The behavior of the Hilbert scheme of points under the derived McKay correspondence

by

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Abstract

In this thesis, we completely determine the image of structure sheaves of zero-dimensional, torus invariant, closed subschemes on the minimal, crepant resolution Y of the Kleinian quotient singularity $X = \mathbb{C}^2/\mathbb{Z}/n$, under the Fourier-Mukai equivalence of categories, between derived category of coherent sheaves on Y and $\mathbb{Z}/n$-equivariant derived category of coherent sheaves on $\mathbb{C}^2$. As a consequence, we obtain a combinatorial correspondence between partitions and $\mathbb{Z}/n$-colored skew partitions.

Preface

All the research ideas and methods explained in this thesis are the results of fruitful discussions between Professor Jim Bryan and Ehsan Mohyedin Kermani. The computations and the manuscript preparation were conduced by Ehsan Mohyedin Kermani with invaluable guidance from Jim Bryan throughout this process.

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Acknowledgments

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Chapter 1

Introduction

In 1980, McKay [14] found an interesting link between representation theory and algebraic geometry as a bijection between the set of non-trivial irreducible representations of a finite group G of $\mathrm{SL}(2,\mathbb{C})$ and the components of the exceptional divisor of the minimal resolution Y of the Kleinian singularity $X = \mathbb{C}^2/G$. In particular, he showed that the McKay quiver 2.5 coincides with resolution graph 2.3 of Y. Later on, in 1983, Gonzalez-Sprinberg-Verdier [8] formulated McKay's result in terms of K-theory. That is, they geometrically constructed an isomorphism between $K(Y)$ and $K^G(\mathbb{C}^2) \cong \mathrm{Rep}(G)$ where $K^G(\mathbb{C}^2)$ is the G-equivariant K-theory on $\mathbb{C}^2$ and $\mathrm{Rep}(G)$ is the representation ring of G. Via this isomorphism a non-trivial irreducible representation corresponds to the structure sheaf of an exceptional component 2.5. Using the Chern character isomorphism $K(Y) \cong H^\star(Y)$, one recovers the classical McKay correspondence and it is often described as the following bijection

$$\text{Geometric basis of } H^\star(Y,\mathbb{Z}) \longleftrightarrow \text{Set of irreducible representations of } G.$$

Finally, in 2001, Bridgeland-King-Reid [1] promoted the above correspondence to an equivalence of derived categories

$$\mathbf{D}(Y) \xrightarrow{\sim} \mathbf{D}^G(\mathbb{C}^2).$$

The above equivalence is then given by pulling back and pushing forward to and from the universal object and this is called the Fourier-Mukai transform.

A key point in Bridgeland-King-Reid is to realize the resolution $Y \longrightarrow X$ as a moduli space, namely $Y = \mathrm{GHilb}(\mathbb{C}^2)$, G-Hilbert scheme which will be described in 2.7.

The main question we would like to answer is to completely classify the image of the Hilbert scheme of points of the minimal, crepant resolution Y under the Fourier-Mukai transform. In other words, we want to investigate where the structure sheaves of zero-dimensional closed subschemes of Y (viewed as objects in $\mathbf{D}(Y)$) go under the equivalence. It has been extensively studied where certain sheaves go under Fourier-Mukai equivalence but always in the other direction [12], i.e. it has been computed where certain sheaves go on the orbifold. Our main result is a complete and explicit description of the image of the structure sheaves of torus invariant, closed subschemes of Y in the case where $G \cong \mathbb{Z}/n$.

We find that the structure sheaves of points on Y do go to sheaves on $\mathbb{C}^2$ (as opposed to more general objects in $\mathbf{D}^{\mathbb{Z}/n}(\mathbb{C}^2)$), but not necessarily the structure sheave of subschemes. Instead, they all go to quotients of a certain universal quasi-coherent sheaf which will be constructed in section 3.2.

Our construction induces as interesting combinatorial correspondence between partitions and certain $\mathbb{Z}/n$-colored skew partition. We illustrate this correspondence with examples and diagrams in 3.2.3.

Chapter 2

Background

2.1 Finite subgroups of SL$(2, \mathbb{C})$

The following is a concise account of the classification of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$. Throughout, we will assume that G is a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$.

By taking the standard Hermitian inner product on $\mathbb{C}^2$ defined by $\langle z, w\rangle = z \cdot \overline{w}$ and averaging by G, we arrive at a G-invariant Hermitian inner product on $\mathbb{C}^2$. This implies that G is conjugate to a finite subgroup of $\mathrm{SU}(2,\mathbb{C})$. In fact, this argument shows that the classification of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ is equivalent to the classification of finite subgroups of $\mathrm{SU}(2,\mathbb{C})$.

We define a surjective group homomorphism $\mathrm{SU}(2,\mathbb{C}) \longrightarrow \mathrm{SO}(3,\mathbb{R})$ using the algebra of quaternions $\mathbb{H}$ as follows:

Let $q = a+bi+cj+dk \in \mathbb{H}$ which can be written in the form $q = z_1+z_2j$, where $z_1 = a + bi$ and $z_2 = c + di$ are complex numbers. Let $\mathbb{H}_1$ be the group of quaternions with norm 1. There is a natural group isomorphism

$$\varphi : \mathbb{H}_1 \longrightarrow \mathrm{SU}(2,\mathbb{C}), \quad z_1 + z_2 j \mapsto \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}$$

Then we identify $\mathbb{R}^3$ with pure quaternions $bi + cj + dk$ and define the action of $\mathbb{H}_1 \cong \mathrm{SU}(2)$ on $\mathbb{R}^3$ by

$$q \cdot q_0 \mapsto q \cdot q_0 \cdot q^{-1}, \quad q \in \mathbb{H}_1, q_0 \in \mathbb{R}^3$$

now because the quaternion norm is multiplicative and coincides with Euclidean norm on $\mathbb{R}^3$, thus $p : \mathbb{H}_1 \to \mathrm{O}(3)$ is a well-defined group homomorphism.

Moreover, write $q \in \mathbb{H}_1$ as $q = \cos\theta + \sin\theta q_1$, where q_1 is a pure quaternion of norm 1. It is also immediate that the action of q on $\mathbb{R}^3$ is the rotation defined by the axis q_1 and the angle θ. Using the definition of $\mathrm{SO}(3,\mathbb{R})$, we can define a surjective homomorphism $p : \mathbb{H}_1 \to \mathrm{SO}(3,\mathbb{R})$. The kernel of p is the center of $\mathrm{SU}(2,\mathbb{C})$ which is $\{\pm 1\}$.

Hence there is a short exact sequence of groups

$$\begin{tikzcd}
1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathrm{SU}(2) \arrow[r] \arrow[d, equal, "\varphi"] & \mathrm{SO}(3) \arrow[r] \arrow[d] & 1 \\
& & \mathbb{H}_1 \arrow[r, "p"] & \mathrm{O}(3) &
\end{tikzcd}$$

The classification of finite subgroups of $\mathrm{SU}(2,\mathbb{C})$ is now equivalent to the classification of finite subgroups of $\mathrm{SO}(3,\mathbb{R})$, since any finite subgroup G of $\mathrm{SU}(2,\mathbb{C})$ is mapped to a finite subgroup of $\tilde{G}$ of $\mathrm{SO}(3,\mathbb{R})$, and conversely any finite subgroup $\tilde{G}$ of $\mathrm{SO}(3,\mathbb{R})$ can be lifted to a finite subgroup G of $\mathrm{SU}(2,\mathbb{C})$ by the above diagram. Classification of finite subgroups of $\mathrm{SO}(3,\mathbb{R})$ is very well-known and consists of the three families of groups, namely, the symmetries of a regular polyhedron (tetrahedral of order 12, octahedral of order 24 and icosahedral of order 60), the dihedral groups (of order $2n$) and the cyclic groups (of order n.)

By lifting the described subgroups we arrive at

1. Cyclic subgroup C_n of order n.
2. Binary dihedral group of order $4n$,
3. Binary tetrahedral group of order 24,
4. Binary octahedral group of order 48,
5. Binary icosahedral group or order 120.

2.2 Kleinian singularities

In this thesis, we are interested in working with quotient varieties $\mathbb{C}^2/G$ so we should first define what we mean by $\mathbb{C}^2/G$ and try to find its defining equations.

Definition The quotient variety $X = \mathbb{C}^2/G = \mathrm{Spec}\mathbb{C}[a,b]^G$ is called a *Kleinian* singularity (also known as *Du Val* singularity, a *simple surface* singularity or *rational double point.*)

We will see that X can be embedded in $\mathbb{C}^3$ as a hypersurface with an isolated singularity at the origin, that is, there is only one defining equation in the ring of invariants for various subgroups.

The action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ is defined by left multiplication which induces the same action on subgroups of $\mathrm{SL}(2,\mathbb{C})$

$$\begin{pmatrix} c & d \\ e & f \end{pmatrix} : (a,b) \mapsto (ca+db, ea+fb)$$

The induced action on the coordinate ring $\mathbb{C}[a,b]$ of $\mathbb{C}^2$ is

$$\begin{pmatrix} c & d \\ e & f \end{pmatrix} : p(a,b) \mapsto p(fa-db, -ea+cb)$$

and we are looking for the ring of invariants i.e. $\mathbb{C}[a,b]^G \subset \mathbb{C}[a,b]$.

As an example, when $G = \mathbb{Z}/n$ the ring of invariants $\mathbb{C}[a,b]^{\mathbb{Z}/n}$ can be computed as follows

Let $\epsilon_n = e^{2\pi i/n}$, then $\mathbb{Z}/n$ is acting on $\mathbb{C}[a,b]$ by $g_n = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{pmatrix}$,

$$g_m \cdot (a,b) \mapsto (\epsilon_n a, \epsilon_n^{-1} b)$$

The monomials a^n, b^n, ab can be taken as the generators of the ring of invariants, therefore, $\mathbb{C}[a,b]^{\mathbb{Z}/n} = \mathbb{C}[a^n, b^n, ab]$. It can be seen that the only defining equation in this case is $xy - z^n = 0$ where $x = a^n, y = b^n, z = ab$. The actions of other cases are illustrated as follows

1. $B\mathbb{D}_{4n}$ is a binary dihedral group of order $4n$ generated by $g_{2n} = \begin{pmatrix} \epsilon_{2n} & 0 \\ 0 & \epsilon_{2n}^{-1} \end{pmatrix}, h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$,

$$g_{2n} \cdot (x, y) \mapsto (\epsilon_{2n} a, \epsilon_{2n}^{-1} b), \quad h \cdot (a, b) \mapsto (ib, -ia)$$

Remark The group generated by $g_{2n} = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{pmatrix}, h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, is conjugate to the cyclic group of order n.

