 + (y_{kj} - y_{kj-1})^2},
\]
\[
z_g - \bar{z} \leq z_{kj} \leq z_g + \bar{z},
\]

**Other constraints**

\[
z_{km_k} = z_{(k+1)1}, \quad k = 1, 2, \ldots, N.
\]

4.3 NOMAD: Nonlinear optimization with the MADS algorithm

NOMAD \([\text{AAC}^+]\) is a black box optimization software that implements the Mesh Adaptive Direct Search (MADS) algorithm under general nonlinear constraints \([\text{LD11}]\). When the objective functions and constraint functions defining the optimization problem are typically simulations which possess no exploitable properties such as derivatives, and they may fail to evaluate at some trial points, even feasible ones, the term blackbox problem is used to denote this problem class \([\text{LD11}]\). Direct search methods are
meant for such a context since they use only function evaluations to drive their exploration in the space of variables and they can deal with missing function values. MADS is an iterative algorithm where at each iteration a finite number of trial points are generated, and the infeasible trial points are discarded. The objective function values at the feasible trial points are compared with the current best feasible objective function value found so far [AD06].

4.4 Solution procedure

The solution procedure for solving the three-dimensional alignment optimization problem is designed to use the derivative-free optimization algorithm. In our numerical tests, we use NOMAD [ALT09, LD11], but any derivative-free optimization algorithm would work. Although the surrogate cost model in each grid cell is continuously differentiable, the overall cost model is nondifferentiable because of two reasons. First, if the terrain corresponding to adjacent grid cells is approximated by different linear functions (or planes), the volume of earthwork is not necessarily equal. Thus, the cost function in each grid cell are not necessarily equal, and this results in a sharp turning on the boundary of the grid cells. Second, at a point where the ground changes from cut to fill or fill to cut, the cost function might have a sharp turning due to the difference in cost penalty parameters for ground cut and ground fill. Moreover, the surrogate cost model is nonconvex, this can be seen by noting the solution alignments of two initial roads, depicted in Figure 5.8. Thus, a derivative-free optimization algorithm is an appealing method to solve the problem.

\[
\text{Cost} = (\theta C_E + (1 - \theta) C_P),
\]

where \(0 \leq \theta \leq 1\).

Given a feasible initial alignment, we run the solver for different values \(\theta\) and solved the problem. Thus, the optimization algorithm finds the optimal decision vectors for both vertical and horizontal alignments simultaneously.
Chapter 5

Numerical results

This chapter presents a numerical experiment that was employed to investigate the proposed model. An actual ground profile in California with an area of 500 meters by 1000 meters is considered. The contours and three-dimensional view of the map are displayed in Figure 5.1 and Figure 5.2. (We note that the contour map scale is in the ratio of 1 m along the $x$-axis is equal to 2 m along the $y$-axis, and 1 m along the $z$-axis.) Based on the ground profile, two example scenarios are generated and tested. In the first scenario, a single alignment is generated to examine the effect of the cost penalty parameters on the solution found. In the second example, two different alignments are generated to investigate whether the problem is convex or not. This is tested by looking at the optimal solutions of both alignments. In the second example, each alignment is tested on two different parameters. Based on discussions with our industrial partner (Softree Technical systems Inc.), the model parameters and design constants are fixed as presented in Table 5.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_p$</td>
<td>0.3</td>
</tr>
<tr>
<td>$C_c$</td>
<td>1</td>
</tr>
<tr>
<td>$C_f$</td>
<td>0.5</td>
</tr>
<tr>
<td>$C_w$</td>
<td>2</td>
</tr>
<tr>
<td>$G_{\text{max}}$ (%)</td>
<td>15</td>
</tr>
<tr>
<td>$r_{\text{min}}$ (m)</td>
<td>20</td>
</tr>
<tr>
<td>$m_k$</td>
<td>5</td>
</tr>
<tr>
<td>N</td>
<td>4</td>
</tr>
<tr>
<td>Road width (m)</td>
<td>5</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>1</td>
</tr>
</tbody>
</table>

Note that, for experimental purposes we set $m_k = 5$ for all $k$, and we varied the value of parameter $\theta$ that we introduced in Equation (4.1).
Figure 5.1: Three-dimensional view of the terrain. Note that the aspect ratio for the $x$-axis, the $y$-axis, and the $z$-axis is, respectively, 10:10:1.

Figure 5.2: The contour map. The $x$-axis and $y$-axis are in tens of meters.
5.1 Experimental setup

Three artificial roads that satisfy the basic design and geometric constraints were generated. All experiments were tested with a MATLAB R2013b code performed on a Dell workstation equipped with an Intel(R) Core(TM) i7 2.8GHz processor, and a 16 GB of RAM using 64-bit Windows operating system. The two examples differ in that, in the first experiment, a single alignment is solved for several parameters values, while in the second experiment, two different alignments are solved for the same parameter. In fact, we tested the two alignments on two different parameters, $\theta = 0$ and $\theta = 0.5$. In this second experiment, we used the same set of constraints and stopping criteria for both alignments. Each case study demonstrates the ability of the model to solve a three-dimensional road alignment optimization problem in a reasonable amount of time.

