

Alon's Second Eigenvalue Conjecture: Simplified and Generalized

by

David-Emmanuel Kohler

M.Sc., Ecole Polytechnique Fédérale de Lausanne, 2006

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate Studies

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

July 2013

© David-Emmanuel Kohler 2013

Abstract

For a fixed graph, B , we study a probability model, $\mathcal{C}_n(B)$, of random covering maps $\pi: G \rightarrow B$ of degree n . Specifically, we study spectral properties of G 's new eigenvalues of the adjacency matrix of G , and of the Hashimoto matrix of G (i.e., the adjacency matrix of the directed line graph of G).

Our main theorem says that if B is d -regular, then for every $\varepsilon > 0$, the probability that $G \in \mathcal{C}_n(B)$ has a new adjacency eigenvalue greater than $2\sqrt{d-1} + \varepsilon$ tends to zero as n tends to infinity. This matches the generalized Alon-Boppana lower bound, in the sense that any degree n cover, G , of a fixed graph, B , must have a new eigenvalue of at least $\rho(A_{\widehat{B}}) - o_n(1)$, where $\rho(A_{\widehat{B}})$ is the spectral radius of the universal cover, \widehat{B} , of B , and where $o_n(1)$ is a function that tends to zero as n tends to infinity. When the base graph, B , is d -regular, it is well known that $\rho(A_{\widehat{B}}) = 2\sqrt{d-1}$ and hence explains the context of our result.

For general base graphs, B , Friedman conjectured in that the new eigenvalue bound holds with $2\sqrt{d-1}$ replaced with $\rho(A_{\widehat{B}})$. We refer to this conjecture as the generalized Alon conjecture; Alon stated this conjecture in the case where B has one vertex, i.e., the case of random d -regular graphs on n vertices. However, for some non-regular base graphs B , we cannot yet prove any non-trivial new eigenvalue upper bound with high probability.

We use trace methods, as pioneered by Broder and Shamir for random, d -regular graphs; these methods were eventually refined by Friedman to prove the original Alon conjecture, i.e., in the case where B has one vertex. Our methods involve several significant simplifications of the methods of Friedman.

Preface

This dissertation is original and constitutes independent work by the author, D. Kohler and his supervisor J. Friedman. The author and his supervisor intend to submit these results in a future article.

Table of Contents

Abstract	ii
Preface	iii
Table of Contents	iv
List of Symbols	vi
Acknowledgements	viii
Dedication	x
1 Introduction	1
1.1 Motivation and Historical Context	1
1.2 The Generalized Alon Conjecture	4
1.3 Outline of Our Trace Method	7
2 Preliminaries	14
2.1 Essential Graph Terminology	14
2.2 Adjacency Eigenvalues	16
2.3 Hashimoto Eigenvalues	18
2.4 Graph Coverings	22
2.5 Variable-Length Graphs	25
2.6 The Broder-Shamir Model	25
2.7 Polyexponential Functions	27
2.8 Functions of Bounded Growth	29
2.9 B -Ramanujan Functions	33

3	Walk Sums	38
3.1	Potential Walks	38
3.2	Walk Sums	41
3.3	Asymptotic Expansion	44
3.4	Types and Forms	48
4	Tangles and the Certified Trace	56
4.1	Tangles	56
4.2	Finitely Certifiable Partial Orders	64
4.3	The Certified Trace	71
4.4	Asymptotic Expansion	74
5	The Side-Stepping Lemma	80
5.1	The Shift Operator	80
5.2	Statement of the Side-Stepping Lemma	82
5.3	Proof of the First Exception Bound	87
5.4	The Proof of the Limit Formula	92
5.5	The End of The Proof of Lemma 5.2.5	100
6	Proof of the Main Result	102
6.1	Certified Traces In Graphs With Tangles	103
6.2	End of the Proof	109
7	Conclusion	113
	Bibliography	114

List of Symbols

Symbols and notations are ordered alphabetically. In the case of Greek symbols, they are ordered in their English names e.g. the Greek letter chi, χ , is understood as starting with the letter *c*.

A_G	adjacency matrix of the graph G	16
\widehat{B}	the universal cover of the base graph B	4
$\rho_B^{\text{new}}(H_G)$	new Hashimoto spectral radius	24
$\mathcal{C}_n(B)$	the Broder-Shamir model	25
$\chi(G)$	Euler characteristic of G	21
CertSNBC	certified SNBC walk collection	71
CertTr	certified trace	71
CertTr $_{<r}$	truncated certified trace	71
d_{\max}	maximum degree of a graph	21
$\mathbb{E}_n[X]$	expectation in $\mathcal{C}_n(B)$ of the r.v. X	26
\mathcal{F}	a form	49
$\mathcal{F}(w; \vec{t})$	form of the potential walk $(w; \vec{t})$	49
G	a graph	14
Graph $(w; \vec{t})$	the graph traced out by a potential walk	39
H_G	Hashimoto matrix of the graph G	18
$\mathbb{I}_{\text{TF}(r,B)[n]}$	indicator function of $\mathcal{C}_n(B) \setminus \text{Tangle}_{r,B}[n]$	62
Line (G)	directed line graph of G	18
NB (e_1, e_2, k)	# of non-backtracking k -walks from e_1 to e_2	50
Occurs $_B$	connected graphs that can occur in $\mathcal{C}_n(B)$	7
ord $(w; \vec{t})$	the order of a potential walk	40
Prob $_n[\mathcal{E}]$	probability in $\mathcal{C}_n(B)$ of the event \mathcal{E}	26
$\rho(A_G) = \rho_A(G)$	spectral radius of A_G	16

$\rho(H_G) = \rho_H(G)$	spectral radius of H_G	18
$\rho_B^{\text{new}}(A_G)$	new adjacency spectral radius	23
SNBC	strictly non-backtracking closed walks collection .	42
$\text{Spec}(A_G)$	spectrum of adjacency eigenvalues	16
$\text{Spec}(H_G)$	spectrum of Hashimoto eigenvalues	18
$\text{Spec}_B^{\text{new}}(A_G)$	new adjacency spectrum of the covering G	23
$\text{Spec}_B^{\text{new}}(H_G)$	new Hashimoto spectrum	24
\mathcal{T}	a type	48
$\mathcal{T}(w; \vec{t})$	type of the potential walk $(w; \vec{t})$	49
$[\vec{t}]_{n,w}$	the (n, w) -equivalence symmetry class of \vec{t}	40
Tangle_B	set of tangles that can occur in $\mathcal{C}_n(B)$	61
$\text{Tangle}_{r,B}$	set of tangles in Tangle_B of order less than r	61
$\text{HasTangle}_{r,B}^{\min}$	set of minimal tangles in $\text{HasTangle}_{r,B}$	61
$\text{Tr}(M)$	trace of the matrix M	16
$\text{Tree}_d(\psi)$	universal d -regular cover with a ψ -inclusion	57
$\text{Types}[r]$	types of order less than r	49
\mathcal{W}	a walk collection	41
$\mathcal{W}(k, n)$	(k, n) -walks in \mathcal{W}	41
$\mathcal{W}_n(G, k)$	(k, n) -modified trace of \mathcal{W}	42
$W_{\mathcal{F}}(\mathcal{W}, k, \vec{m})$	consistent walks in \mathcal{F} of multiplicity \vec{m}	51
$W_{\mathcal{T}}(\vec{m}, \vec{k}(\mathcal{F}))$	consistent (\vec{m}, \vec{k}) -walks in \mathcal{T}	51
$(w; \vec{t})$	a potential walk	38
$[w; \vec{t}]_n$	equivalence class of potential walks	41
$\text{WalkSum}(\mathcal{W}, k, n)$	(k, n) walk sum of \mathcal{W}	42

Acknowledgements

The writing of this thesis is also the product of the help and support of several people, some of whom I would like to acknowledge here.

First, I would like to thank my parents, Adia-Pia and Kurt, for their unconditional love and support. Thanks to my siblings, Jeremy, Evan and Megan, my aunt, Alexandra and my grandfather Henri for their invaluable presence in my life and all their love.

I thank my supervisor, Dr. Joel Friedman, for his patience, mathematical expertise and unwavering support. It has been a privilege to work with him and discover the beauty of his work, that combines so many different fields of mathematics. I thank as well my supervision committee members as well as Dr. Brian Marcus and Dr. Patrick Brosnan for their support at the earlier stages of my degree.

I would like to acknowledge my mathematical colleagues: Dr. David Steinberg and Dr. Warren Code for their friendship in the past several years. I am grateful to Dr. Fok-Shuen Leung, Dr. Mark Mac Lean and Dr. Djun Kim for their support and many discussions on the teaching of mathematics. And many thanks to my many friends, who happen to be mathematicians as well: Dr. Anthony Arnold, Dr. Olivier Isely, Dr. Peter Jossen, Varvara Karpova and Dr. Simon Rose.

There are too many friends in Switzerland, in Canada and elsewhere that I would like to acknowledge. Special thoughts for the extraordinary Nina Vogt who has seen me doing mathematics for over a decade now. I cannot express enough gratitude to Laura Tremblay-Boyer for her support and friendship. For their friendship, cooking skills, coaching, love and overall awesomeness, thanks to Cesar Oboni, Nick Hermes, Kate Moore, Patrick Thompson and Bruce Yoder; Xavier Alexandre and Oliver Prospero;

Carley Kennedy and Kristen McIlhenney; Sévrine Knuchel; Lou Legros Lefevre; Nicole Huk; Roselynn Verwoord; Helena Palmqvist; Christopher Ryan and Cacilda Jethá. A huge thank you to Marlène Oberson, Beth Atwood, and Jessica Mill for being in my life. I love you all.

Finally, I'd like to acknowledge Aaron Kafka and his wonderful coffee shop, *Kafka's Coffee and Tea*, where most of this thesis was written in the company of many an espresso macchiato.

Dedication

To Dr. Henri Woog, who has given me my love for mathematics and all sciences. It is a privilege for me to be his grandson.

Chapter 1

Introduction

The main goal of this thesis is to study spectral properties of random covers of a fixed graph, B . We develop tools to prove what we call a *generalized Alon conjecture* for arbitrary base graph, B , formulated in [Fri03]. In this thesis, we succeed in proving the generalized Alon conjecture for the case where the base graph, B , is any d -regular graph.

Our approach follows the Broder-Shamir trace method of [BS87], with its refinements developed by Friedman in [Fri91, Fri08], which we adapt to the more general situation of random, degree n covering maps of a fixed graph, B . However, our proofs significantly simplify the arguments of [Fri08]; in particular, we replace the cumbersome *selective trace* by a much simpler modified trace that we call a *certified trace*; there are several other new ideas that simplify the methods of [Fri08]. We review these conjectures and their historical context. We shall assume some terminology of algebraic graph theory to be given in chapter 2.

1.1 Motivation and Historical Context

This thesis is inscribed in a large body of research that can be attached to the study of expanders. Expansion requires that a graph is simultaneously sparse and highly connected. The recent survey by Hoory, Linial and Wigderson [HLW06] gives a brief overview of the more than four decades of work that has been done around the concept of expansion. The universality of this notion is becoming more evident as more connections are made, primarily in combinatorics, geometry, probability and algebra.

We will touch briefly upon some points of interest to describe the moti-

vation of our work. Let us begin by a definition of expansion.

Definition 1.1.1. Let G be a d -regular graph on n vertices, its *edge expansion ratio*, denote $h(G)$ is defined as:

$$h(G) = \min_{|S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}$$

where ∂S is the number of edges connecting vertices in the set S to its complement.

Definition 1.1.2. A sequence of d -regular graphs $\{G_i\}_{i \geq 1}$ of size increasing with i is a *family of expander graphs* if there exists $\varepsilon > 0$ such that $h(G_i) > \varepsilon$ for all $i \geq 1$.

The construction of explicit families of expander graphs presents numerous difficulties and is itself an interesting problem. Let us approach this from an algebraic point of view. If G is a graph on n vertices, its adjacency matrix is real and symmetric and thus has n real eigenvalues which we denote by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and that is called the *spectrum* of the graph G . If G is d -regular, then $\lambda_1 = d$ (it is the Perron-Frobenius eigenvalue of the graph) and the graph's second eigenvalue is closely related to its edge expansion ratio. The following theorem is well known.

Theorem 1.1.3. *Let G be a d -regular graph on n vertices, then*

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}$$

Let $\lambda(G) = \max\{|\lambda_i| \mid \lambda_i \neq \pm d\}$, the quantity $d - \lambda(G)$ is called the *spectral gap* of the graph and provides an estimate on its expansion. In particular, a d -regular graph has an edge expansion ratio bounded away from zero if and only if its spectral gap is bounded away from zero. Also of interest is the expander mixing lemma which shows that a small second eigenvalue implies that the graph is nearly random in the sense that its edges are spread out.

Theorem 1.1.4 (Expander Mixing Lemma). *Let G be a d -regular graph on n vertices, then for all sets of vertices $S, T \subseteq V_G$ we have:*

$$\left| E_G(S, T) - \frac{d|S||T|}{n} \right| \leq \lambda(G) \sqrt{|S||T|}$$

At this point, we are interested in how big can the spectral gap be and as such, estimates of $\lambda(G)$. The Alon-Boppana bound [Al91] gives the following tight bound

Theorem 1.1.5 (Alon-Boppana). *Let G be a d -regular graph on n vertices, then*

$$\lambda(G) \geq 2\sqrt{d-1} - o_n(1)$$

Since bipartite or disconnected regular graphs have $\lambda(G) = d$, there are no interesting deterministic upper bounds on $\lambda(G)$. Of interest are graphs such that $\lambda(G) \leq 2\sqrt{d-1}$ which are called *Ramanujan graphs*. Explicit constructions of families of Ramanujan graphs are hard to come by and offers many interesting problems. An open question is to prove the existence of infinitely many d -regular Ramanujan graphs for all $d \geq 3$. A major result, due to Lubotzky-Phillips-Sarnak [LPS88] and independently by Margulis [Mar88] that arbitrarily large d -regular Ramanujan graphs exist when $d-1$ is prime, and moreover they can be explicitly constructed. Morgenstern [Mor94] extended this to the case when $d-1$ is a prime power.

During the writing of this thesis, we learned of a preprint by Marcus, Spielman and Srivastava [MSS13] which establishes the existence of infinite families of d -Ramanujan bipartite graphs for all $d \geq 3$.

Theorem 1.1.6 (Lubotzky-Phillips-Sarnak, Margulis, Morgenstern). *Let p be a prime and k a positive integer, then there exist infinitely many $(p^k + 1)$ -regular Ramanujan graphs.*

Despite the difficulty to explicitly construct Ramanujan graphs, it is conjectured that they are not rare. Experimental evidence by Miller-Novikoff-Sabelli [MNS08] indicates that the probability that a random d -regular graph on n vertices is Ramanujan tends to 27% as n tends to infinity. Relaxing this condition to *nearly Ramanujan* graphs, Alon [Al86] conjectured:

Conjecture 1.1.7 (Alon). *Let G be a random d -regular graph on n vertices, then*

$$\lambda(G) \leq 2\sqrt{d-1} + o_n(1) \quad \text{a.a.s.}^1$$

In [Fri08], Friedman proved this conjecture.

Theorem 1.1.8 (Friedman). *Let G be a d -regular graph on n vertices, then, for any $\varepsilon > 0$,*

$$\lambda(G) \leq 2\sqrt{d-1} + \varepsilon \quad \text{a.a.s.}$$

The Alon-Boppana bound shows that the constant $2\sqrt{d-1}$ cannot be improved.

1.2 The Generalized Alon Conjecture

A significant generalization of this conjecture makes sense of the $2\sqrt{d-1}$ constant. Consider the universal cover, \widehat{B} , of any d -regular graph. \widehat{B} is the infinite d -regular tree and its spectral radius, $\rho(A_{\widehat{B}})$ satisfies

$$\rho(A_{\widehat{B}}) = 2\sqrt{d-1}$$

When considering d -regular graphs for d even, one can make the following simple observation: any d -regular graph, G , is a covering map of the graph consisting of one vertex and $d/2$ whole-loops. From this, one can extend the probabilistic model developed in [BS87] to formulate a natural model, $\mathcal{C}_n(B)$, for any choice of base graph B . This model constructs a random graph, G , admitting a covering map $\pi: G \rightarrow B$ of degree n constructed via the choice of $|E_B|$ independent and uniformly chosen elements of the symmetric group, \mathcal{S}_n , of permutations of n elements. B may have multiple edges and self-loops (if B has half-loops, then we insist that n be even and that we choose from those elements of \mathcal{S}_n that are perfect matchings). For any covering $\pi: G \rightarrow B$, the multiset of its adjacency eigenvalues can be partitioned into two sets: *old* eigenvalues arising from the base graph B

¹Asymptotically almost surely i.e. with probability converging to 1 as n tends to infinity.

and the remaining *new* eigenvalues depending on G . We let $\text{Spec}_B^{\text{new}}(A_G)$ denote the multiset of new eigenvalues of G and let $\rho_B^{\text{new}}(A_G)$ denote the largest new eigenvalue in absolute value of the covering graph G . We remark that for any positive integer k , we have

$$\sum_{\lambda \in \text{Spec}_B^{\text{new}}(A_G)} \lambda^k = \text{Tr}(A_G^k) - \text{Tr}(A_B^k)$$

and so it follows that the new eigenvalues of covering graph G depend only on the graphs B and G and not on the particular covering map $\pi: G \rightarrow B$. In [Fri03], the following conjecture is studied.

Conjecture 1.2.1 (The generalized Alon conjecture). *For any fixed base graph, B , let $\rho(A_{\widehat{B}})$ be the spectral radius of the adjacency operator on the universal cover, \widehat{B} , of the base graph. Consider the probability space, $\mathcal{C}_n(B)$, of random covering graphs of degree n of the graph B . Then the probability that a random covering graph, G , of degree n satisfies*

$$\rho_B^{\text{new}}(A_G) \geq \rho(A_{\widehat{B}}) + \varepsilon$$

tends to zero as n tends to infinity, for any $\varepsilon > 0$.

A weakened form of this conjecture was proven by Friedman [Fri03] with $\rho(A_{\widehat{B}})$ replaced by $\sqrt{d\rho(A_{\widehat{B}})}$. Linial and Puder [LP10], improved this to $3d^{1/3}\rho^{2/3}$ where $\rho = \rho(A_{\widehat{B}})$. Lubetzky, Sudakov and Vu in [LSV11] improved this to order of $\max\{\lambda, \sqrt{d}\} \log(d)$ where λ is the absolute value of the largest non-trivial adjacency eigenvalue of B . Addario-Berry and Griffiths in [ABG10] improved this to $c\sqrt{d}$ where c is 430656. Recently, Puder [Pud12] has improved this to $\rho + 0.84$, for B regular and to 3ρ for arbitrary B .

The main result of this thesis is to prove conjecture 1.2.1 for any d -regular graph B . Specifically, we prove the following theorem.

Theorem 1.2.2. *Let B be a d -regular graph with $d \geq 3$ and consider the Broder-Shamir probability model, $\mathcal{C}_n(B)$ of random covering graphs of B*

of degree n . Then for any $\varepsilon > 0$, we have

$$\text{Prob}_{G \in \mathcal{C}_n(B)}[\rho_B^{\text{new}}(A_G) \geq \rho(A_{\hat{B}}) + \varepsilon] \leq C/n \quad (1.1)$$

for some constant $C = C(B, \varepsilon)$.

As mentioned above, Friedman's famous proof of the Alon Conjecture in [Fri08] can be here see as the specific case where the base graph, B , is the graph with one vertex and $d/2$ self-loops.

We shall prove theorem 1.2.2 by using the fundamental ideas of Broder-Shamir [BS87], later refined by Friedman [Fri91, Fri08]. These methods enable to use estimates for the expected number of closed, non-backtracking walks of a given length in a random graph. This leads us to theorems regarding the eigenvalues of the Hashimoto matrix, H_G , of a graph G . For d -regular graphs, there is a direct translation between Hashimoto eigenvalues and adjacency eigenvalues, which allows us to prove theorem 1.2.2.

The Hashimoto eigenvalues of a covering graph have properties similar to those of the adjacency eigenvalues. Our trace method allows to prove the following theorem on the Hashimoto eigenvalues of random covering graphs.

Theorem 1.2.3. *Let B be a d -regular graph with $d \geq 3$ and consider the Broder-Shamir probability model, $\mathcal{C}_n(B)$ of random covering graphs of B of degree n . Then for any $\varepsilon > 0$, we have*

$$\text{Prob}_{G \in \mathcal{C}_n(B)}[\rho_B^{\text{new}}(H_G) \geq \rho(H_{\hat{B}}) + \varepsilon] \leq C/n \quad (1.2)$$

for some constant $C = C(B, \varepsilon)$.

Kotani and Sunada, in [KS00], show that if G is any graph of maximum degree, d_{\max} , then any Hashimoto eigenvalue, μ , that is not real satisfies

$$|\mu| \leq \sqrt{d_{\max} - 1}$$

and each adjacency eigenvalue, λ , of absolute value greater than $2\sqrt{d_{\max} - 1}$ gives rise to a real Hashimoto eigenvalue greater than $\sqrt{d_{\max} - 1}$. When

B is d -regular, it is known that $\rho(H_{\widehat{B}}) = \sqrt{d-1}$ and hence, Kotani and Sunada's result shows that theorem 1.2.3 implies theorem 1.2.2. We believe it is possible to use their result to prove a weakened generalized Alon conjecture, where $\rho(A_{\widehat{B}})$ is replaced by $2\sqrt{d_{\max}-1}$.

1.3 Outline of Our Trace Method

A lot of the techniques and theorems in this thesis hold for arbitrary base graph B . The case when B is regular is an important special case, one of the limited cases of B for which we establish Conjecture 1.2.1. However, many of the results in the following chapters are valid for arbitrary B .

The techniques and theorems in this thesis apply to a number of models of covering graphs of a base graph B of degree n . However, in this thesis, we assume the Broder-Shamir model mentioned above and to be described in more details in the next chapter. This means that for certain choices of a base graph B , the model insists that n be an even integer (in the case where B has half-loops).

Definition 1.3.1. Consider the Broder-Shamir model, $\mathcal{C}_n(B)$, and let:

- (1) Occurs_B denote the connected graphs, H , such that for n sufficiently large, H is isomorphic to a subgraph of $\mathcal{C}_n(B)$ with positive probability;
- (2) $\text{Tangle}_{r,B} = \{\psi \in \text{Occurs}_B \mid \psi \text{ connected, } \rho(H_\psi) \geq \sqrt{d-1} \text{ and } -\chi(\psi) < r\}$ where $\chi(\psi)$ is the Euler characteristic of the graph ψ ;
- (3) $\Psi_{r,B} = \text{Minimal elements of } \text{Tangle}_{r,B}$;
- (4) $\text{HasTangle}_{r,B}[n] = \{G \in \mathcal{C}_n(B) \mid G \text{ has a subgraph isomorphic to an element of } \text{Tangle}_{r,B}\}$;
- (5) $\mathbb{I}_{\text{HasTangle}(r,B)[n]}(G)$ the indicator function of the event $\text{HasTangle}_{r,B}[n]$.
- (6) $\mathbb{I}_{\text{TF}(r,B)[n]}(G) = 1 - \mathbb{I}_{\text{HasTangle}(r,B)[n]}$ the characteristic function of the complement of the event $\text{HasTangle}_{r,B}[n]$;

Most of the work in this thesis is devoted to an asymptotic series for the expected value of

$$\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}(H_G^k),$$

i.e., the expected value of $\text{Tr}(H_G^k)$, where we replace the trace by zero when G contains an element of $\text{Tangle}_{r,B}$ as a subgraph. The series is in powers of $1/n$, and the coefficients are *polyexponentials in k* , defined as follows.

Definition 1.3.2. We say that a function, $P = P(\vec{k})$, defined on a tuple of non-negative integers, \vec{k} , is *polyexponential* if

$$P(\vec{k}) = \sum_i \alpha_i g_{i,1}(k_1) \dots g_{i,t}(k_t)$$

with $\alpha_i \in \mathbb{C}$ and where the functions $g_{i,j}$ are *powerexponentials*, that is function of the type

$$g(k) = \nu^k k^s$$

for some non-negative integer s and complex number ν . We call the set of all complex numbers ν in these functions the base of P and denote it by $L = L(P)$.

Given a graph, B , set $\mu_1 = \rho(H_B)$, the Perron-Frobenius eigenvalue of the Hashimoto matrix of B , and set $L = L_B$ to be those eigenvalues of H_B that are greater than $(\mu_1)^{1/2}$ in absolute value. We say that a function, $P = P(k)$, defined on non-negative integers, is *B-Ramanujan* if there is a polyexponential function, $Q(k)$, with bases L , such that for any $\tilde{\varepsilon} > 0$ there is a $C > 0$ for which

$$|P(k) - Q(k)| \leq C k^C ((\mu_1)^{1/2} + \tilde{\varepsilon})^k$$

for all k ; we call Q the *principal part* or *polyexponential part* of P . (The principle part is therefore defined only modulo a term smaller than $k^C ((\mu_1)^{1/2} + \tilde{\varepsilon})^k$ for each $\tilde{\varepsilon} > 0$.)

For example, if $B = W_{d/2}$, then $\mathcal{C}_n(B)$ is the Broder-Shamir model of a random d -regular graph on n vertices, for $d \geq 4$ even. Then the eigenvalues

of H_B are precisely $d-1, 1, -1$ (which feature prominently in [Fri91, Fri08]), and $L = \{d-1\}$. Notice that the definition of B -Ramanujan is less restrictive than the definition of d -Ramanujan in [Fri08], since our definition allows a $\tilde{\varepsilon} > 0$, whereas in [Fri08] we essentially take $\tilde{\varepsilon} = 0$. The $\tilde{\varepsilon} > 0$ turns out to greatly simplify our estimates, and does not weaken any conclusions regarding the generalized Alon conjecture.

Theorem 1.3.3. *Let r be any positive integers. Then we have*

$$\mathbb{E}_{G \in \mathcal{C}_n(B)}[\mathbb{I}_{\text{TF}(r,B)}(G) \text{Tr}(H_G^k)] = P_0(k) + P_1(k)n^{-1} + \dots + P_{r-1}(k)n^{1-r} + \text{err}_r(n, k) \quad (1.3)$$

where the $P_i(k)$ are B -Ramanujan functions in k , and there is a C for which

$$|\text{err}_r(n, k)| \leq Ck^C(d-1)^k n^{-r}$$

In the above theorem, if we knew that the principle part of each $P_i(k)$ was zero, then we could apply the methods detailed in [Fri08] to conclude the generalized Alon conjecture, i.e., Conjecture 1.2.1, for all d -regular graphs, B , by applying (1.3) to a single value of k which is proportional to $\log n$. However, it turns out that the fact this (1.3) holds for many values of k can be exploited to show that the fact that $P_0(k) = \text{Tr}(H_B^k)$ (which has essentially been known since [Fri03]) already implies the generalized Alon Conjecture, for B d -regular. The idea is that the principle part of each $P_i(k)$ is annihilated by an appropriate polynomial of the *shift in k* operator, and we apply a polynomial of the shift operator to both sides of (1.3); we thereby show that the coefficients $P_i(k)$ are accounted precisely by H_G eigenvalues *very near* the eigenvalues of H_B . This idea was called the Side-Stepping Lemma in [Fri08] -- side-stepping the fact that the principle parts of the $P_i(k)$ are not known to vanish; in this article we give a somewhat stronger version of this lemma which allows us to prove the generalized Alon conjecture.

We emphasize that the above results on the generalized Alon Conjecture would not follow by an application of (1.3) of one value of k for each n ; rather we apply a polynomial of the *shift k* operator, meaning that we

simultaneously apply (1.3) to some values $k, k + 1, \dots, k + \delta$, where δ is a constant depending on ε, r, B , and then take k an appropriate value, roughly proportional to $\log n$.

Furthermore, we mention that if we do not remove graphs, G , that contain *tangles* as above, but rather we directly apply the inequality

$$\text{Prob}_{G \in \mathcal{C}_n(B)}[\rho \leq \text{Spec}_B^{\text{new}}(A_G)] \rho^k \leq \mathbb{E}_{G \in \mathcal{C}_n(B)}[\text{Tr}(A_G^k) - \text{Tr}(A_B^k)]$$

to one value of k for each n , then we cannot get an interesting bound with $\rho = 2(d - 1)^{1/2} + \varepsilon$ for arbitrary $\varepsilon > 0$; i.e., we cannot use the trace method as is to conclude the generalized Alon conjecture.

1.3.1 Series Expansions

We now try to give a rough idea of how to prove (1.3), with $P_i(k)$ being B -Ramanujan. We will have to be rather vague until we define precise notions of *potential walks*, and the *form* and *type* of a potential walk, in Chapter 3.

