Harnack Inequality for Nondivergent Linear Elliptic Operators on Riemannian Manifolds

A Self-contained Proof

by

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Abstract

In this paper, a self-contained proof is given to a well-known Harnack inequality of second order nondivergent uniformly elliptic operators on Riemannian manifolds with the condition that $\mathcal{M}^{-}[\mathbf{R}(v)] \geqslant 0$, following the ideas of M. Safonov [5]. Basically, the proof consists of three parts: 1) Critical Density Lemma, 2) Power-Decay of the Distribution Functions of Solutions, and 3) Harnack Inequality.

Preface

Chapter 2. Preliminaries is a summary of the introductions on Riemannian manifolds in Cabré [1] and S. Kim [2].

Chapter 3. Critical Density Lemma follows the results and the proofs also by Cabré [1] and S. Kim [2] with more comments and explanations for clarity.

In Chapter 4, The ideas of Safonov [5] is followed for a covering lemma on Riemannian manifolds and a power-decay property of the distribution functions of solutions.

Chapter 5. Harnack Inequality follows the ideas of Cabré [1], Caffarelli [4], and Caffarelli and Cabré [3].

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Do homage to the Son ... (*Psalm* 2 : 12)

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Dedication

To the only true triune God in Jesus Christ , our Savior and Lord ,
to my families in His Church,
in memory of my father, Yeon-Sik Mun,
to my mother, Young-Sook Kim,
and to my wife and our first son, Sang-Eun and John Giju.

Chapter 1

Introduction

There are many results about Harnack inequality on Riemannian manifolds. First, Yau, S-T [9] proved Harnack inequality for positive harmonic functions on Riemannian manifolds with nonnegative Ricci curvature. Also, Saloff-Coste [10] and Grigor'yan [11] obtained that the volume doubling property of measure on manifolds and a kind of weak Poincaré inequalty give Harnack inequality for solutions of divergence parabolic equations on Riemannian manifolds. Especially, for the defintion of second-order, linear, nondivergent uniformly elliptic operators $\mathcal{L}$, Cabré [1] and Stroock [12] can referred to, and the definition will be noted in Section 2.2.

In this paper, a self-contained proof of Harnack inequality on Riemannian manifolds with the condition that $\mathcal{M}^-[\mathbf{R}(\upsilon)] \geqslant 0$ (See Section 2.1), which is stronger–we cannot guarantee whether it is strictly or not–than nonnegative Ricci curvature condition, but weaker than nonnegative sectional curvature, is given. This result was earlier proven by S.Kim [2]; however, S.Kim only gave the proofs of Critical Density Lemma part and refered to Cabré [1] for the remaining parts: Power-decay of distribution functions of solutions, and Harnack inequality.

In fact, Cabré [1] proved Harnack inequality for nondivergent elliptic operators on Riemannian manifolds with nonnegative sectional curvatures, and S. Kim [2] proved a similar result with the condition that $\mathcal{M}^-[\mathbf{R}(\upsilon)] \geqslant 0$. The difference between the conditions on Riemannian manifolds needs different proofs only for Critical Density Lemma part; in fact, the essense of S. Kim [2] improved from Cabré [1] was the computation

$$\mathcal{L}d_y(x) \leq \frac{a_{\mathcal{L}}}{d_y(x)} \quad \text{with } a_{\mathcal{L}} = (n-1)\Lambda$$

for any $x \in \mathbf{M}\backslash[Cut(y) \cup \{y\}]$ under the geometric condition $\mathcal{M}^-[\mathbf{R}(\upsilon)] \geqslant 0$, and using it to get Critical Density Lemma. On the other hand, Cabré used

$$\mathbf{D}^2 d_y(x)(\xi,\xi) \leqslant \frac{1}{d_y(x)}|\xi|^2$$

for any $x \in \mathbf{M} \backslash [Cut(y) \cup \{y\}]$ and any $\xi \in \mathbf{T_x M}$ under the condition of nonnegative sectional curvature to get the lemma.

The strategy used by Cabré and S.Kim is basically based on the proof of Harnack inequality in $\mathbb{R}^n$ by Caffarelli [4]. In Cabré's paper, for the proof of Power-decay of distribution functions of solutions part, he followed Caffarelli's arguments and applied a kind of Calderón-Zygmund decomposition on Riemannian manifolds with nonnegative sectional curvature condition, using a result of M. Christ [6] which is highly nontrivial. S. Kim also directly followed the arguments of Cabré for that part since nonnegative Ricci curvature condition is sufficient to get the doubling-property of volume (see Lemma 2.3) essential for the decomposition, and the condition that $\mathcal{M}^- [\mathbf{R}(\upsilon)] \geqslant 0$ implies the nonnegativity of Ricci curvature (see Section 2.1).

For the part, a more elementary approach using a covering lemma by Safonov (see Lemma 4.2) is applied, and a simpler proof than those of Cabré and S. Kim is given in this paper; this is a small improvement. In fact, there is a similar result using a covering lemma, by Aimar and Forzani, and Toledano [7] in a more general and abstract setting, e.g. homogeneous spaces. However, our proof is simpler and more direct, focusing on the case of Riemannian manifolds.

With respect to the parts of Critical Density Lemma and Power-decay property of distribution functions, Safonov [5] started with Alexandrov-Bakelman-Pucci estimate whose proof is well-known in $\mathbb{R}^n$, and obtained a Growth Lemma which is less restrictive for applications and a Double-section Lemma which make the proof for Harnack simple and direct. However, it was difficult for us to get a lemma which is silimar to Alexandrov-Bakelman-Pucci estimate. Thus, except the part for Power-decay property of distribution functions, we basically followed the ideas and arugments of Cabré, and S. Kim; that is, our paper can be regarded as a self-contained exposition of the results of Safonov, Cabré, and S. Kim.

Chapter 2

Preliminaries

2.1 Riemannian Geometry

Notation

The notation for some concepts on Riemannian manifolds is given as the following. Let $\mathbf{M}$ be a smooth n-dimensional complete Riemannian manifold with a metric g. And the geodesic distance between points x and y on $\mathbf{M}$ is denoted by $d(x,y)$ or $d_y(x)$ or $d_x(y)$, the Riemannian measure of $\mathbf{M}$ by $d\mathbf{V}$, the tangent space of $\mathbf{M}$ at x by $\mathbf{T_x M}$, and the Riemannian curvature tensor by $\mathbf{R}(X,Y)Z$. For convenience, the geodesic distance is sometimes called just by distance.

Since the geodesic parametrized by the arc-length and exponential mapping on Riemannian manifolds will be often used later on, a summary of them including the concept of cut-points of a point x in $\mathbf{M}$ is given.

Exponential Functions

If the exponential map $\exp_x : \mathbf{T_x M} \longrightarrow \mathbf{M}$ is considered, for any $v \in \mathbf{T_x M}$ with $|v| = 1$, a function $v(t) = \exp_x(tv)$ can be set. Then, $v(t)$ is the geodesic parametrized by arc-length, that is, with unit-speed which satisfies $v(0) = x$ and $v'(0) = v$. Here, a constant t_0 is defined by

$$t_0 = t_0(x,v) = \sup\{\ s > 0\ \mid\ v(t) \text{ is the minimal geodesic from } x \text{ to } v(s)\}\,.$$

When $t_0 < \infty$, the point $v(t_0)$ is called the cut point of x along $v(t)$, and the set $Cut(x)$ is defined by

$$Cut(x) = \{\text{cut point of } x \text{ along } \omega(t) = \exp_x(t\omega)\mid\ \omega \in \mathbf{T_x M} \text{ and } |\omega| = 1\}\,.$$

Then, it is well-known that $Cut(x)$ has zero n-dimensional Riemannian measure. An another set G_x is defined by

$$G_x = \{\ t\omega\ \mid\ 0 \leqslant t < d(x, \exp_x(t_0(x,\omega)\omega)) \text{ for } \omega \in \mathbf{T_x M} \text{ with } |\omega| = 1\}\,.$$

Then, it is also well-known that

$$\exp_x : G_x \longrightarrow \exp_x(G_x) \text{ is a diffeomorphism.}$$

Also, a concept of second-variation of vector-fields on Riemannian manifolds is necessary to get a estimate of differential operators acted on the distance function under a geometric condition (see Lemma 2.1).

Curvatures and Morse Index Form

The Riemannian curvature tensor is defined by

$$\mathbf{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where ∇ is the Levi-Civita connection. And, for a unit tangent vector v in $\mathbf{T_x M}$, the Ricci transformation $\mathbf{R}(v) : \mathbf{T_x M} \longrightarrow \mathbf{T_x M}$ is defined by

$$\mathbf{R}(v)X = \mathbf{R}(X,v)v.$$

For a given geodesic $\sigma : [0,l] \longrightarrow \mathbf{M}$ parametrized by the arc-length, the Morse index form $I(V,W)$ is defined by

$$I(V,W) = \int_0^l \left\{ \langle \nabla_{\sigma'} V, \nabla_{\sigma'} W \rangle - g \left\langle R(\sigma', V)W, \sigma' \right\rangle \right\} \, dt,$$

where V, W are piecewise smooth vector fields along σ.

From now on, specific Riemannian manifolds satisfying the following condition are considered in Chapter 3-5.

Geometric Assumptions

It is noted that

$$d_p \text{ is smooth on } \mathbf{M} \backslash \left[Cut(p) \cup \{p\} \right].$$

To express the condition of $\mathbf{M}$ under which Harnack inequalty is proven in this paper, the Pucci's extremal operator for a symmetric endomorphism $\mathbf{A}$ on $\mathbf{T_x M}$ is introduced:

$$\mathcal{M}^- \left[\mathbf{A}, \lambda, \Lambda \right] = \mathcal{M}^- \left[\mathbf{A} \right] = \lambda \sum_{\kappa_j > 0} \kappa_j + \Lambda \sum_{\kappa_j < 0} \kappa_j,$$

where κ_j are the eigenvalues of $\mathbf{A}$. The sufficient condition for Harnack inequality found by S. Kim [2] was the following:

$$\mathcal{M}^- [\mathbf{R}(v)] \geqslant 0,$$

for any $x \in \mathbf{M}$ and $v \in \mathbf{T_x M}$ with $|v| = 1$, where $\mathbf{R}(v)$ is the Ricci transformation on $\mathbf{T_x M}$ in Section 2.1. It is noted that the condition is stronger than the condition of nonnegative Ricci curvature, i.e. $\mathcal{M}^- [\mathbf{R}(v)] \geqslant 0$ on $\mathbf{M}$ implies that $\mathbf{M}$ has nonnegative Ricci curvature. This can be easily checked by the following:

$$\mathcal{M}^- [\mathbf{R}(v)] \leqslant \operatorname{tr} (\mathbf{A}(\mathbf{x}) \circ \mathbf{R}(v))$$

if $\mathbf{A}(\mathbf{x})$ is uniformly elliptic (see Section 2.2); especially, when $\mathbf{A}(\mathbf{x}) = \lambda \mathbf{Id}$, nonnegative Ricci curvature condition for $\mathbf{M}$ is satisfied. Moreover, it is also noted that any Riemannian manifold with nonnegative sectional curvature trivially satisfies the condition

$$\mathcal{M}^- [\mathbf{R}(v)] \geqslant 0,$$

thus the condition is stronger than nonnegative Ricci curvature condition; however, weaker than nonnegative sectional curvature condition. The condition will be always assumed for any Riemannian manifold $\mathbf{M}$ in Chapter 3-5.

2.2 Differential Operators on Riemannian Manifolds

The definitions of some elementary differential operators on Riemannian manifolds are summarized.

Gradient and Hessian of Functions on Manifolds

It is noted that the Hessian of a function u at a point x in $\mathbf{M}$ is defined as an endomorphism of $\mathbf{T_x M}$ by

$$\mathbf{D}^2 u \cdot \xi = \mathbf{D}_\xi \nabla u \qquad \forall \xi \in \mathbf{T_x M},$$

where $\mathbf{D}$ denotes the Levi-Civita connection in $\mathbf{M}$ and $\nabla u(x)$ is the gradient of u at x.