The MATLAB version of NOMAD [AAC+] is used to solve the problems. The parameters of the solver that are set during the solution process include: Maximum time (maxtime), maximum number of function evaluation (maxfeval), and minimum mesh size.

5.2 Case study 1

An artificial three-dimensional road alignment that satisfies the constraint set is generated. The road profile is calculated from the terrain data. The horizontal alignment is shown in Figure 5.3.

Based on the surrogate cost model developed in Chapter 3, and the multi-objective optimization problem formulation in Chapter 4, we solved the problem for five different values of $\theta$: $\theta \in \{0, 0.25, 0.5, 0.75, 1\}$. We used four intersection points and maximum gradient of 12%. Because we fixed the number of vertical grades between two consecutive intersection points, the number of variables depends on the number of intersection points only, and this makes the number of variables equal to forty.
5.2. Case study 1

We used two stopping criteria, the number of black box evaluation and the mesh size, which are set, for this case study, to 10,000 and 0.5, respectively.

5.2.1 Experimental results 1

The result of the experiment for case study 1 indicates that, when the value of $\theta$ is close to zero, the optimal road alignment, as expected, chooses the shortest route. That is, the length of the road is minimized when $\theta$ values is zero. On the other hand, when $\theta$ values are close to one, the optimal route minimizes the earthwork volume, which may result in a longer route. This result indicates that, there are two conflicting objectives in the problem, the length of the road, and the amount of earthwork. Therefore, the road optimization problem is indeed multi-objective optimization problem. The optimal horizontal alignment corresponding to each of the parameters is
5.2. Case study 1

depicted in Figure 5.4.

Figure 5.4: The optimal horizontal alignments for case study 1. The x-axis and y-axis are in tens of meters.

As it can be seen from Figure 5.4, the optimal horizontal alignment corresponding to $\theta = 0$ is not straight because of the box constraints at each intersection point. The optimal vertical alignments for alignments corresponding to $\theta = 0$ and $\theta = 1$ are depicted in Figure 5.5 and Figure 5.6, respectively. The search stopped because of the maximum number of black box evaluations. Table 5.2 presents the numerical result of the experiment.
5.3 Case study 2

In this experiment, we set the number of function evaluation to 70,000 and the minimum mesh size to 0.05, in order to see if the two alignments converge to the same or near the same solution. The inputs and model parameters that we used in case study 1 are also used for this example. For
5.3. Case study 2

Table 5.2: Numerical results for case study 1.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>objective value</th>
<th>black-box evaluations</th>
<th>mesh size</th>
<th>solution time (s)</th>
<th>$C_p \times 10^3$</th>
<th>$C_E \times 10^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1105.1</td>
<td>10,000</td>
<td>0.5</td>
<td>358.62</td>
<td>1.19</td>
<td>334.8</td>
</tr>
<tr>
<td>0.25</td>
<td>1046.26</td>
<td>10,000</td>
<td>0.5</td>
<td>414.38</td>
<td>4.03</td>
<td>7.39</td>
</tr>
<tr>
<td>0.5</td>
<td>2165.04</td>
<td>10,000</td>
<td>0.5</td>
<td>429.43</td>
<td>4.24</td>
<td>6.53</td>
</tr>
<tr>
<td>0.75</td>
<td>2348.4</td>
<td>10,000</td>
<td>0.5</td>
<td>461.91</td>
<td>4.30</td>
<td>6.56</td>
</tr>
<tr>
<td>1</td>
<td>2885.06</td>
<td>10,000</td>
<td>0.5</td>
<td>436.36</td>
<td>4.40</td>
<td>1.42</td>
</tr>
</tbody>
</table>

instance, we used four intersection points and maximum gradient of 12%. The horizontal alignment is shown in Figure 5.7. As mentioned earlier, this case study examines whether the problem is convex or not.

Figure 5.7: The horizontal alignments for case study 2. The $x$-axis and $y$-axis are in tens of meters.
5.3. Case study 2

5.3.1 Experimental results 2

The outcome of this experiment shows that, when $\theta = 0.5$, both alignments converge to different solutions. The optimal horizontal alignments are shown in Figure 5.8. Even for the case when $\theta = 0$, the vertical alignments are not the same, the optimal alignments are shown in Figure 5.9 and Figure 5.10. Then it suffices to conclude that the problem is nonconvex. Table 5.3 and Table 5.4 present the numerical results for both alignments.