The fact that (1.3) holds for some functions, $P_i(k)$, is a consequence of the methods of [Fri91], and one main idea is already evident in [BS87]. The point is that a walk, w , in G , traces out a subgraph, $\text{Graph}(w)$, in G , and one can study $\text{Tr}(H_G^k)$ on the basis of the *rough geometry* of $\text{Graph}(w)$. Specifically, $\text{Tr}(H_G^k)$ counts the number of strictly non-backtracking walks in G , and by disregarding the walks, w , for which $-\chi(\text{Graph}(w)) \geq r$ (where χ is the Euler characteristic), we lose a factor of at most roughly

$$n^{-r} \binom{k}{2r} \text{Tr}(H_B^k).$$

The point is that if $-\chi(\text{Graph}(w)) \geq r$, then the event that w occurs involves at least r *coincidences*, each coincidence contributing roughly a multiple of $1/n$ to its expected value (see Chapter 3).

Furthermore, we can express the $P_i(k)$ in terms of the *type* of a (length k , closed, strictly non-backtracking) walk, w , in G . Each walk, w , in G , traces out a subgraph, $\text{Graph}(w)$ of G ; given a covering map, $\pi: G \rightarrow B$,

π restricts to a map from $\text{Graph}(w)$ to B .

The basic idea of Broder and Shamir is to organize the walks, w , by the graph T associated to w by setting V_T to be the initial vertex of w and all vertices of $\text{Graph}(w)$ of degree at least 3; assuming w is non-backtracking, all vertices of $\text{Graph}(w)$ have degree at least 2, and the vertices not in V_T can be contracted to get a graph T whose vertices are V_T and whose edges represent beaded paths (whose interior vertices are all of degree 2) in $\text{Graph}(w)$. By comparison, if w backtracks, then $\text{Graph}(w)$ can have vertices of degree 1 and the geometry of $\text{Graph}(w)$ can be more complicated. This idea is used in [Fri91], but is essentially the same as the case treated in Broder-Shamir; they consider non-backtracking closed walks that are not necessarily strictly non-backtracking, and there are two types---a cycle and a cycle with a *tail*---of Euler characteristic zero, that they address for their case of $r = 1$.

For a walk, w , in G , we remember some additional data beyond the graph, T , obtained by collapsing the degree two paths in $\text{Graph}(w)$; namely we remember

- (1) the vertices of B lying below (i.e., via the projection $\pi: G \rightarrow B$ restricted to $\text{Graph}(w) \subset G$) the vertices of $\text{Graph}(w)$ that we remember in T (i.e., the vertices of T , which are the initial vertex and all vertices of degree at least three),
- (2) a lettering, meaning, for each edge incident to a vertex of T in $\text{Graph}(w)$, we remember to which edge in B it is mapped, and
- (3) the order in which all type edges and vertices are first encountered in w (so the edges and vertices of T are ordered sets), and the direction in which each edge is first traversed.

We call this data the *type* of w , denoting it \mathcal{T} , which consists of an underlying graph, T , along with the B structure of the 1-neighbourhood of all T vertices in w , and the ordering of the vertices and edges of T , and orientation of edges.

It turns out that, as a fairly straightforward extension of the ideas of [Fri91], we have that the $P_i(k)$ are linear combinations of sums of the form:

$$\sum_{\vec{m} \cdot \vec{k} = k} W_{\mathcal{T}}(\vec{m}, \vec{k}) P(\vec{k}), \quad (1.4)$$

where

(1)

$$\vec{m}, \vec{k} \in \mathbb{Z}_{\geq 1}^{E_T},$$

i.e., \vec{m}, \vec{k} are vectors indexed over E_T of positive integers;

(2) $P(\vec{k})$ is a polyexponential in bases $\text{Spec}(H_B)$, where \vec{k} is a vector indexed over E_T ; and

(3) $W_{\mathcal{T}}(\vec{m}, \vec{k})$ counts the number of walks in T that correspond to walks contributing to closed, strictly non-backtracking walks of length k , where edge $f \in E_T$ corresponds to a path of length k_f in $\text{Graph}(w)$ and m_f is the number of times $f \in E_T$ is traversed (in either direction). For now, $W_{\mathcal{T}}(\vec{m}, \vec{k})$ is independent of \vec{k} .

The problem with (1.4) is that the function $W_{\mathcal{T}}(\vec{m}, \vec{k})$ is often too large to prove that the sum in (1.4) is B -Ramanujan. In fact, it is known that if you don't remove the tangles, then $P_i(k)$ will not be Ramanujan for $B = W_{d/2}$ and i of size roughly $d^{1/2} \log d$ (see [Fri03]).

To remedy this, we note that (1.3) can be generalized to

$$\mathbb{E}_{G \in \mathcal{C}_n(B)}[\mathcal{W}_n(G, k)] = P_0^{\mathcal{W}}(k) + P_1^{\mathcal{W}}(k)n^{-1} + \cdots + P_{r-1}^{\mathcal{W}}(k)n^{1-r} + \text{err}_r(n, k) \quad (1.5)$$

where $\mathcal{W}(G, k)$ counts strictly non-backtracking closed walks subject to certain types of restrictions (we will leave this vague for now, but see these are formalized as *walk sums* in [Fri08] and in chapter 3 in this article), and where $P_i^{\mathcal{W}}(k)$ are functions given by linear combinations of sums similar to (1.4), namely

$$\sum_{\vec{m} \cdot \vec{k} = k} W_{\mathcal{T}}^{\mathcal{W}}(\vec{m}, \vec{k}) P(\vec{k}), \quad (1.6)$$

with $P(\vec{k})$ are before, but now $W_{\mathcal{T}}^{\mathcal{W}}(\vec{m}, \vec{k})$ counts walks in T , the underlying graph of \mathcal{T} , corresponding to strictly non-backtracking closed walks with the restrictions of \mathcal{W} .

The idea is that if \mathcal{W} counts only the strictly closed, non-backtracking walks, w , such that don't trace out a tangle, then $W_{\mathcal{T}}^{\mathcal{W}}(\vec{m}, \vec{k})$ should be small enough so that (1.6) converges to a B -Ramanujan function. In [Fri08] this was achieved with \mathcal{W} being a *selective trace*, which is a fairly technical concept; in this paper we instead use a *certified trace*, which simply counts the w of with $-\chi(\mathcal{T}) < r$ for which $\rho(H_{\text{Graph}(w)}) < (d-1)^{1/2}$. (Note that $\text{Graph}(w)$ and its associated type, \mathcal{T} , with underlying graph T , have the same Euler characteristic, $\chi(\text{Graph}(w)) = \chi(\mathcal{T})$).

The point is that in this case $W_{\mathcal{T}}^{\mathcal{W}}(\vec{m}, \vec{k})$ is simply the original $W_{\mathcal{T}}(\vec{m})$ if $\text{VLG}(T, \vec{k})$, i.e., the graph formed from T by replacing each edge, f with a path of length k_f , has $\rho(H_{\text{VLG}(T, \vec{k})}) < (d-1)^{1/2}$. What this means is that now we sum in (1.6) with the additional constraint in \vec{k} that $\rho(H_{\text{VLG}(T, \vec{k})}) < (d-1)^{1/2}$. We call this \mathcal{W} a *certified trace*. One crucial aspect of the certified trace, like other simplifications such as types, is that it leads to dealing with finite sums.

This sum can be seen to be a good approximation to $\text{Tr}(H_G^k)$, for graphs, G , that do not contain tangles from $\text{Tangle}_{\varepsilon, r, B}$ as subgraphs. Unfortunately, this sum seems unpredictable if G does contain such a tangle. As a remedy, one can see that if we form the same sum, but this time insist that G contains a tangle, we also get terms analogous to (1.6) that converge to B -Ramanujan functions.

Subtracting the two walk sums, we get

$$\mathbb{E}_{G \in \mathcal{C}_n(B)}[\mathbb{I}_{\text{TF}}(G) \text{Tr}(H_G^k)] = P_0(k) + P_1(k)n^{-1} + \dots + P_{r-1}(k)n^{1-r} + \text{err}_r(n, k),$$

for B -Ramanujan functions $P_i(k)$, and at this point we can apply a side-stepping lemma, as described in the previous section, and standard calculations to show that $P_0(k) = \text{Tr}(H_B^k)$, and conclude the generalized Alon Conjecture in the case of regular B .

Chapter 2

Preliminaries

Since we do not always use standard terminology and since some concepts have not yet crystallized in the community; we define in this chapter the essential objects that we will be working with. We layout definitions for directed graphs, graphs, their adjacency and Hashimoto eigenvalues. We define graph coverings, variable-length graphs and the Broder-Shamir probability model for random graph coverings of a fixed based graph. We finish this chapter with a description of polyexponential and Ramanujan functions.

2.1 Essential Graph Terminology

In this paper, we are working with a broader definition of what is usually called a graph. Mainly, we allow self-loops and multiple edges. We formalize this below.

Definition 2.1.1. A *directed graph* (or *digraph*) is a tuple $G = (V, E^{\text{dir}}, t, h)$ where V and E^{dir} are disjoint sets -- the vertex and directed edge sets - and $t: E^{\text{dir}} \rightarrow V$ is the *tail map* and $h: E^{\text{dir}} \rightarrow V$ is the *head map*. A directed edge e is called a *self-loop* if $t(e) = h(e)$, that is, if its tail is its head. Note that our definition also allows for *multiple edges*, that is distinct directed edges with identical tails and heads. Unless specifically mentioned, we will only consider directed graphs which have finitely many vertices and directed edges.

The notation $G = (V_G, E_G, t_G, h_G)$ is convenient when discussing more than one graph at a time, or if we want to emphasize the graph G .

A graph arises from a directed graph by adding an involution that pairs up edges.

Definition 2.1.2. An *undirected graph* (or simply a *graph*) is a tuple $G = (V, E^{\text{dir}}, t, h, \iota)$ where $(V, E^{\text{dir}}, t, h)$ is a directed graph and where $\iota: E^{\text{dir}} \rightarrow E^{\text{dir}}$, called the *opposite map*, is an involution on the set of directed edges (that is, $\iota^2 = \text{id}_{E^{\text{dir}}}$ is the identity) satisfying $t\iota = h$. The directed graph $G = (V, E^{\text{dir}}, t, h)$ is called the *underlying directed graph* of the graph G . If e is an edge, we denote by $e^{-1} = \iota(e)$ and call it the *opposite edge*. A self-loop e is called a *half-loop* if $\iota(e) = e$ and a *whole loop* otherwise.

The opposite map induces an equivalence relation on the directed edges of the graph and we call the quotient set, E , the *undirected edges* of the graph G (or simply its *edges*). Given an edge of a graph, an *orientation* of that edge is the choice of a representative directed edge in the equivalence relation (given by the opposite map).

Next, we define morphisms of undirected and directed graphs.

Definition 2.1.3. A *morphism of directed graphs*, $\varphi: G \rightarrow H$ is a pair $\varphi = (\varphi_V, \varphi_E)$ for which $\varphi_V: V_G \rightarrow V_H$ is a map of vertices and $\varphi_E: E_G^{\text{dir}} \rightarrow E_H^{\text{dir}}$ is a map of directed edges satisfying $h_H(\varphi_E(e)) = \varphi_V(h_G(e))$ and $t_H(\varphi_E(e)) = \varphi_V(t_G(e))$ for all $e \in E_G^{\text{dir}}$. When confusion is unlikely to arise, we simply write φ instead of φ_V or φ_E .

This forms the category of directed graphs (with the obvious identities and composition of morphisms). This category has products and coproducts (disjoint unions).

We can extend this to undirected graphs by requiring that $\iota_H(\varphi(e)) = \varphi(\iota_G(e))$ for all $e \in E_G^{\text{dir}}$. Note that in this case, the terminal object is the vertex with a half-loop (half-loops can only be mapped to half-loops, whereas whole-loops can be mapped to either whole or half loops).

Definition 2.1.4. An *oriented graph* is an undirected graph, G , endowed with an orientation of each of its edges. In this context, when referring to an edge $e \in E_G$ we always assume it represents its underlying directed edge and hence extend the language of directed edges to this edge (so

it has a tail and a head map) and we denote by e^{-1} its opposite directed edge.

Walks on graphs offer interesting questions on their own, see [Lov93] for a survey on the topic. We recall their definition and our terminology here.

Definition 2.1.5. Let $G = (V, E^{\text{dir}}, t, h)$ be a directed graph and $k \geq 0$ an integer. A *walk of length k in G* is an alternating sequence of vertices and directed edges,

$$v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k$$

for which $t(e_i) = v_{i-1}$ and $h(e_i) = v_i$. We say that a walk is:

- *closed* if $v_0 = v_k$,

Furthermore, if G is an undirected graph, say that a walk is:

- *non-backtracking* (or *irreducible*) if $\iota(e_{i+1}) \neq e_i$ for all $i = 1, \dots, k-1$,
- *strictly non-backtracking* (or *strongly irreducible*) if it is not closed and non-backtracking; or if it is closed, non-backtracking and $\iota(e_k) \neq e_1$.

When $k > 0$ it is easier to identify a walk with its sequence of edges only.

Our main interest lies in the algebraic properties of graphs. We review some elementary definitions of algebraic graph theory in the next sections.

2.2 Adjacency Eigenvalues

In this section, we describe our terminology and review fundamental properties of the adjacency matrix and its eigenvalues.

Definition 2.2.1. Given a directed graph G and an ordering of its vertices $V = \{v_1, v_2, \dots, v_n\}$, its *adjacency matrix*, A_G , is the $n \times n$ square matrix indexed on the vertices whose (v_i, v_j) entry is the number of directed edges

whose tail is the vertex v_i and head is the vertex v_j . The adjacency matrix of an undirected graph is simply the adjacency matrix of its underlying directed graph. The ordering of the vertices is usually understood and since the algebraic properties considered are invariant under matrix similarity, it is customary to simply talk about the adjacency matrix of the graph itself.

The *adjacency eigenvalues* (or simply the *eigenvalues*) of a directed graph G are the eigenvalues of its adjacency matrix. When the graph is undirected, its adjacency matrix is symmetric and hence the spectral theorem guarantees that all its eigenvalues are real. In that case, we denote them by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, ordered in decreasing order and duplicating eigenvalues by their multiplicity. We denote by $\text{Spec}(A_G)$ the multiset of these eigenvalues that we call the *adjacency spectrum* and define its *adjacency spectral radius*, $\rho(A_G)$ or $\rho_A(G)$ to be the largest adjacency eigenvalue in absolute value. We reserve the notation $\lambda_i(G)$ to denote the eigenvalues of the adjacency matrix of a graph G .

Of interest to us, are d -regular graphs. We recall their definition and then some important and well-known properties of their adjacency eigenvalues.

Definition 2.2.2. Let $G = (V, E^{\text{dir}}, t, h)$ be a directed graph and $v \in V$ be one of its vertices. We define the *in-degree* (respectively *out-degree*), denoted by $\deg_h(v)$ (respectively $\deg_t(v)$), to be the number of directed edges whose head (respectively tail) is the vertex v . In the case of an undirected graph, the involution ι guarantees that the in-degree always equals the out-degree and hence we simply call it the *degree* of the vertex v and denote it by $\deg(v)$.

Let d be a positive integer, we say that a graph is *d -regular* if the degree of each of its vertices is d .

Using Perron-Frobenius theory, one obtains the following well-known properties of the adjacency eigenvalues.

Proposition 2.2.3. *Let G be a graph with adjacency eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then*

- (1) Its first eigenvalue, λ_1 , is the largest in absolute value and has a positive eigenvector. This eigenvalue is often called the Perron-Frobenius eigenvalue.
- (2) The Perron-Frobenius eigenvalue satisfies $\lambda_1 \leq \Delta(G)$ where $\Delta(G)$ is the maximal degree of the graph.

Furthermore, if the graph is d -regular, then

- (3) The Perron-Frobenius eigenvalue satisfies $\lambda_1 = d$, and the vector $(1, \dots, 1)$ is an eigenvector.
- (4) The multiplicity of the Perron-Frobenius eigenvalue is exactly the number of connected components of the graph.
- (5) Any eigenvector for an eigenvalue that is not the Perron-Frobenius eigenvalue has its vector components sum to zero.

One of the main combinatorial interest of the adjacency matrix is its connection to walks in a graph. We summarize these well-known properties in the following proposition.

Proposition 2.2.4. *Let G be a graph, A_G its adjacency matrix, and k a positive integer, then*

- (1) $(A_G^k)_{i,j}$ is the number of walks of length k starting at the vertex v_i and ending at the vertex v_j .
- (2) $\text{Tr}(A_G^k)$ is the number of closed walks of length k in the graph G .

2.3 Hashimoto Eigenvalues

Definition 2.3.1. Let $G = (V_G, E_G^{\text{dir}}, t_G, h_G, \iota_G)$ be a graph, we define its associated *directed line graph*, denoted $\text{Line}(G)$, to be the directed graph $\text{Line}(G) = (V_L, E_L^{\text{dir}}, t_L, h_L)$ given as follow. Its vertex set, V_L , is the set

E_G^{dir} of directed edges in the directed graph G . Its set of directed edges is defined by

$$E_L^{\text{dir}} = \left\{ (e_1, e_2) \in E_G^{\text{dir}} \times E_G^{\text{dir}} \mid h_G(e_1) = t_G(e_2) \text{ and } \iota_G(e_1) \neq e_2 \right\}$$

that is, non-backtracking walks of length 2. The tail and head maps are simply defined to be the projections in each component, that is by $t_L(e_1, e_2) = e_1$ and $h_L(e_1, e_2) = e_2$.

The adjacency matrix of the directed line graph of a graph G is called the *Hashimoto matrix* of G and we denote it by H_G . The eigenvalues of the Hashimoto matrix are called the *Hashimoto eigenvalues* of G and we denote them by $\mu_i(G)$, for $i = 1, \dots, |E_G^{\text{dir}}|$, reserving $\mu_1(G)$ to denote the Perron-Frobenius eigenvalue of the directed line graph. Since the oriented line graph is not an undirected graph, those eigenvalues are not always real. We denote by $\text{Spec}(H_G)$ the multiset of these eigenvalues and call it the *Hashimoto spectrum* of the graph G . We define its *Hashimoto spectral radius*, $\rho(H_G)$ or $\rho_H(G)$ to be the largest Hashimoto eigenvalue in absolute value.

Remark 2.3.2. In the case where the starting graph G is undirected, there is another construction, often referred to as the *line graph*, which takes for vertices the undirected edges of G . This yields a different construction than the one presented above and we do not use it in our work.

Our main interest in the Hashimoto matrix is that it encodes strictly non-backtracking walks. We review here the main properties that we will be using in our work.

Proposition 2.3.3. *Let G be a directed graph and $\text{Line}(G)$ its associated directed line graph. Then*

- (1) *A walk of length k in $\text{Line}(G)$ corresponds to a non-backtracking walk of length $k + 1$ in G ,*
- (2) *a closed walk in $\text{Line}(G)$ corresponds to a closed strictly non-backtracking walk in G*

and hence, for any positive integer k we have

(3) $(H_G^k)_{i,j}$ is the number of strictly non-backtracking walks of length $k+1$ starting at the vertex $t_G(e_i)$ and ending at the vertex $h_G(e_j)$.

(4) $\text{Tr}(H_G^k)$ is the number of closed strictly non-backtracking walks of length k in the graph G .

Using the Ihara zeta-function, Bass proved show a relationship between the adjacency and Hashimoto eigenvalues. The Ihara zeta-function is a zeta function associated with a finite graph. It closely resembles the Selberg zeta-function, and is used to relate closed paths to the spectrum of the adjacency matrix. The Ihara zeta-function was first defined in [Iha66] by Yasutaka Ihara in 1966 in the context of discrete subgroups and then in [Sun86] by Toshikazu Sunada in the context of graphs in 1986.

Definition 2.3.4. The *Ihara-Zeta function* of a graph G is the function ζ_G defined on the complex numbers and given by

$$\zeta_G(t) = \frac{1}{\det [I - tH_G]}$$

If μ is a Hashimoto eigenvalue of G , then $t = 1/\mu$ is a pole of ζ_G .

In [Bas92], Bass proves that the Ihara-Zeta function can be expressed in terms of the adjacency matrix.

Theorem 2.3.5 (Bass). *Let G be a graph without half-loops, H_G its Hashimoto matrix, A_G its adjacency matrix and D_G the degree matrix, then*

$$\zeta_G(t) = \frac{(1 - t^2)^{\chi(G)}}{\det [I - tA_G + t^2(D_G - I)]}$$

Using this theorem, we can link the adjacency eigenvalues to the Hashimoto eigenvalues when the graph G is d -regular.

Proposition 2.3.6. *Let G be a d -regular graph with no half-loops and denote by $\lambda_1 \geq \dots \geq \lambda_n$ its adjacency eigenvalues. Then its Hashimoto*

eigenvalues are given as follow: For each adjacency eigenvalue λ , there are 2 corresponding Hashimoto eigenvalues (up to multiplicity), $\mu_{1,2}(\lambda)$, given as solutions of the equation

$$\mu^2 - \lambda\mu + d - 1 = 0$$

Furthermore, there are Hashimoto eigenvalues 1 and -1 which each come with additional multiplicity $n(d - 2)/2$.

Proof. Using Bass formula, we obtain that

$$\det[I - tH_G] = (1 - t^2)^{-\chi(G)} \det[I - tA_G + t^2(D_G - I)]$$

Since G is d -regular, we have that $D_G - I = (d - 1)I$ and hence if λ is an adjacency eigenvalue, that is a solution of the equation $\det[\lambda I - A_G] = 0$ then any solution, $\mu = 1/t$ of the equation $\mu^2 - \lambda\mu + (d - 1) = 0$ will be a solution of the equation $\det[\mu I - H_G] = 0$ and so yield a Hashimoto eigenvalue. For the second part of our assertion, just observe that if G is d -regular and doesn't have half-loops, then

$$-\chi(G) = -(|V_G| - |E_G|) = -n + \frac{nd}{2} = \frac{n(d - 2)}{2}$$

which justifies the additional multiplicities of the Hashimoto eigenvalues 1 and -1 . \square

More generally, Kotani and Sunada, in [KS00], show the following result which is valid in the general case when the graph is not d -regular. Our bound on adjacency eigenvalues relies on this result and allows us to work with Hashimoto eigenvalues, which show to be easier to handle.

Theorem 2.3.7 (Kotani and Sunada). *If G is any graph of maximum degree d_{\max} , then any Hashimoto eigenvalue, μ that is not real satisfies*

$$|\mu| \leq \sqrt{d_{\max} - 1}$$

and if λ is an adjacency eigenvalue satisfying $|\lambda| > 2\sqrt{d_{\max} - 1}$ gives rise

to a real Hashimoto eigenvalue greater than $\sqrt{d_{\max} - 1}$.

2.4 Graph Coverings

Definition 2.4.1. A morphism of directed graphs $\nu: H \rightarrow G$ is a *covering map* if ν is an epimorphism (it is surjective on both the vertex and edge sets of G) and a local isomorphism, that is for any vertex $w \in V_H$, the edge morphism ν_E induces a bijection between $t_H^{-1}(w)$ and $t_G^{-1}(\nu(w))$ and a bijection between $h_H^{-1}(w)$ and $h_G^{-1}(\nu(w))$; that is, a bijection between incident edges at this vertex. We call G the *base graph* and H a *graph covering of G* .

If $\nu: H \rightarrow G$ is a covering map and G is connected, then the *degree* of ν , denoted $[H: G]$, is the number of preimages of a vertex or edge in G under ν (which does not depend on the vertex or edge). If G is not connected, we insist that the the number of preimages of ν of a vertex or edge is the same, i.e., the degree is independent of the connected component, and we will write this number as $[H: G]$.

A morphism of graphs is a *covering map* if the morphism on underlying directed graphs is a covering map.

We are particularly interested in the behaviour of the eigenvalues of graph coverings. The following proposition shows that adjacency eigenvalues of the base graph lift to their coverings.

Proposition 2.4.2. *Let B be a connected base graph and $\pi: G \rightarrow B$ a covering map of degree n . Then if $\phi: V_B \rightarrow \mathbb{C}$ is an eigenvector of the adjacency matrix, A_B , of the base graph with associated eigenvalue λ , then $\pi^*\phi = \phi \circ \pi: V_G \rightarrow \mathbb{C}$ is an eigenvector of the adjacency matrix, A_G , of the covering graph G with the same associated eigenvalue.*

Proof. Since ϕ is an eigenvector of the base graph, we have that for each vertex $v \in V_B$

$$\sum_{e \in t_B^{-1}(v)} \phi(h_B(e)) = \lambda \phi(v)$$

Let us just check that $\pi^*\phi$ is an eigenvector of the covering graph. Let $w \in V_G$, we have

$$\sum_{f \in t_G^{-1}(w)} (\pi^*\phi)(h_G(f)) = \sum_{f \in t_G^{-1}(w)} \phi(\pi(h_G(f)))$$

Since π is a covering map, it induces a bijection between $t_G^{-1}(w)$ and $t_B^{-1}(\pi(w))$ hence

$$= \sum_{\substack{e \in t_B^{-1}(\pi(w)) \\ \pi(f)=e}} \phi(\pi(h_G(f)))$$

Since π is a morphism of graphs, it commutes with the tail map and so

$$\begin{aligned} &= \sum_{\substack{e \in t_B^{-1}(\pi(w)) \\ \pi(f)=e}} \phi(h_B(\pi(f))) \\ &= \sum_{e \in t_B^{-1}(\pi(w))} \phi(h_B(e)) \\ &= \lambda\phi(\pi(w)) = \lambda(\pi^*\phi)(w) \end{aligned}$$

□

This fact yields a notion of "new" and "old" eigenvalues among adjacency eigenvalues of the graph covering.

Definition 2.4.3. If $\pi: G \rightarrow B$ is a covering map, we partition the multiset of its adjacency eigenvalues in two:

$$\text{Spec}(A_G) = \text{Spec}_B^{\text{new}}(A_G) \sqcup \text{Spec}_B^{\text{old}}(A_G)$$

We call these respectively the *new adjacency eigenvalues* and *old adjacency eigenvalues* of G , where $\text{Spec}_B^{\text{old}}(A_G) = \text{Spec}(A_B)$ is the set of eigenvalues that are lifted from the base graph B . We denote by $\rho_B^{\text{new}}(A_G)$ the largest new adjacency eigenvalue in absolute value of the graph G .

We want to establish the same type of relationship for Hashimoto eigenvalues. First, we observe that covering maps induce covering maps on directed line graphs.

Proposition 2.4.4. *Let $\pi: G \rightarrow B$ be a covering map. Then π induces a covering map $\pi_{\text{Line}}: \text{Line}(G) \rightarrow \text{Line}(B)$.*

Proof. The induced map $\pi_{\text{Line}}: \text{Line}(G) \rightarrow \text{Line}(B)$ is simply given by

$$\pi_{\text{Line}}(f) = \pi(f) \quad \text{and} \quad \pi_{\text{Line}}(f_1, f_2) = (\pi(f_1), \pi(f_2))$$

for any vertex in the directed line graph of G , $f \in V_{\text{Line}(G)} = E_G^{\text{dir}}$, (which are directed edges in G); and for any edge, $(f_1, f_2) \in E_{\text{Line}(G)}$. This is clearly an epimorphism of directed graphs. To show that it is a local isomorphism, consider a vertex f , in $\text{Line}(G)$, which is simply a directed edge in G . The one-to-one correspondence between $t_{\text{Line}(G)}^{-1}(f)$ and $t_{\text{Line}(B)}^{-1}(\pi_{\text{Line}}(f))$ is given by the one-to-one correspondence between $h_G^{-1}(w)$ and $h_B^{-1}(\pi(w))$ given by the covering π at the vertex $w = t_G(f)$ and reciprocally for the head when looking at the correspondence on tails at the vertex $h_G(f)$. \square

This implies "new" and "old" Hashimoto eigenvalues for graph coverings.

Definition 2.4.5. If $\pi: G \rightarrow B$ is a covering map, then the Hashimoto eigenvalues of the base graph B lift to the covering graph G . We can hence partition the multiset of Hashimoto eigenvalues of G in two:

$$\text{Spec}(H_G) = \text{Spec}_B^{\text{new}}(H_G) \sqcup \text{Spec}_B^{\text{old}}(H_G)$$

called respectively the *new Hashimoto eigenvalues* and *old Hashimoto eigenvalues* of G and where $\text{Spec}_B^{\text{old}}(H_G) = \text{Spec}(H_B)$ is the set of Hashimoto eigenvalues that are lifted from the base graph B . We denote by $\rho_B^{\text{new}}(H_G)$ the largest new Hashimoto eigenvalue in absolute value of the graph G .

2.5 Variable-Length Graphs

We are interested in the various subgraphs that are traced out by random walks. If for example, a walk visits new vertices at each step and finally comes back to its starting point, we call this a simple loop. The length of that loop is not as important as its fundamental shape and this motivates our usage of variable-length graphs which are graphs with weighted edges. This concept allows to handle graphs which have large or infinite parts of them being paths or regular trees.