Similarly for

2. $B\mathbb{T}_{24}$ is a binary tetrahedral group of order 24 generated by $g_{2n} = \begin{pmatrix} \epsilon_{2n} & 0 \\ 0 & \epsilon_{2n}^{-1} \end{pmatrix}, h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, k = \frac{1}{1-i} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$
3. $B\mathbb{O}_{48}$ is a binary tetrahedral group of order 48 generated by $g_8 = \begin{pmatrix} \epsilon_8 & 0 \\ 0 & \epsilon_8^{-1} \end{pmatrix}, h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, k = \frac{1}{1-i} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$
4. $B\mathbb{I}_{120}$ is a binary icosahedral group of order 120 generated by $g_{10} = \begin{pmatrix} \epsilon_{10} & 0 \\ 0 & \epsilon_{10}^{-1} \end{pmatrix}, h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, l = \frac{1}{\sqrt{5}} \begin{pmatrix} \epsilon_5 - \epsilon_5^4 & \epsilon_5^2 - \epsilon_5^3 \\ \epsilon_5^2 - \epsilon_5^3 & -\epsilon_5 + \epsilon_5^4 \end{pmatrix}$.

The complete account of finding the defining equations in these cases have been fully illustrated in [3]. The following table contains the equations of X for various groups up to conjugacy.

Conjugacy class	Equation	Dynkin graph
$\mathbb{Z}/n$	$x^2 + y^2 + z^n = 0$	A_{n-1}
$B\mathbb{D}_{4n}$	$x^2 + y^2 z + z^{n+1} = 0$	D_{n+2}
$B\mathbb{T}_{24}$	$x^2 + y^3 + z^4 = 0$	E_6
$B\mathbb{O}_{48}$	$x^2 + y^3 + yz^3 = 0$	E_7
$B\mathbb{I}_{120}$	$x^2 + y^3 + z^5 = 0$	E_8

Table 2.1: Equations of Kleinian singularities

Explicit generator for the ring of invariants $\mathbb{C}[a, b]^G$ can be found in [3, p.7-13] and the above relations can be checked accordingly.

2.3 Minimal, crepant resolution and resolution graph

Definition A *minimal* resolution Y of X is a resolution such that every other resolution factors through Y.

Remark Note that a surface has a unique minimal resolution, however, for higher dimensional varieties, minimal resolutions are not necessarily unique.

In order to construct a minimal resolution Y, we can find an arbitrary resolution for X and contract (-1)-curves to produce Y successively. The exceptional locus of $\varphi : Y \longrightarrow X$ consists of (-2)-curves E_i intersecting transversely. If we associate a vertex to each curve E_i and join two vertices if and only if the corresponding curves intersect in Y, we arrive at the *resolution graph* of X.

It is now worth mentioning here that the McKay's result [14] gives a nice correspondence between the resolution graph of a Kleinian singularity $\mathbb{C}^2/G$ and the Dynkin diagram of G. The Dynkin diagrams of $A_{n-1}, D_{n+2}, E_6, E_7$ and E_8 are listed below in Figure 2.1.

Definition A crepant resolution $Y \longrightarrow X$ is a resolution which the pullback of the canonical divisor K_X of X coincides with the canonical divisor K_Y of Y.

Remark The minimal resolution of $X = \mathbb{C}^2/G$ exists and is crepant due to the fact that a 2-form $f(a,b)da \wedge db$ on $\mathbb{C}^2/G$ is invariant under the $\mathrm{SL}(2,\mathbb{C})$ action.

Remark 2.4 Toric geometry

The Complete account of Toric geometry can be found in [5].

Throughout, V is an n-dimensional vector space over $\mathbb{R}$. Recall that a subset $C \subset V \cong \mathbb{R}^n$ is *convex* if given any two elements $x_1, x_2 \in C$, then $\lambda x_1 + (1-\lambda)x_2 \in C$ for any $0 \leq \lambda \leq 1$ and C is a *cone* if given $x \in C$,

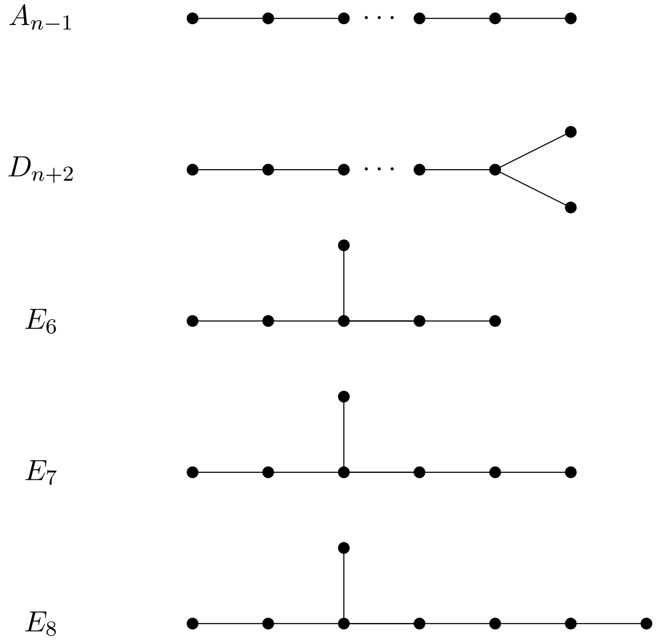

Figure 2.1: ADE Dynkin diagrams

then $ax \in C$ for all $a \geq 0$. If $C \subset \mathbb{R}^n$ is a convex cone, then $C \cap \mathbb{Z}^n$ is a semigroup under addition. The convex cone generated by $x_1, \cdots, x_k \in V$ is the smallest convex cone containing $x_1, \cdots, x_k$ and is defined and denoted by $C = \langle x_1, \cdots, x_k \rangle = \{a_1 x_1 + a_2 x_2 + \cdots + a_k x_k \mid \; a_i \geq 0\}$.

Definition A convex cone is *strongly convex* if it does not contain a non-zero linear subspace and is *simplicial* if it can be generated by linearly independent vectors.

Definition Let $y^* \in V^\star$ be a linear functional on V. Then y is a support of a cone C if $y(x) \geq 0$ for all $x \in C$, and is written as $y|_C \geq 0$. Then $H_y = \{y = 0\}$ is the support hyper plane of C and $H_y \cap C$ is called a *face* of C. A one-dimensional face is called *ray*.

Definition The *dual* cone $C^\vee$ of a cone C is the set of all $y \in V^\star$ with support of C.

Definition A cone $C \in \mathbb{R}^n$ is called rational if it can be generated by rational elements in $\mathbb{Q}^n$.

Remark If C is strongly convex and rational, then it is generated the first lattice points on its rays.

Lemma 2.4.1 (Gordon)**.** *If $C \subset \mathbb{R}^n$ is a rational cone then the semigroup $S = C \cap \mathbb{Z}^n$ is finitely generated. Consequently, the group ring $\mathbb{C}[S]$ is a finitely generated $\mathbb{C}$-algebra.*

Let $N \cong \mathbb{Z}^n$ and $M = \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$. Suppose that $\sigma \subset N_{\mathbb{R}} := N \otimes \mathbb{R} \cong \mathbb{R}^n$ is a strongly rational cone. For $\sigma^\vee$ the corresponding semigroup $\sigma^\vee \cap M$, the semigroup ring $\mathbb{C}[\sigma^\vee \cap M]$ and the corresponding variety $\mathrm{Spec}\mathbb{C}[\sigma^\vee \cap M]$ are denoted by S_σ, A_σ and U_σ, respectively.

Definition A fan Δ in $N_{\mathbb{R}}$ is a finite, non-empty set of rational strongly convex cones in $N_{\mathbb{R}}$, such that

1. If $\sigma \in \Delta$ and $\tau \leq \sigma$ (i.e. τ is a face of σ) then $\tau \in \Delta$.
2. If $\sigma, \tau \in \Delta$ then $\sigma \cap \tau \in \Delta$.

Definition The *toric variety* $X(\Delta)$ associated to the fan Δ is

$$\coprod_{\sigma \in \Delta} U_\sigma / \{U_{\sigma_1}, U_{\sigma_2} \text{ glued along } U_{\sigma_1 \cap \sigma_2}\}.$$

Definition A morphism of cones is $\varphi : (N_1, \sigma_1) \longrightarrow (N_2, \sigma_2)$ where $\varphi : N_1 \longrightarrow N_2$ is a group homomorphism and $\varphi \otimes \mathbb{R} : N_1 \otimes \mathbb{R} \longrightarrow N_2 \otimes \mathbb{R}$ maps σ_1 to σ_2. The induced morphism $\psi : U_{\sigma_1} \longrightarrow U_{\sigma_2}$ is called a *toric morphism.*

Definition A cone $\sigma \subset N_{\mathbb{R}}$ generated by $\langle v_1, \cdots, v_k \rangle$ is *non-singular* if $v_1, \cdots, v_k$ can be extended to a basis of $N_{\mathbb{R}}$.

Remark We will use the fact that if σ is a simplicial cone in $N_{\mathbb{R}}$ and $\dim \sigma = n = \mathrm{rk} N_{\mathbb{R}}$, then σ is non-singular if and only if $\det(v_1, \cdots, v_k) = \pm 1$, and U_σ is non-singular if and only if σ is non-singular.

2.4.1 Toric varieties as quotients

Let G be a finite group acting on $\mathbb{C}^n = \mathrm{Spec}\mathbb{C}[x_1, \cdots, x_n]$. For example, let $G = \langle \epsilon_n \rangle \cong \mathbb{Z}/n$ acting on $\mathbb{C}^2$ by

$$\epsilon_n.(x, y) \mapsto (\epsilon_n x, \epsilon_n^{-1} y)$$

then $X = \mathbb{C}^2/G = \mathrm{Spec}\mathbb{C}[x, y]^G$. Under the action, $x^a y^b$ is invariant if and only if $\epsilon_n^a x^a \epsilon_n^{-b} y^b = x^a y^b$ or $a \equiv b \pmod n$. Let $M' \subset M$ be the lattice of invariant monomials, and $S'_\sigma = \sigma \cap M'$. Then $\mathbb{C}[S'_\sigma] = \mathbb{C}[S_\sigma]^G$ and $U'_\sigma = \mathrm{Spec}\mathbb{C}[S'_\sigma] = \mathbb{C}^2/G$.

