Nonlocally Related Partial Differential Equation Systems, the Nonclassical Method and Applications

by

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Abstract

Symmetry methods are important in the analysis of differential equation (DE) systems. In this thesis, we focus on two significant topics in symmetry analysis: nonlocally related partial differential equation (PDE) systems and the application of the nonclassical method.

In particular, we introduce a new systematic symmetry-based method for constructing nonlocally related PDE systems (inverse potential systems). It is shown that each point symmetry of a given PDE system systematically yields a nonlocally related PDE system. Examples include applications to nonlinear reaction-diffusion equations, nonlinear diffusion equations and nonlinear wave equations. Moreover, it turns out that from these example PDEs, one can obtain nonlocal symmetries (including some previously unknown nonlocal symmetries) from some corresponding constructed inverse potential systems.

In addition, we present new results on the correspondence between two potential systems arising from two nontrivial and linearly independent conservation laws (CLs) and the relationships between local symmetries of a PDE system and those of its potential systems.

We apply the nonclassical method to obtain new exact solutions of the nonlinear Kompaneets (NLK) equation

$$u_t = x^{-2}\left(x^4\left(\alpha u_x + \beta u + \gamma u^2\right)\right)_x,$$

where $\alpha > 0$, $\beta \geq 0$ and $\gamma > 0$ are arbitrary constants. New time-dependent exact solutions for the NLK equation

$$u_t = x^{-2}\left(x^4\left(\alpha u_x + \gamma u^2\right)\right)_x,$$

for arbitrary constants $\alpha > 0$, $\gamma > 0$ are obtained. Each of these solutions is expressed in terms of elementary functions. We also consider the behaviours of these new solutions for initial conditions of physical interest. More specifically, three of these families of solutions exhibit quiescent behaviour and the other two families of solutions exhibit blow-up behaviour in finite time. Consequently, it turns out that the corresponding nontrivial stationary solutions are unstable.

Preface

Chapter 4 is based on joint work with my supervisor George Bluman. The development of the symmetry-based method is a result of close collaboration with him. Theorem 4.2.1 and Corollary 4.2.4 were worked out jointly. In addition, I was responsible for the constructions of the inverse potential systems listed in this thesis, the classifications of point symmetries of such inverse potential systems and the proof of Proposition 4.4.4. A version of Chapter 4 has been submitted for publication.

Chapter 5 is based on joint work with George Bluman and Shou-fu Tian. A version of Chapter 5 is published [36]. I was responsible for the computation of "nonclassical symmetries" and the new stationary solutions $\mathfrak{F}_5$ and $\mathfrak{F}_6$ of the nonlinear Kompaneets (NLK) equation. I wrote the first draft of the "nonclassical analysis" part of the manuscript.

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Chapter 1

Introduction

The use of symmetries to investigate the solvability of equations can be traced to the middle of the nineteenth century when Galois established the relationship between the solvability of polynomial equations by radicals and their symmetry groups. Motivated by Galois' work, Sophus Lie developed the theory of continuous groups, i.e., Lie groups, to deal with the solvability of differential equations (DEs) by quadratures [66, 67].

A symmetry of a DE system is a transformation which maps the solutions of the DE system to other solutions. In this thesis, our interest is limited to symmetries that are connected local Lie groups (continuous symmetries), which can be characterized by their infinitesimal generators. Throughout this thesis, "symmetry" means "continuous symmetry". One important type of symmetry is a local symmetry. Local symmetries include:

- point symmetries: in evolutionary form, the components of an infinitesimal generator for dependent variables depend at most linearly on the first derivatives of dependent variables;
- contact symmetries (exist only for scalar DEs): in evolutionary form, the component of an infinitesimal generator for the dependent variable depends at most on first derivatives of the dependent variable;
- higher-order symmetries: in evolutionary form, the components of an infinitesimal generator for dependent variables depend at most on finite order derivatives of dependent variables.

An important feature of a point symmetry is that one is able to find such a symmetry systematically by Lie's algorithm. Lie's algorithm for finding the point symmetries of a DE system is presented in [21, 25, 28, 29, 39, 53, 75, 76, 80, 87]. In finding the point symmetries of a DE system, one need only find the components of their infinitesimal generators. The invariance conditions yield a system of linear determining equations, which can be solved explicitly through various existing software packages. There are some popular programs for solving large over-determined DE systems, e.g., DIFFGROB2 [69, 70], standard_form [84], rif

[85], CRACK [94], etc. Lie's algorithm can be extended to find contact or higher-order symmetries, in which one needs to take the differential consequences of the given DE system into consideration.

Once one obtains a symmetry of a DE system, various applications arise.

- One can construct new solutions from known solutions.
- One can reduce the order of a given ordinary differential equation (ODE). Moreover, one can obtain the solutions of the given ODE from those of the reduced ODE.
- One can construct invariant solutions for a given partial differential equation (PDE) system.

From the knowledge of the contact (point) symmetries, one is able to

- determine whether a given scalar PDE (PDE system) can be invertibly mapped into a linear scalar PDE (PDE system), and find such a mapping when it exists [25, 29, 31, 62];
- determine whether a linear PDE with variable coefficients can be invertibly mapped into a linear PDE with constant coefficients, and find such a mapping when it exists [18, 25, 29, 30].

In 1918, for a variational system, Emmy Noether introduced a method for constructing conservation laws (CLs) from point symmetries of its action functional [73]. In 1921, Bessel-Hagen extended Noether's work to include divergence symmetries, which leave invariant the action functional to within a divergence term [14]. In [9], Anco and Bluman introduced a systematic procedure (direct method) to construct CLs for a given DE system. The direct method includes and extends Noether's theorem, since it can be applied to any DE system. Moreover, the direct method is coordinate-independent.

Topologically, continuous symmetries are not limited to local symmetries [2, 25, 29, 39, 61, 75, 91, 92]. A symmetry that is not a local symmetry is called a nonlocal symmetry. A special kind of nonlocal symmetry is a symmetry whose infinitesimal generator depends on the integrals of the dependent variables. However, it is hard to find such a special kind of nonlocal symmetry of a given PDE system by applying Lie's algorithm directly to it. A way to seek nonlocal symmetries of a given PDE system is through application of Lie's algorithm to a nonlocally related PDE system of the given PDE system. Two PDE systems are *equivalent* and *nonlocally related* if they have the following properties:

(1) Any solution of either PDE system yields a solution of the other PDE system.

(2) The solutions of either PDE system yield all solutions of the other PDE system.

(3) The correspondence between the solutions of these two PDE systems is not one-to-one.

Throughout this thesis, "nonlocally related" means "equivalent and nonlocally related".

Nonlocally related PDE systems play a crucial rule in the nonlocal analysis of a given PDE system, since one could extend local analysis methods to nonlocal ones by applying local analysis methods to nonlocally related PDE systems. Prior to the new work presented in this thesis, there are two known systematic methods to construct nonlocally related PDE systems [7, 19, 23–26, 29, 32, 43].

(1) The CL-based method: Use k nontrivial local CLs of a given PDE system to construct a k-plet potential system of the given PDE system.

(2) Exclude some dependent variables from a given PDE system to construct subsystems.

Since one is able to find local CLs, provided they exist, of a given PDE system systematically through the direct method, it is straightforward for one to construct potential systems of the given PDE system systematically. To obtain subsystems, one can exclude some dependent variables by cross-differentiation or direct substitution from the given PDE system. It is important to remark that all potential systems arising from nontrivial CLs of a given PDE system are nonlocally related to the given PDE system. However, not all subsystems are nonlocally related to the given PDE system.

In the framework of nonlocally related PDE systems, nonlocal symmetries of a given PDE system $\mathbf{R}\{x,t;u\}$ can arise from point symmetries of any PDE system in a tree of nonlocally related PDE systems that includes $\mathbf{R}\{x,t;u\}$. When such nonlocal symmetries can be found for a given PDE system, one can use such symmetries systematically to possibly generate further exact solutions from its known solutions, to construct new invariant solutions, to find nonlocal linearizations, or to find additional nonlocal CLs.

Finding exact solutions is an essential topic in the field of DEs. One can use symmetries to find invariant solutions of a given DE system. Lie's classical method is based on the construction of invariants for a symmetry of a given DE system. If a given PDE system admits a symmetry group, then the invariant solutions corresponding to this symmetry can be obtained by solving a reduced PDE system with fewer independent variables than the given PDE system. In [15], (also see [27]), Lie's classical method for finding invariant solutions was generalized to the nonclassical method. In the nonclassical method, invariant solutions arise from "nonclassical symmetries", which keep only some subsets of the solutions invariant. By construction, the nonclassical method includes Lie's classical method and the direct method introduced by Clarkson and Kruskal in [45].

In this thesis, we focus on nonlocally related PDE systems, the nonclassical method and their applications. In particular, the following new results are obtained.

- *We present a relationship between two potential systems arising from two nontrivial and linearly independent local CLs of a given PDE system.*

- *We find a correspondence between local symmetries of a given PDE system and those of its potential systems.*

- *We introduce a new systematic symmetry-based method for constructing nonlocally related PDE systems and show that such nonlocally related PDE systems yield nonlocal symmetries for specific examples. Moreover, some nonlocal symmetries are previously unknown.*

- *We apply the nonclassical method to obtain new exact solutions of the dimensional nonlinear Kompaneets (NLK) equation* [60] *given by*

$$u_t = x^{-2}\left(x^4\left(\alpha u_x + \beta u + \gamma u^2\right)\right)_x, \tag{1.1}$$

  where $\alpha > 0$, $\beta \geq 0$ and $\gamma > 0$ are arbitrary constants. These new exact solutions are expressible in terms of elementary functions and allow one to study stability properties with respect to initial data.

In Chapter 2, we give a brief introduction to symmetries, CLs and their applications.

In Chapter 3, we present the known framework for constructing nonlocally related PDE systems and their applications. Two different cases are discussed: PDE systems with two independent variables and PDE systems with three or more independent variables. We present the known CL-based method for constructing nonlocally related PDE systems (potential systems) for these two cases. We also state the method for constructing subsystems of a given PDE system. In addition, we present an extended procedure for constructing a tree of nonlocally related PDE systems. Various examples are shown in this chapter. Moreover, we illustrate how to use nonlocally related PDE systems to find nonlocal symmetries and nonlocal CLs of a given PDE system. Two new results are presented. In particular, we show that for two potential systems written in Cauchy-Kovalevskaya form, arising from two nontrivial and linearly independent local CLs of a given PDE system, the potential variable of one system cannot be expressed as a local function in terms of the independent variables, dependent variables and their derivatives of the other system. Furthermore, we investigate relationships between symmetries of subsystems and those of potential systems. We prove that any local symmetry of a PDE system with precisely n local CLs can be obtained by projection of some local symmetry of its n-plet potential system.

In Chapter 4, we present a new systematic symmetry-based method for constructing nonlocally related PDE systems. It is shown that any point symmetry of a given PDE system yields a nonlocally related PDE system (inverse potential system). It turns out that the nonlocally related PDE systems arising from point symmetries can also yield nonlocal symmetries of a given PDE system. Some examples are listed to illustrate this new method.

In Chapter 5, firstly, we review Lie's classical method for constructing invariant solutions of a given PDE system. Following this, we give an introduction to the nonclassical method. Finally, we use the nonclassical method to construct exact solutions of the NLK equation. It is shown that the nonclassical method can yield further exact solutions beyond those arising from point symmetries of the NLK equation. Moreover, the new solutions are shown to be expressible in terms of elementary functions. The properties of such new solutions are exhibited. It turns out that these new solutions yield five families of solutions with initial conditions of physical interest. In particular, three of these families of solutions exhibit quiescent behaviour, i.e., $\lim_{t\to\infty} u(x,t) = 0$, and the other two families of solutions exhibit blow up behaviour, i.e., $\lim_{t\to t^*} u(x,t^*) = \infty$ for some finite t^* depending on a constant in their initial conditions. Moreover, new stationary solutions are presented.

In Chapter 6, conclusions and some open problems are proposed.

Throughout the thesis, we use the software package GeM to do necessary computations [40].

Chapter 2

Symmetries, Conservation Laws and Applications

2.1 Introduction

In this chapter we review the basic ideas of local symmetries and CLs. Lie's algorithm for finding local symmetries of a DE system is discussed. We also present equivalence transformations and symmetry classifications for a class of PDEs. As an end of this chapter, we state the direct method for constructing CLs and some connections between symmetries and CLs.

2.2 Symmetries of DE systems

2.2.1 Lie groups and local groups of transformations

In applications, symmetries of a DE system are often Lie groups of transformations acting on the solution manifold of the given DE.

Definition 2.2.1 An *r-parameter Lie group* is an r-dimensional smooth manifold G that is also a group with the property that the multiplication map

$$m: G \times G \to G, \quad m(g, h) = g \cdot h, \quad g, h \in G,$$

and the inversion

$$i: G \to G, \quad i(g) = g^{-1}, \quad g \in G,$$

are smooth.

One significant application of Lie groups involves actions by Lie groups on special manifolds.

Definition 2.2.2 A *transformation group* acting on a smooth manifold M is determined by a Lie group G and a smooth map $\Theta: G \times M \to M$, denoted by $\Theta(g, x) = g \cdot x$, which satisfies

(1) $e \cdot x = x$, where e is the identity of G and $x \in M$.

(2) $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ for all $x \in M$, $g, h \in G$.

In many cases, we are only interested in *local* group action, i.e., for a given $x \in M$, $g \cdot x$ is only defined for elements g that lie in a small neighborhood of the identity e.

Definition 2.2.3 A *local group of transformations* acting on a smooth manifold M is determined by a Lie group G, an open subset $\mathcal{D}$, with $\{e\} \times M \subset \mathcal{D} \subset \{G\} \times M$, and a smooth map $\Theta\colon \mathcal{D} \to M$, denoted by $\Theta(g, x) = g \cdot x$, which satisfies

(1) For all $x \in M$, $e \cdot x = x$.

(2) If $(h, x) \in \mathcal{D}$, $(g, h \cdot x) \in \mathcal{D}$, and $(g \cdot h, x) \in \mathcal{D}$, then $g \cdot (h \cdot x) = (g \cdot h) \cdot x$.

(3) If $(g, x) \in \mathcal{D}$, then $(g^{-1}, g \cdot x) \in \mathcal{D}$ and $g^{-1} \cdot (g \cdot x) = x$.

Among transformation groups, a one-parameter group of transformations is an important kind that plays a significant role in various fields.

Definition 2.2.4 A (*smooth*) *local one-parameter group of transformations* (also called a *local flow*) acting on a smooth manifold M is a local group of transformations $\Theta\colon \mathcal{D} \to M$, where $\mathcal{D} \subset \mathbb{R} \times M$ is the *flow domain*, with the properties:

(1) $\Theta(\varepsilon, \Theta(\tau, x)) = \Theta(\varepsilon + \tau, x)$, $x \in M$, for all $\varepsilon, \tau \in \mathbb{R}$ such that both sides of the equation are defined.

(2) $\Theta(0, x) = x$, $x \in M$.

If $\mathcal{D} = \mathbb{R} \times M$, Θ is called a *global one-parameter group of transformations* (or a *global flow*).

In order to distinguish the point $x \in M$ and the parameter ε, we use the notation $\Theta(x; \varepsilon)$ to denote a one-parameter group of transformations.

Example 2.2.5 Consider the map $\Theta\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$\Theta(\varepsilon, x, y) = \Theta(x, y; \varepsilon) = (x \cos \varepsilon - y \sin \varepsilon, x \sin \varepsilon + y \cos \varepsilon).$$

Then Θ is a global one-parameter group of transformations denoting the rotation group on a plane.

Theorem 2.2.6 If $\Theta\colon \mathcal{D} \to M$ is a smooth local one-parameter group of transformations, for each $x \in M$, define a vector by

$$\mathbf{X}|_x = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Theta(x;\varepsilon). \tag{2.1}$$

Then the assignment $x \mapsto \mathbf{X}|_x$ is a smooth vector field on M, which is called the *infinitesimal generator* of Θ.

Proof. See [64] for the proof.

Suppose the dimension of M is n. In local coordinates, by Taylor's formula, for ε in a small neighborhood of 0,

$$\Theta(x;\varepsilon) = x + \varepsilon\xi(x) + \mathrm{O}(\varepsilon^2),$$

where $\xi = (\xi^1, \ldots, \xi^n)$ is given by

$$\xi^i(x) = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \Theta^i(x;\varepsilon), \quad i = 1, \ldots, n. \tag{2.2}$$

Thus the infinitesimal generator of Θ is given by

$$\mathbf{X} = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i}. \tag{2.3}$$

The quantities ξ^i, $i = 1, \ldots, n$, are called the *infinitesimals* of the one-parameter group of transformations Θ. The transformation

$$\bar{x} = x + \varepsilon\, \xi(x) \tag{2.4}$$

defines the *infinitesimal transformation* of Θ.

Example 2.2.7 Consider the rotation group Θ in Example 2.2.5. According to formula (2.3), the infinitesimal generator of Θ is given by

$$\mathbf{X} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

On the other hand, the infinitesimal generators can be used to characterize a one-parameter group of transformations. One can obtain the one-parameter group of transformations generated by a smooth vector $\mathbf{X}$ through solving an ODE system. We use the notation $\exp(\varepsilon\mathbf{X})$ to denote the one-parameter group of transformations generated by $\mathbf{X}$, i.e.,

$$\exp(\varepsilon\mathbf{X}) = \Theta(x;\varepsilon).$$

One can show that

$$\exp(\varepsilon \mathbf{X})x = x + \varepsilon\xi(x) + \frac{\varepsilon^2}{2}\mathbf{X}(\xi)(x) + \cdots = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!}\mathbf{X}^k(x), \tag{2.5}$$

where $\mathbf{X}$ is given by (2.3), $\xi = (\xi^1, \ldots, \xi^n)$, $\mathbf{X}(\xi) = (\mathbf{X}(\xi^1), \ldots, \mathbf{X}(\xi^n))$, and $\mathbf{X}^k = \mathbf{X}\mathbf{X}^{k-1}$.

In local coordinates, the one-parameter group of transformations $\Theta(x;\varepsilon)$ can be determined from its infinitesimal generator through Lie's First Fundamental Theorem.

Theorem 2.2.8 (Lie's First Fundamental Theorem) A one-parameter group of transformations $\Theta(x;\varepsilon)$ is equivalent to the solution of the initial value problem for a system of first order ODEs

$$\frac{d}{d\varepsilon}\Theta(x;\varepsilon) = \xi\left(\Theta(x;\varepsilon)\right), \quad \Theta(x;0) = x. \tag{2.6}$$

Proof. See [21, 29, 48, 75, 80] for the proof.

Example 2.2.9 Consider the infinitesimal generator

$$\mathbf{X} = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}. \tag{2.7}$$

Here the one-parameter group of transformations $\Theta(\varepsilon) = (\bar{t}(\varepsilon), \bar{x}(\varepsilon), \bar{u}(\varepsilon))$ generated by $\mathbf{X}$ satisfies

$$\begin{aligned} \frac{d\bar{t}}{d\varepsilon} &= 0,\\ \frac{d\bar{x}}{d\varepsilon} &= 2\bar{t},\\ \frac{d\bar{u}}{d\varepsilon} &= -\overline{xu}, \end{aligned} \tag{2.8}$$

with initial value $\Theta(0) = (t, x, u)$. Solving the equations (2.8), one finds that the one-parameter group of transformations Θ generated by $\mathbf{X}$ is given by

$$\begin{aligned} \bar{t} &= t,\\ \bar{x} &= x + 2\varepsilon t,\\ \bar{u} &= ue^{-\varepsilon x-\varepsilon^2 t}. \end{aligned} \tag{2.9}$$

The following shows that, by choosing proper local coordinates, a vector field near a regular point x_0, i.e., $\mathbf{X}_{x_0} \neq 0$, can be expressed in a simple canonical form [21, 25, 29, 39, 53, 64, 75, 76, 80].

Theorem 2.2.10 Suppose $\mathbf{X}$ is a smooth vector field on a smooth manifold M, and $\mathbf{X}_{x_0} \neq 0$ at a point $x_0 \in M$. Then there exist smooth coordinates $(y^1, \ldots, y^n)$ on some neighborhood of x_0 such that $\mathbf{X}$ has the coordinate representation (canonical form) $\dfrac{\partial}{\partial y^n}$.

If $y = f(x)$ is a change of coordinates, then the vector field

$$\mathbf{X} = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i} \tag{2.10}$$

has the expression

$$\mathbf{X} = \sum_{j=1}^{n} \sum_{i=1}^{n} \xi^i(f^{-1}(y)) \frac{\partial f^j}{\partial x^i}(f^{-1}(y)) \frac{\partial}{\partial y^j} \tag{2.11}$$

in the y coordinates. Suppose the corresponding canonical coordinates of the vector field $\mathbf{X}$ are given by

$$\begin{aligned} y^1 &= f^1(x), \\ y^2 &= f^2(x), \\ &\cdots \\ y^n &= f^n(x). \end{aligned} \tag{2.12}$$

Since the canonical form of $\mathbf{X}$ is given by $\dfrac{\partial}{\partial y^n}$, according to the formula (2.11), $f(x) = (f^1(x), \ldots, f^n(x))$ satisfies the following first order linear PDE system

$$\begin{aligned} \sum_{i=1}^{n} \xi^i(x) \frac{\partial f^\mu(x)}{\partial x^i} &= \frac{\partial y^\mu}{\partial y^n} = 0, \quad \mu = 1, \ldots, n-1, \\ \sum_{i=1}^{n} \xi^i(x) \frac{\partial f^n(x)}{\partial x^i} &= \frac{\partial y^n}{\partial y^n} = 1. \end{aligned} \tag{2.13}$$

The canonical form of a vector field is essential in the symmetry-based method for constructing nonlocally related PDE systems, which will be presented in Chapter 4.

2.2.2 Lie algebra and Lie bracket

Definition 2.2.11 A *Lie algebra* is a vector space $\mathfrak{g}$ endowed with an operation $[\ ,\ ]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket* for $\mathfrak{g}$, that satisfies the following properties for all $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{g}$:

(1) Bilinearity: For $a, b \in \mathbb{R}$,

$$[a\mathbf{X} + b\mathbf{Y}, \mathbf{Z}] = a[\mathbf{X}, \mathbf{Y}] + b[\mathbf{X}, \mathbf{Z}],$$
$$[\mathbf{X}, a\mathbf{Y} + b\mathbf{Z}] = a[\mathbf{X}, \mathbf{Y}] + b[\mathbf{X}, \mathbf{Z}].$$

(2) Antisymmetry:

$$[\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}].$$

(3) Jacobi identity:

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] = 0.$$

Definition 2.2.12 Let $\mathbf{X}$ and $\mathbf{Y}$ be two smooth vector fields on a smooth manifold M. The *Lie bracket* of $\mathbf{X}$ and $\mathbf{Y}$ is a smooth vector field $[\mathbf{X}, \mathbf{Y}]: C^\infty(M) \to C^\infty(M)$, where $C^\infty(M)$ denotes the set of all smooth real-valued functions on M, defined by

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f)) \quad f \in C^\infty(M). \tag{2.14}$$

An important property of the Lie bracket is given by the following theorem.

Theorem 2.2.13 Suppose $F\colon M \to N$ is a diffeomorphism and $\mathbf{X}, \mathbf{Y} \in \mathcal{T}(M)$, where $\mathcal{T}(M)$ denotes the set of all smooth vector fields on M. Then $F_*[\mathbf{X}, \mathbf{Y}] = [F_*(\mathbf{X}), F_*(\mathbf{Y})]$, where F_* is the pushforward associated with F.

Proof. See [64] for the proof.

Suppose $\mathfrak{g}$ is an r-dimensional Lie algebra. Let $\{\mathbf{X}_1, \ldots, \mathbf{X}_r\}$ be a basis of $\mathfrak{g}$, then $[\mathbf{X}_i, \mathbf{X}_j] \in \mathfrak{g}$, i.e., there are specific constants c^k_{ij}, $i, j, k = 1, \ldots, r$, called the *structure constants* of $\mathfrak{g}$, such that

$$[\mathbf{X}_i, \mathbf{X}_j] = \sum_k^r c^k_{ij}\mathbf{X}_k, \quad i, j = 1, \ldots, r. \tag{2.15}$$

The structure constants have the following properties.

Theorem 2.2.14 (**Lie's Third Fundamental Theorem**) The structure constants satisfy

(1) Antisymmetry:

$$c^k_{ij} = -c^k_{ji}, \tag{2.16}$$

(2) Jacobi identity:

$$\sum_{k=1}^r \left(c^k_{ij}c^m_{kl} + c^k_{li}c^m_{kj} + c^k_{jl}c^m_{ki}\right) = 0. \tag{2.17}$$

Proof. See [48] for the proof.

2.2.3 Jet spaces and prolongations

The basic objects we consider in this thesis are DE systems. In order to apply infinitesimal methods to DE systems, it is necessary to extend the basic space of independent variables $x = (x^1, \ldots, x^n)$ and dependent variables $u = (u^1, \ldots, u^m)$ to a space including the derivatives of u. Let $\mathcal{X} \simeq \mathbb{R}^n$ denote the space of n independent variables and $\mathcal{U} \simeq \mathbb{R}^m$ denote the space of m dependent variables.

Consider a smooth function $f\colon \mathcal{X} \to \mathcal{U}$, i.e., $u = f(x) = (f^1(x), \ldots, f^m(x))$. For each i, there are

$$p_k \equiv \binom{n+k-1}{k}$$

different k-th order partial derivatives of $f^i(x)$. Let

$$\partial_{\boldsymbol{J}} g(x) = \frac{\partial^k g(x)}{\partial x^{j_1} \partial x^{j_2} \cdots \partial x^{j_k}}$$

for every smooth scalar-valued function $g(x) = g(x^1, \ldots, x^n)$, where $\boldsymbol{J} = (j_1, \ldots, j_k)$ is an *unordered* k-tuple of integers with $1 \le j_\kappa \le n$ for $\kappa = 1, \ldots, k$. We call $\boldsymbol{J}$ an *unordered multi-index* of order k, i.e., $|\boldsymbol{J}| = k$. We use the notation

$$\partial^k f(x) = (\partial^k f^1(x), \ldots, \partial^k f^m(x))$$

to denote the k-th order partial derivatives of $f(x)$. In particular, $\partial f(x) = \partial^1 f(x)$. Hence, there are $m \cdot p_k$ different k-th order partial derivatives of $f(x)$. It follows that one needs $m \cdot p_k$ different coordinates $u^i_{\boldsymbol{J}}$, $i = 1, \ldots, m$, $|\boldsymbol{J}| = k$ to represent all different k-th order partial derivatives

$$u^i_{\boldsymbol{J}} = u^i_{j_1,\ldots,j_k} = \frac{\partial^k u^i}{\partial x^{j_1} \partial x^{j_2} \cdots \partial x^{j_k}} = \partial_{\boldsymbol{J}} f^i(x)$$

of a function $u = f(x)$. Let $\mathcal{U}_k \simeq \mathbb{R}^{m \cdot p_k}$ be the space of all k-th order partial derivatives of u. Consequently, the space of all the derivatives of u up to l is $\mathcal{U}^{(l)} \equiv \mathcal{U} \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_l$, whose dimension is

$$m + mp_1 + \cdots + mp_l = m\binom{n+l}{l} \equiv mp^{(l)}.$$

We denote a point in $\mathcal{U}^{(l)}$ by $u^{(l)}$. For the smooth function $u = f(x)$, the l-th prolongation of f, denoted by $u^{(l)} = f^{(l)}(x)$, is defined by the equations

$$u^i_{\boldsymbol{J}} = \partial_{\boldsymbol{J}} f^i(x), \quad i = 1, \ldots, m,$$

where $0 \le |\boldsymbol{J}| \le l$. (By convention, u^i_0 denotes the component u^i of u.)

Definition 2.2.15 If $M \subset \mathcal{X} \times \mathcal{U}$ is an open set, the l-jet space of M is given by

$$M^{(l)} \equiv M \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_l.$$

A (*local*) *point transformation* acting on $M \subset \mathcal{X} \times \mathcal{U}$ is defined by a (local) diffeomorphism on M:

$$(\overline{x}, \overline{u}) = (F(x,u), G(x,u)) \quad (x,u) \in M. \tag{2.18}$$

Let $K \subset \mathcal{X}$ be an open set, and $f \colon K \to \mathcal{U}$ be a continuous function. The *graph* of f is given by

$$\Gamma(f) = \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m : x \in K \text{ and } u = f(x)\}.$$

The *l-th order prolongation of a graph* $\Gamma(f)$ is given by

$$\Gamma^{(l)}(f) = \left\{(x, u, \partial u, \ldots, \partial^l u) : \ x \in K, \ (u, \partial u, \ldots, \partial^l u) = (f(x), \partial f(x), \ldots, \partial^l f(x)\right\}.$$

If τ is a local point transformation on M:

$$\begin{aligned} \overline{x} &= \tau^1(x,u), \\ \overline{u} &= \tau^2(x,u), \end{aligned} \tag{2.19}$$

and $u = f(x)$ is a smooth function, then τ acts on $u = f(x)$ by acting on its graph. Hence, it is natural to extend τ to a map $\tau^{(l)}$ acting on the l-th jet space of M, which maps the derivatives of $u = f(x)$ to the derivatives of the transformed function $\overline{u} = \overline{f}(\overline{x})$. The *l-th order prolongation of* τ is given by $\tau^{(l)}$ which satisfies

$$\tau^{(l)}\left(\Gamma^{(l)}(f)\right) = \Gamma^{(l)}(\overline{f}). \tag{2.20}$$

An *l-th order* (*local*) *contact transformation* is a (local) diffeomorphism of $M^{(l)}$ onto itself:

$$\begin{aligned} \overline{x}^i &= F^i(x, u^{(l)}), \quad i = 1, \ldots, n, \\ \overline{u}^j_{\boldsymbol{J}} &= G^j_{\boldsymbol{J}}(x, u^{(l)}), \quad j = 1, \ldots, m, \quad |\boldsymbol{J}| \le l, \end{aligned} \tag{2.21}$$

for $(x, u^{(l)}) \in M^{(l)}$ with the contact conditions:

$$d\overline{u}^j_{\boldsymbol{J}} - \sum_{i=1}^{n} \overline{u}^j_{\boldsymbol{J},i} d\overline{x}^i = du^j_{\boldsymbol{J}} - \sum_{i=1}^{n} u^j_{\boldsymbol{J},i} dx^i = 0, \quad |\boldsymbol{J}| < l. \tag{2.22}$$

For example, the Legendre transformation

$$\begin{aligned} \overline{x} &= u_x, \\ \overline{u} &= u - x u_x, \\ \overline{u}_{\overline{x}} &= -x, \end{aligned}$$

is a first order contact transformation that is not a first order prolongation of a point transformation. The following significant theorem is due to Bäcklund [12].

Theorem 2.2.16 If there is more than one dependent variable, $m > 1$, then every contact transformation is the prolongation of a point transformation. If $m = 1$, there exist first order contact transformations that are not first order prolongations of point transformations. However, every l-th order contact transformation is the l-th order prolongation of a first order contact transformation.

Proof. See [12, 76] for the proof.

Consider the infinitesimal generator $\mathbf{X}$ of a local one-parameter group of transformations $\exp(\varepsilon\mathbf{X})$ on $M \subset \mathcal{X} \times \mathcal{U}$. We define the l-th order prolongation of $\mathbf{X}$ by

$$\mathbf{X}^{(l)}\Big|_{(x,u^{(l)})} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \exp(\varepsilon\mathbf{X})^{(l)}(x, u^{(l)}), \tag{2.23}$$

In local coordinates, $\mathbf{X}^{(l)}$ can be computed by an explicit formula.

Definition 2.2.17 The *total derivative* with respect to x_i is given by the differential operator

$$D_i = D_{x_i} = \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \sum_{\boldsymbol{J}} u^j_{\boldsymbol{J},i} \frac{\partial}{\partial u^j_{\boldsymbol{J}}}, \tag{2.24}$$

where the summation over the multi-indices $\boldsymbol{J}$ is over all $\boldsymbol{J}'s$ with $|\boldsymbol{J}| \geq 0$.

Theorem 2.2.18 Let

$$\mathbf{X} = \sum_{i=1}^{n} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j(x, u)\frac{\partial}{\partial u^j}$$

be a smooth vector field on M. The l-th order prolongation of $\mathbf{X}$ is the smooth vector field

$$\mathbf{X}^{(l)} = \mathbf{X} + \sum_{j=1}^{m} \sum_{\boldsymbol{J}} \eta^j_{\boldsymbol{J}}(x, u^{(l)})\frac{\partial}{\partial u^j_{\boldsymbol{J}}} \tag{2.25}$$

with the coefficients

$$\eta^j_{\boldsymbol{J}}(x, u^{(l)}) = D_{\boldsymbol{J}}\left(\eta^j - \sum_{i=1}^{n} \xi^i u^j_i\right) + \sum_{i=1}^{n} \xi^i u^j_{\boldsymbol{J},i}, \tag{2.26}$$

where the summation in (2.25) over the multi-indices $\boldsymbol{J}$ is over all unordered multi-indices $\boldsymbol{J} = (j_1, \ldots, j_k)$ with $1 \leq j_\kappa \leq n$ for $\kappa = 1, \ldots, k$, $1 \leq k \leq l$, and $D_{\boldsymbol{J}} = D_{j_1} D_{j_2} \cdots D_{j_k}$.

Proof. See [21, 25, 29, 39, 53, 75, 76, 80] for the proof.

Let $\mathbf{X}$ and $\mathbf{Y}$ be two smooth vector fields on $M \subset \mathcal{X} \times \mathcal{U}$, then their prolongations have the properties [75, 76]:

(1) Linearity: For $c_1, c_2 \in \mathbb{R}$,

$$(c_1\mathbf{X} + c_2\mathbf{Y})^{(l)} = c_1\mathbf{X}^{(l)} + c_2\mathbf{Y}^{(l)}.$$

(2) Lie bracket property:

$$[\mathbf{X}, \mathbf{Y}]^{(l)} = \left[\mathbf{X}^{(l)}, \mathbf{Y}^{(l)}\right].$$

2.2.4 Infinitesimal methods for symmetries

Before discussing symmetries of a DE system, it is necessary to consider a simpler case: symmetries of a system of algebraic equations. Consider a system of algebraic equations defined for x in some manifold M:

$$F_\sigma(x) = 0, \quad \sigma = 1, \ldots, s, \tag{2.27}$$

where $F_\sigma(x) \in C^\infty(M)$, $\sigma = 1, \ldots, s$.

Definition 2.2.19 Let M be a manifold and $S \subset M$. A local group of transformations G acting on M is a *symmetry group* of S if whenever $x \in S$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in S$.

Let $S_F = \{x : F_\sigma(x) = 0, \sigma = 1, \ldots, s\}$ be the set of solutions of the algebraic equation system (2.27). A local group of transformations G is a *symmetry group* of the algebraic equation system (2.27) if it is a symmetry group of S_F.

Definition 2.2.20 Let M and N be two manifolds, and let G be a local group of transformations acting on M. A function $f \colon M \to N$ is a *G-invariant function* if $f(g \cdot x) = f(x)$ for all $x \in M$ and all $g \in G$ such that $g \cdot x$ is defined. If $N = \mathbb{R}$, f is called an *invariant* of G.

Theorem 2.2.21 Let G be a connected group of transformations acting on a manifold M. Then $\zeta \in C^\infty(M)$ is an invariant for G if and only if for every infinitesimal generator $\mathbf{X}$ of G,

$$\mathbf{X}(\zeta) = 0, \quad \text{for all } x \in M. \tag{2.28}$$

Proof. See [75, 76, 80] for the proof.

Definition 2.2.22 Let M be a smooth manifold, $\zeta^1, \ldots, \zeta^k \in C^\infty(M)$ are *functionally dependent* if for arbitrary $x_0 \in M$ there exists a neighborhood U_{x_0} of x_0 and a function $F(y^1, \ldots, y^k) \in C^\infty(\mathbb{R}^k)$ with $F \not\equiv 0$ on any open subset of $\mathbb{R}^k$, such that

$$F\left(\zeta^1(x), \ldots, \zeta^k(x)\right) = 0, \tag{2.29}$$

for all $x \in U_{x_0}$. Otherwise, $\zeta^1, \ldots, \zeta^k$ are *functionally independent*.