Definition 2.5.1. A *variable-length graph* is a tuple $G = (V, E^{\text{dir}}, t, h, \iota, \ell)$ where $(V, E^{\text{dir}}, t, h, \iota)$ is a graph and where each edge, $e \in E$, is assigned a weight, $\ell(e)$, called the *length* of the edge, that can be any positive integer. The length of a walk in a variable-length graph is the sum of all the lengths of the edges in the walk. If $\vec{k} \in \mathbb{Z}_{\geq 0}^{|E|}$ is a vector of positive integers for each edge of G , we denote by $\text{VLG}(G, \vec{k})$ the variable-length graph based on G with edge lengths given by \vec{k} .

All that we will be using in this thesis about variable-length graphs can be found in [Fri08], section 3.2.

2.6 The Broder-Shamir Model

For a graph, B , and integer n , we give a model of random cover of B of degree n which slightly generalizes the model used in [Fri03].

Definition 2.6.1. Let $B = (V_B, E_B^{\text{dir}}, h_B, t_B, \iota_B)$ be a graph that we refer to as the *base graph* of the model. A *permutation assignment of B* is a map $\sigma: E_B^{\text{dir}} \rightarrow \mathcal{S}_n$ satisfying

$$\sigma_{\iota e} = (\sigma_e)^{-1}$$

where \mathcal{S}_n denotes the set of permutations of n elements and where we denote by σ_e the permutation associated to a directed edge $e \in E_B^{\text{dir}}$. We associate a graph covering of degree n , denoted $B[\sigma]$, to each permutation

assignment. This covering is constructed as follow. The vertex set and edge set respectively are

$$V = V_B \times \{1, \dots, n\} \quad \text{and} \quad E^{\text{dir}} = E_B^{\text{dir}} \times \{1, \dots, n\}$$

with the head and tail maps

$$t(e, i) = (t_B(e), i) \quad \text{and} \quad h(e, i) = (h_B(e), \sigma_e(i))$$

And the covering map, $\pi[\sigma]: B[\sigma] \rightarrow B$ is simply the projection on the first component. The *Broder-Shamir model of degree n of B* , we mean the uniform distribution over all permutation assignments, $\sigma: E_B^{\text{dir}} \rightarrow \mathcal{S}_n$ and we use $\mathcal{C}_n(B)$ to denote the induced probability space of covering graphs of degree n of the base graph B . If d is a positive even integer and B is the graph with one vertex and $d/2$ whole loops, the corresponding Broder-Shamir model is the one originally described by Broder and Shamir in [BS87] and then used by Friedman in [Fri03] and [Fri08].

Unless explicitly mentioned, all probabilities and expectations are computed on the Broder-Shamir model. If X is a random variable and \mathcal{E} an event in the probability space $\mathcal{C}_n(B)$, we will denote their respective expectation and probabilities by

$$\mathbb{E}_n[X] = \mathbb{E}_{G \in \mathcal{C}_n(B)}[X]$$

and $\text{Prob}_n[\mathcal{E}]$ (we might drop the mention of n when obvious in the context).

In Chapter 4, we will need to describe the collection of all graphs that occur as a subgraph of a random covering, $G \in \mathcal{C}_n(B)$, for some n large enough. We denote that set by $\text{Occurs}(B)$.

2.7 Polyexponential Functions

Definition 2.7.1. A *powerexponential function* is a function of the form

$$\begin{aligned} g: \mathbb{Z}_{\geq 0} &\longrightarrow \mathbb{C} \\ k &\longmapsto \mu^k k^s \end{aligned}$$

for some non-negative integer s and complex number μ . We call μ the *base* of the function and s its *power*. If $\vec{k} = (k_1, \dots, k_t)$ is a t -dimensional variable, a *t -variate powerexponential function* is a product

$$g(\vec{k}) = g_1(k_1) \dots g_t(k_t)$$

of powerexponential functions. We define a *polyexponential function* to be a \mathbb{C} -linear combination of powerexponential functions, and similarly, we define *t -variate polyexponential functions*.

The fundamental example of polyexponential function is the function that is the i, j -th entry of a k -th power of a fixed matrix. By the Jordan normal form, we know that this function is a polyexponential function, with bases being the eigenvalues of the matrix.

In our work, we will only be interested in the case where the matrix is the Hashimoto matrix of a fixed base graph B . For example, in [Fri08], Friedman considered the Hashimoto matrix of the graph $W_{d/2}$ consisting of one vertex and $d/2$ whole-loops. In this case, the number of strictly non-backtracking walks of length k in $W_{d/2}$ is

$$g(k) = (d-1)^k + (d-1) + (d-2)(-1)^k$$

which is a polyexponential function of bases $d-1, 1, -1$, which are the eigenvalues of the Hashimoto matrix of the graph $W_{d/2}$.

Definition 2.7.2. Let g_1 and g_2 be two functions defined on the non-negative integers. We define their *convolution*, $g_1 * g_2$, to be the function on non-

negative integers given by

$$(g_1 * g_2)(k) = \sum_{j=0}^k g_1(j)g_2(k-j)$$

for any $k \geq 0$.

First, we make the following observation.

Theorem 2.7.3. *The convolution of two polyexponential functions is polyexponential.*

Proof. It suffices to do this for powerexponential functions. Let

$$g_1(k) = \alpha^k k^s \quad \text{and} \quad g_2(k) = \beta^k k^t$$

be two powerexponential functions, for $\alpha, \beta \in \mathbb{C}$ and non-negative integers s, t . In the case $\alpha = \beta$, we have

$$\begin{aligned} (g_1 * g_2)(k) &= \sum_{j=0}^k g_1(j)g_2(k-j) \\ &= \alpha^k \sum_{j=0}^k (k-j)^s j^t \end{aligned}$$

We now examine this sum

$$\begin{aligned} \sum_{j=0}^k (k-j)^s j^t &= \sum_{j=0}^k \sum_{i=0}^s \binom{s}{i} k^i (-1)^{s-i} j^{s+t-i} \\ &= \sum_{i=0}^s (-1)^{s-i} \binom{s}{i} k^i \sum_{j=0}^k j^{s+t-i} \end{aligned}$$

Sums of the type $\sum_{j=0}^k j^p$ are known to be polynomials in k of degree $p+1$ and so we can conclude that $(g_1 * g_2)(k) = \alpha^k p(k)$ for some polynomial

$p(k)$ of degree at most $s + t + 1$.

For the case where $\alpha \neq \beta$, we remark that the polynomial

$$g(x, y) = \sum_{j=0}^k x^{k-j} y^j$$

satisfies

$$g(x, y) = \frac{x^{k+1} - y^{k+1}}{x - y}$$

Differentiating s times in x and t times in y yields

$$\sum_{j=0}^k \binom{k-j}{s} \binom{j}{t} x^{k-j-s} y^{j-t} = x^k h_{s,t}(x, y) + y^k h_{t,s}(x, y)$$

for rational functions, $h_{s,t}$, whose denominator contains only powers of $x - y$. Expressing z^s (with z an indeterminate) as a linear combination of binomial coefficients

$$\binom{z}{0}, \dots, \binom{z}{s}$$

and setting z to $k - j$, and doing similarly for z^t with $z = j$, yields the theorem.

□

2.8 Functions of Bounded Growth

We are interested in results about the convolutions of two (or more) functions, when one or both functions are arbitrary functions satisfying some growth restrictions.

Definition 2.8.1. We say that a function $g = g(k)$ defined on non-negative integers has *growth bounded by ρ* for some real $\rho > 0$, provided that there exists a positive constant c for which

$$|g(k)| \leq C k^C \rho^k$$

Theorem 2.8.2. *The convolution of any finite number of functions of growth ρ is again of growth ρ .*

Proof. It is sufficient to consider the convolution of two functions, g_1, g_2 ; the general claim follows by repeated applications of the result for two functions. So consider

$$g(k) = \sum_{\substack{k_1+k_2=k \\ k_1, k_2 \geq 0}} g_1(k_1)g_2(k_2)$$

with g_1, g_2 having growth bounded by ρ ; for some constants C_1, C_2 we have

$$|g_1(k)| \leq C_1 k^{C_1} \rho^k \quad \text{and} \quad |g_2(k)| \leq C_2 k^{C_2} \rho^k$$

There are $k + 1$ ways of writing k as the sum of two non-negative integers, k_1, k_2 . For each such pair, (k_1, k_2) , we have

$$|g_1(k_1)g_2(k_2)| \leq C_1 C_2 k_1^{C_1} k_2^{C_2} \rho^{k_1+k_2} \leq C_1 C_2 k^{C_1+C_2} \rho^k$$

Since there are at most $k + 1$ pairs, (k_1, k_2) , we have

$$|g(k)| \leq C_1 C_2 (k + 1) k^{C_1+C_2} \rho^k$$

and so it follows that g has growth bounded by ρ . □

For reasons that go back to [Fri91], we will need to convolve polynomials and polyexponentials with functions for which we only have a growth rate bound.

Definition 2.8.3. For any polynomial, $P = P(x)$, with real or complex coefficients, we define its *coefficient norm*, $\|P\|$ as the largest absolute value among its coefficients in its expansion by powers of x ; i.e.,

$$\|s_0 + x s_1 + \cdots + x^t s_t\| = \max_i |s_i|$$

Theorem 2.8.4. *For every non-negative integer D there is a constant, $C_2 = C_2(D)$, for which the following holds. Let $Q = Q(x)$ be any polynomial of*

degree at most D , and $h(r)$ a function defined on non-negative integers, r , for which

$$|h(r)| \leq C_1 r^D \rho^r$$

for some positive constants, C_1 and ρ with $\rho < 1$. Then the infinite sum,

$$q(x) = \sum_{s=0}^{\infty} Q(x-s)h(s)$$

converges in coefficient norm, and has degree at most that of Q , and satisfies

$$\|q\| \leq C_1 C_2 (1-\rho)^{-2D} \|Q\|$$

The same is true of

$$q_k(x) = \sum_{s=k+1}^{\infty} Q(x-s)h(s)$$

except the norm coefficient bound of q_k is given by

$$\|q_k\| \leq C_1 C_2 k^{2D} (1-\rho)^{-2D} \rho^k \|Q\|$$

Hence

$$|(Q * h)(k) - q(k)| = |q_k(k)| \leq C_1 C_2 \|Q\| k^{3D} \rho^k (1-\rho)^{-2D}$$

Proof. The statement on the convergence and norms of q and q_h follow from Lemma 8.8 of [Fri08]; the fact that the sum begins with $r = 0$ just affects the constants slightly. For ease of reading, we review the proof, which goes back to [Fri91] (as the last step of the proof of Sublemma 2.16). The idea is that we can restrict ourselves to the cases where $Q(x) = x^i$ for some i between 0 and D . Then we write

$$\sum_{r=0}^{\infty} Q(x-r)h(r) = \sum_{r=0}^{\infty} (x-r)^i h(r) \tag{2.1}$$

and we expand the $(x-r)^i$ via the binomial theorem, and use an identity

such as

$$\sum_{r=0}^{\infty} \binom{r}{\tau} \rho^r = \frac{\rho^\tau}{(1-\rho)^{\tau+1}}$$

to show the convergence to q for q_k , and obtain the coefficient norm estimate.

For the last statement, we see that

$$\sum_{r=0}^{\infty} Q(k-r)h(r)$$

is absolutely convergent, and hence

$$\begin{aligned} (Q * h)(k) &= \sum_{r=0}^k Q(x-r)h(r) \\ &= \sum_{r=0}^{\infty} Q(k-r)h(r) - \sum_{r=k+1}^{\infty} Q(k-r)h(r) \\ &= q(k) - q_k(k) \end{aligned}$$

But for $k \geq 1$, we have $q_k(x)$ is bounded in absolute value by $D + 1$ times its largest monomial, and hence

$$\begin{aligned} |q_k(k)| &\leq \|q_k\| (D+1)k^D \leq C_1 C_2 k^{2D} (1-\rho)^{-2D} \rho^k \|Q\| (D+1)k^D \\ &\leq C_1 \|Q\| \rho^k (1-\rho)^{-2D} C_2 k^{3D} \end{aligned}$$

□

Corollary 2.8.5. Fix a $d > 2$, a $D > 0$, and an $\varepsilon > 0$. Then for each $\mu \in \mathbb{C}$ with $|\mu| > \sqrt{d-1} + \varepsilon$ there exists a constant, C_2 , for which the following is true. If $P(k) = k^i \mu^k$ with integer i with $0 \leq i \leq D$, and $h(r)$ satisfies $|h(r)| \leq C_1 r^D (\sqrt{d-1} + \varepsilon)^r$, for some constant $C_1 > 0$, then we have that

$$p(x) = \sum_{i=0}^{\infty} P(x-i)h(i)$$

converges to a polyexponential function $p(k)$, and

$$|(P * h)(k) - p(k)| \leq C_2(\sqrt{d-1} + \varepsilon)^k$$

Proof. We have

$$(P * h)(k) = \mu^k(\tilde{P} * \tilde{h})$$

where

$$\tilde{P}(k) = k^i, \quad \tilde{h}(k) = \mu^{-k}h(k)$$

Hence

$$|\tilde{h}(k)| \leq C_1 r^D \rho^k$$

where

$$\rho \leq \frac{\sqrt{d-1} + \varepsilon}{|\mu|}$$

Hence $1 - \rho$, for a given d and ε , is bounded away from zero. Now we apply theorem 2.8.4. \square

2.9 B -Ramanujan Functions

Definition 2.9.1. Let B be a graph and g a function defined on non-negative integers, k . We say that g is B -Ramanujan if there is a polyexponential function P , with bases in the Hashimoto eigenvalues of B such that for every $\varepsilon > 0$ there exists a constant C for which

$$|g(k) - P(k)| \leq C(\rho(H_{\hat{B}}) + \varepsilon)^k$$

For example in our work, when the base graph, B , is d -regular, then $\rho(H_{\hat{B}}) = \sqrt{d-1}$. And when the base graph consists of $d/2$ whole-loops (for d even) on one vertex, then its Hashimoto eigenvalues are $d-1, 1, -1$ and so a B -Ramanujan function is a d -Ramanujan function in the sense of [Fri08].

Definition 2.9.2. Let g_1, \dots, g_s be functions on non-negative integers. Let $\vec{m} = (m_1, \dots, m_s)$ be a s -tuple of positive integers. The *weighted convolution of the g_i of weights \vec{m}* is denoted by $(g_1 * \dots * g_s)_{\vec{m}}$ and is defined to

be

$$(g_1 * \dots * g_s)_{\vec{m}}(k) = \sum_{\substack{\vec{m} \cdot \vec{k} = k \\ k \geq 0}} g(k_1) \dots g(k_s)$$

And if $\vec{k}_0 = (k_1^0, \dots, k_s^0)$ is a s -tuple of non-negative integers, we define the *truncated weighted convolution*, denoted by $(g_1 * \dots * g_s)_{\vec{m}}^{\vec{k}_0}$, to be

$$(g_1 * \dots * g_s)_{\vec{m}}^{\vec{k}_0}(k) = \sum_{\substack{\vec{m} \cdot \vec{k} = k \\ k \geq k_0}} g(k_1) \dots g(k_s)$$

This changes very little for the analysis, but is what we will be using for our certified trace.

Example 2.9.3. Consider the case where $g_1(k) = g_2(k) = (d-1)^k$ and $\vec{m} = (1, 2)$, then the identity on partial sums of geometric series

$$1 + x + \dots + x^t = \frac{x^{t+1} - 1}{x - 1}$$

with $x = d - 1$ easily gives

$$(g_1 * g_2)_{\vec{m}}(k) = \begin{cases} \frac{1}{d-2} ((d-1)^{k+1} - (d-1)^{k/2}) & \text{if } k \text{ is even,} \\ \frac{1}{d-2} ((d-1)^k - (d-1)^{(k+1)/2}) & \text{if } k \text{ is odd.} \end{cases}$$

This is not an exact polyexponential function, but it is a B -Ramanujan function.

Theorem 2.9.4. Let $\vec{m} \geq \vec{1}$ and $\vec{k}_0 \geq \vec{1}$. Let g_1, \dots, g_s be B -Ramanujan functions, then the (truncated) weighted convolution

$$(g_1 * \dots * g_s)_{\vec{m}}^{\vec{k}_0}$$

is B -Ramanujan as well.

Proof. First, observe that a change of variable

$$\kappa_1 = k_1 - k_1^0 + 1 \quad \kappa_2 = k_2 - k_1^0 + 1$$

allows us, without loss of generality, to remove the truncated condition. So, we assume that $\vec{k}_0 = \vec{1}$.

If the function g_i is B -Ramanujan, then we can write

$$g_i(k) = \sum_{\ell \in L} p_{\ell,i}(k) \ell^k + \text{error}_i(k)$$

where $p_{\ell,i}$ is a polynomial and where the error bound is such that for any $\varepsilon > 0$, we have

$$|\text{error}_i(k)| \leq C(\sqrt{d-1} + \varepsilon)^k$$

for some constant $C = C(i, \varepsilon)$. The set L is always finite, since it is the set of Hashimoto eigenvalues that are larger than $\sqrt{d-1}$ in absolute value. Hence, without any loss of generality, we can assume that

$$g_i(k) = p_{\ell,i}(k) \ell^k \quad \text{or} \quad g_i(k) = \text{error}_i(k) \quad \text{with} \quad |\text{error}_i(k)| \leq C(\sqrt{d-1} + \varepsilon)^k$$

and treat each of the three corresponding cases for $i = 1, 2$. Note that in either case, we have that

$$|g_i(k)| \leq \tilde{C} k^{\tilde{C}} (d-1)^k$$

for some constant \tilde{C} , since $d-1$ is the largest possible Hashimoto eigenvalue.

Let us consider now, the case where both g_i are error terms. In this case, we have

$$\begin{aligned} |(g_1 * g_2)(k)| &\leq \sum_{m_1 k_1 + m_2 k_2 = k} |\text{error}_1(k_1) \text{error}_2(k_2)| \\ &\leq \sum_{m_1 k_1 + m_2 k_2 = k} C_1 (\sqrt{d-1} + \varepsilon)^{k_1 + k_2} \\ &\leq C_2 \cdot \#\{(k_1, k_2) \in \mathbb{Z}_{\geq 1}^2 \mid m_1 k_1 + m_2 k_2 = k\} \cdot (\sqrt{d-1} + \tilde{\varepsilon})^k \end{aligned}$$

for some $\tilde{\varepsilon}$ sufficiently small and since $k_1 + k_2 \leq m_1 k_1 + m_2 k_2 = k$ and we

roughly bound

$$\#\{(k_1, k_2) \in \mathbb{Z}_{\geq 1}^2 \mid m_1 k_1 + m_2 k_2 = k\} \leq \binom{k+1}{2} k^2 \leq C_2 k^{C_2}$$

to obtain that

$$|(g_i * g_2)(k)| \leq C(\sqrt{d-1} + \varepsilon)^k$$

for any $\varepsilon > 0$ (for an appropriate choice of C and $\tilde{\varepsilon}$) making it an error term of a B -Ramanujan function and thus B -Ramanujan itself. For each of the two remaining cases, we need to look at three sub cases: (1) when $m_1 = m_2 = 1$, (2) when $m_1, m_2 \geq 2$ and (3) when $m_1 = 1, m_2 \geq 2$.

Consider (1), when $m_1 = m_2 = 1$. Then if g_1 and g_2 are both polyexponential functions, theorem 2.7.3 shows that their convolution is polyexponential as well and hence a B -Ramanujan function. In the case where g_1 is polyexponential while g_2 is an error term, corollary 2.8.5 shows that their convolution is a B -Ramanujan function.

We can treat both cases when $m_1, m_2 \geq 2$ simply using the fact that $g_i(k) \leq \tilde{C}k^{\tilde{C}}(d-1)^k$. Indeed, we have

$$\begin{aligned} |(g_i * g_2)(k)| &\leq \sum_{m_1 k_1 + m_2 k_2 = k} |g_1(k_1)g_2(k_2)| \\ &\leq \sum_{m_1 k_1 + m_2 k_2 = k} C_1 k_1^{C_1} k_2^{C_1} (d-1)^{k_1+k_2} \\ &\leq (d-1)^{k/2} \sum_{m_1 k_1 + m_2 k_2 = k} C_1 k_1^{C_1} k_2^{C_1} \end{aligned}$$

since if $m_1, m_2 \geq 2$ we have that $k_1 + k_2 \leq k/2$ and so for some constant C_2 we have

$$\begin{aligned} |(g_i * g_2)(k)| &\leq (d-1)^{k/2} C_2 k^{C_2} \\ &\leq C_3(\sqrt{d-1} + \varepsilon)^k \end{aligned}$$

for some other constant C_3 and any $\varepsilon > 0$; which shows that it is an error term and hence a B -Ramanujan function as well.

Finally, let us consider (3), when $m_1 = 1, m = m_2 \geq 2, g_1(k) = p(k)\ell^k$

is a polyexponential function and we will treat both cases for g_2 by simply assuming that

$$|g_2(k)| \leq Ck^C(d-1)^k$$

for some constant C . Then we have that

$$(g_1 * g_2)(k) = \sum_{k_1 + mk_2 = k} p(k_1)\ell^{k_1} g_2(k_2) = \sum_{i=1}^{k/m-r} g_2(i)p(k-mi)\ell^{k-mi}$$

for some $r = r(k)$ such that $k/m - r$ is the largest possible value of k_2 . We now rewrite this sum as

$$\ell^k \left[\sum_{i \geq 1} p(k-mi) \frac{g_2(i)}{\ell^{mi}} - \sum_{i \geq k/m-r+1} p(k-mi) \frac{g_2(i)}{\ell^{mi}} \right]$$

since $|g_2(k)| \leq Ck^C(d-1)^k$, $m \geq 2$ and $\ell > \sqrt{d-1}$, we have that

$$\left| \frac{g_2(i)}{\ell^{mi}} \right| \leq C_1 k^{C_1} \left(\frac{d-1}{\ell^2} \right)^k \leq C_1 k^{C_1} \rho^k$$

for $\rho < 1$. We can now apply theorem 2.8.4 to show that the convolution is a B -Ramanujan function. \square

Chapter 3

Walk Sums

In this chapter, we show that the expected value of the k^{th} -Hashimoto trace has an expansion in $1/n$ which satisfies, for any integer r ,

$$\mathbb{E}_n[\text{Tr}(H_G^k)] = f_0(k) + \frac{f_1(k)}{n} + \dots + \frac{f_{r-1}(k)}{n^{r-1}} + \frac{\text{error}(r, k)}{n^r}$$

where the f_i are functions of k and with an upper bound on the error term of

$$|\text{error}(r, k)| \leq ck^{4r}(d-1)^k$$

Furthermore, such an expansion is obtained for a broader notion of trace that we call a *walk sum*.

To obtain this result, we observe that the trace of the k^{th} power of the Hashimoto matrix is the number of strictly non-backtracking closed walks of length k . We generalize this notion with the definition of a walk sum, which counts the number of walks satisfying certain properties, and in this context, show that the expected number of walks has the desired asymptotic expansion. This follows and expands on the theory that was developed by Friedman in [Fri91] and then in [Fri08].

3.1 Potential Walks

We consider a probabilistic theory describing walks on a random cover of the base graph B . Consider the probability space $\mathcal{C}_n(B)$. For the purpose of this chapter, the base graph does not need to be d -regular.

Definition 3.1.1. A *potential (k, n) -walk* is a pair $(w; \vec{t})$ consisting of a walk,

called the *base walk*

$$w = (v_0, e_1, v_1, e_2, \dots, e_k, v_k)$$

in the base graph B of length k ; and a vector, called the *trajectory*

$$\vec{t} = (t_0, \dots, t_k)$$

with each $t_i \in \{1, \dots, n\}$. We refer to k as the *length* and n as the *size* of the potential walk. Given $G \in \mathcal{C}_n(B)$, we say that a potential (k, n) -walk $(w; \vec{t})$ is *feasible in G* if the vector \vec{t} represents the actual sheets that are visited when the walk w is lifted to G , that is, with the above notations, if

$$\sigma_{e_i}(t_{i-1}) = t_i \quad \forall i = 1, \dots, k$$

where $G = B[\sigma]$ and $\sigma = \{\sigma_e\}_{e \in E_B}$. We denote by $\mathcal{E}(w; \vec{t})$ the event in $\mathcal{C}_n(B)$ that the potential walk is feasible in G and denote by $P(w; \vec{t})$ the probability of this event. A potential (k, n) -walk is *feasible* if $P(w; \vec{t}) \neq 0$.

Lemma 3.1.2. *A potential (k, n) -walk $(w; \vec{t})$ is feasible if and only if the following two feasibility conditions hold:*

(1) *for any i, j such that $\sigma_{e_i} = \sigma_{e_j}$, we have*

$$t_{i-1} = t_{j-i} \iff t_i = t_j$$

(2) *and for any i, j such that $\sigma_{e_i} = \sigma_{e_j}^{-1}$, we have*

$$t_{i-1} = t_j \iff t_i = t_{j-1}$$

Definition 3.1.3. The *associated graph* of a potential (k, n) -walk $(w; \vec{t})$, denoted by $\text{Graph}(w; \vec{t})$, is the subgraph that is traced out in any random covering $G \in \mathcal{C}_n(B)$ in which the potential walk is feasible. If $e \in E_B$ is an edge in the base graph, we define $a_e = a_e(w; \vec{t})$ to be the number of edges in the associated graph of the potential walk in the fibre of the edge e . And

if $v \in V_B$ is a vertex in the base graph, we define $b_v = b_v(w; \vec{t})$ to be the number of vertices in the associated graph of the potential walk which is in the fibre of the vertex v . Hence we have that

$$\sum_{e \in E_B} a_e = |E_{\text{Graph}(w; \vec{t})}| \quad \text{and} \quad \sum_{v \in V_B} b_v = |V_{\text{Graph}(w; \vec{t})}|$$

and we denote by $\chi(w; \vec{t}) = |V_{\text{Graph}(w; \vec{t})}| - |E_{\text{Graph}(w; \vec{t})}|$ the *Euler characteristic* of the associated graph. Following the work of Friedman, we define the *order* of a potential walk $(w; \vec{t})$ to be the non-negative integer $\text{ord}(w; \vec{t}) = -\chi(w; \vec{t})$.

We will eventually regroup potential walks with similar graphs under our notion of forms and types. First we describe an obvious equivalence class among potential walks and their associated graph.

Definition 3.1.4. Fix a base walk $w = (v_0, e_1, v_1, \dots, e_k, v_k)$. We say that two trajectories of length k and size at most n , \vec{t} and \vec{s} , *differ by a symmetry* and denote this by $\vec{t} \sim_w \vec{s}$ if they only differ by a permutation of the vertices in each fibre; that is if there exists a collection of permutations of n elements $\tau = \{\tau_v\}_{v \in V_B}$ such that $\vec{s} = \tau(\vec{t})$ that is if

$$s_i = \tau_{v_i}(t_i) \quad i = 0, \dots, k$$

This is a modification of the original definition in [Fri08] that accounts for the multiple vertices of the base graph. Indeed, consider the following two examples. First, if the base walk is a loop, then the trajectories $(1, 1)$ and $(1, 2)$ are not equivalent since the first one yields a loop and the second an edge; but if the base walk is not a loop, then these trajectories are equivalent since they both yield an edge (they refer to vertices in different fibres of the base graph). With this definition we guarantee that if $\vec{t} \sim_w \vec{s}$ then $P(w; \vec{t}) = P(w; \vec{s})$. Define the (n, w) -*equivalence symmetry class* of \vec{t} to be

$$[\vec{t}]_{n, w} = \{\vec{s} \mid \vec{s} \sim_w \vec{t} \text{ and } s_i \leq n, i = 1, \dots, k\}$$

We may omit the n subscript of n is understood. We denote by $[w; \vec{t}]_n$ the

equivalence class of all potential walk with base walk w whose trajectories are in the equivalence class $[\vec{t}]_{n,w}$, that is

$$[w; \vec{t}]_n = \{(w; \vec{s}) \mid \vec{s} \in [\vec{t}]_{n,w}\}$$

and we also say that these potential walk *differ by a symmetry*. Clearly, if two potential (k, n) -walks differ by a symmetry, then their associated graphs are canonically isomorphic (the isomorphism being given by the collection of permutation) and hence the quantities a_e and b_v are identical in the equivalence class for all e and v .

3.2 Walk Sums

We make the following observation. If H_G is the Hashimoto matrix of a random covering graph G , then

$$\mathbb{E}_n \left[\text{Tr}(H_G^k) \right] = \sum_{(w; \vec{t}) \in \mathcal{W}} P(w; \vec{t})$$

where \mathcal{W} is the set of all potential (k, n) -walks $(w; \vec{t})$ whose walk w is a closed strictly non-backtracking walk of length k in the base graph B . We want to generalize this to allow the use of some other "traces", that is sums of $P(w; \vec{t})$ over appropriate w 's and \vec{t} 's.

