Decomposition of free fields and structural stability of dynamical systems for renormalization group analysis

by

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Master of Science in Physics, Eidgenössische Technische Hochschule Zürich, 2009

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate Studies

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

August 2013

Abstract

The main results of this thesis concern the spatial decomposition of Gaussian fields and the structural stability of a class of dynamical systems near a non-hyperbolic fixed point. These are two problems that arise in a renormalization group method for random fields and self-avoiding walks developed by Brydges and Slade. This renormalization group program is outlined in the introduction of this thesis with emphasis on the relevance of the problems studied subsequently.

The first original result is a new and simple method to decompose the Green functions corresponding to a large class of interesting symmetric Dirichlet forms into integrals over symmetric positive semi-definite and finite range (properly supported) forms that are smoother than the original Green function. This result gives rise to multiscale decompositions of the associated free fields into sums of independent smoother Gaussian fields with spatially localized correlations. Such decompositions are the point of departure for renormalization group analysis. The novelty of the result is the use of the finite propagation speed of the wave equation and a related property of Chebyshev polynomials. The result improves several existing results and also gives simpler proofs.

The second result concerns structural stability, with respect to contractive third-order perturbations, of a certain class of dynamical systems near a non-hyperbolic fixed point. We reformulate the stability problem in terms of the well-posedness of an infinite-dimensional nonlinear ordinary differential equation in a Banach space of carefully weighted sequences. Using this, we prove the existence and regularity of flows of the dynamical system which obey mixed initial and final boundary conditions. This result can be applied to the renormalization group map of Brydges and Slade, and is an ingredient in the analysis of the long-distance behavior of four dimensional weakly self-avoiding walks using this approach.

Preface

Chapter 1 is an introduction and motivation for the problems studied in the remainder of the thesis. No originality is claimed and, to give an informative exposition, we explain a number of ideas from a number of references mentioned, but without explicit reference to the origin of each single idea.

Chapter 2, in slightly modified form, has been accepted for publication in the journal *Probability Theory and Related Fields*; see reference [8].

Chapter 3 is based on joint work with David Brydges and Gordon Slade; a version of it has been accepted for publication in the journal *Annales Henri Poincaré*; see reference [11].

Chapter 4 discusses ideas developed together with David Brydges and Gordon Slade.

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Appendices

List of Symbols

Chapter 1

(X, E)	graph with vertices X and edges E
w_t	position of a walk on a graph at time t
L_x	local time of a walk, see (1.7)
ϕ_x	field on a graph, for example, $\phi : X \to \mathbb{R}$
$H_t(L)$	energy (or Hamilton function) of a walk of length t as a function of its local time, see (1.9)

Chapter 2

φ	a smooth function on $\mathbb{R}$ with rapid decay, $\varphi \geq 0$, and $\mathrm{supp}(\hat{\varphi}) \subset [-1, 1]$ where $\hat{\varphi}$ is the Fourier transform, see Lemma 2.2.5
$\mathcal{E}$	Dirichlet form
L	generator of Dirichlet form, see (2.20)
Φ	Green form corresponding to a Dirichlet form, see (2.24)
a	coefficients of a generator, see Examples 2.1.3–2.1.4
$\hat{f}(\xi)$	Fourier transform of a function f
$(P_{\gamma,\theta})$	finite propagation speed condition of wave equation associated to generator of Dirichlet form L
$(P^*_{\theta,B})$	discrete finite propagation speed condition associated to L

Chapter 3

j	discrete time parameter
$X_j = \mathbf{K}_j \oplus \mathbb{R}$	state space of dynamical system at time j
$x_j = (K_j, V_j)$	position of dynamical system at time j
Φ_j	evolution map $\Phi_j : X_j \to X_{j+1}$ of dynamical system at time j
$\bar{\varphi}_j$	quadratic part of evolution map, see (3.1)
$\bar{g}_j$	solution to the recursion relation $\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2$
X^w	space of sequences with weight w, see Definition 3.3.1
w, r	specific choices of weights for sequences, see (3.95)

Acknowledgments

I thank my research advisers, David Brydges and Gordon Slade, without whom this thesis would not have been possible. I feel fortunate that I was introduced to their area of research, and their support throughout the course of this thesis, in academic and non-academic matters, was indispensable for its success.

I also thank Joel Feldman for serving on my thesis committee, Tyler Helmuth for proofreading the introduction, and the members of the Probability Group at the University of British Columbia for having provided an inspiring research environment. Part of the research that led to this thesis was carried out during stays at the Institute Henri Poincaré in Paris and the Department of Mathematics and Statistics at McGill University in Montreal, and I thank these institutions for their hospitality during my stays.

I also gratefully acknowledge financial support from several sources, including the University of British Columbia, the Government of British Columbia, and the Fondation Mathématique Science de Paris.

Finally, I thank my family and friends for their emotional and financial support during the course of my studies, without which I could not have carried these out.

Chapter 1

Introduction

1.1 Outline and preliminaries

1.1.1 Outline

The main results of this thesis concern the spatial decomposition of Gaussian fields and the structural stability of a class of dynamical systems near a non-hyperbolic fixed point, and are given in Chapters 2 and Chapter 3, respectively. The primary motivation for the study of the these problems is an application to a *renormalization group method* for the analysis of four-dimensional weakly self-avoiding walks developed by Brydges and Slade. However, the results of Chapter 2–3 are not specific to the application to self-avoiding walks, and we expect that they may also be useful for renormalization group analysis of other models.

The aim of the present chapter is to sketch the background of the problems studied in Chapter 2–3, in particular their advent in the renormalization group context. In Section 1.2, some aspects of random polymer models are introduced; these models of phenomena from polymer chemistry are our primary motivation. Their relation to the problems studied in this thesis is indirect, however, via random fields which are introduced in Section 1.3. Random fields are related to a broad range of models of statistical mechanics, for example the description of interfaces describing phase separation and models for ferromagnetism. In the description of random polymers, they appear as the local time of a perturbed Markov process. The main results of this thesis are discussed in Section 1.4.

In statistical mechanics, the behavior at *large distances* of a model is of main interest. For random polymer and random field models, the large distance behavior is notoriously difficult to study, however, because both classes of models involve a large number of strongly coupled degrees of freedom. The renormalization group, which is discussed in Section 1.4, is a program to study the large-distance behavior of random fields, pioneered in this sense by the theoretical physicist Wilson. The mathematical realization of Wilson's ideas has been a major challenge since their seminal proposal. We discuss some of the difficulties involved in it, and then sketch important aspects of one of several approaches to resolve these difficulties, initiated by Brydges and Yau, and much generalized and improved in recent work of

Brydges and Slade, based on work of many others. The emphasis of this discussion is on how, specifically, the problems studied in the main part of this thesis pertain to this program, but we also aim to give an introduction to the general ideas.

1.1.2 Preliminaries

General notation. We use the usual *Landau notation*:

$$f(t) = o(g(t)) \quad \text{as } t \to T \quad \text{if } \lim_{t\to T} f(t)/g(t) \to 0; \tag{1.1}$$

$$f(t) = O(g(t)) \quad \text{as } t \to T \quad \text{if } \limsup_{t\to T} |f(t)/g(t)| < \infty; \tag{1.2}$$

and also the usual *asymptotic notation*:

$$f \sim g \quad \text{as } t \to T \quad \text{if } f(t) = g(t)(1 + o(1)) \text{ as } t \to T \tag{1.3}$$

where T is often 0 or ∞.

Limits are abbreviated by $f(t\pm) = \lim_{s\to t\pm} f(s)$. The indicator function 1_z is given by $1_z = 1$ if condition z is satisfied and $1_z = 0$ otherwise. The symbols C and c will mostly denote constants whose values are allowed to change between two occurances. The dependence of a constant on a parameter is sometimes emphasized by a subscript. The letter d is reserved for the dimension of the relevant physical space, i.e., of $\mathbb{Z}^d$ or $\mathbb{R}^d$, and for metrics (which of the two should be clear from the context). The expectation value of a random variable, ϕ, is denoted by $\mathbf{E}(\phi)$.

Graphs. It will be convenient at various places to use the language of graphs, but we do not use any non-trivial results from graph theory. We say $\Gamma = (X, E)$ is a (simple) *graph* if X is a finite or countable set of *vertices* and $E \subset P_2(X)$ is a set of (undirected) *edges*, where $P_2(X)$ denotes the set of subsets of X with exactly two elements. The words simple and undirected will be implicit from now on. Vertices will typically be denoted by the letters x and y and edges by the letter e. The edge connecting two vertices x and y is written as $xy = yx = \{x, y\}$. The *graph distance* $d(x, y)$ between two vertices $x, y \in X$ is the number of edges of the shortest path between two vertices x and y, if there is one, and ∞ otherwise. That x and y are neighbors, $xy \in E$, is denoted by $x \sim y$. All graphs will be assumed to be locally finite, i.e., for any $x \in X$ there are only finitely many $y \in X$ with $x \sim y$.

The graphs of primary relevance for this thesis are *lattice graphs* which can be embedded in $\mathbb{R}^d$ or in the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The most important examples are the Euclidean (or hypercubic) lattice $\mathbb{Z}^d$ with nearest-neighbor edges, i.e., $xy \in E(\mathbb{Z}^d)$ if $|x_i - y_i| = 1$ for exactly one $i \in \{1, \dots, d\}$ and otherwise $|x_i - y_i| = 0$, and the discrete torus of side length n, denoted $\mathbb{Z}_n^d$, for which $xy \in E(\mathbb{Z}_n^d)$ if $|x_i - y_i| = 1 \mod n$ for exactly one $i \in \{1, \dots, d\}$ and otherwise $|x_i - y_i| = 0 \mod n$.

1.2 Random polymers

1.2.1 The simple random walk

Let $\Gamma = (X, E)$ be a graph. A *walk* on Γ of length $t \in (0, \infty]$ is a right-continuous path $w : [0, t) \to X$ with finitely many jumps in finite intervals, i.e., $d(w_s, w_{s-}) \leq 1$ for all $s \in (0, t)$ with equality for only finitely many s in each bounded subinterval of $(0, t)$. Let W_t denote the set of all walks of length t. An important subclass of walks are *discrete walks*, denoted $W_t^* \subset W_t$, for which a jump happens at s if and only if s is an integer and $0 < s < t$. For $t < \infty$, each walk $w \in W_t$ can be specified uniquely by an integer $n \geq 0$, a finite sequence $t_0 = 0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = t$, and an element $w^* \in W_n^*$ as

$$w_s = w_n^* \quad \text{if } s \in [t_n, t_{n+1}). \tag{1.4}$$

The walk w^* is called the *skeleton walk* of w and $(t_1, \ldots, t_n)$ are the *jump times*.

Let $W_{x,t} = \{w \in W_t : w_0 = x\}$ and $W_{x,t}^* = W_t^* \cap W_{x,t}$ be the sets of (continuous and discrete) walks starting at x. There are several natural probability measures on $W_{x,t}$ that arise as restrictions of measures on $W_{x,\infty}$ and $W_{x,\infty}^*$ (stochastic processes). The *simple random walk* is the discrete Markov process which chooses uniformly from its neighbors at each step. The constant-speed simple random walk is a continuous Markov process with skeleton walk given by the simple random walk and the times between two jumps distributed independently with exponential distribution with parameter 1. The variable-speed simple random walk likewise has the simple random walk as skeleton walk, but the waiting times between two jumps now have exponential distribution with parameter given by the degree of the vertex before the jump. This can also be interpreted as that each edge has a supply of exponential clocks with parameter 1 and that the next jump is along the edge whose clock rings first. The two continuous processes only differ by rescaling of the time when the graph is regular, i.e., when all vertices have the same degree as is in particular the case for the graph of main interest, $\Gamma = \mathbb{Z}^d$.

To illustrate what is understood, consider (any of) the simple random walks on $\mathbb{Z}^d$. Then, for any $t \geq 0$,

$$\lambda^{-1/2} w_{\lambda t} \to N_t \quad (\lambda \to \infty) \tag{1.5}$$

where the convergence is in distribution and N_t is a vector of independent Gaussian random variables with mean 0 and variance t (or $2dt$ for the variable-speed walk). This result is essentially the classical central limit theorem. It shows that w_t grows typically like $\sqrt{t}$ as $t \to \infty$. A less precise way of measuring this is the statement that $\mathbf{E}|w_t|^2 \sim t$ as $t \to \infty$. But much more is understood. It is also a well-known

result that if $(B_t)_t$ is the *Wiener process*, a continuous random path $[0,\infty) \to \mathbb{R}^d$ (thus not in W_∞) with Gaussian distribution defined by $B_0 = 0$, $\mathbf{E}(B_t) = 0$, and $\mathbf{E}(B_t^i B_s^j) = \delta_{ij} \min\{s,t\}$ for $i,j \in \{1,\dots,d\}$, that then the convergence of (1.5) holds on a space of paths $[0,\infty) \to \mathbb{R}^d$, i.e., for all t simultaneously in a sense. Proofs of the latter result, Donsker's invariance principle, can be found in many textbooks on advanced probability theory; see e.g. reference [105]. It describes the behavior at *large distances* of the paths of $(w_t)_t$.

1.2.2 Polymers and local time

In polymer science, a *linear polymer* is a long chain of molecules (monomers). The simplest mathematical model for a linear polymer is the *uniform ensemble*, the uniform probability measures on $W^*_{x,t}$, for some graph (in particular for $\mathbb{Z}^d$), but it is not a well motivated approximation. For example, polymers should not be able to intersect themselves due to the finite extent of each molecule. A model that takes this into consideration is the *strictly self-avoiding walk*, the uniform measure on (discrete) walks conditioned on the event that walks do not intersect themselves.

For regular graphs, the uniform ensemble and the simple random walk are the same. This has turned out to be an important observation for the study of random polymers. Moreover, in some aspects, the continuous-time random walks have favorable analytic properties over the discrete-time random walk. For regular graphs, the constant- and variable-speed walks are identical, up to rescaling of time by the constant vertex degree, their jump sequences are Poisson processes, and the skeleton walks are simple random walks, thus uniform when conditioned on the number of jumps. In view of the last aspect, the continuous-time random walks are natural variants of the uniform ensemble.

In reference [43], den Hollander gives a broad overview of mathematical models for *random polymers*. Like the strictly self-avoiding walk, these polymer models for example suppress self-intersections by giving smaller weight to intersecting paths with respect to a reference measure. Natural choices for the reference measure $P^0_{x,t}$ are any of the simple random walk models on the regular graphs $\mathbb{Z}^d$ where $d = 1, 2, \dots$. These models are then defined by an energy or *Hamilton* function, $H_t : W_t \to \mathbb{R}$, assigning an energy cost to every path, as a probability measure $P^H_{x,t}$ on $W_{x,t}$ by

$$P^H_{x,t}(dw) = \frac{1}{Z}\, e^{-H_t(w)}\, P^0_{x,t}(dw) \tag{1.6}$$

where $Z = Z^H_{x,t}$ is a normalizing constant, called the *partition function*. The measure $P^H_{x,t}$ can be viewed as a kind of *Gibbs measure* on walks. For a number of interesting models, the energy function is a functional of the *local time*. The local

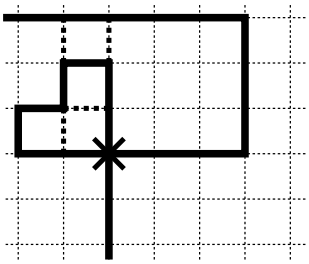

Figure 1.1: Polymer with one self-intersection and several self-contacts.

time of a walk w is given by:

$$L_x^t(w) = \int_0^t 1_{w_s=x}\, ds. \tag{1.7}$$

where we recall the indicator function: $1_{a=b} = 1$ if $a = b$ and $1_{a=b} = 0$ otherwise. To say that H is a functional of the local time means that there is $H : M_+(X) \to \mathbb{R}$, where $M_+(X) = \{m : X \to \mathbb{R}_+ : \sum_{x\in X} m_x < \infty\}$, such that[1]

$$H_t(w) = H(L^t(w)) \quad (w \in W_t). \tag{1.8}$$

For example, an interesting class of Hamilton functions is given by

$$H^{\beta,\gamma}(L) = \beta \sum_{x\in X} L_x^2 - \gamma \sum_{x\in X} \sum_{y\in X:y\sim x} L_x L_y \quad (\beta, \gamma \geq 0). \tag{1.9}$$

This model is known under a number of names. If $\gamma = 0$, it is called the *weakly self-avoiding walk*, *soft polymer*, *discrete Edwards model*, and *Domb-Joyce model* [15, 43, 88], and *with self-attraction* is added to the name if $\gamma > 0$ [43]. The repulsive force ($\beta > 0$) models the effect that polymers should not intersect themselves by suppressing self-intersections of walks, as can be seen from the elementary identity

$$\sum_{x\in X} L_x^t(w)^2 = \int_0^t \int_0^t 1_{w_{s_1}=w_{s_2}}\, ds_1\, ds_2. \tag{1.10}$$

The (optional) attractive force ($\gamma > 0$) models the effect of a solution in which the polymer is immersed, by making it energetically beneficial for a polymer to be in contact with itself (rather than the solution). This can be understood from

$$\sum_{x\in X} \sum_{y\in X:y\sim x} L_x^t(w) L_y^t(w) = \int_0^t \int_0^t 1_{w_{s_1}\sim w_{s_2}}\, ds_1\, ds_2. \tag{1.11}$$

[1]Observe that $\sum_{x\in X} L_x^t(w) = t < \infty$ for $w \in W_t$ and thus $L^t(W_t) \subset M_+(X)$.

Note that the *strictly self-avoiding walk* is obtained in the limit $\gamma = 0$, $\beta \to \infty$ of the discrete-parameter version of (1.6); see e.g. references [82,88]. It can also be related to the continuous-parameter model, but then the relation is more subtle [26].

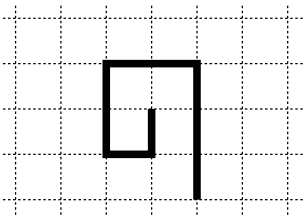

Figure 1.2: A trapped self-avoiding walk.

Unlike the simple random walks, random polymer models like (1.6) are almost never stochastic processes. For example, it is easy to see that strictly self-avoiding walks can get trapped as shown in Figure 1.2. The parameter t of the measures $P^H_{x,t}$ can therefore *not* be interpreted as *time*, but it is rather a measure of the *lengths* of the polymers described by the measures. In analogy to the classical theory of gases in statistical mechanics, the measures $P^H_{x,t}$ describe *ensembles* of walks (which take the role of particle configurations of a gas) with fixed length (taking the role of a fixed number of particles in the gas).

As a consequence, the standard tools for the analysis of stochastic process are not available to study the measures (1.6), making their analysis decidedly more difficult than that of simple random walks. It turns out that random polymer models depend sensitively on the presence of an interaction given as in (1.6). For example, it is believed (but only proved in dimension $d = 1$ so far; but see Section 4.2) that even arbitrarily small values of $\beta > 0$ can change the asymptotic behavior of the walks drastically compared to the case $\beta = 0$. On the other hand, the behavior for all $\beta > 0$ is believed to be similar.

1.2.3 Asymptotic behavior and universality

From now on, the discussion will be restricted to polymer models on the Euclidean lattice $\mathbb{Z}^d$; we also consider only spatially homogeneous interactions, i.e., interactions that are invariant under translations like (1.9). To simplify the notation, we then set the starting point to 0 and drop it from the notation, for example in (1.6).

The perhaps most interesting mathematical problem about random polymers is to determine the typical growth of the distance between the starting and endpoint with its length, t. For the simple random walk, this, and almost any other question, are very well understood, for example by (1.5). However, for *self-interacting* ran-

dom polymers ($H \neq 0$), it is in general a difficult (open) problem to determine the growth of the *end-to-end distance* $\mathbf{E}_t|w_t|^2$. It is a general conjecture that the end-to-end distance is asymptotically described by a power law, i.e., that for $\beta, \gamma \geq 0$, there are constants $c > 0$ and $\nu \geq 0$ such that

$$\mathbf{E}_t^H |w_t|^2 \sim ct^{2\nu} \quad \text{as } t \to \infty \tag{1.12}$$

where $\mathbf{E}_t^H(F)$ is the expectation value of a random variable $F = F(w)$ under P_t^H. For the simple random walk, the exponent is $\nu = \frac{1}{2}$, in any dimension. It is believed that, for general polymers, the constant $c > 0$ depends on all of d, β, and γ, but that the exponent ν is *universal*, i.e., constant for appropriate ranges of β and γ and also independent of the lattice of a given dimension d. It does in general depend on d. In Figure 1.3, the conjectured phase diagram for the weakly self-avoiding walk with self-attraction is shown; it was conjectured by v.d. Hofstad and Klenke [110].

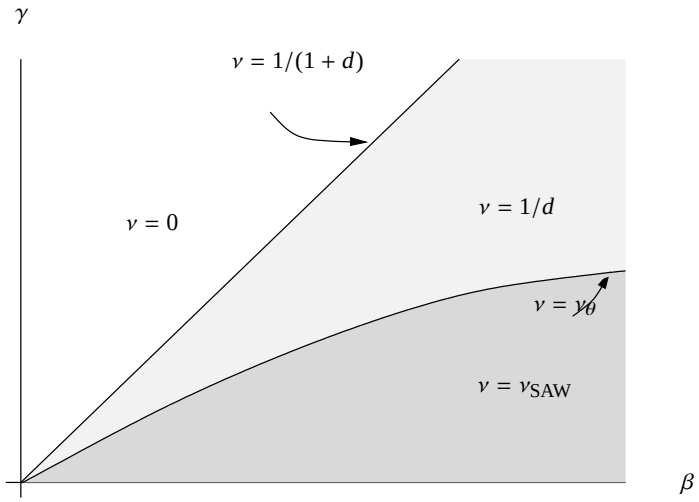

Figure 1.3: The phase diagram conjectured (for discrete-time) in $d \geq 2$, from [110]

The kind of universality described in the last paragraph is one of the paradigms in equilibrium statistical mechanics, yet in general only understood in few specific examples mathematically. For the self-avoiding walk, without self-attraction, seminal results by Brydges and Spencer [20] and by Hara and Slade [73–75] provide an essentially complete picture in dimensions five and higher. In particular, these results include the result $\mathbf{E}_t^H |w_t|^2 \sim ct$ which is the same behavior as for the simple random walk, except for the constant. In dimension two, there is strong evidence

that the long-distance behavior of (strictly) self-avoiding walks is described by the so-called *Schramm-Loewner-Evolution* [83]. This is a subject of intense research, but proofs are not known at the time this thesis is written. The *weakly* self-avoiding walk, even without self-attraction, seems even more difficult to understand in two dimensions, but it is believed to be in the same *universality class* as the strictly self-avoiding walk in any dimension. The term *universality class* refers to the class of models that share the same scaling limit (or at least the same *critical exponents*). The validity of the former conjecture is known only for dimension one [111] and, as discussed, in dimensions five and above without self-attraction. For the physically most interesting dimension three, only numerical estimates of the values for the critical exponents are known [41]. Dimension four is expected to be *critical*, in the sense that the behavior of self-avoiding walks changes from behavior similar to that of the simple random walk to complex behavior as d gets smaller through 4.

The critical dimension. Many models of discrete equilibrium statistical mechanics can be defined, by the "same" specification like (1.9), on an (essentially) arbitrary graph. It is a paradigm of statistical mechanics that when such models are defined on $\mathbb{Z}^d$, there is a critical dimension, d_c, such that for $d < d_c$, the behavior is *complex*, meaning for self-avoiding walks, for example, that it is different from that of the simple random walk, while for $d > d_c$, the model has the so-called *mean-field* behavior, meaning for self-avoiding walks that the behavior is the same as that of the simple random walk. The term *mean-field* stems from analogy with models of ferromagnetism, but it is standard terminology for more general models. For self-avoiding walk models, there is overwhelming evidence that the critical dimension is $d_c = 4$. In the critical dimension, the behavior is expected to be that of the mean-field model with universal *logarithmic corrections*. For example, for self-avoiding walks (with additional small self-attraction allowed), it is conjectured that, in $d = 4$, in the phase where $\beta > 0$, $\gamma \ll \beta$,

$$\mathbf{E}_t^H |w_t|^2 \sim ct(\log t)^{\frac{1}{4}} \quad (t \to \infty); \tag{1.13}$$

see e.g. references [17, 28, 29, 50, 88]. The exponent $\frac{1}{4}$ is expected not to depend on the "details" of the model. Brydges and Slade have developed methods by which a proof of (1.13) seems within reach (but not reached, see also Section 4.2). The work of this thesis is a contribution to this program which we will therefore discuss in some detail.

1.2.4 The two-point function as a Laplace transform

Let $\mathbf{E}_a(F)$ be the expectation value of a random variable $F = F(w)$ with respect to the simple random walk probability distribution P_a^0 on walks starting at a, let

$$c_t^H(a,b) = \mathbf{E}_a(e^{-H(L^t)}1_{w_t=b}) \quad (a,b \in X) \tag{1.14}$$

be the *probability weight function* of the endpoint b for the ensemble of walks of length t that start at a, and set $c_t^H(x) = c_t^H(0,x)$ on $\mathbb{Z}^d$. The main goal in the study of random polymers is to understand this function "very well," in the limit $t \to \infty$. For example, this would enable one to understand

$$\mathbf{E}_t^H|w_t|^2 = \frac{\sum_x c_t^H(x)|x|^2}{\sum_x c_t^H(x)}. \tag{1.15}$$

An approach to understanding $c_t^H(x)$ is via its Laplace transform in t,

$$G_\mu^H(x) = \int_0^\infty \mathbf{E}(e^{-H(L^t)}1_{w_t=x})e^{-\mu t}\,dt \quad (\mu \in \mathbb{R}), \tag{1.16}$$

which is called the *two-point function* for the random polymer described by Hamiltonian H. To recover information about $c_t(x)$, as $t \to \infty$, from $G_\mu(x)$, it is particularly important to understand $G_\mu(x)$ as the minimal value, $\mu = \mu_c$, above which the Laplace transform converges is approached.

To illustrate this, it is instructive to consider the simple random walk with, say, variable-speed. In this case, the two-point function is the *Green function* of $-\Delta + \mu$ where Δ is the graph Laplace operator given by

$$\Delta f(x) = \sum_{y:y\sim x}(f(y) - f(x)). \tag{1.17}$$

By use of the Fourier transform, it is straightforward to establish the exact relations

$$\sum_x G_\mu(x) = \frac{1}{\mu}, \quad \sum_x |x|^2 G_\mu(x) = \frac{2d}{\mu^2} \qquad (\mu > 0). \tag{1.18}$$

In particular, $\mu_c = 0$ and the Laplace transforms of the numerator and the denominator in (1.15) can be inverted explicitly to obtain

$$\mathbf{E}_t|w_t|^2 = 2d \cdot t \tag{1.19}$$

as explained in [28, p. 526]. Even though it may not be the most efficient way to compute $E_t|w_t|^2$ for the simple random walk by analysis of the two-point function,

as the result is elementary there, this approach has proven fruitful for the analysis of interacting models as we will explain (see also [28, 29]).

The two-point function is however also of independent interest. For the simple random walk, it is possible to determine the asymptotic behavior of the two-point function for fixed value of μ. For reference, we record from [64] that if[2] $d > 2$,

$$G_\mu(x) \sim \begin{cases} \dfrac{c}{|x|^{d-2}} & (\mu = 0) \\ \dfrac{c_\mu}{|x|^{(d-1)/2}} e^{-M(\mu) b(x/|x|)\cdot x} & (\mu > 0) \end{cases} \qquad (|x| \to \infty). \tag{1.20}$$

where $b : S^{d-1} \to \mathbb{R}^d$ and the rate of exponential decay satisfies $M = M(\mu) \sim \sqrt{\mu}$ as $\mu \downarrow 0$. It is related to the divergence of (1.18); see e.g. [88, Appendix A]. The parameter $\mu \geq 0$ is also called the *killing rate* of the simple random walk because it has an interpretation in terms of random walks that die (stop) after a finite random time, if $\mu > 0$. In the context of the next section, μ is also called the square of the *mass* and we write $\mu = m^2$ also in the context of the simple random walks.

It turns out that questions about random polymers are related to questions about random fields.

1.3 Random fields and local time

1.3.1 Generalities

Let X be a countable set. It should be thought of as a spatial configuration of points; in the main examples, it is the vertex set of a graph, $\Gamma = (X, E)$. Let us call any map $\phi : X \to \mathbb{R}$ a real-valued *field* on X. It is also of interest to consider vector-valued fields or more generally maps $\phi : X \to M$ that take values in a manifold M, but most of the discussion will be restricted to the simplest case of real-valued fields, $M = \mathbb{R}$. The space of fields is $M^X = \{\phi : X \to M\}$.

Random fields, or probability measures on M^X, are one of the main structures of interest in equilibrium statistical mechanics, in particular with X an infinite set or in the limit when X tends to an infinite set. Examples of random fields in statistical mechanics include spin models, i.e., models of ferromagnetism in which a random field describes spins of particles located at the vertices of a graph, the description of dislocations of particles from a crystal, the modelling of phase interfaces, height functions of some configuration models (e.g. dimers), the local time of Markov (or more general random) processes, and more.

[2] The formula for $\mu > 0$ also holds for $d = 2$, but for $\mu = 0$, the homogeneous function $1/|x|^{d-2}$ is replaced by $-\log|x|$. For simplicity, we restrict to $d > 2$.

In general, it is non-trivial to define random fields on an infinite set X so that their definition often proceeds through an approximation by finite sets.

1.3.2 Gaussian fields

A class of random fields of fundamental importance are *Gaussian fields*. These are special in many ways: they can be defined essentially directly on infinite sets (and also in the continuum), many properties are accessible by elementary calculations, and they play an important role in the study of a number of non-Gaussian fields.

Let X be a *finite* set and $C = (C_{xy})_{x,y\in X}$ be a symmetric positive semi-definite matrix with real entries indexed by X, i.e., $C_{xy} = C_{yx}$ for all $x, y \in X$ and

$$\sum_{x,y\in X} f_x C_{xy} f_y \geq 0 \quad \text{for all } f \in \mathbb{R}^X. \tag{1.21}$$

The Gaussian measure P_C on $\mathbb{R}^X$ with *mean* 0 and *covariance* C is uniquely defined by the Fourier transform:

$$\int e^{i\phi f}\, P_C(d\phi) = e^{-\frac{1}{2}fCf} \quad \text{for all } f \in \mathbb{R}^X \tag{1.22}$$

where

$$\phi f = \sum_{x\in X} f_x \phi_x, \quad fCf = \sum_{x,y\in X} f_x C_{xy} f_y. \tag{1.23}$$

In particular, when C is a strictly positive definite matrix, i.e., if equality in (1.21) holds only if $f_x = 0$ for all $x \in X$, then the inverse matrix $L = C^{-1}$ exists and the Gaussian measure P_C is equivalently given by the density

$$P_C(d\phi) = \frac{e^{-\frac{1}{2}\phi L\phi}}{\sqrt{\det(2\pi C)}}\, \lambda_X(d\phi) \tag{1.24}$$

where λ_X denotes the $|X|$-dimensional Lebesgue measure on $\mathbb{R}^X$. We then say that P_C is a non-degenerate Gaussian measure. The matrix C is the covariance matrix or two-point function of P_C in the sense that

$$\mathbf{E}_C(\phi_x\phi_y) := \int \phi_x\phi_y\, P_C(d\phi) = C_{xy} \tag{1.25}$$

where we have introduced the notation $\mathbf{E}_C(F)$ for the integral or *expectation* of a random variable F with respect to the Gaussian measure P_C.

Wick's formula

The moments of P_C are given explicitly in terms of C by:

$$\mathbf{E}_C\left(\prod_{i=1}^{2p}\phi_{x_i}\right)=\sum_{P}\prod_{j=1}^{p}C_{x_{n_j}x_{m_j}} \tag{1.26}$$

where the sum ranges over all pairings P of $1,\dots,2p$ into p distinct unordered pairs $\{n_1,m_1\},\dots,\{n_p,m_p\}$; the odd moments vanish [104, Proposition 1.2].

Consistency

Gaussian fields are consistent, in the sense that if ϕ is a Gaussian field on X with covariance matrix $C=(C_{xy})_{x,y\in X}$, then for any subset $Y\subset X$, the restriction of ϕ to Y is also a Gaussian field with covariance $(C_{xy})_{x,y\in Y}$; this follows from (1.22).

The consistency implies the existence of Gaussian fields on infinite index sets. A matrix C on an infinite index set X is positive definite if, for every finite subset $Y\subset X$, the restriction of C to Y is positive definite. For any positive definite matrix C indexed by a set X, Kolmogorov's extension theorem [61, Theorem 10.18] implies that there exists a random field ϕ on X such that, for each finite $Y\subset X$, the restriction of ϕ to Y is a Gaussian field with covariance the restriction of C to Y.

Free fields

Now suppose that X is the vertex set of a graph $\Gamma=(X,E)$. A random field ϕ on X is called a *Markov field* on Γ if, for any $A\subset X$, $\{\phi_x : x\in A\}$ is independent of $\{\phi_x : d(A,x)>1\}$ conditionally on $\{\phi_x : d(x,A)=1\}$. Markov random fields play an important role in statistical mechanics because the Markov property describes *local* interactions. It is not difficult to see that a non-degenerate Gaussian field on a finite graph is Markovian if and only if the matrix $L=C^{-1}$ is local in the sense that $L_{xy}=0$ if $d(x,y)>1$; see e.g. [98, Theorems 2.1–2.2].

Let $\mathcal{E}(\phi,\phi)=\phi L\phi$ denote the quadratic form associated to such an L. Note that every quadratic form compatible with the locality requirement is given by functions $\alpha: E\to\mathbb{R}$ and $\mu: X\to\mathbb{R}$ as

$$\mathcal{E}(\phi,\phi)=\phi L\phi=\sum_{e\in E}\alpha_e(\nabla\phi)_e^2+\sum_{x\in X}\mu_x\phi_x^2 \tag{1.27}$$
$$\left(=2\sum_{xy\in E}\alpha_{xy}\phi_x\phi_y+\sum_{x\in X}\left(\mu_x+2\sum_{y:y\sim x}\alpha_{xy}\right)\phi_x^2\right)$$

where

$$(\nabla\phi)_{xy}^2=(\phi_x-\phi_y)^2. \tag{1.28}$$

A quadratic form of this form is called a *Dirichlet form* on X when $\alpha, \mu \geq 0$, but there are also interesting situations in which the last requirement is relaxed and only positive definiteness of $\mathcal{E}$ is required [3, 4]. The inverse matrix $C = L^{-1}$ is called the *Green function* of $\mathcal{E}$. A Gaussian field whose covariance is the Green function of a Dirichlet form $\mathcal{E}$ is called the *free field* associated to $\mathcal{E}$. Much interest is already in the simplest case where α and μ are both constant, say, $\alpha_e = 1$ for all $e \in E$ and $\mu_x = m^2 \geq 0$ for all $x \in X$. Then such a field is called the discrete free field on Γ with mass m. This terminology has its roots in quantum field theory [104].

Local perturbations of Gaussian fields

It turns out that a number of interesting problems can be studied through (approximately) *local perturbations* of Gaussian fields, in particular local perturbations of free fields. By a local perturbation, we shall understand a random field given on a *finite graph* by a measure of the form

$$P_{C,Z_0}(d\phi) = \frac{1}{Z} Z_0(\phi)\, P_C(d\phi) \tag{1.29}$$

where *local* means that Z_0 is a product of *local field functionals*[3],

$$Z_0(\phi) = \prod_{x \in X} Z_{0,x}(\phi), \tag{1.30}$$

i.e., $Z_{0,x}$ depends on $\{\phi_y : d(x, y) \leq 1\}$ only. The most interesting examples are given by homogeneous perturbations for which $Z_{0,x}$ is the same functional for all x which is analogous to the requirement that α and μ are constant in (1.27).

The term "perturbation" might suggests that fields described by such measures are very similar to free fields, in particular when "$Z_{0,x} \approx 1$," but it turns out that the large distance behavior can be drastically different, in a way very much analogous to the behavior of polymer models discussed in the last paragraph of Section 1.2.2. This is no coincidence. In Section 1.3.3, we will sketch how, in terms of a generalized notion of Gaussian field, random polymers are models that can be described in terms of such local perturbations. This description is closely related to *spin models*, subsequently discussed briefly in Section 1.3.4.

[3] We use the term *field functional* rather than random variable for several reasons. It emphasizes the point of view that the former are defined on the fields themselves rather than a probability space. For example, it will become useful to evaluate field functionals on deterministic fields. The second reason is that, in a generalized context involving differential forms (Fermions) introduced later, the notion of random variable does not exist while the notion of field functional still does.

1.3.3 Local time of Markov processes and free fields

The local time of a Markov process on a graph $\Gamma = (X, E)$ is a random field on X, given (for every $t \geq 0$) by (1.7). It is of considerable interest for random polymers. For example, the ratio of weight functions $c_t^H(a,b)/c_t^0(a,b)$ of Section 1.2.4 is the expectation of a functional of the random field L^t under the conditional probability distribution $P_a(\,\cdot\,|w_t = b)$ of the simple random walk.

The distribution of the local time of a Markov process is difficult to study directly, but it is known that, for continuous-time Markov processes, the local time[4] is closely related to the free field associated to the Dirichlet form of the Markov process. (The connection of Dirichlet forms and Markov processes is discussed in the next subsection.) These relations go back to Symanzik [107], Brydges, Fröhlich, and Spencer [18], and Dynkin [51–54], and there are also a number of more recent results [108]. For example, Dynkin's so-called *isomorphism theorem* states [108]

$$\mathbf{E}_C(\phi_a\phi_b F(\tfrac{1}{2}\phi^2)) = \int_0^\infty (\mathbf{E}_C \otimes \mathbf{E}_a)(F(\tfrac{1}{2}\phi^2 + L^t)1_{w_t=b})e^{-\mu t}\,dt \qquad (1.31)$$

where C is the covariance of the free field ϕ with mass $m^2 = \mu > 0$, i.e., the Green function of the variable-speed simple random walk w_t killed at rate μ, $\mathbf{E}_C$ is the expectation functional of the field ϕ, and $\mathbf{E}_a$ is the expectation of the simple random walk w_t started at $w_0 = a$.

Parisi and Sourlas [92, 93] and McKane [89] discovered a more direct relationship involving *supersymmetry*; see also Luttinger [87]. In notation to be introduced below, the so-called τ*-isomorphism* [17, 30] can be stated as

$$\mathbf{E}_C(\bar{\phi}_a\phi_b F(\bar{\phi}\phi + \bar{\psi}\psi)) = \int_0^\infty \mathbf{E}_a(F(L^t)1_{w_t=b})e^{-\mu t}\,dt \qquad (1.32)$$

where the pair (ϕ, ψ) a *supersymmetric* Gaussian field with the same covariance C. Thus, if the square of the free field is replaced by the square of the supersymmetric field on the left-hand side, $\frac{1}{2}\phi^2 + L^t$ is replaced by only L^t on the right-hand side. The *supersymmetric partner* ψ of the complex free field ϕ decouples the two sides.

Dirichlet forms, random walks, and free fields

The theory of *Dirichlet forms* is concerned with far-reaching generalizations of the quadratic form (1.27); see reference [63]. Dirichlet forms stand in close connection to continuous-parameter Markov processes. For example, the Dirichlet form (1.27) with constant coefficients, $\alpha_e = 1$, $\mu_x = m^2 \geq 0$, is associated to the variable-speed

[4] For continuous-time Markov processes, the local time is also often called the occupation time to distinguish it from the local time of the skeleton Markov chain.

simple random walk on the graph Γ. Indeed, $(\alpha_e)_{e\in E}$ with $\alpha_e = 1$ can be viewed as the adjacency matrix $(A_{xy})_{x,y\in X}$ of Γ, defined by

$$A_{xy} = 1_{xy\in E} = \begin{cases} 1 & (xy \in E), \\ 0 & (xy \notin E). \end{cases} \tag{1.33}$$

Let $D_{xx} = \sum_{y\sim x} A_{xy}$ be the number of neighbors of the vertex x, and set $D_{xy} = 0$ if $x \neq y$. The generator of the form (1.27) with $m^2 = 0$ can then be written as the graph Laplace operator

$$L = -\Delta = D - A. \tag{1.34}$$

Standard theory of Markov process implies that there is a Markov process $(w_t)_{t\geq 0}$ on X with $\mathbf{E}_x(1_{w_t=y}) = [e^{-Lt}]_{xy}$. The two-point function of this process is

$$\begin{aligned} G_{m^2}(x,y) &= \int_0^\infty \mathbf{E}_x(1_{w_t=y}) e^{-m^2 t}\, dt \\ &= \int_0^\infty \left[e^{(\Delta - m^2)t}\right]_{xy} dt = \left[(-\Delta + m^2)^{-1}\right]_{xy}. \end{aligned} \tag{1.35}$$

Thus the two-point functions of the simple random walk and the two-point function of the free field are the same. The connections between a Markov process and the corresponding free field go much further, however.

Complex and supersymmetric Gaussian fields

A natural variant of (real) Gaussian fields are *complex Gaussian fields*. In general, a complex field is merely a two-component real field, but we restrict to *symmetric* complex Gaussian fields which means that the real and imaginary parts of the field are independent real Gaussian fields with the same covariance [79]. The symmetric complex field is then determined by

$$\mathbf{E}(\bar{\phi}_x \phi_y) = C_{xy}, \quad E(\phi_x \phi_y) = E(\bar{\phi}_x \bar{\phi}_y) = 0 \tag{1.36}$$

and C is called its covariance. (The real and imaginary components of ϕ both have covariance $\frac{1}{2}C$ in the usual sense.)