![Figure 5.8: Two optimal horizontal alignments computed from different starting alignments. Hence the problem is nonconvex. The x-axis and y-axis are in tens of meters.](image)

<table>
<thead>
<tr>
<th>Optimal alignment</th>
<th>obj. value</th>
<th>BBE</th>
<th>min. mesh size</th>
<th>solution (s)</th>
<th>Stopped due to</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>95.18</td>
<td>26,364</td>
<td>0.05</td>
<td>3,317</td>
<td>min. mesh size</td>
</tr>
<tr>
<td>2</td>
<td>156.14</td>
<td>19,686</td>
<td>0.05</td>
<td>3,131</td>
<td>min. mesh size</td>
</tr>
</tbody>
</table>
5.3. Case study 2

Figure 5.9: The optimal horizontal alignments. Note that the $x$-axis and $y$–axis are in tens of meters.

Figure 5.10: The optimal horizontal alignments.
5.4 Implication of the results

Table 5.4: Numerical results for case study 2 when $\theta = 0$.

<table>
<thead>
<tr>
<th>Optimal alignment</th>
<th>obj. value</th>
<th>BBE size</th>
<th>min. mesh size</th>
<th>solution (s)</th>
<th>Stopped due to</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1222.61</td>
<td>63,771</td>
<td>0.05</td>
<td>4,692</td>
<td>min. mesh size</td>
</tr>
<tr>
<td>2</td>
<td>1267.95</td>
<td>76,931</td>
<td>0.05</td>
<td>4,833</td>
<td>min. mesh size</td>
</tr>
</tbody>
</table>

5.4 Implication of the results

The two case studies results show that, the problem, as multiobjective optimization problems is dependent on the values of the cost penalty parameters. The results, further, imply that role of the length dependent penalty parameter is in conflict with other penalty parameters. For high values of paving cost penalty parameter and smaller values of other parameters, the problem tends to minimizes the cost (or objective) value by minimizing the length of the road. On the other hand, for small values of paving cost penalty parameters, and higher values of the other parameters, the problem minimizes the objective value by minimizing the volume of earthwork between the end points.

Moreover, the numerical results of the experiment indicates that, the problem is nonconvex. This is shown by the result obtained from Experiment 5.8. In this experiment, the number of black box evaluations and the minimum mesh size are set to let the program run until the two different alignments converge. The results indicate that, even for the extreme case, when $\theta = 0$, the solution alignments are different.

Therefore, the optimal alignment is highly dependent on the choice of the cost penalty parameters.
Chapter 6

Conclusion and Future work

In this thesis, a surrogate cost model is developed to solve a three-dimensional road alignment optimization problem. Optimal vertical and horizontal alignments are determined simultaneously. The NOMAD solver is used to solve the problems, but any derivative-free optimization solver could be applied. Results from the two case studies are presented which show that, the problem is multiobjective in terms of cost minimization. As a result, the cost items can be classified into, those which depend on the length of the road and those depending on the volume of the earthwork. They are, usually, conflicting because, during the optimization process, the length dependent costs prefer the shortest route between the two end points, while the cost items that depend on the amount of earthwork choose a route with the minimum amount of earthwork. Therefore, optimal solutions are very dependent on the values of cost penalty parameters. Furthermore, the time of solution for each experiment test shows that, the surrogate model is solved in a reasonable amount of time, which is expected as it is only an approximation of the numerically expensive true model.

6.1 Future work

We have identified four cost penalty parameters, namely, the ground cut cost penalty parameter $C_c$, ground fill cost penalty parameter $C_p$, waste material cost penalty parameter $C_w$, and the paving cost penalty parameter $C_p$. The surrogate can only approximate the cost of earthwork, waste material, and pavement. Any additional costs items such as transportation cost can be incorporated into the model. Even though the overall surrogate cost is not differentiable, it is almost differentiable, that is, the set of points at which the cost function is non-differentiable is finite. So, for future the gradient of the cost model can be calculated to make use of gradient based algorithms. In this thesis, the values of these parameters are obtained from industry partners and are fixed. However, we are also aware that the optimal choice of this parameters should be done systematically, since their values are highly affecting the optimal alignments.
Bibliography

[AAC+] M.A. Abramson, C. Audet, G. Couture, J.E. Dennis, Jr., S. Le Digabel, and C. Tribes. The NOMAD project. Software available at http://www.gerad.ca/nomad. → pages 41, 45


[JS03] J. Jong and P. Schonfeld. An evolutionary model for simultaneously optimizing three-dimensional highway alignments. *Trans-
Bibliography


[OEC73] OECD. *Optimization of road alignment by the use of computers*. ProQuest, 1973. → pages 2


[SJK06] P.M. Schonfeld, J. Jong, and E. Kim. *Intelligent road design*, volume 19. WIT Press, 2006. → pages 1, 2, 3, 4, 5, 7, 8, 9