Definition 3.2.1. A *walk collection*, $\mathcal{W} = \{\mathcal{W}(k, n)\}_{k, n \geq 1}$ is a collection where for any two positive integers k and n the elements of the set $\mathcal{W}(k, n)$ are potential (k, n) -walks satisfying the following conditions.

- (1) *Symmetry*: The walk collection contains all trajectories that differ by a symmetry; that is, if $(w; \vec{t}) \in \mathcal{W}(k, n)$ then $[w; \vec{t}]_n \subseteq \mathcal{W}(k, n)$.
- (2) *Size invariance*: The walk collection reflects the fact that the size of a walk isn't uniquely determined; that is, if $(w; \vec{t}) \in \mathcal{W}(k, n)$ then $(w; \vec{t}) \in \mathcal{W}(k, m)$ for all $m \geq \max\{t_i \mid \vec{t} = (t_0, t_1, \dots, t_k)\}$

- (3) *Strictly non-backtracking walks*: All the potential walks in the walk collection are strictly non-backtracking; that is, the base walk w of any potential (k, n) -walk $(w; \vec{t})$ in the collection is strictly non-backtracking (see definition 2.1.5).
- (4) *Closed walks*: All the potential walks in the walk collection are closed; that is, the base walk w of any potential (k, n) -walk $(w; \vec{t})$ in the collection is closed and its trajectory must satisfy $t_0 = t_k$.

Given a walk collection, \mathcal{W} , its associated (k, n) -*modified trace* is the random variable, \mathcal{W}_n , defined on $\mathcal{C}_n(B)$, where

$$\mathcal{W}_n(G, k) = \#\{(w; \vec{t}) \in \mathcal{W}(k, n) \mid (w; \vec{t}) \text{ is feasible in } G\}$$

if $G \in \mathcal{C}_n(B)$. Given a walk collection, \mathcal{W} , its associated (k, n) -*walk sum* is

$$\text{WalkSum}(\mathcal{W}, k, n) = \sum_{(w; \vec{t}) \in \mathcal{W}(k, n)} P(w; \vec{t})$$

Note that this sum is always a finite sum (unless the base graph B has infinitely many edges).

Example 3.2.2. We denote by $\text{SNBC} = \{\text{SNBC}(k, n)\}_{k, n \geq 1}$ the walk collection of all strictly non-backtracking closed walks of length k and size n . Its associated (k, n) -modified trace is simply the k^{th} Hashimoto trace:

$$\text{SNBC}_n(G, k) = \text{Tr}(H_G^k)$$

(thought of as a random variable on $\mathcal{C}_n(B)$) and its associated (k, n) -walk sum is simply the expected value of the k^{th} Hashimoto trace:

$$\text{WalkSum}(\text{SNBC}, k, n) = \mathbb{E}_n[\text{Tr}(H_G^k)]$$

This walk collection is the largest possible, that is, any other walk collection consists of subsets of this one.

Since walk collections are symmetric, we can regroup the terms of the above sum by grouping trajectories that differ by a symmetry.

Definition 3.2.3. Let

$$\mathbb{E}_{\text{Symm}}(w; \vec{t})_n = \sum_{\vec{s} \in [\vec{t}]_{n,w}} P(w; \vec{s})$$

then we have

$$\text{WalkSum}(\mathcal{W}, k, n) = \sum_{(w; [\vec{t}]_{n,w}) \in \mathcal{W}(k, n)} \mathbb{E}_{\text{Symm}}(w; \vec{t})_n \quad (3.1)$$

Now we comment on $\mathbb{E}_{\text{Symm}}(w; \vec{t})_n$ and will later use this to work out an asymptotic expansion of a walk sum.

Proposition 3.2.4. *Let $(w; \vec{t})$ be a potential (k, n) -walk then we have*

$$\mathbb{E}_{\text{Symm}}(w; \vec{t})_n = \prod_{v \in V_B} \frac{n!}{(n - b_v)!} \prod_{e \in E_B} \frac{(n - a_e)!}{n!} \quad (3.2)$$

with a_e and b_v as in definition 3.1.3, provided that B has no half-loops; for each $e \in E_B$ that is a half-loop, we replace

$$\frac{(n - a_e)!}{n!} \quad \text{by} \quad \frac{(n - 2a_e)!!}{(n - 1)!!}$$

Proof. Since all potential walks considered differ by a symmetry, the probability that they occur in a random cover is the same. Without loss of generality, we can assume that $\vec{t} \in [\vec{t}]_{n,w}$ and hence we have

$$\begin{aligned} \mathbb{E}_{\text{Symm}}(w; \vec{t})_n &= \sum_{\vec{s} \in [\vec{t}]_{n,w}} P(w; \vec{s}) \\ &= |[\vec{t}]_{n,w} | \cdot P(w; \vec{t}) \end{aligned}$$

and we evaluate each term separately.

First to compute the size of the equivalence class $[\vec{t}]_{n,w}$, we only need to consider how we can shuffle the vertices in the fibre of a vertex of the

base graph. If there are b_v elements of the trajectory \vec{t} in the fibre of the vertex v then there are $n(n-1)\dots(n-b_v+1)$ ways to choose those vertices among n possible and this is true for the fibre of each vertex $v \in V_B$ of the base graph.

A random covering $B[\sigma]$ is given by the choice of a collection of permutations $\sigma = \{\sigma_e\}_{e \in E_B}$. Assume B has no half-loops; if there are a_e edges of the associated graph to the potential walk that are in the fibre of the base edge e , then it means that there are a_e values of the permutation σ_e that are determined by the walk. The other $n - a_e$ values can thus be chosen at random. This implies that there are $(n - a_e)!$ permutations out of the possible $n!$ which satisfy the conditions imposed by the potential walk on the choice of the permutation σ_e . Doing this for each edge of the base graph concludes the proof. If B has half-loops, then we replace the factorials by double factorials, where $n!! = n(n-2)(n-4)\dots$ since σ_e is a random perfect matching. \square

3.3 Asymptotic Expansion

Again, begin with the assumption that B has no half-loops. We can rewrite equation 3.2 as

$$n^{\chi(w;\vec{t})} \prod_{v \in V_B} n(n-1)\dots(n-b_v+1) \prod_{e \in E_B} \frac{1}{n(n-1)\dots(n-a_e+1)}$$

Consider the power series expansions about $x = 0$ of

$$(1-x)(1-2x)\dots(1-mx) \quad \text{and} \quad \frac{1}{(1-x)(1-2x)\dots(1-mx)}$$

The x^i coefficient is a polynomial of degree at most $2i$ in m . Using this for $x = 1/n$, we can deduce that there must exist polynomials p_0, p_1, \dots in the variables a_e and b_v for $e \in E_B$ and $v \in V_B$ such that

$$\mathbb{E}_{\text{symm}}(w; \vec{t})_n = n^{\chi(w;\vec{t})} \sum_{i \geq 0} p_i(\vec{a}, \vec{b}) \frac{1}{n^i} \quad (3.3)$$

Definition 3.3.1. We call the polynomials p_0, p_1, \dots the *expansion polynomials* of the potential walk.

We now comment on this expansion and in particular, its truncated form.

Theorem 3.3.2. For any potential (k, n) -walk $(w; \vec{t})$ with $k \leq n/2$ and any integer $r \geq 1$ we have

$$\mathbb{E}_{\text{symm}}(w; \vec{t})_n = n^{\chi(w; \vec{t})} \left(p_0 + \frac{p_1}{n} + \dots + \frac{p_{r-1}}{n^{r-1}} + \frac{\text{error}(k, n)}{n^r} \right) \quad (3.4)$$

where the p_i are the expansion polynomials of the potential walk $(w; \vec{t})$ in the variables $\{a_e\}_{e \in E_V}, \{b_v\}_{v \in V_B}$ and with

$$|\text{error}(k, n)| \leq ck^{2r+2}$$

where c is a constant that depends only on r .

Proof. Consider a function of the form

$$g(x) = (1 - \alpha_1 x) \dots (1 - \alpha_s x) (1 - \beta_1 x)^{-1} \dots (1 - \beta_t x)^{-1}$$

where α_i and β_j are positive constants. The i^{th} -derivative of this function satisfies the bound

$$\left| \frac{g^{(i)}(x)}{i!} \right| \leq (1 - x \beta_{\max})^{-i} \left(\sum \alpha_j + \sum \beta_j \right)^i$$

where β_{\max} is the maximum of the β_k (by equation (6) in [Fri91] on page 339). Using Taylor's theorem about $x = 0$ we obtain that there must exist some $\xi \in [0, 1/n]$ such that

$$|\text{error}(k, n)| \leq (1 - \xi \beta_{\max})^{-r} \left(\sum \alpha_j + \sum \beta_j \right)^r$$

On the interval $[0, 1[$ we have that $(1 - x)^{-1} \leq e^{x/(1-x)}$ and since $\beta_{\max} \leq k$ we have $(1 - \xi \beta_{\max})^{-r} \leq e^{r\xi k/(n-k)}$. If $k \leq n/2$ we have that $e^{r\xi k/(n-k)} \leq e^r$, hence a constant that only depends on r . The $\sum \alpha_j$ represents the sum for

all $v \in V_B$ of all sums of $0, 1, \dots, b_v - 1$, and the $\sum \beta_j$ represents the sum of all $e \in E_B$ of $0, 1, \dots, a_e - 1$, both of which are at most $0 + 1 + \dots + k - 1 = \binom{k}{2}$. Since $2\binom{k}{2} \leq k^2$ we get the second part of the claimed error term. \square

We now study the error term of the truncated expansion.

Definition 3.3.3. Let \mathcal{W} be a walk collection and $r \geq 1$ a positive integer. We define its r -truncated (k, n) -modified trace to be the random variable, $\mathcal{W}_{n, < r}$, defined on $\mathcal{C}_n(B)$ with

$$\mathcal{W}_{n, < r}(G, k) = \#\{(w; \vec{t}) \in \mathcal{W}(k, n) \mid (w; \vec{t}) \text{ is feasible in } G \text{ and } \text{ord}(w; \vec{t}) < r\}$$

and we define its associated r -truncated (k, n) -walk sum to be

$$\text{WalkSum}_{< r}(\mathcal{W}, k, n) = \sum_{\substack{(w; \vec{t}) \in \mathcal{W}(k, n) \\ \text{ord}(w; \vec{t}) < r}} P(w; \vec{t})$$

The following lemma is essentially the observation, dating to Broder and Shamir, of looking at walks and noticing that if a walk is of order i , it has $i + 1$ coincidences.

Lemma 3.3.4. For any walk collection, \mathcal{W} , and any positive integer, $r \geq 1$, we have that

$$\text{WalkSum}(\mathcal{W}, k, n) = \text{WalkSum}_{< r}(\mathcal{W}, k, n) + O_n \left(|V_B| \binom{k}{r+1} (d-1)^k n^{-r} \right)$$

for any $k \leq n/3$.

Proof. We can use here the same method as in [Fri93]. Each walk w such that $\text{ord}(w) \geq r$ has at least $r + 1$ coincidences. Consider the first $r + 1$ coincidences that occur in w . There are $\binom{k}{r+1}$ places among the steps of the walk and each coincidence represents an event of probability at most $1/(n - k)$ in a random permutation (and $1/(n - 2k)$ in a random perfect matching). Each such w begins at one of the $n|V_B|$ vertices of the random graph. \square

We can use this now to prove an expansion theorem for any walk sum.

Theorem 3.3.5. *Let B be a base graph, let \mathcal{W} be a walk collection, and fix an integer $r \geq 1$. Then for all $k \leq n/2$ we have*

$$\text{WalkSum}(\mathcal{W}, k, n) = f_0(k) + \frac{f_1(k)}{n} + \dots + \frac{f_{r-1}(k)}{n^{r-1}} + \text{error}(r, n, k)$$

where

$$f_i(k) = \sum_{j=0}^{r-1} \sum_{(w; \vec{t}) \in \mathcal{W}^j(k)} p_{i-j}(w; \vec{t})$$

and where $\mathcal{W}^j(k)$ is the set of all equivalence classes of potential walks of length k and of order j ; with the error term satisfying

$$|\text{error}(r, n, k)| \leq ck^{4r}(d-1)^k n^{-r}$$

where c is a constant that depends on r only.

Proof. The proof of this theorem mostly follows the proof of theorem 5.9 in [Fri08]. We review it briefly and indicate the one minor variation that accommodates it for a general base graph. Friedman first shows that for a given base walk w of length $k \leq n/2$, we have

$$\sum_{\substack{[\vec{t}]_{n,w} \\ \text{ord}(w; \vec{t}) \geq r}} P(w; \vec{t}) \leq ck^{2r+2} n^{-r}$$

for some constant c depending only on r . Then he shows that if w is a non-backtracking base walk of length k , then there are at most ck^{2r+2} equivalence classes $[\vec{t}]_{n,w}$ whose order with w is at most $r-1$. Combining equations 3.1 and 3.4, and reordering by powers of n and then by the order of the potential walks considered, truncating for potential walks of order at least r . We introduce an error of at most $ck^{2r+2} n^{-r}$ per word by ignoring potential walks of order at least r . Each word has at most ck^{2r+2} associated potential walk classes of order at most $r-1$ and truncating results in an error of at most ck^{2r+2} (given in theorem 3.3.2). The number of strongly

non-backtracking base walks is at most $(d - 1)^k$ which yields the desired error bound. \square

Such an expansion would give us the Alon conjecture if we could show that the coefficients, $f_i(k)$ are B -Ramanujan functions for all values of r . In chapter 4, we show why this doesn't hold and how the notion of *tangle* is responsible for this obstruction.

3.4 Types and Forms

We finish this chapter with a section that describes how to rearrange the walk sum by examining more closely the shapes that the walks can take. Friedman first introduced the notions of types and forms in [Fri91] and then refined them in [Fri08]. We maintain the language and simplify the definitions here. The main difference is that we first define types as essentially variable-length graphs satisfying some technical conditions such that if we fix any positive integer, r , there are only finitely many types of order at most r . From there, we define forms and rewrite the walk sums accordingly. This will play a crucial role in chapter 4 where we will use this to obtain the desired B -Ramanujan asymptotic expansions.

Definition 3.4.1. A *type* is a tuple $\mathcal{T} = (T, \gamma_V, \gamma_E, \mathcal{L})$, where T is an oriented graph, γ_V an ordering of the vertices of T , γ_E an ordering of the edges of T and $\mathcal{L} = (\mathcal{L}^t, \mathcal{L}^h)$, called the *lettering* of the type, where $\mathcal{L}^t, \mathcal{L}^h: E_T \rightarrow E_B$ are maps of oriented graphs called the *lettering at the tail* and the *lettering at the head* respectively. Additionally, we require the following two conditions to hold:

- (1) All the vertices, except possibly for the first one (in the ordering given by the type) are of degree at least 3, and
- (2) the lettering is *consistent*, that is, for any vertex $v \in V_T$ and any edges $f_1, f_2 \in E_T$ such that $t_T(f_1) = t_T(f_2)$ (respectively $h_T(f_1) = h_T(f_2)$) we have that $\mathcal{L}^t(f_1) \neq \mathcal{L}^t(f_2)$ (respectively $\mathcal{L}^h(f_1) \neq \mathcal{L}^h(f_2)$). In other

words, the lettering at the tail for two edges sharing their tail vertex is distinct, and similarly at the head.

Definition 3.4.2. A *form of type* \mathcal{T} is a tuple $\mathcal{F} = (F, \mathcal{E})$ where F is a variable-length graph based on the underlying graph of the type \mathcal{T} and where \mathcal{E} is a labelling of the edges: if $f \in E_F$ is an edge of F , then $\mathcal{E}(f) = (e_1, \dots, e_{\ell(f)})$ is a walk in the base graph B , where $\ell(f)$ is the length of the edge f and we request that the labelling be consistent with the lettering of the type \mathcal{T} , that is, we demand that

$$e_1 = \mathcal{L}^t(f) \quad \text{and} \quad e_{\ell(f)} = \mathcal{L}^h(f)$$

We write $\mathcal{F} \in \mathcal{T}$ to mean that the form \mathcal{F} is of type \mathcal{T} .

Given a potential walk, $(w; \vec{t})$, there is a unique type, $\mathcal{T}(w; \vec{t})$ and form, $\mathcal{F}(w; \vec{t})$, associated to it. The type is obtained by considering the associated graph to the walk, removing all beaded paths, preserving only the initial vertex and other vertices of degree at least three. The vertices and edges are ordered as they occur in the walk. The edges of the form take their length from their corresponding beaded paths in the associated graphs and inherit their labelling from the corresponding walk in the base graph, which then becomes the lettering of the type.

Recall that we defined the order of a potential walk, $(w; \vec{t})$ to be the opposite of its Euler characteristic. Since, this only depends on the number of edges and vertices and is stable when removing beaded paths, we can define the order of a form or a type to be the order of any of its associated potential walk and we denote it by $\text{ord}(w; \vec{t}) = \text{ord}(\mathcal{F}(w; \vec{t})) = \text{ord}(\mathcal{T}(w; \vec{t}))$.

Proposition 3.4.3. *For any positive integer r , there are only finitely many types \mathcal{T} of order less than r . We denote that finite set, $\text{Types}[r]$.*

Proof. Besides the initial vertex which might be of degree 1, all other vertices in a type are of degree at least 3. Applying this to the handshaking

lemma, we obtain

$$2|E_T| = \sum_{v \in V_T} \deg(v) \geq 1 + 3(|V_T| - 1)$$

and since we require that $|E_T| - |V_T| \leq r$ we have

$$|E_T| \leq 3r + 2 \quad \text{and} \quad |V_T| \leq 2r + 2$$

which implies that there are finitely many graphs on which to base a type of order at most r . There are clearly only finitely many ways to order the vertices and the edges as well as adding a lettering (since the alphabet is finite and fixed by the based graph B). \square

We are going to reorganize walk sums by types and forms.

Definition 3.4.4. Let $\mathcal{F} = \mathcal{F}(w; \vec{t})$ be a form associated to some potential walk $(w; \vec{t})$ with underlying oriented graph F . We say that a walk in F is *consistent in \mathcal{F}* if it is strictly non-backtracking, if it reaches every vertex and edge of F , and if it reaches them in the order specified by the orderings on \mathcal{F} .

Proposition 3.4.5. *Let $(w; \vec{t})$ be a potential walk and let $\mathcal{F} = \mathcal{F}(w; \vec{t})$ be its associated form. There is a natural one-to-one correspondence between*

- (1) *the set of potential walk classes $[w; \vec{t}]_n$ whose form is \mathcal{F} , and*
- (2) *the set of consistent walks in \mathcal{F} .*

Proof. This is immediate: A consistent walk uniquely defines a base walk and an equivalence class of trajectories and reciprocally. \square

Lemma 3.4.6. *Let \mathcal{T} be a type, with underlying graph T and let e_1 and e_2 be two directed edges in T . Define $\text{NB}(e_1, e_2, k)$ to denote all walks of length k from e_1 to e_2 in T . For each such walk, w and each edge, $e \in E_B$, of the graph B , let $a_e(w)$ be the number of times e appears in the walk w . Then*

for any polynomial $Q = Q(\{a_e\}_{e \in E_T})$ in the variables a_e , then we have that

$$\sum_{w \in \text{NB}(e_1, e_2, k)} Q(a_e(w))$$

is a B -polyexponential function in k .

Proof. This is proven just as in [Fri93]. \square

Definition 3.4.7. Fix a walk collection, \mathcal{W} , a type, \mathcal{T} with underlying graph T . For every form, \mathcal{F} , of type \mathcal{T} , and every $\vec{m} \in \mathbb{Z}_{\geq 1}^{E_T}$, let $W_{\mathcal{F}}(\mathcal{W}, k, \vec{m})$ be the number of consistent potential walk classes, $[w; \vec{t}]_n \in \mathcal{W}(k, n)$, of form \mathcal{F} , that traverse each edge, $f \in E_T$, m_f times. We say that \mathcal{W} is *multiplicity and length determined* if $W_{\mathcal{F}}(\mathcal{W}, k, \vec{m})$ is just a function of $\mathcal{T}, \vec{m}, \vec{k}$ where $\vec{k} = \vec{k}(\mathcal{F})$ is the lengths of the E_T -edges in the form \mathcal{F} , i.e.

$$W_{\mathcal{F}}(\mathcal{W}, k, \vec{m}) = W_{\mathcal{T}}(\vec{m}, \vec{k}(\mathcal{F})) \quad (3.5)$$

Example 3.4.8. If \mathcal{W} is the walk collection SNBC of all strictly non-backtracking closed walks, then it is multiplicity and length determined. Furthermore, it is a function of \vec{m} alone. Furthermore if the sub-collection, \mathcal{W}_0 , of SNBC of all walks, w , for which the number of edges in $\text{Graph}(w)$ is a prime number, then it is multiplicity and length determined; in fact, more generally, for every type, \mathcal{T} , with underlying graph T , say one is given a subset $\mathcal{K}(\mathcal{T}) \in \mathbb{Z}_{\geq 1}^{E_T}$; let \mathcal{W}_0 be the walk sub-collection of SNBC of walks w for which the form of the walk has vector of length, $\vec{k} = \vec{k}(\mathcal{F})$, on the E_T edges that lie in $\mathcal{K}(\mathcal{T})$, then the walk sub-collection, \mathcal{W}_0 , is multiplicity and length determined.

Definition 3.4.9. We say that a walk collection, \mathcal{W} , *decouples*, if for every type, \mathcal{T} with underlying graph T , the walk collection \mathcal{W} satisfies the following property: if two forms of type \mathcal{T} induce the same edge lengths on E_T , then the first form belongs to \mathcal{W} if and only if the second one does too. In other words, to each type, \mathcal{T} , there corresponds a subset of $\mathbb{Z}_{\geq 1}^{E_T}$ such that a form of type \mathcal{T} belongs to \mathcal{W} if and only if the induced vector of edge lengths on T is in that subset.

This allows us to state the main theorem of this section, namely an expansion of the walk sum in terms of types.

Theorem 3.4.10. *Given a walk collection \mathcal{W} that is multiplicity and length determined, and a positive integer r , we have an expansion, called the type-form expansion of \mathcal{W} :*

$$\text{WalkSum}_{<r}(\mathcal{W}, k, n) = \sum_{\mathcal{T} \in \text{Types}[r]} n^{-\text{ord}(\mathcal{T})} \sum_{i=0}^{r-\text{ord}(\mathcal{T})} Q_{\mathcal{T},i}(k) n^{-i} + \text{error}(r, n, k)$$

where the $Q_{\mathcal{T},i}$ are functions satisfying

$$Q_{\mathcal{T},i}(k) = \sum_{\substack{\vec{m}, \vec{k} \in \mathbb{Z}_{\geq 1}^{E_{\mathcal{T}}} \\ \vec{m} \cdot \vec{k} = k}} W_{\mathcal{T}}(\vec{m}, \vec{k}) P_i(\vec{k})$$

where the $P_i(\vec{k})$ are B -polyexponential functions, and $W_{\mathcal{T}}(\vec{m}, \vec{k})$ is the function given in equation 3.5. Finally, the error term satisfies a similar bound to theorem 3.3.5, that is

$$|\text{error}(r, n, k)| \leq ck^{4r} (d-1)^k n^{-r}$$

Proof. Given a form \mathcal{F} , we denote by $W_{\mathcal{F}}(\mathcal{W}, k, n)$ the number of potential walk classes, $[w; \vec{t}]_n$ of length k in the walk collection whose form is \mathcal{F} . The previous proposition shows that this is the same as the number of consistent walks of length k in the form.

Given a potential walk, $(w; \vec{t})$, its associated form, \mathcal{F} , contains information that is equivalent to the a_e and b_v and hence, $\mathbb{E}_{\text{Symm}}(w; \vec{t})_n$ only depends on the form. So we define $\mathbb{E}[\mathcal{F}]_n = \mathbb{E}_{\text{Symm}}(w; \vec{t})_n$. This allows to rewrite our walk sums as

$$\text{WalkSum}(\mathcal{W}, k, n) = \sum_{\mathcal{T}} \sum_{\mathcal{F} \in \mathcal{T}} W_{\mathcal{F}}(\mathcal{W}, k, n) \mathbb{E}[\mathcal{F}]_n$$

Further on, we will only sum over types of at most some order, so the sum will effectively be finite.

For a fixed type, \mathcal{T} , define $\mathbb{E}[\mathcal{T}]_{n,k}$ to mean

$$\mathbb{E}[\mathcal{T}]_{n,k} = \sum_{\mathcal{F} \in \mathcal{T}} W_{\mathcal{F}}(\mathcal{W}, k, n) \mathbb{E}[\mathcal{F}]_n$$

Since we usually do not work with more than one walk collection at a time, we drop its mention from the notation.

For a $\vec{k} = \{k_e\}_{e \in E_{\mathcal{T}}}$, let $\mathcal{T}(\vec{k})$ denote the set of all forms, \mathcal{F} , of type \mathcal{T} , such that the length of the edges of the forms is given by \vec{k} . For each edge $e \in E_{\mathcal{T}}$ of the type, fix an integer $m_e \geq 1$, and denote $\vec{m} = \{m_e\}_{e \in E_{\mathcal{T}}}$. This allows us to write

$$\mathbb{E}[\mathcal{T}]_{n,k} = \sum_{\vec{m} \cdot \vec{k} = k} W_{\mathcal{T}}(\vec{m}, \vec{k}) \sum_{\mathcal{F} \in \mathcal{T}(\vec{k})} \mathbb{E}[\mathcal{F}]_n$$

Since each of the $\mathbb{E}[\mathcal{F}]_n$ have a $1/n$ expansion, by adding expansions, we get an asymptotic expansion

$$\sum_{\mathcal{F} \in \mathcal{T}(\vec{k})} \mathbb{E}[\mathcal{F}]_n = n^{-\text{ord}(\mathcal{T})} \sum_{\mathcal{F} \in \mathcal{T}(\vec{k})} \left(P_0(\mathcal{F}) + \frac{P_1(\mathcal{F})}{n} + \dots \right)$$

where

$$P_i(\mathcal{F}) = p_i(\vec{a}(\mathcal{F}), \vec{b}(\mathcal{F}))$$

where the p_i are the expansion polynomials in equation 3.3. Note that the vector $\vec{b}(\mathcal{F})$ is an affine linear function of the variables $\vec{a}(\mathcal{F})$ for all forms \mathcal{F} of a given type \mathcal{T} , so we write

$$P_i(\mathcal{F}) = P_{i,\mathcal{T}}(\vec{a}(\mathcal{F}))$$

Rearranging the sum yields

$$\mathbb{E}_n[\mathcal{T}]_{n,k} = n^{-\text{ord}(\mathcal{T})} (Q_0(k) + Q_1(k)n^{-1} + \dots)$$

where

$$Q_i(k) = \sum_{\substack{\vec{m}, \vec{k} \\ \vec{m} \cdot \vec{k} = k}} W_{\mathcal{T}}(\vec{m}, \vec{k}) \sum_{\mathcal{F} \in \mathcal{T}(\vec{k})} P_{i, \mathcal{T}}(\vec{a}(\mathcal{F}))$$

For each form, \mathcal{F} , of type \mathcal{T} , let $A_{e,f}(\mathcal{F})$ be the number of occurrences of a label $e \in E_B$ on an edge $f \in E_T$. We have that

$$\vec{a}(\mathcal{F}) = \{A_{e,f}(\mathcal{F})\}_{e \in E_B, f \in E_T}$$

for all $e \in E_B$,

$$a_e(\mathcal{F}) = \sum_{f \in E_T} A_{e,f}(\mathcal{F})$$

and for a form $\mathcal{F} \in \mathcal{T}(\vec{k})$, we have that

$$k_f = \sum_{e \in E_B} A_{e,f}(\mathcal{F})$$

We claim that

$$\sum_{\mathcal{F} \in \mathcal{T}(\vec{k})} P_{i, \mathcal{T}}(\vec{a}(\mathcal{F}))$$

is polyexponential in k ; indeed, it suffices to show that for any non-negative integers, $s_{e,f}$, with $e \in E_B$, $f \in E_T$, we have

$$\sum_{\mathcal{F} \in \mathcal{T}(\vec{k})} \prod_{f \in E_T} \prod_{e \in E_B} (A_{e,f})^{s_{e,f}}$$

is polyexponential in \vec{k} . But since \mathcal{W} decouples, the above equals

$$\prod_{f \in E_T} \left(\sum_{w_f \in \text{NB}(e_1(f), e_2(f), k_f)} \left(\prod_{e \in E_B} (A_{e,f})^{s_{e,f}} \right) \right)$$

where e_1 and e_2 are determined by the lettering, which by Lemma 3.4.6 is just

$$\prod_{f \in E_T} P_f(k_f)$$

for some polyexponential functions, P_f , with bases in the spectrum of H_B .
Hence

$$\sum_{\mathcal{F} \in \mathcal{T}(\vec{k})} P_{i,\mathcal{T}}(\vec{a}(\mathcal{F}))$$

is polyexponential in k .