Second-order Nondivergent Linear Uniformly Elliptic Operators on Manifolds

First, let $\mathbf{A}(\mathbf{x})$ be a positive definite symmetric endomorphism of $\mathbf{T_x M}$. It is assumed that $\mathbf{A}(\mathbf{x})$ satisfies the uniformly ellipticity with some positive constants Λ and λ:

$$\lambda \left|\xi\right|^2 \leqslant g \left\langle \mathbf{A}(\mathbf{x})\xi, \xi \right\rangle \leqslant \Lambda \left|\xi\right|^2 \qquad \forall x \in \mathbf{M},\ \forall \xi \in \mathbf{T_x M},$$

where $\left|\xi\right|^2 = g \left\langle \xi, \xi \right\rangle$.
Next, a second-order, nondivergent, linear, uniformly elliptic operator $\mathcal{L}$ is defined by

$$\mathcal{L}u = tr \left(\mathbf{A}(\mathbf{x}) \circ \mathbf{D}^2 u\right) = tr \left\{ \xi \longmapsto \mathbf{A}(\mathbf{x}) \nabla_\xi \nabla u \right\},$$

where tr is the trace of endomorphism, $\circ$ is composition of endomorphisms, and $\mathbf{D}^2 u$ is the Hessian of a function u.

2.3 Lemmas on Riemannian Manifolds

This section is basically for some computations in the proof of Lemma 3.2. All the lemmas in this section directly refer to Cabré [1] and S.Kim [2]; however, some of the proofs are included for clarity.

The following lemma provides a boundness of an elliptic operator $\mathcal{L}$ operated on the distance function under a geometric condition.

Lemma 2.1. *(S. Kim [2]) Let* $\mathbf{M}$ *satisfy* $\mathcal{M}^-[\mathbf{R}(v)] \geqslant 0$ *on* $\mathbf{M}$ *(see Section 2.1.) Let* p *be a point on* $\mathbf{M}$ *and* $x \in \mathbf{M}\setminus[Cut(p) \cup \{p\}]$*. Then, it is obtained that*

$$\mathcal{L}d_p(x) \leq \frac{a_{\mathcal{L}}}{d_p(x)}, \quad \textit{where } a_{\mathcal{L}} = (n-1)\Lambda.$$

Proof. Let $\sigma : [0, l] \longrightarrow \mathbf{M}$ be the minimal geodesic parametrized by arc-length from $p = \sigma(0)$ to $x = \sigma(l)$, and choose an orthnormal basis $\{\epsilon_j\}_{j=1}^n$ on $\mathbf{T_x M}$ satisfying that $\epsilon_1 = \sigma'(l)$ and $\{\epsilon_j\}_{j=1}^n$ are eigenvectors of $\mathbf{D}^2 d_p$. Here, by parallel transport along $\sigma(t)$, $\{\epsilon_j\}_{j=1}^n$ can be extended to $\{\epsilon_j(t)\}_{j=1}^n$ with a parameter $t \in [0, l]$. If the Jacobi fields along $\sigma(t)$, $V_j(t)$ is defined, satisfying

$$\begin{aligned} &1)\ V_j(0) = 0 \text{ and } V_j(l) = e_j, \\ &2)\ \left[V_j(t), \sigma'(t)\right] = 0, \end{aligned}$$

then it is obtained that

$$\left\langle \mathbf{D}^2 d_p(\epsilon_j), \epsilon_j \right\rangle = \left\langle \nabla_{\sigma'} V_j, V_j \right\rangle(l) = I(V_j, V_j).$$

Here, since a Jacobi field minimizes the Morse index form among all vector fields along the same geodesic with the same boundary data, it is obtained that

$$I(V_j, V_j) \leqslant I(\frac{t}{l}\epsilon_j(t), \frac{t}{l}\epsilon_j(t)).$$

Thus, it can be computed that

$$
\begin{aligned}
\mathcal{L}d_p(x) &= \sum_{j=2}^{n} a_{jj}\left\langle \mathbf{D}^2 d_p(\epsilon_j), \epsilon_j \right\rangle \\
&\leqslant \sum_{j=2}^{n} a_{jj} I(V_j, V_j) \\
&\leqslant \sum_{j=2}^{n} a_{jj} I(\frac{t}{l}\epsilon_j(t), \frac{t}{l}\epsilon_j(t)) \\
&= \sum_{j=2}^{n} a_{jj} \int_0^l \left|\frac{1}{l}\right|^2 - \int_0^l (\frac{t}{l})^2 \sum_{j=2}^{n} a_{jj}\left\langle R(\sigma', \epsilon_j)\epsilon_j, \sigma' \right\rangle \\
&\leqslant \sum_{j=2}^{n} a_{jj} \int_0^l \left|\frac{1}{l}\right|^2 - \int_0^l (\frac{t}{l})^2 \mathcal{M}^- \left[R(\sigma')\right],
\end{aligned}
$$

and since $\mathcal{M}^- \left[R(v)\right] \geqslant 0$

$$
\leqslant \frac{(n-1)\Lambda}{l} = \frac{(n-1)\Lambda}{d_p(x)}.
$$

This finishes the proof of the lemma. □

For the proof of Lemma 3.2, a computation of the Jacobian of the exponential mapping of the gradients of smooth functions on manifolds will be necessary.

Lemma 2.2. *(Cabré [1]) Let v be a smooth function in an open set Ω of $\mathbf{M}$. For the map $\phi : \Omega \longrightarrow \mathbf{M}$ defined by*

$$
\phi(x) = \exp_x \nabla v(x),
$$

whenever $\nabla v(x) \in G_x$ for some $x \in \Omega$, the following is satisfied

$$
Jac\ \phi(x) = Jac \exp_x(\nabla v(x)) \cdot \left|\det \mathbf{D}^2(v + \frac{d_y^2}{2})(x)\right|,
$$

where $y = \phi(x)$ and $Jac \exp_x(\nabla v(x))$ denotes the Jacobian of the exponential mapping evaluated at $\nabla v(x) \in \mathbf{T_x M}$.

Let $\mathbf{d}$ be the exterior differentiation on $\mathbf{M}$ in the following proof.

Proof. Set a geodesic $\zeta(t)$ satisfying $\zeta(0) = x$ and $\zeta'(0) = \xi$ and a family of geodesics with a parameter s is considered by

$$\delta_s(t) = \exp_{\zeta(t)} s\nabla v(\zeta(t)).$$

Here, it is noted that $\delta_0(t) = \zeta(t)$ and $\delta_1(t) = \exp_{\zeta(t)} \nabla v(\zeta(t))$. Now, some Jacobi fields are considered. First, a Jacobi field $J(s)$ along $\exp_x s\nabla v(x)$ is defined by

$$J(s) = \frac{\partial}{\partial t} |_{t=0} \ \delta_s(t).$$

Then, $J(s)$ satisfies

$$J(0) = \xi, \ J(1) = \mathbf{d}\left\{exp_x \nabla v(x)\right\} \cdot \xi \text{ and } \mathbf{D}_s J(0) = \mathbf{D}^2 v(x) \cdot \xi$$

since $\mathbf{D}_s \frac{\partial \delta}{\partial t} |_{s=0} = \mathbf{D}_t \frac{\partial \delta}{\partial s} |_{s=0} = \mathbf{D}_t \nabla v(\zeta(t)) = \mathbf{D}^2 v(\zeta(t)) \cdot \zeta'(t)$, where $\mathbf{d}$ is the exterior differentiation on $\mathbf{M}$. Next, an another Jacobi field $J_\xi(s)$ along $\exp_x s\nabla v(x)$ is set, satisfying

$$J_\xi(0) = \xi \text{ and } J_\xi(1) = 0,$$

and an another Jacobi field $\widetilde{J}_\xi$ also along $\exp_x s\nabla v(x)$ by

$$\widetilde{J}_\xi = J - J_\xi,$$

then $\widetilde{J}_\xi$ naturally satisfies the following:

$$\widetilde{J}_\xi(0) = 0 \text{ and } \mathbf{D}_s \widetilde{J}_\xi(0) = \mathbf{D}^2 v(x) \cdot \xi - \mathbf{D}_s J_\xi(0),$$

and also

$$\mathbf{d}\left\{\exp_x \nabla v(x)\right\} \cdot \xi = J(1) = \widetilde{J}_\xi(1) = \mathbf{d} \exp_x |_{\nabla v(x)} \cdot \mathbf{D}_s \widetilde{J}_\xi(0).$$

Here, consider an another family of geodesics $\chi_s(t) = \exp_{\zeta(t)} s \exp^{-1}_{\zeta(t)} y$ satisfying

$$\chi_0(t) = \zeta(t) \text{ and } \chi_1(t) = y.$$

Then, it is obtained that

$$\frac{\partial}{\partial t} |_{t=0} \ \chi_s(t) = J_\xi(s),$$

and

$$\begin{aligned} -\mathbf{D}_s J_\xi(0) &= -\mathbf{D}_s \frac{\partial}{\partial t} |_{t=0} \ \chi \ |_{s=0} = \mathbf{D}_t \left\{ -\exp^{-1}_{\zeta(t)} y \right\} |_{t=0} \\ &= \mathbf{D}_t \nabla (\frac{d_y^2}{2})(\zeta(t)) \ |_{t=0} = \mathbf{D}^2 \frac{d_y^2}{2}(x) \cdot \xi. \end{aligned}$$

Thus, it follows that

$$\mathbf{D}_s\widetilde{J}_\xi(0) = \mathbf{D}^2(v + \frac{d_y^2}{2})(x) \cdot \xi$$

□

Note. For Lemma 2.2, any curvature condition is not necessary, but $\nabla v(x) \in G_x$ should be checked to use it.

The doubling-property of measure under a geometric condition–nonnegative Ricci curvature–will be used to get a covering lemma(see Lemma 4.2).

Lemma 2.3. *(Gromov) Let* $\mathbf{M}$ *be an* n*-dimensional Riemannian manifold with nonnegative Ricci curvature. For any balls in* $\mathbf{M}$, $\mathbf{M}$ *satisfies the volume doubling property:*

$$|B_{2R}(x)| \leq 2^n |B_R(x)|.$$

Proof. Since the proof of this lemma is purely geometric, Chavel [13] is referred to. □

Chapter 3

Critical Density Lemma

The following theorem is called Critical Density Lemma since it gives an important result that a sufficient measure–Critical Density–of a cuf-off set by a constant in a ball implies the lower-boundness of the function in a larger ball.

Theorem 3.1. *(modified Critical Density) Let u be a nonnegative smooth function in a ball $B_R(x_0)$ and satisfy $\mathcal{L}u \leqslant 0$ in the same ball. Then, u has the following property:*

$$\frac{\left|\{u \geq 1\} \cap B_{\frac{R}{8}}(x_0)\right|}{\left|B_{\frac{R}{8}}(x_0)\right|} \geqslant \xi_1 \qquad \Longrightarrow \qquad \inf_{B_{\frac{R}{4}}(x_0)} u \geqslant \beta_1,$$

$$\textit{where} \quad \xi_1 = 1 - \{\frac{n\lambda}{\frac{1}{2}\left(C_{\frac{1}{2}} + a_{\mathcal{L}} + \Lambda\right)}\}^n \quad \textit{and} \quad \beta_1 = \frac{1}{\frac{57}{32} + C_{\frac{1}{2}}}.$$

Note. For convenience, every universal constant which depends only on dimension n and ellipticity constants λ, Λ is collected in Appendix A.

Remark. 1. $C_{\frac{1}{2}}$ is to be given as C_δ with $\delta = \frac{1}{2}$ in the proof of Theorem 3.1, Step 1, pp20-22, and $a_{\mathcal{L}} = (n-1)\Lambda$ is given in Lemma 2.2.
2. It suffices to check that

$$C_{\frac{1}{2}} \geqslant n\lambda$$

to guarantee that $\xi_1 > 0$, and it is trivial from the definition of C_δ.
3. Theorem 3.1 can be restated as the following:

$$\text{For any } 0 < \beta_1 \leqslant \frac{1}{\frac{57}{32} + C_{\frac{1}{2}}} < 1, \text{ there exists a constant } \xi_1 = \xi_1(n, \lambda, \Lambda)$$

$$\text{such that} \qquad \inf_{B_{\frac{R}{4}}(x_0)} u \leqslant \beta_1 \qquad \Longrightarrow \qquad \frac{\left|\{u \geq 1\} \cap B_{\frac{R}{8}}(x_0)\right|}{\left|B_{\frac{R}{8}}(x_0)\right|} \leqslant \xi_1.$$

The proof of the theorem is given in the end of this section after some lemmas which are necessary to prove it. The following lemma is also called Critical Density Lemma–only the radius of ball where the infimum is taken is different–, essential for Theorem 3.1, and the proof of it is due to Cabré [1] and S. Kim [2] on Riemannian manifolds. Here, a detailed proof is included for clarity. In $\mathbb{R}^n$, Safonov [5] proved a similar result named Growth Lemma using ABP-estimate, and the lemma has a larger extent of constant β_1, that is, $\beta_1 \in (0,1)$. This benefit comes from the application of the ABP-estimate on $\mathbb{R}^n$.