Theorem 2.2.23 Let $\exp(\varepsilon\mathbf{X})$ be a one-parameter group of transformations acting on an n-dimensional smooth manifold M, and let $x_0 \in M$ be a regular point for $\mathbf{X}$. Then $\exp(\varepsilon\mathbf{X})$ has precisely $m-1$ functionally independent local invariants $\zeta^1(x), \ldots, \zeta^{n-1}(x)$ defined in a neighborhood of x_0. Moreover, any other local invariant of $\exp(\varepsilon\mathbf{X})$ defined near x_0 is of the form

$$\zeta(x) = F\left(\zeta^1(x), \ldots, \zeta^{m-1}(x)\right), \tag{2.30}$$

for some smooth function F.

Proof. See [75, 76] for the proof.

Theorems 2.2.21 and 2.2.23 provide a method for constructing invariants of a one-parameter group of transformations near a regular point x_0. Let

$$\mathbf{X} = \sum_{i=1}^{n} \xi^i(x)\frac{\partial}{\partial x^i}$$

be the infinitesimal generator of a one-parameter group of transformations $\exp(\varepsilon\mathbf{X})$. According to Theorem 2.2.21, a local invariant $\zeta(x)$ of $\exp(\varepsilon\mathbf{X})$ satisfies

$$\mathbf{X}(\zeta) = \sum_{i=1}^{n} \xi^i(x)\frac{\partial\zeta}{\partial x^i} = 0. \tag{2.31}$$

Theorem 2.2.23 implies there exist $n-1$ functionally independent local invariants of $\exp(\varepsilon\mathbf{X})$ near x_0. The general solution of the linear homogeneous first order PDE (2.31) can be obtained through solving the corresponding characteristic system of ODEs

$$\frac{dx^1}{\xi^1(x)} = \frac{dx^2}{\xi^2(x)} = \cdots = \frac{dx^n}{\xi^n(x)}. \tag{2.32}$$

If the general solution of (2.32) is given by

$$\zeta^1(x^1, \ldots, x^n) = c_1, \ldots, \zeta^{n-1}(x^1, \ldots, x^n) = c_{n-1},$$

where c_i's are constants, then $\zeta^1, \ldots, \zeta^{n-1}$ are functionally independent local invariants.

Example 2.2.24 Consider the one parameter rotation group SO(2) with the infinitesimal generator $\mathbf{X} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. The solution for its corresponding characteristic system

$$\frac{dx}{-y} = \frac{dy}{x} \tag{2.33}$$

is given by $x^2 + y^2 = c$. Thus, $\zeta = x^2 + y^2$ is a local invariant of SO(2), and any local invariant of SO(2) is a function of $\zeta = x^2 + y^2$. Geometrically, the distance between a point and the origin is invariant under the rotation group SO(2).

Theorem 2.2.23 also assures one that there exist $n-1$ functionally independent solutions $(f^1(x), \ldots, f^{n-1}(x))$ of the PDE system (2.13) which are used to obtain corresponding canonical coordinates of a vector field $\mathbf{X}$. In order to obtain $f^n(x)$, one introduces a new variable v and solves the following linear homogeneous first order PDE

$$\sum_{i=1}^{n} \xi^i(x)\frac{\partial \zeta}{\partial x^i} + \frac{\partial \zeta}{\partial v} = 0, \tag{2.34}$$

using the method of characteristics. Since

$$\sum_{i=1}^{n} \xi^i(x)\frac{\partial f^n(x)}{\partial x^i} = 1$$

if and only if $\zeta = v - f^n(x)$ is a solution of (2.34), one can obtain $f^n(x)$ from the solution of (2.34). Thus the $n-1$ functionally independent functions $(f^1(x), \ldots, f^{n-1}(x))$, together with the function $f^n(x)$, yield the corresponding canonical coordinates of $\mathbf{X}$.

Example 2.2.25 Consider the vector field $\mathbf{X} = -y\dfrac{\partial}{\partial x}+x\dfrac{\partial}{\partial y}$. Suppose the canonical coordinates of $\mathbf{X}$ are given by

$$\begin{aligned} z &= f(x, y), \\ w &= g(x, y). \end{aligned} \tag{2.35}$$

From Example 2.2.24, one obtains $f(x, y) = x^2 + y^2$. To obtain $g(x, y)$, one first finds the invariants of $\mathbf{Y} = \mathbf{X} + \dfrac{\partial}{\partial v}$, i.e., solves the ODEs

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dv}{1}. \tag{2.36}$$

Since $r = \sqrt{x^2 + y^2}$ is an invariant of $\mathbf{Y}$, one replaces x by $\sqrt{r^2 - y^2}$ in (2.36). This yields the following equation

$$\frac{dy}{\sqrt{r^2 - y^2}} = \frac{dv}{1}. \tag{2.37}$$

Integrating (2.37) leads to another invariant of $\mathbf{Y}$ given by

$$\zeta = v - \arcsin\frac{y}{r} = v - \arctan\frac{y}{x}.$$

Thus the canonical coordinates of $\mathbf{X}$ are given by

$$\begin{aligned} z &= x^2 + y^2, \\ w &= \arctan \frac{y}{x}, \end{aligned} \tag{2.38}$$

and $\mathbf{X} = \dfrac{\partial}{\partial w}$ in (z, w) coordinates.

Theorem 2.2.26 Suppose the algebraic equation system (2.27) is of maximal rank, i.e., the Jacobian matrix $\left(\dfrac{\partial F_\sigma}{\partial x^k}\right)$ is of rank s for every $x \in S_F$. Then G is a symmetry group of the algebraic equation system (2.27) if and only if for every infinitesimal generator $\mathbf{X}$ of G,

$$\mathbf{X}(F_\sigma(x)) = 0, \qquad \sigma = 1, \dots, s, \tag{2.39}$$

whenever

$$F_\sigma(x) = 0, \qquad \sigma = 1, \dots, s.$$

Proof. See [75] for the proof.

For the case of a DE system, consider a DE system $\mathbf{R}\{x; u\}$ of s DEs of order l with n independent variables $x = (x^1, \dots, x^n)$ and m dependent variables $u = (u^1, \dots, u^m)$ given by

$$R^\sigma(x, u^{(l)}) = R^\sigma(x, u, \partial u, \dots, \partial^l u) = 0, \quad \sigma = 1, \dots, s. \tag{2.40}$$

A *solution* of the DE system $\mathbf{R}\{x; u\}$ (2.40) is a smooth function $u = f(x)$ satisfying

$$R^\sigma(x, f^{(l)}(x)) = 0, \quad \sigma = 1, \dots, s,$$

when x is in the domain of f, i.e.,

$$\Gamma_f^{(l)} = \left\{(x, f^{(l)}(x))\right\} \subset S_{\mathbf{R}\{x;u\}} = \left\{(x, u^{(l)}) : R^\sigma(x, u^{(l)}) = 0, \quad \sigma = 1, \dots, s.\right\}$$

Definition 2.2.27 A *point symmetry* of the DE system $\mathbf{R}\{x; u\}$ (2.40) is a local one-parameter group of transformations G that leaves invariant the solution manifold of $\mathbf{R}\{x; u\}$ (2.40), i.e., if $u = f(x)$ is a solution of $\mathbf{R}\{x; u\}$ (2.40), and the transformed function $\overline{f} = g \cdot f$ is defined for $g \in G$, then $\overline{u} = \overline{f}(\overline{x})$ is also a solution of $\mathbf{R}\{x; u\}$ (2.40).

In order to apply the infinitesimal method to the DE system $\mathbf{R}\{x; u\}$ (2.40), it is necessary to impose some additional conditions on $\mathbf{R}\{x; u\}$ (2.40).

Definition 2.2.28 Consider the DE system $\mathbf{R}\{x; u\}$ (2.40). The system is of *maximal rank* if the rank of its Jacobian matrix

$$J(x, u^{(l)}) = \left(\frac{\partial R^\sigma}{\partial x^i}, \frac{\partial R^\sigma}{\partial u^j_{\boldsymbol{J}}}\right)_{s\times(n+mp^{(l)})} \tag{2.41}$$

with respect to the variables $(x, u^{(l)})$ is s whenever $R^\sigma(x, u^{(l)}) = 0$, $\sigma = 1, \ldots, s$.

The maximal rank condition eliminates the redundancy of a DE system.

Theorem 2.2.29 Suppose the DE system $\mathbf{R}\{x; u\}$ (2.40) is of maximal rank. If every infinitesimal generator $\mathbf{X}$ of a local group of transformations G satisfies

$$\mathbf{X}^{(l)} R^\sigma(x, u^{(l)}) = 0, \quad \sigma = 1, \ldots, s, \tag{2.42}$$

whenever

$$R^\sigma(x, u^{(l)}) = 0, \quad \sigma = 1, \ldots, s,$$

then G is a point symmetry of $\mathbf{R}\{x; u\}$ (2.40).

Proof. See [21, 25, 29, 39, 53, 75, 76, 80] for the proof.

Theorem 2.2.29 provides a systematic way to find point symmetry of a DE system with maximal rank. However, there is no assurance that all point symmetries are found. In order to obtain a necessary and sufficient condition, the given DE system must satisfy another condition.

Definition 2.2.30 Consider the DE system $\mathbf{R}\{x; u\}$ (2.40). The system is *locally solvable* at the point $(x_0, u_0^{(l)}) \in S_{\mathbf{R}\{x;u\}}$ if there exists a smooth solution $u = f(x)$, defined in a neighborhood of x_0, which satisfies $u_0^{(l)} = f^{(l)}(x_0)$. A system is said to be *locally solvable* if it is locally solvable at every point in $S_{\mathbf{R}\{x;u\}}$.

Definition 2.2.31 A DE system is *nondegenerate* if at every point $(x_0, u_0^{(l)}) \in M^{(l)}$ it is both of maximal rank and locally solvable. A DE system is *totally nondegenerate* if it and all its differential consequences are nondegenerate.

Throughout the thesis, unless stated otherwise, all DE systems are assumed to be totally nondegenerate.

Theorem 2.2.32 Suppose the DE system $\mathbf{R}\{x; u\}$ (2.40) is nondegenerate. A local group of transformations G is a point symmetry of $\mathbf{R}\{x; u\}$ (2.40) if and only if

$$\mathbf{X}^{(l)} R^\sigma(x, u^{(l)}) = 0, \quad \sigma = 1, \ldots, s, \tag{2.43}$$

whenever

$$R^\sigma(x, u^{(l)}) = 0, \quad \sigma = 1, \ldots, s,$$

for every infinitesimal generator $\mathbf{X}$ of G.

Proof. See [21, 25, 29, 39, 53, 75, 76, 80] for the proof.

According to the properties of prolonged vector fields, the set of all infinitesimal symmetries of a nondegenerate PDE system forms a Lie algebra.

Definition 2.2.33 An l-th order DE system is *regular* if it is of maximal rank, analytic and contains all its differential consequences up to order l, i.e., no further differential consequences of order l or less can be obtained from the DE system through differentiation or taking integrability conditions.

Theorem 2.2.34 A regular DE system is nondegenerate.

Proof. See [68] and references therein for the proof.

Theorem 2.2.32 provides us a systematic way to find the point symmetries of an l-th order nondegenerate DE system.

Algorithm 2.2.35 (Lie's algorithm for finding point symmetries): Consider an l-th order nondegenerate DE system $\mathbf{R}\{x;u\}$.

1. Let the infinitesimal generator $\mathbf{X}$ of the point symmetries be of the form

$$\mathbf{X} = \sum_{i=1}^{n} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j(x,u)\frac{\partial}{\partial u^j},$$

   where the infinitesimals $\xi^i(x,u)$ and $\eta^j(x,u)$ are unknown functions of x and u to be determined.

2. Find the l-th order prolongation $\mathbf{X}^{(l)}$ of $\mathbf{X}$ according to Theorem 2.2.32.

3. Apply the l-th order prolongation $\mathbf{X}^{(l)}$ to $\mathbf{R}\{x;u\}$, and eliminate the dependencies among the derivatives of u arising from $\mathbf{R}\{x;u\}$ itself.

4. Set the coefficients of the remaining derivatives of u to be zero. This step yields a system of linear PDEs for the unknown functions $\xi^i(x,u)$ and $\eta^j(x,u)$, called the *determining equations* of the infinitesimals of the point symmetries of $\mathbf{R}\{x;u\}$.

5. Solve the determining equations explicitly to obtain the general solutions of $\xi^i(x,u)$ and $\eta^j(x,u)$.

6. Exponentiate the infinitesimal generator $\mathbf{X}$ to obtain the global symmetry groups.

Example 2.2.36 Consider the nonlinear reaction-diffusion equation

$$u_t - u_{xx} = u^3. \tag{2.44}$$

Since the nonlinear reaction-diffusion equation (2.44) is totally nondegenerate, one can apply Algorithm 2.2.35 to find its all point symmetries. Let

$$\mathbf{X} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \tag{2.45}$$

be the infinitesimal generator of a point symmetry of the nonlinear reaction-diffusion equation (2.44). Then $\mathbf{X}$ is an infinitesimal point symmetry of the nonlinear reaction-diffusion equation (2.44) if and only if its second order prolongation $\mathbf{X}^{(2)}$ satisfies

$$\mathbf{X}^{(2)}(u_t - u_{xx} - u^3)\Big|_{u_t = u_{xx}+u^3} = 0. \tag{2.46}$$

According to Step 4 in Algorithm 2.2.35, one obtains the following determining equations

$$\begin{aligned}
&u^3\tau_{uu} - \eta_{uu} + \xi_{xu} = 0,\\
&2\xi_x - u^3\tau_u + \tau_{xx} - \tau_t = 0,\\
&2u^3\tau_{xu} - \xi_t - u^3\xi_u + \xi_{xx} - 2\eta_{xu} = 0,\\
&\eta_t - \eta_{xx} - 3u^2\eta_u + u^3\eta_u - u^3\tau_t - u^6\tau_u + u^3\tau_{xx} = 0,\\
&\tau_x = 0, \;\; \tau_u = 0, \;\; \tau_{uu} = 0, \;\; \xi_{uu} = 0, \;\; \xi_u + \tau_{xu} = 0.
\end{aligned} \tag{2.47}$$

Direct computation shows that the nonlinear reaction-diffusion equation (2.44) only has the three point symmetries given by the infinitesimal generators

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial x}, \;\; \mathbf{X}_2 = \frac{\partial}{\partial t},\\
&\mathbf{X}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.
\end{aligned} \tag{2.48}$$

Moreover, the corresponding one-parameter groups of transformations G_i generated by $\mathbf{X}_i$ are given by

$$\begin{aligned}
&G_1 : (\overline{x}, \overline{t}, \overline{u}) = (x+\varepsilon, t, u),\\
&G_2 : (\overline{x}, \overline{t}, \overline{u}) = (x, t+\varepsilon, u),\\
&G_3 : (\overline{x}, \overline{t}, \overline{u}) = (e^{\varepsilon}x, e^{2\varepsilon}t, e^{-\varepsilon}u).
\end{aligned} \tag{2.49}$$

2.2.5 Contact and higher-order symmetries

The infinitesimals for a point symmetry depend only on x and u. A natural extension of the notion of point symmetry is by allowing the infinitesimals to depend on derivatives of u. We use the notation $P[u]$ to denote P as a smooth function depending on x, u and derivatives of u.

Definition 2.2.37 A *generalized vector field* is an expression of the form

$$\mathbf{X} = \sum_{i=1}^{n} \xi^i[u]\frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j[u]\frac{\partial}{\partial u^j}, \tag{2.50}$$

where ξ^i and η^j are smooth functions.

Analogous to the prolongation of a smooth vector field, the *l-th prolongation* of a generalized vector field $\mathbf{X}$ is given by

$$\mathbf{X}^{(l)} = \mathbf{X} + \sum_{j=1}^{m} \sum_{1\leq|\boldsymbol{J}|\leq l} \eta^j_{\boldsymbol{J}}[u]\frac{\partial}{\partial u^j_{\boldsymbol{J}}} \tag{2.51}$$

with the coefficients

$$\eta^j_{\boldsymbol{J}}[u] = D_{\boldsymbol{J}}\left(\eta^j - \sum_{i=1}^{n} \xi^i u^j_i\right) + \sum_{i=1}^{n} \xi^i u^j_{\boldsymbol{J},i}, \tag{2.52}$$

with the same notation as in Theorem 2.2.14. In particular, the *infinite prolongation* of (2.50) is the infinite sum

$$\mathbf{X}^{(\infty)} = \mathbf{X} + \sum_{j=1}^{m} \sum_{|\boldsymbol{J}|\geq 1} \eta^j_{\boldsymbol{J}}[u]\frac{\partial}{\partial u^j_{\boldsymbol{J}}}, \tag{2.53}$$

where $\eta^j_{\boldsymbol{J}}[u]$ is given by (2.52).

Theorem 2.2.38 If the number of dependent variables is one, i.e., $m = 1$, a generalized vector field $\mathbf{X}$ is an infinitesimal generator of a one-parameter group of a (first order) contact transformation if and only if $\xi^i[u] = \xi^i(x, u, \partial u)$ and $\eta^1[u] = \eta^1(x, u, \partial u)$ satisfy

$$\frac{\partial \eta^1[u]}{\partial u_j} - \sum_{i=1}^{n} u_i \frac{\partial \xi^i[u]}{\partial u_j} = 0, \quad j = 1, \ldots, n. \tag{2.54}$$

Proof. See [25, 76] for the proof.

Definition 2.2.39 A generalized vector field $\mathbf{X}$ is a *higher-order infinitesimal symmetry* (or an infinitesimal generator of a *higher-order symmetry*) of a DE system

$$R^\sigma[u] = 0, \quad \sigma = 1, \dots, s, \tag{2.55}$$

if and only if

$$\mathbf{X}^{(\infty)} R^\sigma[u] = 0, \quad \sigma = 1, \dots, s, \tag{2.56}$$

on any solution of (2.55). In particular, for $m = 1$, if $\mathbf{X}$ is an infinitesimal generator of a one-parameter group of contact transformations, we call $\mathbf{X}$ a *contact infinitesimal symmetry*.

Definition 2.2.40 A *local symmetry* is a point symmetry, contact symmetry or higher-order symmetry.

In practice, it is useful to consider the generalized vector field with $\xi^i[u] = 0$.

Definition 2.2.41 A generalized vector field

$$\mathbf{X}_Q = \sum_{j=1}^{m} Q^j[u] \frac{\partial}{\partial u^j}$$

is called an *evolutionary generalized vector field*. The m-tuple of differential functions $Q[u] = (Q^1[u], \dots, Q^m[u])$ is called its *characteristic*.

The *evolutionary form* of a generalized vector field $\mathbf{X}$ (2.50) is given by

$$\hat{\mathbf{X}} = \sum_{j=1}^{m} \left(\eta^j - \sum_{i=1}^{n} \xi^i u_i^j \right) \frac{\partial}{\partial u^j}. \tag{2.57}$$

The m-tuple of differential functions

$$Q[u] = (Q^1[u], \dots, Q^m[u]) = \left(\eta^1 - \sum_{i=1}^{n} \xi^i u_i^1, \dots, \eta^m - \sum_{i=1}^{n} \xi^i u_i^m \right) \tag{2.58}$$

is the corresponding *characteristic* of the generalized vector field $\mathbf{X}$.

Consider the one-parameter group of point transformations generated by the infinitesimal generator

$$\mathbf{X} = \sum_{i=1}^{n} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j(x, u) \frac{\partial}{\partial u^j}. \tag{2.59}$$

given by

$$\begin{aligned} \bar{x} &= \exp(\varepsilon \mathbf{X})x, \\ \bar{u} &= \exp(\varepsilon \mathbf{X})u, \end{aligned} \tag{2.60}$$

with $x = (x^1, \ldots, x^n)$ and $u = (u^1, \ldots, u^m)$. Let $u = f(x)$ be a surface in $\mathcal{X} \times \mathcal{U}$ space. The one-parameter group of point transformations $\exp(\varepsilon\mathbf{X})$ maps $u = f(x)$ into a family of surfaces $u = g(x; \varepsilon)$ in $\mathcal{X} \times \mathcal{U}$ space. According to the property of one-parameter group of transformations, one obtains

$$\begin{aligned} x &= \exp(-\varepsilon\mathbf{X})\bar{x} = \bar{x} - \varepsilon\xi(\bar{x}, f(\bar{x})) + O(\varepsilon^2), \\ u &= \exp(-\varepsilon\mathbf{X})\bar{u} = \bar{u} - \varepsilon\eta(\bar{x}, f(\bar{x})) + O(\varepsilon^2), \end{aligned} \tag{2.61}$$

where $\xi = (\xi^1, \ldots, \xi^n)$ and $\eta = (\eta^1, \ldots, \eta^m)$. Substituting equations (2.61) into $u = f(x)$, one obtains

$$\begin{aligned} \bar{u} - \varepsilon\eta(\bar{x}, f(\bar{x})) + O(\varepsilon^2) &= f(\bar{x} - \varepsilon\xi(\bar{x}, f(\bar{x})) + O(\varepsilon^2)) \\ &= f(\bar{x}) - \varepsilon \sum_{i=1}^{n} \frac{\partial f(\bar{x})}{\partial \bar{x}^i} \xi^i(\bar{x}, f(\bar{x})) + O(\varepsilon^2). \end{aligned} \tag{2.62}$$

It follows that

$$\bar{u} = f(\bar{x}) + \left(\eta(\bar{x}, f(\bar{x})) - \sum_{i=1}^{n} \frac{\partial f(\bar{x})}{\partial \bar{x}^i} \xi^i(\bar{x}, f(\bar{x})) \right) \varepsilon + O(\varepsilon^2). \tag{2.63}$$

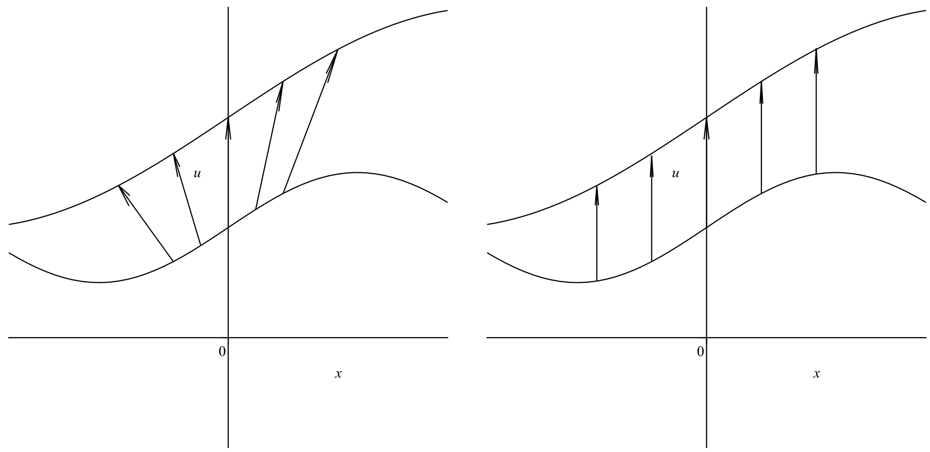
Replacing $\bar{x}$ by x, $\bar{u}$ by u in (2.63), one obtains the image surfaces $u = g(x; \varepsilon)$. Keeping x invariant, the following one-parameter group of transformations

$$\begin{aligned} \bar{x}^i &= x^i, \quad i = 1, \ldots, n, \\ \bar{u}^j &= u + \left(\eta(x, u) - \sum_{k=1}^{n} u^j_k \xi^k(x, u) \right) \varepsilon + O(\varepsilon^2) \quad j = 1, \ldots, m, \end{aligned} \tag{2.64}$$

maps $u = f(x)$ into the same image surfaces $u = g(x; \varepsilon)$. It turns out that the infinitesimal generator $\mathbf{X}$ and its evolutionary form $\hat{\mathbf{X}}$ determine the same action on surfaces. Figure 2.1 illustrates the action of $\exp(\varepsilon\mathbf{X})$ and $\exp(\varepsilon\hat{\mathbf{X}})$.

In general, a generalized vector field $\mathbf{X}$ and its evolutionary form $\hat{\mathbf{X}}$ are equivalent in symmetry analysis [21, 25, 29, 39, 53, 75].

Theorem 2.2.42 A generalized vector field $\mathbf{X}$ is an infinitesimal symmetry of a DE system if and only if its evolutionary form $\hat{\mathbf{X}}$ is.



(a) The action of $\exp(\varepsilon \mathbf{X})$ (b) The action of $\exp(\varepsilon \hat{\mathbf{X}})$

Figure 2.1: The action of $\exp(\varepsilon \mathbf{X})$ and $\exp(\varepsilon \hat{\mathbf{X}})$.

One can extend Lie's algorithm for finding point symmetries to the algorithm for finding local symmetries of a DE system through replacing the symmetry by its evolutionary form and letting the characteristics depend on x, u and a fixed order of derivatives. Then one applies its infinite prolongation to the given DE system. In eliminating the dependencies among the derivatives of u, it is necessary to take the differential consequences of the given DE system into consideration.

Symmetry is one of the main tools in the analysis of DEs. In the ODE case, using a continuous symmetry to integrate an ODE is one of the most important applications. It turns out that one can use a continuous symmetry to reduce the order of a given ODE. If an ODE has a one-parameter symmetry group, then the order of the ODE can be reduced by one. If an ODE has an r-parameter symmetry group and its corresponding Lie algebra is solvable, the order of the ODE can be reduced by r. Moreover, one can obtain the solutions of a given ODE from the solutions of the corresponding reduced ODE. The reduction of the order of an ODE through a symmetry can be obtained in two different ways: canonical coordinates or differential invariants.

In the PDE case, besides constructing new solutions from known ones, one can use symmetries to construct invariant solutions, which will be discussed in Chapter 5. Moreover, one can use the knowledge of symmetries to construct invertible mappings relating PDEs. In [62] (also see [25, 30, 31]), Kumei and Bluman introduced an algorithm to determine whether there exists a smooth invertible mapping that maps a nonlinear PDE system to a linear PDE system based on symmetries and find such a mapping when it exists. In [17, 18] (also see [25, 30]), Bluman presented a symmetry-based algorithm to determine whether a linear PDE with variable coefficients can be invertible mapped to a linear PDE with constant coefficients and find such a mapping when it exists.

2.2.6 Equivalence transformations and symmetry classification

If a given DE system involves some constitutive functions and/or parameters, there exist some special transformations of the system, which preserve the differential structure of the DEs in the family. Such transformations are important in the symmetry analysis of the system. Consider a family $\mathfrak{F}_K$ of DE systems $\mathbf{R}\{x, u; K\}$:

$$R^{\sigma}(x, u^{(l)}; K) = 0, \quad \sigma = 1, \ldots, s, \tag{2.65}$$

involving p constitutive functions and/or parameters $K = (K_1, \ldots, K_p)$.

Definition 2.2.43 An *equivalence transformation* of a family $\mathfrak{F}_K$ of DE systems is a transformation that maps a DE system $\mathbf{R}\{x, u; K\} \in \mathfrak{F}_K$ to another DE system $\bar{\mathbf{R}}\{\bar{x}, \bar{u}; \bar{K}\} \in \mathfrak{F}_K$.

For example, a one-parameter group of equivalence transformations of a family $\mathfrak{F}_K$ of DE systems is a one-parameter group of equivalence transformations

$$\begin{aligned}
\bar{x}^i &= \phi^i(x, u; \varepsilon), \quad i = 1, \ldots, n, \\
\bar{u}^j &= \psi^j(x, u; \varepsilon), \quad j = 1, \ldots, m, \\
\bar{K}_\nu &= \theta_\nu(x, u, K; \varepsilon), \quad \nu = 1, \ldots, p,
\end{aligned} \tag{2.66}$$

which maps a DE system $\mathbf{R}\{x, u; K\} \in \mathfrak{F}_K$ to another DE system $\bar{\mathbf{R}}\{\bar{x}, \bar{u}; \bar{K}\} \in \mathfrak{F}_K$.

Example 2.2.44 Consider the nonlinear reaction-diffusion equation

$$u_t - u_{xx} = Q(u), \tag{2.67}$$

where $Q(u)$ is an arbitrary constitutive function. In order to find the one-parameter groups of equivalence transformations of the nonlinear reaction-diffusion equation (2.67), one treats the constitutive function $Q(u)$ as a new dependent variable and apply Algorithm 2.2.35 to the nonlinear reaction-diffusion equation (2.67). According to the transformations (2.66), one can suppose the infinitesimal generator is given by

$$\mathbf{X} = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u} + \zeta(x, t, u, Q)\frac{\partial}{\partial Q}. \tag{2.68}$$

Applying (2.68) to the nonlinear reaction-diffusion equation (2.67), one obtains the infinitesimal generators given by

$$\begin{aligned}
\mathbf{X}_1 &= uF\frac{\partial}{\partial u} + (QF + uF')\frac{\partial}{\partial Q},\\
\mathbf{X}_2 &= G\frac{\partial}{\partial u} + (G_t - G_{xx})\frac{\partial}{\partial Q},\\
\mathbf{X}_3 &= 2H\frac{\partial}{\partial x} - xuH'\frac{\partial}{\partial u} - x(uH'' + QH')\frac{\partial}{\partial Q},\\
\mathbf{X}_4 &= \frac{1}{2}xP'\frac{\partial}{\partial x} + P\frac{\partial}{\partial t} - \frac{1}{8}x^2uP''\frac{\partial}{\partial u}\\
&\quad + \left(\frac{1}{4}uP'' - \frac{1}{8}x^2uP''' - \frac{1}{8}x^2QP'' - QP'\right)\frac{\partial}{\partial Q},
\end{aligned} \tag{2.69}$$

where $F = F(t)$, $G = G(x,t)$, $H = H(t)$ and $P = P(t)$ are arbitrary functions. Since the constitutive function Q depends only on u, for the above $\mathbf{X}_i$ to generate equivalence transformations, F, G, H and P must satisfy $F(t) = C_1$, $G(x,t) = C_2$, $H' = 0$ and $P'' = 0$, where C_1 and C_2 are constants. Therefore, the infinitesimal generators (2.69) become

$$\begin{aligned}
\mathbf{Y}_1 &= u\frac{\partial}{\partial u} + Q\frac{\partial}{\partial Q},\\
\mathbf{Y}_2 &= \frac{\partial}{\partial u}, \quad \mathbf{Y}_3 = \frac{\partial}{\partial x}, \quad \mathbf{Y}_4 = \frac{\partial}{\partial t}\\
\mathbf{Y}_5 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2Q\frac{\partial}{\partial Q}.
\end{aligned} \tag{2.70}$$

The five-parameter group of equivalence transformations arising from the infinitesimal generators (2.70) is given by

$$\begin{aligned}
\overline{x} &= a_1x + a_2,\\
\overline{t} &= a_1^2t + a_3,\\
\overline{u} &= a_4u + a_5,\\
\overline{Q}(\overline{u}) &= \frac{a_4}{a_1^2}Q(u).
\end{aligned} \tag{2.71}$$

A *symmetry classification problem* of a family $\mathfrak{F}_K$ of DE systems with constitutive functions and/or parameters is to classify DE systems in the family into some subfamilies with the property that all DE systems in the same subfamily admit the same symmetries. It is common to use the group of equivalence transformations to simplify the point symmetry classification problem [1, 2, 56, 68, 80, 82]. Therefore, a point symmetry classification table is usually presented modulo the group of equivalence transformations of the given family of DE systems.

2.3 Conservation laws and the direct method

2.3.1 Conservation laws

Consider a DE system $\mathbf{R}\{x;u\}$ with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$.

Definition 2.3.1 A *conservation law* of $\mathbf{R}\{x;u\}$ is a divergence expression

$$\operatorname{div}(\Phi[u]) = D_1\Phi^1[u] + \cdots + D_n\Phi^n[u] = 0 \tag{2.72}$$

holding for every solution $u = f(x)$ of $\mathbf{R}\{x;u\}$. The functions $\Phi^i[u]$, $i = 1, \ldots, n$, are called the *fluxes* of the CL.

Remark 2.3.2 If one of the independent variables of a PDE system is time t, a CL of the PDE system is of the form

$$D_t\Psi[u] + \sum_{i=1}^{n} D_i\Phi^i[u] = 0, \tag{2.73}$$

where $x = (x^1, \ldots, x^n)$ are n spatial variables; $\Psi[u]$ is the *density* of the CL (2.73), and $\Phi^i[u]$, $i = 1, \ldots, n$, are the *spatial fluxes* of the CL (2.73).

A CL could trivially hold in two different cases.

(1) The n-tuple $\Phi[u]$ in (2.72) vanishes for all solutions of the given DE system. This type of triviality is called the *first* kind of triviality.

(2) The divergence expression $\operatorname{div}(\Phi[u]) \equiv 0$, i.e., $\operatorname{div}(\Phi[u]) = 0$ holds for all functions $u = f(x)$. This type of triviality is called the *second* kind of triviality. Such n-tuple $\Phi[u]$ yields a *null divergence*.

There is a useful characterization of a null divergence as seen in the following theorem.

Theorem 2.3.3 Suppose $\Phi[U] = (\Phi^1[U], \ldots, \Phi^n[U])$ is an n-tuple of smooth functions depending on x, $U = (U^1, \ldots, U^m)$ and their derivatives. Then $\Phi[U]$ yields a null divergence if and only if there exist smooth functions $Q_{ij}[U]$, $i, j = 1, \ldots, n$, such that

$$Q_{ij}[U] = -Q_{ji}[U], \quad i, j = 1, \ldots, n, \tag{2.74}$$

and

$$\Phi^i[U] = \sum_{j=1}^{n} D_jQ_{ij}[U], \quad i = 1, \ldots, n, \tag{2.75}$$

hold for all functions $U(x) = (U^1(x), \ldots, U^m(x))$.

Proof. See [75] for the proof.

Definition 2.3.4 A CL $\operatorname{div}(\Phi[u]) = 0$ of a given PDE system is *trivial* if its fluxes are of the form $\Phi^i[u] = A^i[u] + B^i[u]$, where $A^i[u]$ are the fluxes of a first kind trivial CL and $B^i[u]$ are the fluxes of a second kind trivial CL, $i = 1, \ldots, n$. Two CLs are *equivalent* if they differ by a trivial CL.

Throughout this thesis, unless stated otherwise, a "CL" means a "nontrivial CL".

A set of κ CLs $\left\{\operatorname{div}\left(\Phi_j[u]\right) = 0\right\}_{j=1}^{\kappa}$ is said to be *linearly dependent* if there exists a set of constants $\left\{a^j\right\}_{j=1}^{\kappa}$, not all zero, such that the linear combination $\operatorname{div}\left(\sum_{j=1}^{\kappa} a^j \Phi_j[u]\right) = 0$ is a trivial CL.

Consider a totally nondegenerate DE system $\mathbf{R}\{x; u\}$ with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$ given by

$$R^\sigma[u] = R^\sigma(x, u^{(l)}) = 0, \quad \sigma = 1, \ldots, s. \tag{2.76}$$

Definition 2.3.5 A *multiplier (characteristic)* of a CL $\operatorname{div}(\Phi[u]) = 0$ of the DE system $\mathbf{R}\{x; u\}$ (2.76) is an s-tuple $\Lambda[U] = (\Lambda_1[U], \ldots, \Lambda_s[U])$ such that

$$\sum_{\sigma=1}^{s} \Lambda_\sigma[U] R^\sigma[U] \equiv \operatorname{div}(\Phi[U]) \tag{2.77}$$

holds for all functions $U(x)$. A multiplier is *trivial* if it vanishes for all solutions of the DE system. Two multipliers are *equivalent* if they differ by a trivial multiplier.

In fact, any CL of a totally nondegenerate DE system arises from multipliers to within a trivial CL. Since the DE system $\mathbf{R}\{x; u\}$ (2.76) is totally nondegenerate, $\operatorname{div}(\Phi[u]) = 0$ is a CL of the system if and only if there exist functions $Q_\sigma^{\boldsymbol{J}}$ such that

$$\operatorname{div}(\Phi[U]) \equiv \sum_{\sigma, \boldsymbol{J}} Q_\sigma^{\boldsymbol{J}}[U] D_{\boldsymbol{J}} R^\sigma[U] \tag{2.78}$$

holds for all functions $U(x)$. After integrating by parts, one obtains

$$\operatorname{div}(\Phi[U]) \equiv Q[U] \cdot R[U] + \operatorname{div}(\Psi[U]), \quad R = (R^1, \ldots, R^s), \tag{2.79}$$

where $Q[U] = (Q_1[U], \ldots, Q_s[U])$ with entries $Q_\sigma[U] = \sum_{\boldsymbol{J}} (-D)_{\boldsymbol{J}} Q_\sigma^{\boldsymbol{J}}[U]$, and $\Psi[U] = (\Psi^1[U], \ldots, \Psi^n[U])$ depends linearly on $R^\sigma[U]$ and their derivatives. Therefore, $\operatorname{div}(\Psi[u]) = 0$ is a trivial CL of the given system.