□

Chapter 4

Tangles and the Certified Trace

Tangles are low-probability events that prevent our asymptotic expansion of the expected trace to have B -Ramanujan coefficients. This was first observed by Friedman in [Fri08] and led him to define a *selective trace*, which despite being quite cumbersome, showed to be effective to settle the Alon Conjecture. In our work, we simplify his selective trace and develop a certified trace in this chapter. Before doing this, we review his arguments and revisit the language, which we have simplified as well.

In [Fri08], Theorem 2.12, Friedman shows that there exists an absolute constant (independent of d), C , such that the expansion functions, f_i , of Theorem 3.3.5, cannot be B -Ramanujan for all $i \leq C\sqrt{d} \log(d)$. This is due to the presence of rare events, called *tangles*, which we describe again in the next section. This observation led Friedman to develop a modified trace (the selective trace of [Fri08]) that essentially avoids those tangles. We here introduce another modified trace, the *certified trace*, which shows to be simpler to use than the selective trace. The main result of this chapter is Theorem 4.4.1, which shows that we can obtain a B -Ramanujan expansion of the certified trace of arbitrary length.

4.1 Tangles

We mainly use the theory developed by Friedman. We are modifying some of the language and notation, so we revisit the whole theory here for simplicity and clarity.

In [Fri08], Friedman shows that when d -regular graphs contain some specific subgraphs, their second largest eigenvalue cannot be bounded

by the desired bound of $2\sqrt{d-1}$. We review here his arguments and generalize them to the context of random covers of a d -regular base graph.

Theorem 4.1.1. *Fix $d \geq 3$, a d -regular base graph B and a connected graph ψ whose maximum degree is at most d . Then any covering graph G of degree n of the base graph B which contains ψ as a subgraph satisfies*

$$\rho_B^{\text{new}}(A_G) \geq \rho(A_{\text{Tree}_d(\psi)}) - o_n(1)$$

where $\text{Tree}_d(\psi)$ is the universal cover of d -regular graphs with a ψ -inclusion (for more details on how to construct this universal cover, see [Fri08] definition 3.12).

Proof. This is Theorem 4.2 in [Fri08]. We repeat here the main points of his proof for completeness. Fix an $\varepsilon > 0$, we will show that for n large enough, any graph G on n vertices that contains ψ has $\rho_B^{\text{new}}(A_G) \geq \rho(A_{\text{Tree}_d(\psi)}) - \varepsilon$. First, there is a finitely supported function on the vertices of $\text{Tree}_d(\psi)$ such that

$$\|Af\| \geq \|f\|(\rho - \varepsilon)$$

where A is the adjacency operator of $\text{Tree}_d(\psi)$ and without loss of generality, we can assume that f is non-negative, non zero and that

$$Af(v) \geq (\rho - \varepsilon)f(v) \quad \forall v \in V_{\text{Tree}_d(\psi)}$$

If we let $\pi: \text{Tree}_d(\psi) \rightarrow G$ denote the covering map, we can define π_*f to be the function on G defined by

$$(\pi_*f)(v) = \sum_{\pi(w)=v} f(w)$$

And show that

$$A_G(\pi_*f) \geq (\rho - \varepsilon)(\pi_*f)$$

So the Rayleigh quotient, \mathcal{R}_{A_G} of π_*f satisfies

$$\mathcal{R}_{A_G}(\pi_*f) \geq \rho - \varepsilon$$

On the other hand, the support, N , of π_*f is of size no greater than that of f and this size is bounded independently of G and n . Hence, we can conclude that if g is the characteristic function of the set of vertices of G at distance at least 2 to the support, N , of π_*f , then its Rayleigh quotient satisfies

$$\mathcal{R}_{A_G}(g) \geq d - o_n(1)$$

and since π_*f is orthogonal to both g and A_Gg , then we can use the Min-Max theorem and conclude the proof. \square

In most cases, the infinite tree $\text{Tree}_d(\psi)$ has spectral radius $2\sqrt{d-1}$ but in some rare cases the radius is larger than that. In what is dubbed a *curious theorem*, Friedman shows that this can be characterized in terms of Hashimoto eigenvalues. For completeness, we restate his result here.

Theorem 4.1.2. *Let $d \geq 3$ and let ψ be a connected graph whose maximum degree is at most d , then*

$$\begin{aligned} \rho_A(\text{Tree}_d(\psi)) = 2\sqrt{d-1} &\iff \rho_H(\psi) \leq \sqrt{d-1} \\ \rho_A(\text{Tree}_d(\psi)) > 2\sqrt{d-1} &\iff \rho_H(\psi) > \sqrt{d-1} \end{aligned}$$

Proof. This is Theorem 3.13 in [Fri08]. We repeat here the main points of his proof for completeness. Consider G , the variable-length graph based on the graph ψ that realizes $\text{Tree}_d(\psi)$. Then $\lambda_1(\text{Tree}_d(\psi))$, the spectral radius of the d -regular tree completion of the the graph ψ , is the inverse of the infimum of all the positive roots, z , of the equation $\det[I - Z_G(z)] = 0$ In other words, we have

$$\frac{1}{\lambda_1(\text{Tree}_d(\psi))} = \inf \{z \mid \det[I - Z_G(z)] = 0\}$$

Where $Z_G(z)$ is the adjacency matrix of G .

Lemma 4.1.3. *For any connected graph ψ and the variable length graph*

$G = \text{Tree}_d(\psi)$ we have

$$Z_G(z) = zA + \frac{dI - D}{d-1}S_d(z)$$

where $A = A_\psi$ is the adjacency matrix of the graph ψ , $D = D_\psi$ is its degree matrix and

$$S_d(z) = \frac{1 - \sqrt{1 - 4(d-1)z^2}}{2}$$

Lemma 4.1.4. *The largest irreducible eigenvalue of the graph ψ , $\lambda_{\text{Irred}}(\psi)$ is the inverse of the smallest root, y of the equation*

$$\det[I - yA + y^2(D - I)] = 0$$

Proof. This is a straight consequence of Bass' formula:

$$\det[H - \lambda I] = (\lambda^2 - 1)^{\chi(\psi)} \cdot \det[(\lambda^2 + d - 1)I - \lambda A]$$

where $\lambda_{\text{Irred}}(\psi)$ is the largest solution of the equation $\det[H - \lambda I] = 0$ by substituting $y = 1/\lambda$ we get the equivalence of the solutions. \square

Now consider the change of variables

$$y = y(z) = \frac{z}{1 - S_d(z)} \iff z = z(y) = \frac{y}{(d-1)y^2 + 1}$$

which is valid if

$$|z| < \frac{1}{2\sqrt{d-1}} \quad \text{and} \quad |y| < \frac{1}{\sqrt{d-1}}$$

Notice that this change of variable is increasing as long as it converges. Then we have

$$I - Z_G(z) = (1 - S_d(z))(I - yA + y^2(D - I))$$

which holds for

$$|z| < \frac{1}{2\sqrt{d-1}}$$

We can now finish the proof. First, take the case where $\lambda_1(\text{Tree}_d(\psi)) > 2\sqrt{d-1}$ then there is a z_0 such that

$$z_0 < \frac{1}{2\sqrt{d-1}}$$

which is a root of the equation $\det[I - Z_G(z)] = 0$. Let

$$y_0 = y(z_0) = \frac{z_0}{1 - S_d(z_0)}$$

which is clearly a root of the equation $\det[I - yA + y^2(D - I)] = 0$ and since $y_0 < (d-1)^{-1/2}$ we can conclude that

$$\lambda_{\text{irred}}(\psi) > \sqrt{d-1}$$

Finally, for the case where $\lambda_1(\text{Tree}_d(\psi)) = 2\sqrt{d-1}$ then this means that if z_0 is a root of $\det[I - Z_G(z)] = 0$ then we have that

$$\frac{1}{z_0} \leq 2\sqrt{d-1}$$

Assume by contradiction that $\lambda_{\text{irred}}(\psi) > \sqrt{d-1}$ then there exists a y_0 such that $y_0 < (d-1)^{-1/2}$ and that is a root of the equation

$$\det[I - yA + y^2(D - I)] = 0$$

But then, our correspondence of eigenvalues gives an associated eigenvalue z_0 such that

$$z_0 < \frac{1}{2\sqrt{d-1}}$$

which contradicts our hypothesis. Hence we can claim that

$$\lambda_{\text{irred}}(\psi) \leq \sqrt{d-1}$$

which concludes the proof of this property. \square

We formalize this with the following definition. Despite the appear-

ances, the case of graphs whose largest irreducible eigenvalue is precisely $\sqrt{d-1}$ is critical as well for reasons we will expose later on.

Definition 4.1.5. A connected graph ψ whose maximum degree is at most $d \geq 3$ is called a d -*tangle* (or simply a *tangle*) if

$$\rho(H_\psi) \geq \sqrt{d-1}$$

We say that ψ is d -*critical* if $\rho(H_\psi) = \sqrt{d-1}$. We denote the set of all tangles that occur in some element of the Broder-Shamir model by Tangle_B . Using the notation developed in section 2.6, we can write

$$\text{Tangle}_B = \{\psi \in \text{Occurs}(B) \mid \rho(H_\psi) \geq \sqrt{d-1}\}$$

The *order* of a tangle ψ is the opposite of its Euler characteristic, that is, we define the order to be the non-negative integer: $\text{ord}(\psi) = |E_\psi| - |V_\psi|$ and we let $\text{Tangle}_{r,B}$ be the subset of all tangles of order less than r , that is

$$\text{Tangle}_{r,B} = \{\psi \in \text{Tangle}_B \mid \text{ord}(\psi) < r\}$$

And finally, we consider the event, $\text{HasTangle}_{r,B}[n]$, where a random covering graph, G , contains a subgraph that is isomorphic to a tangle of order less than r , that is

$$\text{HasTangle}_{r,B}[n] = \{G \in \mathcal{C}_n(B) \mid \exists H < G, H \in \text{Tangle}_{r,B}\}$$

For the trace method to yield the Alon conjecture, one must then avoid these tangles. First, we show that if we restrict ourselves to tangles up to some fixed order, then there are only finitely many of them to consider.

Theorem 4.1.6. *Consider the set $\text{Tangle}_{r,B}$ of all tangles of order less than r , and let $\text{HasTangle}_{r,B}^{\min}$ be the minimal set of $\text{Tangle}_{r,B}$, that is the set of all tangles of order less than r that are minimal with respect to inclusion. Then the set $\text{HasTangle}_{r,B}^{\min}$ is finite.*

Proof. This is lemma 9.2 of [Fri08]. We repeat here the main points of his

proof for completeness. Assume by contradiction that the set is not finite. Then there must exist a type \mathcal{T} , with underlying graph T and an infinite sequence of tangles of type \mathcal{T} , which each correspond to a choice $\vec{k} \in \mathbb{Z}_{\geq 1}^{E_T}$ of lengths of the edges of T . Let us refer to that infinite sequence by $\{\vec{k}_i\}_{i \geq 1}$, without loss of generality, for each edge $f \in E_T$, we can assume that the corresponding sequence of lengths for this edge is either constant or tends to infinity. Furthermore, there must exist at least one edge whose length sequence tends to infinity. Denote by ψ_∞ this limiting graph, where we discard the edges with length tending to infinity. Since tangles have their Hashimoto spectral radius at least $\sqrt{d-1}$, the same must hold for the limiting graph and hence ψ_∞ is a tangle as well. Furthermore, the order of ψ_∞ is at most that of the tangle in the sequence and clearly is included in all of them, which contradicts their assumed minimality. \square

Definition 4.1.7. We will denote by $\mathbb{I}_{\text{TF}(r,B)[n]}$ the indicator function of the complement of the event $\text{Tangle}_{r,B}[n]$, that is

$$\mathbb{I}_{\text{TF}(r,B)[n]}(G) = 1 \iff G \notin \text{Tangle}_{r,B}[n]$$

A lot of work in this thesis consists of obtaining a B -Ramanujan expansion of $\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]} \text{Tr}(H_G^k)]$ for any choice of r , then our side-stepping lemma, described in chapter 5, will allow us to conclude the Alon conjecture.

Tangles are rare events in the sense of the following theorem.

Theorem 4.1.8. *Let ψ be a tangle of order r . Then the expected number of occurrences of ψ in an element G of $\mathcal{C}_n(B)$ is $n^{-r} + O_n(n^{-r-1})$ and the probability that at least one occurrence occurs is at most $n^{-r}/c + O_n(n^{-r-1})$ where c is a constant depending on the tangle.*

Proof. This is Theorem 4.7 in [Fri08]. We repeat here the main points of his proof for completeness. Let $V_\psi = \{u_1, \dots, u_s\}$ and for a tuple \vec{t} of distinct integers between 1 and n , let $\iota_{\vec{t}}$ denote the event that the map ι mapping u_i to the vertex at height t_i above the matching vertex of the base graph B , is an occurrence of ψ in $G \in \mathcal{C}_n(B)$. If a_e is the number of edges of ψ

labelled with the edge $e \in E_B$, then the event $\iota_{\vec{t}}$ involves setting a_e values of the corresponding permutations which will occur with probability at most

$$\prod_{e \in E_B} \frac{(n - a_e)!}{n!} = n^{|E_\psi|} + O_n \left(n^{-|E_\psi|-1} \right)$$

Since the sum of the a_e is $|E_\psi|$. For any vertex $v \in V_B$, denote by b_v the number of vertices of ψ in the fibre of v , then the number of choices for $\iota_{\vec{t}}$ is

$$\prod_{v \in V_B} \frac{n!}{(n - b_v)!} = n^{|V_\psi|} + O_n \left(n^{|V_\psi|-1} \right)$$

and so we can conclude that the expected number of occurrences of ψ is

$$n^{-\text{ord}(\psi)} + O_n \left(n^{-\text{ord}(\psi)-1} \right)$$

The remaining part of the proof is obtained by an inclusion-exclusion argument, observing that the order of subgraph is always at least that of the graph containing it (and the order is strictly less if the graph is properly contained); that if a graph has two connected components, then both must be of non-negative order; and that the if a tangle is made of two distinct copies of a smaller tangle, than its order is strictly larger than that of the smaller tangle. \square

We now comment on the probability that a random covering contains a tangle.

Proposition 4.1.9. *Consider the event $\text{HasTangle}_{r,B}^{\min}[n]$ of all the random graphs G which contain an element of $\text{HasTangle}_{r,B}^{\min}$. Then, for r sufficiently large enough we have*

$$\text{Prob}_n \left[\text{HasTangle}_{r,B}^{\min}[n] \right] = Cn^{-\eta}$$

for some $\eta = \eta_{\text{fund}}(B)$.

Proof. We can use Theorem 4.1.8 and obtain that this is true for $\eta_{\text{fund}}(B)$ being the smallest order of a tangle in $\text{HasTangle}_{r,B}^{\min}$. \square

Corollary 4.1.10. *We have an expansion*

$$\text{Prob}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) = 1] = 1 - q_1 n^{-1} - q_2 n^{-2} - \dots$$

where q_i are some constants.

Proof. We simply apply the inclusion-exclusion principle to proposition 4.1.9. □

4.2 Finitely Certifiable Partial Orders

In this section, we revisit cones in partially ordered sets. The main theorem of this section is Theorem 4.2.14, which shows that under some assumptions (that we call being *finitely certifiable*), some infinite sums can be dealt with using only finitely many *cone sums*. This result will be used in the following section to define a *certifiable trace*.

Definition 4.2.1. Let (P, \leq) be a partially ordered set. For $p \in P$ we define the *cone at p* to be

$$\text{Cone}(p) = \{t \in P \mid t \geq p\}$$

and we define the *open cone at p* to be

$$\text{Cone}^*(p) = \{t \in P \mid t > p\}$$

Definition 4.2.2. Let (P, \leq) be a partially ordered set and let $p, q \in P$ be two distinct elements of the set. We say that p and q have a *maximum* (or *least upper bound*) if there exists an element, denoted $\max(p, q)$, such that

$$\text{Cone}(p) \cap \text{Cone}(q) = \text{Cone}(\max(p, q))$$

We say that (P, \leq) *has maximums* if every pair of distinct elements $p, q \in P$ have a maximum.

Definition 4.2.3. A subset $Q \subseteq P$ of a partially ordered set (P, \leq) is an *upper set* if $q \in Q$ implies that $\text{Cone}(q) \subseteq Q$. We say that (P, \leq) has *finitely*

generated upper sets if every upper set, $Q \subseteq P$, has a finite number of minimal elements, which we denote $\text{Minimal}(Q)$; that is, if every upper set is the union of finitely many cones.

Definition 4.2.4. We say that a partially ordered set (P, \leq) is *Noetherian* if every infinite sequence of upper sets in P

$$Q_1 \subseteq Q_2 \subseteq \dots$$

stabilizes, that is there exists a positive integer N such that $Q_i = Q_j$ for all $i, j \geq N$.

We make this simple observation.

Theorem 4.2.5. *A partially ordered set is Noetherian if and only if it has finitely generated upper sets.*

Proof. Assume that (P, \leq) is Noetherian. Let $Q \subseteq P$ be an upper set and assume by contradiction that $\text{Minimal}(Q)$ is infinite. Then, there exists a countable sequence of distinct minimal elements q_1, q_2, \dots . Consider the sets

$$Q_i = \bigcup_{j=0}^i \text{Cone}(q_j)$$

Each Q_i is an upper set since it is a union of cones and clearly $Q_i \subseteq Q_{i+1}$ for all $i \geq 1$ but this sequence cannot stabilize since it is constituted of minimal elements, which contradicts (P, \leq) being Noetherian.

Assume now that (P, \leq) has finitely generated upper sets and consider an infinite sequence of upper sets $Q_1 \subseteq Q_2 \subseteq \dots$

Consider the set

$$Q = \bigcup_{i \geq 0} Q_i$$

Being the union of upper sets makes it an upper set as well, and hence it has a finite set of minimal elements, $\text{Minimal}(Q) = \{q_1, \dots, q_n\}$. Since there are finitely many of them, they must all belong to some Q_j for j large enough and thus have the sequence stabilize which shows that (P, \leq) is Noetherian. \square

Definition 4.2.6. A partially ordered set (P, \leq) is *finitely certifiable* if it has maximums and is Noetherian.

Proposition 4.2.7. Let (P, \leq) be a finitely certifiable partially ordered set, and let $Q \subseteq P$ be an upper set of P . Then (Q, \leq) is finitely certifiable.

Proof. Since Q is an upper set, we are guaranteed that the maximum of any two elements in Q is in Q as well. And similarly, if $Q_1 \subseteq Q_2 \subseteq \dots$ is a sequence of upper sets in Q , then since it stabilizes in P it does in Q as well. \square

Definition 4.2.8. Let (P_1, \leq_1) and (P_2, \leq_2) be two partially ordered set. Define their *product* to be the partially ordered set $(P_1 \times P_2, \leq)$ where

$$(p_1, p_2) \leq (q_1, q_2) \iff p_1 \leq_1 q_1 \text{ and } p_2 \leq_2 q_2$$

We want to show now that the finite product of finitely certifiable partially order sets is finitely certifiable as well. For this, we first show the following lemma.

Lemma 4.2.9. Let (P, \leq) be a Noetherian partially ordered set, and let $P' \subseteq P$ be any infinite subset of P . Then there is an infinite sequence p_1, p_2, \dots of elements of P for which

$$p_1 < p_2 < \dots$$

that is, for all i we have $p_i \neq p_{i+1}$ and $p_i \leq p_{i+1}$. In particular, there is no infinite subset of incomparable elements.

Proof. Let Q be the union of the cones of the elements of the set P' . Since Q is an upper set, it has a finite set of minimal elements, $\text{Minimal}(Q)$, which has to be a subset of P' . So there must exist at least one element of $\text{Minimal}(Q)$, say p_1 , whose cone meets infinitely many elements of P' . Let P'' be those elements of P' in the cone of p_1 that are not equal to p_1 . Applying the same argument to the set P'' yields an element, p_2 , in P'' whose cone meets an infinite number of elements of P'' and clearly we have that $p_1 < p_2$. Iterating this argument yields the desired infinite sequence. \square

Theorem 4.2.10. *Using the notation of definition 4.2.8, the product of two finitely certifiable partially ordered sets is finitely certifiable.*

Proof. Let Q be an upper set of (P, \leq) and assume by contradiction that the set $\text{Minimal}(Q)$ is infinite. This implies that there is an infinite sequence

$$(p_1^1, p_2^1), (p_1^2, p_2^2), (p_1^3, p_2^3), \dots$$

of distinct elements of $\text{Minimal}(Q)$. Then we can find an infinite sequence of distinct elements in either P_1 or P_2 . Without loss of generality, let us assume that the set

$$P' = \{p_1^1, p_1^2, \dots\} \subseteq P_1$$

is infinite and that its elements are all distinct, that is, for all $i \neq j$ we have $p_1^i \neq p_1^j$. Passing to a subsequence and using lemma 4.2.9, we can assume without loss of generality that we have

$$p_1^1 < p_1^2 < p_1^3 < \dots$$

But then we can conclude that for each $i \neq j$ we have $p_2^i \neq p_2^j$ or else one of (p_1^i, p_2^i) and (p_1^j, p_2^j) would not be minimal. Hence the set $\{p_2^1, p_2^2, \dots\} \subseteq P_2$ is an infinite set. Using lemma 4.2.9 again and passing to a subsequence, we can assume without loss of generality that we have

$$p_2^{\tau(1)} < p_2^{\tau(2)} < p_2^{\tau(3)} < \dots$$

for some permutation, τ , of the integers. But this is impossible: consider j_0 such that $\tau(j_0) = 1$ then for any $j > j_0$ we have

$$p_2^1 = p_2^{\tau(j_0)} < p_2^{\tau(j)}$$

and hence

$$(p_1^1, p_2^1) < (p_1^{\tau(j)}, p_2^{\tau(j)})$$

which contradicts the minimality of $(p_1^{\tau(j)}, p_2^{\tau(j)})$. □

Our main interest in this paper consists of the following example.

Example 4.2.11. Let $\mathbb{Z}_{\geq 0}^t$ be the set of non-negatives t -tuples and equip it with the partial order

$$(a_1, \dots, a_t) \leq (b_1, \dots, b_t) \iff a_i \leq b_i \quad \forall i$$

then this partially ordered set is finitely certifiable.

Proof. Using Theorem 4.2.10, we simply need to show that the partially ordered set $(\mathbb{Z}_{\geq 0}^t, \leq)$ is finitely certifiable. The existence of maximums is obvious. Since this set is actually well-ordered, an upper set is always of the form $[a, +\infty[$ for some element a which its unique minimal element; which implies that it is a Noetherian partially ordered set. \square

We now study functions on partially ordered sets and aim to conclude that when they are finitely certifiable, functions can be expressed as finite linear combinations.

Definition 4.2.12. Let (P, \leq) be a partially ordered set and let $L^1(P)$ denote the set of functions $f: P \rightarrow \mathbb{C}$ for which

$$\|f\|_1 = \sum_{p \in P} |f(p)|$$

is finite. For each function $f \in L^1(P)$ and element $p \in P$ define the *cone sum of f at p* to be

$$\hat{f}(p) = \sum_{q \in \text{Cone}(p)} f(q)$$

We now state an inclusion-exclusion theorem, giving Möbius-type coefficients.

Theorem 4.2.13. *Let (P, \leq) be a finitely certifiable partially ordered set and let $f \in L^1(P)$. Then for all $p \in P$, there are integers $\mu(p)$ for which*

- (1) $\mu(p) = 0$ for all but finitely many elements, $p \in P$, and

(2) for any function $f \in L^1(P)$ we have

$$\sum_{p \in P} f(p) = \sum_{p \in P} \mu(p) \hat{f}(p)$$

Specifically, $\mu(p)$ is the sum of the number of times p appears as the maximum of an odd number of minimal points of P , minus the number of times it appears as the maximum of an even (non-zero) number of minimal points.

Proof. Let $\varepsilon > 0$, we will show that

$$\left| \sum_{p \in P} f(p) - \sum_{p \in P} \mu(p) \hat{f}(p) \right| < \varepsilon$$

for some coefficients, $\mu(p)$, that we will describe below.

We begin by replacing f by a function which is very close but has the advantage of having a finite support, which we need in order to rearrange what would otherwise be infinite sums. Since

$$\sum_{p \in P} |f(p)|$$

is finite, there exists a function f_ε of finite support such that $\|f - f_\varepsilon\|_1 < \varepsilon$.

Now, we observe that $|\hat{f}(q) - \hat{f}_\varepsilon(q)| < \varepsilon$, since

$$|\hat{f}(q) - \hat{f}_\varepsilon(q)| \leq \sum_{p \in \text{Cone}(q)} |f(p) - f_\varepsilon(p)| \leq \|f - f_\varepsilon\|_1 < \varepsilon$$

Now since f_ε has a finite support, we can use the finite inclusion-exclusion principle on the cones of the minimal elements of P to obtain

$$\begin{aligned} \sum_{p \in P} f_\varepsilon(p) &= \sum_{1 \leq i_1 \leq n} \hat{f}_\varepsilon(p_{i_1}) - \sum_{1 \leq i_1 < i_2 \leq n} \hat{f}_\varepsilon(\max(p_{i_1}, p_{i_2})) + \dots \\ &\quad + (-1)^{n+1} \sum_{1 \leq i_1 < \dots < i_n \leq n} \hat{f}_\varepsilon(\max(p_{i_1}, \dots, p_{i_n})) \end{aligned}$$

This sum being finite, we can rearrange it and write as

$$\sum_{p \in P} \mu(p) \widehat{f}_\varepsilon(p)$$

where μ is a function on P whose support is included in the set of all possible combinations of maximums of minimal elements of P , that is

$$\text{support}(\mu) \subseteq \{\max(p_{i_1}, \dots, p_{i_k}) \mid k = 1, \dots, n \quad i_j \in \{1, \dots, n\}\}$$

As claimed in the theorem's statement, one can see that $\mu(p)$ is the sum of the number of times p appears as the maximum of an odd number of minimal points of P , minus the number of times it appears as the maximum of an even (non-zero) number of minimal points.

We now have

$$\begin{aligned} \left| \sum_{p \in P} f(p) - \sum_{p \in P} \mu(p) \widehat{f}_\varepsilon(p) \right| &\leq \left| \sum_{p \in P} f(p) - f_\varepsilon(p) \right| + \left| \sum_{p \in P} f_\varepsilon(p) - \mu(p) \widehat{f}_\varepsilon(p) \right| \\ &\quad + \left| \sum_{p \in P} \mu(p) \widehat{f}_\varepsilon(p) - \mu(p) \widehat{f}(p) \right| \\ &\leq \|f - f_\varepsilon\|_1 + 0 + \varepsilon \sum_{p \in P} |\mu(p)| < C\varepsilon \end{aligned}$$

For some fixed constant C which only depends on P and thus concludes the proof. \square

This allows to state the main theorem of this section. It follows immediately from the above theorems and its application suggests a very simple modified trace that we will describe in the next section.

Theorem 4.2.14. *Let (P, \leq) be a finitely certified partially ordered set, let Q be an upper set of P and let $f \in L^1(P)$. Then there exists Möbius coefficients, $\mu_Q: Q \rightarrow \mathbb{Z}$, such that*

$$\sum_{q \in Q} f(q) = \sum_{q \in Q} \mu(q) \widehat{f}(q)$$

where $\mu_Q(q) = 0$ for all but finitely many elements of Q .

Proof. This is immediate, applying Theorem 4.2.13 to Q itself since proposition 4.2.7 guarantees it is a finitely certifiable partially ordered set itself. \square

4.3 The Certified Trace

Definition 4.3.1. Consider the collection of sets

$$\text{CertSNBC} = \{\text{CertSNBC}(k, n)\}_{k, n \geq 1}$$

defined by

$$\text{CertSNBC}(k, n) = \{(w; \vec{t}) \in \text{SNBC}(k, n) \mid \rho_H(\text{Graph}(w; \vec{t})) < \sqrt{d-1}\}$$

that is, all the potential walks that are closed, strictly non-backtracking, of order less than r and which are not tangles.

Proposition 4.3.2. *The collection of sets CertSNBC is a walk collection.*

Proof. This is immediate since both the Hashimoto spectral radius is preserved up to symmetry and size increase. \square

Definition 4.3.3. We define the k th certified trace of size n , $\text{CertTr}(G, k)$, of a graph $G \in \mathcal{C}_n(B)$ to be the (k, n) -modified trace associated to CertSNBC and we write its associated walk sum as

$$\mathbb{E}_n[\text{CertTr}(G, k)] = \text{WalkSum}(\text{CertSNBC}(k, n))$$

and for any positive integer, $r \geq 1$, we define the *truncated certified trace* to be the associated truncated walk sum

$$\mathbb{E}_n[\text{CertTr}_{<r}(G, k)] = \text{WalkSum}_{<r}(\text{CertSNBC}(k, n))$$

which only sums certified potential walks of order less than r .