Suppose that $N' = \mathrm{Hom}(M', \mathbb{Z})$ and consider the following short exact sequence of abelian groups

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

and apply the contravariant left exact functor $\mathrm{Hom}(-, \mathbb{Z})$ to get

$$0 \longrightarrow N \longrightarrow N' \longrightarrow \mathrm{Ext}^1(M'/M, \mathbb{Z})$$

that is,

$$N \subset N' = \{(\alpha, \beta) |\ \langle (\alpha, \beta), m' \rangle \in \mathbb{Z},\ \ \forall m' \in M'\} = \{(\alpha, \beta) | n\alpha \text{ or } \alpha{+}\beta \text{ or } n\beta \in \mathbb{Z}\},$$

since M' is generated by $(n, 0), (1, 1), (0, n)$.

Theorem 2.4.2 (Fulton [5, p. 34])**.** *Let (σ, N) be a singular simplicial cone generated by $v_1, \cdots, v_n$. Let $N' \subset N$ be generated as $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$, then (σ, N') is non-singular and in fact $U'_\sigma = \mathbb{A}^n$. Suppose $G = N/N'$, therefore G acts on $\mathbb{A}^n$ and $U_\sigma = \mathbb{A}^n/G$.*

2.4.2 Resolution of singularities

One of the important features of working with toric varieties is that we can easily find the explicit defining equations of a resolution of a singular toric variety by looking at its singular fan.

Let X be a singular variety. Recall that a resolution Y for X is a proper, birational map $f : Y \longrightarrow X$ where Y is a non-singular variety. In the toric case, let Δ be a singular fan i.e. there exists a singular cone σ in Δ. Desingularization of Δ is carried out by the so called *subdivision* process [5, p. 45]. Indeed

Theorem 2.4.3 (Fulton [5, p. 48])**.** *Every Δ has a non-singular subdivision.*

In our previous example which we will use it later, the toric fan associated to the minimal resolution of X consists of $n+1$ rays (then n fans) from the origin to the points $(\frac{j}{n}, \frac{n-j}{n})$ for $0 \leq j \leq n$ in C.

Each chart Y_i of Y (the resolution) corresponds to the fan C_i generated by the lattice points $(\frac{i}{n}, \frac{n-i}{n}), (\frac{i-1}{n}, \frac{n-i+1}{n})$, where $1 \leq i \leq n$. Then the associated toric variety is

$$Y_i = \mathrm{Spec}\mathbb{C}[x^{-(n-i)}y^i, x^{n-i+1}y^{-(i-1)}].$$

2.5 The classical McKay correspondence

Let G be a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, and let ρ be the 2-dimensional representation of G induced by the inclusion $G \subset \mathrm{SL}(2, \mathbb{C})$. Suppose that $\{\rho_0, \rho_1, \cdots, \rho_k\}$ be the set of irreducible representations of G, where ρ_0 is the trivial representation. For any $0 \leq j \leq k$, the representation $\rho \otimes \rho_j$ decomposes into a direct sum of irreducibles as

$$\rho \otimes \rho_j = \bigoplus_{i=0}^{k} a_{ij} \rho_i$$

where $a_{ij} = \dim_{\mathbb{C}} \mathrm{Hom}_G(\rho_i, \rho \otimes \rho_j)$.

Definition The *McKay quiver* of G is a directed multi-graph with vertices indexed by irreducible representations and the vertex i is connected to j by exactly a_{ij} number of edges.

Proposition 2.5.1 ([14])**.** *Using representation theory, we have*

1. $a_{ij} = a_{ji}$.

2. *McKay quiver is a connected quiver.*

3. $a_{ii} = 0$.

4. $a_{ij} \in \{0, 1\}$.

In fact, by the above Proposition, we can view the McKay quiver of G as a simple, undirected graph called *McKay graph* and denoted by $\tilde{\Gamma}_G$. The subgraph consisting of non-trivial irreducible representations is denoted by Γ_G.

Theorem 2.5.2 (McKay [14])**.** *With the above notations, the McKay graph $\tilde{\Gamma}_G$ is an extended Dynkin graphs of $\tilde{A}\tilde{D}\tilde{E}$ type. Furthermore, the subgraph Γ_G is one of the graphs A_n, D_n, E_6, E_7, E_8 2.1 which each arises as the resolution graph of $\mathbb{C}^2/G$.*

This result establishes a one-to-one correspondence between the prime divisors D_i of the crepant resolution $\varphi : Y \longrightarrow X = \mathbb{C}^2/G$ and the non-trivial irreducible representations of $G \subset \mathrm{SL}(2, \mathbb{C})$.

Knowing that the exceptional divisor classes $[D_i]$ define a basis for $H_2(Y, \mathbb{Z})$, then adding the class of a point, we arrive at a basis for $H_\star(Y, \mathbb{Z})$. Correspondingly, adding the trivial representation, we obtain

$$\text{Set of irreducible representations of } G \longleftrightarrow \text{ Basis of } H_\star(Y, \mathbb{Z}).$$

The first geometric interpretation of the McKay theorem at the K-theory level was given by Gonzalez-Sprinberg and Verdier in 1983 [8]. They associated the so called *tautological vector bundles* on Y to each irreducible representation of G, which will be described quickly as follows;

Definition For an irreducible representation $\rho_i : G \longrightarrow \mathrm{GL}(V_i)$, let M_i denote the $\mathcal{O}_X$-module defined by

$$M_i := \mathrm{Hom}_{\mathbb{C}[G]}(V_i, \mathbb{C}[a, b]).$$

Because M_i is a G-invariant $\mathbb{C}[a,b]$-module, the associated $\mathcal{O}_X$-module is a coherent sheaf on X.

Definition Let $\mathcal{F}_i := \varphi^\star M_i/\mathrm{Tors}_{\mathcal{O}_Y}$ where $\mathrm{Tors}_{\mathcal{O}_Y}$ is the $\mathcal{O}_Y$-torsions of $\varphi^\star M_i$, then the locally free sheaf $\mathcal{F}_i$ is called *tautological bundle* on Y associated to ρ_i.

By case by case analysis, Gonzalez-Sprinberg and Verdier [8] proved that the non-trivial tautological vector bundles $\{\mathcal{F}_i\}$ satisfy

$$c_1(\mathcal{F}_i) \cdot [D_j] = \deg(\mathcal{F}_i|_{D_j}) = \delta_{ij},$$

where $c_1(\mathcal{F}_i)$ is the first Chern class of $\mathcal{F}_i$ and δ_{ij} is the Kronecker delta symbol.

At the K-theory level, they proved that $\Theta : K^G(\mathbb{C}^2) \longrightarrow K(Y)$ is an isomorphism of abelian groups, where $K^G(\mathbb{C}^2)$ is the Grothendieck ring of G-equivariant coherent sheaves on $\mathbb{C}^2$, $K(Y)$ is the Grothendieck ring of Y and Θ is defined by

$$\Theta(-) := (\pi_{Y\star}(\mathcal{M} \otimes \pi^\star_{\mathbb{C}^2}(-)))^G$$

where $\pi_Y, \pi_{\mathbb{C}^2}$ are projections of $Y \times \mathbb{C}^2$ to Y and $\mathbb{C}^2$ respectively, and $\mathcal{M}$ is the structure sheaf of the reduced fiber product $(Y \times_{\mathbb{C}^2/G} \mathbb{C}^2)_{\mathrm{red}}$.

Moreover, they showed that for $K^G_c(\mathbb{C}^2)$ the G-equivariant K-theory with compact support on $\mathbb{C}^2$,

$$K^G_c(\mathbb{C}^2) \cong \mathrm{Rep}(G)$$

where $\mathrm{Rep}(G)$ is the representation ring of G, and the isomorphism is given by $\Theta(\mathcal{O}_0 \otimes \rho_i) = [\mathcal{O}_{D_i}]$ where $\mathcal{O}_0$ is the skyscraper sheaf of the origin $(0,0) \in \mathbb{C}^2$ and ρ_i is a non-trivial irreducible representation of G [2].

Bridgeland-King-Reid [1] have expanded the geometric interpretation by enlarging the framework to the derived category of coherent sheaves.

2.6 On derived category

One of the main motivations in constructing derived category for an *abelian category*, such as category of coherent sheaves on a scheme is that for a morphism of schemes $\psi : Y \to X$ we are interested in pulling back $\psi^\star\mathcal{F}$ and pushing forward $\psi_\star\mathcal{G}$ of sheaves where $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-module and $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-module. The issue with these functors is that they are not exact, that is, the pull back $\psi^\star$ is right exact but not left exact and the push forward $\psi_\star$ is left exact but not right exact. The first remedy was to introduce right and left ith derived functor $L^i\psi_\star, R^i\psi^\star$ respectively, within the category under consideration. However, there are disadvantages in developing theories using these objects. The main remedy was to first enlarge the category by adding more objects and then considering morphisms which behave perfectly at the cohomology and K-theory levels, in order to capture as much data as possible.

The construction of derived category $\mathbf{D}(\mathcal{A})$ associated to an abelian category $\mathcal{A}$ will be briefly explained.

Like in many mathematical constructions, the aim is to set up a paradigm in which capturing more data becomes more efficient and it also helps simplify some previous constructions. Let $\mathcal{A}$ be an abelian category. We can obtain the derived category $\mathbf{D}(\mathcal{A})$ in the following steps [7]:

1. Consider the category of chain complexes $\mathbf{Kom}(\mathcal{A})$ in $\mathcal{A}$, where its objects are chain complexes and its morphisms are chain maps.

2. Identify chain homotopic morphisms in $\mathbf{Kom}(\mathcal{A})$, to arrive at the homotopy category of chain complexes $\mathbf{K}(\mathcal{A})$.

3. Construct $\mathbf{D}(\mathcal{A})$ by localizing $\mathbf{K}(\mathcal{A})$ at the set of quasi-isomorphisms in $\mathcal{A}$.