2.3.2 The direct method

Definition 2.3.6 The *Euler operator* with respect to u^α, $1 \le \alpha \le m$, is defined by

$$E_{u^\alpha} = \sum_{\boldsymbol{J}} (-D)_{\boldsymbol{J}} \frac{\partial}{\partial u^\alpha_{\boldsymbol{J}}}, \tag{2.80}$$

where the summation is over all multi-indices $\boldsymbol{J} = (j_1, \ldots, j_k)$ with $1 \le j_\kappa \le n$ for $\kappa = 1, \ldots, k$, $|\boldsymbol{J}| \ge 0$, and $(-D)_{\boldsymbol{J}} = (-1)^k D_{\boldsymbol{J}} = (-D_{j_1})(-D_{j_2}) \cdots (-D_{j_k})$ for $\boldsymbol{J} = (j_1, \ldots, j_k)$.

One of the most important properties of the Euler operators is characterized in the following theorem [9, 25, 75].

Theorem 2.3.7 The equations $E_{U_\alpha}(F[U]) \equiv 0$, $\alpha = 1, \ldots, m$, hold for arbitrary function $U(x)$ if and only if $F[U] = \operatorname{div}(\Phi[U])$ for some m-tuple of differential functions $\Phi[U] = (\Phi^1[U], \ldots, \Phi^m[U])$.

In [9] (also see [25]), Anco and Bluman introduce a systematic way, called *the direct method*, for constructing CLs for the DE system $\mathbf{R}\{x; u\}$ (2.76).

Algorithm 2.3.8 (The direct method for constructing CLs):

1. Let $\Lambda_\sigma[U] = \Lambda_\sigma(x, U^{(k)})$, $\sigma = 1, \ldots, s$, be the k-th order multipliers for CLs of the DE system $\mathbf{R}\{x; u\}$ (2.76). Eliminate the dependence of the derivatives of U due to the DE system $\mathbf{R}\{x; u\}$ (2.76).

2. Solve the system of determining equations generated by

$$E_{U^\alpha}\left(\sum_{\sigma=1}^{s} \Lambda_\sigma[U] R^\sigma[U]\right) \equiv 0, \quad \alpha = 1, \ldots, m, \tag{2.81}$$

   explicitly to obtain its general solution sets $\{\Lambda_\sigma[U]\}_{\sigma=1}^{s}$.

3. Find the corresponding fluxes for each solution set $\{\Lambda_\sigma[U]\}_{\sigma=1}^{s}$. In particular, find an n-tuple of functions $\Phi[U] = (\Phi^1[U], \ldots, \Phi^n[U])$ satisfying the identity

$$\operatorname{div}(\Phi[U]) \equiv \sum_{\sigma=1}^{s} \Lambda_\sigma[U] R^\sigma[U]. \tag{2.82}$$

4. Replace U and their derivatives by u and their derivatives in (2.82) to obtain a CL of the DE system $\mathbf{R}\{x; u\}$ (2.76) given by

$$\operatorname{div}(\Phi[u]) = 0. \tag{2.83}$$

Remark 2.3.9 For a PDE system with n independent variables $(t,x) = (t, x^1, \ldots, x^{n-1})$ and m dependent variables $(u^1, \ldots, u^m)$ in *Cauchy-Kovalevskaya form*:

$$\frac{\partial^{l_j} u^j}{\partial t^{l_j}} = f^j(x, t, u, \partial u, \ldots, \partial^l u), \quad 1 \le l_j \le l, \quad j = 1, \ldots, m, \tag{2.84}$$

where, for each j, the orders of all derivatives with respect to t appearing in f^j are lower than l_j, it suffices to consider specific forms of multipliers. In particular, one can rewrite the PDE system (2.84) into the following equivalent form

$$\frac{\partial \tilde{u}^\nu}{\partial t} = \tilde{f}^\nu(t, x, \tilde{u}, \partial_x \tilde{u}, \ldots, \partial_x^l \tilde{u}), \quad \nu = 1, \ldots, m', \tag{2.85}$$

where $\partial_x^i \tilde{u}$ denotes the i-th order partial derivatives of $\tilde{u}$ with respect to x ($\partial_x \tilde{u} = \partial_x^1 \tilde{u}$). It follows that all derivatives with respect to t can be expressed in terms of t, x, $\tilde{u}$ and derivatives of $\tilde{u}$ with respect to x from the equations (2.85). According to Theorem 2.3.10, all nontrivial local CLs of (2.85), up to equivalence classes, arise from the multipliers of the form $\{\Lambda_\sigma[\tilde{U}] = \Lambda_\sigma(x, t, \tilde{U}, \partial_x \tilde{U}, \ldots, \partial_x^k \tilde{U})\}_{\sigma=1}^{m'}$.

The correspondence between equivalence classes of CLs and those of multipliers of a PDE system in Cauchy-Kovalevskaya form is stated in the following theorem [9, 25, 75].

Theorem 2.3.10 Suppose a given PDE system is in Cauchy-Kovalevskaya form. Let L and $\tilde{L}$ be two CLs of the given PDE system determined from the multipliers Λ and $\tilde{\Lambda}$, respectively. Then L and $\tilde{L}$ are equivalent CLs if and only if Λ and $\tilde{\Lambda}$ are equivalent multipliers.

Once one obtains the set of multipliers for a local CL, a problem is how to find the corresponding fluxes. The method that is used to find fluxes depends on specific problems [4, 6, 8, 9, 42, 93]. For multipliers in simple forms, an effective way is integration by parts. For multipliers in complicated forms, Anco and Bluman introduced the following integral formula for finding the corresponding fluxes [9].

Theorem 2.3.11 For a given set of local CL multipliers $\{\Lambda_\sigma[U] = \Lambda_\sigma(x, U^{(k)})\}_{\sigma=1}^{s}$ of the DE system (2.76), its corresponding fluxes are given by the following integral formulas:

$$\begin{aligned}
\Phi^i[U] = \Phi^i[\tilde{U}] \\
&+ \int_0^1 \Big(S^i\big[U - \tilde{U}, \Lambda[\lambda U + (1-\lambda)\tilde{U}]; R[\lambda U + (1-\lambda)\tilde{U}]\big] \\
&\quad + \tilde{S}^i\big[U - \tilde{U}, R[\lambda U + (1-\lambda)\tilde{U}]; \Lambda[\lambda U + (1-\lambda)\tilde{U}]\big]\Big)\, d\lambda, \\
i = 1, \ldots, n,
\end{aligned} \tag{2.86}$$

with

$$S^i[V,W;R[U]] = \sum_{p=0}^{l-1}\sum_{q=0}^{l-p-1}\sum_{\rho=1}^{m}\sum_{\boldsymbol{I},\boldsymbol{J}}(-1)^q D_{\boldsymbol{I}} V^\rho D_{\boldsymbol{J}}\left(\sum_{\sigma=1}^{s} W_\sigma \frac{\partial R^\sigma[U]}{\partial U^\rho_{\boldsymbol{J},i,\boldsymbol{I}}}\right), \qquad (2.87)$$

and

$$\tilde{S}^i[\tilde{V},\tilde{W};\Lambda[U]] = \sum_{p=0}^{k-1}\sum_{q=0}^{k-p-1}\sum_{\rho=1}^{m}\sum_{\boldsymbol{I},\boldsymbol{J}}(-1)^q D_{\boldsymbol{I}} \tilde{V}^\rho D_{\boldsymbol{J}}\left(\sum_{\sigma=1}^{s} \tilde{W}^\sigma \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho_{\boldsymbol{J},i,\boldsymbol{I}}}\right), \qquad (2.88)$$

for arbitrary functions $V = (V^1(x),\ldots,V^m(x))$, $W = (W_1(x),\ldots,W_s(x))$, $\tilde{V} = (\tilde{V}^1(x),\ldots,\tilde{V}^m(x))$ and $\tilde{W} = (\tilde{W}_1(x),\ldots,\tilde{W}_s(x))$, where $\boldsymbol{J} = (j_1,\ldots,j_q)$ and $\boldsymbol{I} = (i_1,\ldots,i_p)$ are ordered multi-indices such that $1 \le j_1 \le \cdots \le j_q \le i \le i_1 \le \cdots \le i_p \le n$.

Example 2.3.12 Consider the nonlinear diffusion equation

$$u_t - (uu_x)_x = 0. \qquad (2.89)$$

Applying the Euler operator E_U to the function $\Lambda(x,t,U)\,(U_t - (UU_x)_x)$ yields the expression

$$E_U\,(\Lambda(x,t,U)\,(U_t - (UU_x)_x)) \equiv 0, \qquad (2.90)$$

for arbitrary $U(x,t)$. Splitting equation (2.90) with respect to arbitrary U_t, U_x and U_{xx} leads to

$$\begin{aligned} &-\Lambda_U - U\Lambda_{UU} = 0,\\ &-2U\Lambda_U = 0,\\ &-2U\Lambda_{xU} = 0,\\ &-\Lambda_t - U\Lambda_{xx} = 0. \end{aligned} \qquad (2.91)$$

The general solution of equations (2.91) is $\Lambda(x,t,U) = c_1 x + c_2$, where c_1 and c_2 are arbitrary constants. Thus there are two linearly independent multipliers of the form $\Lambda = \Lambda(x,t,U)$: $\Lambda_1 = 1$ and $\Lambda_2 = x$. The corresponding CLs are given by

$$\begin{aligned} \Lambda_1 = 1 :&\quad D_t u - D_x(uu_x) = 0,\\ \Lambda_2 = x :&\quad D_t(xu) - D_x\left(xuu_x - \frac{u^2}{2}\right) = 0. \end{aligned} \qquad (2.92)$$

The next theorem shows that it is possible to use the direct method to find all local CLs for an evolution equation of specific form [54].

Theorem 2.3.13 Consider the (1+1)-dimensional scalar evolution equation with two independent variables (x, t) and one dependent variable u of even order $2l$ given by

$$u_t = F(x, t, u, \partial_x u, \ldots, \partial_x^{2l} u). \tag{2.93}$$

If a CL of the equation (2.93) is given by

$$D_t \Psi[u] + D_x \Phi[u] = 0, \tag{2.94}$$

then the maximal order of a derivative of u in $\Psi[u]$ is l.

A notable result relating CLs and symmetry groups was obtained by Noether [73]. Noether showed that each CL of a DE system admitting a variational principle arises from a point symmetry (variational symmetry) of the action functional. Boyer [38] extended Noether's theorem by taking higher-order symmetries into consideration. In particular, a one-parameter higher-order transformation (in evolutionary form) is a variational symmetry of an action functional if the corresponding Lagrangian is invariant to within a divergence term under such a transformation.

However, there are some limitations of Noether's theorem for finding CLs of a given DE system. First of all, the given DE system is restricted to Euler-Lagrange equations for some variational problem. In addition, it is sometimes difficult to determine the variational symmetry for a given system of Euler-Lagrange equations. It is incorrect that all symmetries of a system of Euler-Lagrange equations are variational symmetries of the corresponding action functional. In order to check whether a symmetry of a system of Euler-Lagrange equations is a variational symmetry of the corresponding action functional, one must firstly determine the corresponding Lagrangian of the action functional. Finally, Noether's theorem is coordinate-dependent since an invertible transformation may transform a DE system admitting a variational principle to a DE system that has no such property. However, any invertible transformation maps a CL of a given DE system to a CL of the transformed one, i.e., CLs are coordinate-independent. Thus an ideal method for finding CLs should be coordinate-independent.

The direct method for finding CLs is superior to Noether's theorem in the sense that it is free of all the above limitations. The direct method can be applied to any DE system, whether it is variational or not. Moreover, it does not require one to find the variational symmetries and Lagrangian when the given DE system is a system of Euler-Lagrange equations. In fact, the direct method directly generates the multipliers for CLs of any given DE system. Most importantly, the direct method is coordinate-independent.

The next theorem shows that a divergence expression is mapped to a divergence expression by a point transformation [25, 35, 81].

Theorem 2.3.14 Under the point transformation

$$
\begin{aligned}
x^i &= x^i(z, W), \quad i = 1, \ldots, n, \\
U^j &= U^j(z, W), \quad j = 1, \ldots, m,
\end{aligned}
\tag{2.95}
$$

where $U = (U^1(x), \ldots, U^m(x))$, $z = (z^1, \ldots, z^n)$ and $W = (W^1(z), \ldots, W^m(z))$, there exists an n-tuple $\Psi[W] = (\Psi^1[W], \ldots, \Psi^n[W])$ such that

$$
J[W]\mathrm{div}(\Phi[U]) = \widetilde{\mathrm{div}}(\Psi[W]), \tag{2.96}
$$

where $\Phi[U] = (\Phi^1[U], \ldots, \Phi^n[U])$, $J[W] = \frac{D(x^1,\ldots,x^n)}{D(z^1,\ldots,z^n)}$ and $\widetilde{\mathrm{div}}$ is the divergence operator on (z, W) space.

Remark 2.3.15 Similar to Theorem 2.3.14, it is straightforward to show that if $m = 1$, then any contact transformation maps a divergence expression into a divergence expression.

The following theorem illustrates how the fluxes change under a symmetry in evolutionary form [75].

Theorem 2.3.16 Let $\mathrm{div}(\Phi[u]) = 0$ be a CL of a totally nondegenerate DE system $\mathbf{R}\{x; u\}$. If $\hat{\mathbf{X}}$ is a symmetry in evolutionary form of $\mathbf{R}\{x; u\}$, then the induced n-tuple $\tilde{\Phi} = \hat{\mathbf{X}}^{(\infty)}(\Phi[u])$, with entries $\tilde{\Phi}^i[u] = \hat{\mathbf{X}}^{(\infty)}(\Phi^i[u])$, also yields a CL: $\mathrm{div}(\tilde{\Phi}[u]) = 0$.

Besides the basic applications of CLs, a CL could also be used to determine whether a nonlinear PDE can be mapped into a linear PDE through an invertible transformation. In [10], Anco, Bluman and Wolf presented an algorithm for determining whether there exist an invertible transformation that maps a nonlinear PDE into a linear PDE through CLs. Moreover, one can use this algorithm to explicitly find such an invertible transformation provided it exists.

Chapter 3

Nonlocally Related PDE Systems and Applications

3.1 Introduction

In Chapter 2, we presented algorithms to find local (point, contact or higher-order) symmetries for a given DE system. Since a symmetry of a DE system is a transformation that keeps its solution manifold invariant, it is possible that the infinitesimals of a continuous symmetry need not depend only on independent and dependent variables and their derivatives. Such symmetries are not local symmetries, and are called *nonlocal symmetries*. A special kind of nonlocal symmetry is one whose infinitesimals depend on integrals of the dependent variables. However, it is not possible to find such nonlocal symmetries through a direct application of Lie's algorithm to the given DE system. In addition, there is the problem of how to use such symmetries.

As stated in Chapter 1, two PDE systems are *equivalent* and *nonlocally related* if they have the following properties.

(1) Any solution of either PDE system yields a solution of the other PDE system.

(2) The solutions of either PDE system yield all solutions of the other PDE system.

(3) The correspondence between the solutions of these two PDE systems is not one-to-one.

Nonlocally related PDE systems are important in the analysis of a given PDE system. In particular, one may be able to obtain new exact solutions for a given PDE system through solutions of its nonlocally related PDE systems. Bluman, Kumei and Reid [32] introduced a systematic CL-based method for constructing nonlocally related PDE systems of a PDE system with two independent variables. In [23], an extended procedure for the construction of a tree of nonlocally related PDE systems was presented. It turns out that one can obtain nonlocal symmetries and nonlocal CLs for a given PDE system through its nonlocally related PDE systems (see [25] and references therein).

In [7], a systematic CL-based method for constructing nonlocally related PDE systems of a PDE system with three or more independent variables was presented. However, in this case, nonlocally related PDE systems arising from divergence-type CLs are invariant under gauge transformations, hence are under-determined. But it turns out that one can also obtain nonlocal CLs from such nonlocally related PDE systems. Unlike the situation for two independent variables, it is shown that, in the case of three or more independent variables, nonlocally related PDE systems arising from divergence-type CLs cannot yield nonlocal symmetries of a given PDE system. In order to find nonlocal symmetries of a given PDE system, it is necessary to add gauge constraints to such nonlocally related PDE systems. Conservation laws of a PDE system with three or more independent variables are not limited to divergence-type CLs. There exist curl-type CLs (*lower-degree CLs*) of a PDE system with three or more independent variables. A systematic method for constructing nonlocally related PDE systems of a PDE system with three or more independent variables through lower-degree CLs is presented in [25, 43, 44]. In addition, it is shown that CLs with degree one can yield determined potential systems.

In this chapter, we present the known CL-based method for constructing nonlocally related PDE systems (potential systems) and the method for constructing subsystems. We also state the method for seeking nonlocal symmetries and nonlocal CLs of a given PDE system through its nonlocally related PDE systems. Moreover, at the end of this chapter, we investigate relationships between symmetries of a given PDE system and those of its potential systems.

Two new results are presented in this chapter. Consider a given PDE system $\mathbf{R}\{x,t;u\}$ with two independent variables (x,t) and m dependent variables $u=(u^1,\dots,u^m)$.

- It is shown that for two potential systems $\mathbf{S}^1\{x,t;u,v\}$ and $\mathbf{S}^2\{x,t;u,w\}$ of $\mathbf{R}\{x,t;u\}$ written in Cauchy-Kovalevskaya form, arising from two nontrivial linearly independent local CLs of $\mathbf{R}\{x,t;u\}$, the potential variable of one system cannot be expressed as a local function in terms of the independent variables, dependent variables and their derivatives of the other system.

- It is shown that if $\mathbf{R}\{x,t;u\}$ has precisely a finite number n of local CLs, then any local symmetry of $\mathbf{R}\{x,t;u\}$ can be obtained by projection from a local symmetry of its corresponding n-plet potential system.

3.2 CL-based method for constructing nonlocally related PDE systems in 2D

3.2.1 Potential systems and subsystems

Consider a PDE system $\mathbf{R}\{x,t;u\}$ with two independent variables (x,t) and m dependent variables $u = (u^1,\ldots,u^m)$ given by

$$R^\sigma[u] = 0, \quad \sigma = 1,\ldots,s. \tag{3.1}$$

In [32], Bluman, Kumei and Reid introduced a systematic way to construct nonlocally related PDE systems of a PDE system $\mathbf{R}\{x,t;u\}$ (3.1) based on its CLs.

Suppose $\mathbf{R}\{x,t;u\}$ (3.1) has a nontrivial CL given by

$$D_t\Psi[u] + D_x\Phi[u] = 0. \tag{3.2}$$

By introducing a *potential variable* v, one obtains a pair of potential equations

$$\begin{aligned} v_x &= \Psi[u], \\ v_t &= -\Phi[u]. \end{aligned} \tag{3.3}$$

Definition 3.2.1 A system of PDEs $\mathbf{S}\{x,t;u,v\}$ consisting of the given PDE system $\mathbf{R}\{x,t;u\}$ and the pair of potential equations (3.3) arising from a CL of $\mathbf{R}\{x,t;u\}$ is a *potential system* of $\mathbf{R}\{x,t;u\}$.

Remark 3.2.2 If the given PDE system $\mathbf{R}\{x,t;u\}$ is a scalar PDE and the CL (3.2) arises from multipliers that do not involve u and its derivatives, it is redundant to add $\mathbf{R}\{x,t;u\}$ to the potential system $\mathbf{S}\{x,t;u,v\}$. One can deduce the PDE in $\mathbf{R}\{x,t;u\}$ from the pair of potential equations through integrability conditions.

Remark 3.2.3 The given PDE system $\mathbf{R}\{x,t;u\}$ and its potential system $\mathbf{S}\{x,t;u,v\}$ are equivalent. Without loss of generality, one can consider the case when the given PDE system $\mathbf{R}\{x,t;u\}$ is a scalar PDE. Suppose $u = f(x,t)$ is a solution of $\mathbf{R}\{x,t;u\}$. Since $D_t\Psi[u] + D_x\Phi[u] = 0$ is a CL of $\mathbf{R}\{x,t;u\}$, due to the integrability condition $v_{xt} = v_{tx}$, there exists a function $g(x,t)$ such that $(u,v) = (f(x,t), g(x,t))$ is a solution of $\mathbf{S}\{x,t;u,v\}$. Thus any solution of $\mathbf{R}\{x,t;u\}$ yields a solution of $\mathbf{S}\{x,t;u,v\}$. Conversely, if $(u,v) = (f(x,t), g(x,t))$ is a solution of $\mathbf{S}\{x,t;u,v\}$, by projection, $u = f(x,t)$ solves $\mathbf{R}\{x,t;u\}$. Hence $\mathbf{R}\{x,t;u\}$ and $\mathbf{S}\{x,t;u,v\}$ are equivalent. Moreover, if $(u,v) = (f(x,t), g(x,t))$ is a solution of $\mathbf{S}\{x,t;u,v\}$, so is $(u,v) = (f(x,t), g(x,t) + C)$, where C is arbitrary constant. It follows that the relationship between the solutions of the given PDE system $\mathbf{R}\{x,t;u\}$ and those of its potential system $\mathbf{S}\{x,t;u,v\}$ is not one-to-one. Hence the given PDE system $\mathbf{R}\{x,t;u\}$ and its potential system $\mathbf{S}\{x,t;u,v\}$ are nonlocally related.

Remark 3.2.4 The potential variable v is a *nonlocal variable* of $\mathbf{R}\{x,t;u\}$, i.e., v cannot be expressed as a local function of the variables in $\mathbf{R}\{x,t;u\}$ and their derivatives. Suppose v is a local variable of $\mathbf{R}\{x,t;u\}$, then $v = F[u]$ on the solutions of $\mathbf{R}\{x,t;u\}$ for some local function F. Since

$$\begin{aligned} v_x &= \Psi[u], \\ v_t &= -\Phi[u], \end{aligned}$$

it follows that

$$\begin{aligned} D_x(F[u]) &= \Psi[u], \\ D_t(F[u]) &= -\Phi[u], \end{aligned}$$

on the solutions of $\mathbf{R}\{x,t;u\}$. Consequently, on the solutions of $\mathbf{R}\{x,t;u\}$, the CL (3.2) can be rewritten into

$$D_t\Psi[u] + D_x\Phi[u] = D_t\left(D_x(F[u])\right) + D_x\left(D_t(F[u])\right) = 0,$$

which implies that the CL (3.2) is a trivial CL. This contradicts the assumption that the CL (3.2) is nontrivial. Hence v is a nonlocal variable.

Since each potential system arising from a nontrivial CL is nonlocally related to the given PDE system, we also use the terminology *nonlocally related CL-based system* (*nonlocally related CL system*) to denote a potential system.

Example 3.2.5 Consider the nonlinear diffusion equation

$$u_t = (K(u)u_x)_x\,. \tag{3.4}$$

The nonlinear diffusion equation (3.4) is in a CL form. By introducing a potential variable, one obtains the potential system given by

$$\begin{aligned} v_x &= u, \\ v_t &= K(u)u_x. \end{aligned} \tag{3.5}$$

Moreover, the nonlinear diffusion equation (3.4) has another CL given by

$$(xu)_t - (x(L(u))_x - L(u))_x = 0, \tag{3.6}$$

where $L'(u) = K(u)$. Based on the CL (3.6), one can construct another potential system of the nonlinear diffusion equation (3.4) given by

$$\begin{aligned} \alpha_x &= xu, \\ \alpha_t &= x(L(u))_x - L(u). \end{aligned} \tag{3.7}$$

For potential systems arising from equivalent CLs, the following theorem shows that such potential systems are locally related [25].

Theorem 3.2.6 If two potential systems $\mathbf{S}^1\{x, t; u, v\}$ and $\mathbf{S}^2\{x, t; u, w\}$ of a given PDE system $\mathbf{R}\{x, t; u\}$ arise from two equivalent CLs of $\mathbf{R}\{x, t; u\}$, then $\mathbf{S}^1\{x, t; u, v\}$ and $\mathbf{S}^2\{x, t; u, w\}$ are locally related. In particular, $w = v + F[u]$ for some function $F[u]$.

The following new theorem concerns the relationship between two potential variables arising from two nontrivial and linearly independent local CLs.

Theorem 3.2.7 Suppose two potential systems $\mathbf{S}^1\{x, t; u, v\}$ and $\mathbf{S}^2\{x, t; u, w\}$ of a given PDE system $\mathbf{R}\{x, t; u\}$ arise from two nontrivial and linearly independent local CLs, where v and w are potential variables. If $\mathbf{S}^1\{x, t; u, v\}$ and $\mathbf{S}^2\{x, t; u, w\}$ are in Cauchy-Kovalevskaya form, then w is a nonlocal variable of $\mathbf{S}^1\{x, t; u, v\}$ and v is a nonlocal variable of $\mathbf{S}^2\{x, t; u, w\}$.

Proof. In order to show w is a nonlocal variable of $\mathbf{S}^1\{x, t; u, v\}$, it suffices to show that w cannot be expressed as a local function of the variables in $\mathbf{S}^1\{x, t; u, v\}$ and their derivatives. Without loss of generality, one can assume that $\mathbf{R}\{x, t; u\}$ is a scalar PDE: $R[u] = 0$. Suppose $\mathbf{S}^1\{x, t; u, v\}$ arises from the CL

$$D_t\Psi^1[u] - D_x\Phi^1[u] = 0, \tag{3.8}$$

with corresponding multiplier $\Lambda_1 = \Lambda_1[U]$, i.e.,

$$\Lambda_1[U]R[U] = D_t\Psi^1[U] - D_x\Phi^1[U], \tag{3.9}$$

for arbitrary U. Since $\mathbf{S}^1\{x, t; u, v\}$ is in Cauchy-Kovalevskaya form, $\mathbf{S}^1\{x, t; u, v\}$ is given by

$$\begin{aligned} v_x &= \Psi^1[u], \\ v_t &= \Phi^1[u]. \end{aligned} \tag{3.10}$$

Suppose $\mathbf{S}^2\{x, t; u, w\}$ arises from the CL

$$D_t\Psi^2[u] - D_x\Phi^2[u] = 0, \tag{3.11}$$

with corresponding multiplier Λ_2, i.e.,

$$\Lambda_2[U]R[U] = D_t\Psi^2[U] - D_x\Phi^2[U], \tag{3.12}$$

for arbitrary U. Since $\mathbf{S}^2\{x, t; u, v\}$ is in Cauchy-Kovalevskaya form, $\mathbf{S}^2\{x, t; u, w\}$ is given by

$$\begin{aligned} w_x &= \Psi^2[u], \\ w_t &= \Phi^2[u]. \end{aligned} \tag{3.13}$$

Assume w can be expressed by a local function of the variables in $\mathbf{S}^1\{x,t;u,v\}$ and their derivatives, i.e., $w = F[u,v]$ for some local function F. Then the CL (3.11) is a trivial CL of $\mathbf{S}^1\{x,t;u,v\}$, since

$$D_t\Psi^2[u] - D_x\Phi^2[u] = D_t(D_xF[u,v]) - D_x(D_tF[u,v]) \tag{3.14}$$

on solutions of $\mathbf{S}^1\{x,t;u,v\}$.

On the other hand,

$$\begin{aligned} D_t\Psi^2[U] - D_x\Phi^2[U] = \Lambda_2 R[U] = \frac{\Lambda_2}{\Lambda_1}\left(D_t(\Psi^1[U]-V_x) - D_x(\Phi^1[U]-V_t)\right) \\ = D_t\left(\frac{\Lambda_2}{\Lambda_1}(\Psi^1[U]-V_x)\right) - D_x\left(\frac{\Lambda_2}{\Lambda_1}(\Phi^1[U]-V_t)\right) \\ - D_t\left(\frac{\Lambda_2}{\Lambda_1}\right)(\Psi^1[U]-V_x) + D_x\left(\frac{\Lambda_2}{\Lambda_1}\right)(\Phi^1[U]-V_t), \end{aligned} \tag{3.15}$$

for arbitrary U and V. The identity (3.15) implies the multipliers for the CL (3.11) with respect to $\mathbf{S}^1\{x,t;u,v\}$ are $D_t\left(\frac{\Lambda_2}{\Lambda_1}\right)$ and $-D_x\left(\frac{\Lambda_2}{\Lambda_1}\right)$.

Theorem 2.3.10 shows that for a PDE system in Cauchy-Kovalevskaya form, a CL is trivial if and only if its multipliers are trivial. Since the CL (3.11) is a second kind trivial CL of $\mathbf{S}^1\{x,t;u,v\}$, it follows that $D_t\left(\frac{\Lambda_2}{\Lambda_1}\right) \equiv 0$ and $-D_x\left(\frac{\Lambda_2}{\Lambda_1}\right) \equiv 0$. Consequently, $\Lambda_2 = c\Lambda_1$ for some constant c. Hence the CLs (3.8) and (3.11) are linearly dependent. It turns out that w is not a local function of the variables in $\mathbf{S}^1\{x,t;u,v\}$ and their derivatives. Thus w is a nonlocal variable of $\mathbf{S}^1\{x,t;u,v\}$.

Similarly, one can show that v is a nonlocal variable of $\mathbf{S}^2\{x,t;u,w\}$. □

If the PDE system $\mathbf{R}\{x,t;u\}$ (3.1) has n linearly independent local CLs:

$$D_t\Psi^i[u] + D_x\Phi^i[u] = 0, \quad i = 1,\ldots,n, \tag{3.16}$$

one can introduce n potential variables v^i with the potential equations:

$$\begin{aligned} v^i_x &= \Psi^i[u], \\ v^i_t &= -\Phi^i[u]. \end{aligned} \tag{3.17}$$

Let $\mathcal{P}^i$ denote the potential equations (3.17). Through the potential equations (3.17), one can obtain n potential systems $\mathbf{S}^{(1)}\{x,t;u,v^i\} = \mathbf{R}\{x,t;u\} \cup \mathcal{P}^i$.

Definition 3.2.8 A *k-plet potential system* ($1 \le k \le n$) of a PDE system $\mathbf{R}\{x,t;u\}$ with n linearly independent local CLs is the potential system

$$\mathbf{S}^{(k)}\{x,t;u,v^{i_1},\ldots,v^{i_k}\} = \mathbf{R}\{x,t;u\} \cup \mathcal{P}^{i_1} \cup \cdots \cup \mathcal{P}^{i_k}. \tag{3.18}$$

In particular, for $k = 1,2,3,4$, such k-plet potential systems are called *singlets*, *couplets*, *triplets* and *quadruplets*, respectively.

Example 3.2.9 Consider the nonlinear diffusion equation (3.4). A couplet potential system of (3.4) is given by

$$\begin{aligned} v_x &= u, \\ v_t &= K(u)u_x, \\ \alpha_x &= xu, \\ \alpha_t &= x(L(u))_x - L(u). \end{aligned} \tag{3.19}$$

For a PDE system $\mathbf{R}\{x, t; u\}$ (3.1) with n linearly independent local CLs, one can construct $2^n - 1$ potential systems:

- n singlets: $\mathbf{S}^{(1)}\{x, t; u, v^i\}$, $i = 1, \dots, n$.
- $\frac{1}{2}n(n-1)$ couplets: $\mathbf{S}^{(2)}\{x, t; u, v^i, v^j\}$ $i, j = 1, \dots, n$ and $i \neq j$.
- $\vdots$
- One n-plet: $\mathbf{S}^{(n)}\{x, t; u, v^1, \dots, v^n\}$.

Definition 3.2.10 For a PDE system $\mathbf{R}\{x, t; u\}$ with n linearly independent local CLs, the set of all $2^n - 1$ potential systems arising from n potential variables v^1, $\dots$, v^n is called a *combination potential system*, denoted by $\mathbb{P}_{v^1 \dots v^n}$.

3.2.2 Subsystems

Another effective way to construct equivalent PDE systems of a PDE system $\mathbf{R}\{x, t; u\}$ with two independent variables (x, t) and $m \geq 2$ dependent variables $u = (u^1, \dots, u^m)$ is through excluding some dependent variables of $\mathbf{R}\{x, t; u\}$.

Definition 3.2.11 Consider a PDE system $\mathbf{R}\{x, t; u\}$ with two independent variables (x, t) and $m \geq 2$ dependent variables $u = (u^1, \dots, u^m)$. A *subsystem* of $\mathbf{R}\{x, t; u\}$ is a PDE system obtained by excluding some dependent variables of $\mathbf{R}\{x, t; u\}$ and has the properties:

(1) Any solution of the subsystem yields a solution of $\mathbf{R}\{x, t; u\}$.

(2) The solutions of the subsystem yield all solutions of $\mathbf{R}\{x, t; u\}$.

Example 3.2.12 Consider the potential system (3.5) of the nonlinear diffusion equation (3.4). By excluding u from (3.5), one obtains the subsystem of (3.5) given by

$$v_t = K(v_x)v_{xx}. \tag{3.20}$$

Example 3.2.13 Consider the Lagrange system of gas dynamics given by

$$\begin{aligned} q_x - v_y &= 0,\\ v_x + p_y &= 0,\\ p_x + B(p,q)v_y &= 0, \end{aligned} \tag{3.21}$$

where $B(p,q)$ is the constitutive function. By excluding v from (3.21), one obtains the subsystem given by

$$\begin{aligned} q_{xx} + p_{yy} &= 0,\\ p_x + B(p,q)q_x &= 0. \end{aligned} \tag{3.22}$$

The following theorem states when a subsystem of $\mathbf{R}\{x,t;u\}$ is nonlocally related to $\mathbf{R}\{x,t;u\}$.

Theorem 3.2.14 A subsystem $\hat{\mathbf{R}}\{x,t;u^1,\ldots,u^{m-1}\}$, obtained by excluding the dependent variable u^m from the PDE system $\mathbf{R}\{x,t;u\}$, is nonlocally related to $\mathbf{R}\{x,t;u\}$ if and only if u^m cannot be directly expressed from the PDEs of $\mathbf{R}\{x,t;u\}$ in terms of x, t, the dependent variables u^1, ..., u^{m-1} of $\hat{\mathbf{R}}\{x,t;u^1,\ldots,u^{m-1}\}$ and their derivatives. Otherwise the subsystem $\hat{\mathbf{R}}\{x,t;u^1,\ldots,u^{m-1}\}$ is locally related to $\mathbf{R}\{x,t;u\}$.

Proof. See [23, 25] for the proof.

From Theorem 3.2.14, one concludes that the PDE (3.20) is a locally related subsystem of the PDE system (3.5), since the excluded variable u can be expressed in terms of x, t, v and its derivatives from the PDEs of the PDE system (3.5). In particular, $u = v_x$. But the PDE system (3.22) is a nonlocally related subsystem of the Lagrange system of gas dynamics (3.21), since the excluded variable v cannot be expressed as a local function of x, y, p, q and their derivatives from the PDEs of the PDE system (3.21).

3.2.3 Procedure for constructing a tree of nonlocally related PDE systems

For a given PDE system $\mathbf{R}\{x,t;u\}$, a basic procedure for the construction of a tree of nonlocally related PDE systems is as follows.

Procedure 3.2.15 (A Tree Construction Procedure)

1. **Construction of potential systems.** For each CL of $\mathbf{R}\{x,t;u\}$, one introduces a potential variable. If n linearly independent CLs are found, one can construct 2^n-1 potential systems. This yields up to 2^n-1 nonlocally related PDE systems. Denote the resulting tree by $\mathcal{T}_1$.

2. **Continuation of construction of potential systems.** For each potential system in $\mathcal{T}_1$, find its CLs using any method. Repeat Step 1 for such potential systems to obtain potential systems of each potential system. Repeat this step to obtain more potential systems. This leads to a tree $\mathcal{T}_2$ containing nonlocally related PDE systems.

3. **Construction of subsystems.** For each PDE system in $\mathcal{T}_3$, by excluding its dependent variables one by one when possible to generate its subsystems. Eliminate locally related PDE systems. This step could result in a larger tree of nonlocally related PDE systems denoted by $\mathcal{T}_3$.

Remark 3.2.16 It is redundant to construct potential systems of the new resulting subsystems in $\mathcal{T}_3$ for the reason that the set of all local CLs of a PDE system includes all local CLs of its subsystems [25].

Remark 3.2.17 It may be difficult to determine whether two such systems are nonlocally related. However, by construction, all PDE systems constructed by Procedure 3.2.15 are equivalent in the sense that the solutions of any such PDE system can be obtained from the solutions of any other such PDE system. Thus redundant (locally related) systems in a tree do not lead to incorrect results.