Lemma 3.2. *(Critical Density) Let u be a nonnegative smooth function in a ball $B_{4R}(x_0)$ and $\inf_{B_R(x_0)} u \leqslant 1$. Then, the following inequality is satisfied:*

$$\left|B_{\frac{R}{4}}\right| \leqslant \frac{1}{(n\lambda)^n} \int_{\{u \leqslant \frac{57}{32}\} \cap B_{3R}(x_0)} \{(R^2 \mathcal{L}u + a_{\mathcal{L}} + \Lambda)^+\}^n \, d\mathbf{V},$$

where $f^+(x) = max\{f(x), 0\}$ and $a_{\mathcal{L}} = (n-1)\Lambda$.

Proof. From now on, B_R will be used in every chapter to notify $B_R(x_0)$ for simplicity if there is no risk of confusion. First, a point y in $B_{\frac{R}{4}}$ is arbitrarily chosen, and a continuous function w_y is defined by

$$w_y(x) = R^2 u(x) + \frac{1}{2} d_y^2(x), \quad .$$

Step 1.

A minimum of w_y in B_{4R} is achived at a point z_0 in B_{3R}.

It is noted that

$$\inf_{B_R} w_y \leqslant R^2 + \frac{\left(R + \frac{R}{4}\right)^2}{2} = R^2 + \frac{25R^2}{32} = \frac{57R^2}{32} < 2R^2,$$

and, since $u \geqslant 0$ in B_{4R}, it is obtained that

$$w_y(x) \geqslant \frac{(3R - R)^2}{2} = 2R^2 \quad \text{in } B_{4R} \backslash B_{3R}.$$

Thus, it is concluded that the minimum of $w_y(x)$ in $\overline{B}_{3R}$ is achieved at a point of B_{3R}, and that is also the minimum of $w_y(x)$ in B_{4R}. That is,

$$\inf_{B_{4R}} w_y = \inf_{B_{3R}} w_y = w_y(z_0) \quad \text{for some } z_0 \text{ in } B_{3R}.$$

So, the claim of **Step 1** is proved.

Here, it is noted that for any y in $B_{\frac{R}{4}}$ there exists such z_0, and a set A of such z_0's is defined by

$$A = \left\{ z \in B_{3R} \mid z \text{ is a minimum point of } w_y(x) \text{ in } B_{3R} \text{ for } y \in B_{\frac{R}{4}} \right\}.$$

Step 2. The two points on **M**, y and z_0, which were considered in **Step 1** have the following relation:

$$y = \exp_{z_0} \nabla \left(R^2 u \right) (z_0) .$$

From **Step 1**, it is easily noted that for $\forall z_0 \in A$,

$$w_y (z_0) \leqslant w_y (x) = R^2 u (x) + \frac{1}{2} d_y^2 (x) \qquad \forall x \in B_{4R}.$$

First, an arbitrary geodesic γ parametrized by the arc-length is considered, satisfying $\gamma(0) = z_0$. Then, for any t,

$$d_y (\gamma(t)) \leqslant t + d_y (z_0) , \text{ thus,}$$

$$w_y(z_0) \leqslant R^2 u(\gamma(t)) + \frac{1}{2} d_y^2(\gamma(t)) \leqslant R^2 u(\gamma(t)) + \frac{1}{2} \{t + d_y(z_0)\}^2 .$$

Here, it is noted that both inequalities become equality when $t = 0$, that is, if a function f is set by

$$f(t) = R^2 u(\gamma(t)) + \frac{1}{2} \{t + d_y(z_0)\}^2 - w_y(z_0),$$

then f has its minimum when $t = 0$. Thus, if f is differentiated with respect to t at $t = 0$,

$$\begin{aligned} 0 \leqslant f'(t) \mid_{t=0} &= g \left\langle \nabla(R^2 u)(\gamma(t)), \gamma'(t) \right\rangle \mid_{t=0} + \{t + d_y(z_0)\} \mid_{t=0} \\ &= g \left\langle \nabla(R^2 u)(z_0), \gamma'(0) \right\rangle + d_y(z_0). \\ &\Leftrightarrow g \left\langle \nabla(R^2 u)(z_0), -\gamma'(0) \right\rangle \leqslant d_y(z_0). \end{aligned}$$

Here, since the geodesic γ was arbitrarily chosen only to satisfy $\gamma(0) = z_0$ with unit-speed, (3) is satisfied for any unit vector $\upsilon \in M_{z_0}$ instead of $-\gamma'(0)$, that is,

$$g \left\langle \nabla(R^2 u)(z_0), \upsilon \right\rangle \leqslant d_y(z_0) \qquad \forall \upsilon \in M_{z_0} \text{ with } |\upsilon| = 1. \tag{3.1}$$

Next, an another minimal geodesic η parametrized by arc-length is considered; however, joining z_0 and y by the condition that $\eta(0) = z_0$ and $\eta(d_y(z_0)) = y$. Then, it is obtained that

$$d_y(z_0) = d_y(\eta(t)) + t \qquad \text{for } 0 \leqslant t \leqslant d_y(z_0), \text{ thus,}$$

$$w_y(z_0) \leqslant w_y(\eta(t)) = R^2 u(\eta(t)) + \frac{1}{2} d_y^2(\eta(t)) \quad = R^2 u(\eta(t)) + \frac{1}{2}\left\{d_y(z_0) - t\right\}^2.$$

Here, it is also noted that this inequality become equality when $t = 0$, that is, if a function h is defined by

$$h(t) = R^2 u(\eta(t)) + \frac{1}{2}\left\{d_y(z_0) - t\right\}^2 - w_y(z_0),$$

then h has its minimum when $t = 0$. Thus, by differentiation, it is computed that

$$\begin{aligned} 0 \leqslant h'(t)\mid_{t=0} &= g\left\langle \nabla(R^2 u)(\eta(t)), \eta'(t)\right\rangle\mid_{t=0} + \left\{d_y(z_0) - t\right\}\mid_{t=0} \\ &= g\left\langle \nabla(R^2 u)(z_0), \eta'(0)\right\rangle - d_y(z_0). \end{aligned}$$

$$\Leftrightarrow \; g\left\langle \nabla(R^2 u)(z_0), \eta'(0)\right\rangle \geqslant d_y(z_0). \tag{3.2}$$

Here, since $|\eta'(0)| = 1$, from (3.1) and (3.2) it is concluded that

$$\begin{aligned} &g\left\langle \nabla(R^2 u)(z_0), \eta'(0)\right\rangle = d_y(z_0) \\ &\Rightarrow \; \nabla(R^2 u)(z_0) = d_y(z_0)\eta'(0). \end{aligned}$$

Then by the definition of exponential mapping on Riemannian manifold

$$\Rightarrow \; \exp_{z_0} \nabla(R^2 u)(z_0) = \eta(d_y(z_0)) = y.$$

So, the claim of **Step 2** is proved.

If a smooth map $\psi : B_{3R} \longrightarrow \mathbf{M}$ is defined by $\psi(z) = \exp_z \nabla(R^2 u)(z)$, then we have proved that for any $y \in B_{\frac{R}{4}}$ there is at least one $z \in A$ such that $\psi(z) = y$. Thus, by virtue of the area formula it is obtained that

$$|B_{\frac{R}{4}}| \leqslant \int_A \text{Jac } \psi(z) \; d\mathbf{V}(z).$$

And, from **Step 1**, it is easily noted that

$$A \subset \left\{ u \leqslant \frac{57}{32} \right\} \cap B_{3R}.$$

Hence, for the proof of this lemma, it suffices to show

Step 3.

$$\text{Jac } \psi(z) \leqslant \frac{1}{(n\lambda)^n} \{(R^2 \mathcal{L}u(z) + a_{\mathcal{L}} + \Lambda)^+\}^n \text{ for any } z \in A.$$

Let $z_1 \in A$ and take $y_1 \in B_{\frac{R}{4}}$ such that $w_{y_1}(z_1) = \inf_{B_{3R}} w_{y_1}$, that is, $y_1 = \psi(z_1) = \exp_{z_1} \nabla \left(R^2 u\right)(z_1)$. Here, there are two different cases: 1) z_1 is not a cut point of y_1, or 2) z_1 is a cut point of y_1.

Case 1) This is the easier case. If z_1 is not a cut point of y_1–i.e. $\nabla(R^2 u)(z_1) \in G_{z_1}$–, then by Lemma 2.2 it is attained that

$$\text{Jac } \psi(z_1) \leqslant \left| \det \mathbf{D}^2 \left(R^2 u + \frac{d_{y_1}^2}{2} \right)(z_1) \right| = |\det \mathbf{D}^2 w_{y_1}(z_1)|.$$

Here, it is used that

$$\mathrm{Jac} \exp_x(\upsilon) \leqslant 1$$

for any $x \in \mathbf{M}$ and $\upsilon \in G_x$. Li [17] (see Bishop Comparison Theorem), S. Kim [2] can be referred to for details, and the sketch of it is the following:

1)Let $J(r, \psi)d\psi$ be the area element of the geodesic sphere $\partial B_r(x)$,

2)$J(r, \psi)d\psi = r^{n-1} A(r, \psi)d\psi$ where $A(r, \psi)$ is the Jacobian of the map, $\exp_x$ at $r\psi \in \mathbf{T_x M}$,

3)By the Laplace Comparison Theorem under nonnegative Ricci curvature condition (see Schoen[15] and Schoen and Yau[16]), it is obtained that $\Delta d_p(x) \leqslant \dfrac{n-1}{d_p(x)}$,

4)In Li[17], it is computed that $\dfrac{J'(r,\psi)}{J(r,\psi)} = \Delta r$, and by 3) $\Delta r \leqslant \dfrac{n-1}{r}$,

5)From 4) it is obtained that $A(r, \psi)$ is nondecreasing with respect to r,

6)Since $\lim_{r \to 0} A(r, \psi) = 1$, it is concluded that $A(r, \psi) \leqslant 1$.

Since w_{y_1} achieves its minimum at z_1, $\mathbf{D}^2 w_{y_1}(z_1) \geqslant 0$. Therefore, by using the well-known inequality

$$\det \mathbf{A} \cdot \det \mathbf{B} \leqslant \left\{ \frac{\operatorname{tr}(\mathbf{A} \circ \mathbf{B})}{n} \right\}^n \quad \text{where } \mathbf{A}, \mathbf{B} \text{ are symmetric and nonnegative,}$$

it is concluded that

$$\begin{aligned}
\text{Jac } \psi(z_1) &\leqslant \det \mathbf{D}^2 w_{y_1}(z_1) \\
&= \frac{1}{\det \mathbf{A}(z_1)} \det \mathbf{A}(z_1) \cdot \det \mathbf{D}^2 w_{y_1}(z_1) \\
&\leqslant \frac{1}{\lambda^n} \det \mathbf{A}(z_1) \cdot \det \mathbf{D}^2 w_{y_1}(z_1) \\
&\leqslant \frac{1}{(n\lambda)^n} \left[\operatorname{tr} \left\{ \mathbf{A}(z_1) \circ \mathbf{D}^2 w_{y_1}(z_1) \right\} \right]^n \\
&= \frac{1}{(n\lambda)^n} \{ \mathcal{L} w_{y_1}(z_1) \}^n \\
&= \frac{1}{(n\lambda)^n} \{ R^2 \mathcal{L} u(z_1) + \mathcal{L}(\frac{d_{y_1}^2}{2})(z_1) \}^n \\
&\leqslant \frac{1}{(n\lambda)^n} \{ (R^2 \mathcal{L} u(z_1) + a_{\mathcal{L}} + \Lambda)^+ \}^n,
\end{aligned}$$

where, in the last step, by Lemma 2.1 it is computed that

$$\mathcal{L}(\frac{d_{y_1}^2}{2}) = d_{y_1} \mathcal{L} d_{y_1} + \langle \mathbf{A} \nabla d_{y_1}, \nabla d_{y_1} \rangle \leqslant a_{\mathcal{L}} + \Lambda |\nabla d_{y_1}|^2,$$

where

$$\nabla d_{y_1}(z) = - \frac{\exp_z^{-1} y_1}{\left| \exp_z^{-1} y_1 \right|} \quad \text{if } z \neq y_1.$$

Thus, the claim of **Step 3** for the case of when z_1 is not a cut point of y_1 is proved.

