Minimal Hypersurfaces of the Round Sphere

by

Pamela Sargent

B.Sc., Mount Allison University, 2011

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in
The Faculty of Graduate and Postdoctoral Studies
(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA
(Vancouver)
August 2013
© Pamela Sargant 2013
Abstract

The purpose of this thesis is to discuss a conjectured classification concerning the index of non-totally geodesic minimal hypersurfaces of the $n$-dimensional standard sphere of radius one $S^n$. We briefly discuss the basic theory of minimal submanifolds before turning our attention to minimal submanifolds and hypersurfaces in $S^n$. We present some results of Simons which show that any minimal submanifold of $S^n$ is unstable, and how the totally geodesic $S^k \subset S^n$ are characterized by their index. We then present a related conjecture which claims that the Clifford hypersurfaces are also characterized by their index in a similar way, discuss the most recent developments related to the conjecture, and give Urbano’s proof of the conjecture for the special case when $n = 3$. 
Preface

The topic of this thesis was jointly chosen by the author and her supervisor, Dr. Ailana Fraser.

This thesis surveys a collection of known results. While the content is not original, the organization and presentation, to the best of the author’s knowledge, are. Moreover, the author has provided a few modifications in the proof of the first main result and many details in the proof of the second main result that are not presented in the original proof.
# Table of Contents

Abstract ................................. ii  
Preface .................................. iii  
Table of Contents ........................ iv  
Acknowledgements ........................ vi  

1 Introduction ............................ 1  
  1.1 Some Preliminaries ................. 2  
      1.1.1 Notation and Conventions .... 2  
      1.1.2 Riemannian Submanifolds .... 3  
      1.1.3 Minimal Submanifolds ......... 5  
  1.2 Operators on Riemannian Manifolds 6  
      1.2.1 Basic Differential Operators on Riemannian Manifolds 7  
      1.2.2 The Jacobi Operator and the Morse Index .... 8  
      1.2.3 Spectral Theory of Strongly Elliptic Operators .... 10  

2 Classification Results for Minimal Submanifolds of $S^n$ 13  
  2.1 Minimal Submanifolds of $S^n$ .... 13  
      2.1.1 Isometric and Minimal Immersions into $S^n$ .... 13  
      2.1.2 Simons Classification Result .... 16  
  2.2 Non-totally Geodesic Minimal Hypersurfaces of $S^n$ .... 25  
      2.2.1 A Classification Conjecture .... 25  
      2.2.2 Urbano's Result: Proof of the Conjecture when $n = 3$ 28  

3 Conclusion ............................ 44
Acknowledgements

I would like to thank my supervisor, Dr. Ailana Fraser, who introduced me to this field. I have benefited greatly from her expertise, encouragement and patience, and this thesis would not have been achieved without her.
Chapter 1

Introduction

This thesis focuses on the study of minimal submanifolds, a central topic in geometric analysis. Minimal submanifolds are critical points of the volume functional. The study of minimal surfaces, 2-dimensional minimal submanifolds, has involved such prolific mathematicians as Lagrange and Euler, as well as the physicist Plateau. It was Plateau who investigated the surface obtained by dipping a wire frame into a soap solution, a physical example of a minimal surface, and it was his work that inspired the famous Plateau’s problem: given a simple, closed curve, is there a minimal surface whose boundary corresponds to that curve? The general solution to this problem was found in the 1930s independently by Rado and Douglas, and Douglas later won the Fields medal for his work on the problem. Some other notable examples of minimal surfaces are helicoids (the geometric shape of DNA and double-spiral staircases) and catenoids (minimal surfaces obtained by rotating catenaries about their directrices). The study of minimal surfaces, besides being important in its own right, has physical applications in fluid interface problems and navigation problems, deep connections to fundamental questions in general relativity and, in fact, played a crucial role in the celebrated proof of the century-old Poincare conjecture.

The classical case of minimal submanifolds in $\mathbb{R}^n$ is a subject that has also been studied for centuries. Another natural setting for studying minimal surfaces is in Riemannian manifolds, and one interesting case is the $n$-dimensional sphere $S^n$. Here, there is a fundamental difference from the case of $\mathbb{R}^n$: every minimal submanifold in $\mathbb{R}^n$ is necessarily non-compact, but in $S^n$ there exist closed minimal submanifolds. In particular, minimal surfaces in $S^3$ are bountiful: in 1970, Lawson [8] proved that a closed orientable surface of any genus can be realized as an embedded minimal surface.
in $S^3$.

Here we will often deal with minimal hypersurfaces of $S^n$, i.e. minimal submanifolds of $S^n$ of dimension $n - 1$. More specifically, we will be concerned with problems regarding minimal submanifolds and hypersurfaces of $S^n$ and their index. Despite their name, minimal submanifolds are critical points of the volume functional which, in general, are not necessarily locally volume minimizing. The index of a minimal submanifold corresponds to the index of the hessian of the volume functional, and, intuitively speaking, is the number of independent directions in which one can deform the minimal submanifold to decrease its volume. So, a minimal submanifold is locally volume minimizing if and only if its index is 0. We will show that there is a characterization of the minimal submanifolds of $S^n$ which minimize the index, and present a related conjecture which claims that there is also a characterization of the non-totally geodesic minimal hypersurfaces of $S^n$ which minimize the index. We will state known partial results related to the conjecture and conclude with a proof of the conjecture in the special case when $n = 3$.

1.1 Some Preliminaries

The purpose of this first chapter is to briefly introduce some preliminary definitions, notations and conventions, and some of the basic theory of Riemannian submanifolds. We conclude with the first and second variation formulas for volume and the definition of a minimal submanifold.

1.1.1 Notation and Conventions

Let $(M, g), (M', g')$ be Riemannian manifolds with Levi-Civita connections $\nabla, \nabla'$ respectively. By $C^\infty(M, M')$ we mean the collection of smooth functions $f : M \to M'$, and $C^k(M, M')$ will denote the collection of functions $f : M \to M'$ which are $k$ times continuously differentiable. In the special case that $M' = \mathbb{R}$, we will simply write $C^\infty(M)$ or $C^k(M)$. Let $E \xrightarrow{\pi} M$ be a vector bundle over $M$. We denote the collection of smooth sections of $E$...
1.1. Some Preliminaries

by \( \Gamma(E) \), i.e.

\[
\Gamma(E) = \{ X \in C^\infty(M, E) \mid \pi \circ X = id_M \}.
\]

We take the convention of defining the Riemannian curvature endomorphism on \( M \), \( R_M : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) \) by

\[
R_M(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,
\]

and the Riemannian curvature tensor (field), \( R_M : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \mathbb{R} \) by

\[
R_M(X,Y,Z,W) = g(R_M(X,Y)Z,W).
\]

Unless otherwise stated, \( S^n \) will denote the standard \( n \)-dimensional sphere of radius one with the induced metric from \( \mathbb{R}^{n+1} \).

1.1.2 Riemannian Submanifolds

If \( M^k \subset \bar{M}^n \) is an immersed submanifold, then for every \( p \in M \), the tangential and normal projections \((\cdot)^T : T_p\bar{M} \to T_pM, (\cdot)^N : T_p\bar{M} \to N_pM \) naturally decompose the tangent space of \( \bar{M} \):

\[
T_{p \bar{M}} = T_{p M} \oplus N_{p M} \quad \text{for} \ p \in M.
\]

An immersed manifold \( \Phi : M \to \bar{M} \) in a Riemannian manifold \( \bar{M} \) naturally inherits a Riemannian manifold structure from the ambient space: if \( \bar{M} \) has Riemannian metric \( \bar{g} \) and Levi-Civita connection \( \bar{\nabla} \), then \( g = \Phi^*\bar{g} \) is a Riemannian metric with Levi-Civita connection given by

\[
\nabla_X Y = d\Phi^{-1} \left( (\nabla_{d\Phi(X)}d\Phi(Y))^T \right).
\]

Thinking of \( M \subset \bar{M} \), we will often simply write \( g = \bar{g}|_M \) and \( \nabla = (\nabla)^T \).

Then

\[
\nabla = (\nabla)^T + (\nabla)^N = \nabla + \nabla^N,
\]

3
with $\nabla^N := (\nabla)^N$.

**Definition 1.1.2.1.** The second fundamental form of $M$ induced by $\overline{\nabla}$ is the vector-valued bilinear form $A$ defined by

$$A(X, Y) = (\nabla_X Y)^N \quad X, Y \in T_p M.$$  

It is easy to see that the $A$ is symmetric, i.e. $A(X, Y) = A(Y, X)$, since

$$A(X, Y) = (\nabla_X Y)^N = (\nabla_Y X + [X, Y])^N \quad (\overline{g} \text{ is torsion free})$$

$$= A(Y, X) + 0 \quad ([X, Y] \in T_p M).$$

Let $p \in M$ and let $\{E_i\}_{i=1}^k$ be an orthonormal basis for $T_p M$.

**Definition 1.1.2.2.** The mean curvature vector $H$ at $p \in M$ is given by

$$H = \sum_{i=1}^k A(E_i, E_i).$$

**Definition 1.1.2.3.** The squared norm of the second fundamental form at $p \in M$ is

$$\|A\|^2 = \sum_{i,j}^k \|A(E_i, E_j)\|^2.$$  

Later we will need to distinguish the following special type of submanifold.

**Definition 1.1.2.4.** A Riemannian submanifold $M \subset \overline{M}$ is said to be totally geodesic if one of the following equivalent conditions holds:

(i) Every $g$-geodesic in $M$ is also a $\overline{g}$-geodesic in $\overline{M}$.

(ii) The second fundamental form of $M$ vanishes identically, i.e. $A \equiv 0$.

The next propositions relate the second fundamental form of $M$ to the curvatures of $M$ and $\overline{M}$. The first tells us how the difference in the curvature of the submanifold and the curvature of the ambient manifold relates to the second fundamental form. The second tells us how the covariant derivative of
1.1. Some Preliminaries

the second fundamental form also gives us information about the curvature endomorphism of $M$.

**Proposition 1.1.2.1.** (Gauss Equation) For any $X,Y,Z,W \in T_p M$

\[ R_{M}(X,Y,Z,W) = R_{\tilde{M}}(X,Y,Z,W) - \langle A(X,W), A(Y,Z) \rangle + \langle A(X,Z), A(Y,W) \rangle, \]

where $\langle \cdot, \cdot \rangle$ denotes the metric on $\tilde{M}$.

**Proposition 1.1.2.2.** (Codazzi Equation) For any $X,Y,Z \in \Gamma(TM)$,

\[ (R_{M}(X,Y)Z)^N = (\nabla_X A)(Y,Z) - (\nabla_Y A)(X,Z), \]

where $(\nabla_X A)(Y,Z) = \nabla_X^N A(Y,Z) - A(\nabla_X Y,Z) - A(Y,\nabla_X Z)$.

1.1.3 Minimal Submanifolds

Let $\tilde{M}$ be a Riemannian $n$-manifold and $\Phi : M \to \tilde{M}$ be an immersion, where $M$ is a compact oriented $k$-manifold with (possibly empty) boundary $\partial M$. Let $F : I \times M \to M$, $I = (-1,1)$, be a smooth variation of $\Phi$, i.e. $F$ is a smooth mapping such that

(i) For each $t \in I$, the map $\Phi_t := F(t,\cdot) : M \to \tilde{M}$ is an immersion.

(ii) $\Phi_0 := F(0,\cdot) = \Phi$.

(iii) For each $t \in I$, $\Phi_t|_{\partial M} = \Phi|_{\partial M}$.