Remark 3.2.18 Suppose the given PDE system $\mathbf{R}\{x,t;u\}$ has n ($n \geq 2$) linearly independent CLs. Let v^i, $i = 1, \dots, n$ be the corresponding potential variables. It is shown that the linear combinations of such potential variables:

$$w = \sum_{i=i_1}^{i_k} a_i v^i, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n, \quad 1 \leq k \leq n, \tag{3.23}$$

could also yield potential systems that are nonlocally related to $\mathbf{R}\{x,t;u\}$ as well as $\mathbf{S}^{(1)}\{x,t;u,v^i\}$ for arbitrary constants a_i with at least two of them not zero [25, 57]. It follows that a set of n CLs could yield a spectrum of singlet potential systems.

Remark 3.2.19 In Procedure 3.2.15, subsystems are obtained by directly excluding dependent variables of a given PDE system $\mathbf{R}\{x,t;u\}$. In addition, it turns out one can employ hodograph transformations on a PDE system before excluding its dependent variables [20, 24, 25]. By excluding dependent variables from the transformed PDE system, it is possible to generate additional nonlocally related subsystems. More generally, any point transformation could be applied to a PDE system before excluding its dependent variables as long as one is able to exclude some dependent variables from the transformed PDE system.

According to Remarks 3.2.18 and 3.2.19, it is straightforward to modify Steps 2 and 3 in Procedure 3.2.15 to obtain further nonlocally related PDE systems in a tree.

For Step 2 in Procedure 3.2.15, there is an important result to avoid redundant computation in finding new CLs of a potential system [25, 26, 63].

Theorem 3.2.20 A CL of any potential system $\mathbf{S}\{x, t; u, v\}$ is equivalent to a local CL of the given PDE system $\mathbf{R}\{x, t; u\}$ if and only if this CL arises from multipliers that do not essentially depend on the potential variable v, modulo the equivalence class.

According to Theorem 3.2.20, additional CLs of a potential system can only arise from multipliers that include the potential variable v. Therefore, it is necessary to consider multipliers with an essential dependence on at least one potential variable intoduced in Step 2 of Procedure 3.2.15.

Example 3.2.21 Consider the nonlinear wave equation

$$u_{tt} = \left(c^2(u)u_x\right)_x, \tag{3.24}$$

where $c(u)$ is an arbitrary constitutive function. In this example, we use Procedure 3.2.15 to construct a tree of nonlocally related PDE systems of the nonlinear wave equation (3.24).

Using the direct method, one can show that there are four multipliers of the form $\Lambda = \Lambda(x, t, U)$ for arbitrary $c(u)$ [24]. The corresponding CLs are given by

$$\Lambda_1 = 1 : \ u_{tt} - \left(c^2(u)u_x\right)_x = 0, \tag{3.25}$$

$$\Lambda_2 = t : \ D_t\left(tu_t - u\right) - D_x\left(tc^2(u)u_x\right) = 0, \tag{3.26}$$

$$\Lambda_3 = x : \ D_t\left(xu_t\right) - D_x\left(xc^2(u)u_x - \int c^2(u)du\right) = 0, \tag{3.27}$$

and

$$\Lambda_4 = xt : \ D_t\left(x\left(tu_t - u\right)\right) - D_x\left(t\left(xc^2(u)u_x - \int c^2(u)du\right)\right) = 0. \tag{3.28}$$

By introducing potential variables, one obtains four corresponding singlet potential systems given by

$$\begin{cases} v_x = u_t, \\ v_t = c^2(u)u_x. \end{cases} \tag{3.29}$$

$$\begin{cases} w_x = tu_t - u, \\ w_t = tc^2(u)u_x. \end{cases} \tag{3.30}$$

$$\begin{cases} \alpha_x = xu_t, \\ \alpha_t = xc^2(u)u_x - \displaystyle\int c^2(u)du. \end{cases} \tag{3.31}$$

and

$$\begin{cases} \beta_x = x\left(tu_t - u\right), \\ \beta_t = t\left(xc^2(u)u_x - \displaystyle\int c^2(u)du\right). \end{cases} \tag{3.32}$$

The hodograph transformation

$$\begin{cases} x = x(u, v), \\ t = t(u, v), \end{cases} \tag{3.33}$$

maps the potential system (3.29) into an invertibly equivalent linear PDE system

$$\begin{cases} x_v = t_u, \\ x_u = c^2(u)t_v. \end{cases} \tag{3.34}$$

In order to obtain more nonlocally related PDE systems, one seeks local CLs of the PDE system (3.34). If one considers the multipliers of the form $(\Lambda^1, \Lambda^2) = (\Lambda_1(u, v, X, T), \Lambda_2(u, v, X, T))$, through the direct method, one can show that there are only four multipliers holding for all $c(u)$. These multipliers and their corresponding CLs are given by

$$(\Lambda_1^1, \Lambda_1^2) = (1, 0) : \ x_v - t_u = 0, \tag{3.35}$$

$$(\Lambda_2^1, \Lambda_2^2) = (0, 1) : \ x_u - \left(c^2(u)t\right)_v = 0, \tag{3.36}$$

$$(\Lambda_3^1, \Lambda_3^2) = (-X, T) : \ (tx)_u - \left(\frac{x^2 + c^2(u)t^2}{2}\right)_v = 0, \tag{3.37}$$

and

$$(\Lambda_4^1, \Lambda_4^2) = (-v, u) : \ (ux + vt)_u - \left(vx + uc^2(u)t\right)_v = 0. \tag{3.38}$$

From the above four CLs, one obtains four potential systems of the potential system (3.34) given by

$$\begin{cases} x_v = t_u, \\ x_u = c^2(u)t_v, \\ p_v = t, \\ p_u = x. \end{cases} \tag{3.39}$$

$$\begin{cases} x_v = t_u, \\ x_u = c^2(u)t_v, \\ q_v = x, \\ q_u = c^2(u)t. \end{cases} \tag{3.40}$$

$$\begin{cases} x_v = t_u, \\ x_u = c^2(u)t_v, \\ r_v = tx, \\ r_u = \dfrac{x^2 + c^2(u)t^2}{2}. \end{cases} \tag{3.41}$$

and

$$\begin{cases} x_v = t_u, \\ x_u = c^2(u)t_v, \\ \rho_v = ux + vt, \\ \rho_u = vx + uc^2(u)t. \end{cases} \tag{3.42}$$

Now we construct subsystems. After excluding the dependent variable x or t from the potential system (3.34), one obtains two subsystems:

$$x_{vv} = \left(c^{-2}(u)x_u\right)_u, \tag{3.43}$$

and

$$t_{uu} = c^2(u)t_{vv}. \tag{3.44}$$

In [24], it was shown that the nonlinear wave equation (3.24), the potential systems (3.29)–(3.32), the PDEs (3.43) and (3.44) are mutually nonlocally related. Let $\mathcal{T}_1$ denote these nonlocally related PDE systems.

One can show that the potential systems (3.39)–(3.42) are mutually nonlocally related and nonlocally related to each PDE system in $\mathcal{T}_1$.

Moreover, excluding the dependent variable x or t from the four potential systems of the potential system (3.34) leads to additional nonlocally related PDE systems. Take the PDE system (3.39) for example. Since the CL (3.35) is equivalent to the following CL

$$D_u(ut_u - t) - D_v\left(uc^2(u)t_v\right) = 0, \tag{3.45}$$

the PDE system (3.39) is locally related to the following PDE system

$$\begin{cases} x_v = t_u, \\ x_u = c^2(u)t_v, \\ \gamma_v = ut_u - t, \\ \gamma_u = uc^2(u)t_v. \end{cases} \tag{3.46}$$

Excluding x from the PDE system (3.46), one obtains the following PDE system

$$\begin{cases} t_{uu} = \left(c^2(u)t_v\right)_v, \\ \gamma_v = ut_u - t, \\ \gamma_u = uc^2(u)t_v. \end{cases} \tag{3.47}$$

In [24], it was shown that the PDE system (3.47) is nonlocally related to each PDE system in $\mathcal{T}_1$. Moreover, one can show that the PDE system (3.47) is nonlocally related to potential systems (3.39)–(3.42).

In summary, a tree of nonlocally related PDE systems involving the nonlinear wave equation (3.24) is shown in Figure 3.1.

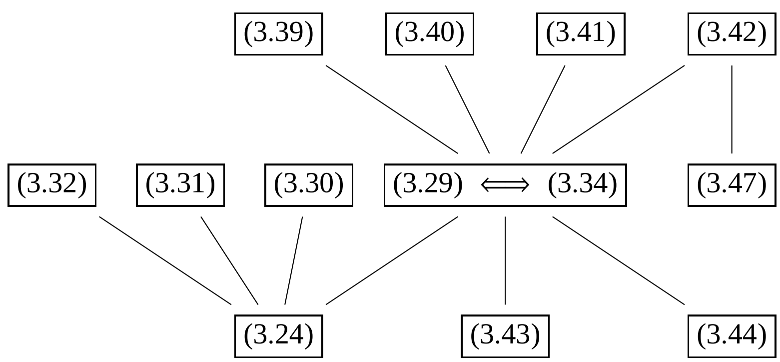


Figure 3.1: A tree of nonlocally related PDE systems for the nonlinear wave equation (3.24).

One can obtain a larger tree of nonloncally related PDE systems if one takes into account the k-plet potential systems.

3.3 Nonlocal symmetries

In the previous section, we showed that one can obtain nonlocal CLs of a given PDE system $\mathbf{R}\{x,t;u\}$ (3.1) from its potential systems. For example, the local CL (3.37) of the PDE system (3.34) is a nonlocal CL of the PDE (3.44). In particular, nonlocal CLs arising from potential systems must have multipliers involving at least one potential variable. However, all local CLs of a subsystem of $\mathbf{R}\{x,t;u\}$ are local CLs of $\mathbf{R}\{x,t;u\}$ [25]. Analogous to nonlocal CLs, symmetries of a given PDE system are not limited to local symmetries. Symmetries that are not local symmetries are called *nonlocal symmetries*. How to find nonlocal symmetries is a significant problem in symmetry analysis. Lie's algorithm provides a simple and applicable way to find local symmetries. However, there does not exist a uniform way to find nonlocal symmetries. In this section, we present a systematic procedure to seek nonlocal symmetries of $\mathbf{R}\{x,t;u\}$ from its nonlocally related PDE systems. It is shown that unlike nonlocal CLs, both potential systems and subsystems in a tree of nonlocally related PDE systems can yield nonlocal symmetries of $\mathbf{R}\{x,t;u\}$.

Let $\mathbf{S}\{x,t;u,v\}$ with $v=(v^1,\ldots,v^k)$ be a k-plet potential system of the PDE system $\mathbf{R}\{x,t;u\}$ (3.1). Suppose $\mathbf{S}\{x,t;u,v\}$ has a point symmetry given by

$$\begin{cases} \bar{x}=x+\varepsilon\xi(x,t,u,v)+\mathrm{O}(\varepsilon^2), \\ \bar{t}=t+\varepsilon\tau(x,t,u,v)+\mathrm{O}(\varepsilon^2), \\ \bar{u}^i=u^i+\varepsilon\eta^i(x,t,u,v)+\mathrm{O}(\varepsilon^2), \quad i=1,\ldots,m, \\ \bar{v}^j=v^j+\varepsilon\zeta^j(x,t,u,v)+\mathrm{O}(\varepsilon^2), \quad j=1,\ldots,k, \end{cases} \tag{3.48}$$

with infinitesimal generator

$$\mathbf{X}=\xi(x,t,u,v)\frac{\partial}{\partial x}+\tau(x,t,u,v)\frac{\partial}{\partial t}+\sum_{i=1}^{m}\eta^i(x,t,u,v)\frac{\partial}{\partial u^i}+\sum_{j=1}^{k}\zeta^j(x,t,u,v)\frac{\partial}{\partial v^j}. \tag{3.49}$$

The symmetry (3.48) leaves the solution manifold of $\mathbf{S}\{x,t;u,v\}$ invariant. Since the solution sets of $\mathbf{S}\{x,t;u,v\}$ and $\mathbf{R}\{x,t;u\}$ are equivalent, by projection, the one-parameter group of transformations (3.48) leads to a mapping that maps any solution of $\mathbf{R}\{x,t;u\}$ to a solution of $\mathbf{R}\{x,t;u\}$. Thus the one-parameter group of transformations (3.48) induces a symmetry of $\mathbf{R}\{x,t;u\}$ with corresponding infinitesimal generator

$$\tilde{\mathbf{X}}=\xi(x,t,u,v)\frac{\partial}{\partial x}+\tau(x,t,u,v)\frac{\partial}{\partial t}+\sum_{i=1}^{m}\eta^i(x,t,u,v)\frac{\partial}{\partial u^i}. \tag{3.50}$$

If the infinitesimals $(\xi(x,t,u,v),\tau(x,t,u,v),\eta^i(x,t,u,v))$ do not depend explicitly on the nonlocal variable v, i.e., $(\xi(x,t,u,v),\tau(x,t,u,v),\eta^i(x,t,u,v))=(\xi(x,t,u),\tau(x,t,u),\eta^i(x,t,u))$, then $\tilde{\mathbf{X}}$ becomes

$$\tilde{\mathbf{X}}=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\sum_{i=1}^{m}\eta^i(x,t,u)\frac{\partial}{\partial u^i}, \tag{3.51}$$

which implies $\mathbf{X}$ only yields a point symmetry of $\mathbf{R}\{x,t;u\}$.

If the infinitesimals $(\xi(x,t,u,v),\tau(x,t,u,v),\eta^i(x,t,u,v))$ essentially depend on the nonlocal variable v, then the one-parameter group of transformations (3.48) defines a nonlocal symmetry of $\mathbf{R}\{x,t;u\}$.

Definition 3.3.1 The infinitesimal generator (3.50), obtained by projection of some infinitesimal point symmetry of a k-plet potential system $\mathbf{S}\{x, t; u, v\}$ of $\mathbf{R}\{x, t; u\}$, generates a *potential symmetry* of $\mathbf{R}\{x, t; u\}$ if the infinitesimals of (3.50) depend explicitly on one or more components of v.

From the above discussion, one has proved the following theorem [25, 29, 32].

Theorem 3.3.2 A potential symmetry of $\mathbf{R}\{x, t; u\}$ is a nonlocal symmetry of $\mathbf{R}\{x, t; u\}$.

Since the solution sets of all systems in a tree of related PDE systems are equivalent, local symmetries of any system yield symmetries of the other systems in the tree. However, for locally related subsystem (in the sense of Theorem 3.2.14), one has the following theorem [25].

Theorem 3.3.3 Any local symmetry of a locally related subsystem $\hat{\mathbf{R}}\{x, t; u^1, \ldots, u^{m-1}\}$ of a PDE system $\mathbf{R}\{x, t; u\}$ is a projection of some local symmetry of $\mathbf{R}\{x, t; u\}$ onto the variable space of $\hat{\mathbf{R}}\{x, t; u^1, \ldots, u^{m-1}\}$.

The correspondence between the solutions of a given PDE system and those of its nonlocally related subsystems is not one-to-one. It is possible that nonlocal symmetries of a given PDE system can arise from local symmetries of its nonlocally related subsystems.

Example 3.3.4 Consider the nonlinear diffusion equation (3.4) for example. The point symmetry classification of the nonlinear diffusion equation (3.4), modulo the equivalence transformations

$$\begin{aligned}
\bar{t} &= a_4 t + a_1, \\
\bar{x} &= a_5 x + a_2, \\
\bar{u} &= a_6 u + a_3, \\
\bar{K} &= \frac{a_5^2}{a_4} K,
\end{aligned} \tag{3.52}$$

where are $a_1, \ldots, a_6$ are arbitrary constants with $a_4 a_5 a_6 \neq 0$, is presented in Table 3.1 [79].

The point symmetry classification of the potential system (3.5), modulo its

Table 3.1: Point symmetry classification for the nonlinear diffusion equation (3.4)

$K(u)$	#	admitted point symmetries
arbitrary	3	$\mathbf{X}_1 = \frac{\partial}{\partial x}$, $\mathbf{X}_2 = \frac{\partial}{\partial t}$, $\mathbf{X}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$
u^μ $(\mu \neq 0)$	4	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 = x\frac{\partial}{\partial x} + \frac{2}{\mu}u\frac{\partial}{\partial u}$
e^u	4	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_5 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}$
$u^{-\frac{4}{3}}$	5	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$ $(\mu = -\frac{4}{3})$, $\mathbf{X}_6 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$

equivalence transformations

$$\begin{aligned}
\bar{t} &= a_1 t + a_2,\\
\bar{x} &= a_3 x + a_4 v + a_5,\\
\bar{u} &= \frac{a_6 + a_7 u}{a_3 + a_4 u},\\
\bar{v} &= a_6 x + a_7 v + a_8,\\
\bar{K} &= \frac{(a_3 + a_4 u)^2}{a_1} K,
\end{aligned} \tag{3.53}$$

where $a_1, \ldots, a_8$ are arbitrary constants with $a_1(a_3a_7 - a_4a_6) \neq 0$, is listed in Table 3.2 [25, 82].

Table 3.2: Point symmetry classification for the potential system (3.5)

$K(u)$	#	admitted point symmetries
arbitrary	4	$\mathbf{Y}_1 = \frac{\partial}{\partial x}$, $\mathbf{Y}_2 = \frac{\partial}{\partial t}$, $\mathbf{Y}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$, $\mathbf{Y}_4 = \frac{\partial}{\partial v}$
u^μ $(\mu \neq 0)$	5	$\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Y}_5 = x\frac{\partial}{\partial x} + \frac{2}{\mu}u\frac{\partial}{\partial u} + \left(1 + \frac{2}{\mu}\right)v\frac{\partial}{\partial v}$
e^u	5	$\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Y}_6 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u} + (2x + v)\frac{\partial}{\partial v}$
u^{-2}	∞	$\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4, \mathbf{Y}_5$ $(\mu = -2)$, $\mathbf{Y}_7 = -xv\frac{\partial}{\partial x} + (xu + v)u\frac{\partial}{\partial u} + 2t\frac{\partial}{\partial v}$, $\mathbf{Y}_8 = -x(2t + v^2)\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} + u(6t + 2xuv + v^2)\frac{\partial}{\partial u}$ $+4tv\frac{\partial}{\partial v}$, $\mathbf{Y}_\infty = F(v,t)\frac{\partial}{\partial x} - u^2G(v,t)\frac{\partial}{\partial u}$, where $(F(v,t), G(v,t))$ is an arbitrary solution of the linear system: $F_t = G_v$, $F_v = G$
$\frac{1}{1+u^2}e^{\lambda \arctan u}$	5	$\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4$, $\mathbf{Y}_9 = v\frac{\partial}{\partial x} + \lambda t\frac{\partial}{\partial t} - (1 + u^2)\frac{\partial}{\partial u} - x\frac{\partial}{\partial v}$

Moreover, we present the point symmetry classification of the couplet potential system (3.19) in Table 3.3 [25], modulo its equivalence transformations

$$\begin{aligned}
\bar{t} &= a_8^2 a_4 t + a_1,\\
\bar{x} &= a_7^{-1} a_8 x + a_5 a_7^{-1} a_8,\\
\bar{u} &= a_7^2 u + a_6 a_7^2,\\
\bar{v} &= a_6 a_7 a_8 x + a_7 a_8 v + a_2,\\
\bar{w} &= a_8^2 w + a_8^2 a_5 v + \tfrac{1}{2} a_6 a_8^2 x^2 + a_5 a_6 a_8^2 x + a_3,\\
\bar{K} &= a_4^{-1} a_7^{-2} K,
\end{aligned} \tag{3.54}$$

and

$$\begin{aligned}
\bar{t} &= t,\\
\bar{x} &= x - a_9 v,\\
\bar{u} &= \frac{u}{1 - a_9 u},\\
\bar{v} &= v,\\
\bar{w} &= -\frac{a_9}{2} v^2 + w,\\
\bar{K} &= (1 - a_9 u)^2 K,
\end{aligned} \tag{3.55}$$

where $a_1, \ldots, a_9$ are arbitrary constants with $a_4 a_7 a_8 \neq 0$.

Comparing Tables 3.1, 3.2 and 3.3, it is immediate to conclude that for some special cases of $K(u)$, the potential system (3.5) and the couplet potential system (3.19) yield nonlocal symmetries of the nonlinear diffusion equation (3.4). For example, when $K(u) = \frac{1}{1+u^2} e^{\lambda \arctan u}$, $\mathbf{Y}_9$ and $\mathbf{Z}_{12}$ yield the same nonlocal symmetry of the nonlinear diffusion equation (3.4). In addition, the couplet potential system (3.19) could also yield nonlocal symmetries of the potential system (3.5). For example, the symmetry $\mathbf{Z}_8$ yields a nonlocal symmetry of the potential system (3.5) when $K(u) = u^{-\frac{4}{3}}$; the symmetry $\mathbf{Z}_9$ yields a nonlocal symmetry of the potential system (3.5) when $K(u) = u^{-\frac{2}{3}}$.

Proposition 3.3.5 The symmetry $\mathbf{X}_6$ yields a nonlocal symmetry of the potential system (3.5) with $K(u) = u^{-\frac{4}{3}}$.

Proof. Suppose the symmetry $\mathbf{X}_6$ yields a local symmetry of the potential system (3.5) with $K(u) = u^{-\frac{4}{3}}$. Consequently, there must exist a differential function $f[u, v]$ such that, in evolutionary form, $\tilde{\mathbf{X}}_6 = (-3xu - x^2 u_x)\frac{\partial}{\partial u} + f[u, v]\frac{\partial}{\partial v}$ is a local symmetry of the potential system (3.5). Since $v_x = u$, $v_t = u^{-\frac{4}{3}} u_x$ and $u_t = (u^{-\frac{4}{3}} u_x)_x$, one can restrict $f[u, v]$ to be of the form $f(x, t, u, v, u_x, u_{xx}, ...)$ depending on x, t, u

Table 3.3: Point symmetry classification for the potential system (3.19)

$K(u)$	#	admitted point symmetries
arbitrary	5	$\mathbf{Z}_1=\frac{\partial}{\partial x}+v\frac{\partial}{\partial \alpha}$, $\mathbf{Z}_2=\frac{\partial}{\partial t}$, $\mathbf{Z}_3=\frac{\partial}{\partial v}$, $\mathbf{Z}_4=\frac{\partial}{\partial \alpha}$, $\mathbf{Z}_5=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}+v\frac{\partial}{\partial v}+2\alpha\frac{\partial}{\partial \alpha}$
u^{μ} $(\mu\neq 0)$	6	$\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, $\mathbf{Z}_5$, $\mathbf{Z}_6=x\frac{\partial}{\partial x}+\frac{2}{\mu}u\frac{\partial}{\partial u}+\left(1+\frac{2}{\mu}\right)v\frac{\partial}{\partial v}+2\alpha\left(1+\frac{1}{\mu}\right)\frac{\partial}{\partial \alpha}$
e^{u}	6	$\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, $\mathbf{Z}_5$, $\mathbf{Z}_7=x\frac{\partial}{\partial x}+2\frac{\partial}{\partial u}+(2x+v)\frac{\partial}{\partial v}+(x^2+2\alpha)\frac{\partial}{\partial \alpha}$
$u^{-\frac{4}{3}}$	7	$\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, $\mathbf{Z}_5$, $\mathbf{Z}_6$ $(\mu=-\frac{4}{3})$, $\mathbf{Z}_8=x^2\frac{\partial}{\partial x}-3xu\frac{\partial}{\partial u}-\alpha\frac{\partial}{\partial v}$
$u^{-\frac{2}{3}}$	7	$\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, $\mathbf{Z}_5$, $\mathbf{Z}_6$ $(\mu=-\frac{2}{3})$, $\mathbf{Z}_9=(xv-\alpha)\frac{\partial}{\partial x}-3uv\frac{\partial}{\partial u}-v^2\frac{\partial}{\partial v}-v\alpha\frac{\partial}{\partial \alpha}$
u^{-2}	∞	$\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, $\mathbf{Z}_5$, $\mathbf{Z}_6$ $(\mu=-2)$, $\mathbf{Z}_{10}=-(xv+\alpha)\frac{\partial}{\partial x}+(2xu+v)u\frac{\partial}{\partial u}+2t\frac{\partial}{\partial v}$ $-v\alpha\frac{\partial}{\partial \alpha}$, $\mathbf{Z}_{11}=-(6xt+xv^2+2v\alpha)\frac{\partial}{\partial x}+4t^2\frac{\partial}{\partial t}$ $+u(10t+4xuv+2u\alpha+v^2)\frac{\partial}{\partial u}$ $+4tv\frac{\partial}{\partial v}-(2t+v^2)\alpha\frac{\partial}{\partial \alpha}$, $\mathbf{Z}_{\infty}=F(v,t)\frac{\partial}{\partial x}-u^2G(v,t)\frac{\partial}{\partial u}+H(v,t)\frac{\partial}{\partial v}$, where $(F(v,t),G(v,t),H(v,t))$ is an arbitrary solution of the linear system: $F_v=G$, $H_v=F$, $H_t=G$
$\frac{1}{1+u^2}e^{\lambda\arctan u}$	6	$\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, $\mathbf{Z}_5$, $\mathbf{Z}_{12}=v\frac{\partial}{\partial x}+\lambda t\frac{\partial}{\partial t}-(1+u^2)\frac{\partial}{\partial u}-x\frac{\partial}{\partial v}+\frac{v^2-x^2}{2}\frac{\partial}{\partial \alpha}$

and the partial derivatives of u with respect to x. Firstly, suppose $f[u,v]$ is of the form $f(x,t,u,v,u_x)$. Applying $\tilde{\mathbf{X}}_6^{(\infty)}$ to the potential system (3.5), one obtains

$$\begin{aligned} &f_x + f_u u_x + f_v v_x + f_{u_x} u_{xx} = -3xu - x^2 u_x,\\ &f_t + f_u u_t + f_v v_t + f_{u_x} u_{tx} = \tfrac{4}{3}(3xu + x^2 u_x)u^{-\frac{7}{3}} u_x + D_x(-3xu - x^2 u_x)u^{-\frac{4}{3}} \end{aligned} \tag{3.56}$$

on every solution of the potential system (3.5). After making appropriate substitutions and equating the coefficients of the term u_{xx}, one obtains $f_{u_x} = 0$. By similar reasoning, one can show that $f(x,t,u,v,u_x,u_{xx},...)$ has no dependence on any partial derivative of u with respect to x. Hence $f[u,v]$ is of the form $f(x,t,u,v)$. Consequently, if $\mathbf{X}_6$ yields a local symmetry of the nonlinear diffusion equation (4.45) with $K(u) = u^{-\frac{4}{3}}$, then $\tilde{\mathbf{X}}_6$ must be a point symmetry of the corresponding potential system (3.5).

Comparing Tables 3.1 and 3.2, one immediately sees that symmetry $\mathbf{X}_6$ does not yield a point symmetry of the corresponding potential system (3.5). This follows from the fact that when $K(u) = u^{-\frac{4}{3}}$, the potential system (3.5) has no point symmetry whose infinitesimal components corresponding to the variables (x,t) are the same as those for $\mathbf{X}_6$. Hence $\mathbf{X}_6$ yields a nonlocal symmetry of the potential system (3.5) with $K(u) = u^{-\frac{4}{3}}$.

Remark 3.3.6 Proposition 3.3.5 shows that a local symmetry of a subsystem of a given PDE system can yield a nonlocal symmetry of the given PDE system, since the point symmetry $\mathbf{X}_6$ of the nonlinear diffusion equation (3.4), as a subsystem of (3.5), yields a nonlocal symmetry of the potential system (3.5).

3.4 Nonlocally related systems in three or more dimensions

In previous sections we showed how to construct nonlocally related PDE systems for a PDE system with two independent variables, and how to use such nonlocally related PDE systems to find nonlocal symmetries and nonlocal CLs for the given PDE system. Now consider a PDE system $\mathbf{R}\{x;u\}$ with n ($n \geq 3$) independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$, given by

$$R^{\sigma}[u] = 0, \quad \sigma = 1, \ldots, s. \tag{3.57}$$

The situation in the case of $n \geq 3$ independent variables is more complicated than in the case of two independent variables, since there exist several different types of CLs in higher dimensions. In this section, we present a systematic method for the construction of nonlocally related PDE systems of $\mathbf{R}\{x;u\}$ (3.57) using divergence-type CLs [7, 25, 43, 44]. For nonlocally related PDE systems arising from lower-degree CLs, one can refer to [25, 43, 44] for more details.

Suppose $\mathbf{R}\{x; u\}$ (3.57) has a local CL given by

$$\operatorname{div}(\Phi[u]) = \sum_{i=1}^{n} D_i \Phi^i[u] = 0. \tag{3.58}$$

By introducing n^2 potential variables v^{jk} $(j, k = 1, \ldots, n)$ with $v^{jk} = -v^{kj}$, one obtains n potential equations

$$\sum_{j=1}^{n} D_j v^{ij} = \Phi^i[u], \quad i = 1, \ldots, n. \tag{3.59}$$

Since $v^{jk} = -v^{kj}$, the potential equations (3.59) only involve $\frac{n(n-1)}{2}$ potential variables, say v^{jk} $(j < k)$. It is straightforward to show that the system of potential equations (3.59) is equivalent to the CL (3.58). Note that the system of potential equations (3.59) is under-determined. In particular, the system of potential equations (3.59) is invariant under the transformations

$$v^{ij} \to v^{ij} + D_k w^{ijk}, \tag{3.60}$$

where w^{ijk} are $\frac{n(n-1)(n-2)}{6}$ arbitrary functions that are components of a totally antisymmetric tensor. Hence the system of potential equations (3.59) has an infinite number of point symmetries

$$\mathbf{X}_{\text{gauge}} = \sum_{i,j,k} D_k w^{ijk} \frac{\partial}{\partial v^{ij}}, \tag{3.61}$$

which are called *gauge symmetries*.

The system of potential equations (3.59) together with the given PDE system $\mathbf{R}\{x; u\}$ (3.57) yields a *potential system* $\mathbf{S}\{x; u, v\}$ of $\mathbf{R}\{x; u\}$, given by

$$\begin{aligned} &R^\sigma[u] = 0, \quad \sigma = 1, \ldots, s. \\ &\sum_{j=1}^{n} D_j v^{ij} = \Phi^i[u], \quad i = 1, \ldots, n. \end{aligned} \tag{3.62}$$

As in the case of two dependent variables, one can show that the potential system $\mathbf{S}\{x; u, v\}$ (3.62) is nonlocally related to $\mathbf{R}\{x; u\}$ (3.57).

Example 3.4.1 Consider a PDE system $\mathbf{R}\{x, y, z; u\}$ with three independent variables. Suppose $\mathbf{R}\{x, y, z; u\}$ has a local CL

$$\operatorname{div}(\Phi^i[u]) = 0. \tag{3.63}$$

Hence one can introduce three potential variables $v = (v^1, v^2, v^3)$ to obtain three potential equations

$$\begin{aligned} v^3_y - v^2_z &= \Phi^1[u], \\ v^1_z - v^3_x &= \Phi^2[u], \\ v^2_x - v^1_y &= \Phi^3[u]. \end{aligned} \tag{3.64}$$

Therefore, a potential system $\mathbf{S}\{x, y, z; u, v\}$ of $\mathbf{R}\{x, y, z; u\}$ is given by $\mathbf{R}\{x, y, z; u\}$ and the system of potential equations (3.64). The potential system $\mathbf{S}\{x, y, z; u, v\}$ is invariant under the transformations

$$(v^1, v^2, v^3) \to (v^1, v^2, v^3) + (D_x w[u], D_y w[u], D_z w[u]). \tag{3.65}$$

It follows that the gauge symmetries of $\mathbf{S}\{x, y, z; u, v\}$ are given by

$$\mathbf{X}_{\text{gauge}} = D_x w[u]\frac{\partial}{\partial v^1} + D_y w[u]\frac{\partial}{\partial v^2} + D_z w[u]\frac{\partial}{\partial v^3}. \tag{3.66}$$

Due to the under-determined property of the potential system, the situation for seeking nonlocal symmetries is different in three or more independent variables case. The following important theorem shows nonlocal symmetries cannot arise from the under-determined potential system $\mathbf{S}\{x; u, v\}$ (3.62) [7, 25].

Theorem 3.4.2 Each local symmetry of an under-determined potential system $\mathbf{S}\{x; u, v\}$ (3.62) projects onto a local symmetry of $\mathbf{R}\{x; u\}$ (3.57).

In order to eliminate the gauge freedom of the potential system $\mathbf{S}\{x; u, v\}$ (3.62), it is necessary to add gauge constraints to $\mathbf{S}\{x; u, v\}$. The choice of gauge constraints will depend on particular problems. But the corresponding gauge-constrained (determined) potential system $\tilde{\mathbf{S}}\{x; u, v\}$ must have the property: all solutions of $\mathbf{S}\{x; u, v\}$ can be obtained from the solutions of $\tilde{\mathbf{S}}\{x; u, v\}$. Examples of gauge constraints for the potential system (3.64):

- divergence gauge: $\operatorname{div}(v) = v^1_x + v^2_y + v^3_z = 0$,
- algebraic gauge: $v^k = 0$, $k = 1$ or 2 or 3,
- Poincaré gauge: $xv^1 + yv^2 + zv^3 = 0$.

If x represents the time, i.e, $x = t$, the following gauge constraints are frequently used in applications:

- Lorentz gauge: $v^1_t - v^2_y - v^3_z = 0$,
- Cronstrom gauge: $tv^1 - yv^2 - zv^3 = 0$.

It is shown that a determined potential system could yield nonlocal symmetries of a given PDE system with three or more independent variables [7, 25, 43, 44]. However, not all determined potential systems can yield nonlocal symmetries of a given PDE system. For example, consider the wave equation

$$u_{tt} = u_{xx} + u_{yy}. \tag{3.67}$$

A determined potential system is obtained by adding the Lorentz gauge to its under-determined potential system. Hence the determined potential system is given by

$$\begin{aligned} v^3_x - v^2_y &= u_t, \\ v^1_y - v^3_t &= -u_x, \\ v^2_t - v^1_x &= -u_y, \\ v^1_t - v^2_y - v^3_z &= 0. \end{aligned} \tag{3.68}$$

It turns out that there exist point symmetries of the potential system (3.68) that project onto nonlocal symmetries of the wave equation (3.67) [25, 44]. However, it is also shown that if one replaces the Lorentz gauge by the divergence gauge, the algebraic gauge, the Poincaré gauge, or the Cronstrom gauge, no nonlocal symmetries of the wave equation (3.67) arise from these determined potential systems [25, 44].

Similar to the case of two independent variables, one can construct a tree of nonlocally related PDE systems of $\mathbf{R}\{x; u\}$ (3.57). In seeking additional CLs for the potential systems, there is a similar result to Theorem 3.2.20 [25, 26, 63].

Theorem 3.4.3 Suppose $\mathbf{R}\{x; u\}$ (3.57) has a CL (3.58). Let $\mathbf{S}\{x; u, v\}$ be the potential system of $\mathbf{R}\{x; u\}$ consisting of $\mathbf{R}\{x; u\}$ and the potential equations (3.59). Then each CL of $\mathbf{S}\{x; u, v\}$, arising from multipliers that do not involve the potential variables v, is equivalent to a local CL of $\mathbf{R}\{x; u\}$.

It is important to note that Theorem 3.4.3 does not hold for a potential system with a gauge constraint [25]. Moreover, unlike the situation for nonlocal symmetries, nonlocal CLs can arise from both determined and under-determined potential systems [11, 25].

3.5 Relationships between local symmetries of PDE systems

As discussed in Section 3.3, an effective way to seek nonlocal symmetries of a given PDE system is to apply Lie's algorithm to PDE systems in a tree of nonlocally related PDE systems. It has been shown that both potential systems and subsystems can yield nonlocal symmetries. Does there exist any relationship between local symmetries of a given PDE system and those of its potential systems?

In next new theorem, a correspondence between local symmetries of a given PDE system having precisely n linearly independent local CLs and those of its potential systems is presented.

Theorem 3.5.1 Suppose a PDE system $\mathbf{R}\{x,t;u\}$ with two independent variables (x,t) and m dependent variables $u=(u^1,\ldots,u^m)$ given by

$$R^\sigma[u]=0, \quad \sigma=1,\ldots,s, \tag{3.69}$$

has precisely n linearly independent local CLs. Then any local symmetry of the PDE system $\mathbf{R}\{x,t;u\}$ (3.69) can be obtained by projection of some local symmetry of its n-plet potential system.