In summary, objects of $\mathbf{D}(\mathcal{A})$ are chain complexes and morphisms are chain maps where two morphisms are equal if and only if they induce isomorphisms on cohomology groups, i.e. are quasi-isomorphic.

Some useful examples that we will use later are as follows. The category of coherent sheaves on Y and the category of coherent sheaves on Y with compact support are denoted by $\mathbf{Coh}(Y)$, $\mathbf{Coh}_c(Y)$ and the category of coherent sheaves on $[\mathbb{C}^2/G]$ 3 is denoted by $\mathbf{Coh}([\mathbb{C}^2/G])$ which is isomorphic to $\mathbf{Coh}^G(\mathbb{C}^2)$ the category of G-equivariant coherent sheaves on $\mathbb{C}^2$.

Bridgeland-King-Reid [1] result is an equivalence of derived categories between $\mathbf{D}^b(\mathbf{Coh}(Y))$ and $\mathbf{D}^b(\mathbf{Coh}([\mathbb{C}^2/G]))$ where $\mathbf{D}^b$ is the derived category of bounded complexes and the equivalence is given by the so called Fourier-Mukai transform Φ which we will define and use later.

2.7 Hilbert scheme of points

For a given non-singular, complex variety (manifold) X, the configuration space of m points moving around in X is an interesting geometric object. The configuration space of m-*ordered* points in X is apparently X^m, and the configuration space of m-*unordered* points is $\mathrm{Sym}^m X := X^m/S_m$, the mth symmetric product of X, where S_m is the symmetric group of m-letters. In both spaces, points that correspond to m-tuples of points that are not pairwise distinct are of special interests. In X^m, they have non-trivial isotropic groups with respect to S_m-action and in $\mathrm{Sym}^m X$ they are singular (except for the case where X is one-dimensional.)

The smooth compactifications of the configuration space of m-tuples of distinct points are important. Fulton-MacPherson [6] constructed a nice compactification for ordered m-tuples [13].

Let X be a smooth quasi-projective scheme of finite type over $\mathbb{C}$. Let $Z \subset X$ be a zero-dimensional closed subscheme. The length of Z is the length of the artinian $\mathbb{C}$-algebra $H^0(Z, \mathcal{O}_Z)$, i.e. $l(Z) := \dim_{\mathbb{C}} H^0(\mathcal{O}_Z)$. The Hilbert scheme of m points on X is the set of all zero-dimensional closed subschemes $Z \subset X$ of length m and is denoted by $\mathrm{Hilb}^m(X)$.

If $x \in Z$ is a closed point, the multiplicity of x in Z is defined as $\dim_{\mathbb{C}}(\mathcal{O}_{Z,x})$. We can associate the cycle $|Z|$ to any Z which corresponds to the underlying set counted with multiplicities. That is, $|Z|$ is a point in $\mathrm{Sym}^m X$ and is defined by

$$|Z| := \sum_{x \in X} \dim_{\mathbb{C}}(\mathcal{O}_{Z,x}) \cdot x \in \mathrm{Sym}^m X$$

The Hilbert-Chow morphism is then

$$\rho : \mathrm{Hilb}^m(X) \longrightarrow \mathrm{Sym}^m X$$

which sends Z to $|Z|$.

Briefly, lets discuss why $\mathrm{Hilb}^m(X)$ has a scheme structure. Let X be defined as above. A flat family of proper subschemes in X parameterized by scheme S is a closed subscheme $Z \subset S \times X$ such that the projection $Z \longrightarrow S$ is flat and proper. For a closed point $s \in Z$, denote the fiber of Z over s by Z_s. Now, given such a family and a morphism $f : S' \longrightarrow S$, the family $Z' := (f \times id_X)^{-1}(Z) \subset S' \times X$ is flat and proper over S'. Thus we have defined a functor

$$\mathrm{hilb}(X) : \mathbf{Sch}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

which associates to S the set of all flat families of proper subschemes in X parameterized by S, where **Sch** and **Set** are categories of schemes and sets, respectively.

For any proper subscheme $Z \subset X$ the Hilbert polynomial of Z is defined by $P_Z(n) := \chi(\mathcal{O}_Z \otimes \mathcal{O}_X(nH))$ where H is an ample Cartier divisor and χ is the Euler characteristic. It can be seen that for a flat family $Z \subset S \times X$ the function $s \mapsto P_{Z_s} \in \mathbb{Q}[T]$ where $s \in S$, is locally constant. This means that given a polynomial P, the functor

$$\mathrm{hilb}^P(X) : S \longrightarrow \{Z \subset S \times X |\ Z \text{ is proper and flat over } S,\ P(Z_s) = P, \forall s \in S\}$$

is an open and closed subfunctor of $\mathrm{hilb}(X)$. In particular, the functor $\mathrm{hilb}^m X$ associated to the constant polynomial $P = m \in \mathbb{N}$, parameterizes all zero-dimensional subschemes of length m. In general

Theorem 2.7.1 (Grothendieck [9])**.** *For a quasi-projective (projective) scheme*

X, the functor $hilb^P(X)$ is representable by a quasi-projective (projetive) scheme $Hilb^P(X)$.

Using Yoneda's lemma, we can deduce that there exists a universal subscheme

$$\mathcal{Z}_P \subset \mathrm{Hilb}^P(X) \times X,$$

flat over $\mathrm{Hilb}^P(X)$, such that for any $Z \in \mathrm{hilb}^P(X)(S)$, there is a unique morphism $f : S \longrightarrow \mathrm{Hilb}^P(X)$ with $Z \cong (f \times id_X)^{\star}(\mathcal{Z}_P)$.

Theorem 2.7.2 (Fogarty [4])**.** *Let X be a smooth connected quasi-projective surface, the the Hilbert scheme $Hilb^m(X)$ is smooth and connected of dimension $2m$.*

We are interested in Hilbert scheme of points on $\mathbb{C}^2$ which has an elementary description as follows

$$\mathrm{Hilb}^m(\mathbb{C}^2) = \{I \overset{\text{ideal}}{\subset} \mathbb{C}[a,b] \mid \dim_{\mathbb{C}} \mathbb{C}[a,b]/I = m\}.$$

Remark Consider the Hilbert-Chow morphism $\mathrm{Hilb}^m(\mathbb{C}^2) \longrightarrow \mathbb{C}^2/S_m$ which sends a zero-dimensional closed subscheme Z of length m on $\mathbb{C}^2$ to its associated effective divisor $|Z|$ in $\mathbb{C}^2/S_m$. Let $\mathrm{Hilb}^{|G|}(\mathbb{C}^2)^G$ denote the G fixed point set of $\mathrm{Hilb}^{|G|}(\mathbb{C}^2)$. In fact, $\mathrm{Hilb}^{|G|}(\mathbb{C}^2)^G$ parameterizes G-invariant subschemes, so the Hilbert-Chow morphism restricts to a map

$$\mathrm{Hilb}^{|G|}(\mathbb{C}^2)^G \longrightarrow \mathbb{C}^2/G \subset \mathrm{Sym}^{|G|}(\mathbb{C}^2).$$

This map is a bijection on the set of points parameterizing free orbits of G and hence is a birational map. Moreover, $\mathrm{Hilb}^{|G|}(\mathbb{C}^2)^G$ is non-singular by [11, Lemma 9.1].

If we define $\mathrm{GHilb}(\mathbb{C}^2)$, the G-Hilbert scheme, to be the component of $\mathrm{Hilb}^{|G|}(\mathbb{C}^2)^G$ which contains the free orbits of G, we find that

$$\mathrm{GHilb}(\mathbb{C}^2) \longrightarrow \mathbb{C}^2/G$$

is a resolution of singularities by [11, Theorem 9.3].

2.8 Main question

As discussed previously, we have denoted the irreducible component of $\mathrm{Hilb}^{|G|}(\mathbb{C}^2)^G$ containing free G-orbits by $\mathrm{GHilb}(\mathbb{C}^2)$. Points of $\mathrm{GHilb}(\mathbb{C}^2)$ correspond to G-invariant subschemes $Z \subset \mathbb{C}^2$ with $H^0(Z, \mathcal{O}_Z) \cong \mathrm{Reg}$, where Reg is the regular representation ring of G.

We can generalize $\mathrm{GHilb}(\mathbb{C}^2)$ as follows. The scheme whose points correspond to G-invariant subschemes $Z \subset \mathbb{C}^2$ with $H^0(Z, \mathcal{O}_Z) \cong \rho$, for any representation ρ of G. Precisely,

$$\mathrm{GHilb}^{\rho}(\mathbb{C}^2) = \{I \overset{\text{ideal}}{\subset} \mathbb{C}[a,b] \mid I \text{ is } G \text{ invariant and } \mathbb{C}[a,b]/I \cong \rho\}.$$

Therefore, when $\rho = \mathrm{Reg}$, we obtain $\mathrm{GHilb}^{\mathrm{Reg}}(\mathbb{C}^2) = \mathrm{GHilb}(\mathbb{C}^2)$.

For reasons which will become clear later on, we change our notations $\mathrm{GHilb}(\mathbb{C}^2)$ and $\mathrm{GHilb}^{m\mathrm{Reg}}(\mathbb{C}^2)$ to $\mathrm{Hilb}^{\mathrm{Reg}}([\mathbb{C}^2/G])$ and $\mathrm{Hilb}^{\mathrm{mReg}}([\mathbb{C}^2/G])$, respectively.

As mentioned before, Y, the minimal resolution, can be identified with $\mathrm{GHilb}(\mathbb{C}^2)$ and is clearly isomorphic to $\mathrm{Hilb}^1(Y)$. Therefore, the main question we would like to answer is related to a generalization of the described isomorphism

$$\mathrm{Hilb}^1(Y) \cong \mathrm{Hilb}^{\mathrm{Reg}}([\mathbb{C}^2/G]).$$

That is, we would like to study the relation between the Hilbert scheme of m points of $\mathrm{Hilb}^1(Y)$ i.e. $\mathrm{Hilb}^m(\mathrm{Hilb}^1(Y)) = \mathrm{Hilb}^m(Y)$ and $\mathrm{Hilb}^{m\mathrm{Reg}}([\mathbb{C}^2/G])$.