Let $t$ be the coordinate on $I$ and let $E$ be the section of $T(M) \oplus N(M)$ given by $E = dF\left( \frac{\partial}{\partial t} \big|_{t=0} \right)$. We will suppress the time dependence and let $dV$ denote the volume element of the metric induced by $\Phi_t$ so that the volume of $M$ at time $t$, Area$(t)$, is given by

\[ \text{Area}(t) = \int_M dV. \]

Then we have
1.2. Operators on Riemannian Manifolds

Theorem 1.1.3.1. (The first variation formula)

\[ \frac{d \text{Area}}{dt} \bigg|_{t=0} = \int_M \text{div}_M E \, dV = - \int_M \langle H, E \rangle \, dV. \]  (1.1)

Theorem 1.1.3.2. (The second variation formula)

\[ \frac{d^2 \text{Area}}{dt^2} \bigg|_{t=0} = - \int_M |\langle A(\cdot, \cdot), E \rangle|^2 \, dV \]
\[ + \int_M |\nabla^N_M E|^2 \, dV - \int_M \text{Tr}_M \langle R_M(\cdot, E), E \rangle \, dV. \]  (1.2)

Definition 1.1.3.1. Let \( \Phi : M \to \bar{M} \) be an immersion. We say that an (immersed) manifold \( M \) is a minimal submanifold of \( \bar{M} \) if \( \frac{d \text{Area}}{dt} \big|_{t=0} = 0 \) for every smooth variation of \( \Phi \).

Lemma 1.1.3.3. \( M \) is a minimal submanifold of \( \bar{M} \) if and only if the mean curvature vector vanishes.

Proof of 1.1.3.3: It is clear from the first variation formula (1.1) that if \( H \equiv 0 \), then \( M \) is a minimal submanifold of \( \bar{M} \).

Now suppose \( M \) is a minimal submanifold of \( \bar{M} \) and that \( H(p) \neq 0 \) for some \( p \in M \). Then, by continuity, \( \|H\| > \frac{1}{2} \|H(p)\| \) on some neighbourhood \( U \subset M \) of \( p \). Let \( \phi : M \to \mathbb{R} \) be a smooth bump function such that \( \phi|_U \equiv 1 \), and let \( F \) be a smooth variation of \( \Phi \) with variation field \( \phi H \) (e.g. \( F(t, q) = \exp_q (t\phi(q)H(q)) \)). Then we have that

\[ 0 = \frac{d \text{Area}}{dt} \bigg|_{t=0} = - \int_M \langle H, \phi H \rangle \, dV_0 \leq -\frac{1}{2} \|H(p)\|^2 \int_U \, dV_0. \]

Since \( \int_U \, dV_0 > 0 \), this gives us that \( \|H(p)\| \leq 0 \), a contradiction. \( \square \)

1.2 Operators on Riemannian Manifolds

In this chapter we discuss some specific operators on Riemannian manifolds. We begin with the definitions of some common differential operators and make note of a few results that we will make use of later in Chapter 2.1.
Then we will move on to discuss a specific elliptic operator on minimal submanifolds which comes from the second variation formula: the Jacobi operator. We will introduce the ideas of the index and stability of a minimal submanifold, and outline the general spectral theory for elliptic operators on compact Riemannian manifolds.

1.2.1 Basic Differential Operators on Riemannian Manifolds

**Definition 1.2.1.1.** Let $f \in C^\infty(M)$. We define:

(i) The *gradient of* $f$, $\text{grad } f \in \Gamma(TM)$ to be the vector field characterized by the equation

$$df(X) = \langle \text{grad } f, X \rangle.$$ 

(ii) The *Hessian of* $f$, $\text{Hess } f$, to be the symmetric $(0, 2)$-tensor field such that, for $X, Y \in \Gamma(TM)$

$$\text{Hess } f(X, Y) = \langle \nabla_X \text{grad } f, Y \rangle = XYf - (\nabla_X Y)f.$$ 

(iii) The *Laplacian of* $f$, $\Delta f \in C^\infty(M)$ to be the function given by

$$\Delta f = \text{Tr}(\text{Hess } f) = \sum_{i=1}^{k} \nabla_{E_i}(\nabla_{E_i}f) - (\nabla_{E_i}E_i)f,$$

where $\{E_i\}_{i=1}^k$ is a local orthonormal frame for $TM$.

A simple calculation shows that the Laplacian satisfies a type of product rule:

$$\Delta(fg) = f\Delta g + g\Delta f + 2(\text{grad } f, \text{grad } g). \quad (1.3)$$

We will also make use of the following Green’s formula:

**Lemma 1.2.1.1.** Let $f \in C^2(M)$, $g \in C^1(M)$ be functions such that $h(\text{grad } f)$ has compact support. Then

$$\int_M h\Delta f + \langle \text{grad } h, \text{grad } f \rangle dV = 0.$$
1.2. Operators on Riemannian Manifolds

See [4] (pg. 6) for more details. Note that (1.3) and Lemma 1.2.1.1 together show that, if \( \text{grad} f \) has compact support, then

\[
0 = \int_M 1 \cdot \Delta f + \langle \text{grad} f, \text{grad} \ 1 \rangle \ dV = \int_M \Delta f \ dV.
\]

In particular, if \( M \) is compact, then (1.4) holds for any \( f \in C^2(M) \).

1.2.2 The Jacobi Operator and the Morse Index

We can alternatively write the second variation formula (1.2) as

\[
\frac{d^2 \text{Area}}{dt^2} \bigg|_{t=0} = - \int_M \langle E, LE \rangle dV.
\]

where \( L \) is the self-adjoint Jacobi operator of the second variation which acts on a normal vector field \( X \) to \( M \) by

\[
LX = \Delta_N^M X - R(X) + \tilde{A}(X).
\]

Here, if \( \{ E_i \}_{i=1}^k \) is an orthonormal frame for \( TM \), \( \tilde{A} \) is the Simons’s operator defined by

\[
\tilde{A}(X) = \sum_{i,j=1}^k g(A(E_i, E_j), X)A(E_i, E_j),
\]

\( \Delta_N^M \) is the Laplacian on the normal bundle

\[
\Delta_N^M X = \sum_{i=1}^k \nabla_{E_i}^N \nabla_{E_i}^N X - \sum_{i=1}^k \nabla_{(\nabla_{E_i} E_i)^T}^N X,
\]

and

\[
\mathcal{R}(X) := \text{Tr}[R_{\bar{M}}(\cdot, X) \cdot] = \sum_{i=1}^k R_{\bar{M}}(E_i, X)E_i.
\]

If \( M \) is an orientable hypersurface of \( \bar{M} \), then \( M \) has a trivial normal bundle (the normal bundle has a global orthonormal frame) and the Jacobi operator simplifies to an operator on functions: because we can write any normal vector field as a function times the unit normal vector field, we can
identify normal vector fields with functions, i.e. if $X = fN$, then we can identify $X$ with $f$, and

$$Lf = \Delta_M f + \|A\|^2 f + \text{Ric}_M(N,N)f.$$ (1.9)

We will say that $\lambda$ is a (Dirichlet) eigenvalue for $L$ on $\Omega \subset M$ if there is a non-trivial normal vector field $X \in \Gamma(NM)$ such that $X|_{\partial\Omega} = 0$ and

$$LX + \lambda X = 0.$$ (1.10)

**Definition 1.2.2.1.** The Morse index (or just index) of a compact minimal submanifold $M$, denoted $\text{ind}(M)$, is the number of negative eigenvalues of the Jacobi operator $L$ acting on the space of smooth sections of the normal bundle which vanish on the boundary (counted with multiplicity).

There is a quadratic form, $Q$, associated to the Jacobi operator that is given by

$$Q(X, X) = -\int_M \langle X, LX \rangle dV = -\int_M \left\langle X, \Delta_M^N X - \text{R}(X) + \tilde{A}(X) \right\rangle dV,$$ (1.11)

and $\text{ind}(M)$ is the index of $Q$. In the case that $M$ is a hypersurface, this simplifies to

$$Q(f, f) = \int_M \|\nabla f\|^2 - (\|A\|^2 + \text{Ric}(N,N))f^2 dV.$$ (1.12)

It is sometimes useful to work with this quadratic form instead of the Jacobi operator.

It follows from the second variation formula (1.2) that we could have alternatively defined the Morse index of $M$ to be the index of $M$ as a critical point for the volume functional, and so the Morse index, in some sense, describes how stable the minimal submanifold is: it gives the number of independent directions in which the minimal submanifold can be deformed to make its volume decrease. Thus, a stable minimal submanifold, one which truly minimizes (locally) the volume functional, has index zero. If the index
1.2. Operators on Riemannian Manifolds

is positive, the \( M \) is said to be \textit{unstable}. We will see later that there are no stable minimal submanifolds in the standard sphere \( S^n \).

1.2.3 Spectral Theory of Strongly Elliptic Operators

Here we give a brief overview of some standard results concerning the general theory for strongly elliptic operators. Recall that an \textit{elliptic differential operator} of order \( m \) on an \( n \)-dimensional manifold \( M \) is an operator \( P \) that in local coordinates has the form

\[
P(x, D)u = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u,
\]

and whose principal symbol

\[
P_m(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha
\]

is invertible for nonzero \( \xi \in \mathbb{R}^n \). Here \( D = (D_1, \ldots, D_n) \), \( D_j = \frac{1}{i} \frac{\partial}{\partial x_j} \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a multi-index so that \( D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \). A \textit{strongly elliptic operator} is an elliptic differential operator with

\[
\frac{1}{2} (P_m(x, \xi) + P_m(x, \xi)^*) \geq C \|\xi\|^m,
\]

where \( P_m(x, \xi)^* = P_m(x, \xi)^T \).

The Jacobi operator is known to be strongly elliptic (see [16] pg. 65) and so we will later apply some of these results to the Jacobi operator when we prove a special case of the main conjecture.

If \( L \) is a strongly elliptic operator and \( M \) is compact, then the spectrum of \( L \) is composed entirely of eigenvalues \( \{\lambda_i\}_{i=1}^\infty \). If we arrange the eigenvalues according to size

\[
\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots
\]

then \( \{\lambda_i\}_{i=1}^\infty \) are discrete and \( \lambda_k \to \infty \). Moreover, the multiplicity of each eigenvalue is finite. In fact, the first eigenvalue has multiplicity one and there is an eigenfunction associated to the first eigenvalue that is strictly
1.2. Operators on Riemannian Manifolds

positive.

The eigenspaces $E_i$ corresponding to $\lambda_i$ are mutually orthogonal with respect to the inner product on $L^2(M)$, and $\bigoplus_{i=1}^{\infty} E_i$ is dense in both $L^2(M)$ and the completion of $C^\infty(M)$ in $L^2(M)$.

If $f_i$ is an eigenfunction corresponding to $\lambda_i$, then for any function $u \in W^{1,2}(M, \mathbb{R})$ with $\|u\|_{L^2} = 1$ and $\langle u, f_i \rangle_{L^2} = 0$ for $i = 1, 2, \ldots, j - 1$, we have that

$$Q(u, u) \geq \lambda_j,$$

with equality holding if and only if $Lu + \lambda_j u = 0$.

Sometimes bounds on the index of a minimal submanifold can be obtained directly from getting bounds on the number of negative eigenvalues of $L$, though this is more feasible in certain situations than in others. If the Ricci curvature and squared norm of the second fundamental form are constant, then finding the eigenvalues of the Jacobi operator turns into the problem of finding the eigenvalues of the Laplacian on $M$, a more thoroughly-studied operator. In some special cases, such as when we have $(M, g) = (S^k, g_0)$, with $g_0$ the metric induced from $\mathbb{R}^{k+1}$, the eigenvalues of the Laplacian and their multiplicities are known precisely and the index can be calculated.

**Theorem 1.2.3.1.** The eigenvalues of the Laplacian on $(S^k, g_0)$ are given by

$$\lambda_j := (j - 1)(k + j - 2),$$

with multiplicity

$$\dim \mathcal{P}_{j-1} - \dim \mathcal{P}_{j-3} = \binom{k + j - 1}{j - 1} - \binom{k + j - 3}{j - 3},$$

where $\mathcal{P}_j$ is the space of homogeneous polynomials of degree $j$ on $\mathbb{R}^{k+1}$.

The proof is given by Sakai in [15] (pg. 272).

The following theorem shows that the spectrum of the Laplacian on a product manifold (endowed with the product metric) is completely determined by the spectra of the Laplacian on each of the component manifolds.
Here $\text{Spec}(M, g)$ denotes the spectrum of the Laplacian on the Riemannian manifold $(M, g)$.