The whole idea of this definition is that for graphs which do not contain tangles, the truncated certified trace is a convenient approximation of the Hashimoto trace. We state this more precisely in the next proposition.

Proposition 4.3.4. *For any positive integer $r \geq 1$ and $k \leq n/2$ we have*

$$\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}(H_H^k)] = \mathbb{E}_n[\text{CertTr}_{<r}(G, k)] + O_n\left(k^{cr}(d-1)^k n^{-(r+1)}\right)$$

for some constant c .

Proof. Modifying Theorem 3.3.5 slightly, we have that any walk sum can be written as

$$\text{WalkSum}(\mathcal{W}, k, n) = f_0(k) + \frac{f_1(k)}{n} + \dots + \frac{f_r(k)}{n^r} + \text{error}(r, k)$$

for any choice of an integer $r \geq 1$ and any $k \leq n/2$, where

$$f_i(k) = \sum_{j=0}^{r-1} \sum_{(w; \vec{t}) \in \mathcal{W}^j(k)} p_{i-j}(w; \vec{t})$$

and where $\mathcal{W}^j(k)$ is the set of all equivalence classes of potential walks of length k and of order j ; with the error term satisfying

$$|\text{error}(r, k)| \leq ck^{4(r+1)}(d-1)^k n^{-(r+1)}$$

Our truncated certified trace only considers potential walks of order less than r and since we are only considering the case for random graphs, G , that are free of any tangles, we can conclude that the expansions of $\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}(H_H^k)]$ and $\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{CertTr}_{<r}(G, k)]$ have to agree up to the term of order r and so we are simply left with the error term as claimed. \square

We now observe that if a graph is a tangle, then making its edges longer (assume it is a variable-length graph) will eventually disappear the

Proposition 4.3.5. *Let T be the graph of a type \mathcal{T} . Then the set of vectors*

$$\{\vec{k} \in \mathbb{Z}_{\geq 1}^{|E_T|} \mid \text{VLG}(T, \vec{k}) \notin \text{Tangle}_B\}$$

is an upper set of $\mathbb{Z}_{\geq 1}^{|E_T|}$. In other words, if a choice of edge lengths, \vec{k} , for a type do not make the resulting graph a tangle, then any larger choice of edge lengths, $\vec{k}' \geq \vec{k}$ will guarantee that it isn't a tangle either.

Proof. This is fairly straightforward and is but only a slight modification of the argument of Theorem 4.1.6. Given a variable-length graph, making its edges longer will either leave its Hashimoto spectral radius unchanged or lower it. In both cases, if it wasn't a tangle in the first place, it will remain that way. \square

Corollary 4.3.6. *For each type, \mathcal{T} , there is a finite number of tangle certificates, that is, a finite family of vectors, $\vec{k}_{0,1}, \dots, \vec{k}_{0,s}$ such that*

$$\text{VLG}(T, \vec{k}) \notin \text{Tangle}_B \iff \vec{k} \geq \vec{k}_{0,i} \text{ for some } i = 1, \dots, s$$

where T is the underlying graph of the type \mathcal{T}

We now have all the elements required to state the main theorem of this section.

Theorem 4.3.7. *Consider the type-form expansion described in Theorem 3.4.10. The walk sum of the certified trace can be written as a finite linear combination of functions of the form*

$$\text{Term}[\mathcal{T}, \vec{k}_0, r](k) = \sum_{\vec{k} \geq \vec{k}_0} \sum_{\vec{m} \cdot \vec{k} = k} W_{\mathcal{T}}(\vec{m}, \vec{k}) P_i(\vec{k})$$

where $W_{\mathcal{T}}(\vec{m}, \vec{k})$ counts the number of strictly non-backtracking walks in the type \mathcal{T} with edge lengths defined by \vec{k} and that traverses each edge $f \in E_T$ of the type m_f times; and $P_i(\vec{k})$ is a polyexponential function.

Proof. Since CertSNBC is clearly multiplicity and length determined, we

can apply Theorem 3.4.10 to obtain an expansion

$$\text{CertTr}_{<r}(\mathcal{W}, k, n) = \sum_{\mathcal{T} \in \text{Types}[r]} n^{-\text{ord}(\mathcal{T})} \sum_{i=0}^{r-\text{ord}(\mathcal{T})} Q_{\mathcal{T},i}(k) n^{-i} + \text{error}(r, n, k)$$

where the $Q_{\mathcal{T},i}$ are functions satisfying

$$Q_{\mathcal{T},i}(k) = \sum_{\substack{\vec{m}, \vec{k} \in \mathbb{Z}_{\geq 1}^{E_{\mathcal{T}}} \\ \vec{m} \cdot \vec{k} = k}} W_{\mathcal{T}}(\vec{m}, \vec{k}) P_i(\vec{k})$$

where the $P_i(\vec{k})$ are B -polyexponential functions, and with error term

$$|\text{error}(r, n, k)| \leq ck^{4r} (d-1)^k n^{-r}$$

Now, we simply apply Theorem 4.2.14 to the upper set

$$\{\vec{k} \in \mathbb{Z}_{\geq 1}^{|E_{\mathcal{T}}|} \mid \text{VLG}(T, \vec{k}) \notin \text{Tangle}_B\}$$

and obtain that there are finitely many certificates, \vec{k}_0 , such that we can rewrite the $Q_{\mathcal{T},i}(k)$ as a finite linear combination of cone sums of the form

$$\text{Term}[\mathcal{T}, \vec{k}_0, r](k) = \sum_{\vec{k} \geq \vec{k}_0} \sum_{\vec{m} \cdot \vec{k} = k} W_{\mathcal{T}}(\vec{m}, \vec{k}) P_i(\vec{k})$$

□

4.4 Asymptotic Expansion

Our goal here is to prove that we have a B -Ramanujan expansion for the certified trace and prove the following theorem.

Theorem 4.4.1. *The terms of the form*

$$\text{Term}[\mathcal{T}, \vec{k}_0, r](k) = \sum_{\vec{k} \geq \vec{k}_0} \sum_{\vec{m} \cdot \vec{k} = k} W_{\mathcal{T}}(\vec{m}, \vec{k}) P_i(\vec{k}) \quad (4.1)$$

given in Theorem 4.3.7 are *B-Ramanujan functions*. In other words, the truncated certified trace admits a *B-Ramanujan*.

These types of sums are similar to those in Theorem 8.2 of [Fri08] and Theorem 2.18 of [Fri91]. In both those theorems, a key idea is to divide the vector \vec{m} by those components equal to 1 and those components that are at least 2. Here we notice that by a similar division, into components at most M and components at least $M + 1$, for a fixed value of M , we greatly simplify the computations. The role of tangles on the asymptotic expansion becomes clear as well.

Fix an integer M such that $M \geq 1/\varepsilon$ and fix a partition of the edges of the type

$$E_T = E_{T,1} \amalg E_{T,2}$$

We call the edges in $E_{T,1}$ the *short* and the edges in $E_{T,2}$ *long*. We write $\vec{m} \in ML(k, M) = ML(k, M, E_{T,1}, E_{T,2})$ if

- (1) $\vec{m} \in \{\vec{m} \mid \vec{k} \cdot \vec{k} = k, \vec{k} \geq \vec{k}_0, \text{ for } \vec{k}_0 \text{ a certificate}\},$
- (2) $m_f \leq M$ if $f \in E_{T,1},$
- (3) $m_f \geq M + 1$ if $f \in E_{T,2}.$

and so we can rewrite equation (4.1) as

$$\text{Term}[\mathcal{T}, \vec{k}_0, r](k) = \sum_{E_{T,1} \amalg E_{T,2} = E_T} \sum_{(\vec{m}, \vec{k}) \in ML(k, M, E_{T,1}, E_{T,2})} W_{\mathcal{T}}(\vec{m}) q_i(\vec{k})$$

For each $(\vec{m}, \vec{k}) \in ML(k, M)$ define

$$K_1(\vec{m}, \vec{k}) = \sum_{f \in E_{T,1}} m_f k_f \quad \text{and} \quad K_2(\vec{m}, \vec{k}) = \sum_{f \in E_{T,2}} m_f k_f \quad (4.2)$$

and so it follows that $K_1(\vec{m}, \vec{k}) + K_2(\vec{m}, \vec{k}) = k$ for all $(\vec{m}, \vec{k}) \in ML(k, M)$. Since there are finitely many partitions of the edges of the type, we only

need to consider expressions of the form

$$\text{Term}(k, E_{T,1}, E_{T,2}) = \sum_{(\vec{m}, \vec{k}) \in ML(k, M)} W_{\mathcal{T}}(\vec{m}) q_i(\vec{k}) \quad (4.3)$$

and show that they are B -Ramanujan functions, which would prove our Theorem 4.4.1

Theorem 4.4.2. *Let $\mathcal{T} \in \text{Types}[r]$ be a type of order less than r and let $(E_{T,1}, E_{T,2})$ be a partition of the edges of the type. Then for any $\varepsilon > 0$ and integer $M \geq 1/\varepsilon$ we have that the function $\text{Term}(k, E_{T,1}, E_{T,2})$, as described above, is B -Ramanujan.*

Proof. We distinguish two cases for this proof:

(1) When $E_{T,1} = \emptyset$

(2) When $E_{T,1} \neq \emptyset$

So, let us start with the first case: we consider the term in (4.3), in the case where $E_{T,1} = \emptyset$ and $E_{T,2} = E_T$. For every $(\vec{m}, \vec{k}) \in ML(k, M)$ we have

$$W_{\mathcal{T}}(\vec{m}) \leq Ck^C(d-1)^{k/2},$$

since each walk in $W_{\mathcal{T}}(\vec{m})$ gives a walk of length k in $\text{Path}(T, \vec{k})$, which corresponds to walk of length at most k in $\text{Path}(T, \vec{k}^0)$, and the number of such walks is bounded by $Ck^C(d-1)^{k/2}$ since the largest Hashimoto eigenvalue of $\text{Path}(T, \vec{k}^0)$ is at most $(d-1)^{-1/2}$.

Now we note the following crude bound.

Lemma 4.4.3. *The number of pairs (\vec{m}, \vec{k}) for which $\vec{m}, \vec{k} \geq \vec{1}$ and $\vec{m} \cdot \vec{k} = k$ is at most*

$$\binom{k + |E_T| - 1}{|E_T| - 1} k^{|E_T|}.$$

Proof. For each (\vec{m}, \vec{k}) , set $s_f = m_f k_f$ for each $f \in E_T$. Then the sum of the s_f is k , and there are $\binom{k + |E_T| - 1}{|E_T| - 1}$ ways of representing k as the sum of integers $\{s_f\}_{f \in E_T}$ with $s_f \geq 0$. Furthermore, each s_f can be represented as at most k products $m_f k_f$, with $m_f, k_f \geq 1$. \square

Finally, we notice that if $m_f \geq M + 1$ for all $f \in E_T$, and $\vec{m} \cdot \vec{k} = k$, then we have

$$S = \sum_{f \in E_T} k_f \leq k/(M + 1),$$

and so

$$P(\vec{k}) \leq \prod_{f \in E_T} (d - 1)^{k_f} C k_f^C \leq C' k^{C'} (d - 1)^{\sum_{f \in E_T} k_f} \leq C' k^{C'} (d - 1)^{k/(M+1)}.$$

It follows that, in the notation of (4.3), we have

$$\begin{aligned} |\text{Term}(\emptyset, E_T)| &\leq |ML(k, M, \emptyset, E_T)| \max_{\vec{m} \in ML(k, M, \emptyset, E_T)} |W_{\mathcal{T}}(\vec{m})| \max_{\vec{k} \in ML(k, M, \emptyset, E_T)} P(\vec{k}) \\ &\leq C k^C (d - 1)^{k/2} C k^C C' k^{C'} (d - 1)^{(k/2) + (k/(M+1))}. \\ &\leq C'' k^{C''} (d - 1)^{k((1/2) + \varepsilon)}. \end{aligned}$$

It follows that for any $\tilde{\varepsilon} > 0$ and some (new) constant, C , and sufficiently large M we have

$$|\text{Term}(\emptyset, E_T)| \leq C k^C (d - 1 + \tilde{\varepsilon})^{k/2}.$$

It suffices to give a similar bound for $\text{Term}(E_{T,1}, E_{T,2})$ for the other partitions of E_T as $E_{T,1} \amalg E_{T,2}$. But this case alone shows how the certification seems essential, in that if $W_{\mathcal{T}}(\vec{m})$ could be as large as $(d - 1 + \delta)^{k/2}$ for some fixed $\delta > 0$, we could not give a good enough bound just on $\text{Term}(\emptyset, E_T)$.

A Specialized Bound For $W_{\mathcal{T}}(\vec{m})$

To bound $\text{Term}(E_{T,1}, E_{T,2})$ when $E_{T,1}$ is not empty, we will use Theorem 2.8.4. To do so we would like to bound $W_{\mathcal{T}}(\vec{m})$ for each $(\vec{m}, \vec{k}) \in ML(k, M)$, in terms of K_1, K_2 given by (4.2). Specifically, we which for m_f fixed for $f \in E_{T,1}$, gives a bound for $W_{\mathcal{T}}(\vec{m})$ in terms of the exponent K_2 , rather than k . Here is what we will need.

Lemma 4.4.4. *Fix $\mathcal{T}, \vec{k}^0, M$, and a partition $E_{T,1} \amalg E_{T,2}$ of E_T . Then there*

exists a constant, C , for which

$$W_{\mathcal{T}}(\vec{m}) \leq CK_2^C (d-1)^{K_2/2}$$

provided that $(\vec{m}, \vec{k}) \in ML(k, M, E_{T,1}, E_{T,2})$

Proof. Consider a walk, w , in \mathcal{T} . Such a walk traverses a bounded number of edges in $E_{T,1}$, namely at most $B = M|E_{T,1}|$ edges. Let w^0 be the walk in $E_{T,2}$ edges taken before the first $E_{T,1}$ edge is taken in w , and for $1 \leq i \leq B$, let w^i be the walk taken between the i -th and $(i+1)$ -th occurrence of an $E_{T,1}$ edge in w . (Some of the w^i will be empty when w traverses two or more $E_{T,1}$ edges in a row; and some of the w^i will be empty if w traverses fewer than B edges of $E_{T,1}$.) Let K_2^i denote the number of edges traversed in w^i when transferred to the graph $\text{Path}(T, \vec{k})$. First, we have

$$K_2^0 + K_2^1 + \cdots + K_2^B = K_2.$$

Second, for fixed K_2^i , the number of walks in $\text{Path}(T, \vec{k})$ of length at most K_2^i is bounded by

$$C(K_2^i)^C (d-1)^{K_2^i/2} \quad (4.4)$$

Finally, between w^i and w^{i+1} , we traverse one edge of $E_{T,1}$, which can happen in at most a constant number of ways (namely one less than the maximum degree of T). It follows that

$$W_{\mathcal{T}}(\vec{m}) \leq \sum_{\substack{K_2^i \geq 0 \\ K_2^0 + \cdots + K_2^B = k}} C^B \prod_{i=0}^B [C(K_2^i)^C (d-1)^{K_2^i/2}], \quad (4.5)$$

where the C^B term reflects the B choices of $E_{T,1}$ element, and the product over i comes from the (4.4). We bound the right-hand-side of (4.5) as follows:

$$\prod_{i=0}^B [C(K_2^i)^C (d-1)^{K_2^i/2}] \leq (CK_2^C)^{B+1} (d-1)^{(K_2^0 + \cdots + K_2^B)/2} \leq C' K_2^{C'} (d-1)^{K_2/2}.$$

Finally the number of ways of writing K_2 as a sum of the K_2^i , i.e., $B + 1$ non-negative integers is exactly $\binom{K_2+B}{B+1}$, a polynomial in K_2 . Hence (4.5) yields

$$W_{\mathcal{T}}(\vec{m}) \leq \binom{K_2+B}{B+1} C' K_2^{C'} (d-1)^{K_2/2}.$$

□

It remains to show an asymptotic expansion for $W_{\mathcal{T}}(\vec{m})P(\vec{k})$ for B -Ramanujan P . Let us partition \vec{m} and \vec{k} by the $E_{T,1} \amalg E_{T,2}$ partition; namely, let

$$\vec{k}^i = \{k_f\}_{f \in E_{T,i}},$$

so we may write $\vec{k} = (\vec{k}^1, \vec{k}^2)$, and similarly $\vec{m} = (\vec{m}^1, \vec{m}^2)$. It suffices to show that for $P_1(\vec{k}^1)$ and $P_2(\vec{k}^2)$ polyexponentials, we have that

$$g(E_{T,1}, E_{T,2}, k) = \sum_{(\vec{m}, \vec{k}) \in ML(k, M, E_{T,1}, E_{T,2})} W_{\mathcal{T}}(\vec{m}) P_1(\vec{k}^1) P_2(\vec{k}^2)$$

is B -Ramanujan. So for any integer, K_1 , and any $\vec{m}^1 \in \{1, \dots, M\}^{E_{T,1}}$, let

$$Q_1(K_1; \vec{m}^1) = \sum_{\substack{\vec{m}^1 \cdot \vec{k}^1 = K_1 \\ \vec{m}^1 \geq \vec{1}, \vec{k}^1 \geq \vec{k}^{1,0}}} P_1(\vec{k}^1),$$

and

$$Q_2(K_2) = \sum_{\substack{\vec{m}^2 \cdot \vec{k}^2 = K_2 \\ \vec{m}^2 \geq \vec{1}, \vec{k}^2 \geq \vec{k}^{2,0}}} W_{\mathcal{T}}((\vec{m}^1, \vec{m}^2)) P_2(\vec{k}^2).$$

We have Q_1 is a B -Ramanujan, and Q_2 is of growth at most $(d-1+\varepsilon)^{1/2}$. Hence, for each $\vec{m}^1 \in \{1, \dots, M\}^{E_{T,1}}$ we have $Q_1 * Q_2$ is B -Ramanujan. Hence the sum over all \vec{m}^1 is B -Ramanujan.

This completes the proof of Theorem 4.4.2 and with it, the proof of Theorem 4.4.1 □

Chapter 5

The Side-Stepping Lemma

In this chapter, we prove a new version of the side-stepping lemma in [Fri08]. We improve this lemma to the more general case where we consider polyexponential functions with bases on all the Hashimoto eigenvalues strictly greater than $(d - 1)^{1/2}$ which will allow us to use the B -Ramanujan expansions that we have to conclude theorem 1.2.3.

The key idea is to use the fact that we can have an expansion for several consecutive values of k and apply the shift operator, described in the next section, several times. This side-stepping lemma is stated for an abstract collection of finite probability spaces.

5.1 The Shift Operator

Let S denote the *shift operator in k* , meaning the operator on functions, $f(k)$, defined on non-negative integers, k , taking f to

$$(Sf)(n, k) = f(n, k + 1)$$

For any integer $i \geq 0$ we let S^i denote the i -fold application of S , so that

$$(S^i f)(n, k) = f(n, k + i)$$

For any polynomial (with real or complex coefficients),

$$Q(z) = q_0 + q_1 z + \cdots + q_t z^t$$

we define $Q(S)$ in the natural way, i.e.,

$$Q(S) = q_0 S^0 + q_1 S^1 + \cdots + q_t S^t$$

The utility of polynomials in S is due to the following simple observations.

Proposition 5.1.1. *Let $Q(z)$ be a polynomial, then for $k \geq 0$ we have*

$$(1) (Q(S))(\mu^k) = Q(\mu)\mu^k \text{ for any } \mu \in \mathbb{C},$$

$$(2) (S - 1)(k^D) = O(k^{D-1}),$$

$$(3) (S - 1)^D(f) \equiv 0 \text{ for any function } f(k) = \mu^k p(k) \text{ where } \mu \in \mathbb{C} \text{ and } p \text{ is a polynomial of degree at most } D - 1.$$

Proof. (1) and (2) are straightforward and (3) is a consequence of the following identity,

$$\sum_{k=0}^N \binom{N}{k} (-1)^k k^p = 0$$

for any $N \geq 0$ and any $p < N$. □

When $h(k)$ is a function for which we do not have a simple expression, but is rather bounded by a polyexponential function (such as in our "error terms"), we employ the following crude bound.

Lemma 5.1.2. *Let $h(k)$ be a function for which*

$$|h(k)| \leq C\tau^k k^C,$$

for some positive real numbers C and τ with $\tau \geq 1$. Let $Q(x)$ be any complex polynomial of degree at most $D - 1$. Then, for $k \geq 0$, we have

$$|Q(S)h(k)| \leq CD\|Q\|\tau^{k+D}(k + D)^C$$

Where $\|Q\|$ denotes the largest absolute value of Q 's coefficients. In par-

icular, for fixed Q and τ we have for all $k \geq 0$

$$|Q(S)h(k)| \leq C' \tau^k k^{C'} \quad (5.1)$$

for some constant C' .

Proof. Denote

$$Q(x) = q_0 + q_1x + \cdots + q_{D-1}x^{D-1}$$

and for any non-negative integer i we clearly have

$$|S^i h(k)| \leq C \tau^{k+i} (k+i)^C \leq C \tau^{k+D} (k+D)^C$$

Hence

$$\begin{aligned} |Q(S)h(k)| &\leq \sum_{i=0}^{D-1} |q_i| |S^i h(K)| \\ &\leq D \|Q\| C \tau^{k+D} (k+D)^C \end{aligned}$$

Clearly the last equation of the lemma, (5.1), follows. \square

5.2 Statement of the Side-Stepping Lemma

The main lemma of this section, Lemma 5.2.5, generalizes the Side-Stepping Lemma of [Fri08], both of which can be viewed as abstract lemmas in probability theory. Of course, these abstract lemmas are motivated by our trace methods, and we will need Lemma 5.2.5 to prove the Hashimoto version of the generalized Alon conjecture for all base graphs, B , which are d -regular. Also, Lemma 5.2.5, especially (5.4), is, arguably, more direct and simpler than the Side-Stepping Lemma in [Fri08].

This chapter involves a number of positive, real constants, depending on various parameters, that need to be chosen sufficiently large; we will generally use the letters C, C', C'' , for various of these constants, when no confusion is likely to arise.

Definition 5.2.1. By a *finite probability space* we mean a pair, (Ω, ν) , where Ω is a finite set, and a probability measure, ν , which we view as a function $\nu: \Omega \rightarrow \mathbb{R}$, such that $\nu(\omega) \geq 0$ for each $\omega \in \Omega$, and

$$\sum_{\omega \in \Omega} \nu(\omega) = 1.$$

We will view ν as giving a measure to each subset $E \subset \Omega$ in the usual way, i.e.,

$$\nu(E) = \sum_{\omega \in E} \nu(\omega).$$

By a *complex random variable* on (Ω, ν) we mean a function $\mu: \Omega \rightarrow \mathbb{C}$; similarly we define a *real random variable*.

We often write

$$\text{Prob}_\Omega[E] \text{ for } \nu(E)$$

and, for a random variable, $\mu = \mu(\omega)$,

$$\mathbb{E}_{\omega \in \Omega}[\mu(\omega)] \text{ for } \sum_{\omega \in \Omega} \nu(\omega)\mu(\omega)$$

As such, the measure ν will often be omitted in notation, being implicit in the notation $\text{Prob}_\Omega[E]$ and $\mathbb{E}_{\omega \in \Omega}[\mu(\omega)]$.

Definition 5.2.2. Let τ_0, τ_1, γ be positive real numbers for which $\tau_0 < \tau_1$; let L be a finite collection of real numbers, each of whose absolute value is strictly larger than τ_0 and at most τ_1 . By an *abstract partial trace with parameters* $(\tau_0, \tau_1, \gamma, L)$ we mean the following data for each positive integer, n :

1. a finite probability space, (Ω_n, ν_n) ;
2. complex random variables $\mu_i = \mu_i(\omega)$ on (Ω_n, ν_n) such that for all $i = 1, \dots, \gamma n$ and $\omega \in \Omega_n$ we have

$$\mu_i(\omega) \in R_1 \cup R_2,$$

where

$$R_1 = \{z \in \mathbb{C} \mid |z| \leq \tau_0\}, \quad R_2 = \{z \in \mathbb{R} \mid \tau_0 < |z| \leq \tau_1\};$$

3. there are polyexponentials $P_0(k), P_1(k), \dots$ with bases in L , for which for any integer $r > 0$ we have

$$\sum_{\omega \in E_n} \nu_n(\omega) \sum_{i=1}^{\gamma n} \mu_i^k(\omega) = P_0(k) + P_1(k)n^{-1} + \dots + P_{r-1}(k)n^{1-r} + \text{err}_r(n, k) \quad (5.2)$$

for all positive integers k, n with $k \leq n/2$, where $\text{err}_r(n, k)$ is a function such that for any $\varepsilon > 0$ there is a constant, C (depending on ε and r) for which

$$|\text{err}_r(n, k)| \leq Ck^C \left(\tau_1^k n^{-r} + (\tau_0 + \varepsilon)^k \right).$$

Example 5.2.3. We will apply the above setting to $\Omega_n = \mathcal{C}_n(B)$ with the following parameters when B is d -regular: $\tau_0 = (d-1)^{1/2}$, $\tau_1 = d-1$, $\gamma = |E_B^{\text{dir}}|$; the choice of $\mu_i(\omega)$ is a bit intricate: we will fix an $\varepsilon' > 0$ and take the $\mu_i(\omega)$ be either (1) all zero, if G contains a (d, ε') -tangle, or (2) the Hashimoto eigenvalues of G , except that the old Hashimoto eigenvalues will be set to zero. Note that the ε' in this paragraph is different from the ε in Lemma 5.2.5.

We remark that above setting essentially permits the random variables $\mu_i(\omega)$, to range over $i = 1, \dots, f(n)$ for any function, $f(n)$, bounded by γn for some fixed γ ; this can be done simply by introducing new "dummy" random variables, $\mu_i(\omega) = 0$ for $i = f(n) + 1, \dots, \gamma n$.

We shall prove a lemma which will apply (5.2) for numerous values of k to show that it is "unlikely" for $\omega \in \Omega_n$ that some $\mu_i(\omega)$ is larger than $\tau_0 + \varepsilon$ in absolute value but not near any $\ell \in L$. We shall also obtain important information on the "dominant" part of the $P_i(k)$ corresponding to the ℓ^k term for each $\ell \in L$. To describe this, we need some further terminology.

Definition 5.2.4. Consider an abstract partial trace as defined above with parameters (τ_0, τ_1, C, L) and with Ω_n, ν_n, μ_i as above. For any $\varepsilon, \delta > 0$ and

integer n , set

$$\text{Out}(\delta, \varepsilon) = \{z \in \mathbb{R} \mid \tau_0 + \varepsilon < |z| \leq \tau_1 \text{ and } |z - \ell| > \delta \text{ for all } \ell \in L\}$$

and

$$\text{Exception}_n(\delta, \varepsilon) = \{\omega \in \Omega_n \mid \mu_i(\omega) \in \text{Out}_n(\delta, \varepsilon) \text{ for some } i\}$$

and

$$\text{AbsoluteException}_n(\varepsilon) = \{\omega \in \Omega_n \mid |\mu_i(\omega)| > \tau_0 + \varepsilon \text{ for some } i\}.$$

Also set

$$\text{Near}_n(\ell, \delta) = \mathbb{E}_{\omega \in \Omega_n} \left[\left| \{i \mid |\mu_i(G) - \ell| \leq \delta\} \right| \right],$$

i.e., the expected number of $\mu_i(\omega)$ which lie within δ of ℓ .

Lemma 5.2.5 (Improved Side-Stepping Lemma). *Consider an abstract partial trace with parameters (τ_0, τ_1, C, L) as above, with $\Omega_n, \nu_n, \mu_i, P_i$ as above. For any $\alpha, \varepsilon > 0$ there exists a $\theta > 0$ for which*

$$\text{Prob}_n[\text{Exception}_n(n^{-\theta}, \varepsilon)] \leq n^{-\alpha} \quad (5.3)$$

for n sufficiently large.

Furthermore, write

$$P_i(k) = \sum_{\ell \in L} p_{\ell,i}(k) \ell^k,$$

i.e., let $p_{\ell,i}(k)$ be the polynomial coefficient of ℓ^k in $P_i(k)$. Assume that for some j and $\ell \in L$ we have that

$$p_{\ell,0}(k) = \cdots = p_{\ell,j-1}(k) = 0.$$

Then $p_{\ell,j}(k)$ is a constant (i.e., independent of k), and this constant is the limit

$$p_{\ell,j} = \lim_{n \rightarrow \infty} n^j \text{Near}_n(\ell, n^{-\theta}) \quad (5.4)$$

for any $\theta > 0$ sufficiently small (i.e., there is a $\theta_0 > 0$ such that (5.4) holds for any θ with $0 < \theta < \theta_0$).