Let us consider the symmetric complex Gaussian measure on $\mathbb{C}^X$ with strictly positive definite covariance matrix C for a *finite* set X. Then, with $L = C^{-1}$, the expectation of a random variable $F : \mathbb{C}^X \to \mathbb{C}$ is given by

$$\mathbf{E}_C(F(\phi)) = \frac{1}{\det(2\pi i C)} \int_{\mathbb{C}^X} F(\phi)\, \exp\left(-\sum_{x,y\in X} \phi_x L_{xy} \bar{\phi}_y\right) d\bar{\phi}\, d\phi \tag{1.37}$$

which is interpreted as follows: in terms of two real fields, u and v, ϕ and $\bar{\phi}$ are given by $\phi_x = u_x + iv_x$ and $\bar{\phi}_x = u_x - iv_x$, and the measure $d\bar{\phi}\, d\phi$ is a shorthand for $d\bar{\phi}_{x_1}\, d\phi_{x_1} \cdots d\bar{\phi}_{x_n}\, d\phi_{x_n}$, if $X = \{x_1, \ldots, x_n\}$, where

$$d\bar{\phi}_i\, d\phi_i = 2i\, du_x\, dv_x \tag{1.38}$$

with $du_x\, dv_x$ the usual Lebesgue measures on $\mathbb{C} \cong \mathbb{R}^2$.

Now observe that the probability density of the complex Gaussian measure is the *top degree part* of the differential form

$$\gamma_C = \exp\left(- \sum_{x,y \in X} \phi_x L_{xy} \bar{\phi}_y - \frac{1}{2\pi i} \sum_{x,y \in X} d\phi_x L_{xy} d\bar{\phi}_y \right). \tag{1.39}$$

Here differential forms are multiplied with the anticommuting wedge product (suppressed in the notation above), and the exponential function is defined by expansion into a power series (which is unambiguous because the argument has even degree). An interesting property of this formula is that the normalization factor of the measure does not appear explicitly. The expectation (1.37) can now be written as

$$\mathbf{E}_C(F(\phi)) = \int_{\mathbb{C}^X} F \gamma_C \tag{1.40}$$

with the convention that the integral of a differential form is the integral of the top degree part of the form only, in the usual sense of integrals of differential forms.

Observe that, while equation (1.37) only has an interpretation for ordinary random variables $F(\phi)$, i.e., differential forms of degree 0, equation (1.40) has a natural interpretation when F is a more general differential form, namely as the integral of the top degree part of the differential form $F\gamma_C$. Differential forms can then be viewed as functionals of the field ϕ_x and the differential form $\psi_x = (2\pi i)^{-1/2} d\phi_x$.[5]

ϕ_x and ψ_x appear in a (formally) symmetric way in the formula for γ_C. In the terminology of quantum mechanics, ϕ has the interpretation of a *Boson* field, while ψ can be interpreted as a *Fermion* field. The formal symmetry between ϕ and ψ is called a *supersymmetry* and has several fascinating implications which we will not discuss, but see references [17, 30]. We still call the pair (ϕ_x, ψ_x) the *supersymmetric Gaussian field* with covariance C. The identification of Fermion fields with differential forms in this context is due to Le Jan [85, 86].

To exemplify in which ways supersymmetric Gaussian fields behave like ordinary Gaussian fields, let us mention that the sum of two supersymmetric Gaussian fields can again be interpreted as a supersymmetric Gaussian field whose covariance is the sum of the covariances [34, Proposition 2.6]. The covariance is

$$\mathbf{E}(\bar{\phi}_x \phi_y) = \mathbf{E}(\bar{\psi}_x \psi_y) = -\mathbf{E}(\psi_y \bar{\psi}_x) = C_{xy}. \tag{1.41}$$

[5] The complex square root function is fixed in an arbitrary way.

This can be generalized to a version of *Wick's formula* for the moments:

$$\mathbf{E}_C\left(\prod_{i=1}^{p}\bar{\phi}_{x_i}\phi_{y_i}\prod_{j=1}^{q}\bar{\psi}_{u_j}\psi_{v_j}\right)=\left(\sum_{\pi\in S_p}\prod_{i=1}^{p}C_{x_i y_{\pi(i)}}\right)\left(\sum_{\pi\in S_q}(-1)^{|\pi|}\prod_{j=1}^{q}C_{u_j v_{\pi(j)}}\right) \tag{1.42}$$

where S_n is the symmetric group of order n, and $(-1)^{|\pi|}$ is the sign of a permutation $\pi \in S_n$. More details are given in [29,30], but the upshot is that again, as in (1.26), all moments can be calculated in a simple way in terms of the covariance.

Local time and supersymmetry

Finally, we can discuss the connection between random walks and supersymmetry, discovered by Parisi and Sourlas [92, 93] and McKane [89], in the form stated in reference [33]. To explain it, define differential forms τ_x, $x \in X$, on $\mathbb{C}^X$ by

$$\tau_x = \bar{\phi}_x\phi_x + \frac{1}{2\pi i}d\bar{\phi}_x d\phi_x = \bar{\phi}_x\phi_x + \bar{\psi}_x\psi_x. \tag{1.43}$$

For $F : \mathbb{R}^X \to \mathbb{R}$ smooth, it is natural to define a differential form $F(\tau)$ as the *finite* Taylor series around the degree 0 part of τ which is $\bar{\phi}\phi = |\phi|^2$:

$$F(\tau) = \sum_{m=1}^{|X|}\frac{1}{m!}\sum_{x_1,\dots,x_m\in X}F_{x_1\cdots x_m}(\bar{\phi}\phi)\prod_{j=1}^{m}\frac{1}{2\pi i}d\bar{\phi}_{x_j}d\phi_{x_j} \tag{1.44}$$

where $F_{x_1\cdots x_m}(t)$ is the mth derivative of $F(t)$ in direction $(e_{x_1},\dots,e_{x_m})$. The Taylor series is finite because differential forms on a finite dimensional space have a maximal degree (the dimension of the space). It is unambiguous because the differential form τ is even.

Theorem 1.3.1. *Let X be a finite set, $(w_t)_{t\geq 0}$ be a continuous-time Markov process on X, and C be the Green function of $(w_t)_{t\geq 0}$ with killing rate $m^2 > 0$:*

$$C_{xy} = \int_0^\infty \mathbf{E}_x(1_{w_t=y})e^{-m^2 t}\,dt. \tag{1.45}$$

Then, for any smooth $F : \mathbb{R}_+^X \to \mathbb{R}$ that does not grow too rapidly,

$$\int_0^\infty \mathbf{E}_x(F(L^t)1_{w_t=y})e^{-m^2 t}\,dt = \mathbf{E}_C(F(\tau)\bar{\phi}_x\phi_y). \tag{1.46}$$

Proof. See [30, Propositions 2.7 and 4.4]. □

Theorem 1.3.1 with $F = \prod_x e^{-g\tau_x^2 - (\mu - m^2)\tau_x}$ for some $m^2 > 0$ implies that the two-point function of the continuous-time weakly self-avoiding walk on a finite graph is equal to the two-point function of a local perturbation of the supersymmetric free field on the same graph, in the sense of Section 1.3.2 with the Gaussian measure P_C replaced by the supersymmetric Gaussian "measure" γ_C. If we write g instead of β and set $\gamma = 0$, the two-point function (1.16) is thus, more explicitly,

$$G_\mu(a, b) = \mathbf{E}_C(\bar{\phi}_a \phi_b Z_0) \tag{1.47}$$

where $C = [-\Delta + m^2]^{-1}$ and

$$Z_0 = \prod_{x \in X} e^{-g\tau_x^2 - (\mu - m^2)\tau_x} \tag{1.48}$$

is a local perturbation. In fact, there is some flexibility in the split of perturbation and Gaussian measure, for example, by choice of m^2. It turns out that this split can be made use of in the context of the renormalization group, and that then, it is also necessary to consider a more general splitting, $C = (1 + z)[-\Delta + m^2]^{-1}$ with

$$Z_0 = \prod_{x \in X} e^{-g\tau_x^2 - (\mu - zm^2)\tau_x - z\tau_{\Delta x}} \tag{1.49}$$

and

$$\tau_{\Delta,x} = \frac{1}{2}\left[\phi_x(\Delta\bar{\phi})_x + (\Delta\phi_x)\bar{\phi}_x + \psi_x(\Delta\psi)_x + (\Delta\psi)_x\bar{\psi}_x\right]. \tag{1.50}$$

The study of the perturbation (1.48) is actually also very interesting when the Fermionic (differential form) part of τ is dropped, and then such perturbations have been studied extensively, as *spin models* which are models of ferromagnetism.

1.3.4 Spin models

Let $\Gamma = (X, E)$ be a finite graph. A *spin model* on Γ is real- or vector-valued random field on Γ with distribution given by [60]

$$P(d\phi) = \frac{1}{Z} e^{-H(\phi)} \prod_{x \in X} \rho(d\phi_x) \tag{1.51}$$

where Z is a normalizing constant, ρ is a probability measure on $\mathbb{R}^N$ called *a priori measure* of the spin model, $\alpha : E \to [0, \infty)$ are *pair interactions*, and

$$H(\phi) = -\sum_{xy \in E} \alpha_{xy}\, \phi_x \cdot \phi_y. \tag{1.52}$$

The best-known case is when α is constant, i.e., $\alpha_e = \alpha > 0$, and the a priori measure is given by the uniform (surface) measure of the unit sphere $S^{N-1} \subset \mathbb{R}^N$. These so-called *N-vector models* include the *Ising model* ($N = 1$), the *rotor* or *XY model* ($N = 2$), and the *Heisenberg model* ($N = 3$). Much attention has also been devoted to the ϕ^4 models, given by $\alpha_e = 1$ and a priori measure

$$\rho(d\phi_x) = e^{-g|\phi_x|^4 - s|\phi_x|^2}. \tag{1.53}$$

The ϕ^4 models include the N-vector models as limits with $g \to \infty$ and $s \propto -g$; see references [60, 100]. They can be written in exact analogy to (1.48) as

$$dP = \frac{1}{Z} Z_0 \, dP_C, \quad Z_0 = \prod_{x \in X} e^{-g|\phi_x|^4 - \mu|\phi_x|^2}. \tag{1.54}$$

where P_C is the Gaussian measure with covariance[6] given by $C = [-\Delta + m^2]^{-1}$ and $\mu = s - 1 - m^2$.

Spin models and walks

The relation (1.54) with $N = 2$ components is the same as (1.46) with τ_x replaced by its 0-degree part, $|\phi_x|^2$. Thus the weakly self-avoiding walk model is a supersymmetric version of the two-component ϕ^4-model. It is known that spin models also have interpretations in terms of walks, but with additional loops [18, 60]. In fact, the discovery of the relations between walks and fields departed from this direction in the study of field theories in terms walks and loops [107].

De Gennes [42] also argued that the self-avoiding walk is described by the limit $N \to 0$ of the N-vector model (also see [30, 88]), but this limit does not have a meaning at the level of probability measures. The supersymmetric version is a way of giving rigorous meaning to it, in the context of the weakly self-avoiding walk. The essential idea is that the Fermion components of τ count, in a sense, *negatively* to the number of components due to the minus sign in equation (1.42), in this sense giving "$N = 2 - 2 = 0$." For a more complete discussion, see reference [30].

Behavior of spin models

In view of the connection between spin models and interacting walks (with loops), it is not surprising that many qualitative features of the weakly self-avoiding walk are shared by the spin models. In the context of spin models, the critical value μ_c has an instructive interpretation. For example, consider the ϕ^4 model with $N = 1$

[6] Each component is an independent Gaussian field with this covariance, in the vector-valued case.

and $g > 0$ fixed (or the Ising model). It is known, see e.g. reference [6], that there is $\mu_c > -\infty$ such that its infinite volume limits on $\mathbb{Z}^d$, $d \geq 2$, satisfy

$$\chi(\mu) = \sum_x |\mathbf{E}(\phi_0 \phi_x)| \begin{cases} < \infty & (\mu > \mu_c), \\ = \infty & (\mu < \mu_c). \end{cases} \tag{1.55}$$

The field ϕ_x can be interpreted as a kind of *spin* of a particle (an arrow) located at vertex x. For $\mu < \mu_c$, (1.55) means that the spins are *ordered*. This corresponds to the ferromagnetic phase of a magnet in which most spins point in the same direction. On the other hand, the case $\mu > \mu_c$ corresponds to a disordered phase. The variation of μ corresponds to a variation in inverse temperature. The *critical point* $\mu = \mu_c$ corresponds to the critical temperature of the phase transition between the ordered and the disordered phase. For $N > 1$, the picture is similar, but much more delicate due to the *continuous* $O(N)$-symmetry of the model on finite graphs. This continuous symmetry is "spontaneously" broken in the ordered phase in the infinite volume limit [62], giving a different magnitude of difficulty to the problem.

1.4 The renormalization group

1.4.1 The concept of renormalization in statistical mechanics

Random polymer models on the Euclidean lattice are expected to have *scaling limits*. The fundamental example of this is the convergence of the simple random walk to the Wiener process (1.5). This is a statement about large distances and times related by diffusive scaling. The basic idea of *renormalization* is to study the large-distance behavior of a model by reduction of the degrees of freedom of the model by a version of *coarse graining*, i.e., disregarding information about the behavior at small distances, say, smaller than $0 \ll L \ll \infty$. The fundamental hypothesis of the renormalization idea is that, after coarse graining and rescaling, the model should be similar to the original model with modified parameters. The combination of the two operations of coarse graining and rescaling is called a *renormalization group transformation*. However, concrete formulations of such transformations for models of self-avoiding walks on the Euclidean lattices, in any dimension, defined directly in terms of walks and amenable to analysis, seem not to be understood.[7]

The renormalization group concept is, however, much better understood in the context of (near) *critical* random fields, in particular if these are *local perturbations*

[7] On hierarchical groups, the work of Brydges, Evans, and Imbrie [17,28,29] has an interpretation in terms of walks, and there is also work in preparation by Ohno in which renormalization of self-avoiding walks on hierarchical lattices is studied directly in terms of walks.

of a Gaussian field on $\mathbb{Z}^d$, i.e., described in finite volume by measures

$$P_{C,Z_0}(d\phi) = \frac{1}{Z} Z_0(\phi)\, P_C(d\phi) \tag{1.56}$$

where P_C is a Gaussian measure on $\mathbb{Z}^d$, $Z_0(\phi)$ is a local perturbation, in the sense discussed in Section 1.3.4, and Z is the normalization constant $Z = \mathbf{E}_C(Z_0)$. In this context (but not only in this), the renormalization group has been used successfully to study the long-distance behavior of a number of such models. It also provides an approach to a renormalization group study of random polymers via (1.46). The term *critical* random field refers, for example, to a spin model at the critical point; see Section 1.3.4. In the context of models of walks, the near critical behavior is related to the behavior of *long* polymers as discussed in Section 1.2.4.

Let us mention two historically important ideas for the renormalization group study of random fields: Kadanoff [80] proposed the intuitively appealing idea to replace a random field in an $L \times L \times \cdots \times L$ block of points in $\mathbb{Z}^d$ by an effective *block spin* field, constructed for example by averaging the field in that block. He claimed that this block spin field should behave in a similar way as the original field, but did not provide arguments to justify such an approximation. Wilson later argued, still non-rigorously but with deep insight, how a variant of this idea may be justified. He was awarded the Nobel Prize in Physics in 1982 for his contributions [113]. Following the introduction of [1], let us sometimes refer to the mathematical realization of Wilson's ideas as *Wilson's program*. There has been quite remarkable progress in the realization of approaches like Wilson's renormalization group. We do not attempt to provide a comprehensive list of references, but let us only mention a few relevant references: Benfatto et al. [13], Feldman et al. [58], Gawedzki and Kupiainen [67], and Brydges and Yau [22]. Unfortunately, these works all involve numerous technical challenges, and it seems unlikely that the full capacity of the renormalization group idea has been attained yet. Nonetheless, it is one of the most powerful tools available for the study of random fields.

We will give a short heuristic account of our interpretation of the challenges of Wilson's program and also sketch very briefly aspects of the approach initiated by Brydges and Yau [22], in a further developed form of Brydges and Slade [10, 34–38]. The latter authors conceptualized, simplified, and generalized the approach in significant aspects to study weakly self-avoiding walks via (1.46). The method of Brydges and Yau has, however, also been applied to a number of other models, including the dipole and Coulomb gases [44–46, 48, 49, 55], gradient interface models [3], as well as problems from quantum field theory [1, 16, 32, 47]. Introductions to concepts of the method are given in [12, 24, 25, 106]. Our discussion is inspired by many of the references previously mentioned and by the general expositions on the renormalization group [14, 67, 100, 113]. The focus of

our discussion is on the relation to the problems studied in this thesis.

1.4.2 Progressive integration, dynamical systems, and coordinates

Let us consider a random field that is a local perturbation of the free field, (1.56), with covariance given by the Green function $C = [-\Delta + m^2]^{-1}$ of the graph Laplace operator on $\Lambda \subset \mathbb{Z}^d$. The perturbation Z_0 makes sense only if it depends on a finite set Λ and then $m^2 > 0$ may be required, but the goal is to analyze such measures P_{C,Z_0} in the limit $\Lambda \to \mathbb{Z}^d$ and $m^2 \downarrow 0$; we will, however, not devote much attention to the details of these limits.

In principle, the measure P_{C,Z_0} can of course be studied in terms of

$$\mathbf{E}_C(F Z_0), \tag{1.57}$$

for enough field functionals F which we call *observables*. For instance, with $F = 1$, (1.57) expresses the normalization factor in (1.56), and with $F = \phi_a \phi_b$, it gives the unnormalized two-point function. It is well-known, however, that it can be useful to study a measure in terms of a transform, e.g., its Laplace or Fourier transformation. Let us denote the Laplace transform of the *unnormalized* measure $Z_0\, dP_C$ by

$$Z^f := \mathbf{E}_C(e^{-\phi f} Z_0(\phi)) =: \mathbf{E}_C(Z_0^f(\phi)). \tag{1.58}$$

To study the *large distance behavior* of the field, the class of test functions f should be insensitive to fluctuations at short distances. For example, a *scaling limit* would be determined by increasingly *smooth* $f = f^\varepsilon$ given by $f_x^\varepsilon = \varepsilon^\alpha \bar{f}(\varepsilon x)$, $(x \in \mathbb{Z}^d)$, for some exponent $\alpha > 0$ and $\bar{f} \in C_c^\infty(\mathbb{R}^d)$, in the limit $\varepsilon \downarrow 0$. It is, however, also interesting to consider pointwise asymptotics of correlation functions, for example with $f = f^{ab} = \sigma_a \delta_a + \sigma_b \delta_b$ as $d(a,b) \to \infty$, where σ_c are constants $(c = a, b)$, and $(\delta_c)_x = 1$ if $c = x$ and $(\delta_c)_x = 0$ otherwise. Then the normalized two-point function is the derivative of $\log Z^f$ with respect to σ_a and σ_b.

The accurate analysis of expectations like (1.57)–(1.58) is however highly non-trivial because the free field is *strongly correlated*: for example, see (1.20), (1.18),

$$\mathbf{E}_C((\phi_x - \mathbf{E}_C(\phi_x)(\phi_y - \mathbf{E}_C(\phi_y))) = \mathbf{E}_C(\phi_x \phi_y) \to 0 \tag{1.59}$$

so slowly that $\sum_y |\mathbf{E}_C(\phi_x \phi_y)| \to \infty$ as $\Lambda \to \mathbb{Z}^d$ and $m \downarrow 0$. The crucial property that will facilitate the analysis is that the perturbation Z_0 is local, i.e., a product

$$Z_0 = \prod_{x \in \Lambda} Z_{0,x} \tag{1.60}$$

where each $Z_{0,x}$ is a local field functional; see Section 1.3.2. This factorization property provides the important structure, as we will sketch, for the iterative analysis of such expectations by a particular form of coarse graining.

The fundamental idea is to decompose the free field ϕ into a sum of two independent Gaussian fields, $\phi = \phi_s + \phi_l$, corresponding to *small* and *large* distances. The coarse graining step is then implemented by taking the expectation of the field ϕ_s which is called the *fluctuation field* because it captures the small distance fluctuations that are to be eliminated. Wilson's renormalization group program involves iteration of this procedure and rescaling of the underlying physical space after each step. The motivation is that this *renormalization group transformation*, the combination of coarse graining and rescaling, should bring a *critical* model approximately back to its original form so that the transformation can be iterated to obtain an effective description for increasingly large distances.

In practice, it can be convenient to omit the rescaling step and instead consider "increasingly smooth" test functions, as discussed below (1.58). Furthermore, the iterated decomposition of the Gaussian field, or equivalently of its covariance, into small and large distance contributions can be implemented by *a priori decomposition* of the initial covariance,

$$C = C_1 + C_2 + \cdots \tag{1.61}$$

into a sum of covariances corresponding to geometrically increasing length scales. This idea goes back to Wilson, but was perhaps first explicitly formulated by Benfatto et al. [13]. The somewhat vague term *length* or *distance scale* means that each C_j should account for the fluctuations of the free field in an exponential range of distances $L^{j-1} \lesssim |x| \lesssim L^j$ for a fixed $L > 1$. This is discussed in the next section.

From a pragmatic point of view, the covariance decomposition $C = C_1 + C_2 + \cdots$ allows to evaluate the expectation $\mathbf{E}_C(Z_0^f(\phi))$ progressively, in terms of a sequence of field functionals Z_j^f which are integrated with respect to the Gaussian fields with covariance $C_{j+1} + \cdots$, defined by

$$Z_{j+1}^f(\phi) := E_{j+1} Z_j^f(\phi) := \mathbf{E}_{C_{j+1}}^{\phi'} \left(Z_j^f(\phi + \phi') \right), \tag{1.62}$$

where the expectation on the right-hand side is that of the fluctuation field ϕ'. E_j is thus the *convolution operator* of the Gaussian measure with covariance C_j. It then follows that the expectation is given by[8]

$$Z^f = Z_\infty^f(0) := \lim_{j \to \infty} Z_j^f(0). \tag{1.63}$$

The progressive integration (E_j) can be regarded as a *time-dependent* dynamical system, with the scale parameter j in the role of "time:" if $\mathbf{N}$ is an appropriate

[8]The limit requires some mild assumptions on the decomposition. Moreover, in practice, it can be more convenient to stop the iteration after finitely many steps, when the decomposition has reached the size of the finite set Λ; we will ignore such details.

space of field functionals and $\mathbf{N}_j \subset \mathbf{N}$ a subspace of field functionals which are integrable with respect to the Gaussian measure with covariance $C_j + C_{j+1} + \cdots$, then $E_{j+1} : \mathbf{N}_j \to \mathbf{N}_{j+1} \subseteq \mathbf{N}$. This picture, in itself, is not a simplification of the problem since the dynamical system $(E_j)_j$ is enormously complicated and time-dependent, and it must be understood in the limit $\Lambda \to \mathbb{Z}^d$. To analyze particular aspects of this dynamical system, one must find appropriate coordinates in which an aspect of consideration becomes tractable, uniformly in Λ.

In particular, it is natural to consider the evolution of the perturbation Z_0 only, without f. For example, by an elementary calculation for Gaussian measures,

$$Z^f = Z_\infty^f(0) = e^{fCf} Z_\infty(Cf) \tag{1.64}$$

where Z_∞ on the right-hand side does not have a superscript f. Thus, in principle, i.e., given sufficient knowledge about Z_∞, the general case can be reduced to it.

The goal of the next subsections is to outline how *coordinates* x_j can be found in which the action of the Gaussian convolution with covariance C_{j+1} on Z_j is expressed in a much simpler form by a map Φ_j acting on x_j:

$$E_{j+1}(\hat{Z}_j(x_j)) = \hat{Z}_{j+1}(\Phi_j(x_j)) \tag{1.65}$$

for some coordinate maps $\hat{Z}_j$ that map an "abstract" coordinate x_j to a field functional $\hat{Z}_j(x_j) = \hat{Z}_j(x_j, \phi)$. In his pioneering work, Wilson argued how this should be possible and, with the previously mentioned rescaling step, his dynamical system is approximately autonomous. In the rigorous approach of Brydges and Slade [10,34–37], it has turned out useful to allow the coordinate spaces to depend on the scale j. Thus there is a sequence of spaces X_j such that $x_j \in X_j$ and the evolution maps are given as $\Phi_j : X_j \to X_{j+1}$, but approximate invariance under rescaling must, of course, still play a role. Finding such coordinates x_j, rigorously, is at the heart of the difficulties of the renormalization group.

Let us mention again, with the more specific context that has now been introduced, that the main results of this thesis are the following.

- **Chapter 2** provides a new method for decomposition of Green functions that give decompositions of free fields with particularly useful properties for the analysis of the renormalization group transformations that they induce.
- **Chapter 3** is the analysis of a class of general dynamical systems $\Phi = (\Phi_j)$ that arise as coordinates of the renormalization group map for four-dimensional weakly self-avoiding walks [10, 37].

The outline of this subsection will be expanded with further details in the following subsections. In Appendix A, we provide some concrete details how the covariance decomposition of Chapter 2 gives rise to the assumptions of Chapter 3.

1.4.3 Decomposition of the free field

The starting point for the renormalization group, in the form discussed in the previous section, is a decomposition of the free field, or equivalently the decomposition of its covariance C, into *distance scales*:

$$C = C_1 + C_2 + \cdots . \tag{1.66}$$

The covariance should here be regarded as an infinite (in the limit $\Lambda \to \mathbb{Z}^d$) symmetric matrix $(C_{xy})_{x,y\in\mathbb{Z}^d}$ that is *positive definite* in the sense that

$$\sum_{x,y\in\mathbb{Z}^d} f_x C_{xy} f_y \geq 0 \quad \text{for all finitely supported } f : \mathbb{Z}^d \to \mathbb{R}. \tag{1.67}$$

The decomposition (1.66) must be such that each term C_j satisfies (1.67), in order for the C_j to be the covariances associated to Gaussian fields, and, at the same time, the covariances C_j must "capture" the distance scales $L^{j-1} \lesssim |x| \lesssim L^j$ for some fixed $L > 1$, where $L^j = L \times \cdots \times L$. These are two competing constraints.

In Chapter 2, in particular in Theorem 2.1.2 and Example 2.1.3, we prove that, if C is the Green function of a quadratic form in general class (containing Dirichlet forms on a general graph, not necessarily $\mathbb{Z}^d$), then a strong form of the decomposition of the above kind is possible. There exists $\phi_t(x, y),\ t > 0$ such that

$$C_{xy} = \int_0^\infty \phi_t(x, y)\ \frac{dt}{t} \tag{1.68}$$

where ϕ_t is *positive definite*, for each $t > 0$. The use of the scale-invariant measure dt/t on $[0, \infty)$ in (1.68), rather than the Lebesgue measure dt, is not important but a natural choice. The kernel ϕ_t satisfies the *finite range property*

$$\phi_t(x, y) = 0 \quad \text{if } d(x, y) > t, \tag{1.69}$$

and *natural estimates*. For example, if C is the Green function associated to the lattice Laplace operator, then, for all multi-indices $l_x, l_y \in \mathbb{N}_0^{\{\pm1,\dots,\pm d\}}$,

$$\left|\nabla_x^{l_x} \nabla_y^{l_y} \phi_t(x, y)\right| \leq C t^{-(d-2)-|l_x|_1-|l_y|_1} \tag{1.70}$$

where negative components of l denote discrete gradients in the negative coordinate directions and $|l|_1 = \sum_{i=1}^d (l_i + l_{-i})$. Moreover, ϕ_t is then also translation-invariant, i.e., $\phi_t(x, y) = \phi_t(0, y - x)$, and symmetric, i.e., $\phi_t(0, x) = \phi_t(0, -x)$.

To obtain a discrete decomposition, as in (1.66), the integral (1.68) can be split into integrals over finite intervals. For example, for any $L > 1$, set

$$[C_j]_{xy} = \begin{cases} \int_0^{\frac{1}{2}L} \phi_t(x,y)\,\frac{dt}{t} & (j=1), \\ \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t(x,y)\,\frac{dt}{t} & (j>1). \end{cases} \tag{1.71}$$

The properties of ϕ_t immediately imply

$$\begin{cases} C_j \text{ is positive definite;} \\ C_j \text{ has the finite range property: } [C_j]_{xy} = 0 \text{ if } d(x,y) > \frac{1}{2}L^j; \\ C_j \text{ is translation-invariant: } [C_j]_{x+a,y+a} = [C_j]_{xy}; \\ C_j \text{ satisfies } |[\nabla_x^{l_x}\nabla_y^{l_y} C_j]_{xy}| \le O(L^{-(d-2+|l_x|_1+|l_y|_1)(j-1)}). \end{cases} \tag{1.72}$$

In addition, we show that the ϕ of the Euclidean lattice has a *scaling limit*. For the discrete decomposition, this means that there exists $c \in C_c^\infty(B_{\frac{1}{2}}(0))$ such that

$$[C_j]_{xy} = L^{-(d-2)j} c(L^{-j}(x-y)) + O(L^{-(d-2+1)j}). \tag{1.73}$$

An analogous result also holds for all discrete gradients of C_j. The existence of the scaling limit implies that certain functions of C_j can be computed very precisely in the limit $j \to \infty$, as illustrated in Appendix A.

As hinted at, the two constraints that C_j is positive definite and finite range are non-trivial to satisfy simultaneously. It is a natural question if covariance decompositions in which C_j is localized exponentially, e.g., for some $c > 0$,

$$|[C_j]_{xy}| \le O(L^{-(d-2)(j-1)} e^{-cL^{-(j-1)}|x-y|}), \tag{1.74}$$

would be equally useful. It is much easier to find decompositions with this relaxed localization property. The answer is that such decompositions are almost as useful, and, in fact, they have been used in earlier results on the renormalization group, see in particular [13, 65]. The use of the finite range property, originally proposed by Brydges [90], leads to simplifications of the method and, in some aspects, slightly better results.

1.4.4 Formal perturbation theory

Physicists have long understood that the evolution $Z_j \to Z_{j+1} = E_{j+1} Z_j$ becomes *formally* simple when expressed as an exponential function. Let $\tilde{V}_j = -\log Z_j$ be

the *effective potential*. In particular, for the weakly self-avoiding walk model, by (1.49),

$$\tilde{V}_0 = \sum_{x\in\Lambda} (g_0\tau_x^2 + \mu_0\tau_x + z_0\tau_{\Delta,x}) \tag{1.75}$$

is parametrized by the three *coupling constants* (g_0, μ_0, z_0). Formally, by which we mean by expanding the exponential function into a power series without paying attention to its convergence,

$$\begin{aligned}\tilde{V}_{j+1} &= -\log(E_{j+1}(\exp(-\tilde{V}_j))) \\ &\approx E_{j+1}(\tilde{V}_j) + \tfrac{1}{2}(E_{j+1}(\tilde{V}_j)^2 - E_{j+1}(\tilde{V}_j^2)) + \cdots\end{aligned} \tag{1.76}$$

where $\approx$ means in the sense of a formal power series in $\tilde{V}_j$. This relation is called the *cumulant expansion* and also *perturbation expansion* in the physics literature.

If $\tilde{V}_j$ was a polynomial (or formal power series) of the field, as for $\tilde{V}_0$ in (1.75), then the terms of each order of on the right-hand side of (1.76) could be calculated explicitly in terms of the covariance by Wick's formula (1.42). Ignoring a number of problems with (1.76), Wilson observed that in this formal series of monomials of the field, a few terms seem to be much more important than the others. He argued that, in dimensions four and above[9], $\tilde{V}_j$ can be approximated by a polynomial of the same form as $\tilde{V}_0$. Effectively, this reduces the complexity from an infinite number of variables to three variables, (g_j, μ_j, z_j), parametrizing $\tilde{V}_j$ as in (1.75).

First-order perturbation theory and local field monomials

To explain Wilson's argument, some terminology is convenient. A field functional M is a *local field monomial*, localized at $x \in \Lambda$, if M can be expressed as a monomial in ϕ_x and $\nabla\phi_x$, and corresponding terms in other fields (such as the Fermionic field ψ). Moreover, P is a *local field polynomial* if there is $X \subset \Lambda$ and local field monomials M_x for $x \in X$ such that $P = \sum_{x\in X} M_x$. For example, ϕ_x^2 is a local field monomial and $\sum_{x\in\Lambda}\phi_x^2$ is a local field polynomial. In particular, $\tilde{V}_0$ is a local field polynomial.

Then, to explain a fundamental idea, suppose that $\tilde{V}_{j+1}$ is given by the first term of the right-hand side of (1.76) only, i.e., $\tilde{V}_{j+1} = E_{j+1}\tilde{V}_j$. Observe that

$$E_j(\tau_x) = \tau_x, \tag{1.77}$$

$$E_j(\tau_x^2) = \tau_x^2 + 2[C_j]_{xx}\tau_x = \tau_x^2 + 2[C_j]_{00}\tau_x, \tag{1.78}$$

$$E_j(\tau_{\Delta,x}) = \tau_{\Delta,x}, \tag{1.79}$$

[9] In fact, he also considers dimension "$4-\varepsilon$," but we will not be concerned with this case. Below we outline the considerations in general dimension d, but for field theories with "quadric interactions," these will only be useful in $d \geq 4$ which is our main interest.

by the definitions of τ and τ_Δ, (1.43) and (1.50), Wick's formula (1.42), and translation-invariance of C_j, i.e., $[C_j]_{xx} = [C_j]_{00}$. The exact expressions of the right-hand sides in (1.77) rely on the differential form parts of τ_x and $\tau_{\Delta,x}$, but for many other purposes one can think of τ_x and $\tau_{\Delta,x}$ simply as their degree 0 parts, $\bar{\phi}_x\phi_x$ and $\frac{1}{2}(\bar{\phi}_x(\Delta\phi)_x + (\Delta\bar{\phi})_x\phi_x)$, and we will then do so, and also replace the complex field by a real field if the distinction is not important.[10] It follows that, in the linear approximation, all $\tilde{V}_j$ are local field monomials of the same form as $\tilde{V}_0$ with (g_0, μ_0, z_0) replaced by $(\tilde{g}_j, \tilde{\mu}_j, \tilde{z}_j)$ determined by the recursion relation

$$(\tilde{g}_{j+1}, \tilde{\mu}_{j+1}, \tilde{z}_{j+1}) = (\tilde{g}_j, \tilde{\mu}_j + 2[C_{j+1}]_{00}\tilde{g}_j, \tilde{z}_j). \tag{1.80}$$

Now observe that, according to the discussion about the decomposition of the Green function in the previous section,

$$\mathrm{Var}(\nabla_x^l \phi_{j+1,x}) = [\nabla_x^l \nabla_y^l C_{j+1}]_{xy}\big|_{x=y} \approx cL^{-(d-2+2|l|_1)j} = cL^{-2([\phi]+|l|_1)j} \tag{1.81}$$

for any multi-index l. The constant $[\phi] := \frac{1}{2}(d-2)$ on the right-hand side is called the *dimension* of ϕ. A measure of the *typical magnitude* of a field is the square root of its variance and, in this sense,

$$|\phi_{j+1,x}| \approx L^{-[\phi]j}. \tag{1.82}$$

Moreover, by (1.81), each discrete derivative ∇ of ϕ_{j+1} decreases this typical magnitude by an additional factor of L^{-j} (up to an absolute constant). The *dimension* $[M]$ of a local field monomial M is defined so that $|M(\phi_{j+1})| \approx L^{-[M]j}$ according to this heuristic, i.e., by adding a summand $[\phi]$ for each factor of ϕ and a summand of 1 for each discrete gradient ∇. For example,

$$[\phi^4] = 4[\phi] = 2(d-2), \quad [(\nabla\phi)^2] = 2[\phi] + 2 = d. \tag{1.83}$$

In dimensions $d > 2$, the typical magnitude of a fluctuation field decreases as j increases, by (1.82), but at the same time its range increases like L^j. For a scaling limit, the natural "effective size" of a field monomial is that of its sum over a block B of approximate diameter L^j, i.e.,

$$\sum_{x\in B} |M(\phi_{j+1,x})| \approx L^{(d-[M])j}. \tag{1.84}$$

This gives rise to the following classification of local field monomials:

[10] The expressions with differential forms are simpler than those of their degree 0 parts alone. This is because of cancellations due to *supersymmetry*. It corresponds to the cancellation of "loops" in the random walk representation; see Section 1.3.4.

- if $[M] > d$, the heuristic magnitude of M *contracts* (M is *irrelevant*);
- if $[M] < d$, the heuristic magnitude of M *expands* (M is *relevant*);
- if $[M] = d$, the heuristic magnitude of M *remains the same* (M is *marginal*).

It is therefore natural to consider the coupling constants (g_j, μ_j, z_j) with respect to the "normalized" field monomials[11] $L^{(d-4)j}\tau_x^2$, $L^{-2j}\tau_x$, and $\tau_{\Delta,x}$. In the formal first order approximation, the evolution of these is given by:

$$(g_{j+1}, \mu_{j+1}, z_{j+1}) = (L^{-(d-4)}g_j, L^2(\mu_j - 2L^{2j}[C_{j+1}]_{00}g_j), z_j). \tag{1.85}$$

These heuristic considerations lead to the following predictions for the large distance behavior of the perturbed field. In dimension five and higher, the only non-contracting local field monomials compatible with the symmetries[12] of the model are τ and τ_Δ; in particular $g_j \to 0$, and the large distance behavior is expected to be that of the free field. In dimension three and lower, there are several relevant local field monomials, finitely many in dimension three, for example τ and τ^2, and infinitely many in dimension two, and in both cases the large distance behavior is expected to be non-trivial (different from the free field). In dimension four, there is only one relevant field monomial, τ, and only two marginal field monomials, τ^2 and τ_Δ, and the first-order approximation is not sufficient to (heuristically) determine the long-distance behavior. A second-order analysis reveals that the long distance behavior should be like that of the free field, but in a much more subtle way than in dimensions above four.

Higher-order perturbation theory and approximation by local polynomials

The immediate difficulty encountered when trying to *formally* include higher-order terms of (1.76) in the previously described heuristic procedure is that such terms are not local field monomials. For example, an (important, as it will turn out) term arising at second-order is

$$-g_j^2 \sum_{x,y} [C_{j+1}]_{xy}^2 \bar{\phi}_x \phi_x \bar{\phi}_y \phi_y. \tag{1.86}$$

This term involves ϕ_x and ϕ_y with $d(x, y) \approx L^{j+1}$ and is therefore not a *local* field polynomial. However, such terms which arise in (1.76) can always be replaced by

[11] For simplicity, we refer to, e.g., $\tau_x = \bar{\phi}_x\phi_x + \bar{\psi}_x\psi_x$ as a "monomial," even though it is actually a sum of two monomials in the fields in the previously introduced terminology.

[12] The model is symmetric under Euclidean transformation that preserve the lattice and under a so-called supersymmetry [10, 35].

a local field polynomial and a *contracting* non-local remainder term. For example, for the above term, one can make the replacement

$$\sum_{x,y}[C_{j+1}]^2_{xy}\bar{\phi}_x\phi_x\bar{\phi}_y\phi_y \rightsquigarrow C^{(2)}_{j+1}\sum_x(\bar{\phi}_x\phi_x)^2 \tag{1.87}$$

where we have introduced the abbreviation

$$C^{(2)}_j = \sum_y [C_j]^2_{xy} \tag{1.88}$$

which is independent of x, by translation-invariance of C_j. The right-hand side of (1.87) is again a local field monomial and, as such, it can be included as a second-order correction to the flow of coupling constants $(g_j, \mu_j, z_j) \mapsto (g_{j+1}, \mu_{j+1}, z_{j+1})$. The above term results in a contribution to g_{j+1} like $g_{j+1} = g_j - \beta_j g_j^2 + \cdots$ with $\beta_j > 0$. More details of the resulting equations are given in Appendix A.

The difference between the right- and left-hand sides of (1.87) is

$$\sum_{x,y}[C_{j+1}]^2_{xy}(\bar{\phi}_x\phi_x)(\bar{\phi}_y\phi_y - \bar{\phi}_x\phi_x). \tag{1.89}$$

This term "contracts" in dimensions $d \geq 4$, roughly, since the difference between a local field monomial at two points decays faster than the individual monomials, by (1.72), if the distance between the points remains fixed. To illustrate this, consider (1.89) with $y = x + re$ where e is a unit lattice vector and r an integer with $|r| \leq O(L^{j+1})$; the latter restriction on r is because of the finite range condition that $C_j(x, y) = 0$ if $d(x, y) \geq cL^j$. Then

$$\bar{\phi}_x\phi_x(\bar{\phi}_y\phi_y - \bar{\phi}_x\phi_x) = \sum_{k=0}^{r-1}\bar{\phi}_x\phi_x(\nabla_e(\bar{\phi}\phi))_{x+ke}. \tag{1.90}$$

This term, at scale $l > j$, i.e., if tested with fluctuation covariance C_l, has the effective size $O(rL^{(d-4[\phi]-1)l}) = O(L^{-(l-j)}L^{(d-4[\phi])l})$ which decreases exponentially in l (because r and therefore j remain fixed). This argument can be made for each term appearing in (1.76). Brydges and Slade developed a systematic treatment [35].

1.4.5 Dynamical systems

That the space of relevant and marginal spatially homogeneous local field polynomials has *finite dimension* (in dimension four and above), and that every term in (1.76) can be approximated by such a local field polynomial with a "contracting error," is the principal idea of Wilson's renormalization group. Wilson argues [113]

that the contractive terms should not influence the *critical behavior* of the model, determined by the evolution of the relevant and marginal terms, in our example, the three dimensional system $(g_j, z_j, \mu_j) \mapsto (g_{j+1}, z_{j+1}, \mu_{j+1})$.

There are numerous mathematical difficulties encountered when trying to justify this picture given by formal perturbation theory; these are discussed (in part) in Section 1.4.6. Formal perturbation theory suggests that there should be *coordinates* $x_j = (K_j, V_j)$ determining Z_j where $V_j = (g_j, z_j, \mu_j)$ is the three dimensional vector describing the marginal and relevant monomials of formal perturbation theory and K_j is an infinite-dimensional vector capturing all of the irrelevant directions. The evolution of V_j should approximately be given by a "localized" version of (1.76) as illustrated in (1.87), while K_j should be contractive in some sense.