**Theorem 1.2.3.2.** Let $(M, g)$ and $(N, h)$ be Riemannian manifolds. On the product manifold $(M \times N, g \times h)$,

$$\text{Spec}(M \times N, g \times h) = \{ \lambda + \mu \mid \lambda \in \text{Spec}(M, g), \mu \in \text{Spec}(N, h) \}.$$  

The details are given by Berger in [2] (pg. 143).
Chapter 2

Classification Results for Minimal Submanifolds of $S^n$

In this chapter we specifically focus on minimal submanifolds of $S^n$. We begin by deriving a condition for which an isometric immersion $\Phi : M \to S^n$ is actually a minimal immersion. Following Simons’s original exposition [16], we prove that out of all $k$-dimensional minimal submanifolds of $S^n$, the standard embedded $S^k \subset S^n$ are the ones which minimize the index. We discuss a related conjecture which claims that amongst the non-totally geodesic minimal hypersurfaces, the Clifford hypersurfaces minimize the index. Finally, we present Urbano’s proof of this conjecture in the special case when $n = 3$.

2.1 Minimal Submanifolds of $S^n$

Here we present some properties of minimal submanifolds of $S^n$.

2.1.1 Isometric and Minimal Immersions into $S^n$

Later we will use the following proposition from Lawson’s book [9] (pg. 15) which tells us how the Laplacian acts on the component functions of isometric and minimal immersions into the standard sphere $S^n$.

**Lemma 2.1.1.1.** If $\psi : M^m \to S^n \subset \mathbb{R}^{n+1}$ is an isometric immersion, then

$$\Delta_M \psi = \bar{H} = H - m\psi,$$

where $H, \bar{H}$ are the mean curvature vectors of $M$ in $S^n, \mathbb{R}^{n+1}$ respectively.
2.1. Minimal Submanifolds of $S^n$

Hence, $\psi$ is a minimal immersion if and only if

$$\Delta_M \psi = -m \psi.$$ 

Note 2.1.1.1. Here we think of $\psi$ as the vector $\psi_i \frac{\partial}{\partial x^i} \in \mathbb{R}^{n+1}$ and by $\Delta_M \psi$ we mean the vector $(\Delta_M \psi_1, \Delta_M \psi_2, \ldots, \Delta_M \psi_{n+1})$.

Proof of 2.1.1.1: Let $D, \nabla$ and $\nabla$ denote the covariant differentiation operators on $\mathbb{R}^{n+1}, S^n$ and $M$ respectively, and let $\{E_i\}_{i=1}^m$ be a local orthonormal frame tangent to $M$. Then, if $\{y^1, \ldots, y^m\}$ and $\{x^1, \ldots, x^{n+1}\}$ are local coordinates on $M$ and $\mathbb{R}^{n+1}$ respectively, then we first note that, for $f \in C^\infty(\mathbb{R}^{n+1})$,

$$d\psi(E_k)(f) = E_k(f \circ \psi) = E_k^i \frac{\partial f}{\partial y^i} \frac{\partial \psi_j}{\partial x^j},$$

where $E_k = E_k^i \frac{\partial}{\partial x^i}$, and so we get that $d\psi(E_k) = E_k^i (\psi_j) \frac{\partial}{\partial x^j}$. From this we see that

$$E_k(\psi_i) \frac{\partial}{\partial x^i} = d\psi_i(E_k) \frac{\partial}{\partial x^i} = d\psi(E_k) = E_k,$$

so $E_k(\psi_i) = E_k^i$. Hence

$$E_kE_k(\psi_i) \frac{\partial}{\partial x^i} = E_k(E_k^i) \frac{\partial}{\partial x^i} = dE_k^i(E_k) \frac{\partial}{\partial x^i} = dE_k(E_k) = DE_k E_k.$$

Also,

$$(\nabla E_k E_k)(\psi_i) \frac{\partial}{\partial x^i} = d\psi_i (\nabla E_k E_k) \frac{\partial}{\partial x^i} = d\psi (\nabla E_k E_k) = \nabla E_k E_k.$$

Therefore,

$$\Delta_M \psi = \sum_{k=1}^m E_k E_k(\psi) - (\nabla E_k E_k) \psi = \sum_{k=1}^m DE_k E_k - \nabla E_k E_k$$

$$= \sum_{k=1}^m (DE_k E_k)^\perp = \bar{H},$$

(2.1)

where $(\cdot)^\perp$ denotes the projection onto the space normal to $T_p M$ in $T_p \mathbb{R}^{n+1}$. 

14
2.1. Minimal Submanifolds of $S^n$

Now

\[
H = \sum_{k=1}^{m} (\nabla E_k E_k)^N = \sum_{k=1}^{m} \left( (D E_k E_k)^T \right)^N = \sum_{k=1}^{m} \left( (D E_k E_k)^N \right)^T = (\bar{H})^T,
\]

where $(\cdot)^N$ denotes the projection onto the space normal to $T_pM$ in $T_pS^n$, and $(\cdot)^T$ denotes the tangential projection onto $T_pM$. From this and (2.1) we see that

\[
\Delta_M \psi = \bar{H} = (\bar{H})^T + (\bar{H})^\perp = H + \lambda \psi,
\]

for some function $\lambda$. Together $\|\psi\| \equiv 1$ and (1.3) give us that

\[
0 = \frac{1}{2} \Delta_M \|\psi\|^2 = \langle \psi, \Delta_M \psi \rangle + \|\text{grad}_M \psi\|^2 = \lambda + \|\text{grad}_M \psi\|^2.
\]

Thus $\lambda = -\|\text{grad}_M \psi\|^2$. Now

\[
\|\text{grad}_M \psi\|^2 = \sum_{i=1}^{n+1} \|\text{grad}_M \psi_i\|^2 = \sum_{i=1}^{n+1} \sum_{j=1}^{m} \langle \text{grad}_M \psi_i, E_j \rangle^2
\]

\[
= \sum_{i=1}^{n+1} \sum_{j=1}^{m} \langle \text{grad}_M \psi_i, E_j \rangle^2
\]

\[
= \sum_{i=1}^{n+1} \sum_{j=1}^{m} \langle d\psi_i(E_j) \rangle^2.
\]

If we choose the local coordinates $\{x^i\}$ so that the vectors $\{\frac{\partial}{\partial x^i}\}_{p}$ are orthonormal, then (at $p$) this gives us that

\[
\|\text{grad}_M \psi\|^2 = \sum_{i=1}^{n+1} \sum_{j=1}^{m} (d\psi_i(E_j))^2 = \sum_{i=1}^{n+1} \sum_{j=1}^{m+1} \left( d\psi_i(E_j) \frac{\partial}{\partial x^i} \cdot d\psi_i(E_j) \frac{\partial}{\partial x^j} \right)
\]

\[
= \sum_{j=1}^{m} \|d\psi(E_j)\|^2
\]

\[
= \sum_{j=1}^{m} 1 = m.
\]
and so we get that $\Delta_M \psi = \bar{H} = H - m\psi$.

2.1.2 Simons Classification Result

The easiest example of a closed minimal submanifold of $S^n$ is the embedded totally geodesic $S^k \subset S^n$, but these are by no means the only ones: in 1970, Lawson [8] showed that every compact, orientable surface can be minimally embedded into $S^3$. However, it is well known that the embedded, totally geodesic $S^k$ are the only immersed totally geodesic minimal submanifolds of $S^n$. We will briefly outline the proof of this fact before turning our attention to the proof of the following main result: in 1968, Simons [16] proved that any immersed minimal submanifold $M^k$ of $S^n$ is unstable ($\text{ind}(M) \geq 1$), and that $\text{ind}(M) = 1$ if and only if $M$ is diffeomorphic to $S^{n-1}$ embedded in the standard way as a totally geodesic submanifold. In fact, he proved that if $M^k$ is a $k$-dimensional minimal submanifold of $S^n$, then $\text{ind}(M) \geq n-k$ with equality holding if and only if $M$ is diffeomorphic to $S^k$ (embedded into $S^n$ in the standard, totally geodesic way), classifying the totally geodesic spheres as those which minimize the index.

Since we will be primarily concerned with characterizing the minimal submanifolds $M$ of $S^n$ which minimize the index, note that, henceforth, we will assume that $M$ is connected. If $M$ is not connected, then each of its connected components is itself a minimal submanifold of $S^n$ and the index of $M$ is the sum of the indices of its connected components. Since, as we will show, each connected minimal submanifold of $S^n$ is unstable, it follows that the index of $M$ is strictly greater than the index of any of its connected components, and so characterizing the minimal submanifolds $S^n$ which minimize the index will be the same as characterizing the connected minimal submanifolds of $S^n$ which minimize the index.

**Lemma 2.1.2.1.** Let $M^k$ be a closed totally geodesic immersed submanifold of $S^n$. Then $M$ is isometric to $S^k$.

Proof of 2.1.2.1: Let $p \in M$ and $v \in T_p M$. As $M$ is assumed to be totally geodesic, for sufficiently small $t$, $\exp^M_p(tv)$ is a geodesic in $S^n$ and therefore a piece of a great circle. Since $M$ is closed, it is compact and therefore
2.1. Minimal Submanifolds of $S^n$

complete, and $\partial M = \emptyset$. Hence $\exp^M_p(vt)$ is defined for all $t$ and must sweep out the entire great circle. So, letting $v$ vary throughout $T_pM$ and $t \in \mathbb{R}$, $\exp^M_p(vt)$ sweeps out a $k$-dimensional sphere. Since $M$ is connected, we must therefore have that $M \cong S^k$.

We now present the proof of the main classification result.

**Theorem 2.1.2.2.** (Simons, 1968) Let $M$ be a compact, closed $k$-dimensional minimal submanifold immersed in $S^n$. Then $\text{ind}(M) \geq n - k$, with equality holding if and only if $M$ is diffeomorphic to $S^k$ embedded into $S^n$ in the standard way as a totally geodesic submanifold.

**Proof of 2.1.2.2:** For the sake of brevity, by $S^k$ we will always mean the totally geodesic minimal embedding $S^k \hookrightarrow S^n$.

We start by showing that $\text{ind}(S^k) = n - k$. First, for any $k$-dimensional minimal submanifold we have that for any $V \in \Gamma(NS^k)$,

$$R(V) := \text{Tr}[R_{S^n}(\cdot, V) \cdot] = -kV.$$  \hspace{1cm} (2.2)

To show this, we make note that, since $S^n$ has constant sectional curvature $K \equiv 1$, it follows that

$$R_{S^n}(X, Y, Y, X) = 1 \cdot (\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2)$$

for all $X, Y \in \Gamma(TS^n)$. Moreover,

$$\langle X, W \rangle \langle Y, Z \rangle - \langle X, Y \rangle \langle W, Z \rangle$$

is a covariant 4-tensor on $T_pS^n$ with the same symmetries as the curvature tensor. By a standard result in Riemannian geometry it must therefore be equal to the curvature tensor, i.e.

$$R_{S^n}(X, Y, Z, W) = \langle X, W \rangle \langle Y, Z \rangle - \langle X, Y \rangle \langle W, Z \rangle$$

for all $X, Y, Z, W \in \Gamma(TS^n)$ (for proof see [10] (pg. 146) ). Hence it follows
2.1. Minimal Submanifolds of $S^n$

that

$$R_{S^n}(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y,$$

for all $X, Y, Z \in \Gamma(TS^n)$. So, if $\{E_i\}_{i=1}^k$ is an orthonormal frame for $TS^k$, then

$$\mathcal{R}(V) = \sum_{i=1}^k (R_{S^k}(E_i, V)E_i)^N = \sum_{i=1}^k ((\langle V, E_i \rangle E_i - \langle E_i, V \rangle)E_i)^N = -kV.$$

Since $A \equiv 0$ in the case of $S^k$ ($S^k$ is totally geodesic), this means that (1.6) becomes

$$L = \Delta_{S^k} + k,$$

and so it is clear that the eigenspaces of $\Delta_{S^k}$ with eigenvalue $\lambda$ correspond to eigenspaces of $L$ with eigenvalue $\lambda - k$. Now, note that the normal bundle of $S^k$ is trivial: we can rotate, if necessary, so that $S^k = \{(x_1, \ldots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1} | x_{k+2} = \ldots = x_{n+1} = 0\}$. Then the vectors $\{\frac{\partial}{\partial x_{k+2}}, \ldots, \frac{\partial}{\partial x_n}\}$ form a global frame for the normal bundle of $S^k$ in $S^n$. From this it follows that we can choose a global parallel frame $\{V_1, \ldots, V_{n-k}\}$ for $NS^k$. Then, for any $V \in \Gamma(NS^k)$, we can write $V = v^iV_i$ for some function $v^i \in C^\infty(S^k)$, and, since $\{V_i\}$ is a parallel frame,

$$\Delta_{S^k}V = \Delta_{S^k}(v^iV_i) = \Delta_{S^k}(v^i)V_i.$$

Hence, $\Delta_{S^k}(V) + \lambda V = 0$ if and only if

$$\left(\Delta_{S^k}(v^1), \ldots, \Delta_{S^k}(v^{n-k})\right) + \lambda \left(v^1, \ldots, v^{n-k}\right) = \vec{0}. \quad (2.3)$$

Now, on functions, from Lemma 1.2.3.1 we know that $\Delta_{S^k}$ has a 1-dimensional nullspace and a $(k + 1)$-dimensional eigenspace corresponding to the eigenvalue $k$. Moreover, from Lemma 1.2.3.1 it is also clear that all of the other eigenvalues are all strictly greater than $k$. Thus, $\text{ind}(S^k) = n - k$.