Case 2) When z_1 is a cut point of y_1, we can reduce this kind of critical situation to the previous non-critical case by using upper barrier technique due to Calabi [14] as the followings:

(**Upper Barrier Technique**) Since $y_1 = \exp_{z_1} \nabla(R^2 u)(z_1)$, z_1 is not a cut point of $y_s = \psi_s(z_1) := \exp_{z_1} \nabla(s R^2 u)(z_1)$ for $0 \leqslant \forall s < 1$. By continuity, Jac $\psi(z_1) = \lim_{s \to 1}$ Jac $\psi_s(z_1)$. As before,

$$\text{Jac } \psi_s(z_1) \leqslant \left| \det \mathbf{D}^2 \left(s R^2 u + \frac{d_{y_s}^2}{2} \right) (z_1) \right|.$$

Since

$$\liminf_{s\to 1}\left|\det \mathbf{D}^2\left(sR^2u+\frac{d_{y_s}^2}{2}\right)(z_1)\right| = \liminf_{s\to 1}\left|\det \mathbf{D}^2\left(R^2u+\frac{d_{y_s}^2}{2}\right)(z_1)\right|$$
$$= \liminf_{s\to 1}|\det \mathbf{D}^2 w_{y_s}(z_1)|,$$

it only remains to prove that

$$\liminf_{s\to 1}|\det \mathbf{D}^2 w_{y_s}(z_1)| \leqslant \frac{1}{(n\lambda)^n}\{(R^2\mathcal{L}u+a_{\mathcal{L}}+\Lambda)^+\}^n.$$

Here, since it cannot be guaranteed that $\mathbf{D}^2 w_{y_s}(z_1)$ is nonnegative, the above inequality between determinant and trace to $\mathbf{D}^2 w_{y_s}(z_1)$ is not directly applied. By the way, since

$$d_{y_s}(y_1)\to 0 \quad \text{as } s\to 1,$$

by passing to the limit as $s\to 1$, the inequality can be applied to

$$\mathbf{D}^2 w_{y_s}(z_1)+Nd_{y_s}(y_1)\mathbf{Id} \quad \text{for some } N$$

instead of $\mathbf{D}^2 w_{y_s}(z_1)$, only if it is possible to check that

$$\mathbf{D}^2 w_{y_s}(z_1)+Nd_{y_s}(y_1)\mathbf{Id} \quad \text{is nonnegative.}$$

Here, let $-k^2(k>0)$ be a lower bound of sectional curvature along the minimal geodesic connecting z_1 and y_1. Then, the nonnegative definiteness of $\mathbf{D}^2 w_{y_s}(z_1)+Nd_{y_s}(y_1)\mathbf{Id}$ is clear if it is noted that Hessian comparison theorem (see Schoen [15], and Schoen and Yau [16]) states that

$$\mathbf{D}^2 d_{y_s}(z_1) \leqslant k\ \coth(k\ d_{y_s}(z_1))\ \mathbf{Id} \leqslant N\ \mathbf{Id}$$

uniformly in $s\in(\frac{1}{2},1)$ for some N, and an auxilary function is considered by

$$R^2u(z)+\frac{1}{2}\{d_{y_s}(z)+d_{y_s}(y_1)\}^2 = w_{y_s}(z)+d_{y_s}(y_1)d_{y_s}(z)+\frac{1}{2}d_{y_s}(y_1)^2$$

which is smooth near z_1 and has a local minimum at z_1, so that its Hessian at z_1 is nonnegative. Thus, the following is obtained

$$0\leqslant \mathbf{D}^2 w_{y_s}(z_1)+d_{y_s}(y_1)\mathbf{D}^2 d_{y_s}(z_1)\leqslant \mathbf{D}^2 w_{y_s}(z_1)+Nd_{y_s}(y_1)\mathbf{Id}.$$

Now the previously mentioned relation between determinant and trace is applied to this nonnegative definite endomorphism for $s \in (\frac{1}{2}, 1)$, and it is obtained that

$$\begin{aligned}
0 &\leqslant \liminf_{s\to 1} |\det \mathbf{D}^2 w_{y_s}(z_1)| \\
&= \liminf_{s\to 1} \left|\det \left\{\mathbf{D}^2 w_{y_s}(z_1) + N d_{y_s}(y_1)\mathbf{Id}\right\}\right| \\
&\leqslant \frac{1}{\lambda^n} \liminf_{s\to 1} \det \mathbf{A}(z_1) \cdot \det \left\{\mathbf{D}^2 w_{y_s}(z_1) + N d_{y_s}(y_1)\mathbf{Id}\right\} \\
&\leqslant \frac{1}{\lambda^n} \liminf_{s\to 1} \left[\frac{\mathrm{tr}\ \left\{\mathbf{A}(z_1) \circ \left(\mathbf{D}^2 w_{y_s}(z_1) + N d_{y_s}(y_1)\mathbf{Id}\right)\right\}}{n}\right]^n \\
&= \frac{1}{(n\lambda)^n} \liminf_{s\to 1} \left[\mathrm{tr}\ \{\mathbf{A}(z_1) \circ \mathbf{D}^2 w_{y_s}(z_1)\}\right]^n \text{ since } d_{y_s}(y_1) \longrightarrow 0 \text{ as } s \longrightarrow 0 \\
&= \frac{1}{(n\lambda)^n} \liminf_{s\to 1} \{\mathcal{L} w_{y_s}(z_1)\}^n \\
&= \frac{1}{(n\lambda)^n} \liminf_{s\to 1} \left\{R^2 \mathcal{L} u(z_1) + \mathcal{L}\left(\frac{d_{y_s}^2}{2}\right)(z_1)\right\}^n,
\end{aligned}$$

and by the same computation with the last line of **Case 1**)

$$\leqslant \frac{1}{(n\lambda)^n} \{(R^2 \mathcal{L} u(z_1) + a_{\mathcal{L}} + \Lambda)^+\}^n.$$

This finishes the proof of the theorem. □

Remark. 1. It is noted that any ball $B_{3R+\epsilon}(x_0)$ for any $\epsilon > 0$ instead of $B_{4R}(x_0)$ might be used since it makes no change in the proof; just the fact that a little bit bigger than $B_{3R}(x_0)$ is sufficient.
2. The condition that $u \geqslant 0$ in $B_{4R} \backslash B_{3R}$ is enough for the proof instead of the condition that $u \geqslant 0$ in B_{4R}. 3. Under an additional condition of $\mathcal{L}u \leqslant 0$ a different statement of Lemma 3.2, which looks similar to the result of Theorem 3.1, can be obtained:

If u is nonnegative, smooth in B_{4R} and $\inf_{B_R} u \leqslant 1$, then

$$\left(\frac{\frac{1}{4}}{3}\right)^n |B_{3R}| = \left|B_{\frac{R}{4}}\right| \leqslant \frac{1}{(n\lambda)^n} \int_{\{u \leqslant \frac{57}{32}\} \cap B_{3R}} \{(R^2 \mathcal{L} u + a_{\mathcal{L}} + \Lambda)^+\}^n \, d\mathbf{V},$$

and also if $\mathcal{L}u \leqslant 0$ in B_{3R}, which is a condition of Theorem 3.1

$$\leqslant \frac{1}{(n\lambda)^n} (a_{\mathcal{L}} + \Lambda)^n \left|\left\{u \leqslant \frac{57}{32}\right\} \cap B_{3R}\right|.$$

Thus, under the additional condition that $\mathcal{L}u \geqslant 0$ in B_{3R} we get the following inequality similar to Theorem 3.1:

$$\begin{aligned}
&\Longleftrightarrow \quad \frac{\left|\{u \leqslant \frac{57}{32}\} \cap B_{3R}\right|}{|B_{3R}|} \geqslant \left\{\frac{n\lambda}{12(a_{\mathcal{L}} + \Lambda)}\right\}^n \\
&\Longleftrightarrow \quad \frac{\left|\{u \geqslant \frac{57}{32}\} \cap B_{3R}\right|}{|B_{3R}|} \leqslant 1 - \left\{\frac{n\lambda}{12(a_{\mathcal{L}} + \Lambda)}\right\}^n.
\end{aligned}$$

Since $\mathcal{L}$ is linear and constants are independent of the radius of ball R, this implies that

Corollary 3.3. *Let u be a nonnegative smooth function in a ball B_R and satisfy $\mathcal{L}u \leqslant 0$ in the same ball. Then, u has the following property:*

$$\frac{\left|\{u \geq 1\} \cap B_{\frac{R}{2}}\right|}{\left|B_{\frac{R}{2}}\right|} \geqslant \xi_0 \qquad \Longrightarrow \qquad \inf_{B_{\frac{R}{6}}} u \geqslant \beta_0,$$

$$\textit{where} \qquad \xi_0 = 1 - \{\frac{n\lambda}{12\,(a_{\mathcal{L}} + \Lambda)}\}^n \ \textit{and}\ \beta_0 = \frac{32}{57}.$$

Remark. 1. It is noted that any ball $B_{\frac{R}{2}+\epsilon}$ for any $\epsilon > 0$ may be used instead of B_R.
2. It is mentioned that the main difference between Theorem 3.1 and Corollary 3.3 is the following:

1) Theorem 3.1: the radius of the ball where infima are taken is bigger than the radius of the ball where measure is computed.
2) Corollary 3.3: the radius of the ball where infima are taken is smaller than the radius of the ball where measure is computed.

Thus, to prove Theorem 3.1 it suffices to find a proper auxiliary function v, so that v might reduce the radius of the ball on which the measure is computed, that is, where the integration is done. Then, the situation of Corollary 3.3 might change into that of Theorem 3.1. Here, we can consider an application of Lemma 3.2 to a modified function $u+v$ instead of u, where v roughly satisfies the following:

$$R^2\mathcal{L}v + a_{\mathcal{L}} + \Lambda \leqslant 0 \text{ a.e. in } B_{3R} \backslash B_{\delta R},$$

where $B_{\delta R}$ is a smaller ball than B_R on which the infimum was taken. Then, it is obtained that

$$\frac{1}{(n\lambda)^n}\int_{\{u+v\leqslant\frac{57}{32}\}\cap B_{3R}(x_0)}\{(R^2\mathcal{L}(u+v)+a_{\mathcal{L}}+\Lambda)^+\}^n\ d\mathbf{V},$$

when $\mathcal{L}u\leqslant 0$ in B_{4R}

$$\begin{aligned}
&=\frac{1}{(n\lambda)^n}\int_{\{u+v\leqslant\frac{57}{32}\}\cap B_{3R}(x_0)}\{(R^2\mathcal{L}v+a_{\mathcal{L}}+\Lambda)^+\}^n\ d\mathbf{V},\\
&=\frac{1}{(n\lambda)^n}\int_{\{u+v\leqslant\frac{57}{32}\}\cap B_{\delta R}(x_0)}\{(R^2\mathcal{L}v+a_{\mathcal{L}}+\Lambda)^+\}^n\ d\mathbf{V}.
\end{aligned}$$

It is also noted that in order to apply Lemma 3.2 to $u+v$, it would be better for v to basically satisfy the following:

1) For $u+v\geqslant 0$ in $B_{4R}\backslash B_{3R}$, $\quad v\geqslant 0$ in $B_{4R}\backslash B_{3R}$ when $u\geqslant 0$ in $B_{4R}\backslash B_{3R}$,
2) For $\inf\limits_{B_R}(u+v)\leqslant 1$ in B_R, $\quad v\leqslant 0$ in B_R when $\inf\limits_{B_R} u\leqslant 1$ in B_R.