From the moduli space point of view, $\mathrm{Hilb}^m(Y)$ can be considered as a subspace of the space of objects of $\mathbf{D}(Y)$ parameterizing one term complexes consisting of the structure sheaves of closed subschemes of length m on Y. Likewise, $\mathrm{Hilb}^{m\mathrm{Reg}}([\mathbb{C}^2/G])$ is parameterizing finite substackes of the quotient stack $[\mathbb{C}^2/G]$, thus $\mathrm{Hilb}^{m\mathrm{Reg}}([\mathbb{C}^2/G])$ is also parameterizing some objects of $[\mathbb{C}^2/G]$.

We can observe that the birational morphism from $Y \dashrightarrow X$ induces a birational morphism

$$\Phi \; : \; \mathrm{Hilb}^m(Y) \dashrightarrow \mathrm{Hilb}^{m\mathrm{Reg}}([\mathbb{C}^2/G])$$

for $m > 1$, (we will explain the exact definition of Φ in the next chapter) because m distinct points of Y away from the exceptional set map to m distinct points in $X \setminus \{0\}$ where the preimage of m points of $X \setminus \{0\}$ is in fact, m distinct $\mathbb{Z}/n$-orbits in $\mathbb{C}^2 \setminus \{0\}$ under the projection $\mathbb{C}^2 \setminus \{0\} \longrightarrow X \setminus \{0\}$. Now the question is

Question: *What is the image of* Φ?

In other words, what sort of objects in $\mathbf{D}[\mathbb{C}^2/G]$ are being parameterized by $\Phi(\mathrm{Hilb}^m(Y))$?

We will provide an explicit answer to our question to the case of $G = \mathbb{Z}/n$ in the next chapter.

Chapter 3

The main results

In this chapter, we will use the described framework and concepts in chapter one to introduce new concepts in order to explicitly answer our main question.

From now on, $G = \mathbb{Z}/n$. Let $X = \mathbb{C}^2/\mathbb{Z}/n = \mathrm{Spec}T$ where $T = \mathbb{C}[x,y,z]/(xy - z^n)$ for $n \geq 2$, and $Y \cong \mathrm{GHilb}(\mathbb{C}^2)$ be the minimal (crepant) resolution of X and let $\mathfrak{X} = [\mathbb{C}^2/\mathbb{Z}/n]$ be the quotient stack whose coarse space is X. Let $\mathcal{Z} \subset Y \times \mathfrak{X}$ be the universal closed subscheme associated to the Hilbert scheme Y. We have the following diagram

$$\begin{tikzcd} & \hat{\mathcal{Z}} \arrow[dl, "\hat{p}"'] \arrow[dr, "\hat{q}"] & \\ Y \arrow[dr, "\tau"'] & & \mathfrak{X} \arrow[dl, "\hat{\pi}"] \\ & \mathrm{X} & \end{tikzcd}$$

where $\hat{p}$ is finite and flat, $\hat{q}, \tau$ are birational and $\hat{\pi}$ is finite.

The *Fourier-Mukai transform* [1] Φ from $\mathbf{D}(Y)$ the (bounded) derived category of coherent sheaves on Y, and $\mathbf{D}(\mathfrak{X})$ the (bounded) derived category of coherent sheaves on the stack resolution of X, is defined by

$$\Phi = \mathbf{R}\hat{q}_\star \circ \mathbf{L}\hat{p}^\star : \mathbf{D}(Y) \longrightarrow \mathbf{D}(\mathfrak{X})$$

Note that, since p is already exact we have $\mathbf{L}\hat{p}^\star = \hat{p}^\star$ and in our case, according to Bridgeland-King-Reid [1] Φ is an equivalence of categories.

In this thesis, we will be working with derived category of coherent sheaves with *compact support* so from now on by $\mathbf{D}(Y)$ we mean the derived category of coherent sheaves with compact support on Y.

For the computational purposes, we use the equivalence of categories between $\mathbf{D}(\mathfrak{X})$ and the (bounded) derived category of $\mathbb{Z}/n$-equivariant coherent sheaves on $\mathbb{C}^2$, i.e. $\mathbf{D}^{\mathbb{Z}/n}(\mathbb{C}^2)$. Thus, not only we can replace $\mathfrak{X}$ by $\mathbb{C}^2$, but we can also use the fact that $\hat{\mathcal{Z}} = [\mathcal{Z}/G]$, where $\mathcal{Z}$ is the universal closed subscheme of $Y = \mathrm{GHilb}(\mathbb{C}^2)$ due to the isomorphism $\mathbf{D}(\hat{\mathcal{Z}}) \cong \mathbf{D}^{\mathbb{Z}/n}(\mathcal{Z})$, to replace $\hat{\mathcal{Z}}$ by $\mathcal{Z}$. Then, the above diagram is altered into

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y \arrow[dr, "\tau"'] & & \mathbb{C}^2 \arrow[dl, "\pi"] \\ & \mathrm{X} & \end{tikzcd}$$

and

$$\Phi = \mathbf{R}q_\star \circ \mathbf{L}p^\star : \mathbf{D}(Y) \longrightarrow \mathbf{D}^{\mathbb{Z}/n}(\mathbb{C}^2) \cong \mathbf{D}(\mathfrak{X})$$

We want to study the image of torus invariant structure sheaves of zero-dimensional subschemes (sheaves supported at points) on Y, considered as one term complexes in $\mathbf{D}(Y)$, under Φ. As discussed in the previous chapter, Y has $n-1$ exceptional divisors (with n coordinate charts Y_i) $E_i \cong \mathbb{P}^1$, and since $Y \setminus \{\text{exceptional divisors}\}$ is isomorphic to $X \setminus \{0\}$, the image of those sheaves (one term complex of sheaves with compact support) supported completely outside of the exceptional divisors E_is, will be sent into sheaves, therefore, we are left to investigate the image of sheaves supported on the exceptional curves.

To simplify the problem, we will consider torus-invariant, zero-dimensional, length m sheaves supported at the origin of the coordinate chart Y_i. We will use the fact that torus-invariant, zero-dimensional, length m sheaves supported at the origin correspond to length m monomial ideals of the ring of regular functions on each chart and the latter corresponds to the partition of m which is of combinatorial interest. We also go back and forth between length m torus-invariant, zero dimensional structure sheaves on Y_i and length m torus-invariant, zero-dimensional subschemes of Y_i.

Since our computations rely on coordinates, in the first step we should use toric geometry techniques in order to write the defining equations on each coordinate chart of Y as follows

3.1 Coordinates on the universal closed subscheme of the Hilbert scheme

For our computational purposes, we need coordinate charts on the universal closed subscheme $\mathcal{Z}$. To obtain such coordinates, we prove the following theorem

Theorem 3.1.1. *With the same objects as above, one can isomorphically replace $\mathcal{Z}$ on each chart with $(Y_i \times_X \mathbb{C}^2)_{red}$ for $1 \leq i \leq n$. Equivalently, $\mathcal{Z} \cong (Y \times_X \mathbb{C}^2)_{red}$.*

Proof. By the definition of the universal closed subscheme, a point in $\mathcal{Z}$ is a pair (Z, p) where $Z \subset \mathbb{C}^2$ is a G-invariant subscheme with $H^0(\mathcal{O}_Z) \cong \mathrm{Reg}$ (the regular representation of G) and $p \in Z$.

By the definition of the fiber product, a point in $Y \times_X \mathbb{C}^2$ is a pair (y, q) where $y \in Y, q \in \mathbb{C}^2$ and $\tau(y) = \pi(q)$. Since $Y \cong \mathrm{GHilb}(\mathbb{C}^2)$, a point $y \in Y$ corresponds uniquely to a G-invariant closed subscheme $Z \subset \mathbb{C}^2$ with $H^0(\mathcal{O}_Z) \cong \mathrm{Reg}$. The condition $\tau(y) = \pi(q)$ is then the condition $q \in Z$. Thus, we see that $\mathcal{Z}$ and $Y \times_X \mathbb{C}^2$ coincide pointwise, so to prove the isomorphism $\mathcal{Z} \cong (Y \times_X \mathbb{C}^2)_{\mathrm{red}}$, it suffices to show that $\mathcal{Z}$ is reduced. Because $\mathcal{Z} \longrightarrow Y$ is the universal family, $\mathcal{Z}$ is flat over Y and thus the associated points on $\mathcal{Z}$ are pulled back from the associated points of Y [16, p. 632]. Since Y is smooth, the only associated point is itself and so likewise for $\mathcal{Z}$. Then, since $\mathcal{Z}$ is generically reduced, hence is globally reduced. $\square$

Hence, we have the following diagram

$$\begin{tikzcd} & (Y_i \times_X \mathbb{C}^2)_{\mathrm{red}} \arrow[dl] \arrow[dr] & \\ Y_i \arrow[dr] & & \mathbb{C}^2 \arrow[dl] \\ & X & \end{tikzcd}$$

By definition of the fiber product, we obtain the defining equations for $Y_i \times_X \mathbb{C}^2$ as follows.

Let

$$\begin{aligned} S_i &= \mathbb{C}[u_i, v_i] \\ R &= \mathbb{C}[a, b] \\ T &= \mathbb{C}[x, y, z]/(xy - z^n) \end{aligned}$$

where $u_i = x^{-(n-i)}y^i, v_i = x^{n-i+1}y^{-(i-1)}$ and

$$\begin{aligned} Y_i &= \mathrm{Spec} S_i \\ \mathbb{C}^2 &= \mathrm{Spec} R \\ X &= \mathrm{Spec} T \end{aligned}$$

Therefore,

$$Y_i \times_X \mathbb{C}^2 = \mathrm{Spec}(S_i \otimes_T R)$$

and can be easily seen that

$$Y_i \times_X \mathbb{C}^2 = \mathrm{Spec}\mathbb{C}[u_i, v_i, a, b]/(a^n - u_i^{n-i+1}v_i^{n-i}, b^n - u_i^{i-1}v_i^i, u_iv_i - ab)$$

for $1 \leq i \leq n$.

Proposition 3.1.2. *With the same above notations*

$$(Y_i \times_X \mathbb{C}^2)_{red} \cong Spec\mathbb{C}[u_i, v_i, a, b]/(a^i - u_i b^{n-i}, b^{n-i+1} - v_i a^{i-1}, u_i v_i - ab)$$

for $1 \le i \le n$.