We will prove the second half of the theorem through a sequence of
lemmas which together establish that the index form $Q$ takes a special form when restricted to a specific subspace of $\Gamma(NM)$. In particular, we will show that for every $u$ in this specific subspace of $\Gamma(NM)$,

$$Q(u, u) = -k \int_M \|u\|^2 \, dV.$$  

It is clear from this that the index form is negative definite on this particular subspace, and so the result will then follow once we show that the dimension of this subspace is greater than or equal to $n - k$, with equality if and only if $A \equiv 0$.

As before, let $D, \nabla, \nabla$ denote the Levi-Civita connections for $\mathbb{R}^{n+1}$, $S^n$ and $M$ respectively, and $\nabla^N, \nabla^N$ denote the connections on the normal bundles of $M$ in $S^n$ and of $S^n$ in $\mathbb{R}^{n+1}$ respectively.

Given a parallel vector field in $\mathbb{R}^{n+1}$, we can take the tangential projection onto $S^n$ to get a vector field on $S^n$. The collection of all such vector fields, $\xi$, forms an $(n+1)$-dimensional subspace of $\Gamma(TS^n)$.

**Lemma 2.1.2.3.** Let $Z \in \xi$, $p \in S^n$. Then there is $\lambda \in C^\infty(S^n)$ such that, for any $X \in T_pS^n$

$$\nabla_X Z = \lambda X.$$  

**Proof of 2.1.2.3:** Let $Z = W^T$, where $W$ is a parallel vector field on $\mathbb{R}^{n+1}$ so that

$$\nabla_X Z = (D_X Z)^T = (D_X W^T)^T = (D_X (W - W^N))^T = - (D_X W^N)^T,$$

where $W^N$ is the component of $W$ normal to $S^n$. If we let $N$ be the unit normal to $S^n$, then $W^N = -\lambda N$ for some function $\lambda$, and so putting this into (2.4) we get that

$$\nabla_X Z = (D_X \lambda N)^T = (X(\lambda)N + \lambda D_X N)^T = \lambda X,$$

where, as a consequence of the standard coordinate vector fields on $\mathbb{R}^{n+1}$
2.1. Minimal Submanifolds of $S^n$

\{\partial_i := \frac{\partial}{\partial x^i}\}_{i=1}^{n+1} \text{ being parallel, we have that}

$$D_X N = X^i D_{\partial_i} (x^j \partial_j) = X^i \partial_i = X.$$  \hfill \Box

Now, since $Z \in \xi$ is a vector field on $S^n$, restricting it to $M$ and taking the normal and tangential projections gives vector fields $Z^N \in \Gamma(NM)$ and $Z^T \in \Gamma(TM)$ respectively.

**Lemma 2.1.2.4.** For $Z \in \xi$, the vector fields $Z^N \in \Gamma(NM)$ and $Z^T \in \Gamma(TM)$ satisfy

$$\nabla_X Z^N = -A(X, Z^T),$$

$$\nabla_X Z^T = - (\nabla_X Z^N)^T + \lambda X,$$

where $X \in T_p M$ and $\lambda$ is independent of $X$.

**Proof of 2.1.2.4:** Taking into account Lemma 2.1.2.3 we have that

$$\nabla_X Z^N = (\nabla_X Z^N)^N = (\nabla_X (Z - Z^T))^N = (\lambda X - \nabla_X Z^T)^N = -A(X, Z^T).$$

Again, using Lemma 2.1.2.3 we see that

$$\nabla_X Z^T = (\nabla_X Z^T)^T = (\nabla_X (Z - Z^N))^T = \lambda X - (\nabla_X Z^N)^T.$$  \hfill \Box

Let $\{E_i\}_{i=1}^k$ be a local orthonormal frame for $TM$ for which $\nabla_{E_i} E_j|_p = 0$ for $i, j = 1, \ldots, k$. Though we will not write it explicitly, note that the following calculations will all be computed at $p$.

**Lemma 2.1.2.5.** Let $Z \in \xi$ and consider $Z^N \in \Gamma(NM)$. Then

$$\|\nabla^N Z^N\|^2 = \sum_{i=1}^k \langle Z^N, A(E_i, \nabla_{E_i} Z^T) \rangle + E_i(\langle Z^N, \nabla_{E_i} Z^N \rangle).$$
2.1. Minimal Submanifolds of $S^n$

Proof of 2.1.2.5: Using Lemma 2.1.2.4 we have that

\[ \|\nabla^N Z^N\|^2 = \sum_{i=1}^{k} \langle \nabla^N_{E_i} Z^N, \nabla^N_{E_i} Z^N \rangle \]

\[ = \sum_{i=1}^{k} E_i (Z^N, \nabla^N_{E_i} Z^N) - (Z^N, \nabla_{E_i} \nabla^N_{E_i} Z^N) \]

\[ = \sum_{i=1}^{k} E_i (Z^N, \nabla^N_{E_i} Z^N) + (Z^N, \nabla^N_{E_i} (A(E_i, Z^T))) \]

Now

\[ \nabla^N_{E_i} (A(E_i, Z^T)) = (\nabla E_i A) (E_i, Z^T) + A(\nabla E_i E_i, Z^T) + A(E_i, \nabla E_i Z^T) \]

\[ = (\nabla E_i A) (E_i, Z^T) + A(E_i, \nabla E_i Z^T). \] (2.5)

From the Codazzi Equation (1.1.2.2),

\[ (\nabla E_i A) (Z^T, E_i) = (\nabla Z^T A) (E_i, E_i) + (R_{S^n}(E_i, Z^T) E_i)^N, \]

and, again using that $S^n$ has constant curvature,

\[ (R_{S^n}(E_i, Z^T) E_i)^N = ((Z^T, E_i) E_i - (E_i, E_i) Z^T)^N = 0. \]

Also,

\[ (\nabla Z^T A) (E_i, E_i) = \nabla^N_{Z^T} (A(E_i, E_i)) - A(E_i, \nabla Z^T E_i) - A(\nabla Z^T E_i, E_i) \]

\[ = \nabla^N_{Z^T} (A(E_i, E_i)), \]

since $\nabla Z^T E_i = \sum_{j=1}^{k} (Z^T, E_j) \cdot \nabla E_j E_i = 0$. Thus

\[ \sum_{i=1}^{k} (\nabla Z^T A) (E_i, E_i) = \sum_{i=1}^{k} \nabla^N_{Z^T} (A(E_i, E_i)) \]

\[ = \nabla^N_{Z^T} \sum_{i=1}^{k} A(E_i, E_i) = \nabla^N_{Z^T} H = 0, \]
2.1. Minimal Submanifolds of $S^n$

and so we get that $\langle \nabla E_i A, (Z^T, E_i) \rangle = 0$. Therefore (2.5) becomes

$$\nabla^N_{E_i} (A(E_i, Z^T)) = A(E_i, \nabla E_i Z^T).$$

Hence,

$$\|\nabla^N Z^N\|^2 = \sum_{i=1}^k E_i \langle Z^N, \nabla^N_{E_i} Z^N \rangle + \langle Z^N, \nabla^N_{E_i} (A(E_i, Z^T)) \rangle$$

$$= \sum_{i=1}^k \langle Z^N, A(E_i, \nabla E_i Z^T) \rangle + E_i \langle Z^N, \nabla^N_{E_i} Z^N \rangle$$

$$= -\langle \nabla^N Z^N, Z^N \rangle.$$

Lemma 2.1.2.6. Let $Z \in \xi$ and consider $Z^N \in \Gamma(NM)$. Then

$$\langle \Delta^N_{M} Z^N, Z^N \rangle = -\langle \tilde{A}(Z^N), Z^N \rangle.$$
2.1. Minimal Submanifolds of $S^n$

Now, using Lemma 2.1.2.4 we also have that

$$
\nabla_{E_i}Z^T = \sum_{j=1}^{k} \langle E_j, \nabla_{E_i}Z^T \rangle E_j = \sum_{j=1}^{k} -\langle E_j, (\nabla_{E_i}Z^N)^T \rangle E_j + \lambda \delta_{i,j} E_j
$$

$$
= \sum_{j=1}^{k} (\nabla_{E_i}E_j, Z^N) E_j + \lambda \delta_{i,j} E_j
$$

$$
= \sum_{j=1}^{k} (A(E_i, E_j), Z^N) E_j + \lambda \delta_{i,j} E_j.
$$

So, since $A$ is bilinear,

$$
A(E_i, \nabla_{E_i}Z^T) = \sum_{j=1}^{k} \langle A(E_i, E_j), Z^N \rangle A(E_i, E_j) + \lambda \delta_{i,j} A(E_i, E_j),
$$

and therefore

$$
\sum_{i=1}^{k} \langle Z^N, A(E_i, \nabla_{E_i}Z^T) \rangle = \sum_{i,j=1}^{k} \langle A(E_i, E_j), Z^N \rangle^2 + \lambda \delta_{i,j} \langle A(E_i, E_j), Z^N \rangle
$$

$$
= \langle \tilde{A}(Z^N), Z^N \rangle + \sum_{i=1}^{k} \lambda \langle A(E_i, E_i), Z^N \rangle
$$

$$
= \langle \tilde{A}(Z^N), Z^N \rangle + \lambda \langle H, Z^N \rangle
$$

$$
= \langle \tilde{A}(Z^N), Z^N \rangle.
$$

Hence, (2.6) becomes

$$
\langle \Delta_M^N Z^N, Z^N \rangle = -\langle \tilde{A}(Z^N), Z^N \rangle.
$$

\[\square\]

**Lemma 2.1.2.7.** For $Z \in \xi$, considering $Z^N \in \Gamma(NM)$ we have that

$$
Q(Z^N, Z^N) = -k \int_M \|Z^N\|^2 dV.
$$
Proof of 2.1.2.7: Since (2.2) holds for any \( k \)-dimensional minimal submanifold of \( S^n \), (1.6) becomes

\[
LZ^N = \Delta_M (Z^N) + kZ^N + \tilde{A}(Z^N).
\]

Hence, the desired result follows from applying Lemma 2.1.2.6.

Let \( \xi^N = \{ Z^N \in \Gamma(NM) \mid Z \in \xi \} \). Then it follows from Lemma 2.1.2.7 that \( Q(\cdot, \cdot) \) is negative definite on \( \xi^N \).

**Lemma 2.1.2.8.** We have that \( \dim \xi^N \geq n - k \) with equality if and only if \( M \) is isometric to \( S^k \) (embedded in the standard way as a totally geodesic submanifold).