In Chapter 3, the following abstract version of this set-up is considered. We assume that there is a sequence of Banach spaces $\mathbf{K}_j$ such that $K_j \in \mathbf{K}_j$, that the joint evolution of (K_j, V_j) is described by an evolution map

$$\Phi_j : \mathbf{K}_j \times \mathbb{R}^3 \mapsto \mathbf{K}_{j+1} \times \mathbb{R}^3 \tag{1.91}$$

of the form

$$\Phi_j(K_j, V_j) = (\psi_j(K_j, V_j), \bar{\varphi}_j(V_j) + \rho_j(K_j, V_j)) \tag{1.92}$$

with ψ_j and ρ_j *contractive* in K_j and *third-order* in V_j, and $\bar{\varphi}_j$ a *quadratic* polynomial of V_j. The quadratic polynomials $\bar{\varphi}_j$ describe the formal second-order perturbation theory of the relevant and marginal directions and therefore depend on V_j only; ρ_j describes higher-order contributions which can either be due to the relevant and marginal coordinates or due to the contracting directions. The maps Φ_j are allowed to have a *weak* scale-dependence. In addition, we assume that the $\bar{\varphi}$ do not have constant parts which allows for the interpretation $\Phi_j(0, 0) = (0, 0)$ such that $0 = (0, 0)$ can be considered a kind of *fixed point*[13] of the dynamical system $\Phi = (\Phi_j)_j$. This corresponds to the fact that the evolution of Z_j is trivial if $Z_0 = 1$.

The main interest is in the long-time behavior of this dynamical system, as this is related to the large distance behavior of the fields. For a dynamical system near a *hyperbolic* fixed point, the structure of the flows near the fixed point are well understood. A dynamical system $\Phi : X \to X$ on a Banach space X has a hyperbolic fixed point 0 if the spectrum of $D\Phi(0)$ is bounded away from 1. Informally stated, the *stable manifold theorem* [99, Theorem 6.1] asserts that if Φ is a hyperbolic C^r map (for some integer $r > 0$), then there exists a decomposition $X = X_s \oplus X_u$ such that, near 0, the domain of attraction $M \subset X$ is the graph of a C^r map $X_s \to X_u$, and that the convergence under iteration of Φ of points on M to 0 is exponentially

[13] 0 is a different element in every space $X_j = \mathbf{K}_j \times \mathbb{R}^3$, but we will neglect this point in the present discussion.

fast. This gives rise to a schematic phase portrait as shown in Figure 1.4. In the context of the renormalization group, the choice of V_0 on the stable manifold of a fixed point corresponds to a *critical model*, whose scaling limit is the same as that of the perturbed Gaussian measure. (This is known as *infrared asymptotic freedom* in the physics literature. It will be discussed again in Section 1.4.8.)

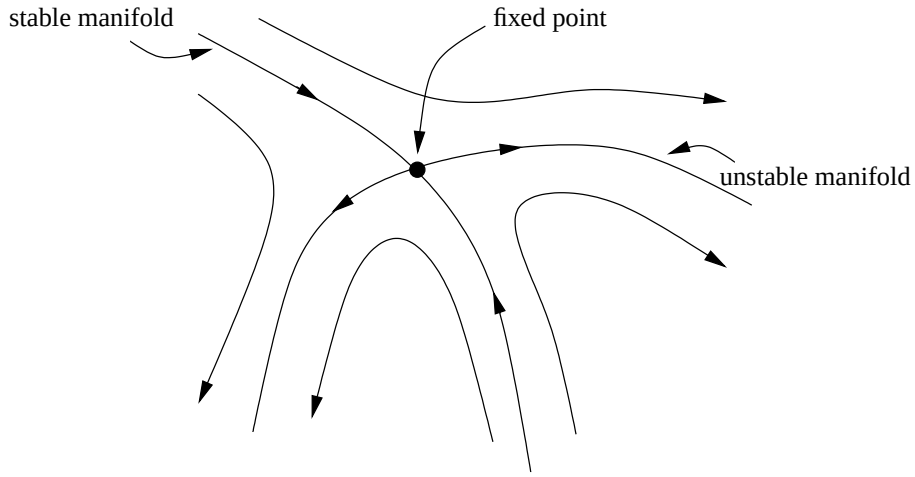

Figure 1.4: Schematic phase portrait of the renormalization group.

The "fixed point" of the dynamical system arising in the renormalization group analysis of the four dimensional weakly self-avoiding walk, outlined above, is *not hyperbolic*; the reason is that τ^2 is marginal. The analysis of the (local) long-time behavior of non-hyperbolic fixed points is more subtle than that of hyperbolic ones and depends on specific properties of the dynamical system. For example, a small change of the value of a single coefficient of the quadratic term $\bar{\varphi}$ above can change the long-time behavior in an important way; see e.g. Example 3.1.6.

In Chapter 3, we study dynamical systems of the form (1.92), and prove that, for the class of dynamical systems considered, an analog of a stable manifold theorem holds. The exponentially fast convergence along the stable trajectory of the stable manifold theorem is replaced in our result by a polynomial bound with logarithmic correction (which is likely optimal). Informally said, we show that, for sufficiently small V_0 and K_0, there is a codimension two manifold of (K_0, V_0) such that the solution to $(K_{j+1}, V_{j+1}) = \Phi_j(K_j, V_j)$ exists for all $j \in \mathbb{N}$ and is a perturbation of the solution to the analogous two dimensional manifold for the recursion $\bar{V}_{j+1} = \bar{\varphi}_j(\bar{V}_j)$ which can be studied by elementary means.

In Appendix A, we provide the explicit expression of the quadratic part, φ_{pt}, of

the dynamical system that arises in the renormalization group map for the weakly self-avoiding walk [10]. It is expressed in terms of the covariance decomposition of Chapter 2. It turns out that φ_{pt} is not exactly of the form of $\bar{\varphi}$ studied in Chapter 3. We provide an explicit transformation which expresses φ_{pt} in terms of a map $\bar{\varphi}$ to which the result of Chapter 3 can be applied.

1.4.6 The error coordinate and polymer gases

Finally, we provide some indication how the error coordinate K_j can be found. This is, of course, the major mathematical difficulty in implementing Wilson's program. In essence, this amounts to obtaining an approximate version of (1.76) with a *useful remainder estimate*.

This was first achieved for the ϕ^4 model, in a somewhat different formulation, by Gawedzki and Kupiainen [65–67]; this model has also been studied by different approaches, see e.g. [58]. An infamous difficulty, known as the *large field problem*, is that (1.76) can only be a good approximation when V and, thus ϕ, are small. This problem, in its simplest form, is already present in perturbations of the standard one-dimensional Gaussian measure. For example,

$$I(g) = \int_{\mathbb{R}} e^{-gt^4} e^{-\pi t^2}\, dt \tag{1.93}$$

is a singular function of g at $g = 0$ because e^{-gt^4} is not integrable for $g < 0$. Large fields turn out to cause difficulties for the applicability of certain expansion methods, but their probability is very small (in a large deviation sense). The solution of Gawedzki and Kupiainen to the large field problem involves a separate treatment of small and large fields, in which the small field contribution gives rise to similar effective action as the formal analysis of Section 1.4.4, while the large field contribution is very small. Brydges and Yau [22] developed a different solution in which no distinction between small and large fields has to be made, by use of well-chosen weights on the space of field functionals.

The main issue, however, is that the perturbations Z_j involve an unbounded number of variables (as $\Lambda \to \mathbb{Z}^d$) and that it is difficult to estimate the error to a formal approximation like (1.76) in a uniform way. This difficulty has historically been handled by *cluster expansions* [22, 65–67]. There, an important role is played by a *polymer gas*[14] which, informally said, can describe the irrelevant directions of the formal analysis. The previously mentioned references use covariance decompositions $C = C_1 + \cdots$ in which the C_j are only exponentially localized, rather than

[14] As a warning, we emphasize that the polymers that appear in the polymer gas are not the same kind of polymers as in Section 1.2.

finite range, discussed in Section 1.4.3. The use of the finite range property allows a simplified treatment without cluster expansion [25, 27, 90].

Polymer gases

The simplest version of a *polymer gas* is defined as follows; see [23, 68]. Let $\mathbf{P}_0(\Lambda)$ be the set of finite subsets of Λ; for later convenience, we include the empty set $\emptyset$ in $\mathbf{P}_0(\Lambda)$ although, at the moment, it would be more natural not to do so. Suppose that, for each *polymer* $Y \in \mathbf{P}_0(\Lambda)$, there is a weight $K(Y)$, called *polymer activity*. The partition function of the polymer gas with activity K is given by

$$\begin{aligned} Z &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\substack{Y_1,\dots,Y_N \in \mathbf{P}_0 \\ \text{disjoint}, Y_i \neq \emptyset}} K(Y_1) \cdots K(Y_N) \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{Y_1,\dots,Y_N \in \mathbf{P}_0} K(Y_1) \cdots K(Y_N) \prod_{i \neq j} e^{-v(Y_i, Y_j)}, \end{aligned} \tag{1.94}$$

with the hard core interaction

$$v(Y_i, Y_j) = \begin{cases} 0 & (Y_i \cap Y_j = \emptyset), \\ \infty & (\text{otherwise}). \end{cases} \tag{1.95}$$

$K(Y)$ appears in analogy to the activity of the "particle" at Y in the grand canonical partition function of a gas, which is why it is called polymer activity.

A simplification is the *connected polymer gas* with configuration space given by *connected* polymers $\mathbf{CP}_0 \subset \mathbf{P}_0$. This requires a notion of connected polymer with the property that each $Y \in \mathbf{P}_0$ has a unique disjoint decomposition $Y = Y_1 \cup \cdots \cup Y_N$ into connected polymers in $\mathbf{CP}_0$. The activities K can then naturally be extended from $\mathbf{CP}_0$ to $\mathbf{P}_0$ by

$$K(Y) = K(Y_1) \cdots K(Y_N) \tag{1.96}$$

with the convention $K(\emptyset) = 1$. We identify polymer activities defined on $\mathbf{CP}_0$ with such defined on $\mathbf{P}_0$ satisfying (1.96) and call them *connected polymer activities*. The partition function (1.94) with $\mathbf{P}_0$ replaced by $\mathbf{CP}_0$ then has the simple form

$$Z = 1 + \sum_{Y \in \mathbf{P}_0, Y \neq \emptyset} K(Y) = \sum_{Y \in \mathbf{P}_0} K(Y). \tag{1.97}$$

The first expression shows that "$Z \approx 1$" if K is "small," but observe that $\mathbf{P}_0$ has $2^{|\Lambda|}$ elements, so that, for the sum to be small, $K(X)$ must be very small for most X.

For the further development, it turns out convenient to introduce an algebraic structure on polymer activities, introduced in [22, 25]. Define a commutative and associative product on polymer activities by

$$(F \circ G)(X) = \sum_{Y \in \mathbf{P}_0(X)} F(Y)G(X \setminus Y) \tag{1.98}$$

where $\mathbf{P}_0(X)$ denotes the polymers contained in X. Let 1 denote the constant polymer activity given by $1(Y) = 1$ for all $Y \in \mathbf{P}_0$. Then the partition function of a connected polymer gas is given by

$$Z = (K \circ 1)(\Lambda). \tag{1.99}$$

1.4.7 Polymer representation

In the use of polymer gases to control the renormalization group, the polymer activities $K(X)$ are *local field functionals*. More precisely, the space of field functionals $\mathbf{N}$ is considered a commutative algebra with subalgebras $\mathbf{N}_0(Y) \subset \mathbf{N}$ of field functionals that only depend on the field in $Y \in \mathbf{P}_0$ and a "small neighborhood" of Y. The polymer activities are then *local* in the sense that $K(Y) \in \mathbf{N}_0(Y)$.

The simplest example is the *trivial polymer activity*, denoted $K = 1_\emptyset$, and defined by $1_\emptyset(X) = 1$ if $X = \emptyset$ and $1_\emptyset(X) = 0$ else. $1_\emptyset$ is the unit of the product $\circ$. The initial partition function can then be written as

$$Z_0 = I_0(\Lambda) = (1_\emptyset \circ I_0)(\Lambda) \tag{1.100}$$

where $I_0 : \mathbf{P}_0 \to \mathbf{N}$ is given by $I(X) = \prod_x e^{-V_{0,x}}$.

If the covariance decomposition has the finite range property, see Section 1.4.3, it turns out that all Z_j can be expressed in a similar way, but to obtain a useful representation, the class of polymers must be restricted to reflect the increasingly long range nature of the remaining fluctuation fields. More specifically, if Λ is a finite torus or cube of side length L^N for some integers L and N, let $\mathbf{B}_j(\Lambda)$ be a set of mutually *disjoint blocks* of side length L^j with the property that their union equals Λ. Let $\mathbf{P}_j(\Lambda)$ be the set of finite unions of blocks in $\mathbf{B}_j(\Lambda)$; these are called scale j polymers. Then everything discussed in the previous section about polymer gases has a straightforward scale j generalization, given by replacing $\mathbf{P}_0$ with $\mathbf{P}_j$, and $\mathbf{N}_0(Y)$ with $\mathbf{N}_j(Y)$ which are field functionals that are allowed to depend on Y and a small neighborhood of blocks in $\mathbf{B}_j$ near Y. In particular, the circle product $\circ$ then depends on j, although we will not emphasize this in the notation.

Brydges and Slade [25,37] show that, if the finite range decomposition is given in terms of the same parameter $L > 1$, then Z_j can be written as

$$Z_j = (K_j \circ I_j)(\Lambda) = I_j(\Lambda) + \sum_{X \in \mathbf{P}_j(\Lambda), X \neq \emptyset} K_j(X) I_j(\Lambda \setminus X) \tag{1.101}$$

where I_j and K_j and $\circ$ are defined on scale j polymers, I_j is to second order essentially given by (1.76), and K_j represents all of the higher order terms of formal perturbation theory in a rigorous fashion. To understand the significance that polymers must be at the correct length scale, observe that nth order terms of the formal approximation (1.76) have range $O(nL^j)$. This is easily understood by the example of the the second-order term (1.86). The polymer gas description becomes useful if it can be arranged in such a way that nth order terms correspond, approximately, to polymer activities $K(X)$ on polymers with $O(n)$ blocks so that $K(X)$ can be expected to be smaller and smaller when X is large. This compensates "loss of locality" by smallness.

Finite range property

To illustrate how the finite range property is helpful in obtaining the representation (1.101), we recall that the finite range property $[C_j]_{xy} = 0$ if $d(x, y) \geq cL^j$ has the consequence that, if $\phi_j = (\phi_{j,x})_x$ is a Gaussian field with such a covariance, then $\phi_{j,x}$ and $\phi_{j,y}$ are *independent* if $d(x, y) \geq cL^j$. In particular, if $Y_1, \dots, Y_N \in \mathbf{P}_j$ do not touch each other, then

$$E_j \prod_{i=1}^{N} K(Y_i) = \prod_{i=1}^{N} E_j K(Y_i). \tag{1.102}$$

Now suppose that a local field functional $I_{1,x} = I_{1,x}(\phi_2+\phi_3+\cdots)$, independent of the first fluctuation field, ϕ_1, is given in some way, and let $\delta I_{0,x} = I_{0,x} - I_{1,x}$ where $I_{0,x} = I_{0,x}(\phi_1 + \phi_2 + \cdots)$ does depend on ϕ_1. Then

$$Z_0 = I_0(\Lambda) = \prod_{x\in\Lambda} I_{0,x} = \prod_{x\in\Lambda} (I_{1,x} + \delta I_{0,x}) = (\delta I_0 \circ I_1)(\Lambda). \tag{1.103}$$

The expectation of Z_0 with respect to C_1 can be written as

$$Z_1 = E_1 Z_0 = \sum_{X\in\mathbf{P}_0(\Lambda)} \left(\prod_{x\in\Lambda\setminus X} I_{1,x} \right) E_1 \left(\prod_{x\in X} \delta I_{0,x} \right) = (\tilde{K}_1 \circ I_1)(\Lambda), \tag{1.104}$$

where

$$I_1(X) = \prod_{x\in X} I_{1,x}, \quad \tilde{K}_1(X) = E_1 \left(\prod_{x\in X} \delta I_{0,x} \right) \tag{1.105}$$

and the product $\circ$ on scale 0. However, $\tilde{K}_1(X)$ depends on the field in a neighborhood of X of range $O(L^1)$ (or more generally $O(L^j)$ at scale j). To write (1.104)

in terms of scale 1 polymers, one can restrict I_1 to $\mathbf{B}_1$ and "coarsen" $\tilde{K}_1$ by setting

$$K_1(U) = \sum_{X \in \mathbf{P}_0(U): \overline{X} = U} \left(\prod_{x \in U \setminus X} I_{1,x} \right) E_1 \left(\prod_{x \in X} \delta I_{0,x} \right) \tag{1.106}$$

for $U \in \mathbf{P}_1$, where the closure $\overline{X} \in \mathbf{P}_{j+1}$ of a polymer $X \in \mathbf{P}_j$ is the smallest scale $j+1$ polymer such that $\overline{X} \supseteq X$. The finite range property of E_1 implies that $K_1(U)$ only depends on the field in U and in $\mathbf{B}_1$-blocks touching U; the appropriate choice of $\mathbf{N}_j$ is such that $K_1(U) \in \mathbf{N}_1(U)$.

The representation $Z_j = K_j \circ I_j$ is *far from unique*. There are many choices of K_j and I_j that satisfy $Z_j = K_j \circ I_j$. It is crucial to choose the I_j correctly to capture the important directions, and the K_j such that K_{j+1} contracts compared to K_j in an appropriate norm. The details of this are quite delicate [25, 37]. The representation $Z_j = K_j \circ I_j$ bridges between the representations as an effective action, i.e., as an exponential and as a polymer gas. It resembles the expression $e^{-V_j} + K_j$ sufficiently well to serve as a replacement, but gives at the same time the flexibility to measure the non-locality of the error.

1.4.8 Conclusion

The renormalization group, in the sense sketched in the previous subsections, can provide a complete description of the evolution of a local perturbation of a Gaussian field, $Z_{j+1} = E_{j+1} Z_j$, induced by a finite range decomposition of its covariance

$$C = C_1 + C_2 + \cdots , \tag{1.107}$$

in terms of tractable coordinates $x_j = (K_j, V_j)$ defining field functionals $\hat{Z}_j(K_j, V_j)$ such that, with $V_j = (g_j, \mu_j, z_j)$,

$$E_j \cdots E_1 Z_0(V_0) = \hat{Z}_j(K_j, V_j) \approx \prod_{x \in \Lambda} e^{-g_j \tau_x^2 - \mu_j \tau_x - z_j \tau_{\Delta,x}} . \tag{1.108}$$

The coordinates x_j lie on the trajectory of a dynamical system Φ,

$$\Phi_j(K_j, V_j) = (K_{j+1}, V_{j+1}). \tag{1.109}$$

The long-time properties of the dynamical system Φ can be used to establish properties of the large distance behavior of the fields. For example, if V_0 is chosen carefully, the flow V_j converges to the fixed point 0; this choice describes *critical models*. The phenomenon $V_j \to 0$ is called *infrared asymptotic freedom*. The term *infrared* means that it concerns the large distance (short "wavelength") limit,

while *freedom* refers to the fact that $V = 0$ describes a free field. Together with detailed estimates on K_j, guaranteeing that its contribution is sufficiently small, the convergence $V_j \to 0$ can, for example, be used to prove that the critical model has the same scaling limit as the perturbed Gaussian field (in an appropriate sense). In addition, the trajectories of Φ close to the critical V_0 reveal information about the approach of the critical point, again with appropriate (non-trivial) estimates on the remainder part K_j.

In the next two chapters, two aspects of this program are studied in detail, the decomposition of Gaussian fields and the analysis of a class of dynamical systems that arises in the renormalization group study of the weakly self-avoiding walk. As mentioned, we provide some explicit details of the connection between the results of Chapter 2 and Chapter 3 in Appendix A.

Chapter 2

Decomposition of free fields

2.1 Introduction and main result

2.1.1 The Newtonian potential

Let us place the result of this chapter into context via an example. Consider the Newtonian potential, the Green function of the Laplace operator on $\mathbb{R}^d$ given by

$$\Phi(x) = C_d \begin{cases} |x|^{-(d-2)} & (d \geq 3) \\ \log 1/|x| & (d = 2) \end{cases} \quad \text{for all } x \in \mathbb{R}^d,\ x \neq 0. \tag{2.1}$$

For $d \geq 3$ and *any* measurable function $\varphi : [0, \infty) \to \mathbb{R}$ such that $t^{d-3}\varphi(t)$ is integrable, the Newtonian potential can be written, up to a constant, as

$$|x|^{-(d-2)} = \int_0^\infty t^{-(d-2)}\, \varphi(|x|/t)\, \frac{dt}{t} \quad \text{for all } x \in \mathbb{R}^d,\ x \neq 0. \tag{2.2}$$

This is true because both sides are radially symmetric and homogeneous of degree $-(d-2)$, where homogeneity of the right-hand side simply follows from the change of variables formula. In particular, φ can be chosen smooth with compact support and such that $\varphi(|x|)$ is a positive semi-definite function on $\mathbb{R}^d$. The last condition means that $\varphi(|x|)$ is positive as a quadratic form: for any $f \in C_c^\infty(\mathbb{R}^d)$, that is, $f : \mathbb{R}^d \to \mathbb{R}$ smooth with compact support,

$$\Phi_t(f, f) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(|x - y|/t)\, f(x)\, f(y)\, dx\, dy \geq 0. \tag{2.3}$$

Similarly, if $d = 2$, and $\varphi : [0, \infty) \to \mathbb{R}$ is any absolutely continuous function with $\varphi(0) = 1$ and such that $\varphi'(t)$ is integrable, then

$$\log 1/|x| = \int_0^\infty (\varphi(|x|/t) - \varphi(1/t))\, \frac{dt}{t} \quad \text{for all } x \in \mathbb{R}^2,\ x \neq 0. \tag{2.4}$$

Indeed, for $x \neq 0$,

$$\begin{aligned} \log 1/|x| = \varphi(0) \log 1/|x| &= -\int_0^\infty \varphi'(s) \log 1/|x|\, ds \\ &= \int_0^\infty \varphi'(s) \int_{s/|x|}^s \frac{dt}{t}\, ds, \end{aligned} \tag{2.5}$$

and thus, since φ' is integrable, by Fubini's theorem,

$$\log 1/|x| = \int_0^\infty \int_t^{t|x|} \varphi'(s)\, ds\, \frac{dt}{t} = \int_0^\infty (\varphi(t|x|) - \varphi(t))\, \frac{dt}{t}, \tag{2.6}$$

showing (2.4) after the change of variables $t \mapsto 1/t$. Now suppose again that φ is chosen such that $\varphi(|x|)$ is a positive semi-definite function on $\mathbb{R}^2$. Then the function $\mathbb{R}^2 \ni x \mapsto \varphi(|x|/t) - \varphi(1/t)$ is positive as a quadratic form on the domain of smooth and compactly supported functions with vanishing integral:

$$\begin{aligned} \Phi_t(f, f) &:= \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\varphi(|x-y|/t) - \varphi(1/t))\, f(x) f(y)\, dx\, dy \\ &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi(|x-y|/t)\; f(x) f(y)\, dx\, dy \geq 0 \end{aligned} \tag{2.7}$$

for all $f \in C_c^\infty(\mathbb{R}^2)$ with $\int f\, dx = 0$.

The above shows that the Newtonian potentials (2.1) can be decomposed into integrals of compactly supported and positive semi-definite functions, with the appropriate restriction of the domain for $d = 2$.

Let us recall at this point that the positivity of a quadratic form has the important implication that it entails the existence of a corresponding Gaussian process, discussed briefly in Section 2.1.4. However, it is also of interest in mathematical physics for different reasons [71].

2.1.2 Finite range decompositions of quadratic forms

It is an open problem to characterize the class of positive quadratic forms, $S : D(S) \times D(S) \to \mathbb{R}$, that admit decompositions into integrals (or sums) of positive quadratic forms of finite range: for all $f, g \in D(S)$, $t > 0$,

$$\begin{cases} S(f,g) = \displaystyle\int_0^\infty S_t(f,g)\, \frac{dt}{t}, \\ S_t : D(S) \times D(S) \to \mathbb{R}, \\ S_t(f, f) \geq 0, \\ S_t(f,g) = 0 \;\text{ if } d(\mathrm{supp}(f), \mathrm{supp}(g)) > \theta(t), \end{cases} \tag{2.8}$$

where $\theta : (0,\infty) \to (0,\infty)$ is increasing and d is a distance function. The condition of *finite range*, the last condition in (2.8), generalizes the property of compact support of the function φ in (2.3) to quadratic forms that are not defined by a convolution kernel. The difficulty in decomposing quadratic forms in such a way is to achieve the two conditions of positivity and finite range simultaneously. Note

that by splitting up the integral, one can obtain a decomposition into a sum from (2.8), and conversely, a decomposition into a sum can be written as an integral (without regularity in t).

For applications, not only the existence, but also the regularity of the decomposition (2.8) is important. Let (X, μ) be a metric measure space, i.e., a locally compact complete separable metric space X with a Radon measure μ on X with full support (i.e., μ is strictly positive), $C_c(X)$ the space of continuous functions on X with compact support, and $C_b(X)$ the space of bounded and continuous functions on X. Let us say that the decomposition (2.8) is regular if $C_c(X) \cap D(S)$ is S-dense in $D(S)$ and if every S_t has a bounded continuous kernel $s_t \in C_b(X \times X)$:

$$S_t(f,g) = \int s_t(x,y) f(x) g(y)\, d\mu(x)\, d\mu(y) \quad \text{for all } f, g \in C_c(X) \cap D(S). \tag{2.9}$$

For the decompositions (2.2), (2.4), the kernels are of course given in terms of the smooth function φ by the explicit formula

$$\phi_t(x,y) = t^{-(d-2)} \varphi(|x-y|/t) \quad \text{for all } x, y \in \mathbb{R}^d,\ t > 0. \tag{2.10}$$

Note that for $d = 2$ the second term in (2.4) could be omitted by (2.7), with the understanding that the quadratic form is restricted to functions with vanishing integral. It follows in particular that

$$|\phi_t(x,y)| \le C t^{-(d-2)} \quad \text{uniformly in all } x, y \in \mathbb{R}^d. \tag{2.11}$$

This reflects the decay of the Newtonian potential. Moreover, for all integers $l_x, l_y \ge 0$, the derivatives of the kernel ϕ_t decay according to

$$|D_x^{l_x} D_y^{l_y} \phi_t(x,y)| \le C_l t^{-(d-2)} t^{-l_x - l_y}, \tag{2.12}$$

reflecting that $|D^l \Phi(x)| \le C_l |x|^{-(d-2-l)}$ for all $x \in \mathbb{R}^d$, $x \neq 0$.

The main result of this chapter is a rather simple construction of decompositions (2.8) with estimates like (2.11) for quadratic forms that arise by duality with Dirichlet forms in a large class. We call such forms *Green forms*, motivated by the Newtonian potential, or Green function, that is a special case; this is explained in Section 2.1.3.

The main idea of our method is that (2.8) can be achieved by applying formulae like (2.2) to the spectral representation of the Green form, and then exploiting finite propagation speed properties of appropriate wave flows. These are generalizations of the fact that if $u(t,x)$ is a solution to

$$\partial_t^2 u - \Delta u = 0, \quad u(0,x) = u_0(x),\ \partial_t u(0,x) = 0 \tag{2.13}$$

with compactly supported initial data u_0 that then

$$\operatorname{supp}(u(t,\cdot)) \subseteq N_t(\operatorname{supp}(u_0)) \tag{2.14}$$

where $N_t(U) = \{x \in X : d(x,U) \le t\}$ for any $U \subset X$.

The idea of exploiting properties of the wave equation in the context of probability theory is not new. For example, Varopoulos [112] used the finite propagation speed of the wave equation to obtain Gaussian bounds on the heat kernel of general Markov chains, by decomposing it into an integral over compactly supported parts. Our objective is slightly different in that we are interested in the constraint of positive definite decompositions.

Decompositions of singular functions into sums or integrals of smooth and compactly supported functions have a history in analysis. For example, Fefferman's celebrated proof of pointwise almost everywhere convergence of the Fourier series [56] uses a decomposition of $1/x$ on $\mathbb{R}$ like (2.2), albeit without using positive semi-definiteness. Hainzl and Seiringer [71], motivated by applications to quantum mechanics such as [57], decompose general radially symmetric functions, without assuming a priori that they are positive definite, into weighted integrals over *tent functions*. These, like $\varphi(|x|)$ in (2.2), are positive semi-definite. They state sufficient conditions for the weight to be non-negative, and thus obtain decompositions like (2.2) for a class of radially symmetric potentials including $e^{-m|x|}/|x|$ on $\mathbb{R}^3$. Special cases and similar results have also appeared in earlier works of Pólya [94] and of Gneiting [69, 70].

These results, like (2.2), make essential use of radial symmetry. One example of particular interest for probability theory—where radial symmetry is not given—is the Green function of the discrete Laplace operator:

$$\Delta_{\mathbb{Z}^d} u(x) = \sum_{e \in \mathbb{Z}^d : |e|_1 = 1} (u(x+e) - u(x)) \quad \text{for any } u : \mathbb{Z}^d \to \mathbb{R},\ x \in \mathbb{Z}^d. \tag{2.15}$$

Brydges, Guadagni, and Mitter [27] showed that also in this discrete case, the corresponding Green function, or more generally the resolvent, admits a decomposition like (2.8) into a sum (instead of an integral) of positive semi-definite lattice functions with estimates analogous to (2.12). Brydges and Talarczyck [21] gave a related construction which applies to quite general elliptic operators on domains in $\mathbb{R}^d$, but estimates on the kernels of this decomposition are only known when the coefficients are constant. Their construction was adapted by Adams, Kotecký, and Müller [4] to show that the Green functions of constant coefficient discrete elliptic systems on $\mathbb{Z}^d$ admit decompositions with estimates analogous to (2.12) and that the decomposition obtained this way is analytic as a function of the (constant) coefficients. These results are based on constructions that average Poisson kernels.

Our method, sketched earlier, is different from that of [4, 21, 27, 31] and yields simpler proofs of their results about constant coefficient elliptic operators—both in discrete and continuous context. It furthermore naturally yields a decomposition into an integral instead of a sum (with integrand smooth in t), and gives effective estimates for decompositions of Green functions of variable coefficient operators.

2.1.3 Duality and spectral representation of the Green form

Let us now introduce the general set-up in which our result is framed more precisely. For motivation, we first return to the quadratic forms defined by the Newtonian potentials (2.1):

$$\Phi(f,g) := \int_{\mathbb{R}^d\times\mathbb{R}^d} \Phi(x-y)\,f(x)g(y)\,dx\,dy, \qquad f,g\in D(\Phi) \tag{2.16}$$

where

$$\begin{cases} D(\Phi) = C_c^\infty(\mathbb{R}^d) & (d\geq 3)\\ D(\Phi) = \{f\in C_c^\infty(\mathbb{R}^2) : \int_{\mathbb{R}^2} f\,dx = 0\} & (d=2). \end{cases} \tag{2.17}$$

These quadratic forms are not bounded on $L^2(\mathbb{R}^d)$, as is most apparent when $d=2$. They are closely related to the Dirichlet forms given by

$$\mathcal{E}(u,v) := \int_{\mathbb{R}^d} \nabla u\cdot\nabla v\,dx, \qquad u,v\in C_c^\infty(\mathbb{R}^d). \tag{2.18}$$

The correspondence between the two is *duality*: for all $f\in D(\Phi)$,

$$\sqrt{\Phi(f,f)} = \sup\left\{\int_{\mathbb{R}^d} fu\,dx : u\in C_c^\infty(\mathbb{R}^d), \mathcal{E}(u,u)\leq 1\right\}. \tag{2.19}$$

This set-up admits the following natural generalization: Let (X,μ) be a metric measure space and $L^2(X)$ be the Hilbert space of equivalence classes of real-valued square μ-integrable functions on X with inner product $(u,v) = (u,v)_{L^2}$. Let $\mathcal{E} : D(\mathcal{E})\times D(\mathcal{E})\to\mathbb{R}$ be a closed positive quadratic form on $L^2(X)$ with $D(\mathcal{E})\subseteq L^2(X)$ a dense linear subspace. It is sometimes convenient to assume that $\mathcal{E}$ is regular, i.e., that $C_c(X)\cap D(\mathcal{E})$ is $\mathcal{E}$-dense in $D(\mathcal{E})$. That $\mathcal{E}$ is closed means that $D(\mathcal{E})$ is a Hilbert space with inner product $\mathcal{E}(u,v)+m^2(u,v)_{L^2}$ for any $m^2>0$. For the example (2.18), the domain of the form closure $D(\mathcal{E})$ of $C_c^\infty(\mathbb{R}^d)$ is the usual Sobolev space $H^1(\mathbb{R}^d)$ and $(u,v)_{H^1} = \mathcal{E}(u,v)+(u,v)_{L^2}$ is the usual Sobolev inner product.

It follows [96] from closedness that $\mathcal{E}$ is the quadratic form associated to a unique self-adjoint operator $L : D(L)\to L^2(X)$,

$$\mathcal{E}(u,v) = (u,Lv) \quad \text{for } u\in D(\mathcal{E}),\ v\in D(L), \tag{2.20}$$

where $D(L) \subseteq D(\mathcal{E})$ is a dense linear subspace in $L^2(X)$. The self-adjointness of L gives rise to a spectral family and functional calculus. This means in particular that for any Borel measurable $F : [0, \infty) \to \mathbb{R}$, there is a self-adjoint operator, denoted $F(L) : D(F(L)) \to L^2(X)$, where

$$F(L) := \int_0^\infty F(\lambda)\, dP_\lambda, \tag{2.21}$$

$$D(F(L)) := \left\{ u \in L^2(X) : \int_0^\infty F(\lambda)^2\, d(u, P_\lambda u) < \infty \right\} \tag{2.22}$$

with P_λ the spectral family associated to L, and $(u, P_\lambda u)$ is the spectral measure associated to L and $u \in L^2(X)$. In these terms, $\mathcal{E}$ has the representation

$$\mathcal{E}(u, u) = \|L^{\frac{1}{2}} u\|_{L^2(X)} = \int_{\mathrm{spec}(L)} \lambda\, d(u, P_\lambda u), \quad u \in D(\mathcal{E}) = D(L^{\frac{1}{2}}), \tag{2.23}$$

where $\mathcal{E}(u, v)$ is defined by the polarization identity, if $u \neq v$. Similarly, the corresponding Green form can be defined by polarization and

$$\Phi(f, f) = \|L^{-\frac{1}{2}} f\|_{L^2(X)} = \int_{\mathrm{spec}(L)} \lambda^{-1}\, d(u, P_\lambda u), \quad f \in D(\Phi) = D(L^{-\frac{1}{2}}). \tag{2.24}$$

This representation will be our starting point for the decomposition of the Green form. Before stating the result and its proof, let us sketch how the decomposition problem arises in probability theory.

2.1.4 Gaussian fields and statistical mechanics

The linear space $D(\mathcal{E})$ is complete under the metric induced by the inner product $\mathcal{E}(u, v) + m^2(u, v)_{L^2}$ for any $m^2 > 0$, but it is generally not complete for $m^2 = 0$. It may however be completed to a Hilbert space abstractly; we denote this Hilbert space by $(H_\mathcal{E}, (\cdot, \cdot)_\mathcal{E})$. Similarly, we can complete the domain $D(\Phi)$ to a Hilbert space under the quadratic form Φ; this Hilbert space is denoted by $(H_\Phi, (\cdot, \cdot)_\Phi)$. $H_\mathcal{E}$ and H_Φ are dual in the following sense: The L^2 inner product can be restricted to

$$\langle \cdot, \cdot \rangle : D(\Phi) \times D(\mathcal{E}) \to \mathbb{R}, \quad \langle f, u \rangle = (f, u) = (L^{-\frac{1}{2}} f, L^{\frac{1}{2}} u) \tag{2.25}$$

which extends to a bounded bilinear form on $H_\Phi \times H_\mathcal{E}$. L acts by definition isometric from $D(\mathcal{E})$ to $D(\Phi)$, with respect to the norms of $H_\mathcal{E}$ and H_Φ, and it extends to an isometric isometry from $H_\mathcal{E}$ to H_Φ. Thus H_Φ is identified with the dual space of $H_\mathcal{E}$ naturally, via the extension of the L^2 pairing $\langle \cdot, \cdot \rangle$.

Remark 2.1.1. To give some insight into the interpretation of the spaces $H_\mathcal{E}$ and H_Φ, let us mention how $H_\mathcal{E}$ can be characterized in the case of the Newtonian potential [40]:

$$H_\mathcal{E} \cong \{f : \mathbb{R}^d \to \mathbb{R} \text{ measurable :} \\ \text{there exists an } \mathcal{E}\text{-Cauchy sequence } f_n \in D(\mathcal{E}) \text{ with } f_n \to f \text{ a.e.}\}/\sim_d \quad (2.26)$$

where $\sim_d$ is the usual identification of functions that are equal almost everywhere when $d \geq 3$. For $d = 2$, $\sim_d$ in contrast identifies functions that may differ by a constant almost everywhere. (It is therefore sometimes said that the massless free field does not exist in two dimensions, but that its gradient does. The massless free field is the free field corresponding to Φ in the terminology explained below.) To understand this distinction, take a smooth cut-off function φ_1 on $\mathbb{R}^2$, e.g. with $\varphi_1 \equiv 1$ on $B_1(0)$ and $\varphi_1 \equiv 0$ on $B_2(0)^c$, set $\varphi_n(x) = \varphi_1(x/n)$, and note that $\mathcal{E}(\varphi_n, \varphi_n) = n^{d-2}\mathcal{E}(\varphi_1, \varphi_1)$. Thus, (φ_n) is bounded in $H_\mathcal{E}$ whenever $d \leq 2$, and then (by the Banach-Alaoglu theorem) there is $\psi \in H_\mathcal{E}$ such that $\varphi_n \to \psi$ weakly along a subsequence in $H_\mathcal{E}$; however, $\varphi_n \to 1$ pointwise, so that $\psi \equiv 1 \in H_\mathcal{E}$. Now $\mathcal{E}(1, 1) = 0$ implies that the constant functions must be in the same equivalence class as the zero function.

It is well-known that any separable real Hilbert space $(H, (\cdot, \cdot)_H)$ defines a Gaussian process indexed by H [105]. This is a probability space (Ω, P) and a unitary map $f \in H \to \langle f, \phi\rangle \in L^2(P)$ such that the random variables $\langle f, \phi\rangle$ are Gaussian with variance $(f, f)_H$. Note that $\langle f, \phi\rangle$ is merely a symbolic notation for the random variable on $L^2(P)$ that corresponds to $f \in H$. It cannot in general be interpreted as the pairing of $f \in H$ with a random element $\phi(\omega) \in H$ defined for $\omega \in \Omega$; see e.g. [101]. In particular, if $(H, (\cdot, \cdot)_H)$ is the Hilbert space $(H_\Phi, (\cdot, \cdot)_{H_\Phi})$, this process is called the *free field* or the *Gaussian free field* (corresponding to Dirichlet form $\mathcal{E}$ or Green function Φ).

This is a generalization of the context introduced in Section 1.3.2 where X is a countable set and $\delta_x \in H$, $(x \in X)$ so that the field $\phi_x = \langle \delta_x, \phi\rangle$ has a pointwise interpretation.

2.1.5 Main result

Let (X, μ) be a metric measure space. In addition, suppose that $d : X \times X \to [0, \infty]$ is an extended pseudometric on X. (*Extended* means that $d(x, y)$ may be infinite and *pseudo* that $d(x, y) = 0$ for $x \neq y$ is allowed. Example 2.1.4 below gives an example of interest where d is not the metric of X.)

Let $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ be a regular closed symmetric form on $L^2(X)$ as in Section 2.1.3 and denote by $L : D(L) \to L^2(X)$ its self-adjoint generator.

Theorem 2.1.2 assumes that $(X, \mu, d, \mathcal{E})$ satisfies one of the following two *finite propagation speed* conditions that we now introduce: For $\gamma > 0$, $B > 0$, and an increasing function $\theta : (0, \infty) \to (0, \infty)$, let us say that $(X, \mu, d, \mathcal{E})$ satisfies $(P_{\gamma,\theta})$ respectively $(P^*_{\theta,B})$ if:

$$\operatorname{supp}(\cos(L^{\frac{1}{2}\gamma}t)u) \subseteq N_{\theta(t)}(\operatorname{supp}(u)) \quad \text{for all } u \in C_c(X),\ t > 0, \tag{$P_{\gamma,\theta}$}$$

respectively

$$\begin{aligned} &\mathcal{E}(u,u) \le B\|u\|_{L^2(X)} \quad \text{for all } u \in L^2(X), \\ &\operatorname{supp}(L^n u) \subseteq N_{\theta(n)}(\operatorname{supp}(u)) \quad \text{for all } u \in C_c(X),\ n \in \mathbb{N}, \end{aligned} \tag{$P^*_{\theta,B}$}$$

where as before $N_t(U) = \{x \in X : d(x, U) \le t\}$ for any $U \subset X$. The left-hand side of $(P_{\gamma,\theta})$ is defined in terms of functional calculus for the self-adjoint operator L.

Note that if $L = -\Delta_{\mathbb{R}^d} = -\sum_{i=1}^d \partial^2_{x_i}$ is the standard Laplace operator of $\mathbb{R}^d$, then $u(t, x) = [\cos(L^{\frac{1}{2}}t)u_0](x)$ is a solution to the standard wave equation (2.13), and the condition $(P_{\gamma,\theta})$ with $\gamma = 1$ and $\theta(t) = t$ is the finite propagation speed property (2.14). The property holds for more general elliptic operators and elliptic systems (not necessarily of second order), however; see Example 2.1.4 below. Similarly, if $L = -\Delta_{\mathbb{Z}^d}$ is the discrete Laplace operator (2.15), then $(P^*_{\theta,B})$ holds with $B = 2d$ and $\theta(n) = n$, since $Lu(x)$ only depends on $u(y)$ when x and y are nearest neighbors. As for the property $(P_{\gamma,\theta})$, the condition $(P^*_{\theta,B})$ remains true for more general discrete Dirichlet forms; see Examples 2.1.4–2.1.5.