Now the above strategy is implemented in detail.

Proof. **of Theorem 3.1**.
Step 1. For $0<\delta<1$, there is a continuous function v_δ in B_{4R} satisfying the following properties:

$$\begin{aligned}
&1)\ v_\delta \text{ is smooth in } B_{4R}\backslash Cut(x_0),\\
&2)\ R^2\mathcal{L}v+a_{\mathcal{L}}+\Lambda\leqslant 0 \text{ a.e. in } B_{3R}\backslash B_{\delta R},\\
&3)\ v\geqslant 0 \text{ in } B_{4R}\backslash B_{3R},\\
&4)\ v\leqslant 0 \text{ in } B_R,\\
&5)\ R^2\mathcal{L}v\leqslant C_\delta \text{ a.e. in } B_{3R}, \text{ and}\\
&6)\ -v_\delta\leqslant C_\delta \text{ in } B_{4R},
\end{aligned}$$

where C_δ depends only on δ.

A function v_δ is defined by

$$v_\delta(x)=I_\delta(\frac{d_{x_0}(x)}{R})$$

where I_δ is a smooth increasing function on $\mathbb{R}^+$ satisfying that

$$1)\ I'_\delta(0) = 0, \tag{3.3}$$

$$2)\ I_\delta(r) = (\frac{3}{2})^\beta - (\frac{3}{r})^\beta \text{ for } r \geqslant \delta. \tag{3.4}$$

with β to be chosen later.
It is trivial that v_δ is continuous on B_{4R}. Also, v_δ is smooth in $B_{4R}\backslash Cut(x_0)$ since d_{x_0} is smooth in $\mathbf{M}\backslash(Cut(x_0)\cup\{x_0\})$ and $I'_\delta(0) = 0$. Thus, $-v_\delta$ should be bounded from above by some constant $C_{1,\delta}$. From (3.4), 3) and 4) of the above claim are also trivial. For the remained 2) and 5), we need to compute $\mathcal{L}v_\delta$ under the condition of the smoothness of v_δ in $B_{4R}\backslash Cut(x_0)$:

$$\begin{aligned}\mathcal{L}v_\delta &= \frac{1}{R}I'_\delta(\rho)\mathcal{L}d_{x_0} + \frac{1}{R^2}I''_\delta(\rho)g\left\langle \mathbf{A}\nabla d_{x_0}, \nabla d_{x_0}\right\rangle \\ &= \frac{1}{R^2}\frac{I'_\delta(\rho)}{\rho}d_{x_0}\mathcal{L}d_{x_0} + \frac{1}{R^2}I''_\delta(\rho)g\left\langle \mathbf{A}\nabla d_{x_0}, \nabla d_{x_0}\right\rangle,\end{aligned}$$

where for $\delta \leqslant \rho < 3$ it is noted that

$$\begin{aligned}&i)\ I'_\delta(\rho) = \frac{\beta}{3}(\frac{3}{\rho})^{\beta+1}, \\ &ii)\ I''_\delta(\rho) = -\frac{\beta(\beta+1)}{3^2}(\frac{3}{\rho})^{\beta+2}, \\ &iii)\ \lambda \leqslant g\left\langle \mathbf{A}\nabla d_{x_0}, \nabla d_{x_0}\right\rangle \leqslant \Lambda, \\ &iv)\ d_{x_0}\mathcal{L}d_{x_0} \leqslant a_{\mathcal{L}}.\end{aligned}$$

Thus, in $B_{3R}\backslash\{B_{\delta R}\cup Cut(x_0)\}$, it is also obtained that

$$\begin{aligned}\mathcal{L}v_\delta \leqslant& \frac{1}{R^2}\frac{I'_\delta(\rho)}{\rho}d_{x_0}\mathcal{L}d_{x_0} + \frac{1}{R^2}I''_\delta(\rho)g\left\langle \mathbf{A}\nabla d_{x_0}, \nabla d_{x_0}\right\rangle \\ \leqslant& \frac{1}{R^2}\frac{\beta}{3^2}(\frac{3}{\rho})^{\beta+2}a_{\mathcal{L}} - \frac{1}{R^2}\frac{\beta(\beta+1)}{3^2}(\frac{3}{\rho})^{\beta+2}\lambda \\ \leqslant& \frac{1}{R^2}\frac{\beta}{3^2}(\frac{3}{\rho})^{\beta+2}\left(a_{\mathcal{L}} - (\beta+1)\lambda\right) \\ \leqslant& \frac{1}{R^2}\frac{\beta}{9}(\frac{3}{\delta})^{\beta+2}\left(a_{\mathcal{L}} - (\beta+1)\lambda\right).\end{aligned}$$

Here, since $\frac{3}{\delta} > 1$ and $a_{\mathcal{L}} - (\beta+1)\lambda < 0$ for a sufficient large β, the last term can be made smaller than $\frac{-(a_{\mathcal{L}}+\Lambda)}{R^2}$ by choosing a large β. Thus, 2) of the claim is proven. Moreover, in $B_{4R}\backslash Cut(x_0)$, basically it is computed that

$$\mathcal{L}v_\delta \leqslant \frac{a_{\mathcal{L}}}{R^2}\sup_{0<\rho<4}\left|\frac{I'_\delta(\rho)}{\rho}\right| + \sup_{0<\rho<4}\frac{\Lambda}{R^2}\left|I''_\delta(\rho)\right| = C_{2,\delta} < \infty.$$

Thus, 5) of the claim is obtained with $C_\delta = \max\{C_{1,\delta}, C_{2,\delta}, n\lambda\}$ where $n\lambda$ is taken to guarantee that $C_\delta \geqslant n\lambda$ in Remark 2 of Theorem 3.1, and the proof of the claims of **Step 1** is finished.

Since the above v_δ is guaranteed to be smooth only in $B_{4R} \backslash Cut(x_0)$, but might not be smooth in B_{4R} itself, an approximating process by smooth functions to v_δ in B_{4R} is necessary in order to apply Lemma 3.2.

Step 2. There exist a smooth bump function ξ such that

$$\begin{aligned} &i)\ 0 \leqslant \xi \leqslant 1 \text{ in } \mathbf{M},\\ &ii)\ \xi \equiv 1 \text{ in } B_{3R},\\ &iii)\ supp(\xi) \subset B_{4R}, \end{aligned}$$

and a sequence of smooth functions $\{w_k\}$ in $\mathbf{M}$ satisfying the followings:

$$\begin{aligned} &i)\ w_k \longrightarrow \xi v_\delta \quad \text{uniformly in } \mathbf{M},\\ &ii)\ \mathbf{D}^2 w_k \longrightarrow \mathbf{D}^2 v_\delta \quad \text{a.e. in } B_{3R},\\ &iii)\ \mathbf{D}^2 w_k \leqslant C\ Id \quad \text{in } \mathbf{M} \text{ for some } C \text{ independent of } k. \end{aligned}$$

In regard to the proof of this step, Cabré [1] is referred to for simplicity since its arguments just consist of applications of an approximation of the identity and partition of unity.

Step 3.

If Lemma 3.2 is applied to $u + w_k$, then Theorem 3.1 is proven.

First, if $u+w_k$ is approximated by $\frac{u+w_k+\epsilon_k}{1+2\epsilon_k}$ with a sequence $\{\epsilon_k\}$ converging to 0, then it might assumed that $u+w_k$ satisfies the hypotheses of Theorem 3.1:

1) $u + w_k$ be a nonnegative smooth function in a ball $B_{4R}(x_0)$,
2) $\inf_{B_R(x_0)} u + w_k \leqslant 1$.

This is because a proper sequence $\{\epsilon_k\} \longrightarrow 0$ can be chosen by the following steps:

1) For $x \in B_R,$ that is, $w_k(x) \longrightarrow v_\delta(x) \leqslant 0,$
a sufficiently large k_1 is picked such that $w_k(x) \geqslant 0$ for all $k \leqslant k_1$,
and a sequence is set by $\epsilon_{1,k} = \sup_{B_R} w_k \geqslant w_k(x)$ for $k \leqslant k_1$
and $\epsilon_{1,k} = 0$ for $k \geqslant k_1$

$$\Longrightarrow \quad \inf_{B_R} \frac{u + w_k + \epsilon_{1,k}}{1 + 2\epsilon_{1,k}} \leqslant \frac{1 + 2\epsilon_{1,k}}{1 + 2\epsilon_{1,k}} = 1,$$

2) For $x \in B_{4R} \backslash B_{3R},$ that is, $w_k(x) \longrightarrow v_\delta(x) \geqslant 0,$
a sufficiently large k_2 is chosen such that $w_k(x) \leqslant 0$ for all $k \leqslant k_2$,
and a sequence is defined by $\epsilon_{2,k} = \sup_{B_{4R} \backslash B_{3R}} (-w_k) \geqslant -w_k(x)$ for $k \leqslant k_1$
and $\epsilon_{2,k} = 0$ for $k \geqslant k_2$

$$\Longrightarrow \quad \frac{u + w_k + \epsilon_{2,k}}{1 + 2\epsilon_{2,k}} \geqslant 0 \text{ in } B_{4R} \backslash B_{3R},$$

3) A sequence is set by $\epsilon_k = \max\left\{\epsilon_{1,k}, \epsilon_{2,k}\right\}.$

It is also noted that

1) For any $\epsilon > 0,$ there is a sufficiently large k such that
$$\left\{u + w_k \leqslant \frac{57}{32}\right\} \cap B_{3R} \ \subset \ \left\{u + v_\delta \leqslant \frac{57}{32} + \epsilon\right\} \cap B_{3R},$$
2) $\left\{\left(R^2 \mathcal{L}(u + w_k) + a_{\mathcal{L}} + \Lambda\right)^+\right\}^n$ is uniformly bounded in $\mathbf{M}$
since $\mathbf{D}^2 w_k \leqslant C \ Id.$

Thus, by applying the Lebesgue dominated convergence theorem as $k \longrightarrow \infty$ and Lemma 3.2, and with the conditions of v_δ in Step 1 and $\mathcal{L}u \leqslant 0$, the following inequalities are computed:

$$\begin{aligned}\left|B_{\frac{R}{4}}\right| &\leqslant \frac{1}{(n\lambda)^n}\int_{\{u+v_\delta\leqslant\frac{57}{32}\}\cap B_{3R}(x_0)} \{(R^2\mathcal{L}(u+v_\delta)+a_{\mathcal{L}}+\Lambda)^+\}^n\; d\mathbf{V}\\ &= \frac{1}{(n\lambda)^n}\int_{\{u+v_\delta\leqslant\frac{57}{32}\}\cap B_{3R}(x_0)} \{(R^2\mathcal{L}(v_\delta)+a_{\mathcal{L}}+\Lambda)^+\}^n\; d\mathbf{V}\\ &= \frac{1}{(n\lambda)^n}\int_{\{u+v_\delta\leqslant\frac{57}{32}\}\cap B_{\delta R}(x_0)} \{(R^2\mathcal{L}(v_\delta)+a_{\mathcal{L}}+\Lambda)^+\}^n\; d\mathbf{V}\\ &\leqslant \frac{1}{(n\lambda)^n}\int_{\{u+v_\delta\leqslant\frac{57}{32}\}\cap B_{\delta R}(x_0)} \{(C_\delta+a_{\mathcal{L}}+\Lambda)^+\}^n\; d\mathbf{V}\\ &\leqslant \left(\frac{C_\delta+a_{\mathcal{L}}+\Lambda}{n\lambda}\right)^n\left|\left\{u+v_\delta\leqslant\frac{57}{32}\right\}\cap B_{\delta R}(x_0)\right|\\ &\leqslant \left(\frac{C_\delta+a_{\mathcal{L}}+\Lambda}{n\lambda}\right)^n\left|\left\{u\leqslant\frac{57}{32}+C_\delta\right\}\cap B_{\delta R}(x_0)\right|\end{aligned}$$

Here, the inequality can be expressed like below:

$$\begin{aligned}&\Longleftrightarrow \quad \frac{\left|\left\{u\leqslant\frac{57}{32}+C_\delta\right\}\cap B_{\delta R}\right|}{|B_{\delta R}|} \geqslant \left\{\frac{n\lambda}{4\delta(C_\delta+a_{\mathcal{L}}+\Lambda)}\right\}^n\\ &\Longleftrightarrow \quad \frac{\left|\left\{u\geqslant\frac{57}{32}+C_\delta\right\}\cap B_{\delta R}\right|}{|B_{\delta R}|} \leqslant 1-\left\{\frac{n\lambda}{4\delta(C_\delta+a_{\mathcal{L}}+\Lambda)}\right\}^n.\end{aligned}$$

Thus, since $\mathcal{L}$ is linear and all the constants are independent of the radius of ball R, by letting $\delta=\frac{1}{2}$ the proof of (Theorem 3.1) is finished:

If u is a nonnegative smooth function in a ball $B_R(x_0)$ and satisfy $\mathcal{L}u\leqslant 0$ in the same ball. Then, u has the following property:

$$\frac{\left|\{u\geq 1\}\cap B_{\frac{R}{8}}(x_0)\right|}{\left|B_{\frac{R}{8}}(x_0)\right|}\geqslant \xi_1 \qquad \Longrightarrow \qquad \inf_{B_{\frac{R}{4}}(x_0)} u\geqslant\beta_1,$$

$$\text{where}\quad \xi_1 = 1-\{\frac{n\lambda}{\frac{1}{2}\left(C_{\frac{1}{2}}+a_{\mathcal{L}}+\Lambda\right)}\}^n \quad\text{and}\quad \beta_1=\frac{1}{\frac{57}{32}+C_{\frac{1}{2}}}.$$

□

Chapter 4

Power Decay of the Distribution Functions of Solutions

This is a chapter to get a power-decay property of distribution functions of solutions of nondivergent uniformly elliptic partial differential operators.