Proof. Let the n local CLs of $\mathbf{R}\{x,t;u\}$ (3.69) be given by $D_t\Psi^j[u]+D_x\left(-\Phi^j[u]\right)=0$ for some densities $\Psi^j[u]$ and fluxes $-\Phi^j[u]$, $j=1,\ldots,n$. Then the corresponding n-plet potential system of $\mathbf{R}\{x,t;u\}$ (3.69) is given by

$$\begin{aligned} &v^j_x=\Psi^j[u],\\ &v^j_t=\Phi^j[u], \quad j=1,\ldots,n,\\ &R^\sigma[u]=0, \quad \sigma=1,\ldots,s. \end{aligned} \tag{3.70}$$

Suppose $\hat{\mathbf{X}}=\displaystyle\sum_{i=1}^m \eta^i[u]\frac{\partial}{\partial u^i}$ is a local symmetry of $\mathbf{R}\{x,t;u\}$ (3.69). It suffices to prove that there exist functions $\zeta^j[u,v]$, $j=1,\ldots,n$, such that $\hat{\mathbf{Y}}=\hat{\mathbf{X}}+\displaystyle\sum_{j=1}^n \zeta^j[u,v]\frac{\partial}{\partial v^j}$ is a local symmetry of the n-plet potential system (3.70).

Applying the corresponding infinite prolongation

$$\hat{\mathbf{Y}}^{(\infty)}=\hat{\mathbf{X}}^{(\infty)}+\sum_{j=1}^n \zeta^j[U,V]\frac{\partial}{\partial V^j}+\sum_{\boldsymbol{J}} D_{\boldsymbol{J}}\zeta^j[U,V]\frac{\partial}{\partial V^j_{\boldsymbol{J}}}$$

to the functions

$$\begin{aligned} &V^j_x-\Psi^j[U],\\ &V^j_t-\Phi^j[U], \quad j=1,\ldots,n,\\ &R^\sigma[U], \quad \sigma=1,\ldots,s, \end{aligned} \tag{3.71}$$

one obtains

$$\begin{aligned}
&D_x\zeta^j[U,V] - \hat{\mathbf{X}}^{(\infty)}\Psi^j[U],\\
&D_t\zeta^j[U,V] - \hat{\mathbf{X}}^{(\infty)}\Phi^j[U], \quad j = 1,\dots,n, \\
&\hat{\mathbf{X}}^{(\infty)}R^\sigma[U], \quad \sigma = 1,\dots,s.
\end{aligned} \tag{3.72}$$

Then $\hat{\mathbf{Y}}$ is a local symmetry of the n-plet potential system (3.70) if and only if (3.72) vanishes on any solution $(U,V) = (u,v) = (u(x,t), v(x,t))$ of the n-plet potential system (3.70). From Theorem 2.2.32, $\hat{\mathbf{X}}$ is a local symmetry of $\mathbf{R}\{x,t;u\}$ (3.69) if and only if its infinite prolongation $\hat{\mathbf{X}}^{(\infty)}$ satisfies

$$\hat{\mathbf{X}}^{(\infty)}R^\sigma[u] = 0, \quad \sigma = 1,\dots,s, \tag{3.73}$$

on any solution of $\mathbf{R}\{x,t;u\}$ (3.69). Hence the equations (3.73) hold on any solution of the n-plet potential system (3.70). Therefore, it suffices to prove that there exist some functions $\zeta^j[u,v]$ so that the equations

$$\begin{aligned}
D_x\zeta^j[u,v] &= \hat{\mathbf{X}}^{(\infty)}\Psi^j[u],\\
D_t\zeta^j[u,v] &= \hat{\mathbf{X}}^{(\infty)}\Phi^j[u], \quad j = 1,\dots,n,
\end{aligned} \tag{3.74}$$

hold on any solution of the n-plet potential system (3.70).

Theorem 2.3.16 shows that a symmetry maps fluxes (densities) to fluxes (densities). Since $\hat{\mathbf{X}}$ is a local symmetry of $\mathbf{R}\{x,t;u\}$ (3.69), the entries $\tilde{\Psi}^j[u] = \hat{\mathbf{X}}^{(\infty)}\Psi^j[u]$ and $-\tilde{\Phi}^j[u] = \hat{\mathbf{X}}^{(\infty)}\left(-\Phi^j[u]\right)$, $j = 1,\dots,n$, must be densities and fluxes of CLs of $\mathbf{R}\{x,t;u\}$ (3.69), i.e., for each $j = 1,\dots,n$, $D_t\tilde{\Psi}^j[u] + D_x\left(-\tilde{\Phi}^j[u]\right) = 0$ is a CL of $\mathbf{R}\{x,t;u\}$ (3.69). Since $\mathbf{R}\{x,t;u\}$ (3.69) has precisely n linear independent local CLs, it follows that, for each $j = 1,\dots,n$,

$$\begin{aligned}
\hat{\mathbf{X}}^{(\infty)}\Psi^j[u] = \tilde{\Psi}^j[u] &= \sum_{k=1}^n a_k^j\Psi^k[u] + T^j[u],\\
\hat{\mathbf{X}}^{(\infty)}\Phi^j[u] = \tilde{\Phi}^j[u] &= \sum_{k=1}^n a_k^j\Phi^k[u] + S^j[u],
\end{aligned} \tag{3.75}$$

for some constants a_k^j, $k = 1,\dots,n$, and $D_tT^j[u] - D_xS^j[u] = 0$ is a trivial CL of $\mathbf{R}\{x,t;u\}$ (3.69). In particular, for each $j = 1,\dots,n$, $T^j[u] = A^j[u] + B^j[u]$ and $S^j[u] = F^j[u] + G^j[u]$, where $D_tA^j[u] - D_xF^j[u] = 0$ is a first kind trivial CL and $D_tB^j[u] - D_xG^j[u] = 0$ is a second kind trivial CL. It follows that $(A^j[u], F^j[u])$ vanish on any solution of $\mathbf{R}\{x,t;u\}$ (3.69), and hence $(A^j[u], F^j[u])$ vanish on any solution of the n-plet potential system (3.70). Since $(B^j[u], G^j[u])$ yield a null divergence, according to Theorem 2.3.3, there exists some function $H^j[u]$ such that $B^j[u] = D_x H^j[u]$ and $G^j[u] = D_t H^j[u]$.

Now let $\zeta^j[U, V]$ be the functions

$$\zeta^j[U, V] = \sum_{k=1}^{n} a_k^j V^k + H^j[U], \quad j = 1, \dots, n. \tag{3.76}$$

Then, on any solution $(U, V) = (u, v) = (u(x, t), v(x, t))$ of the n-plet potential system (3.70), one has

$$\begin{aligned}
D_x\zeta^j[u, v] &= \sum_{k=1}^{n} a_k^j v_x^k + D_x H^j[u] = \sum_{k=1}^{n} a_k^j \Psi^k[u] + B^j[u] \\
&= \sum_{k=1}^{n} a_k^j \Psi^k[u] + B^j[u] + A^j[u] = \sum_{k=1}^{n} a_k^j \Psi^k[u] + T^j[u] \\
&= \hat{\mathbf{X}}^{(\infty)}\Psi^j[u], \\
D_t\zeta^j[u, v] &= \sum_{k=1}^{n} a_k^j v_t^k + D_t H^j[u] = \sum_{k=1}^{n} a_k^j \Phi^k[u] + G^j[u] \\
&= \sum_{k=1}^{n} a_k^j \Phi^k[u] + G^j[u] + F^j[u] = \sum_{k=1}^{n} a_k^j \Phi^k[u] + S^j[u] \\
&= \hat{\mathbf{X}}^{(\infty)}\Phi^j[u], \quad j = 1, \dots, n.
\end{aligned} \tag{3.77}$$

Hence, $\zeta^j[u, v]$, $j = 1, \dots, n$, given by (3.76) when $(U, V) = (u, v)$ is a solution of the n-plet potential system (3.70), satisfy the equations (3.74).

By construction,

$$\hat{\mathbf{Y}} = \hat{\mathbf{X}} + \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_k^j v^k + H^j[u] \right) \frac{\partial}{\partial v^j}$$

is a local symmetry of the n-plet potential system (3.70), whose projection is the local symmetry $\hat{\mathbf{X}} = \sum_{i=1}^{m} \eta^i[u] \frac{\partial}{\partial u^i}$ of the given PDE system $\mathbf{R}\{x, t; u\}$ (3.69). □

Remark 3.5.2 The proof of Theorem 3.5.1 shows how to directly construct the local symmetry $\hat{\mathbf{Y}}$ of the n-plet potential system (3.70) for *any* local symmetry $\hat{\mathbf{X}}$ of a given PDE system $\mathbf{R}\{x, t; u\}$ (3.69) which has precisely n local CLs.

Corollary 3.5.3 Consider a PDE system $\mathbf{R}\{x, t; u\}$ with two independent variables (x, t) and m dependent variables $u = (u^1, \dots, u^m)$ given by

$$R^\sigma[u] = 0, \quad \sigma = 1, \dots, s. \tag{3.78}$$

Suppose $\hat{\mathbf{X}}$ is a local symmetry in evolutionary form of $\mathbf{R}\{x,t;u\}$ and $D_t\Psi^i[u] - D_x\Phi^i[u] = 0$, $i = 1, \ldots, k$, are linearly independent local CLs of $\mathbf{R}\{x,t;u\}$ (3.78). If for each $\nu = 1, \ldots, k$, the CL

$$D_t\left(\hat{\mathbf{X}}^{(\infty)}\Psi^\nu[u]\right) - D_x\left(\hat{\mathbf{X}}^{(\infty)}\Phi^\nu[u]\right) = 0$$

is equivalent to the CL

$$D_t\left(\sum_{i=1}^{k} a_i^\nu \Psi^i[u]\right) - D_x\left(\sum_{i=1}^{k} a_i^\nu \Phi^i[u]\right) = 0,$$

for some constants a_i^ν, $i = 1, \ldots, k$, then $\hat{\mathbf{X}}$ can be obtained by projection of some local symmetry of the corresponding k-plet potential system given by

$$\begin{aligned} &v_x^i = \Psi^i[u], \\ &v_t^i = \Phi^i[u], \quad i = 1, \ldots, k, \\ &R^\sigma[u] = 0, \quad \sigma = 1, \ldots, s. \end{aligned} \tag{3.79}$$

Proof. The proof in Theorem 3.5.1 can be directly extended to the k-plet potential system (3.79). □

Remark 3.5.4 Both Theorem 3.5.1 and Corollary 3.5.3 hold for PDE systems with three or more independent variables, since one can directly extend the above proofs to such PDE systems.

Example 3.5.5 Consider the nonlinear diffusion equation

$$u_t = \left(u^{-\frac{4}{3}} u_x\right)_x. \tag{3.80}$$

From Tables 3.1 and 3.3, one sees that all point symmetries of the nonlinear diffusion equation (3.80) can be obtained by projection of some point symmetry of its 2-plet potential system (3.19). For this example, we illustrate how to use Theorem 3.5.1 to obtain this conclusion.

Consider the point symmetry $\mathbf{X} = -x^2\frac{\partial}{\partial x} + 3xu\frac{\partial}{\partial u}$ of the nonlinear diffusion equation (3.80), whose evolutionary form is given by $\hat{\mathbf{X}} = (3xu + x^2u_x)\frac{\partial}{\partial u}$. Using the direct method and Theorem 2.3.13, one can show that the nonlinear diffusion equation (3.80) has exactly two linearly independent local CLs given by

$$u_t - \left(u^{-\frac{4}{3}} u_x\right)_x = 0, \tag{3.81}$$

$$(xu)_t - \left(xu^{-\frac{4}{3}}u_x + 3u^{-\frac{1}{3}}\right)_x = 0. \tag{3.82}$$

From Theorem 3.5.1, the point symmetry $\hat{\mathbf{X}}$ of the nonlinear diffusion equation (3.80) can be obtained by projection of some local symmetry $\hat{\mathbf{Y}}$ of its 2-plet potential system given by

$$\begin{aligned} v_x &= u, \\ v_t &= u^{-\frac{4}{3}}u_x, \\ \alpha_x &= xu, \\ \alpha_t &= xu^{-\frac{4}{3}}u_x + 3u^{-\frac{1}{3}}. \end{aligned} \tag{3.83}$$

Here $\Psi^1[u] = u$, $\Phi^1[u] = u^{-\frac{4}{3}}u_x$, $\Psi^2[u] = xu$, $\Phi^2[u] = xu^{-\frac{4}{3}}u_x + 3u^{-\frac{1}{3}}$. In particular, one can explicitly find $\hat{\mathbf{Y}}$ without applying the Lie's algorithm to the 2-plet potential system (3.83). Applying the corresponding infinite prolongation $\hat{\mathbf{X}}^{(\infty)}$ to the fluxes of the CL (3.81), one obtains

$$\begin{aligned} \hat{\mathbf{X}}^{(\infty)}\left(\Psi^1[u]\right) &= \hat{\mathbf{X}}^{(\infty)}(u) = 3xu + x^2u_x = xu + x^2u_x + 2xu \\ &= \Psi^2[u] + T^1[u], \\ \hat{\mathbf{X}}^{(\infty)}\left(\Phi^1[u]\right) &= \hat{\mathbf{X}}^{(\infty)}(u^{-\frac{4}{3}}u_x) = 3u^{-\frac{1}{3}} + xu^{-\frac{4}{3}}u_x - \tfrac{4}{3}x^2u^{-\frac{7}{3}}u_x^2 + x^2u^{-\frac{4}{3}}u_{xx}, \\ &= \Phi^2[u] + S^1[u], \end{aligned} \tag{3.84}$$

where $D_tT^1[u] - D_xS^1[u] = 0$ is the trivial CL given by

$$\begin{aligned} &D_t(x^2u_x + 2xu) - D_x\left(-\tfrac{4}{3}x^2u^{-\frac{7}{3}}u_x^2 + x^2u^{-\frac{4}{3}}u_{xx} - x^2u_t + x^2u_t\right) \\ &= D_t(D_x(x^2u)) - D_x\left(x^2\left(-\tfrac{4}{3}u^{-\frac{7}{3}}u_x^2 + x^2u^{-\frac{4}{3}}u_{xx} - u_t\right) + D_t(x^2u)\right) = 0. \end{aligned} \tag{3.85}$$

Applying the corresponding infinite prolongation $\hat{\mathbf{X}}^{(\infty)}$ to the fluxes of the CL (3.82), one obtains

$$\begin{aligned} \hat{\mathbf{X}}^{(\infty)}\left(\Psi^2[u]\right) &= \hat{\mathbf{X}}^{(\infty)}(xu) = 3x^2u + x^3u_x = T^2[u], \\ \hat{\mathbf{X}}^{(\infty)}\left(\Phi^2[u]\right) &= \hat{\mathbf{X}}^{(\infty)}(xu^{-\frac{4}{3}}u_x + 3u^{-\frac{1}{3}}) = x^3u^{-\frac{4}{3}}u_{xx} - \tfrac{4}{3}x^3u^{-\frac{7}{3}}u_x^2 = S^2[u], \end{aligned} \tag{3.86}$$

where $D_tT^2[u] - D_xS^2[u] = 0$ is the trivial CL given by

$$D_t(D_x(x^3u)) - D_x\left(D_t(x^3u) + x^3(u^{-\frac{4}{3}}u_{xx} - \frac{4}{3}u^{-\frac{7}{3}}u_x^2 - u_t)\right) = 0. \tag{3.87}$$

It follows that the constants and the functions $H^j[u]$, $j = 1, 2$, in (3.76) are given by

$$a_1^1 = 0,\ a_2^1 = 1,\ a_1^2 = 0,\ a_2^2 = 0,\ H^1[u] = x^2u,\ H^2[u] = x^3u.$$

Consequently, $\hat{\mathbf{Y}} = \hat{\mathbf{X}} + (\alpha + x^2 u)\frac{\partial}{\partial v} + x^3 u \frac{\partial}{\partial \alpha} = \hat{\mathbf{X}} + (\alpha + x^2 v_x)\frac{\partial}{\partial v} + x^2 \alpha_x \frac{\partial}{\partial \alpha}$, which is the evolutionary form for the point symmetry $\mathbf{Y} = \mathbf{X} + \alpha \frac{\partial}{\partial v}$.

Besides being useful for obtaining nonlocal CLs and nonlocal symmetries of a given PDE system $\mathbf{R}\{x; u\}$, nonlocally related CL systems of $\mathbf{R}\{x; u\}$ have various other important applications. For instance, one can use nonlocal symmetries to construct new solutions, which are not invariant solutions of local symmetries, of $\mathbf{R}\{x; u\}$ arising from invariant solutions of its nonlocally related PDE systems [41]. In [37], it was shown that new solutions can also arise from the "nonclassical symmetries" of nonlocally related PDE systems of $\mathbf{R}\{x; u\}$. In addition, one can construct nonlocal mappings that linearize a given PDE system through its nonlocally related CL systems [25]. The well-known Hopf-Cole transformation (see [52]) that linearizes Burgers' equation can be obtained through its potential system [59, 90]. One can also construct a nonlocal mapping that maps a scalar PDE with variable coefficients to a linear PDE with constant coefficients through its nonlocally related CL systems [33, 34]. Moreover, in [5], Anco and Bluman used nonlocal symmetries to obtain CLs of a given PDE.

3.6 Summary

In this chapter, we presented the known CL-based method for constructing nonlocally related CL systems of a given PDE system. Such nonlocally related CL systems are important in obtaining nonlocal CLs, nonlocal symmetries and new exact solutions of a given PDE system. Besides the known results, we introduced two new results in this chapter. Theorem 3.2.7 showed that for two potential systems written in Cauchy-Kovalevskaya form, arising from two nontrivial and linearly independent local CLs, the potential variable in one system is a nonlocal variable of the other system. Theorem 3.5.1 showed that any local symmetry of a PDE system having precisely n local CLs must be a projection of some local symmetry of its corresponding n-plet potential system.

Chapter 4

Symmetry-based Method for Constructing Nonlocally Related PDE Systems

4.1 Introduction

In Chapter 3, we presented the CL-based method for constructing nonlocally related CL systems of a given PDE system and the method for constructing subsystems. Moreover, some important applications of nonlocally related PDE systems, such as using nonlocally related PDE systems to find nonlocal symmetries and nonlocal CLs, were presented.

In the CL-based method, to construct nonlocally related PDE systems of a given scalar PDE, it is necessary that the given PDE has at least one nontrivial CL. A natural problem is how to construct nonlocally related PDE systems for a scalar PDE that has no nontrivial CL. For example, consider the nonlinear reaction-diffusion equation

$$u_t - u_{xx} = Q(u), \tag{4.1}$$

where the reaction term $Q(u)$ is an arbitrary constitutive function.

According to Theorem 2.3.13, one can find all local CLs of the nonlinear reaction-diffusion equation (4.1) by the direct method. In particular, it suffices to consider multipliers of the form $\Lambda = \Lambda(x, t, U, U_x, U_{xx}, U_{xxx})$. Applying the direct method, one obtains that $\Lambda(U_t - U_{xx} - Q(U))$ is a divergence form if and only if Λ satisfies the following equations:

$$\begin{aligned}
&\Lambda_U = 0, \quad \Lambda_{U_x} = 0, \quad \Lambda_{U_{xx}} = 0, \quad \Lambda_{U_{xxx}} = 0,\\
&\Lambda\Lambda_{tx} + \Lambda\Lambda_{xxx} - \Lambda_x(\Lambda_{xx} + \Lambda_t) = 0,\\
&\Lambda\Lambda_{tt} + \Lambda\Lambda_{txx} - \Lambda_t(\Lambda_{xx} + \Lambda_t) = 0,\\
&\Lambda_t + \Lambda_{xx} + \Lambda Q'(U) = 0.
\end{aligned} \tag{4.2}$$

From equations (4.2), one immediately concludes that, for any nonlinear function $Q(u)$, the nonlinear reaction-diffusion equation (4.1) has no nontrivial local

CL. Consequently, it is impossible to construct a nonlocally related PDE system of the nonlinear reaction-diffusion equation (4.1) via the CL-based method.

In this chapter, we introduce a new systematic method to construct nonlocally related PDE systems which can be applied to a PDE system that has no nontrivial CL. In particular, we show that, for any given PDE system, a nonlocally related PDE system arises naturally from each point symmetry of the given PDE system. Consequently, one can extend a tree of nonlocally related PDE systems by adding this method to Procedure 3.2.15 for the construction of a tree of nonlocally related PDE systems.

Unlike the CL-based method, the symmetry-based method can be directly applied to a PDE system with three or more independent variables. More importantly, nonlocal related PDE systems arising from the symmetry-based mathod are determined. In addition, by various examples, it is shown that a nonlocally related PDE system arising from a point symmetry of a given PDE system could also yield nonlocal symmetries of the given PDE system.

4.2 Nonlocally related PDE systems arising from point symmetries

Consider a PDE system $\mathbf{R}\{x, t; u\}$ of order l with two independent variables (x, t) and m dependent variables $u = (u^1, \ldots, u^m)$ given by

$$R^\sigma[u] = R^\sigma(x, t, u, \partial u, \partial^2 u, \ldots, \partial^l u) = 0, \quad \sigma = 1, \ldots, s. \tag{4.3}$$

Suppose the PDE system $\mathbf{R}\{x, t; u\}$ (4.3) has a point symmetry with infinitesimal generator $\mathbf{X} = \xi(x, t, u)\dfrac{\partial}{\partial x} + \tau(x, t, u)\dfrac{\partial}{\partial t} + \sum_{i=1}^{m} \eta^i(x, t, u)\dfrac{\partial}{\partial u^i}$. By introducing the canonical coordinates corresponding to $\mathbf{X}$:

$$\begin{aligned} X &= X(x, t, u), \\ T &= T(x, t, u), \\ U^i &= U^i(x, t, u), \quad i = 1, \ldots, m, \end{aligned} \tag{4.4}$$

satisfying

$$\begin{aligned} \mathbf{X}X &= 0, \\ \mathbf{X}T &= 0, \\ \mathbf{X}U^1 &= 1, \\ \mathbf{X}U^i &= 0 \quad i = 2, \ldots, m, \end{aligned} \tag{4.5}$$

one maps $\mathbf{X}$ into the canonical form $\mathbf{Y} = \frac{\partial}{\partial U^1}$ while the PDE system $\mathbf{R}\{x,t;u\}$ (4.3) becomes the invertibly equivalent PDE system $\hat{\mathbf{R}}\{X,T;U\}$ in terms of the canonical coordinates (X,T,U). Since an invertible transformation maps a symmetry of a PDE system to a symmetry of the transformed system, $\mathbf{Y}$ is the infinitesimal generator of a point symmetry of $\hat{\mathbf{R}}\{X,T;U\}$. Consequently, $\hat{\mathbf{R}}\{X,T;U\}$ is invariant under translations in U^1. It follows that $\hat{\mathbf{R}}\{X,T;U\}$ is of the form

$$\hat{R}^\sigma(X,T,\hat{U},\partial U,\ldots,\partial^l U)=0, \quad \sigma=1,\ldots,s, \tag{4.6}$$

where $\hat{U}=(U^2,\ldots,U^m)$.

Introducing two new variables α and β, related to the first partial derivatives of U^1, one obtains the equivalent PDE system $\tilde{\mathbf{R}}\{X,T;U,\alpha,\beta\}$ given by

$$\begin{aligned}
&\alpha = U^1_T,\\
&\beta = U^1_X,\\
&\tilde{R}^\sigma(X,T,\hat{U},\alpha,\beta,\partial\hat{U},\ldots,\partial^{l-1}\alpha,\partial^{l-1}\beta,\partial^l\hat{U})=0, \quad \sigma=1,\ldots,s,
\end{aligned} \tag{4.7}$$

where $\tilde{R}^\sigma(X,T,\hat{U},\alpha,\beta,\partial\hat{U},\ldots,\partial^{l-1}\alpha,\partial^{l-1}\beta,\partial^l\hat{U})=0$ is obtained from $\hat{R}^\sigma(X,T,\hat{U},$ $\partial U,\ldots,\partial^l U)=0$ after making the appropriate substitutions. The PDE system $\tilde{\mathbf{R}}\{X,T;U,\alpha,\beta\}$ (4.7) is called the *intermediate system* of $\hat{\mathbf{R}}\{X,T;U\}$ (4.6).

According to Theorem 3.2.14, the PDE system $\tilde{\mathbf{R}}\{X,T;U,\alpha,\beta\}$ (4.7) is locally related to the PDE system $\hat{\mathbf{R}}\{X,T;U\}$ (4.6), and hence locally related to the given PDE system $\mathbf{R}\{x,t;u\}$ (4.3).

Excluding the dependent variable U^1 from the PDE system $\tilde{\mathbf{R}}\{X,T;U,\alpha,\beta\}$ (4.7), one obtains the equivalent PDE system $\check{\mathbf{R}}\{X,T;\hat{U},\alpha,\beta\}$

$$\begin{aligned}
&\alpha_X = \beta_T,\\
&\tilde{R}^\sigma(X,T,\hat{U},\alpha,\beta,\partial\hat{U},\ldots,\partial^{l-1}\alpha,\partial^{l-1}\beta,\partial^l\hat{U})=0, \quad \sigma=1,\ldots,s.
\end{aligned} \tag{4.8}$$

According to Theorem 3.2.14, since the PDE system $\check{\mathbf{R}}\{X,T;\hat{U},\alpha,\beta\}$ (4.8) is obtained by excluding U^1 from the PDE system $\tilde{\mathbf{R}}\{X,T;U,\alpha,\beta\}$ (4.7), which cannot be expressed as a local function of α, β and their derivatives, it follows that the PDE system $\check{\mathbf{R}}\{X,T;\hat{U},\alpha,\beta\}$ (4.8) is nonlocally related to the PDE system $\tilde{\mathbf{R}}\{X,T;U,\alpha,\beta\}$ (4.7). In particular, if $(\alpha,\beta,U^2,\ldots,U^m) = (f(x,t),g(x,t),$ $h^2(x,t),\ldots,h^m(x,t))$ solves the PDE system $\check{\mathbf{R}}\{X,T;\hat{U},\alpha,\beta\}$ (4.8), there exists a spectrum of functions $U^1=h^1(x,t)+C$, where C is an arbitrary constant, such that $(\alpha,\beta,U^1,U^2,\ldots,U^m) = (f(x,t),\, g(x,t),\, h^1(x,t)+C, h^2(x,t),\ldots,h^m(x,t))$ solves the intermediate system $\tilde{\mathbf{R}}\{X,T;U,\alpha,\beta\}$ (4.7). By projection, $(U^1,U^2,\ldots,U^m) = (h^1(x,t)+C,h^2(x,t),\ldots,h^m(x,t))$ is a solution of the PDE system $\hat{\mathbf{R}}\{X,T;U\}$ (4.6). Thus the correspondence between the solutions of the PDE system $\check{\mathbf{R}}\{X,T;\hat{U},\alpha,\beta\}$

(4.8) and those of the PDE system $\hat{\mathbf{R}}\{X, T; U\}$ (4.6) is not one-to-one. It follows that the PDE system $\check{\mathbf{R}}\{X, T; \hat{U}, \alpha, \beta\}$ (4.8) is nonlocally related to the PDE system $\hat{\mathbf{R}}\{X, T; U\}$ (4.6), and hence nonlocally related to the given PDE system $\mathbf{R}\{x, t; u\}$ (4.3).

Since the way one constructs the PDE system $\check{\mathbf{R}}\{X, T; \hat{U}, \alpha, \beta\}$ (4.8) is in the reverse direction of the construction for a potential system, we call the PDE system $\check{\mathbf{R}}\{X, T; \hat{U}, \alpha, \beta\}$ (4.8) an *inverse potential system*. Since the inverse potential system arising from a point symmetry of a given PDE system is nonlocally related to the given PDE system, we use the terminology *nonlocally related symmetry-based system* to denote an inverse potential system.

Based on the above discussions, one has proved the following theorem.

Theorem 4.2.1 Any point symmetry of a given PDE system $\mathbf{R}\{x, t; u\}$ (4.3) yields a nonlocally related PDE system (inverse potential system) of the given PDE system $\mathbf{R}\{x, t; u\}$ (4.3) given by the PDE system $\check{\mathbf{R}}\{X, T; \hat{U}, \alpha, \beta\}$ (4.8).

Remark 4.2.2 *Connection between the symmetry-based method and the CL-based method.* The symmetry-based method to obtain a nonlocally related PDE system does not require the existence of a nontrivial local CL of a given PDE system. Thus the new method is complementary to the CL-based method for constructing nonlocally related PDE systems. In particular, for the CL-based method, the constructed system is a potential system of the given PDE system. For the symmetry-based method, since the intermediate system is locally related to the given PDE system, one can treat the intermediate system as the starting PDE system. In this sense, for the symmetry-based method, the starting PDE system is a potential system of the final constructed system (the inverse potential system). It follows that the symmetry-based method is in the reverse direction of the CL-based method.

Remark 4.2.3 *The situation for a PDE system with at least three independent variables.* The symmetry-based method can be adapted to a PDE system which has at least three independent variables. Without loss of generality, consider a scalar PDE $\mathbf{R}\{x; u\}$ with $n \geq 3$ independent variables $x = (x^1, \ldots, x^n)$ and one dependent variable u:

$$R(x, u, \partial u, \partial^2 u, \ldots, \partial^l u) = 0. \tag{4.9}$$

Suppose the scalar PDE $\mathbf{R}\{x; u\}$ (4.9) has a point symmetry with the infinitesimal generator $\mathbf{X}$. The canonical coordinates corresponding to $\mathbf{X}$:

$$\begin{aligned} &X^i = X^i(x, u), \quad i = 1, \ldots, n, \\ &U = U(x, t, u), \end{aligned} \tag{4.10}$$

satisfying

$$\begin{aligned} &\mathbf{X}X^i = 0, \quad i = 1, \dots, n, \\ &\mathbf{X}U = 1, \end{aligned} \tag{4.11}$$

maps $\mathbf{X}$ into the canonical form $\mathbf{Y} = \frac{\partial}{\partial U}$. In terms of (X, T, U) coordinates, the given scalar PDE $\mathbf{R}\{x; u\}$ (4.9) becomes the invertibly related PDE $\hat{\mathbf{R}}\{X; U\}$ $(X = (X^1, \dots, X^n))$ of the form

$$\hat{R}(X, \partial U, \partial^2 U, \dots, \partial^l U) = 0. \tag{4.12}$$

Introducing the new variables $\alpha = (\alpha^1, \dots, \alpha^n)$, related to the first partial derivatives of U, one obtains the equivalent locally related intermediate system $\tilde{\mathbf{R}}\{X; U, \alpha\}$ given by

$$\begin{aligned} &\alpha^i = U_{X^i}, \quad i = 1, \dots, n, \\ &\tilde{R}(X, \alpha, \partial\alpha \dots, \partial^{l-1}\alpha) = 0, \end{aligned} \tag{4.13}$$

where $\tilde{R}(X, \alpha, \partial\alpha, \dots, \partial^{l-1}\alpha) = 0$ is obtained from $\hat{R}(X, \partial U, \partial^2 U, \dots, \partial^l U) = 0$ after making the appropriate substitutions. Excluding U from the PDE system $\tilde{\mathbf{R}}\{X; U, \alpha\}$ (4.13), one obtains the inverse potential system $\check{\mathbf{R}}\{X; \alpha\}$

$$\begin{aligned} &\alpha^i_{X^j} - \alpha^j_{X^i} = 0, \quad i, j = 1, \dots, n, \\ &\tilde{R}(X, \alpha, \partial\alpha, \dots, \partial^{l-1}\alpha) = 0. \end{aligned} \tag{4.14}$$

By construction, one can show that the inverse potential system $\check{\mathbf{R}}\{X; \alpha\}$ (4.14) is nonlocally related to the scalar PDE $\hat{\mathbf{R}}\{X; U\}$ (4.12), hence nonlocally related to the scalar PDE $\mathbf{R}\{x; u\}$ (4.9). Moreover, since the inverse potential system $\check{\mathbf{R}}\{X; \alpha\}$ (4.14) has curl-type CLs, it could possibly yield nonlocal symmetries of the scalar PDE $\mathbf{R}\{x; u\}$ (4.9) from local symmetries of the inverse potential system $\check{\mathbf{R}}\{X; \alpha\}$ (4.14) [25, 43, 44].

Corollary 4.2.4 Consider an evolutionary scalar PDE $\mathbf{R}\{x, t; u\}$, invariant undertranslations in u, given by

$$u_t = F(x, t, \partial_x u, \dots, \partial^l_x u). \tag{4.15}$$

Let $\beta = \partial_x u = u_x$. Then the scalar PDE

$$\beta_t = D_x F(x, t, \beta, \partial_x \beta, \dots, \partial_x^{l-1}\beta) \tag{4.16}$$

is a locally related subsystem of an inverse potential system of the PDE $\mathbf{R}\{x, t; u\}$ (4.15).

Proof. Introducing the new variables α and β, related to the first partial derivatives of u, one obtains the following locally related intermediate system $\tilde{\mathbf{R}}\{x,t;u,\alpha,\beta\}$ of the given PDE $\mathbf{R}\{x,t;u\}$ (4.15):

$$\begin{aligned}
&\alpha = u_t,\\
&\beta = u_x,\\
&\alpha = F(x,t,\beta,\partial_x\beta,\ldots,\partial_x^{l-1}\beta).
\end{aligned} \tag{4.17}$$

Excluding the dependent variable u from the intermediate system $\tilde{\mathbf{R}}\{x,t;u,\alpha,\beta\}$ (4.17), one obtains the inverse potential system $\check{\mathbf{R}}\{X,T;\hat{U},\alpha,\beta\}$

$$\begin{aligned}
&\alpha_x = \beta_t,\\
&\alpha = F(x,t,\beta,\partial_x\beta,\ldots,\partial_x^{l-1}\beta).
\end{aligned} \tag{4.18}$$

From the previous discussion, the PDE system $\check{\mathbf{R}}\{X,T;\hat{U},\alpha,\beta\}$ (4.18) is nonlocally related to (4.17). Furthermore, one can exclude the dependent variable α from the PDE system $\check{\mathbf{R}}\{X,T;\hat{U},\alpha,\beta\}$ (4.18) to obtain the scalar PDE $\dot{\mathbf{R}}\{x,t;\beta\}$

$$\beta_t = D_x F(x,t,\beta,\partial_x\beta,\ldots,\partial_x^{l-1}\beta). \tag{4.19}$$

Since the excluded variable α can be expressed from the equations of the PDE system $\check{\mathbf{R}}\{X,T;\hat{U},\alpha,\beta\}$ (4.18) in terms of β and its derivatives, the PDE $\dot{\mathbf{R}}\{x,t;\beta\}$ (4.19) is locally related to the inverse potential system $\check{\mathbf{R}}\{X,T;\hat{U},\alpha,\beta\}$ (4.18). □

4.3 Examples of inverse potential systems arising from point symmetries

In the previous section we introduced a new systematic symmetry-based method to construct nonlocally related PDE systems (inverse potential systems) of a given PDE system. Such equivalent PDE systems are nonlocally related to the given PDE system. In this section, we illustrate this method by several examples.

4.3.1 Nonlinear reaction-diffusion equation

Consider the nonlinear reaction-diffusion equation (4.1). As stated in Section 4.1, the nonlinear reaction-diffusion equation (4.1) has no local CL for any nonlinear term $Q(u)$. Thus it is impossible to construct nonlocally related PDE systems of the nonlinear reaction-diffusion equation (4.1) by the CL-based method.

In contrast, the nonlinear reaction-diffusion equation (4.1) has point symmetries. Thus one can construct nonlocally related PDE systems of the the nonlinear reaction-diffusion equation (4.1) through the symmetry-based method introduced in Section 4.2. The point symmetry classification of the nonlinear reaction-diffusion (4.1) is presented in [47, 49] and exhibited in Table 4.1, modulo the group of equivalent transformations (2.71).