Proof of 2.1.2.8: It is clear that at each \( p \in M \), \( \xi(p) := \{ X(p) \mid X \in \xi \} \) spans \( T_pS^n \), and so for each \( p \in M \), \( \xi^N \) spans \( N_pM \subset T_pS^n \). Hence, \( \dim \xi^N \geq \dim N_pM = n - k \).

Now suppose that \( \dim \xi^N = n - k \), and let \( \eta \) be the kernel of homomorphism \( \xi \to \xi^N \) so that if \( X \in \eta \), then \( X^T = X \) at every \( p \in M \), and define \( \eta(p) := \{ X(p) \mid X \in \eta \} \). Now if we let \( \eta_p \) be the kernel of the homomorphism \( \xi \to N_pM \) defined by \( Z \mapsto Z^N(p) \) \( (p \in M) \), then it is clear that \( \eta(p) \subseteq \eta_p \). So since \( \dim \eta_p = (n + 1) - (n - k) \), and since our assumption \( \dim \xi^N = n - k \) implies that \( \dim \eta = (n + 1) - (n - k) \), we must have that \( \eta = \eta_p \). This means that the map \( Z \mapsto Z^T(p) \), which clearly maps \( \eta_p \) onto \( T_pM \), must map \( \eta \) onto \( T_pM \). Hence, for any \( Z \in T_pM \), there exists \( \tilde{Z} \in \xi \) such that \( \tilde{Z}(p) = Z \), and \( \tilde{Z} \) is everywhere tangent to \( M \). From this and Lemma 2.1.2.4 it follows that, for any \( X, Z \in T_pM \),

\[
A(X, Z) = -\nabla^N_X \tilde{Z}^N = 0,
\]

and so we can conclude that \( A \equiv 0 \). Thus \( M \) is totally geodesic and therefore isometric to \( S^k \) by Lemma 2.1.2.1.

\[ \square \]
2.2 Non-totally Geodesic Minimal Hypersurfaces of $S^n$

If we restrict our attention to non-totally geodesic minimal hypersurfaces of $S^n$, then it is conjectured that the index also characterizes the Clifford hypersurfaces in a way similar to Simon’s classification result for minimal hypersurfaces of index one: the Clifford hypersurfaces are thought to minimize the index amongst non-totally geodesic minimal hypersurfaces. Indeed we will prove that this is the case when $n = 3$.

2.2.1 A Classification Conjecture

Here we introduce the Clifford hypersurfaces of $S^n$ and some of their properties before turning to the classification conjecture.

Definition 2.2.1.1. By a Clifford hypersurface of $S^n$ we mean a product $S^k \left( \sqrt{\frac{k}{n-1}} \right) \times S^l \left( \sqrt{\frac{l}{n-1}} \right)$, where $k + l = n - 1$.

To show that the Clifford hypersurfaces are, in fact, minimal, we follow Lawson’s exposition [9].

Lemma 2.2.1.1. The Clifford hypersurfaces are minimal submanifolds of $S^n$.

The proof can be found in [9] (pg. 23).

The next theorem gives a characterization of the Clifford hypersurfaces in terms of the square norm of their second fundamental form. We will use this result in calculating their index, as well as in the proof of the conjecture in the case $n = 3$ (Theorem 2.2.2.5). The proof can be found in [5] (see the Main theorem, pg. 60).

Theorem 2.2.1.2. (Chern, Do Carmo, Kobayashi (1970); Lawson (1969)) 
The Clifford hypersurfaces are the only minimal hypersurfaces of $S^n$ with $\|A\|^2 \equiv n - 1$.

As mentioned above, it is thought that the Clifford hypersurfaces are also characterized by their index.
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

**Conjecture 2.2.1.1.** Let $M$ be a closed, orientable, non-totally geodesic minimal hypersurface of $S^n$. Then $\text{ind}(M) \geq n + 2$ with equality holding if and only if $M$ is a Clifford hypersurface.

We now show that one direction of the conjecture is true: the Clifford hypersurfaces have index $n + 2$.

**Lemma 2.2.1.3.** If $M = S^k \left( \sqrt{\frac{k}{n-1}} \right) \times S^l \left( \sqrt{\frac{l}{n-1}} \right)$ is a Clifford hypersurface where $k + l = n - 1$, then $\text{ind}(M) = n + 2$.

**Proof of 2.2.1.3:** It follows from Theorem 2.2.1.2 that $L = \Delta_M + 2(n - 1)$, and so the eigenvalues of $L$ are in one-to-one correspondence with the eigenvalues of $\Delta_M$: $\Delta_M f + \mu f = 0$ if and only if $Lf + \lambda f = 0$ where $\lambda = \mu - 2(n - 1)$. Now, since $M$ is a product manifold (with the product metric), we can use Lemma 1.2.3.2 to calculate the eigenvalues of the Laplacian on $M$ if we know the eigenvalues of the Laplacian on $S^k \left( \sqrt{\frac{k}{n-1}} \right)$ and $S^l \left( \sqrt{\frac{l}{n-1}} \right)$.

From Lemma 1.2.3.1 we know that the eigenvalues of the Laplacian on $S^k(1)$ are

$$\sigma_j = (j - 1)(k + j - 2), \; j = 1, 2, 3, \ldots$$

$$= 0, \; k, \; 2(k + 1), \ldots$$

with multiplicity

$$\dim \mathcal{P}_{j-1} - \dim \mathcal{P}_{j-3} = \binom{k + j - 1}{j - 1} - \binom{k + j - 3}{j - 3}$$

$$= 1, \; k + 1, \; \binom{k + 2}{2} - 1, \ldots$$

(2.8)

where $\mathcal{P}_j$ is the space of homogeneous polynomials of degree $j$ on $\mathbb{R}^{k+1}$.

Now let $f : (S^k(1), h) \to (S^k(r), g)$ be defined by $x \mapsto rx$, where $g$ is the metric induced from $\mathbb{R}^{k+1}$ and $h$ is the pullback metric $h = f^*g$. Clearly $h = r^2g$ is a scaling of the metric on $S^k(1)$ induced from $\mathbb{R}^{k+1}$, so we can find the eigenvalues of the Laplacian on $S^k(r)$ by figuring out how the eigenvalues of the Laplacian on $S^k(1)$ change when we scale the metric. Using the local
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

formula for the Laplacian with respect to the metric $h$, $\Delta_h$, we see that

$$\Delta_h f = \frac{1}{\sqrt{H}} \frac{\partial}{\partial x^i} \left( h^{ij} \sqrt{H} \frac{\partial}{\partial x^j} f \right) = \frac{1}{r^k \sqrt{G}} \frac{\partial}{\partial x^i} \left( r^{-2} g^{ij} r^k \sqrt{G} \frac{\partial}{\partial x^j} f \right)$$

$$= \frac{1}{r^2} \Delta_g f,$$

where $\sqrt{H} = \sqrt{|\det(h_{ij})|}$. This shows that

$$\Delta_g f + \lambda f = 0 \iff \Delta_h f + \frac{\lambda}{r^2} f = 0,$$

and so scaling the metric by a factor of $r^2$ has the effect of scaling the eigenvalues by a factor of $\frac{1}{r^2}$. Hence, from (2.7) and (2.8) the first three eigenvalues of the Laplacian on $S^k \left( \sqrt{\frac{k}{n-1}} \right)$ and $S^l \left( \sqrt{\frac{l}{n-1}} \right)$ are $\eta_{k,1} = 0$ (multiplicity 1), $\eta_{k,2} = \frac{k(n-1)}{k} = n-1$ (multiplicity $k+1$), $\eta_{k,3} = \frac{2(k+1)(n-1)}{k}$ (multiplicity $(k+2) - 1$) and $\eta_{l,1} = 0$ (multiplicity 1), $\eta_{l,2} = \frac{l(n-1)}{l} = n-1$ (multiplicity $l+1$), $\eta_{l,3} = \frac{2(l+1)(n-1)}{l}$ (multiplicity $(l+2) - 1$) respectively. Therefore, Lemma 1.2.3.2 implies that the first three eigenvalues of the Laplacian on $M$ are $\mu_1 = 0$ (multiplicity one), $\mu_2 = n-1$ (multiplicity $(k+1)+(l+1) = n+1$) and $\mu_3 = \min \left\{ 2(n-1), \frac{2(k+1)(n-1)}{k}, \frac{2(l+1)(n-1)}{l} \right\} = 2(n-1)$. From this it follows that the first two eigenvalues of $L$ on $M$ are $\lambda_1 = -2(n-1) \text{ (multiplicity 1)}$ and $\lambda_2 = -(n-1) \text{ (multiplicity } n+1\text{), and that all the other eigenvalues are non-negative. Hence } \text{ind}(M) = n+2.$

The other direction of the conjecture is much less clear. Though it is known to be true in the case $n = 3$ (see §2.2.2), there are several things used in the proof of this special case which are no longer available when one moves to higher dimensions, preventing the proof from generalizing. Perdomo [14] has shown that the conjecture is true under an additional symmetry assumption which all known minimal hypersurfaces of spheres satisfy. Namely, he has proved the following theorem:

**Theorem 2.2.1.4.** (Perdomo, 2001) *Let $M$ be a compact, oriented non-totally geodesic minimal hypersurface of $S^n$, and let $O_M(n+1)$ be the subgroup of the orthogonal group $O(n+1)$ consisting of orthogonal transforma-
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

... which fix $M$, i.e.

$$O_M(n + 1) = \{ \gamma \in O(n + 1) \mid \gamma(M) = M \}.$$ 

If $O_M(n + 1)$ fixes only the origin of $\mathbb{R}^{n+1}$, then $\text{ind}(M) \geq n + 2$ with equality if and only if $M$ is a Clifford hypersurface.

His proof heavily relies on the imposed symmetry condition, and it seems that the only obvious way to extend his result to prove the full conjecture would be to somehow show that all minimal submanifolds of $S^n$ of index $n + 2$ actually satisfy the symmetry condition.

Motivated by the behaviour of the index functional of the space of minimal hypersurfaces with constant squared norm of the second fundamental form, Perdomo [14] has also conjectured that $\text{ind}(M) \geq 2n + 3$ for all non-totally geodesic minimal hypersurfaces of $S^n$ which are not Clifford. This is true for all known examples of minimal hypersurfaces of $S^n$, and Perdomo shows that it is also true when one also assumes that the symmetry condition in Theorem 2.2.1.4 is satisfied and that the hypersurfaces has an even unit normal vector $N$ in the sense that the functions $f_a(p) = \langle a, N(p) \rangle$ are all even functions.

2.2.2 Urbano’s Result: Proof of the Conjecture when $n = 3$

In 1990, Urbano proved that the conjecture is true in the special case of $n = 3$. We now mention those results that will be used in the proof that have not already been stated.

**Lemma 2.2.2.1.** (Almgren, 1966) Let $f : S^2 \to S^3 \subset \mathbb{R}^4$ be a real analytic minimal immersion. Then $f$ embeds $S^2$ in $S^3$, and $f(S^2) = S^3 \cap \{ x \in S^3 \mid x \cdot v = 0 \}$ for some $v \in S^3$.

The proof can be found in [1] (pg. 279).

**Lemma 2.2.2.2.** (Obata, 1962) A complete Riemannian manifold $M^n$ with $n \geq 2$ admits a non-constant function $\phi$ such that $\text{Hess} \phi(\cdot, \cdot) = -c^2 \phi(\cdot, \cdot)$ if and only if $M$ is isometric with a sphere $S^n \left( \frac{1}{c} \right)$ in $\mathbb{R}^{n+1}$.
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

The proof can be found in [13] (pg. 334).

**Theorem 2.2.2.3.** (Gauss-Bonnet) Suppose $M$ is a compact two-dimensional Riemannian manifold. Then

$$\int_M KdA = 2\pi \cdot \chi(M),$$

where $\chi(M)$ is the Euler characteristic of $M$.