Finally, if j is any positive integer for which

$$P_0(k) = \dots = P_{j-1}(k) = 0,$$

then for each $\varepsilon > 0$ there is a $C = C_\varepsilon$ for which

$$\text{Prob}_n[\text{AbsoluteException}_n(\varepsilon)] \leq C_\varepsilon n^{-j}. \quad (5.5)$$

Lemma 5.2.5 will be proven in the sections that follow. The proofs are based on the simple idea of applying certain polynomials of the "shift operator in k " to (5.2). The basic idea behind these proofs is very simple: we let S denote the "shift operator in k ," taking a function, $f(n, k)$, to the shifted function $(Sf)(n, k) = f(n, k+1)$; we begin our proof by establishing (5.3), by applying

$$Q(S) = \prod_{\ell \in L} (S - \ell)^D$$

to both sides of (5.2), where D is an even integer that is sufficiently large to annihilate the $P_i(k)$ for $i = 0, \dots, r-1$ (this is explained in the next section); we then prove the second part, regarding the $p_{\ell,i}(k)$, by applying, for each $\ell \in L$,

$$Q_{\ell,D}(S) = \prod_{\ell' \in L, \ell' \neq \ell} (S - \ell')^D$$

to both sides of (5.2), to isolate the effect of the μ_i near ℓ .

A very similar set of ideas appears in [Fri08], Chapter 11, where the side-stepping lemma there is proven only in the case $L = \{\tau_1\}$, which suffices for random d -regular graphs, i.e., base graph $B = W_{d/2}$ or B being one vertex and d half-loops. The proof of Lemma 5.2.5 is primarily complicated by the fact that L can have two or more elements, unlike the case in [Fri08].

5.3 Proof of the First Exception Bound

In this section we will prove (5.3). In other words, fix an $\varepsilon > 0$; we will show that for any $\alpha > 0$, there exists a $\theta = \theta(\alpha) > 0$ for which (5.3) holds. First, let us describe $\theta = \theta(\alpha)$ explicitly, in rather unmotivated terms.

Let ϕ be any real number with $\phi > 2\alpha$. Let ϕ', ϕ'' be any real numbers satisfying

$$\phi' > \frac{\phi + 1}{\log \frac{\tau_0 + \varepsilon}{\tau_0 + (\varepsilon/2)}} \quad (5.6)$$

and

$$\phi'' > \frac{\phi + 2}{\log \frac{\tau_0 + \varepsilon}{\tau_0}}. \quad (5.7)$$

Let $\rho > 0$ be any real number for which

$$\rho < (1/2) \log(\tau_1/\tau_0). \quad (5.8)$$

Let r be any positive integer for which

$$r/2 > \max(\phi, \phi'/\rho, \phi''/\rho). \quad (5.9)$$

let P_0, \dots, P_{r-1} be as in Definition 5.2.2, and D be an even integer bounding the polynomial degree of the $p_{\ell,i}$ ranging over all $\ell \in L$ and $i = 0, \dots, r-1$; we take θ to be any positive real number for which

$$\theta < \alpha/(|L|D); \quad (5.10)$$

hence θ , and $D, r, \rho, \phi, \phi', \phi''$, depend only on $\alpha, \varepsilon, \tau_0, \tau_1, L$. So θ can be viewed as a function depending only on $\alpha, \varepsilon, \tau_0, \tau_1, L$. For the rest of this section we will prove that this value of θ satisfies (5.3). even integer.

Our general approach is to note that

$$Q(S) = \prod_{\ell \in L} (S - \ell)^D,$$

satisfies $Q(S)P_i(k) = 0$ for all i ; we shall apply $Q(S)$ to both sides of (5.2).

For the rest of this section, the letters C, C', C'' will refer to various constants that are independent of k and n , but may depend on any of $\alpha, \varepsilon, \tau_0, \tau_1, L$, and the above $\phi, \phi', \phi'', \rho, r, D, Q, \theta$, which are functions of $\alpha, \varepsilon, \tau_0, \tau_1, L$.

Let us begin with a few immediate remarks about $Q(S)$:

1. for $\mu \in \mathbb{R}$, $Q(\mu)$ is real and non-negative;
2. we have $|Q(\mu)|$, for any μ with $|\mu| \leq \tau_1$ is bounded by a constant depending on Q and τ_1 ;
3. for $\mu \notin B_\delta(L)$ we have

$$|Q(\mu)| \geq \delta^{L|D|};$$

4. for any real $\tau > 0$, if $h(k) = \mu^k$ with $|\mu| \leq \tau$, then

$$|Q(S)h(k)| \leq C\tau^k$$

for some C depending only on Q ;

5. furthermore, if $h(k)$ is any function bounded by $Ck^C\tau^k$ for some C , then we have

$$|Q(S)h(k)| \leq C'k^{C'}\tau^k$$

for a constant, C' , depending only on C and Q .

We are now ready to apply $Q(S)$ to both sides of (5.2). Beginning the with left-hand-side, we write:

$$Q(S)\mathbb{E}_{\omega \in \Omega_n} \left[\sum_{i=1}^{\gamma n} \mu_i^k(\omega) \right] = \mathbb{E}_{\omega \in \Omega_n} \left[\sum_{i=1}^{\gamma n} Q(S)\mu_i^k(\omega) \right];$$

note that for each i and ω for which $\mu_i(\omega) \notin \mathbb{R}$ we have $|\mu_i(\omega)| \leq \tau_0$, and hence

$$|Q(S)(\mu_i^k(\omega))| \leq C'\tau_0^k,$$

where C' is independent of n, k . Since there are γn values of i in the above, we have

$$\begin{aligned} \sum_{i=1}^{\gamma n} Q(S)\mu_i^k(\omega) &\geq \sum_{i, \mu_i \in \mathbb{R}} Q(S)\mu_i^k(\omega) - \sum_{i, \mu_i \notin \mathbb{R}} |Q(S)\mu_i^k(\omega)| \\ &\geq \sum_{i, \mu_i \in \mathbb{R}} Q(S)\mu_i^k(\omega) - C'' n \tau_0^k \end{aligned}$$

for some constant C'' independent of n, k . Furthermore, since $Q(\mu) \geq 0$ for μ real, we have for any $j = 1, \dots, \gamma n$ and $\omega \in \Omega_n$ for which $\mu_j = \mu_j(\omega)$ is real

$$Q(\mu_j)\mu_j^k \leq \sum_{i, \mu_i \in \mathbb{R}} Q(S)\mu_i^k(\omega) \leq C'' n \tau_0^k + Q(S) \sum_{i=1}^{\gamma n} Q(S)\mu_i^k(\omega).$$

But for each $\omega \in \text{Exception}_n(\delta, \varepsilon)$ we have $\mu_j(\omega) \in \text{Out}(\delta, \varepsilon)$ for some j , and hence

$$Q(\mu_j(\omega))\mu_j^k(\omega)$$

is non-negative, and bounded from below by

$$\delta^{|L|D}(\tau_0 + \varepsilon)^k.$$

It follows that for $\omega \in \text{Exception}_n(\delta, \varepsilon)$ we have

$$\begin{aligned} \text{Prob}_n[\text{Exception}_n(\delta, \varepsilon)] \delta^{|L|D}(\tau_0 + \varepsilon)^k &\leq \mathbb{E}_{\omega \in \Omega_n} \left[\sum_{i, \mu_i(\omega) \in \mathbb{R}} Q(S)\mu_i^k(\omega) \right] \\ &\leq C'' n \tau_0^k + Q(S) \text{LHS}, \end{aligned}$$

where LHS is the left-hand-side of (5.2), i.e.,

$$\text{LHS} = \mathbb{E}_{\omega \in \Omega_n} \left[\sum_{i=1}^{\gamma n} \mu_i^k(\omega) \right].$$

But the left-hand-side of (5.2) equals its right-hand-side, RHS, and $Q(S)$

applied to any $P_i(k)$ vanishes; hence

$$\text{Prob}_n[\text{Exception}_n(\delta, \varepsilon)] \delta^{|L|D} (\tau_0 + \varepsilon)^k \leq C'' n \tau_0^k + Q(S) \text{RHS} \quad (5.11)$$

$$\leq C'' n \tau_0^k + \left| Q(S) (h_1(k) + h_2(n, k)) \right|, \quad (5.12)$$

for any k, n with $k + D \leq n/2$, where

$$|h_1(k)| \leq C k^C (\tau_0 + (\varepsilon/2))^k, \quad |h_2(n, k)| \leq C k^C \tau_1^k n^{-r}$$

for some constant, C , independent of n, k . Lemma 5.1.2 implies that

$$|Q(S)h_1(k)| \leq C k^C (\tau_0 + (\varepsilon/2))^k \quad (5.13)$$

and

$$|Q(S)h_2(n, k)| \leq C k^C \tau_1^k n^{-r} \quad (5.14)$$

for a new constant, C , independent of n, k . Combining these estimates with (5.11) and (5.12) we have

$$\text{Prob}_n[\text{Exception}_n(\delta, \varepsilon)] \leq \delta^{-|L|D} C \left(\frac{n \tau_0^k + k^C (\tau_0 + (\varepsilon/2))^k + k^C \tau_1^k n^{-r}}{(\tau_0 + \varepsilon)^k} \right), \quad (5.15)$$

with C independent of n, k .

We claim that setting k to be

$$k = f(n) = \text{the smallest even integer greater than } \max(\phi', \phi'') \log n, \quad (5.16)$$

for sufficiently large n we have that (5.15) implies

$$\text{Prob}_n[\text{Exception}_n(\delta, \varepsilon)] \leq 3 \delta^{-|L|D} n^{-\phi} \quad (5.17)$$

If so, then (5.3) follows by taking $\delta = n^{-\theta}$, for then, in view of (5.10)

$$\text{Prob}_n[\text{Exception}_n(n^{-\theta}, \varepsilon)] \leq 3 n^{-|L|D\theta} n^{-\phi} \leq n^{-\phi+\alpha}$$

which, since $\phi > 2\alpha$, is at most $n^{-\alpha}$ for sufficiently large n .

So we will finish this section, i.e., the proof of (5.3), by establishing (5.17).

To show (5.17), from (5.15) it suffices to show that each of the three expressions

$$\frac{Cn\tau_0^k}{(\tau_0 + \varepsilon)^k}, \quad \frac{Ck^C(\tau_0 + (\varepsilon/2))^k}{(\tau_0 + \varepsilon)^k}, \quad \text{and} \quad \frac{Ck^C\tau_1^k n^{-r}}{(\tau_0 + \varepsilon)^k}$$

is bounded by $n^{-\phi}$ for n sufficiently large. For the first expression, we note that

$$\frac{Cn\tau_0^k}{(\tau_0 + \varepsilon)^k} \leq Cn(\tau_0/(\tau_0 + \varepsilon))^k \leq Cn(\tau_0/(\tau_0 + \varepsilon))^{\phi'' \log n} \leq Cnn^{-\phi-2} = Cn^{-\phi-1},$$

using (5.16) and (5.7), which is therefore bounded by $n^{-\phi}$ for sufficiently large n . For the second expression, we note that

$$\frac{Ck^C(\tau_0 + (\varepsilon/2))^k}{(\tau_0 + \varepsilon)^k} \leq Ck^C \left(\frac{\tau_0 + (\varepsilon/2)}{\tau_0 + \varepsilon} \right)^{\phi' \log n} \leq Ck^C n^{-\phi-1},$$

using (5.16) and (5.6), which is therefore bounded by $n^{-\phi}$ for sufficiently large n . For the third expression, we note that

$$k \leq \max(\phi', \phi'') \log n + 2$$

by virtue of (5.16), and hence

$$k \leq \rho(r/2) \log n$$

for sufficiently large n , by virtue of (5.9), and hence

$$\frac{Ck^C\tau_1^k n^{-r}}{(\tau_0 + \varepsilon)^k} \leq Ck^C n^{-r} (\tau_1/\tau_0)^k \leq Ck^C n^{-r} n^{r/2} = Ck^C n^{-r/2},$$

using (5.8), and by virtue of (5.9), $Ck^C n^{-r/2}$ is bounded by $n^{-\phi}$ for sufficiently large n .

Hence we conclude (5.17), which, as mentioned just below this equation, implies (5.3).

5.4 The Proof of the Limit Formula

In this section we will prove the limit formula in (5.4). Since we have established (5.3), we will use (5.3) and perform a slight variant of the technique in the previous section. However, there is an added subtlety which we now explain. Given r , let D_0 be a positive integer for which bounds the degrees of the polynomials which are the coefficients of the $P_i(k)$ over all $i = 0, \dots, r - 1$. For each $\ell \in L$, and each even, positive integer D with $D \geq D_0$, consider

$$Q_{\ell,D}(z) = \prod_{\ell' \in L \setminus \{\ell\}} (z - \ell')^D. \quad (5.18)$$

We shall apply $Q_{\ell,D}(S)$ to both sides of (5.2), the rough new ideas being:

1. $Q_{\ell,D}(S)P_i(k)$ annihilates the part of $P_i(k)$ involving a polynomial in k times $(\ell')^k$, for any $\ell' \in L$ with $\ell' \neq \ell$;
2. $Q_{\ell,D}(S)P_i(k)$ does not annihilate the part of $P_i(k)$ involving a polynomial in k times ℓ^k ; and
3. for μ within $n^{-\theta}$ of any $\ell' \in L$ with $\ell' \neq \ell$, we have that $Q(\mu)$ is bounded by roughly $n^{-D\theta}$.

The new ideas (1) and (2) indicate that our choice of $Q_{\ell,D}$ can isolate the ℓ^k terms in the $P_i(k)$; the new idea (3) is subtle, in that we will fix $j, r, \alpha > 0$ first, deducing a value $\theta > 0$ for which (5.3) holds, and then will we choose D sufficiently large so that $D\theta$ is large enough, compared to j .

Let us, as in the previous section, now fix a number of variables in a somewhat unmotivated fashion. First, we fix an $\varepsilon > 0$ sufficiently small so that $\tau_0 + \varepsilon$ is strictly less than the absolute value of any element of L . It follows that for sufficiently small $\delta > 0$, we have that the closed balls of radius δ about each element of L are disjoint, and also disjoint from the set

of complex numbers of absolute value at most $\tau_0 + \varepsilon$. Fix an $\ell \in L$ and an integer $j \geq 1$ for which

$$p_{\ell,0}(k) = \cdots = p_{\ell,j-1}(k) = 0.$$

Let

$$\kappa = \frac{(j+2) \log(\tau_1/|\ell|)}{\log\left(|\ell|/((\tau_0 + \varepsilon))\right)}$$

which is non-negative, and equals zero iff $|\ell| = \tau_1$. (This illustrates the fact that the case $|\ell| = \tau_1$ is, in a sense, easier than $|\ell| < \tau_1$.) Choose α so that

$$\alpha - 1 - j - 2 > \kappa \tag{5.19}$$

and choose r so that

$$r - j - 1 > \kappa; \tag{5.20}$$

let $\theta > 0$ be chosen so that (5.3) holds for sufficiently large n . Choose an even integer D so that

1. D is greater than the degree of all polynomials $p_{\ell',i}$ with $i = 0, \dots, j-1$ and $\ell' \in L \setminus \{\ell\}$, and

2.

$$\theta D - j - 2 > \kappa. \tag{5.21}$$

Given the above, we can find a real number $\nu > 0$ for which

$$\log\left(|\ell|/((\tau_0 + \varepsilon))\right) > \nu - (j+2) > (j+2) \log(\tau_1/|\ell|) \min(X). \tag{5.22}$$

Where $X = \{\alpha - 1 - j - 2, r - j - 1, \theta D - j - 2\}$. Let $Q_{\ell,D}(z)$ be as in (5.18). Our strategy will be so apply $Q_{\ell,D}(S)$ to both sides of (5.2), and then choose

$$k = f(n) = \text{the smallest even integer greater than } \nu \log n. \tag{5.23}$$

We claim that (5.4) will follow. Before going through this calculation, let us note for future use that the choice of ν as above, (5.22), shows that

$$\max(\delta^D, n^{-\alpha+1}, n^{r-1}) \tau_1^k, (\tau_0 + \varepsilon)^k \quad \text{are both} \quad O(n^{-j-2}) \ell^k \quad (5.24)$$

where $\delta = n^{-\theta}$, for a constant in the $O(n)$ notation that is independent of n, k .

Again, we use C, C' to denote various constants that are independent of n, k , but may depend upon $L, \gamma, \ell, \tau_0, \tau_1, \varepsilon, j$, and therefore depending on $\alpha, r, \theta, D, \kappa, \nu$ as above.

We shall make some simple observations about $Q_{\ell, D}(z)$, analogous to the ones made of $Q(z)$ in the last section. There is only one set of new estimates, which we state as a lemma.

Lemma 5.4.1. *The following bounds hold for, say, all $\delta \leq 1$:*

1. *if for some $\ell' \in L$ with $\ell' \neq \ell$ we have $|z - \ell'| \leq \delta$, then*

$$|Q_{\ell, D}(z)| \leq (2\tau_1 + \delta)^{(|L|-2)D} \delta^D \leq C\delta^D,$$

where C is a constant independent of n, k .

2. *for any τ_0 , we have that $|z| \leq \tau_0$ implies that*

$$|Q_{\ell, D}(z)| \leq (\tau_0 + \tau_1)^{(|L|-1)D} \leq C,$$

where C is a constant independent of n, k .

Proof. For (1), we see that if $\ell'' \in L$ with $\ell'' \neq \ell', \ell$, then

$$|z - \ell''| \leq |z| + |\ell''| \leq (\tau_1 + \delta) + \tau_1 = 2\tau_1 + \delta;$$

hence

$$|Q_{\ell, D}(z)| \leq |z - \ell'|^D \prod_{\ell'' \in L \setminus \{\ell, \ell'\}} |z - \ell''|^D \leq \delta^D (2\tau_1 + \delta)^{(|L|-2)D}.$$

For (2) we note that $\ell' \in L$ implies that

$$|z - \ell'| \leq |z| + |\ell'| \leq \tau_0 + \tau_1,$$

and (2) follows. □

We compile a list of simple observations about $Q_{\ell,D}(z)$, analogous to the ones made of $Q(z)$ in the last section:

1. for $\mu \in \mathbb{R}$, $Q_{\ell,D}(\mu)$ is real and non-negative;
2. for any μ with $|\mu| \leq \tau_1$ we have

$$|Q_{\ell,D}(\mu)| \leq (2\tau_1)^{(|L|-1)D} \leq C \quad (5.25)$$

where C is a constant independent of n, k (by (2) of Lemma 5.4.1);

3. $Q_{\ell,D}(\ell) \neq 0$;
4. for real z with $|z - \ell| \leq \delta$ and $\delta \leq 1$, we have

$$Q_{\ell,D}(z) = Q_{\ell,D}(\ell) + O(\delta), \quad (5.26)$$

where the constant in the $O()$ is independent of n, k (and, in fact, depends only on L and D), which follows using the mean-value-theorem, with the $O()$ constant being any bound on the derivative $Q'_{\ell,D}(z)$ over all z within 1 of ℓ ;

5. for any real $\tau > 0$, if $h(k) = \mu^k$ with $|\mu| \leq \tau$, then

$$|Q_{\ell,D}(S)h(k)| \leq C\tau^k$$

for some C depending only on L and D ; and

6. furthermore, if $h(k)$ is any function bounded by $Ck^C\tau^k$ for some C , then we have

$$|Q_{\ell,D}(S)h(k)| \leq C'k^{C'}\tau^k$$

for a constant, C' , depending only on C, L , and D .

In addition to these simple estimates, we will use Lemma 5.4.1.

Returning to Lemma 5.2.5, the goal of the rest of this section is to establish (5.4). As in the previous section, we let LHS and RHS, respectively, be the left-hand-side and right-hand-side of (5.2). Again, the basic idea is to apply $Q_{\ell,D}(S)$ to LHS = RHS. As before, we see that

$$Q_{\ell,D}(S)\text{LHS} = \mathbb{E}_{\omega \in \Omega_n} \left[\sum_i Q_{\ell,D}(\mu_i(\omega)) \mu_i^k(\omega) \right], \quad (5.27)$$

and we wish to estimate this expectation (for various values of n, k).

For $\delta > 0$ sufficiently small (ultimately we will take $\delta = n^{-\theta}$) for each i and $\omega \in \Omega_n$ we have that exactly one of the following holds:

1. for some $\ell' \in L \setminus \{\ell\}$ we have $|\mu_i(\omega) - \ell'| \leq \delta$;
2. we have $|\mu_i(\omega) - \ell| \leq \delta$;
3. we have $|\mu_i(\omega)| \leq \tau_0 + \varepsilon$;
4. we have

$$\mu_i(\omega) \in \text{Out}(\delta, \varepsilon).$$

Assume that δ is sufficiently small for this to hold. In view of (5.27), let us estimate the contribution to the expected value of $Q_{\ell,D}(\mu)\mu^k$ for $\mu = \mu_i(\omega)$ in each of the above four cases.

For case (1), i.e., $|\mu_i(\omega) - \ell'| \leq \delta$ with $\ell' \in L \setminus \{\ell\}$, use the estimate

$$\mathbb{E}_{\omega \in \Omega_n} \left[\sum_{i \text{ s.t. } |\mu_i(\omega) - \ell'| \leq \delta} Q_{\ell,D}(\mu_i^k(\omega)) \mu_i^k(\omega) \right] \leq \text{Near}_n(\ell', \delta) \delta^D C(\ell')^k,$$

using Lemma 5.4.1, where C is independent of n, k . Summing over all $\ell' \in L \setminus \{\ell\}$, and using the crude bound

$$\sum_{\ell' \in L} \text{Near}_n(\ell', \delta) \leq \gamma n,$$

we conclude that

$$\mathbb{E}_{\omega \in \Omega_n} \left[\sum_{i \text{ s.t. } |\mu_i(\omega) - \ell'| \leq \delta \text{ for some } \ell' \in L \setminus \{\ell\}} Q_{\ell, D}(\mu_i^k(\omega)) \mu_i^k(\omega) \right] \quad (5.28)$$

$$\leq \gamma n \delta^D (2\tau_1 + \delta)^{(|L|-2)D} \max_{\ell'} (\ell')^k \leq C n \delta^D \tau_1^k,$$

where C is independent of n, k .

For case (2), i.e., $|\mu_i(\omega) - \ell| \leq \delta$, with $\delta > 0$ sufficiently small, we note that

$$\begin{aligned} |\mu_i^k(\omega) - \ell^k| &\leq |\mu_i(\omega) - \ell| |\mu_i^{k-1}(\omega) + \mu_i^{k-2}(\omega)\ell + \dots + \ell^{k-1}| \\ &\leq |\mu_i(\omega) - \ell| k(\tau_1 + \delta)^k \leq \delta k(\ell + \delta)^k = \delta k \ell^k (1 + (\delta/\ell))^k. \end{aligned}$$

Hence we have

$$\mu_i^k(\omega) = \ell^k (1 + \delta k O(1)),$$

with the $O(1)$ being at most e , provided that

$$(1/k) \geq \ell/\delta;$$

given k as in (5.23), this holds for all n sufficiently large. Then (5.26) implies that

$$\begin{aligned} \mathbb{E}_{\omega \in \Omega_n} \left[\sum_{i \text{ s.t. } |\mu_i(\omega) - \ell| \leq \delta} Q_{\ell, D}(\mu_i^k(\omega)) \mu_i^k(\omega) \right] \\ = \text{Near}_n(\ell, \delta) Q_{\ell, D}(\ell) (1 + O(\delta)) \ell^k (1 + \delta k O(1)), \end{aligned}$$

and so

$$\mathbb{E}_{\omega \in \Omega_n} \left[\sum_{i \text{ s.t. } |\mu_i(\omega) - \ell| \leq \delta} Q_{\ell, D}(\mu_i^k(\omega)) \mu_i^k(\omega) \right] = \text{Near}_n(\ell, \delta) Q_{\ell, D}(\ell) \ell^k (1 + \delta k O(1)). \quad (5.29)$$

where the constant in the $O(1)$ is independent of n, k for $\delta = n^{-\theta}$ and k as in (5.23).

For case (3), (5.25) implies that

$$\mathbb{E}_{\omega \in \Omega_n} \left[\sum_{i \text{ s.t. } |\mu_i(\omega)| \leq \tau_0 + \varepsilon} Q_{\ell, D}(\mu_i^k(\omega)) \mu_i^k(\omega) \right] \leq n\gamma(2\tau_1)^{(|L|-1)D} (\tau_0 + \varepsilon)^k \quad (5.30)$$

$$\leq nC(\tau_0 + \varepsilon)^k \quad (5.31)$$

with C independent of n, k .

Finally, for case (4), we use the estimate

$$\begin{aligned} \mathbb{E}_{\omega \in \Omega_n} \left[\sum_{i \text{ s.t. } \mu_i(\omega) \in \text{Out}(\delta, \varepsilon)} Q_{\ell, D}(\mu_i^k(\omega)) \mu_i^k(\omega) \right] \\ \leq n\gamma C \tau_1^k \text{Prob}_n[\text{Exception}_n(\delta, \varepsilon)], \end{aligned}$$

since $\mu_i(\omega) \in \text{Out}(\delta, \varepsilon)$ implies that $\omega \in \text{Exception}_n(\delta, \varepsilon)$, and in this case there are at most γn values of i for which $\mu_i(\omega)$ lies in $\text{Out}(\delta, \varepsilon)$, and for each such i we have $Q_{\ell, D}(\mu_i^k(\omega))$ is at most a constant, C , by (5.25). Hence, using (5.3), we have

$$\begin{aligned} \mathbb{E}_{\omega \in \Omega_n} \left[\sum_{i \text{ s.t. } \mu_i(\omega) \in \text{Out}(\delta, \varepsilon)} Q_{\ell, D}(\mu_i^k(\omega)) \mu_i^k(\omega) \right] \\ \leq Cn\tau_1^k n^{-\alpha}. \end{aligned} \quad (5.32)$$

for n sufficiently large, and C independent of n, k .

Combining (5.28)--(5.32), and (5.27), we get that for $\delta = n^{-\theta}$, for all k with $k \leq \ell/\delta$ and n sufficiently large we have and

$$\begin{aligned} Q_{\ell, D}(S)\text{LHS} - \text{Near}_n(\ell, \delta)Q_{\ell, D}(\ell)\ell^k(1 + \delta k O(1)) \\ = O(n)(\delta^D \tau_1^k + (\tau_0 + \varepsilon)^k + n^{1-\alpha} \tau_1^k), \end{aligned} \quad (5.33)$$

where the constant in the $O(n)$ is independent of n, k . But (5.24) implies that (5.33) is $O(\ell^k n^{-j-1})$ for k as in (5.23), and hence for k as such we have

$$Q_{\ell,D}(S)\text{LHS} - \text{Near}_n(\ell, \delta)Q_{\ell,D}(\ell)\ell^k(1 + \delta k O(1)) = O(\ell^k n^{-j-1}). \quad (5.34)$$

As in the previous section, we apply $Q_{\ell,D}(S)$ to RHS, and find

$$Q_{\ell,D}(S)\text{RHS} = \sum_{i=0}^{r-1} Q_{\ell,D}(S)P_i(k) + O(n^{-j-1})\ell^k \quad (5.35)$$

provided that for any C independent of n, k we have

$$k^C(\tau_0 + (\varepsilon/2))^k + k^C \tau_1^k n^{-r} = O(n^{-j-1})\ell^k$$

(for the same reasoning as (5.13), (5.14), (5.12)); but this is implied by (5.24). Hence, dividing (5.35) by ℓ^k , we have

$$\begin{aligned} Q_{\ell,D}(\ell) \sum_{i=0}^{r-1} p_{\ell,i}(k)n^{-i} &= \ell^{-k}Q_{\ell,D}(S)\text{RHS} + O(n^{-j-1}) \\ &= \ell^{-k}Q_{\ell,D}(S)\text{LHS} + O(n^{-j-1}) \\ &= \text{Near}_n(\ell, \delta)Q_{\ell,D}(\ell)(1 + \delta k O(1)) + O(n^{-j-1}) \end{aligned}$$

for k as in (5.23). Given that $p_{\ell,0}(k), \dots, p_{\ell,j-1}$ all vanish, we have, upon dividing by $n^{-j}Q_{\ell,D}(\ell)$, that

$$p_{\ell,j}(k) + O(n^{-1}) = n^j \text{Near}_n(\ell, n^{-\theta}) + O(n^{-1}).$$

But $n^j \text{Near}_n(\ell, n^{-\theta})$ is independent of k , and k is proportional to $\log n$. Hence taking $n \rightarrow \infty$ we conclude (5.4).

We note that the only requirement on our choice of $\theta > 0$ is that θ satisfies (5.3) for our chosen value of α (in (5.19)). Hence if θ_0 is such a value of θ , then for any θ such that $0 < \theta < \theta_0$, then

$$\text{Prob}_n[\text{Exception}_n(n^{-\theta}, \varepsilon)] \leq \text{Prob}_n[\text{Exception}_n(n^{-\theta_0}, \varepsilon)] \leq n^{-\alpha},$$

for n sufficiently large. Hence any θ with $0 < \theta < \theta_0$ also satisfies (5.3) for α (in (5.19)), and hence satisfies (5.4).