Theorem 4.1. *(Power-decay) Let B_R be a ball in $\mathbf{M}$ and $\mathcal{L}$ be as defined in Section 2.2. Let u be a nonnegative smooth function in a ball B_R satisfying $\mathcal{L}u \leqslant 0$ in the same ball. Then, for any nonnegative integer k, u has the following property :*

$$\frac{\left|\{u \geq M^k\} \cap B_{\frac{R}{8}}\right|}{\left|B_{\frac{R}{8}}\right|} \geqslant \xi^k \qquad \Longrightarrow \qquad \inf_{B_{\frac{R}{8}}} u \geqslant 1,$$

$$\textit{where} \quad M = \frac{1}{\beta_1} = \frac{57}{32} + C_{\frac{1}{2}} \quad \textit{and} \quad \xi = \frac{1}{1 + \frac{1-\xi_1}{5^n}}.$$

The proof of this theorem is given in the end of this chapter, and Appendix A can be referred to for any universal constants.

Now a covering lemma on Riemannian manifold is proven, and it will be used to prove Power decay of the distribution functions of nonnegative supersolutions of $\mathcal{L}$.

Lemma 4.2. *(Covering) Let $\mathbf{M}$ be a Riemannian manifold, A is an measurable set on $\mathbf{M}$ with respect to the Riemannian measure on $\mathbf{M}$, and $B_0 = B_R(x_0)$ is a ball containing A on $\mathbf{M}$. A positive number δ is chosen so that*

$$|A| < \delta |B_0| \quad \textit{where } |\cdot| \textit{ denotes the Riemannian measure on } \mathbf{M}.$$

A family of balls, $\mathscr{S}$ is set by

$$\mathscr{S} = \left\{ B \ \mid \ B \subset B_0 \ \textit{and} \ \frac{|B \cap A|}{|B|} \geqslant \delta \right\} \ \textit{where } B \textit{ are balls on } M.$$

Also, an another set Γ *is defined by* $\Gamma = \cup_{B \in \mathscr{S}} B$. *Then,* $A \subset \Gamma$ *a.e.* x *and* $|\Gamma| \geq (1 + c_1)|A|$ *where* $c_1 = c_1(\delta, n) = \frac{1-\delta}{5^n}$.

Proof. The proof of Safonov [5] in $\mathbb{R}^n$ is followed with languages of Riemannian geometry. A function ϕ_A is defined by

$$\phi_A(x, r) = \frac{|B_r(x) \cap A|}{|B_r(x)|}.$$

Step 1.

$$A \subset \Gamma \text{ a.e. } x.$$

First, it is mentioned that a similar covering lemma in metric spaces equipped with a measure satisfying a doubling property, which implies the Lebesgue differentiation theorem in those spaces, was proved in Coifman and Weiss [8]. Then, we can apply the Lebesgue differentiation theorem on Riemmanian manifolds with nonnegative Ricci curvature, which is a metric space equipped with the doubling property of measure, to the charateristic function of A, χ_A. If $x \notin \Gamma$, then we can find a sequence of small balls S_i which contain x, shrink to one point, and satisfy

$$\frac{|S_i \cap A|}{|S_i|} < \delta < 1.$$

Thus, by the Lebesque differentiation theorem on Riemannian manifolds applied to χ_A,

$$\chi_A(x) = \lim_{diam(S_i) \to 0} \frac{1}{|S_i|} \int_{S_i} \chi_A,$$

and also

$$= \lim_{diam(S_i) \to 0} \frac{|S_i \cap A|}{|S_i|} < \delta.$$

for a.e. $x \notin \Gamma$. Thus, for a.e. $x \notin \Gamma$, $x \notin A$, that is $A \subset \Gamma$ a.e.x.

For every $B_r(x) \in \mathscr{S}$, since $|B_r(x) \cap A| \geq \delta |B_r(x)|$ and $|A| < \delta |B_0|$, by the continuity of $\phi_A(x, r)$ as a function of x and r, a ball $B_{\tilde{r}}(\tilde{x})$ can be chosen such that $\phi_A(\tilde{x}, \tilde{r}) = \delta$. In detail, this can be done by finding an interpolation ball between $B_0 = B_R(x_0)$ and $B_r(x)$. The interpolation ball B_t with a parameter t is defined by

$$B_t = B_{tr+(1-t)R}(\gamma(t))$$

where $\gamma(t)$ is the minimal geodesic conneting x_0 and x, and satisfying $\gamma(0) = x_0$ and $\gamma(1) = x$. Here, it is noted that $B_0 = B_R(x_0)$, $B_1 = B_r(x)$, and

Step 2.

$$B_1 \subset B_t \subset B_0 \text{ for any } t \in [0, 1].$$

This is because

$$\begin{aligned}
\text{for } p \in B_1,\ \ d(p, \gamma(t)) &\leqslant d(p, x) + d(x, \gamma(t)) \\
&< r + d(x, \gamma(t)) \text{ since } p \in B_1 \\
&\leqslant r + (1-t)d(x, x_0) \text{ since } \gamma(t) \text{ is the minimal} \\
&\quad\ \text{geodesic satisfying } \gamma(0) = x_0 \text{ and } \gamma(1) = x \\
&< r + (1-t)(R-r) \text{ since } B_r(x) \subset B_R(x_0) \\
&= tr + (1-t)R \\
\Longrightarrow p \in B_t,
\end{aligned}$$

and

$$\begin{aligned}
\text{for } q \in B_t,\ \ d(q, x_0) &\leqslant d(q, \gamma(t)) + d(\gamma(t), x_0) \\
&< tr + (1-t)R + d(\gamma(t), x_0) \text{ since } q \in B_t \\
&\leqslant tr + (1-t)R + td(x, x_0) \text{ since } \gamma(t) \text{ is the minimal} \\
&\quad\ \text{geodesic satisfying } \gamma(0) = x_0 \text{ and } \gamma(1) = x \\
&< tr + (1-t)R + t(R-r) = R \\
\Longrightarrow q \in B_0.
\end{aligned}$$

Here, an another function ψ is set by

$$\psi(t) = \frac{|B_t \cap A|}{|B_t|}$$

with the condition that $\psi(0) < \delta$ and $\psi(1) \geqslant \delta$. Then, by the continuity of ψ, there exists a number $0 < t_0 \leq 1$ such that $\psi(t_0) = \delta$. Thus, the above $\tilde{x}$ and $\tilde{r}$ can be picked by

$$\tilde{x} = \gamma(t_0) \quad \text{and} \quad \tilde{r} = t_0 r + (1-t_0)R,$$

and a refinement of $\mathscr{S}$, $\widetilde{\mathscr{S}}$ can be considered by

$$\widetilde{\mathscr{S}} = \left\{ B_{t_0} = B_{t_0 r + (1-t_0)R}(\gamma(t_0)) \ \mid\ B_r(x) \subset \mathscr{S} \right\},$$

where $\gamma(t)$ and t_0 are taken by the above process for each ball $B_r(x)$ in $\mathscr{S}$. Then, for every $B \in \widetilde{\mathscr{S}}$ it is trivial that

$$\frac{|B \cap A|}{|B|} = \delta,$$

so $\widetilde{\mathscr{S}}$ is called as a refinement of $\mathscr{S}$. And a set $\widetilde{\Gamma}$ is defined by

$$\widetilde{\Gamma} = \cup_{B \in \widetilde{\mathscr{S}}} B.$$

Then, by the claim of **Step 2** and a trivial fact that $\widetilde{\Gamma} \subset \Gamma$, it is obtained that $\Gamma = \tilde{\Gamma}$. Thus, a modified way for the construction of Γ is attained.

Step 3.

$$\frac{|\widetilde{\Gamma}|}{|A|} \geqslant 1 + \frac{(1-\delta)}{5^n}.$$

With the above $\widetilde{\mathscr{S}}$, a Vital Covering of A can be considered; that is, there exist distjoint balls $B_i = B_{r_i}(x_i) \in \widetilde{\mathscr{S}}$ such that

$$A \subset \cup_i B_{5r_i}(x_i).$$

Here, by virtue of the doubling property of measure, it is computed that

$$|A| \leqslant |\cup_i B_{5r_i}(x_i)| \leqslant \sum_i |B_{5r_i}(x_i)| = 5^n \sum_i |B_{r_i}(x_i)|.$$

Then, since

$$1)\ B_i's \text{ are disjoint,}$$

$$2)\ B_i's \text{ satisfy } \frac{|B_i \cap A|}{|B_i|} = \delta,$$

it is obtained that

$$|\widetilde{\Gamma} - A| \geqslant \sum_i |(\widetilde{\Gamma} - A) \cap B_i| = \sum_i |B_i - A| = (1-\delta) \sum_i |B_i| \geqslant \frac{(1-\delta)}{5^n}|A|.$$

Thus, it follows that

$$\frac{|\widetilde{\Gamma}|}{|A|} = 1 + \frac{|\tilde{\Gamma}/A|}{|A|} \geqslant 1 + \frac{(1-\delta)}{5^n}.$$

This proves the theorem with a constant $c_1 = c_1(n, \delta) = \frac{1-\delta}{5^n}$. □

Note. The condition of Riemannian manifold is considered only for the Lebesgue Differentiation Theorem, and the doubling property of Riemannian measure. So, its proof is basically the same with that in $\mathbb{R}^n$, and this lemma essentially simplify our proof of Harnack inequality on Riemannian manifolds, comparing with the proof of Cabré who used a Calderon-Zygmund decomposition theorem in Riemannian manifolds.

The above covering lemma explains a uniform growth, which is essential for the proof of Theorem 4.1, of the measure of cut-off sets by increasing lower bounds. Thus, if it is iterated, uniform increments of the measures of cut-off sets is obtained successively. The following are details of that idea.