Proof. First, we show that $I \subset J \subset \sqrt{I}$ where

$$I = (a^n - u_i^{n-i+1} v_i^{n-i}, b^n - u_i^{i-1} v_i^i, u_i v_i - ab),$$

and

$$J = (a^i - u_i b^{n-i}, b^{n-i+1} - v_i a^{i-1}, u_i v_i - ab).$$

The inclusion $I \subset J$ is obvious because

$$a^n - u_i^{n-i+1} v_i^{n-i} = a^{n-i}(a^i - u_i b^{n-i}) + u_i(a^{n-i} b^{n-i} - u_i^{n-i} v_i^{n-i}),$$

$$b^n - u_i^{i-1} v_i^i = b^{i-1}(b^{n-i+1} - v_i a^{i-1}) + v_i(a^{i-1} b^{i-1} - u_i^{i-1} v_i^{i-1}),$$

and

$$a^n - u_i^{n-i+1} v_i^{n-i}, b^n - u_i^{i-1} v_i^i$$

are zero in $\mathbb{C}[u_i, v_i, a, b]/J$.
For the second inclusion $J \subset \sqrt{I}$, we claim that

$$(a^i - u_i b^{n-i})^n, (b^{n-i+1} - v_i a^{i-1})^n \in I.$$

In fact,

$$(a^i - u_i b^{n-i})^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a^{ki} u_i^{n-k} b^{(n-i)(n-k)}.$$

When $ki > (n-i)(n-k)$ or $k > n-i$ each monomial can be written in the form

$$(ab)^{(n-i)(n-k)} a^{ki-(n-i)(n-k)} u_i^{n-k} = (u_i v_i)^{(n-i)(n-k)} a^{n(k+i-n)} u_i^{n-k}$$

in $\mathbb{C}[u_i, v_i, a, b]/I$. Likewise, when $ki \leq (n-i)(n-k)$ or $k \leq n-i$ we can write each monomial of the form $a^{ki} u_i^{n-k} b^{(n-i)(n-k)}$ into

$$(ab)^{ki} u_i^{n-k} b^{(n-i)(n-k)-ki} = (u_i v_i)^{ki} u_i^{n-k} b^{n(n-i-k)}$$

in $\mathbb{C}[u_i, v_i, a, b]/I$.
Therefore, using the relations $a^n = u_i^{n-i+1} v_i^{n-i}$ and $b^n = u_i^{i-1} v_i^i$, we obtain

$$\begin{aligned}
(a^i - u_i b^{n-i})^n &= \sum_{0 \leq k \leq n-i} (-1)^{n-k} \binom{n}{k} u_i^{i(n-i+1)} v_i^{i(n-i)} \\
&\quad + \sum_{n-i < k \leq n} (-1)^{n-k} \binom{n}{k} u_i^{i(n-i+1)} v_i^{i(n-i)} \\
&= u_i^{i(n-i+1)} v_i^{i(n-i)} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \\
&= u_i^{i(n-i+1)} v_i^{i(n-i)} (1-1)^n \\
&= 0
\end{aligned}$$

in $\mathbb{C}[u_i, v_i, a, b]/I$.

Similarly,

$$(b^{n-i+1} - v_i a^{i-1})^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b^{k(n-i+1)} v_i^{n-k} a^{(n-k)(i-1)},$$

and when $k(n_i + 1) > (n-k)(i-1)$ or $k > i-1$ the monomial can be rewritten in the form

$$b^{k(n-i+1)-(n-k)(i-1)}(ab)^{(n-k)(i-1)}v_i^{n-k} = b^{n(k-i+1)}(u_iv_i)^{(n-k)(i-1)}v_i^{n-k}$$

and when $k(n_i+1) \leq (n-k)(i-1)$ or $k \leq i-1$,

$$(ab)^{k(n-i+1)}a^{n(i-1-n)}v_i^{n-k} = (u_iv_i)^{k(n-i+1)}a^{n(i-1-n)}v_i^{n-k}$$

in $\mathbb{C}[u_i, v_i, a, b]/I$. Then using the relations $a^n = u_i^{n-i+1}v_i^{n-i}$ and $b^n = u_i^{i-1}v_i^i$, we arrive at

$$\begin{aligned}
(b^{n-i+1} - v_ia^{i-1})^n &= \sum_{0\leq k\leq i-1}(-1)^{n-k}\binom{n}{k}u_i^{(i-1)(n-i+1)}v_i^{i(n-i+1)} \\
&+ \sum_{i-1<k\leq n}(-1)^{n-k}\binom{n}{k}u_i^{(i-1)(n-i+1)}v_i^{i(n-i+1)} \\
&= u_i^{(i-1)(n-i+1)}v_i^{i(n-i+1)}(1-1)^n \\
&= 0
\end{aligned}$$

in $\mathbb{C}[u_i, v_i, a, b]/I$.

Now, it is enough to show that J is self-radical i.e. $J = \sqrt{J}$. We will be using the fact a projective map is flat if and only if the Hilbert polynomial of its fibers coincide for every fiber [16, corollary 24.7.2] and [15, theorem 1.4]. We will show that the fibers of the map

$$q : \mathrm{Spec}\mathbb{C}[u_i, v_i, a, b]/J \longrightarrow \mathrm{Spec}\mathbb{C}[u_i, v_i]$$

are finite sets and thus the Hilbert polynomial of each fiber is constant and is equal to the length of that fiber.

First of all, q is a finite morphism, therefore, is quasi-finite, i.e. its fibers are finite sets. It is also projective, because we can embed $\mathrm{Spec}\mathbb{C}[u_i, v_i, a, b]/J$ into $\mathrm{Proj}_{\mathbb{C}[u_i,v_i]}\mathbb{C}[a, b, c]$ by homogenizing J to $\overline{J}$, for example, when $i \leq n-i$

$$\overline{J} = (a^ic^{n-2i} - u_ib^{n-i}, b^{n-i+1} - v_ia^{i-1}c^{n-2i+2}, u_iv_ic^2 - ab).$$

To see the number of elements of fibers is constant on every fiber, we claim that $1, a, \cdots, a^{i-1}, b, \cdots, b^{n-i}$ is a $\mathbb{C}$-basis of $\mathbb{C}[u_i, v_i, a, b]/J$, for fixed u_i, v_i. Any monomial element of $\mathbb{C}[u_i, v_i, a, b]$ is of the form $a^l b^k u_i^r v_i^s$. If $l \geq i$ or $k \geq n-i+1$ we can use the relations $a^i = u_i b^{n-i}$ and $b^{n-i+1} = v_i a^{i-1}$, to reduce l, k to the cases where $l < i$ and $b < n-i+1$. Furthermore, for fixed u_i, v_i the monomial $a^l b^k u_i^r v_i^s$ is considered only having a, b variables and if $k \leq l \leq i-1$, we can replace $a^l b^k$ by $a^{l-k}(u_i v_i)^k$ using the relation $ab = u_i v_i$ and by $b^{k-l}(u_i v_i)^l$ when $l \leq k \leq n-i$. $\square$

3.2 Main result

As described earlier, let $\mathfrak{I}$ be a monomial ideal sheaf of a length m subscheme supported at the origin of the coordinate chart Y_i where $\mathcal{O}_{Y_i}/\mathfrak{I}$ is the structure sheaf of the corresponding torus-invariant, zero dimensional, closed subscheme of length m supported at the origin of Y_i. We have the following short exact sequence

$$0 \longrightarrow \mathfrak{I} \longrightarrow \mathcal{O}_{Y_i} \longrightarrow \mathcal{O}_{Y_i}/\mathfrak{I} \longrightarrow 0$$

if we apply Φ to the above short exact sequence we will obtain the long exact sequence

$$0 \longrightarrow h^0(\Phi(\mathfrak{I})) \longrightarrow h^0(\Phi(\mathcal{O}_{Y_i})) \xrightarrow{f} h^0(\Phi(\mathcal{O}_{Y_i}/\mathfrak{I})) \longrightarrow$$

$$\longrightarrow h^1(\Phi(\mathfrak{I})) \longrightarrow h^1(\Phi(\mathcal{O}_{Y_i})) \longrightarrow h^1(\Phi(\mathcal{O}_{Y_i}/\mathfrak{I})) \longrightarrow 0$$

Proposition 3.2.1. *$\Phi(\mathcal{O}_{Y_i}/\mathfrak{I})$ is a sheaf, i.e. one term complex of sheaves.*

Proof. The support of $\mathcal{O}_{Y_i}/\mathfrak{I}$ is zero dimensional, p is a finite morphism also q maps the finite set $\mathrm{Supp}(\mathcal{O}_{Y_i}/\mathfrak{I})$ to $\mathbb{C}^2$, therefore, $\Phi(\mathcal{O}_{Y_i}/\mathfrak{I})$ has no higher cohomology, that is, $\mathbf{R}q_\star(p^\star(\mathcal{O}_{Y_i}/\mathfrak{I})) = q_\star(p^\star(\mathcal{O}_{Y_i}/\mathfrak{I}))$. Hence, $\Phi(\mathcal{O}_{Y_i}/\mathfrak{I})$ is a sheaf. $\square$

3.2.1 Statement of the main result

As we proceed, it is necessary to mention that the ideal sheaves of torus invariant subschemes of $\mathbb{C}^2$ are associated to monomial ideals of $\mathbb{C}[a,b]$ and monomial ideals also correspond to partitions. The illustration of the later correspondence is the following.

Each partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d)$ (where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 1$) corresponds to a monomial ideal

$$I_\lambda = (v^{\lambda_1}, uv^{\lambda_2}, \cdots, u^{d-1}v^{\lambda_d}, u^d)$$

on the chart Y_i. We can associate the so called *Ferrer diagram* to the monomial $\mathbb{C}$-basis of $\mathbb{C}[u_i, v_i]/I_\lambda$. For example, let $m = 5$ and $\lambda = (3, 2)$ the monomial ideal is $(v_i^3, u_i v_i^2, u_i^2)$ and its $\mathbb{C}$-basis Ferrer diagram is depicted below.