**Lemma 2.2.2.4.** Let $M$ be a 2-dimensional Riemannian manifold with local conformal parameter $z = x + iy$ and let $f: M \to \bar{M}$ be a conformal map. Then

$$\text{Area}(f(M)) = \frac{1}{2} E(f),$$

where $\text{Area}(f(M))$ is the area of $f(M) \subset \bar{M}$ and $E(f)$ is the energy of $f$,

$$E(f) = \int_M \langle df, df \rangle.$$

**Proof of 2.2.2.4:** Let $g$, also written as $\langle \cdot, \cdot \rangle$, be the metric on $\bar{M}$ so that $f^*g$ is the metric on $M$. Let $\{E_1, E_2\}$ be a local orthonormal frame for $TM$, and let $\{\omega^1, \omega^2\}$ be the coframe for $T^*M$ dual to $\{E_1, E_2\}$. Then, since $f$ is conformal and therefore preserves angles, the energy of $f$ is simply

$$E(f) = \int_M \left( \langle df(E_1), df(E_1) \rangle + \langle df(E_2), df(E_2) \rangle \right) d\omega^1 \wedge d\omega^2. \quad (2.9)$$

Moreover, since $E_1$ and $E_2$ each have unit length and $f$ is conformal,

$$\langle df(E_1), df(E_1) \rangle = \langle df(E_2), df(E_2) \rangle = : \Lambda.$$

Hence, (2.9) simplifies to

$$E(f) = 2 \int_M \Lambda d\omega^1 \wedge d\omega^2. \quad (2.10)$$

29
On the other hand the area is given by
\[
\text{Area}(f(M)) = \int_M dV = \int_M \sqrt{\det(f^*g)} d\omega^1 \wedge d\omega^2;
\]
where \((f^*g)_{ij} = \langle df(E_i), df(E_j) \rangle\). Then since \(\det(f^*g) = \Lambda^2\), using (2.10) it is easy to see that
\[
\text{Area}(f(M)) = \int_M \Lambda d\omega^1 \wedge d\omega^2 = \frac{1}{2} E(f).
\]

Amongst the various tools that Urbano uses, some are are only available in the specific case when \(M\) is a surface; the Gauss-Bonnet Formula and the result of Almgren are only available in dimension two and don’t allow the proof to be directly generalized to higher dimensions.

**Theorem 2.2.2.5.** (Urbano, 1990) Let \(M\) be a compact orientable non-totally geodesic minimal surface is \(S^3\). Then \(\text{ind}(M) \geq 5\), and the equality holds if and only if \(M\) is the Clifford torus.

The proof is roughly broken into three parts. First we argue that \(\text{ind}(M) \geq 5\) by showing that \(-2\) is an eigenvalue of \(L\) and construct an eigenspace corresponding to the eigenvalue \(-2\). Using the fact that \(M\) is assumed to be non-totally geodesic, we show that this eigenspace has dimension 4. Since the first eigenvalue of \(L\) is simple, this gives us the desired result. Next we assume that \(\text{ind}(M) = 5\). Our previous calculations still hold and therefore tell us that the second eigenvalue is \(-2\) and that all others are non-negative. We use the Hersch trick to construct a conformal transformation \(F_g\) of \(M\) for which the component functions of \(F_g \circ \Phi\) are orthogonal to the first eigenfunction, where \(\Phi : M \to S^3\) is a minimal immersion. We then show that equality actually holds in (1.13), and using Lemma 2.1.1.1 we can then show that \(\|A\|^2 \equiv 2\). It then follows from Theorem 2.2.1.2 that \(M\) must be the Clifford torus. The final step in the proof is to show that the Clifford torus actually has index 5, but this follows from Lemma 2.2.1.3.

As with Simons’s classification result, the details of the proof will be
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

established through a sequence of lemmas.

**Proof of 2.2.2.5:** Let $\Phi : M \to S^3$ be a minimal immersion, $N$ denote the unit normal vector field to $M$, and let $\nabla, \nabla$ and $D$ denote the covariant differentiation operators on $M, S^3$ and $\mathbb{R}^4$ respectively ($\nabla_X Y = (\nabla_X Y)^TM$, $\nabla_X Y = (D_X Y)^{T_{S^3}}$).

**Lemma 2.2.2.6.** For any vector $a \in \mathbb{R}^4$, the function $f_a = \langle a, N \rangle$ is an eigenfunction of $L$ with eigenvalue $-2$.

**Proof of 2.2.2.6:** Let $p \in M$ and $\{E_1, E_2\}$ be an orthonormal frame in $TM$ (defined in a neighbourhood of $p$) such that $\nabla_{E_i} E_j(p) = 0$ for $i, j = 1, 2$. Then, $\{E_1, E_2, N, \Phi\}$ is a local orthonormal frame for $\mathbb{R}^4$, and

$$\Delta_M f_a = \sum_{i=1}^2 E_i E_i(f_a) - \nabla_{E_i} E_i f_a = \sum_{i=1}^2 E_i E_i \langle a, N \rangle$$

(note that this calculation is evaluated at $p$ and that we will continue to suppress the $p$ for convenience). Since $a \in \mathbb{R}^4$, $\langle \cdot, \cdot \rangle$ here denotes the metric on $\mathbb{R}^4$, and so by the compatibility of the $\mathbb{R}^4$ metric with $D$ we have that

$$\Delta_M f_a = \sum_{i=1}^2 E_i (\langle D_{E_i} N, a \rangle + \langle N, D_{E_i} a \rangle) = \sum_{i=1}^2 E_i \langle D_{E_i} N, a \rangle,$$

and

$$D_{E_i} N = \langle D_{E_i} N, E_1 \rangle E_1 + \langle D_{E_i} N, E_2 \rangle E_2 + \langle D_{E_i} N, N \rangle N + \langle D_{E_i} N, \Phi \rangle \Phi.$$

Since $N$ is a unit vector, $\langle D_{E_i} N, N \rangle = \frac{1}{2} E_i \langle N, N \rangle = 0$. Also, if $X$ and $Y$ are orthogonal vector fields on $\mathbb{R}^4$, then from the compatibility of the connection with the inner product we have that for any vector field $Z$ on $\mathbb{R}^4$.

$$\langle D_Z X, Y \rangle = Z \langle X, Y \rangle - \langle X, D_Z Y \rangle = -\langle X, D_Z Y \rangle. \quad (2.11)$$
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

Now $\langle D_{E_i}N, \Phi \rangle = -\langle N, D_{E_i}\Phi \rangle = -\langle N, E_i \rangle = 0$, so

$$D_{E_i}N = 2 \sum_{j=1}^{2} \langle D_{E_i}N, E_j \rangle E_j = -2 \sum_{j=1}^{2} \langle N, D_{E_i}E_j \rangle E_j \quad \text{(by (2.11))}$$
$$= -\sum_{j=1}^{2} h_{ij} E_j,$$

(2.12)

where $h_{ij} = \langle N, \nabla_{E_i} E_j \rangle = \langle N, D_{E_i}E_j \rangle$ is the scalar second fundamental form of $M$ in $S^3$. Thus,

$$\Delta_M f_a = -\sum_{i,j=1}^{2} E_i (h_{ij} \langle E_j, a \rangle)$$
$$= \sum_{i,j=1}^{2} E_i (h_{ij} \langle E_j, a \rangle + h_{ij} \langle D_{E_i}E_j, a \rangle) + 0.$$  \hspace{1cm} (2.13)

From the Codazzi Equation (Lemma 1.1.2.2) we know that, since $R_{R^4} = 0$,

$$0 = \langle R_{R^4}(E_i, E_j)E_k, N \rangle = \langle (D_{E_j}\bar{A})(E_i, E_k), N \rangle - \langle (D_{E_i}\bar{A})(E_j, E_k), N \rangle,$$

(2.14)

where $\bar{A}$ is the second fundamental form on $M$ as a submanifold of $R^4$ and

$$\langle (D_{E_i}\bar{A})(E_j, E_k), N \rangle = E_i \langle \bar{A}(E_j, E_k), N \rangle - \langle \bar{A}(\nabla_{E_i} E_j, E_k), N \rangle$$
$$- \langle \bar{A}(E_j, \nabla_{E_i} E_k), N \rangle - \langle \bar{A}(E_j, E_k), \bar{A}(E_i, N) \rangle.$$

From (2.12) we see that $D_{E_i}N$ is tangent to $M$, so

$$\langle \bar{A}(E_j, E_k), \bar{A}(E_i, N) \rangle = \langle (D_{E_j}E_k)^\perp, (D_{E_i}N)^\perp \rangle = \langle (D_{E_j}E_k)^\perp, 0 \rangle = 0.$$

Also, since $\nabla_{E_i} E_j = 0 \ (at \ p)$ for all $i,j$, and

$$E_i (h_{jk}) = E_i \langle D_{E_j}E_k, N \rangle = E_i \langle (D_{E_j}E_k)^\perp, N \rangle = \langle (D_{E_i}\bar{A})(E_j, E_k), N \rangle,$$
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

(2.14) yields

$$E_i(h_{jk}) = \langle (D_{E_i} \vec{A})(E_j, E_k), N \rangle = \langle (D_{E_j} \vec{A})(E_i, E_k), N \rangle = E_j(h_{ik}).$$

Hence, the Codazzi Equation and the symmetric property of the second fundamental form imply that

$$E_i(h_{ij}) = E_i(h_{ji}) = E_j(h_{ii}).$$

Therefore

$$\sum_{i,j=1}^{2} E_i(h_{ij}) \langle E_j, a \rangle = \sum_{i,j=1}^{2} E_j(h_{ii}) \langle E_j, a \rangle = \sum_{j=1}^{2} E_j(h_{11} + h_{22}) \langle E_j, a \rangle = 0,$$

since $M$ is minimal and therefore has zero mean curvature by Lemma 1.1.3.3. Thus, from (2.13) we find that

$$\Delta_M f_a = - \sum_{i,j} h_{ij} \langle D_{E_i} E_j, a \rangle.$$  \hfill (2.16)

Now,

$$D_{E_i} E_j = \langle D_{E_i} E_j, E_1 \rangle E_1 + \langle D_{E_i} E_j, E_2 \rangle E_2 + \langle D_{E_i} E_j, N \rangle N + \langle D_{E_i} E_j, \Phi \rangle \Phi$$

$$= \langle D_{E_i} E_j, N \rangle N + \langle D_{E_i} E_j, \Phi \rangle \Phi,$$

(2.17)

since $\langle D_{E_i} E_j, E_k \rangle = \langle \nabla_{E_i} E_j, E_k \rangle = 0$. Also, using (2.11) we see that

$$\langle D_{E_i} E_j, \Phi \rangle = - \langle E_j, D_{E_i} \Phi \rangle = - \langle E_j, E_i \rangle = - \delta_{ji},$$

(2.18)
and so (2.16) becomes

$$\Delta_M f_a = -\sum_{i,j}^2 h_{ij} (h_{ij} \langle N, a \rangle - \delta_{ji} \Phi)$$

$$= -\|A\|^2 f_a + \left(\sum_{i=1}^2 h_{ii}\right) \langle \Phi, a \rangle$$

$$= -\|A\|^2 f_a,$$

again using the minimality of $M$ and Lemma 1.1.3.3. Hence,

$$Lf_a - 2f_a = (\Delta_M f_a + \|A\|^2 f_a + 2f_a) - 2f_a = (-\|A\|^2 + \|A\|^2 + 2 - 2)f_a = 0.$$

Lemma 2.2.2.6 and the bilinearity of the metric imply that $V = \{f_a \mid a \in \mathbb{R}^4\}$ forms a subspace of the eigenspace of $L$ associated to the eigenvalue $-2$, and clearly $\dim V \leq 4$.

If $\dim V \leq 3$, then the kernel of the linear transformation $\mathbb{R}^4 \rightarrow V$, $a \mapsto f_a$ is non-empty, and so there is a non-zero $a \in \mathbb{R}^4$ for which $f_a \equiv 0$.

**Lemma 2.2.2.7.** If $a \in \mathbb{R}^4$, $a \neq \vec{0}$, is such that $f_a \equiv 0$, then $g = \langle \Phi, a \rangle$ satisfies $\text{Hess}_Mg(\cdot, \cdot) = -g(\cdot, \cdot)$.