Let us introduce a further condition: The heat kernel bound $(H_{\alpha,\omega})$ holds when the heat semigroup $(e^{-tL})_{t>0}$ has continuous kernels p_t for all $t > 0$ and there is $\alpha > 0$ and a bounded function $\omega : X \to \mathbb{R}_+$ such that

$$p_t(x, x) \le \omega(x)t^{-\alpha/2} \quad \text{for all } x \in X. \tag{$H_{\alpha,\omega}$}$$

Criteria for $(H_{\alpha,\omega})$ are classic; see e.g. [91] for second-order elliptic operators and also the discussion in the examples below.

Theorem 2.1.2. *Suppose $(X, \mu, d, \mathcal{E})$ satisfies $(P_{\gamma,\theta})$ or $(P^*_{\theta,B})$. Then the corresponding Green form* (2.24) *admits a finite range decomposition* (2.8) *with $S = \Phi$ and $S_t = \Phi_t$ such that the Φ_t are bounded quadratic forms with*

$$|\Phi_t(f, g)| \le C_{\gamma,B}t^{2/\gamma}\|f\|_{L^2(X)}\|g\|_{L^2(X)} \quad \textit{for all } f, g \in L^2(X). \tag{2.27}$$

Moreover, $(H_{\alpha,\omega})$ implies that the Φ_t have continuous kernels ϕ_t that satisfy

$$|\phi_t(x, y)| \le C_{\alpha,\gamma,B}\sqrt{\omega(x)\omega(y)}t^{-(\alpha-2)/\gamma}. \tag{2.28}$$

2.1.6 Examples

Example 2.1.3 (Elliptic operators with constant coefficients). Let $a = (a_{ij})_{1\le i,j\le d}$ be a strictly positive definite matrix in $\mathbb{R}^{d\times d}$ and

$$\mathcal{E}_a(u,v) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} (D_i u(x)) a_{ij} (D_j v(x))\, dx, \qquad u,v \in C_c^\infty(\mathbb{R}^d), \tag{2.29}$$

$$\mathcal{E}_a^*(u,v) = \sum_{i,j=1}^{d} \sum_{x\in\mathbb{Z}^d} (\nabla_i u(x)) a_{ij} (\nabla_j v(x)), \qquad u,v \in C_c(\mathbb{Z}^d), \tag{2.30}$$

where $D_i u(x)$ is the partial derivative of $u(x)$ in direction $i = 1,\dots,d$,

$$\nabla_i u(x) = u(x+e_i) - u(x) \tag{2.31}$$

with e_i the unit vector in the positive ith direction, and $C_c(\mathbb{Z}^d)$ is the space of functions $u : \mathbb{Z}^d \to \mathbb{R}$ with finite support. For $m^2 \ge 0$, further set

$$\mathcal{E}_{a,m^2}(u,v) = \mathcal{E}_a(u,v) + m^2 \int_{\mathbb{R}^d} u(x)v(x)\, dx \tag{2.32}$$

and define $\mathcal{E}^*_{a,m^2}$ analogously. Assume that the eigenvalues of a are contained in the interval $[B_-^2, B_+^2]$, and in the discrete case also that $m^2 \in [0, M_+^2]$ for some $B_-^2, B_+^2, M_+^2 > 0$; these assumptions are only important for uniformity in the constants below.

In the continuous context, let d be the Euclidean distance on $X = \mathbb{R}^d$ and μ be the Lebesgue measure. It follows that $(X,\mu,d,\mathcal{E})$ satisfies $(P_{\gamma,\theta})$ with $\gamma = 1$, $\theta(t) = B_+ t$; see Example 2.1.4 for more details. In the discrete context, let d be the infinity distance on $X = \mathbb{Z}^d$, i.e., $d(x,y) = \max_{i=1,\dots d} |x_i - y_i|$, and μ be the counting measure. Then $(P^*_{\theta,B})$ holds with $B = B_+ + M_+^2$ and $\theta(n) = n$.

Theorem 2.1.2 implies that the Green functions associated to $\mathcal{E}_{a,m^2}$ and $\mathcal{E}^*_{a,m^2}$ admit finite range decompositions. We denote their kernels by $\phi_t(x,y;a,m^2)$ and $\phi_t^*(x,y;a,m^2)$. In addition to (2.28), it is not difficult to obtain estimates on the decay of the derivatives of ϕ_t and ϕ_t^*, like (2.12), in this situation of constant coefficients. Since these estimates are of interest for applications, we provide the details in Section 2.3.2 (in a slightly more general context). We show that there are constants $C_{l,k} > 0$ depending only on B_- and B_+, and in the discrete case also on M_+, such that

$$|D_a^{l_a} D_{m^2}^{l_{m^2}} D_y^{l_y} D_x^{l_x} \phi_t(x,y;a,m^2)| \le C_{l,k} t^{-(d-2)-l_x-l_y+2l_{m^2}} (1+m^2t^2)^{-k} \tag{2.33}$$

and

$$|D_a^{l_a} D_{m^2}^{l_{m^2}} \nabla_y^{l_y} \nabla_x^{l_x} \phi_t^*(x,y,t;a,m^2)| \le C_{l,k} t^{-(d-2)-l_x-l_y+2l_{m^2}} (1+m^2t^2)^{-k} \tag{2.34}$$

for all integers l_a, l_{m^2}, l_x, l_y, and k such that

$$l_{m^2} < \tfrac{1}{2}(d + l_x + l_y), \tag{2.35}$$

and that the following approximation result holds: There is $c > 0$ such that

$$\nabla_x^{l_x}\nabla_y^{l_y}\phi_t^*(x,y;a,m^2) = c^{d-2}D_x^{l_x}D_y^{l_y}\phi_t(cx,cy;a,c^{-2}m^2) + O(t^{-(d-2)-l_x-l_y-1}(1+m^2t^2)^{-k}). \tag{2.36}$$

This reproduces and generalizes many results of [4, 27]. More precisely, we verify that there exists a smooth function $\bar{\phi} : \mathbb{R}^d \times [B_-^2, B_+^2] \times [0,\infty) \to \mathbb{R}$ supported in $|x| \le B_+$ such that

$$\phi_t(x,y;a,m^2) = t^{-(d-2)}\bar{\phi}\left(\frac{x-y}{t};a,m^2t^2\right) \tag{2.37}$$

which has the same structure as (2.10) when $m^2 = 0$; this is scale invariance. Moreover, by (2.36), the discrete Green function has a scaling limit and the error is of the order of the rescaled lattice spacing $O(t^{-1})$. This result improves [31].

Example 2.1.4 (Elliptic operators and systems with variable coefficients). Let $M \in \mathbb{N}$ and $a_{ij} : \mathbb{R}^d \to \mathbb{R}^{M\times M}$, $i,j = 1,\dots,d$, be the smooth coefficients of a uniformly elliptic system (or in particular, if $M = 1$, of a uniformly elliptic operator):

$$B_-^2|\xi|^2 \le \sum_{k,l=1}^{M}\sum_{i,j=1}^{d} a_{ij}^{kl}(x)\xi_i^k\xi_j^l \le B_+^2|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^{dM},\ x \in \mathbb{R}^d, \tag{2.38}$$

with $B_-, B_+ > 0$. Let us write $\mathbf{u} = (\mathbf{u}^1,\dots,\mathbf{u}^M) \in \mathbb{R}^{dM}$ with $\mathbf{u}^i \in \mathbb{R}^d$, $i = 1,\dots,M$. Let

$$\mathcal{E}(\mathbf{u},\mathbf{v}) = \sum_{i,j=1}^{d}\int_{\mathbb{R}^d}(D_i\mathbf{u}^k(x))a_{ij}^{kl}(x)(D_j\mathbf{u}^l(x))\,dx, \quad \mathbf{u},\mathbf{v} \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^M) \tag{2.39}$$

and analogously in the discrete case (as in (2.29), (2.30)).

To apply Theorem 2.1.2, (X,μ,d) is defined by $X = \mathbb{R}^d \times \{1,\dots,M\}$, μ is the product of the Lebesgue measure on $\mathbb{R}^d$ and the counting measure on $\{1,\dots,M\}$, and the distance is given by $d((x,i),(y,j)) = d(x,y)$. In particular, d is only a pseudometric on X. We may use the identification of $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^M$ and $u : X \to \mathbb{R}$ by $u(x,i) = \mathbf{u}^i(x)$.

It suffices to verify the condition (P_{1,B_+t}) for smooth, compactly supported $\mathbf{u}_0 : \mathbb{R}^d \to \mathbb{R}^M$. For such a $\mathbf{u}_0$, set, by using spectral theory for self-adjoint operators:

$$\mathbf{u}(t) := \cos((L+m^2)^{\frac{1}{2}}t)\mathbf{u}_0. \tag{2.40}$$

Then, since $\mathbf{u}_0$ is smooth, $\mathbf{u}(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^M$ is smooth jointly in (t,x), and

$$\partial_t^2 \mathbf{u} + L\mathbf{u} + m^2\mathbf{u} = 0, \quad \partial_t \mathbf{u}(0) = 0, \ \mathbf{u}(0) = \mathbf{u}_0 \tag{2.41}$$

holds in the classical sense. If $M = 1$, $m^2 = 0$, and a is the $d \times d$ identity matrix, $(P_{1,t})$ is the finite propagation speed of the wave equation.

Similarly, in the general situation, the property (P_{1,B_+t}) can be deduced from the finite propagation speed of first order hyperbolic systems. This is well-known, but the explicit reduction for the case of (2.41) with (2.39) is difficult is to find in the literature. Let us therefore sketch how to convert (2.41) to a hyperbolic system for readers interested in this case. For example, one can define $\mathbf{v} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{(d+2)M}$ by:

$$\mathbf{v}_0^k = \partial_t \mathbf{u}^k, \quad \mathbf{v}_i^k = \sum_{j=1}^{d} \sum_{l=1}^{M} a_{ij}^{kl} \partial_{x_j} \mathbf{u}^l, \quad \mathbf{v}_{d+1}^k = m\mathbf{u}^k, \tag{2.42}$$

where $i = \{1, \dots, d\}$ and $k \in \{1, \dots, M\}$. It follows that $\mathbf{v}$ satisfies

$$\mathbf{S}\partial_t \mathbf{v} + \sum_{j=1}^{d} \mathbf{A}_j \partial_{x_j} \mathbf{v} + \mathbf{B}\mathbf{v} = 0, \quad \mathbf{v}(0) = (\mathbf{0}, (aD\mathbf{u}_0)^1, \dots, (aD\mathbf{u}_0)^d, m\mathbf{u}_0) \tag{2.43}$$

where $\mathbf{S}, \mathbf{A}_j, \mathbf{B} : \mathbb{R}^d \to R^{(d+2)M \times (d+2)M}$ are defined as the block matrices

$$\mathbf{S} = \begin{pmatrix} 1_{M\times M} & 0_{dM\times M} & 0_{M\times M} \\ 0_{M\times dM} & a^{-1} & 0_{M\times dM} \\ 0_{M\times M} & 0_{dM\times M} & 1_{M\times M} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0_{1\times 1} & 0_{d\times 1} & m \\ 0_{1\times d} & 0_{d\times d} & 0_{1\times d} \\ -m & 0_{d\times 1} & 0_{1\times 1} \end{pmatrix} \otimes 1_{M\times M}, \tag{2.44}$$

and

$$\mathbf{A}_i = \begin{pmatrix} 0 & -\delta_{1i} & \cdots & -\delta_{di} & 0 \\ -\delta_{1i} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ -\delta_{di} & 0 & \dots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \otimes 1_{M\times M}, \quad i = 1, \dots, d. \tag{2.45}$$

It is immediate that this system is symmetric uniformly hyperbolic, by the symmetry and uniform ellipticity of the matrix a. The property (P_{1,B_+t}) now follows from the finite propagation speed of linear hyperbolic systems; see e.g. [7, 84].

Nash showed [91] that $(H_{d,\omega})$ holds when $M = 1$. In [77, 81], conditions are given for $(H_{d,\omega})$ to hold when $M > 1$. In particular, this includes the constant coefficient case. The latter case can be treated by using the Fourier transform; see Section 2.3.2.

Example 2.1.5 (Random walk on graphs). Let (X, E) be a (locally finite) graph, with vertex set X and edge set $E \subset P_2(X)$, where X is a countable (or finite) set and $P_2(X)$ are the subsets of X with two elements. Let $d : X \times X \to [0, \infty]$ be the graph distance on (X, E), i.e., $d(x, y)$ is the (unweighted) length of the shortest path from x to y.

Suppose that edge weights $\mu_{xy} = \mu_{yx} \geq 0$, $x, y \in X$ are given. These induce a natural measure, also denoted μ, on X by:

$$\mu_x = \sum_{y \in X} \mu_{xy}, \quad \mu(A) = \sum_{x \in A} \mu_x \quad \text{for all } A \subseteq X. \tag{2.46}$$

The associated Dirichlet form is

$$\mathcal{E}(u, u) = \tfrac{1}{2} \sum_{xy \in E} \mu_{xy}(u(x) - u(y))^2 \quad \text{for all } u \in D(E) = L^2(\mu) \tag{2.47}$$

and its generator is given by

$$Lu(x) = \mu_x^{-1} \sum_{y \in X} \mu_{xy}(u(x) - u(y)) \quad \text{for all finitely supported } u : X \to \mathbb{R}. \tag{2.48}$$

L is called the *probabilistic Laplace operator* associated to the simple random walk on the weighted graph (X, μ) with transition probabilities μ_{xy}/μ_x. Let us remark that a probabilistic interpretation (or a maximum principle) does not hold in general for Examples 2.1.3–2.1.4 (when a is non-diagonal or vector-valued).

The Dirichlet form (2.47) is bounded on $L^2(\mu)$ with operator norm 2 so that the property $(P^*_{\theta,B})$ holds with $\theta(n) = n$ and $B = 2$, and Theorem 2.1.2 is applicable.

For applications, it is often useful to add a killing rate to the random walk: The probabilistic Green density with killing rate $\kappa \in (0, 1)$ is defined by:

$$G^\kappa(x, y) = \sum_{n \geq 0} p^n(x, y)\kappa^n = (\kappa L + (1 - \kappa))^{-1}(x, y) = (L^\kappa)^{-1}(x, y) \tag{2.49}$$

where $p^n(x, y)$ is the kernel of the operator P^n on $L^2(\mu)$. Note that (2.49) only converges for $\kappa = 0$ when the random walk is transient, but that L^{-1} still makes sense as a quadratic form on its appropriate domain when the random walk is recurrent, as in (2.16), (2.17) for $d = 2$. Note further that $\operatorname{spec}(L^\kappa) \subseteq [0, 2]$ for all $\kappa \in [0, 1]$, so that Theorem 2.1.2 is applicable uniformly in $\kappa \in [0, 1]$.

Closely related to the killed Green function G^κ is the resolvent kernel of L. The resolvent of L is defined on $L^2(\mu)$ by $G_{m^2} = (L + m^2)^{-1}$ for $m^2 > 0$. It is related to the killed Green density by:

$$G^\kappa = \kappa^{-1} G_{(1-\kappa)/\kappa}. \tag{2.50}$$

One difference compared with the killed Green function is that $L+m^2$ is not bounded uniformly in $m^2 \geq 0$. To achieve the condition $(P^*_{\theta,B})$ for fixed $B > 0$, it is therefore necessary to restrict to $m^2 \leq M_+^2$ with $M_+^2 = B - 2$.

Remark 2.1.6. Other examples which Theorem 2.1.2 is applicable to include Dirichlet spaces that satisfy a Davies-Gaffney estimate [103] such as weighted manifolds and quadratic forms corresponding to powers of elliptic operators like Δ^2.

2.1.7 Remarks

Remark 2.1.7. Theorem 2.1.2 also gives the decomposition into sums as in [4, 21, 27]: Suppose that the assumptions of Theorem 2.1.2 are satisfied and, for notational simplicity, that the resulting decomposition has a kernel. Then, for any $L > 1$,

$$\Phi(x,y) = \sum_{j\in\mathbb{Z}} C_j(x,y) \quad \text{for all } x, y \in X \times X \tag{2.51}$$

where the functions $C_j : X \times X \to [0,\infty)$, $j \in \mathbb{Z}$ are given by

$$C_j(x,y) := \int_{L^{j-1}}^{L^j} \phi_t(x,y)\, \frac{dt}{t} \quad \text{for all } x, y \in X. \tag{2.52}$$

They satisfy the following properties:

$$C_j \text{ is the kernel of a positive semi-definite form,} \tag{2.53}$$

$$C_j(x,y) = 0 \quad \text{for all } x, y \in X \text{ with } d(x,y) \geq L^j, \tag{2.54}$$

and, if $(H_{\alpha,\omega})$ holds,

$$|C_j(x,y)| \leq c_\alpha(x,y) \begin{cases} L^{-(\alpha-2)(j-1)} & (\alpha > 2) \\ L^{(2-\alpha)j} & (\alpha < 2) \\ \log(L) & (\alpha = 2) \end{cases} \tag{2.55}$$

with $c_\alpha(x,y)$ is independent of L. Thus, $(C_j)_{j\in\mathbb{Z}}$ is a finite range decomposition into discrete scales of the Green function Φ. Similarly, gradient estimates such as (2.33), (2.34), (2.36) in Example 2.1.3 have obvious discrete versions.

Remark 2.1.8. More generally than in Theorem 2.1.2, we may consider a *family* of symmetric forms, $(\mathcal{E}^s)_{s\in Y}$, where Y is a domain in a Banach space, with generators L^s. Let us assume that $\mathcal{E}^s$ is smooth in s, in the following sense: There exists a projection-valued measure P on a measurable space M and a function $V : M \times Y \to (0,\infty)$, smooth in Y, such that

$$F(L^s) = \int_{\mathrm{spec}(L^s)} F(\lambda)\, dP^s_\lambda = \int_M F(V(s,\tau))\, dP_\tau. \tag{2.56}$$

An example of this condition is $\mathcal{E}^s(f,f) = \mathcal{E}(f,f) + s(f,f)$ so that $V(s,\lambda) = \lambda + s$ and $(L^s)^{-1}$ is the resolvent of L; similarly, the killed Green function of Example 2.1.5 can be expressed in this way. Then the family of kernels ϕ^s is continuous in s, and if $(H_{\alpha,\omega})$ holds for $s = 0$, and $V(\lambda, s) \geq z^2(s)V(\lambda,0)+m^2(s)$, then

$$|\phi_t^s(x,y)| \leq C_{\alpha,\gamma,l}\sqrt{\omega(x)\omega(y)}(z(s)t)^{-(\alpha-2)/\gamma}(1+tm(s))^{-l}. \tag{2.57}$$

This can be verified by a straightforward adaption of the proof of Theorem 2.1.2.

2.2 Proof of main result

2.2.1 Spectral decomposition

The starting point for the proof is the spectral representation of the Green form (2.24):

$$\Phi(f,f) = \int_{\mathrm{spec}(L)} \lambda^{-1}\, d(f, P_\lambda f) \quad \text{for all } f \in D(\Phi), \tag{2.58}$$

where $f \in D(\Phi)$ implies that the integral can be restricted to $\mathrm{spec}(L) \setminus 0$. The main result follows by decomposition of the function $\lambda^{-1} : \mathrm{spec}(L) \setminus 0 \to \mathbb{R}_+$. Different decompositions are needed under the two conditions $(P_{\gamma,\theta})$, $(P^*_{\theta,B})$. The main idea of the proof is that decompositions with good properties exist. The result that we prove after using it to deduce Theorem 2.1.2 is summarized in the following lemma.

Lemma 2.2.1 (Spectral decomposition)**.** *Suppose that L satisfies $(P_{\gamma,\theta})$ or $(P^*_{\theta,B})$; in the second case, we assume that $\gamma = 1$. Then there exists a smooth family of functions $W_t \in C^\infty(\mathbb{R})$, $t > 0$, such that for all $\lambda \in \mathrm{spec}(L) \setminus 0$, $t > 0$, and all integers l,*

$$\lambda^{-1} = \int_0^\infty t^{\frac{2}{\gamma}} W_t(\lambda)\, \frac{dt}{t}, \tag{2.59}$$

$$W_t(\lambda) \geq 0, \tag{2.60}$$

$$(1 + t^{\frac{2}{\gamma}}\lambda)^l W_t(\lambda) \leq C_l, \tag{2.61}$$

and that for all $u \in C_c(X)$,

$$\mathrm{supp}(W_t(L)u) \subseteq N_{\theta(t)}(\mathrm{supp}(u)). \tag{2.62}$$

Remark 2.2.2. More precisely, we will give explicit formulae for W_t that imply

$$(1+t^2\lambda)^l\,\lambda^m\left|\frac{\partial^m}{\partial\lambda^m}W_t(\lambda)\right|\le C_{l,m} \tag{2.63}$$

for all m and l, improving (2.61). This improvement is used in Section 2.3.2.

Proof of Theorem 2.1.2. It follows from (2.59) that, for any $f\in D(\Phi)$,

$$\begin{aligned}\Phi(f,f)&=\int_{\mathrm{spec}(L)}\left(\int_0^\infty t^{\frac{2}{\gamma}}W_t(\lambda)\,\frac{dt}{t}\right)\,d(f,P_\lambda f)\\&=\int_0^\infty t^{\frac{2}{\gamma}}\left(\int_{\mathrm{spec}(L)}W_t(\lambda)\,d(f,P_\lambda f)\right)\,\frac{dt}{t}\\&=\int_0^\infty t^{\frac{2}{\gamma}}(f,W_t(L)f)\,\frac{dt}{t}.\end{aligned} \tag{2.64}$$

The exchange of the order of the two integrals in the equation above is justified by non-negativity of the integrand, by (2.60). The latter also implies that $(f,W_t(L)f)\ge 0$ for all $f\in L^2(X)$. The polarization identity allows to recover $\Phi(f,g)$ for all $f,g\in D(\Phi)$. Finally, (2.62) completes the verification of (2.8) for Φ_t defined by

$$\Phi_t(f,g)=t^{\frac{2}{\gamma}}(f,W_t(L)g). \tag{2.65}$$

It remains to prove that $(H_{\alpha,\omega})$ implies (2.28). The semigroup property and the continuity of p_t imply that $p_t\in C_b(X,L^2(X))$ with

$$\|p_t(x,\cdot)\|_{L^2(X)}=\int_X p_t(x,y)p_t(y,x)\,d\mu(y)=p_{2t}(x,x), \tag{2.66}$$

$$\|p_t(x,\cdot)-p_t(y,\cdot)\|_{L^2(X)}=p_{2t}(x,x)+p_{2t}(y,y)-2p_{2t}(x,y)\to 0\quad\text{as } x\to y. \tag{2.67}$$

This implies that $e^{-tL}:L^2(X)\to C_b(X)$ is a bounded linear operator ($e^{-tL}f(x)=(p_t(x,\cdot),f)$). Duality then also implies continuity of $e^{-tL}:C_b(X)^*\to L^2(X)$ (with respect to the strong topology on $C_b(X)^*$). Let $M(X)\subseteq C_b(X)^*$ be the space of signed finite Radon measures on X equipped with the weak-* topology. Let $m_i\in M(X)$ with $m_i\to 0$. Then:

$$\begin{aligned}\|e^{-tL}m_i\|_{L^2(X)}&=\left(\int_X\left(\int_X p_t(x,y)\,dm_i(y)\right)^2\,d\mu(x)\right)^{\frac{1}{2}}\\&=\left(\int_X\int_X(p_t(y,\cdot),p_t(z,\cdot))\,dm_i(y)\,dm_i(z)\right)^{\frac{1}{2}}\to 0\end{aligned} \tag{2.68}$$

which means that $e^{-tL} : M(X) \to L^2(X)$ is continuous (because X is separable and therefore the weak-* topology of $M(X)$ is metrizable). This implies that $(1 + t^{2/\gamma}L)^{-l} : M(X) \to L^2(X)$ is likewise continuous for all $l > \alpha/4$. To see this, we use the relation

$$(1 + t^{2/\gamma}\lambda)^{-l} = \Gamma(l)^{-1} \int_0^\infty e^{-s} s^{l-1} e^{-st^{2/\gamma}\lambda}\, ds \tag{2.69}$$

which holds by the change of variables formula and the definition of Euler's gamma function. The spectral theorem thus implies that, for any $u \in L^2(X)$,

$$\|(1 + t^{2/\gamma}L)^{-l}u\|_{L^2(X)} \leq \Gamma(l)^{-1} \int_0^\infty e^{-s} s^{l-1} \|e^{-st^{2/\gamma}L}u\|_{L^2(X)}\, ds. \tag{2.70}$$

Since μ has full support, $L^2(X) \cap M(X)$ is dense in $M(X)$ (where $L^p(X)$ is always with respect to μ), and the claimed continuity of $(1 + t^{2/\gamma}L)^{-l} : M(X) \to L^2(X)$ follows from (2.68). In particular, the pointwise bound for p_t implies that for $l > \alpha/4$,

$$\begin{aligned} \|(1 + t^{2/\gamma}L)^{-l}\delta_x\|_{L^2(X)} &\leq \Gamma(l)^{-1} \int_0^\infty e^{-s} s^{l-1} \|e^{-st^{2/\gamma}L}\delta_x\|_{L^2(X)}\, ds \\ &\leq \Gamma(l)^{-1}\sqrt{\omega(x)}t^{-\alpha/2\gamma} \int_0^\infty e^{-s} s^{l-1-\alpha/4}\, ds \\ &= C\sqrt{\omega(x)}t^{-\alpha/2\gamma}. \end{aligned} \tag{2.71}$$

Let $\kappa_t(\lambda) = W_t(\lambda)^{1/2}$. Then (2.61) and the spectral theorem also imply that

$$\|\kappa_t(L)(1 + t^{2/\gamma}L)^l\|_{L^2(X)\to L^2(X)} = \sup_{\lambda>0} \kappa_t(\lambda)(1 + t^{2/\gamma}\lambda)^l \leq C_l, \tag{2.72}$$

uniformly in $t > 0$. It follows from (2.71) that $\kappa_t(L) : M(X) \to L^2(X)$ with

$$\|\kappa_t(L)\delta_x\|_{L^2} \leq C\sqrt{\omega(x)}t^{-\alpha/2\gamma}. \tag{2.73}$$

Finally, by the Cauchy-Schwarz inequality,

$$|\phi_t(x, y)| = t^{2/\gamma}(\kappa_t(L)\delta_y, \kappa_t(L)\delta_x) \leq t^{2/\gamma}\|\kappa_t(L)\delta_y\|_{L^2(X)}\|\kappa_t(L)\delta_x\|_{L^2(X)} \tag{2.74}$$

which, with (2.73), proves (2.28). The continuity of ϕ_t is implied by the continuity of $\kappa_t(L) : M(X) \to L^2(X)$ and of δ_x in $x \in X$ (in the weak-* topology). □

Remark 2.2.3. The decay for ϕ^s claimed in (2.57) can be obtained by a straightforward generalization of the above argument, replacing (2.69) by

$$(1 + t^{2/\gamma}z^2\lambda + t^{2/\gamma}m^2)^{-l} = \Gamma(l)^{-1} \int_0^\infty e^{-s} s^{l-1} e^{-st^{2/\gamma}m^2} e^{-sz^2t^{2/\gamma}\lambda}\, ds. \tag{2.75}$$

Remark 2.2.4. Furthermore, by (2.61), the operators $W_t(L)$ are smoothing for $t > 0$, in the general sense that, for any $t > 0$,

$$W_t(L) : L^2(X) \to C^\infty(L), \quad \text{where } C^\infty(L) := \bigcap_{n=0}^{\infty} D(L^n) \subset L^2(X) \tag{2.76}$$

is the set of C^∞-vectors for L; see [95]. Standard elliptic regularity estimates imply e.g. that $C^\infty(L) = C^\infty(X)$ when E is the quadratic form associated to an elliptic operator with smooth coefficients.

2.2.2 Proof of Lemma 2.2.1

To complete the proof of Theorem 2.1.2, it remains to demonstrate Lemma 2.2.1. We first prove it under condition $(P_{\gamma,\theta})$ in Lemma 2.2.5 below; this proof is quite straightforward using the assumption and (2.2). Then we prove Lemma 2.2.1 in the situation of condition $(P^*_{\theta,B})$ in Lemma 2.2.7; here additional ideas are required.

To fix conventions, let us define the Fourier transform of an integrable function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\hat{\varphi}(k) = (2\pi)^{-1} \int_{\mathbb{R}} \varphi(x) e^{-ikx}\, dx \quad \text{for all } k \in \mathbb{R}. \tag{2.77}$$

Lemma 2.2.5 (Lemma 2.2.1 under $(P_{\gamma,\theta})$)**.** *For any $\varphi : \mathbb{R} \to [0,\infty)$ such that $\hat{\varphi}$ is smooth and symmetric with* $\operatorname{supp}(\hat{\varphi}) \subseteq [-1,1]$*, and for any $\gamma > 0$, there is $C > 0$ such that*

$$W_t(\lambda) := C\varphi(\lambda^{\frac{1}{2}\gamma} t) \tag{2.78}$$

satisfies (2.59), (2.60), (2.61)*, and also* (2.63)*, for all $\lambda > 0$, $t > 0$; and if $(P_{\gamma,\theta})$ holds, then (W_t) also satisfies* (2.62)*.*

Remark 2.2.6. It is not difficult to see that such φ exist. For example, if $\hat{\kappa}$ is a smooth real-valued function with support in $[-\frac{1}{2}, \frac{1}{2}]$, then $\varphi = |\kappa|^2$ satisfies the assumptions. For simplicity, let us assume sometimes in the following that φ is chosen such that $C = 1$ when Lemma 2.2.1 is applied.

Proof. Note that for any $\varphi : [0,\infty) \to \mathbb{R}$ with $t\varphi(t)$ integrable, there is $C > 0$ such that

$$\lambda^{-1} = C \int_0^\infty t^{\frac{2}{\gamma}} \varphi(\lambda^{\frac{1}{2}\gamma} t)\, \frac{dt}{t} \quad \text{for all } \lambda > 0. \tag{2.79}$$

This simply follows (as in (2.2)) because the right-hand side is homogeneous in λ of degree -1, which is immediate by rescaling of the integration variable. This shows (2.59); (2.60) is obvious by assumption; and (2.61) follows since $\hat{\varphi}$

is smooth. The improved estimate (2.63) follows from the chain rule (or Faà di Bruno's formula) and

$$\lambda^{m-\frac{1}{2}\gamma}\left|\frac{\partial^m}{\partial\lambda^m}\lambda^{\frac{1}{2}\gamma}\right| \le C_{\gamma,m} \tag{2.80}$$

for non-negative integers m, using that $\mathrm{supp}(\hat{\varphi}) \subseteq [-1,1]$ implies that φ is smooth. Moreover, since $\mathrm{supp}(\hat{\varphi}) \subset [-1,1]$, and since $\hat{\varphi}$ is smooth,

$$W_t(L)u = C\int_{-1}^{1}\hat{\varphi}(s)\cos(L^{\frac{1}{2}\gamma}ts)u\,ds \quad \text{for all } u \in L^2(X), \tag{2.81}$$

where the integral is the Riemann integral, i.e., the strong limit of its Riemann sums (with values in L^2). Therefore (2.62) follows from $(P_{\gamma,\theta})$. □

The previous proof makes essential use of the finite propagation speed of the wave equation $(P_{\gamma,\theta})$ to prove (2.62). This property fails for discrete Dirichlet forms such as (2.30) where we instead know the property $(P^*_{\theta,B})$ that polynomials of degree n of the generator have finite range $\theta(n)$.

This leads to the following problem. Find polynomials W_t^*, $t > 0$, of degree at most t satisfying the properties (2.60), (2.61), (2.63) such that the decomposition formula (2.59) for $1/\lambda$ holds. In the proof of Lemma 2.2.5, the verification of (2.61) (and (2.63)) and of the decomposition formula (2.59) are directly linked to the "ballistic" scaling of the wave equation: $W_t(\lambda) = W_1(\lambda t^2)$. To construct polynomials satisfying such "ballistic" estimates, we are led by the following remarkable discovery of Carne [39]: The Chebyshev polynomials T_k, $k \in \mathbb{Z}$, defined by

$$T_k(\theta) = \cos(k\arccos(\theta)) \quad \text{for all } \theta \in [-1,1],\ k \in \mathbb{Z}, \tag{2.82}$$

are solutions to the discrete (in space and time) wave equation in the following sense: Let $\nabla_+ f(n) = f(n+1) - f(n)$ and $\nabla_- f(n) = f(n-1) - f(n)$ be the discrete (forward and backward) time differences. Then, as polynomials in X,

$$\nabla_-\nabla_+ T_n(X) = \nabla_+\nabla_- T_n(X) = 2(X-1)T_n(X). \tag{2.83}$$

In particular, when $2(X-1) = -L$ or equivalently $X = 1 - \frac{1}{2}L$, then $v(n,x) = [T_n(1-\frac{1}{2}L)u](x)$ solves the following "Cauchy problem" for the discrete wave equation:

$$-\nabla_+\nabla_- v + Lv = 0, \quad v(0) = u, (\nabla_+ v - \nabla_- v)(0) = 0. \tag{2.84}$$

The analogy between the discrete- and the continuous-time wave equations is like that between the discrete- and the continuous-time random walk. It turns out that the structure of Chebyshev polynomials allows to prove the following lemma.

Lemma 2.2.7 (Lemma 2.2.1 under $(P^*_{\theta,B})$)**.** *Let $\varphi : \mathbb{R} \to [0,\infty)$ satisfy the assumptions of Lemma 2.2.5. Then $W_t^* : [0,4] \to [0,\infty)$, defined by*

$$W_t^*(\lambda) := \sum_{n\in\mathbb{Z}} \varphi(\arccos(1-\tfrac{1}{2}\lambda)t - 2\pi nt) \quad \text{for all } \lambda \in [0,4],\, t > 0, \tag{2.85}$$

is the restriction of a polynomial in λ of degree at most t to $[0,4]$, with coefficients smooth in t, and, for any $\varepsilon > 0$, (2.59)*,* (2.60)*,* (2.61)*,* (2.62)*, and* (2.63) *hold for all $\lambda \in (0, 4-\varepsilon]$, $t > 0$.*

Proof. The proof verifies that W_t^* as defined in (2.85) has the asserted properties. Let

$$\varphi_t^*(x) := \sum_{n\in\mathbb{Z}} \varphi(xt - 2\pi nt) = \sum_{k\in\mathbb{Z}} t^{-1}\hat{\varphi}(k/t)\cos(kx) \tag{2.86}$$

where the second equality follows by symmetry of $\hat{\varphi}$, the change of variables formula, and a version of the Poisson summation formula which is easily verified, for sufficiently nice φ. Then the claim (2.59) can be expressed as

$$\lambda^{-1} = \int_0^\infty t^2 \varphi_t^*(\arccos(1-\tfrac{1}{2}\lambda))\,\frac{dt}{t} \quad \text{for all } \lambda \in (0,4]. \tag{2.87}$$

Let $x = \arccos(1-\frac{1}{2}\lambda)$ or equivalently $\lambda = 2(1-\cos x) = 4\sin^2(\frac{1}{2}x)$. In terms of this change of variables, (2.87) and thus the claim (2.85) are then equivalent to

$$\tfrac{1}{4}\sin^{-2}(\tfrac{1}{2}x) = \int_0^\infty t^2 \varphi_t^*(x)\,\frac{dt}{t} \quad \text{for all } x \in (0,\pi]. \tag{2.88}$$

The left-hand side defines a meromorphic function on $\mathbb{C}$ with poles at $2\pi\mathbb{Z}$. Its development into partial fractions is (see e.g. [5, page 204])

$$\tfrac{1}{4}\sin^{-2}(\tfrac{1}{2}x) = \sum_{n\in\mathbb{Z}} (x-2\pi n)^{-2} \quad \text{for all } x \in \mathbb{C}\setminus 2\pi\mathbb{Z}. \tag{2.89}$$

It follows, by (2.79) with $\gamma = 1$ and $\lambda = (x-2\pi n)^2$, assuming $C = 1$, that

$$\tfrac{1}{4}\sin^{-2}(\tfrac{1}{2}x) = \sum_{n\in\mathbb{Z}} \int_0^\infty t^2 \varphi((x-2\pi n)t)\,\frac{dt}{t} \quad \text{for all } x \in (0,\pi]. \tag{2.90}$$

The order of the sum and the integral can be exchanged, by non-negativity of the integrand, thus showing (2.88) and therefore (2.59).

To verify that W_t^* is the restriction of a polynomial, we note that by (2.85), (2.86), and $\operatorname{supp}(\hat{\varphi}) \subseteq [-1,1]$,

$$\begin{aligned} W_t^*(\lambda) = \varphi_t^*(\arccos(1-\tfrac{1}{2}\lambda)) &= \sum_{k\in\mathbb{Z}} t^{-1}\hat{\varphi}(k/t)\cos(k\arccos(1-\tfrac{1}{2}\lambda)) \\ &= \sum_{k\in\mathbb{Z}\cap[-t,t]} t^{-1}\hat{\varphi}(k/t)T_k(1-\tfrac{1}{2}\lambda) \end{aligned} \tag{2.91}$$

where $T_k, k \in \mathbb{Z}$, are the Chebyshev polynomials defined by (2.82). This shows that $W_t^*(\lambda)$ is indeed the restriction of a polynomial in λ of degree at most t to the interval $\lambda \in [0,4]$. In particular, (2.62) is a trivial consequence of $(P^*_{\theta,B})$ which states that polynomials in L of degree n have range at most $\theta(n)$.

Finally, we verify the estimate (2.63) and thus in particular (2.61). To this end, we note that, in analogy to (2.80), for $\lambda \in [0, 4-\varepsilon]$ and non-negative integers m,

$$\lambda^{m-\frac{1}{2}} \left| \frac{\partial^m}{\partial \lambda^m} \arccos(1-\tfrac{1}{2}\lambda) \right| \leq C_{\varepsilon,m}. \tag{2.92}$$

For example, for $m = 1$,

$$\frac{\partial}{\partial \lambda} \arccos(1-\tfrac{1}{2}\lambda) = \tfrac{1}{2}(\lambda - \tfrac{1}{4}\lambda^2)^{-\frac{1}{2}} \leq \varepsilon^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \quad \text{for } \lambda \in [0, 4-\varepsilon]. \tag{2.93}$$

Therefore (2.63) follows, by the chain rule (or Faà di Bruno's formula), from

$$(1+t^2(1-\cos(x))^l t^{-m} \left| \frac{\partial^m}{\partial x^m} \varphi_t^*(x) \right| \leq C_{l,m} \tag{2.94}$$

which we will now show. The argument is essentially a discrete version of the classic fact that the Fourier transform acts continuously on the Schwartz space of smooth and rapidly decaying functions on $\mathbb{R}$. To show (2.94), first note that

$$(1-\cos(x))e^{ikx} = e^{ikx} - \tfrac{1}{2}e^{i(k+1)x} - \tfrac{1}{2}e^{i(k-1)x} =: \Delta_k e^{ikx} \tag{2.95}$$

and thus by induction, for any $l \in \mathbb{N}$,

$$\begin{aligned} (1-\cos(x))^l e^{ikx} &= (1-\cos(x))^{l-1} \Delta_k e^{ikx} \\ &= \Delta_k (1-\cos(x))^{l-1} e^{ikx} = \Delta_k^l e^{ikx}. \end{aligned} \tag{2.96}$$

It follows by (2.86) and summation by parts that

$$\begin{aligned} (1+t^2(1-\cos(x))^l t^{-m} \frac{\partial^m}{\partial x^m} \varphi_t^*(x) &= \sum_{k\in\mathbb{Z}} t^{-1} \hat{\varphi}(k/t)(ik/t)^m [(1+t^2\Delta_k)^l e^{ikx}] \\ &= \sum_{k\in\mathbb{Z}} [(1+t^2\Delta_k)^l t^{-1} \hat{\varphi}(k/t)(ik/t)^m] e^{ikx}. \end{aligned} \tag{2.97}$$

Let $h(s) = \frac{1}{2}(|s|-1)1_{|s|\leq 1}$ for $s \in \mathbb{R}$. Then, for any smooth $f : \mathbb{R} \to \mathbb{R}$,

$$\Delta_k^n f(k) = (h^{*n} * D^{2n} f)(k), \tag{2.98}$$

where $*$ denotes convolution of two functions on $\mathbb{R}$, $h^{*n} = h * h * \cdots * h$, and Df is the derivative of f. Indeed,

$$\begin{aligned}\Delta_k f(k) &= -\tfrac{1}{2}\int_0^1 [Df(k+t) - Df(k-t)]\,dt \\ &= -\tfrac{1}{2}\int_0^1 \int_{-t}^{t} D^2 f(k+s)\,ds\,dt \\ &= \int_{\mathbb{R}} D^2 f(s) h(s-k)\,ds = (h * D^2 f)(k), \end{aligned} \tag{2.99}$$

and (2.98) then follows by induction:

$$\Delta_k^{n+1} f = \Delta(h^{*n} * D^{2n} f) = h * D^2(h^{*n} * D^{2n} f) = h * h^{*n} * D^2 D^{2n} f. \tag{2.100}$$

It then follows using the facts that $\sum_{k\in\mathbb{Z}} |h^{*n}(k-s)| \le C_n$, uniformly in $s \in \mathbb{R}$, and that $\hat{\varphi}$ is smooth and of rapid decay,

$$\begin{aligned} & t^{-1} \sum_{k\in\mathbb{Z}} \left| (1 + t^2\Delta_k^2)^l [\hat{\varphi}(k/t)(ik/t)^m] \right| \qquad (2.101) \\ &= \sum_{n=0}^{l} C_{l,n} t^{-1} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} |h^{*n}(k-s)|\, |[D^{2n}((\cdot)^m \hat{\varphi})](s/t)|\,ds \\ &\le \sum_{n=0}^{l} C_{l,n} t^{-1} \int_{\mathbb{R}} |[D^{2n}((\cdot)^m \hat{\varphi})](s/t)|\,ds \\ &= \sum_{n=0}^{l} C_{l,n} \int_{\mathbb{R}} |[D^{2n}((\cdot)^m \hat{\varphi})](s)|\,ds \le C_{m,l} \end{aligned}$$

and thus (2.94), and therefore (2.63), follow from this inequality and (2.97). □

Proof of Lemma 2.2.1. Lemma 2.2.1 under $(P_{\gamma,\theta})$ is an immediate consequence of Lemma 2.2.5; under $(P^*_{\theta,B})$, it follows from Lemma 2.2.7 with appropriate rescaling to achieve $\lambda \le 3$, i.e., by setting $W_t(\lambda) = c^{-1} W_t^*(c\lambda)$ for some $c > 0$. □

2.3 Extensions

2.3.1 Discrete approximation

In view of the discussion about Chebyshev polynomials before Lemma 2.2.7, it is not surprising that the functions W_t^* of Lemma 2.2.7 approximate the W_t of Lemma 2.2.5. In Proposition 2.3.1 below, we show that this is indeed the case with natural error $O(t^{-1})$ as $t \to \infty$. This result is used in Section 2.3.2 to prove (2.36).