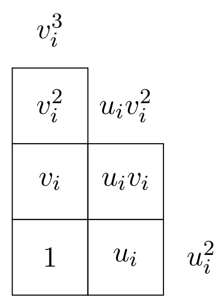

Figure 3.1: Ferrer diagram for $m = 5$ and $\lambda = (3, 2)$

Let λ be a partition of m and let $\mathcal{Z}_{\lambda,i} \subset Y_i$ be a zero-dimensional, torus invariant, closed subscheme of length m supported at the origin of the chart Y_i with the structure sheaf $\mathcal{O}_{\mathcal{Z}_{\lambda,i}}$ whose ideal sheaf is determined by λ. We would like to describe the image under Φ, that is, $\Phi(\mathcal{O}_{\mathcal{Z}_{\lambda,i}}) \in \mathbf{Coh}^{\mathbb{Z}/n}(\mathbb{C}^2)$, where $\mathbf{Coh}^{\mathbb{Z}/n}(\mathbb{C}^2)$ is isomorphic to the category of finitely generated $\mathbb{Z}/n$-invariant, $\mathbb{C}[a,b]$-modules.

Theorem 3.2.2. *The image of* $\mathbb{C}[u_i, v_i]$ *the associated module to* $\mathcal{O}_{Y_i}$ *under* $\Phi = q_\star \circ p^\star$ *is the following* R*-module*

$$M_i := \bigoplus_k t_k R/(a^{i-1}t_k - b^{n-i+1}t_{k-1},\ \forall k \geq 1 \text{ and } a^i t_k - b^{n-i}t_{k-1},\ \forall k \leq 0)$$

where $R = \mathbb{C}[a,b]$.

Remark Obviously the above R-module is *not finitely generated* because $\mathcal{O}_{Y_i}$ is *not of compact support.*

The main result is

Theorem 3.2.3.

$$\Phi(\mathcal{O}_{\mathcal{Z}_{\lambda,i}}) = M_i/\mathcal{J}_{\lambda,i}$$

where $\mathcal{J}_{\lambda,i}$ *is the* R*-submodule of* M_i *determined combinatorially by* λ *by the procedure described in 3.2.3.*

3.2.2 Computational technique and proof of the result

The underlying computational technique that we have exploited is demonstrated in the following diagram

$$\begin{tikzcd} & S_i \otimes_T R\text{-modules} \arrow[dr, "q_\star"] & \\ S_i\text{-modules} \arrow[ur, "p^\star"] & & R\text{-modules} \end{tikzcd}$$

where $T = \mathbb{C}[x,y,z]/(xy - z^n)$ and an $S_i = \mathbb{C}[u_i, v_i]$-module is considered as a $\mathbb{C}[u_i,v_i] \otimes_{\mathbb{C}[x,y,z]/(xy-z^n)} \mathbb{C}[a,b]$-module through the way $\mathbb{C}[u_i,v_i]$ sits inside $\mathbb{C}[u_i,v_i] \otimes_{\mathbb{C}[x,y,z]/(xy-z^n)} \mathbb{C}[a,b]$ by $p^\star$ and then using the existing relations it is sent to a $\mathbb{C}[a,b]$-module under $q_\star$.

Proof. 3.2.2 Define a surjective R-module homomorphism

$$M_i \longrightarrow q_\star(p^\star(\mathbb{C}[u_i, v_i]))$$

sending $t_0 \mapsto 1, t_k \mapsto \Phi(v_i^k)$ and $t_{-k} \mapsto \Phi(u_i^k)$ for $k \in \mathbb{N}$. It is well-defined because of these relations $a^i = u_i b^{n-i}$ and $b^{n-i+1} = v_i a^{i-1}$. The injectivity boils down to the fact that the mentioned relations together with $u_i v_i = ab$ exhaust all the possible relations among u_i, v_i, a and b as can be seen from Proposition 3.1.2. □

Proof. 3.2.3 Let $\mathcal{I}_{\lambda,i}$ be the ideal sheaf of $\mathcal{O}_{\mathcal{Z}_{\lambda,i}}$, that is, $\mathcal{O}_{\mathcal{Z}_{\lambda,i}} = \mathcal{O}_{Y_i}/\mathcal{I}_{\lambda,i}$. Let $I_{\lambda,i}$ be the associated ideal of S_i to $\mathcal{I}_{\lambda,i}$. Then, we have the following short exact sequence of S_i-modules

$$0 \longrightarrow I_{\lambda,i} \longrightarrow S_i \longrightarrow S_i/I_{\lambda,i} \longrightarrow 0$$

Since p is a flat morphism, pulling back the above sequence by p preserves the exactness and it will be

$$0 \longrightarrow I_{\lambda,i} \otimes_T R \longrightarrow S_i \otimes_T R \longrightarrow S_i/I_{\lambda,i} \otimes_T R \longrightarrow 0$$

Now, because $q_\star$ is not right exact, applying $q_\star$ to our sequence we will get

$$0 \longrightarrow q_\star(I_{\lambda,i} \otimes_T R) \longrightarrow q_\star(S_i \otimes_T R) \longrightarrow q_\star(S_i/I_{\lambda,i} \otimes_T R) \longrightarrow R^1 q_\star(I_{\lambda,i} \otimes_T R)$$

where $R^1 q_\star(I_{\lambda,i} \otimes_T R)$ is the first right derived functor of $q_\star(I_{\lambda,i} \otimes_T R)$.

Lemma 3.2.4. $R^1 q_\star(I_{\lambda,i} \otimes_T R) = 0$.

Proof. In other words, we want to show that $R^1 q_\star(\mathcal{I}_{\lambda,i} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathbb{C}^2}) = 0$, which holds, since $q_\star(\mathcal{I}_{\lambda,i} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathbb{C}^2})$ is a quasi-coherent $\mathcal{O}_{\mathbb{C}^2}$-module on an affine variety.

□

3.2.3 Combinatorial corollary and illustrations

First of all, to get more sense of how we found M_i (the image of $\mathbb{C}[u_i, v_i]$ under Φ) in 3.2.2 and later the combinatorial consequent, we represent the Ferrer diagram of M_i. The method we used was thematically the same in all the coming illustrations.

At the outset, the $\mathbb{C}$-basis of $\mathbb{C}[u_i, v_i]$ is $\{u_i^r v_i^s \mid r, s \geq 0\}$ with the Ferrer diagram

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
v_i^3	$u_i v_i^3$	$u_i^2 v_i^3$	$u_i^3 v_i^3$	$\cdots$
v_i^2	$u_i v_i^2$	$u_i^2 v_i^2$	$u_i^3 v_i^2$	$\cdots$
v_i	$u_i v_i$	$u_i^2 v_i$	$u_i^3 v_i$	$\cdots$
1	u_i	u_i^2	u_i^3	$\cdots$

Figure 3.2: Ferrer diagram of $\mathbb{C}[u_i, v_i]$

under Φ it goes to the Ferrer diagram Figure 3.3. and the discovered rule which is: we move to the right by multiplication to a and to the up by multiplication to b.

To understand $\Phi(\mathcal{O}_{\mathcal{Z}_{\lambda,i}})$ better, first note that the origin of Y_i, that is, $\mathbb{C}[u_i, v_i]/(u_i, v_i)$ maps to $\mathbb{C}[a, b]/(a^i, b^{n-i}, ab)$ which its Ferrer diagram is L-shaped depicted in Figure 3.4.

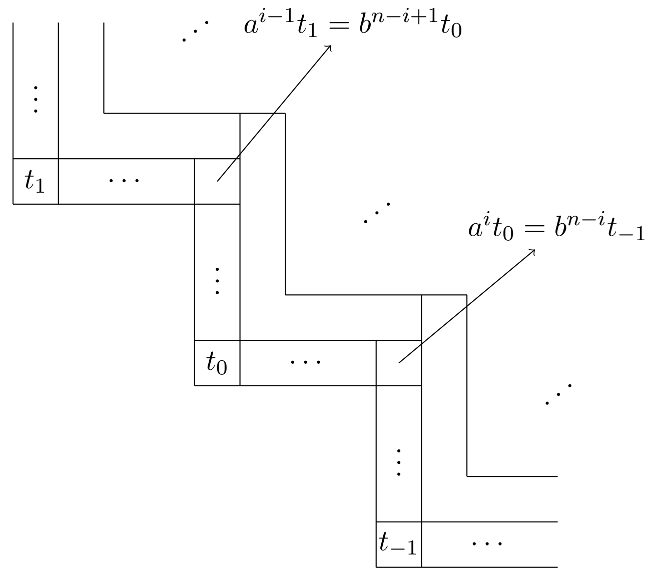

Figure 3.3: Ferrer diagram of M_i

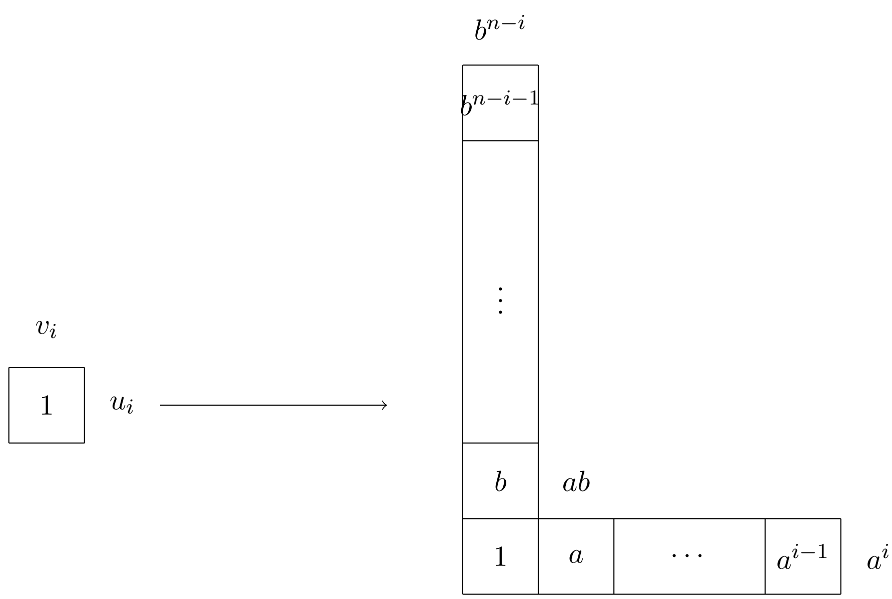

Figure 3.4: Image of the origin

Then, for any monomial ideals I in a chart Y_i, to find the image under Φ, we first find the Ferrer diagram of the monomial $\mathbb{C}$-basis of $\mathbb{C}[u_i, v_i]/I$ and stick to the following set of rules:

We start with the box 1 (the origin). Either there is a box attached to its right hand side or on top of it