5.5 The End of The Proof of Lemma 5.2.5

It remains to prove (5.5), having established (5.3) and (5.4). First we note that (5.4) implies that for any $\ell \in L$ and $\varepsilon > 0$ (and given $L, \gamma, \tau_0, \tau_1, j$) there is a $C > 0$ for which

$$\text{Near}_n(\ell, n^{-\theta}) \leq C(n^{-j}) \quad (5.36)$$

for any fixed $\theta > 0$ for which (5.3) holds with α given in (5.19). Hence this also holds with θ replaced by any smaller, positive value of θ .

Proof. Given an $\varepsilon > 0$, we can therefore choose a $\theta > 0$ for which

1. for this ε , and with $\alpha = j$, we have (5.3) holds; and
2. we have that (5.36) holds for all $\ell \in L$.

We have

$$\text{Prob}_n[\{\omega \mid |\mu_i(\omega) - \ell| \leq n^{-\theta} \text{ for some } i\}] \leq \text{Near}_n(\ell, n^{-\theta}) \leq C n^{-j}$$

for a constant, C , independent of n, k . Hence, summing over all ℓ , we have

$$\text{Prob}_n[E_n \leq C n^{-j}]$$

where

$$E_n = \{\omega \mid |\mu_i(\omega) - \ell| \leq n^{-\theta} \text{ for some } i \text{ and some } \ell \in L\}.$$

Since θ also satisfies (5.3) with $\alpha = j$, we have

$$\text{Prob}_n[\text{Exception}_n(n^{-\theta}, \varepsilon)] \leq n^{-j},$$

for n sufficiently large. But clearly

$$\begin{aligned} & \text{Prob}_n[\text{AbsoluteException}_n(\varepsilon)] \\ & \leq \text{Prob}_n[\text{Exception}_n(n^{-\theta}, \varepsilon)] + \text{Prob}_n[E_n], \end{aligned}$$

since if $\omega \in \Omega_n$ has $|\mu_i(\omega)| > \tau_0 + \varepsilon$ for some i , then either $\mu_i(\omega)$ lies in $\text{Out}(n^{-\theta}, \varepsilon)$ or $\mu_i(\omega)$ is within $n^{-\theta}$ of some $\ell \in L$. Hence

$$\begin{aligned} & \text{Prob}_n[\text{AbsoluteException}_n(\varepsilon)] \\ & \leq \text{Prob}_n[\text{Exception}_n(n^{-\theta}, \varepsilon)] + \text{Prob}_n[E_n] \leq Cn^{-j}. \end{aligned}$$

□

Chapter 6

Proof of the Main Result

We can now finish our proof of the regular case of the proof of the Alon conjecture in the Broder-Shamir model, $\mathcal{C}_n(B)$, where B is any connected d -regular graph. In section 4.1, we define the event

$$\text{Tangle}_{r,B}[n] = \{G \in \mathcal{C}_n(B) \mid \exists H < G, H \in \text{Tangle}_{r,B}\}$$

where a random covering contains a tangle of order less than r as a subgraph; and denoted by $\mathbb{I}_{\text{TF}(r,B)[n]}$ the indicator function of the complement of this event, that is

$$\mathbb{I}_{\text{TF}(r,B)[n]}(G) = 1 \iff G \notin \text{Tangle}_{r,B}[n]$$

In order to be able to use our side-stepping lemma, we need to prove that we have a B -Ramanujan expansion for $\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}(H_G^k)]$. To do this, we will show that this is the case for our truncated certified trace since we have that

$$\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}(H_G^k)] = \mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{CertTr}_{<r}(G, k)] + O_n(k^{2r} n^{-r}) \quad (6.1)$$

Given the B -Ramanujan expansion for the certified trace (Theorem 4.4.1), it is enough to show that $\mathbb{E}_n[(1 - \mathbb{I}_{\text{TF}(r,B)[n]}(G)) \text{CertTr}(G, k)]$ admits a B -Ramanujan expansion.

Theorem 6.0.1. *We have a B -Ramanujan expansion,*

$$\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}(H_G^k)] = P_0(k) + \frac{P_1(k)}{n} + \dots + \frac{P_{r-1}(k)}{n^{r-1}} + \text{error}(r, n, k)$$

where

$$|\text{error}(r, n, k)| \leq Ck^C (d-1)^k n^{-r}$$

Proof. Since $\mathbb{I}_{\text{TF}(r,B)[n]}(G) + (1 - \mathbb{I}_{\text{TF}(r,B)[n]}(G)) = 1$, we have that

$$\begin{aligned} \mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{CertTr}_{<r}(G, k)] &= \mathbb{E}_n[\text{CertTr}_{<r}(G, k)] \\ &\quad - \mathbb{E}_n[(1 - \mathbb{I}_{\text{TF}(r,B)[n]}(G)) \text{CertTr}_{<r}(G, k)] \end{aligned}$$

The first term of the right-hand side has a B -Ramanujan expansion by Theorem 4.4.1 and the second one by Theorem 6.0.2. The sum of B -Ramanujan expansions is B -Ramanujan. Using equation (6.1) concludes the proof. \square

Theorem 6.0.2. *We have a B -Ramanujan expansion,*

$$\begin{aligned} \mathbb{E}_n[(1 - \mathbb{I}_{\text{TF}(r,B)[n]}(G)) \text{CertTr}_{<r}(G, k)] &= P_0(k) + \frac{P_1(k)}{n} + \dots \\ &\quad + \frac{P_{r-1}(k)}{n^{r-1}} + \text{error}(r, n, k) \end{aligned}$$

where

$$|\text{error}(r, n, k)| \leq Ck^C (d-1)^k n^{-r}$$

Proof. The proof of this theorem is in the next section. Take Theorem 6.1.2 with $\Psi = \text{HasTangle}_{r,B}^{\min}$. \square

So let us state and prove the last missing expansion theorem in the next section.

6.1 Certified Traces In Graphs With Tangles

In this section we widen the notion of a *type* to prove the following theorem, based on the ideas of Chapter 9 of [Fri08]. The main difference here is that we will view a *tangle* as a finite graph with a given morphism to B .

Definition 6.1.1. By a B -graph, ψ , we mean a finite graph, ψ along with a morphism $\psi \rightarrow B$. We view Occurs_B as a set of B -graphs.

Theorem 6.1.2. *Let Ψ be any finite collection of B -graphs, such that each ψ in Ψ satisfies $-\chi(\psi) \geq 1$. Let $I_\Psi(G)$ be the function that is one or zero according to whether or not G has a subgraph isomorphic to an element of Ψ . Then for any positive integer r and base graph, B , we have*

$$\mathbb{E}_n[I_\Psi(G)\text{CertTr}_{<r}(G, k)] = P_0(k) + P_1(k)n^{-1} + \cdots + P_{r-1}(k)n^{1-r} + \text{err}_r(n, k),$$

where the $P_i(k)$ are B -Ramanujan, and there is a C independent of n, k for which

$$|\text{err}(n, k)| \leq Ck^{2r+2}(d-1)^k n^{-r}$$

Our proof is a straightforward adaptation of Chapter 9 of [Fri08] to this slightly more general situation. Let us give the rough idea, before giving a rigorous proof.

We have established this for Ψ being the empty set, where the above expectation has $P_i(k)$ that are given by a finite number of terms

$$\sum_{\substack{\vec{m}, \vec{k}=k \\ \vec{k} \geq \vec{k}^0}} W_{\mathcal{T}}^{\mathcal{Y}}(\vec{m}) P(\vec{k})$$

for types, \mathcal{T} , of order (i.e., minus Euler characteristic) at most r . To multiply the certified trace by $I_\Psi(G)$ can be seen as taking the information in a type, \mathcal{T} , and augmenting it by further information that describes how G is supposed to contain one or more elements of Ψ as a subgraph. This is handled by inclusion/exclusion, and since $-\chi(\psi) \geq 1$ for $\psi \in \Psi$, it follows that the inclusion/exclusion can be stopped after a finite number of levels.

Our interest in the above theorem is the following corollary, obtained by setting Ψ to be Ψ_r , the set of minimal elements of tangles of order less than r ; each such element, ψ , has all vertices of degree at least two, and must have some vertex of degree three, or else ψ is a cycle, which has $\rho(H_\psi) = 1$ which is impossible, since a tangle, ψ , has $\rho(H_\psi) \geq (d-1)^{1/2} > 1$. Hence each element of Ψ_r has all vertices of degree at least two, and at least one vertex of degree three, and hence each such element has order at least one. We therefore conclude.

Theorem 6.1.3. *For any positive integer r and base graph, B , we have*

$$\mathbb{E}_{G \in \mathcal{C}_n(B)} [\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{CertSNBC}_r(G, k)]$$

and

$$\mathbb{E}_{G \in \mathcal{C}_n(B)} [(1 - \mathbb{I}_{\text{TF}(r,B)[n]}(G)) \text{CertSNBC}_r(G, k)]$$

both have asymptotic expansions of the form

$$P_0(k) + P_1(k)n^{-1} + \cdots + P_{r-1}(k)n^{1-r} + \text{err}_r(n, k),$$

where the $P_i(k)$ are B -Ramanujan, and there is a C independent of n, k for which

$$|\text{err}(n, k)| \leq Ck^{2r+2}(d-1)^k n^{-r}$$

Another consequence of this technique is the following theorem.

Theorem 6.1.4. *Let Ψ be any finite collection of graphs, such that each ψ in Ψ satisfies $-\chi(\psi) \geq 1$. Let $I_\Psi(G)$ be the function that is one or zero according to whether or not G has a subgraph isomorphic to an element of Ψ . Then for any positive integer r and base graph, B , we have that*

$$\text{Prob}_n[G \text{ contains a } B\text{-subgraph isomorphic to some element of } \Psi]$$

$$= \mathbb{E}_n[I_\Psi(G)] = p_1 n^{-1} + p_2 n^{-2} + \cdots + p_{r-1} n^{1-r} + O(n^{-r})$$

for some constants p_i .

At this point we simply adapt the proof of Theorem 9.3 of [Fri08] to this situation.

Define a *potential graph specialization* as a pair, (Ω, σ) , where Ω is a B -graph, and $\sigma: V_\Omega \rightarrow \{1, \dots, n\}$ is a map. Given a potential walk, $(w; \vec{t})$, we define

$$\text{Prob}_n[(w; \vec{t}), (\Omega, \sigma)]$$

as the probability that $(w; \vec{t})$ and (Ω, σ) occur as subgraphs in G , and we let

$$\mathbb{E}_n[(w; \vec{t}), (\Omega, \sigma)]$$

be the expected number of times that $(w; \vec{t})$ and (Ω, σ') occurs in G , where \vec{t}', σ' are just \vec{t}, σ , respectively, with the numbers $\{1, \dots, n\}$ permuted. We also write

$$\mathbb{E}_n[(\Omega, \sigma)]$$

just for the expected number of times that a (Ω, σ') as above occurs in G . We have that both

$$\mathbb{E}_n[(w; \vec{t}), (\Omega, \sigma)], \quad \mathbb{E}_n[(\Omega, \sigma)]$$

are given by asymptotic expansions in $1/n$ with coefficients $p_i(\vec{a})$, where $\vec{a} = \vec{a}((w; \vec{t}), (\Omega, \sigma))$ or $\vec{a} = \vec{a}((\Omega, \sigma))$ is a vector indexed on the edges $e \in E_B$ that counts the number of occurrences of e in the union of the data that $(w; \vec{t})$ and (Ω, σ) occur in G .

Let the *derived graphs of Ψ* , denoted Ψ^+ , be the B -graphs in Occurs_B which can be expressed as the union of graphs isomorphic to a finite union of elements of Ψ , and let $\Psi^+_{<r}$ be those elements of Ψ^+ of order less than r . For any two finite graphs, ψ, ψ' , let $N(\psi, \psi')$ denote the number of inclusions of ψ in ψ' as B -graphs, i.e., the number of B -subgraphs of ψ' isomorphic to ψ . Clearly $N(\psi, \psi')$ is finite (and $N(\psi, \psi)$ is the number of B -automorphisms of ψ). We view Ψ^+ as a partially ordered set via $\psi \leq \psi'$ if $N(\psi, \psi') > 0$, i.e., ψ is a B -subgraph of ψ' (if $\psi \leq \psi'$ and $\psi' \leq \psi$ then the inclusion of ψ into ψ' must be surjective, and hence ψ' and ψ are the same B -graph).

Möbius inversion yields the following lemma (see Proposition 9.5 in [Fri08]).

Lemma 6.1.5. *There exist real numbers $\{\mu_\Omega\}_{\Omega \in \Phi^+}$ such that for any $\Omega' \in \Psi^+$ we have*

$$\sum_{\Omega \leq \Omega'} N(\Omega, \Omega') \mu_\Omega = 1.$$

It follows that

$$\begin{aligned} & \mathbb{E}_n[I_\Phi(G) \text{CertSNBC}_r(G, k)] \\ &= \sum_{(w; \vec{t}) \in \mathcal{W}} \sum_{(\Omega, \sigma) \in \Psi^+} \mathbb{E}_n[(w; \vec{t}), (\Omega, \sigma)] \mu_\Omega, \end{aligned} \quad (6.2)$$

and similarly

$$\begin{aligned} & \mathbb{E}_n[I_\Phi(G)] \\ &= \sum_{(\Omega, \sigma), \Omega \in \Psi^+} \mathbb{E}_n[(\Omega, \sigma)] \mu_\Omega \end{aligned} \quad (6.3)$$

(compare (37) and (38) of [Fri08]).

Lemma 6.1.6. *Let Ψ be a set of B -graphs each of order at least one. For each r , $\Psi^+ < r$, is a finite set. Furthermore, in (6.2), we may replace Ψ^+ with $\Psi^+_{<r}$, with a difference of at most $C(d-1)^k n^{-r} k^{2r+2}$; similarly, we may do the same in (6.3), with a difference of $O(n^{-r})$.*

Proof. See the proof of Lemma 9.6 of [Fri08] □

Hence, to prove Theorems 6.1.2 and 6.1.4, it suffices to prove an asymptotic expansion, for any fixed $\Omega \in \Psi^+_{<r}$ for

$$\sum_{(w, \vec{t}) \in \text{CertSNBC}_r} \sum_{\sigma} \mathbb{E}_n[(w; \vec{t}), (\Omega, \sigma)]$$

and the same without the (w, \vec{t}) .

Theorem 6.1.4 follows immediately from the fact that each

$$\mathbb{E}_n[(\Omega, \sigma)]$$

is simply a power series in $1/n$, and each $\Omega \in \Phi^+$ is of order at least one. It remains to show Theorem 6.1.2, we requires us to incorporate the original notion of a type with the new data from Ω .

So we fix $\Omega \in \Psi^+_{<r}$, and we define an Ω -specialization of a form, Γ , as in Definition 9.7 of [Fri08], and an Ω -type as in Definition 9.8 of [Fri08]: namely,

1. an Ω -specialization of a form, Γ , is an inclusion, $V_\Gamma \cup V_\Omega \rightarrow \{1, \dots, n\}$, where the notation $V_\Gamma \cup V_\Omega$ means that we allow V_Γ and V_Ω to share vertices (i.e., viewing V_Γ and V_Ω as different sets, we impose on a subset of V_Ω , an identification for each element of this subset with

an element of V_Γ ; we see that $\mathbb{E}_n[(w; \vec{t}), (\Omega, \sigma)]$ depends only on the Ω -specialized form of $(w; \vec{t}), (\Omega, \sigma)$, and so we may write

$$\mathbb{E}_n[(w; \vec{t}), (\Omega, \sigma)] = \mathbb{E}_n[\Gamma; \Omega]_n,$$

which has an asymptotic expansion with coefficients $p_i(\vec{a})$ with $\vec{a} = \vec{a}(\Gamma)$;

2. an Ω -type is a type, \mathcal{T} , as before, except that we have the additional information that Ω is a subgraph of T , the underlying graph of \mathcal{T} , and that now V_T contains not only the initial vertex of the walk and vertices of degree at least three, but also contains all vertices and edges of Ω ; hence T may be disconnected (if (w, \vec{t}) and (Ω, σ) represent a disconnected graph), and the beaded paths of $\text{Graph}(w)$ can contain edges and vertices of Ω , which are retained in the Ω -type.

It should be noted that Ω -types, \mathcal{T} , have two types of edges, namely (1) edges belonging to Ω , which are fixed, with a fixed association of each fixed edge to an edge of B , and (2) "variable" edges, which represent beaded paths in the walk or form (that are free of Ω edges).

To each Ω -type, \mathcal{T} , we may associate its type, \mathcal{T}' , obtained by forgetting the edges of Ω , i.e., derived from the walk (w, \vec{t}) from which it arises; each edge of the underlying graph of \mathcal{T}' corresponds to a path in the edges of \mathcal{T} , some fixed and some variable. The certification of a walk, (w, \vec{t}) , now means that each edge in the type, \mathcal{T}' , which corresponds to some path in \mathcal{T} , has a lower bound on its length. But the set of variable edge lengths in \mathcal{T} satisfying these lower bounds on each \mathcal{T}' edge is an upset in functions $\mathbb{Z}_{\geq 1}^{E'_T}$, where E'_T are the variable edges in \mathcal{T} . Hence we get an expansion for the $P_i(k)$ in terms of a sums

$$\sum_{\vec{m}, \vec{k}} W_{\mathcal{T}}^{\mathcal{Y}}(\vec{m}) P(\vec{k}), \tag{6.4}$$

where \mathcal{T} is an Ω -type, and the \vec{m}, \vec{k} vary over the variable edges of \mathcal{T} , with $\vec{k} \geq \vec{k}^0$, and $W_{\mathcal{T}}^{\mathcal{Y}}(\vec{m})$ satisfies the same bounds, via certification, that does

$W_{\mathcal{T}}^{\mathcal{Y}}(\vec{m})$. Furthermore, any original type, \mathcal{T}' , can be associated to at most finitely many Ω -types, \mathcal{T} , since we obtain \mathcal{T} from \mathcal{T}' by including Ω into the edges and vertices of \mathcal{T}' , adding only a finite additional number of edges and vertices (i.e., those of (Ω, σ) which do not lie on the graph of (w, \vec{t})).

Hence the total number of Ω -types with original type of order less than r is finite, and hence only a finite number of sums as in (6.4) are involved in the $P_i(k)$ for Theorem 6.1.2. Hence we conclude Theorem 6.1.2.

6.2 End of the Proof

We now proceed to complete our proof of the Alon conjecture for d -regular base graphs. Let us review our strategy here briefly. Due to the presence of tangles, we cannot expect a B -Ramanujan expansion of $\mathbb{E}_n[\text{Tr}(H_G^k)]$. This lead us to show Theorem 6.0.1 that we have a B -Ramanujan expansion of

$$\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}(H_G^k)]$$

To do this, we defined our certified trace, which essentially agrees up to any given order, r with the Hashimoto trace for random graphs that are tangle-free. Most of this thesis was devoted to show that this certified trace admits a B -Ramanujan expansion.

Finally, our side-stepping lemma, allows us to make some conclusion out of this expansion if we can show that some of the coefficients of the expansion vanish. It has been known since [Fri03] that the principal part of the coefficient of order 0 vanishes:

Lemma 6.2.1. *Consider the B -Ramanujan expansion of Theorem 6.0.1.*

$$\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}(H_G^k)] = P_0(k) + \frac{P_1(k)}{n} + \dots + \frac{P_{r-1}(k)}{n^{r-1}} + \text{error}(r, n, k)$$

with

$$|\text{error}(r, n, k)| \leq Ck^C (d-1)^k n^{-r}$$

then there exists a constant C_0 , independent of n, k such that

$$|P_0(k) - \text{Tr}(H_B^k)| \leq C_0 k^{C_0} (d-1)^{k/2}$$

Proof. This is essentially calculated in [Fri03]. The point is that the $P_k(0)$ is the sum of the number of walks of length k in a type, \mathcal{T} , whose underlying graph, T , has order 0. Since T is a connected graph with all vertices of degree at least two, we see that T must consist of a single vertex and a single self-loop (hence there is only one order that can be imposed on it). The additional type information (beyond T and its ordering of vertices and edges) is the lettering. Since $P_k(0)$ is the leading term of the series for T , it follows that $P_k(0)$ equals the expected number of strictly non-backtracking closed walks of length k' in B , summed over all k' divisible by k . Hence

$$P_k(0) = \sum_{k' \text{ divides } k} \text{Tr}(H_B^{k'}).$$

But if k' is a positive integer dividing k , then either $k' = k$, or else k' is between 1 and $k/2$. Hence

$$|P_k(0) - \text{Tr}(H_B^k)| \leq (k/2) \max_{1 \leq k' \leq k/2} \text{Tr}(H_B^{k'}) \leq (k/2) |E_B^{\text{dir}}| (d-1)^{k/2}.$$

□

This, along with our side-stepping lemma allows us to prove our Theorem 1.2.3:

Proof. We want to apply the side-stepping lemma here. For this, we need a polyexponential expansion of the "new" Hashimoto trace, when the graph doesn't contain any tangles. Recall that the "new trace", that is, the sum of all new eigenvalues is given by

$$\text{Tr}_B^{\text{new}}(H_G^k) = \sum_{\lambda \in \text{Spec}_B^{\text{new}}(H_G)} \lambda^k = \text{Tr}(H_G^k) - \text{Tr}(H_B^k)$$

In other words, we are interested in the expansion of

$$\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}_B^{\text{new}}(H_G^k)]$$

So

$$\begin{aligned} \mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}_B^{\text{new}}(H_G^k)] &= \mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}(H_G^k)] \\ &\quad - \mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}(H_B^k)] \\ &= \left(\sum_{i=0}^{r-1} \frac{P_i(k)}{n^i} + \text{error}(r, n, k) \right) - \text{Tr}(H_B^k) \text{Prob}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) = 1] \end{aligned}$$

Corollary 4.1.10 gives us that

$$\text{Prob}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) = 1] = 1 - q_1 n^{-1} - q_2 n^{-2} - \dots$$

hence we can rewrite this as

$$\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}_B^{\text{new}}(H_G^k)] = \sum_{i=0}^{r-1} \widehat{P}_i(k) n^{-i} + \text{error}(r, n, k)$$

for $\widehat{P}_i(k) = P_i(k) - q_i \text{Tr}(H_B^k)$, and where the error bound satisfies

$$|\text{error}(r, n, k)| \leq C k^C (d-1)^k n^{-r}$$

We want to rewrite this as an expansion of polyexponential functions in order to apply our side-stepping lemma. If \widetilde{P}_i denotes the principal part of \widehat{P}_i , then we can rewrite this as

$$\mathbb{E}_n[\mathbb{I}_{\text{TF}(r,B)[n]}(G) \text{Tr}_B^{\text{new}}(H_G^k)] = \sum_{i=0}^{r-1} \widetilde{P}_i(k) n^{-i} + \text{error}(r, n, k)$$

with a new error term satisfying

$$|\text{error}(r, n, k)| \leq C k^C (d-1)^k n^{-r} + C k^C (\sqrt{d-1} + \varepsilon)^k$$

Notice that $\widehat{P}_0(k) = P_0(k) - \text{Tr}(H_B^k)$ and by lemma 6.2.1 we have

$$|P_0(k) - \text{Tr}(H_B^k)| \leq Ck^C(d-1)^{k/2}$$

hence $\widetilde{P}_0(k) = 0$ and we can apply the side-stepping lemma to conclude that for any $\varepsilon > 0$ there is a constant C for which

$$\text{Prob}_n[\rho_B^{\text{new}}(H_G) \geq \sqrt{d-1} + \varepsilon] \leq C/n$$

which completes our proof. \square

As mentioned in the introduction on page 6, for regular graphs, Theorem 1.2.3 implies the Alon Conjecture for regular graphs, that is our Theorem 1.2.2.

Chapter 7

Conclusion

We have successfully proved the generalized Alon conjecture for all regular graphs. This generalizes the result of Friedman [Fri08] and improves on the work of [Fri03, LP10, LSV11, ABG10, Pud12]. In an article in preparation, the author and Joel Friedman, will give a weakened generalized Alon conjecture for some non-regular base graphs based on the ideas of this thesis.

To solve the generalized Alon conjecture for arbitrary base graph, one could try to develop analogous results for walks that allow backtracking. The Broder-Shamir [BS87] approach is necessarily more complicated when backtracking is allowed, but we believe that a coincidence theory can still be adapted to this situation; however, there seem to be some serious difficulties in extending some of our methods to walks that can backtrack.

There are many generalizations of the Alon conjecture to other classes of random graphs [FJR⁺96] or to other matrices associated to a graph [Pud12].

The recent result of [MSS13] shows that new eigenvalues of random coverings lead to d -regular Ramanujan graphs for all values of d . This gives stronger motivation to consider spectral aspects of random covering graphs.

Bibliography

- [ABG10] Louigi Addario-Berry and Simon Griffiths, *The spectrum of random lifts*, ArXiv e-prints **1012.4097** (2010).
- [Alo86] Noga Alon, *Eigenvalues and expanders*, *Combinatorica* **6** (1986), no. 2, 83--96.
- [Bas92] Hyman Bass, *The Ihara-Selberg zeta function of a tree lattice*, *International Journal of Mathematics* **3** (1992), no. 6, 717--797.
- [BS87] Andrei Broder and Eli Shamir, *On the second eigenvalue of random regular graphs*, 28th Annual Symposium on Foundations of Computer Science (1987), 286--294.
- [FJR⁺96] Joel Friedman, Antoine Joux, Yuval Roichman, Jacques Stern, and Jean-Pierre Tillich, *The action of a few random permutations on r -tuples and an application to cryptography*, STACS, 1996, pp. 357--386.
- [Fri91] Joel Friedman, *On the second eigenvalue and random walks in random d -regular graphs*, *Combinatorica* **6** (1991), no. 2, 331--362.
- [Fri93] _____, *Some geometric aspects of graphs and their eigenfunctions*, *Duke Mathematics Journal* **69** (1993), no. 3, 487--525.
- [Fri03] _____, *Relative expanders or weakly relative ramanujan graphs*, *Duke Mathematical Journal* **118** (2003), no. 1, 19--35.

- [Fri08] _____, *A proof of Alon's second eigenvalue conjecture and related problems*, *Memoirs of the American Mathematical Society* **195** (2008), no. 910.
- [HLW06] Shlomo Hoory, Nathan Linial, and Avi Wigderson, *Expander graphs and their applications*, *Bulletin of the American Mathematical Society* **43** (2006), no. 4, 439--561.
- [Iha66] Yasutaka Ihara, *On discrete subgroups of the two by two projective linear group over p -adic fields*, *Journal of the Mathematical Society of Japan* **18** (1966), 219--235.
- [KS00] Motoko Kotani and Toshikazu Sunada, *Zeta functions of finite graphs*, *Journal of Mathematical Sciences, The University of Tokyo* **7** (2000), no. 1, 7--25.
- [Lov93] László Lovász, *Random walks on graphs: A survey*, *Paul Erdős is Eighty* **2** (1993), 1--46.
- [LP10] Nati Linial and Doron Puder, *Word maps and spectra of random graph lifts*, *Random Structures and Algorithms* **37** (2010), no. 1, 100--135.
- [LPS88] Alexander Lubotzky, Ralph Phillips, and Peter Sarnak, *Ramanujan graphs*, *Combinatorica* **8** (1988), no. 3, 261--277.
- [LSV11] Eyal Lubetzky, Benny Sudakov, and Van Vu, *Spectra of lifted Ramanujan graphs*, *Advances in Mathematics* **227** (2011), no. 4, 1612--1645.
- [Mar88] Gregory Margulis, *Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators*, *Problems of Information Transmission* **24** (1988), no. 1, 39--46.
- [MNS08] Steven Miller, Tim Novikoff, and Anthony Sabelli, *The distribution of the largest nontrivial eigenvalues in families of random regular graphs*, *Experimental Mathematics* **17** (2008), no. 2, 231--244.

- [Mor94] Moshe Morgenstern, *Existence and explicit construction of $q+1$ regular ramanujan graphs for every prime power q* , Journal of Combinatorial Theory, Series B **62** (1994), no. 1, 44--62.
- [MSS13] Adam Marcus, Daniel Spielman, and Nikhil Srivastava, *Interlacing families I: Bipartite Ramanujan graphs of all degrees*, ArXiv e-prints **1304.4132** (2013).
- [Nil91] Alon Nilli, *On the second eigenvalue of a graph*, Discrete Mathematics **91** (1991), no. 2, 207--210.
- [Pud12] Doron Puder, *Expansion of random graphs: New proofs, new results*, ArXiv e-prints **1212.5216** (2012).
- [Sun86] Toshikazu Sunada, *L-functions in geometry and some applications*, Lecture Notes in Mathematics **1201** (1986), 266--284.