Table 4.1: Point symmetry classification for the reaction-diffusion equation (4.1)

$Q(u)$	#	admitted point symmetries
arbitrary	2	$\mathbf{X}_1 = \frac{\partial}{\partial x}$, $\mathbf{X}_2 = \frac{\partial}{\partial t}$
$u^a (a \neq 0, 1)$	3	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 = u\frac{\partial}{\partial u} - (a-1)t\frac{\partial}{\partial t} - \frac{a-1}{2}x\frac{\partial}{\partial x}$
e^u	3	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_4 = \frac{\partial}{\partial u} - t\frac{\partial}{\partial t} - \frac{1}{2}x\frac{\partial}{\partial x}$
$u \ln u$	4	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_5 = ue^t\frac{\partial}{\partial u}, \mathbf{X}_6 = 2e^t\frac{\partial}{\partial x} - xue^t\frac{\partial}{\partial u}$

(I) The case when $Q(u)$ is arbitrary

For arbitrary $Q(u)$, the nonlinear reaction-diffusion equation (4.1) has the exhibited two point symmetries: $\mathbf{X}_1$ and $\mathbf{X}_2$. Therefore, using the symmetry-based method one can use interchanges of x and u and also t and u to construct two inverse potential systems of the nonlinear reaction-diffusion equation (4.1).

(I-a) Inverse potential system arising from $\mathbf{X}_1$

After an interchange of the variables x and u, the nonlinear reaction-diffusion equation (4.1) becomes the invertibly related PDE given by

$$x_t = \frac{x_{uu} - Q(u)x_u^3}{x_u^2}. \tag{4.20}$$

Corresponding to the invariance of PDE (4.20) under translations of its dependent variable x, one obtains the following locally related intermediate system for the nonlinear reaction-diffusion equation (4.1) by introducing two new variables:

$$\begin{aligned} v &= x_u, \\ w &= x_t, \\ w &= \frac{v_u - Q(u)v^3}{v^2}. \end{aligned} \tag{4.21}$$

Excluding x from the intermediate system (4.21), one obtains the inverse potential system of the PDE system (4.21) given by

$$\begin{aligned} v_t &= w_u, \\ w &= \frac{v_u - Q(u)v^3}{v^2}. \end{aligned} \tag{4.22}$$

In addition, one can exclude w from the PDE system (4.22) to get the scalar PDE

$$v_t = \left(\frac{v_u - Q(u)v^3}{v^2}\right)_u. \tag{4.23}$$

By construction, the scalar PDE (4.23) is a locally related subsystem of the PDE system (4.22). Moreover, since the scalar PDE (4.23) is in a CL form and the nonlinear reaction-diffusion equation (4.1) has no local CL, from Remark 2.3.15, it follows that there is no invertible transformation that relates the scalar PDE (4.23) and the nonlinear reaction-diffusion equation (4.1). Therefore, the scalar PDE (4.23) is nonlocally related to the nonlinear reaction-diffusion equation (4.1).

(I-b) Inverse potential system arising from $\mathbf{X}_2$

After an interchange of the variables t and u, the nonlinear reaction-diffusion equation (4.1) becomes

$$t_u^2 - Q(u)t_u^3 + t_u^2 t_{xx} - 2t_x t_u t_{xu} + t_x^2 t_{uu} = 0, \tag{4.24}$$

which is not in solved form and has mixed derivatives.

Corresponding to the invariance of PDE (4.24) under translations of its dependent variable t, one introduces two new variables $\alpha = t_x$ and $\beta = t_u$ to obtain the locally related intermediate system of the nonlinear reaction-diffusion equation (4.1) given by

$$\begin{aligned} &\alpha = t_x, \\ &\beta = t_u, \\ &\beta^2 - Q(u)\beta^3 + \beta^2\alpha_x - 2\alpha\beta\alpha_u + \alpha^2\beta_u = 0. \end{aligned} \tag{4.25}$$

Excluding t from the intermediate system (4.25), one obtains another inverse potential system of the nonlinear reaction-diffusion equation (4.1) given by

$$\begin{aligned} &\alpha_u - \beta_x = 0, \\ &\beta^2 - Q(u)\beta^3 + \beta^2\alpha_x - 2\alpha\beta\alpha_u + \alpha^2\beta_u = 0, \end{aligned} \tag{4.26}$$

which is nonlocally related to the nonlinear reaction-diffusion equation (4.1).

The constructed inverse potential systems for the nonlinear reaction-diffusion equation (4.1) ($Q(u)$ is arbitrary) are illustrated in Figure 4.1.

$$\begin{tikzcd}
 & \boxed{(4.1)} \arrow[dl] \arrow[dr] & \\
\boxed{(4.22), (4.23)} & & \boxed{(4.26)}
\end{tikzcd}$$

Figure 4.1: The constructed inverse potential systems for the nonlinear reaction-diffusion equation (4.1) ($Q(u)$ is arbitrary), with the arrows pointing to the inverse potential systems.

(II) Inverse potential system arising from $\mathbf{X}_3$ when $Q(u) = u^3$

When $Q(u) = u^a$, $(a \neq 0, 1)$, the nonlinear reaction-diffusion equation (4.1) has one additional point symmetry $\mathbf{X}_3$. For simplicity, we consider the case when $a = 3$, i.e., $Q(u) = u^3$. Canonical coordinates induced by $\mathbf{X}_3$ are given by

$$\begin{aligned} X &= xu, \\ T &= \frac{t}{x^2}, \\ U &= -\ln x. \end{aligned} \tag{4.27}$$

In (X, T, U) coordinates, the corresponding nonlinear reaction-diffusion equation (4.1) becomes the invertibly related PDE

$$\begin{aligned} &-3U_X^2 - 2XU_X^3 - X^3U_X^3 - U_X^2U_T + 10TU_X^2U_T + U_{XX} - 4TU_TU_{XX} \\ &+ 4T^2U_T^2U_{XX} + 4T^2U_X^2U_{TT} + 4TU_XU_{TX} - 8T^2U_XU_TU_{TX} = 0. \end{aligned} \tag{4.28}$$

Accordingly, introducing the new variables $\alpha = U_X$ and $\beta = U_T$, one obtains the locally related intermediate system of the nonlinear reaction-diffusion equation (4.1) given by

$$\begin{aligned} &\alpha = U_X, \\ &\beta = U_T, \\ &-3\alpha^2 - 2X\alpha^3 - X^3\alpha^3 - \alpha^2\beta + 10T\alpha^2\beta + \alpha_X - 4T\beta\alpha_X \\ &+ 4T^2\beta^2\alpha_X + 4T^2\alpha^2\beta_T + 4T\alpha\beta_X - 8T^2\alpha\beta\beta_X = 0. \end{aligned} \tag{4.29}$$

Excluding U from the intermediate system (4.29), one obtains an additional inverse potential system of the corresponding nonlinear reaction-diffusion equation (4.1) given by

$$
\begin{aligned}
&\alpha_T = \beta_X,\\
&-3\alpha^2 - 2X\alpha^3 - X^3\alpha^3 - \alpha^2\beta + 10T\alpha^2\beta + \alpha_X - 4T\beta\alpha_X\\
&+4T^2\beta^2\alpha_X + 4T^2\alpha^2\beta_T + 4T\alpha\beta_X - 8T^2\alpha\beta\beta_X = 0,
\end{aligned}
\tag{4.30}
$$

which is nonlocally related to the nonlinear reaction-diffusion equation (4.1). Moreover, comparing the number of point symmetries of the PDE system (4.29) and the PDE system (4.26), one is able to show there is no invertible transformation relating these two systems. Hence, the PDE system (4.29) is nonlocally related to the PDE system (4.26).

The constructed inverse potential systems for the nonlinear reaction-diffusion equation (4.1) ($Q(u) = u^3$) are illustrated in Figure 4.2.

$$
\begin{tikzcd}
 & \boxed{(4.1)} \arrow[dl] \arrow[d] \arrow[dr] & \\
\boxed{(4.22), (4.23)} & \boxed{(4.30)} & \boxed{(4.26)}
\end{tikzcd}
$$

Figure 4.2: The constructed inverse potential systems for the nonlinear reaction-diffusion equation (4.1) ($Q(u) = u^3$), with the arrows pointing to the inverse potential systems.

(III) Inverse potential system arising from $\mathbf{X}_4$ when $Q(u) = e^u$

When $Q(u) = e^u$, the nonlinear reaction-diffusion equation (4.1) admits one additional point symmetry $\mathbf{X}_4$. Canonical coordinates induced by $\mathbf{X}_4$ are given by

$$
\begin{aligned}
X &= u + 2\ln x,\\
T &= \frac{t}{x^2},\\
U &= -2\ln x.
\end{aligned}
\tag{4.31}
$$

In (X, T, U) coordinates, the corresponding nonlinear reaction-diffusion equation (4.1) becomes the invertibly related PDE

$$
\begin{aligned}
&-2U_X^2 - 2U_X^3 - e^X U_X^3 - U_X^2 U_T + 6TU_TU_X^2 + 4U_{XX} - 8TU_TU_{XX}\\
&+4T^2U_T^2U_{XX} + 4T^2U_X^2U_{TT} + 8TU_XU_{TX} - 8T^2U_XU_TU_{TX} = 0.
\end{aligned}
\tag{4.32}
$$

It follows that the introduction of the new variables $\phi = U_X$ and $\psi = U_T$ yields the locally related intermediate system of the nonlinear reaction-diffusion equation (4.1) given by

$$\begin{aligned}
&\phi = U_X,\\
&\psi = U_T,\\
&-2\phi^2 - 2\phi^3 - e^X\phi^3 - \phi^2\psi + 6T\phi^2\psi + 4\phi_X - 8T\psi\phi_X\\
&+4T^2\psi^2\phi_X + 4T^2\phi^2\psi_T + 8T\phi\psi_X - 8T^2\phi\psi\psi_X = 0.
\end{aligned} \tag{4.33}$$

Excluding U from the intermediate system (4.33), one obtains a third inverse potential system of the corresponding nonlinear reaction-diffusion (4.1) given by

$$\begin{aligned}
&\phi_T = \psi_X,\\
&-2\phi^2 - 2\phi^3 - e^X\phi^3 - \phi^2\psi + 6T\phi^2\psi + 4\phi_X - 8T\psi\phi_X\\
&+4T^2\psi^2\phi_X + 4T^2\phi^2\psi_T + 8T\phi\psi_X - 8T^2\phi\psi\psi_X = 0,
\end{aligned} \tag{4.34}$$

which is nonlocally related to the nonlinear reaction-diffusion equation (4.1). Similar to the situation in (II), one can show that the PDE system is nonlocally related to the PDE system (4.26).

The constructed inverse potential systems for the nonlinear reaction-diffusion equation (4.1) ($Q(u) = e^u$) are illustrated in Figure 4.3.

$$\begin{tikzcd}
 & \boxed{(4.1)} \arrow[dl] \arrow[d] \arrow[dr] & \\
\boxed{(4.22),(4.23)} & \boxed{(4.34)} & \boxed{(4.26)}
\end{tikzcd}$$

Figure 4.3: The constructed inverse potential systems for the nonlinear reaction-diffusion equation (4.1) ($Q(u) = e^u$), with the arrows pointing to the inverse potential systems.

(IV) The case when $Q(u) = u \ln u$

When $Q(u) = u \ln u$, the nonlinear reaction-diffusion equation (4.1) has two additional point symmetries $\mathbf{X}_5$ and $\mathbf{X}_6$.

(IV-a) Inverse potential system arising from $\mathbf{X}_5$

Canonical coordinates induced by $\mathbf{X}_5$ are given by

$$\begin{aligned} X &= x, \\ T &= t, \\ U &= e^{-t} \ln u. \end{aligned} \tag{4.35}$$

In (X, T, U) coordinates, the corresponding nonlinear reaction-diffusion equation (4.1) becomes

$$U_T = U_{XX} + e^T U_X^2. \tag{4.36}$$

Thus one introduces the new variables $p = U_X$ and $q = U_T$ to obtain the locally related intermediate system of the nonlinear reaction-diffusion equation (4.1) given by

$$\begin{aligned} p &= U_X, \\ q &= U_T, \\ q &= p_X + e^T p^2. \end{aligned} \tag{4.37}$$

Excluding U from the intermediate system (4.37), one obtains the inverse potential system of the corresponding nonlinear reaction-diffusion (4.1) given by

$$\begin{aligned} p_T &= q_X, \\ q &= p_X + e^T p^2. \end{aligned} \tag{4.38}$$

In addition, excluding q from the inverse potential system (4.37), one obtains the locally related subsystem of the inverse potential system (4.38) given by

$$p_T = p_{XX} + 2e^T p p_X, \tag{4.39}$$

which is in a CL form. The PDE (4.39) is in a CL form and the nonlinear reaction-diffusion equation (4.1) has no local CL. Hence the PDE (4.39) is nonlocally related to the nonlinear reaction-diffusion equation (4.1).

(IV-b) Inverse potential system arising from $\mathbf{X}_6$

Canonical coordinates induced by $\mathbf{X}_6$ are given by

$$\begin{aligned} X &= e^{\frac{x^2}{4}} u, \\ T &= t, \\ U &= \frac{1}{2} e^{-t} x. \end{aligned} \tag{4.40}$$

In (X, T, U) coordinates, the corresponding nonlinear reaction-diffusion equation (4.1) becomes

$$U_T = \frac{e^{-2T} U_{XX} + 2XU_X^3 - 4X \ln X U_X^3}{4U_X^2}. \tag{4.41}$$

Hence, introducing the variables $r = U_X$ and $s = U_T$, one obtains the locally related intermediate system of the corresponding nonlinear reaction-diffusion equation (4.1) given by

$$\begin{aligned} r &= U_X, \\ s &= U_T, \\ s &= \frac{e^{-2T} r_X + 2Xr^3 - 4X \ln X r^3}{4r^2}. \end{aligned} \tag{4.42}$$

Excluding U from the intermediate system (4.42), one obtains the inverse potential system of the corresponding nonlinear reaction-diffusion (4.1) given by

$$\begin{aligned} r_T &= s_X, \\ s &= \frac{e^{-2T} r_X + 2Xr^3 - 4X \ln X r^3}{4r^2}. \end{aligned} \tag{4.43}$$

In addition, excluding s from the inverse potential system (4.42), one obtains the locally related subsystem of the inverse potential system (4.43) given by

$$r_T = \left(\frac{e^{-2T} r_X + 2Xr^3 - 4X \ln X r^3}{4r^2} \right)_X . \tag{4.44}$$

which is in a CL form. The PDE (4.44) is in a CL form and the nonlinear reaction-diffusion equation (4.1) has no local CL. Thus the PDE (4.44) is nonlocally related to the nonlinear reaction-diffusion equation (4.1).

The constructed inverse potential systems for the nonlinear reaction-diffusion equation (4.1) ($Q(u) = u \ln u$) are illustrated in Figure 4.4.

$$\begin{tikzcd}
 & \boxed{(4.1)} \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\boxed{(4.38), (4.39)} & \boxed{(4.22), (4.23)} & \boxed{(4.26)} & \boxed{(4.43), (4.44)}
\end{tikzcd}$$

Figure 4.4: The constructed inverse potential systems for the nonlinear reaction-diffusion equation (4.1) ($Q(u) = u \ln u$), with the arrows pointing to the inverse potential systems.

4.3.2 Nonlinear diffusion equation

Consider the scalar nonlinear diffusion equation

$$v_t = K(v_x) v_{xx}, \tag{4.45}$$

where $K(v_x)$ is an arbitrary constitutive function. The point symmetry classification of the locally related PDE system (3.5) of the nonlinear diffusion equation (4.45) is listed in Table 3.2, modulo its group of equivalence transformations. By projection of the symmetries in Table 3.2, one obtains that, for arbitrary $K(v_x)$, there are are four point symmetries of the nonlinear diffusion equation (4.45), namely, $\mathbf{Y}_1 = \frac{\partial}{\partial x}$, $\mathbf{Y}_2 = \frac{\partial}{\partial t}$, $\mathbf{Y}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$ and $\mathbf{Y}_4 = \frac{\partial}{\partial v}$.

(I) Inverse potential system arising from $\mathbf{Y}_1$

Since the nonlinear diffusion equation (4.45) is invariant under translations of its independent variable x, one can interchange x and v to generate an invertibly related PDE of the nonlinear diffusion equation (4.45) given by

$$x_t = \frac{K\left(\frac{1}{x_v}\right) x_{vv}}{x_v^2}. \tag{4.46}$$

Introducing new variables $w = x_v$ and $y = x_t$, one obtains the locally related intermediate system of the nonlinear diffusion equation (4.45) given by

$$\begin{aligned} w &= x_v, \\ y &= x_t, \\ y &= \frac{K\left(\frac{1}{w}\right) w_v}{w^2}. \end{aligned} \tag{4.47}$$

Excluding x from the PDE system (4.47), one obtains the inverse potential system of the nonlinear diffusion equation (4.45) given by

$$\begin{aligned} w_t &= y_v, \\ y &= \frac{K\left(\frac{1}{w}\right) w_v}{w^2}. \end{aligned} \tag{4.48}$$

Moreover, one can exclude the variable y from the PDE system (4.48) to obtain the locally related subsystem of the inverse potential system (4.48) given by

$$w_t = \left(\frac{K\left(\frac{1}{w}\right) w_v}{w^2}\right)_v. \tag{4.49}$$

(II) Inverse potential system arising from $\mathbf{Y}_2$

Since the nonlinear diffusion equation (4.45) is invariant under translations of its independent variable t, one can interchange t and v to obtain an invertibly related PDE of the PDE (4.45) given by

$$t_v^2 - K\left(-\frac{t_x}{t_v}\right)\left(2t_v t_x t_{xv} - t_x^2 t_{vv} - t_v^2 t_{xx}\right) = 0. \tag{4.50}$$

Introducing new variables $\alpha = t_v$ and $\beta = t_x$, one obtains the locally related intermediate system of the nonlinear diffusion equation (4.45) given by

$$\begin{aligned} &\alpha = t_v,\\ &\beta = t_x,\\ &\alpha^2 - K\left(-\frac{\beta}{\alpha}\right)\left(2\alpha\beta\alpha_x - \beta^2\alpha_v - \alpha^2\beta_x\right) = 0. \end{aligned} \tag{4.51}$$

Excluding t from the PDE system (4.51), one obtains the inverse potential system of the nonlinear diffusion equation (4.45) given by

$$\begin{aligned} &\alpha_x = \beta_v,\\ &\alpha^2 - K\left(-\frac{\beta}{\alpha}\right)\left(2\alpha\beta\alpha_x - \beta^2\alpha_v - \alpha^2\beta_x\right) = 0. \end{aligned} \tag{4.52}$$

(III) Inverse potential system arising from $\mathbf{Y}_3$

Since the nonlinear diffusion equation (4.45) is invariant under the scaling symmetry generated by $\mathbf{Y}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$, one can use the corresponding canonical coordinate transformation given by

$$\begin{aligned} X &= \frac{t}{x^2},\\ T &= \frac{v}{x},\\ V &= \ln x \end{aligned} \tag{4.53}$$

to map the nonlinear diffusion equation (4.45) into the invertibly related PDE

$$\begin{aligned} &- V_XV_T^2 + K\left(\frac{1 + TV_T + 2XV_X}{V_T}\right)\Big(-4XV_TV_{TX} + V_{TT} + 4XV_XV_{TT} - V_T^2\\ &- 8X^2V_XV_TV_{TX} + 4X^2V_X^2V_{TT} + 2XV_XV_T^2 + 4X^2V_T^2V_{XX}\Big) = 0. \end{aligned} \tag{4.54}$$

Introducing new variables $\phi = V_X$ and $\psi = V_T$, one obtains the locally related intermediate system of the nonlinear diffusion equation (4.45) given by

$$\begin{aligned} &\phi = V_X,\\ &\psi = V_T,\\ &-\phi\psi^2 + K\left(\frac{1 + T\psi + 2X\phi}{\psi}\right)\Big(-4X\psi\psi_X + \psi_T + 4X\phi\psi_T - \psi^2\\ &-8X^2\phi\psi\psi_X + 4X^2\phi^2\psi_T + 2X\phi\psi^2 + 4X^2\psi^2\phi_X\Big) = 0. \end{aligned} \tag{4.55}$$

Excluding V from the intermediate system (4.55), one obtains the inverse potential system of the nonlinear diffusion equation (4.45) given by

$$\begin{aligned} &\phi_T = \psi_X, \\ &-\phi\psi^2 + K\left(\frac{1+T\psi+2X\phi}{\psi}\right)\Big(-4X\psi\psi_X + \psi_T + 4X\phi\psi_T - \psi^2 \\ &-8X^2\phi\psi\psi_X + 4X^2\phi^2\psi_T + 2X\phi\psi^2 + 4X^2\psi^2\phi_X\Big) = 0. \end{aligned} \tag{4.56}$$

(IV) Inverse potential system arising from $\mathbf{Y}_4$

From its invariance under translations of its dependent variable v, one can apply directly the symmetry-based method to the equation (4.45). Letting $u = v_x$, $z = v_t$, one obtains the corresponding locally related intermediate system of the nonlinear diffusion equation (4.45) given by

$$\begin{aligned} u &= v_x, \\ z &= v_t, \\ z &= K(u)u_x. \end{aligned} \tag{4.57}$$

Excluding v from the intermediate system (4.57), one obtains the inverse potential system of the nonlinear diffusion equation (4.45) given by

$$\begin{aligned} u_t &= z_x, \\ z &= K(u)u_x. \end{aligned} \tag{4.58}$$

Excluding z from the PDE system (4.58), one obtains the locally related subsystem of the inverse potential system (4.58) given by the nonlinear diffusion equation

$$u_t = (K(u)\, u_x)_x\,. \tag{4.59}$$

Remark 4.3.1 In fact, the above procedure is in the reverse direction of Example 3.2.5 and Example 3.2.12, in which the given PDE is the nonlinear diffusion equation (4.59). In particular, in Example 3.2.5 and Example 3.2.12, we start with the nonlinear diffusion equation (4.59), and use the CL-based method to obtain the nonlinear diffusion equation (4.45). Conversely, in the above example, the nonlinear diffusion equation (4.45) is the starting PDE. We finally construct the nonlocally related nonlinear diffusion equation (4.59) through the symmetry-based method.

(IV) Inverse potential system for the the nonlinear diffusion equation (4.59)

Now take as the given PDE the nonlinear diffusion equation (4.59). The point symmetry classification for the nonlinear diffusion equation (4.59) is presented in Table 3.1, modulo its group of equivalence transformations. There are three point symmetries of the nonlinear diffusion equation (4.59) for arbitrary $K(u)$: $\mathbf{X}_1 = \frac{\partial}{\partial x}$, $\mathbf{X}_2 = \frac{\partial}{\partial t}$ and $\mathbf{X}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$. Therefore, one can construct three inverse potential systems of the nonlinear diffusion equation (4.59) through the symmetry-based method. Take $\mathbf{X}_1$ for example. From its invariance under translations in x, one can employ the hodograph transformation interchanging x and u to obtain the invertibly related PDE of the nonlinear diffusion equation (4.59):

$$x_t = -\left(\frac{K(u)}{x_u}\right)_u. \tag{4.60}$$

Accordingly, let $p = x_u$ and $q = x_t$, one obtains the following locally related intermediate system of the nonlinear diffusion equation (4.59):

$$\begin{aligned} p &= x_u, \\ q &= x_t, \\ q &= -\left(\frac{K(u)}{p}\right)_u. \end{aligned} \tag{4.61}$$

Excluding the variable x from the PDE system (4.61), one obtains the inverse potential system of the nonlinear diffusion equation (4.59) given by

$$\begin{aligned} p_t &= q_u, \\ q &= -\left(\frac{K(u)}{p}\right)_u. \end{aligned} \tag{4.62}$$

Finally, after excluding the variable q from the PDE system (4.62), one obtains the locally related subsystem of the inverse potential system (4.62) given by

$$p_t = -\left(\frac{K(u)}{p}\right)_{uu}. \tag{4.63}$$

The constructed inverse potential system for the nonlinear diffusion equation (4.45) are indicated in Figure 4.5.

4.3.3 Nonlinear wave equation

As a third example, we construct a further nonlocally related PDE system of the nonlinear wave equation

$$u_{tt} = (c^2(u)u_x)_x, \tag{4.64}$$

$$\begin{tikzcd}
 & \boxed{(4.45)} \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\boxed{(4.58), (4.59)} \arrow[d] & \boxed{(4.48), (4.49)} & \boxed{(4.52)} & \boxed{(4.56)} \\
\boxed{(4.62), (4.63)} & & &
\end{tikzcd}$$

Figure 4.5: The constructed inverse potential system for the nonlinear diffusion equation (4.45), with the arrows pointing to the inverse potential systems.

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with an arbitrary constitutive function $c(u)$.

In Section 3.2, a tree of equivalent PDE systems was constructed for the nonlinear wave equation (4.64). We now use point symmetries of the following potential system of the nonlinear wave equation (4.64):

$$\begin{aligned} v_x &= u_t, \\ v_t &= c^2(u)u_x \end{aligned} \tag{4.65}$$

to obtain nonlocally related PDE systems of the nonlinear wave equation (4.64). For arbitrary $c(u)$, the potential system (4.65) has the following point symmetries: $\mathbf{Y}_1 = \frac{\partial}{\partial t}$, $\mathbf{Y}_2 = \frac{\partial}{\partial x}$, $\mathbf{Y}_3 = \frac{\partial}{\partial v}$, $\mathbf{Y}_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$ and $\mathbf{Y}_\infty$, where $\mathbf{Y}_\infty$ represent the infinite number of point symmetries arising from the linearization of the potential system (4.65) through the hodograph transformation (interchange of independent and dependent variables).

Due to its invariance under translations in v and t, the PDE system (4.65) has a point symmetry with the infinitesimal generator $\frac{\partial}{\partial v} - \frac{\partial}{\partial t}$. Corresponding canonical coordinates yield an invertible point transformation of the form:

$$\rho : \begin{cases} X = x, \\ T = u, \\ U = t + v, \\ V = v. \end{cases} \tag{4.66}$$

The transformation (4.66) maps the potential system (4.65) into the invertibly related PDE system

$$\begin{aligned} V_X U_T - V_T U_X - 1 &= 0, \\ V_T + c^2(T)U_X - c^2(T)V_X &= 0, \end{aligned} \tag{4.67}$$

which is invariant under translations in U and V.

First of all, by excluding the dependent variable V from the PDE system (4.67), one obtains the following subsystem given by

$$\begin{aligned} &U_{TT} + c^2(T)\left(c^2(T)U_{XX} - U_{XX}U_T^2 - U_{TT}U_X^2 - 2U_{TX} + 2U_{TX}U_TU_X\right) \\ &- 2c(T)c'(T)\left(U_X - U_X^2U_T\right) = 0. \end{aligned} \tag{4.68}$$

which, in turn, is an equivalent PDE for the nonlinear wave equation (4.64).

Secondly, by excluding the dependent variable U from the PDE system (4.67), one obtains another subsystem given by

$$c(T)\left(V_X^2V_{TT} - 2V_XV_TV_{TX} + V_{XX}V_T^2 - c^3(T)V_{XX}\right) - 2c'(T)V_X^2V_T = 0, \tag{4.69}$$

which, in turn, is another equivalent PDE for the nonlinear wave equation (4.13).

Remark 4.3.2 In terms of (x, u, v) coordinates, the equation (4.69) becomes

$$c(u)\left(v_x^2v_{uu} - 2v_xv_uv_{ux} + v_{xx}v_u^2 - c^2(u)v_{xx}\right) - 2c'(u)v_x^2v_u = 0. \tag{4.70}$$

It is straightforward to check the equation (4.70) is invertibly related to the PDE (3.43) after interchanging x and v. However, in next section we will show that the equation (4.68) is nonlocally related to any PDE system constructed in Example 3.2.21.

4.4 Examples of nonlocal symmetries arising from the symmetry-based method

In the previous section, we constructed several inverse potential systems for the nonlinear reaction-diffusion equation (4.1), the nonlinear diffusion equations (4.45) and (4.59), and the nonlinear wave equation (4.64). For the nonlinear reaction-diffusion equation (4.1), one can show that each point symmetry of the constructed inverse potential systems yields no nonlocal symmetry of the nonlinear reaction-diffusion equation (4.1). In this section, it is shown that for the nonlinear diffusion equations (4.45) and (4.59), and the nonlinear wave equation (4.64), nonlocal symmetries can arise from some of the constructed inverse potential systems. Most importantly, some previously unknown nonlocal symmetries are obtained for the nonlinear wave equation (4.64).

4.4.1 Nonlocal symmetries of nonlinear diffusion equation

As shown in Proposition 3.3.5, the nonlinear diffusion equation (4.59) has a point symmetry $\mathbf{X}_6$ that induces a nonlocal symmetry of the PDE system (3.5). Since the nonlinear diffusion equation (4.45) is locally related to the PDE system (3.5), it follows that $\mathbf{X}_6$ also yields a nonlocal symmetry of the nonlinear diffusion equation (4.45).

Now consider the class of scalar PDEs (4.49). The equivalence transformations for this class of PDEs arise from the six generators

$$\begin{aligned}
&\mathbf{E}_1 = \frac{\partial}{\partial v}, \quad \mathbf{E}_2 = \frac{\partial}{\partial w} + \frac{2K}{w}\frac{\partial}{\partial K}, \quad \mathbf{E}_3 = w\frac{\partial}{\partial w} + 2K\frac{\partial}{\partial K}, \\
&\mathbf{E}_4 = v\frac{\partial}{\partial v} + 2K\frac{\partial}{\partial K}, \quad \mathbf{E}_5 = t\frac{\partial}{\partial t} - K\frac{\partial}{\partial K}, \quad \mathbf{E}_6 = \frac{\partial}{\partial t}.
\end{aligned} \tag{4.71}$$

Thus the group of equivalence transformations for the class of PDEs (4.49) is given by

$$\begin{aligned}
\bar{v} &= a_3 v + a_1, \\
\bar{t} &= a_5 t + a_6, \\
\bar{w} &= a_4 w + a_2, \\
\bar{K} &= \frac{a_3^2(a_4 w + a_2)^2}{a_5 w^2} K,
\end{aligned} \tag{4.72}$$

where $a_1, \ldots, a_6$ are arbitrary constants with $a_3 a_4 a_5 \neq 0$.

In Table 4.2, we present the point symmetry classification of the PDE (4.49), modulo its group of equivalence transformations (4.72).

By similar reasoning as in the proof of Proposition 3.3.5, one can show that, for $K(u) = u^{-\frac{2}{3}}$, the point symmetry $\mathbf{V}_5$ of the PDE (4.49) yields a nonlocal symmetry of the PDE system (4.47), which is locally related to the nonlinear diffusion equation (4.45). Hence $\mathbf{V}_5$ yields a nonlocal symmetry of the nonlinear diffusion equation (4.45).

Moreover, comparing Tables 3.1 and 4.2, one also sees that when $K(u) = u^{-\frac{2}{3}}$, since its infinitesimal component for the variable u has an essential dependence on the variable v, the symmetry $\mathbf{V}_5$ of the PDE (4.49) yields a nonlocal symmetry of the nonlinear diffusion equation (4.59), which cannot be obtained through its potential system (3.5). By similar reasoning, when $K(u) = u^{-2}$, one can show that the symmetries $\mathbf{V}_6$, $\mathbf{V}_7$ and $\mathbf{V}_\infty$ of the PDE (4.49) yield nonlocal symmetries of the nonlinear diffusion equation (4.59).

Remark 4.4.1 Comparing Tables 3.3 and 4.2, for the nonlinear diffusion equation (4.59), one concludes that the nonlocal symmetries yielded by $\mathbf{V}_5$, $\mathbf{V}_6$, $\mathbf{V}_7$ and $\mathbf{V}_\infty$

Table 4.2: Point symmetry classification for the PDE (4.49)

$K(1/w)$	$K(u)$	#	admitted point symmetries in (t,v,w) coordinates
arbitrary	arbitrary	3	$\mathbf{V}_1=\frac{\partial}{\partial t}$, $\mathbf{V}_2=\frac{\partial}{\partial v}$, $\mathbf{V}_3=2t\frac{\partial}{\partial t}+v\frac{\partial}{\partial v}$
$w^{-\mu}$	u^{μ}	4	$\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4=(2+\mu)v\frac{\partial}{\partial v}-2w\frac{\partial}{\partial w}$
$w^{\frac{2}{3}}$	$u^{-\frac{2}{3}}$	5	$\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4\ (\mu=-\frac{2}{3})$, $\mathbf{V}_5=3vw\frac{\partial}{\partial w}-v^2\frac{\partial}{\partial v}$
w^2	u^{-2}	∞	$\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4\ (\mu=-2)$, $\mathbf{V}_6=-vw\frac{\partial}{\partial w}+2t\frac{\partial}{\partial v}$, $\mathbf{V}_7=4t^2\frac{\partial}{\partial t}+4vt\frac{\partial}{\partial v}-(2t+v^2)w\frac{\partial}{\partial w}$, $\mathbf{V}_\infty=G(t,v)\frac{\partial}{\partial w}$, where $G(t,v)$ satisfies $G_t=G_{vv}$

Table 4.2: Point symmetry classification for the PDE (4.49) (continued)

$K(1/w)$	$K(u)$	#	admitted point symmetries in (t,v,u) coordinates
arbitrary	arbitrary	3	$\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3$
$w^{-\mu}$	u^{μ}	4	$\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4=(2+\mu)v\frac{\partial}{\partial v}+2u\frac{\partial}{\partial u}$
$w^{\frac{2}{3}}$	$u^{-\frac{2}{3}}$	5	$\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4\ (\mu=-\frac{2}{3})$, $\mathbf{V}_5=-3uv\frac{\partial}{\partial u}-v^2\frac{\partial}{\partial v}$
w^2	u^{-2}	∞	$\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4\ (\mu=-2)$, $\mathbf{V}_6=uv\frac{\partial}{\partial u}+2t\frac{\partial}{\partial v}$, $\mathbf{V}_7=4t^2\frac{\partial}{\partial t}+4vt\frac{\partial}{\partial y}+(2t+v^2)u\frac{\partial}{\partial u}$, $\mathbf{V}_\infty=-u^2G(t,v)\frac{\partial}{\partial u}$, where $G(t,v)$ satisfies $G_t=G_{vv}$

correspond to the nonlocal symmetries yielded by $\mathbf{Z}_9$, $\mathbf{Z}_{10}$, $\mathbf{Z}_{11}$ and $\mathbf{Z}_\infty$ respectively.

In addition, consider the PDE (4.63). The equivalence transformations of the class of PDEs (4.63) arise from the six generators

$$\begin{aligned}&\mathbf{E}_1=\frac{\partial}{\partial u},\quad \mathbf{E}_2=u\frac{\partial}{\partial u}+2K\frac{\partial}{\partial K},\quad \mathbf{E}_3=p\frac{\partial}{\partial p}+2K\frac{\partial}{\partial K},\\&\mathbf{E}_4=t\frac{\partial}{\partial t}-K\frac{\partial}{\partial K},\quad \mathbf{E}_5=\frac{\partial}{\partial t},\quad \mathbf{E}_6=u^2\frac{\partial}{\partial u}-3up\frac{\partial}{\partial p}-2Ku\frac{\partial}{\partial K}.\end{aligned}\tag{4.73}$$

Hence, the five-parameter group of equivalence transformations of the PDE class (4.63), arising from the first five generators of (4.73), is given by

$$\begin{aligned}&\bar{u}=a_2u+a_1,\\&\bar{t}=a_4t+a_5,\\&\bar{p}=a_3p,\\&\bar{K}=\frac{a_2^2a_3^2}{a_4}K,\end{aligned}\tag{4.74}$$

where $a_1, \ldots, a_5$ are arbitrary constants with $a_2 a_3 a_4 \neq 0$.

The generator $\mathbf{E}_6$ yields the additional one-parameter group of equivalence transformations given by

$$\begin{aligned} \bar{u} &= \frac{u}{1 - a_6 u}, \\ \bar{t} &= t, \\ \bar{p} &= (1 - a_6 u)^3 p, \\ \bar{K} &= (1 - a_6 u)^2 K, \end{aligned} \tag{4.75}$$

where a_6 is an arbitrary constant.

In Table 4.3, we present the point symmetry classification of the PDE (4.63), modulo its group of equivalence transformations.