Proposition 2.3.1 (Discrete approximation)**.** *Let φ be as in Lemma 2.2.5 and 2.2.7, with associated functions W_t and W_t^* for $\gamma = 1$. Then, for any integer l,*

$$|W_t^*(\lambda) - W_t(\lambda)| \le C_l (1 \vee t)^{-1} (1 + t^2\lambda)^{-l} \quad \text{for all } \lambda \in [0, 4]. \tag{2.102}$$

In particular, $W_t^*(\lambda/t^2) \to C\varphi(\lambda^{\frac{1}{2}})$ as $t \to \infty$.

Proof. Note that it suffices to restrict to $t \ge 1$, since for $t \le 1$, the claim follows from (2.61). The left-hand side of (2.102) is then proportional to the absolute value of

$$\varphi(\arccos(1 - \tfrac{1}{2}\lambda)t) - \varphi(\lambda^{\frac{1}{2}}t) + \sum_{n \in \mathbb{Z}\setminus\{0\}} \varphi(\arccos(1 - \tfrac{1}{2}\lambda)t + 2\pi n t). \tag{2.103}$$

We estimate the difference of the first two terms in (2.103) and the sum separately, and show that each of them satisfies (2.102). The first two terms can be written as

$$\varphi(\arccos(1 - \tfrac{1}{2}\lambda)t) - \varphi(\lambda^{\frac{1}{2}}t) = (\arccos(1 - \tfrac{1}{2}\lambda) - \lambda^{\frac{1}{2}}) t \zeta_t(\lambda) \tag{2.104}$$

with

$$\zeta_t(\lambda) = \int_0^1 \varphi'(s \arccos(1 - \tfrac{1}{2}\lambda)t + (1 - s)\lambda^{\frac{1}{2}}t)\, ds. \tag{2.105}$$

The bounds

$$\sqrt{2\lambda} = \arccos(1 - \lambda) + O(\lambda) \quad \text{as } \lambda \to 0+, \tag{2.106}$$

$$\sqrt{2\lambda} \le \arccos(1 - \lambda) \le \tfrac{\pi}{2}\sqrt{2\lambda} \quad \text{for all } \lambda \in [0, 2], \tag{2.107}$$

and the rapid decay of φ' therefore imply that

$$|\zeta_t(\lambda)| \le C_l (1 + \lambda t^2)^{-l} \tag{2.108}$$

and

$$\varphi(\arccos(1 - \tfrac{1}{2}\lambda)t) - \varphi(\lambda^{\frac{1}{2}}t) \le C_l t^{-1} (1 + t^2\lambda)^{-l}. \tag{2.109}$$

To estimate the sum in (2.103), we can use the rapid decay of φ with the inequality $x + y \ge 2(xy)^{1/2}$ to obtain that

$$\begin{aligned} \sum_{n \in \mathbb{Z}\setminus\{0\}} \varphi(xt + 2\pi n t) &\le C_l \sum_{n \in \mathbb{Z}\setminus\{0\}} (1 + xt + 2\pi n t)^{-l} \\ &\le C_l (1 + xt)^{-l/2} t^{-l/2} \sum_{n>0} n^{-l/2} \le C_l (1 + xt)^{-l/2} t^{-l/2} \end{aligned} \tag{2.110}$$

for any $l > 2$, with the constant changing from line to line. In particular, upon substituting $x = \arccos(1 - \frac{1}{2}\lambda)$, this bound and (2.107) imply

$$\sum_{n \in \mathbb{Z}\setminus\{0\}} \varphi(\arccos(1 - \tfrac{1}{2}\lambda)t + 2\pi n t) \le C_l t^{-2l} (1 + t^2\lambda)^{-l}. \tag{2.111}$$

The claim then follows by adding (2.109) and (2.111). □

2.3.2 Estimates for systems with constant coefficients

In this section, we verify the assertions of Example 2.1.3. We work in the slightly more general context of second-order elliptic systems (instead of operators) with constant coefficients. These are defined as in Example 2.1.4, and we now show that claims of Example 2.1.3 hold mutadis mutandis. The analysis is straightforward, with aid of the Fourier transform. It reproduces several results of [4, 31]. Note that by writing $L = \frac{1}{c^2}[\frac{1}{c^2}L]^{-1}$ and considering $\frac{1}{c^2}L$ instead of L, we may assume that the coefficients, a, are bounded such that $(P^*_{\theta,B})$ holds with $B = 3$ (for example).

Spectral measures

The spectral measures corresponding to the vector-valued case of (2.29) are given in terms of the Fourier transform as follows. For $F : [0,\infty) \to \mathbb{R}$,

$$(v, F(L)u) = \sum_{k,l=1}^{M} \int_{\mathbb{R}^d} \left[F\left(\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \right) \right]_{kl} \overline{\hat{v}}^k(\xi)\hat{u}^l(\xi)\, d\xi \tag{2.112}$$

where $\hat{u} = (\hat{u}^1, \dots, \hat{u}^M)$ is the Fourier transform of $u = (u^1, \dots, u^M)$, separately for each component,

$$a(\xi) := \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j = \left(\sum_{i,j=1}^{d} a^{kl}_{ij}\xi_i\xi_j \right)_{k,l=1,\dots,M} \tag{2.113}$$

are symmetric positive definite $M \times M$ matrices, for all $\xi \in \mathbb{R}^d$, and the matrices $F(a(\xi))$ are defined in terms of the spectral decomposition of $a(\xi)$. Similarly, for the (vector-valued case of the) discrete Dirichlet form (2.30),

$$(v, F(L)u) = \sum_{k,l=1}^{M} \int_{[-\pi,\pi]^d} \left[F\left(\sum_{i,j=1}^{d} a_{ij}(1 - e^{i\xi_i})(1 - e^{-i\xi_j}) \right) \right]_{kl} \overline{\hat{v}}^k(\xi)\hat{u}^l(\xi)\, d\xi \tag{2.114}$$

where here $\hat{u}$ is the component-wise discrete Fourier transform. Let us also write

$$a^*(\xi) := \sum_{i,j=1}^{d} a_{ij}(1-e^{i\xi_i})(1-e^{-i\xi_j}) = \left(\sum_{i,j=1}^{d} a^{kl}_{ij}(1 - e^{i\xi_i})(1 - e^{-i\xi_j}) \right)_{k,l=1,\dots,M}. \tag{2.115}$$

We will often use, without mentioning this further, that the spectra of $a(\xi)$ and $a^*(\xi)$ are bounded from above and from below by $|\xi|^2$.

Estimates

Let us introduce the following notation for derivatives: For a function $u : \mathbb{R}^d \to \mathbb{R}$, we regard the lth derivative, $D^l u(x)$, as an l-linear form, and $|D^l u(x)|$ is a norm of the form $D^l u(x)$. In terms of the Fourier transform, we denote by $\hat{D}^l(\xi)$ the corresponding "multiplier" operator from functions to l-linear forms, and by $|\hat{D}^l(\xi)|$ its norm. Similarly, for a discrete function $u : \mathbb{Z}^d \to \mathbb{R}$, the lth order discrete difference in *positive coordinate direction* is denoted by $\nabla^l u(x)$ and has Fourier multiplier $\hat{\nabla}^l(\xi)$. In particular, when $l = 1$,

$$\hat{D}(\xi) \cong (i\xi_1, \dots, i\xi_d), \quad \hat{\nabla}(\xi) \cong (e^{i\xi_1} - 1, \dots, e^{i\xi_d} - 1). \tag{2.116}$$

Furthermore, k and p will denote integers that may be chosen arbitrarily, and C constants that can change from instance to instance and may depend on k and p, as well as $l = (l_x, l_y, l_a, l_{m^2})$, B_+, B_-, and M_+, but not on x, ξ, and m.

Proof of (2.37),(2.33),(2.34). It follows by the change of variables $\xi \mapsto t\xi$, from the fact that $a(\xi)$ is homogeneous of degree 2, and from $W_t(\lambda) = W_1(\lambda t^2)$ that

$$\begin{aligned} \phi_t(x, y; a, m^2) &= t^2 \int_{\mathbb{R}^d} W_t(a(\xi) + m^2) e^{i(x-y)\cdot\xi} \, d\xi \\ &= t^{-(d-2)} \bar{\phi}(\frac{x-y}{t}; a, m^2 t^2) \end{aligned} \tag{2.117}$$

with

$$\bar{\phi}(x; a, m^2) := \int_{\mathbb{R}^d} W_1(a(\xi) + m^2) e^{i(x-y)\cdot\xi} \, d\xi \tag{2.118}$$

which is supported in $|x| \le B_+$. This verifies (2.37). Furthermore, (2.33) is a straightforward consequence of (2.117) by differentiation and (2.63). Let us omit the details and only verify them explicitly in the discrete case (2.34): The (derivatives of the) decomposition kernel ϕ_t^* can here be expressed as

$$D_a^{l_a} D_{m^2}^{l_{m^2}} \nabla_x^{l_x} \nabla_y^{l_y} \phi_t^*(x, y; a, m^2) = t^{-(d-2)-l_x-l_y+2l_{m^2}} \bar{\phi}_{t;l}^*(x - y; a, m^2) \tag{2.119}$$

with

$$\bar{\phi}_{t;l}^*(x; a, m^2) = t^{d+l_x+l_y-2l_{m^2}} \int_{[-\pi,\pi]^d} D_a^{l_a} D_{m^2}^{l_{m^2}} W_t^*(a^*(\xi) + m^2) \bar{\hat{\nabla}}^{l_y} \hat{\nabla}^{l_x} e^{ix\cdot\xi} \, d\xi. \tag{2.120}$$

Thus (2.63), $|\hat{\nabla}(\xi)| \le C|\xi|$, and $\eta \cdot a^*(\xi)\eta \ge C|\xi|^2|\eta|^2$ for $\eta \in \mathbb{R}^M$ imply

$$\begin{aligned} |\bar{\phi}_{t;l}^*(x; a, m^2)| &\le C \int_{[-\pi,\pi]^d} (1 + C|\xi|^2 t^2 + m^2 t^2)^{-k-p} (t|\xi|)^{l_x+l_y-2l_{m^2}} \, t^d d\xi \\ &\le C(1 + m^2 t^2)^{-k} \int_{\mathbb{R}^d} (1 + C|\xi|^2)^{-p} |\xi|^{l_x+l_y-2l_{m^2}} \, d\xi \end{aligned} \tag{2.121}$$

and therefore that the integral converges if $\frac{1}{2}(d + l_x + l_y) > l_{m^2}$ and p is chosen sufficiently large. It follows that

$$|\overline{\phi}^*_{t;l}(x; a, m^2)| \leq C(1 + m^2t^2)^{-k} \tag{2.122}$$

verifying the claim. □

Proof of (2.36).

$$\nabla_x^{l_x}\nabla_y^{l_y}\phi_t^*(x,y) - D_x^{l_x}D_y^{l_y}\phi_t(x,y) = t^2\int_{[-\pi,\pi]^d} W_t^*(a^*(\xi))\hat{\nabla}^{l_x}\overline{\hat{\nabla}}^{l_y}e^{i\xi\cdot(x-y)}\,d\xi \tag{2.123}$$
$$- t^2\int_{\mathbb{R}^d} W_t(a(\xi))\hat{D}^{l_x}\overline{\hat{D}}^{l_y}e^{i\xi\cdot(x-y)}\,d\xi.$$

To simplify notation, we will write $\hat{D}^l = \hat{D}^{l_x}\overline{\hat{D}}^{l_y} = \hat{D}^{l_x}\otimes\overline{\hat{D}}^{l_y}$ if $l = (l_x, l_y)$, and similarly for ∇. Then the difference (2.123) may be estimated as follows. Proposition 2.3.1 implies

$$\int_{[-\pi,\pi]^d} |W_t^*(a^*(\xi)+m^2) - W_t(a^*(\xi)+m^2)||\hat{D}^l(\xi)|\,d\xi$$
$$\leq Ct^{-1}\int_{\mathbb{R}^d}(1+C|\xi|^2t^2+m^2t^2)^{-p-k}|\xi|^l\,d\xi \leq Ct^{-d-l-1}(1+m^2t^2)^{-k} \tag{2.124}$$

where we have assumed in the second inequality above that p was chosen sufficiently large so that the integral is convergent. Similarly, we may proceed for the other differences, always choosing p large enough in the estimates. Using (2.63) with $m = 1$ and $|a^*(\xi) - a(\xi)| = O(|\xi|^3)$, which follows from Taylor's theorem, we obtain

$$\int_{[-\pi,\pi]^d} |W_t(a^*(\xi)+m^2) - W_t(a(\xi)+m^2)||\hat{D}^l(\xi)|\,d\xi$$
$$\leq C\int_{\mathbb{R}^d}|\xi|(1+C|\xi|^2t^2+m^2t^2)^{-p-k}|\xi|^l\,d\xi \leq Ct^{-d-l-1}(1+m^2t^2)^{-k}. \tag{2.125}$$

Taylor's theorem similarly implies $|\hat{\nabla}^l(\xi) - \hat{D}^l(\xi)| \leq C|\xi|^{l+1}$ so that, by (2.61),

$$\int_{[-\pi,\pi]^d} |W_t^*(a^*(\xi)+m^2)||\hat{\nabla}^l(\xi) - \hat{D}^l(\xi)|\,d\xi$$
$$\leq C\int_{\mathbb{R}^d}(1+C|\xi|^2t^2+m^2t)^{-p-k}|\xi|^{l+1}\,d\xi \leq Ct^{-d-l-1}(1+m^2t^2)^{-k}. \tag{2.126}$$

Finally, we obtain by (2.61) that

$$\int_{\mathbb{R}^d\setminus[-\pi,\pi]^d} |W_t(a(\xi)+m^2)||\hat{D}^l(\xi)|\,d\xi \\ \leq C\int_{\mathbb{R}^d\setminus[-\pi,\pi]^d} (1+C|\xi|^2t^2+m^2t^2)^{-p-k}|\xi|^l\,d\xi \leq Ct^{-2p}(1+m^2t^2)^{-k}. \tag{2.127}$$

The combination of the previous four inequalities gives (2.36). □

Chapter 3

Structural stability of a class of dynamical systems

3.1 Introduction and main result

3.1.1 Introduction

Let $\mathbf{V} = \mathbb{R}^3$ with elements $V \in \mathbf{V}$ written $V = (g, z, \mu)$ and considered as a column vector for matrix multiplication. For each $j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, we define the *quadratic flow* $\bar{\varphi}_j : \mathbf{V} \to \mathbf{V}$ by

$$\bar{\varphi}_j(V) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \eta_j & \gamma_j & \lambda_j \end{pmatrix} V - \begin{pmatrix} V^T q_j^g V \\ V^T q_j^z V \\ V^T q_j^\mu V \end{pmatrix}, \tag{3.1}$$

with the quadratic terms of the form

$$q_j^g = \begin{pmatrix} \beta_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q_j^z = \begin{pmatrix} \theta_j & \frac{1}{2}\zeta_j & 0 \\ \frac{1}{2}\zeta_j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.2}$$

and

$$q_j^\mu = \begin{pmatrix} \upsilon_j^{gg} & \frac{1}{2}\upsilon_j^{gz} & \frac{1}{2}\upsilon_j^{g\mu} \\ \frac{1}{2}\upsilon_j^{gz} & \upsilon_j^{zz} & \frac{1}{2}\upsilon_j^{z\mu} \\ \frac{1}{2}\upsilon_j^{g\mu} & \frac{1}{2}\upsilon_j^{z\mu} & 0 \end{pmatrix}. \tag{3.3}$$

All entries in the above matrices are real numbers. We assume that there exists a $\lambda > 1$ such that $\lambda_j \geq \lambda$ for all j, together with assumptions that ensure that for most values of j we have $\beta_j \geq c > 0$ and $\zeta_j \leq 0$. Our hypotheses on the parameters of $\bar{\varphi}$ are stated precisely in Assumptions (A1–A2) below. The significance of the assumption $c > 0$ is explained in Section 3.1.3 below.

The quadratic flow $\bar{\varphi}$ defines a time-dependent discrete-time 3-dimensional dynamical system. It is triangular, in the sense that the equation for g does not depend on z or μ, the equation for z depends only on g, and the equation for μ depends on

g and z. Moreover, the equation for z is linear in z, and the equation for μ is linear in μ. This makes the analysis of the quadratic flow elementary.

Our main result concerns structural stability of the dynamical system $\bar{\varphi}$ under a class of infinite-dimensional perturbations. Let $(\mathbf{K}_j)_{j\in\mathbb{N}_0}$ be a sequence of Banach spaces and $X_j = \mathbf{K}_j \oplus \mathbf{V}$. We write $x_j \in X_j$ as $x_j = (K_j, V_j) = (K_j, g_j, z_j, \mu_j)$. A norm on X_j is given by

$$\|x_j\|_{X_j} = \max\{\|K_j\|_{\mathbf{K}_j}, \|V_j\|_{\mathbf{V}}\} = \max\{\|K_j\|_{\mathbf{K}_j}, |g_j|, |z_j|, |\mu_j|\}. \tag{3.4}$$

We identify $\mathbf{K}_j$ and $\mathbf{V}$ with subspaces of X_j, so that $\|K_j\|_{\mathbf{K}_j} = \|K_j\|_{X_j}$ and $\|V\|_{\mathbf{V}} = \|V\|_{X_j}$ with this norm on X_j. However, we will only make use of the norm of the K- and V-components in X_j separately, but never of $\|x_j\|_{X_j}$. (The reason is that the two components will need to be re-weighted.) Suppose that we are given maps $\psi_j : X_j \to \mathbf{K}_{j+1}$ and $\rho_j : X_j \to \mathbf{V}$. Then we define $\Phi_j : X_j \to X_{j+1}$ by

$$\Phi_j(K_j, V_j) = (\psi_j(K_j, V_j), \bar{\varphi}_j(V_j) + \rho_j(K_j, V_j)). \tag{3.5}$$

This is an infinite-dimensional perturbation of the 3-dimensional quadratic flow $\bar{\varphi}$, which breaks triangularity and which involves the spaces $\mathbf{K}_j$ in a nontrivial way. We will impose estimates on ψ_j and ρ_j below, which make Φ a third-order perturbation of $\bar{\varphi}$.

We give hypotheses under which there exists a sequence $(x_j)_{j\in\mathbb{N}_0}$ with $x_j \in X_j$ which is a *global flow* of Φ, in the sense that

$$x_{j+1} = \Phi_j(x_j) \quad \text{for all } j \in \mathbb{N}_0, \tag{3.6}$$

obeying the boundary conditions that (K_0, g_0) is fixed, $z_j \to 0$, and $\mu_j \to 0$. Moreover, within an appropriate space of sequences, this global flow is unique.

As we have discussed in more detail in Chapter 1, this result provides an essential ingredient in a renormalisation group analysis of the 4-dimensional continuous-time weakly self-avoiding walk [9, 19, 38], where the boundary condition $\mu_j \to 0$ is the appropriate boundary condition for the study of a *critical* trajectory. It is this application that provides our immediate motivation to study the dynamical system Φ, but we expect that the methods developed here will have further applications to dynamical systems arising in renormalisation group analyses in statistical mechanics.

3.1.2 Dynamical system

We think of $\Phi = (\Phi_j)_{j\in\mathbb{N}_0}$ as the *evolution map* of a discrete time-dependent dynamical system, although it is more usual in dynamical systems to have the spaces

X_j be identical. Our application in [9, 19, 38] requires the greater generality of j-dependent spaces.

In the case that Φ is a time-independent dynamical system, i.e., when $\Phi_j = \Phi$ and $X_j = X$ for all $j \in \mathbb{N}_0$, its fixed points are of special interest: $x^* \in X$ is a fixed point of Φ if $x^* = \Phi(x^*)$. The dynamical system is called *hyperbolic* near a fixed point $x^* \in X$ if the spectrum of $D\Phi(x^*)$ is disjoint from the unit circle [99]. It is a classic result that for a hyperbolic system there exists a splitting $X \cong X_s \oplus X_u$ into a *stable* and an *unstable manifold* near x^*. The stable manifold is a submanifold $X_s \subset X$ such that $x_j \to x^*$ in X, exponentially fast, when (x_j) satisfies (3.6) and $x_0 \in X_s$. This result can be generalised without much difficulty to the situation when the Φ_j and X_j are not necessarily identical, viewing "0" as a fixed point (although 0 is the origin in different spaces X_j). The hyperbolicity condition must now be imposed in a uniform way [25, Theorem 2.16].

By definition, $\bar{\varphi}_j(0) = 0$, and we will make assumptions below which can be interpreted as a weak formulation of the fixed point equation $\Phi_j(0) = 0$ for the dynamical system defined by (3.5). Despite this technical condition, will simply refer to 0 as a fixed point of Φ. This fixed point 0 is not hyperbolic due to the two unit eigenvalues of the matrix in the first term of (3.1). Thus the g- and z-directions are *centre* directions, which neither contract nor expand in a linear approximation. On the other hand, the hypothesis that $\lambda_j \geq \lambda > 1$ ensures that the μ-direction is *expanding*, and we will assume below that $\psi_j : X_j \to \mathbf{K}_{j+1}$ is such that the K-direction is *contractive* near the fixed point 0. The behaviour of dynamical systems near non-hyperbolic fixed points is much more subtle than for the hyperbolic case. A general classification does not exist, and a nonlinear analysis is required.

3.1.3 Main result

In Section 3.2, we give an elementary proof that there exists a unique global flow $\bar{V} = (\bar{g}, \bar{z}, \bar{\mu})$ of the quadratic flow $\bar{\varphi}$ with boundary conditions $\bar{g}_0 = g_0$ (always assumed sufficiently small) and $(\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0)$, where we are writing, e.g., $\bar{z}_\infty = \lim_{j\to\infty} \bar{z}_j$. Our main result is that, under the assumptions stated below, there exists a unique global flow of Φ with small initial conditions (K_0, g_0) and final conditions $(z_\infty, \mu_\infty) = (0, 0)$, and that this flow is a small perturbation of $\bar{V}$.

The sequence $\bar{g} = (\bar{g}_j)$ plays a prominent role in the analysis. Determined by the sequence (β_j), it obeys

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2, \qquad \bar{g}_0 = g_0 > 0. \tag{3.7}$$

We regard $\bar{g}$ as a known sequence (only dependent on the initial condition g_0). The following examples are helpful to keep in mind.

Example 3.1.1. (i) Constant $\beta_j = b > 0$. In this case, it is not difficult to show that $\bar{g}_j \sim g_0(1 + g_0 bj)^{-1} \sim (bj)^{-1}$ as $j \to \infty$ (e.g., by applying (3.41) below with $\psi(t) = t^{-2}$).

(ii) Abrupt cut-off, with $\beta_j = b$ for $j \le J$ and $\beta_j = 0$ for $j > J$, with $J \gg 1$. In this case, $\bar{g}_j$ is approximately the constant $(bJ)^{-1}$ for $j > J$. In particular, $\bar{g}_j$ does not go to zero as $j \to \infty$.

Example 3.1.1 prompts us to make the following general definition of a cut-off time for bounded sequences β_j. Let $\|\beta\|_\infty = \sup_{j\ge 0} |\beta_j| < \infty$, and let $n_+ = n$ if $n \ge 0$ and otherwise $n_+ = 0$. Given a fixed $\Omega > 1$, we define the Ω-*cut-off time* j_Ω by

$$j_\Omega = \inf\{k \ge 0 : |\beta_j| \le \Omega^{-(j-k)_+}\|\beta\|_\infty \text{ for all } j \ge 0\}. \tag{3.8}$$

The infimum of the empty set is defined to equal ∞, e.g., if $\beta_j = b$ for all j. By definition, $j_\Omega \le j_{\Omega'}$ if $\Omega \le \Omega'$. To abbreviate the notation, we write

$$\chi_j = \Omega^{-(j-j_\Omega)_+}. \tag{3.9}$$

The evolution maps Φ_j are specified by the real parameters η_j, γ_j, λ_j, β_j, θ_j, ζ_j, $\upsilon_j^{\alpha\beta}$, together with the maps ψ_j and ρ_j on X_j. Throughout this paper, we fix $\Omega > 1$ and make Assumptions (A1–A2) on the real parameters and Assumption (A3) on the maps, all stated further below. The constants in all estimates are permitted to depend on the constants in these assumptions, including Ω, but *not* on j_Ω and $g_0 > 0$. Furthermore, we consider the situation when the parameters of $\bar{\varphi}_j$ are continuous maps from a metric space M_{ext} of external parameters, $m \in M_{\text{ext}}$, into $\mathbb{R}$, that the maps ψ_j and ρ_j similarly have continuous dependence on m, and that j_Ω is allowed to depend on m, but that Assumptions (A1–A3) hold with the constants independent of m. Corollary 3.1.7 below then shows that the solutions to (3.6) constructed in Theorem 3.1.4 below also depend continuously on m.

In Section 3.2, as a preliminary result to the proof of the main result, we prove the following Proposition 3.1.2 concerning flows of the three-dimensional quadratic dynamical system $\bar{\varphi}$. Its proof is elementary.

Assumption (A1). *The sequence β:* The sequence (β_j) is bounded: $\|\beta\|_\infty < \infty$. There exists $c > 0$ such that $\beta_j \ge c$ for all but c^{-1} values of $j \le j_\Omega$.

Assumption (A2). *The other parameters of $\bar{\varphi}$:* There exists $\lambda > 1$ such that $\lambda_j \ge \lambda$ for all $j \ge 0$. There exists $c > 0$ such that $\zeta_j \le 0$ for all but c^{-1} values of $j \le j_\Omega$. Each of ζ_j, η_j, γ_j, θ_j, ζ_j, $\upsilon_j^{\alpha\beta}$ is bounded in absolute value by $O(\chi_j)$, with a constant that is independent of both j and j_Ω.

Note that when $j_\Omega < \infty$, Assumption (A1) permits the possibility that eventually $\beta_k = 0$ for large k. The simplest setting for the assumptions is in the situation when $j_\Omega = \infty$, for which $\chi_j = 1$ for all j. Our applications include situations in which β_j approaches a positive limit as $j \to \infty$, but also situations in which β_j is approximately constant in j over a long initial interval $j \le j_\Omega$ and then abruptly decays to zero.

Proposition 3.1.2. *Assume (A1–A2). If $\bar{g}_0 > 0$ is sufficiently small, then there exists a unique global flow $\bar{V} = (\bar{V})_{j\in\mathbb{N}_0} = (\bar{g}_j, \bar{z}_j, \bar{\mu}_j)_{j\in\mathbb{N}_0}$ of $\bar{\varphi}$ with initial condition $\bar{g}_0$ and $(\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0)$. This flow satisfies the estimates*

$$\chi_j \bar{g}_j = O\left(\frac{\bar{g}_0}{1 + \bar{g}_0 j}\right), \quad \bar{z}_j = O(\chi_j \bar{g}_j), \quad \bar{\mu}_j = O(\chi_j \bar{g}_j), \tag{3.10}$$

with constants independent of j_Ω and $\bar{g}_0$. Furthermore, if the maps $\bar{\varphi}_j$ depend continuously on an external parameter such that (A1–A2) hold with uniform constants, then $\bar{V}_j$ is continuous in this parameter, for every $j \in \mathbb{N}_0$.

We now define domains $D_j \subset X_j$ on which we assume the perturbation (ψ_j, ρ_j) to be defined, and an assumption which states estimates for (ψ_j, ρ_j). The domain and estimates depend on an initial condition $g_0 > 0$ and a possible external parameter m. Theorem 3.1.4 below shows existence and uniqueness of solutions to (3.6) with this initial condition, and existence and differentiability of solutions for initial conditions in a neighborhood of g_0.

For parameters $r, u > 0$ and sufficiently small $g_0 > 0$, let $(\bar{g}_j, \bar{z}_j, \bar{\mu}_j)_{j\in\mathbb{N}_0}$ be the sequence determined by Proposition 3.1.2 with initial condition $\bar{g}_0 = g_0$, and define the domain $D_j = D_j(g_0, r, u) \subset X_j$ by

$$\begin{aligned} D_j = \{x_j \in X_j : \|K_j\|_{\mathbf{K}_j} &\le r\chi_j \bar{g}_j^3, \\ |g_j - \bar{g}_j| &\le u\bar{g}_j^2 |\log \bar{g}_j|, \\ |z_j - \bar{z}_j| &\le u\chi_j \bar{g}_j^2 |\log \bar{g}_j|, \\ |\mu_j - \bar{\mu}_j| &\le u\chi_j \bar{g}_j^2 |\log \bar{g}_j|\}. \end{aligned} \tag{3.11}$$

Note that if β_j depends on an external parameter m, then D_j also depends on this parameter through $\bar{g}_j = \bar{g}_j(m)$. For statements concerning continuity in m, we will assume that Φ_j is defined on the union of these domains over $m \in M_{\text{ext}}$.

Throughout this chapter, we denote by $D_\alpha \phi$ the Fréchet derivative of a map ϕ with respect to the component α, and by $L^m(X_j, X_{j+1})$ the space of bounded m-linear maps from X_j to X_{j+1}. The following Assumption (A3) depends on positive parameters $(g_0, r, u, \kappa, \Omega, R, M)$.

Assumption (A3). *The perturbation:* The maps $\psi_j : D_j \to \mathbf{K}_{j+1} \subset X_{j+1}$ and $\rho_j : D_j \to \mathbf{V} \subset X_{j+1}$ are three times continuously Fréchet differentiable, there exist $\kappa \in (0, \Omega^{-1})$, $R \in (0, r(1-\kappa\Omega))$, and $M > 0$ such that, for all $x_j = (K_j, V_j) \in D_j$,

$$\|\psi_j(0,V_j)\|_{\mathbf{K}_{j+1}} \le R\chi_{j+1}\bar{g}_{j+1}^3, \quad \|\rho_j(x_j)\|_{\mathbf{V}} \le M\chi_{j+1}\bar{g}_{j+1}^3, \tag{3.12}$$

$$\|D_K\psi_j(x_j)\|_{L(\mathbf{K}_j,\mathbf{K}_{j+1})} \le \kappa, \quad \|D_K\rho_j(x_j)\|_{L(\mathbf{K}_j,\mathbf{V})} \le M, \tag{3.13}$$

and such that, for both $\phi = \psi$ and $\phi = \rho$ and $2 \le n + m \le 3$,

$$\|D_V\phi_j(x_j)\|_{L(\mathbf{V},X_{j+1})} \le M\chi_j\bar{g}_{j+1}^2, \tag{3.14}$$

$$\|D_V^m D_K^n \phi_j(x_j)\|_{L^{n+m}(X_j,X_{j+1})} \le M(\chi_j\bar{g}_{j+1}^3)^{1-n}(\bar{g}_{j+1}^2|\log\bar{g}_{j+1}|)^{-m}. \tag{3.15}$$

The bounds (3.12) guarantee that Φ is a third-order perturbation of $\bar{\varphi}$. Moreover, since $\kappa < 1$, the ψ-part of (3.13) ensures that the K-direction is contractive for Φ. (3.15) imposes bounds on the second and third derivatives of ψ and ρ which permit these derivatives to be quite large.

The following elementary Lemma 3.1.3 shows that a sequence $(\bar{K}_j)_{j\in\mathbb{N}_0}$ can be defined inductively by $\bar{K}_{j+1} = \psi_j(\bar{K}_j, \bar{V}_j)$, assuming that r is large enough. Denote by $\pi_K D_j$ the projection of D_j onto $\mathbf{K}_j$, i.e.,

$$\pi_K D_j = \{K_j \in \mathbf{K}_j : \|K_j\|_{\mathbf{K}_j} \le r\chi_j\bar{g}_j^3\}. \tag{3.16}$$

Lemma 3.1.3. *Assume Assumption (A3), let $r^* \in (R/(1-\kappa\Omega), r]$, and assume that $g_0 > 0$ is sufficiently small. Then $\psi_j(D_j(g_0, r^*, u)) \subseteq \pi_K D_{j+1}(g_0, r^*, u)$.*

Proof. The triangle inequality and the first bounds of (3.12)–(3.13) imply

$$\begin{aligned}\|\psi_j(K_j,V_j)\|_{\mathbf{K}_{j+1}} &\le \|\psi_j(0,V_j)\|_{\mathbf{K}_{j+1}} + \|\psi_j(K_j,V_j) - \psi_j(0,V_j)\|_{\mathbf{K}_{j+1}} \\ &\le R\chi_{j+1}\bar{g}_{j+1}^3 + r^*\kappa\Omega(1+O(g_0))\chi_{j+1}\bar{g}_{j+1}^3 \\ &\le r^*\chi_{j+1}\bar{g}_{j+1}^3,\end{aligned} \tag{3.17}$$

where the last inequality uses that $\bar{g}_j^3/\bar{g}_{j+1}^3 = 1 + O(g_0)$ whose elementary verification is given in Lemma 3.2.1(i) below, and that $g_0 > 0$ is sufficiently small. □

The sequence $\bar{x} = (\bar{K}_j, \bar{V}_j)_{j\in\mathbb{N}_0}$ is a flow of the dynamical system $\bar{\Phi} = (\psi, \bar{\varphi})$ in the sense of (3.6), with initial condition $(\bar{K}_0, \bar{g}_0) = (K_0, g_0)$ and final condition $(\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0)$. We consider this sequence as a function $\bar{x} : (K_0, g_0) \mapsto \bar{x} = \bar{x}(K_0, g_0)$ of the initial condition (K_0, g_0). Our main result is the following Theorem 3.1.4 about flows x of the dynamical system $\Phi = (\psi, \bar{\varphi} + \rho) = \bar{\Phi} + (0, \rho)$ of interest, as perturbations of the flows $\bar{x}$ of $\bar{\Phi}$.

Theorem 3.1.4. *Assume (A1–A3) with parameters* $(g_0, r, u, \kappa, \Omega, R, M)$*, let* $r_* \in (R/(1-\kappa\Omega), r)$*,* $b \in (0,1)$*. There exists* $u_* > 0$ *such that for all* $u \geq u_*$*, there exists* $g_* > 0$ *such that if* $g_0 \in (0, g_*]$ *and* $\|K_0\|_{\mathbf{K}_0} \leq r_* g_0^3$*, the following conclusions hold.*

(i) There exists a global flow $x = (K_j, V_j)_{j\in\mathbb{N}_0}$ *of* $\Phi = (\psi, \bar{\varphi} + \rho)$ *with initial condition* (K_0, g_0) *and final condition* $(z_\infty, \mu_\infty) = (0,0)$ *such that, with* $\bar{x} = \bar{x}(K_0, g_0)$*, the following estimates hold:*

$$\|K_j - \bar{K}_j\|_{\mathbf{K}_j} \leq b(r - r_*)\chi_j \bar{g}_j^3, \tag{3.18}$$

$$|g_j - \bar{g}_j| \leq bu\bar{g}_j^2 |\log \bar{g}_j|, \tag{3.19}$$

$$|z_j - \bar{z}_j| \leq bu\chi_j \bar{g}_j^2 |\log \bar{g}_j|, \tag{3.20}$$

$$|\mu_j - \bar{\mu}_j| \leq bu\chi_j \bar{g}_j^2 |\log \bar{g}_j|. \tag{3.21}$$

The sequence x *is the unique solution to* (3.6) *which obeys the boundary conditions and the bounds* (3.18)–(3.21)*.*

(ii) There is a neighbourhood $\mathsf{I} = \mathsf{I}(K_0, g_0) \subset \mathbf{K}_0 \oplus \mathbb{R}$ *of* (K_0, g_0) *such that, for initial conditions* $(K_0', g_0') \in \mathsf{I}$*, there also exists a global flow* x' *of* Φ *with* $(z_\infty', \mu_\infty') = (0,0)$*, and* (3.18)–(3.21) *hold with* x *replaced by* x' *and* $\bar{x}$ *replaced by* $\bar{x}' = \bar{x}(K_0', g_0')$*. Moreover, for all* $j \in \mathbb{N}_0$*, the maps* $(K_j, V_j) : \mathsf{I} \to \mathbf{K}_j \oplus \mathbf{V}$ *are continuously Fréchet differentiable, and*

$$\frac{\partial z_0}{\partial g_0} = O(1), \quad \frac{\partial \mu_0}{\partial g_0} = O(1). \tag{3.22}$$

Remark 3.1.5. (i) For $j_\Omega = \infty$ and with (3.10), the bounds (3.18) and (3.19)–(3.21) imply $\|K_j\|_{\mathbf{K}_j} = O(j^{-3})$ and $V_j = O(j^{-2}\log j)$, respectively. However, the latter bounds do not reflect that $K_j, V_j \to 0$ as $g_0 \to 0$, while the former do. Furthermore, (3.10) implies $\chi_j \bar{g}_j \to 0$ as $j \to \infty$ (also when $j_\Omega < \infty$), and thus (3.18) and (3.20)–(3.21) imply $K_j \to 0$, $z_j \to 0$, $\mu_j \to 0$ as $j \to \infty$. More precisely, these estimates imply $z_j, \mu_j = O(\chi_j \bar{g}_j)$ so that z_j and μ_j decay exponentially after the Ω-cut-off time j_Ω; we interpret this as indicating that the boundary condition $(z_\infty, \mu_\infty) = (0,0)$ is essentially achieved already at j_Ω.

(ii) We do not give a proof, but we expect that the error bounds in (3.18)–(3.21) have optimal decay as $j \to \infty$. Some indication of this can be found in Remark 3.3.2 below.

Theorem 3.1.4 is an analogue of a *stable manifold theorem* for the non-hyperbolic dynamical system defined by (3.5). It is inspired by [25, Theorem 2.16] which however holds only in the hyperbolic setting. Irwin [78] showed that the stable manifold theorem for hyperbolic dynamical systems is a consequence of the

implicit function theorem in Banach spaces (see also [99, 102]). Irwin's approach was inspired by Robbin [97], who showed that the local existence theorem for ordinary differential equations is a consequence of the implicit function theorem. By contrast, in our proof of Theorem 3.1.4, we directly apply the local existence theorem for ODEs, without explicit mention of the implicit function theorem. This turns out to be advantageous to deal with the lack of hyperbolicity.

Our choice of $\bar{\varphi}$ in (3.1) has a specific triangular form. One reason for this is that (3.1) accommodates what is required in our application in [9,19,38]. A second reason is that additional nonzero terms in $\bar{\varphi}$ can lead to the failure of Theorem 3.1.4. The condition that β_j is mainly non-negative is important for the sequence $\bar{g}_j$ of (3.7) to remain bounded. The following example shows that for the ζ_j term in the flow of $\bar{z}$, our sign restriction on ζ_j is also important, since positive ζ_j can lead to violation of a conclusion of Theorem 3.1.4.

Example 3.1.6. Suppose that $\zeta_j = \theta_j = \beta_j = 1$, that $\rho = 0$, and that $\bar{g}_0 > 0$ is small. For this constant β sequence, $j_\Omega = \infty$ (for any $\Omega > 1$) and hence $\chi_j = 1$ for all j. As in Example 3.1.1, $\bar{g}_j \sim j^{-1}$. By (3.1) and (3.7),

$$\bar{z}_{j+1} = \bar{z}_j(1 - \bar{g}_j) - \bar{g}_j^2 = \bar{z}_j \frac{\bar{g}_{j+1}}{\bar{g}_j} - \bar{g}_j^2. \tag{3.23}$$

Let $\bar{y}_j = \bar{z}_j/\bar{g}_j$. Since $\bar{g}_j/\bar{g}_{j+1} = (1 - \bar{g}_j)^{-1} \geq 1$, we obtain $\bar{y}_j \geq \bar{y}_{j+1} + \bar{g}_j$ and hence

$$\bar{y}_j \geq \bar{y}_{n+1} + \sum_{l=j}^{n} \bar{g}_l. \tag{3.24}$$

Suppose that $\bar{z}_j = O(\bar{g}_j)$, as in (3.20). Then $\bar{y}_j = O(1)$ and hence by taking the limit $n \to \infty$ we obtain

$$\bar{y}_j \geq \limsup_{n\to\infty} \left(\bar{y}_{n+1} + \sum_{l=j}^{n} \bar{g}_l \right) \geq -C + \sum_{l=j}^{\infty} \bar{g}_l. \tag{3.25}$$

However, since $\bar{g}_j \sim j^{-1}$, the last sum diverges. This contradiction implies that the conclusion $\bar{z}_j = O(\bar{g}_j)$ of (3.20) is impossible.

Because of its triangularity, an exact analysis of the flows of $\bar{\varphi}$ with the boundary conditions of interest is straightforward: the three equations for g, z, μ can be solved successively and we do this in Section 3.2 below. Triangularity does not hold for Φ, and we prove that the flows of Φ with the same boundary conditions nevertheless remain close to the flow of $\bar{\varphi}$ in Section 3.3.

3.1.4 Continuity in external parameter

The uniqueness statement of Theorem 3.1.4 implies the following Corollary 3.1.7 regarding continuous dependence on an external parameter of the solution to (3.6) given by Theorem 3.1.4.