**Proof of 2.2.2.7:**
Let $\{E_1, E_2\}$ be as in the proof of Lemma 2.2.2.6. Then

$$\text{Hess}_Mg(E_i, E_j) = E_iE_j(g) - (\nabla_{E_i}E_j)(g)$$

$$= E_iE_j\langle \Phi, a \rangle$$

$$= E_i(\langle DE_i\Phi, a \rangle + 0)$$

$$= E_i\langle E_i, a \rangle$$

$$= \langle DE_iE_i, a \rangle.$$

From (2.17) and (2.18) we know that

$$DE_iE_j = h_{ij}N - \delta_{ij}\Phi,$$
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

and so

$$\text{Hess}_M g(E_i, E_j) = h_{ij} \langle N, a \rangle - \delta_{ij} \langle \Phi, a \rangle = -\delta_{ij} g$$

since $f_a = \langle N, a \rangle = 0$ by choice of $a$. The desired result follows from the bilinearity of Hess$_M$ and $\langle \cdot, \cdot \rangle$. □

Now, from Obata’s Lemma 2.2.2.2 we have that either $M$ is isometric to a unit sphere or $g \equiv 0$. Since Almgren showed that the all minimal spheres in $S^3$ are totally geodesic (Lemma 2.2.2.1), it must be that $g \equiv 0$. Now, rotating $M$ (if necessary) through an isometry $r : S^3 \to S^3$ so that $a$ coincides with the standard basis vector $e_4 \in \mathbb{R}^4$ we see that

$$0 = g = \langle \tilde{\Phi}, a \rangle = \langle \tilde{\Phi}, e_4 \rangle = \Phi_4,$$

where $\tilde{\Phi} = r \circ \Phi$ and $\Phi_4 = (r \circ \Phi)_4$ is the fourth component function of $\tilde{\Phi}$. Hence,

$$1 = \Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \Phi_4^2 = \Phi_1^2 + \Phi_2^2 + \Phi_3^2,$$

and so we see that $\tilde{\Phi}(M)$, and therefore $\Phi(M)$, lies in an equator of $S^3$ and is therefore totally geodesic. Hence, $\dim V \not\leq 3$ and therefore $\dim V = 4$.

Since the first eigenvalue of $L$ is simple, we conclude that it cannot be $-2$ and that $\text{ind}(M) \geq 5$.

Now, if $\text{ind}(M) = 5$, then the second eigenvalue of $L$ is $-2$. Let $\rho > 0$ be an eigenfunction for the first eigenvalue $\lambda_1$.

**Lemma 2.2.2.8.** There is a conformal transformation $F_g$ of $S^3$ such that

$$\int_M \rho \cdot (F_g \circ \Phi)_i dA = 0, \quad i = 1, 2, 3, 4.$$

**Proof of 2.2.2.8:** To prove the lemma we follow the exposition of Montiel and Ros found in [11] (pg. 154). Consider the conformal transformations of $S^3$ of the form

$$F_g(p) = \frac{p + (\mu \langle p, g \rangle + \lambda)g}{\lambda(1 + \langle p, g \rangle)},$$

(2.19)

where $\lambda = (1 - \|g\|^2)^{-\frac{1}{2}}, \mu = (\lambda - 1)\|g\|^{-2}$ and $g \in B^4$ is fixed.
The collection of these conformal transformations can be extended to 
\( \bar{B}^4 = \{ x \in \mathbb{R}^4 \mid \|x\| \leq 1 \} \) by letting 
\( g \in \bar{B}^4 \). Note that if \( p \) is such that \( \langle p, g \rangle \neq -1 \), then

\[
F_g(p) = \frac{p + \lambda(\|g\|^{-2}(p, g) + 1)g - \langle p, g \rangle\|g\|^{-2}g}{\lambda(1 + \langle p, g \rangle)}
\]

\[
= g + \sqrt{1 - \|g\|^2} \left( \frac{p - \langle p, g \rangle g}{1 + \langle p, g \rangle} \right)
\]

\[
= g.
\]

Consider the map \( H : \bar{B}^4 \to \bar{B}^4 \) whose component functions are given
by

\[
H_i(g) = \frac{1}{\int_M \rho \, dA} \int_M \rho \cdot (F_g \circ \Phi)_i \, dA,
\]

for \( i = 1, 2, 3, 4 \). We have that

\[
\|H(g)\| = \frac{1}{\int_M \rho \, dA} \left\| \int_M \rho \cdot (F_g \circ \Phi) \, dA \right\|
\]

\[
\leq \frac{1}{\int_M \rho \, dA} \int_M |\rho| \cdot \|F_g \circ \Phi\| \, dA
\]

\[
= \frac{1}{\int_M \rho \, dA} \int_M \rho \, dA = 1 \quad \text{(since } \rho \geq 0, \|F_g \circ \Phi\| = 1),
\]

and if \( g \in S^3 \), (2.20) shows that \( F_g = g \) except on a set of measure zero, so

\[
H(g) = \frac{1}{\int_M \rho \, dA} \int_M \rho \cdot g \, dA = g.
\]

Thus \( H \) maps \( \bar{B}^4 \) to itself and is the identity on the boundary \( \partial \bar{B}^4 = S^3 \), so
by a standard topological argument \( H \) must be surjective. Hence there is a
\( g \in B^4 \) such that \( H(g) = 0 \) and therefore

\[
\int_M \rho \cdot (F_g \circ \Phi) \, dA = 0.
\]

\[\square\]
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

Together Lemma 2.2.2.8 and (1.13) imply that

$$Q((F_g \circ \Phi)_i, (F_g \circ \Phi)_i) \geq -2 \cdot \int_M (F_g \circ \Phi)_i^2 \, dA \quad (2.21)$$

for $i = 1, 2, 3, 4$, with equality holding if and only if $(F_g \circ \Phi)_i$ is an eigenfunction of $L$ associated to the eigenvalue $-2$.

Note that $(F_g \circ \Phi)_i \neq 0$; if for some $i$, $(F_g \circ \Phi)_i \equiv 0$, then as argued before $F_g \circ \Phi(M)$ lies in an equator of $S^3$. Hence there is a non-zero vector $v \in \mathbb{R}^4$ such that $F_g \circ \Phi(M)$ lies is the hyperspace orthogonal to $v$, and so $M$ lies in the hyperspace orthogonal to $F_g^{-1}(v)$ (since $F_g$ preserves angles) and therefore lies in an equator of $S^3$ and is totally geodesic. This means that if we get equality in (2.21), then we can divide both sides of the equation by $\|(F_g \circ \Phi)_i\|_{L^2}^2 \neq 0$ and apply (1.13) to get that the $L^2$-normalized component functions, and hence the unnormalized component functions, are eigenfunctions with eigenvalue $-2$.

Hence,

$$\int_M \|\nabla (F_g \circ \Phi)_i\|^2 - (\|A\|^2 + 2)(F_g \circ \Phi)_i^2 \, dA \geq -2 \int_M (F_g \circ \Phi)_i^2 \, dA$$

and so

$$\int_M \|\nabla (F_g \circ \Phi)_i\|^2 \, dA \geq \int_M \|A\|^2(F_g \circ \Phi)_i^2 \, dA,$$

which gives

$$\int_M \|\nabla (F_g \circ \Phi)\|^2 \, dA = \sum_{i=1}^4 \int_M \|\nabla (F_g \circ \Phi)_i\|^2 \, dA \geq \sum_{i=1}^4 \int_M \|A\|^2(F_g \circ \Phi)_i^2 \, dA = \int_M \|A\|^2 \, dA,$$

since $\sum_{i=1}^4 (F_g \circ \Phi)_i^2 \equiv 1$.

**Lemma 2.2.2.9.**

$$2A(f(M)) = \int_M \|\nabla (F_g \circ \Phi)\|^2 \, dA + 2 \int_M \left(\frac{\langle N, g \rangle}{1 + \langle \Phi, g \rangle}\right)^2 \, dA,$$
where $A(f(M))$ is the area of $M$.

Proof of 2.2.2.9: For $g \in \mathbb{B}^4$, let $F_g$ denote the conformal transformation of $S^3$ as in (2.19). Then, thinking of $F_g$ as a map $\mathbb{R}^4 \to \mathbb{R}^4$, we have that the $j$th component of $dF_g \left( \frac{\partial}{\partial x^i} \right)$ is

$$
\left( dF_g \left( \frac{\partial}{\partial x^i} \right) \right)_j = \frac{1}{\lambda} \left[ (\delta_{ij} + \mu g_i g_j)(\langle p, g \rangle + 1) - (p_j + (\mu \langle p, g \rangle + \lambda)g_j)g_i \right]
$$

$$
= (\langle p, g \rangle + 1)^{-2} \lambda^{-1} [\delta_{ij} \langle p, g \rangle + \delta_{ij} + g_i g_j(\mu - \lambda) - p_i g_i]
$$

$$
= (\langle p, g \rangle + 1)^{-2} \lambda^{-2} \left[ \lambda \delta_{ij} \langle p, g \rangle + \lambda \delta_{ij} + g_i g_j(1 - \lambda)g_i^2 - \lambda p_j g_i \right],
$$

so

$$
dF_g(v) = v_i dF_g \left( \frac{\partial}{\partial x^i} \right) = \sum_{i,j=1}^{4} v_i (\langle p, g \rangle + 1)^{-2} \lambda^{-2} \left[ \lambda \delta_{ij} \langle p, g \rangle
$$

$$
+ \lambda \delta_{ij} + g_i g_j(1 - \lambda)|g|^2 - \lambda p_j g_i \right] \frac{\partial}{\partial x^j}
$$

$$
= (\langle p, g \rangle + 1)^{-2} \lambda^{-2} \left[ \lambda v \langle p, g \rangle
$$

$$
+ \lambda v + \langle v, g \rangle g(1 - \lambda)|g|^2 - \lambda p \langle v, g \rangle \right].
$$

From this we get that, for $v, w \in T_p S^3$,

$$
\langle dF_g(v), dF_g(w) \rangle
$$

$$
= (\langle p, g \rangle + 1)^{-2} \lambda^{-4} \left[ \lambda(\langle p, g \rangle + 1) \left\{ \lambda(\langle p, g \rangle + 1) \langle v, w \rangle
$$

$$
+ \langle w, g \rangle(1 - \lambda)|g|^2 \langle v, g \rangle - \lambda \langle w, g \rangle \langle p, v \rangle \right\}
$$

$$
+ \langle v, g \rangle(1 - \lambda)|g|^2 \left\{ \lambda(\langle p, g \rangle + 1) \langle g, w \rangle
$$

$$
+ \langle w, g \rangle(1 - \lambda)|g|^2 \langle g, g \rangle \lambda \langle w, g \rangle \langle p, g \rangle \right\}
$$

$$
- \lambda \langle v, g \rangle \left\{ \lambda(\langle p, g \rangle + 1) \langle p, w \rangle
$$

$$
+ \langle w, g \rangle(1 - \lambda)|g|^2 \langle p, g \rangle - \lambda \langle w, g \rangle \langle p, p \rangle \right\} \right].
$$
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

\[
\begin{align*}
\&= (\langle p, g \rangle + 1)^{-4} \lambda^4 \left[ \langle v, w \rangle \lambda^2 (\langle p, g \rangle + 1)^2 \\
&\quad + \langle v, g \rangle \langle w, g \rangle \left\{ \lambda (\langle p, g \rangle + 1 - \lambda) \| g \|^2 - 2 \langle v, g \rangle \langle w, g \rangle \right\} \\
&\quad + (1 - \lambda) \| g \|^2 \left( 1 - \lambda (1 - \lambda) \lambda \| g \|^2 + 2 \lambda \| p \|^2 \right) \right] \\
&= \frac{1 - \| g \|^2}{(\langle p, g \rangle + 1)^2} \langle v, w \rangle.
\end{align*}
\]

So, for an isometric immersion $\psi : M \to S^3$, the area of $F_g \circ \psi$, $\text{Area}(g)$, is given by

\[ \text{Area}(g) = \int_M \frac{1 - \| g \|^2}{(\langle \psi, g \rangle + 1)^2} \, dA. \quad (2.22) \]