Table 4.3: Point symmetry classification for the PDE (4.63)

$K(u)$	#	admitted point symmetries
arbitrary	2	$\mathbf{W}_1 = \frac{\partial}{\partial t}$, $\mathbf{W}_2 = 2t\frac{\partial}{\partial t} + p\frac{\partial}{\partial p}$
u^μ	3	$\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3 = 2u\frac{\partial}{\partial u} + (\mu - 2)p\frac{\partial}{\partial p}$
e^u	3	$\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_4 = 2\frac{\partial}{\partial u} + p\frac{\partial}{\partial p}$
$\frac{1}{1+u^2}e^{\lambda \arctan u}$	3	$\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_5 = 2(1 + u^2)\frac{\partial}{\partial u} - p(6u - \lambda)\frac{\partial}{\partial p}$
u^{-2}	4	$\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ $(\mu = -2)$, $\mathbf{W}_6 = u^2\frac{\partial}{\partial u} - 3pu\frac{\partial}{\partial p}$

Similar to the situation in Proposition 3.3.5, when $K(u) = \frac{1}{1+u^2}e^{\lambda \arctan u}$, the point symmetry $\mathbf{W}_5$ of the PDE (4.63) yields a nonlocal symmetry of the corresponding intermediate system (4.61), which is locally related to the nonlinear diffusion equation (4.59). Hence $\mathbf{W}_5$ yields a nonlocal symmetry of the nonlinear diffusion equation (4.59) with $K(u) = \frac{1}{1+u^2}e^{\lambda \arctan u}$. By similar reasoning, the symmetry $\mathbf{W}_6$ also yields a nonlocal symmetry of the nonlinear diffusion equation (4.59) with $K(u) = u^{-2}$.

Taking the equivalence transformation (4.75) into consideration, one can obtain more nonlocal symmetries for the nonlinear diffusion equation (4.59) from the corresponding PDE (4.63). In particular, the equivalence transformation (4.75) maps u^μ into $\bar{u}^\mu(1 + a_6\bar{u})^{-(\mu+2)}$, e^u into $(1 + a_6\bar{u})^{-2}e^{\frac{\bar{u}}{1+a_6 u}}$. Moreover, the symmetries $\mathbf{W}_3$ and $\mathbf{W}_4$ are mapped into $\bar{\mathbf{W}}_3$ and $\bar{\mathbf{W}}_4$ respectively. One can show that when $K(u) = u^\mu(1 + a_6 u)^{-(\mu+2)}$, $\bar{\mathbf{W}}_3 = 2u(1 + a_6 u)\frac{\partial}{\partial u} - p(6a_6 u - \mu + 2)\frac{\partial}{\partial p}$; when $K(u) = (1 + a_6 u)^{-2}e^{\frac{u}{1+a_6 u}}$, $\bar{\mathbf{W}}_4 = 2(1 + a_6 u)^2\frac{\partial}{\partial u} - p(6a_6^2 u + 6a_6 - 1)\frac{\partial}{\partial p}$. Similar to the situation in Proposition 3.3.5, one can show that $\bar{\mathbf{W}}_3$ and $\bar{\mathbf{W}}_4$ yield nonlocal symmetries of the corresponding nonlinear diffusion equations (4.59).

Remark 4.4.2 Comparing Tables 3.2 and 4.3, for the nonlinear diffusion equation (4.59), one concludes that when $K(u) = \frac{1}{1+u^2}e^{\lambda \arctan u}$, the nonlocal symmetry yielded by $\mathbf{W}_5$ corresponds to the nonlocal symmetry yielded by $\mathbf{Y}_9$. When $K(u) = u^{-2}$, the nonlocal symmetry yielded by $\mathbf{W}_6$ corresponds to a nonlocal symmetry yielded by $\mathbf{Y}_\infty$.

4.4.2 Nonlocal symmetries of nonlinear wave equation

We now use the PDE (4.68) to find previously unknown nonlocal symmetries of the nonlinear wave equation (4.64).

In [3], the point symmetry classification is given for the class of nonlinear wave equations (4.64), which is presented in Table 4.4, modulo its group of equivalence transformations:

$$\begin{aligned} \overline{x} &= a_1 x + a_4, \\ \overline{t} &= a_2 t + a_5, \\ \overline{u} &= a_3 u + a_6, \\ \overline{c} &= \frac{a_1}{a_2} c, \end{aligned} \tag{4.76}$$

where $a_1, \ldots, a_6$ are arbitrary constants with $a_1 a_2 a_3 \neq 0$.

Table 4.4: Point symmetry classification for the nonlinear wave equation (4.64)

$c(u)$	#	admitted point symmetries
arbitrary	3	$\mathbf{X}_1 = \frac{\partial}{\partial x}$, $\mathbf{X}_2 = \frac{\partial}{\partial t}$, $\mathbf{X}_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$
u^μ $(\mu \neq 0)$	4	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 = \mu x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$
e^u	4	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_5 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$
u^{-2}	5	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$ $(\mu = -2)$, $\mathbf{X}_6 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}$
$u^{-\frac{2}{3}}$	5	$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$ $(\mu = -\frac{2}{3})$, $\mathbf{X}_7 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$

The equivalence transformations for the PDE class (4.68) arise from the five generators

$$\begin{aligned} &\mathbf{E}_1 = \frac{\partial}{\partial T}, \quad \mathbf{E}_2 = \frac{\partial}{\partial X}, \quad \mathbf{E}_3 = \frac{\partial}{\partial U}, \\ &\mathbf{E}_4 = T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X} + U\frac{\partial}{\partial U}, \quad \mathbf{E}_5 = -T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X} + c\frac{\partial}{\partial c}. \end{aligned} \tag{4.77}$$

Correspondingly, the five-parameter group of equivalence transformations for the class of PDEs (4.68) is given by

$$\begin{aligned}\overline{T} &= \frac{a_4}{a_5}T + a_1,\\ \overline{X} &= a_4 a_5 X + a_2,\\ \overline{U} &= a_4 U + a_3,\\ \overline{c} &= a_5 c,\end{aligned} \tag{4.78}$$

where $a_1, \ldots, a_5$ are arbitrary constants with $a_4 a_5 \neq 0$.

The point symmetry classification of the PDE (4.68), modulo its equivalence transformations (4.78), is presented in Table 4.5.

Table 4.5: Point symmetry classification for the PDE (4.68)

$c(T)$	$c(u)$	#	admitted point symmetries in (X, T, U) coordinates
arbitrary	arbitrary	3	$\mathbf{W}_1 = \frac{\partial}{\partial U}$, $\mathbf{W}_2 = \frac{\partial}{\partial X}$, $\mathbf{W}_3 = (X + \int^T c^2(\xi)d\xi)\frac{\partial}{\partial X} + U\frac{\partial}{\partial U}$
T^μ	u^μ	4	$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ $(c(T) = T^\mu)$, $\mathbf{W}_4 = T\frac{\partial}{\partial T} + (2\mu + 1)X\frac{\partial}{\partial X} + (\mu + 1)U\frac{\partial}{\partial U}$
e^T	e^u	4	$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ $(c(T) = e^T)$, $\mathbf{W}_5 = \frac{\partial}{\partial T} + 2X\frac{\partial}{\partial X} + U\frac{\partial}{\partial U}$
T^{-2}	u^{-2}	5	$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ $(c(T) = T^{-2})$, $\mathbf{W}_4$ $(\mu = -2)$, $\mathbf{W}_6 = U^2\frac{\partial}{\partial U} + TU\frac{\partial}{\partial T} - \frac{U}{T^3}\frac{\partial}{\partial X}$
$T^{-\frac{2}{3}}$	$u^{-\frac{2}{3}}$	5	$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ $(c(T) = T^{-\frac{2}{3}})$, $\mathbf{W}_4$ $(\mu = -\frac{2}{3})$, $\mathbf{W}_7 = (XT - 3T^{\frac{2}{3}})\frac{\partial}{\partial T} + (XT^{-\frac{1}{3}} - \frac{X^2}{3})\frac{\partial}{\partial X}$

Table 4.5: Point symmetry classification for the PDE (4.68) (continued)

$c(T)$	$c(u)$	#	(x, u) components of admitted symmetries
arbitrary	arbitrary	3	$\check{\mathbf{W}}_2 = \frac{\partial}{\partial x}$, $\check{\mathbf{W}}_3 = (x + \int^u c^2(\xi)d\xi)\frac{\partial}{\partial x}$
T^μ	u^μ	4	$\check{\mathbf{W}}_2, \check{\mathbf{W}}_3$ $(c(u) = u^\mu)$, $\check{\mathbf{W}}_4 = u\frac{\partial}{\partial u} + (2\mu + 1)x\frac{\partial}{\partial x}$
e^T	e^u	4	$\check{\mathbf{W}}_2, \check{\mathbf{W}}_3$ $(c(u) = e^u)$, $\check{\mathbf{W}}_5 = \frac{\partial}{\partial u} + 2x\frac{\partial}{\partial x}$
T^{-2}	u^{-2}	5	$\check{\mathbf{W}}_1, \check{\mathbf{W}}_2, \check{\mathbf{W}}_3$ $(c(u) = u^{-2})$, $\check{\mathbf{W}}_4$ $(\mu = -2)$, $\check{\mathbf{W}}_6 = u(t + v)\frac{\partial}{\partial u} - \frac{t+v}{u^3}\frac{\partial}{\partial x}$
$T^{-\frac{2}{3}}$	$u^{-\frac{2}{3}}$	5	$\check{\mathbf{W}}_1, \check{\mathbf{W}}_2, \check{\mathbf{W}}_3$ $(c(u) = u^{-\frac{2}{3}})$, $\check{\mathbf{W}}_4$ $(\mu = -\frac{2}{3})$, $\check{\mathbf{W}}_7 = (xu - 3u^{\frac{2}{3}})\frac{\partial}{\partial u} + (xu^{-\frac{1}{3}} - \frac{x^2}{3})\frac{\partial}{\partial x}$

Remark 4.4.3 In order to determine whether a symmetry $\mathbf{W}$ of the PDE (4.68) yields a nonlocal symmetry of the nonlinear wave equation (4.64), it requires us to trace back to the nonlinear wave equation (4.64) using the PDE system (4.67). Because one excludes the dependent variable V from the potential system (4.67), one needs to investigate how the variable V changes under the action induced by $\mathbf{W}$. Since $\rho_*^{-1}(\frac{\partial}{\partial V}) = \frac{\partial}{\partial v} - \frac{\partial}{\partial t}$, where ρ^{-1} is the inverse of the transformation (4.66), the infinitesimal components for the variables x and u remain invariant when tracing back. This is why we only present the (x, u) components of admitted symmetries in Table 4.5 (continued).

Proposition 4.4.4 The symmetries $\mathbf{W}_6$ and $\mathbf{W}_7$ yield nonlocal symmetries of the PDE system (4.65).

Proof. If the symmetry $\mathbf{W}_6$ yields a local symmetry $\tilde{\mathbf{W}}_6$ of the potential system (4.65) with $c(u) = u^{-2}$, then, in evolutionary form, $\tilde{\mathbf{W}}_6 = \left(U^2 - TUU_T + \frac{U}{T^3}U_X\right)\frac{\partial}{\partial U}$ $+F[U,V]\frac{\partial}{\partial V}$, where the differential function $F[U,V]$ must depend on X, T, U, V and the partial derivatives of U and V with respect to X and T. By applying $\tilde{\mathbf{W}}_6$ to the corresponding PDE system (4.67) which is invertibly related to the potential system (4.65), one can show that $F[U,V]$ must be of the form $F(X, T, U, V, U_X, U_T)$. Applying $\tilde{\mathbf{W}}_6^{(\infty)}$ to the corresponding PDE system (4.67) and making appropriate substitutions, one can prove that the resulting determining equation system is inconsistent. Hence $\mathbf{W}_6$ yields a nonlocal symmetry of the potential system (4.65) with $c(u) = u^{-2}$.

By similar reasoning, it turns out that $\mathbf{W}_7$ also yields a nonlocal symmetry of the potential system (4.65) with $c(u) = u^{-\frac{2}{3}}$. □

When $c(u)$ is arbitrary, in (x, t, u, v) coordinates, $\mathbf{W}_3 = (x + \int^u c^2(\xi)d\xi)\frac{\partial}{\partial x} + (t + v)\frac{\partial}{\partial t}$. One can show that $\mathbf{W}_3$ is a point symmetry of the potential system (4.65), whose infinitesimal component for the variable t has an essential dependence on v. By projection, $\mathbf{W}_3$ yields a nonlocal symmetry of the nonlinear wave equation (4.64).

When $c(u) = u^{-2}$, the infinitesimal components for the variables (x, u) of the symmetry $\mathbf{W}_6$ depend on the variable v. By Remark 4.4.3, $\mathbf{W}_6$ yields a nonlocal symmetry of the nonlinear wave equation (4.64).

When $c(u) = u^{-\frac{2}{3}}$, if the symmetry $\mathbf{W}_7$ yielded a local symmetry $\tilde{\mathbf{W}}_7$ of the nonlinear wave equation (4.64), then $\tilde{\mathbf{W}}_7 = \check{\mathbf{W}}_7 + f[u]\frac{\partial}{\partial t}$, where the function $f[u]$ depends on x, t, u and its derivatives. Since $\rho_*^{-1}(\frac{\partial}{\partial V}) = \frac{\partial}{\partial v} - \frac{\partial}{\partial t}$, when tracing back to the PDE system (4.65), the infinitesimal component for the variable v must be equal to $-f[u]$. Thus $\mathbf{W}_7$ would also yield a local symmetry of the PDE system (4.65), which is a contradiction since $\mathbf{W}_7$ yields a nonlocal symmetry of the PDE system

(4.65). Hence, $\mathbf{W}_7$ yields a nonlocal symmetry of the nonlinear wave equation (4.64).

Remark 4.4.5 One can show that the symmetries $\mathbf{W}_4$ and $\mathbf{W}_5$ yield point symmetries $\tilde{\mathbf{W}}_4 = \mathbf{W}_4 + (\mu+1)V\frac{\partial}{\partial V}$ and $\tilde{\mathbf{W}}_5 = \mathbf{W}_5 + V\frac{\partial}{\partial V}$ of the PDE system (4.67) respectively since in terms of (x,t,u,v) coordinates, $\tilde{\mathbf{W}}_4 = u\frac{\partial}{\partial u} + (2\mu+1)x\frac{\partial}{\partial x} + (\mu+1)t\frac{\partial}{\partial t} + (\mu+1)v\frac{\partial}{\partial v} = (2\mu+1)\mathbf{Y}_4 - \mathbf{Y}_5$ and $\tilde{\mathbf{W}}_5 = \frac{\partial}{\partial u} + 2x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v} = \mathbf{Y}_4 + \mathbf{Y}_7$. Hence, by projection, $\mathbf{W}_4$ and $\mathbf{W}_5$ yield point symmetries of the nonlinear wave equation (4.64).

Remark 4.4.6 Comparing the symmetries listed in [24], one sees that the symmetries $\mathbf{W}_6$ and $\mathbf{W}_7$ yield previously unknown nonlocal symmetries of the nonlinear wave equation (4.64).

4.5 Summary

In this chapter, we introduced a new systematic symmetry-based method for constructing nonlocally related PDE systems (*inverse potential systems*) of a given PDE system. The starting point for this method is *any* point symmetry of the given PDE system. In the case of three or more independent variables, the symmetry-based method directly yields determined nonlocally related systems for a given PDE system, unlike the situation in the CL-based method where one must append gauge constraints.

The symmetry-based method was shown to yield previously unknown nonlocally related systems for nonlinear reaction-diffusion, nonlinear diffusion and nonlinear wave equations. In addition, it was shown that nonlocally related symmetry-based systems could yield nonlocal symmetries of a given PDE system. Moreover, some previously unknown nonlocal symmetries were obtained for the nonlinear wave equation.

Chapter 5

New Exact Nonclassical Solutions of the NLK Equation

5.1 Introduction

An exact solution is of great interest for researchers, since it plays an essential role in the analysis of a PDE system. A significant application of symmetries of a given PDE system is the finding of exact solutions of the PDE system. The method of using a local symmetry to construct exact solutions of a PDE system is called *Lie's classical method* [21, 25, 29, 39, 53, 75, 80]. Moreover, reduction through a point symmetry could lead to the solution of a boundary value problem for a given PDE system [16, 21]. In [15, 27], Lie's classical method was generalized to *the nonclassical method*, in which one searches for "nonclassical symmetries" of a given PDE system. In particular, "nonclassical symmetries" are local symmetries of an augmented PDE system consisting of the given PDE system, the invariant surface condition and their differential consequences. It follows that "nonclassical symmetries" leave only submanifolds of solutions invariant. Consequently, the nonclassical method turns out to be useful for finding further specific solutions in addition to those obtained by Lie's classical method.

In this chapter, we first present the basic ideas of both Lie's classical method and the nonclassical method. Then we apply the nonclassical method to obtain previously unknown exact solutions of the dimensional NLK equation [60] given by

$$u_t = x^{-2}\left(x^4\left(\alpha u_x + \beta u + \gamma u^2\right)\right)_x, \tag{5.1}$$

where $\alpha > 0, \beta \geq 0$ and $\gamma > 0$ are arbitrary constants.

The NLK equation (5.1), also known as the photon diffusion equation, was first presented by Kompaneets [60], and in dimensionless form, after appropriate scalings of x, t and u, can be written as

$$u_t = x^{-2}\left(x^4\left(u_x + u + u^2\right)\right)_x, \tag{5.2}$$

and

$$u_t = x^{-2}\left(x^4\left(u_x + u^2\right)\right)_x, \tag{5.3}$$

when $\beta \neq 0$ and the case with dominating induced scattering $\beta = 0$ ($u^2 \gg u$), respectively. By construction, the solutions obtained by the nonclassical method of course include the solutions obtained by Ibragimov [55] through classical symmetry reductions. Correspondingly, these new solutions yield five families of solutions with initial conditions of physical interest. It is shown that three of these families of solutions exhibit quiescent behaviour, i.e., $\lim\limits_{t\to\infty} u(x,t) = 0$, and that the other two families of solutions exhibit blow up behaviour, i.e., $\lim\limits_{t\to t^*} u(x,t^*) = \infty$ for some finite t^* depending on a constant in their initial conditions. Moreover, we consider nontrivial stationary solutions of (5.3). We exhibit four families of stationary solutions not presented explicitly in [50] for the NLK equations (5.2) and (5.3). We show that two of these families of stationary solutions are unstable.

5.2 Lie's classical method

5.2.1 The invariant form method

In this section, we present the invariant form method for constructing invariant solutions of a given PDE system [21, 25, 29, 39, 53, 75]. Consider a PDE system $\mathbf{R}\{x;u\}$ of order l with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$, given by

$$R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^l u) = 0, \quad \sigma = 1, \ldots, s. \tag{5.4}$$

Suppose the PDE system $\mathbf{R}\{x;u\}$ (5.4) has a point symmetry with the infinitesimal generator

$$\mathbf{X} = \sum_{i=1}^{n} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j(x,u)\frac{\partial}{\partial u^j}. \tag{5.5}$$

Definition 5.2.1 A solution $u = f(x)$, with $u^\mu = f^\mu(x)$, $\mu = 1, \ldots, m$, of the PDE system $\mathbf{R}\{x;u\}$ (5.4) is an *invariant solution* arising from the point symmetry (5.5) if $u^\mu = f^\mu(x)$ is an invariant surface of the point symmetry (5.5) for each $\mu = 1, \ldots, m$.

From Definition 5.2.1, a function $u = f(x)$ is an invariant solution of the PDE system $\mathbf{R}\{x;u\}$ (5.4) arising from the point symmetry (5.5) if and only if $u = f(x)$ satisfies the following two conditions:

$$\mathbf{X}(u^\mu - f^\mu(x))\Big|_{u=f(x)} = 0, \quad \mu = 1, \ldots, m. \tag{5.6}$$

$$R^\sigma(x, u, \partial u, \ldots, \partial^l u)\Big|_{u=f(x)} = 0, \quad \sigma = 1, \ldots, s. \tag{5.7}$$

In order to obtain the invariants of the point symmetry (5.5) of the PDE system $\mathbf{R}\{x;u\}$ (5.4), one can employ the characteristic method stated in Chapter 2, i.e., solve the following characteristic equations:

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^m}{\eta^m(x,u)}. \tag{5.8}$$

Suppose one obtains $n+m-1$ corresponding functionally independent invariants given by

$$\omega^1(x,u),\ \ldots,\ \omega^{n-1}(x,u),\ \zeta^1(x,u),\ \ldots,\ \zeta^m(x,u), \tag{5.9}$$

with

$$\frac{\partial(\omega^1,\ldots,\omega^{n-1})}{\partial(x^{i_1},\ldots,x^{i_{n-1}})} \neq 0,\ \ \frac{\partial(\zeta^1,\ldots,\zeta^m)}{\partial(u^1,\ldots,u^m)} \neq 0. \tag{5.10}$$

By introducing the new independent variables $y=(y^1,\ldots,y^n)$ and dependent variables $v=(v^1,\ldots,v^m)$:

$$\begin{aligned} y^i &= \omega^i(x,u), \quad i=1,\ldots,n-1,\\ y^n &= \omega^n(x,u),\\ v^\mu &= \zeta^\mu(x,u), \quad \mu=1,\ldots,m, \end{aligned} \tag{5.11}$$

with

$$\mathbf{X}\omega^n(x,u) = 1,$$

one obtains the canonical coordinates corresponding to the point symmetry (5.5). The point transformation corresponding to the canonical coordinates (5.14) maps the point symmetry (5.5) into the canonical form

$$\tilde{\mathbf{X}} = \frac{\partial}{\partial y^n}. \tag{5.12}$$

Suppose the PDE system $\mathbf{R}\{x;u\}$ (5.4) becomes the transformed PDE system $\mathbf{S}\{y;v\}$ in the canonical coordinates (5.14). Since the transformed PDE system $\mathbf{S}\{y;v\}$ has the translation point symmetry with the infinitesimal generator $\tilde{\mathbf{X}}$, it follows that the transformed PDE system $\mathbf{S}\{y;v\}$ does not depend explicitly on the new independent variable y^n. Consequently, the transformed PDE system $\mathbf{S}\{y;v\}$ has particular solutions of the form

$$v^\mu = h^\mu(y^1,\ldots,y^{n-1}), \quad \mu=1,\ldots,m. \tag{5.13}$$

By the assumption (5.10), one can solve $x^{i_1},\ \ldots,\ x^{i_{n-1}}$ and $u^1,\ \ldots,\ u^m$ from (5.14) in terms of $y=(y^1,\ldots,y^{n-1})$, $v=(v^1,\ldots,v^m)$ and the remaining variable x^{i_n}, i.e.,

$$\begin{aligned} x^{i_k} &= \alpha^{i_k}(x^{i_n},y,v), \quad k=1,\ldots,n-1,\\ u^\mu &= \beta^\mu(x,v) = \tilde{\beta}^\mu(x^{i_n},y,v), \quad \mu=1,\ldots,m. \end{aligned} \tag{5.14}$$

Therefore, for each solution $v = h(y)$, where $h(y) = (h^1(y), \dots, h^m(y))$, of the transformed PDE system $\mathbf{S}\{y; v\}$, there is a corresponding implicit solution

$$u^\mu = \tilde{\beta}^\mu(x^{i_n}, \omega(x,u), h(\omega(x,u))), \quad \mu = 1, \dots, m, \tag{5.15}$$

of the given PDE system $\mathbf{R}\{x; u\}$ (5.4), where $\omega(x,u) = (\omega^{i_1}(x,u), \dots, \omega^{i_{n-1}}(x,u))$. Moreover, one can show that the solution (5.15) is invariant under the point symmetry (5.5). In particular, if the point symmetry is of the form

$$\mathbf{X} = \sum_{i=1}^{n} \xi^i(x)\frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j(x,u)\frac{\partial}{\partial u^j}, \tag{5.16}$$

then one can choose $y = \omega(x)$. It follows that the solution (5.15) becomes

$$u^\mu = \tilde{\beta}^\mu(x^{i_n}, \omega(x), h(\omega(x))), \quad \mu = 1, \dots, m, \tag{5.17}$$

which defines an explicit invariant solution of the given PDE system $\mathbf{R}\{x; u\}$ (5.4). From the above discussion, one concludes that the invariant solutions of a given PDE system can be found by solving a reduced DE system involving fewer independent variables.

Example 5.2.2 Consider the heat equation

$$u_t = u_{xx}, \tag{5.18}$$

which has the point symmetry

$$\mathbf{X} = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}. \tag{5.19}$$

The invariants of $\mathbf{X}$ are given by

$$\omega = t, \quad \zeta = ue^{\frac{x^2}{4t}}, \tag{5.20}$$

for $t > 0$. By introducing the canonical coordinates corresponding to $\mathbf{X}$:

$$\begin{aligned} y_1 &= t, \\ y_2 &= \frac{x}{2t}, \\ v &= ue^{\frac{x^2}{4t}}, \end{aligned} \tag{5.21}$$

one obtains the invertibly related PDE of (5.18) given by

$$4y_1^2 v_{y_1} + 2y_1 v - v_{y_2 y_2} = 0. \tag{5.22}$$

In order to obtain the invariant solution corresponding to **X**, one seeks solutions of PDE (5.22) of the form $v = h(y_1)$. Consequently, the PDE (5.22) reduces to the ODE

$$2y_1^2 h_{y_1} + y_1 h = 0. \tag{5.23}$$

The solution of the reduced ODE (5.23) is given by

$$h(y_1) = \frac{c}{\sqrt{y_1}}, \tag{5.24}$$

where c is an arbitrary constant. Hence the invariant solution corresponding to **X** is given by

$$u = h(y_1)e^{-\frac{x^2}{4t}} = \frac{c}{\sqrt{t}}e^{-\frac{x^2}{4t}}. \tag{5.25}$$

One can replace the point symmetry (5.5) by any local symmetry with the infinitesimal generator

$$\hat{\mathbf{X}} = \sum_{j=1}^{m} Q^j[u]\frac{\partial}{\partial u^j} \tag{5.26}$$

to obtain invariant solutions arising from a local symmetry (5.26). One can refer to [29] for more details of the extension of Lie's classical method for finding invariant solutions to local symmetries.

The invariant form method can be applied to a boundary value problem for a PDE system provided the symmetry of the given PDE system also leaves invariant the boundary and the boundary conditions [16, 21, 29]. Moreover, this method can also be applied to the nonlocally related PDE systems of a given PDE system, which possibly yields further exact solutions of the given PDE system [41].

5.2.2 The direct substitution method

If one is unable to solve the characteristic equations (5.8), one can employ the direct substitution method to resolve such a dilemma [25]. Without loss of generality, one can assume that $\xi^n(x,u) \neq 0$. From the condition (5.6), it follows that

$$\eta^\mu(x,u) - \sum_{i=1}^{n} \xi^i(x,u)\frac{\partial u^\mu}{\partial x^i} = 0, \quad \mu = 1, \ldots, m. \tag{5.27}$$

Solving for $\frac{\partial u^\mu}{\partial x^n}$ from the system of equations (5.27), one obtains

$$\frac{\partial u^\mu}{\partial x^n} = \frac{\eta^\mu(x,u)}{\xi^n(x,u)} - \sum_{i=1}^{n-1} \frac{\xi^i(x,u)}{\xi^n(x,u)}\frac{\partial u^\mu}{\partial x^i} = 0, \quad \mu = 1, \ldots, m. \tag{5.28}$$

Thus the terms involving derivatives of u with respect to x^n appearing in the PDE system $\mathbf{R}\{x;u\}$ (5.4) can be expressed in terms of x, u and derivatives of u with respect to $\hat{x} = (x^1, \ldots, x^{n-1})$. Then one obtains a reduced DE system involving $n-1$ independent variables $\hat{x}$, m dependent variables u, and the parameter variable x^n. A solution $u = f(\hat{x}; x^n)$ of the reduced DE system yields an invariant solution of the PDE system $\mathbf{R}\{x;u\}$ (5.4) provided the equations (5.28) are satisfied.

5.3 The nonclassical method

In the nonclassical method, for a given PDE system one seeks symmetries that leave only a submanifold of the solution manifold invariant. Such a "nonclassical symmetry" maps solution surfaces not in the submanifold to surfaces that are not solutions of the PDE system.

Consider a PDE system $\mathbf{R}\{x;u\}$ (5.4). In the nonclassical method, instead of seeking local symmetries of the PDE system $\mathbf{R}\{x;u\}$ (5.4), one seeks local symmetries that leave invariant a submanifold of the solution manifold of the PDE system $\mathbf{R}\{x;u\}$ (5.4). In particular, one seeks functions $\xi^i(x,u)$, $\eta^j(x,u)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, so that

$$\mathbf{X} = \sum_{i=1}^{n} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j(x,u)\frac{\partial}{\partial u^j} \tag{5.29}$$

is a "symmetry" ("nonclassical symmetry") of the submanifold, which is the augmented PDE system $\mathbf{A}\{x;u\}$ consisting of the given PDE system $\mathbf{R}\{x;u\}$ (5.4), the invariant surface condition equations

$$\mathbf{I}^\nu(x,u,\partial u) \equiv \eta^\nu(x,u) - \sum_{i=1}^{n} \xi^i(x,u)\frac{\partial u^\nu}{\partial x^i} = 0, \quad \nu = 1, \ldots, m, \tag{5.30}$$

and their differential consequences. Consequently, one obtains an overdetermined set of nonlinear determining equations for the unknown functions $\xi^i(x,u)$, $\eta^j(x,u)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

For any given "nonclassical symmetry", one can employ either the invariant form or the direct substitution method to find the invariant solution corresponding to such a "nonclassical symmetry".

Definition 5.3.1 A solution $u = f(x)$ of a given PDE system $\mathbf{R}\{x;u\}$ (5.4) is a *nonclassical solution* if $u = f(x)$ arises from a "nonclassical symmetry" of $\mathbf{R}\{x;u\}$ (5.4), and does not arise as an invariant solution of the given PDE system $\mathbf{R}\{x;u\}$ (5.4) with respect to its local symmetries.

Indeed, for any functions $\xi^i(x,u)$, $\eta^j(x,u)$, $i = 1,\dots,n$, $j = 1,\dots,m$, the following expressions

$$\mathbf{X}^{(1)}\mathbf{I}^\nu(x,u,\partial u) = \sum_{j=1}^{m}\left(\frac{\partial\eta^\nu}{\partial u^j} - \sum_{i=1}^{n}\frac{\partial\xi^i}{\partial u^j}\frac{\partial u^\nu}{\partial x^i}\right)\cdot\mathbf{I}^j, \quad \nu = 1,\dots,m, \tag{5.31}$$

where $\mathbf{X}^{(1)}$ is the first order prolongation of $\mathbf{X}$, vanish on $\mathbf{I}^\nu(x,u,\partial u) = 0$, $\nu = 1,\dots,m$. Therefore, the nonclassical method includes Lie's classical method.

In the nonclassical method, invariance of a given PDE system $\mathbf{R}\{x;u\}$ (5.4) is replaced by invariance of the augmented PDE system $\mathbf{A}\{x;u\}$. Consequently, it is possible to find symmetries leaving invariant the augmented PDE system $\mathbf{A}\{x;u\}$ which are not symmetries of a given PDE system $\mathbf{R}\{x;u\}$ (5.4). In turn, this can lead to further exact solutions of a given PDE system $\mathbf{R}\{x;u\}$ (5.4).

When a given PDE is a scalar PDE with two independent variables, one needs only to consider two essential cases when solving the determining equations for $(\xi(x,t,u), \tau(x,t,u), \eta(x,t,u))$. Let $x^1 = x$, $x^2 = t$, $\xi^1 = \xi(x,t,u)$, $\xi^2 = \tau(x,t,u)$. If the infinitesimal generator $\mathbf{X} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}$ generates a "nonclassical symmetry" of the PDE system $\mathbf{R}\{x;u\}$ (5.4), then so does $\mathbf{Y} = p(x,t,u)\mathbf{X}$, where $p(x,t,u)$ is any smooth function. It follows that if $\tau \neq 0$, one can set $\tau \equiv 1$, so that only two cases need to be considered: $\tau \equiv 1$ and $\tau \equiv 0$, $\xi \equiv 1$.

In [45], Clarkson and Kruskal introduced a method (the direct method) to obtain exact solutions of a scalar PDE with two independent variables. In the direct method, one aims to find exact solutions of the form

$$u(x,t) = \theta(x,t,w(z)) \ \text{ with } \ z = z(x,t), \tag{5.32}$$

where θ and z are functions of the indicated variables. By substituting (5.32) into the given PDE, one obtains an ODE. After solving the resulting ODE, one can obtain exact solutions for the given PDE. In [74], it was shown that the nonclassical method is more general than the direct method. Further discussions of the nonclassical method can be found in [46, 47, 51, 65, 77, 78].

Other discussions and extensions of obtaining exact solutions of a given PDE system appear in [22, 41, 83, 89]

5.4 Nonclassical analysis of the NLK equation

In this section, we use the nonclassical method to obtain new exact solutions of the NLK equation [36].

5.4.1 Invariant solutions of the NLK equation

As stated in Ibragimov [55], the NLK equation (5.1) describes an interaction of free electrons and electromagnetic radiation, specifically, the interaction of a low-energy homogeneous photon gas with a rarefied electron gas through *Compton scattering*. In equation (5.1), u is the density of the photon gas (photon number density), t is a dimensionless time and $x = \frac{h\nu}{kT}$, where h is *Planck's constant* and ν is the *photon frequency*. Then $h\nu$ denotes the *photon energy*. T is the *electron temperature* and k is *Boltzmann's constant*. The terms u and u^2 in equation (5.1) correspond to *spontaneous scattering* (Compton effect) and *induced scattering*, respectively [71, 95]. The Kompaneets model has been investigated in many publications, and some numerical and analytical solutions have been obtained for the NLK equation (5.1) [13, 50, 55, 71, 72, 86, 88, 95, 96].

By applying Lie's classical method to the NLK equation (5.3), one is able to obtain its corresponding invariant solutions. In [55], it was shown that the NLK equation (5.3) has two point symmetries

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \quad \mathbf{X}_2 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

Using these two point symmetries, Ibragimov obtained two sets of invariant solutions given by

$$u(x,t) = \frac{1}{x(1 - a_1 e^{2t})}, \tag{5.33}$$

where a_1 is an arbitrary constant, and

$$u(x,t) = \frac{\phi(z)}{x} \text{ with } z = xe^{a_2 t}, \tag{5.34}$$

where a_2 is an arbitrary constant and $\phi(z)$ satisfies the ODE

$$z\phi'' + (2\phi + 2 - a_2)\phi' + \frac{2}{z}(\phi^2 - \phi) = 0.$$

5.4.2 Nonclassical symmetries of the NLK equation

The nonclassical method is now applied to the NLK equations (5.2) and (5.3) respectively. Here the invariant surface condition equation becomes

$$\xi(x,t,u)u_x + \tau(x,t,u)u_t = \eta(x,t,u). \tag{5.35}$$

(I) Nonclassical symmetries of the NLK equation (5.2)

(I-a) The case when $\tau \equiv 1$

The nonclassical method applied to (5.2) yields the following determining equation system for the infinitesimals $(\xi(x,t,u), \eta(x,t,u))$.

$$\begin{aligned}
&2\xi_x\eta - 8xu\eta - \frac{2\xi\eta}{x} - 4x\eta + 4u^2\xi + 4u\xi + \eta_t - x^2\eta_{xx} - 4x\eta_x + 4xu^2\eta_u \\
&- 2x^2u\eta_x - x^2\eta_x - 8xu^2\xi_x + 4xu\eta_u - 8xu\xi_x = 0, \\
&4\xi - 2x^2\eta + \frac{2\xi^2}{x} - \xi_t - 2x^2u\xi_x - 2x^2\eta_{xu} + 2\xi_u\eta - 12xu^2\xi_u - 12xu\xi_u \\
&- 4x\xi_x + x^2\xi_{xx} - x^2\xi_x - 2\xi\xi_x = 0, \\
&2x^2\xi_{xu} - 2x^2\xi_u - 4x^2u\xi_u - 8x\xi_u - x^2\eta_{uu} - 2\xi_u\xi = 0, \\
&x^2\xi_{uu} = 0.
\end{aligned} \tag{5.36}$$

The solution of the determining equation system (5.36) is given by

$$\begin{cases} \xi(x,t,u) = 0, \\ \eta(x,t,u) = 0. \end{cases} \tag{5.37}$$

Hence the corresponding "nonclassical symmetry" is

$$\mathbf{Y}_1 = \frac{\partial}{\partial t},$$

which directly results from the point symmetry $\mathbf{X}_1$.