Corollary 3.1.7. *Assume that the Φ_j depend continuously on an external parameter $m \in M_{\text{ext}}$ and that Assumptions (A1–A3) hold uniformly in m. Let $x(m) = (K(m), V(m))$ be the solution for external parameter m given by Theorem 3.1.4. Then $x_j(m)$ is continuous in m for each $j \in \mathbb{N}_0$.*

Proof. Theorem 3.1.4 implies that $V_0(m)$ is bounded uniformly in $m \in M_{\text{ext}}$. This implies that there exists some limit point V_0^* of $V_0(m')$ as $m' \to m$. Let $x_j^* = (V_j^*, K_j^*)$ be the flow of $\Phi(m, \cdot)$ started with this V_0^* and $K_0^* = K_0$ independent of m. By Proposition 3.1.2, $\bar{V}_0(m)$ is continuous in $m \in M_{\text{ext}}$. The continuity of $\bar{K}_j(m)$ follows inductively from this and the assumed continuity of the ψ_j and ρ_j. This continuity and (3.18)–(3.21) imply that any limit point x^* must satisfy

$$\|K_j^* - \bar{K}_j(m)\|_{\mathbf{K}_j} \le b(r - r_*)\chi_j(m)\bar{g}_j(m)^3, \tag{3.26}$$

$$|g_j^* - \bar{g}_j(m)| \le bu\bar{g}_j(m)^2|\log \bar{g}_j(m)|, \tag{3.27}$$

$$|\mu_j^* - \bar{\mu}_j(m)| \le bu\chi_j(m)\bar{g}_j(m)^2|\log \bar{g}_j(m)|, \tag{3.28}$$

$$|z_j^* - \bar{z}_j(m)| \le bu\chi_j(m)\bar{g}_j(m)^2|\log \bar{g}_j(m)|. \tag{3.29}$$

The uniqueness assertion of Theorem 3.1.4 implies that $x_j^* = x_j(m)$, and therefore that V_0 is continuous in m. The continuity of x_j now follows immediately from the continuity of the Φ_j. □

3.2 The quadratic flow

In this section, we prove that, for the quadratic approximation $\bar{\varphi}$, there is a unique solution $\bar{V} = (\bar{V}_j)_{j\in\mathbb{N}_0} = (\bar{g}_j, \bar{z}_j, \bar{\mu}_j)_{j\in\mathbb{N}_0}$ to the flow equation

$$\bar{V}_{j+1} = \bar{\varphi}_j(\bar{V}_j) \quad \text{with fixed small } \bar{g}_0 > 0 \text{ and with } (\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0). \tag{3.30}$$

Due to the triangular nature of $\bar{\varphi}$, we can obtain very detailed information about the sequence $\bar{V}$.

3.2.1 Flow of $\bar{g}$

We start with the analysis of the sequence $\bar{g}$, which obeys the recursion

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2, \qquad \bar{g}_0 > 0. \tag{3.31}$$

The following lemma collects the information we will need about $\bar{g}$.

Lemma 3.2.1. *Assume (A1). The following statements hold if $\bar{g}_0 > 0$ is sufficiently small, with all constants independent of j_Ω and $\bar{g}_0$.*

(i) For all j, $\bar{g}_j > 0$,

$$\bar{g}_j = O(\inf_{k\le j} \bar{g}_k), \quad \text{and} \quad \bar{g}_j \bar{g}_{j+1}^{-1} = 1 + O(\chi_j \bar{g}_j) = 1 + O(\bar{g}_0). \tag{3.32}$$

Moreover, for any j and k, $\bar{g}_j$ is non-increasing in β_k.

(ii) For $n \ge 1$ and $m \ge 0$, there exists $C_{n,m} > 0$ such that for all $k \ge j \ge 0$,

$$\sum_{l=j}^{k} \chi_l \bar{g}_l^n |\log \bar{g}_l|^m \le C_{n,m} \begin{cases} |\log \bar{g}_k|^{m+1} & n = 1 \\ \chi_j \bar{g}_j^{n-1} |\log \bar{g}_j|^m & n > 1, \end{cases} \tag{3.33}$$

and there exists $C > 0$ such that for all $j \ge 0$,

$$\chi_j \bar{g}_j \le \frac{C\bar{g}_0}{1 + \bar{g}_0 j}. \tag{3.34}$$

(iii) (a) For $\gamma \ge 0$ and $j \ge 0$, there exist constants $c_j = 1 + O(\chi_j \bar{g}_j)$ (depending on γ) such that, for all $l \ge j$,

$$\prod_{k=j}^{l} (1 - \gamma \beta_k \bar{g}_k)^{-1} = \left(\frac{\bar{g}_j}{\bar{g}_{l+1}}\right)^{\gamma} (c_j + O(\chi_l \bar{g}_l)). \tag{3.35}$$

(b) For $\zeta_j \le 0$ except for c^{-1} values of $j \le j_\Omega$, $\zeta_j = O(\chi_j)$, and $j \le l$, (with a constant independent of j and l),

$$\prod_{k=j}^{l} (1 - \zeta_k \bar{g}_k)^{-1} \le O(1). \tag{3.36}$$

(iv) Suppose that $\bar{g}$ and $\mathring{g}$ each satisfy (3.31)*. Let $\delta > 0$. If $|\mathring{g}_0 - \bar{g}_0| \le \delta \mathring{g}_0$ then $|\mathring{g}_j - \bar{g}_j| \le \delta \mathring{g}_j (1 + O(\bar{g}_0))$ for all j.*

Proof. (i) By (3.31),

$$\bar{g}_{j+1} = \bar{g}_j (1 - \beta_j \bar{g}_j). \tag{3.37}$$

Since $\beta_j = O(\chi_j)$, by (3.37) the second statement of (3.32) is a consequence the first, so it suffices to verify the first statement of (3.32). Assume inductively that $\bar{g}_j > 0$ and that $\bar{g}_j = O(\inf_{k\le j} \bar{g}_k)$. It is then immediate from (3.37) that $\bar{g}_{j+1} > 0$ if

$\bar{g}_0$ is sufficiently small depending on $\|\beta\|_\infty$, and that $\bar{g}_{j+1} \le \bar{g}_j$ if $\beta_j \ge 0$. By (A1), there are at most c^{-1} values of $j \le j_\Omega$ for which $\beta_j < 0$. Therefore, by choosing $\bar{g}_0$ sufficiently small depending on $\|\beta\|_\infty$ and c, it follows that $\bar{g}_j \le O(\inf_{k\le j} \bar{g}_k)$ for all $j \le j_\Omega$ with a constant that is independent of j_Ω.

To advance the inductive hypothesis for $j > j_\Omega$, we use $1 - t \le e^{-t}$ and $\sum_{l=j_\Omega}^{\infty} |\beta_l| \le \sum_{n=1}^{\infty} \Omega^{-n} = O(1)$ to obtain, for $j \ge k \ge j_\Omega$,

$$\bar{g}_j \le \bar{g}_k \exp\left[-\sum_{l=k}^{j-1} \beta_l \bar{g}_l\right] \le \bar{g}_k \exp\left[C\bar{g}_k \sum_{l=k}^{j-1} |\beta_l|\right] \le O(\bar{g}_k). \tag{3.38}$$

This shows that $\bar{g}_j = O(\inf_{j_\Omega \le k \le j} \bar{g}_k)$. However, by the inductive hypothesis, $\bar{g}_{j_\Omega} = O(\inf_{k\le j_\Omega} \bar{g}_k)$ for $j \le j_\Omega$, and hence for $j > j_\Omega$ we have $\bar{g}_j = O(\inf_{k\le j} \bar{g}_k)$ as claimed. This completes the verification of the first bound of (3.32) and thus, as already noted, also of the second.

The monotonicity of $\bar{g}_j$ in β_k can be proved as follows. Since it is obvious that $\bar{g}_j$ does not depend on β_k if $k \ge j$, we may assume that $k < j$. Moreover, by replacing j by $j + k$, we can assume that $k = 0$. Let $\bar{g}'_j = \partial \bar{g}_j / \partial \beta_0$. Clearly, $\bar{g}'_0 = 0$ and therefore

$$\bar{g}'_1 = -\bar{g}_0^2 < 0. \tag{3.39}$$

Assuming that $\bar{g}'_j < 0$ by induction, it follows that for $j \ge 1$,

$$\bar{g}'_{j+1} = \bar{g}'_j(1 - 2\beta_j \bar{g}_j) < 0, \tag{3.40}$$

and the proof of monotonicity is complete.

(ii) We first show that if $\psi : \mathbb{R}_+ \to \mathbb{R}$ is absolutely continuous, then

$$\sum_{l=j}^{k} \beta_l \psi(\bar{g}_l)\bar{g}_l^2 = \int_{\bar{g}_{k+1}}^{\bar{g}_j} \psi(t)\, dt + O\left(\int_{\bar{g}_{k+1}}^{\bar{g}_j} t^2 |\psi'(t)|\, dt\right). \tag{3.41}$$

To prove (3.41), we apply (3.31) to obtain

$$\sum_{l=j}^{k} \beta_l \psi(\bar{g}_l)\bar{g}_l^2 = \sum_{l=j}^{k} \psi(\bar{g}_l)(\bar{g}_l - \bar{g}_{l+1}) = \sum_{l=j}^{k} \int_{\bar{g}_{l+1}}^{\bar{g}_l} \psi(\bar{g}_l)\, dt. \tag{3.42}$$

The integral can be written as

$$\int_{\bar{g}_{l+1}}^{\bar{g}_l} \psi(\bar{g}_l)\, dt = \int_{\bar{g}_{l+1}}^{\bar{g}_l} \psi(t)\, dt + \int_{\bar{g}_{l+1}}^{\bar{g}_l} \int_{t}^{\bar{g}_l} \psi'(s)\, ds\, dt. \tag{3.43}$$

The first term on the right-hand side of (3.41) is then the sum over l of the first term on the right-hand side of (3.43), so it remains to estimate the double integral. By Fubini's theorem,

$$\int_{\bar g_{l+1}}^{\bar g_l}\int_t^{\bar g_l}\psi'(s)\,ds\,dt = \int_{\bar g_{l+1}}^{\bar g_l}\int_{\bar g_{l+1}}^{s}\psi'(s)\,dt\,ds = \int_{\bar g_{l+1}}^{\bar g_l}(s-\bar g_{l+1})\psi'(s)\,ds. \tag{3.44}$$

By (3.31) and (3.32), for $s\in[\bar g_{l+1},\bar g_l]$ we have

$$|s-\bar g_{l+1}| \le |\bar g_l - \bar g_{l+1}| = |\beta_l|\bar g_l^2 \le (1+O(\bar g_0))|\beta_l|\bar g_{l+1}^2 \le O(s^2). \tag{3.45}$$

This permits us to estimate (3.44) and conclude (3.41).

Direct evaluation of the integrals in (3.41) with $\psi(t)=t^{n-2}|\log t|^m$ gives

$$\sum_{l=j}^{k}\beta_l\bar g_l^n|\log\bar g_l|^m \le C_{n,m}\begin{cases}|\log\bar g_{k+1}|^{m+1} & n=1\\ \bar g_j^{n-1}|\log\bar g_j|^m & n>1.\end{cases} \tag{3.46}$$

To deduce (3.33), we only consider the case $n>1$, as the case $n=1$ is similar. Suppose first that $j\le j_\Omega$ (and $j_\Omega<\infty$). Assumption (A1) implies

$$1 \le \frac{\beta_l}{c} + \left(1+\frac{|\beta_l|}{c}\right)1_{\beta_l<c} \le O(\beta_l) + O(1_{\beta_l<c}) \tag{3.47}$$

and therefore that

$$\sum_{l=j}^{k}\chi_l\bar g_l^n|\log\bar g_l|^m \le \sum_{l=j}^{j_\Omega}O(\beta_l)\bar g_l^n|\log\bar g_l|^m + \sum_{l=j}^{j_\Omega}O(1_{\beta_l<c})\bar g_l^n|\log\bar g_l|^m + \sum_{l=j_\Omega+1}^{k}\Omega^{-(l-j_\Omega)_+}\bar g_l^n|\log\bar g_l|^m. \tag{3.48}$$

By (3.46), the first term is bounded by $O(\bar g_j^{n-1}|\log\bar g_j|^m)$. The second term obeys the same bound, by (A1) and (3.32), as does the last term due to the exponential decay. This proves (3.33) for the case $j\le j_\Omega$. On the other hand, if $j>j_\Omega$, then again using the exponential decay of χ_l and (3.32), we obtain

$$\sum_{l=j}^{k}\chi_l\bar g_l^n|\log\bar g_l|^m \le C\chi_j\bar g_j^n|\log\bar g_j|^m \le C\bar g_0\chi_j\bar g_j^{n-1}|\log\bar g_j|^m. \tag{3.49}$$

This completes the proof of (3.33) for the case $n>1$.

To prove (3.34), let $c > 0$ be as in Assumption (A1) and set $\hat{g}_{j+1} = \hat{g}_j - c\hat{g}_j$ with $\hat{g}_0 = \bar{g}_0$. Let $j_0 = -1$ and denote by $0 \le j_1 < j_2 < \dots$ the sequence of j such that $\beta_j < c$. By induction, we show that $\bar{g}_{j_k+1} \le (1 + O(\bar{g}_0))^k \hat{g}_{j_k+1}$. This is trivial for $k = 0$. To advance the induction, note that, since $\bar{g}_j$ is monotone in β, $\bar{g}_j \le (1 + O(\bar{g}_0))^k \hat{g}_j$ for $j \le j_{k+1}$, and therefore

$$\begin{aligned}\bar{g}_{j_{k+1}+1} = \bar{g}_{j_{k+1}}(1 - \beta_{j_{k+1}}\bar{g}_{j_{k+1}}) &\le (1 - \beta_{j_{k+1}}\bar{g}_{j_{k+1}})(1 + O(\bar{g}_0))^k \hat{g}_{j_{k+1}} \\ &= \frac{1 - \beta_{j_{k+1}}\bar{g}_{j_{k+1}}}{1 - c\hat{g}_{j_{k+1}}}(1 + O(\bar{g}_0))^k \hat{g}_{j_{k+1}+1}. \end{aligned} \tag{3.50}$$

The induction is advanced since

$$\frac{1 - \beta_{j_{k+1}}\bar{g}_{j_{k+1}}}{1 - c\hat{g}_{j_{k+1}}} = 1 + O(\bar{g}_0). \tag{3.51}$$

By Assumption (A1), $m = \max\{k : j_k \le j_\Omega\}$ is bounded so that, for $j \le j_\Omega$,

$$\chi_j \bar{g}_j = \bar{g}_j \le (1 + O(\bar{g}_0))^m \hat{g}_j \le (1 + O(\bar{g}_0))\hat{g}_j. \tag{3.52}$$

For $j > j_\Omega$, we use that, for $\bar{g}_0$ sufficiently small,

$$\Omega^{-1} \le 1 - c\bar{g}_0 \le 1 - c\hat{g}_j = \frac{\hat{g}_{j+1}}{\hat{g}_j} \tag{3.53}$$

and that, by (3.32), $\bar{g}_j = O(\bar{g}_{j_\Omega})$ which together imply

$$\chi_j \bar{g}_j \le O(\Omega^{-(j-j_\Omega)}\bar{g}_{j_\Omega}) \le O(\Omega^{-(j-j_\Omega)}\hat{g}_{j_\Omega}) \le O\left(\prod_{l=j_\Omega}^{j-1} \frac{\hat{g}_{l+1}}{\hat{g}_l}\right)\hat{g}_{j_\Omega} = O(\hat{g}_j). \tag{3.54}$$

The proof of (3.34) is concluded by the observation that $\hat{g}_j$ satisfies the bound claimed, as can be seen by applying (3.41) with $\psi(t) = t^{-2}$.

(iii-a) By Taylor's theorem and (3.31), there exists $r_k = O(\beta_k \bar{g}_k)^2$ such that

$$(1 - \gamma\beta_k\bar{g}_k)^{-1} = (1 - \beta_k\bar{g}_k)^{-\gamma}(1 + r_k) = \left(\frac{\bar{g}_k}{\bar{g}_{k+1}}\right)^\gamma (1 + r_k). \tag{3.55}$$

Let

$$c_{j,l} = \prod_{k=j}^{l}(1 + r_k). \tag{3.56}$$

With the bounds $1 + t \le e^{|t|}$ and $\beta_k = O(\chi_k)$, we obtain

$$\begin{aligned} \left|c_{j,l} - 1\right| &= \left|\sum_{k=j}^{l} r_k \prod_{m=k+1}^{l} (1 + r_m)\right| \\ &\le \sum_{k=j}^{l} O(\chi_k \bar{g}_k^2) \exp\left[\sum_{m=k+1}^{l} O(\chi_m \bar{g}_m^2)\right] \le O(\chi_j \bar{g}_j). \end{aligned} \tag{3.57}$$

In particular, $c_{j,l} = 1 + O(\bar{g}_0) = O(1)$ uniformly in j and l. Similarly, we obtain that, with $\tilde{r}_k = (1 + r_k)^{-1} - 1 = O(\chi_k \bar{g}_k^2)$, for $n \ge l$,

$$\begin{aligned} |c_{j,l} - c_{j,n}| &= c_{j,n} \left|\prod_{k=l}^{n} (1 + r_k)^{-1} - 1\right| \\ &= c_{j,n} \left|\sum_{k=l}^{n} \tilde{r}_k \prod_{m=k+1}^{n} (1 + \tilde{r}_m)\right| \le O(\chi_l \bar{g}_l). \end{aligned} \tag{3.58}$$

In particular, $(c_{j,l})_l$ is a Cauchy sequence, $c_j = \lim_{l\to\infty} c_{j,l}$ exists, and with (3.57), $c_j = 1 + O(\chi_j \bar{g}_j)$. It also follows that $|c_{j,l} - c_j| \le O(\chi_l \bar{g}_l)$ as claimed, and the proof is complete.

(iii-b) Since $\zeta_j \le 0$ for all but c^{-1} values of $j \le j_\Omega$, by (3.32) with $\bar{g}_0$ sufficiently small, $\prod_{k=j}^{l} (1 - \zeta_k \bar{g}_k)^{-1} \le O(1)$ for $l \le j_\Omega$, with a constant independent of j_Ω. For $j \ge j_\Omega$, we use $1/(1 - x) \le 2e^x$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$ to obtain

$$\prod_{k=j}^{l} (1 - \zeta_k \bar{g}_k)^{-1} \le 2 \exp\left[\sum_{k=j}^{l} \zeta_k \bar{g}_k\right] \le 2 \exp\left[C \bar{g}_j \sum_{k=j_\Omega}^{\infty} |\chi_k|\right] \le O(1). \tag{3.59}$$

The bounds for $l \le j_\Omega$ and $j \ge j_\Omega$ together imply (3.36).

(iv) If $|\mathring{g}_j - \bar{g}_j| \le \delta_j \mathring{g}_j$ then by (3.31),

$$|\mathring{g}_{j+1} - \bar{g}_{j+1}| = |\mathring{g}_j - \bar{g}_j|(1 - \beta_j(\mathring{g}_j + \bar{g}_j)) \le \delta_{j+1} \mathring{g}_{j+1} \tag{3.60}$$

with

$$\delta_{j+1} = \delta_j \frac{1 - \beta_j(\mathring{g}_j + \bar{g}_j)}{1 - \beta_j \mathring{g}_j} = \delta_j \left(1 - \frac{\beta_j \bar{g}_j}{1 - \beta_j \mathring{g}_j}\right). \tag{3.61}$$

In particular, if $\beta_j \ge 0$, then $\delta_{j+1} \le \delta_j$. By (A1), there are at most c^{-1} values of $j \le j_\Omega$ for which $\beta_j < 0$, and hence $\delta_j \le \delta(1 + O(\bar{g}_0))$ for $j \le j_\Omega$. The desired estimate therefore holds for $j \le j_\Omega$. For $j \ge l > j_\Omega$, as in (3.38) we have

$$\prod_{k=l}^{j} (1 + O(\beta_k \bar{g}_k)) \le \exp\left[O(\bar{g}_l) \sum_{k=l}^{j} \chi_k\right] \le 1 + O(\bar{g}_0), \tag{3.62}$$

and thus the claim remains true also for $j > j_\Omega$. □

3.2.2 Flow of $\bar{z}$ and $\bar{\mu}$ and proof of Proposition 3.1.2

We now establish the existence of unique solutions to the $\bar{z}$ and $\bar{\mu}$ recursions with boundary conditions $\bar{z}_\infty = \bar{\mu}_\infty = 0$, and obtain estimates on these solutions.

Lemma 3.2.2. *Assume (A1–A2). If $\bar{g}_0$ is sufficiently small then there exists a unique solution to* (3.30) *obeying $\bar{z}_\infty = \bar{\mu}_\infty = 0$. This solution obeys $\bar{z}_j = O(\chi_j \bar{g}_j)$ and $\bar{\mu}_j = O(\chi_j \bar{g}_j)$. Furthermore, if the maps $\bar{\varphi}_j$ depend continuously on an external parameter $m \in M_{\text{ext}}$ such that (A1–A2) hold with uniform constants, then $\bar{z}_j$ and $\bar{\mu}_j$ are continuous in M_{ext}.*

Proof. By (3.1), $\bar{z}_{j+1} = \bar{z}_j - \zeta_j \bar{g}_j \bar{z}_j - \theta_j \bar{g}_j^2$, so that

$$\bar{z}_j = \prod_{k=j}^{n} (1 - \zeta_k \bar{g}_k)^{-1} \bar{z}_{n+1} + \sum_{l=j}^{n} \prod_{k=j}^{l} (1 - \zeta_k \bar{g}_k)^{-1} \theta_l \bar{g}_l^2. \tag{3.63}$$

In view of (3.36), whose assumptions are satisfied by (A2), the unique solution to the recursion for $\bar{z}$ which obeys the boundary condition $\bar{z}_\infty = 0$ is

$$\bar{z}_j = \sum_{l=j}^{\infty} \prod_{k=j}^{l} (1 - \zeta_k \bar{g}_k)^{-1} \theta_l \bar{g}_l^2, \tag{3.64}$$

and by (A2), (3.33), and (3.36),

$$|\bar{z}_j| \le \sum_{l=j}^{\infty} O(\chi_l) \bar{g}_l^2 \le O(\chi_j \bar{g}_j). \tag{3.65}$$

To verify continuity of $\bar{z}_j$ in an external parameter, let $\bar{z}_{j,n} = \sum_{l=j}^{n} \prod_{k=j}^{l} (1 - \zeta_k \bar{g}_k)^{-1} \theta_l \bar{g}_l^2$. Clearly, since $\bar{g}_j$ is continuous in M_{ext} for any $j \ge 0$, $\bar{z}_{j,n}$ is also continuous, for any $j \le n$. (3.33)–(3.34) of Lemma 3.2.1(ii) imply that $|\bar{z}_j - \bar{z}_{j,n}| \le O(\chi_n \bar{g}_n) \to 0$ uniformly, as $n \to \infty$, and thus, as a uniform limit of continuous functions, it follows that $\bar{z}_j$ must be continuous in M_{ext}.

For $\bar{\mu}$, we first define

$$\sigma_j = +\eta_j \bar{g}_j + \gamma_j \bar{z}_j - \upsilon_j^{gg} \bar{g}_j^2 - \upsilon_j^{gz} \bar{g}_j \bar{z}_j - \upsilon_j^{zz} \bar{z}_j^2, \quad \tau_j = \upsilon_j^{g\mu} \bar{g}_j + \upsilon_j^{z\mu} \bar{z}_j, \tag{3.66}$$

so that the recursion for $\bar{\mu}$ can be written as

$$\bar{\mu}_{j+1} = (\lambda_j - \tau_j) \bar{\mu}_j + \sigma_j. \tag{3.67}$$

Alternatively,

$$\bar{\mu}_j = (\lambda_j - \tau_j)^{-1} (\bar{\mu}_{j+1} - \sigma_j). \tag{3.68}$$

Given $\alpha \in (\lambda^{-1}, 1)$, we can choose $\bar{g}_0$ sufficiently small that

$$\tfrac{1}{2}\lambda^{-1} \le (\lambda_j - \tau_j)^{-1} \le \alpha. \tag{3.69}$$

The limit of repeated iteration of (3.68) gives

$$\bar{\mu}_j = -\sum_{l=j}^{\infty} \left(\prod_{k=j}^{l} (\lambda_k - \tau_k)^{-1} \right) \sigma_l \tag{3.70}$$

as the unique solution which obeys the boundary condition $\mu_\infty = 0$. Geometric convergence of the sum is guaranteed by (3.69), together with the fact that $\sigma_j \le O(\chi_j \bar{g}_j) \le O(1)$. To estimate (3.70), we use

$$|\bar{\mu}_j| \le \sum_{l=j}^{\infty} \alpha^{l-j+1} O(\chi_l \bar{g}_l). \tag{3.71}$$

Since $\alpha < 1$, the first bound of (3.32) and monotonicity of χ imply that

$$|\bar{\mu}_j| \le O(\chi_j \bar{g}_j). \tag{3.72}$$

The proof of continuity of $\bar{\mu}_j$ in M_{ext} is analogous to that for $\bar{z}_j$. The proof is complete. □

Proof of Proposition 3.1.2. (3.10) follows immediately from Lemma 3.2.1(ii) and Lemma 3.2.2. Since $\bar{g}_j$ is defined by a finite recursion, its continuity in $m \in M_{\text{ext}}$ is trivial. The continuity of $\bar{z}_j$ and $\bar{\mu}_j$ was proved in Lemma 3.2.2. □

3.2.3 Differentiation of quadratic flow

The following lemma gives estimates on the derivatives of the components of $\bar{V}_j$ with respect to the initial condition $\bar{g}_0$. We write f' for the derivative of f with respect to $\bar{g}_0$. These estimates will be an ingredient in the proof of Theorem 3.1.4(ii).

Lemma 3.2.3. *For each $j \ge 0$, $\bar{V}_j = (\bar{g}_j, \bar{z}_j, \bar{\mu}_j)$ is twice differentiable with respect to the initial condition $\bar{g}_0 > 0$, and the derivatives obey*

$$\bar{g}_j' = O\left(\frac{\bar{g}_j^2}{\bar{g}_0^2}\right), \qquad \bar{z}_j' = O\left(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^2}\right), \qquad \bar{\mu}_j' = O\left(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^2}\right), \tag{3.73}$$

$$\bar{g}_j'' = O\left(\frac{\bar{g}_j^2}{\bar{g}_0}\right), \qquad \bar{z}_j'' = O\left(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^3}\right), \qquad \bar{\mu}_j'' = O\left(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^3}\right). \tag{3.74}$$

Proof. Differentiation of (3.7) gives

$$\bar{g}'_{j+1} = \bar{g}'_j(1 - 2\beta_j \bar{g}_j), \tag{3.75}$$

from which we conclude by iteration and $\bar{g}'_0 = 1$ that for $j \geq 1$,

$$\bar{g}'_j = \prod_{l=0}^{j-1}(1 - 2\beta_l \bar{g}_l). \tag{3.76}$$

Therefore, by (3.35),

$$\bar{g}'_j = \left(\frac{\bar{g}_j}{\bar{g}_0}\right)^2 (1 + O(\bar{g}_0)). \tag{3.77}$$

For the second derivative, we use $\bar{g}''_0 = 0$ and $\bar{g}''_{j+1} = \bar{g}''_j(1 - 2\beta_j \bar{g}_j) - 2\beta_j \bar{g}'^2_j$ to obtain

$$\bar{g}''_j = -2\sum_{l=0}^{j-1} \beta_l \bar{g}'^2_l \prod_{k=l}^{j-2}(1 - 2\beta_k \bar{g}_k). \tag{3.78}$$

With the bounds of Lemma 3.2.1, this gives

$$\bar{g}''_j = O\left(\frac{\bar{g}_j}{\bar{g}_0}\right)^2 \sum_{l=0}^{j-1} \beta_l \bar{g}'^2_l = O\left(\frac{\bar{g}'^2_j}{\bar{g}_0}\right). \tag{3.79}$$

For $\bar{z}$, we define $\sigma_{j,l} = \prod_{k=j}^{l}(1 - \zeta_k \bar{g}_k)^{-1}$. Then (3.64) becomes $\bar{z}_j = \sum_{l=j}^{\infty} \sigma_{j,l}\theta_l \bar{g}'^2_l$. By (3.36), $\sigma_{j,l} = O(1)$. It then follows from (A2), (3.77), and Lemma 3.2.1(ii,iii-b) that

$$\sigma'_{j,l} = \sigma_{j,l} \sum_{k=j}^{l}(1 - \zeta_k \bar{g}_k)^{-1}\zeta_k \bar{g}'_k = \sum_{k=j}^{l} O(\zeta_k \bar{g}'_k) = O\left(\chi_j \frac{\bar{g}_j}{\bar{g}'^2_0}\right). \tag{3.80}$$

We differentiate (3.64) and apply (3.77) and Lemma 3.2.1(ii) to obtain

$$\bar{z}'_j = \sum_{l=j}^{\infty} \sigma'_{j,l}\theta_l \bar{g}'^2_l + 2\sum_{l=j}^{\infty} \sigma_{j,l}\theta_l \bar{g}_l \bar{g}'_l = O\left(\chi_j \frac{\bar{g}'^2_j}{\bar{g}'^2_0}\right). \tag{3.81}$$

Similarly, $\sigma''_{j,l} = O(\bar{g}'^2_j/\bar{g}'^4_0)$ and

$$\bar{z}''_j = \sum_{l=j}^{\infty} \sigma''_{j,l}\theta_l \bar{g}'^2_l + 4\sum_{l=j}^{\infty} \sigma'_{j,l}\theta_l \bar{g}_l \bar{g}'_l + 2\sum_{l=j}^{\infty} \sigma_{j,l}\theta_l(\bar{g}_l \bar{g}''_l + \bar{g}'^2_l) = O\left(\chi_j \frac{\bar{g}'^2_j}{\bar{g}'^3_0}\right) \tag{3.82}$$

using the fact that $\bar{g}_j^3/\bar{g}_0^4 = O(\bar{g}_j^2/\bar{g}_0^3)$ by (3.32). It is straightforward to justify the differentiation under the sum in (3.81)–(3.82).

For $\bar{\mu}_j$, we recall from (3.69)–(3.70) that

$$\bar{\mu}_j = -\sum_{l=j}^{\infty}\left(\prod_{k=j}^{l}(\lambda_k - \tau_k)^{-1}\right)\sigma_l, \tag{3.83}$$

with τ_j and σ_l given by (3.66), and with $0 \le (\lambda_j - \tau_j)^{-1} \le \alpha < 1$. This gives

$$\bar{\mu}_j' = -\sum_{l=j}^{\infty}\left(\prod_{k=j}^{l}(\lambda_k - \tau_k)^{-1}\right)\left(\sigma_l' + \sum_{i=j}^{l}(\lambda_i - \tau_i)^{-1}\tau_i'\right). \tag{3.84}$$

The first product is bounded by α^{l-j+1}, and this exponential decay, together with (3.66), (3.65), and the bounds just proved for $\bar{g}'$ and $\bar{z}'$, lead to the upper bound $|\bar{\mu}_j'| \le O(\chi_j \bar{g}_j^2 \bar{g}_0^{-2})$ claimed in (3.73). Straightforward further calculation leads to the bound on $\bar{\mu}_j''$ claimed in (3.74) (the leading behaviour can be seen from the $\bar{z}_j''$ contribution to the σ_l'' term). □

3.3 Proof of main result

In this section, we prove Theorem 3.1.4. We begin in Section 3.3.1 with a sketch of the main ideas, without entering into details. The remainder of Section 3.3 expands the sketch into a complete proof.

3.3.1 Proof strategy

Two difficulties in proving Theorem 3.1.4 arflow-e: (i) from the point of view of dynamical systems, the evolution map Φ is not hyperbolic; and (ii) from the point of view of nonlinear differential equations, a priori bounds that any solution to (3.6) must satisfy are not readily available due to the presence of both initial and final boundary conditions.

Our strategy is to consider the one-parameter family of evolution maps $\Phi = (\Phi^t)_{t\in[0,1]}$ defined by

$$\Phi^t(x) = \Phi(t,x) = (\psi(x), \bar{\varphi}(x) + t\rho(x)) \quad \text{for } t \in [0,1], \tag{3.85}$$

with the t-independent boundary conditions that K_0 and g_0 are given and that $z_\infty = 0$ and $\mu_\infty 0$). This family interpolates between the problem $\Phi^1 = \Phi$ we are interested in, and the simpler problem $\Phi^0 = \bar{\Phi} = (\psi, \bar{\varphi})$. The unique solution

for $\bar{\Phi}$ is $\bar{x}_j = (\bar{K}_j, \bar{V}_j)$, where $\bar{V}$ is the unique solution of $\bar{\varphi}$ from Section 3.2, and where $\bar{K}_j$ is defined inductively for $j \geq 0$ by

$$\bar{K}_{j+1} = \psi_j(\bar{V}_j, \bar{K}_j), \qquad \bar{K}_0 = K_0. \tag{3.86}$$

We refer to $\bar{x}$ as the *approximate flow*.

We seek a t-dependent global flow x which obeys the generalisation of (3.6) given by

$$x_{j+1} = \Phi_j^t(x_j). \tag{3.87}$$

Assuming that $x_j = x_j(t)$ is differentiable in t for each $j \in \mathbb{N}_0$, we set

$$\dot{x}_j = \frac{\partial}{\partial t} x_j. \tag{3.88}$$

Then differentiation of (3.87) shows that a family of flows $x = (x_j(t))_{j\in\mathbb{N}_0, t\in[0,1]}$ must satisfy the infinite nonlinear system of ODEs

$$\dot{x}_{j+1} - D_x\Phi_j(t, x_j)\dot{x}_j = \rho_j(x_j), \quad x_j(0) = \bar{x}_j. \tag{3.89}$$

Conversely, any solution $x(t)$ to (3.89), for which each x_j is continuously differentiable in t, gives a global flow for each Φ^t.

We claim that (3.89) can be reformulated as a well-posed nonlinear ODE

$$\dot{x} = F(t, x), \quad x(0) = \bar{x}, \tag{3.90}$$

in a Banach space of sequences $x = (x_0, x_1, \dots)$ with carefully chosen weights, and for a suitable nonlinear functional F. To see this, consider the *linear* equation

$$y_{j+1} - D_x\Phi_j(t, x_j)y_j = r_j, \tag{3.91}$$

where the sequences x and r are held fixed. Its solution with the same boundary conditions as stated below (3.85) is written as $y = S(t, x)r$. Then we define F, which we consider as a map on sequences, by

$$F(t, x) = S(t, x)\rho(x). \tag{3.92}$$

Thus $y = F(t, x)$ obeys the equation $y_{j+1} - D_x\Phi_j(t, x_j)y_j = \rho_j(x)$, and hence (3.90) is equivalent to (3.89) with the same boundary conditions.

The main work in the proof is to obtain good estimates for $S(t, x)$, in the Banach space of weighted sequences, which allow us to treat (3.90) by the standard theory of ODE. We establish bounds on the solution simultaneously with existence, via the weights in the norm. These weights are useful to obtain bounds on the solution, but they are also essential in the formulation of the problem as a well-posed ODE.

As we will see in more detail in Section 3.3.4, the occurrence of $D_x\Phi_j(t, x_j)$ in (3.89), rather than the naive linearisation $D_x\Phi_j(0)$ at the "fixed point" $x = 0$, replaces the eigenvalue 1 in the upper left corner of the square matrix in (3.1) by a smaller eigenvalue $1 - 2\beta_j g_j < 1$. This helps address difficulty (i) mentioned above. Also, the weights guarantee that a solution in the Banach space obeys the final conditions $(z_\infty, \mu_\infty) = (0, 0)$, thereby helping to solve difficulty (ii).

3.3.2 Sequence spaces and weights

We now introduce the Banach spaces of sequences used in the reformulation of (3.89) as an ODE. These are weighted l^∞-spaces.

Definition 3.3.1. Let X^* be the space of sequences $x = (x_j)_{j\in\mathbb{N}_0}$ with $x_j \in X_j$. For each $\alpha = K, g, z, \mu$ and $j \in \mathbb{N}_0$, we fix a positive weight $w_{\alpha,j} > 0$. We write $x_j \in X_j = \mathbf{K}_j \oplus \mathbf{V}$ as $x_j = (x_{\alpha,j})_{\alpha=K,g,z,\mu}$. Let

$$\|x_j\|_{X_j^w} = \max_{\alpha=K,g,z,\mu} (w_{\alpha,j})^{-1}\|x_{\alpha,j}\|_{X_j}, \quad \|x\|_{X^w} = \sup_{j\in\mathbb{N}_0} \|x_j\|_{X_j^w}, \tag{3.93}$$

and

$$X^w = \{x \in X^* : \|x\|_{X^w} < \infty\}. \tag{3.94}$$

It is not difficult to check that X^w is a Banach space for any positive weight sequence w. Different choices of weights w will be needed. These are all defined in terms of the sequence $\mathring{g} = (\mathring{g}_j)_{j\in\mathbb{N}_0}$ which is the same as the sequence $\bar{g}$ for a *fixed* $\mathring{g}_0$; i.e., given $\mathring{g}_0 > 0$, it satisfies $\mathring{g}_{j+1} = \mathring{g}_j - \beta_j\mathring{g}_j^2$. We define the two weights $\mathsf{w} = \mathsf{w}(\mathring{g}_0, r, u)$ and $\mathsf{r} = \mathsf{r}(\mathring{g}_0, r, u)$ by

$$\mathsf{w}_{\alpha,j} = \begin{cases} (r - r_*)\mathring{g}_j^3\chi_j & \alpha = K \\ u\mathring{g}_j^2|\log\mathring{g}_j| & \alpha = g \\ u\mathring{g}_j^2|\log\mathring{g}_j|\chi_j & \alpha = z, \mu, \end{cases} \qquad \mathsf{r}_{\alpha,j} = \begin{cases} (r - r_*)\mathring{g}_j^3\chi_j & \alpha = K \\ u\mathring{g}_j^3\chi_j & \alpha = g \\ u\mathring{g}_j^3\chi_j & \alpha = z, \mu, \end{cases} \tag{3.95}$$

where (χ_j) is the Ω-dependent sequence defined by (3.9). Furthermore, we recall that $\bar{x} = (\bar{K}, \bar{V}) = \bar{x}(K_0, g_0)$ denotes the sequence in X^* uniquely determined from the boundary conditions $(\bar{K}_0, \bar{g}_0) = (K_0, g_0)$ and $(\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0)$ via $\bar{V}_{j+1} = \bar{\varphi}_j(\bar{V}_j)$ and $\bar{K}_{j+1} = \psi_j(\bar{K}_j, \bar{V}_j)$, whenever the latter is well-defined. Given an initial condition $(\mathring{K}_0, \mathring{g}_0)$, let $\mathring{x} = \bar{x}(\mathring{K}_0, \mathring{g}_0)$.

Denoting the closed ball of radius s in X^w by $s\mathsf{B}$, observe that, if $\mathring{g}_0 = g_0$ and $\mathring{K}_0 = K_0$, the bounds (3.18)–(3.21) are equivalent to $x \in \mathring{x} + b\mathsf{B}$, and that, by definition, the projection of $\mathring{x} + \mathsf{B}$ onto the the jth sequence element is contained in the domain D_j. We will always assume that $g_0 = \bar{g}_0$ and $\mathring{g}_0$ are close, but not necessarily that they are equal. The use of $\mathring{g}$ rather than $\bar{g}$ permits us to vary the

initial condition $g_0 = \bar{g}_0$ without changing the Banach spaces $X^{\mathsf{w}}, X^{\mathsf{r}}$. The use of g_0-dependent weights rather than, e.g., the weight $j^{-2}\log j$ for $j_\Omega = \infty$ (see Remark 3.1.5(i)) allows us to obtain estimates with good behaviour as $g_0 \to 0$. Note that the weight $\mathsf{w}_{g,j}$ does not include a factor χ_j, and thus does not go to 0 when $j_\Omega < \infty$ (see Example 3.1.1(ii)).

Remark 3.3.2. The weights w apply to the sequence $\dot{x}$ (see (3.88)). As motivation for their definition, consider the explicit example of $\rho_j(x_j) = \chi_j g_j^3$. In this case, the g equation becomes simply

$$g_{j+1} = g_j - \beta_j g_j^2 + t\chi_j g_j^3. \tag{3.96}$$

With the notation $\dot{g}_j = \frac{\partial}{\partial t} g_j^t$, differentiation gives

$$\dot{g}_{j+1} = \dot{g}_j(1 - 2\beta_j g_j + 3t\chi_j g_j^2) + \chi_j g_j^3. \tag{3.97}$$

Thus, by iteration, using $\dot{g}_0 = 0$, we obtain

$$\dot{g}_j = \sum_{l=0}^{j-1} \chi_l g_l^3 \prod_{k=l+1}^{j-1} (1 - 2\beta_k g_k + 3t\chi_k g_k^3). \tag{3.98}$$

For simplicity, consider the case $t = 0$, for which $g = \bar{g}$. In this case, it follows from (3.35), (3.32), and (3.46) that

$$\dot{g}_j \le O(1) \sum_{l=0}^{j-1} \left(\frac{\bar{g}_j}{\bar{g}_{l+1}}\right)^2 \chi_l \bar{g}_l^3 = O(1) g_j^2 \sum_{l=0}^{j-1} \chi_l \bar{g}_l \le O(\bar{g}_j^2 |\log \bar{g}_j|), \tag{3.99}$$

which produces the weight $\mathsf{w}_{g,j}$ of (3.95). (It can also be verified using (3.41) that if we replace χ_j by β_j in the above then no smaller weight will work.)

3.3.3 Reduction to a linear equation with nonlinear perturbation

For given sequences $x, r \in X^*$, we now consider the equation

$$y_{j+1} - D_x \Phi_j(t, x_j) y_j = r_j. \tag{3.100}$$

For x and r fixed, (3.100) is an inhomogeneous linear equation in y. Lemma 3.3.3 below, which lies at the heart of the proof of Theorem 3.1.4, obtains bounds on solutions to (3.100), including bounds on its x-dependence. The latter will allow us to use the standard theory of ODE in Banach spaces to treat the original nonlinear equation, where x and r are both functionals of the solution y, as a perturbation of the linear equation.