Proof. **of Theorem 4.1**.
For convenience, it is recalled that

$$M = \frac{1}{\beta_1} = \frac{57}{32} + C_{\frac{1}{2}} \quad \text{and} \quad \xi = \frac{1}{1 + \frac{1-\xi_1}{5^n}},$$
$$\text{where } \xi_1 = 1 - \{\frac{n\lambda}{\frac{1}{2}\left(C_{\frac{1}{2}} + a_{\mathcal{L}} + \Lambda\right)}\}^n.$$

When $k = 0$, this is trivial since

$$\frac{\left|\{u \geq 1\} \cap B_{\frac{R}{8}}\right|}{\left|B_{\frac{R}{8}}\right|} \geqslant 1 \qquad \Longrightarrow \qquad \left|\{u \geq 1\} \cap B_{\frac{R}{8}}\right| = \left|B_{\frac{R}{8}}\right|$$
$$\Longleftrightarrow \qquad \inf_{B_{\frac{R}{8}}} u \geqslant 1.$$

It suffices to prove that for any positive integer k,

$$\frac{\left|\{u \geq M^k\} \cap B_{\frac{R}{8}}\right|}{\left|B_{\frac{R}{8}}\right|} \geqslant \xi^k \qquad \Longrightarrow \qquad \frac{\left|\{u \geq M^{k-1}\} \cap B_{\frac{R}{8}}\right|}{\left|B_{\frac{R}{8}}\right|} \geqslant \xi^{k-1}$$

because then it is concluded by induction that

$$\frac{\left|\{u \geq M^k\} \cap B_{\frac{R}{8}}\right|}{\left|B_{\frac{R}{8}}\right|} \geqslant \xi^k \implies \cdots \implies \left|\{u \geqslant 1\} \cap B_{\frac{R}{8}}\right| = \left|B_{\frac{R}{8}}\right|$$
$$\Longleftrightarrow \inf_{B_{\frac{R}{8}}} u \geqslant 1.$$

First, it is assumed that

$$\frac{\left|\{u \geq M^k\} \cap B_{\frac{R}{8}}\right|}{\left|B_{\frac{R}{8}}\right|} \geqslant \xi^k.$$

Here, let $\Gamma = B_{\frac{R}{8}} \cap \{u \geq M^k\}$ for simplicity, then it is assumed that

$$|\Gamma| \geqslant \xi^k \left|B_{\frac{R}{8}}\right|.$$

Next, a covering of Γ, which consists of small balls contained in $B_{\frac{R}{8}}$, is considered; moreover, simultaneously whose sufficient portions intersect with Γ; that is, a colletion of such balls $\mathscr{F}$ is defined by

$$\mathscr{F} = \left\{B \subset B_{\frac{R}{8}} \;\middle|\; \frac{|\Gamma \cap B|}{|B|} \geqslant \xi_1\right\}, \quad \text{where } \xi_1 \text{ is from Theorem 3.1.}$$

Then, by Theorem 3.1 and linearity of $\mathcal{L}$, it is obtained that for any $B \in \mathscr{F}$

$$\inf_B u \geqslant M^k \left(\frac{1}{\frac{57}{32} + C_{\frac{1}{2}}}\right) = M^{k-1} \quad \text{since } M = \frac{57}{32} + C_{\frac{1}{2}}.$$

Thus, if $\Gamma_1 := \cup_{B \in \mathscr{F}} B$ is considered, by Lemma 4.2 it is noted that

1) $\Gamma \subset \Gamma_1$ almost every x in $B_{\frac{R}{2}}$.

2) $\Gamma_1 \subset \left\{x \in B_{\frac{R}{8}} \;\middle|\; u(x) \geqslant M^{k-1}\right\} = \{u \geq M^{k-1}\} \cap B_{\frac{R}{8}}$.

3) $|\Gamma_1| \geqslant (1 + \frac{1-\xi_1}{5^n}) |\Gamma| \geqslant (1 + \frac{1-\xi_1}{5^n}) \xi^k \left|B_{\frac{R}{8}}\right| = \xi^{k-1} \left|B_{\frac{R}{8}}\right|$
since $\xi = \dfrac{1}{1 + \frac{1-\xi_1}{5^n}}$.

From 2) and 3), it is obtained that

$$\frac{\left|\{u \geq M^{k-1}\} \cap B_{\frac{R}{8}}\right|}{\left|B_{\frac{R}{8}}\right|} \geqslant \xi^{k-1}$$

and finish the proof. $\square$

A variant of Theorem 4.1 as a power decay expression is given for an application in the next section.

Corollary 4.3. *($\frac{b}{t^d}$-expression) Let u be a nonnegative smooth function in a ball B_R satisfying $\mathcal{L}u \leqslant 0$ in the same ball. Moreover, suppose that* $\inf_{B_{\frac{R}{8}}} u \leqslant 1$*. Then, for any number $t \geqslant 1$, u has the following property :*

$$\frac{\left|\{u \geq t\} \cap B_{\frac{R}{8}}\right|}{\left|B_{\frac{R}{8}}\right|} \leqslant \frac{b}{t^d},$$

$$\textit{where} \quad b = \frac{1}{\xi} = (1 + \frac{1-\xi_1}{5^n}) \quad \textit{and} \quad d = log_M \frac{1}{\xi}.$$

Note. Appendix A is referred to for any universal constants.

Proof. The contrapositive statement of (Theorem 4.1) is considered:

$$\inf_B u \leqslant 1 \qquad \Longrightarrow \qquad \frac{\left|\{u \geqslant M^k\} \cap B\right|}{|B|} \leqslant \xi^k,$$

where $B = B_{\frac{R}{8}}$.
Then, for any $M^k \leqslant t < M^{k+1}$,

$$\begin{aligned} \frac{|\{u \geqslant t\} \cap B|}{|B|} &\leqslant \frac{\left|\{u \geqslant M^k\} \cap B\right|}{|B|} \\ &\leqslant \xi^k = \frac{\xi_{k+1}}{\xi} = b\ \xi^{k+1} \text{ with } b = \frac{1}{\xi} \end{aligned}$$

, and if let $d = log_M \frac{1}{\xi} \Longleftrightarrow \frac{1}{\xi} = (M^d)$, it is obtained

$$t < M^{k+1} \iff \frac{1}{\xi^{k+1}} = (M^d)^{k+1} > t^d \iff \xi^{k+1} < \frac{1}{t^d}.$$

Thus, it is computed

$$\frac{|\{u \geqslant t\} \cap B|}{|B|} \leqslant \frac{b}{t^d}.$$

and finish the proof. □

Chapter 5

Harnack Inequality

This is a chapter to prove Harnack inequality.

Theorem 5.1. *(Harnack Inequality) Let* $\mathbf{M}$ *be a smooth* n*-dimensional complete Riemannian manifold which satisfies* $\mathcal{M}^{-}[R(v)] \geqslant 0$*, and* u *be a nonnegative smooth solution of* $\mathcal{L}u = 0$ *in a ball* $B_R(x_0)$ *on* M*. Then* u *has the following property:*

$$\sup_{B_{\frac{R}{2}}(x_0)} u \leq C \inf_{B_{\frac{R}{2}}(x_0)} u,$$

where C *is a universal constant depending only on* $\lambda, \Lambda, n,$ *and* $a_{\mathcal{L}}$*.*

The geometric assumption of $\mathbf{M}$, $\mathcal{M}^{-}[R(v)] \geqslant 0$, is mentioned for clarity, and the proof of Harnack inequality is given in the end of this chapter after some auxiliary lemmas.

Lemma 5.2. *(a Lower Bound) Let* u *be a nonnegative smooth function in a ball* $B_R(x_0)$ *and satisfy* $\mathcal{L}u = 0$ *in the same ball. Moreover, suppose that* $\inf_{B_{\frac{R}{8}}(x_0)} u \leqslant 1$*. Then, for* $k \geqslant \widetilde{k}$ *(see Remark of Lemma 5.2),* u *has the following property:*

$$\text{If there is } x_1 \in B_{\frac{R}{8}}(x_0) \text{ such that } \begin{aligned} &1)\ d(x_1, \partial B_{\frac{R}{8}}(x_0)) \geqslant r_k = \frac{P}{(LN^k)^{\frac{d}{n}}}, \\ &2)\ u(x_1) \geqslant LN^{k-1}, \end{aligned}$$

$$\text{then} \quad \sup_{B_{r_k}(x_1)} u \geqslant LN^k,$$

$$\text{where} \quad L = 2b^{\frac{1}{d}} = 2 + 2\{\frac{2n\lambda}{5\left(C_{\frac{1}{2}} + a_{\mathcal{L}} + \Lambda\right)}\}^n, \quad N = \frac{L}{L - \frac{1}{2}} > 1,$$

$$P = 2R\left\{\frac{2^d b}{1 - \frac{b}{L^d}}\right\}^{\frac{1}{n}}, \quad \widetilde{k} \text{ is mentioned in the following remark.}$$

Remark. If, in fact, $r_k = \frac{P}{(LN^k)^{\frac{d}{n}}} > \frac{R}{8}$, then the above lemma is meaningless; so, in the beginning it is necessary to check that

$$r_k = \frac{P}{(LN^k)^{\frac{b}{n}}} \leqslant \frac{R}{8} \quad \text{for sufficiently large } k. \tag{5.1}$$

We check that (5.1) is true for some $\widetilde{k}$:

$$\begin{aligned} r_k = \frac{P}{(LN^k)^{\frac{b}{n}}} &\leqslant \frac{R}{8} \\ \Leftrightarrow 2R\left(\frac{2^d b}{1-\frac{b}{L^d}}\right)^{\frac{1}{n}} \frac{1}{(LN^k)^{\frac{b}{n}}} &\leqslant \frac{R}{8} \\ \Leftrightarrow 16\frac{2^d b}{1-\frac{b}{L^d}} &\leqslant L^d N^{kd} \\ \Leftrightarrow 16\frac{2^d b L^d}{L^d - b} &\leqslant L^d N^{kd} \\ \Leftrightarrow 162^d b L^d &\leqslant N^{kd}\left(L^d - b\right). \end{aligned}$$

Now, since b, d, L, and $N > 1$ are constants and $L^d - b > 0$, it is trivial that there exists a large $\widetilde{k}$ satisfies the claim.

Proof. The idea of Caffarelli [4] is followed for the proof. Suppose that there is a point x_1 satisfying the hypotheses of this lemma, but $\sup_{B_{r_k}(x_1)} u \leqslant LN^k$, then a contradition finishes the proof. First, a function w is defined by

$$w(x) = \frac{LN^k - u(x)}{LN^k - LN^{k-1}}.$$

Step 1.

$$\frac{\left|\{w \geqslant t\} \cap B_{\frac{r_k}{8}}(x_1)\right|}{\left|B_{\frac{r_k}{8}}(x_1)\right|} \leqslant \frac{b}{t^d} \quad \text{for any positive number } t.$$

This trivially comes from Corollary 4.3 since w satisfies the following properties:

1) $w(x)$ is nonnegative smooth function in $B_{r_k}(x_1)$,
2) $\mathcal{L}w \leqslant 0$ in $B_{r_k}(x_1)$,
3) $w(x_1) = \dfrac{LN^k - u(x_1)}{LN^k - LN^{k-1}} \leqslant \dfrac{LN^k - LN^{k-1}}{LN^k - LN^{k-1}} = 1 \implies \inf\limits_{B_{r_k}(x_1)} w \leqslant 1.$

Step 2.

$$\left(2R\left\{\frac{2^d b}{1-\frac{b}{L^d}}\right\}^{\frac{1}{n}}=\right)P \leqslant R\left\{\frac{2^d b}{1-\frac{b}{L^d}}\right\}^{\frac{1}{n}}, \text{ thus a contradiction happens.}$$

First, an estimate for the measure of the ball $B_{\frac{r_k}{8}}(x_1)$ can be computed:

$$\left|B_{\frac{r_k}{8}}(x_1)\right| \leqslant \underbrace{\left|\left\{u \leqslant \frac{LN^k}{2}\right\} \cap B_{\frac{r_k}{8}}(x_1)\right|}_{(1)} + \underbrace{\left|\left\{u \geqslant \frac{LN^k}{2}\right\} \cap B_{\frac{r_k}{8}}(x_1)\right|}_{(2)} \tag{5.2}$$

Here, (1) and (2) can be computed by Step 1 and Corollary 4.3, respectively:

$$(1) = \left|\{w \geqslant L\} \cap B_{\frac{r_k}{8}}(x_1)\right| \leqslant \frac{b}{L^d}\left|B_{\frac{r_k}{8}}\right|$$

since $B_{2\frac{r_k}{8}}(x_1) \subset B_R(x_0)$ and $u(x) \leqslant \frac{LN^k}{2} \iff w(x) = \frac{LN^k-u(x)}{LN^k-LN^{k-1}} \geqslant \frac{\frac{LN^k}{2}}{LN^k-LN^{k-1}} = \frac{\frac{N}{2}}{N-1} = L$ because $N = \frac{L}{L-\frac{1}{2}}$,

$$(2) \leqslant \left|\left\{u \geqslant \frac{LN^k}{2}\right\} \cap B_{\frac{R}{8}}(x_0)\right| \leqslant \frac{b}{(\frac{LN^k}{2})^d}\left|B_{\frac{R}{8}}\right|$$

since $B_{\frac{r_k}{8}}(x_1) \subset B_{\frac{R}{8}}(x_0)$ and $\inf_{B_{\frac{R}{8}}(x_0)} u \leqslant 1$.
Next, it is obtained that

$$\begin{aligned}
(5.2) &\implies (1-\frac{b}{L^d})\left|B_{\frac{r_k}{8}}\right| \leqslant \frac{b}{(\frac{LN^k}{2})^d}\left|B_{\frac{R}{8}}\right|,\\
&\implies (1-\frac{b}{L^d})w_n(\frac{r_k}{8})^n \leqslant \frac{2^d b}{(LN^k)^d}w_n(\frac{R}{8})^n, \text{ where } r_k = \frac{P}{(LN^k)^{\frac{d}{n}}},\\
&\implies (1-\frac{b}{L^d})\frac{P^n}{(LN^k)^d} \leqslant \frac{2^d b}{(LN^k)^d}R^n,\\
&\implies P \leqslant R\left\{\frac{2^d b}{1-\frac{b}{L^d}}\right\}^{\frac{1}{n}}.
\end{aligned}$$

Thus, the claim of **Step 2** is proved, and a contradiction happens.