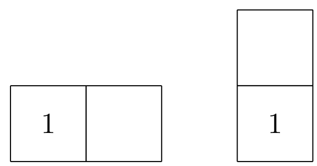

Figure 3.5: Attachment possibilities

and the image will be constructed by attaching the *L-shaped image* of *the origin* to the *right hand side* of the leftmost box or to the *top of it* as follows

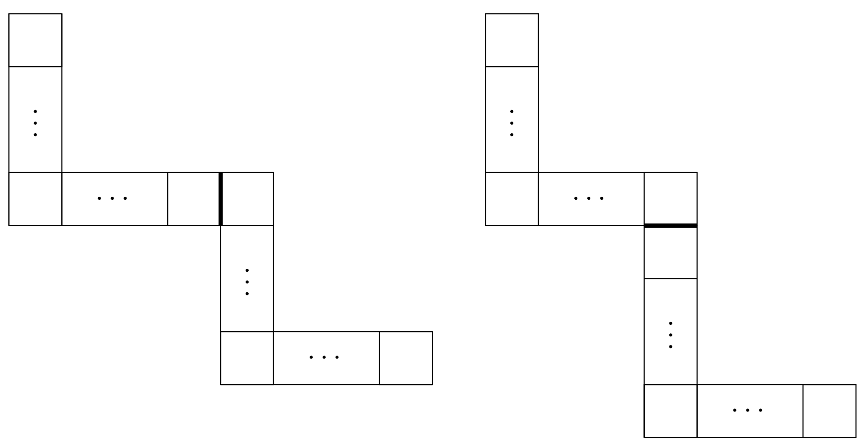

Figure 3.6: Attaching rules

and we continue this process for other boxes in the Ferrer diagram of $\mathbb{C}[u_i, v_i]/I$.

For example, let $n = 4, i = 2$ and $m = 5$ with $\lambda = (3, 2)$. Thus, on Y_2, $\mathbb{C}[u_2, v_2]/(u_2, v_2)$ is mapped to $\mathbb{C}[a, b]/(a^2, b^3, ab)$ and in order to find the image of $\mathbb{C}[u_2, v_2]/(u_2^2, u_2v_2^2, v_2^3)$ we should follow the above rules step by step.

First, determine the image of the origin

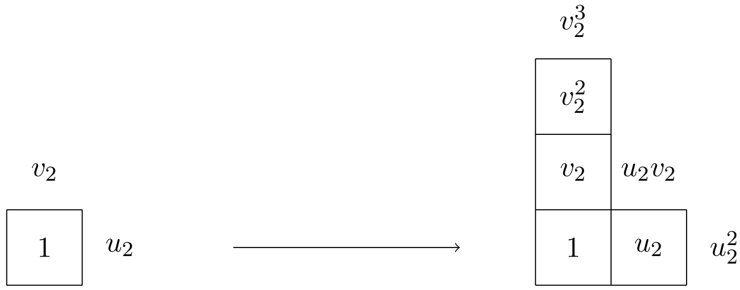

Figure 3.7: Ferrer diagram of the origin for $n = 4, i = 2$ and $m = 5$ with $\lambda = (3, 2)$.

then find the Ferrer diagram of the $\mathbb{C}$-basis of $\mathbb{C}[u_2, v_2]/(u_2^2, u_2v_2^2, v_2^3)$

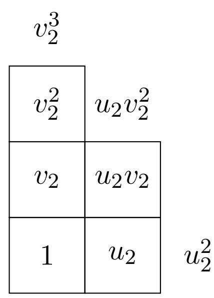

Figure 3.8: Ferrer diagram of $\mathbb{C}[u_2, v_2]/(u_2^2, u_2v_2^2, v_2^3)$

and finally, use the attaching rules to construct the Ferrer diagram of the image shown below in Figure 3.9. Therefore, the image is

$$(a^4, a^2b^2, ab^5, b^8)/(a^6, a^5b, a^4b^4, a^2b^6, ab^9, b^{11})$$

which is also isomorphic as an $R = \mathbb{C}[a, b]$-module to

$$M_2/\mathcal{J}_{\lambda,2}$$

where $\mathcal{J}_{\lambda,2}$ is a submodule of M_2 with relations coming from the monomials $a^6, a^5b, a^4b^4, a^2b^6, ab^9, b^{11}$.

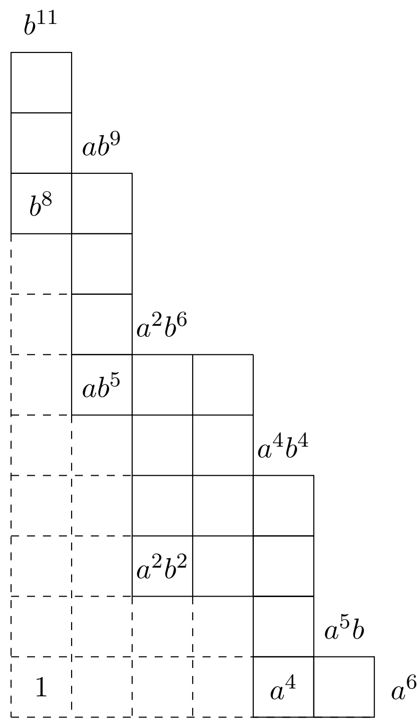

Figure 3.9: Image of $\mathbb{C}[u_2, v_2]/(u_2^2, u_2v_2^2, v_2^3)$

Moreover, we have established the subsequent combinatorial result

Corollary 3.2.5. *The above procedure establishes a correspondence between partitions of m and $\mathbb{Z}/n$-colored skew partitions of mn.*

We color the boxes according to the initial coloring of the image of the origin as can be seen in the coming example.

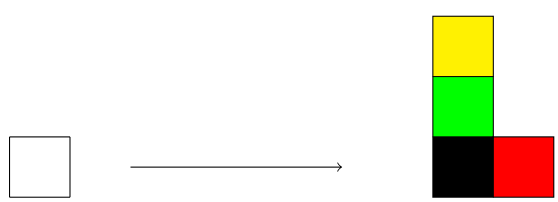

Figure 3.10: $\mathbb{Z}/4$-coloring

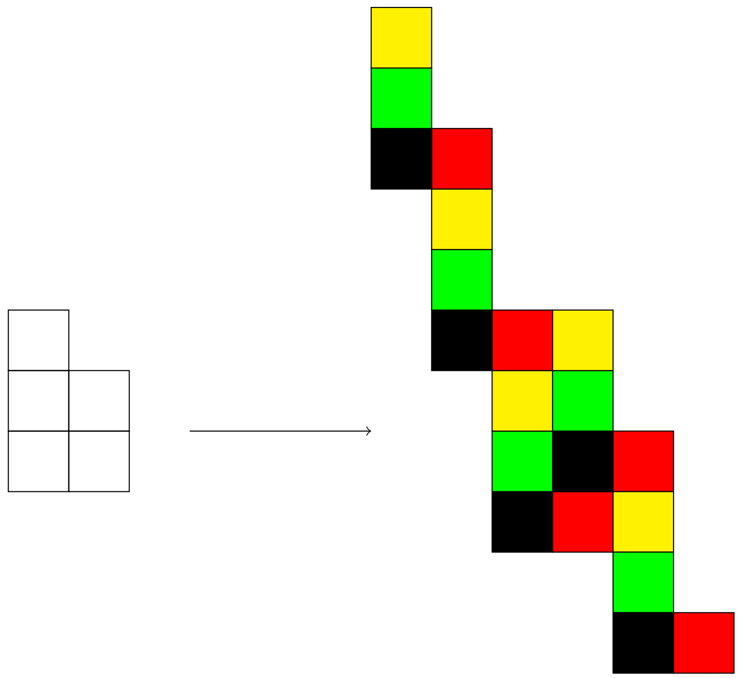

Figure 3.11: $\mathbb{Z}/4$-colored skew partition

Chapter 4

Conclusion

Recall that the question we set out to answer was : "Where do the structure sheaves of zero-dimensional, closed subschemes on Y go under the Fourier-Mukai equivalence $\mathbf{D}(Y) \longrightarrow \mathbf{D}^G(\mathbb{C}^2)$?"

We have used explicit coordinate charts on Y to fully answer this question in the case where $G = \mathbb{Z}/n$ and the subschemes are torus invariant. We have heavily exploited the combinatorics of the $\mathbb{C}$-basis of a monomial ideal associated to the ideal sheaf of a zero-dimensional, torus invariant closed subscheme of Y_i, in the associated Ferrer diagram. Our method involved first determining the Ferrer diagram of the image of the origin of each chart Y_i and then by the combinatorial attaching rules, constructing the image using the Ferrer diagram of the image of the origin. We have also explained how to construct the image from Ferrer diagrams in an example. Nevertheless, in order to obtain a unifying picture, we have found the universal R-module M_i on each chart, so that the formulation of our result becomes easier and later, we have proven that the image is obtained by taking the quotient of M_i by the ideal determined from our combinatorial rules.

It remains to address our question in the case of general zero-dimensional closed subschemes of Y or even when G is not necessarily cyclic. One possible approach is related to Haiman's result [10] which gives a system of coordinates for Hilbert scheme in general, and our method could be applied directly to get a more general result.

Moreover, we believe that our combinatorial outcome has not been pushed as much as it could. As discussed, the combinatorial Fourier-Mukai gives us the correspondence between partitions of m and $\mathbb{Z}/n$-colored skew partitions of mn. Also, since $\mathrm{Hilb}^m(Y)$ and $\mathrm{Hilb}^{m\mathrm{Reg}}([\mathbb{C}^2/\mathbb{Z}/n])$ are holomorphic symplectic, their Betti numbers coincide, therefore, they do have the same

Euler characteristic. Meanwhile, we know that the Euler characteristic is the number of torus fixed points. Thus, there is a correspondence between m partitions of total size n and $\mathbb{Z}/n$-colored partitions of size mn with m boxes of each color. Hence, there should be a set of rules by which given a $\mathbb{Z}/n$-colored partition of mn with m boxes of each color, one can reconstruct a $\mathbb{Z}/n$-colored skew partition of mn and vice versa. Our correspondence comes close to this: we associate to m partitions of size n, m $\mathbb{Z}/n$-colored skew partitions with a total of m boxes of each color. This suggests that there should be some way of combining our skew partitions into a single $\mathbb{Z}/n$-colored partition with m boxes of each color. Discovering this process is a project for the future.

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