In addition to the decomposition $X_j = \mathbf{K}_j \oplus \mathbf{V}_j$, with $x_j \in X_j$ written $x_j = (K_j, V_j)$, it will be convenient to also use the decomposition $X_j = E_j \oplus F_j$ with $E_j = \mathbf{K}_j \oplus \mathbb{R}$ and $F_j = \mathbb{R} \oplus \mathbb{R}$, for which we write $x_j = (u_j, v_j)$ with $u_j = (K_j, g_j)$ and $v_j = (z_j, \mu_j)$. We denote by π_α the projection operator onto the α-component of the space in which it is applied, with α in any of $\{K, V\}$, $\{u, v\} = \{(K, g), (z, \mu)\}$, or $\{K, g, z, \mu\}$.

Recall that the spaces of sequences X^w are defined in Definition 3.3.1 and the specific weights w and r in (3.95).

Lemma 3.3.3. *Assume (A1–A3). There exists a constant C_S, independent of r and u, and a constant $C_S' = C_S'(r, u)$, such that if $\mathring{g}_0 > 0$ is sufficiently small, the following hold for all $t \in [0, 1]$, $x \in \mathring{x} + \mathsf{B}$.*

(i) For $r \in X^{\mathsf{r}}$, there exists a unique solution $y = S(t, x)r \in X^{\mathsf{w}}$ of (3.100) *with boundary conditions $\pi_u y_0 = 0$, $\pi_v y_\infty = 0$.*

(ii) The linear solution operator $S(t, x)$ satisfies

$$\|S(t, x)\|_{L(X^{\mathsf{r}}, X^{\mathsf{w}})} \leq C_S. \tag{3.101}$$

(iii) As a map $S : [0, 1] \times (\mathring{x} + \mathsf{B}) \to L(X^{\mathsf{w}}, X^{\mathsf{r}})$, the solution operator is continuously Fréchet differentiable and satisfies

$$\|D_x S(t, x)\|_{L(X^{\mathsf{w}}, L(X^{\mathsf{r}}, X^{\mathsf{w}}))} \leq C_S'. \tag{3.102}$$

Lemma 3.3.3 needs to be supplemented with information about the initial condition $\bar{x}$ and the perturbation ρ for the analysis of (3.90) with (3.92). (Note that the sequence $\bar{x}$ serves as initial condition, at $t = 0$, for the ODE (3.89), not as initial condition for the flow equation (3.5).) Some information about $\bar{x}$ is already contained in Lemma 3.2.2. For ρ, we define $\rho : \mathring{x} + \mathsf{B} \to X^*$ by

$$(\rho(x))_0 = 0, \quad (\rho(x))_{j+1} = \rho_j(x_j), \tag{3.103}$$

where ρ_j is the map of (3.5). The map $\psi : \mathring{x} + \mathsf{B} \to X^*$ is defined analogously. The next lemma expresses immediate consequences of Assumption (A3) for ρ and ψ in terms of the weighted spaces. Although the proof of Theorem 3.1.4 only directly requires the estimates for ρ, we will also need bounds on ψ to prove Lemma 3.3.3, so for convenience we combine both in a single lemma.

Lemma 3.3.4. *Assume (A3), let $\omega > \kappa\Omega$, and assume that $\mathring{g}_0 > 0$ is sufficiently small. Then $(\psi, \rho) : \mathring{x} + \mathsf{B} \to X^{\mathsf{r}}$ is twice continuously Fréchet differentiable,*

$$\|\rho(x)\|_{X^{\mathsf{r}}} \leq M/u, \tag{3.104}$$

and there exists a constant $C = C(r, u)$ such that

$$\|D_K\rho(x)\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le C, \quad \|D_V\rho(x)\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le O(\mathring{g}_0|\log\mathring{g}_0|),$$
$$\|D_K\psi(x)\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le \omega, \quad \|D_V\psi(x)\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le O(\mathring{g}_0|\log\mathring{g}_0|), \tag{3.105}$$

and

$$\|D_x^2\rho(x)\|_{L^2(X^{\mathsf{w}},X^{\mathsf{r}})} \le C, \quad \|D_x^2\psi(x)\|_{L^2(X^{\mathsf{w}},X^{\mathsf{r}})} \le C. \tag{3.106}$$

We defer the proofs of Lemmas 3.3.3–3.3.4 to Sections 3.3.4 and 3.3.5, respectively. Given these, we now prove Theorem 3.1.4(i).

Proof of Theorem 3.1.4(i). Let C_S be the constant of Lemma 3.3.3, define $u_* = C_S M/(\frac{1}{2}b \wedge (1-b))$, and assume $u > u_*$. For $t \in [0,1]$ and $x \in \mathring{x} + \mathsf{B}$, let

$$F(t,x) = S(t,x)\rho(x). \tag{3.107}$$

Let $(\mathring{K}_0, \mathring{g}_0) = (K_0, g_0)$. Lemmas 3.3.3–3.3.4 imply that if $\mathring{g}_0 > 0$ is sufficiently small, $F : [0,1] \times (\mathring{x} + \mathsf{B}) \to X^{\mathsf{w}}$ is continuously Fréchet differentiable and

$$\|F(t,x)\|_{X^{\mathsf{w}}} \le \|S(t,x)\|_{L(X^{\mathsf{r}},X^{\mathsf{w}})}\|\rho(x)\|_{X^{\mathsf{r}}} \le C_S M/u \le \frac{1}{2}b \wedge (1-b). \tag{3.108}$$

Similarly, by the product rule, it follows that there is C such that

$$\|D_xF(t,x)\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} \le \|[D_xS(t,x)]\rho(x)\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} + \|S(t,x)[D_x\rho(x)]\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} \le C, \tag{3.109}$$

and thus, in particular, that F is Lipschitz continuous on $x \in \mathring{x} + \mathsf{B}$.

The theorem now follows from the well-known local existence theory for ODE in Banach spaces. Indeed, for $y \in \mathsf{B}$, let

$$\mathring{F}(t,y) = F(t, \mathring{x} + y). \tag{3.110}$$

Let $X_0^{\mathsf{w}} = \{y \in X^{\mathsf{w}} : \pi_u y_0 = 0\}$, $\mathsf{B}_0 = \mathsf{B} \cap X_0^{\mathsf{w}}$. Then the statement about boundary conditions of Lemma 3.3.3(i) and (3.108) imply that $\mathring{F}(t, \frac{1}{2}b\mathsf{B}_0) \subseteq \mathring{F}(t, \mathsf{B}_0) \subseteq \frac{1}{2}b\mathsf{B}_0$. With (3.108)–(3.109), the local existence theory for ODEs on Banach spaces [2, Chapter 2, Lemma 1] implies that the initial value problem

$$\dot{y} = \mathring{F}(t,y), \quad y(0) = 0 \tag{3.111}$$

has a unique C^1-solution $y : [0,1] \to X_0^{\mathsf{w}}$ with $y(t) \in \frac{1}{2}b\mathsf{B}_0$ for all $t \in [0,1]$. (The length of the existence interval of the initial value problem (3.111) in $\frac{1}{2}b\mathsf{B}$ is

bounded from below by $\frac{1}{2}b/(\frac{1}{2}b \wedge (1-b)) \geq 1$ because $\|\mathring{F}(t,y)\| \leq \frac{1}{2}b \wedge (1-b)$ when $\|y\| \leq \frac{1}{2}b$. It does not depend on the Lipschitz constant of $\mathring{F}$.)

In particular, as discussed around (3.90), it follows that $x = \mathring{x} + y(1)$ is a solution to (3.6). By construction, $\pi_u x_0 = \pi_u \mathring{x}_0 = (\mathring{K}_0, \mathring{g}_0) = (K_0, g_0)$. Also, $\pi_v y_\infty(1) = 0$ because $y(1) \in X^{\mathsf{w}}$, and since $\pi_v \mathring{x}_\infty = 0$, it is also true that $\pi_v x_\infty = 0$. Thus x satisfies the required boundary conditions. The estimates (3.18)–(3.21) are an immediate consequence of $\|y\|_{X^{\mathsf{w}}} \leq \frac{1}{2}b$, with (3.95).

To prove uniqueness, suppose that x' is a solution to (3.6) with boundary conditions $(K_0', g_0') = (K_0, g_0)$ and $(z_\infty', \mu_\infty') = (0,0)$, and such that (3.18)–(3.21) hold (with x replaced by x', and with $\bar{x}$ as before). Let $\mathring{x} = \bar{x}$ as before. By assumption, $x' - \mathring{x} \in b\mathsf{B}_0$. It follows that $F : [0,1] \times (x' + (1-b)\mathsf{B}_0) \to X^{\mathsf{w}}$ is Fréchet differentiable and $\|F(t,x)\|_{X^{\mathsf{w}}} \leq 1-b$ for all $t \in [0,1]$ and for all $x \in x' + (1-b)\mathsf{B}_0 \subset \mathring{x} + \mathsf{B}_0$ as discussed around (3.107)–(3.109). In particular there is a unique solution $x'(t)$ for $t \in [0,1]$ to $\dot{x}' = F(t,x')$ with $x'(1) = x'$ and $x'(t) \subset \mathring{x} + \mathsf{B}_0$, by considering the ODE backwards in time, which is equally well-posed. It follows that $x'(0)$ is a flow of $\Phi^0 = \bar{\Phi}$ with the same boundary conditions as $\mathring{x}$. The uniqueness of such flows, by Lemma 3.2.2, implies that $x'(0) = \mathring{x}$, and the uniqueness of solutions to the initial value problem (3.111) in $\mathring{x} + \mathsf{B}_0$ then also that $x = x'$ as claimed. This completes the proof of Theorem 3.1.4(i). □

To prove Theorem 3.1.4(ii), we need to know that the initial condition $\bar{x}$ is differentiable in a small ball $\mathring{x} + \delta\mathsf{B}$. The smoothness of $\bar{x}$ is addressed in the following lemma, whose proof is deferred to Section 3.3.5.

Lemma 3.3.5. *Assume (A1–A3), and let $\underline{\delta} > 0$ and $\mathring{g}_0 > 0$ both be sufficiently small. Then there exists a neighbourhood $\bar{\mathsf{I}} = \bar{\mathsf{I}}_\delta \subset \mathbf{K}_0 \oplus \mathbb{R}_+$ of $(\mathring{K}_0, \mathring{g}_0)$ such that $\bar{x} : \bar{\mathsf{I}} \to \mathring{x} + \delta\mathsf{B}$ is continuously Fréchet differentiable with*

$$\|D_{g_0}\bar{x}\|_{X^{\mathsf{w}}} \leq O(\mathring{g}_0^{-2}|\log \mathring{g}_0|^{-1}). \tag{3.112}$$

Proof of Theorem 3.1.4(ii). For fixed initial condition $(\mathring{K}_0, \mathring{g}_0) = (K_0, g_0) = u_0$ obeying the hypothesis of Theorem 3.1.4(i), let $\bar{\mathsf{I}}$ be the neighbourhood of u_0 defined by Lemma 3.3.5 with $\delta < \frac{1}{2}b$. By Lemma 3.3.5, $\bar{x} : \bar{\mathsf{I}} \to \mathring{x} + \delta\mathsf{B} \subset X^{\mathsf{w}}$ is continuously Fréchet differentiable. It follows from [2, Chapter 2, Lemma 4] that

$$\dot{y} = \mathring{F}(t,y), \quad y(0) = \bar{x}(u_0) - \mathring{x} \tag{3.113}$$

has a unique C^1-solution $y : [0,1] \times \bar{\mathsf{I}} \to X_0^{\mathsf{w}}$ with $\|y(t)\|_{X^{\mathsf{w}}} \leq \frac{1}{2}b$. [2, Chapter 2, Lemma 4] and Lemma 3.3.5 also imply

$$\left\|D_{g_0}y(t,K_0,g_0)\right\|_{X^{\mathsf{w}}} \leq C\left\|D_{g_0}\bar{x}(K_0,g_0)\right\|_{X^{\mathsf{w}}} \leq O(\mathring{g}_0^{-2}|\log \mathring{g}_0|^{-1}). \tag{3.114}$$

Let $x(u_0) = \mathring{x} + y(1, u_0)$. It follows as previously that $x(u_0) = (u(u_0), v(u_0))$ is a solution to (3.6) with boundary conditions $u(u_0) = u_0$ and $v_\infty(u_0) = 0$. Moreover, the differentiability in the sequence space X^{w} implies in particular that, as elements of the spaces X_j, each $x_j = (K_j, V_j)$ is a C^1 function of u_0. Also (3.114) with (3.95) immediately implies that

$$\frac{\partial z_0}{\partial g_0} = O(1), \quad \frac{\partial \mu_0}{\partial g_0} = O(1). \tag{3.115}$$

To prove (3.18)–(3.21) for $x(u_0)$ with $u_0 \in \mathsf{I} \subseteq \overline{\mathsf{I}}$, we use that $\|x(u_0) - \mathring{x}\| \leq \frac{1}{2}b$ and $\|\bar{x}(u_0) - \mathring{x}\|_{X^{\mathsf{w}}} \leq \delta$ imply

$$\|K_j - \bar{K}_j\|_{\mathbf{K}_j} \leq \|K_j - \mathring{K}_j\|_{\mathbf{K}_j} + \|\mathring{K}_j - \bar{K}_j\|_{\mathbf{K}_j} \leq (\tfrac{1}{2}b + \delta)(r - r_*)\mathring{g}_j^3 \tag{3.116}$$

and analogously that

$$|g_j - \bar{g}_j| \leq (\tfrac{1}{2}b + \delta)u\mathring{g}_j^2 |\log \mathring{g}_j^2| \tag{3.117}$$

$$|z_j - \bar{z}_j| \leq (\tfrac{1}{2}b + \delta)u\chi_j\mathring{g}_j^2 |\log \mathring{g}_j^2| \tag{3.118}$$

$$|\mu_j - \bar{\mu}_j| \leq (\tfrac{1}{2}b + \delta)u\chi_j\mathring{g}_j^2 |\log \mathring{g}_j^2|. \tag{3.119}$$

Since $(\frac{1}{2}b + \delta) < b$, by assuming that $|\mathring{g}_0 - \bar{g}_0|$ is sufficiently small, i.e., shrinking $\overline{\mathsf{I}}$ to a smaller neighborhood I if necessary, we obtain with (3.73) that

$$(\tfrac{1}{2}b + \delta)\mathring{g}_j^2 |\log \mathring{g}_j^2| \leq b\bar{g}_j^2 |\log \bar{g}_j^2|. \tag{3.120}$$

This completes the proof of Theorem 3.1.4(ii). □

It now remains only to prove Lemmas 3.3.3–3.3.5. We begin with Lemma 3.3.3, which lies at the heart of the proof.

3.3.4 Proof of Lemma 3.3.3

The proof proceeds in three steps. The first two steps concern an approximate version of the equation and the solution of the approximate equation, and the third step treats (3.100) as a small perturbation of this approximation.

Step 1. Approximation of the linear equation

Define $\bar{\Phi}_j^0 : X_j \to X_{j+1}$ by extending $\bar{\varphi}_j$ trivially to the K-component, i.e., $\bar{\Phi}_j^0 = (0, \bar{\varphi}_j)$ with respect to the decomposition $X_{j+1} = \mathbf{K}_{j+1} \oplus \mathbf{V}$. Thus $\Phi(t, x) =$

$\bar{\Phi}^0(x) + (\psi(x), t\rho(x))$. Explicit computation of the derivative of $\bar{\varphi}_j$ of (3.5), using (3.1), shows that

$$D\bar{\Phi}^0_j(x_j) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1-2\beta_j g_j & 0 & 0 \\ \hline 0 & -\tilde{\xi}_j & 1-2\zeta_j g_j & 0 \\ 0 & \tilde{\eta}_j & -\tilde{\gamma}_j & \tilde{\lambda}_j \end{array}\right), \tag{3.121}$$

with

$$\begin{aligned} \tilde{\eta}_j &= \eta_j - 2\upsilon_j^{gg} g_j - \upsilon_j^{gz} z_j - \upsilon_j^{g\mu} \mu_j, \\ \tilde{\gamma}_j &= \gamma_j - \upsilon_j^{gz} g_j - 2\upsilon_j^{zz} z_j - \upsilon_j^{z\mu} \mu_j, \\ \tilde{\lambda}_j &= \lambda_j - \upsilon_j^{g\mu} g_j - \upsilon_j^{z\mu} z_j, \\ \tilde{\xi}_j &= 2\theta_j g_j + 2\zeta_j z_j. \end{aligned} \tag{3.122}$$

The block matrix structure in (3.121) is with respect to the decomposition $X_j = E_j \oplus F_j$ introduced in Section 3.3.3.

The matrix $D\bar{\Phi}^0_j(x_j)$ depends on $x_j \in X_j$, but it is convenient to approximate it by the constant matrix

$$L_j = D\bar{\Phi}^0_j(\mathring{x}_j) = \begin{pmatrix} A_j & 0 \\ B_j & C_j \end{pmatrix}, \tag{3.123}$$

where the blocks A_j, B_j, and C_j of L_j are defined by evaluating the blocks of the matrix (3.121) at $\mathring{x}_j$ rather than at x_j (given explicitly in (3.129) below). We will study the equation

$$y_{j+1} = L_j y_j + r_j, \tag{3.124}$$

which approximates (3.100). Lemma 3.3.6 below provides a useful reformulation of (3.124). For its statement, we define linear operators $H : D(H) \to X^*$ and $U : D(U) \to X^*$ (where $D(H)$ and $D(U)$ are the subspaces of X^* on which the infinite sums converge) by

$$\pi_u H = 0, \quad (\pi_v H x)_j = -\sum_{l=j}^{\infty} C_j^{-1} \cdots C_l^{-1} B_l \pi_u x_l, \tag{3.125}$$

and

$$\begin{aligned} (\pi_u U x)_j &= \sum_{l=0}^{j-1} A_{j-1} \cdots A_{l+1} \pi_u x_l, \\ (\pi_v U x)_j &= -\sum_{l=j}^{\infty} C_j^{-1} \cdots C_l^{-1} \pi_v x_l. \end{aligned} \tag{3.126}$$

It follows from the definitions (recalling $\pi_K A_j = 0 = A_j \pi_K$) that

$$\pi_K H = 0 = H\pi_K, \quad \pi_V H = H = H\pi_V, \quad \pi_K U = U\pi_K, \quad \pi_V U = U\pi_V. \tag{3.127}$$

The empty product in the formula for $\pi_u U x$ is interpreted as the identity, so the term in the sum corresponding to $l = j-1$ is simply $\pi_u x_j$.

Lemma 3.3.6. *Assume (A1–A2) and that $\mathring{g}_0 > 0$ is sufficiently small. If $r \in D(U)$ and $y \in D(H)$ satisfies $\pi_u y_0 = 0$ and $\pi_v y_\infty = 0$, then* (3.124) *holds if and only if*

$$y = Hy + Ur, \tag{3.128}$$

holds.

The proof is straightforward, but requires an estimate on the product of the matrices C_j which we will prove first. Products of the C_j and A_j will also play an important role in the analysis of the operators H and U in the following section, so that it is convenient to prove a more precise statement about them now than what is needed for the proof of Lemma 3.3.6. Let us first record explicitly the blocks of L_j:

$$A_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 - 2\beta_j \mathring{g}_j \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 & -\mathring{\xi}_j \\ 0 & \mathring{\eta}_j \end{pmatrix}, \quad C_j = \begin{pmatrix} 1 - 2\zeta_j \mathring{g}_j & 0 \\ -\mathring{\gamma}_j & \mathring{\lambda}_j \end{pmatrix} \tag{3.129}$$

with $\mathring{\eta}_j$, $\mathring{\gamma}_j$, $\mathring{\lambda}_j$, and $\mathring{\xi}_j$ as in (3.122) with x replaced by $\mathring{x}$.

Lemma 3.3.7. *Assume (A1–A2). Let $\alpha \in (\lambda^{-1}, 1)$. Then for $\mathring{g}_0 > 0$ sufficiently small (depending on α), the following hold.*

(i) Uniformly in all $l \le j$,

$$A_j \cdots A_l = \begin{pmatrix} 0 & 0 \\ 0 & O(\mathring{g}_{j+1}^2 / \mathring{g}_l^2) \end{pmatrix}. \tag{3.130}$$

(ii) Uniformly in all j,

$$B_j = \begin{pmatrix} 0 & O(\mathring{g}_j \chi_j) \\ 0 & O(\chi_j) \end{pmatrix}. \tag{3.131}$$

(iii) Uniformly in all $l \ge j$,

$$C_j^{-1} \cdots C_l^{-1} = \begin{pmatrix} O(1) & 0 \\ O(\chi_j) & O(\alpha^{l-j+1}) \end{pmatrix}. \tag{3.132}$$

Proof. (i) It follows immediately from (3.129) that

$$A_j \cdots A_l = \prod_{k=l}^{j} (1 - 2\beta_k \mathring{g}_k)\pi_g, \tag{3.133}$$

and thus (3.35) implies (i).

(ii) It follows directly from (3.129) and Lemma 3.2.2 that (3.131) holds.

(iii) Note that

$$\begin{pmatrix} c_1 & 0 \\ b_1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} c_n & 0 \\ b_n & a_n \end{pmatrix} = \begin{pmatrix} c^* & 0 \\ b^* & a^* \end{pmatrix} \tag{3.134}$$

with

$$a^* = a_1 \cdots a_n, \quad b^* = \sum_{i=1}^{n} a_1 \cdots a_{i-1} b_i c_{i+1} \cdots c_n, \quad c^* = c_1 \cdots c_n. \tag{3.135}$$

We apply this formula with the inverse matrices

$$C_j^{-1} = \begin{pmatrix} (1 - 2\zeta_j \mathring{g}_j)^{-1} & 0 \\ (1 - 2\zeta_j \mathring{g}_j)^{-1} \mathring{\gamma}_j \mathring{\alpha}_j & \mathring{\alpha}_j \end{pmatrix} \tag{3.136}$$

where $\mathring{\alpha}_j = \mathring{\lambda}_j^{-1}$. Thus

$$C_j^{-1} \cdots C_l^{-1} = \begin{pmatrix} \mathring{\tau}_{j,l} & 0 \\ \mathring{\sigma}_{j,l} & \mathring{\alpha}_{j,l} \end{pmatrix} \tag{3.137}$$

with

$$\mathring{\alpha}_{j,l} = \mathring{\alpha}_j \cdots \mathring{\alpha}_l, \qquad \mathring{\tau}_{j,l} = \prod_{k=j}^{l} (1 - 2\zeta_k \mathring{g}_k)^{-1}, \tag{3.138}$$

$$\mathring{\sigma}_{j,l} = \sum_{i=1}^{l-j+1} \left(\prod_{k=j+i}^{l} (1 - 2\zeta_k \mathring{g}_k)^{-1} \right) \mathring{\gamma}_{j+i-1} \left(\prod_{k=j}^{j+i-2} \mathring{\alpha}_k \right). \tag{3.139}$$

The product defining $\mathring{\tau}_{j,l}$ is $O(1)$ by (3.36). Assume that $\mathring{g}_0$ is sufficiently small that, with Lemma 3.2.2 and (A2), $\mathring{\alpha}_m < \alpha$ for all m. Then $\mathring{\alpha}_{j,l} \le O(\alpha^{l-j+1})$. Similarly, since $\mathring{\gamma}_m \le O(\chi_m)$,

$$|\mathring{\sigma}_{j,l}| \le \sum_{i=1}^{l-j+1} \alpha^i O(\chi_{j+i-1}) \le O(\chi_j). \tag{3.140}$$

This completes the proof. □

Proof of Lemma 3.3.6. The u-component of (3.124) is given by

$$u_{j+1} = A_j u_j + \pi_u r_j. \tag{3.141}$$

By induction, under the initial condition $u_0 = 0$ this recursion is equivalent to

$$u_j = \pi_u y_j = \sum_{l=0}^{j-1} A_{j-1} \cdots A_{l+1} \pi_u r_l, \tag{3.142}$$

which is the same as the u-component of (3.128).

The v-component of (3.124) states that

$$v_{j+1} = B_j u_j + C_j v_j + \pi_v r_j, \tag{3.143}$$

and this is equivalent to

$$v_j = C_j^{-1} v_{j+1} - C_j^{-1} B_j u_j - C_j^{-1} \pi_v r_j. \tag{3.144}$$

By induction, for any $k \geq j$, the latter is equivalent to

$$v_j = C_j^{-1} \cdots C_k^{-1} v_{k+1} - \sum_{l=j}^{k} C_j^{-1} \cdots C_l^{-1} (B_l u_l + \pi_v r_l). \tag{3.145}$$

By Lemma 3.3.7(iii), with some $\alpha \in (\lambda^{-1}, 1)$ and with $\mathring{g}_0$ sufficiently small, $\|C_0^{-1} \cdots C_k^{-1}\|$ is uniformly bounded. Thus, if $y_j = (u_j, v_j)$ satisfies (3.124) and $v_j \to 0$, then $C_0^{-1} \cdots C_k^{-1} v_{k+1} \to 0$ and hence

$$v_j = -\sum_{l=j}^{\infty} C_j^{-1} \cdots C_l^{-1} (B_l u_l + \pi_v r_l), \tag{3.146}$$

which is the same as the v-component of (3.128). Conversely, suppose that y_j satisfies (3.128) and $v_j \to 0$. It is also straightforward to conclude that (3.146) implies (3.145) and thus that the v-component of y satisfies (3.124). □

Step 2. Solution of the approximate equation

We now prove existence, uniqueness, and bounds for the solution to the approximate equation (3.124).

Lemma 3.3.8. *Assume (A1–A2) and that $\mathring{g}_0 > 0$ is sufficiently small. For each $r \in X^{\mathsf{r}}$ and $x \in \mathring{x} + \mathsf{B}$, there exists a unique solution $y = S^0 r \in X^{\mathsf{w}}$ to* (3.124)

obeying the boundary conditions $\pi_u y_0 = 0$, $\pi_v y_\infty = 0$. The solution operator S^0 is block diagonal w.r.t. the decomposition $x = (K, V)$, with

$$S^0 = \begin{pmatrix} 1 & 0 \\ 0 & S^0_{VV} \end{pmatrix}, \tag{3.147}$$

and there is a constant $C_{S^0} > 0$ such that, uniformly in small $\mathring{g}_0$,

$$\|S^0_{VV}\|_{L(X^{\mathsf{r}}, X^{\mathsf{w}})} \leq C_{S^0}. \tag{3.148}$$

The constant C_{S^0} is independent of u and r.

Proof. According to Lemma 3.3.6, it suffices to prove that there is a unique solution in X^{w} to (3.128) (instead of (3.124)) which obeys the required boundary conditions. Observe that as a block matrix with respect to the decomposition $x = (u, v)$, with $H_{vu} = \pi_v H \pi_u$, the operator $1 - H$ is triangular of the form

$$1 - H = \begin{pmatrix} 1 & 0 \\ -H_{vu} & 1 \end{pmatrix}. \tag{3.149}$$

We will prove that H_{vu} is a bounded operator in $L(X^{\mathsf{w}}, X^{\mathsf{w}})$. It follows that $1 - H$ has a bounded inverse on X^{w} given by the block matrix

$$(1 - H)^{-1} = \begin{pmatrix} 1 & 0 \\ H_{vu} & 1 \end{pmatrix}. \tag{3.150}$$

We further show that U is a bounded operator in $L(X^{\mathsf{r}}, X^{\mathsf{w}})$. This implies that the unique solution in X^{w} of (3.124) is given by

$$y = S^0 r = (1 - H)^{-1} U r \tag{3.151}$$

and, since $\pi_u (1 - H)^{-1} = \pi_u$ and $\pi_K U = \pi_K$, that (3.147)–(3.148) hold.

The boundary condition $\pi_v y_\infty = 0$ is a consequence of $y \in X^{\mathsf{w}}$, and the initial condition $\pi_u y_0 = 0$ is implicit in the equation (3.128). The claim that $\pi_K S^0 = S^0 \pi_K$ and $\pi_V S^0 = S^0 \pi_V$ then follows from (3.127). Since $\pi_u S^0 r = \pi_u U r$, the cases $\alpha = K, g$ of (3.148) follow from the bounds claimed for U.

To complete the proof, we require estimates for $\pi_\alpha U$ for $\alpha \in \{K, g, z, \mu\}$, and on $\pi_\alpha H$ for $\alpha = z, \mu$. Thus there are six estimates in all. Their treatment is similar, and uses Lemma 3.2.1(ii), which gives that for all $k \geq j \geq 0$ and $m \geq 0$,

$$\sum_{l=j}^{k} \chi_l \mathring{g}_l^n |\log \mathring{g}_l|^m \leq C_{n,m} \begin{cases} |\log \mathring{g}_k|^{m+1} & n = 1 \\ \chi_j \mathring{g}_j^{n-1} |\log \mathring{g}_j|^m & n > 1. \end{cases} \tag{3.152}$$

(i) Bound for K-component. By definition, since $\pi_K A_l = 0$, we have $\pi_K U = \pi_K$. Therefore,

$$\|\pi_K U r\|_{X^{\mathsf{w}}} \le \sup_j \|\pi_K r_j\|_{X_j^{\mathsf{w}}} \le \sup_j [\mathsf{w}_{K,j}^{-1}\mathsf{r}_{K,j}]\,\|r\|_{X^{\mathsf{r}}} = \|r\|_{X^{\mathsf{r}}}. \tag{3.153}$$

(ii) Bound for g-component. By Lemma 3.3.7(i), (3.95), (3.32), and (3.152),

$$\begin{aligned}\|\pi_g U r\|_{X^{\mathsf{w}}} &\le \sup_j \sum_{l=0}^{j-1} \|\pi_g A_{j-1}\cdots A_{l+1} r_l\|_{X_j^{\mathsf{w}}} \le \sup_j \sum_{l=0}^{j-1} \mathsf{w}_{V,j}^{-1}\mathsf{r}_{V,l} O(\mathring{g}_j/\mathring{g}_l)^2 \|r\|_{X^{\mathsf{r}}} \\ &\le c\|r\|_{X^{\mathsf{r}}} \sup_j |\log \mathring{g}_j|^{-1} \sum_{l=0}^{j-1} \chi_l \mathring{g}_l \le c\|r\|_{X^{\mathsf{r}}}. \end{aligned} \tag{3.154}$$

(iii) Bound for z-component. By Lemma 3.3.7(iii), (3.95), and (3.152),

$$\begin{aligned}\|\pi_z U r\|_{X^{\mathsf{w}}} &\le \sup_j \sum_{l=j}^{\infty} \|\pi_z C_j^{-1}\cdots C_l^{-1} r_l\|_{X_l^{\mathsf{w}}} \\ &\le c \sup_j \mathsf{h}_V \mathsf{w}_{V,j}^{-1} \sum_{l=j}^{\infty} \chi_l \mathring{g}_l^3 \|r\|_{X^{\mathsf{r}}} \le c|\log \mathring{g}_0|^{-1}\|r\|_{X^{\mathsf{r}}}. \end{aligned} \tag{3.155}$$

Similarly, by Lemma 3.3.7(ii-iii), (3.95), and (3.152),

$$\begin{aligned}\|\pi_z H\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} &\le \sup_j \sum_{l=j}^{\infty} \|\pi_z C_j^{-1}\cdots C_l^{-1} B_l\|_{L(X_l^{\mathsf{w}},X_j^{\mathsf{w}})} \\ &\le c \sup_j \mathsf{w}_{V,j}^{-1} \sum_{l=j}^{\infty} \chi_l \mathring{g}_l \mathsf{w}_{V,l} \le c. \end{aligned} \tag{3.156}$$

(iv) Bound for μ-component. Using Lemma 3.3.7(iii), we obtain

$$\begin{aligned}\|\pi_\mu U r\|_{X^{\mathsf{w}}} &\le \sup_j \Big[\sum_{l=j}^{\infty} \|\pi_\mu C_j^{-1}\cdots C_l^{-1} r_l\|_{X_j^{\mathsf{w}}} \Big] \\ &\le c \sup_j u\mathsf{w}_{V,j}^{-1} \Big[\sum_{l=j}^{\infty} \chi_l \mathring{g}_l^3 + \sum_{l=j}^{\infty} \alpha^{l-j+1} \chi_l \mathring{g}_l^3 \Big] \|r\|_{X^{\mathsf{r}}} \\ &\le c|\log \mathring{g}_0|^{-1}\|r\|_{X^{\mathsf{r}}}, \end{aligned} \tag{3.157}$$

where we used (3.152) and also that $\sum_{l=j}^{\infty} \alpha^{l+1-j} \chi_l \mathring{g}_l^3 \le c\chi_j \mathring{g}_j^3$ in the last step. To bound $\|\pi_\mu H\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})}$, we argue similarly as for $\pi_\mu U r$, and use Lemma 3.3.7 to

obtain

$$\|\pi_\mu H\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} \le \sup_j \sum_{l=j}^{\infty} \|\pi_\mu C_j^{-1}\cdots C_l^{-1}B_l\|_{L(X_l^{\mathsf{w}},X_j^{\mathsf{w}})}$$
$$\le c \sup_j \mathsf{w}_{V,j}^{-1}\left[\sum_{l=j}^{\infty} \mathring{g}_j \chi_j \mathsf{w}_{V,l} + \sum_{l=j}^{\infty} \alpha^{l+1-j}\chi_j \mathsf{w}_{V,l}\right] \le c. \quad (3.158)$$

This proves the required bounds for $\alpha = \mu$ and thus completes the proof. □

Step 3. Solution of the linear equation

We now prove Lemma 3.3.3, which involves solving the equation (3.100).

Proof of Lemma 3.3.3. Fix $\omega \in (\kappa\Omega, 1)$.
(i) We define

$$W_j(t,x_j) = D_x\Phi_j(t,x_j) - L_j$$
$$= [D_x\bar{\Phi}_j^0(x_j) - D_x\bar{\Phi}_j^0(\mathring{x})] + D_x(\psi_j(x_j), t\rho_j(x_j)), \quad (3.159)$$

and rewrite (3.100) as

$$y_{j+1} = D_x\Phi_j(t,x_j)y_j + r_j = L_j y_j + W_j(t,x_j)y_j + r_j. \quad (3.160)$$

It will be convenient to combine the $W_j(t,x)$ to an operator on sequences via $(W(t,x))_0 = 0$ and $(W(t,x))_{j+1} = W_j(t,x)$. This operator can be written as a block matrix with respect to the decomposition $x = (K,V)$ as

$$W(t,x) = \begin{pmatrix} W_{KK} & W_{KV} \\ W_{VK} & W_{VV} \end{pmatrix}, \quad (3.161)$$

with $W_{\alpha\beta} = \pi_\alpha W(t,x)\pi_\beta$. We claim that $W : [0,1]\times(\mathring{x}+\mathsf{B}) \to L(X^{\mathsf{w}},X^{\mathsf{r}})$, that W is continuously Fréchet differentiable, and that if $x \in \mathring{x}+\mathsf{B}$ then,

$$\|W_{KK}\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le \omega, \quad \|W_{VK}\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le C,$$
$$\|W_{KV}\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le o(1), \quad \|W_{VV}\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le o(1), \quad (3.162)$$

as $\mathring{g}_0 \to 0$, and

$$\|D_x W_j(t,x_j)\|_{L(X_j^{\mathsf{w}},L(X_j^{\mathsf{w}},X_{j+1}^{\mathsf{r}}))} \le C. \quad (3.163)$$

To see this, note that the first term on the right-hand side of (3.159) only depends on the V-components, and is continuously Fréchet differentiable since, by (3.121),

$D^2\bar{\Phi}^0_j$ is a constant matrix for each j with coefficients bounded by $O(\chi_j)$. Therefore, for $x \in \mathring{x} + \mathsf{B}$,

$$\begin{aligned}\|[D\bar{\Phi}^0_j(\mathring{x}_j) - D\bar{\Phi}^0_j(x_j)]\pi_V\|_{L(X^{\mathsf{w}}_j, X^{\mathsf{r}}_{j+1})} &\le c\chi_j \mathsf{r}^{-1}_{V,j+1}\mathsf{w}^2_{V,j}\|\mathring{x}_j - x_j\|_{X^{\mathsf{w}}_j} \\ &= O(u\mathring{g}_0|\log \mathring{g}_0|^2). \end{aligned} \tag{3.164}$$

This contributes to the bounds (3.162), with $\mathring{g}_0$ taken small enough. The second term on the right-hand side of (3.159), as well as its derivative, have been bounded in Lemma 3.3.4, completing the proof of (3.163).

By the assumption that $y \in X^{\mathsf{w}}$, Lemma 3.3.8, and (3.162), the equation (3.160) with the boundary conditions of Lemma 3.3.3(i) is equivalent to

$$y = S^0(W(t,x)y + r). \tag{3.165}$$

(ii) To solve this equation, we use that if A and B are bounded operators on a Banach space such that A has a bounded inverse A^{-1} and $\|A^{-1}B\| < 1$, then $A - B$ has a bounded inverse. (Indeed, $A - B = A(1 - A^{-1}B)$ and the inverse of $1 - A^{-1}B$ is given by the Neumann series.) As in (3.147), we write S^0 as a block matrix with respect to the decomposition $x = (K, V)$ as

$$S^0 = \begin{pmatrix} 1 & 0 \\ 0 & S^0_{VV} \end{pmatrix}. \tag{3.166}$$

Let

$$A = \begin{pmatrix} 1 - W_{KK} & 0 \\ -S^0_{VV}W_{VK} & 1 - S^0_{VV}W_{VV} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & W_{KV} \\ 0 & 0 \end{pmatrix} \tag{3.167}$$

such that $1 - S^0W(t,x) = A - B$. Then (3.162) with $\mathring{g}_0$ sufficiently small implies $\|W_{KK}\|_{L(X^{\mathsf{w}}, X^{\mathsf{w}})} < 1$ and $\|S^0_{VV}W_{VV}\|_{L(X^{\mathsf{w}}, X^{\mathsf{w}})} < 1$. Thus A is a block matrix of the form

$$A = \begin{pmatrix} A_{KK} & 0 \\ A_{VK} & A_{VV} \end{pmatrix} \tag{3.168}$$

where A_{KK} and A_{VV} have inverses in $L(X^{\mathsf{w}}, X^{\mathsf{w}})$, and it follows that A has the bounded inverse on X^{w} given by the block matrix

$$A^{-1} = \begin{pmatrix} A^{-1}_{KK} & 0 \\ A^{-1}_{VV}A_{VK}A^{-1}_{KK} & A^{-1}_{VV} \end{pmatrix}. \tag{3.169}$$

Moreover, (3.162) with $\mathring{g}_0$ sufficiently small implies that $\|A^{-1}B\|_{L(X^{\mathsf{w}}, X^{\mathsf{w}})} < 1$ and thus that $1 - S^0W(t,x)$ has a bounded inverse in $L(X^{\mathsf{w}}, X^{\mathsf{w}})$. It follows that the solution operator is given by

$$S(t,x) = (1 - S^0W(t,x))^{-1}S^0. \tag{3.170}$$

(iii) By (3.170), continuous Fréchet differentiability in x of $S(t,x)$ follows from the continuous Fréchet differentiability of $S^0W(t,x)$, which itself follows from part (i) and from $D_xS^0W(t,x) = S^0D_xW(t,x)$ by linearity of S^0. Explicitly,

$$D_xS(t,x) = (1 - S^0W(t,x))^{-1}D_xS^0W(t,x)(1 - S^0W(t,x))^{-1}S^0. \tag{3.171}$$

By (3.163),

$$\|D_xS^0W(t,x)\|_{L(X^{\mathsf{w}},L(X^{\mathsf{w}},X^{\mathsf{w}}))} \le C\|D_xW(t,x)]\|_{L(X^{\mathsf{w}},L(X^{\mathsf{w}},X^{\mathsf{r}}))} \le C. \tag{3.172}$$

Together with the boundedness of the operators $(1 - S^0W(t,x))^{-1}$ and S^0, this proves (3.102) and completes the proof. □

3.3.5 Proofs of Lemmas 3.3.4–3.3.5

Proof of Lemma 3.3.4. We begin with the verification of the bounds on the first derivatives in (3.104). By assumptions (3.13)–(3.14), together with (3.32), the definition of the weights (3.95), and for (3.174) also the fact that $\chi_j/\chi_{j+1} \le \Omega$ by (3.9), we obtain for $x \in \mathring{x} + \mathsf{B}$,

$$\|D_V\psi_j(x_j)\|_{L(X_j^{\mathsf{w}},X_{j+1}^{\mathsf{r}})} \le M\chi_j\mathring{g}_j^2\mathsf{r}_{K,j+1}^{-1}\mathsf{w}_{V,j} \le O(\mathring{g}_0|\log\mathring{g}_0|), \tag{3.173}$$

$$\|D_K\psi_j(x_j)\|_{L(X_j^{\mathsf{w}},X_{j+1}^{\mathsf{r}})} \le \kappa\mathsf{r}_{K,j+1}^{-1}\mathsf{w}_{K,j} \le \kappa\Omega(1 + O(\mathring{g}_0)), \tag{3.174}$$

$$\|D_V\rho_j(x_j)\|_{L(X_j^{\mathsf{w}},X_{j+1}^{\mathsf{r}})} \le M\chi_j\mathring{g}_j^2\mathsf{r}_{V,j+1}^{-1}\mathsf{w}_{V,j} \le O(\mathring{g}_0|\log\mathring{g}_0|), \tag{3.175}$$

$$\|D_K\rho_j(x_j)\|_{L(X_j^{\mathsf{w}},X_{j+1}^{\mathsf{r}})} \le M\mathsf{r}_{V,j+1}^{-1}\mathsf{w}_{K,j} \le O(1), \tag{3.176}$$

which establishes the bounds on the first derivatives in (3.104), choosing $\mathring{g}_0$ small enough. The bounds on the second derivatives are also immediate consequences of Assumption (A3). Let ϕ denote either ψ or ρ. Then (3.15) and the definition of the weights (3.95) imply that, for $2 \le n + m \le 3$,

$$\|D_K^nD_V^m\phi\|_{L^{n+m}(X^{\mathsf{w}},X^{\mathsf{r}})} \le C. \tag{3.177}$$

In addition, these bounds on the second and third derivatives imply that

$$\|\phi(x+y) - \phi(x) - D\phi(x)y\|_{X^{\mathsf{r}}} \le C\|y\|_{X^{\mathsf{w}}}^2, \tag{3.178}$$

$$\|D\phi(x+y) - D\phi(x) - D^2\phi(x)y\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le C\|y\|_{X^{\mathsf{w}}}^2, \tag{3.179}$$

and hence that $\phi : \mathring{x} + \mathsf{B} \to X^{\mathsf{r}}$ is indeed twice Fréchet differentiable. The above bound on the third derivatives also implies continuity of this differentiability. The ρ-bound is equivalent to Assumption (A3) since

$$\|\rho_j(x_j)\|_{X_{j+1}^{\mathsf{r}}} = \mathsf{r}_{V,j+1}^{-1}M\chi_{j+1}\mathring{g}_{j+1}^3 = M/u. \tag{3.180}$$

This completes the proof. □

Proof of Lemma 3.3.5. Let

$$\bar{\mathsf{I}} = ([\tfrac{1}{2}\mathring{g}_0, 2\mathring{g}_0] \times \mathbf{K}_0) \cap \bar{x}^{-1}(\mathring{x} + \delta\mathsf{B}). \tag{3.181}$$

We will show that $\bar{\mathsf{I}}$ is a neighbourhood of $(\mathring{K}_0, \mathring{g}_0)$ and that $\bar{x} : \bar{\mathsf{I}} \to \mathring{x} + \delta\mathsf{B}$ is continuously Fréchet differentiable. Since $\bar{x}^{-1}(\mathring{x}+\delta\mathsf{B}) = \bar{V}^{-1}(\mathring{x}+\delta\mathsf{B}) \cap \bar{K}^{-1}(\mathring{x}+\delta\mathsf{B})$, it suffices to show that each of $\bar{V}^{-1}(\mathring{x} + \delta\mathsf{B})$ and $\bar{K}^{-1}(\mathring{x} + \delta\mathsf{B})$ is a neighbourhood of $(\mathring{K}_0, \mathring{g}_0)$, and that each of $\bar{V}$ and $\bar{K}$ is continuously Fréchet differentiable on $\bar{\mathsf{I}}$ as maps with values in subspaces of X^{w}.