Now it is concluded that if u satisfies all of the hypotheses of this lemma and all constants are defined as above, then u should satisfy that

$$\sup_{B_{r_k}(x_1)} u \geqslant LN^k.$$

This finishes the proof. $\square$

Remark. 1. L may be an arbitrary constant satisfying

$$1)\ L > \frac{1}{2} \text{ since } N = \frac{L}{L - \frac{1}{2}} > 0,$$

$$2)\ L \geqslant b^{\frac{1}{d}} \text{ since } 1 - \frac{b}{L^d} \geqslant 0.$$

Thus, L can be defined by $L = 2b^{\frac{1}{d}} = 2 + 2\{\frac{2n\lambda}{5\left(C_{\frac{1}{2}} + a_{\mathcal{L}} + \Lambda\right)}\}^n > 2$.

2. $P = 2R\left\{\frac{2^d b}{1 - \frac{b}{L^d}}\right\}^{\frac{1}{n}}$ depends on the radius of the ball; however, another constants are independent of R. Thus, it is necessary to recognize the reason why we can get a universal constant independent of R in the following lemma in spite of such P. This independency of constants is essential to obtain Harnack inequality.

Lemma 5.3. *(almost Harnack) Let u be a nonnegative smooth function in a ball B_R and satisfy $\mathcal{L}u = 0$ in the same ball. Moreover, suppose that $\inf_{B_{\frac{R}{8}}(x_0)} u \leqslant 1$. Then, u has the following property in $B_{\frac{R}{16}}(x_0)$:*

$$\sup_{B_{\frac{R}{16}}(x_0)} u \leqslant C_0, \quad \text{where } C_0 \text{ is a universal constant.}$$

Proof. Since $r_k = \frac{P}{(LN^k)^{\frac{d}{n}}}$ in Lemma 5.2, a sufficiently large positive integer $k_0 > \widetilde{k}$ (see Remark of Lemma 5.2) can be chosen such that

$$\sum_{k \geqslant k_0} 2r_k \leqslant \frac{R}{16}. \tag{5.3}$$

A Claim

$$\sup_{B_{\frac{R}{16}}(x_0)} u < LN^{k_0 - 1} = C_0.$$

Suppose that the claim is false, that is, there exist a point $x_{k_0} \in B_{\frac{R}{16}}(x_0)$ such that $u(x_{k_0}) \geqslant LN^{k_0-1}$.
First, since $B_{r_{k_0}}(x_{k_0}) \subset B_{\frac{R}{16}+\frac{R}{16}}(x_0) = B_{\frac{R}{8}}(x_0)$ by (5.3), in virtue of Lemma 5.2, an another point $x_{k_0+1} \in B_{r_{k_0}}(x_{k_0}) \subset B_{\frac{R}{8}}(x_0)$ can be picked such that

$$1) \sup_{B_{r_{k_0}}(x_{k_0})} u = u(x_{k_0+1}) \geqslant LN^{k_0},$$

$$2) B_{r_{k_0+1}}(x_{k_0+1}) \subset B_{\frac{R}{8}}(x_0) \text{ by (5.3) again.}$$

In detail, 2) comes from the fact that, for any integer $k \geqslant k_0$,

$$\begin{aligned} d(x_0, x_k) &\leqslant d(x_0, x_{k_0}) + \sum_{k \geqslant k_0} d(x_k, x_{k+1}) \\ &\leqslant \frac{R}{16} + \sum_{k \geqslant k_0} 2r_k \leqslant \frac{R}{16} + \frac{R}{16} \leqslant \frac{R}{8}. \end{aligned}$$

Next, the same process can be iterated to construct a sequence of points $\{x_k\}_{k \geqslant k_0}$ in $B_{\frac{R}{8}}(x_0)$ such that, for any $k \geqslant k_0$,

$$1) \sup_{B_{r_k}(x_k)} u = u(x_{k+1}) \geqslant LN^{k},$$

$$2) B_{r_{k+1}}(x_{k+1}) \subset B_{\frac{R}{8}}(x_0) \text{ also by (5.3).}$$

This implies that

$$u(x_k) \longrightarrow \infty \text{ in } B_{\frac{R}{8}}(x_0) \quad \text{as} \quad k \longrightarrow \infty;$$

however, it is contradictory to the condition that u is smooth in $B_R(x_0)$, so that, u is bounded in $B_{\frac{R}{8}}(x_0)$. Thus, it is concluded that if u satisfies all the condition in this lemma, then

$$\sup_{B_{\frac{R}{16}}(x_0)} u < C_0 = LN^{k_0-1}, \text{ where } k_0 \text{ satisfies } \sum_{k \geqslant k_0} 2r_k = \sum_{k \geqslant k_0} 2\frac{P}{(LN^k)^{\frac{d}{n}}} \leqslant \frac{R}{16}.$$

Finally, to guarantee that C_0 is a universal constant which is independent of u and the radius R, it is recalled that

$$L = 2b^{\frac{1}{d}}, N = \frac{L}{L - \frac{1}{2}} > 1, \text{ and } P = 2R\left\{\frac{2^d b}{1 - \frac{b}{L^d}}\right\}^{\frac{1}{n}}.$$

$\square$

Remark. The previous lemma is almost a Harnack inequality since it is easy to get a Harnack inequality from it by the following simple argument:
Let u be a nonnegative smooth function in a ball B_R and satisfy $\mathcal{L}u = 0$ in the same ball. Here, an auxiliary function is defined by

$$\widetilde{u}(x) = \frac{u(x)}{\inf_{B_{\frac{R}{16}}(x_0)} u + \epsilon} \quad \text{for an arbitrarily given } \epsilon > 0,$$

then $\widetilde{u}(x)$ naturally satisfies all the hypotheses of Lemma 5.3 since

$$\inf_{B_{\frac{R}{8}}(x_0)} \widetilde{u} = \frac{\inf_{B_{\frac{R}{8}}(x_0)} u}{\inf_{B_{\frac{R}{16}}(x_0)} u + \epsilon} \leqslant 1,$$

so that we have

$$\begin{aligned} \sup_{B_{\frac{R}{16}}(x_0)} u &\leqslant C_0 \left\{ \inf_{B_{\frac{R}{16}}(x_0)} u + \epsilon \right\} \\ &\leqslant C_0 \inf_{B_{\frac{R}{16}}(x_0)} u \quad \text{since } \epsilon > 0 \text{ is arbitrary.} \end{aligned}$$

Now the proof of a different statement of Harnack inequality, Theorem 5.1, is given below.

Proof. **of Theorem 5.1**.
It suffices to prove that

$$u(x) \leqslant C\ u(y) \quad \text{for any } x \text{ and } y \text{ in } B_{\frac{R}{2}}(x_0) \text{ for a universal constant } C.$$

Let x and y be any two points in $B_{\frac{R}{2}}(x_0)$, and consider the minimal geodesic from x to y , $\kappa(t)$, which is parametrized by the arc-length satisfying

$$\kappa(0) = x,\ \kappa(l) = y,\ \text{and } l < R = \text{the diameter of } B_{\frac{R}{2}}(x_0).$$

Then, a colletion of small balls $\left\{ B_j = B_{\frac{R}{2\cdot 16}}(x_j) = B_{\frac{R}{32}}(x_j) \right\}$ can be constructed by the following steps:

1) A positive integer i is chosen such that
$$t_0 = 0 < t_1 = \frac{R}{32} < t_2 = 2\frac{R}{32} < \cdots < t_i = i\frac{R}{32} < t_{m+1} = l \leqslant (i+1)\frac{R}{32}$$
, then $i\dfrac{R}{32} < l \implies i < \dfrac{32l}{R} < 32$.
2) Let $x_0 = x, x_j = \kappa(t_j),$ and $x_{i+1} = \kappa(t_{i+1}) = y$ on the geodesic curve.

Then, the balls successively intersect each other since $\kappa(t)$ is a unit-speed curve, so that, by using the above remark of (Lemma 6.3) iteratively, it is concluded that

$$u(x) = u(x_0) \leqslant C_0 u(x_1) \leqslant \cdots \leqslant {C_0}^i u(x_{i+1}) = {C_0}^i u(y) < {C_0}^{32} u(y).$$

This finishes the proof with a universal constant $C = {C_0}^{32}$. $\square$

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Appendix A

Some Universal Constants

All the constants depend only on dimension n and differential operator $\mathcal{L}$:

1) (Lemma 2.2: Upper Boundness of $\mathcal{L}d_p$) $a_{\mathcal{L}} = (n-1)\Lambda$,

   (in the proof of Theorem 3.1, Step 1, pp20-22) $C_{\frac{1}{2}} = C_\delta$ with $\delta = \dfrac{1}{2}$,

2) (Theorem 3.1: Critical Density)

$$\xi_1 = 1 - \{\frac{n\lambda}{\frac{1}{2}\left(C_{\frac{1}{2}} + a_{\mathcal{L}} + \Lambda\right)}\}^n, \quad \text{and} \quad \beta_1 = \frac{1}{\frac{57}{32} + C_{\frac{1}{2}}},$$

3) (Theorem 4.1: Power Decay)

$$M = \frac{57}{32} + C_{\frac{1}{2}}, \quad \xi = \frac{1}{1 + \{\frac{2n\lambda}{5\left(C_{\frac{1}{2}} + a_{\mathcal{L}} + \Lambda\right)}\}^n},$$

4) (Corollary 4.3: $\dfrac{b}{t^d}$-expression)

$$b = 1 + \{\frac{2n\lambda}{5\left(C_{\frac{1}{2}} + a_{\mathcal{L}} + \Lambda\right)}\}^n, \quad d = log_{\left(\frac{57}{32} + C_{\frac{1}{2}}\right)}(1 + \{\frac{2n\lambda}{5\left(C_{\frac{1}{2}} + a_{\mathcal{L}} + \Lambda\right)}\}^n),$$

5) (Corollary 5.2: Lower Bound)

$$L = 2 + 2\{\frac{2n\lambda}{5\left(C_{\frac{1}{2}} + a_{\mathcal{L}} + \Lambda\right)}\}^n, \quad N = \frac{L}{L - \frac{1}{2}} > 1, \quad P = 2R\left(\frac{2^d b}{1 - \frac{b}{L^d}}\right)^{\frac{1}{n}},$$

6) (Theorem 5.1: Harnack Inequality)

$$C = L^{32} N^{32(k_0 - 1)}, \text{ where } k_0 \text{ satisfies } \sum_{k \geqslant k_0} \frac{1}{N^k} \leqslant \frac{L}{64^{\frac{n}{d}}\left(\frac{2^d b}{1 - \frac{b}{L^d}}\right)^{\frac{1}{d}}}.$$