(I-b) The case when $\tau \equiv 0$ and $\xi \equiv 1$

In this case, the determining equation for $\eta(x,t,u)$ is given by

$$\begin{aligned}
&4xu\eta_u - x^2\eta^2\eta_{uu} - 2x^2\eta\eta_{xu} - 4u - 4u^2 - 4\eta + \eta_t - x^2\eta_{xx} - 6x\eta_x - 6x\eta \\
&- 2x^2\eta^2 - x^2\eta_x + 4xu^2\eta_u - 2x^2u\eta_x - 12xu\eta - 2x\eta\eta_u = 0.
\end{aligned} \tag{5.38}$$

One is unable to find the general solution of (5.38). Hence one must consider ansätze to obtain particular solutions of (5.38). If one considers an ansatz of the form $\eta = f(x,t) + g(x,t)u + h(x,t)u^2$, one obtains

$$\eta(x,t,u) = \frac{b_1}{x^4} - u - u^2, \tag{5.39}$$

where b_1 is an arbitrary constant. The corresponding "nonclassical symmetry" is

$$\mathbf{Y}_2 = \frac{\partial}{\partial x} + (\frac{b_1}{x^4} - u - u^2)\frac{\partial}{\partial u}.$$

(II) Nonclassical symmetries of the NLK equation (5.3)

(II-a) The case when $\tau \equiv 1$

The nonclassical method applied to (5.3) yields the following determining equation system for the infinitesimals $(\xi(x,t,u), \eta(x,t,u))$.

$$\begin{aligned}
&4u^2\xi - 8xu^2\xi_x - \frac{2\xi\eta}{x} - 2x^2u\eta_x - 8xu\eta - x^2\eta_{xx} + \eta_t \\
&+ 4xu^2\eta_u + 2\xi_x\eta - 4x\eta_x = 0, \\
&x^2\xi_{xx} - 12xu^2\xi_u - 2x^2u\xi_x - 2\xi_x\xi - \xi_t - 2x^2\eta + 4\xi \\
&+ \frac{2\xi^2}{x} + 2\xi_u\eta - 2x^2\eta_{xu} - 4x\xi_x = 0, \\
&2x^2\xi_{xu} - 4x^2u\xi_u - 8x\xi_u - 2\xi\xi_u - x^2\eta_{uu} = 0, \\
&x^2\xi_{uu} = 0.
\end{aligned} \tag{5.40}$$

The solutions of the determining equation system (5.40) are given by

$$\begin{cases} \xi(x,t,u) = b_2x, \\ \eta(x,t,u) = -b_2u, \end{cases} \tag{5.41}$$

where b_2 is an arbitrary constant, and

$$\begin{cases} \xi(x,t,u) = -2x^2u, \\ \eta(x,t,u) = 4xu^2 - 2u. \end{cases} \tag{5.42}$$

The solution (5.41) yields the "nonclassical symmetry"

$$\mathbf{Y}_3 = b_2x\frac{\partial}{\partial x} + \frac{\partial}{\partial t} - b_2u\frac{\partial}{\partial u},$$

which directly results from the point symmetry $\mathbf{X}_1 + b_2\mathbf{X}_2$. The solution (5.42) yields the "nonclassical symmetry"

$$\mathbf{Y}_4 = -2x^2u\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + (4xu^2 - 2u)\frac{\partial}{\partial u},$$

which does not result from any point symmetry of (5.3).

(II-b) The case when $\tau \equiv 0$ **and** $\xi \equiv 1$

In this case, the determining equation for $\eta(x,t,u)$ is given by

$$\begin{aligned}
&-4\eta - 4u^2 - 6x\eta_x - 2x\eta\eta_u - 12xu\eta - 2x^2\eta^2 - 2x^2u\eta_x + \eta_t \\
&+ 4xu^2\eta_u - x^2\eta_{xx} - 2x^2\eta\eta_{xu} - x^2\eta^2\eta_{uu} = 0.
\end{aligned} \tag{5.43}$$

If one considers an ansatz of the form $\eta = f(x,t) + g(x,t)u + h(x,t)u^2$, the determining equation (5.43) has solutions

$$\eta(x,t,u) = \frac{b_4 e^{-2t}}{x^2(b_3 + b_4 e^{-2t} - x)} + \frac{(x - 2b_3 - 2b_4 e^{-2t})u}{x(b_3 + b_4 e^{-2t} - x)}, \tag{5.44}$$

$$\eta(x,t,u) = \frac{1}{x^2(1 + b_5 e^{2t})} - \frac{2u}{x}, \tag{5.45}$$

$$\eta(x,t,u) = \frac{b_6}{x^4} - u^2, \tag{5.46}$$

where b_3, b_4, b_5 and b_6 are arbitrary constants.

Hence, the corresponding "nonclassical symmetries" are given by

$$\mathbf{Y}_5 = \frac{\partial}{\partial x} + \left[\frac{b_4 e^{-2t}}{x^2(b_3 + b_4 e^{-2t} - x)} + \frac{(x - 2b_3 - 2b_4 e^{-2t})u}{x(b_3 + b_4 e^{-2t} - x)}\right]\frac{\partial}{\partial u},$$

$$\mathbf{Y}_6 = \frac{\partial}{\partial x} + \left[\frac{1}{x^2(1 + b_5 e^{2t})} - \frac{2u}{x}\right]\frac{\partial}{\partial u},$$

and

$$\mathbf{Y}_7 = \frac{\partial}{\partial x} + (\frac{b_6}{x^4} - u^2)\frac{\partial}{\partial u}.$$

5.4.3 New exact solutions of the NLK equation

It is obvious that the invariant solutions arising from $\mathbf{Y}_1$ and $\mathbf{Y}_3$ are those obtained by Ibragimov [55], given by solutions (5.33) and (5.34). Moreover, the invariant solution corresponding to $\mathbf{Y}_2$ is the stationary solution obtained by Dubinov [50] for the NLK equation (5.2).

Consider the "nonclassical symmetry" $\mathbf{Y}_4$ of the NLK equation (5.3). Using the direct substitution method, one seeks solutions of the PDE system

$$\begin{cases} u_t = 4x(u_x + u^2) + x^2(u_{xx} + 2uu_x), & (5.47) \\ u_t = 2x^2 uu_x + (4xu^2 - 2u). & (5.48) \end{cases}$$

After equating the right hand sides of (5.47) and (5.48), one obtains

$$4xu_x + x^2 u_{xx} + 2u = 0. \tag{5.49}$$

The solution of (5.49) is given by

$$u(x,t) = \frac{A(t) + B(t)x}{x^2}, \tag{5.50}$$

where $A(t)$ and $B(t)$ are arbitrary functions. Substituting (5.50) into (5.47), one obtains an ordinary differential equation (ODE) system for $A(t)$ and $B(t)$:

$$\begin{cases} A'(t) + 2A(t) - 2A(t)B(t) = 0, & (5.51) \\ B'(t) + 2B(t) - 2B(t)^2 = 0. & (5.52) \end{cases}$$

From (5.52), one obtains $B(t) \equiv 0$ or $B(t) = \frac{1}{1+c_1 e^{2t}}$, where c_1 is an arbitrary constant. In particular, there are three families of solutions when $B(t) \not\equiv 0$. In terms of an arbitrary constant t_0, $-\infty < t_0 < \infty$, these solutions are given by

$$B(t) = \frac{1}{2}\left[1 - \tanh(t + t_0)\right], \quad \text{where } \ 0 < B(t) < 1;$$

$$B(t) = \frac{1}{2}\left[1 - \coth(t + t_0)\right], \quad \text{where } \ \begin{cases} B(t) < 0, & \text{if } \ t > -t_0, \\ B(t) > 1, & \text{if } \ t < -t_0; \end{cases}$$

$$B(t) \equiv 1.$$

If $B(t) \not\equiv 0$, one has $A(t) = -cB(t)$, where c is an arbitrary constant. If $B(t) \equiv 0$, one has $A(t) = c_2 e^{-2t}$, where c_2 is an arbitrary constant. Therefore, there are four families of solutions of (5.3).

$$\mathfrak{F}_1: \quad u(x,t) = \frac{x-c}{2x^2}\left[1 - \tanh(t + t_0)\right]; \tag{5.53}$$

$$\mathfrak{F}_2: \quad u(x,t) = \frac{x-c}{2x^2}\left[1 - \coth(t + t_0)\right]; \tag{5.54}$$

$$\mathfrak{F}_3: \quad u(x,t) = \frac{x-c}{x^2}; \tag{5.55}$$

$$\mathfrak{F}_4: \quad u(x,t) = \frac{c_2}{x^2 e^{2t}}. \tag{5.56}$$

The first two families of solutions $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are new and cannot be obtained through classical symmetry reductions.

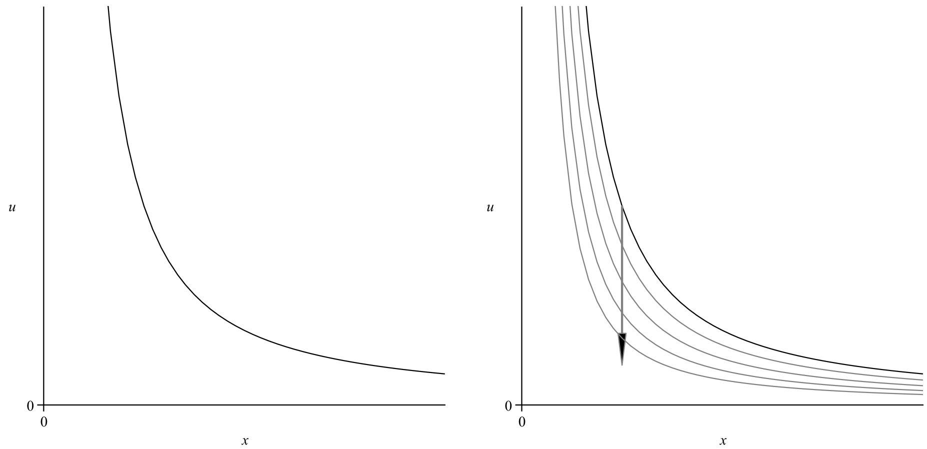
The corresponding initial conditions $u(x,0) = U(x)$ are given by the following.

(I) The family $\mathfrak{F}_1$:

(I-a) $U(x) = \frac{b(x-c)}{x^2}$ with $0 < b < 1$, $c \le 0$, on the domain $0 < x < \infty$. Such a $U(x)$ is illustrated in Figure 5.1 (a). The corresponding solutions of (5.3) are given by

$$u(x,t) = \frac{x-c}{2x^2}\left[1 - \tanh(t + t_0)\right] \tag{5.57}$$

with constants $t_0 = \frac{1}{2}\ln\left(\frac{1}{b} - 1\right)$, $0 < b < 1$ and $c \leq 0$, valid for $x > 0, t > 0$. For each value of x, the solution $u(x,t)$ is monotonically decreasing as a function of t. Moreover $\lim_{t\to\infty} u(x,t) = 0$ for any $x > 0$. The evolution of a solution $u(x,t)$ is illustrated in Figure 5.1 (b).



(a) (b)

Figure 5.1: (a) $U(x) = \frac{b(x-c)}{x^2}$, $0 < b < 1$, $c \leq 0$, $x > 0$. In (b), $u(x,t)$ is given by (5.57) for $x > 0$, $t > 0$, with the arrow pointing in the direction of increasing t.

(I-b) $U(x) = \frac{b(x-c)}{x^2}$ with $0 < b < 1$, $c > 0$, on the domain $x \geq c$. Such a $U(x)$ is illustrated in Figure 5.2 (a). The corresponding solutions of (5.3) are given by

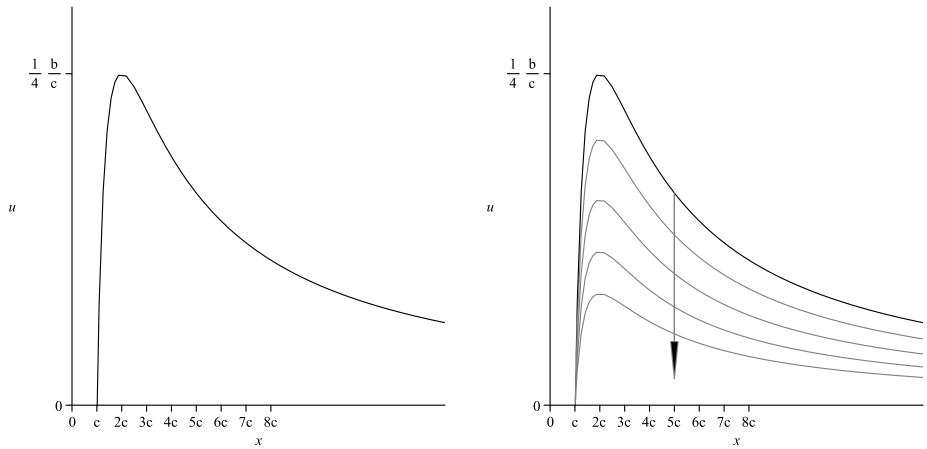
$$u(x,t) = \frac{x-c}{2x^2}\left[1 - \tanh(t + t_0)\right] \tag{5.58}$$

with constants $t_0 = \frac{1}{2}\ln\left(\frac{1}{b} - 1\right)$, $0 < b < 1$ and $c > 0$, valid for $x \geq c$, $t > 0$. For each value of x, the solution $u(x,t)$ is monotonically decreasing as a function of t. Moreover $\lim_{t\to\infty} u(x,t) = 0$ for any $x \geq c$. The evolution of a solution $u(x,t)$ is illustrated in Figure 5.2 (b).

(II) The family $\mathfrak{F}_2$:

(II-a) $U(x) = \frac{b(x-c)}{x^2}$ with $b < 0$, $c > 0$, on the domain $0 < x \leq c$. Such a $U(x)$ is illustrated in Figure 5.3 (a). The corresponding solutions of (5.3) are given by

$$u(x,t) = \frac{x-c}{2x^2}\left[1 - \coth(t + t_0)\right] \tag{5.59}$$



(a) (b)

Figure 5.2: (a) $U(x) = \frac{b(x-c)}{x^2}$, $0 < b < 1$, $c > 0$, $x \geq c$. In (b), $u(x,t)$ is given by (5.58) for $x \geq c$, $t > 0$, with the arrow pointing in the direction of increasing t.

with constants $t_0 = \frac{1}{2}\ln\left(1 - \frac{1}{b}\right)$, $b < 0$ and $c > 0$, valid for $0 < x \leq c, t > 0$. For each value of x, the solution $u(x,t)$ is monotonically decreasing as a function of t. Moreover $\lim\limits_{t\to\infty} u(x,t) = 0$ for $0 < x \leq c$. The evolution of a solution $u(x,t)$ is illustrated in Figure 5.3 (b).

(II-b) $U(x) = \frac{b(x-c)}{x^2}$ with $b > 1$, $c > 0$, on the domain $x \geq c$. Such a $U(x)$ is illustrated in Figure 5.4 (a). The corresponding solutions of (5.3) are given by

$$u(x,t) = \frac{x-c}{2x^2}\left[1 - \coth(t + t_0)\right] \tag{5.60}$$

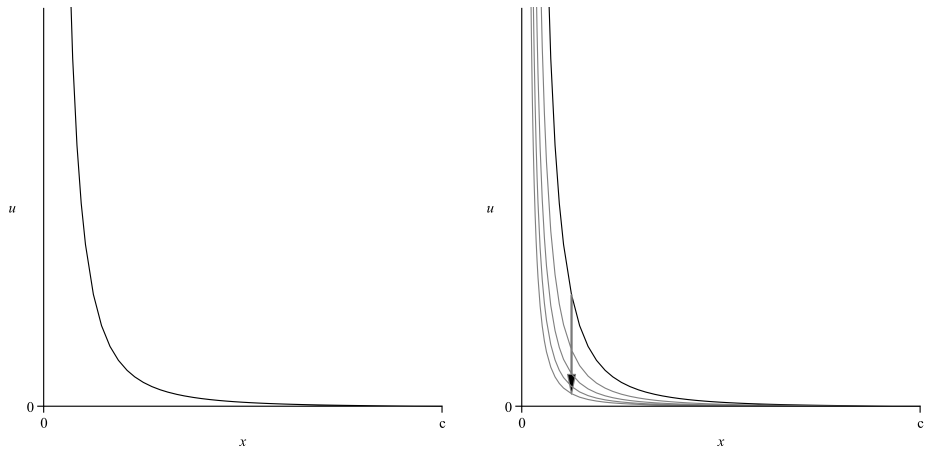
with constants $t_0 = \frac{1}{2}\ln\left(1 - \frac{1}{b}\right)$, $b > 1$ and $c > 0$, valid for $x \geq c, 0 < t < -t_0$. For each value of x, the solution $u(x,t)$ is monotonically increasing as a function of t. Moreover $\lim\limits_{t\to -t_0} u(x,t) = \infty$ for each value of $x \geq c$. The evolution of a solution $u(x,t)$ is illustrated in Figure 5.4 (b).

(II-c) $U(x) = \frac{b(x-c)}{x^2}$ with $b > 1$, $c \leq 0$, on the domain $x > 0$. Such a $U(x)$ is illustrated in Figure 5.5 (a). The corresponding solutions of (5.3) are given by

$$u(x,t) = \frac{x-c}{2x^2}\left[1 - \coth(t + t_0)\right] \tag{5.61}$$

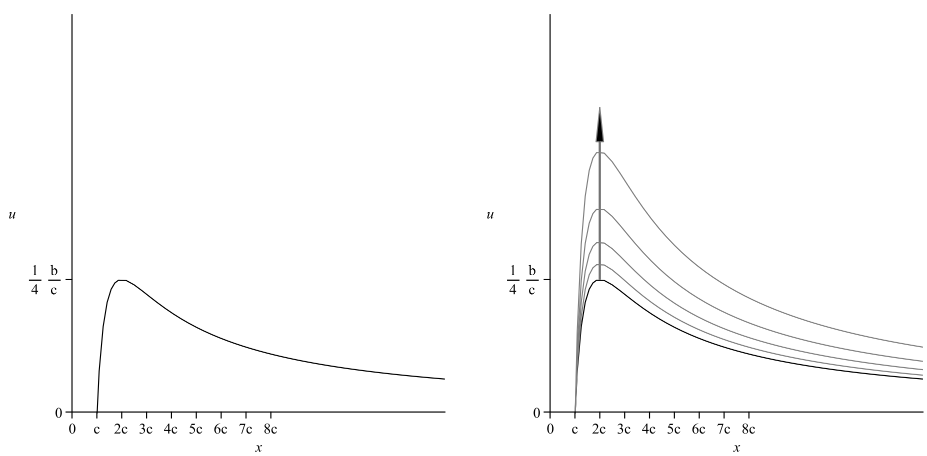
with constants $t_0 = \frac{1}{2}\ln\left(1 - \frac{1}{b}\right)$, $b > 1$ and $c \leq 0$, valid for $x > 0, 0 < t < -t_0$. For each value of x, the solution $u(x,t)$ is monotonically increasing as a function of t.

Moreover $\lim_{t\to -t_0} u(x,t) = \infty$ for each value of $x > 0$. The evolution of a solution $u(x,t)$ is illustrated in Figure 5.5 (b).



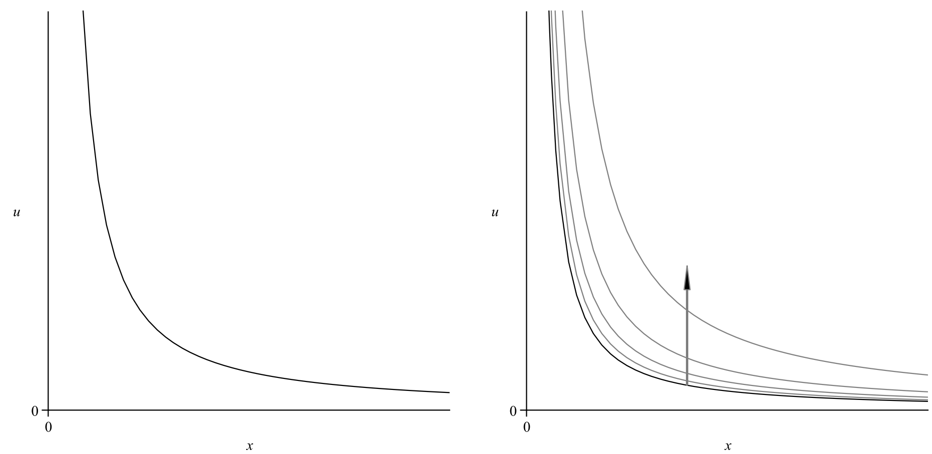
(a) (b)

Figure 5.3: (a) $U(x) = \frac{b(x-c)}{x^2}$, $b < 0$, $c > 0$, $0 < x \leq c$. In (b), $u(x,t)$ is given by (5.59) for $0 < x \leq c$, $t > 0$, with the arrow pointing in the direction of increasing t.



(a) (b)

Figure 5.4: (a) $U(x) = \frac{b(x-c)}{x^2}$, $b > 1$, $c > 0$, $x \geq c$. In (b), $u(x,t)$ is given by (5.60) for $0 < x \leq c$, $0 < t < -t_0$, with the arrow pointing in the direction of increasing t.



(a) (b)

Figure 5.5: (a) $U(x) = \frac{b(x-c)}{x^2}$, $b > 1$, $c \leq 0$, $x > 0$. In (b), $u(x,t)$ is given by (5.61) for $0 < x \leq c$, $0 < t < -t_0$, with the arrow pointing in the direction of increasing t.

5.4.4 Stationary solutions

Stationary solutions of the NLK equation (5.1) were found in [50] in terms of the doubly degenerate Heun's function (HeunD) and its derivative (HeunD'). A stationary solution $u(x,t) \equiv V(x)$ of the NLK equation (5.3) satisfies the ODE

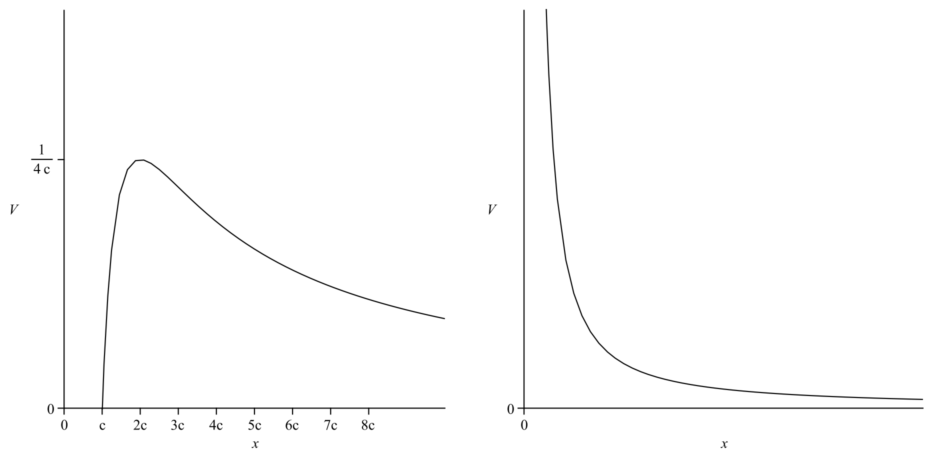
$$V'(x) + V^2 = \frac{c_3}{x^4} \tag{5.62}$$

for some constant c_3 which represent the photon flux in the frequency domain. One can show that a nontrivial stationary solution

$$V(x) = \frac{b(x-c)}{x^2} \tag{5.63}$$

satisfies equation (5.62) for some constant Q if and only if $b = 1$ and c is an arbitrary constant. Consequently, $c_3 = c^2$. Interestingly, the explicit family of solutions $V(x) = \frac{x-c}{x^2}$ is not exhibited in [50]. For $c > 0$, $V(x)$ is exhibited in Figure 5.6 (a); for $c \leq 0$, $V(x)$ is exhibited in Figure 5.6 (b).

From the solutions obtained in the last subsection, one sees that all of these nontrivial stationary solutions are unstable since a slight change in the initial condition will lead to a solution blowing up in finite time or decaying to the trivial stationary solution $u(x,t) \equiv 0$ as $t \to \infty$.



(a) (b)

Figure 5.6: The stationary solution $V(x) = \frac{x-c}{x^2}$, in (a) $c > 0$, in (b) $c \leq 0$.

Moreover, if one applies the "nonclassical symmetry" $\mathbf{Y}_7$ to the NLK equation (5.3), one obtains two more families of explicit stationary solutions, $\mathfrak{F}_5$ and $\mathfrak{F}_6$.

The family of stationary solutions $\mathfrak{F}_5$, in terms of an arbitrary positive constant a, is given by

$$\mathfrak{F}_5 : \quad V(x) = \frac{x + a\tan\left(\frac{a}{x}\right)}{x^2}, \tag{5.64}$$

valid on the domains:

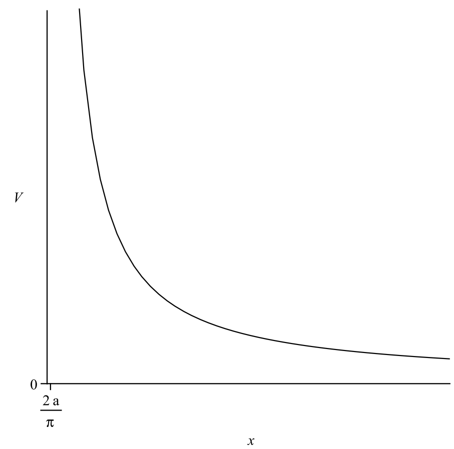
(1) $x > \frac{2a}{\pi}$, illustrated in Figure 5.7 (a);

(2) $\frac{2a}{(2k+1)\pi} < x \leq x_k$, where $x_k \in \left(\frac{2a}{(2k+1)\pi}, \frac{2a}{(2k-1)\pi}\right)$ satisfies $x_k + a\tan\left(\frac{a}{x_k}\right) = 0$, $k = 1, 2, \ldots$, illustrated in Figure 5.7 (b).

The family of stationary solutions $\mathfrak{F}_6$, in terms of an arbitrary positive constant a, is given by

$$\mathfrak{F}_6 : \quad V(x) = \frac{x - a\tanh\left(\frac{a}{x}\right)}{x^2}, \tag{5.65}$$

valid on the domain $x \geq \delta$, where δ is the unique positive solution of the equation $\delta - a\tanh\left(\frac{a}{\delta}\right) = 0$. The maximum value of $V(x)$ occurs at $x = \sigma = \frac{2a}{1+\mathrm{LambertW}(e^{-1})}$, in terms of the Lambert W function. Such a solution is illustrated in Figure 5.8.



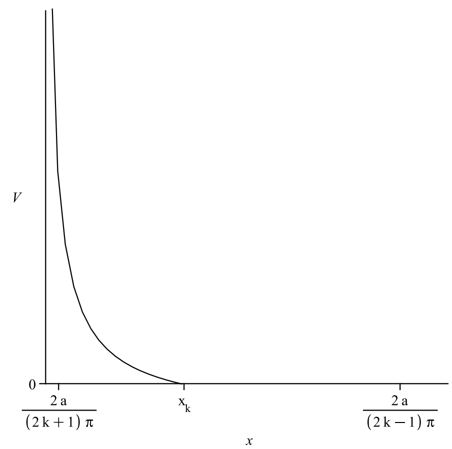


Figure 5.7: The stationary solution $V(x) = \frac{x+a\tan\left(\frac{a}{x}\right)}{x^2}$, in (a) $x > \frac{2a}{\pi}$, in (b) $\frac{2a}{(2k+1)\pi} < x \leq x_k$.

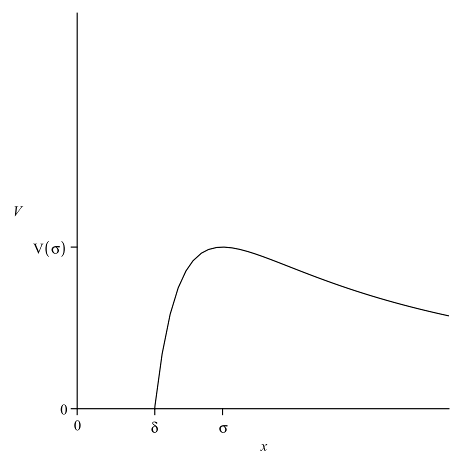


Figure 5.8: The stationary solution $\frac{x-a\tanh\left(\frac{a}{x}\right)}{x^2}$, $x \geq \delta$.

5.5 Summary

In this chapter, we presented Lie's classical method and the nonclassical method for the construction of exact solutions of a PDE system. Then we used the nonclassical method to obtain some previously unknown solutions of the NLK equation (5.3)

[36]. These solutions do not arise as invariant solutions of the NLK equation (5.3) with respect to its point symmetries. Moreover, the newly obtained exact solutions are explicit solutions of (5.3) expressed in terms of elementary functions. It was further observed that these solutions explicit both quiescent and blow up behaviour depending on their initial conditions. It was also shown that related stationary solutions are unstable.

Chapter 6

Concluding Chapter

6.1 Conclusions

In this thesis, we presented the basic ideas of symmetries, CLs and their applications. In particular, we focused on nonlocally related PDE systems and their applications, and the application of the nonclassical method. The following new results were obtained.

(1) In Theorem 3.2.7, for two potential systems $\mathbf{S}^1\{x,t;u,v\}$ and $\mathbf{S}^2\{x,t;u,w\}$, arising from two nontrivial and linearly independent local CLs of a given PDE system $\mathbf{R}\{x,t;u\}$, we showed that if $\mathbf{S}^1\{x,t;u,v\}$ and $\mathbf{S}^2\{x,t;u,w\}$ are in Cauchy-Kovalevskaya form, then the potential variable v is a nonlocal variable of $\mathbf{S}^2\{x,t;u,w\}$ and the potential variable w is a nonlocal variable of $\mathbf{S}^1\{x,t;u,v\}$.

(2) In Section 3.5, we investigated the relationship between local symmetries of a given PDE system and those of its potential systems. In particular, in Theorem 3.5.1, we proved that any local symmetry of a given PDE system having precisely n local CLs is a projection of some local symmetry of its n-plet potential system.

(3) In Chapter 4, we introduced a new systematic symmetry-based method to construct nonlocally related PDE systems (inverse potential systems) for a given PDE system. The symmetry-based method is complementary to the well known CL-based method. Most importantly, the symmetry-based method can be directly applied to any PDE system that has a point symmetry, no matter whether it has nontrivial local CLs, and no matter how many independent variables it involves. In addition, by applying the symmetry-based method, we constructed previously unknown nonlocally related PDE systems (inverse potential systems) for the nonlinear reaction-diffusion equations, the nonlinear diffusion equations and the nonlinear wave equations. Moreover, we also showed that for these example equations, one can obtain nonlocal symmetries (including previously unknown nonlocal symmetries) from some of their inverse potential systems.

(4) In Chapter 5, we applied the nonclassical method to the NLK equation. Consequently, we obtained four families of solutions, two of which are new and cannot be obtained by the classical symmetry reductions. Moreover, we analyze the behaviour of the solutions with initial condition $U(x) = \frac{b(x-c)}{x^2}$. It has been shown that a slight change of c in the initial condition will lead to significant change of the solutions of the NLK equation. In particular, for three cases of $U(x) = \frac{b(x-c)}{x^2}$, the solutions of the NLK equation exhibit quiescent behaviour, and for two cases of $U(x) = \frac{b(x-c)}{x^2}$, the solutions of the NLK equation exhibit blow up behaviour. Finally, we obtained some new stationary solutions of the NLK equation.

6.2 Future work

Besides the results presented in the thesis, there are still some open problems that arise from the work presented in this thesis.

6.2.1 To determine whether two PDE systems are nonlocally related

In Chapters 3 and 4, we presented two different systematic methods for the construction of nonlocally related PDE systems of a given PDE system: the CL-based method and the symmetry-based method. In [23], an extended procedure for the construction of a tree of nonlocally related PDE systems was introduced. However, as stated in Remark 3.2.17, it may be difficult to determine whether two resulting systems are nonlocally related. The first step to solve this problem would be to investigate the existence of an invertible transformation that relates two PDE system in a resulting tree. There are several possible methods to deal with the problem.

(1) To investigate the number of point symmetries (contact symmetries) of such PDE systems (scalar PDEs). If two PDE systems (scalar PDEs) are related by an invertible transformation, then they must have the same number of point symmetries (contact symmetries). It follows that if two PDE systems (scalar PDEs) in a resulting tree have a different number of point symmetries (contact symmetries), then they cannot be mapped to each other by an invertible transformation.

(2) To investigate the number of multipliers or local CLs of such PDE systems. If two PDE systems are related by an invertible transformation, then they must have the same number of multipliers or local CLs. It follows that if two PDE systems in a resulting tree have a different number of multipliers

or local CLs, then they cannot be mapped to each other by an invertible transformation.

(3) Cartan's method of equivalence (see [58, 76] and references therein for more details).

In general, to investigate whether two PDE systems in a tree are nonlocally related, one needs to investigate whether there exists a local function that relates the two PDE systems. Is the existence of a local function related to the existence of an invertible transformation? Can we find necessary and sufficient conditions to determine whether two PDE systems are nonlocally related?

6.2.2 The existence of subsystems

Nonlocally related PDE systems can arise from subsystems of a given PDE system. However, not all PDE systems can generate a subsystem through excluding some dependent variables. Can we find necessary and sufficient conditions so that a given PDE system generates subsystems by excluding dependent variables? This problem is also related to the extended procedure for constructing nonlocally related PDE systems. To obtain more nonlocally related PDE systems, one can employ invertible transformations acting on a given PDE system. Which invertible transformations can lead to PDE systems that have subsystems? Is there a relationship between the existence of a nonlocally related subsystem and the existence of a symmetry of the PDE system yielding the subsystem?

6.2.3 The relationship of symmetries of a given PDE system and those of its potential systems

In [19], a conjecture about the construction of potential systems and the the relationship of symmetries of a given PDE system and those of its potential systems was presented. In the case of two independent variables, the conjecture is as follows.

Conjecture 6.2.1 (1) The process of obtaining potential systems $\mathbf{S}^{(1)} = \mathbf{S}^{(1)}\{x, t; u, v^{(1)}\}$, $\mathbf{S}^{(2)} = \mathbf{S}^{(2)}\{x, t; u, v^{(1)}, v^{(2)}\}$, $\ldots$, $\mathbf{S}^{(N)} = \mathbf{S}^{(N)}\{x, t; u, v^{(1)}, v^{(2)}, \ldots, v^{(N)}\}$ of a given PDE system $\mathbf{R}\{x, t; u\}$ terminates at some finite N where either

- $\mathbf{S}^{(N)}$ can be linearized by some invertible point transformation; or
- $\mathbf{S}^{(N)}$ has no further conservation law.

(2) The group of all point symmetries of $\mathbf{S}^{(N)}$ yields, through projections, all point symmetries of any subsystem of $\mathbf{S}^{(N)}$ including $\mathbf{R}\{x, t; u\}$, $\mathbf{S}^{(1)}$, $\ldots$, $\mathbf{S}^{(N-1)}$.

For the first conjecture, a key step would be to develop a counterpart of Theorem 2.3.13, i.e., to determine whether one can find all local CLs of a PDE system. For the second conjecture, we presented a related result in Section 3.5. In particular, we proved that if a given PDE system $\mathbf{R}\{x, t; u\}$ has precisely n local CLs, then the local symmetries of its n-plet potential system yield all local symmetries of $\mathbf{R}\{x, t; u\}$. A related question is whether one can find a relationship between local symmetries of each k-plet ($1 \leq k \leq n$) potential system of $\mathbf{R}\{x, t; u\}$ and those of the n-plet potential system. More specifically, can each local symmetry of any k-plet potential system be obtained by projection of some local symmetry of the n-plet potential system? If the answer to this question is no, does there exist a potential system whose local symmetries includes all local symmetries of each k-plet potential system?

6.2.4 The application of the obtained nonlocal symmetries

In Chapters 3 and 4, we obtained some nonlocal symmetries of each given PDE. Can these nonlocal symmetries yield new exact solutions of each corresponding given PDE? Can we use such nonlocal symmetries to obtain more useful inverse potential systems? It is meaningful to continue to investigate these problems in the future.

6.2.5 Nonlocal symmetries for PDE systems with three or more independent variables

The symmetry-based method for constructing nonlocally related PDE systems can be directly applied to PDE systems with three or more independent variables. For the CL-based method, in order to obtain nonlocal symmetries of a given PDE system with three or more independent variables from potential systems arising from divergence-type CLs, it is necessary to add gauge constraints to such underdetermined potential systems. However, there is no known systematic procedure to determine which gauge constraints yield nonlocal symmetries. Since the inverse potential systems constructed by the symmetry-based method are determined, it is expected that for some PDE systems with three or more independent variables, one should be able to directly obtain nonlocal symmetries from local symmetries of corresponding inverse potential systems. For physical examples of interest, it would be worthwhile to investigate whether the symmetry-based method can generate inverse potential systems whose local symmetries can directly yield nonlocal symmetries for a given PDE system with three or more independent variables.

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