Now, for fixed $g \in \mathcal{B}^4$, define $f : M \to \mathbb{R}$ by $f = \langle \psi, g \rangle + 1$. Then, if we let $\{E_1, E_2\}$ be a local orthonormal frame for $TM$ such that $\nabla_{E_i} E_j \big|_p = 0$, we have that (at $p$)

\[
\Delta_M \log f = \sum_{i=1}^2 E_i E_i (\log f) = \sum_{i=1}^2 E_i \left[ \frac{1}{f} E_i (f) \right] \\
= \frac{1}{f^2} \left[ E_1 (f)^2 + E_2 (f)^2 \right] + \frac{1}{f} \Delta_M (f) \\
= \frac{1}{f^2} \left[ -E_1 (f)^2 - E_2 (f)^2 + f \Delta_M (f) \right]. \quad (2.23)
\]

Also,

\[ E_i (f) = E_i (\langle \psi, g \rangle + 1) = \langle D_{E_i} \psi, g \rangle + 0 + 0 = \langle E_i, g \rangle. \quad (2.24) \]

Now $g = g^N + g^T$, where $g^N$ is the component of $g$ normal to $M$ (and tangent to $S^3$) and $g^T$ is the component tangent to $M$, and $g^T = \langle g, E_1 \rangle E_1 + \langle g, E_2 \rangle E_2$. Hence

\[ \| g^T \|^2 = \langle g, E_1 \rangle^2 + \langle g, E_2 \rangle^2. \quad (2.25) \]
Also,
\[
\Delta_M f = \sum_{i=1}^{2} E_i E_i \langle \psi, g \rangle = \sum_{i=1}^{2} \langle D_{E_i} D_{E_i} \psi, g \rangle = \langle \Delta_M \psi, g \rangle. \tag{2.26}
\]

Taking this into account and using Lemma 2.1.1.1 we see that
\[
f \Delta_M(f) = ((\psi, g) + 1) \langle \Delta_M \psi, g \rangle
= ((\psi, g) + 1)(-2\psi + H, g)
= -2(\psi, g)^2 - 2(\psi, g) + f \langle H, g \rangle,
\tag{2.27}
\]
where \(H\) is the mean curvature vector of \(M\) in \(S^3\). So (2.23), (2.24), (2.25) and (2.27) give
\[
\Delta_M \log f = f^{-2}[-2(\psi, g)^2 - 2(\psi, g) + f \langle H, g \rangle - \|g^T\|^2]. \tag{2.28}
\]

Since \(\|g^T\|^2 = \|g\|^2 - \|g^N\|^2 - (\psi, g)^2\), (2.28) simplifies to
\[
\Delta_M \log f = f^{-2} [-2(\psi, g)^2 - 2(\psi, g) + f \langle H, g \rangle - \|g^T\|^2]
= f^{-2} [-(1 + (\psi, g))^2 + f \langle H, g \rangle + \|g^N\|^2 + (1 - \|g\|^2)]
= -1 + \frac{1 - \|g\|^2}{f^2} + \frac{f \langle H, g \rangle + \|g^N\|^2}{f^2}. \tag{2.29}
\]

Therefore, using (1.4) and (2.22) we have that
\[
0 = \int_M \Delta_M \log f \, dA = \int_M -1 + \frac{1 - \|g\|^2}{f^2} + \frac{f \langle H, g \rangle + \|g^N\|^2}{f^2} \, dA
= -\text{Area}(f(M)) + \text{Area}(g) + \int_M \frac{f \langle H, g \rangle + \|g^N\|^2}{f^2} \, dA.
\]

If \(\psi\) is, moreover, a minimal immersion, then \(H \equiv 0\) and the equation simplifies to
\[
\text{Area}(f(M)) = \text{Area}(g) + \int_M \frac{\|g^N\|^2}{f^2} \, dA. \tag{2.30}
\]

So, for our minimal immersion \(\Phi : M \to S^3\), (2.30) holds. Now, since \(F_g\)
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

is conformal, from Lemma 2.2.2.4 we get that the area, $\text{Area}(g)$, is equal to half of the energy of $F_g \circ \Phi$ and so

$$2\text{Area}(f(M)) = \int_M \| \nabla (F_g \circ \Phi) \|^2 dA + 2 \int_M \frac{\|g^N\|^2}{f^2} dA.$$ 

From Lemma 2.2.2.9 it is easy to see that

$$2\text{Area}(f(M)) \geq \int_M \| \nabla (F_g \circ \Phi) \|^2 dA \geq \int_M \| A \|^2 dA, \quad (2.31)$$

with equality if and only if $\langle N, g \rangle \equiv 0$ and we have equality in (2.21).

**Lemma 2.2.2.10.** If $K$ is the Gauss curvature of $M$, then

$$\| A \|^2 = 2 - 2K.$$

**Proof of 2.2.2.10:** Let $\{E_1, E_2\}$ be as before. Then

$$\| A \|^2 = \sum_{i,j=1}^{2} \langle A(E_i, E_j), A(E_i, E_j) \rangle = \sum_{i,j=1}^{2} \left( \langle (\nabla E_i E_j)^N, N \rangle \right)^2$$

$$= h_{11}^2 + h_{12}^2 + h_{21}^2 + h_{22}^2,$$

where $h_{ij} = \langle \nabla E_i E_j, N \rangle$. Since $M$ is minimal and $A$ is symmetric this simplifies to

$$\| A \|^2 = 2(h_{12}^2 - h_{11}h_{22}). \quad (2.32)$$

Moreover, using the Gauss equation (1.1.2.1) we see that

$$h_{11}h_{22} = \langle A(E_1, E_1), N \rangle \langle A(E_2, E_2), N \rangle$$

$$= \langle A(E_1, E_1), A(E_2, E_2), N \rangle N$$

$$= \langle A(E_1, E_1), A(E_2, E_2) \rangle$$

$$= \langle \tilde{R}(E_1, E_2) E_1, E_2 \rangle - \langle R(E_1, E_2) E_1, E_2 \rangle + \langle A(E_1, E_2), A(E_1, E_2) \rangle$$

$$= \langle \tilde{R}(E_1, E_2) E_1, E_2 \rangle - \langle R(E_1, E_2) E_1, E_2 \rangle + h_{12}^2.$$
2.2. Non-totally Geodesic Minimal Hypersurfaces of $S^n$

Hence, using this with (2.32) we get

$$2(h_{12}^2 - h_{11}h_{22}) = 2\left( \frac{\langle \bar{R}(E_1, E_2)E_2, E_1 \rangle}{\|E_1\|^2\|E_2\|^2 - \langle E_1, E_2 \rangle^2} - \frac{\langle R(E_1, E_2)E_2, E_1 \rangle}{\|E_1\|^2\|E_2\|^2 - \langle E_1, E_2 \rangle^2} \right)$$

$$= 2 - 2K.$$

From the Gauss-Bonnet formula (Lemma 2.2.2.3) we get

$$\int_M K \, dA = 2\pi \cdot \chi(M) = 2\pi(2 - 2\gamma),$$

where $\gamma$ is the genus of $M$. Therefore, (2.31) and Lemma 2.2.2.10 together give

$$2\text{Area}(f(M)) \geq \int_M \|A\|^2 \, dA = \int_M 2 - 2K \, dA$$

$$= 2\text{Area}(f(M)) - 2\int_M K \, dA$$

$$= 2\text{Area}(f(M)) - 4\pi(2 - 2\gamma),$$

and so we see that $\gamma \leq 1$. Again, Almgren’s result (Lemma 2.2.2.1) shows that if $\gamma = 0$, then $M$ is totally geodesic, so it must be that $\gamma = 1$ and equality holds in the above inequalities. That is, $(N, g) \equiv 0$ and equality holds in (2.21).

If we let $f = \langle \Phi, g \rangle$, then it follows from Lemma 2.2.2.7 that $\text{Hess } f(\cdot, \cdot) = -f(\cdot, \cdot)$. Using the same argument as before, we can conclude that $g = 0$ and so $F_g \circ \Phi = \Phi$. Therefore, having equality in (2.21) implies that the component functions $\Phi_i, i = 1, 2, 3, 4$, are eigenfunctions of $L$ with eigenvalue $-2$. Hence,

$$L\Phi_i - 2\Phi_i = 0 \quad \text{for } i = 1, 2, 3, 4,$$

so

$$\Delta_M \Phi_i = -\|A\|^2\Phi_i \quad \text{for } i = 1, 2, 3, 4.$$

However, from Lemma 2.1.1.1 we also know that $\Phi : M^2 \to S^3$ is a minimal
immersion if and only if $\Delta_M \Phi = -2\Phi$ ($\Delta_M \Phi_i = -2\Phi_i$ for $i = 1, 2, 3, 4$).
Therefore

$$\Phi_i(\|A\|^2 - 2) = 0 \quad \text{for } i = 1, 2, 3, 4,$$

and since for any $p \in M$ there is an $i$ such that $\Phi_i(p) \neq 0$, we must have that $\|A\|^2 \equiv 2$. By Theorem 2.2.1.2 this means that $M$ must be the Clifford torus.

It follows from Lemma 2.2.1.3 that the index of the Clifford torus is 5.

\qed
Chapter 3

Conclusion

In this thesis we have given a brief introduction to minimal submanifolds, focusing most of our attention on minimal submanifolds of $S^n$. In particular, we gave a detailed proof of a characterization result due to Simons [16]: out of all $k$-dimensional minimal submanifolds of $S^n$, the embedded, totally geodesic $S^k$ are those which minimize the index. This led us to introduce Conjecture 2.2.1.1 and discuss Perdomo’s [14] partial results with an additional symmetry assumption. Finally, we presented Urbano’s proof of the conjecture in the special case when $n = 3$, and noted the tools used in the proof that prevent it from extending to higher dimensions.

Since relatively few examples of minimal hypersurfaces of $S^n$ are known, it is hard to say how strong Perdomo’s additional symmetry assumption actually is. One possible course for further research would be to try to construct more examples of minimal hypersurfaces in $S^n$, especially ones of low index or ones that do not satisfy Perdomo’s symmetry condition. It is possible that this could lead one to better understand how this symmetry condition and the index are linked. Even though the conjecture is known to be true in the case when $n = 3$, it may still be worthwhile to try to construct minimal surfaces in this special case. Lawson [8] has proved a strong existence result for minimal surfaces in $S^3$ that we also do not have in higher dimensions, so focusing efforts on the case when $n = 3$ may be more fruitful. One could hope to extend methods used to construct minimal surfaces in $S^3$ to higher dimensions.

Another possible direction would be to look at the analogous problems for the free boundary problem in $B^n$, the standard $n$-dimensional unit ball in $\mathbb{R}^n$. Moving to the free boundary situation would change the problems slightly: we would instead be considering minimal submanifolds with bound-
ary in the boundary of the ball, and meeting the boundary of the ball orthogonally. These submanifolds still arise variationally as critical points of the area functional, but among submanifolds in the ball whose boundaries lie on the boundary of the ball but are free to vary on the boundary. An interesting open problem is the analogue of Urbano’s result: every free boundary minimal surface $S \subset B^3$ satisfies $\text{ind}(S) \geq 4$ with equality holding if and only if $S$ is the critical catenoid. One could also ask whether some analogous form of Lawson’s existence result is true in the free boundary setting: which compact orientable surfaces with boundary can be realized as properly embedded free boundary minimal surfaces in $B^3$. A classical result of Nitsche [12] showed that the equatorial disk is the only immersed minimal disk in $B^3$ with free boundary, and it is conjectured that, for the case of the annulus, the critical catenoid is the unique properly embedded minimal annulus in $B^3$ with free boundary. This conjecture can be viewed as an analog for free boundary minimal surfaces in $B^3$ of a famous conjecture of Lawson that asserts that the Clifford torus is the unique embedded minimal torus in $S^3$, recently settled by Brendle. As is the case with $S^n$, few examples of minimal submanifolds are known in the free boundary setting, so it would be interesting to work on construction problems there as well.
Bibliography