We begin with $\bar{V}$. Let $\bar{V}_j'$ denote the derivative of $\bar{V}_j$ with respect to g_0, and let $\bar{V}' = (\bar{V}_j')$ denote the sequence of derivatives. It is straightforward to conclude from Lemmas 3.2.3 and 3.2.1(iv) and (3.95) that

$$\|\bar{V}'\|_{X^{\mathsf{w}}} \le O(\mathring{g}_0^{-2}|\log \mathring{g}_0|^{-1}), \tag{3.182}$$

and hence that $\bar{V}' \in X^{\mathsf{w}}$ if $g_0 \in \bar{\mathsf{I}}_g \subseteq [\frac{1}{2}\mathring{g}_0, 2\mathring{g}_0]$, and similarly that $\bar{V}^{-1}(\mathring{x} + \delta\mathsf{B})$ contains a neighbourhood of $\mathring{g}$. That $\bar{V}'$ is actually the derivative of $\bar{V}$ in the space X^{w} can be deduced from the fact that the sequence $\bar{V}''(g_0)$ is uniformly bounded in X^{w} for $g_0 \in \bar{\mathsf{I}}_g$ (though not uniform in $\mathring{g}_0$). In fact, by Lemma 3.2.3,

$$\|\bar{V}_j(g_0 + \varepsilon) - \bar{V}_j(g_0) - \varepsilon\bar{V}_j'(g_0)\|_{X_j^{\mathsf{w}}} \le O(\varepsilon^2) \sup_{0<\varepsilon'<\varepsilon} \|\bar{V}_j''(g + \varepsilon')\|_{X_j^{\mathsf{w}}}. \tag{3.183}$$

The continuity of $\bar{V}'$ in X^{w} follows similarly.

For $\bar{K}$, we first note that $\|D_{K_0}\bar{K}_0\|_{L(\mathbf{K}_0,\mathbf{K}_0)} = 1$, $\|D_{g_0}\bar{K}_0\|_{\mathbf{K}_0} = 0$. By (A3) and induction,

$$\|D_{K_0}\bar{K}_{j+1}\|_{L(\mathbf{K}_0,\mathbf{K}_{j+1})} \le \kappa\|D_{K_0}\bar{K}_j\|_{L(\mathbf{K}_0,\mathbf{K}_j)} \le \kappa^{j+1}. \tag{3.184}$$

Since $\kappa < \Omega^{-1} < 1$, and since $\mathring{g}_{j+1}/\mathring{g}_j \to 1$ by (3.32), we obtain

$$\|D_{K_0}\bar{K}_{j+1}\|_{L(\mathbf{K}_0,\mathbf{K}_{j+1})} \le O(\mathring{g}_0^{-3}\mathsf{w}_{K,j+1}). \tag{3.185}$$

Similarly, by (3.14) and Lemma 3.2.3,

$$\begin{aligned}\|D_{g_0}\bar{K}_{j+1}\|_{\mathbf{K}_{j+1}} &\le \kappa\|D_{g_0}\bar{K}_j\|_{\mathbf{K}_j} + O(\chi_j\bar{g}_j^2)\|D_{g_0}\bar{V}_j\|_{\mathbf{V}} \\ &\le \kappa\|D_{g_0}\bar{K}_j\|_{\mathbf{K}_j} + O(\chi_j\bar{g}_j^4/\bar{g}_0^2).\end{aligned} \tag{3.186}$$

By induction as in the proof of Lemma 3.1.3, again using $\kappa < \Omega^{-1}$, we conclude

$$\|D_{g_0}\bar{K}_{j+1}\|_{\mathbf{K}_{j+1}} \le O(\chi_j\bar{g}_j^4/\bar{g}_0^2) \le O(\mathring{g}_0^{-1}\mathsf{w}_{K,j+1}). \tag{3.187}$$

These bounds imply that $\bar{K}^{-1}(\mathring{x} + \delta\mathsf{B})$ contains a neighbourhood of $(\mathring{K}_0, \mathring{g}_0)$ and also that the component-wise derivatives of $\bar{K}$ with respect to g_0 and K_0 are respectively in $X^{\mathsf{w}} \cong L(\mathbb{R}, X^{\mathsf{w}})$ and $L(\mathbf{K}_0, X^{\mathsf{w}})$.

To verify that the component-wise derivative of $\bar{K}$ is the Fréchet derivative in X^{w}, it again suffices to obtain bounds on the second derivatives in X^{w}, as in (3.183). For example, since $D^2_{K_0}\bar{K}_0 = 0$, $D_{K_0}\bar{V}_j = 0$, and

$$D^2_{K_0}\bar{K}_{j+1} = D_K\psi(\bar{K}_j, \bar{V}_j)D^2_{K_0}\bar{K}_j + D^2_K\psi(\bar{K}_j, \bar{V}_j)D_{K_0}\bar{K}_j D_{K_0}\bar{K}_j, \tag{3.188}$$

it follows from (3.184) and induction that, for $(K_0, g_0) \in \bar{\mathsf{I}}$ with $\bar{\mathsf{I}} \subset \mathbf{K}_0 \oplus \mathbb{R}$ chosen sufficiently small,

$$\|D^2_{K_0}\bar{K}_{j+1}\| \le \kappa\|D^2_{K_0}\bar{K}_j\| + C\kappa^{2j} \le C(1 + j\kappa)\kappa^j \le O(\mathring{g}_0^{-3}\mathsf{w}_{K,j+1}). \tag{3.189}$$

Thus the component-wise derivative $D^2_{K_0}\bar{K}$ is uniformly bounded $L^2(\mathbf{K}_0, X^{\mathsf{w}})$ for $(K_0, g_0) \in \bar{\mathsf{I}}$. Similarly, slightly more complicated recursion relations than (3.188) for $D^2_{g_0}\bar{K}_j$ and $D_{g_0}D_{K_0}\bar{K}_j$ show that the component-wise second derivative of $\bar{K}$ is uniformly bounded in $L^2(\mathbf{K}_0 \oplus \mathbb{R}, X^{\mathsf{w}})$ for $\bar{\mathsf{I}}$ sufficiently small. This shows as in (3.183) that $\bar{K}$ is continuously Fréchet differentiable from $\bar{\mathsf{I}}$ to X^{w}.

We have thus shown that $\bar{x}$ is continuously Fréchet differentiable from a neighbourhood $\bar{\mathsf{I}}$ of $(\mathring{K}_0, \mathring{g}_0)$ to X^{w}, and (3.112) follows from (3.182), (3.187). □

Chapter 4

Outlook

4.1 The weakly self-avoiding walk with contact attraction

In Section 1.2, the weakly self-avoiding walk with additional contact self-attraction was introduced, see (1.9), but the subsequent discussion focused on the special case without self-attraction, $\gamma = 0$. For the model with self-attractive interaction, there is the conjectured phase diagram of Figure 1.3 which, in particular, predicts the same behavior as for $\gamma = 0$ also for sufficiently small $\gamma > 0$. However, even small self-attraction makes the analysis more difficult than the weakly self-avoiding walk already is because the energy functional then loses the superadditivity property. For $\gamma = 0$,

$$H(L + L') = \beta \sum_x (L_x + L'_x)^2 \geq \beta \sum_x (L_x^2 + L_x'^2) = H(L) + H(L'). \qquad (4.1)$$

This superadditivity implies, for example, that $c_t = \sum_x c_t(x)$ is submultiplicative, i.e., $c_{t+s} \leq c_t c_s$, and therefore that there is μ_c such that $\frac{1}{t} \log c_t \to \mu_c$; see e.g. [88] or [12]. The subadditivity (4.1) does not hold if $\gamma > 0$.

As a result of the failure of (4.1), little is known if $\gamma > 0$. For example, the results about the (weakly or strictly) self-avoiding walk in dimension five and higher obtained with the lace expansion do not easily extend to small $\gamma > 0$. The unique exception is a result by Ueltschi [109] who studies a model of the strictly self-avoiding walk with additional small self-attraction, in dimension five and higher, but relies on very particular exponentially decaying step weights (instead of nearest neighbor steps). The special step distribution helps in the analysis, for example by making c_t submultiplicative, but is an undesirable feature otherwise.

Although superadditivity of H fails for $\gamma > 0$, it has been observed [110] that the attractive force can be written as

$$\begin{aligned} \sum_x \sum_{y:y\sim x} L_x^t L_y^t &= 2d \sum_x (L_x^t)^2 + \sum_x L_x^t (\Delta L^t)_x \\ &= 2d \sum_x (L_x^t)^2 - \frac{1}{2} \sum_x (\nabla L^t)_x^2 \end{aligned} \qquad (4.2)$$

so that

$$H^{\beta,\gamma}(L) = (\beta - 2d\gamma)\sum_x L_x^2 + \frac{\gamma}{2}\sum_x (\nabla L)_x^2 \geq H^{\beta-2d\gamma,0}(L). \tag{4.3}$$

In terms of the renormalization group approach sketched in Section 1.4, the term $(\nabla L)^2$ is *irrelevant*. This is the basis for our work in preparation, with Brydges and Slade, in which we extend the result [38] to small $\gamma > 0$, thus showing that the two-point function is asymptotic to a multiple for $|x|^{-(d-2)}$ in dimension $d \geq 4$.

4.2 Logarithmic corrections to scaling behavior

A long-term goal of the renormalization group program for four dimensional weakly self-avoiding walks is to prove the conjecture (1.13) for the weakly self-avoiding walk, or more generally that, for any $p \geq 0$,

$$\left(\mathbf{E}_t^H |w_t|^p\right)^{\frac{1}{p}} \sim c_p t^{\frac{1}{2}} (\log t)^{\frac{1}{8}} \quad (t \to \infty). \tag{4.4}$$

A step towards this goal, interesting in itself, is to establish that the so-called susceptibility $\chi(\mu) = \sum_x G_\mu(x)$ has a related logarithmic correction,

$$\chi(\mu_c + T^{-1}) \sim cT(\log T)^{\frac{1}{4}} \quad (T \to \infty) \tag{4.5}$$

where μ_c is the smallest real number such that $\chi(\mu) < \infty$ for $\mu > \mu_c$. In work in preparation with Brydges and Slade, we utilize results from Chapters 2–3, together with [10, 34–37], to establish (4.5).

4.2.1 End-to-end distance and Laplace transforms

A *heuristic* argument (a version of Fisher's scaling relation for the critical exponents that applies in the critical dimension, see e.g. [15, 88]) predicts that if

$$\mathbf{E}_t^H |w_t|^2 \sim ct(\log t)^{2\nu} \quad (t \to \infty), \tag{4.6}$$

$$\chi(\mu + \varepsilon) \sim \varepsilon^{-1}(-\log \varepsilon)^{\gamma} \quad (\varepsilon \downarrow 0), \tag{4.7}$$

$$G_{\mu_c}(x) \sim c|x|^{-(d-2)}(\log |x|)^{-\eta} \quad (|x| \to \infty), \tag{4.8}$$

then the exponents of the logarithms should be related by

$$\gamma = 2\nu - \eta. \tag{4.9}$$

It has been proved that $\eta = 0$ [38] and we can prove that $\gamma = \frac{1}{4}$. Then (4.9) leads to the prediction $\nu = \frac{1}{8}$ as in (4.4).

Let us give some indication in which way (4.5) is a natural step in the direction of proving (4.4). The left-hand side of (4.4) is

$$\left(E_t^H |w_t|^p\right)^{\frac{1}{p}} = \left(\frac{\sum_x c_t(x)|x|^p}{\sum_x c_t(x)}\right)^{\frac{1}{p}}. \tag{4.10}$$

It would suffice to establish the more general claim that

$$e^{-\mu_c t} \sum_x c_t(x)|x|^p \sim c_p t^{\frac{p}{2}} (\log t)^{\frac{1}{4}+\frac{p}{8}} \quad (t \to \infty). \tag{4.11}$$

This would in particular include

$$\sum_x c_t(x) \sim c e^{\mu_c t} (\log t)^{\frac{1}{4}} \quad (t \to \infty). \tag{4.12}$$

An approach to proving (4.11) is given by proving related asymptotic behavior of its Laplace transform, which is given in terms of the two-point function (1.16) by

$$\int_0^\infty \left(\sum_x c_t(x)|x|^p\right) e^{-\mu t}\, dt = \sum_x G_\mu(x)|x|^p. \tag{4.13}$$

The asymptotics (4.11) are related to the asymptotics of the Laplace transform near its the critical point μ_c. For example, equation (4.11) implies that

$$\sum_x G_{\mu_c+\frac{1}{T}}(x)|x|^p \sim c'_p \left(T(\log T)^{\frac{1}{4}}\right)^{1+\frac{p}{2}} \quad (T \to \infty). \tag{4.14}$$

For $p = 0$, this is the same as (4.5). Equation (4.14) follows from (4.11) by a direct calculation: indeed, with $t = sT$,

$$\int_0^\infty e^{-(\mu_c+\frac{1}{T})t} \left(\sum_x c_t(x)|x|^p\right) dt = T \int_0^\infty e^{-s} e^{-\mu_c sT} \left(\sum_x c_{sT}(x)|x|^p\right) ds, \tag{4.15}$$

and, using (4.11), it is possible to conclude that

$$e^{-\mu_c sT} \sum_x c_{sT}(x)|x|^p \sim c_p T^{\frac{p}{2}} s^{\frac{p}{2}} (\log T)^{\frac{1}{4}+\frac{p}{8}} \quad (T \to \infty). \tag{4.16}$$

This implies (4.14) with c'_p given by

$$c'_p = c_p \int_0^\infty e^{-s} s^{\frac{p}{2}}\, ds = c_p \Gamma\left(1 + \tfrac{p}{2}\right). \tag{4.17}$$

The converse, that (4.14) implies (4.11), is not true in general. However, *Tauberian theory* [59, Chapter XIII] shows that (4.14) implies that (4.11) holds asymptotically in *Cesaro mean*, i.e.,

$$\frac{1}{T}\int_0^T e^{-\mu_c t}\left(\sum_x c_t(x)|x|^p\right)\,dt \sim c_p T^{\frac{p}{2}}(\log T)^{\frac{1}{4}+\frac{p}{8}} \quad (T\to\infty). \tag{4.18}$$

To conclude (4.11) rather than the averaged version (4.18), further information is needed such as, e.g., eventual monotonicity of the integrand in (4.18), or related asymptotics as $z = \frac{1}{T} \to 0$ for z in a region of the complex plane. The latter approach presumably requires major extensions to the argument which shows (4.5), but in the simpler case of weakly self-avoiding walks on a four dimensional *hierarchical* lattice, this was successfully carried by Brydges and Imbrie [28].

4.2.2 The renormalization group approach

The renormalization group method can be used to establish that the long-distance behavior of the weakly self-avoiding walk is, *in a suitable sense*, related to that of a free field. The *critical* model, $\mu = \mu_c$, is described by a *massless* free field, $m^2 = 0$, and subcritical models, $\mu > \mu_c$, are related to *massive* free fields, $m^2 > 0$. For example, we can show that there is a function $\mu = \mu(m^2)$ such that

$$\chi(\mu(m^2)) \sim \frac{c}{m^2} \quad (m^2 \downarrow 0), \tag{4.19}$$

i.e., the susceptibility of the weakly self-avoiding walk with parameter $\mu = \mu(m^2)$ is similar to that of the free field with mass m^2. It turns out important to establish the relation between μ and m^2 in the non-critical case. We can show that the right-inverse $m^2(\mu) = \inf\{m^2 > 0 : \mu(m^2) = \mu\}$ satisfies

$$m^2(\mu_c + \varepsilon) \sim c\varepsilon(-\log\varepsilon)^{-\frac{1}{4}} \quad (\varepsilon \downarrow 0). \tag{4.20}$$

These two properties allow to conclude (4.5).

To exemplify in which ways the results of Chapters 2 and 3 enter the proof of (4.20), let us mention that the coefficients β_j of Appendix A, in particular (A.8), given in terms of the decomposition of the Green function with $m^2 > 0$, satisfy

$$\sum_{j=0}^{\infty}\beta_j \sim c(-\log m^2) \quad (m^2 \downarrow 0). \tag{4.21}$$

Using this, it can be shown that $\bar{g}_j \to \bar{g}_\infty$ as $j \to \infty$ with $\bar{g}_\infty \sim c(-\log m^2)^{-1}$ as $m^2 \downarrow 0$. This is origin of the logarithm in (4.20). The power $\frac{1}{4}$ is a consequence of the explicit structure of the μ-equation of the recursion (A.8).

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Appendix A

Perturbation theory and coordinates of the renormalization group

In this appendix, the second-order part of the renormalization group map for the weakly self-avoiding walk model is considered, i.e., the map $\bar{\varphi}$ of Section 1.4.5.

The map $\bar{\varphi}$ is defined in terms of a map φ_{pt} that arises from formal perturbation theory, but does not satisfy the condition (3.1) imposed on the map $\bar{\varphi}$ of Chapter 3 itself. The remedy to this issue is an (explicit) coordinate change, exhibited in this appendix, that transforms φ_{pt} into a map $\bar{\varphi}$ to which Chapter 3 can be applied. The maps are defined in terms of the decomposition of the Green function of Chapter 2.

This provides an explicit connection between Chapters 2 and 3.

A.1 Flow of coupling constants

Let $C = C_1 + C_2 + \cdots$ be a positive definite decomposition of the Green function, and use the convenient short-hand notation, with j fixed,

$$C = C_j, \quad w = w_j = \sum_{l=1}^{j} C_l. \tag{A.1}$$

By translation-invariance, we can identify C and w with functions of one variable, for example, $C_x = C_{0x}$. Let

$$V_x = g\tau_x^2 + \nu\tau_x + z\tau_{\Delta,x} \tag{A.2}$$

be the (local) interaction polynomial for the weakly self-avoiding walk model. (For the definitions of τ and τ_Δ, see (1.43) and (1.50).) In [10], a new local interaction polynomial $V_{\mathrm{pt},x}$ is defined, in terms of V, C, and w, describing the effect of (formal) second-order perturbation theory. The details of the specification of V_{pt} are not important for the current discussion, so we only state the result: V_{pt} is essentially of the same form as (A.2) with coefficients $g_{\mathrm{pt}}, \nu_{\mathrm{pt}}, z_{\mathrm{pt}}$ given by polynomials

of degree two in g, ν, z. To express the coefficients of the polynomials, it is convenient to introduce the following abbreviations: for a function $f = f(\nu, w)$, set

$$\delta[f] = f(\nu + 2C_0 g, w + C) - f(\nu, w). \tag{A.3}$$

Moreover, for a function $q : \mathbb{Z}^d \to \mathbb{R}$, set

$$(\Delta q)_x = \tfrac{1}{2} \sum_{e \in \mathbb{Z}^d : |e|_1 = 1} (q_{x+e} - q_x), \tag{A.4}$$

$$(\nabla q)_x^2 = \tfrac{1}{2} \sum_{e \in \mathbb{Z}^d : |e|_1 = 1} (q_{x+e} - q_x)^2, \tag{A.5}$$

and

$$q^{(n)} = \sum_x q_x^n. \tag{A.6}$$

All functions q below arise in terms of the covariance decomposition, e.g., $q = w$, and satisfy:

$$\sum_x q_x x_i = 0, \quad \sum_x q_x x_i x_j = q^{(**)} \delta_{ij} \quad (i, j = 1, \dots, d). \tag{A.7}$$

Then the coefficients are given by:

$$\begin{cases} g_{\mathrm{pt}} = g - 8g^2 \delta[w^{(2)}] - 4g\delta[\nu w^{(1)}], \\ \\ \nu_{\mathrm{pt}} = \nu + 2C_0 g - 4g^2(\delta[w^{(3)}] - 3w^{(2)} C_0) - 2g(\nu + 2C_0 g)\delta[w^{(2)}] \\ \qquad\qquad - \delta[\nu^2 w^{(1)}] + 2g(z + y)\delta[(w\Delta w)^{(1)}] + 8g\nu w^{(1)} C_0, \\ \\ z_{\mathrm{pt}} = z - 2g^2 \delta[(w^3)^{(**)}] - \tfrac{1}{2}\delta[\nu^2 w^{(**)}] - 2z\delta[\nu w^{(1)}]. \end{cases} \tag{A.8}$$

A.2 Bounds on the coefficients

From now on, assume that the covariance decomposition $C = \sum_{j=1}^{\infty} C_j$ is given by

$$[C_j]_x = \begin{cases} \displaystyle\int_0^{\frac{1}{2}L} \phi_t^*(x)\, \frac{dt}{t} & (j = 1) \\ \displaystyle\int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t^*(x)\, \frac{dt}{t} & (j > 1) \end{cases} \tag{A.9}$$

where ϕ_t^* is as given in Example 2.1.3. In particular, (A.9) implies the finite range property

$$[C_j]_x = 0 \quad \text{if } d(x,0) > \tfrac{1}{2}L^j \tag{A.10}$$

and the bounds

$$|[\nabla^\alpha C_j]_x| \le O(L^{-(d-2+|\alpha|_1)(j-1)}). \tag{A.11}$$

Natural estimates on the coefficients in (A.8) are given in terms of the variable $\mu = L^{2j}\nu$ instead of ν and $\mu_{\mathrm{pt}} = L^{2(j+1)}\nu_{\mathrm{pt}}$. Let $\varphi_{\mathrm{pt}}(g, z, \mu) = (g_{\mathrm{pt}}, z_{\mathrm{pt}}, \mu_{\mathrm{pt}})$.

Proposition A.2.1. *The coefficients of the polynomials φ_{pt} are bounded by $O((1 + m^2L^{2j})^{-k})$ for any $k \in \mathbb{R}$ and continuous in $m^2 \in [0, \delta)$ for some $\delta > 0$.*

Proof. The proof uses (A.10)–(A.11) and is given in reference [10]. □

The previous result is similar to Assumption (A2) of Chapter 3. (We will show below that $O((1 + m^2L^{2j})^{-k})$ can be bounded by $O(\chi_j)$.) However, the map $\bar{\varphi}$ of Chapter 3 is assumed to be *triangular* which φ_{pt} is *not*. This is will be addressed in the next subsection. In addition, for the applicability of the result of Theorem 3.1.4, a positive lower bound on the coefficient of the g^2-term in the g-equation is crucial to satisfy assumption (A1). This is a consequence of Lemma A.2.2 below, in which we verify that the sequence of coefficients has a positive limit if $m^2 = 0$.

Lemma A.2.2. *Let $d = 4$, $m^2 = 0$. Then there is $\beta_\infty > 0$ such that*

$$\beta_j := 8\delta_j[w^{(2)}] = \beta_\infty + O(L^{-j}). \tag{A.12}$$

Remark A.2.3. The constant β_∞ can be determined exactly:

$$\beta_\infty = \frac{\log(L)}{\pi^2}. \tag{A.13}$$

Proof of Lemma A.2.2. Denote the covariance decomposition by $C_j(x)$, $x \in \mathbb{Z}^4$. By (2.36), there is $c_0 \in C_c(\mathbb{R}^4)$ such that with $c_j(x) = L^{-2j}c_0(L^{-j}x)$,

$$C_j(x) = c_j(x) + O(L^{-3j}). \tag{A.14}$$

Let us first verify

$$(C_j, C_{j+l}) - \langle c_0, c_l\rangle = O(L^{-j}L^{-2l}) \tag{A.15}$$

where we use the notation $(F, G) = \sum_{x\in\mathbb{Z}^4} F(x)G(x)$ whenever $F, G : \mathbb{Z}^4 \to \mathbb{R}$ and $\langle f, g\rangle = \int_{\mathbb{R}^4} fg\, dx$ for $f, g : \mathbb{R}^4 \to \mathbb{R}$. Let $R_j = C_j - c_j$. Then:

$$(C_j, C_{j+l}) = (c_j, c_{j+l}) + (c_j, R_{j+l}) + (c_{j+l}, R_j) + (R_j, R_{j+l}). \tag{A.16}$$

Riemann sum approximation shows

$$(c_j, c_{j+l}) - \langle c_0, c_l\rangle = L^{-4j} \sum_{y\in L^{-j}\mathbb{Z}^d} c(y)c_l(y) - \int_{\mathbb{R}^d} c(y)c_l(y)\, dy$$
$$= O(L^{-j})\|\nabla(cc_l)\|_{L^\infty} = O(L^{-2l-j}). \tag{A.17}$$

The remaining terms are easily bounded using $|\mathrm{supp}(C_j)|, |\mathrm{supp}(R_j)| = O(L^{4j})$:

$$(c_j, R_{j+l}) \le O(L^{4j})\|c_j\|_{L^\infty(\mathbb{Z}^4)}\|R_{j+l}\|_{L^\infty(\mathbb{Z}^4)} \le O(L^{-j}L^{-3l}), \tag{A.18}$$
$$(c_{j+l}, R_j) \le O(L^{4j})\|c_{j+l}\|_{L^\infty(\mathbb{Z}^4)}\|R_j\|_{L^\infty(\mathbb{Z}^4)} \le O(L^{-j}L^{-2l}), \tag{A.19}$$
$$(R_j, R_{j+l}) \le O(L^{4j})\|R_j\|_{L^\infty(\mathbb{Z}^4)}\|R_{j+l}\|_{L^\infty(\mathbb{Z}^4)} \le O(L^{-2j}L^{-3l}), \tag{A.20}$$

and (A.15) follows. From this we can now deduce:

$$\sum_{k=1}^{j}(C_k, C_{j+1}) = \sum_{k=1}^{j}\langle c_0, c_{j+1-k}\rangle + \sum_{k=1}^{j} O(L^{-k}L^{-2(j-k)})$$
$$= \sum_{k=1}^{j}\langle c_0, c_k\rangle + O(L^{-j}), \tag{A.21}$$
$$(C_{j+1}, C_{j+1}) = \langle c_0, c_0\rangle + O(L^{-j}), \tag{A.22}$$

and thus, using $\langle c_0, c_k\rangle = \langle c_0, c_{-k}\rangle$,

$$w_{j+1}^{(2)} - w_j^{(2)} = 2(w_j, C_{j+1}) + (C_{j+1}, C_{j+1}) \tag{A.23}$$
$$= \sum_{k=-j}^{j}\langle c_0, c_k\rangle + O(L^{-j}). \tag{A.24}$$

Note that with $\|c_{-k}\|_{L^\infty} \le L^{2k}\|c_0\|_{L^\infty}$ and $\mathrm{supp}(c_{-k}) \subset B_{CL^{-k}}$,

$$\sum_{k=j+1}^{\infty}|\langle c_0, c_k\rangle| = \sum_{k=j+1}^{\infty}|\langle c_0, c_{-k}\rangle| \le \|c_0\|_{L^\infty}\sum_{k=j+1}^{\infty} L^{2k}\int_{B_{CL^{-k}}}|c_0(x)|\, dx$$
$$\le \|c_0\|_{L^\infty}^2\sum_{k=j+1}^{\infty} O(L^{-2k}) \le O(L^{-2j}). \tag{A.25}$$

Thus, with $\beta_\infty = 8\sum_{k=-\infty}^{\infty}\langle c_0, c_k\rangle$, we have obtained

$$8(w_{j+1}^{(2)} - w_j^{(2)}) = \beta_\infty + O(L^{-j}). \tag{A.26}$$

That $\beta_\infty > 0$ can be seen from the fact that $\hat{c}_k \ge 0$ and Plancherel's theorem. □

Proof of Remark A.2.3. By (A.21), it follows that

$$\beta_\infty = 8\langle c_0, v\rangle \quad \text{with } v = \sum_{k\in\mathbb{Z}} c_k. \tag{A.27}$$

The Fourier transforms of c and v are

$$\hat{c}_0(\xi) = \frac{1}{|\xi|^2}\int_{L^{-1}|\xi|}^{|\xi|} \rho(t)\, dt, \quad \hat{v}(\xi) = \frac{1}{|\xi|^2} \tag{A.28}$$

where ρ is a non-negative function with $\int_0^\infty \rho\, dt = 1$. Observe that the claim for $\hat{v}$ follows from the claim for $\hat{c}$; the latter claim is verified at the end of the proof. (A.28) implies, by Plancherel's theorem, radial symmetry, and Fubini's theorem,

$$\begin{aligned}\langle c_0, v\rangle &= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4} |\xi|^{-4}\left(\int_{L^{-1}|\xi|}^{|\xi|}\rho(t)\, dt\right) d\xi \\ &= \frac{\omega_3}{(2\pi)^4}\int_0^\infty \left(\int_{L^{-1}r}^{r}\rho(t)\, dt\right)\frac{dr}{r} \\ &= \frac{\omega_3}{(2\pi)^4}\int_0^\infty \left(\int_t^{Lt}\frac{dr}{r}\right)\rho(t)\, dt \end{aligned}\tag{A.29}$$

where $\omega_3 = 2\pi^2$ is the surface measure of the 3-sphere ($\subset \mathbb{R}^4$). The inner integral in the last equation is equal to $\log(L)$. Thus, with $\int_0^\infty \rho\, dt = 1$,

$$\beta_\infty = \frac{8\omega_3}{(2\pi)^4}\log(L) = \frac{\log(L)}{\pi^2} \tag{A.30}$$

as claimed. To verify (A.28), use that by (2.36)–(2.37), there is $k > 0$ such that

$$\phi_t^*(x) = (t/k)^{-(d-2)}\bar{\phi}(kx/t) + O(t^{-(d-2+1)}) \tag{A.31}$$

where, denoting the Fourier transform of $\bar{\phi}$ by $\tilde{\phi}$, see (2.118), (2.78),

$$\tilde{\phi}(\xi) = \int_0^\infty t^2\varphi(|\xi|t)\,\frac{dt}{t}, \quad \int_0^\infty t^2\varphi(t)\,\frac{dt}{t} = 1. \tag{A.32}$$

In particular, the function c in (A.14) is more explicitly given by

$$\hat{c}_0(\xi) = \frac{1}{k^2}\int_{\frac{1}{2}L^{-1}}^{\frac{1}{2}} t^2\varphi\left(\frac{|\xi|t}{k}\right)\frac{dt}{t} = \frac{1}{|\xi|^2}\int_{L^{-1}|\xi|}^{|\xi|}\rho(t)\, dt \tag{A.33}$$

as claimed where ρ is given by

$$\rho(t) = \left(\frac{t}{2k}\right)^2\varphi\left(\frac{t}{2k}\right)\frac{1}{t}. \tag{A.34}$$

This completes the proof. □

A.3 Transformation

As discussed, the map $\varphi_{\mathrm{pt}}(g, z, \mu) = (g_{\mathrm{pt}}, z_{\mathrm{pt}}, \mu_{\mathrm{pt}})$ does not have the right form to apply the result of Chapter 3. In Proposition A.3.1, we show that the coordinates can be brought to the form expected in Chapter 3 by a simple transformation.

Proposition A.3.1. *Define* $\bar{\varphi} : \mathbb{R}^3 \to \mathbb{R}^3$ *by* $(\bar{g}, \bar{z}, \bar{\mu}) = \bar{\varphi}(g, z, \mu)$ *with* $\bar{\mu} = L^{2(j+1)}\bar{\nu}$, $\mu = L^{2j}\nu$, *and*

$$\bar{g} = g - 8g^2\delta[w^{(2)}], \tag{A.35}$$

$$\bar{z} = z - 2g^2\delta[(w^3)^{(**)}], \tag{A.36}$$

$$\begin{aligned}\bar{\nu} = \nu &+ 2C_{0,0}g - 4g^2(\delta[w^{(3)}] - 3w^{(2)}C_{0,0} + C_{0,0}\delta[w^{(2)}]) \\ &- 2g\nu\delta[w^{(2)}] + 2gz\delta[(w\Delta w)^{(1)}].\end{aligned} \tag{A.37}$$

Then the coefficients of the polynomials $\bar{\varphi}$ *are bounded by* $O((1 + m^2L^{2j})^{-k})$ *for an arbitrary* k *and* $m^2 \in [0, \delta)$. *Define* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *by* $T(g, z, \mu) = (g_T, z_T, \mu_T)$, *with* $\mu = L^{2j}\nu$, $\mu_T = L^{2j}\nu$, *where*

$$g_T = g + 4g\nu w^{(1)}, \tag{A.38}$$

$$z_T = z + 2z\nu w^{(1)} + \tfrac{1}{2}\nu^2 w^{(**)}, \tag{A.39}$$

$$\nu_T = \nu + \nu^2 w^{(1)}. \tag{A.40}$$

Then $T(V) = V + O(|V|^2)$. *Let* $T_+ = T_{j+1}$. *There exists a ball* $B \subset \mathbb{R}^3$ *independent of* j *and* $m^2 \in [0, \delta)$ *such that, on* B,

$$T_+ \circ \varphi_{\mathrm{pt}} \circ T^{-1} = \bar{\varphi} + \rho_{\mathrm{pt}} \tag{A.41}$$

where ρ_{pt} *is an analytic function on* B *with* $\rho_{\mathrm{pt}}(g, z, \mu) = O((1 + m^2L^{2j})^{-k}(|g| + |z| + |\mu|)^3)$ *uniformly in* j *and* $m^2 \in [0, \delta)$, *for any* k.

Remark A.3.2. The transformation T is simple and explicit, but we believe that its existence may have a deeper origin that we have not unravelled. Formally, i.e., without consideration of the formal third-order error, different covariance decompositions induce dynamical systems like (1.92) whose three-dimensional parts can be of slightly different form. Some of the monomials that appear in the polynomials $\bar{\varphi}_j$ are essentially independent of the decomposition. On the other hand, some decompositions of the Green function have the special property that

$$\sum_x [C_j]_x = 0 \tag{A.42}$$

which is not true for the finite range decomposition discussed in Chapter 2. It can be seen that the terms in (1.43) involving $w^{(1)}$ would thus vanish with such a decomposition. Is it possible that the existence of such a transformation expresses an invariance property of the dynamical system under coordinates induced by different covariance decompositions?

Note that the map $\bar{\varphi}$ has the form assumed for $\bar{\varphi}$ in Chapter 3. The next corollary illustrates how the result of Chapter 3 is used in the study of the weakly self-avoiding walk, except that in the real application, the error coordinate is non-trivial.

Corollary A.3.3. *Fix any $\Omega > 1$. The maps $\bar{\varphi}$ then satisfy Assumptions (A1–A2) of Chapter 3. Moreover, Assumption (A3) can be satisfied with $\rho = \rho_{\mathrm{pt}}$ and $\psi = 0$.*

Sketch of proof. (i) Set $j_m = [\log_L m]$. We first show that for any $c < \log L/\pi^2$, there is $n < \infty$ such that the number of $j \le j_m$ with $\beta_j < c$ is bounded by n, uniformly in $m^2 \in [0, \delta)$. To prove this, we first note that (A.13) implies that, if $m^2 = 0$, for every $c + \varepsilon < \log L/\pi^2$, there is n_0 such that the number of j such that $\beta_j < c + \varepsilon$ is bounded by n_0. We now prove the claim for $m^2 > 0$. It can be shown using Example 2.1.3 that there are constants c' and q independent of L such that

$$\left| \frac{\partial}{\partial m^2} \beta_j(m^2) \right| \le c' L^q L^{2j}, \tag{A.43}$$

but we omit the proof. This implies that for $j \le j_m - q - p$, with p large enough,

$$|\beta_j(0) - \beta_j(m^2)| \le c' L^q L^{2j} m^2 \le c' L^{-p} \le \varepsilon. \tag{A.44}$$

It follows that the number of $j \le j_m$ such that $\beta_j < c$ can be bounded by $n_0 + p + q$. (ii) We now verify Assumptions (A1)–(A2) of Chapter 3 for $\bar{\varphi}$. Let $\Omega > 1$,

$$j_\Omega = \inf\{k \ge 0 : |\beta_j| \le \Omega^{-(j-k)} \|\beta\|_\infty \text{ for all } j\}, \quad \text{and } \chi_j = \Omega^{-(j-j_\Omega)_+}. \tag{A.45}$$

Let k be such that $L^{2k} \ge \Omega$. Then

$$(1 + m^2 L^{2j})^{-k} \le L^{-2k(j-j_m)_+} \le \Omega^{-(j-j_m)_+}. \tag{A.46}$$

(i) implies that $\|\beta\|_\infty > c > 0$ uniformly in $m^2 \in (0, \delta)$. By Proposition A.3.1 and (A.46), there is a constant C such that

$$|\beta_j| \le C\Omega^{-(j-j_m)_+} \le \frac{C}{c} \Omega^{-(j-j_m)_+} \|\beta\|_\infty \le \Omega^{-(j-j_\Omega)_+} \|\beta\|_\infty \tag{A.47}$$

with $j_\Omega \le j_m + \log_\Omega C - \log_\Omega c$. In particular, the number of $j \le j_\Omega$ with $\beta_j < c$ is bounded by $n_\Omega = n + \log_\Omega C - \log_\Omega c$ where n is as in (i), uniformly in $m^2 \in [0, \delta)$. This proves Assumption (A1) and Assumption (A2) is then a consequence of Proposition A.3.1 with $(1 + m^2 L^{2j})^{-k} = O(\chi_j)$. □

Sketch of proof of Proposition A.3.1. The bounds on the coefficients of the maps $\bar{\varphi}$, given in (A.35)–(A.36), follow from Proposition A.2.1 and $w^{(**)} = O(L^{4j})$ and $w^{(1)} = O(L^{2j})$. The last two bounds are a straightforward with the properties of the covariance decomposition that $|C_j| \le O(L^{-2j})$ and $C_j(x) = 0$ for $x \ge cL^j$. Indeed,

$$w_j^{(1)} = \sum_{l=1}^{j} \sum_x [C_l]_x = \sum_{l=1}^{j} O(L^{2l}) = O(L^{2j}), \tag{A.48}$$

$$w_j^{(**)} = \sum_{l=1}^{j} \sum_x |x|^2 [C_l]_x = \sum_{l=1}^{j} O(L^{4l}) = O(L^{4j}). \tag{A.49}$$

These bounds similarly imply $T = \mathrm{id} + O((|g| + |z| + |\mu|)^2)$ uniformly in j.

Let $w_+ = w + C$ and $\nu_+ = \nu + 2C_0 g$. Then (A.8) can be written as

$$g_{\mathrm{pt}} + 4g\nu_+ w_+^{(1)} = (g + 4g\nu w^{(1)}) - 8^2 \delta[w^{(2)}]g^2, \tag{A.50}$$

$$\begin{aligned} \nu_{\mathrm{pt}} + \nu_+^2 w_+^{(1)} = {} & (\nu + \nu^2 w^{(1)}) + 2C_{0,0}(g + 4g\nu w^{(1)}) \\ & - 4g^2(\delta[w^{(3)}] - 3w^{(2)}C_{0,0}) \\ & - 2g(\nu + 2C_0 g)\delta[w^{(2)}] \\ & + 2g(z + y)\delta[(w\Delta w)^{(1)}], \end{aligned} \tag{A.51}$$

$$z_{\mathrm{pt}} + 2z\nu_+ w_+^{(1)} + \tfrac{1}{2}\nu_+^2 w_+^{(**)} = (z + 2z\nu w^{(1)} + \tfrac{1}{2}\nu^2 w^{(**)}) - 2g^2 \delta[(w^3)^{(**)}]. \tag{A.52}$$

Expressing ν and ν_{pt} as $\nu = L^{-2j}\mu$ and $\nu_{\mathrm{pt}} = L^{-2(j+1)}\mu_{\mathrm{pt}}$, the right- and left-hand sides of (A.50)–(A.52) equal $\bar{\varphi} \circ T(g, z, \mu) + O((|g| + |z| + |\mu|)^3)$ respectively $T_+ \circ \hat{\varphi}(g, z, \mu) + O((|g| + |z| + |\mu|)^3)$, with both bounds uniform in j. This and $T_+((g, z, \mu) + r) = T_+(g, z, \mu) + O(r)$ imply the claim. □